Swampland Program: Cobordism, tadpoles, and the dark dimension

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München 2023

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Dissertation
an der Fakultät der Physik
der Ludwig–Maximilians–Universität
München

vorgelegt von
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aus Thessaloniki

München, den 19. Juni 2023

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Tag der mündlichen Prüfung: 24. Juli 2023

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Abstract

A consistent gravitational effective field theory is not guaranteed to have a UV completion to Quantum Gravity. If it does, it belongs to the very restricted *Landscape*, while if it does not, it belongs to the *Swampland*. The Swampland Program aims at formulating an interconnected set of qualitative criteria, the *Swampland Conjectures*, which all theories in the Landscape should respect. The conjectures have profound implications for low-energy physics. String theory, as a theory of Quantum Gravity, is inherently related to the Swampland program. On the one hand, String Theory provides necessary evidence for formulating sensible conjectures. On the other hand, Swampland conjectures can uncover previously unidentified objects and consistency conditions in String Theory.

The Dark Dimension proposal approaches the Cosmological Constant problem with insights from the Swampland Program combined with observational data. It makes a concrete prediction that our universe should have one extra mesoscopic dimension of a few micrometers. We propose a first string realization for the Dark Dimension proposal, using a common feature of string compactifications in the presence of higher-form fluxes, a strongly warped throat.

The Cobordism Conjecture is related to the absence of global symmetries in Quantum Gravity and prohibits global cobordism charges. In practice, we start with a suitable approximation which admits such charges and then appropriately trivialize them, breaking or gauging the global symmetry. In the case of gauging, we study the interplay of K-theory and cobordism charges and how this behaves under dimensional reduction to give charge neutrality conditions, commonly known as tadpole cancelation conditions. In the context of breaking, we work in the geometric framework of dynamical cobordism, which relates dynamical tadpoles to cobordism. We identify a formerly unknown, cobordism-predicted 7-brane and provide its explicit description.

Zusammenfassung

Bei einer konsistenten Formulierung einer gravitativen effektiven Feldtheorie ist es nicht garantiert, dass sie eine UV-Vervollständigung zur Quantengravitation hat. Wenn dies der Fall ist, gehört sie zur sehr eingeschränkten *Landscape*, wenn nicht, gehört sie zum *Swampland*. Das Swampland-Programm zielt darauf ab, eine Reihe miteinander verbundener qualitativer Kriterien zu formulieren, die Swampland-Vermutungen, die alle Theorien in der Landscape erfüllen sollten. Die Vermutungen haben tiefgreifende Auswirkungen auf die Niederenergiephysik. Die Stringtheorie als eine Theorie der Quantengravitation ist von Natur aus mit dem Swampland-Programm verbunden. Einerseits liefert die Stringtheorie die notwendigen Hinweise für die Formulierung sinnvoller Vermutungen. Andererseits können Swampland-Vermutungen bisher unentdeckte Objekte und Konsistenzbedingungen in der Stringtheorie aufdecken.

Das Konzept der Dunklen Dimension begegnet dem Problem der kosmologischen Konstante mit Erkenntnissen aus dem Swampland-Programm in Kombination mit Beobachtungsdaten. Es führt zu einer konkreten Vorhersage, dass unser Universum eine zusätzliche mesoskopische Dimension von einigen Mikrometern haben sollte. Wir schlagen eine erste String-Realisierung für den Vorschlag der Dunklen Dimension vor, indem wir ein gemeinsames Merkmal von String-Kompaktifizierungen in Gegenwart von Flüssen höherer Formen, eine *Warped Throat* Geometrie, nutzen.

Die Kobordismus-Vermutung hängt mit der Abwesenheit globaler Symmetrien in der Quantengravitation zusammen und verbietet globale Kobordismus-Ladungen. In der Praxis beginnen wir mit einer geeigneten Annäherung, die solche Ladungen zulässt und trivialisieren sie dann in geeigneter Weise, indem wir die globale Symmetrie brechen oder eichen. Im Falle der Eichung untersuchen wir das Zusammenspiel von K-Theorie und Kobordismus-Ladungen und wie es sich bei einer Dimensionsreduktion verhält, um Ladungsneutralitätsbedingungen zu erhalten, die allgemein als *Tadpole-Cancellation*-Bedingungen bekannt sind. Im Kontext der Symmetriebrechung arbeiten wir im Rahmen des dynamischen Kobordismus, der *Dynamical Tadpoles* mit dem Kobordismustheorie in Beziehung setzt. Wir identifizieren eine bisher unbekannte, durch Kobordismus vorhergesagte 7-Brane und beschreiben sie explizit.

Chapter 1

Introduction

1.1 The need for Quantum Gravity

The beginning of the previous century was marked by the development of groundbreaking theories: Quantum Mechanics [1], Special Relativity [2], and General Relativity (GR) [3]. Special Relativity and Quantum Mechanics were soon made compatible in the framework of Quantum Field Theory, and the Standard Model (SM) has proven to be exceptionally successful in explaining and unifying electromagnetism, weak and strong fundamental interactions. However, any attempt to directly include gravity in a perturbative QFT framework breaks down at the Planck scale $M_p \simeq 10^{19}\ GeV$ due to the presence of irrelevant operators that render the theory non-renormalizable [4].

One might wonder why there is a need for such a unified theory of Quantum Gravity. Apart from the continuous strive in theoretical physics for identifying the bigger, more complete picture, we have clear indications that the Standard Model or General Relativity alone do not suffice to explain all observations. The observed positive and small value of the cosmological constant [5] is such a puzzle.

Semiclassical gravity is an approximation that treats matter as quantum fields, while spacetime is regarded as classical. Non-renormalization issues are avoided by truncating the theory at the one-loop level. Despite not being a full theory of Quantum Gravity, semiclassical gravity has led to important insights regarding cases where the quantum nature of gravity becomes significant. These include cosmological perturbations [6] and black hole physics.

Let us elaborate on the latter, focusing on four spacetime dimensions. Black holes arise as solutions to the GR equations of motion, parametrized by their mass, angular momentum, and charge. Black holes have been observed recently through gravitational waves [7, 8] and direct imaging [9, 10]. Due to their large sizes, these observed black holes are likely of astrophysical origin, yet primordial black holes [11,12], i.e., black holes created in the very early universe due to fluctuations, have not been excluded [13, 14].

The radius r_H of a black hole of mass M is $r_H \sim l_p M/M_p$, with $l_p \simeq 8 \cdot 10^{-35}$ m the Planck length and M_p the Planck mass. For any black hole of at least a few solar masses, the curvature close to the horizon $\mathcal{R} \sim 1/r_H^2$ is very small. Hence semiclassical gravity is a good approximation. Work by Bekenstein and Hawking in this framework leads to a groundbreaking thermodynamical description of black holes [12, 15–18], assigning to them the so-called *Bekenstein-Hawking* entropy

$$S_{BH} = \frac{A}{4G}, \tag{1.1}$$

where $A \sim (^M/_{M_p})^2 l_p^2$ is the surface area of the black hole. At this point, we can already see that semiclassical gravity is a useful, yet incomplete, approach to Quantum Gravity. S_{BH} signals that the black hole should be comprised of adequately many microstates, but semiclassical gravity does not provide such a description. Luckily, we have a better option at hand.

1.2 String Theory as a theory of Quantum Gravity

The fundamental degree of freedom in perturbative string theory is a spatially extended one-dimensional string moving through a d-dimensional spacetime, delineating a two-dimensional world-sheet. The action is given in analogy to the length of the world-line for the point particle, as the area of the world-sheet. We distinguish between closed and open strings depending on whether the endpoints of the string are identified or not. This is a remarkably meaningful boundary condition, as it influences the spectrum of fields described by the oscillation modes of each string. What is of interest to us is that the quantized closed string spectrum includes a massless spin-two mode, while the open string spectrum contains a massless spin-one mode.

Historically, it was first noticed that the open string spin-one mode interacted according to Yang-Mills theory in a specific low-energy limit [19]. Soon after, it was realized by Scherk and Schwarz [20] that the massless closed string spin-two mode was indeed a graviton since, at low energies, its interactions precisely reproduced GR. Combining these two facts, the authors proposed that string theory should be interpreted as the "quantum theory of gravity, unified with the other forces".

There are some undisputable arguments in favor of string theory as the theory of Quantum Gravity, apart from the coexistence of GR and Yang-Mills theory in the same framework. In particular, the issue of the non-renormalizability of GR is resolved; in string theory, there are no point interactions. The interactions now are described by the joining and splitting of strings, and the Feynman diagrams schematically transform as in figure 1.1. In general, for the scattering of n closed strings, the world-sheet topologically becomes some compact Riemann manifold of genus g, with n infinite cylinders glued to it. The corresponding diagrams are (UV-)finite for each loop order, hence renormalized [21].

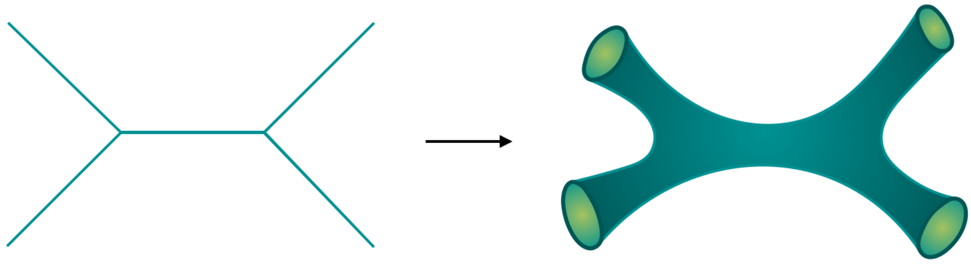

Figure 1.1: Sketch of field theory (left) versus string theory (right) scattering process.

String theory has also been remarkably successful in tackling black hole physics. The Bekenstein-Hawking entropy has been reproduced in the seminal paper of Strominger and Vafa [22], where an explicit counting of microstates for certain supersymmetric black holes in terms of BPS solitons was performed.

1.3 From the Landscape to the Swampland

The landscape . . .

String theory has to satisfy several non-trivial checks to be consistent. We will briefly review some of them in the next chapters 2 and 3, but let us already sketch the picture at this point.

The simplest string theory is the bosonic string theory. A plethora of conditions need to be imposed to quantize the theory, ranging from the closure of the quantum algebra to the vanishing of the Weyl anomaly, and they all unequivocally necessitate that bosonic string theory lives in $d = 26$ dimensions. Setting aside the extra dimensions, attempts for sensible phenomenological descriptions using just the bosonic string are plagued by two main problems: firstly, the presence of a closed string-tachyon, signaling some important instability, and secondly, the lack of any fermions in the spectrum.

Both these issues can successfully be treated with the introduction of fermions in the world-sheet action. Typically this happens in a supersymmetric fashion, where one adds terms in the world-sheet action so that it is invariant under supersymmetry (SUSY) transformations. This time, tachyonic modes can be removed through some appropriate projection, which simultaneously may render the spacetime spectrum supersymmetric. The consistency of the superstring imposes $d = 10$.

Customarily, one tries to construct a four-dimensional effective field theory by "hiding away" the extra dimensions through compactification. The specifics of this process, like the shape and size of the compactification manifold or fluxes going through it, leave a strong imprint on the four-dimensional physics and are responsible for the physical properties of the EFT, such as the amount of supersymmetry, couplings, and masses of fields. Each different set of choices gives a different EFT, and all these EFT together form the *String Landscape*. Depending on the starting framework, the estimations for the size of the landscape range from 10^{500} [23] to $10^{272.000}$ [24]. Perhaps even more impressively, very large numbers of models with MSSM-like gauge groups have been identified [25], or, more recently, even the exact chiral spectrum of the Standard Model was found in a quadrillion models [26].

The sheer size of the landscape initially led to the idea that *any* consistent quantum gauge theory can be found somewhere in the string landscape - the issue was effectively weeding out the physically non-interesting models until the phenomenologically relevant could be identified.

. . . and the Swampland

A radical change of perspective was initiated in 2005 [27]. The fundamental idea was that not all consistent-looking low-energy effective field theories can be UV-completed to Quantum Gravity. Those that can form the *Landscape*, while the much broader set of those that cannot form the *Swampland*. The Swampland Program [27] aims at distinguishing these two sets of theories via *swampland conjectures*, i.e., quantitative criteria that only theories in the landscape satisfy, with far-reaching implications.

For instance, two conjectures with striking cosmological implications are the (refined) de Sitter Swampland Conjecture (dSC) [28, 29] and the Trans-Planckian Censorship Conjecture (TCC) [30]. The dSC is mainly motivated by the lack of well-controlled dS constructions in string theory and proposes that no de Sitter extrema (resp. minima) are allowed in an EFT that can be UV-completed to QG. The TCC states that no subplanckian modes may stretch enough to become transplackian and allows for dS minima if sufficiently short-lived. These conjectures can contradict certain models that describe the accelerated expansion of the universe, such as slow-roll single-field inflation [31, 32].

The Swampland Program goes far beyond merely formulating conjectures. It encompasses a vibrant research activity, including sharpening the conjectures, uncovering hidden interrelations, and studying their implications. Perhaps even more importantly, the Swampland Program aims to bring us closer to the true nature of Quantum Gravity.

1.4 Outline of the thesis

This thesis aims to showcase how the Swampland Program can help us better understand Quantum Gravity. For this reason, the main part of this thesis will focus on three papers that concern different corners of the Program. Without entering any details, let us briefly sketch the areas of interest for the rest of this thesis. In chapter 5, we will approach the phenomenological side of the Swampland, discussing the string theoretical realization of the Dark Dimension proposal [33], which uses insights from the Swampland and string theory to interpret observational and experimental data, leading to a potentially experimentally testable proposal for the existence of an extra dimension of mesoscopic size. In chapter 6, motivated by the conjectured absence of (a certain topological version of) global symmetries in Quantum Gravity, we perform a formal analysis that explores the mathematical structure of string theory and its imprint on physics. Finally, in chapter 7, we study a certain non-supersymmetric version of string theory, where, using a Swampland Conjecture as a guiding principle, we find evidence for a new, formerly unknown object.

The material is organized in the remaining chapters as follows:

Chapter 2 is a concise introduction to the superstring theories used throughout this work. We discuss the 10d massless bosonic spectrum; we take a first look at the simplest compactification example. We discuss D-branes, their classification schemes, and the effective theories for the bosonic fields and the localized objects. Experts may safely skip this chapter, yet its inclusion is deemed necessary both for completeness and for setting the conventions used in the remainder of the thesis. The only part of the chapter that goes beyond standard textbook-level material is the D-brane and K-theory section 2.5.

Chapter 3 concerns compactifications to lower dimensions and, in particular, $d = 4$. We start by introducing compactification manifolds and their necessary properties, and we present the 4d supergravity spectra. The complicated issue of moduli stabilization will be discussed, both at tree-level and beyond, and two attempts at constructing de Sitter vacua in string theory are outlined. Chapter 2 and 3 are complementary to each other and serve to illustrate how the Landscape of string theory is populated and, at the same time, that there are serious constraints in string model building that stem from the quantum nature of gravity.

Chapter 4 aims to familiarize the reader with the fundamental ideas of the Swampland Program and the Conjectures that will become relevant in the main part of the thesis. In particular, we will discuss the No Global Symmetries Conjecture and the related Cobordism Conjecture, which will play a significant role in later chapters. Moreover, we will introduce the conjectures and concepts necessary for formulating the Dark Dimension proposal, mostly focusing on the Distance Conjecture and some of its implications.

Chapters 5, 6, and 7 concern research conducted by the author and collaborators. The three works presented here vary in their goals, methods, and level of formality, but all fit nicely within the Swampland Program.

Chapter 5 is our most phenomenologically oriented chapter and focuses on a very exciting recent development in the Swampland Program, the Dark Dimension proposal [33], which makes use of swampland ideas and experimental data to confront the cosmological constant problem, making a concrete, experimentally verifiable prediction. Our work was the first attempt at a string realization of the proposal, using certain features of strongly warped compactifications. We point out difficulties in rendering such a realization compatible with observational data.

Chapters 6 and 7 aim at a better understanding of the trivialization of global symmetries in string theory. Both examine the Cobordism Conjecture [34], which reflects the *uniqueness* of Quantum Gravity. These two chapters focus on the two complementary ways of trivializing global charges, gauging and breaking, respectively.

In particular, in chapter 6, which is the most formal part of the thesis, we use the Cobordism Conjecture and a proposed correspondence between cobordism and K-theory [35], that K-theory and cobordism charges should combine to give tadpole cancelation conditions. We examine whether this correspondence persists when considering nontrivial manifolds relevant for dimensional reduction. Section 6.4 is devoted to computing the K-theory and cobordism groups of such manifolds, using the Atiyah-Hirzebruch spectral sequence, while we finish the chapter with the physical interpretation of the results in section 6.5. The upshot of this chapter is that not only does the cobordism conjecture hold in this generalized setup, but also the proposed correspondence to K-theory behaves as required under dimensional reduction.

In chapter 7, we use the Dynamical Cobordism framework of [36, 37] to study a particular example of a domain wall in a non-supersymmetric string theory. The backreacted spacetime solution has a finite spatial extent, and cobordism-breaking defects are expected to generate this behavior. We provide an explicit description of these defects and realize that these are novel, previously unidentified items in string theory, showcasing that Swampland ideas point to fruitful unexplored territory within string theory.

Finally, some concluding thoughts are presented in chapter 8.

Relevant Publications

Chapters 5, 6, and 7 of this thesis rely heavily on work published in the following papers:

- **The Dark Dimension in a Warped Throat** [38]
R. Blumenhagen, M. Brinkmann, A. Makridou.
ArXiV:2208.01057, published in: Phys.Lett.B 838 (2023) 137699.

- **Dimensional Reduction of Cobordism and K-theory** [39]
R. Blumenhagen, N. Cribiori, C. Kneissl, A. Makridou.

ArXiV:2208.01656, published in: JHEP 03 (2023) 181

- **Dynamical cobordism of a domain wall and its companion defect 7-brane** [40]
R. Blumenhagen, N. Cribiori, C. Kneissl, A. Makridou.
ArXiV:2205.09782, published in: JHEP 08 (2022) 204.

The following papers were also published during the author's doctoral studies but are not included in the present thesis:

- **Swampland conjectures for an almost topological gravity theory** [41]
R. Álvarez-García, R. Blumenhagen, C. Kneissl, A. Makridou, L. Schlechter.
ArXiV:2107.07546, published in: Phys.Lett.B 825 (2022) 136861.

- **dS quantum breaking, swampland conjectures and thermal strings** [42]
R. Blumenhagen, C. Kneissl, A. Makridou.
ArXiV:2011.13956, published in: JHEP 10 (2021) 157.

Chapter 2

A Superstring Theory Primer

The present chapter summarizes the necessary background material on which concepts of the following chapters will be based and sets our conventions. The reader familiar with superstring theory may safely skip this chapter, as the level will mostly not exceed standard textbook material, except possibly for the K-theory classification of D-branes. The treatment here mainly follows the lines of the textbook [43], while [21, 44–47] have also been useful.

It is beyond the scope of this thesis to provide a full introduction to string theory, so a certain level of familiarity with basic string and conformal field theory concepts, such as modular and conformal invariance, is assumed.

2.1 The different superstring theories

There exists a plethora of conditions leading to a particular effective description of string theory. Some of them are intrinsically related to the consistency of the theory, either at the classical or quantum level. Such a requirement is the *modular invariance* of the theory/partition function. On a similar footing, we also have anomaly cancellation - this is also related to the so-called *tadpole cancellation.* The consistency of theory is also intertwined with conditions often required for gauge theories, such as unitarity. As the totality of these necessary consistency conditions is considered, one is forced to define string theories that can only live in a certain number of spacetime dimensions. On the other hand, one can go beyond simply requiring a *consistent* theory and try to find a *phenomenologically relevant* theory. In this endeavor, among the usually desired properties of the theory, one can find the presence of chiral fermions, (partial) supersymmetry breaking, a gauge group that is compatible with the Standard Model, and of course the existence of only four (non-compact) spacetime dimensions. It is unclear how all these “optional” requirements can be simultaneously fulfilled, i.e., how to realize every aspect of the Standard Model is unknown. However, a clear roadmap exists that ensures the fulfillment of all the necessary conditions on which more involved constructions are based.

Type	Sector	Spectrum
IIA	NS-NS	G_{MN}, B_{MN}, ϕ
	R-R	C_1, C_3
IIB	NS-NS	G_{MN}, B_{MN}, ϕ
	R-R	$C_4^\dagger, C_2, C_0$
Het($E_8 \times E_8$)		G_{MN}, B_{MN}, ϕ
		C_1 in adj. of $E_8 \times E_8$
Het($SO(32)$)		G_{MN}, B_{MN}, ϕ
		C_1 in adj. of $SO(32)$
I	NS-NS	G_{MN}, ϕ
	R-R	C_2
	open	A_M in adj. of $SO(32)$

Table 2.1: Massless bosonic spectrum of the five superstring theories. The indices in the higher-form fields in the R-R sector have been suppressed, and $C_4^\dagger$ has a self-dual field strength.

In this (and the following) chapter, we will sketch the most significant steps in this procedure. Our starting point will be critical superstring theory, defined consistently in 10-dimensional spacetime. Five distinct supersymmetric theories exist, differing in the amount of supersymmetry (i.e., the number of supercharges) they possess and the properties of their spectrum. The theories with $\mathcal{N} = 2$ supersymmetry and 32 supercharges in 10d are the IIA and IIB theories, while type I and the two heterotic theories Het($E_8 \times E_8$), Het($SO(32)$), with gauge groups $E_8 \times E_8$ and $SO(32)$, only preserve 16 supercharges. We summarize the massless bosonic spectrum of these theories, split into the R-R and NS-NS sectors[1], in table 2.1. As this thesis does not involve heterotic constructions, we mostly focus on type I and II cases and refer the interested reader to the textbooks [21, 43–47].

[1]How these sectors (and their naming conventions) arise will be discussed in subsequent sections 2.2.1, 2.2.2.

2.2 Towards the consistent spectrum

This section reviews two basic concepts that enter a string theoretical construction. Given our later focus on D-branes, we start by reviewing the world-sheet boundary conditions in section 2.2.1, while the GSO projection in section 2.2.2 serves as a starting point for our discussion of the Sugimoto model in 7.3.1.

2.2.1 Boundary conditions

The possible world-sheet boundary conditions (BC) follow from the variations of the world-sheet action with respect to the fields. Whether dealing with open or closed string degrees of freedom also plays a role. Parametrizing the world-sheet Σ by the coordinates (τ, σ), one gets the following list of boundary conditions:

- For the closed string bosons X^M, with $M = 0, \ldots, d-1$, where $0 \leq \sigma \leq 2\pi$, $-\infty < \tau < \infty$ and with periodicity condition $X^M(\tau, \sigma) = X^M(\tau, \sigma + 2\pi)$, the world-sheet has the topology of a cylinder. Due to the identification of the ends of the string, the two surface-term contributions to the variation of the action cancel out and no additional condition needs to be imposed.

- For the open string, now $0 \leq \sigma \leq \pi$, so the boundary of the strip-shaped world-sheet has two disjoint components $\partial\Sigma_1$, $\partial\Sigma_2$ at $\sigma = 0, \pi$ respectively. The vanishing of the boundary terms can be achieved by imposing the following BC:

  - *Dirichlet (D) BC*: $\partial_\tau X^M\big|_{\partial\Sigma_i} = 0, \quad i = 1, 2.$
  - *Neumann (N) BC*: $\partial_\sigma X^M\big|_{\partial\Sigma_i} = 0, \quad i = 1, 2.$

  Either of these conditions can be independently imposed at each endpoint, so, in reality, there are four possible combinations of boundary conditions: (NN), (ND), (DN), (DD). Moreover, there is no need to impose the same BC in all the spacetime directions.

  The physical meaning of the two types of BC is clear: an endpoint with a Dirichlet boundary condition is fixed, selecting a privileged point and hence breaking Poincaré invariance. On the other hand, a Neumann boundary condition signifies that no momentum flow is allowed at the end of the string.

At this point, we are ready to introduce the so-called *D-branes*. Consider a configuration in a d-dimensional theory, where $(p+1)$ spacetime dimensions $X^m, m = 0, \ldots, p$, have NN boundary conditions, while the remaining $(d-p-1)$ directions $X^\alpha, \alpha = p+1, \ldots, d-1$, obey DD boundary conditions. The open string endpoints are then fixed in the DD spacetime directions but can move freely in the (p+1) NN directions. Hence we have a (p+1)-dimensional hypersurface on which the string endpoints move. This hypersurface actually defines a Dp-brane, with D standing for Dirichlet. We will discuss D-branes in more detail in section 2.4.

Consistently introducing the world-sheet fermions $\psi^M = \binom{\psi_+^M}{\psi_-^M}$ to the action requires suitable boundary conditions. Since the fermionic part of the action, and hence the relevant equations of motion, are quadratic in the fermionic variations, the fermions can be either periodic or antiperiodic along the length of the string ($\sigma-$direction). Once again, we distinguish between the closed and open string cases, and the fermionic boundary conditions for the chiral fermion ψ_+ and the antichiral fermion ψ_- are as follows:

- For the closed string, the chiral and antichiral fermions do not mix. Hence we have the following:

  – *Ramond (R)* boundary conditions:

  $$\psi_+^M(\sigma) = +\psi_+^M(\sigma + 2\pi), \tag{2.1a}$$
  $$\psi_-^M(\sigma) = +\psi_-^M(\sigma + 2\pi), \tag{2.1b}$$

  – *Neveu-Schwarz (NS)* boundary conditions:

  $$\psi_+^M(\sigma) = -\psi_+^M(\sigma + 2\pi), \tag{2.2a}$$
  $$\psi_-^M(\sigma) = -\psi_-^M(\sigma + 2\pi). \tag{2.2b}$$

  These boundary conditions can be chosen independently for each spinorial component, so, in total, there are four possibilities: (R, R), (R, NS), (NS, R), (NS, NS). Poincaré invariance requires that *all* spacetime dimensions obey the same boundary condition.

- For the open string, there is mixing between the chiral and antichiral fermions, and the admissible boundary conditions are $\psi_+^M(\sigma)\big|_{\partial\Sigma_i} = \pm\psi_-^M(\sigma)\big|_{\partial\Sigma_i}$. One should also impose Neumann or Dirichlet boundary conditions on the fermions at each endpoint. Splitting the d spacetime indices $M = (\alpha, i)$ into $(p+1)$ Neumann directions with $\alpha = 0, \ldots, p$, and $(d-p-1)$ Dirichlet directions with $i = p+1, \ldots, d-1$, we define the matrix $D_{MN} = (\eta_{\alpha\beta}, -\delta_{ij})$. Using this auxiliary matrix, one can summarize the boundary conditions for the open string as

  $$\psi_+^M(0) = D^M{}_N \psi_-^N(0), \tag{2.3a}$$
  $$\psi_+^M(2\pi) = \eta D^M{}_N \psi_-^N(2\pi), \tag{2.3b}$$

  where the η-value in equation (2.3b) above encodes the fermionic boundary conditions:

  – *Ramond (R)* boundary conditions: $\eta = +1$,

  – *Neveu-Schwarz (NS)* boundary conditions: $\eta = -1$.

Implications

Starting from a (supersymmetric) string action, we can schematically summarize the computation of the quantized spectrum in the following steps: Solution of equations of motion and selection of BC, mode expansion of the fields, construction of the creation/annihilation operators and the (super-)Virasoro generators, determination of the ground states, generation of the full Hilbert space by acting on ground states with creation operators and imposing additional physical conditions.

During this quantization procedure, the fermionic boundary conditions become particularly important, as both the value of the ordering constant in the super-Virasoro algebra and the modes entering the mode expansion of the fermionic fields are different for the NS and R sectors. These dissimilarities between the two sectors translate into a major difference in the respective vacuum representations. In particular, the Ramond sector ground state $|0\rangle_R$ is degenerate and turns out to be a representation of the Clifford algebra. When constructing the open string spectrum, states in the NS sector are spacetime bosons, while R states correspond to spacetime fermions. The closed string spectrum is constructed analogously, considering two copies of the open string spectrum and tensoring them. In that case, states in the (R, R) and (NS, NS) sectors are spacetime bosons, while states in the mixed sectors (NS, R) and (R, NS) correspond to spacetime fermions.

2.2.2 GSO Projection

At this point, one can straightforwardly compute the masses of the lowest-lying states. However, the spectra initially constructed this way are pathological due to violating modular invariance and including tachyons. Fortunately, there is a way to tackle these problems systematically, called the Gliozzi-Scherk-Olive (GSO) projection [48, 49]. To perform this projection, which truncates the spectrum and makes it spacetime supersymmetric, one relies on the *world-sheet fermion number* F operator. More specifically, one uses the operator $(-1)^F$, which effectively counts whether the number of fermions in the world-sheet is even or odd.

In practice, the GSO projection selects the allowed eigenvalue of $(-1)^F$ in each state. For the NS sector of the open string, the physically admissible states are those with $(-1)^F = +1$ so that the tachyon is automatically removed. However, in the R sector, there are two consistent choices, namely $(-1)^F = +1$ and $(-1)^F = -1$, giving rise to states of different chiralities.

For the closed string, the spectrum arises by taking the tensor product of two open string spectra. Now, instead of only having to fix the eigenvalues of $(-1)^F$, we also need to fix the eigenvalues of $(-1)^{\bar{F}}$, i.e., we treat the left- and right-movers separately. The NS states have all $(-1)^F = (-1)^{\bar{F}} = +1$, while for the R states the eigenvalues can be either ± 1, so two inequivalent options arise: $(-1)^F = (-1)^{\bar{F}}$ and $(-1)^F = -(-1)^{\bar{F}}$. The former option gives rise to Type IIB superstring theory and the latter to Type IIA superstring theory.

2.3 A few words on dualities

Dualities are among the most striking features of string theory and embody the fact that seemingly different theories can be viewed as just distinct incarnations of the same underlying theory in different regions of the moduli space. Dualities often become evident upon compactifying some spacetime dimensions. We will discuss compactifications down to four dimensions in more detail in chapter 3, but here we illustrate some basic concepts by considering the simplest possible example, the compactification of just a boson on a circle S^1 of radius R.

2.3.1 Compactification on S^1

Suppose we have the closed string bosonic fields X^M, $M = 0, \ldots, d-1$ and consider a compactification on a circle of radius R along the last spatial direction. For the first $d-1$ directions we have $X^m(\sigma + 2\pi, \tau) = X^m(\sigma, \tau), m = 0, \ldots, d-2$, while for the $(d-1)$-direction the string may also wrap the circle w times, so we have the less restrictive periodicity condition $X^{d-1}(\sigma + 2\pi, \tau) = X^{d-1}(\sigma, \tau) + 2\pi R w$, where $w \in \mathbb{Z}$ is called *winding number*. The spacetime momentum along the circle, p^{d-1}, also called internal momentum, now has to be quantized, i.e., $p^{d-1} = \frac{n}{R}$ with $n \in \mathbb{Z}$, to ensure the wavefunction remains single-valued.

The compactification has many profound implications for the theory. It is beyond the scope of the thesis to delve into such details, and we refer to the textbooks cited at the beginning of this chapter for more insights. Here, we mention two very basic facts. First, additional states appear in the theory, purely due to the compactification: in our specific case, the Kaluza-Klein reduction (see for example [50]) gives two new massless vectors, giving rise to a $U(1)_L \times U(1)_R$ symmetry, and a new scalar related to the radius R, which is a *modulus* of the compactification. Moreover, in the first winding sector $n = w = \pm 1$ and satisfying the level-matching constraint, we get additional states with masses related to the compactification radius as

$$\alpha' m^2(R) = \frac{\alpha'}{R^2} + \frac{R^2}{\alpha'} - 2. \tag{2.4}$$

There is a radius of special importance, $R = \sqrt{\alpha'}$, the so-called *self-dual radius*. At this point, states coming from the non-trivial winding sector become massless. In particular, we have four new massless vectors and eight new massless scalars, and the gauge symmetry in the bosonic string case is enhanced to $SU(2)_L \times SU(2)_R$.

2.3.2 Compactification on $S^1/\mathbb{Z}_2$

A straightforward way to reduce the amount of supersymmetry in the compactified theory is by using orbifolds, i.e., spaces of singular geometry, arising as quotients of manifolds by discrete groups. The orbifold action kills some supercharges so that the lower-dimensional effective field theory has the desired amount of supersymmetry. Here we illustrate some of the key features by sketching the case of the simplest example: Compactifying a free boson X on a $\mathbb{Z}_2$-orbifold of the circle.

Consider a free boson $X(\sigma, \tau)$ compactified on S^1, and a $\mathbb{Z}_2$ symmetry acting by

$$I_1 : X(\sigma, \tau) \rightarrow -X(\sigma, \tau). \tag{2.5}$$

Gauging the discrete symmetry and taking the quotient, one identifies the fields $X(\sigma, \tau)$ and $-X(\sigma, \tau)$. This can alternatively be viewed as compactifying on the $\mathbb{Z}_2-$orbifold of the circle, which is a line segment with two fixed points at its ends. This action eliminates the states of the Hilbert space which are not invariant under I_1. However, it turns out that the one-loop partition function constructed out of the remaining I_1-invariant states is not modular invariant. To fix this, one should include the *twisted sector*, which includes new states, localized at the fixed points of the orbifold, contributing to the partition function exactly in the way required to restore modular invariance. Moreover, since there are two fixed points under the projection, the twisted sector states have a two-fold degeneracy. This is a generic fact that will also persist in more involved orbifold or orientifold compactifications.

2.3.3 T-duality

For the circle compactification, as is evident from equation (2.4), the mass spectrum stays invariant under the transformation $R \rightarrow \frac{\alpha'}{R}$, with the simultaneous exchange of the momentum and winding mode quantum numbers $w \leftrightarrow n$. This is a particular incarnation of the famous *T-duality* (see [51,52] for reviews), with the self-dual radius being the fixed point of the transformation. This shows that the inequivalent circle compactifications are those with either $0 < R \leq \sqrt{\alpha'}$, or, alternatively, with $\sqrt{\alpha'} \leq R$.

Going one step further, one should look into how the mode expansions transform under T-duality and generalize this to the superstring. This happens in a systematic way, which we summarize below: For the bosonic modes, the left-moving sector remains invariant, while the right-moving one picks up a minus sign: $(X_L, X_R) \rightarrow (X_L, -X_R)$. Supersymmetry forces the world-sheet fermions to transform similarly as $(\psi_-, \psi_+) \rightarrow (\psi_-, -\psi_+)$. This directly relates to the GSO projection since for T-duality along a single circle the operator $(-1)^F$ picks up an extra sign. The GSO projections for type IIA and type IIB theories are now exchanged, so we have that type IIB compactified on a S^1 of radius R is T-dual to type IIA on S^1 of radius $\frac{\alpha'}{R}$. This can be generalized to a higher number of compact dimensions; The two type II superstrings keep getting exchanged under an odd number of T-dualities, while under an even number of T-dualized dimensions they are mapped to themselves. However, one should keep in mind that when compactifying on manifolds more complicated than S^1 additional consistency conditions arise. For instance, for T^D, only even self-dual momentum lattices can occur. Finally, as a nod to section 2.4, we note that the open string boundary conditions are altered under T-duality. In particular, Dirichlet boundary conditions turn into Neumann boundary conditions and vice versa.

2.3.4 The bigger picture: Duality web and M-theory

Our discussion of T-duality has made clear that a duality is some exact quantum equivalence between two theories that initially seem to be different, by having, for instance, different fundamental degrees of freedom, but in reality, are two incarnations of the same theory in different regimes of the moduli space. Moreover, some clear prescription for transitioning back and forth between the two duality frames should exist.

Through a non-trivial web of dualities, it can be shown that the five superstring theories introduced in section 2.1 are not stand-alone theories but rather form nodes of a so-called duality web, pictured in figure 2.1. Notice that this figure, apart from the five superstring theories, also includes 11-dimensional supergravity. Extensive reviews of the web of string dualities can be found, for instance, in [53–56], while here we only provide a very brief overview.

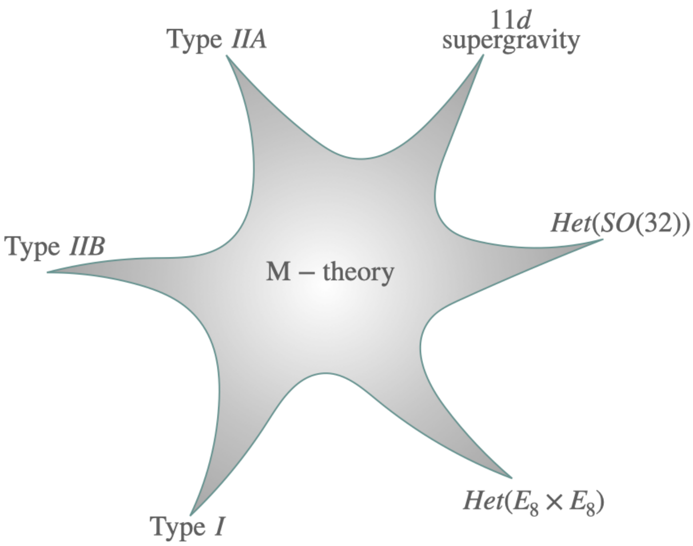

Figure 2.1: Web of superstring theories and M-theory.

T-duality is a perturbative duality, i.e., it holds order-by-order in string perturbation theory. Apart from the two type II theories being T-dual to each other, it turns out that also the two heterotic theories are T-dual upon compactification on a circle, hence related at the perturbative level. Additional connections between theories can be uncovered upon taking into account non-perturbative dualities, which map weakly-coupled theories to strongly-coupled dual theories and vice versa. S-duality is such a non-perturbative duality. Type IIB theory turns out to be self-dual under its action, while type I is S-dual to Het($SO(32)$).

M-theory

Let us step back and assume an 11-dimensional $N = 1$ theory exists. Then, type IIA string theory, as a 10-dimensional non-chiral $N = 2$ theory, can be viewed as the compactification of the postulated 11d theory on a circle. This 11d theory is the famous, yet elusive, *M-theory*. The lowest-lying bosonic states of M-theory are known and form the spectrum of 11d supergravity, which appears already in figure 2.1. They are the metric $G_{\mu\nu}$ and a 3-form field C_3. A 2-brane, called M2-brane, charged under C_3. After the compactification, one additionally gets the scalar modulus corresponding to the radius of the circle, as well as a vector and a 2-form. This is precisely the massless bosonic content of type IIA. Interestingly, the IIA coupling constant relates to the size of the circle, and in particular, the strong coupling limit in IIA corresponds to the circle decompactifying and the 11^{th} dimension opening up. Moreover, D0-branes in type IIA are KK-modes of M-theory along the circle. Similarly, Het($E_8 \times E_8$) can be related to M-theory compactified on a circle modded out by reflection.

F-theory

Since the relation of type IIA and heterotic theories to M-theory is established, one might wonder how type IIB can make contact with M-theory. The answer lies in *F-theory* [57], a non-perturbative formulation of type IIB theory, dual to M-theory.

F-theory provides a geometric interpretation of the $SL(2, \mathbb{Z})$ symmetry of type IIB by identifying it to the geometric symmetry of a torus. More specifically, F-theory is postulated to be a 12-dimensional theory, which, upon compactification on a torus with modulus τ, turns out to give type IIB. However, there are certain subtleties: the torus is not completely geometric since there is no modulus corresponding to its volume, so in a sense, the 12-dimensional theory is not truly physical, but it rather presents a convenient way to study compactifications of type IIB to lower dimensions. The correspondence of F-theory to M-theory also becomes clear upon compactification: M-theory compactified on an elliptically fibered manifold M is equivalent to F-theory compactified on $M \times S^1$.

F-theory has turned out to be a very powerful way to study non-perturbative string compactifications in their geometric regime. Many interesting results can be found in the reviews [58, 59]. For future use in chapter 6, let us mention that F-theory on an elliptically fibered Calabi-Yau fourfold is equivalent to type IIB compactified on the base of the fourfold.

2.4 Branes and basic properties

2.4.1 Branes as charged objects

Up to now, we have only considered D-branes as the hypersurfaces on which open strings end. In reality, they are much more than that, as they are dynamical objects that interact gravitationally and are charged under higher-form fields [60].

Let us consider a generic p-brane[2]. Using the same conventions as for Dp-branes in section 2.2.1, a p-brane is a $(p+1)$-dimensional object with world-volume Σ_{p+1}, where p directions are spatial. Hence this brane couples electrically to a $(p+1)$-form C_{p+1}. The associated field-strength is $F_{p+2} = dC_{p+1}$; hence the (electric) charge of the brane is given by integrating over a $(d-p-2)$-dimensional sphere, as

$$Q_e = \int_{S^{d-p-2}} \star F_{p+2}. \tag{2.6}$$

This is a straightforward generalization of Gauss's law: One recovers the usual point-particle case simply by taking $p = 0$.

Similarly to the usual 0-form gauge symmetry, the p-form gauge symmetry can also have objects magnetically charged under it. The dimensionality of these objects is $(d-p-3)$, where $(d-p-4)$ dimensions extend spatially - we hence have $(d-p-4)-$branes. To see this, it suffices to consider the Hodge dual to the field strength $\star F_{p+2} = \tilde{F}_{d-p-2}$, which in turn corresponds locally to the magnetic $(d-p-3)$-form potential via $\tilde{F}_{d-p-2} = d\tilde{A}_{d-p-3}$. The magnetic charge is now given as

$$Q_m = \int_{S^{p+2}} \star \tilde{F}_{d-p-2} = \int_{S^{p+2}} F_{p+2}. \tag{2.7}$$

Let us briefly summarize what higher-dimensional objects one has to consider for superstring theories. The dimensional analysis shows that 0-forms couple electrically to (-1)-branes and magnetically to 7-branes, 1-forms couple to electrically to 0-branes and magnetically to 6-branes, 2-forms couple electrically to strings and magnetically to 5-branes, 3-forms couple electrically to 2-branes and magnetically to 4-branes, and, finally, 4-forms couple both electrically and magnetically to 3-branes.

In type IIB we are dealing with the even Ramond-Ramond potentials, C_0, C_2, C_4, while type IIA includes the odd fields C_1, C_3. Moreover, both type II theories also include the NS-NS 2-form field B, projected out in type I.

[2]For the time being, we keep the type of the brane open: It could, for instance, be a D-brane or an NS-brane.

The NS-NS 2-form B couples to the string world-sheet, i.e., the fundamental string F1 is charged under B[3]. The magnetic dual to the fundamental string now couples to $\tilde{B}_{10-2-2} = \tilde{B}_6$, i.e., is a 5-brane. This object is conventionally called the NS5-brane.

Unlike the NS-NS gauge fields, the R-R gauge fields do not couple to the string world-sheet, but instead, they couple to higher-dimensional non-perturbative objects, i.e., the D-branes. Type IIA then includes D2-, D4-, D6- and D8-branes, while type IIB includes D(-1)-instantons and D1-, D3-, D5-, D7- and D9-branes. All the aforementioned D-branes are supersymmetric and referred to as *BPS D-branes*.

The type I superstring theory arises by projecting type IIB states onto those invariant under $\Omega : \sigma \to l - \sigma$, i.e., the states in the theory exhibit a world-sheet parity symmetry. This projects away the NS-NS two-form B_2 and the R-R C_4. The BPS spectrum of the type I theory then includes D1-, D5-, and D9-branes. In fact, the consistency of the theory requires including D-branes together with appropriate charge-canceling O-planes.

2.4.2 D-branes and T-duality

One may wonder what happens to D-branes under T-duality. As mentioned in section 2.2.1, T-duality exchanges the Dirichlet and Neumann boundary conditions along the direction it is performed. Hence, what happens to the D-brane depends on how it is spatially located. Let us explain how this works with a simple, concrete example, keeping in mind that the discussion generalizes in the expected fashion.

Suppose we initially consider a type IIB D3-brane and perform T-duality along the single dimension X^9. We expect the brane to transform into a Dp-brane with even p since these are the objects present in type IIA. Two distinct options exist regarding the placement of the D3 in spacetime, namely whether it extends along the T-dualized dimension X^9 or is localized in it. If the D3 extends along X^9, this direction initially has N boundary conditions that will become D boundary conditions. Hence the dimension of the T-dual brane will be reduced by 1, i.e., it is a D2-brane. Conversely, if the brane is initially transverse to X^9, the initial boundary condition is D and flips to N, so the T-dual brane will also extend along that direction. Hence, its dimensionality increases by 1 to a D4-brane.

2.4.3 D-branes and gauge bundles

As we have already seen, D-branes can be viewed as the submanifolds on which open strings end. One can consider several configurations, including stacks of branes, possibly parallel or intersecting. In the case of a stack of N coincident Dp-branes, the endpoints

[3]Here the charged object corresponds to $p = 1$ - it is common to refer to the object as "string" and not as "brane" in such cases. Similarly, for $p = -1$, the charged objects are often referred to as "instantons".

of the string may be in any of the N branes. This is directly reflected in the quantized spectrum of the string, which includes gluons that transform under a $U(N)$ symmetry. The way to quantify this is by using the so-called Chan-Paton factors [61], which practically are $N \times N$ matrices used as a basis in the string wavefunction expansion [62]. More importantly, the gluon configurations are described by a $U(N)$ gauge bundle E, which is a vector bundle that assigns a copy of $\mathbb{C}^N$ at each point of the worldvolume of the stack of the N Dp-branes. This gauge bundle that comes hand in hand with every D-brane will become very significant regarding the classification schemes of D-branes.

2.5 Brane and charge classification schemes

In this section, we will briefly review two classification schemes for D-branes[4], following [63, 64] to a large extent. The homology classification is the older, yet phenomenically simpler, scheme for brane classification. The more recently proposed K-theory classification, while slightly mathematically more involved, is suitable for addressing the shortcoming of the homology classification and, as we will extensively discuss in chapter 6, naturally relates to cobordism.

2.5.1 Homology classification

Consider type II superstring theory, and suppose the spacetime topology is $\mathbb{R}\times M^9$, where $\mathbb{R}$ corresponds to the non-compact time direction and M^9 is a compact manifold. This ansatz allows for an evolution of the metric with time yet disallows topology-changing processes. A p-brane will now extend along the time direction and wrap a p-dimensional submanifold of M, which determines the charge carried by the brane. According to [65], a brane, which carries a homology charge, is consistent if it does not have a boundary, i.e., the submanifold wrapped by the brane does not have a boundary. By definition, a p-submanifold without a boundary is a *p-cycle*. Moreover, the charge needs to be conserved under small, continuous deformations of the p-cycle, i.e., the charge for homotopic cycles is the same. At this point, one could think that a homotopic classification of charges might be appropriate.

However, it turns out that not all non-trivial homotopy classes correspond to conserved charges, i.e., branes initially wrapping those cycles can decay and are unstable. In particular, the p-cycles that correspond to unstable branes are precisely those which are themselves boundaries of $(p+1)$ submanifolds of M. This naturally leads to the *homology classification*, since the homology group of M $H_p(M)$ is defined as the group

[4]The classifications here have been proposed in several versions, regarding D-brane charges, D-brane trajectories, or RR-fields. In this section, for concreteness, we will mostly focus on the classification of D-brane charges.

of p-cycles modulo cycles which are boundaries of $(p+1)-$submanifolds in M.

Homology groups are finitely generated abelian groups. As such, they are sums of copies of the integers and copies of finite-order cyclic groups, according to

$$H_p(M) = b_p\mathbb{Z} \oplus_i k_i\mathbb{Z}_{p_i}, \tag{2.8}$$

where b_p is the pth Betti number (defined in appendix A). The first part of this sum is called the free part, while the second is the torsional part. A non-trivial homology group signals that branes can be charged under this group. However, consider $H_p = \mathbb{Z}_k$. Then, k-coincident p-branes would have a trivial charge, i.e., they can annihilate, which physically translates into the emission of radiation. These torsionally charged branes break all supersymmetry and are often called non-BPS branes.

There is one important subtlety one needs to take into account, and that is the coefficients of the homology group. In the last paragraphs, we implicitly took integer coefficients in the homology group, i.e., we discussed integral homology classes $H_p(M, \mathbb{Z})$. This can be generalized to homology groups with any abelian group G as the coefficients, i.e., $H_p(M, G)$. The choice of G relates to the theory we are dealing with: Integer coefficients become necessary whenever a Dirac quantization condition exists, like in quantum gauge theories. On the other hand, theories such as supergravities, which arise as the classical limits of superstring theories, do not require integral coefficients.

In particular, p-branes in supergravity are classified by real homology $H_p(M; \mathbb{R})$, i.e., homology with real coefficients. Real and integral cohomology are related by tensoring with $\mathbb{R}$, which kills the torsion part of the integral homology. Hence, the non-BPS branes, which carry only torsional charge, are unstable in the classical/supergravity limit.

Finally, let us remark that even though the homology classification has been very successful, there are some finer points it does not address satisfactorily. Some of the charges it classifies correspond to branes that are anomalous [66], hence not physically realized. Moreover, it also contains some charges that are not conserved [67]. Hence, the true group classifying the p-brane charges should schematically be the homology group minus the classes corresponding to anomalous branes, quotiented by the unstable branes. This leads to a K-theoretical classification of charges.

2.5.2 K-theory classification

Basics of K-theory

Let us provide some basic definitions regarding (topological) K-theory, mainly following the reviews [63, 64, 68], supplemented with material by Hatcher [69]. The material presented here is very much physics-oriented and motivated. For the mathematically inclined reader, we point to references such as [70] and the book by Atiyah [71].

For concreteness, the definitions below concern complex K-theory. KO-theory, also known as real K-theory, is defined similarly yet has slightly different properties and additional intricacies. We will comment on that in due time.

K-theory is a generalized cohomology theory classifying vector bundles over a space X. Consider a complex vector bundle E over the manifold X. Given a second bundle F over the same base manifold X, one can construct a new vector bundle $E \oplus F$ over X by taking the direct sum of the vector spaces fiberwise. Hence the "summation" of two vector bundles is straightforward. The "subtraction", which becomes relevant in string theory for brane/antibrane stacks, is more complicated.

Subtracting two bundles E, F should be equivalent to adding a third bundle G to both and then subtracting the direct sums. Denoting the subtraction operation by (E, F), this requirement translates to

$$(E, F) = (E \oplus G, F \oplus G). \tag{2.9}$$

Hence one can define addition and subtraction for pairs of complex vector bundles as follows:

$$\text{Addition} : (E, F) + (E', F') = (E \oplus E', F \oplus F'), \tag{2.10a}$$

$$\text{Subtraction} : (E, F) - (E', F') = (E \oplus F', F \oplus E'). \tag{2.10b}$$

Under the addition operation, the space of pairs of bundles (E, F) gains a group structure. The pair $(0, 0)$, 0 being the trivial bundle of rank 0, is the identity element, while the inverse of an element (E, F) is $(F, E) = -(E, F)$ since $(E, F) + (F, E) = (E \oplus F, F \oplus E) = (G, G) = (0, 0)$. The group defined above, using the equations 2.10, is the *K-theory of* X, denoted $K^0(X)$. The simplest choice is to set $X = pt$, which leads to $K^0(pt) = \mathbb{Z}$.

Up to now, no specifications on the dimensions of the bundle have been set. Imposing additionally that the ranks of the bundles in a pair are the same leads to the *reduced K-theory group* $\widetilde{K}(X)$. More specifically, consider the map

$$\varphi[(E, F)] = \text{rank}(E) - \text{rank}(F), \qquad (E, F) \in K(X)\,. \tag{2.11}$$

The reduced K-theory group is then defined as $\widetilde{K}(X) \equiv \ker \varphi$, where map φ is surjective and the short exact sequence

$$0 \to \widetilde{K}(X) \to K(X) \overset{\varphi}{\to} K(\text{pt}) \to 0, \tag{2.12}$$

is split. One then gets the Splitting Lemma for K-theory

$$K(X) = K(\text{pt}) \oplus \widetilde{K}(X). \tag{2.13}$$

Higher K-theory groups

One can define higher reduced K-theory groups using the reduced suspension Σ. In particular, the reduced higher K-theory groups are given by

$$\widetilde{K}^{-n}(X) = \widetilde{K}(\Sigma^n X)\,, \tag{2.14}$$

for $n \in \mathbb{Z}$, $n \geq 0$. The higher K-theory groups have all negative indices, so the coboundary maps increase dimension. Using the properties of the reduced suspension (see appendix C.4) and the known relation $K(X) = \widetilde{K}(X \sqcup \mathrm{pt})$ one can also define the higher unreduced groups of the point in terms of the reduced groups of spheres S^n:

$$\widetilde{K}(S^n) = \widetilde{K}(\Sigma^n S^0) = K^{-n}(\mathrm{pt}), \tag{2.15}$$

Plugging everything together, one can see that the splitting lemma also holds for the higher groups

$$K^{-n}(X) = K^{-n}(\mathrm{pt}) \oplus \widetilde{K}^{-n}(X). \tag{2.16}$$

Finally, the (higher) K-theory groups satisfy *Bott periodicity*

$$K^{-n}(X) = K^{-n+2}(X). \tag{2.17}$$

As we will see soon, the Bott periodicity is directly reflected in the spectrum of branes in string theory.

Real K-theory

Without entering any subtleties, let us mention that *real K-thory*, conventionally also known as *KO-theory*, is defined analogously to complex K-theory, with the notable difference that it now concerns real bundles. One can similarly define the reduced KO-theory groups and the Splitting Lemma still applies, while the reduced suspension once again defines the higher KO-theory groups. We summarize the relevant relationships below:

$$\begin{aligned} \widetilde{KO}^{-n}(X) &= \widetilde{KO}(\Sigma^n X), \\ \widetilde{KO}(S^n) &= \widetilde{KO}(\Sigma^n S^0) = KO^{-n}(\mathrm{pt}), \\ KO^{-n}(X) &= KO^{-n}(\mathrm{pt}) \oplus \widetilde{KO}^{-n}(X). \end{aligned} \tag{2.18}$$

Finally, we remark that Bott periodicity is also a property of KO-theory, but this time the period is different:

$$KO^{-n}(X) = KO^{-n+8}(X)\,. \tag{2.19}$$

Let us close this section by presenting the K- and KO-theory groups for $0 \leq n \leq 10$ in table 2.2, where Bott periodicity becomes evident.

n	0	1	2	3	4	5	6	7	8	9	10
$KO^{-n}(\mathrm{pt})$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$K^{-n}(\mathrm{pt})$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

Table 2.2: K- and KO-groups of the point up to $n = 10$.

K-theory and charges

Let us now sketch how the K-theory groups of table 2.2 correspond to physical objects, mainly following Witten's arguments in [72], which builds upon the Sen Conjecture [73] and the proposal by Minasian and Moore [74] that K-theory classifies R-R charges in type II theories.

Let us consider type II superstring as our working theory. The simplest case of interest is a p-brane and an anti-p-brane wrapped on the same submanifold W of the spacetime X. The system can be described by the quantized spectrum of the strings extending between the two branes, i.e., we have the following four options $(p,p),(\bar{p},\bar{p}),(\bar{p},p),(p,\bar{p})$. For the two strings starting and ending on the same object, i.e., the $(p,p),(\bar{p},\bar{p})$ cases, the usual GSO projection kills the NS sector tachyon. However, the mixed states $(\bar{p},p),(p,\bar{p})$ require the opposite GSO projection, and the tachyon survives. This signals that the system is unstable, and the brane/antibrane pair is expected to annihilate. This setup can be directly generalized to n p-branes and n anti-p-branes. The absence of D-brane charges for the total system imposes that both stacks carry the same (on a topological level) bundle E, and, once again, the $(p,\bar{p})$-sector tachyon is expected to cause the system to annihilate.

Focusing on the IIB case, we can take $p = 9$ and consider some configuration of $n\ D9/\bar{n}\ \overline{D9}$-branes. Requiring tadpole cancellation fixes $n = \bar{n}$, and we denote the $U(n)$ gauge bundle on the brane stack by E and the one on the antibrane stack by F. The system is now characterized by the pair (E, F)[5].

In order to classify the D-brane charge, we need to classify these pairs of bundles modding out the effects of relevant physical processes, i.e., brane-antibrane creation and annihilation. As we have just explained, a pair of n' 9-branes and $\bar{n}' = n'$ anti-9-branes carrying the same gauge bundle H are equivalent to the vacuum due to the presence of the tachyon. Hence, the pair $(E \oplus H, F \oplus H)$ should describe the same system as (E, F): This is the equivalence relation precisely defining K-theory. More precisely, since tadpole cancellation requires that the ranks of the two bundles E, F are the same, one reaches the conclusion that the tadpole-canceling $9-\bar{9}$ configurations are classified by reduced K-theory $\widetilde{K}(X)$, while we can more generally say that in general type IIB $9-\bar{9}$

[5]This discussion can become more accurate in the language of modules. Here for simplicity, we only mention bundles and we refer to [64] for a more mathematically precise description.

configurations are classified by K-theory $K(X)$[6].

We can now turn our attention to type I and IIA theories. For type I, the generalization is straightforward since the theory also admits spacetime-filling 9-branes. Now, a configuration with n 9-branes and $\bar{n}$ $\bar{9}$-branes requires $n - \bar{n} = 32$ for tadpole cancellation. Moreover, the stacks admit $SO(n)$ and $SO(\bar{n})$ gauge bundles, respectively. In general, $9 - \bar{9}$ configurations are classified by KO-theory $KO(X)$, while imposing tadpole cancellation leads to the reduced group $\widetilde{KO}(X)$.

The situation in type IIA is a bit more complicated since no spacetime-filling branes exist. However, this can be bypassed by considering bundles on $X \times S^1$. Now a type IIA p-brane with p even will wrap an odd-dimensional submanifold $Z \subset X$. This can be extended to $Z' = w \times Z \subset S^1 \times X$ where w is a point in S^1, which is of even codimension in $S^1 \times X$. Hence a brane wrapped on Z' determines an element of the higher K-theory group $K(S^1 \times X) = K^1(X)$, and, in general, D-brane charges of type IIA are classified by $K^{-1}(X)$.

Finally, one needs to consider branes of higher codimensions. To this end, Sen's construction [75] is particularly useful. It allows to view lower-dimensional p-branes with $p < 9$ as bound states of 2^{k-1} $(p + 2k)$-brane/antibrane pairs. The intricacies of such constructions, especially at the global level, go beyond the scope of this thesis, and here we merely present the main result:

- Type IIB Dp-brane charges are classified by $\widetilde{K}(S^{9-p})$.
- Type IIA Dp-brane charges are classified by $\widetilde{K}(S^{10-p}) = \widetilde{K}^{-1}(S^{9-p})$.
- Type I Dp-brane charges are classified by $\widetilde{KO}(S^{9-p})$.

In all the cases above, the p-branes are pointlike with respect to S^n. We summarize the type II and type I brane spectra and their respective K-theory charges in table 2.3.

2.6 Effective actions

The full superstring theories, with their infinite towers of massive states, are particularly hard to work with. In practice, one often works at a specific energy regime where an EFT description is possible. For string phenomenology, one is in a low-energy regime, where "low" now means "much below the string scale M_s", where one can safely disregard the contributions of massive states. Provided the theory is weakly coupled and higher-curvature corrections can be consistently ignored, there is a very convenient EFT description of string theory in the form of *supergravity actions*. At leading order in α', the

[6] To be more precise, one often needs to consider K-theory with compact support since X can be non-compact.

n	0	1	2	3	4	5	6	7	8	9	10
$\widetilde{KO}(S^n)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
Type I D-brane	$D9$	$\widehat{D8}$	$\widehat{D7}$	-	$D5$	-	-	-	$D1$	$\widehat{D0}$	$D\widehat{(-1)}$
$\widetilde{K}(S^n)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
Type IIB D-brane	$D9$	-	$D7$	-	$D5$	-	$D3$	-	$D1$	-	$D(-1)$
Type IIA D-brane	$D10$	-	$D8$	-	$D6$	-	$D4$	-	$D2$	-	D0

Table 2.3: KO- and K-theory classes and respective branes. Hats indicate non-BPS branes.

only dimensionful parameter of the action, the EFT action can be derived by exploiting spacetime and gauge symmetries and imposing conditions such as anomaly cancellation. Imposing that the effective action reproduces the string theory amplitudes in the $\alpha' \to 0$ limit is another way of determining the EFT action. The supergravity actions are not the only relevant EFT actions. One can also find an EFT description for the localized objects of the theory, namely the D-branes: this is the so-called Dirac-Born-Infeld (DBI) action.

2.6.1 10d supergravity actions

Type IIB supergravity

The low-energy EFT corresponding to type IIB string theory is the so-called type IIB supergravity. It is a (2,0) chiral supergravity theory. The 10d massless bosonic field content of type IIB string theory is the graviton G_{MN}, the 2-form B_2 with field strength $H_3 = dB_2$, and the dilaton ϕ in the NS-NS sector, and the even forms $C_n, n = 0, 2, 4$ in the R-R sector. The field strengths are subsequently given by

$$F_1 = dC_0, \tag{2.20a}$$

$$F_3 = dC_2 - C_0 dB_2, \tag{2.20b}$$

$$F_5 = dC_4 - \frac{1}{2} C_2 \wedge dB_2 + \frac{1}{2} B_2 \wedge dC_2, \tag{2.20c}$$

where one should keep in mind that F_5 is self-dual. Using these fields, one can write the supergravity action in the string frame as follows:

$$
\begin{aligned}
S_{\rm IIB} = & \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}\Big[e^{-2\phi}\Big(\mathcal{R}+\quad 4(\nabla\phi)^2-\frac{1}{2}|H_3|^2\Big)-\frac{1}{2}|F_1|^2-\frac{1}{2}|F_3|^2-\frac{1}{4}|F_5|^2\Big] \\
& -\frac{1}{4\kappa_{10}^2}\int C_4\wedge H_3\wedge F_3,
\end{aligned}
\tag{2.21}
$$

where $\mathcal{R}$ is the Ricci scalar, the 10d gravitational coupling κ_{10} relates to α' as

$$
\kappa_{10}^2 = \frac{(4\pi^2\alpha')^4}{4\pi}, \tag{2.22}
$$

and we denote $\sqrt{-G}|F_p|^2 = \sqrt{-G}\frac{1}{p!}F_{M_1\dots M_p}F^{M_1\dots M_p} = F_p\wedge * F_p$, with $*$ being the Hodge star operator for the 10-dimensional metric G. The gauge transformations

$$
\delta B_2 = d\zeta\,,\quad \delta C_0 = 0\quad \delta C_2 = \delta\Lambda_1\,,\quad \delta C_4 = d\Lambda_3 - \frac{1}{2}dB\wedge\Lambda_1 + \frac{1}{2}dC_2\wedge\zeta, \tag{2.23}
$$

leave the action (2.21) invariant, up to a total derivative.

The action (2.21) is composed of two terms, the former being the Ricci scalar and the kinetic terms of the fields, and the latter being a Chern-Simons term, which is topological. Importantly, the NS-NS and the R-R fields coupled to the dilaton differently. Moreover, the self-duality of F_5, $F_5 = *F_5$ needs to be imposed by hand since, for dimensional reasons, it cannot be included in the action. Since the unconstrained F_5 accounts for too many degrees of freedom, an additional factor of $\frac{1}{2}$ in front of the corresponding kinetic term serves to fix this overcounting. Finally, let us note that the Ricci scalar in the above action couples explicitly to the dilaton: this defines the so-called *string frame*. However, it is possible to rescale the metric in order to go to the one conventionally known as *Einstein frame*, where the Ricci scalar appears without coupling to other fields[7]. In particular, the required rescaling is

$$
G^E_{MN} = e^{-\frac{\phi}{2}}G_{MN}. \tag{2.24}
$$

In the Einstein frame, the action explicitly manifests the $SL(2,\mathbb{R})$ symmetry of the theory. To see this, we need to define the complex scalar τ, called *axio-dilaton*, and the three-form G_3 as:

$$
\tau = C_0 + ie^{-\phi}, \tag{2.25a}
$$

$$
G_3 = F_3 - ie^{-\phi}H_3 = dC_2 - \tau dB_2. \tag{2.25b}
$$

In this notation, the IIB supergravity action assumes the form:

$$
\begin{aligned}
S_{\rm IIB} = & \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G^E}\Big(\mathcal{R}^E - \frac{\partial_M\tau\partial^M\bar\tau}{2\big(Im(\tau)\big)^2} - \frac{1}{2}\frac{|G_3|^2}{Im(\tau)} - \frac{1}{4}|F_5|^2\Big) \\
& -\frac{i}{8\kappa_{10}^2}\int\frac{C_4\wedge G_3\wedge\bar G_3}{Im(\tau)}.
\end{aligned}
\tag{2.26}
$$

[7] A very detailed overview of frames and conventions can be found in [76].

In equation (2.26), the superscript E stands for Einstein frame. From now on this index will be dropped since the distinction between the two frames is clear just by looking at the Einstein-Hilbert term.

Type IIA supergravity

Type IIA supergravity is the low-energy limit of type IIA string theory and is a non-chiral (1,1) supergravity theory. The NS-NS part of the spectrum is identical to type IIB, while the R-R forms are now the odd C_1, C_3. The field strengths are given by

$$F_2 = dC_1, \tag{2.27a}$$

$$F_4 = dC_3 - dB_2 \wedge C_1. \tag{2.27b}$$

The IIA supergravity action in the string frame is:

$$\begin{aligned} S_{\text{IIA}} = & \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} \left[e^{-2\phi} \left(\mathcal{R} + 4(\nabla\phi)^2 - \frac{1}{2}|H_3|^2 \right) - |F_2|^2 - \frac{1}{2}|F_4|^2 \right] \\ & - \frac{1}{4\kappa_{10}^2} \int B_2 \wedge dC_3 \wedge dC_3. \end{aligned} \tag{2.28}$$

The gauge transformations which leave (2.28) invariant are:

$$\delta B_2 = d\zeta \,, \quad \delta C_1 = d\Lambda_0 \,, \; \delta C_3 = d\Lambda_2 - dB\Lambda_0 \;. \tag{2.29}$$

As a side remark, there is a possible deformation of the above action. In particular, we can consider *massive* Type IIA supergravity, where a non-vanishing background field strength $F_0 = -m$ is included in the action. This term acts as a mass term, called the Romans mass [77].

Type IIA 10d supergravity can directly arise by dimensionally reducing 11d supergravity. 11d supergravity is considered the low-energy approximation of M-theory, and the relevant fields, arranged in an 11d supersymmetry gravity multiplet, are the graviton $\hat{G}$, the 3-form $\hat{C}_3$, and the gravitino $\hat{\psi}$, where the hats indicate we are dealing with 11d fields. The bosonic piece of the 11d supergravity action is:

$$S_{11\text{d}} = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-\hat{G}} \left(\hat{\mathcal{R}} - \frac{1}{2}|d\hat{C}_3|^2 \right) - \frac{1}{12\kappa_{11}^2} \int \hat{C}_3 \wedge d\hat{C}_3 \wedge d\hat{C}_3. \tag{2.30}$$

Type I supergravity

Let us now discuss type I supergravity. Now, some of the IIB degrees of freedom get projected out due to being odd under world-sheet parity. On the other hand, there is an additional SO(32) gauge field A_M^a and 32 space-filling D9-branes. Hence the supergravity action will be the sum of two terms, the first similar to IIB supergravity, minus the projected-out field contributions, and the second being the super-Yang Mills action for the gauge fields. Note that now the Chern-Simons term is absent due to B_2 being projected out. In the string frame, the action reads

$$\begin{aligned} S_{\mathrm{I}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x\sqrt{-G}\Big[e^{-2\phi}\Big(\mathcal{R} + 4(\nabla\phi)^2\Big) - \frac{1}{2}|F_3|^2\Big] \\ -\frac{1}{2g_{10}^2}\int \sqrt{-G}e^{-\phi}tr_v(|F_{YM}|^2). \end{aligned} \tag{2.31}$$

We have defined $F_{YM} = F_{YM}^a T^a$ as the Yang-Mills field strength, the trace runs over the vector representation of SO(32), and the gravitational and Yang-Mills couplings are related by $\frac{\kappa_{10}^2}{g_{10}^2} = \frac{\alpha'}{4}$. Moreover, the 3-form field strength F_3 receives two Chern-Simons correction terms.

$$F_3 = dC_2 - \frac{\alpha'}{4}(\Omega_{YM} - \Omega_L), \tag{2.32a}$$

$$\Omega_{YM} = tr\big(A \wedge dA - \frac{2i}{3}A \wedge A \wedge A\big), \tag{2.32b}$$

$$\Omega_L = tr\big(\omega \wedge d\omega - \frac{2}{3}\omega \wedge \omega \wedge \omega\big). \tag{2.32c}$$

These two terms combine, ultimately leading to the equation

$$dF_3 = \frac{\alpha'}{4}\big(tr\mathcal{R} \wedge \mathcal{R} - tr\mathcal{F} \wedge \mathcal{F}\big). \tag{2.33a}$$

2.6.2 Dp-brane effective actions

In this subsection, we will discuss the effective action of a BPS Dp-brane, $p \leq 9$. We will focus on the action of a single Dp-brane so that we do not have to worry about non-abelian contributions. The action consists of two terms. These are the *Dirac-Born-Infeld (DBI)* term, which describes the couplings of the open string degrees of freedom to the closed string bulk NS-NS fields and is a generalization of Maxwell theory to objects with higher-dimensional world-volumes, and the *Chern-Simons (CS)* term, which captures the couplings to the R-R higher-form fields. We will discuss each of these terms separately, following [43, 78].

DBI action

Let us start by introducing some necessary quantities. First of all, we have a gauge-invariant field strength

$$2\pi\alpha'\mathcal{F} = B + 2\pi\alpha' F. \tag{2.34}$$

The $p + 1$ spacetime directions parallel to the brane are labeled by the Greek indices $\alpha, \beta, \ldots \in \{0, \ldots, p\}$, while the transverse directions are labeled by latin indices $i, j, \ldots \in$

$\{p+1,\ldots,9\}$. The world-volume of the brane is denoted by $\mathcal{W}$ and is parametrized by the coordinates ξ^α. Moreover, it is embedded in the 10-dimensional spacetime by the functions $X^M(\xi)$, $M=0,\ldots,9$. The bosonic part of the DBI action in the string frame is [79–81]:

$$S_{\text{DBI}} = -T_p \int_{\mathcal{W}} d^{p+1}\xi e^{-\phi(X)} \sqrt{-det\big(g_{\alpha\beta}(X) + 2\pi\alpha' \mathcal{F}_{\alpha\beta}(X)\big)}. \tag{2.35}$$

Here $g_{\alpha\beta} = \partial_\alpha X^M \partial_\beta X^N G_{MN}$ is the pullback of the string-frame spacetime metric on the brane world-volume. As for the quantities entering through $\mathcal{F}_{\alpha\beta}$, $B_{\alpha\beta}$ is also pulled-back to $\mathcal{W}$, while F is just restricted to it. The multiplicative parameter in front of the integral is the brane tension, which differs between type II and type I branes according to

$$T_p^{II} = 2\pi l_s^{-(p+1)}, \tag{2.36a}$$
$$T_p^{I} = \sqrt{2}\pi l_s^{-(p+1)}. \tag{2.36b}$$

The position and the deformations of the brane in spacetime are parametrized by the massless bosonic open string modes. These are the $(p+1)$ gauge field components $A_\alpha(\xi)$ and the $(9-p)$ fluctuations of the transverse coordinates $X^i(\xi)$ respectively.

The two-derivative order Lagrangian can be computed by expanding the determinant in powers of the field strength. Setting for simplicity the scalars and B to zero, we have

$$S_{\text{DBI}} = -T_p \int d^{p+1}\xi \sqrt{-g} e^{-\phi}\big(1 + \frac{1}{4}(2\pi\alpha')^2 F_{\alpha\beta}F^{\alpha\beta} + \ldots\big), \tag{2.37}$$

which, as expected, is the sum of the vacuum energy (term proportional to the brane volume) and the gauge field kinetic term.

Chern-Simons action

The Chern-Simons part S_{CS} of the open string effective action, also known as the Wess-Zumino action, includes all the contributions from the R-R potentials. It is of paramount importance for the consistency of the theory as it enters the Green-Scwharz anomaly cancellation mechanism. In the string frame, it is given by [82, 83]

$$S_{\text{CS}} = -\mu_p \int_{\mathcal{W}} ch(2\pi\alpha'\mathcal{F}) \wedge \sqrt{\frac{\hat{A}(\mathcal{R}_T)}{\hat{A}(\mathcal{R}_N)}} \wedge \bigoplus_q C_q\Big|_{p+1}. \tag{2.38}$$

Above μ_p is the charge of the Dp-brane and is equal (opposite) to the tension T_p of the D-brane (antibrane). Moreover, some characteristic polynomials enter equation (2.38). These are the *Chern character* $ch(2\pi\alpha'\mathcal{F}) = \text{tr}(e^{2\pi\alpha'\mathcal{F}})$ and the *A-roof genus* $\hat{A}$ is the A-roof genus, which can be expressed in terms of the *Pontryagin classes* p_n. Explicit definitions for all the relevant characteristic classes are given in appendix B. The gauge invariant field strength $\mathcal{F}$ is defined as in the DBI action (2.35), and the factor of $2\pi\alpha'$ serves to render it dimensionless. The arguments of $\hat{A}$ are the dimensionless curvature 2-forms for the tangent and normal bundles of the brane, denoted by $\mathcal{R}_T = 4\pi^2\alpha' R_T = l_s^2 R_T$ and $\mathcal{R}_N = 4\pi^2\alpha' R_N = l_s^2 R_N$ respectively.

The direct sum runs over all R-R potentials present in the theory[8], but a (p+1)-form is effectively picked out due to the integration over $\mathcal{W}$. Note that there is no dependence on the metric, explicit or implicit, in the Chern-Simons terms. This fact precisely shows the topological nature of the term, which physically corresponds to the (topological) charge of the D-brane [60].

2.6.3 Op-plane effective actions

Similarly to D-branes, orientifold planes (O-planes) can also be described by their effective action since they have tension and are charged under the relevant RR higher form fields. The most notable difference between the D-brane and the O-plane effective actions is that O-planes do not have any world-volume fields. Following [43], we introduce the notation $Op^{(\epsilon_1,\epsilon_2)}$, with $\epsilon_i = \pm 1$, which allows us to consider orientifold planes with both signs of charge/tension. Whenever the subscript is omitted, it should be assumed that $\epsilon_1 = \epsilon_2 = -1$, i.e., the orientifold plane has negative tension and negative charge, opposite to that of the D-branes. With these conventions, the two parts of the O-plane effective action are [84]

$$S_{DBI}^{Op^{(\epsilon_1,\epsilon_2)}} = -\epsilon_1 2^{p-4} T_p \int_{\mathcal{W}} d^{p+1}\xi e^{-\phi} \sqrt{-det(g_{\alpha\beta})}, \tag{2.39}$$

$$S_{CS}^{Op^{(\epsilon_1,\epsilon_2)}} = \epsilon_2 2^{p-4} \mu_p \int_{\mathcal{W}} \sqrt{\frac{L(\mathcal{R}_T)}{L(\mathcal{R}_N)}} \wedge \bigoplus_q C_q \big|_{p+1}, \tag{2.40}$$

where now the Hirzebruch L-polynomial (for definition see appendix B) enters the square root instead of the A-roof genus, which appears in the D-brane case. Both integrals involve integration over the world-volume $\mathcal{W}$ of the O-plane, which is the fixed locus of the projection creating the O-plane. In contrast to the D-branes, this world-volume is fixed and does not fluctuate; hence no scalars are needed to parameterize it.

Finally, note that there exists a prefactor of 2^{p-4} appearing in the actions (2.39), (2.40). This reflects the fact that multiple Dp-branes are necessary to cancel the charge/tension of an Op-plane fully. One well-known instance is that of $p = 9$ in type I, where one has a single O9-plane extending over the full 10d space, and 32 D9-branes are required to cancel its charge and tension.

[8] Here one usually uses all the fields present in the so-called *democratic formulation*, which is explicitly self-dual and also includes a C_{8-p}-form for every C_p RR form.

Chapter 3

Compactifications and Moduli Stabilization

3.1 Compactifications

Up to this point, we have given a lightning-fast review of some key concepts and elements of string theory, such as circle compactifications, D-branes, and O-planes. The absence of conformal anomaly dictates that bosonic string theory is consistent only in the critical dimension $d_c = 26$, while superstring theory is defined in $d_c = 10$. Reducing the number of dimensions is usually desired for a phenomenologically relevant effective field theory. As we have already seen in the circle example, this can be achieved through compactifications. The details of the particular string compactification are paramount for the physics of the four-dimensional theories, as they influence not only the spectrum but also properties such as the amount of supersymmetry and the effective cosmological constant. The study of string compactifications is interconnected with the richness of the string landscape, so now we delve into a slightly more systematic view of the topic. Useful resources for the study of string compactifications and their physical implications include, among others [43, 47, 85–87].

3.1.1 Compactification manifolds

The starting point in a typical string theory compactification is the assumption that the spacetime can be viewed locally as a product of two manifolds, one being a possibly non-compact four-dimensional spacetime that preserves Poincaré invariance and the other being a compact $(d_c - 4)$-dimensional manifold, according to $\mathcal{M}^{d_c} = \mathcal{M}^4 \times K^{d_c-4}$. Compactness may be avoided if K^{d_c-4} has finite volume, which can be achieved for non-compact manifolds with strong enough warping. The manifold on which we compactify is conventionally called *compactification* or *internal manifold*. Under this simple assumption, the total metric takes the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + \tilde{g}_{mn}(y) dy^m dy^n \ , \tag{3.1}$$

where $X^M = (x^\mu, y^m)$, with x^μ the four non-compact directions and y^m the internal directions. It would be naive to think that any compact manifold can be a compactification manifold - as we have seen, string theory is often remarkably restrictive. In practice, the following constraints should be respected:

- The size of the compact dimensions should be "moderate": They cannot be too large, as multiple large extra dimensions could be at odds with experiments [88]. On the other hand, considering manifolds of size smaller than the string scale would also pose a significant hindrance invalidating the particularly convenient supergravity description and necessitating quantum corrections.
- Supersymmetry necessitates the existence of a Killing spinor ϵ, i.e., a spinor which satisfies $\langle \nabla_M \epsilon \rangle = 0$ with ∇_M the covariant derivative on spinors. On a Riemannian manifold, a necessary condition for this is the triviality of the Ricci tensor $R_{mn} = 0$, i.e., the internal manifold needs to be Ricci-flat. Ricci-flatness is also useful from a CFT viewpoint, as it, e.g., ensures Weyl invariance.
- A Killing spinor is a nowhere-vanishing covariantly constant spinor. A covariantly constant spinor ϵ must be a singlet under the holonomy group $\mathcal{H}$ of the compact manifold. For connected Riemann m-manifolds $\mathcal{H} \subseteq O(m)$, while if we impose that the manifold is also oriented, we have $\mathcal{H} \subseteq SO(m)$. For a six-dimensional internal manifold, if the holonomy group is reduced to $SU(3)$ instead of $SU(4) \backsim SO(6)$, the existence of ϵ is guaranteed.

The so-called *Calabi-Yau manifolds* fulfill all the properties above[1].

Definition:
A **Calabi-Yau (CY)** manifold X of dimension n is a Kähler manifold with vanishing first Chern class $c_1(X) = 0$.

The CY n-folds with n up to 4 are relevant for superstring compactifications. For $n = 1$, there is a unique CY, the torus T^2. For $n = 2$, there is the (trivial) case of $T^2 \times T^2$ and the topologically unique so-called Kummer or K3 surface. For $n = 3$, however, billions of CY_3 manifolds have been enumerated, but whether this set is finite has not been settled yet. Finally, the computational complexity increases dramatically for $n = 4$,

[1]In appendix A complex, Kähler and Calabi-Yau manifolds are discussed in more detail, while appendix B explains characteristic classes, including the Chern classes.

$$
\begin{array}{ccccc}
 & & h^{0,0} & & \\
 & h^{1,0} & & h^{0,1} & \\
h^{2,0} & & h^{1,1} & & h^{0,2} \\
 & h^{2,1} & & h^{1,2} & \\
 & & h^{2,2} & &
\end{array}
=
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & h^{1,1} & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
$$

Figure 3.1: Hodge diamond of a Calabi Yau twofold $K3$.

$$
\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
h^{3,0} & & h^{2,1} & & h^{2,1} & & h^{0,3} \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & & & h^{3,3} & & &
\end{array}
=
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Figure 3.2: Hodge diamond of a Calabi Yau threefold CY_3.

which is of interest for F-theory compactifications. While there is no known analytic expression for the metrics of the non-trivial Calabi-Yau manifolds, recently, there have been systematic attempts (see, e.g., [89–91]) to compute numerical CY metrics using machine learning.

As discussed in appendix A, the topological data for complex manifolds are encoded in their Hodge diamonds. Here we present the Hodge diamonds for K3 and CY_3 surfaces, which will be relevant later in this thesis.

It turns out most Hodge numbers are fixed due to the Calabi-Yau properties. Those not fixed correspond to the so-called *moduli*. In the case of CY_3, the $h^{1,1}$ *Kähler moduli* parametrize the deformations of the Kähler structure of the Ricci-flat metric, while the $h^{2,1}$ moduli parametrize the deformations of the complex structure of the manifold. In the case of the K3 manifold, all of them are fixed, so in this topological sense, K3 is unique.

The Hodge diamond provides information about additional topological quantities. The sum of all Hodge numbers in a row gives the Betti numbers $b_r = \sum_{p+q=r} h^{p,q}$, while the alternating sum of the Betti numbers gives the Euler number $\chi(M)$ of the manifold $\chi(M) = \sum_r (-1)^r b_r$. Specifically for the CY threefold CY_3, it is $\chi(CY_3) = 2(h^{1,1} - h^{1,2})$.

One can notice that the two Hodge diamonds above exhibit a high degree of symmetry. Symmetry upon reflection along a vertical axis through the center of the Hodge diamond corresponds to complex conjugation, and reflection along a horizontal axis corresponds to Hodge duality. In addition there is a *mirror symmetry*, which relates the Hodge numbers of two mirror Calabi-Yau manifolds $X, \tilde{X}$ as $h^{p,q}(X) = h^{n-p,q}(\tilde{X})$, and results in opposite Euler numbers $\chi(X) = -\chi(\tilde{X})$.

3.1.2 Compactification and spectrum

Now that we have introduced the mathematical background of compactification, we are ready to discuss the compactification of type II supergravities on Calabi-Yau threefolds. The supergravity approximation is well justified as long as the energies in play are well below the masses of the massive string states, i.e., $E \ll 1/\alpha'$. Moreover, the compactification manifold's characteristic length scale L should be much larger than the string length, i.e., $L^2 \gg \alpha'$. This allows us to neglect α'-corrections, which become relevant at scales $L \sim \sqrt{\alpha'}$ and could alter the interaction terms present. In the absence of such corrections, the Kaluza-Klein reduction of supergravity, restricting to the massless modes of the wave operator in the internal manifold, is a good approximation to the actual string compactification.

Type IIB

Type IIB is a $\mathcal{N} = 2$ chiral supergravity. In ten dimensions, the spectrum consists of the single gravity multiplet with bosonic content $\{G_{MN}, B_2, C_2, C_4, C_0, \phi\}$, with $M, N = 0, \ldots, 9$, and fermionic content: $\{\psi^-_M, \tilde{\psi}^-_M, \lambda^+, \tilde{\lambda}^+\}$, where the superscripts denote chirality and spinorial indices are suppressed.

To get the four-dimensional spectrum, one practically splits the spacetime indices in a $SU(3)$-covariant way, since this is the holonomy group of the internal manifold, to $M = (\mu, i, \bar{\imath})$. The Greek indices run over the four non-compact dimensions, while the latin indices run in the internal space. The situation for the bosons is quite clear. Looking at the example of C_2, it now potentially splits into $C_{2\mu\nu}, C_{2\mu i}, C_{2\mu}, C_{2i\bar{\jmath}}$. However, we know that no 1-cycles exist in the CY[2]; hence $C_{2\mu i}, C_{2\mu}$ are actually not realized. Moreover, there are $h^{1,1}$ 2-cycles, so we have $h^{1,1}$ 4d bosons $C_{2i\bar{\jmath}}$, depending on which (1,1)-cycle is relevant. A subtlety arises when considering C_4 since one has to take into account the halving of degrees of freedom due to the self-duality condition. The fermions decompose similarly. For instance, the positive chirality dilatino λ^+ decomposes as $\lambda^+ \sim \lambda_\alpha \eta \oplus \bar{\lambda}_{\dot{\alpha}} \eta_{\bar{\imath}} \oplus \lambda_\alpha \eta_{\bar{\imath}\bar{\jmath}} \oplus \bar{\lambda}_{\dot{\alpha}} \eta_{\bar{\imath}\bar{\jmath}\bar{k}}$. Taking everything into account, the 4d spectrum gets arranged

[2] By assumption, the Calabi-Yau manifolds we consider are such that $\pi_1(CY_3) = 0$. In general, there exist Calabi-Yau manifolds with $\pi_1(CY_3) = \mathbb{Z}_n$, for some integer n, i.e. with torsion in $H^1(CY_3;\mathbb{Z})$. Typical examples are the free quotient of Calabi-Yaus without torsion, such as the free quotient of the quintic $\mathbb{P}_4[5]/\mathbb{Z}_5$. For instance, they have been investigated in [92, 93].

in $\mathcal{N}=2$ supergravity multiplets, according to table 3.1, where we only explicitly write out the bosons and not their fermionic superpartners.

$\mathcal{N}=2$ Multiplet	Bosonic components	Multiplicity
Gravity	$g_{\mu\nu}, C^{\dagger}_{4\mu ijk}$	1
Hypermultiplet	$\phi, a, B_{\mu\nu}, C_{2\mu\nu}$	1
Hypermultiplet	$g_{i\bar{\jmath}}, B_{i\bar{\jmath}}, C_{2i\bar{\jmath}}, C^{\dagger}_{4\mu\nu i\bar{\jmath}}$	$h^{1,1}$
Vector multiplet	$C^{\dagger}_{4\mu i\bar{\jmath}\bar{k}}, g_{ij}, g_{\bar{\imath}\bar{\jmath}}$	$h^{2,1}$

Table 3.1: Massless spectrum of type IIB compactified on CY_3 .

Type IIA

The situation in type IIA is similar. The starting spectrum is now a non-chiral supergravity multiplet with bosonic content $\{G_{MN}, B_{MN}, C_{3MNP}, C_{1M}, \phi\}$, with $M, N = 0, \dots, 9$ and fermionic content$\{\psi^{+}_{M}, \tilde{\psi}^{-}_{M}, \lambda^{+}, \lambda^{-}\}$, with spinorial indices again suppressed. The 4d spectrum after the compactification on the CY threefold is given in table 3.2.

$\mathcal{N}=2$ Multiplet	Bosonic components	Multiplicity
Gravity	$g_{\mu\nu}, C_{1\mu}$	1
Hypermultiplet	$\phi, B_{\mu\nu}, C_{3ijk}, C_{3\bar{\imath}\bar{\jmath}\bar{k}}$	1
Hypermultiplet	$g_{ij}, g_{\bar{\imath}\bar{\jmath}}, C_{3i\bar{\jmath}\bar{k}}, C_{3\bar{\imath}jk}$	$h^{2,1}$
Vector multiplet	$C_{3\mu i\bar{\jmath}}, g_{i\bar{\jmath}}, B_{i\bar{\jmath}}$	$h^{1,1}$

Table 3.2: Massless spectrum of type IIA compactified on CY_3 .

Tables 3.1 and 3.2 indicate a deeper structure. In particular, the two spectra are exchanged upon exchanging $h^{1,1}$ and $h^{2,1}$, i.e., type IIB compactified on a CY manifold X is dual to type IIA compactified on the mirror manifold $\tilde{X}$.

3.2 Moduli stabilization

Several massless moduli appear in the 4d spectrum. However, this is incompatible with a sensible phenomenological model since massless bosons would lead to deviations from Newton's law that are not observed [94]. Hence there should be some mechanism that makes them massive enough. This happens by considering additional "ingredients" in the string compactification, such as fluxes, branes, and O-planes, which generate a 4d effective potential for the scalars. This procedure of generating masses for the moduli is called *moduli stabilization.* The so-called *string vacua* are then the extrema of the potential. The value of the potential at the vacuum V_0 determines which geometry describes the 4d spacetime:

- For $V_0 < 0$ we have a vacuum corresponding to an Anti-de-Sitter (AdS) 4d spacetime, which can be either supersymmetric or non-supersymmetric.
- For $V_0 = 0$ we have a vacuum corresponding to a Minkowski 4d spacetime, which also can be supersymmetric or non-supersymmetric.
- For $V_0 > 0$, we have a vacuum corresponding to a de-Sitter (dS) 4d spacetime. In this case, all supersymmetry is necessarily broken.

In any semi-realistic compactification, apart from giving masses to the moduli fields, one should also consider the couplings. Two types of couplings are relevant, gravitational and gauge couplings. Consider a pretty minimalistic setup, where we compactify type II superstring on a CY_3 of volume V_x and include a spacetime-filling Dp-brane, which wraps a $(p-3)$-cycle of volume V_c in the internal manifold. The initial 10d effective action contains an Einstein-Hilbert term $\frac{M_{p,10}^8}{(2\pi)^6}\int d^{10}x\sqrt{-G}R^{(10)}$, where we now explicitly denote that the Planck mass $M_{p,10}$ and the Ricci tensor $R^{(10)}$ are 10d quantities. This term gets dimensionally reduced to 4d as $V_x\int d^4x\sqrt{-G}R^{(4)}$. It is then clear that the 4d Planck Mass $M_{p,4}$ or simply M_p can be expressed in terms of the higher-dimensional Planck mass and the volume of the compactification manifold:

$$M_p^2 \equiv \frac{8\pi}{\kappa_4^2} = \frac{8\pi V_x}{\kappa_{10}^2} \equiv V_x(2\pi)^6 M_{p,10}^8 = \frac{8M_s^2 V_x}{g_s^2 l_s^6}. \tag{3.2}$$

Notice that when the volume of the compactification manifold becomes infinite, the 4d Planck mass follows the same trend. The gravitational coupling then goes to zero, and *gravity decouples.* Similarly, from the dimensional reduction of the DBI action for the Dp-brane, we find that the 4d gauge couplings can be written as

$$\frac{1}{g_{YM}^2} \equiv \frac{V_C}{g_{D_p}^2} = \frac{V_c}{2\pi g_s l_s^{p-3}}. \tag{3.3}$$

Note that the product $M_p g_{YM}^2$ is independent of the string coupling.

3.2.1 Orbifold compactifications

Type II CY orientifold compactifications, i.e., compactifications on Calabi-Yau manifolds on which we perform a discrete orientifold projection, allow for $\mathcal{N}=1$ in four dimensions. The reader should be aware that the CY orbifold compactifications are also very similar in the sense that a usually discrete projection is invoked, but this time without an orientation-reversal piece.

More concretely, for an orientifold, one divides the string theory by $G_\Omega \equiv G \cup \Omega_p S$ where G is a group of target space symmetries and $\Omega_p S$ is a symmetry of the theory on $\mathcal{M}/G$, written as a combination of the world-sheet parity operator Ω_p with some appropriate operator S. In particular, Ω_p acts as

$$\Omega_p : (\sigma, \tau) \to (2\pi - \sigma, \tau), \tag{3.4}$$

in the open string case, so it interchanges the two endpoints of the string, and as

$$\Omega_p : (\sigma, \tau) \to (\pi - \sigma, \tau), \tag{3.5}$$

in the closed string case, so it exchanges the left- and right-moving sectors. As for the operator S, it usually includes some target-space symmetry σ[3], the fixed points of which give the orientifold planes. In principle, one can treat the orientifold Ω_p as two operations in a row, first orbifolding by G and then performing an orientifold projection but treating both steps at once is also possible.

Our discussion will concern orientifold compactification on a Calabi-Yau threefold X, i.e., the total space will be $\mathbb{R}^{1,3} \times X$, and for simplicity, we will set the "orbifold" part of the projection to $G = \mathbb{1}$. X can be described by local complex coordinates $z^i, i = 1, 2, 3$, which allow us to write the holomorphic three-form as $\Omega = dz^1 \wedge dz^2 \wedge dz^3$ and the Kähler form as $\omega = \frac{i}{2} \sum_{i=1}^{3} dz^i \wedge d\bar{z}^i$. Since σ acts qualitatively differently on type IIB and type IIA theories, we discuss them separately.

Type IIB case

In this case, σ is a discrete holomorphic isometry of X, which leaves the metric and the complex structure invariant. Its pullback σ^* is

$$\sigma^*(\omega) = \omega, \quad \sigma^*(\Omega) = \pm\Omega. \tag{3.6}$$

In practice, σ sends the local coordinates to $\pm z^i$. The sign of $\sigma^*(\Omega)$ depends on how many of the three coordinates flip sign, and we distinguish the two cases:

- Even number of sign flips: $\sigma^*(\Omega) = +\Omega$. This means either 0 or 2 directions flip signs. The dimensionality of the fixed points is then 10 or 6 (since we flip the sign of two complex dimensions), respectively, and the O-planes introduced are, accordingly, O9- and O5-planes. To reach this conclusion, we also use that σ acts trivially on the 4d part of the space; hence the O-planes are spacetime-filling.

[3] It should be clear by the context that we are not referring to the world-sheet coordinate σ.

– Odd number of sign flips: $\sigma^*(\Omega) = -\Omega$. We now have 1 or 3 minus signs, which lead to codimension-2 and -6 objects in the compact space. Hence the O-planes accompanying this action are O7 and O3, respectively.

The discussion above generalizes to fermions by supersymmetry. The combination $S\Omega_p$ needs to be an involution to have spacetime-invariant fermions; more specifically, a gravitino must remain after the projection. Hence, in the two cases above, the operator S turns out to be $S = \sigma$ for an even number of sign flips and $S = \sigma(-1)^{F_L}$ for an odd number of sign flips, where F_L is the left-moving spacetime fermion number.

Finally, the choice of orientifold projection, through the tadpole cancellation conditions, dictates which Dp-branes can be present. For type IIB, the projection with O9/05-planes can also have D9/D5-branes, while the 07/O3-plane case is compatible with D7/D3-branes.

The orientifold projection can reduce the amount of supersymmetry to $\mathcal{N} = 1$. Hence the 4d fields should be arranged in such multiplets. This happens naturally since only the fields invariant under the projection remain in the spectrum. In particular, the cohomology groups split into $H^* = H^*_- \oplus H^*_+$, i.e., odd and even subgroups under the $\bar{\sigma}$-action. Moreover, the fields B_2 and C_2 are odd under $\Omega(-1)^{F_L}$, while g, ϕ, C_0 and C_4 are intrinsically even. The spectrum then is arranged in multiplets as shown in table 3.3.

$\mathcal{N} = 1$ Multiplet	Bosonic Field Content	Multiplicity
Gravity	$g_{\mu\nu}$	1
Chiral (Kähler)	$T_\alpha = v^\alpha + iC_\alpha$	$h^+_{1,1}$
Chiral (Complex structure)	$U_A = i\int \Omega_3 \wedge \alpha_A$	$h^-_{2,1}$
Chiral (axio-dilaton)	$S = e^{-\phi} + iC_0$	1
Chiral	$G^a = c^a - iSb^a$	$h^-_{1,1}$

Table 3.3: Massless spectrum of type IIB compactified on a CY orientifold.

Type IIA case

The main difference between type IIA and the previously discussed type IIB case is that now σ is (part of) an antiholomorphic involution. For this reason, it is often encountered in the literature as $\bar{\sigma}$, but we will not adopt this notation. The pullbacks are now

$$\sigma^*(\omega) = -\omega, \qquad \sigma^*(\Omega) = \bar{\Omega}. \tag{3.7}$$

Now σ exchanges the local coordinates z^i with their conjugates $\bar{z}^i$. Unlike the type IIB case, we do not have multiple options[4], and now since three real dimensions flip sign the only O-planes present are O6-planes. Moreover the anti-holomorphic involution is given by $S = \sigma(-1)^{F_L}$ and once again preserves $\mathcal{N} = 1$ supersymmetry.

3.2.2 Introducing fluxes

String compactifications may be generalized in a non-trivial way by including non-vanishing background fluxes. The so-called *flux compactification* serves the aim of stabilizing the moduli, as the fluxes generate tree-level potentials for the scalars. The fluxes induce some backreaction on the compactification manifold, which is no longer Ricci-flat. While in type IIA orientifolds it is possible to stabilize all moduli using only fluxes [95, 96], this is impossible in type IIB. Then one must consider non-perturbative effects and carefully balance them against the tree-level effects. We will discuss such examples in section 3.2.3.

Compactification à la GKP

One of the most prominent examples of flux compactification is type IIB compactification on a Calabi-Yau orientifold with fluxes and D3/D7 branes. This is usually called a *GKP* compactification, named after Giddings, Kachru, and Polchinski [97]. In such a compactification the fluxes can only stabilize the axio-dilaton and the complex structure moduli.

The starting point is the 10d type IIB supergravity action in the Einstein frame (2.26), which we repeat for the reader's convenience, and where one also includes a term S_{loc} for the localized sources, which in this case are D3/D7-branes and O3-planes.

$$\begin{aligned} S_{\text{IIB}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G^E} \Big(\mathcal{R}^E - \frac{\partial_M \tau \partial^M \bar{\tau}}{2\big(Im(\tau)\big)^2} \quad - \frac{1}{2} \frac{|G_3|^2}{Im(\tau)} - \frac{1}{4} |F_5|^2 \Big) \\ - \frac{i}{8\kappa_{10}^2} \int \frac{C_4 \wedge G_3 \wedge \bar{G}_3}{Im(\tau)} + S_{loc}, \end{aligned} \tag{3.8}$$

where $G_3 = dC_2 - \tau dB_2$. Notice that the axio-dilaton multiplet of table 3.3 relates to the 10d axio-dilaton τ through $\tau = iS$.

We consider compactifications with $F_1 = 0$ and 3-form fluxes without sources that consequently define cohomology classes in X since

$$dF_3 = 0, \quad dH_3 = 0. \tag{3.9}$$

[4]This is due to the absence of homology 1- and 5-cycles that would allow for spacetime-filling O4- and O6-planes.

Keeping the description general, with γ any non-trivial 3-cycle in X, the well-definedness of the partition function necessitates a Dirac quantization condition for the fluxes [47]:

$$\frac{1}{2\pi\alpha'}\int_\gamma F_3 \in 2\pi\mathbb{Z}, \qquad \frac{1}{2\pi\alpha'}\int_\gamma H_3 \in 2\pi\mathbb{Z}. \tag{3.10}$$

Moreover, the self-dual $\tilde{F}_5$ form needs to obey the Bianchi identity

$$d\tilde{F}_5 = H_3 \wedge F_3 + 2\kappa_{10}^2 \mu_3 \rho_{3,loc}, \tag{3.11}$$

where $\rho_{3,loc}$ is the (yet unspecified) localized source contribution. The presence of fluxes/branes induces backreaction on the metric. The first term on the right-hand-side of (3.11) can be interpreted as an effective charge induced by the fluxes given by

$$N_{flux} = \frac{1}{l_s^2}\int_X H_3 \wedge F_3. \tag{3.12}$$

Since we are interested in solutions that preserve 4d Poincaré invariance, we can make the following warped ansatz:

$$ds_{10}^2 = e^{A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{-A(y)}\tilde{g}_{mn}dy^m dy^n, \tag{3.13}$$

where x^μ denote the 4d coordinates and y^m the internal coordinates. Similarly, one can postulate that self-dual five-form flux $\tilde{F}_5$ takes the form

$$\tilde{F}_5 = (1+*)d\alpha \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \tag{3.14}$$

with $\alpha(y)$ a function on the compact space, in accordance with the desired Poincaré invariance and the relevant Bianchi identity.

It turns out that the ansatze (3.13), (3.14) can only be made compatible with the equations of motion in the presence of localized sources, in accordance with the no-go theorem of [98]. Moreover, it is necessary to have an imaginary self-dual (ISD) three-form G_3 to maintain the 4d Poincaré invariance. The ISD condition actually serves to relate the two yet undetermined quantities α, A as $\alpha = e^{4A}$.

Integrating equation (3.11), we get a *R-R tadpole cancellation condition* [99,100] for the C_4 R-R form:

$$\frac{1}{2}N_{flux} + N_{D3} - \frac{1}{4}N_{O3} = 0. \tag{3.15}$$

Here N_{D3} is the number of D3-branes, also including the contributions to the D3-charge coming from D7-branes, N_{O3} denotes the negative contributions coming from O-planes, and N_{flux} is the effective charge define in equation (3.12). For ISD G_3, it is always $N_{flux} > 0$, so the presence of the O3-planes is necessary to cancel the C_4-tadpole.

This tadpole cancellation condition can also be expressed in the F-theory language, as [101]

$$\frac{1}{2}\int_{CY_4} G_4 \wedge G_4 + N_{D3} - \frac{1}{24}\chi(CY_4) = 0, \tag{3.16}$$

with $\chi = 6(8 + h^{1,1} + h^{3,1} - h^{2,1})$ the Euler number of the fourfold. While (3.16) does not seem to impose any direct threat for moduli stabilization a priori, recently, the tadpole conjecture [102] (see also [103–107] for more recent developments) postulates that full moduli stabilization for a large number of $h^{3,1}$ complex structure moduli might be obstructed by such tadpole cancellation conditions.

Whenever the supergravity approximation is good, i.e., we are in a large volume regime where the fluxes are dilute, the system has a convenient 4d effective description in the language of $\mathcal{N} = 1$ supergravity, which we review in the "Aside" box below.

Aside: Basics of 4d $\mathcal{N} = 1$ Supergravity

Consider $\mathcal{N} = 1$ supergravity in four dimensions, with chiral multiplets Φ_i and antichiral multiplets $\overline{\Phi}_i$[a]. The EFT description of the theory is determined via the following two quantities:

- Kähler Potential $\mathcal{K}(\Phi_i, \overline{\Phi}_i)$:
  This leads to (possibly non-canonical) kinetic terms of the form $G_{i\bar{\jmath}} \partial_\mu \Phi^i \partial_\nu \overline{\Phi}^{i\bar{\jmath}}$, where the *Kähler metric* is given by
  $$G_{i\bar{\jmath}} = \frac{\partial^2 \mathcal{K}}{\partial \Phi^i \partial \overline{\Phi}^{\bar{\jmath}}}. \tag{3.17}$$
  The naming choice is not accidental: There is a direct geometric interpretation in terms of a Kähler manifold, with the Φ^i's being the coordinates. Canonical kinetic terms correspond to a flat manifold.

- Superpotential $W(\Phi^i)$: This is a holomorphic function of the chiral superfields encoding interaction terms. The superpotential is protected against perturbative corrections by non-renormalization theorems [108] but may receive non-perturbative corrections.

The Kähler potential and the superpotential combine to give the (F-term) scalar potential:
$$V_F(\Phi^i, \overline{\Phi}^{\bar{\imath}}) = e^{\kappa_4^2 \mathcal{K}} (G^{i\bar{\jmath}} D_i W D_{\bar{\jmath}} \overline{W} - 3\kappa_4^2 |W|^2), \tag{3.18}$$
where $\kappa_4 = \frac{8\pi}{M_p^2}$ and $D_i = \partial_i + \kappa_4^2 (\partial_i \mathcal{K})$ is the *Kähler derivative.*

[a]For simplicity, we do not include any gauge fields in the description, but we note the supergravity language can also appropriately describe such interactions.

In our case, the invariance of the equations of motion and the conditions we require to solve them under the rescaling $\tilde{g}_{mn} \to \lambda^2 \tilde{g}_{mn}$ shows that there should be a Kähler modulus T parametrizing the overall volume of the internal manifold via $(ReT)^{3/2} =$

Vol(X). To simplify the description we postulate that this is the only Kähler modulus present.

The Kähler potential, setting $\kappa_4 = 1$, is:

$$\mathcal{K} = \mathcal{K}_T + \mathcal{K}_{CS,S} \tag{3.19a}$$

$$= -3\ln\big[-i(T-\bar{T})\big] - \ln\big[-i(S-\bar{S})\big] - \ln\Big(-i\int_X \Omega\wedge\bar{\Omega}\Big). \tag{3.19b}$$

The expression above is valid at the large-volume regime. We do not include any quantum correction at this stage; whether this is a valid assumption should be checked a posteriori. The superpotential is of Gukov-Vafa- Witten [109, 110] type and can be computed as

$$W = \frac{1}{\kappa_{10}^2}\int_X G_3\wedge\Omega. \tag{3.20}$$

The quantities above combine to give the scalar potential

$$V = e^{\mathcal{K}}\big(G^{a\bar{b}}D_aW\bar{D}_{\bar{b}}\bar{W} - 3|W|^2\big) = e^{\mathcal{K}}G^{i\bar{j}}D_iW\bar{D}_{\bar{j}}\bar{W}, \tag{3.21}$$

The indices a, b initially run over *all* moduli. However, the potential enjoys a *no-scale structure*: the terms contributed by the Kähler derivatives of the superpotential with respect to the Kähler moduli precisely cancel out the $-3|W|^2$ term. This results in a final positive-definite scalar potential that only depends on the axio-dilaton and the complex structure moduli, which we collectively denote by the i, j indices. As such, the Kähler moduli cannot be stabilized. The no-scale structure is intrinsically related to the *classical* Kähler potential: even if more Kähler moduli are present beyond just T, the relation $G^{i\bar{j}}\mathcal{K}_i\mathcal{K}_{\bar{j}}$ still holds. However, α'-corrections destroy this structure.

Clearly, the potential (3.21) is positive-definite and admits Minkowski vacua. In particular, the Minkowski vacua are only achievable when $D_iW = 0$. If, additionally, $D_TW = 0$, these Minkowski vacua are also supersymmetric. These conditions turn out to be in direct correspondence with the nature of G_3: Minkowski vacua are only possible when G_3 is ISD, and supersymmetry additionally requires that G_3 is primitive, i.e., it has only a (2,1) component.

3.2.3 Beyond fluxes: Non-perturbative effects

As we have seen, in type IIB compactifications, it is impossible to stabilize the Kähler moduli due to the no-scale structure of the scalar potential. To this end, one needs to include some additional effect that will generate a T-dependent contribution to the superpotential and the scalar potential. We will provide examples of this type of moduli stabilization by presenting two constructions of very high phenomenological interest. Both constructions follow the same theme, i.e., the starting point in a type IIB CY orientifold, where fluxes are turned on to stabilize the complex structure moduli and the axio-dilaton, some non-perturbative effect is used to stabilize the Kähler moduli, leading to an AdS vacuum. This AdS vacuum is then uplifted to dS via some supersymmetry-breaking contribution. We will delve into the specifics of both constructions in the remainder of this section. Before doing so, we want to emphasize that neither construction is fully controlled. An ongoing debate revolves around the validity of these constructions, and possible weak points have been identified, especially considering the uplift procedure.

KKLT

The KKLT construction [111], named after Kachru, Kallosh, Linde, and Trivedi, is probably the most cited attempt to construct a de Sitter vacuum in a string theoretical framework. It is often referred to as a "three-step procedure", but one should keep in mind that the steps are meant as a way to keep track of the computation but do not reflect the actual physical process, which should happen at once. The three aforementioned steps go as follows:

- **Step 1:** This is just a compactification á la GKP: The scalar potential is given by (3.21). If one does not explicitly demand a supersymmetric vacuum, then G_3 can have two pieces, the primitive (2,1) and the non-supersymmetric (0,3) part. The (0,3)-part leads to a non-zero value W_0 at the minimum of the potential. We assume that the fluxes fix the axio-dilaton and the complex structure moduli to high-enough masses to be considered frozen for the following step.

- **Step 2:** The stabilization of T happens by including an explicitly T-dependent term to the superpotential, which must be of non-perturbative nature due to the non-renormalization property of W. In particular, the non-perturbative contributions are of the form

$$W_{np} = ce^{-2\pi aT}, \tag{3.22}$$

where c is a one-loop determinant, in general, depending on the complex structure moduli, and a is a positive constant depending on the microscopic origin of the correction term. This comes from Euclidean D3-brane instantons [112] or by stacks of D7-branes wrapping 3-cycles, undergoing gluino condensation. Hence, after freezing out the moduli of step 1, the superpotential becomes

$$W(T) = W_0 + ce^{-2\pi aT}, \tag{3.23}$$

and T is stabilized by $D_T W = 0$ at $T = \tau_0$, corresponding to

$$W_0 = -ce^{-2\pi a\tau_0}\big(1 + \frac{4\pi a}{3}\tau_0\big). \tag{3.24}$$

Whether this exponentially small value of W_0 is physically achievable has been questioned. Recent progress has been achieved in [113–116].

After stabilizing the T modulus, the supersymmetric AdS 4d vacuum is

$$V_0 = -\frac{2\pi^2 a^2 c^2 e^{-4\pi a \tau_0}}{3\tau_0}. \tag{3.25}$$

- **Step 3:** The uplift is a delicate process, balancing out the negative value of the AdS potential by some supersymmetry-breaking effect, such that a) the total potential still admits a minimum, and b) the value at this minimum is positive, yet close to zero. The simplest way to do this is to include anti-D3-branes, which contribute to the scalar potential like [117]

$$V_{up} = \frac{D}{(T+\overline{T})^3}, \tag{3.26}$$

with D a constant depending on the number of antibranes and the specifics of the model. It turns out that this term gets exponentially suppressed by the warp factor $e^{4A_{min}}$ since the anti-D3-branes naturally settle at the bottom of any possibly present throat in the compactification manifold. This exponential suppression of the supersymmetry-breaking contribution is crucial for achieving a minimum in the total potential. We will discuss strongly warped throats and their physical implications more extensively in chapter 5. Note that the anti-D3-branes contribute to the (C_4) tadpole condition

$$N_{D3} - N_{\overline{D3}} + N_{flux} - \frac{1}{2} N_{O3} = 0. \tag{3.27}$$

In practice, both the number of antibranes and the number of flux quanta (through the warp factor) influence the D-coefficient in the uplift potential. The full potential finally reads

$$V = \frac{\pi a c e^{-2\pi a \tau}}{\tau^2}\left(\frac{2\pi a c \tau e^{-2\pi a \tau}}{3} + W_0 + c e^{-2\pi a \tau}\right) + \frac{D}{8\tau^3}, \tag{3.28}$$

and for an appropriate selection of fluxes, it seems to lead to a dS vacuum.

Several aspects of the KKLT construction have been scrutinized, and no clear consensus exists in the community regarding its controllability or, in its absence, what is the issue that leads to the failure of the construction. While we acknowledge this is a very interesting discussion with important implications,here we merely point to recent reviews related to dS in string theory [118, 119] and references therein.

Large Volume Scenario

Let us briefly review the Large Volume Scenario (LVS) [120] by comparing its salient features to KKLT. For simplicity, instead of presenting the scenario in its full generality, we only sketch a case where the compactification manifold is of Swiss-cheese type and is parametrized by two Kähler moduli T_b, T_s, corresponding to the overall volume and the size of the "holes", respectively. A manifold that indeed realizes $h^{1,1} = 2$ is $\mathbb{CP}^4$ [120]. For a detailed description of the full scenario and important phenomenological applications, we point the reader to, e.g., [121]. To stabilize the moduli, one proceeds as follows:

- The stabilization of the complex structure moduli and the axio-dilaton at a parametrically large scale happens similarly to KKLT, using 3-form fluxes.
- The Kähler moduli are stabilized via a combination of two effects. On the one hand, the Euclidean D3-brane instantons wrapped on the small cycle contribute a term $W_s = A_s e^{-a_s T_s}$, similarly to KKLT. On the other hand, the novelty of the LVS lies in also including the leading α'-correction to the Kähler potential for the Kähler moduli, which now becomes

$$\mathcal{K} = -2 \log \left(\frac{1}{9\sqrt{2}} (\tau_b^{3/2} - \tau_s^{3/2}) + \frac{\xi}{2 g_s^{3/2}} \right). \tag{3.29}$$

  Here $\xi > 0$ is a constant and $\tau_i = Re(T_i)$. Approximating the overall volume of the internal manifold by $\mathcal{V} \approx \tau_b^{3/2}$, one can explicitly compute the scalar potential in terms of $\tau_s, \mathcal{V}$. The potential consists of three terms, and an AdS minimum can be found. There, $\mathcal{V} \sim \sqrt{\tau_s} e^{a\tau_s} \gg 1$, and all the terms exhibit the same scaling, so:

$$V_{AdS} \sim -\frac{1}{\tau_s} e^{-3a\tau_s} \sim \frac{1}{\mathcal{V}^3}. \tag{3.30}$$

  The LVS AdS vacuum has two main differences compared to the KKLT one. First, since it arises as a balance of three terms, no tuning is necessary for the value of W_0. Second, it is not supersymmetric.
- The uplift to dS happens again by introducing an explicit effect that contributes positively to the scalar potential. If this ingredient is an anti-D3-brane, similarly to the KKLT case, the brane will dynamically go to the tip of the throat, giving a redshifted contribution.

The controllability and stability of the LVS construction are not guaranteed and have been the topic of debate. Part of the criticism originates from within the Swampland Program, which will be the subject of the following chapter.

Chapter 4

The Swampland Program

This chapter constitutes a selective view of the Swampland Program, aiming to expose the reader to the main idea and only some conjectures relevant to our work. The interested reader may benefit from a plethora of excellent lecture notes and reviews of developments in the field [119, 122–127].

4.1 Main idea

Our discussion has only partially uncovered the many possibilities for low-energy effective field theories coming from string theory. In particular, we have seen that for any consistent four-dimensional model, there is a multitude of intertwined parameters and choices that influence the properties of the final theory: We have (superficially) discussed some of them, including the compactification manifold, potential orientifold projections and the accompanying O-planes, D-branes, non-trivial background fluxes. Considering all possible choices, one constructs the *string theory landscape*. Some famous estimates about the size of this landscape have been performed, ranging from 10^{500} vacua for IIB compactifications [23] to even $10^{272.000}$ in the framework of F-theory compactifications [24].

However, one should not forget that consistent vacua do not arise randomly. On the contrary, string theory has a clear-cut set of constraints and consistency requirements that significantly reduce the possible vacua. Let us list a few: modular invariance, conformal invariance, absence of anomalies, tadpole cancellation, and unitarity. Hence string theory is pretty restrictive at a fundamental level. In particular, at a certain energy scale, only a finite number of vacua is postulated to be present [128]. This is not accidental but rather intrinsically related to the fact that string theory is a theory of quantum gravity.

The Swampland Program [27] aims to deepen our understanding of the fundamental constraints accompanying a consistent theory of quantum gravity. In particular, the main idea is that a low-energy effective field theory might seem consistent as a stand-alone theory, but coupling the theory to gravity may uproot its consistency. This naturally leads to the following definitions:

Landscape/Swampland Definition: [27]
Consider the set of consistent effective-field theories, where consistency is based on EFT criteria, such as the absence of anomalies and unitarity. The subset of these theories that *can* be UV-completed to a theory of Quantum Gravity(QG) are said to belong in the **landscape**, while the complementary subset of theories that *do not* admit such a completion form the **swampland**.

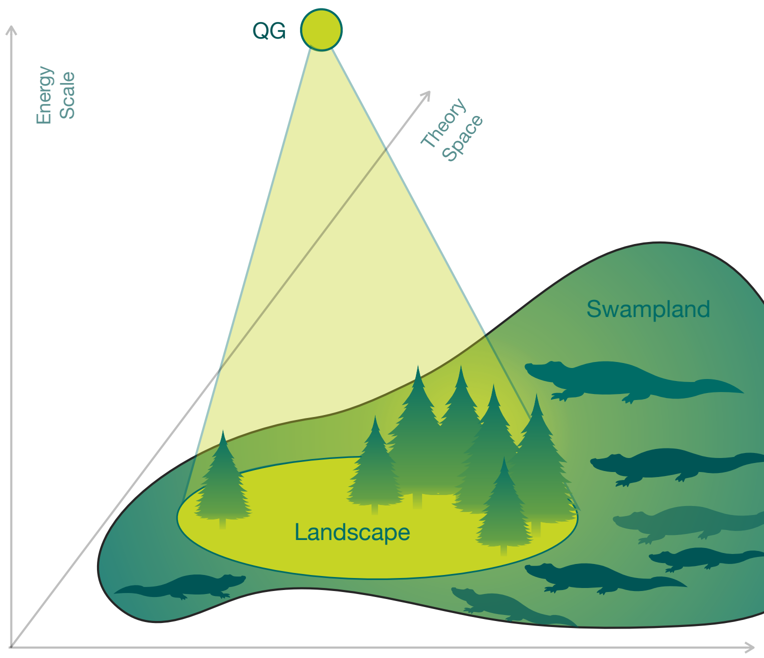

Figure 4.1: Schematic Depiction of the landscape and the swampland.

4.2 Some fundamental conjectures

The Swampland Program culminates in developing an intricate set of interconnected *swampland conjectures*, which are quantitative statements of conjectural nature that are postulated only to be true for theories in the landscape. Interestingly, the conjectures are customarily trivially satisfied upon decoupling gravity. Some of these conjectures are supported by our understanding of black hole physics, which are systems where the quantum nature of gravity becomes evident. Other conjectures are motivated by some universal behavior of well-controlled string theory vacua or the absence of it. Mutual support between conjectures is another factor that lends credence to the whole web of conjectures, hinting at some underlying physical reasoning.

Formulating conjectures is only a part of the Swampland Program. A significant effort is also being made regarding testing the conjectures and understanding their regime of applicability and limitations, properly refining or generalizing them.

Multiple conjectures, e.g., [28–30, 34, 129–145], have been formulated, refined, and generalized to this date. It goes far beyond the scope of this thesis to try even to enumerate all of them, let alone review them. However, allow us a few comments on some general themes that arise.

As shown in figure 4.2, three conjectures can be viewed as the pillars of the Swampland Program. These are the No Global Symmetries Conjecture (see, for instance, [146]), the Distance Conjecture [130], and the Weak Gravity Conjecture [129]. The rich web of conjectures surrounding them has its own very high merit but can usually be related in one or more ways to one or more of these three conjectures. In this thesis, only the first two of the conjectures mentioned will become relevant, so the next part of this chapter will be devoted to presenting their basic features.

4.3 No Global Symmetries Conjecture

The first swampland conjecture we will discuss is the No Global Symmetries Conjecture. This particular idea/statement was already hinted at/formulated well before the beginning of the Swampland Program, starting already in the 1950s and throughout the last century, see, e.g., [147–150], and had the status of a "folk theorem" [146]. While there is no preferred paper to which the conjecture is attributed, [146] is one of the most frequently cited references. The conjecture itself is quite straightforward, yet the implications are very deep.

> **No Global Symmetries Conjecture (see e.g. [146]):**
> A theory coupled to gravity cannot have any exact global symmetries.

4.3.1 Supporting arguments

The statement above stands as a conjecture; however, there have been strong arguments to support it. The most widespread are the black hole argument and the holographic argument, outlined below.

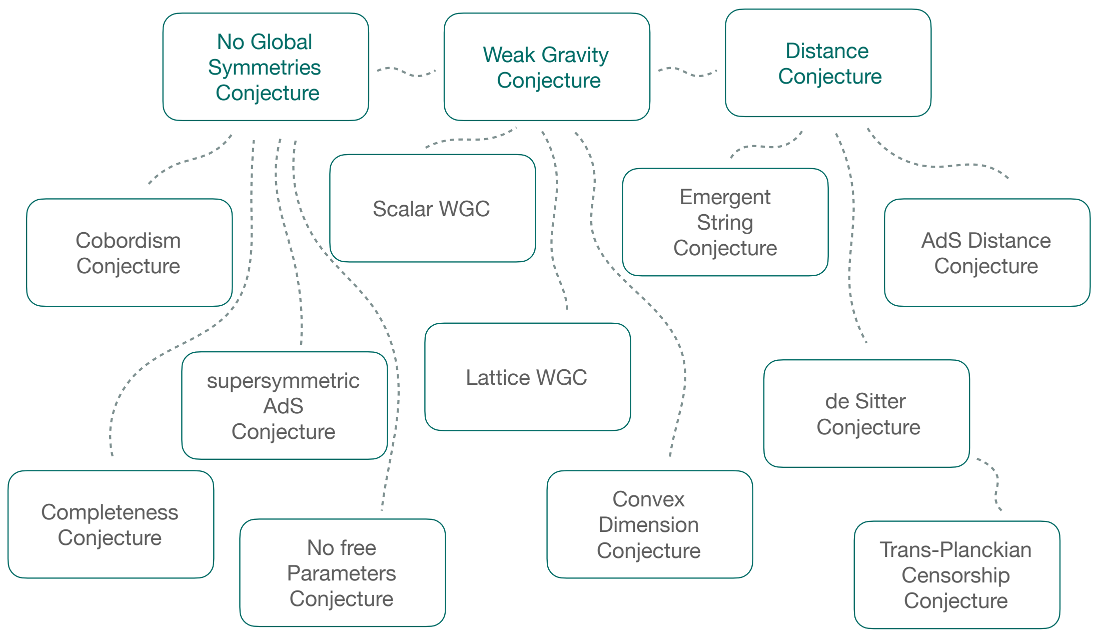

Figure 4.2: The swampland conjectures form an intricate web of interconnected statements. Note that the conjectures depicted above and the interrelations explicitly drawn are very far from complete.

The black hole argument

The first argument against the existence of global charges is the so-called *black hole argument*. In this case, strong support comes from black hole physics/thermodynamics, and one does not need to resort to explicit string theoretical examples. Several variations of this argument for an abelian global symmetry can be found in the literature, e.g., in [34, 129, 146, 149]. A nice overview of the historical evolution of the BH argument can be found in the TASI lecture notes [151]. Here we sketch the argument in [146], which had the novelty of using the covariant entropy bound.

The argument goes as follows: One starts with the assumption that quantum gravity admits a continuous global symmetry - for simplicity, we discuss $U(1)$ here. Hence a particle can be charged with some large charge Q. Suppose that now this Q-charged particle is thrown into an uncharged black hole[1]. Since we are dealing with a quantum gravitational theory, the black hole will start evaporating through Hawking radiation [18]. However, the evaporation process is blind to this global charge, as the horizon may depend only on the mass, angular momentum, and gauge charges [152], and the outgoing radiation spectrum does not carry the global charge, or, equivalently, carries equal amounts of positive and negative global charge. Assuming there is no direct violation of the charge conservation, which would immediately invalidate our starting assumption, the global charge Q remains in the black hole. The black hole will continue shedding its mass and shrinking, and evaporation terminates, leaving behind a remnant of charge Q and size $R \sim \Lambda_{QG}^{-1}$, with Λ_{QG} the cut-off of our effective theory. This happens for an arbitrarily large initial charge Q, resulting in an infinite number of remnants.

The covariant entropy bound [153,154] bounds (from above) the entropy at any finite spacetime region by the entropy of a black hole occupying the same spacetime region. Having infinitely many remnants clearly violates this bound, signaling an inconsistency in the effective theory. One can alternatively argue that a theory with an infinite number of remnants is problematic due to the nullification of the renormalized Planck constant [155] or simply due to our lack of observational evidence for the remnants.

One way to avoid such inconsistencies is to consider a *gauged* continuous symmetry. Consider this time a gauged $U(1)$, as in [151]: the gauge field lines are penetrating the horizon, hence driving the black hole evaporation towards depletion of the charge [149]. The final state is either the total evaporation of the black hole or a black hole with $M = M_{extr}$ saturating the extremality bound [16]. The extremality bound leads to an upper value for the charge within a specific space, which, combined with the charge quantization, leads to only a finite number of states in accordance with the covariant entropy bound. Hence gauging the global symmetry renders it compatible with quantum gravity.

One might wonder if there is any other hidden assumption that can lead to a loophole. Indeed, there is: the quantum gravity theory should allow for a unitary black hole evaporation, consistent with the Bekenstein-Hawking entropy formula [156]. This is not always the case for theories of three or fewer spacetime dimensions, where examples of global symmetries have been identified [156].

Finally, the last limitation of the black hole argument is the type of global symmetry. Up to now, we have only discussed continuous global symmetries. It turns out that the argument also goes through for infinite discrete symmetries, such as $SL(2,\mathbb{Z})$, but it fails for finite discrete symmetries, such as $\mathbb{Z}_n$. Luckily, the holographic argument also covers these discrete symmetries.

[1] This staring setup can also be viewed in a slightly different light: one can put together enough massive particles of charge q_i, such that the system gravitationally collapses into a black hole of charge $\sum_i q_i = Q$.

The holographic argument

The holographic argument [157, 158] applies to theories that admit a holographic description and can be roughly sketched as follows:[2] Assuming a global symmetry in the bulk, a local operator $\phi(x)$ can be charged under it. The global symmetry extends to the boundary CFT. If one splits the boundary into disjoint spatial regions $\{R_i\}$, the action of the unitary operator $U(g)$ representing the symmetry can also be split up into a product of $U(g, R_i)$, each of them localized on some disjoint cover of the boundary spatial slice, as

$$U(g) = U(g, R_1) \cdot U(g, R_2) \cdot \ldots \cdot U(g, R_n) U_{\text{edge}}, \tag{4.1}$$

with U_{edge} an operator with support close to the boundaries of the R_i which takes care of the arbitrariness at those boundaries. However, since the partition into R_i can be chosen arbitrarily, one can have R_i small enough, such that the union of their respective entanglement wedges does not contain the central part of the bulk, see figure 4.3. This means that the charged operator $\phi(x)$ localized at the center of the graph would then commute with U_g, which leads to an obvious contradiction.

Once again, running the previous argument does not lead to contradiction if one starts with a long-range gauge symmetry in the bulk instead of a global one. In that case, Wilson lines would attach any charged operator to the boundary, always intersecting the entanglement wedges, no matter how small. Then $U(g, R_i)$ would be able to detect the charged operator, hence avoiding any inconsistency.

One final comment regarding the holographic argument: There is no direct restriction on the nature of postulated problematic global symmetry - it applies equally well to continuous or discrete p-form global symmetries, at least for $0 \leq p \leq d - 2$.

4.3.2 Generalized higher-form symmetries and their trivialization

The No Global Symmetries conjecture is believed to hold when the notion of symmetry is extended, i.e., not only for ordinary zero-form symmetries but also for generalized symmetries. For instance, it was argued in [156] that one can appropriately modify the black hole argument to include higher-form symmetries [159]. Works regarding this generalized version of the conjecture concern, for instance, higher-form symmetries [160], non-invertible symmetries [161, 162] and Chern-Weil symmetries [163].

Here we will provide a brief overview of higher-form global symmetries [159] since they will prove relevant for our discussion of cobordism in 4.4. Recent pedagogical reviews of such symmetries can be found in [151, 164].

[2] The holographic "dictionary" or the following short section is taken for granted, for more detailed explanations, we refer the interested reader to the original papers.

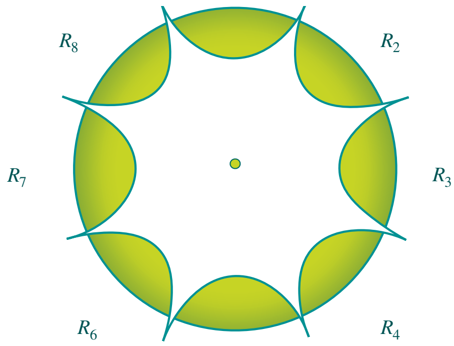

Figure 4.3: The entanglement wedges (colored regions) on which the boundary global symmetry operator is supported does not contain the center of the bulk where the charged operator $\phi(x)$ is located.

Let us first describe a usual 0-form global symmetry in d spacetime dimensions using the conventions of [151]. This symmetry acts on local operators, i.e., operators living on a point. The charged objects are particles with a $(0+1)$-dimensional world-line. The charge is measured on codimension-one surfaces through spacetime, and the symmetry operators are hence of the form $U(\Sigma^{d-1}, g)$, where Σ^{d-1} is a closed $(d-1)$-dimensional manifold. In case the symmetry is continuous, there exists a conserved 1-form current j and a corresponding $(d-1)$-form current J_{d-1}, which is closed, i.e., $dJ_{d-1} = 0$.

Similarly, a *p-form global symmetry* acts on operators supported on p spatial dimensions. The charged objects have $(p+1)$-dimensional world-volumes, and the symmetry operators are of the form $U(\Sigma^{d-p-1}, g)$, i.e., the closed manifold Σ is now of codimension-$(p+1)$. The associated conserved current for a continuous symmetry is a $(p+1)$-form j_{p+1}, while one can describe the symmetry using the closed $(d-p-1)$-form current $dJ_{d-p-1} = 0$.

The question that naturally arises is how to deal with a theory that *seems* to have a (continuous) global symmetry, at least at the energy scales accessible through the EFT. Two possible resolutions trivialize the global symmetry and place the theory back into the landscape: the symmetry can either be *broken* or *gauged*. The former option means that the initially conserved current $dJ_{d-p-1} = 0$ is either not conserved anymore due to the presence of symmetry-breaking defects Δ, i.e., $0 \neq dJ_{d-p-1} = \delta^{(d-p)}(\Delta)$, while for the latter option one couples the current to a $(p+1)$-form gauge field via $A_{p+1} \wedge J_{d-p-1}$, which in turn leads to J_{p+1} being exact.

The upshot of the No Global Symmetries conjecture is that quantum gravity is not compatible with any global symmetries. Suppose we start with an EFT with such a global (not gauge) symmetry, which must be broken. The conjecture does not indicate the energy scale at which the global symmetry should break or how much the deviation from an exact global symmetry should be. In practice, we are dealing with a very wellaccepted conjecture with little predictive power. One can try to circumvent this issue by making a bolder proposal [165], imposing that the symmetry is *strongly* broken. This leads to more precise quantitative statements, similar to other well-known conjectures, such as the Weak Gravity Conjecture or the Distance Conjecture.

There exists, however, another way in which the absence of global symmetries can lead to concrete predictions. This relies on further generalizing the notion of symmetry to include topological charges. To properly discuss this, let us introduce cobordism.

4.4 Cobordism Conjecture and implications

4.4.1 Cobordism

Quantum gravity naturally involves topology-altering processes [166–169], which become particularly relevant at Planck-length scales [170]. Cobordism is a way to define equivalence classes of manifolds permitting certain topology changes. Cobordism is a generalized homology theory classifying compact manifolds of the same dimension, i.e., it can be defined axiomatically similarly to a usual homology theory, with the notable exception that the higher groups of the point do not have to vanish [171, 172].

Definition:
Two smooth, closed, unoriented manifolds M_1, M_2 of real dimension k are **cobordant**, i.e. $M_1 \sim M_2$, if there exists a manifold W of real dimension $k+1$ such that

$$M_1 \sqcup M_2 = \partial W \,, \tag{4.2}$$

i.e., the boundary of W is the disjoint union of M_1, M_2.

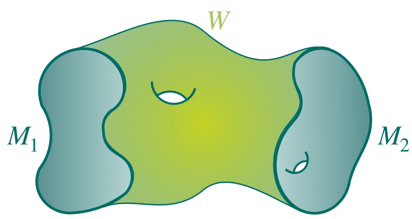

Figure 4.4: The manifolds M_1, M_2 are cobordant, as $M_1 \sqcup M_2 = \partial W$.

A single manifold M, which itself is a boundary of a higher dimensional manifold, is by definition cobordant to the empty set, hence belongs to the trivial cobordism class $[M] = [\emptyset] = 0$. Considering the disjoint union as a group operation, one turns the cobordism equivalence classes into a group. Since any manifold is cobordant to itself, one can see that two copies of the same manifold M are in the trivial cobordism class - it suffices to smoothly fold the cylinder connecting the two copies in half. Hence the inverse element of M under the disjoint union is M itself. These properties are sketched in figure 4.5. One important consequence of using the disjoint union as the group operation is that the cobordism group is always abelian.

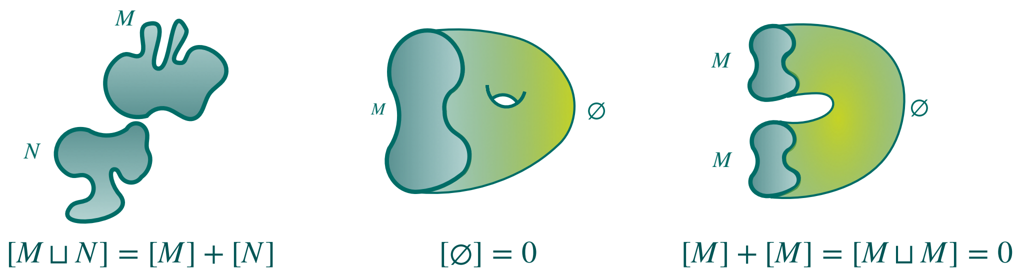

Figure 4.5: The disjoint union is the operation under which cobordism is a group. From left to right: addition, identity element (vacuum), inverse element.

One can systematically restrict the allowed topology changes by considering cobordism groups with a ξ-structure, Ω_k^ξ. The ξ-structure usually encodes restrictions on the allowed tangential structure of the manifolds M, such as having an orientation or admitting fermions. In general, the structure is of paramount importance for physical applications, as it dictates which type of topology transformations are compatible with the kinematics of the physical theory.

Depending on the structure, the definition of the cobordism group needs to be appropriately modified. For instance, the cobordism group operation between two oriented manifolds M, N is $M \sqcup \bar{N}$, i.e., it is necessary to reverse the orientation of the second manifold.

The cobordism groups introduced above as Ω_k^ξ are not the most general cobordism groups that can be defined. One can specify an additional topological space X, of dimension possibly different than k, or even infinite, and can then define $\Omega_k^\xi(X)$, i.e., the *ξ-cobordism groups of* X. Consider continuous maps $f : M_1 \to X$ and $g : M_2 \to X$. The pairs (M_1, f) and (M_2, g) are cobordant if there is a cobordism W, such that $\partial W = M_1 \sqcup M_2$, together with a map $h : W \to X$ appropriately restricting to f and g at the boundary ∂W. This is schematically depicted in figure 4.6. These equivalence classes once again turn into a group under disjoint union, and one gets the group $\Omega_k^\xi(X)$. This more general definition includes our former definition of cobordism groups. In the case $X = pt$, the necessary continuous maps trivially exist, and we have $\Omega_k^\xi \equiv \Omega_k^\xi(pt)$.

The cobordism groups of X are generally larger than those of the point. This statement can be made precise using the Splitting Lemma for abelian groups. Consider the forgetful map

$$\phi : \quad \Omega_k^\xi(X) \to \Omega_k^\xi(\mathrm{pt}) \,, \tag{4.3}$$

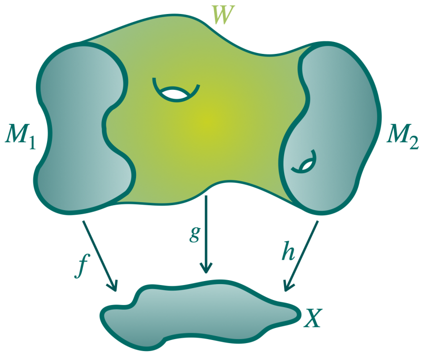

Figure 4.6: The pairs (M_1, f), (M_2, h) are cobordant, as $M_1 \sqcup M_2 = \partial W$ and $g|_{M_1} = f$, $g|_{M_2} = h$ with all the maps continuous.

with $\phi([M, f]) = [M]$. The *reduced cobordism group* $\tilde{\Omega}_n^\xi(X)$ is given by the kernel of this map, i.e., $\ker \phi \equiv \tilde{\Omega}_n^\xi(X)$. By definition, we also have $\tilde{\Omega}_n^\xi(\mathrm{pt}) = 0$. Since ϕ is surjective, one gets the short exact sequence

$$0 \longrightarrow \tilde{\Omega}_n^\xi(X) \longrightarrow \Omega_n^\xi(X) \xrightarrow{\phi} \Omega_n^\xi(\mathrm{pt}) \longrightarrow 0, \tag{4.4}$$

which is split. Therefore, Ω_k^ξ is given by the simplest possible combination of the other two groups, their direct sum, as

$$\Omega_k^\xi(X) = \Omega_k^\xi(\mathrm{pt}) \oplus \tilde{\Omega}_k^\xi(X). \tag{4.5}$$

The above construction is independent of the selected structure. Hence the Splitting Lemma for Cobordism (4.5) holds for any ξ.

The reader might recall from section 2.5.2 that a similar relation exists for the K- and KO-theory groups. This is not accidental but rather indicative of a deeper relation between cobordism and K-theory, which has profound physical implications and will be the main subject of our upcoming chapter 6.

4.4.2 Cobordism and Quantum Gravity

Cobordism is a natural language to describe equivalence classes of quantum gravitational theories since it encompasses the topology-changing processes. Compact manifolds are useful for going from a higher-dimensional theory to a phenomenologically relevant four-dimensional one. Consider, for instance, type IIA and type IIB theories: Type IIB compactified on a circle of radius r is T-dual to type IIA compactified on a circle of dual radius r'. It is natural to assume that these theories belong in some common equivalence class of Ω_1^ξ. Since the web of dualities in string theory goes much beyond T-duality, one can assume this is only part of the bigger picture.

This idea was explored in [34], where the *Cobordism Group of Quantum Gravity* Ω^{QG}, with the superscript standing for the -still unknown- *Quantum Gravity structure*, was introduced. Staring with a d-dimensional theory and compactifying on a k-dimensional compact manifold down to $D = d - k$ dimensions, two complementary definitions of Ω^{QG} were proposed:

- $\Omega^{QG,D}$: This group classifies all D-dimensional theories of quantum gravity, where the D-dimensions can be non-compact. The group interpolates between theories that differ by QG-allowed finite-energy processes.
- Ω_k^{QG}: This group classifies all theories compactified on a k-dimensional manifold. Evidently, if $D = d - k$, this physically should match $\Omega^{QG,D}$. Using Ω_k^{QG} is often more convenient, being much closer to the usual definition of cobordism and avoiding the non-compactness complications. The full, uncompactified quantum-gravitational theory also can be probed by looking at Ω_0^{QG}.

From now on, we will only consider Ω_k^{QG}, and, more generally, Ω_k^{ξ}. Physically, the picture is as follows: The starting point is the d-dimensional theory, and then one compactifies on the compact manifold M_i^k, getting a D-dimensional effective field theory EFT_i. If $M_1 \sim M_2$, a finite-energy domain wall exists between the two EFTs. The trivial class is even more interesting: no D-dimensional EFT corresponds to the empty set, and spacetime ceases existing. Hence, any compactification on $M \sim \emptyset$ leads to a D-dimensional EFT in which a bubble of nothing [173] can pop up and expand, consuming all of the spacetime using a finite amount of energy [174].

Figure 4.7: Cobordant manifolds and associated EFTs.

4.4.3 Cobordism Conjecture

One can define *cobordism invariants* for any cobordism group Ω^{ξ}. These are functions $a : M \to A$, with A some abelian group, such that $M_1 \sim M_2 \implies a_i([M_1]) = a_i(M_1) = a_i(M_2)$ for all i, i.e., the cobordism invariants take the same values for all manifolds in the same cobordism class. The number of cobordism invariants depends on the group and is the same as the number of group generators. While the trivial cobordism class $[\emptyset] = 0$ always corresponds to vanishing cobordism invariants $a_i([\emptyset]) = 0$, any other class is characterized by at least one non-vanishing invariant. This effectively quantifies the obstruction for M with $a(M) \neq 0$ to disappear via a topologically allowed process - alternatively, one can say that the cobordism invariants define topological global charges and can write

$$Q_i(M) = a_i(M). \tag{4.6}$$

Hence, a clear correspondence exists: Any non-trivial cobordism group leads to non-trivial global charges, i.e., some p-form global symmetry. Using the dimensional "dictionary" of section 4.3.2 and noting that the codimension-$(p+1)$ symmetry operators now live on the k-dimensional compactification manifolds, we have $d-p-1=k$. Hence, we have

$$\Omega_k^\xi \neq 0 \leftrightarrow (d-k-1) - \text{form global symmetry}. \tag{4.7}$$

The charged objects should now have $d-k-1$ spatial dimensions, hence are $(d-k)-$ dimensional. According to [34], they have an intuitive interpretation as gravitational solitons [175]: Consider for example $\Omega_k^\xi \neq 0$ and M^k with $\alpha(M) \neq 0$. Consider also the flat space $\mathbb{R}^k$, and cut out spheres S^{k-1} from each initial space, gluing them along the cut-out spheres to form $M^k \# \mathbb{R}^k$. This can be viewed as a defect with $(d-k)$-dimensional world-volume. Hence the gravitational soliton interpretation is justified.

The black hole argument goes through for the cobordism charges [34], with the only difference that now the global charge is topological. Since gauged symmetries evade the argument, one could wonder whether the cobordism charges can be a priori gauged. However, away from the manifold M, there is no way to detect its cobordism charge - the space is locally identical to flat space. Hence, the symmetry is not gauged and poses an inconsistency. This naturally leads to the Cobordism Conjecture.

Cobordism Conjecture [34]:
In a theory of quantum gravity, the cobordism groups Ω_k^{QG} are trivial, i.e.

$$\Omega_k^{QG} = 0. \tag{4.8}$$

What is the relevant structure?

Since we do not know the full Quantum Gravity structure, we generically use an *approximate* quantum gravitational structure, denoted by $\widetilde{QG}$. The cobordism group $\Omega_k^{\widetilde{QG}}$ then classifies manifolds that admit the structure $\widetilde{QG}$, and which are cobordant if their disjoint union is a $\widetilde{QG}$-manifold of one dimension more.

Let us list several well-motivated options for tangential structures $\widetilde{QG}$, which arise before turning on gauge fields. In principle, the structure can also include data about gauge fields or geometric data. A nice overview and more details on a mathematically precise definition of cobordism groups with certain structures can be found in [176]. Luckily, the cobordism groups of the point with the structures mentioned below are known [177, 178], and presented for $0 \leq k \leq 10$ in table 4.1.

- $\Omega_k^{\widetilde{QG}} = \Omega_k^{\mathrm{SO}}$: The SO-structure means that the relevant manifolds are orientable and is ensured by the vanishing of the first Stiefel-Whitney class

$$w_1(TM) = 0. \tag{4.9}$$

All relevant Stiefel-Whitney classes are defined in appendix B. This requirement is usually a bit weak, but upon also considering gauge fields, it is enough to lead to non-trivial results, such as the existence of Dp-branes [34].

- $\Omega_k^{\widetilde{QG}} = \Omega_k^{\mathrm{Spin}}$: This is one of the most straightforward structures one can demand. The Spin tangential structure ensures that the manifolds are spin so that spinors can be consistently defined. This is necessary for any supersymmetric theory, so it is a good starting point for type II supergravities and string theories. The mathematical condition that ensures that a manifold M is spin, and hence can be classified in Ω_k^{Spin} is:

$$w_1(TM) = 0 \ \ \& \ \ w_2(TM) = 0, \tag{4.10}$$

where w_2 is the second Stiefel-Whitney class. Clearly, any orientable manifold with $w_2(TM) = 0$ is also spin.

- $\Omega_k^{\widetilde{QG}} = \Omega_k^{\mathrm{Pin}^+}$: If we want to consider a non-orientable theory, such as M-theory, it is clear that none of the above structures is suitable. It was argued in [179] that the relevant structure, in this case, is Pin^+, defined by

$$w_1(TM) \neq 0 \ \ \& \ \ w_2(TM) = 0, \tag{4.11}$$

and allows for spinors in a non-orientable manifold[3].

- $\Omega_k^{\widetilde{QG}} = \Omega_k^{\mathrm{Spin}^c}$: F-theory relaxes the need for a Spin structure to that of a Spin^c structure. Spin^c is defined as a choice of a lift of w_2 from $\mathbb{Z}_2$-cohomology to $\mathbb{Z}$-cohomology, and for a manifold to be Spin^c one needs

$$W_3(TM) = \beta w_2(TM) = 0. \tag{4.12}$$

W_3 is the third integral Stiefel-Whitney class, and β the Bockstein homomorphism, corresponding to reduction modulo 2. It is clear that all Spin manifolds are $Spin^c$, but the inverse does not hold. Physically, Spin manifolds admit uncharged spinors, while Spin^c manifolds only admit charged spinors.

[3]There is a second structure that allows for such spinors, Pin^-, which comes from the opposite parity projection and requires $w_2(TM) + w_1^2(TM) = 0$.

k	0	1	2	3	4	5	6	7	8	9	10
Ω_k^{SO}	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	0	0	$\mathbb{Z}^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$
Ω_k^{Spin}	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$
$\Omega_k^{\mathrm{Spin}^c}$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}^2$	0	$\mathbb{Z}^2$	0	$\mathbb{Z}^4$	0	$\mathbb{Z}_2 \times \mathbb{Z}^4$
$\Omega_k^{\mathrm{Pin}^+}$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$	0	0	0	$\mathbb{Z}_2 \times \mathbb{Z}_{32}$	0	$\mathbb{Z}_2^3$
$\Omega_k^{\mathrm{String}}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2 \times \mathbb{Z}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_6$

Table 4.1: Cobordism groups Ω_k^{ξ} for various dimensions k and ξ-structures.

- $\Omega_k^{\widetilde{QG}} = \Omega_k^{\mathrm{String}}$: For Heterotic theory, one starts with a Spin structure and on top of it has to satisfy the constraint $dH = \frac{p_1(R)-p_1(F)}{2}$. Whenever the gauge field F vanishes, this leads to a String structure, defined by:

$$\frac{p_1(R)}{2} = 0, \quad w_1(TM) = 0 \;\;\&\;\; w_2(TM) = 0, \tag{4.13}$$

  where p_1 is the first Pontrjagin class.

Much work has been dedicated recently to identifying sensible structure approximations for different string theories. For instance, in [176], it was proposed that one can organize the approximations in terms of a Whitehead tower for the orthogonal group. The situation becomes considerably more involved upon turning on gauge fields, yet even in this case, progress has been made. Recently, for example, an explicit result for cobordism groups $\Omega^{\xi^{het}}$ of heterotic theory with non-trivial B-field was given in [180].

4.4.4 Implications

As shown in table 4.1, the approximate cobordism groups are far from trivial; hence under these approximate structures, global symmetries are present. Identically to usual higher-form symmetries, there are two ways to trivialize the cobordism global symmetry, breaking or gauging. Let us discuss these options in slightly more detail.

- **Breaking:** In this case, one proceeds under the assumption that the reason that the cobordism group $\Omega_k^{\widetilde{QG}}$ does not vanish is having missed one or more $(d-k-1)$-dimensional defect which explicitly breaks the symmetry. Including the defect leads to a modified structure $\widetilde{QG}$ + defect, now with $\Omega_k^{\widetilde{QG}+\text{defect}} = 0$, i.e. there exists a map

$$\Omega_k^{\widetilde{QG}} \to \Omega_k^{\widetilde{QG}+\text{defect}}. \tag{4.14}$$

  Following once again [34], one says that the cobordism charges in the kernel of the map above are *killed*. The formerly closed J_k current now satisfies

$$0 \neq dJ_k = \sum_{\text{def } j} \delta^{(k+1)}(\Delta_{d-k-1,j}) \,, \tag{4.15}$$

where the δ-functions are the Poincaré dual of the $(d-k-1)$-cycles wrapped by the defects.

– **Gauging:** In this case, similarly to the usual gauge symmetries, one imposes that the cobordism charges over a compact manifold have to vanish. Even though $\Omega_k^{\widetilde{QG}} \neq 0$, the trivial class always belongs in this group. Hence all theories that arise by compactifying on these trivial manifolds are not plagued by any inconsistency. The map that describes this procedure is [34]

$$\Omega_k^{\widetilde{QG}+\text{g.fields}} \to \Omega_k^{\widetilde{QG}}, \tag{4.16}$$

and now the trivial classes are in the co-kernel of the map above. Conventionally, one says that the cobordism chances are *co-killed*. The current J_k remains closed, but this time it is also exact,

$$J_k = dF_{k-1} \,. \tag{4.17}$$

Finally, we should note that, in reality, the above procedure is more complicated. For instance, breaking by including a single defect in the structure might not fully trivialize the group but only reduce its rank. Hence, a fully vanishing cobordism group requires additional structure modification, either by gauging or breaking again. In general, the whole procedure is iterative, and the calculation of each step is becoming increasingly mathematically involved.

At this point, the usefulness of the Cobordism Conjecture should be crystal clear. It is a highly predictive statement, which allows us to identify yet unknown defects. Not only can a defect be predicted, but one has additional information about it, such as its dimensionality. Moreover, the whole procedure does not explicitly depend on supersymmetry - hence, the cobordism conjecture is a particularly useful way to probe setups where traditional methods usually lack control.

The Cobordism Conjecture has gained much attention in the past few years, yielding very interesting results. For instance, new non-supersymmetric objects have been predicted [181, 182] and described [183–186], while significant results have been achieved regarding anomaly cancellation using cobordism - see, e.g., [181, 182, 187–190]. The cobordism conjecture has been studied in an AdS background [191]. A connection to the Ricci flow conjecture has been uncovered in [192]. A relation between K-theory and cobordism, useful for gauging cobordism charges, has been established in [35]. Our work [39] carefully examined this proposal and its implications and will be presented in chapter 6. Finally, a dynamical description of cobordism [37], with a direct connection to the distance conjecture [36] has been established, and successfully implemented in a wide range of setups [40, 193–196]. Our work [40], which provides an explicit description of a novel cobordism-predicted defect, will be the main subject of chapter 7.

While there is much more to say about cobordism, let us defer that to chapters 6 and 7, and allow us to continue our journey along the borders of the swampland.

4.5 Distance Conjecture and infinite distances

4.5.1 Distance Conjecture and variations

Before discussing the distance conjecture, we must discuss the concept of moduli space. Any point of the moduli space is, in a sense, a different incarnation of a theory, and one moves along the moduli space by changing the values of the moduli fields. In the case of string theory, and particularly for supersymmetric compactifications, moduli spaces naturally arise during the compactification procedure. The geometry of the compactification manifold and other details of the compactification, such as the positions of the localized objects, are all encoded in the moduli fields. Moreover, this scalar moduli space is equipped with a metric via the kinetic terms of the moduli, encoded in the Kähler potential in supersymmetric cases. Hence the notion of distance is meaningful. Note that at this stage, the moduli fields are massless; hence we do not consider any potential.

Several universal properties have been conjectured to hold for the moduli spaces of theories of quantum gravity [130]: The moduli space $\mathcal{M}$ is of finite volume, yet, if one starts from a point P in $\mathcal{M}$, a different point $Q \in \mathcal{M}$ is postulated to exist such that the geodesic distance between these two points $d(P, Q)$ is larger than any positive number K. In particular, for $K \to \infty$, one talks about an *infinite distance* limit, depicted in figure 4.8. To accommodate infinite distances within a finite volume, $\mathcal{M}$ must have negative curvature close to the infinite distance points.

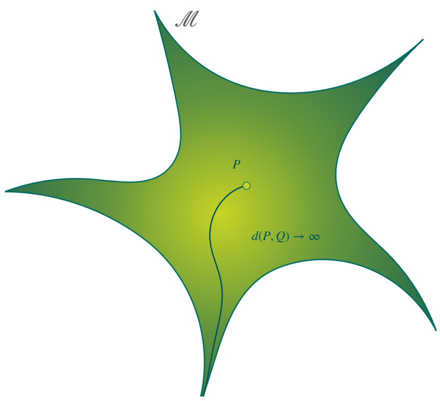

Figure 4.8: Moduli space and infinite distance limit from point P.

The Swampland Distance Conjecture [130] concerns the validity of the effective field theory when transversing large distances in the moduli space. In particular, the most striking behavior appears for these postulated infinite distance limits. Examples of such limits include the volume of the compactification manifold going to infinity or the string coupling g_s going to zero. In fact, we have the most control over our effective theories close to such limits in the asymptotic regime of the moduli space. For instance, the supergravity approximation is only valid in such a regime.

Swampland Distance Conjecture (SDC) [130]:
In the scalar field moduli space of a theory coupled to gravity, when moving from point p to point p_0, there appears an infinite tower of states with mass scaling like

$$M(p) \sim M(p_0)e^{-\lambda d(p_0,p)} \tag{4.18}$$

with $\lambda > 0$ some constant of order 1 in Planck units and $M(p_0)$ some associated mass scale. In an infinite distance limit, the tower becomes exponentially light.

The appearance of this infinite tower of light states signals that the initial theory is no longer a good description of the physical system, and the cut-off of the EFT needs to be appropriately adjusted. When exactly the control over the effective theory is lost, and what the physical realization of the light tower of states is, are not clear a priori. However, several variations of the Distance Conjecture attempt to answer these questions. We find it instructive to provide a partial list of these variations, refinements, and generalizations, as they beautifully illustrate the range of directions in which the Swampland Program extends:

- The Refined Swampland Distance Conjecture (rDC) [197,198] extends the range of validity of SDC also to include non-vanishing potentials, i.e., non-massless moduli fields. Moreover, the exponential behavior is postulated to become important over order-one distances in Planck units.
- The Sharpened Distance Conjecture [199] bounds the value of λ for the lightest tower in (4.18) by $\lambda \geq 1/\sqrt{d-2}$. Alternative (yet un-named) refinements were proposed recently [200], placing an absolute upper bound on the tower mass scale in the interior of moduli space.
- The AdS Distance Conjecture (ADC) [136] builds on the notion of distance between metrics, as it interprets the $|\Lambda| \to 0$ limit for AdS as an infinite distance limit, accompanied by infinite tower becoming light. We will expand on this in section 4.5.2.
- The Generalized Distance Conjecture (GDC) [136] is closely related to ADC and postulates the appearance of an exponentially light tower for large variations of the generalized distance between metric for *all* Einstein spaces, including dS.

• The Ricci flow conjecture [201] considers the Ricci flow for a family of metrics and anticipates a tower becoming exponentially light at a fixed point at an infinite distance.

• The Black Hole Entropy Distance Conjecture [202] extends the Distance Conjecture to black hole spacetimes and expresses the mass scale of the tower in terms of the entropy.

• The Convex Hull Swampland Distance Conjecture [141] is a reformulation of SDC in terms of a convex hull condition for a theory with multiple scalar towers characterized by their charge-to-mass ratio.

• The CFT Distance Conjecture [139] considers infinite distance limits in CFTs, proposing that they correspond to higher-spin points.

• The Gravitino Mass/Distance Conjecture [143,144] postulates that $m_{3/2} \to 0$ is an infinite distance limit, accompanied by an infinite tower of states becoming massless.

• The Cobordism Distance Conjecture [37] interprets infinite field distance limits as running into a cobordism wall of nothing in the theory and gives scaling relations similar to ADC. We will discuss this more extensively in chapter 7.

• The Emergent String Conjecture [137] concerns the light tower's nature, proposing only two possible physical realizations. It will be discussed in section 4.5.3.

4.5.2 AdS Distance Conjecture

Let us elaborate on some of the results of [136], where the near-flat limits of AdS (and other Einstein spaces) were explored. We start with the formulation of the AdS Distance Conjecture (ADC).

Anti-de Sitter Distance Conjecture (ADC) [136]:
For a quantum gravitational theory on a d-dimensional AdS space with cosmological constant Λ, there exists an infinite tower of states with mass scale m, in Planck units, which behaves in the $|\Lambda| \to 0$ limit as:

$$m \sim |\Lambda|^{\alpha}, \tag{4.19}$$

where α is an $\mathcal{O}(1)$ positive constant. The strong version of the conjecture additionally requires $\alpha = \frac{1}{2}$ for supersymmetric vacua.

The value of the constant α has great physical meaning: It was postulated in [136] that it is always $\alpha \geq \frac{1}{2}$ for AdS, which in turn prohibits parametric separation between the scale of AdS and the scale of the massive states. For $\alpha \geq \frac{1}{2}$, if the tower satisfying (4.19) is a KK tower, we cannot have scale separation between the scale of the internal manifold and the AdS radius, i.e., the vacua are not truly four-dimensional. While most examples support this claim, the question of scale separation for AdS vacua is not yet completely settled, as, for instance, the DGKT vacua [95] seem to exhibit scale separation, with $\alpha = 7/18$. Recent works on this topic include [203–219].

The Generalized Distance Conjecture (GDC) was also introduced in [136]. This is a version of the distance conjecture which applies to all tensor fields, and in the case of scalars, one recovers the original SDC. Practically, a field $\mathcal{O}_{M_1 \ldots M_n}$ has an accompanying metric in the field space given by its kinetic terms, which allows the definition of a generalized distance Δ. If the spacetime is a product manifold $M = X_d \times Y_k$, with S a non-compact Einstein manifold and Y the internal manifold,then the limit of large field variation one indeed gets a tower scaling like $m \sim e^{-\alpha\Delta}$. This analysis is independent of the sign of the cosmological constant and applies to de Sitter. Consequently, one expects a scaling relation like ADC to hold for de Sitter, and we have

$$m \sim \Lambda^{\alpha}, \tag{4.20}$$

where α is an order-one positive constant. We will refer to this relation/version of the conjecture as the *de Sitter Distance Conjecture (dSC)*, but we caution the reader that there seems to be no universally adopted name in the wider literature. The main difference to ADC is the value of the scaling constant. In this case, there is a strong physical constraint, unitarity, which is embodied in the form of the Higuchi bound [220] $m_{spin\geq 2} \gtrsim \Lambda^{1/2}$ and imposes $\alpha \leq \frac{1}{2}$ for $\Lambda \ll 1$.

4.5.3 Emergent String Conjecture

The Emergent String Conjecture (ESC) [137] builds on the idea that in the infinite distance limit, where one phenomenically loses control over the initial theory, there exists a dual description. It proposes there are only two alternative realizations for this dual description: either the emergence of a tensionless string or a decompactification. The formulation of the original paper, presented in the box below, makes this statement more precise.

Emergent String Conjecture [137]:
For a d-dimensional theory of quantum gravity, the infinite distance limits in the moduli space fall in either of the two categories:

– *Equi-dimensional infinite distance limit*: There, the theory reduces to a weakly coupled string theory, i.e., the infinite tower of asymptotically massless states forms the particle excitations of a unique weakly coupled tensionless type II or heterotic string. This type of limit is conventionally called **tensionless string limit**.

– *Non-equidimensional infinite distance limit*: There, the theory decompactifies at least partially, i.e., the infinite tower of states corresponds to Kaluza-Klein states along large dimensions, such that the total internal volume diverges in units of the higher-dimensional Planck scale. This type of limit is conventionally called **decompactification limit**.

The Emergent String Conjecture has been tested and verified in many non-trivial setups, see for instance [137, 221–229]. ESC will be used indirectly when explaining the Dark Dimension Proposal [33], which will be the main subject of chapter 5.

4.6 Species scale and emergence

4.6.1 Species scale

When examining the range of validity of a gravitational theory, the Planck mass M_p is naturally considered to be the UV cut-off. However, the appearance of infinite light towers of states, quantified by the Distance Conjecture, directly lowers the cut-off of the effective theory. There is a related well-established scale, the *Species Scale* [230, 231] $\tilde{\Lambda}$, which precisely quantifies the UV cut-off in the presence of multiple species of light particles. For an EFT coupled to gravity in d spacetime dimensions the species scale is given by:

$$\tilde{\Lambda} \approx \frac{M_p}{N_s^{\frac{1}{d-2}}}, \tag{4.21}$$

where N_s is the number of particle species *below* $\tilde{\Lambda}$.

The Species Scale (4.21) can be motivated both perturbatively [230, 232] or non-perturbatively [230, 231]. Consider $d = 4$ for simplicity. The perturbative estimation considers the contributions to the graviton propagator coming from the N_s particle species running in loops, and the species scale arises as the energy scale for which the loop corrections become comparable to the tree-level term. Then, the one-loop graviton propagator reads [231, 233]

$$\pi^{-1}(p^2) = 2p^2\left(1 - \frac{N_s p^2}{120\pi M_p^2}\log\big(-\frac{p^2}{\mu^2}\big)\right), \tag{4.22}$$

with μ some scale related to the renormalization. Ignoring the log-factor[4] the species scale is such that the two terms inside the parenthesis are comparable, i.e., $\frac{N\tilde{\Lambda}^2}{120\pi M_p^2} \sim 1 \leftrightarrow \tilde{\Lambda} \sim \frac{M_p}{\sqrt{N_s}}$, in accordance with (4.21).

At the non-perturbative level, one considers the smallest possible black hole within the EFT, with radius $r_{min} \approx 1/\tilde{\Lambda}$. For a large number of light modes, one can approximate its entropy by $S \sim N_s$. At the same time, the Bekenstein-Hawking entropy [16] for the same black hole can be expressed as $S_{BH} = \frac{M_p^2}{\tilde{\Lambda}^2}$. Equating these two entropy expressions once again leads to (4.21).

In the context of the Swampland, where towers of light states are expected to appear, the number of species effectively counts the number of states with masses $m < \tilde{\Lambda}$, according to the perturbative picture above. As we have seen from the Emergent String Conjecture, we expect two types of towers with different spacing and species scales. For a KK tower signaling decompactification, the species scale is the higher-dimensional Planck scale $M_{p,d+k}$, up to corrections due to the number of species in the theory that do not belong to the tower. For an emergent string tower the species scale turns out to be the string scale itself, up to logarithmic corrections.

There have been more recent developments regarding the species scale within the swampland. In the context of 4d $\mathcal{N} = 2$ theories, it was recently proposed in [234] that a moduli-dependent species scale can be defined as $\tilde{\Lambda} \sim \frac{1}{\sqrt{F_1}}$, with F_1 the one-loop topological free energy [235]. This claim was supported by a black hole argument in [236]. Moreover, the species scale has been analyzed from a thermodynamic perspective in [237]. Modular invariance was recently noticed to be useful for the calculation of a moduli-dependent species scale [238]. Finally, the interplay of the moduli-dependent species scale with other swampland conjectures has led to sensible bounds [239, 240].

4.6.2 Emergence

The Emergence Proposal [241–243] makes use of the ubiquitous presence of towers of light scalars in the asymptotic regimes of the moduli space to provide an underlying physical principle behind some of the most robust swampland conjectures. Integrating out the states in the towers can lead to the Distance Conjecture and the Weak Gravity Conjecture and can hence be viewed as a unifying principle for the swampland web.

[4] Subtleties regarding the log-factor in the definition of the species scale were recently discussed, e.g., in [229, 233].

Emergence Proposal [241–243]
The dynamics (kinetic terms) for all fields are emergent in the infrared by integrating out towers of states down from an ultraviolet scale Λ_s, which is below the Planck scale. In its strong version, the Emergence proposal claims that the tree-level kinetic terms vanish, while the mild version allows for tree-level terms as long as the emergent kinetics term is of the same functional form.

Since emergence will play only a small part in the remainder of this thesis, we refrain from providing more details and refer to a systematic study of the proposal and its implications was performed in [233, 244]. We also find worth mentioning that recently emergence in concrete orientifold compactification with 14 moduli was discussed in [229], while its relation to the stability of dS was uncovered in [245].

Chapter 5

The Dark Dimension in a warped throat

5.1 Preface

The Dark Dimension [33] is a recent idea by Montero, Vafa, and Valenzuela, which confronts the observed value of the cosmological constant with insights from the Swampland Program. In particular, the extremely small positive value of the observed cosmological constant Λ indicates that our universe lies in some asymptotic region of the moduli space, which, in turn, signals the appearance of a light infinite tower of states. Theoretical arguments, in tandem with compliance with experimental and observational data, lead to the prediction of a single mesoscopic extra dimension of the size of a few microns, accompanied by an infinite tower of states with masses scaling like $m \sim \Lambda^{\frac{1}{4}}$. This proposal has a strong potential of being experimentally testable within a few years, so further investigation is warranted.

The first part of this chapter, 5.2, will be a review of the Dark Dimension proposal, effectively tying together the ideas and conjectures laid out in chapter 4 leading to the single extra dimension and its signature $\Lambda^{1/4}$ tower. Moreover, we will very briefly review some implications of the proposal.

The second part of this chapter, 5.3, is based on our work [38], which discusses a possible realization of the Dark Dimension proposal in string theory using a very common feature of flux compactifications, the warped throat. We will start with a review of the salient properties of the conifold [246] and the Klebanov-Strassler [247] solution and how this fits within attempts to construct dS vacua within string theory. In section 5.3.2, we will see how this reproduces the coveted $\Lambda^{1/4}$-scaling, providing an option for the Dark Dimension realization. Possible shortcomings of such a realization are also discussed. Finally, in section 5.4, we close the chapter, commenting on newer developments regarding possible realizations of the Dark Dimension.

5.2 Reviewing the Dark Dimension proposal

The observed positive and small value of the cosmological constant [5] has been a long-standing problem in theoretical physics. Together with the apparent $B-L$ global symmetry [248], this can be interpreted as an indication that our universe is realized close to some asymptotic limit of the moduli space. Assuming this is indeed the case led to the Dark Dimension proposal in [33].

The starting point is that our universe is realized as some quasi-stable vacuum with a positive cosmological constant or even as some slow-rolling potential with $\Lambda > 0$. One of the core features of the Swampland program is the presence of light towers in such asymptotic limits, and the incarnation of the distance conjecture suitable for our setup is the de Sitter Distance Conjecture [136], according to which there exists a tower of light states with mass scale scaling as $m \sim \Lambda^{\alpha}$. Equivalently, this corresponds to the potential scaling like $\Lambda \sim m^{1/\alpha}$, instead of the usual EFT expectation for $\Lambda \sim \Lambda_0 + c \cdot m^{\frac{1}{\alpha}}$, with the Λ_0 being the contribution of the heavy modes. The vanishing of Λ_0 is argued to be intrinsically related to the modular invariance of string theory. The value of α is not arbitrary. It was already noted in [136] that $\alpha \leq 1/2$ is required for compliance with the Higuchi bound, while in [33] the case for $\alpha \geq 1/d$ was made, based on the fact that the one-loop term would generically scale like $m^{1/d}$ and a different scaling would require some "magical" cancellation. Taking everything into account, the expected range for α is $\frac{1}{d} \leq \alpha \leq \frac{1}{2}$.

At this point, one can use experimental input to fix α within this theoretically allowed range of values. Torsion balance experiments [94] study deviations from the $1/r^2$ gravitational attraction law, and the state-of-the-art measurements exclude any deviation for distances larger than 30μm, which means that the scale of a tower compatible with the data must be $m \gtrsim 6.6$ meV [94]. Plugging in the values of $\Lambda = 10^{-122} M_p^4$ and $d = 4$ for our universe necessitates saturating α to its minimum value, $\alpha = 1/4$, to ensure compatibility with experimental data.

In fact, $\Lambda^{1/4} \approx 2.3$meV, the energy scale where we expect the tower to kick in, which is very close to the neutrino scale. This, together with the Emergent String Conjecture, is a strong indication for the nature of the tower: for an emergent string tower, the EFT would already break down at this scale, but we know an EFT description valid for scales much higher than the neutrino scale - the Standard Model itself - and the emergent string option is excluded. Hence, according to the ESC, the tower must be a decompactification tower, and at least one extra dimension should open up.

Once again, we have experimental and observational input, leading to concrete bounds on the size R and the number n of extra dimensions [88]. Physically, the extra dimensions are accompanied by KK-gravitons, which, in turn, leave an imprint on the evolution of astrophysical objects. For instance, the emission of KK gravitons from supernovae (SNe) cores after collapse can shorten the observable signal, leading to bounds for the size extra dimensions [249]. Moreover, part of the KK-gravitons emitted over the age of the universe is thought to eventually decay into photons, producing a detectable background [250]. Finally, the larger-than-expected temperature of old neutron stars may be explained by a halo of KK gravitons around them. The common feature of all these bounds is that the size of the extra dimensions R decreases with the number of extra dimensions n. The most stringent bounds [251] are due to neutron star excess heat:

$$\begin{aligned} R &\leq 4.44 \cdot 10^{-5} m, \quad \text{if } n = 1, \\ R &\leq 1.55 \cdot 10^{-10} m, \quad \text{if } n = 2, \\ R &\leq 2.58 \cdot 10^{-12} m, \quad \text{if } n = 3. \end{aligned} \tag{5.1}$$

Since $\Lambda^{-1/4} \approx 88\mu m$, the only case that can be conciliated with the experimental bounds is that of $n = 1$, i.e., a *single extra dimension*, dubbed *the Dark Dimension* for reasons that will become apparent later. To strike the delicate balance between theoretical prediction and experimental bounds, it is necessary to introduce an explicit prefactor in the scaling relation, as

$$\Lambda^{\frac{1}{4}} = \lambda m, \tag{5.2}$$

with $10^{-4} \leq \lambda \leq 1$. This range of λ ensures the tower does not deviate significantly from the required 1/4-scaling, while simultaneously having a quasi-dS phase in the potential and was explicitly verified in the case of a Casimir contribution [33]. For most applications, we can assume $\lambda \sim 10^{-1} - 10^{-3}$, so the size of the extra dimension is estimated to be around:

$$R \sim 0.1\ \mu m - 10\mu m. \tag{5.3}$$

Finally, the species scale associated with such a decompactification is $\hat{M} = M_{pl,5} = m^{\frac{1}{3}} M_p^{\frac{1}{2}} \sim \lambda^{-\frac{1}{3}} \Lambda^{\frac{1}{12}} M_p^{\frac{2}{3}} \sim 10^9 - 10^{10}\ GeV$.

The idea of extra dimensions of considerable size is not unique to the Dark Dimension scenario. It was originally proposed in the Large Extra Dimensions (LED) model, commonly known as ADD after the authors Arkani-Hamed, Dimopoulos, Dvali [252, 253]. However, there are important differences stemming from the fact that the ADD model aims at solving the electroweak Hierarchy problem and is not concerned with the value of the cosmological constant. For instance, the relevant energy scales are different, and the $n \geq 2$ extra dimensions, in that case, are millimeter-sized.

Some interesting implications

Let us briefly discuss some of its implications to motivate further why the Dark Dimension proposal is worth exploring.

One expects an excess of fermionic modes in the tower since the potential should go to zero from above [254], with masses close to the neutrino scale. It is then natural to interpret the tower as sterile right-handed neutrinos, similarly to [255]. Estimating the mass of active neutrinos as $m_\nu \sim y^2 \langle H \rangle^2 / \hat{M}$ and imposing that the masses of active and sterile neutrinos are of the same order leads to a prediction for the Higgs vacuum expectation value, $\langle H \rangle \sim 1/y\sqrt{\hat{M}/R} \sim 10 - 10^3\ GeV$ for Yukawa coupling $y \sim 10^{-2} - 10^{-3}$, i.e., the electroweak scale can be related to the cosmological constant as $\langle H \rangle \sim \Lambda^{1/6} \sim 10 - 10^3\ GeV$. It was shown recently in [256] that using the Gravitino Conjecture [143, 144] the supersymmetry breaking scale can also be related to the cosmological constant as $M_{\text{SUSY}} \sim \Lambda^{1/8} \sim 10 - 100\ TeV$.

As we have seen, the UV cut-off estimation is around $\hat{M} \sim 10^9 - 10^{10}\ GeV$. This energy scale is below the scale of $10^{11}\ GeV$ where the Higgs potential develops an instability [257], and it is expected that effects from the new dimension opening up will appropriately modify the potential resolving the instability. Another energy scale close to $\hat{M}$ is the so-called Greisen-Zatsepin-Kuzmin (GZK) limit [258,259], at $10^9\ GeV$, where a sharp cut-off in the flux of ultra-high-energy cosmic rays (UHECR) is observed. In [260] it was noted that for $\lambda = 10^{-3}$ the GZK scale precisely matches $\hat{M}$ and the Dark Dimension could be the reason behind the cosmic ray flux cut-off.

Going back to the cosmological side, one of the most attractive properties of the Dark Dimension is that it provides not only one but two complementary dark matter descriptions. In [261], primordial black holes were argued to fully account for the current dark matter density since their decay rate would sufficiently slow down in the presence of the mesoscopic fifth dimension. In [262], dark gravitons, i.e., KK gravitons along the dark dimension, were shown to also lead to the observed dark matter density while simultaneously providing a natural solution for the cosmological coincidence problem [263]. These two proposals were found to be equivalent in [264], using a corpuscular description of black holes as bound states of gravitons [265].

Finally, we want to remark that the Dark Dimension proposal is experimentally testable. There are certain types of experiments whose next generation is expected to be able to detect relevant signals, such as the next generation of torsion balance experiments and the next generation of experiments probing the low- and high-redshift universe. For the latter case, an explicit study of the characteristic modulation of the 21-cm lines due to the dark dimension was performed in [266]. Very recently, Auger data [267] for UHECR were examined with respect to the dark dimension [268].

5.3 Realization in a warped throat

The Dark Dimension proposal assumes that one starts in a quasi-de Sitter phase. An extensive recent review of different approaches to constructing de Sitter in string theory can be found in [118]. An explicit, well-controlled realization of dS in string theory is a greatly challenging, if not impossible task [269], which needs to bypass known no-go theorems [98, 270]. In fact, the Swampland web includes several conjectures challenging dS vacua: The dS Swampland Conjecture [28] prohibits *any* dS vacuum, the refined de Sitter conjecture [29,134] prohibits *stable* dS vacua, while the trans-Planckian censorship conjecture [30] allows for dS minima as long as they are *short-lived enough.* Additional arguments against dS have appeared in [271,272] based on a description of dS as a coherent state of gravitons and have been shown to relate to the aforementioned swampland cojectures [42]. Luckily, the Dark Dimension scenario is indifferent to the specifics of the dS phase- it goes through for stable, metastable, or unstable dS or even quasi-dS solutions. The only real requirement is that one can define Λ. The original paper [33] remained agnostic about the realization of dS. Our work [38] was the first to propose a concrete setup realizing the desired scaling (5.2), using a common feature in string compactifications, a strongly-warped throat. While the throat is a common ingredient in dS uplifts of AdS vacua, in principle, our argument goes through even for rolling potentials as long as the modes in the throat set the energy scale.

In section 3.2.2, we have seen that the presence of fluxes can induce warping in the metric (3.13). A common approximation is to work in the *dilute flux limit*, where the backreaction of the fluxes can be neglected. However, sometimes it is desirable to take the backreaction into account. Such are the cases of KKLT and LVS, where one eventually attempts to uplift the AdS vacuum to dS by including $\overline{D}3$-branes. The contribution of the branes needs to be such that the final positive value of the potential is tiny. This can be naturally achieved by using a highly warped throat, such that the contribution of the anti-branes, which naturally settle at the tip of the throat, gets appropriately redshifted by the warp factor.

5.3.1 Conifold and Klebanov-Strassler throat

Let us introduce the conifold singularity [246] and its smoothened-out deformation, which can be described by the Klebanov-Strassler solution [247].

Basic info on throats

The non-compact singular Calabi-Yau space, called conifold [246], can be described by the quadric in $\mathbb{C}^4$

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 = 0. \tag{5.4}$$

This surface is a cone with base the coset space $T^{1,1} = (SU(2) \times SU(2))/U(1)$, topologically $S^2 \times S^3$. The singularity at the origin $w_i = 0$ can be smoothened by performing a complex deformation, which renders the size of S^3 at the origin finite. This is the *deformed conifold* and is described by

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 = Z, \tag{5.5}$$

where Z is a complex structure modulus parametrizing the size of S^3, called the *conifold modulus*. The 6d metric of the deformed conifold is [246, 273]

$$\begin{aligned} ds_6^2 = \frac{1}{2}|S|^{\frac{2}{3}} K(y) \Bigg[\frac{dy^2 + (g^5)^2}{3K^3(y)} &+ \cosh^2\left(\frac{y}{2}\right)\left((g^3)^2 + (g^4)^2\right) \\ &+ \sinh^2\left(\frac{y}{2}\right)\left((g^1)^2 + (g^2)^2\right)\Bigg], \end{aligned} \tag{5.6}$$

where y is parametrizing the direction along the throat, and all the moduli dependence is absorbed into S, an appropriate dimensionful rescaling of Z and

$$K(y) = \frac{\left(\sinh(2y) - 2y\right)^{1/3}}{2^{1/3}\sinh(y)}. \tag{5.7}$$

The 1-forms $g^i, i = 1, \ldots, 5$ form a base of the cone [273] - they arise from the angular variables in the base $S^2 \times S^3$, and they can be expressed in terms of the auxiliary forms e^i as:

$$g^1 = \frac{e^1 - e^3}{\sqrt{2}}, \quad g^2 = \frac{e^2 - e^4}{\sqrt{2}}, \quad g^3 = \frac{e^1 + e^3}{\sqrt{2}}, \quad g^4 = \frac{e^2 + e^4}{\sqrt{2}}, \quad g^5 = e^5, \tag{5.8}$$

with

$$\begin{aligned} &e^1 \equiv -\sin\theta_1 d\phi_1, \quad e^2 \equiv d\theta_1, \quad e^3 \equiv \cos\psi \sin\theta_2 d\phi_2 - \sin\psi d\theta_2, \\ &e^4 \equiv \sin\psi \sin\theta_2 d\phi_2 + \cos\psi d\theta_2, \quad e^5 \equiv d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2. \end{aligned} \tag{5.9}$$

This geometric description is physical in the context of type IIB orientifold compactifications. The 10d supergravity solution of the deformed conifold was given by Klebanov and Strassler [247] and is a warped metric of the form

$$ds^2 = e^{2A(y)} g_{\mu\nu} dx^\mu dx^\nu + e^{-2A(y)} ds_6^2. \tag{5.10}$$

with ds_6^2 given in equation (5.6). The throat is supported by the introduction of appropriate fluxes. There exist two relevant 3-cycles, the A-cycle, which effectively wraps the S^3 whose radius is parametrized by Z, and its dual B-cycle. Threading these cycles with R-R and NS-NS fluxes, respectively, according to

$$\frac{1}{2\pi\alpha'}\int_A F_3 = 2\pi M, \quad \frac{1}{2\pi\alpha'}\int_B H_3 = -2\pi K, \tag{5.11}$$

induces a D3-tadpole contribution $N_{\text{flux}} = M \cdot K$. Solving the equations of motion leads to the warp factor close to the tip of the throat [247]:

$$e^{-4A(y)} = 2^{\frac{2}{3}} \frac{(\alpha' g_s M)^2}{|S|^{\frac{4}{3}}} \mathcal{I}(y), \tag{5.12}$$

with

$$\mathcal{I}(y) \sim \frac{(g_s M)^2}{|S|^2} \int_y^\infty dx \frac{x \coth x - 1}{\sinh^2 x} (\sinh(2x) - 2x)^{\frac{1}{3}}, \tag{5.13}$$

where we use the conventions of [274]. We express everything in terms of the conifold modulus Z using that $S = (\alpha')^{\frac{3}{2}} \sqrt{g_s^{3/2} \mathcal{V}_w Z}$ [274] and we get:

$$e^{-4A(y)} \approx 2^{\frac{2}{3}} \frac{g_s M^2}{(\mathcal{V}_w |Z|^2)} \mathcal{I}(y), \tag{5.14}$$

with $\mathcal{V}_w$ the warped volume. Clearly, strong warping corresponds to $\mathcal{V}_w |Z|^2 \ll 1$.

5.3.2 Mass scales in the presence of strong warping

The physical setup we want to consider is similar to that of section 3.2.3. We want to uplift an AdS vacuum to dS and need the conifold singularity to appropriately redshift the uplift contribution. We consider a setup with such warping that can be approximated by a long KS throat glued to a bulk unwarped CY_3 at the radial distance y_{UV}, as sketched in figure 5.1. It was shown in [275] that the details of this gluing do not influence the warp factor, which turns out to be at leading order [247, 276]

$$e^{-4A(y)} \approx 1 + 2^{\frac{2}{3}} \frac{g_s M^2}{(\mathcal{V} |Z|^2)} \mathcal{I}(y), \tag{5.15}$$

with $\mathcal{V} \sim \tau^{\frac{3}{2}}$ the total volume. For strong warping, the necessary condition is

$$\mathcal{V} |Z|^2 \ll 1. \tag{5.16}$$

Let us now consider moduli stabilization. Assuming that all other moduli (complex structure and axio-dilaton) have been fixed by fluxes in the bulk, we consider the $N = 1$ low-energy effective action for Z, $\mathcal{V}$ [275, 277]. The Kähler potential is

$$\mathcal{K} = -2 \log(\mathcal{V}) + \frac{2c' g_s M^2 |Z|^{\frac{2}{3}}}{\mathcal{V}^{\frac{2}{3}}} + \dots, \tag{5.17}$$

with the string coupling constant, as usual, related to the vacuum expectation value of the dilaton $g_s = e^{\langle \phi \rangle}$, which is the real part of the axio-dilaton $S = e^{-\phi} + iC_0$. The

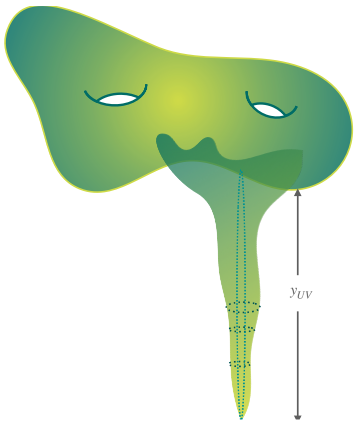

Figure 5.1: Schematic depiction of throat glued to bulk CY_3.

superpotential, with fluxes as in (5.11), is of the Gukov-Vafa-Witten type and for $|Z| \ll 1$ can be approximated by

$$W = -M\frac{Z}{2\pi i}\log Z + iKSZ + \dots, \tag{5.18}$$

where the dots denote terms depending on the remaining moduli. The conifold modulus is hence stabilized at the (exponentially small for appropriate fluxes) value

$$Z \sim \exp\left(-\frac{2\pi K}{g_s M}\right) . \tag{5.19}$$

Note that while the potential admits off-shell corrections [278], they are irrelevant for our discussion since we stay close to the minimum. The conifold mass was computed in [274] and is

$$m_Z \simeq \frac{1}{(g_s M^2)^{\frac{1}{2}}}\left(\frac{|Z|}{\mathcal{V}}\right)^{\frac{1}{3}} . \tag{5.20}$$

Let us consider the other mass scales entering the game. For an isotropic Calabi-Yau threefold, the bulk KK tower has a mass scale

$$M_{\text{KK}} \sim \frac{1}{\tau} \sim \frac{1}{\mathcal{V}^{\frac{2}{3}}} . \tag{5.21}$$

Naively, this would mean that a new dimension would first open up at M_{KK}. However, the warped throat induces a large anisotropy in our setup, which might feature lighter states localized close to the tip of the throat. This is, in fact, true. In [274], the Laplace equation close to the conifold was solved to first approximation, and it was found that there exists a one-dimensional tower of redshifted KK modes mainly supported close to the tip of the KS throat. Their masses scale similarly to the mass of Z as

$$m_{\text{KK}} \sim \frac{1}{(g_s M^2)^{\frac{1}{2}}\, y_{\text{UV}}}\left(\frac{|Z|}{\mathcal{V}}\right)^{\frac{1}{3}} . \tag{5.22}$$

This result was also confirmed numerically in [116, 274]. The numerical analysis showed that the KK masses (5.22) are more or less insensitive to the length of the throat $y_{\rm UV}$ beyond a critical length $y_{\rm UV} > \tilde{y}_{\rm UV} = O(10)$, due to their localization close to the tip of the throat.

Direct comparison of the bulk and throat KK scales shows that the throat modes are much lighter since their masses are suppressed by both the exponentially small $|Z|$ and the factor $g_s M^2$, which has to be large for the effective action to be controllable. Now, effectively one dimension decompactifies once we reach energies $m_{\rm KK}$, so we have a 5d EFT until we hit the energy scale $m_{\rm KK}^{(1)}$ of the next lighter KK tower, which signals the decompactification of even more dimensions. Note that this tower is not necessarily a bulk tower - it can, for instance, be localized the S^3 in the throat, which is also strongly redshifted.

Let us recap the situation: we are considering a generic type IIB orientifold compactification in the vicinity of a conifold singularity. One can stabilize the bulk Kähler moduli, the axio-dilaton, and the Kähler moduli using fluxes and some non-perturbative effect, respectively, and we do not impose any condition regarding supersymmetry on the minimum. The analysis of the EFT in the throat, where fluxes induce strong backreaction, stabilizes the conifold modulus Z to an exponentially small value, and we uncover a light tower of KK modes localized close to the tip of the throat and with masses similar to m_Z. Under these very general assumptions, both the KKLT and LVS AdS minima are described by our setup.

Now is the time to take a leap of faith and perform the uplift, probably the biggest assumption in our analysis. We set aside possible destabilization issues and proceed, assuming the uplift works.

The uplift contribution to the scalar potential for an anti-D3-brane placed at the tip of the KS throat is given by [274, 279]

$$V_{\rm up} \sim \frac{1}{(g_s M^2)} \left(\frac{|Z|}{\mathcal{V}}\right)^{\frac{4}{3}} . \tag{5.23}$$

Simple algebraic manipulations can relate it to the mass of the throat KK modes as

$$m_{\rm KK} \sim \frac{1}{(g_s M^2)^{\frac{1}{4}} y_{\rm UV}} \left|V_{\rm up}\right|^{\frac{1}{4}} . \tag{5.24}$$

Now, imposing that $V_{\rm up} \sim |V_{\rm AdS}|$, which holds for appropriate selection of fluxes, can be used to relate $|Z|$ to $V_{\rm AdS}$ as

$$\frac{|Z|}{\mathcal{V}} \sim (g_s M^2)^{\frac{3}{4}} \left|V_{\rm AdS}\right|^{\frac{3}{4}} . \tag{5.25}$$

If the uplift works, leading to a meta-stable dS minimum, the cosmological constant Λ in the dS minimum is given by $\Lambda = V_{\rm up} + V_{\rm AdS} \gtrsim 0$. The delicate balancing of the AdS cosmological constant and the uplift contribution means that all three terms $V_{\rm AdS}, V_{\rm up}, \Lambda$ scale in the same way with respect to the exponentially small $|Z|/\mathcal{V}$.

In principle, one might expect that Λ could have a parametrically smaller value through sufficient tuning of the warp factor, according to the "old" Landscape philosophy[1]. However, the warp factor (5.14) depends on the fluxes and Z, which in turn is determined by fluxes, according to (5.19). The fluxes compatible with any given compactification are not arbitrary. First, they obey quantization conditions; hence they are only integers. Moreover, as we have seen in equation (3.15), the tadpole cancellation condition is a restrictive condition inherently related to the consistency of the theory, which further limits the possible flux choices. Finally, on a slightly more speculative level, the tadpole conjecture [102] is an additional condition that might hinder arbitrary flux selection. Moreover, as shown in [278], the minimum should not move too far away from its initial position. This suggests that in quantum gravity, the actual controllable tuning is limited. We quantify this expectation by writing

$$\Lambda = \lambda' \, |V_{\rm up}|, \quad \lambda' < 1. \tag{5.26}$$

Finally, we use equation (5.24) to arrive at the relation

$$\Lambda^{\frac{1}{4}} = (g_s M^2)^{\frac{1}{4}} y_{\rm UV} \lambda' \, m_{\rm KK} \,. \tag{5.27}$$

This is precisely the postulated scaling relation (5.2) for the dark dimension scenario, which we repeat here to allow for convenience

$$\Lambda^{\frac{1}{4}} = \lambda m. \tag{5.28}$$

Direct comparison shows that in our setup $\lambda = (g_s M^2)^{\frac{1}{4}} y_{\rm UV} \lambda'$.

Assuming that $\lambda > 10^{-4}$ as argued in [33] supports the conclusion that λ' cannot be arbitrarily tuned, as it leads to

$$\lambda' > \frac{10^{-4}}{(g_s M^2)^{\frac{1}{4}} y_{\rm UV}}. \tag{5.29}$$

As a final remark, we note that the AdS vacuum before the uplift scales similarly to the uplift term and Λ. This means that this AdS vacuum would be scale-separated, in contradiction to the expectation of ADC that $\alpha \geq \frac{1}{4}$ - now $\alpha = \frac{1}{4}$. Once again, the situation regarding scale separation is not crystal clear, especially after recent results that point towards scale separation being achievable, so this is not an a priori inconsistency in the proposed model. What is perhaps even more puzzling in our result is that a dS minimum is behaving exactly in the way predicted by some of the most fundamental

[1] A landscape of AdS vacua could also contribute to the Λ tuning.

principles of the swampland program, the (generalized) Distance Conjecture, while, at the same time, violates the dS conjecture (and possibly TCC, depending on its lifetime). However, our result persists as long as a strongly warped KS throat exists and the final (quasi) dS vacuum or even quintessence potential is dominated by the energy scale in the strongly warped throat. This is also relevant in case the uplifting procedure is not fully controlled due to issues such as [116, 280, 281] for the KKLT scenario and [282, 283] for the LVS.

5.3.3 Intermediate mass scales

While the appearance of the required $\Lambda^{\frac{1}{4}}$ scaling seems undisputed, at least under the assumptions made in our general analysis, there are more factors one needs to consider to ensure the setup's validity and compatibility with experimental bounds.

An issue that needs to be carefully examined on a case-by-case basis is the existence of additional states, particularly towers. More specifically, in the Dark Dimension scenario, the gravitational species scale $\Lambda_{\rm sp}^{\rm grav} = M_{\rm pl}/\sqrt{N} = m_{\rm KK}^{1/3} M_{\rm pl}^{2/3} \sim 10^{9-10}$ GeV is estimated as the scale arising by the decompactification of one extra dimension, i.e., taking into account only the KK tower in this dimension. According to our discussion in section 4.6.1, any additional tower would lower this UV cut-off, as the number of species N_s would increase. In our setup, such a tower that might be lighter than $\Lambda_{\rm sp}^{\rm grav}$ could, for instance, arise from KK modes localized on the S^3 in the throat.

Employing the emergence proposal of section 4.6.2, [274] showed that the effective theory in the throat determined by the Kähler potential (5.17) and the superpotential (5.18) comes with its own cut-off $\Lambda_{\rm sp}^{\rm throat} \sim (g_s M^2 y_{\rm UV})^{\frac{2}{3}} m_{\rm KK}$. This signals that the validity of the EFT stops above $\Lambda_{\rm sp}^{\rm throat}$, where we expect additional non-perturbative states to appear that need to be included in the effective action. Since $\Lambda_{\rm sp}^{\rm throat} < \Lambda_{\rm sp}^{\rm grav}$, these states would also become relevant for the correct determination of the gravitational species scale. This whole discussion makes it clear that we cannot say anything conclusive about the robustness of the warped throat as a means of realizing the Dark Dimension proposal unless we are talking about specific examples where these intermediate mass scales can be computed explicitly.

5.3.4 Example: uplifted LVS

Let us discuss how the $1/4-$scaling appears in a specific, simple realization of LVS on a Swiss-cheese type Calabi Yau with only two Kähler moduli, the volume modulus $\mathcal{V} \simeq \tau_b^{\frac{3}{2}}$ and τ_s. This is exactly the setup discussed in 3.2.3, and we remind the reader that τ_b is stabilized non-perturbatively to a small value, while the volume modulus is stabilized at $\mathcal{V} \sim \sqrt{\tau_s}\, e^{a\tau_s}$. The value of the cosmological constant in the non-supersymmetric AdS minimum scales like

$$V_{\text{AdS}} \sim -\frac{1}{\tau_s}\, e^{-3a\tau_s} \sim \frac{1}{\mathcal{V}^3}\,, \tag{5.30}$$

while the masses of the small and the large Kähler moduli scale as

$$m_{\tau_b} \sim \frac{1}{\mathcal{V}^{\frac{3}{2}}}\,, \qquad m_{\tau_s} \sim \frac{1}{\mathcal{V}}\,. \tag{5.31}$$

For fluxes such that $V_{\text{up}} \sim |V_{\text{AdS}}|$, we express the value of the conifold modulus in terms of the volume

$$|Z| \sim (g_s M^2)^{\frac{3}{4}}\, \mathcal{V}^{-\frac{5}{4}}\,. \tag{5.32}$$

Note that since $\mathcal{V}|Z|^2 \sim \mathcal{V}^{-\frac{3}{2}}$, strong warping is guaranteed in the large volume regime where we are working. The scale of the warped throat KK modes can now be expressed in terms of the volume as $m_{\text{KK}} \sim V_{\text{up}}^{\frac{1}{4}} \sim 1/\mathcal{V}^{\frac{3}{4}}$, while we can estimate the (naive) bulk KK mass scale $M_{\text{KK}} \sim 1/\mathcal{V}^{\frac{2}{3}}$. Hence we have the following hierarchy of mass scales

$$m_{\mathcal{V}} < m_{\tau_s} < m_{\text{KK}} < M_{\text{KK}}\,. \tag{5.33}$$

Both Kähler moduli and most complex structure moduli are lighter than the warped KK scale m_{KK}. The heaviest complex structure modulus is Z. Consistent with our general analysis, the bulk KK modes are heavier than the throat ones. Their masses turn out to differ only by a few orders of magnitude: A first estimation would be

$$\frac{M_{\text{KK}}}{m_{\text{KK}}} \sim \mathcal{V}^{\frac{1}{12}} \sim \Lambda^{-\frac{1}{36}} \sim 2 \cdot 10^3, \tag{5.34}$$

while taking into account the anisotropy of the bulk can give an even better estimation. In that case $\text{Vol} = r_b^5\, r_s$ and we define the bulk KK scale as $M_{\text{KK}} \sim \frac{1}{r_b}$. In this case, the ratio increases by one order of magnitude:

$$\frac{M_{\text{KK}}}{m_{\text{KK}}} \sim \mathcal{V}^{\frac{1}{10}} \sim \Lambda^{-\frac{1}{30}} \sim 10^4. \tag{5.35}$$

This shows that the corresponding length scale in the bulk is only by a factor of 10^{-3}–10^{-4} smaller than the length scale of the throat. Effectively, given the experimental bounds (5.1), we are entering the $n > 1$ regime, which restricts the size of all the decompactified dimensions much more drastically. This puts the uplifted LVS in tension with the astrophysical bounds on KK modes in more than one large extra dimension, and new physics is expected to appear below the species scale computed (naively) via the throat KK modes.

5.4 Recent developments and outlook

We have pointed out that a common aspect of stringy dS constructions, the strongly warped throat, naturally gives rise to the main requirement of realizing the dark dimension scenario, the exponent $\alpha = 1/4$ in the dS distance conjecture. Our assumptions have been fairly general, using a setup where an AdS vacuum is uplifted to dS via an anti-D3-brane localized at the tip of the throat. Our result is expected to persist even under the milder assumption that the energy scale in the strongly warped throat dominates the quasi-dS energy, even if the final dS phase is not strictly a vacuum.

However, the practical realization of the Dark Dimension proposal in such a setup is not so simple. The appearance of additional modes below the gravitational species scale lower the gravity cut-off, and one should be careful to take into account all such possible towers. Moreover, in our concrete example of LVS, the bulk KK modes turned out to be "too light": they did not pose a problem for the validity of the effective theory per se, but their associated length scale conflicted with experimental bounds on the size of extra dimensions.

In the spirit of the Dark Dimension proposal, in [256], the value of the cosmological constant Λ was related to the scale of supersymmetry breaking M_{SUSY}, using the Gravitino Conjecture, leading to a scaling of the form $M_{\text{SUSY}} \sim \Lambda^{\frac{1}{8n}}$, with n between $1 \leq n \leq 2$ leading to a supersymmetry breaking scale compatible with current experimental results - for instance, $n = 1$ corresponds to $M_{\text{SUSY}} \sim \mathcal{O}(10 - 100)\ TeV$. While this is a very interesting result on its own, we want to specifically comment on one result of [256]: it was pointed out that our warped throat Dark Dimension realization in the KKLT case, where AdS minimum is supersymmetric, leads to $n = 1/2$, hence seems to be experimentally excluded.

All in all, a concrete stringy realization of the Dark Dimension scenario, while not currently available, is certainly worth pursuing. Our proposed warped throat realization has shown that achieving the necessary $1/4-$scaling is just the first, not trivial, part of the story. Additional constraints need to be checked and satisfied such that no other directions decompactify simultaneously, the species scale does not get altered too much due to the presence of other states, and the supersymmetry breaking scale remains compatible with experimental constraints.

Finally, we must mention that more setups were identified that exhibit the $\Lambda^{\frac{1}{4}}$-scaling: The setup of [256] corresponds to Heterotic and type II STU models with three chiral fields ϕ_i. Moreover, it has been pointed out [284] that the dS construction of [285] as a negative-curvature compactification of M-theory also provides the required scaling.

Chapter 6

Cobordism Conjecture and Dimensional Reduction

6.1 Preface

As we have seen, non-vanishing cobordism groups generate global symmetries, which must be trivialized to render a given theory compatible with quantum gravity. This happens either by gauging or by breaking the symmetry. This chapter is focused on the gauging of such cobordism groups. In fact, the cobordism groups will not be treated on their own but in tandem with appropriate K/KO-theory groups. Blumenhagen and Cribiori first proposed this joint treatment in [35] physically motivated by the open/closed string duality [286], which can be viewed as a specific incarnation of gauge/gravity duality. Not only does a solid mathematical background supporting this postulated relation exist, but also non-trivial physical constraints are produced as a result.

As is common within the Swampland Program, a conjecture and its implications can be tested under dimensional reduction, and here we have to study how the aforementioned Cobordism/K-theory correspondence behaves under dimensional reduction, i.e., when specifying a manifold on which our starting 10-dimensional theory is compactified. In practice, one needs to go beyond the cobordism and K-theory groups of the point and consider cobordism groups $\Omega^{\xi}(M)$ and $K(M)$ of higher-dimensional manifolds M. This is not arbitrary but has a solid mathematical footing. The proposed K-theory/cobordism interplay relies on a mathematical correspondence between certain versions of cobordism and K-theory. It is deeply rooted in the structure of such generalized (co)homology theories and is based on well-established theorems, such as the Conner–Floyd [287] and the Hopkins–Hovey [288] theorems.

This chapter will have the following structure: We will start by reviewing the proposal of [35] in section 6.2, which will give us the chance to introduce the mathematical formalism relating the cobordism and K-theory groups of the point and what the physical implications of their relation are. To test this proposal against dimensional reduction, we will first explain in section 6.3 what our expectations are for the dimensional reduction of an ordinary higher-form symmetry. Section 6.4 will be the most technical part of the chapter and probably of this thesis. The goal is to compute cobordism and K-theory groups of higher dimensional manifolds, such as spheres, tori, and Calabi-Yau manifolds. To this end, we will introduce the Atiyah-Hirzebruch spectral sequence and apply its homological and cohomological versions to cobordism and K-theory, respectively. We aim to present a pedagogical overview of these techniques, which turn out to be very useful for the study of cobordism and anomalies. However, for the reader who is mostly interested in the physical interpretation, we gather our results in appendices D.1, D.2. Section 6.5 will feature a systematic discussion of our results. In short, using the K-theory/cobordism correspondence, we will replicate the expected symmetry-breaking pattern, leading to sensible tadpole cancellation conditions, while taking into account quantum mechanical effects. We interpret this as strong support towards both the proposal of [35] and the original Cobordism Conjecture [34].

6.2 Cobordism and K-theory interplay

This section reviews some important insights and results first presented in [35].

As discussed in section 2.4, D-branes are inherently related to open strings, as they can be viewed as the submanifolds on which the open strings end. The D-branes charges are classified according to K-theory, and one can view the K-theoretical charge as an obstruction for the brane to decay. This is reminiscent of a global charge, similar to how the non-trivial cobordism charges of compact manifolds prevent them from being consumed by an expanding bubble of nothing.

The K-theory charges carried by D-branes should vanish in a compact space. In [289], it was argued that for uncancelled K-theoretical charges, global gauge anomalies would appear in certain D-brane probes. Arguments in favor of the K-theory charge cancellation from a swampland perspective were presented in [290, 291]. The fact that D-branes source R-R fluxes was used to argue that the fluxes are also classified by K-theory [292]; hence the K-theory charges are gauge charges. For the case of non-BPS branes, where no higher-form R-R field directly couples to them, differential K-theory still allows to associate a discrete gauge symmetry should be associated with them [293]. In short, the global K-theory symmetries should be gauged.

A central part of this chapter will be the assertion made in [35] that whenever a suitable mathematical framework exists, cobordism charges will also be gauge charges, entering the tadpole cancellation conditions jointly with the K-theoretical contributions.

As a starting point, let us discuss which K-theory and cobordism groups will interest us. On the K-theory side, we have already seen that the groups classifying the Dp-brane charges are $K^{-n}(pt) = \tilde{K}(S^n)$, with $n = d - p - 1$ in the case of type IIB theory and $KO^{-n}(pt) = \widetilde{KO}(S^n)$, with $n = d - p - 1$ in the case of type I theory. For cobordism, the situation is not so clear-cut as, in a sense, the cobordism group truly describing string theories is $\Omega^{QG} = 0$. Hence, the suitable group depends on how we approximate this structure. As we have seen in section 4.4.3, a sensible first approximation for type I would be Ω^{Spin}, since the fermionic matter of the theory is neutral. For type IIB/F-theory, we consider $\Omega^{\mathrm{Spin^c}}$. There is a two-fold motivation for the Spin^c structure. On the one hand, it was argued in [34] that F-theory compactifications on elliptically fibered CY induce a Spin^c structure on the base manifolds. On the other hand, in the case of trivial H-flux, the Freed-Witten anomaly is canceled for Spin^c manifolds [66]. Hence our working approximation will be Ω^{Spin} for type I and $\Omega^{\mathrm{Spin^c}}$ for type IIB/F-theory. We gather all the relevant groups in table 6.1.

n	0	1	2	3	4	5	6	7	8	9	10
$\Omega_n^{\mathrm{Spin}}(\mathrm{pt})$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	$2\mathbb{Z}_2$	$3\mathbb{Z}_2$
$\Omega_n^{\mathrm{Spin^c}}(\mathrm{pt})$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$2\mathbb{Z}$	0	$2\mathbb{Z}$	0	$4\mathbb{Z}$	0	$4\mathbb{Z} \oplus \mathbb{Z}_2$
$KO^{-n}(\mathrm{pt})$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$K^{-n}(\mathrm{pt})$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

Table 6.1: Spin and Spin^c cobordism groups of the point, higher KO- and K- theory groups up to $n = 10$.

One can immediately notice the similarities between Ω^{Spin} / KO, and $\Omega^{\mathrm{Spin^c}}$ / K-theory. For $n < 8$ and $n < 4$, respectively, these groups are isomorphic, while for higher values of n, the K-theory groups are always contained in the cobordism ones.

This is not a coincidence but rather relates to a deeper mathematical structure uncovered by Atiyah, Bott, and Shapiro [294]. The *Atiyah-Bott-Shapiro(ABS) orientation* [294] refers to both the orientation of complex K-theory over Spin^c structures and of KO-theory over Spin structures. For our purposes, it suffices to mention that there exist maps

$$\begin{aligned} \alpha: &\quad \Omega_n^{\mathrm{Spin}}(\mathrm{pt}) \to KO^{-n}(\mathrm{pt}) \\ \alpha^c: &\quad \Omega_n^{\mathrm{Spin}^c}(\mathrm{pt}) \to K^{-n}(\mathrm{pt}), \end{aligned} \tag{6.1}$$

which can be used to define isomorphisms between the groups [295]:

$$\Omega_n^{\mathrm{Spin}}(\mathrm{pt})/\ker\alpha \cong KO^{-n}(\mathrm{pt})\,, \tag{6.2}$$

$$\Omega_n^{\mathrm{Spin}^c}(\mathrm{pt})/\ker\alpha^c \cong K^{-n}(\mathrm{pt})\,. \tag{6.3}$$

The explicit form of the above orientations is known at a fixed degree. For the Spin case, it is given by the index of the Dirac operator on M [296]:

$$\alpha_n([M]) = \begin{cases} \hat{A}(M) & n = 8m, \\ \hat{A}(M)/2 & n = 8m+4, \\ \dim H \mod 2 & n = 8m+1, \\ \dim H^+ \mod 2 & n = 8m+2, \\ 0 & \text{otherwise.} \end{cases} \tag{6.4}$$

Here $\hat{A}$ is the A-roof genus, defined in appendix B, and H (H^+) is the space of (positive) harmonic spinors. The expression for the Spin^c case is simpler, and is given by the Todd genus, i.e., the index of the Spin^c Dirac operator:

$$\alpha_n^c([M]) = \mathrm{Td}(M) \equiv \int_M \mathrm{td}_n(M)\,, \tag{6.5}$$

the definition of which can be found in appendix B. Note that in all cases, since the map acts on cobordism equivalence classes, the right-hand-side quantities are cobordism invariants.

6.2.1 Physical implications: Gauging and tadpoles

The proposal of [35] relies on the existence of the ABS orientation and postulates that a combination of K-theory and cobordism charges is gauged, not just K-theory. To make this more concrete, we consider type IIB on a *Spin^c* manifold, but the same should hold for type I on a Spin manifold. The idea is that D-brane defect and cobordism charges couple to the same gauge field, i.e., appear in the same tadpole cancellation condition. Schematically, this happens as

$$dF_{n-1} = J_n^K + a^{(n)}\, J_n^{\text{cobord}}, \tag{6.6}$$

i.e., a linear combination of K-theory and cobordism global charges is gauged, with the coefficient $a^{(n)}$ a priori unknown. Note that on the right-hand side, a contribution will appear *for each* cobordism invariant, not only the one which appears in the ABS orientation. Integrating such an equation leads to a charge neutrality condition, and it was found in [35] that the condition arising this way is sensible and precisely reproduces known string theoretical tadpole cancellation conditions.

The invariants of $\Omega_n^{\mathrm{Spin}^c}(\mathrm{pt})$, for n up to 6, denoted as μ_n^j, are presented below:

$$\begin{aligned} &\mu_0 = \mathrm{td}_0(M) = 1\,, \\ &\mu_2 = \mathrm{td}_2(M) = \frac{1}{2}c_1(M)\,, \\ &\mu_4^1 = \mathrm{td}_4(M) = \frac{1}{12}\left(c_2(M) + c_1^2(M)\right)\,, \qquad \mu_4^2 = c_1^2(M)\,, \\ &\mu_6^1 = \mathrm{td}_6(M) = \frac{1}{24}c_2(M)\,c_1(M)\,, \qquad \mu_6^2 = \frac{1}{2}c_1^3(M)\,. \end{aligned} \tag{6.7}$$

We have simplified the notation above by denoting $c_i(TM)$ simply by $c_i(M)$. Sometimes, we will avoid displaying the argument altogether if the manifold we refer to is clear from the context. The invariants in the left-side column are precisely those given by the ABS orientation, while more invariants may be present depending on the corresponding cobordism group. The cobordism n-form currents can be constructed using these invariants, i.e., $J_n^{\text{cobord}} = \sum_{j\in inv} a_j^{(n)}\mu_n^j$, where the sum must run over all the *all* cobordism invariants. As for K-theory, the currents arise using the delta functions over the localized D-branes, i.e., $J_n^K = \sum_{i\in def} Q_i\delta^{(n)}(\Delta_{10-n,i})$, where $\Delta_{10-n,i}$ is the submanifold wrapped by the i-th Dp-brane (with $p = 9-n$ in type I/IIB) with charge Q_i. The assertion of [35] is that the integrated charge neutrality condition for a manifold $M \in [M]$ is then

$$0 = \int_M dF_{n-1} = \int_M \sum_{i\in\text{def}} Q_i\,\delta^{(n)}(\Delta_{10-n,i}) + \int_M \sum_{j\in\text{inv}} a_j^{(n)}\,\mu_n^j\,. \tag{6.8}$$

Interestingly, (6.8) is valid off-shell for all compact manifolds cobordant with M.

Let us clarify the proposal with two examples, first presented in [35]. The first example will become relevant later in this chapter, where it will arise naturally as part of a higher dimensional analysis.

Example 1: $\Omega_6^{\text{Spin}^c}(pt)$ and $K^{-6}(pt)$

We have $\Omega_6^{\text{Spin}^c}(\text{pt}) = \mathbb{Z}\oplus\mathbb{Z}$, so we expect two cobordism invariants contributing to the current, and it classifies three-form global symmetries in 10d. Using the μ_6^i invariants from equation (6.7) we have $J_6^{\text{cobord}} = a_1^{(6)}c_1c_2/24 + a_2^{(6)}c_1^3/2$. In the K-theory side, $K^{-6}(pt)$ classifies D3-branes. Hence, the integrated tadpole cancellation condition is

$$0 = \int_M \sum_{i\in\text{def}} Q_i\,\delta^{(6)}(\Delta_{4,i}) + \int_M \left(a_1^{(6)}\,\frac{c_1(M)\,c_2(M)}{24} + a_2^{(6)}\,\frac{c_1^3(M)}{2}\right). \tag{6.9}$$

This expression is familiar from F-theory compactifications, and in particular for coefficients $a_1^{(6)} = -12$, $a_2^{(6)} = -30$ it precisely reproduces the known D3-tadpole cancellation condition for F-theory [101] on a smooth Calabi-Yau fourfold elliptically fibered over a base M.

The following example illustrates how this correspondence plays out for type I theory, where torsional classes may appear.

Example 2: $\Omega_2^{\text{Spin}}(pt)$ **and** $KO^{-2}(pt)$

We have $\Omega_2^{\text{Spin}}(pt) = \mathbb{Z}_2$ so the cobordism current will come from the ABS orientation $J_2^{\text{cobord}} = a^{(2)}\alpha_2$. In the KO-theory side, $KO^{-2}(pt)$ classifies non-BPS $\widehat{D7}$-branes. Hence, the tadpole cancellation condition is

$$0 = \int_M \sum_i Q_i\, \delta^{(2)}(\Delta_{8,i}) - a^{(2)}\, \alpha_2(M) \qquad \text{mod } 2\,. \tag{6.10}$$

Now, the value of the coefficient is important, as for $a^{(2)}$, the cobordism contribution decouples. Since $M = S_p^1 \times S_p^1$ is a generator of Ω_2^{Spin} with $\alpha(M) = 1$ and is a valid background for type I theory, we can deduce that $a^{(2)}$ is even.

Note that in the examples above, there is no way to fix the relative coefficients a priori without input from string theory. Such an achievement would be a great step towards a bottom-up realization of string theory.

6.2.2 Fixing a background X

We have already seen that the cobordism and K-theory groups can be defined for higher-dimensional manifolds X. In practice, such a background fixing is natural in string theory, especially within the context of dimensional reduction. Moreover, in the proposed correspondence of [35] is correct, it should behave appropriately under dimensional reduction on X, so specifying this background manifold constitutes a highly non-trivial test.

When considering cobordism groups of the point, $\Omega_n^{\xi}(\text{pt})$, one is looking at global symmetries of the d-dimensional effective theory by scanning through all possible topologies of n-dimensional compact manifolds. Going from pt to X, the cobordism group is generically enlarged, as already expected due to the Splitting Lemma. In particular, the classes $\Omega_n^{\xi}(\text{pt})$ will also be present in $\Omega_n^{\xi}(X)$, but new classes can appear, depending on the topology of X. Intuitively, passing from $\Omega_n^{\xi}(\text{pt})$ to $\Omega_n^{\xi}(X)$, the rank of the group increases, and we can interpret $\tilde{\Omega}_n^{\xi}(X)$ as the part of global symmetries genuinely stemming from having fixed a manifold X.

Up to now, we have only discussed the Atiyah-Bott-Shapiro orientation, mapping cobordism groups of the point to K-groups of the point. Luckily, the actual mathematical structure goes much deeper than that, and a framework suitable for higher-dimensional manifolds exists. This is the Conner-Floyd [287] and the Hopkins-Hovey theorem [288].

Hopkins and Hovey proved that the maps

$$\Omega_*^{\mathrm{Spin}}(X) \otimes_{\Omega_*^{\mathrm{Spin}}} KO_* \to KO_*(X), \tag{6.11}$$
$$\Omega_*^{\mathrm{Spin}^c}(X) \otimes_{\Omega_*^{\mathrm{Spin}^c}} K_* \to K_*(X), \tag{6.12}$$

are isomorphisms for any topological space X, i.e., KO-theory is isomorphic to the extension of scalars of Spin^c cobordism theory. For more details regarding this isomorphism and the extension of scalars we point to [35]. The existence of this isomorphism indicates strongly that the K-theory/cobordism correspondence is extendable to X instead of pt. To check this quantitatively, we need to compute such groups, at least for some treatable classes of background manifolds X, and what kind of physical information they contain. This will be extensively discussed in section 6.4 and 6.5, but for now let us see how these global symmetries behave under dimensional reduction in ordinary cohomology.

6.3 Dimensional reduction of symmetries

To set our expectations for the dimensional reduction in the cobordism/K-theory picture, let us review how global symmetries are dimensionally reduced in the usual cohomology picture. In section 4.3.2, we have already seen that a continuous global p-form symmetry in d dimensions is described by an associated closed current J_n, with $n = d - p - 1$. Breaking requires the introduction of an appropriate defect such that

$$dJ_n = \delta^{(n+1)}(\Delta_p) \neq 0\,, \tag{6.13}$$

where Δ_p is the cycle wrapped by the defect. Gauging requires gauge fields coupling minimally to the current

$$S = \int \left(-\frac{1}{2} F_{p+2} \wedge *F_{p+2} + C_{p+1} \wedge J_n + \dots \right), \qquad F_{p+2} = dC_{p+1}, \tag{6.14}$$

such that the current is trivial in cohomology, i.e., exact

$$J_n = (-1)^p \, d * F_{p+2}\,. \tag{6.15}$$

A dimensional reduction over a compact space X is performed via expanding the various objects, such as currents and gauge fields, in a cohomological basis of X. The expansion coefficients are fields propagating along the non-compact dimensions. Since the charges are quantized, we consider singular cohomology with integer coefficients, $H^p(X;\mathbb{Z})$.

Compactifying the theory on a k-dimensional space X gives a $D = (d-k)$-dimensional effective theory with (broken and gauged) symmetries inherited from the original theory. A p-form symmetry in D dimensions can receive contributions from different $(p+q)$-form symmetries of the d-dimensional theory. These contributions are related to currents J_{n+m}, with $p = D - n - 1$ and $q = k - m$, wrapping $m = 0, 1, \ldots, k$ cycles in X and extending along n directions in the non-compact space. In a cohomology basis $\omega_{(m)a} \in H^m(X; \mathbb{Z})$, with $a = 1, \ldots, b_m$, b_m the Betti numbers, we can decompose the currents as

$$J_{n+m} = \sum_{a=1}^{b_m} j_n^{(m)a} \wedge \omega_{(m)a} \,. \tag{6.16}$$

The set of currents $j_n^{(m)a}$, for $a = 1, \ldots b_m$ and $m = 0, \ldots, k$ correspond to p-form symmetries in D dimensions. If the J_{n+m} are closed, $dJ_{n+m} = 0$, they produce a lattice of global p-form symmetries in D dimensions:

$$dj_n^{(m)a} = 0, \qquad \forall\, a = 1, \ldots b_m, \quad \forall\, m = 0, \ldots k \,. \tag{6.17}$$

Breaking or gauging the global symmetries does not affect this structure, and we generically expect a lattice of broken or gauged symmetries in the lower-dimensional theory arising from different broken or gauged symmetries of the original theory. Let us discuss how this works in more detail.

Breaking the currents J_{n+m} requires forms $\delta^{(n+m+1)}$ in the d-dimensional theory such that $dJ_{n+m} = \delta^{(n+m+1)}(\Delta_{p+q}) \neq 0$, with $p + q = d - n - m - 1$. These forms represent defects wrapping submanifolds $\Delta_{p+q} = \Pi_p \times \Sigma_q$ of the d-dimensional space, where Π_p is a p-dimensional submanifold of the non-compact space and Σ_q is a q-dimensional cycle of X. For the global symmetry to be broken in the D-dimensional theory, we take $p = D - n - 1$ and $q = k - m$ such that the defect has now codimension $n + 1$. The $n = 0$ case, i.e., a $(D-1)$-form global symmetry, corresponds to a domain wall in D dimensions). The cohomology expansion is then

$$\delta^{(n+m+1)}(\Delta_{p+q}) = \sum_{a=1}^{b_m} \delta^{(n+1)}(\Pi_p)^{(m)a} \wedge \omega_{(m)a}. \tag{6.18}$$

Any defect $\delta^{(n+m+1)}$ in d dimensions generates a lattice of codimension $(n+1)$ defects in D dimensions, $\delta^{(n+1)}(\Pi_p)^{(m)a}$, which can break the lattice of global currents (6.17):

$$dj_n^{(m)a} = \delta^{(n+1)}(\Pi_{D-n-1})^{(m)a} \neq 0, \tag{6.19}$$

with $a = 1, \ldots b_m$ and $m = 0, \ldots, k$.

Gauging the currents J_{n+m} means there exist field strengths F_{n+m-1} in the d-dimensional theory such that $J_{n+m} = dF_{n+m-1}$, where F_{n+m-1} is the magnetic dual of the field strength in (6.15). The dimensional reduction of these Bianchi identities can be performed simply by replacing J_{n+m} by F_{n+m-1} in the breaking analysis and repeating the same steps. Hence we find a lattice of $(n-1)$-form field strengths $f_{n-1}^{(m)a}$ in D dimensions, which gauge the n-form currents $j_n^{(m)a}$, thus giving the Bianchi identities

$$j_n^{(m)a} = df_{n-1}^{(m)a} \,. \tag{6.20}$$

In general, the currents $j_n^{(m)a}$ also contain contributions from localized delta functions $\delta^{(n)}(\Pi_{D-n})$, arising from the reduction of D-branes in d dimensions.

After computing the cobordism and K-theory groups of X in the next section, we will show that they exhibit the same pattern explained here for the dimensional reduction of broken and gauged symmetries on X. We will see that the description in terms of cobordism and K-theory provides an organizing principle for the various symmetries in the dimensionally reduced theory, which is not transparent from the above analysis. Indeed, contributions to a given (broken or gauged) p-form symmetry in D-dimensions and its corresponding charged objects will be encoded into $K^{-n}(X)$ and $\Omega_{k+n}^{\mathrm{Spin}^c}(X)$, for $p = D - 1 - n$ and $n \geq 0$. We will see that for $-k \leq n \leq 0$, the corresponding D-brane, respectively gravitational soliton, does not consistently fit into the D-dimensional space so that there does not exist any obvious physical interpretation of the cobordism and K-theory groups. For type I, we have a similar story for $\Omega_{k+n}^{\mathrm{Spin}}(X)$ and $KO^{-n}(X)$. This behavior under compactification further supports the interpretation of K-theory and cobordism groups as higher-form charges in an effective field theory.

In the above dimensional reduction using (de Rham) singular (co)homology without torsion, all objects in D dimensions arise from the naive dimensional reduction along homological cycles in X. The appearance of torsion through the refinement to generalized (co)homology theories can open up new decay channels of non-BPS branes, and new stable torsion branes may appear on X, even if they were not present in d dimensions. Moreover, for wrapped D-branes, quantum effects, such as the Freed-Witten anomaly, can spoil these simple (classical) expectations. The description in terms of cobordism and K-theory automatically takes care of these quantum effects.

6.4 Computing cobordism and K-theory on X

In this section, we compute cobordism and K-theory groups of higher dimensional spaces using a technique known as Atiyah-Hirzebruch spectral sequence. We first provide a pedagogical introduction to spectral sequences for generalized homology and cohomology, and then we use them to compute physically relevant cobordism and K-theory groups, respectively. The results of our computations can be found in appendices D.1 and D.2, while their physical interpretation is discussed in section 6.5.

6.4.1 The Atiyah–Hirzebruch spectral sequence

The Atiyah–Hirzebruch spectral sequence (AHSS) is a tool for calculating generalized (co)homology groups of certain manifolds using the (co)homology groups of some other, usually simpler, manifold. It has been used in a string theoretical context already in [65, 67,92,93,187] among others. In this section, we will briefly review the main steps of these techniques and some relevant mathematical results to provide a self-contained exposition of the subject. Standard references in the mathematical literature are, for example, [171, 172, 297], for introductory material, and [70] for a physics-motivated treatment. A nice, recent review with applications to anomaly cancellation in physics can be found, e.g., in [187]. Supplementary background material is gathered in appendix C. In sections 6.4.2 and 6.4.3, we will use the homological AHSS to determine the cobordism groups $\Omega_n^\xi(X)$ and the cohomological AHSS for K-groups $K^{-n}(X)$, with X a compact manifold of dimension up to ten. In particular, we will have $X = \{S^k, T^k, K3, CY_3\}$, and to $\xi = \text{Spin}, \text{Spin}^c$ due to the clear physical relevance .

Homological spectral sequence

Consider a possibly non-trivial fibration $F \to E \to B$. Having some knowledge of the (generalized) homology of the base B and/or the fiber F, one can use the *Atiyah–Hirzebruch spectral sequence (AHSS) for homology* as a tool to compute the homology of the total space E, $G_n(E)$. The AHSS is based on a filtration of $G_n(E)$, i.e., a sequence of subspaces $\ldots \subset F_p \subset F_{p+1} \subset \ldots$ whose union is $G_n(E)$. In certain cases, it is also possible to use the spectral sequence "backwards" and compute, for example, $G_n(F)$ from the $G_n(E)$. In practice, this is an approximate method, the accuracy of which improves with each iterative step, finally stabilizing after a finite number of steps. Running the process until the final step does not always give $G_n(E)$; instead, it usually produces an associated graded module, which must be solved case-by-case, requiring additional information beyond the AHSS.

A spectral sequence consists of a sequence of objects E^r, called pages, together with endomorphisms d^r that square to zero, called differentials, with r non-negative integers. The pairs (E^r, d^r) are such that the $(r+1)$-st page E^{r+1} is given by the homology of the r-th page E^r,

$$E^{r+1} \cong H(E^r) = \frac{\ker\ d^r\ :\ E^r \to E^r}{\text{Im}\ d^r\ :\ E^r \to E^r}. \tag{6.21}$$

The page E^r and the differential d^r fully determine the following page E^{r+1}, but do not fully determine the differentials d^{r+1}. Intuitively, the spectral sequence calculates a generalized (co)homology by first approximating it with ordinary (co)homology and then refining the approximation through the action of the differentials.

For AHSS, the pages are bi-graded, i.e. $E^r = \oplus_{p,q} E^r_{p,q}$ with $p, q \in \mathbb{Z}$, and the differentials d^r have a bi-degree $(-r, r+1)$, mapping between the bi-graded page elements as $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$.

A pictorial representation of the pages and the relevant differentials is often used, as shown below in the figures 6.1 and 6.2. We assume that all entries outside the first quadrant vanish, ensuring the termination of the homological spectral sequence after a finite number of steps [171]. The horizontal axis refers to the p-value and the vertical to the q-value of an element of the n^{th} page, $E^n_{p,q}$. Usually, one starts the sequence with an explicit expression for the second page, E^2.

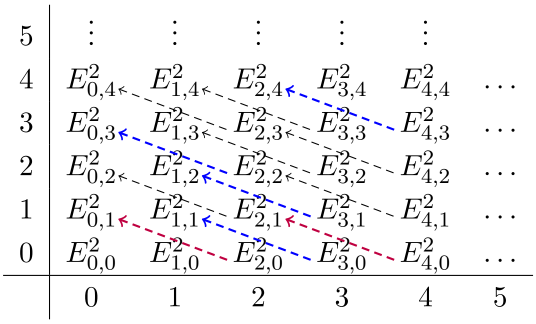

Figure 6.1: Example of a second page E^2 of a first quadrant homological spectral sequence and all possible d^2 differentials. The non-vanishing d^2 are shown by purple and blue arrows.

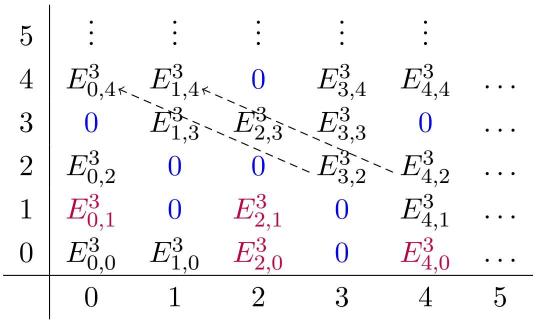

Figure 6.2: Third page E^3 of the same spectral sequence and all possibly non-vanishing d^3 differentials. The blue differentials have (co-)killed the page elements they were acting on, while the purple ones let them partially survive. The black elements, on which no differential acted, carried over intact to the next page, i.e., $E^3_{p,q} \cong E^2_{p,q}$.

Acting on a page with the differential is called *turning the page* and leads to the next page. The only elements that might differ between two consecutive pages are those that non-vanishing differentials act on, while the rest carry over intact. No non-trivial differential can act after a finite number of iterations for a sequence confined to the first quadrant, such as a homological spectral sequence. Hence the sequence stabilizes at the so-called E^∞-page.

One computes the generalised homology groups $G_n(E)$ using all diagonal elements of $E^\infty_{p,q}$, with $p + q = n$. The spectral sequence is said to *converge* to $G_n(E)$, i.e., $E^2_{p,q} \Rightarrow G_{p+q}$. In the simplest case, there is just one element on the diagonal of E^∞, so a direct identification is possible, but usually one has to deal with a non-trivial extension problem, especially when torsion is present. In the general case, one has

$\mathrm{Gr}(G_n(E)) \cong \bigoplus_{p=0}^{n} E^{\infty}_{p,n-p}$ and to obtain $G_n(E)$ extra information is needed. We will discuss specific examples and possible ways around the extension problem in the following sections 6.4.2 and 6.4.3.

Having explained the general idea of a spectral sequence, let us apply it to our initial problem of computing (generalized) homologies of the total space E for a fibration $F \to E \to B$. One can distinguish three types of spectral sequences depending on what kind of structure one has. First, let M be an abelian group and B path-connected. The *homological Serre spectral sequence* is a first quadrant spectral sequence defined as

$$E^2_{p,q} \cong H_p(B; H_q(F; M)) \Rightarrow H_{p+q}(E; M)\,. \tag{6.22}$$

Second, for R a ring and B simply connected, we have the *Leray-Serre spectral sequence*

$$E^2_{p,q} \cong H_p(B; R \otimes H_q(F; R)) \Rightarrow H_{p+q}(E; R)\,. \tag{6.23}$$

The *Leray–Serre–Atiyah–Hirzebruch spectral sequence*, or simply *Atiyah–Hirzebruch Spectral Sequence* is defined for an additive homology theory G_* and a path-connected B

$$E^2_{p,q} \cong H_p(B; G_q(F)) \Rightarrow G_{p+q}(E)\,. \tag{6.24}$$

Note that $H_p(B; G_q(F)) = 0$ for $p < 0$. This spectral sequence can be used for the computation of cobordism groups.

Cohomological Spectral Sequence

As mentioned, for the computation of the K-theory groups $K^{-n}(X)$, we will employ the cohomological version of the AHSS. Indeed, analogously to the discussion in the previous section, one constructs spectral sequences to compute generalized cohomology groups. One starts with a fibration fulfilling certain requirements and uses knowledge of cohomological groups of some space (such as fiber or base) to deduce the generalized cohomology of the desired space (such as the total space). Once again, we have a collection of objects (E_r, d_r), where now the bi-grading of the differential is $(r, -r+1)$, i.e. $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ and the $(r+1)$-st page E_{r+1} given by the cohomology of the E_r page. In the pictorial representation, the pages of a cohomological spectral sequence look very similar to those of a homological one, except that the differential arrows now point in the opposite direction. Another difference is that now the sequence possesses a cup product structure which may allow for a formal computation of the differentials.

The *cohomological Serre spectral sequence* is defined similarly to the homological one. For the usual fibration $F \to E \to B$, with B path-connected and R a ring, there is a first quadrant cohomological spectral sequence of algebras, converging (as a graded algebra) as

$$E_2^{p,q} = H^p(B; H^q(F; R)) \Rightarrow H^{p+q}(E; R)\,. \tag{6.25}$$

If $\pi_1(B) = 0$ and R a field, the previous equation simplifies to

$$E_2^{p,q} = H^p(B) \otimes H^q(F;R) \Rightarrow H^{p+q}(E;R)\,. \tag{6.26}$$

Since K-theory is a generalized cohomology theory, generalizing the Serre spectral sequence is necessary. This is the *Atiyah–Hirzebruch spectral sequence*, defined now for G^* a generalized cohomology theory and the fibration as in 6.25. Namely, there is a half-plane cohomological spectral sequence

$$E_2^{p,q} = H^p(B; G^q(F)) \Rightarrow G^{p+q}(E)\,. \tag{6.27}$$

Trivial fibration and vanishing edge differentials

Besides the extension problem, computing the differentials in a spectral sequence can also be tedious. However, there are instances where one can generally show that they vanish. This is the case for differentials from/to the edge of a given page when the AHSS involves particularly simple fibrations.

Consider the trivial fibration

$$\mathrm{pt} \hookrightarrow X \overset{\mathrm{id}}{\to} X. \tag{6.28}$$

The inclusion pt $\hookrightarrow X$ is split by the constant map $X \to$ pt, implying that

$$G_n(\mathrm{pt}) \to G_n(X) \tag{6.29}$$

is a split injection (G_* being a generalized homology theory). On the other hand, this is also a special case of a map known as the edge homomorphism. Indeed, consider the fibration $F \to E \to B$, which generalizes (6.28). An edge homomorphism is defined as

$$G_n(F) \to H_0(B; G_n(F)) = E^2_{0,n} \to E^\infty_{0,n} \to G_n(E), \tag{6.30}$$

where the last arrow is an injection while the others are surjections. As stated e.g. in Theorem 9.10 of [171], this is equal to the map

$$G_n(F) \to G_n(E), \tag{6.31}$$

induced by the inclusion $F \hookrightarrow E$. For $F =$ pt and $B = E = X$, one should recover the split injection (6.29) and thus

$$E^2_{0,n} \cong E^\infty_{0,n}. \tag{6.32}$$

In other words, in this case, the entries survive to the final page, and any differential acting on them

$$d^r : E^r_{r,q} \to E^r_{0,q+r-1}, \tag{6.33}$$

has to be zero. This observation greatly simplifies the calculation and will directly apply to the upcoming computation of cobordism groups.

6.4.2 Application to cobordism

In this section, we employ the homological version of the AHSS to compute cobordism groups $\Omega_n^\xi(X)$ for non-trivial k-dimensional spaces X. Considering the trivial vibration $\text{pt} \to X \to X$, we use the AHSS to determine $\Omega_n^\xi(X)$ from the known cobordism groups of the point given in table 6.1. The trivial fibration allows us to avoid the following complication: B is assumed to be path-connected but, in general, not simply connected. When $\pi_1(B) \neq 0$, one deals with a system of local coefficients over B with fiber $G_q(F)$ [171] and has to consider ordinary homology with local coefficients in (6.24). However, if the fibration is trivial, this complication can be ignored [171]. Then, the second page of the AHSS is given by

$$E^2_{p,q} = H_p(X;\Omega_q^\xi)\,. \tag{6.34}$$

To avoid cluttering the expressions, in the remainder of this section, we use the shorthand notation $\Omega_n^\xi(\text{pt}) \equiv \Omega_n^\xi$. Additionally, we will only show the parts of the pages relevant to physical applications, i.e., $p, q \leq 10$ (or less).

Computing $\Omega_n^\xi(S^k)$

Before considering higher-dimensional spheres, we start with the straightforward yet illustrative computation of $\Omega_n^\xi(S^2)$. We present here the case where $\xi = \text{Spin}$, while the similarly computed results for $\xi = \text{Spin}^\text{c}$ are relegated to the appendix D.1.

While a direct computation of $H_p(S^2, \Omega_q^{\text{Spin}})$ is straightforward for low q, in general, one turns to the universal coefficient theorem (see appendix C.2), according to which there is a short exact sequence

$$0 \to H_n(S^2;\mathbb{Z}) \otimes \Omega_q^{\text{Spin}} \to H_n(S^2;\Omega_q^{\text{Spin}}) \to \text{Tor}_1(H_{n-1}(S^2;\mathbb{Z}), \Omega_q^{\text{Spin}}) \to 0. \tag{6.35}$$

Recalling the well-known homology groups

$$H_n(S^2;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 2, \\ 0 & \text{otherwise} \end{cases} \tag{6.36}$$

and the fact that $\mathbb{Z}$ is torsion-free, the second page (6.34) can be directly evaluated as

$$E^2_{p,q} = H_p(S^2;\Omega_q^{\text{Spin}}) \cong H_p(S^2;\mathbb{Z}) \otimes \Omega_q^{\text{Spin}} = \begin{cases} \Omega_q^{\text{Spin}} & \text{for } p = 0, 2, \\ 0 & \text{otherwise.} \end{cases} \tag{6.37}$$

Hence the second page of the AHSS takes the following form:

There exist four differentials that could kill some of the page entries. However, they all end on the first column of the page, and thus they vanish according to the edge homomorphism reviewed in section 6.4.1. Thus, we immediately see that $E^2_{p,q} \cong E^3_{p,q}$.

10	$\Omega_{10}^{\text{Spin}}$	0	$\Omega_{10}^{\text{Spin}}$	0	0	0
9	Ω_{9}^{Spin}	0	Ω_{9}^{Spin}	0	0	0
8	Ω_{8}^{Spin}	0	Ω_{8}^{Spin}	0	0	0
7	Ω_{7}^{Spin}	0	Ω_{7}^{Spin}	0	0	0
6	Ω_{6}^{Spin}	0	Ω_{6}^{Spin}	0	0	0
5	Ω_{5}^{Spin}	0	Ω_{5}^{Spin}	0	0	0
4	Ω_{4}^{Spin}	0	Ω_{4}^{Spin}	0	0	0
3	Ω_{3}^{Spin}	0	Ω_{3}^{Spin}	0	0	0
2	Ω_{2}^{Spin}	0	Ω_{2}^{Spin}	0	0	0
1	Ω_{1}^{Spin}	0	Ω_{1}^{Spin}	0	0	0
0	Ω_{0}^{Spin}	0	Ω_{0}^{Spin}	0	0	0
	0	1	2	3	4	5

$=$

10	$3\mathbb{Z}_2$	0	$3\mathbb{Z}_2$	0	0	0
9	$2\mathbb{Z}_2$	0	$2\mathbb{Z}_2$	0	0	0
8	$2\mathbb{Z}$	0	$2\mathbb{Z}$	0	0	0
7	0	0	0	0	0	0
6	0	0	0	0	0	0
5	0	0	0	0	0	0
4	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0
3	0	0	0	0	0	0
2	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	0	0
1	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	0	0
0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0
	0	1	2	3	4	5

Figure 6.3: Second (and final) page of AHSS for $\Omega_n^{\text{Spin}}(S^2)$.

n	0	1	2	3	4	5	6	7	8	9	10
$\Omega_n^{\text{Spin}}(S^2)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$e(\mathbb{Z},\mathbb{Z}_2)$	$\mathbb{Z}_2$	$e(\mathbb{Z}_2,\mathbb{Z})$	0	$\mathbb{Z}$	0	$2\mathbb{Z}$	$2\mathbb{Z}_2$	$e(2\mathbb{Z}$,$3\mathbb{Z}_2)$

Table 6.2: Cobordism groups $\Omega_n^{\text{Spin}}(S^2)$, $n = 0, \ldots, 10$, up to extensions.

On the third page, no differentials can act, as their degree would be larger than any possible degree difference between the non-zero elements. Therefore, $E^2_{p,q} \cong E^\infty_{p,q}$ and we arrive at the results in table 6.2, where we denote by $e(A, B)$ the extension of A by B.

We can try to solve the extension problems using results reviewed in appendix C.1.

- $e(\mathbb{Z}, \mathbb{Z}_2)$: We have $\text{Ext}^1(\mathbb{Z}, \mathbb{Z}_2) = 0$ and thus there is only the trivial extension, $e(\mathbb{Z}, \mathbb{Z}_2) = \mathbb{Z} \oplus \mathbb{Z}_2$.
- $e(\mathbb{Z}_2, \mathbb{Z})$: Equation (C.6) gives $\text{Ext}^1(\mathbb{Z}_2, \mathbb{Z}) = \mathbb{Z}_2$. The two possible extensions are $\mathbb{Z}$ and $\mathbb{Z}_2 \oplus \mathbb{Z}$, so we need additional input to select the correct one. A simple strategy would be to use the splitting lemma (4.5), which tells us that $\Omega_4^{\text{Spin}}(S^2)$ should contain a factor $\Omega_4^{\text{Spin}} = \mathbb{Z}$. However, such a factor is present in both options, so we cannot draw any conclusion. In appendix C.5, we show (indirectly) that for $\Omega_n^\xi(S^k)$ the extension is always trivial, therefore even in this case $e(\mathbb{Z}_2, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2$.
- $e(2\mathbb{Z}, 3\mathbb{Z}_2)$: We have $\text{Ext}^1(2\mathbb{Z}, 3\mathbb{Z}_2) = 2\text{Ext}^1(\mathbb{Z}, 3\mathbb{Z}_2) = 5\text{Ext}^1(\mathbb{Z}, \mathbb{Z}_2) = 0$, so the trivial extension must be chosen.

We summarize our findings in the following table.

n	0	1	2	3	4	5	6	7	8	9	10
$\Omega_n^{\mathrm{Spin}}(S^2)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}$	0	$\mathbb{Z}$	0	$2\mathbb{Z}$	$2\mathbb{Z}_2$	$2\mathbb{Z}\oplus 3\mathbb{Z}_2$

Table 6.3: Cobordism groups $\Omega_n^{\mathrm{Spin}}(S^2)$.

The calculation of $\Omega_n^{\mathrm{Spin}}(S^k)$ for higher k proceeds similarly. Since the only non-vanishing homology classes are $H_0(S^k;\mathbb{Z}) = H_k(S^k;\mathbb{Z}) = \mathbb{Z}$ and the universal coefficient theorem applies, the second page for the trivial fibration pt $\to S^k \to S^k$ looks very similar to the one for S^2, with the non-vanishing entries along the $p = 0, k$ columns. The only possibly non-vanishing differentials are d_k, but they vanish due to the edge homomorphism, and the computation proceeds exactly as before. For S^1, the computation is even simpler since no differential can act for degree reasons. As explained at the beginning of the present section, the fact that $\pi_1(S^1) \neq 0$ does not concern us since we are using a trivial vibration.

The computation of the Spin^c cobordism groups $\Omega_n^{\mathrm{Spin}^c}(S^k)$ is similar. Now the second page is

$$E^2_{p,q} = H_p(S^k;\Omega_q^{\mathrm{Spin}^c}) \cong H_p(S^k;\mathbb{Z}) \otimes \Omega_q^{\mathrm{Spin}^c} = \begin{cases} \Omega_q^{\mathrm{Spin}^c} & \text{for } p = 0, k, \\ 0 & \text{otherwise}, \end{cases} \tag{6.38}$$

and the arguments for the $\Omega_n^{\mathrm{Spin}}(S^k)$ computation still go through. As proven in appendix C.5, for both structures $\xi = \mathrm{Spin}, \mathrm{Spin}^c$ the final result can be compactly written as

$$\Omega_n^\xi(S^k) = \Omega_n^\xi(\mathrm{pt}) \oplus \Omega_{n-k}^\xi(\mathrm{pt})\,. \tag{6.39}$$

Explicitly, the groups for $n, k \leq 10$ are given in the appendix D.1.

Computing $\Omega_n^\xi(T^2)$

For the two-torus, $T^2 = S^1 \times S^1$, we present the computations for $\xi = \mathrm{Spin}$ and $\xi = \mathrm{Spin}^c$ in parallel. Starting from the known homology groups

$$H_n(T^2;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 2, \\ 2\mathbb{Z} & \text{for } n = 1, \\ 0 & \text{otherwise}, \end{cases} \tag{6.40}$$

and using the universal coefficient theorem again (with vanishing Tor_1 group), one can compute the second page

$$E^2_{p,q} = H_p(T^2;\Omega_q^\xi) \cong H_p(T^2;\mathbb{Z}) \otimes \Omega_q^\xi = \begin{cases} \Omega_q^\xi & \text{for } p = 0, 2, \\ 2\Omega_q^\xi & \text{for } p = 1, \\ 0 & \text{otherwise}\,. \end{cases} \tag{6.41}$$

10	$3\mathbb{Z}_2$	$6\mathbb{Z}_2$	$3\mathbb{Z}_2$	0
9	$2\mathbb{Z}_2$	$4\mathbb{Z}_2$	$2\mathbb{Z}_2$	0
8	$2\mathbb{Z}$	$4\mathbb{Z}$	$2\mathbb{Z}$	0
7	0	0	0	0
6	0	0	0	0
5	0	0	0	0
4	$\mathbb{Z}$	$2\mathbb{Z}$	$\mathbb{Z}$	0
3	0	0	0	0
2	$\mathbb{Z}_2$	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	0
1	$\mathbb{Z}_2$	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	0
0	$\mathbb{Z}$	$2\mathbb{Z}$	$\mathbb{Z}$	0
	0	1	2	3

10	$4\mathbb{Z} \oplus \mathbb{Z}_2$	$8\mathbb{Z} \oplus 2\mathbb{Z}_2$	$4\mathbb{Z} \oplus \mathbb{Z}_2$	0
9	0	0	0	0
8	$4\mathbb{Z}$	$8\mathbb{Z}$	$4\mathbb{Z}$	0
7	0	0	0	0
6	$2\mathbb{Z}$	$4\mathbb{Z}$	$2\mathbb{Z}$	0
5	0	0	0	0
4	$2\mathbb{Z}$	$4\mathbb{Z}$	$2\mathbb{Z}$	0
3	0	0	0	0
2	$\mathbb{Z}$	$2\mathbb{Z}$	$\mathbb{Z}$	0
1	0	0	0	0
0	$\mathbb{Z}$	$2\mathbb{Z}$	$\mathbb{Z}$	0
	0	1	2	3

Figure 6.4: Final pages of AHSS for $\Omega_n^{\text{Spin}}(T^2)$ (left) and $\Omega_n^{\text{Spin}^c}(T^2)$ (right).

The second pages for the two structures $\xi = \text{Spin}, \text{Spin}^c$ are shown in figure 6.4. In the Spin case, we have four differentials that could be non-trivial, but they vanish due to the edge homomorphism. In the Spin^c case, no differential can act for degree reasons. Hence the second pages are, in fact, the final pages, and the results are shown in table 6.4, where we used the notation $e(A, B, C) = e(A, e(B, C))$.

n	0	1	2	3	4
$\Omega_n^{\text{Spin}}(T^2)$	$\mathbb{Z}$	$e(2\mathbb{Z}, \mathbb{Z}_2)$	$e(\mathbb{Z}, 2\mathbb{Z}_2, \mathbb{Z}_2)$	$e(\mathbb{Z}_2, 2\mathbb{Z}_2)$	$e(\mathbb{Z}_2, \mathbb{Z})$
$\Omega_n^{\text{Spin}^c}(T^2)$	$\mathbb{Z}$	$2\mathbb{Z}$	$e(\mathbb{Z}, \mathbb{Z})$	$2\mathbb{Z}$	$e(\mathbb{Z}, 2\mathbb{Z})$

n	5	6	7	8	9	10
$\Omega_n^{\text{Spin}}(T^2)$	$2\mathbb{Z}$	$\mathbb{Z}$	0	$2\mathbb{Z}$	$e(4\mathbb{Z}, 2\mathbb{Z}_2)$	$e(2\mathbb{Z}, 4\mathbb{Z}_2, 3\mathbb{Z}_2)$
$\Omega_n^{\text{Spin}^c}(T^2)$	$4\mathbb{Z}$	$e(2\mathbb{Z}, 2\mathbb{Z})$	$4\mathbb{Z}$	$e(2\mathbb{Z}, 4\mathbb{Z})$	$8\mathbb{Z}$	$e(4\mathbb{Z}, 4\mathbb{Z} \oplus \mathbb{Z}_2)$

Table 6.4: Cobordism groups $\Omega_n^{\text{Spin}}(T^2)$ and $\Omega_n^{\text{Spin}^c}(T^2)$, $n = 0, \ldots, 10$, up to extensions.

Two facts are crucial to solve the extension problems. First, the extensions of all free abelian groups are trivial. Second, $e(m\mathbb{Z}, n\mathbb{Z}_k) = m\mathbb{Z} \oplus n\mathbb{Z}_k$ since $\text{Ext}^1(m\mathbb{Z}, n\mathbb{Z}_k) = 0$. However, since $\text{Ext}^1(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$, we cannot decide about $e(\mathbb{Z}_2, \mathbb{Z}_2)$, which is either $2\mathbb{Z}_2$ or $\mathbb{Z}_4$. A similar story applies for $e(\mathbb{Z}_2, \mathbb{Z})$. Up to this point, our results are shown in table 6.5. According to the general proof in appendix C.5, the remaining extension problems should be trivial. There we generically show that the cobordism groups of

n	0	1	2	3	4
$\Omega_n^{\text{Spin}}(T^2)$	$\mathbb{Z}$	$2\mathbb{Z} \oplus \mathbb{Z}_2$	$e(\mathbb{Z}, 2\mathbb{Z}_2, \mathbb{Z}_2)$	$e(\mathbb{Z}_2, 2\mathbb{Z}_2)$	$e(\mathbb{Z}_2, \mathbb{Z})$
$\Omega_n^{\text{Spin}^c}(T^2)$	$\mathbb{Z}$	$2\mathbb{Z}$	$2\mathbb{Z}$	$2\mathbb{Z}$	$3\mathbb{Z}$

n	5	6	7	8	9	10
$\Omega_n^{\text{Spin}}(T^2)$	$2\mathbb{Z}$	$\mathbb{Z}$	0	$2\mathbb{Z}$	$4\mathbb{Z} \oplus 2\mathbb{Z}_2$	$e(2\mathbb{Z}, 4\mathbb{Z}_2, 3\mathbb{Z}_2)$
$\Omega_n^{\text{Spin}^c}(T^2)$	$4\mathbb{Z}$	$4\mathbb{Z}$	$4\mathbb{Z}$	$6\mathbb{Z}$	$8\mathbb{Z}$	$8\mathbb{Z} \oplus \mathbb{Z}_2$

Table 6.5: Cobordism groups $\Omega_n^{\text{Spin}}(T^2)$ and $\Omega_n^{\text{Spin}^c}(T^2)$, $n = 0, \ldots, 10$.

k-dimensional tori have a simple decomposition,

$$\Omega_n^{\xi}(T^k) = \bigoplus_{m=0}^{k} \binom{k}{m} \Omega_{n-m}^{\xi}(\text{pt}), \tag{6.42}$$

for a generic structure ξ, which can be taken to be Spin or Spin^c. The binomial coefficient can be interpreted as the number of m-cycles on T^k. Detailed results with all extensions solved are reported in appendix D.1.

Computing $\Omega_n^{\text{Spin}^c}(K3)$

To determine the cobordism groups of $K3$, we again start with the known result for $H_n(K3;\mathbb{Z})$.

$$H_n(K3;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 4, \\ 22\mathbb{Z} & \text{for } n = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{6.43}$$

where the non-vanishing Betti numbers of $K3$ are $b_0 = b_4 = 1$, $b_2 = 22$. Once again, we compute the second-page entries shown in figure 6.5 using the trivial fibration and the universal coefficient theorem. For Spin^c, all differentials are trivial for degree reasons, so that we can conclude $E^2_{p,q} = E^\infty_{p,q}$ with

$$E^2_{p,q} = H_p(K3; \Omega_q^{\text{Spin}^c}) \cong H_p(K3;\mathbb{Z}) \otimes \Omega_q^{\text{Spin}^c} = \begin{cases} \Omega_q^{\text{Spin}^c} & \text{for } p = 0, 4, \\ 22\, \Omega_q^{\text{Spin}^c} & \text{for } p = 2, \\ 0 & \text{otherwise}. \end{cases} \tag{6.44}$$

Up to $n = 10$, all extension problems are trivial, so we can express the final result as

$$\begin{aligned} \Omega_n^{\text{Spin}^c}(K3) &= \Omega_n^{\text{Spin}^c}(\text{pt}) \oplus \tilde{\Omega}_n^{\text{Spin}^c}(K3) \\ &= \Omega_n^{\text{Spin}^c}(\text{pt}) \oplus 22\,\Omega_{n-2}^{\text{Spin}^c}(\text{pt}) \oplus \Omega_{n-4}^{\text{Spin}^c}(\text{pt}). \end{aligned} \tag{6.45}$$

In this formula, it is understood that cobordism groups with negative indices are set to zero. The groups with $n \leq 10$ are presented in table 6.6.

10	$4\mathbb{Z}\oplus\mathbb{Z}_2$	0	$88\mathbb{Z}\oplus 22\mathbb{Z}_2$	0	$4\mathbb{Z}\oplus\mathbb{Z}_2$	0	0	0
9	0	0	0	0	0	0	0	0
8	$4\mathbb{Z}$	0	$88\mathbb{Z}$	0	$4\mathbb{Z}$	0	0	0
7	0	0	0	0	0	0	0	0
6	$2\mathbb{Z}$	0	$44\mathbb{Z}$	0	$2\mathbb{Z}$	0	0	0
5	0	0	0	0	0	0	0	0
4	$2\mathbb{Z}$	0	$44\mathbb{Z}$	0	$2\mathbb{Z}$	0	0	0
3	0	0	0	0	0	0	0	0
2	$\mathbb{Z}$	0	$22\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0
1	0	0	0	0	0	0	0	0
0	$\mathbb{Z}$	0	$22\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0
	0	1	2	3	4	5	6	7

Figure 6.5: Second (and final) page of the AHSS for the computation of $\Omega_n^{\mathrm{Spin}^c}(K3)$.

n	0	1	2	3	4	5	6	7	8	9	10
$\Omega_n^{\mathrm{Spin}^c}(K3)$	$\mathbb{Z}$	0	$23\mathbb{Z}$	0	$25\mathbb{Z}$	0	$47\mathbb{Z}$	0	$50\mathbb{Z}$	0	$94\mathbb{Z}\oplus\mathbb{Z}_2$

Table 6.6: Cobordism groups $\Omega_n^{\mathrm{Spin}^c}(K3)$, $n = 0, \ldots, 10$.

Computing $\Omega_n^{\mathrm{Spin}^c}(CY_3)$

The cobordism groups of a Calabi-Yau threefold are obtained similarly to those of $K3$. We start with the known result

$$H_n(CY_3;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 6, \\ b_2\,\mathbb{Z} & \text{for } n = 2, 4, \\ b_3\,\mathbb{Z} & \text{for } n = 3, \\ 0 & \text{otherwise}, \end{cases} \tag{6.46}$$

where b_p are the CY_3 Betti numbers, with $b_p = b_{6-p}$. The second page is given by

$$\begin{aligned} E^2_{p,q} &= H_p(CY_3;\Omega_q^{\mathrm{Spin}^c}) \\ &\cong H_p(CY_3;\mathbb{Z})\otimes\Omega_q^{\mathrm{Spin}^c} = \begin{cases} \Omega_q^{\mathrm{Spin}^c} & \text{for } p = 0, 6, \\ b_2\,\Omega_q^{\mathrm{Spin}^c} & \text{for } p = 2, 4, \\ b_3\,\Omega_q^{\mathrm{Spin}^c} & \text{for } p = 3, \\ 0 & \text{otherwise}, \end{cases} \end{aligned} \tag{6.47}$$

and shown explicitly in figure 6.6. This time five non-vanishing columns $E^2_{p,q}$ exist on the second page, the elements of which are given by $b_p\Omega_q^{\mathrm{Spin}^c}$.

10	$4\mathbb{Z}\oplus\mathbb{Z}_2$	0	$b_2(4\mathbb{Z}\oplus\mathbb{Z}_2)$	$b_3(4\mathbb{Z}\oplus\mathbb{Z}_2)$	$b_2(4\mathbb{Z}\oplus\mathbb{Z}_2)$	0	$4\mathbb{Z}\oplus\mathbb{Z}_2$	0
9	0	0	0	0	0	0	0	0
8	$4\mathbb{Z}$	0	$4b_2\mathbb{Z}$	$4b_3\mathbb{Z}$	$4b_2\mathbb{Z}$	0	$4\mathbb{Z}$	0
7	0	0	0	0	0	0	0	0
6	$2\mathbb{Z}$	0	$2b_2\mathbb{Z}$	$2b_3\mathbb{Z}$	$2b_2\mathbb{Z}$	0	$2\mathbb{Z}$	0
5	0	0	0	0	0	0	0	0
4	$2\mathbb{Z}$	0	$2b_2\mathbb{Z}$	$2b_3\mathbb{Z}$	$2b_2\mathbb{Z}$	0	$2\mathbb{Z}$	0
3	0	0	0	0	0	0	0	0
2	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$	0
1	0	0	0	0	0	0	0	0
0	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$	0
	0	1	2	3	4	5	6	7

Figure 6.6: Second (and final) page of the AHSS for the computation of $\Omega_n^{\mathrm{Spin}^c}(CY_3)$. One of the possibly non-vanishing differentials $d^3 : E^3_{6,q} \to E^3_{3,q+2}$ is displayed (for $q = 0$). They eventually vanish for $q \leq 6$.

None of the differentials d_r with even r can act for degree reasons. However, two kinds of third differentials can be non-trivial. The first class is

$$d^3 : E^3_{3,q} \to E^3_{0,q+2}\,, \tag{6.48}$$

which vanishes due to the edge homomorphism (see section 6.4.1). The second class acts as

$$d^3 : E^3_{6,q} \to E^3_{3,q+2}\,, \tag{6.49}$$

which is, in principle, non-vanishing[1]. This differential is trivial up to $q = 6$ according to Lemma 3.1 of [298]. We thus get the results in table 6.7.

6.4.3 Application to K-theory

Next, we perform similar computations for the K- and KO-theory groups on spheres, tori, and Calabi-Yau manifolds. For this purpose, we employ the cohomological version of the AHSS. Our computations will mostly involve K-theory, since the lack of torsional classes significantly simplifies the calculation, but we also include some results for KO-groups in sections 6.4.3 and 6.4.3.

[1] This differential is given by the homological dual of the cohomology operation $Sq^3_{\mathbb{Z}}$, the (integral) third Steenrod square (the operations Sq^i are introduced briefly later on; see also the appendix C.3). Interestingly, its triviality is the homological dual statement of the Freed-Witten anomaly cancellation [65, 67], which we will discuss later on in the K-theory calculations.

n	0	1	2	3	4	5
$\Omega_n^{\mathrm{Spin^c}}(CY_3)$	$\mathbb{Z}$	0	$(b_2+1)\mathbb{Z}$	$b_3\mathbb{Z}$	$(2+2b_2)\mathbb{Z}$	$b_3\mathbb{Z}$

n	6	7	8	9	10
$\Omega_n^{\mathrm{Spin^c}}(CY_3)$	$(3+3b_2)\mathbb{Z}$	$2b_3\mathbb{Z}$	$(5+4b_2)\mathbb{Z}$	$2b_3\mathbb{Z}$	$(6+6b_2)\mathbb{Z}\oplus\mathbb{Z}_2$

Table 6.7: Cobordism groups $\Omega_n^{\mathrm{Spin^c}}(CY_3)$, $n = 0, \dots, 10$.

Computing $K^{-n}(S^k)$

The K-theory groups of spheres S^k are known to be [172]

$$K^{-n}(S^k) = \begin{cases} \mathbb{Z} & \text{for } k \text{ odd}, \\ 2\mathbb{Z} & \text{for } n, k \text{ even}, \\ 0 & \text{otherwise}. \end{cases} \tag{6.50}$$

It is instructive to reproduce these results using the cohomological AHSS (6.27). As usual, we use the trivial fibration pt $\to S^k \to S^k$, and we do not have to worry about local coefficients. Recalling that

$$K^{-n}(\mathrm{pt}) = \begin{cases} \mathbb{Z} & \text{for } n \text{ even}, \\ 0 & \text{otherwise}, \end{cases} \tag{6.51}$$

we have the second page

$$E_2^{p,q} = H^p(S^k; K^q(\mathrm{pt})) = \begin{cases} \mathbb{Z}, & \text{for } q \text{ even}, p = 0, k, \\ 0, & \text{otherwise}. \end{cases} \tag{6.52}$$

Including the bottom quadrant (with $q < 0$) is essential to arrive at reasonable results, respecting Bott periodicity.

For concreteness, consider $X = S^3$. We are interested in the groups $K^{-n}(X)$, with $n > 0$, so the relevant page elements lie on the $p+q = -n$ bands of the final page, which now intersect the axes only once. The d_2 differentials vanish so that $E_3^{p,q} = E_2^{p,q}$, but d_3 may act non-trivially according to

$$d_3 : E_3^{0,q} \to E_3^{3,q-2}, \qquad q \text{ even}. \tag{6.53}$$

Atiyah and Hirzebruch showed this differential [299] is an instance of a cohomological operation known as (integral) Steenrod square ($Sq^i_{\mathbb{Z}}$)

$$Sq^3_{\mathbb{Z}} : H^n(X;\mathbb{Z}) \to H^{n+3}(X;\mathbb{Z}). \tag{6.54}$$

Explicitly, it is given by the composition

$$d_3 = Sq^3_{\mathbb{Z}} = \beta \circ Sq^2 \circ \rho, \tag{6.55}$$

6	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
5	0	0	0	0	0
4	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
3	0	0	0	0	0
2	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
1	0	0	0	0	0
0	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
-1	0	0	0	0	0
-2	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
-3	0	0	0	0	0
-4	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0
-5	0	0	0	0	0
-6	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0

Figure 6.7: Second (and final) page of the AHSS for the computation of $K^{-n}(S^3)$. One of the d_3 differentials is shown explicitly. They all eventually vanish.

where ρ is the reduction modulo 2 and β the Bockstein homomorphism, namely

$$Sq^3_{\mathbb{Z}} : H^n(X;\mathbb{Z}) \xrightarrow{\rho} H^n(X;\mathbb{Z}_2) \xrightarrow{Sq^2} H^{n+2}(X;\mathbb{Z}_2) \xrightarrow{\beta} H^{n+3}(X;\mathbb{Z}). \tag{6.56}$$

We refer the reader to the appendix C.3 for a more precise definition of Steenrod squares and the Bockstein homomorphism and a summary of their main properties.

Fortunately, according to Theorem 4.8 of [70], all differentials (including d_3) vanish since no torsion is involved. This is a consequence of the Chern isomorphism

$$K^0(X)\otimes_{\mathbb{Z}}\mathbb{R} \cong \bigoplus_n H^{2n}(X;\mathbb{R}), \qquad K^{-1}(X)\otimes_{\mathbb{Z}}\mathbb{R} \cong \bigoplus_n H^{2n+1}(X;\mathbb{R}), \tag{6.57}$$

which implies that if there is no torsion in cohomology, the AHSS for K-theory terminates already on the second page. This fact will be used systematically throughout the computations of $K^{-n}(X)$ groups. Moreover, the extension problem is always trivial since only free abelian groups are present. Thus, for every odd value of k we recover $K^{-n}(S^{2k+1}) = \mathbb{Z}$. The situation for even k is simpler as, for degree reasons, no differentials can act, so that $E_2^{p,q} = E_\infty^{p,q}$. We recover then $K^{-2n-1}(S^{2k}) = 0$ and $K^{-2n}(S^{2k}) = 2\mathbb{Z}$. The final result, regardless is k is even or odd, can be expressed as

$$K^{-n}(S^k) = K^{-n}(\text{pt}) \oplus K^{-k-n}(\text{pt})\,. \tag{6.58}$$

Steenrod Squares and Freed-Witten anomalies

The vanishing of the Steenrod square $d_3 = Sq^3_{\mathbb{Z}}$, beyond significantly simplifying the computation of $K^{-n}(S^k)$, has deep physical implications. It is known that type II D-branes in the absence of B field must wrap a Spin^c manifold Y; otherwise, they develop a global Freed–Witten anomaly [66]. Given an element $y \in H^n(X;\mathbb{Z})$, we have

$$Sq^3_{\mathbb{Z}}(y) = W_3(N) \cup y, \tag{6.59}$$

where N is the normal bundle of Y, the codimension-n submanifold Poincaré dual to y, and $\cup$ is the cup product. Since Y is Spin^c, $W_3(Y) = \beta(w_2(Y)) = 0$. However, because X is Spin and Y is oriented by assumption (in type II), one can show that $w_2(N) = w_2(Y)$, implying $W_3(N) = W_3(Y)$ [66, 72]. Hence one can relate a trivial action of d_3 in the AHSS to the absence of Freed–Witten anomalies for a D-brane wrapping Y [65, 67]. Indeed, if $E^4 = \ker d_3/\operatorname{Im} d_3$ is given in terms of the groups $H^n(X;\mathbb{Z})$ without further restrictions, all cohomology classes (and their dual cycles) survive. Otherwise, some are removed when passing from cohomology to K-theory, or they change to a torsion group [65]. Physically, they would correspond to D-branes which are anomalous or unstable.

Computing $K^{-n}(T^k)$

Next, we consider the k-dimensional torus $T^k = (S^1)^k$. One can either compute the groups by using the AHSS, similarly to the sphere computation, or use the known results for the reduced K-theory groups $\widetilde{K}^{-n}(T^k)$ and the decomposition (2.16).

The second approach is straightforward since we have [172]:

$$\widetilde{K}^{-n}(T^k) = \begin{cases} 2^{k-1}\mathbb{Z} & \text{for } n \text{ odd,} \\ (2^{k-1}-1)\mathbb{Z} & \text{for } n \text{ even.} \end{cases} \tag{6.60}$$

Since $K^{-2n}(\mathrm{pt}) = \mathbb{Z}$ and $K^{-2n-1}(\mathrm{pt}) = 0$, it follows immediately that

$$K^{-n}(T^k) = 2^{k-1}\mathbb{Z}, \tag{6.61}$$

for n any integer. For the trivial case $k = 1$, the above result coincides with the sphere computation, i.e., $K^{-n}(T^1) = \mathbb{Z}$, as expected.

The spectral sequence approach starts with the second page

$$E_2^{p,q} = H^p(T^k; K^q(\mathrm{pt}))\,, \tag{6.62}$$

and using the trivial fibration we get the expected result (6.61), upon realizing that once again all differentials vanish since there is no torsion, so $E_2^{p,q} = E_\infty^{p,q}$, and the extension problem is trivial. The final result can be elegantly written as

$$K^{-n}(T^k) = \bigoplus_{m=0}^{k} \binom{k}{m} K^{-m-n}(\mathrm{pt})\,, \tag{6.63}$$

where the binomial coefficient counts the number of m-cycles on T^k, i.e., are the Betti numbers b_m.

Computing $K^{-n}(K3)$

For the computation of the K-theory groups on $K3$, the second page of the sequence is shown in figure 6.8 and explicitly given by

$$E_2^{p,q} = H^p(K3; K^q(\mathrm{pt})) = \begin{cases} \mathbb{Z} & \text{for } p = 0, 4,\ q \text{ even}, \\ 22\,\mathbb{Z} & \text{for } p = 2,\ q \text{ even}, \\ 0 & \text{otherwise}. \end{cases} \tag{6.64}$$

6	$\mathbb{Z}$	0	$22\mathbb{Z}$	0	$\mathbb{Z}$
5	0	0	0	0	0
4	$\mathbb{Z}$	0	$22\mathbb{Z}$	0	$\mathbb{Z}$
3	0	0	0	0	0
2	$\mathbb{Z}$	0	$22\mathbb{Z}$	0	$\mathbb{Z}$
1	0	0	0	0	0
0	$\mathbb{Z}$	0	$22\mathbb{Z}$	0	$\mathbb{Z}$
-1	0	0	0	0	0
-2	$\mathbb{Z}$	0	$22\mathbb{Z}$	0	$\mathbb{Z}$
-3	0	0	0	0	0
-4	$\mathbb{Z}$	0	$22\mathbb{Z}$	0	$\mathbb{Z}$
-5	0	0	0	0	0
-6	$\mathbb{Z}$	0	$22\mathbb{Z}$	0	$0\mathbb{Z}$

Figure 6.8: Second (and final) page of the AHSS for the computation of $K^{-n}(K3)$.

No differentials can act non-trivially on the second page for degree reasons, so the sequence promptly terminates. Thus, the final result reads

$$K^{-n}(K3) = \begin{cases} 0 & \text{for } n \text{ odd}, \\ 24\mathbb{Z} & \text{for } n \text{ even}. \end{cases} \tag{6.65}$$

Note that the factor 24 arises as $b_0 + b_2 + b_4 = 1 + 22 + 1$ with b_m being the Betti numbers of $K3$. Therefore, we can also express the K-theory groups on K3 as

$$K^{-n}(K3) = \bigoplus_{m=0}^{4} b_{4-m}(K3)\, K^{-m-n}(\mathrm{pt})\,. \tag{6.66}$$

Computing $K^{-n}(CY_3)$

The computation of $K^{-n}(CY_3)$ is similar to that of $K3$, so we directly present the second page in figure 6.9. The only possibly non-vanishing differential is $d_3 : E_3^{1,q} \to E_3^{4,q-2}$ and

6	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$
5	0	0	0	0	0	0	0
4	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$
3	0	0	0	0	0	0	0
2	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$
1	0	0	0	0	0	0	0
0	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$
-1	0	0	0	0	0	0	0
-2	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$
-3	0	0	0	0	0	0	0
-4	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$
-5	0	0	0	0	0	0	0
-6	$\mathbb{Z}$	0	$b_2\mathbb{Z}$	$b_3\mathbb{Z}$	$b_2\mathbb{Z}$	0	$\mathbb{Z}$

Figure 6.9: Second (and final) page of AHSS for computation of $K^{-n}(CY_3)$.

vanishes due to lack of torsion. Given the triviality of the extension problem, we get

$$K^{-n}(CY_3) = \begin{cases} b_3\,\mathbb{Z} & \text{if n odd,} \\ (2+2b_2)\,\mathbb{Z} & \text{if n even.} \end{cases} \tag{6.67}$$

Now the factor $(2+2b_2)$ arises as $b_0+b_2+b_4+b_6$, with $b_0 = b_6 = 1$ and $b_2 = b_4$, the Betti numbers of the CY_3. This result can also be found in Corollary 1.9 of [300]. Again, we can elegantly express the K-theory groups on (simply connected) Calabi-Yau threefolds as

$$K^{-n}(CY_3) = \bigoplus_{m=0}^{6} b_{6-m}(CY_3)\,K^{-m-n}(\mathrm{pt})\,. \tag{6.68}$$

KO-groups of spheres and tori

The KO groups can be computed using the AHSS, but the presence of torsion leads to possibly non-vanishing differentials. For spheres S^k, one can use the splitting lemma and express the relevant groups as

$$KO^{-n}(S^k) = \widetilde{KO}(S^{n+k}) \oplus \widetilde{KO}(S^n) = KO^{-n-k}(\mathrm{pt}) \oplus KO^{-n}(\mathrm{pt})\,. \tag{6.69}$$

The full results for $KO^{-n}(S^k)$ for $n, k \leq 10$ are provided in appendix D.2. For tori, it was shown in [301] that

$$KO^{-n}(T^k) = \bigoplus_{m=0}^{k} \binom{k}{m} KO^{-m-n}(\mathrm{pt})\,. \tag{6.70}$$

Computing $KO^{-n}(K3)$

For real K-theory, computations involving higher dimensional manifolds can become complicated due to more involved differentials and extension problems. In fact, in the case of CY_3 the complexity of the computation goes beyond the scope of the present work. We will however discuss a simpler Calabi-Yau case, $K3$. In this case, all differentials are vanishing, and the computations can be performed up to extensions. The second page of the spectral sequence is:

$$E_2^{p,q} = H^p(K3; KO^q(\text{pt})) = \begin{cases} KO^q(\text{pt}) & \text{for } p = 0, 4, \\ 22\, KO^q(\text{pt}) & \text{for } p = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{6.71}$$

Due to the form of the page, only the differentials d_2 and d_4 may be non-vanishing. At

7	$\mathbb{Z}_2$	0	$22\mathbb{Z}_2$	0	$\mathbb{Z}_2$
6	$\mathbb{Z}_2$	0	$22\mathbb{Z}_2$	0	$\mathbb{Z}_2$
5	0	0	0	0	0
4	$\mathbb{Z}$	0	$22\mathbb{Z}$	0	$\mathbb{Z}$
3	0	0	0	0	0
2	0	0	0	0	0
1	0	0	0	0	0
0	$\mathbb{Z}$	0	$22\mathbb{Z}$	0	$\mathbb{Z}$
-1	$\mathbb{Z}_2$	0	$22\mathbb{Z}_2$	0	$\mathbb{Z}_2$
-2	$\mathbb{Z}_2$	0	$22\mathbb{Z}_2$	0	$\mathbb{Z}_2$
-3	0	0	0	0	0
-4	$\mathbb{Z}$	0	$22\mathbb{Z}$	0	$\mathbb{Z}$
-5	0	0	0	0	0
-6	0	0	0	0	0
-7	0	0	0	0	0

Figure 6.10: Second (and final) page of the AHSS for the computation of $KO^{-n}(K3)$.

degree two, we have

$$d_2 : E_2^{p,q} \to E_2^{p+2,q-1}, \tag{6.72}$$

for $p = 0, 2$ and $q = 0, -1$, together with all of its periodic copies. The explicit form of this differential is known to be [302, 303]

$$d_2 = \begin{cases} Sq^2\rho : & H^p(K3; KO^0(\text{pt})) \to H^{p+2}(K3; KO^{-1}(\text{pt}))\,, \\ Sq^2 : & H^p(K3; KO^{-1}(\text{pt})) \to H^{p+2}(K3; KO^{-2}(\text{pt})), \end{cases} \tag{6.73}$$

corresponding to $q = 0, -1$ respectively. Here, $Sq^2 : H^p(X;\mathbb{Z}_2) \to H^{p+2}(X;\mathbb{Z}_2)$ is the second Steenrod square and ρ is the reduction modulo 2. It turns out that d_2 is vanishing for $X = K3$: We discuss the case $q = -1$, but the analysis can be similarly extended to $q = 0$. For any element $y \in H^p(X;\mathbb{Z}_2)$, we can represent $Sq^2(y) = \iota_*(w_2(N)) \cup y$ [304]. Here, N is the normal bundle of the submanifold $Y \subset X$, Poincaré dual to y and $\iota_* : H^p(Y) \to H^p(X)$ the cohomological push-forward. For $p = 0$, the differential d_2 vanishes since y is dual to the whole four dimensional manifold $X = K3$ which is Spin, thus $w_1(N) = w_2(N) = 0$. Alternatively, it vanishes since $Sq^2(y) = 0$ for $y \in H^0(X;\mathbb{Z}_2)$, according to the properties of Sq^i listed in appendix C.3). For $p = 2$, the differential vanishes as well, since from the condition $w_2(X) = w_1(X) = 0$, one can then prove $w_2(N) = 0$ for a two-dimensional manifold Y not necessarily orientable [67]. Alternatively, for $p = 2$ we can write $Sq^2(y) = \nu_2 \cup y$ (see equation (C.22)) and then the second Wu class, $\nu_2 = w_2(X) + w_1(X)^2$, vanishes since $X = K3$ is Spin. Thus, d_2 is trivial.

At degree four, we have the differential

$$d_4 : E_4^{0,-1} \to E_4^{4,-4}. \tag{6.74}$$

Since there cannot be non-trivial homomorphisms[2] $\mathbb{Z}_k \to \mathbb{Z}$ for $k \geq 2$, this differential must vanish and $E_2^{p,q} \cong E_\infty^{p,q}$. Thus, one can read off the $KO^{-n}(K3)$ groups, which we present in table 6.8 up to extensions.

n	0	1	2	3	4	5	6	7
$KO^{-n}(K3)$	$\mathbb{Z} \oplus e(22\mathbb{Z}_2, \mathbb{Z})$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus 22\mathbb{Z}$	0	$2\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus 22\mathbb{Z}$	$22\mathbb{Z}_2$

Table 6.8: KO-groups $KO^{-n}(K3)$, $n = 0, \ldots, 7$, up to extensions. The result can be extrapolated to $n \geq 8$ by Bott periodicity.

6.5 Physical interpretation

In this section, we show that the cobordism and K-theory groups of X we calculated can be interpreted in terms of the dimensional reduction of global symmetries, in the spirit of the discussion of section 6.3. For $X \in \{S^k, T^k, K3, CY_3\}$, all differentials and the extension problems turned out to be trivial, significantly simplifying our analysis. Since this was not the case for KO-theory and Ω^{Spin}, we refrain from making any concrete

[2]This can be seen as follows. Consider the case $\phi : \mathbb{Z}_2 \to \mathbb{Z}$, the generalisation to $k > 2$ being straightforward. ϕ cannot be a non-trivial homomorphism since choosing $\phi(0) = 0$ and $\phi(1) = 1$ leads to the contradiction $0 = \phi(0) = \phi(2) = \phi(1) + \phi(1) = 2$. One has to set $\phi(1) = 0$, hence ϕ is trivial.

comments about these groups at this stage, and we defer this to future work, since a more involved mathematical analysis is necessary.

However, even for the simpler $K/\Omega^{\mathrm{Spin}^c}$ case complications could arise: A first major obstacle would be turning on fluxes. For instance, non-trivial NS-NS three-form flux H necesitates the use of H-twisted K-theory groups $K_H^{-n}(X)$ and the corresponding cobordism groups $\Omega^{\mathrm{Spin}^c,H}(X)$. For manifolds with $W_3 = 0$, i.e., with a Spin^c-structure, the absence of Freed–Witten anomalies implies that the H-flux through a D-brane must vanish. This will result in non-trivial differentials $d_r : E_r^{p,q} \to E^{p+r,q-r+1}$ in the AHSS. Second, even in the purely geometric case, without fluxes, when computing for instance $KO^{-n}(X)$, there could be non-trivial differentials, indicating, e.g., that certain cycles are not Spin. An explicit verification of these expectations is left for future work.

6.5.1 General aspects

The final results of the AHSS computation for all our examples can be expressed in a compact form. In particular, the K-theory groups $K^{-n}(X)$ of a k-dimensional manifold $X \in \{S^k, T^k, K3, CY_3\}$, with $n \geq 0$, turn out to be

$$K^{-n}(X) = \bigoplus_{m=0}^{k} b_{k-m}(X)\, K^{-n-m}(\mathrm{pt})\,. \tag{6.75}$$

This result has a clear interpretation in terms of D-branes: Consider $d = 10$ and compactify the theory on the k-dimensional manifold X, such that the total space is $\mathbb{R}^{1,d-k-1} \times X$. Then, $K^{-n}(X)$ classifies all D-branes that are of codimension-n in the flat space $\mathbb{R}^{1,d-k-1}$. From the d-dimensional point of view, these are given by codimension-$(n+m)$ branes wrapping $(k-m)$-cycles on the compact space X. Hence, the result (6.75) reflects that the dimensional reduction performed following this geometrical reasoning is already the correct answer on these manifolds. The AHSS provides additional information: none of the wrapped D-branes experiences a Freed–Witten anomaly nor that there is an instantonic decay channel.

The relation (6.75) connects to the completeness hypothesis [305]. We can regard the right-hand side of (6.75) as a lattice of charges $(\mathbf{q_1}, \ldots, \mathbf{q_k})$, where each entry $\mathbf{q}_m$ is a charge vector with b_{k-m} components.[3] The lattice sites are populated independently of one another, meaning that, in general, the full spectrum of charges (or rather stable states with that given charge) is complete. To understand the point, consider the simple two-dimensional situation in which the lattice is just $\mathbb{Z} \oplus \mathbb{Z}$. In this case, one not only has stable bound states of branes associated to, say, $(1,0)$ and $(0,1)$, but also to $(1,1)$. Thus,

[3] In principle, this could be slightly inaccurate: the groups of the point might be direct sums, so one of them could correspond to more sites in the lattice. Here, we neglect this complication, since the analysis can be straightforwardly generalized.

what the relation (6.75) means is that any non-vanishing element $(\mathbf{q_1}, \ldots, \mathbf{q_k})$ must be associated with a stable object and, in this sense, the spectrum is complete. In general, the situation might become highly involved, especially in the presence of multicharged or non-BPS branes, but K-theory should give the correct answer.

For cobordism groups we found similarly that for $n \geq 0$ there exists the compact expression

$$\Omega_{n+k}^{\mathrm{Spin}^c}(X) = \bigoplus_{m=0}^{k} b_{k-m}(X)\, \Omega_{n+m}^{\mathrm{Spin}^c}(\mathrm{pt})\,. \tag{6.76}$$

We will comment on the $-k \leq n < 0$ case later, but for $n > -k$ we propose the following intuitive interpretation of this result:

Recalling that in the definition of $\Omega_n(X)$ we introduce continuous maps $f : M \to X$, for every n-dimensional compact manifold M, such that $[M, f] \in \Omega_n(X)$. A non-vanishing term labeled by m in the sum on the right-hand side indicates that the map $f : M \to X$ from the $(n+k)$-dimensional manifold M to the k-dimensional manifold X is such that M is wrapped around a non-trivial $(k-m)$-cycle of X, while the map introduces no other obstruction in the remaining $(n+m)$ directions of M. Since there are b_{k-m} different $(k-m)$-cycles on X, we get b_{k-m} factors of $\Omega_{n+m}^{\mathrm{Spin}^c}(\mathrm{pt})$ in the total cobordism group $\Omega_{n+k}^{\mathrm{Spin}^c}(X)$.

Taking into account that the objects charged under the cobordism groups $\Omega_n(\mathrm{pt})$ are the $(d-n)$-dimensional gravitational solitons mentioned in section 4.4.3, there is a similar interpretation to the K-theory groups: $\Omega_{n+k}^{\mathrm{Spin}^c}(X)$ classifies all gravitational solitons that are of codimension n in the flat space $\mathbb{R}^{1,d-k-1}$. From the full d-dimensional point of view, they are given by the set of all codimension-$(n+m)$ objects wrapping $(k-m)$-cycles on the compact space X.

Concretely, defining a basis $\{\Sigma_m^a\}$ of m-cycles on X, with $a = 1, \ldots, b_m(X)$, and taking into account that $\Omega_{\mathrm{even}}^{\mathrm{Spin}^c}(\mathrm{pt}) = \mathbb{Z}$, for a given m-charge vector

$$\mathbf{q_m} = (q_m^1, \ldots, q_m^{b_m}) \in \mathbb{Z}^{b_m}, \tag{6.77}$$

the map f is such the $(n+k)$-dimensional manifold M_{n+k} is wrapped q_m^a times around the m-cycle Σ_m^a of X - we can think of the m-cycle as shared between M and X.

For all values of the index $n+k$, we want to figure out how to organize the information contained in K-theory and cobordism groups of X to reconstruct tadpole cancellation conditions as a bottom-up approach to string theory. We assume $n \geq 0$ for the time being.

Given the previous results, we can understand how the Hopkins–Hovey isomorphism applies to cobordism and K-theory groups of higher-dimensional manifolds X. First, we bring the K-theory result (6.75) into the same form as (6.76), converting the K-theory group from generalized cohomology to homology:

$$K^{-n}(X) = K_{n+k}(X), \tag{6.78}$$

valid for X a k-dimensional Spin^c manifold. The analogous result holds for real K-theory, namely

$$KO^{-n}(X) = KO_{n+k}(X), \tag{6.79}$$

for X a k-dimensional Spin manifold. Both equations (6.78),(6.79) follow e.g. from Theorem 2.9 of section V of [306], after recalling that a manifold is K-oriented (resp. KO-oriented) iff it is Spin^c (resp. Spin).Therefore, for $n \geq 0$, the ABS orientation can be extended to a map

$$\alpha^c_X : \Omega^{\text{Spin}^c}_{n+k}(X) \to K_{n+k}(X), \tag{6.80}$$

acting as α^c in (6.4) on each term $\Omega^{\text{Spin}^c}_{n+k-m}(\text{pt})$. Dividing by the kernel of this map provides an isomorphism between cobordism and K-theory classes on X, directly inherited from the isomorphism between $\Omega^{\text{Spin}^c}_n(\text{pt})$ and $K_n(\text{pt})$. We expect the analogous result for the relation between Spin cobordism groups $\Omega^{\text{Spin}}_{n+k}(X)$ and the real K-theory classes $KO_{n+k}(X)$ for $X \in \{S^k, T^k\}$, where we could explicitly verify that the groups have the same compact, convenient decomposition.

As expected, there exists a unifying interpretation in terms of global symmetries. In our examples, the groups $K_{n+k}(X)$ and $\Omega^{\text{Spin}^c}_{n+k}(X)$ classify all global $(D-1-n)$-form charges in the non-compact $D = d-k$ dimensions, that arise from the dimensional reduction of global $d-1-n, d-2-n, \ldots, d-1-k-n$ form charges along the $k, k-1, \ldots, 0$ cycles of X. Due to the simple underlying structure, these global symmetries follow the usual rules of dimensional reduction. As we explained in section 6.3, if a global symmetry in D dimensions descends from a global symmetry in d dimensions, then its gauging involves the dimensionally reduced gauge field in d dimensions and also the corresponding dimensionally reduced D-branes (defects). The full tadpole cancellation condition in D dimensions arises from the dimensional reduction of the tadpole cancellation condition in d dimensions. Let us now make this more explicit through an extensive example.

6.5.2 Example: Type IIB on a Calabi-Yau threefold

Consider ten-dimensional type IIB superstring compactified on a Calabi-Yau threefold X. On the K-theory side, we have

$$K^0(X) = b_6 \underbrace{K^0(\text{pt})}_{\mathbb{Z}} \oplus b_4 \underbrace{K^{-2}(\text{pt})}_{\mathbb{Z}} \oplus b_2 \underbrace{K^{-4}(\text{pt})}_{\mathbb{Z}} \oplus b_0 \underbrace{K^{-6}(\text{pt})}_{\mathbb{Z}}, \tag{6.81}$$

with $b_0 = b_6 = 1$. The corresponding D-branes are of codimension zero in the flat $\mathbb{R}^{1,3}$ space, the four terms in the right-hand side of (6.81) correspond respectively to

$D9$-branes wrapping the entire CY_3, $D7$-branes wrapping the b_4 4-cycles of the CY_3, $D5$-branes wrapping the b_2 2-cycles of the CY_3 and finally $D3$-branes being point-like on the CY_3. At the next level, we have

$$K^{-1}(X) = b_3 \underbrace{K^{-4}(\mathrm{pt})}_{\mathbb{Z}}, \tag{6.82}$$

corresponding to a codimension-one brane in $\mathbb{R}^{1,3}$ arising by $D5$-branes wrapping any 3-cycle on the CY_3. As already explained, In accordance with the completeness hypothesis, for all multi-charges, corresponding bound states of the single-charged states should exist.

On the cobordism side we have

$$\Omega_6^{\mathrm{Spin}^c}(X) = b_6 \underbrace{\Omega_0^{\mathrm{Spin}^c}(\mathrm{pt})}_{\mathbb{Z}} \oplus b_4 \underbrace{\Omega_2^{\mathrm{Spin}^c}(\mathrm{pt})}_{\mathbb{Z}} \oplus b_2 \underbrace{\Omega_4^{\mathrm{Spin}^c}(\mathrm{pt})}_{\mathbb{Z}\oplus\mathbb{Z}} \oplus b_0 \underbrace{\Omega_6^{\mathrm{Spin}^c}(\mathrm{pt})}_{\mathbb{Z}\oplus\mathbb{Z}}. \tag{6.83}$$

This corresponds to a $(3b_2 + 3)$-dimensional lattice of $\mathbb{Z}$-valued global 3-form charges in $\mathbb{R}^{1,3}$, arising from the dimensional reduction of the ten-dimensional 9-form, 7-form, 5-form, and 3-form global symmetries along the 6-, 4-, 2-, 0-cycles of the CY_3 respectively.

At the next level

$$\Omega_7^{\mathrm{Spin}^c}(X) = b_3 \underbrace{\Omega_4^{\mathrm{Spin}^c}(\mathrm{pt})}_{\mathbb{Z}\oplus\mathbb{Z}}, \tag{6.84}$$

similarly corresponding to a lattice of global 2-form symmetries coming from the dimensional reduction of the 10-dimensional 5-form symmetry on the 3-cycles of the CY_3.

Tadpole reconstruction

Through the example of type IIB on CY_3, we will explain how to relate the information contained in the cobordism and K-theory groups of X and construct appropriate tadpole cancellation conditions. Fundamentally, the idea is precisely that of [35], but now each of the groups $K^{-n}(X), \Omega_{k+n}^{\mathrm{Spin}^c}(X)$ decomposes to multiple pieces, each of them being a group of the point, hence contributes to multiple cancellation conditions. Let us make this more concrete.

Consider a six-dimensional Spin^c-manifold M_6 classified by $\Omega_6^{\mathrm{Spin}^c}(X)$ but, contrary to the Calabi-Yau X, not necessarily a solution to the string theory equations of motion. In this sense, M_6 can be off-shell. The same manifold must also lie in the right-hand side contributions $b_m\Omega_{6-m}^{\mathrm{Spin}^c}(\mathrm{pt})$. Since a continuous map $f : M_6 \to X$ must exist, the manifold M_6 shares some m-cycles with the fixed background space X. Which m-cycles are shared depends on the non-zero entries in the charge vector (6.77). Then, the magnetic $(6-m)$-form currents are obtained from the cobordism invariants (6.7), for $M = M_6$. We propose that this time the magnetic $(6-m)$-form currents are defined by expanding their right-hand sides into a basis of those $(6-m)$-forms in $H^{6-m}(M_6;\mathbb{Z})$ that also lie in $H^{6-m}(X;\mathbb{Z})$ (again depending on the entries in the charge vector).

$$\hat{\tilde{J}}_{m,i}(M_6) = \sum_{a=1}^{b_m} \alpha^a_{m,i}\, q^a_m\, \Sigma^a_m + \dots, \tag{6.85}$$

where the dots indicate more contributions along m-cycles of M_6 that do not lie in X. Note that the (co)homology of M_6 can, in principle, be bigger than that of X. Since this expansion is also valid for $M_6 \neq X$, we can go slightly off-shell. Generally, topological K-theory and cobordism groups classify all global charges that can be present in the theory, irrespective of properties like supersymmetry or being on-shell.

The ABS orientation, together with the fact that K-theory global symmetries are gauged, lead to the following tadpole cancellation conditions, as in [35].

First, we look at $\Omega_0^{\mathrm{Spin}^c}(\mathrm{pt})$ and $K^0(\mathrm{pt})$. $\Omega_0^{\mathrm{Spin}^c}(\mathrm{pt}) = \mathbb{Z}$ gives rise to a single global 3-form symmetry in 4d, with the trivial magnetic current $\tilde{J}_0(M_6) = \mathrm{td}_0(M_6) = 1$. In 10d, the corresponding 9-form symmetry is gauged with the charged objects being $D9$-branes, classified by $K^0(\mathrm{pt}) = \mathbb{Z}$. This leads to the tadpole condition

$$N\, \delta^{(0)}(M_6) + a^{(0)}\, \mathrm{td}_0(M_6) = 0, \tag{6.86}$$

where $\delta^{(0)}(M_6)$ denotes the 0-form Poincaré dual to the 6-cycle M_6 wrapped by the stack of N $D9$-branes. This 0-form comes from the ten-dimensional delta $\delta^{(0)}(\mathbb{R}^{1,3} \times M_6) = \delta^{(0)}(\mathbb{R}^{1,3}) \wedge \delta^{(0)}(M_6)$. For $a^{(0)} = 0, -32$, this is the usual $D9$-tadpole cancellation condition in type IIB/type I string theory.

Second, we have $\Omega_2^{\mathrm{Spin}^c}(\mathrm{pt})$ and $K^{-2}(\mathrm{pt})$. $b_4\Omega_2^{\mathrm{Spin}^c}(\mathrm{pt}) = b_4\mathbb{Z}$ gives rise to b_4 global 3-form symmetries in 4d, with preserved magnetic 0-form currents $\tilde{j}_0^{(2)a}$ given by the expansion of the ten-dimensional 2-form current $\tilde{J}_2(M_6) = \mathrm{td}_2(M_6)$ in a cohomological basis $\omega_{(2)a} \in H^2(X;\mathbb{Z})$:

$$\tilde{J}_2(M_6) = \sum_{a=1}^{b_4} \tilde{j}_0^{(2)a} \wedge \omega_{(2)a}\,. \tag{6.87}$$

Since $b_4 = b^2$, this is the Poincaré dual to the expansion (6.85) and we have incorporated the charges q_4^a into the coefficients. For a $D7$-brane classified by $K^{-2}(\mathrm{pt})$, wrapping 4-cycles $\Sigma_4 \in H_4(M_6;\mathbb{Z})$ in X (times the flat space $\mathbb{R}^{1,3}$), we can expand its Poincaré dual 2-form as

$$\delta^{(2)}(\mathbb{R}^{1,3} \times \Sigma_4) = \sum_{a=1}^{b_4} \delta^{(0)}(\mathbb{R}^{1,3})^{(2)a} \wedge \omega_{(2)a}\,. \tag{6.88}$$

In 10d, the gauging of the corresponding 7-form global symmetry is associated with a tadpole constraint

$$\sum_{j\in\mathrm{def}} N_j\, \delta^{(2)}(\mathbb{R}^{1,3} \times \Sigma_{4,j}) + a^{(2)}\, \frac{c_1(M_6)}{2} = 0. \tag{6.89}$$

Upon expansion in a cohomological basis of $H^2(X;\mathbb{Z}) = b_4\mathbb{Z}$ this leads to $b_4 = b^2$ tadpole cancellation conditions. For $a^{(2)} = -24$ and M_6 being the base B_3 of an elliptically fibered Calabi-Yau fourfold, (6.89) is the 7-brane tadpole constraint of F-theory. Note that for $M_6 = X$, $c_1(X) = 0$, and the tadpole cancellation condition simplifies. However, the power of our formalism is that we can go off-shell and detect terms that could appear in principle, even if they are absent for the on-shell configurations.

Third, we consider $\Omega_4^{\mathrm{Spin}^c}(\mathrm{pt})$ and $K^{-4}(\mathrm{pt})$. $b_2\Omega_4^{\mathrm{Spin}^c}(\mathrm{pt}) = b_2\,(\mathbb{Z}\oplus\mathbb{Z})$ gives rise to $2b_2$ global 3-form symmetries in 4d. The ABS orientation between K-theory and cobordism is not an isomorphism, but we consider the contributions from all the cobordism invariants, according to [35]. The preserved magnetic 0-form currents $\tilde{j}^{(4)a}_{0,i}$, $i = 1,2$, in $D = 4$ are given by the expansion of the ten-dimensional 4-form currents $\tilde{J}_{4,i}(M_6)$ in a cohomological basis $\hat{\omega}_{(4)a}$ of $H^4(X;\mathbb{Z})$ as

$$\tilde{J}_{4,i}(M_6) = \sum_{a=1}^{b_2} \tilde{j}^{(4)a}_{0,i} \wedge \hat{\omega}_{(4)a}\,. \tag{6.90}$$

$K^{-4}(\mathrm{pt})$ classifies $D5$-branes wrapping 2-cycles $\hat{\Sigma}_2$ on M_6 that are shared with X (times the flat space $\mathbb{R}^{1,3}$). Their Poincaré duals can be expanded similarly to (6.90). The gauging of the 10d 5-form symmetry gives a tadpole condition of the form

$$\sum_{j\in\mathrm{def}} N_j\,\delta^{(4)}(\mathbb{R}^{1,3}\times\hat{\Sigma}_{2,j}) + a_1^{(4)}\left(\frac{c_2(M_6)+c_1^2(M_6)}{12}\right) + a_2^{(4)}\,c_1^2(M_6) = 0. \tag{6.91}$$

Upon expansion in a cohomological basis of $H^4(X;\mathbb{Z}) = b_2\mathbb{Z}$, one obtains $b_2 = b^4$ tadpole cancellation conditions. Two setups are described by (6.91): the type I string on $X = K3\times T^2$ for $a_1^{(4)} = -12$ and $a_2^{(4)} = 3/2$, and the $\Omega\sigma$ orientifold of type IIB on $X = K3\times T^2$ [307,308] for $a_1^{(4)} = -24$ and $a_2^{(4)} = 0$.

Finally, we have $\Omega_6^{\mathrm{Spin}^c}(\mathrm{pt})$ and $K^{-6}(\mathrm{pt})$. $\Omega_6^{\mathrm{Spin}^c}(\mathrm{pt}) = \mathbb{Z}\oplus\mathbb{Z}$ gives two global 3-form symmetries in 4d, with preserved magnetic 0-form currents $\tilde{j}^{(6)}_{0,i}$, $i = 1,2$ (again in $D = 4$) given by the reduction of the ten-dimensional 6-form currents $\tilde{J}_{6,i}(M_6)$ along the volume 6-form of M_6,

$$\tilde{J}_{6,i}(M_6) = \tilde{j}^{(6)}_{0,i}\ \mathrm{vol}(M_6)\,. \tag{6.92}$$

$K^{-6}(\mathrm{pt})$ point-like $D3$-branes on M_6. The gauging of the 10d 3-form symmetry implies a tadpole condition of the general form

$$\sum_{j\in\mathrm{def}} N_j\,\delta^{(6)}(\mathbb{R}^{1,3}\times\mathrm{pt}_j) + a_1^{(6)}\,c_2(M_6)\,\frac{c_1(M_6)}{24} + a_2^{(6)}\,\frac{c_1^3(M_6)}{2} = 0\,. \tag{6.93}$$

For $a_1^{(6)} = -12$ and $a_2^{(6)} = -30$, this tadpole condition corresponds to F-theory compactified on a smooth elliptically fibered Calabi-Yau fourfold with base $M_6 = B_3$. For a Calabi-Yau manifold, such as $M_6 = X$, the two contributions from cobordism vanish, but the off-shell nature of cobordism makes them visible.

At the next level, we find $K^{-1}(X) = b_3 K^{-4}(\mathrm{pt}) = b_3\mathbb{Z}$ and $\Omega_7^{\mathrm{Spin}^c}(X) = b_3\Omega_4^{\mathrm{Spin}^c}(\mathrm{pt}) = b_3(\mathbb{Z}\oplus\mathbb{Z})$ which are associated to 2-form symmetries in 4d, arising from the reduction of the global 5-form symmetries along the b_3 3-cycles of X. For $\Omega_7^{\mathrm{Spin}^c}(X)$, the $2b_3$ preserved magnetic 1-form currents $\tilde{\jmath}_{1,i}^{(3)a}$, with $i = 1, 2$, in $D = 4$ are given by the dimensional reduction of the ten-dimensional 4-form currents $\tilde{J}_{4,i}(M_6)$ along the basis 3-forms $\omega_{(3)a} \in H^3(X;\mathbb{Z})$, and are given schematically by

$$\tilde{J}_{4,i}(M_6) = \sum_{a=1}^{b_3} \tilde{\jmath}_{1,i}^{(3)a} \wedge \omega_{(3)a}\,, \tag{6.94}$$

where the currents $\tilde{\jmath}_{1,i}^{(3)a}$ may be vanishing. The $D5$-brane defects wrapping 3-cycles Σ_3 on M_6 shared with X times a three-dimensional submanifold Π_3 of the flat space $\mathbb{R}^{1,3}$ can be expanded as

$$\delta^{(4)}(\Pi_3 \times \Sigma_3) = \sum_{a=1}^{b_3} \delta^{(1)}(\Pi_3)^{(3)a} \wedge \omega_{(3)a}. \tag{6.95}$$

In 10d, the global symmetry of $K^{-4}(\mathrm{pt})$ is gauged with a magnetic Bianchi identity

$$d\tilde{F}_3 = \sum_{j\in\mathrm{def}} N_j\,\delta^{(4)}(\Pi_{3,j} \times \Sigma_{3,j}) + a_1^{(4)}\,\tilde{J}_{4,1}(M_6) + a_2^{(4)}\,\tilde{J}_{4,2}(M_6)\,. \tag{6.96}$$

Expanding the magnetic field strength as

$$\tilde{F}_3 = \sum_{a=1}^{b_3} \tilde{f}_0^{(3)a} \wedge \omega_{(3)a}, \tag{6.97}$$

leads to b_3 Bianchi identities for the four-dimensional 0-forms

$$d\tilde{f}_0^{(3)a} = \sum_{j\in\mathrm{def}} N_j\,\delta^{(1)}(\Pi_{3,j})^{(3)a} + a_1^{(4)}\,\tilde{\jmath}_{1,1}^{(3)a} + a_2^{(4)}\,\tilde{\jmath}_{1,2}^{(3)a}\,. \tag{6.98}$$

The discussion at the next level, for groups such as $\Omega_8^{\mathrm{Spin}^c}(X)$ and $\Omega_9^{\mathrm{Spin}^c}(X)$, together with their K-theory counterparts, proceeds along the same lines.

6.5.3 Fate of low-dimensional $\Omega_n^{\mathrm{Spin}^c}(X)$

Our discussion up to now has not involved the low-dimensional cobordism groups, $\Omega_{n+k}^{\mathrm{Spin}^c}(X)$ with $-k \le n < 0$. These groups are formally non-vanishing, hence one could expect them to be accompanied by tadpole conditions, as for their higher-dimensional counterparts.

However, it turns out there is a fundamental difference between these groups. To explain it, let us consider which K-theory groups would enter the same tadpoles as the lower-dimensional cobordism groups. These are $K_{n+k}(X) = K^{-n}(X)$ with $-k \leq n < 0$. For concreteness, we pick once again the well-studied example of CY_3, and we consider $K^2(CY_3)$. Extrapolating the relation (6.75) to $n = -2$, we would get

$$K^2(X) = \bigoplus_{m=2}^{6} b_{6-m}(X)\, K^{2-m}(\text{pt}) = b_4(X)\, K^0(\text{pt}) \oplus \dots \,, \tag{6.99}$$

where we removed from the sum the term $K^2(\text{pt})$, associated to $m = 0$, which formally is not vanishing, but is unphysical due to the negative codimension it would require for the brane. In addition, it does not have a counterpart in the cobordism side (6.76). What seems to be physical is the term $K^0(\text{pt})$, which corresponds formally to a $D9$-brane wrapped on a 4-cycle of the CY_3. However, the D9-brane fills completely the 10d spacetime, so in reality, it must wrap a 6-chain in CY_3, i.e., a 4-cycle times a 2-chain. This is, however, a topologically trivial configuration, hence the physical interpretation of these K-theory groups $K_{n+k}(X)$ is questionable. This precisely matches the intuitive picture we have about $K^{-n}(X)$: for $n < 0$, we practically have to do with negative codimension branes in the $D = (d-k)$-dimensional spacetime - this does not seem to have a real physical meaning.

Now that the D-brane picture indicates that these lower-dimensional groups are unphysical, we can check if the cobordism side provides a matching intuition. In fact, we can think of the objects charged under the non-trivial cobordism groups $\Omega^{\text{Spin}^c}_{n+k}(X)$ as gravitational solitons of codimension n in the $(d-k)$-dimensional non-compact space. For $-k \leq n < 0$, even though the group formally does not vanish, the solitons would not fit in our spacetime, hence we expect this group to be unphysical.

All in all, we have strong indications to disregard these groups. In a sense, the cobordism/K-theory correspondence goes through nicely even in this case.

6.6 Summary and outlook

Our analysis strongly supports the Cobordism Conjecture and the subsequent proposal of [35], that whenever an appropriate map exists on the mathematical side, such as the Hopkins-Hovey isomorphism between K-theory and Spin^c cobordism (or KO-theory and Spin cobordism) the global symmetries associated to both groups are gauged simultaneously, though a joint tadpole cancellation condition. We have studied how this proposal behaves when fixing a topological space as a background. To this end, we employed the Atiyah-Hirzebruch spectral sequence to compute cobordism and K-theory groups of simple, higher dimensional manifolds often encountered in string theory, such as tori and CY manifolds. We found that performing the dimensional reduction through the

AHSS gives results consistent with the usual dimensional reduction in cohomology, but additionally takes into account quantum mechanical effects, such as the absence of a Freed-Witten anomaly. These results for K-theory and cobordism, combined nicely into tadpoles, compatible with our known tadpole cancellation conditions in string theory.

This project can be extended in multiple directions. Most straightforwardly, it would be of interest to verify the cobordism/K-theory correspondence in cases with torsion, i.e., calculate the elusive differentials and extension problems for KO-theory and Ω^{Spin}. More complicated structures and incorporating gauge fields in the game would be a possible next, non-trivial, step. This could truly demonstrate the power of the AHSS and the way it takes into account quantum mechanical effects.

In a different direction, one could try to figure out how the relative coefficients arise in the tadpoles, and how they relate to discrete symmetries. Additionally, we could try and identify all the different setups where some version of the Hopkins-Hovey isomorphism might apply. Since we know that cobordism does not rely on the presence of supersymmetry, could this correspondence, for instance, be relevant for some non-supersymmetric theory?

Chapter 7

Dynamical Cobordism: One explicit example

7.1 Preface

Cobordism is a very powerful framework, which allows us to study setups over which we usually have little control. Such a case was exemplified in the previous chapter 6, where the interplay of cobordism and K-theory allowed us to uncover tadpoles in their most general form, including terms that appear only for off-shell configuration. Another case that is less accessible with our conventional methods concerns non-supersymmetric setups. Up to now, we have only discussed supersymmetric string theories and explicit supersymmetry breaking, for instance through the anti-D3-uplift in the warped throat. In reality, there is an additional set of consistent, anomaly-free ten-dimensional string theories that arise from a supersymmetric world-sheet, yet do not enjoy any spacetime supersymmetry. These theories might have undesirable features, such as tachyonic modes, or no spacetime fermions. However, there exist three theories among them which are both tachyon-free and include fermions in their spectrum, so, from a phenomenological standpoint, deserve further attention. In particular, these are the $USp(32)$ theory, commonly also referred to as the Sugimoto model [309], the U(32) type 0'B theory [310,311] and the $SO(16) \times SO(16)$ heterotic theory [312, 313]. These models have attracted attention within the Swampland Program (see e.g., [228, 314]), since they help provide a more complete description of the Landscape.

It turns out that certain non-supersymmetric setups, including non-supersymmetric 10d strings, can indeed be efficiently probed using cobordism.

Supersymmetry-breaking vacua are expected to feature dynamical tadpoles [315, 316]. These tadpoles are fundamentally different from topological tadpoles, such as the R-R we have encountered so far, in the sense that they do not signal a fundamental inconsistency in the theory. Instead, they can be viewed as an indication that we have not identified the true vacuum of the theory - in fact, the true vacuum is not expected to be maximally symmetric, but rather correspond to a spacetime-dependent solution. These solutions generically have peculiar features, such as singularities where the string coupling diverges.

Recently, it was proposed that these spacetime-dependent configurations, sourced by dynamical tadpoles, admit a description in the framework of *Dynamical Cobordism* [36, 37]. This is a complementary framework to the usual topological study of cobordism, and provides a geometrical picture describing the spacetime-ending defects at finite spacetime distance. Locally, these defects are argued to exhibit a universal scaling behaviour, quantified in a single critical exponent [36, 37].

In this chapter, we will use this framework to study the backreaction of a non-supersymmetric, positive-tension domain wall, described aptly within the Dynamical Cobordism framework. The singularities in the solution in the presence of this domain wall will be interpreted as an indication for the existence of an end-of-the-world defect, and we will manage to provide an explicit description of this cobordism-predicted object.

This chapter will have the following structure: In section 7.2 we will review the salient features of the Dynamical Cobordism framework. In section 7.3, we first comment on the Sugimoto model, and then introduce the T-dual version of Blumenhagen-Font [317] which admits a $16 \times \overline{D8} + O8^{++}$ stack that can be viewed as a neutral domain wall. We will study the backreaction of this domain wall following [317] and interpret it with respect to Dynamical Cobordism. In section 7.4 we construct the (local) solution of the dilaton-gravity equations of motion around the cobordism-predicted end-of-the-world (ETW) defect. This solution turns out to not correspond to any known codimension-two object in string theory, signaling a possible novel defect.

7.2 Dynamical cobordism: Idea and scaling relations

Dynamical Cobordism [36, 37, 193] is a framework that gives a geometrical description to the configurations that can end spacetime, as predicted by the Cobordism Conjecture. This is achieved via the study of theories that feature dynamical tadpoles, i.e., potentials without minima that do not admit maximally symmetric solutions. These potentials admit spacetime-dependent solutions, dubbed *dynamical cobordisms* in [36, 37], which feature singularities at finite spacetime distance.

In [36] two universal features of these solutions were identified:

- For a dynamical tadpole governed by an order parameter $\mathcal{T}$, the spacetime-dependent solution of the equations of motion cannot be extended beyond a critical spacetime distance Δ, scaling with respect to $\mathcal{T}$ as

$$\Delta^{-n} \sim \mathcal{T}, \tag{7.1}$$

with n an $\mathcal{O}(1)$ constant.
- The physical mechanism cutting off spacetime at Δ is a cobordism defect of the initial theory.

This behavior was confirmed in several setups, including the Sugimoto Model [309], massive type IIA theory, and our familiar from chapter 5 Klebanov-Strassler solution.

In [37], this behavior was studied with respect to the scalars contributing to the potential. It was realized that one can differentiate two cases: the scalars can either remain at finite distance or go to infinity. The postulated cobordism defect at the singularity will then be either a domain wall interpolating between different theories or a wall of nothing capping of spacetime, respectively. In this work, we will focus on the second case, which is reminiscent of the Distance Conjecture. The following conjecture was proposed:

Cobordism Distance Conjecture [37]:
Every infinite field distance limit of an effective theory consistent with quantum gravity, can be realized by running into a cobordism wall of nothing in (possibly a suitable compactification of) the theory.

It was shown for several examples in [37] that the following scaling laws hold close to the singularity (in Planck units)

$$\Delta \sim e^{-\frac{1}{2}\delta D}, \quad |\mathcal{R}| \sim e^{\delta D}, \tag{7.2}$$

with D the field distance and δ a positive coefficient. In [193] a general formalism for the local effective description of Dynamical Cobordism was provided, precisely reproducing this scaling behavior. Note that the terminology of *End of The World* (ETW) brane was used to describe the singular sources in the effective field theory.

Finally, let us comment on some more recent developments. The Dynamical Cobordism framework proved suitable to describe also time-dependent solutions with a spacelike-singularity denoting the beginning of time [194], small black hole solutions [195], and AdS/CFT [318].

7.3 Non-supersymmetric string theory and the backreacted domain wall

7.3.1 The Sugimoto model and its T-dual

In [309] the consistency conditions for the $D9-\overline{D9}$ system in type I were studied. For n D9- and m anti-D9-branes, and gauge group $USp(n) \times USp(m)$, the theory turned out to be anomaly free for $m-n=32$. The tachyon-free case, commonly referred to as the *Sugimoto model*, corresponds to the pair $(m,n)=(32,0)$, i.e., the gauge groups is $USp(32)$. In practice, this theory is similar to the usual type I string theory, with the significant difference that now the orientifold projection creates O9-planes with positive charge and tension, denoted $O9^{++}$. This is necessary to cancel the R-R tadpole, since the gauge sector includes 32 anti-D9-branes, but comes at the cost of leaving an uncancelled NS-NS tadpole for the dilaton. Moreover, the Möbius amplitude of the model is non-vanishing, so supersymmetry is explicitly broken.

The effective Lagrangian of the Sugimoto model in the Einstein frame is [319]

$$S_E = \frac{1}{2\kappa_{10}^2} \int d^{10}x\sqrt{G}\big[\mathcal{R} - \frac{1}{2}(\partial\Phi)^2\big] - T_9 \int d^{10}x\sqrt{G} \cdot 64 e^{\frac{3\Phi}{2}} + \dots , \tag{7.3}$$

with T_9 the tension.

The equations of motion were shown to have the so-called Dudas-Mourad solution [319], a warped metric with 9d Poincaré symmetry and the tenth dimension spontaneously compactified on an interval. The Dudas-Mourad solution was shown in [37, 193] to satisfy the dynamical cobordism scalings (7.2). While we will not further discuss this solution, we want to remark that it is still a subject of active research, see e.g., [196, 320–323].

In [317] a T-dual version of the Sugimoto model was constructed, by performing a T-duality along the tenth direction. The result was a model with two positively charged $O8^{++}$-planes, with 16 anti-D8-branes at each fixed point to cancel the R-R charge locally. The original action describing this setup included the contributions from the stacks at both fixed points. However, it turned out that the solution was spontaneously compactified on a circle along the tenth dimension, hence due to the periodicity one can equivalently consider an action with only one neutral stack. Similarly to the Dudas-Mourad solution, the Blumenhagen-Font solution featured singularities at finite distance, hence we find it worthwhile to revisit it with respect to Dynamical Cobordism.

7.3.2 Backreacted domain wall

We decide to generalize our setup a bit more, and we consider the backreaction of a gauge-neutral, non-supersymmetric 9-dimensional domain wall carrying only a positive tension and coupling to the dilaton like a D-brane, i.e. with the factor $\exp(-\Phi)$ in its action. Such a domain wall could be a non-BPS D8-brane of type I string theory or as a local R-R tadpole-free $16 \times \overline{D8} + O8^{++}$ stack in the Blumenhagen-Font model, and will be generically addressed as a neutral domain wall.

We consider a configuration where the neutral domain wall, carrying positive tension T, is located at the position $r = 0$ in the transversal directions. At leading order, its supergravity action is

$$S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} \left(\mathcal{R} - \frac{1}{2}(\partial\Phi)^2 \right) - T \int d^{10}x \sqrt{-g}\, e^{\frac{5}{4}\Phi}\, \delta(r), \tag{7.4}$$

where G_{MN} denotes the metric in ten dimensions and $g_{\mu\nu} = \delta_\mu^M \delta_\nu^N G_{MN}$ the induced metric on the nine-dimensional worldvolume of the brane. Moreover, one has $\kappa_{10}^2 = l_s^8/(4\pi)$, with l_s the string length - this is the action of [317] with a single source.

The gravity equation of motion reads

$$\begin{aligned} \mathcal{R}_{MN} - \frac{1}{2} G_{MN} \mathcal{R} - \frac{1}{2} \left(\partial_M \Phi \partial_N \Phi - \frac{1}{2} G_{MN} (\partial\Phi)^2 \right) = \\ - \lambda\, \delta_M^\mu \delta_N^\nu\, g_{\mu\nu} \sqrt{\frac{g}{G}}\, e^{\frac{5}{4}\Phi}\, \delta(r), \end{aligned} \tag{7.5}$$

with $\lambda = \kappa_{10}^2 T$, while the dilaton equation of motion is

$$\partial_M \left(\sqrt{-G}\, G^{MN}\, \partial_N \Phi \right) = \frac{5}{2} \lambda \sqrt{-g}\, e^{\frac{5}{4}\Phi}\, \delta(r)\,. \tag{7.6}$$

These non-linear equations share some features with the backreaction of BPS branes, but are much harder to solve due to the lack of supersymmetry.

7.3.3 Solutions breaking 9D Poincaré symmetry

As expected, there exists no solution preserving 9D Poincaré invariance, but, following [317], we can find a solution preserving 8D Poincaré invariance and featuring a single non-trivial longitudinal direction y. The general ansatz for the metric is

$$ds^2 = e^{2\mathcal{A}(r,y)} ds_8^2 + e^{2\mathcal{B}(r,y)} (dr^2 + dy^2), \tag{7.7}$$

and we separate the dependence of the warp factors $\mathcal{A}$, $\mathcal{B}$ and the dilaton Φ on the coordinates r and y, i.e.

$$\begin{aligned} &\mathcal{A}(r,y) = A(r) + U(y)\,, \quad \mathcal{B}(r,y) = B(r) + V(y)\,, \\ &\Phi(r,y) = \chi(r) + \psi(y)\,. \end{aligned} \tag{7.8}$$

For such a separation of variables, the ansatz (7.7) turns out to be the most general one.

The equations of motion give the following five, a priori independent, equations. The one coming from the variation $\delta G^{\mu\nu}$ is

$$\Big(7A'' + 28(A')^2 + B'' + \tfrac{1}{4}(\chi')^2\Big) + \Big(7\ddot{U} + 28(\dot{U})^2 + \ddot{V} + \tfrac{1}{4}(\dot{\psi})^2\Big) = -\lambda\, e^{B+V}\, e^{\frac{5}{4}\Phi}\,\delta(r)\,. \tag{7.9}$$

The prime denotes the derivative with respect to r and the dot the derivative with respect to y. For δG^{rr} and δG^{yy}, we obtain

$$\begin{aligned} &\Big(28(A')^2 + 8A'B' - \tfrac{1}{4}(\chi')^2\Big) + \Big(8\ddot{U} + 36(\dot{U})^2 - 8\dot{U}\dot{V} + \tfrac{1}{4}(\dot{\psi})^2\Big) = 0,\\ &\Big(8A'' + 36(A')^2 - 8A'B' + \tfrac{1}{4}(\chi')^2\Big) + \Big(28(\dot{U})^2 + 8\dot{U}\dot{V} - \tfrac{1}{4}(\dot{\psi})^2\Big) \\ &\qquad\qquad = -\lambda\, e^{B+V}\, e^{\frac{5}{4}\Phi}\,\delta(r)\,, \end{aligned} \tag{7.10}$$

and for the off-diagonal δG^{ry}

$$-8A'\dot{U} + 8B'\dot{U} + 8A'\dot{V} - \tfrac{1}{2}\chi'\dot{\psi} = 0\,. \tag{7.11}$$

Finally, the dilaton equation of motion is

$$\Big(\chi'' + 8A'\chi'\Big) + \Big(\ddot{\psi} + 8\dot{U}\dot{\psi}\Big) = \frac{5}{2}\lambda\, e^{B+V}\, e^{\frac{5}{4}\Phi}\,\delta(r)\,. \tag{7.12}$$

Summing the two equations in (7.10) gives the simpler equation

$$8\Big(A'' + 8(A')^2\Big) + 8\Big(\ddot{U} + 8(\dot{U})^2\Big) = -\lambda\, e^{B+V}\, e^{\frac{5}{4}\Phi}\,\delta(r)\,. \tag{7.13}$$

We first solve these equations in the bulk and then implement the δ-source via a jump of the first derivatives A', B', χ' at $r = 0$, precisely as in [317], and we present the solutions right away.

Solution 0

This is a physically uninteresting solution, which we merely include for completeness.

$$\begin{aligned} A(r) &= \frac{1}{8}\log\Big|\sin\big[8K(|r| - \tfrac{R}{2})\big]\Big|, \quad B(r) = -\frac{7}{16}\log\Big|\sin\big[8K(|r| - \tfrac{R}{2})\big]\Big|,\\ \chi(r) &= \phi_0, \quad U(y) = \pm K y, \quad V(y) = \pm\frac{9}{2}K y, \quad \psi(y) = 0. \end{aligned} \tag{7.14}$$

The jump conditions fix $\cot(4KR) = 0$, with minimal solution $K = \pi/(8R)$. Consequently, the jumps in $A'(r)$ and $B'(r)$ at $r = 0$ are vanishing separately. This implies $\lambda e^{\frac{5}{4}\phi_0} \sim \cot(4KR) = 0$ and thus the string coupling vanishes. Therefore, this solution does not describe the backreaction of a positive tension object, and from now on we completely disregard it.

Solution I

From (7.13) we see that both expressions in the brackets need to be constant (away from the sources). Hence $A(r)$ and $U(y)$ must be functions of trigonometric and hyperbolic type respectively (or vice versa). All the equations of motion and boundary conditions are satisfied by the following functions

$$\begin{aligned} A(r) = B(r) &= \frac{1}{8}\log\Big|\sin\left[8K(|r|-\tfrac{R}{2})\right]\Big|, \\ \chi(r) &= -\frac{3}{2}\log\Big|\tan\left[4K(|r|-\tfrac{R}{2})\right]\Big| + \phi_0, \\ U(y) &= -K\,y, \qquad \psi(y) = V(y) = 0\,, \end{aligned} \tag{7.15}$$

where the integration constant ϕ_0 is related to the string coupling constant. Moreover, the appearing parameters satisfy

$$\cos(4KR) = \frac{3}{5}\,, \qquad e^{\frac{5}{4}\phi_0} = 3\left(\frac{5}{2}\right)^{\frac{1}{8}}\frac{K}{\lambda} \sim \frac{1}{\lambda R}\,. \tag{7.16}$$

The second relation nicely reflects that the compactness of the r direction is a consequence of the backreaction of the neutral domain wall, since for $\phi_0 \to -\infty$, i.e., $g_s \to 0$, the compact space decompactifies.

The direction transversal to the neutral domain wall is spontaneously compactified on a S^1 of a size proportional to R. One could alternatively think that the r-direction may be an interval of finite proper size, but since there is no distinguished singular point in the y-direction, this would lead to a pair of ETW 8-branes at $r = -R/2$ and $r = R/2$. However, an 8-brane is not consistent with the singularities (7.17) of the solution at these points. Moreover, since the solution (7.15) features only trigonometric functions, we safely conclude the r-direction is circular.

The proper length of the y-direction is infinite, so we cannot find walls of nothing in this solution - equivalently, walls of nothing are infinitely far away and thus cannot be captured by our effective description. A coordinate of finite proper length is necessary for an interpretation in terms of local dynamical cobordism. At $r = \pm R/2$ there appear logarithmic singularities in the warp factors and the dilaton

$$A(r) = B(r) \sim \frac{1}{8}\log\rho\,, \qquad \chi(r) \sim -\frac{3}{2}\log\rho, \tag{7.17}$$

with $\rho = |r| - R/2$. This leads to curvature singularities, where the string coupling diverges. The appearance of such a singularity at $\rho = 0$ was already observed in [317], but back then its physical meaning stood as a puzzle. Now, these singularities are to be expected in the dynamical cobordism framework - we will come back to this issue after discussing Solution II.

Solution II

Let us now review the more involved Solution II. The equations in the bulk still admit three free parameters, α, K, R, which are restricted by implementing the 8-brane boundary conditions at $r = 0$. The r-dependent solutions satisfying the proper jump conditions at $r = 0$ are

$$\begin{aligned} A(r) &= \frac{1}{8} \log \Big| \sin \big[8K(|r| - \tfrac{R}{2})\big] \Big|, \\ \chi(r) &= \frac{\alpha^{\pm}}{8} \log \Big| \sin \big[8K(|r| - \tfrac{R}{2})\big] \Big| \mp 2 \log \Big| \tan \big[4K(|r| - \tfrac{R}{2})\big] \Big| + \phi_0, \\ B(r) &= \frac{\mu}{8} \log \Big| \sin \big[8K(|r| - \tfrac{R}{2})\big] \Big| \mp \frac{\alpha^{\pm}}{8} \log \Big| \tan \big[4K(|r| - \tfrac{R}{2})\big] \Big|, \end{aligned} \tag{7.18}$$

with $r \in [-\frac{R}{2}, \frac{R}{2}]$. The parameters appearing above are defined as

$$\mu = \frac{\alpha^2}{16} + \frac{5\alpha}{4} + 1\,, \qquad \text{for} \quad \alpha^{\pm} = -4(5 \mp 4\sqrt{2})\,, \tag{7.19}$$

where the latter are the positive and the negative root of

$$\alpha^2 + 40\alpha - 112 = 0\,. \tag{7.20}$$

The consistency of the boundary conditions requires

$$\cos(4KR) = \pm \frac{16}{\alpha^{\pm} + 20} = \frac{1}{\sqrt{2}}, \tag{7.21}$$

with the minimal solution $K = \pi/(16R)$. The integration constant ϕ_0 is related to the string coupling constant and has to satisfy

$$e^{\frac{5}{4}\phi_0} = \frac{\pi}{\lambda R} e^{-B - \frac{5}{4}(\chi - \phi_0)}\Big|_{r=0} = \frac{\pi}{\lambda R} \sqrt{2} \left(\sqrt{2} - 1\right)^{2\sqrt{2}}\,. \tag{7.22}$$

Again, the circle parametrized by R decompactifies as the string coupling goes to zero. The solutions for the y-dependent functions are a bit simpler and read

$$U(y) = \frac{1}{8} \log \Big(\cosh \left[8K\, y\right] \Big)\,, \quad \psi(y) = \alpha U(y)\,, \quad V(y) = -\frac{5\alpha}{4} U(y)\,, \tag{7.23}$$

i.e., all three functions are proportional to each other. An integration constant was used to render the solution symmetric around $y = 0$. We will denote the present two solutions as $\text{II}^{\pm}$, with the superscripts referring to the roots $\alpha^{\pm}$ respectively. As we will see shortly, only Solution II^{+} leads to a finite size for the y-direction. For this solution, we show the three r-dependent functions in figure 7.1, and notice they all have a kink at the location $r = 0$ of the neutral domain wall but exhibit a singularity at $r = \pm R/2$.

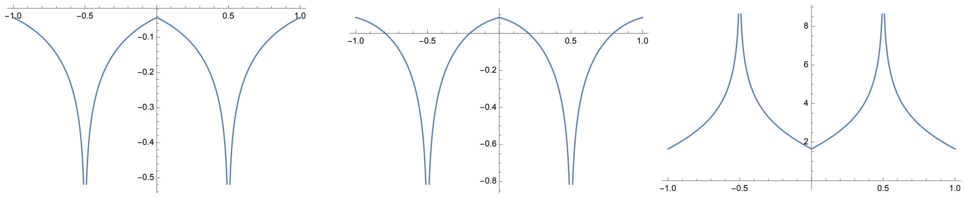

Figure 7.1: For Solution II^+, the three functions $A(r)$, $B(r)$ and $\chi(r)$ are shown from left to right for $R = 1$ in a double period $r \in [-R, R]$.

Introducing the coordinate close to the singularity, $\rho = r - R/2$, and expanding $\sin[8K(|r| - \frac{R}{2})] \simeq 8K\rho$, the behavior of these three functions close to $\rho = 0$ is

$$A(\rho) \simeq \frac{1}{8} \log \rho \,, \quad \chi(\rho) \simeq \frac{1}{8}(\alpha^+ - 16) \log \rho \,, \quad B(\rho) \simeq \frac{1}{8}(\mu - \alpha^+) \log \rho \,. \tag{7.24}$$

Since $(\alpha^+ - 16) < 0$ while $(\mu - \alpha^+) > 0$, for $\rho \to 0$ the warp factors A and B go to zero while the string coupling $g_s = \exp(\Phi)$ goes to infinity, i.e., it is a strong coupling singularity.

7.3.4 The geometry of the solution

In this section, we try to provide an intuitive description of the stringy geometry of the Solutions $\mathrm{II}^\pm$. We mainly work in the string frame, while we will mention some results in the Einstein frame as well.

In the string frame, the proper length of the y-direction (at fixed r) is

$$L_y = \int_{-\infty}^{\infty} dy \, e^{V(y) + \frac{1}{4}\psi(y)} = \int_{-\infty}^{\infty} dy \left(\cosh\left[\tfrac{\pi}{2R} y\right] \right)^{-\frac{\alpha}{8}} . \tag{7.25}$$

This length can only be finite for positive α, so Solution II^- is, similarly to Solution I, incompatible with the Dynamical Cobordism description. For Solution II^+, one finds instead

$$L_y = \frac{2R}{\sqrt{\pi}} \frac{\Gamma\left(\frac{\alpha^+}{16}\right)}{\Gamma\left(\frac{\alpha^+}{16} + \frac{1}{2}\right)} \approx 4.7 \, R \,. \tag{7.26}$$

In the Einstein frame, the situation is qualitatively similar: Solutions I and II^- still exhibit an infinite proper length L_y, while for Solution II^+ it is finite, just rescaled to $\approx 3.9R$.

Since (7.26) is finite, there are two end-of-the-world walls at a finite distance from one another, describable within the same effective description. In view of the cobordism conjecture, which hints at a $(10 - 1 - 1) = 8$-dimensional defect since the compact manifold is 1-dimensional, and by taking the logarithmic singularity at $|r| = R/2$ into account, we expect two corresponding ETW 7-branes located in the (r, y)-plane at

$$\text{ETW}_1 : \ (r, y) = (R/2, -\infty) , \qquad \text{ETW}_2 : \ (r, y) = (R/2, +\infty) . \tag{7.27}$$

In the following, we focus on Solution II$^+$. The string-frame proper length in the r-direction (at fixed y) is

$$L_r = \int_{-R/2}^{R/2} dr\, e^{B(r)+\frac{1}{4}\chi(r)} = 2\int_{-R/2}^{0} dr\, \frac{\left(\sin\left[\frac{\pi}{2R}\left(r+\frac{R}{2}\right)\right]\right)^{\frac{\mu}{8}+\frac{\alpha^+}{32}}}{\left(\tan\left[\frac{\pi}{4R}\left(r+\frac{R}{2}\right)\right]\right)^{\frac{\alpha^+}{8}+\frac{1}{2}}} \approx 2.1\, R \tag{7.28}$$

and, despite a singularity of the integrand at $r = \pm R/2$, is finite. In the Einstein frame, it remains finite ($\approx 0.9R$). The area of the compact (r, y)-space is then finite in the string frame:

$$\text{Area} = \int drdy\, \sqrt{G_{rr}G_{yy}} = L_r\, L_y \approx 9.9\, R^2. \tag{7.29}$$

Taking the y- and r-dependence of the lengths L_r and L_y respectively into account, we get the following picture: In the string frame, the circle gets warped with the y coordinate as

$$R(y) = e^{V(y)+\frac{1}{4}\psi(y)}\, R = e^{-\alpha^+ U(y)}\, R \underset{y\to\pm\infty}{\longrightarrow} 0\, . \tag{7.30}$$

The behavior in the Einstein frame is qualitatively similar, $R(y) = e^{-\frac{5}{4}\alpha^+ U(y)}R$. As for L_y, close to the singularity at $r = R/2$ it is multiplied with the exponential of

$$B(r) + \frac{1}{4}\chi(r) \simeq \frac{1}{8}\left(\frac{(\alpha^+)^2}{16} + \frac{\alpha^+}{2} - 3\right)\log\rho \approx -0.16\log\rho, \tag{7.31}$$

and its proper length diverges at $\rho \to 0$. On the contrary, in the Einstein frame the warp factor becomes

$$B(r) \simeq \frac{1}{8}\left(\frac{(\alpha^+)^2}{16} + \frac{\alpha^+}{4} + 1\right)\log\rho \approx 0.26\log\rho, \tag{7.32}$$

and thus the proper length goes to zero as $\rho \to 0$.

Therefore, at the presumed location of the ETW 7-branes, in the string frame the size of the circle in the r-direction tends to zero, while the length of the interval in the y-direction goes to infinity, such that the area stays finite. This means that topologically this space is the unreduced suspension of the circle, $S(S^1) = S^2$. In figure 7.2, we provide a schematic representation of the solution in the string frame. The two gray circles on the left and on the right-hand side of the upper figure actually have zero size and should be considered points. This is sketched in the bottom part of the figure.

In the Einstein frame, the distance between the two ETW 7-branes at $r = R/2$ goes to zero, and the space in the (r, y)-plane is topologically the reduced suspension $\Sigma(S^1) = S^2$.

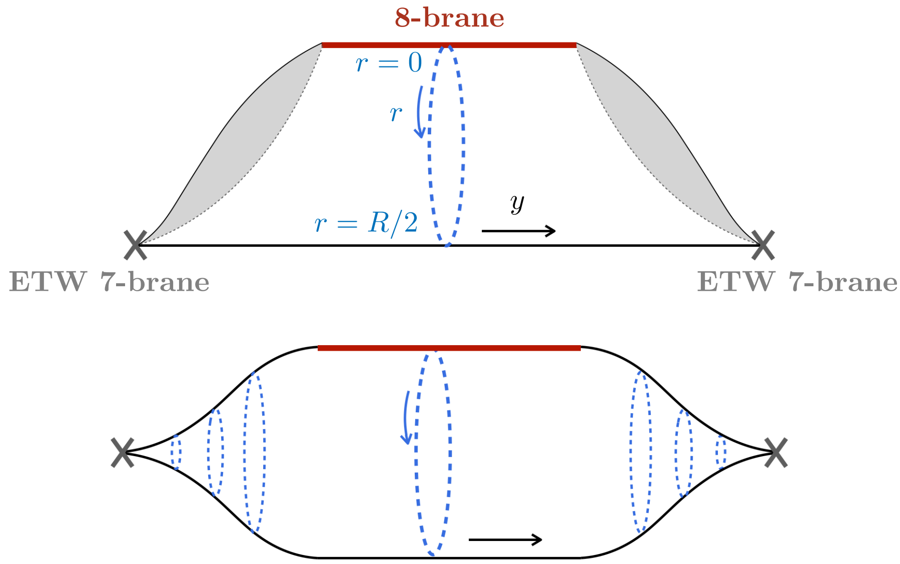

Figure 7.2: Schematic view of the string frame geometry.

Dynamical Cobordism interpretation

To make contact with the Dynamical Cobordism framework of section 7.2, we need to verify that at the singularity the scaling relations (7.1) and (7.2) are satisfied. In our setup, Δ can be expressed in terms of the tension $\mathcal{T} \sim \lambda \exp(\frac{5}{4}\phi_0)$ of the neutral domain wall

$$\Delta \sim L_y \sim \mathcal{T}^{-1}, \tag{7.33}$$

where we used (7.22), so by direct comparison to (7.1) we have $n = 1$.

Consider the ETW 7-brane located at $y = +\infty$ and $|r| = R/2$. The proper length of the circle in the r-direction goes to zero at $y = +\infty$, so all trajectories specified by a fixed r and $y \to +\infty$ do reach the ETW 7-brane location. Note that e.g. trajectories with a fixed finite value of y and $r \to R/2$ do not reach the ETW 7-brane location.

For the string-frame distance close to the location of the ETW brane, we can approximate

$$\Delta \sim \int_y^{\infty} dy' \, e^{-\frac{\pi\alpha}{16R}y'} \sim e^{-\frac{\pi\alpha}{16R}y} \,. \tag{7.34}$$

The field distance is $\psi(y) \sim \frac{\pi\alpha}{16R}y$ and we can define the canonically normalized field $D = \psi(y)/\sqrt{2}$, so we get the scaling relation [1]

$$\Delta \sim e^{-\sqrt{2}D} \,, \tag{7.35}$$

[1] Actually, we have neglected the r-dependence in Δ. Including it leads to a power law correction of the type $\Delta \sim \rho^{(1-2\sqrt{2})} e^{-\sqrt{2}D}$.

so the field-distance scaling relation in the string frame leads to $\delta_s = 2\sqrt{2}$. In the Einstein frame, we have $\Delta \sim e^{-\frac{5\pi\alpha}{64}y} \sim e^{-\frac{5}{4}\sqrt{2}D}$ and thus $\delta_E = \frac{5}{2}\sqrt{2}$.

To verify we have indeed a dynamical cobordism, the critical exponent for the curvature scaling relation in (7.2) must be the same. For the general warped ansatz of the 10D metric (7.7), the Ricci scalar is

$$\mathcal{R} = -e^{-2\mathcal{B}}\Big(16\,\Box\mathcal{A} + 72\,(\nabla\mathcal{A})^2 + 2\,\Box\mathcal{B}\Big). \tag{7.36}$$

Thus, in the string frame, we have

$$|\mathcal{R}| \sim e^{-2\left(V(y)+\frac{1}{4}\psi(y)\right)} \sim e^{2\sqrt{2}D}\,, \tag{7.37}$$

which indeed consistently gives again $\delta_s = 2\sqrt{2}$. Similarly, in the Einstein frame we have $|\mathcal{R}| \sim e^{-2V(y)} \sim e^{\frac{5}{2}\sqrt{2}D}$, i.e., we find again $\delta_E = \frac{5}{2}\sqrt{2}$.

We can conclude then that the scalings (7.2) are indeed satisfied, and the singularities of the solution correspond to spacetime-ending defects.

7.3.5 Cobordism interpretation

Let us discuss how the original cobordism conjecture relates to these spacetime-dependent solutions. Both solutions I and II featured a spontaneous compactification on the circle, a one-dimensional manifold, so the relevant cobordism group is $\Omega_1^\xi \neq 0$, with the structure ξ left open for the time being. We interpret the singularities of all solutions as indications of an inconsistency in the theory, which can be traced back to a non-vanishing global cobordism charge. However, only solution II$^+$ adheres to the dynamical cobordism framework, where the singularity is interpreted as a cobordism defect, rather than an inconsistency.

This non-vanishing cobordism group can be trivialized through breaking or gauging. Breaking predicts in this case an 8-dimensional defect, alternatively viewed as a 7-brane. For gauging, as we have discussed extensively in chapter 6, if there is an ABS orientation relating the cobordism group to an appropriate K-theory group, the cobordism charges can enter the same charge neutrality condition as K-theoretical charges. Gauging, in the absence of the ABS orientation, simply means that the cobordism charge over the compact manifold should vanish, i.e., the manifold should belong in the trivial cobordism class.

Let us go back to the ξ-structure. Perhaps the simplest assumption would be $\xi =$Spin, since we want fermions in our effective theory, and then $\Omega_1^{\text{Spin}} = \mathbb{Z}_2 \neq 0$. Whether this group will be gauged or broken depends on the physical setup we are dealing with: this is either the $\overline{D8}/O8$ stack in the T-dual of the Sugimoto model or a non-BPS type I brane.

The non-BPS $\widehat{D8}$-brane carries a $\mathbb{Z}_2$ KO-theory charge - if the structure is indeed Spin, we would expect the cobordism charge to combine with KO-theoretical contributions in a joint tadpole cancellation condition. However, as explained in example 2 in section 6.2.1, following [35], we expect the coefficient of the cobordism contribution to be even, so the two charges decouple. Then the KO-theoretical charge would have to vanish on a compact space, hence a single non-BPS brane would be inconsistent. We can try to speculate how this relates to the solutions we have found: For solutions I and II^-, the singularities do not have a dynamical interpretation, yet have singularities. We can interpret this as a sign of inconsistency due to the uncancelled KO-charge.

The stack of $\overline{D8}/O8^{++}$ on the other hand could in principle have an accompanying cobordism-breaking defect. We don't expect the cobordism/K-theory correspondence to go through since this T-dual model is not classified by KO-theory. In fact, in the Sugimoto model, the branes are classified by the symplectic K-theory KSp [309]. Since the Blumenhagen-Font solution concerns its T-dual setup, we still expect the charged objects to be classified by KSp, just with an appropriate shift. On the other hand, breaking such a symmetry predicts the existence of 7-brane defects which can be nicely identified with the ETW-branes of Local Dynamical Cobordism. This situation would correspond to Solution II^+, where the singularities are actually at finite distance and spacetime closes off. In this picture, one could try to identify the topological S^1 as the generator of Ω_1^{Spin}. As we have seen, the size of this S^1 shrinks to zero at the positions of the two ETW-branes and thus the boundary conditions for the fermions should be periodic.

7.4 The ETW 7-brane

The cobordism conjecture suggests that the singularity for Solution II^+ can be cured by introducing an appropriate pair of ETW 7-branes. The main restriction is that close to the core, this solution should show be able to close off the singularity found for Solution II^+.

Thus, we are looking for a 7-brane solution to the equations of motion (7.5) and (7.6) that preserves 8D Poincaré symmetry, that has $\log \rho$ singularities close to its core at $\rho \to 0$ and, as figure 7.2 suggests, that is non-isotropic in the two transversal directions. The tension of the brane and its dependence on the dilaton are not determined a priori, but will be determined along the way.

7.4.1 Solution breaking rotational symmetry

We make a non-isotropic ansatz for the Einstein-frame metric

$$ds^2 = e^{2\hat{\mathcal{A}}(\rho,\varphi)} ds_8^2 + e^{2\hat{\mathcal{B}}(\rho,\varphi)} (d\rho^2 + \rho^2 d\varphi^2), \tag{7.38}$$

with a separated dependence of the warp factors and the dilaton on the radial coordinate ρ and the angular coordinate and φ, i.e.

$$\begin{aligned}&\hat{\mathcal{A}}(\rho,\varphi)=\hat{A}(\rho)+\hat{U}(\varphi)\,,\quad \hat{\mathcal{B}}(\rho,\varphi)=\hat{B}(\rho)+\hat{V}(\varphi)\,,\\&\hat{\Phi}(\rho,\varphi)=\hat{\chi}(\rho)+\hat{\psi}(\varphi)\,.\end{aligned}\tag{7.39}$$

The hat sets apart the various quantities here from the similar ones in the 9-dimensional neutral domain wall solution.

The equations of motion are very similar to the ones for the 9d domain wall. For the variation $\delta g^{\mu\nu}$ we get the equation

$$\begin{aligned}&\Big(7\hat{A}''+7\frac{\hat{A}'}{\rho}+28(\hat{A}')^2+\hat{B}''+\frac{\hat{B}'}{\rho}+\tfrac{1}{4}(\hat{\chi}')^2\Big)\\&\qquad+\frac{1}{\rho^2}\Big(7\ddot{\hat{U}}+28(\dot{\hat{U}})^2+\ddot{\hat{V}}+\tfrac{1}{4}(\dot{\hat{\psi}})^2\Big)=-\hat{\lambda}\,e^{a\hat{\Phi}}\,\frac{1}{2\pi\rho}\delta(\rho)\,,\end{aligned}\tag{7.40}$$

where the tension $\hat{\lambda}$ and the parameter a are left undetermined. The prime denotes the derivative with respect to ρ and the dot is the derivative with respect to φ. For the two variations $\delta g^{\rho\rho}$ and $\delta g^{\varphi\varphi}$ we obtain

$$\begin{aligned}&\Big(8\frac{\hat{A}'}{\rho}+28(\hat{A}')^2+8\hat{A}'\hat{B}'-\tfrac{1}{4}(\hat{\chi}')^2\Big)+\frac{1}{\rho^2}\Big(8\ddot{\hat{U}}+36(\dot{\hat{U}})^2-8\dot{\hat{U}}\dot{\hat{V}}+\tfrac{1}{4}(\dot{\hat{\psi}})^2\Big)=0,\\&\Big(8\hat{A}''+36(\hat{A}')^2-8\hat{A}'\hat{B}'+\frac{1}{4}(\hat{\chi}')^2\Big)+\frac{1}{\rho^2}\Big(28(\dot{\hat{U}})^2+8\dot{\hat{U}}\dot{\hat{V}}-\frac{1}{4}(\dot{\hat{\psi}})^2\Big)=0\,,\end{aligned}\tag{7.41}$$

and for the off-diagonal $\delta g^{\rho\varphi}$

$$8\frac{\dot{\hat{U}}}{\rho}-8\hat{A}'\dot{\hat{U}}+8\hat{B}'\dot{\hat{U}}+8\hat{A}'\dot{\hat{V}}-\frac{1}{2}\hat{\chi}'\dot{\hat{\psi}}=0\,.\tag{7.42}$$

Finally, the dilaton equation of motion becomes

$$\Big(\hat{\chi}''+\frac{\hat{\chi}'}{\rho}+8\hat{A}'\hat{\chi}'\Big)+\frac{1}{\rho^2}\Big(\ddot{\hat{\psi}}+8\dot{\hat{U}}\dot{\hat{\psi}}\Big)=2a\lambda\,e^{a\hat{\Phi}}\,\frac{1}{2\pi\rho}\delta(\rho)\,.\tag{7.43}$$

Exactly as before, summing the two equations in (7.41) gives a simpler one: equation

$$8\Big(\hat{A}''+\frac{\hat{A}'}{\rho}+8(\hat{A}')^2\Big)+\frac{8}{\rho^2}\Big(\ddot{\hat{U}}+8(\dot{\hat{U}})^2\Big)=0\,.\tag{7.44}$$

The main difference in comparison to the previous section, is the appearance of some extra $1/\rho$-terms. For instance, they contain the 2D Laplacian in polar coordinates

$$\Box\,F(\rho,\varphi)=\frac{1}{\rho}\partial_\rho(\rho\,\partial_\rho F)+\frac{1}{\rho^2}\partial_\varphi^2 F\,.\tag{7.45}$$

A pair of solutions: *ETW* $7^{\pm}$-branes

Remarkably, the new equations of motion admit a three-parameter bulk solution which is similar to the one from the previous section. Now it is the ρ-dependent functions that are of hyperbolic type

$$\begin{aligned}\hat{A}(\rho) &= \frac{1}{8}\log\left(\cosh\left[8\hat{K}\log\left(\frac{\rho}{\rho_0}\right)\right]\right),\\ \hat{\chi}(\rho) &= \hat{\alpha}\hat{A}(\rho),\\ \hat{B}(\rho) &= -\log\left(\frac{\rho}{\rho_0}\right)+\left(\tfrac{\hat{\alpha}^2}{32}-\tfrac{7}{2}\right)\hat{A}(\rho),\end{aligned} \tag{7.46}$$

where at this stage $\hat{\alpha}$, $\hat{K}$ and the dimensionful parameter ρ_0 are still undetermined. The angle-dependent solutions are instead

$$\begin{aligned}\hat{U}(\varphi) &= \frac{1}{8}\log\left|\cos(8\hat{K}\varphi)\right|,\\ \hat{\psi}(\varphi) &= \frac{\hat{\alpha}}{8}\log\left|\cos(8\hat{K}\varphi)\right| \pm 2\log\left|\tan\left(4\hat{K}\varphi+\frac{\pi}{4}\right)\right|,\\ \hat{V}(\varphi) &= -\left(\tfrac{\hat{\alpha}^2}{32}-\tfrac{9}{2}\right)\hat{U}(\varphi)+\frac{\hat{\alpha}}{16}\hat{\psi}(\varphi),\end{aligned} \tag{7.47}$$

where due to the periodicity of the cosine-function we have $\varphi \in [0, \frac{\pi}{4\hat{K}}]$. The sign in the second line is in fact a free parameter, so in reality, we have a pair of solutions. Flipping this sign is the same as shifting the argument of the tan-function by $-\pi/2$, which is the same as flipping the sign of $\hat{K}$. We denote this pair of solutions as *ETW* $7^{\pm}$*-branes* and from now on we invoke the freedom to choose $\hat{K} > 0$.

Finally, each of the quantities in the solutions (7.46) and (7.47) can be shifted by arbitrary integration constants, which have been omitted here for simplicity but will become relevant soon: getting our desired delta source configuration can be guaranteed by fixing one of these constants.

7.4.2 Brane sources

Given the bulk solution, we would like now to study its behavior at the boundary. To this end, we include the δ-function source on the right-hand side of the equations of motion.

First, we notice that $\hat{f} = \log\rho$ is the 2D Green's function satisfying

$$\Box_\rho \hat{f} \equiv \hat{f}'' + \frac{\hat{f}'}{\rho} = \frac{1}{\rho}\delta(\rho) = 2\pi\delta^2(\vec{y})\,. \tag{7.48}$$

To understand where potential δ-functions can appear, we do not initially specify $\hat{A}(\rho)$, but instead make the ansatz

$$\hat{\chi}(\rho) = \hat{\alpha}\hat{A}(\rho)\,, \qquad \hat{B}(\rho) = \underbrace{-\log(\rho/\rho_0)}_{\tilde{B}(\rho)} + \left(\tfrac{\hat{\alpha}^2}{32}-\tfrac{7}{2}\right)\hat{A}(\rho), \tag{7.49}$$

together with the solution (7.47) for the angle-dependent functions. Inserting this ansatz into the equation of motion shows that only three of them potentially develop δ-terms, according to

$$
\begin{aligned}
\delta G^{\mu\nu}: &\quad \Box_\rho \tilde{B} + \left(\tfrac{\hat{\alpha}^2+112}{32}\right)\left(\Box_\rho \hat{A} + 8(\hat{A}')^2 - \frac{8\hat{K}^2}{\rho^2}\right) = {}''\delta(\rho)''\,, \\
\delta G^{\varphi\varphi}: &\quad 8\left(\Box_\rho \hat{A} + 8(\hat{A}')^2 - \frac{8\hat{K}^2}{\rho^2}\right) = {}''\delta(\rho)''\,, \\
\delta\Phi: &\quad \hat{\alpha}\left(\Box_\rho \hat{A} + 8(\hat{A}')^2 - \frac{8\hat{K}^2}{\rho^2}\right) = {}''\delta(\rho)''\,,
\end{aligned}
\tag{7.50}
$$

with the $\delta G^{\rho\rho}$ and $\delta G^{\rho\varphi}$ gravity equations of motion leading to no source term. Here, with $''\delta(\rho)''$ we indicate potential source terms with generic coefficients.

The term $\Box_\rho \tilde{B}$ in the first equation generates the source term in (7.40) for $a = 0$ and $\hat{\lambda} = 2\pi$. he second potential contribution on the right-hand side of (7.50) is related to $\hat{A}$, and in particular the specific combination

$$
\hat{A}'' + \frac{\hat{A}'}{\rho} + 8(\hat{A}')^2 - \frac{8\hat{K}^2}{\rho^2} = 0\,, \qquad \text{for } \rho \neq 0\,. \tag{7.51}
$$

Since $\hat{A}(\rho) \simeq -\hat{K}\log\rho/\rho_0$ for $\rho \ll \rho_0$, this could lead to another two-dimensional δ-source. However, equation (7.50) indicates that this source comes from an 8-brane wrapping the φ direction. Alternatively, this term could be zero even at the core, $\rho = 0$, an then the right-hand side of the last two equations in (7.50) would vanish identically. This is actually the physical setup we would like to describe, as it would avoid the introduction of yet another 8-brane.

This is indeed a realizable option: notice that the first two terms in (7.51) recombine into $\Box_\rho \hat{A}$ and thus give a δ^2-term. Integrating them over a disc of small radius ε_0 yields

$$
\int_{D_{\varepsilon_0}} d\rho d\varphi\, \rho\, \Box_\rho \hat{A} = -2\pi\hat{K}\,. \tag{7.52}
$$

Hence the remaining two terms in (7.51) need to somehow cancel out the contribution (7.52), to avoid the appearance of additional sources. Note that all four terms scale like $\pm 1/\rho^2$ close to the core. To obtain the desired cancellation, we invoke the freedom to choose arbitrary integration constants in the ansatz. After a redefinition of the coordinates x_μ and ρ, we are left with one physical integration constant, usually identified with $\hat{\phi}_0$. We make this integration constant explicit in our ansatz for $\hat{A}(\rho)$, as

$$
\hat{A}(\rho) = \frac{1}{8}\log\left(\cosh\left[8\hat{K}\log\left(\frac{\rho}{\rho_0}\right)\right]\right) + \frac{\hat{\phi}_0}{\hat{\alpha}}\,. \tag{7.53}
$$

Then, the last two terms in (7.51) can be written as

$$\begin{aligned} 8(\hat{A}')^2 - \frac{8\hat{K}^2}{\rho^2} &= 8\left(\hat{A}' - \frac{\hat{K}}{\rho}\right)\left(\hat{A}' + \frac{\hat{K}}{\rho}\right) \\ &\simeq -\frac{16\hat{K}}{\rho}\,\partial_\rho\left(\hat{A}(\rho) + \hat{K}\log\left(\frac{\rho}{\rho_0}\right)\right), \end{aligned} \tag{7.54}$$

so that integrating over a disc of small radius ε_0 and invoking Stoke's theorem one gets

$$\int_{D_{\varepsilon_0}} d\rho d\varphi\,\rho\,\left(8(\hat{A}')^2 - \frac{1}{8\rho^2}\right) = -\frac{32\pi\hat{K}}{\hat{\alpha}}\hat{\phi}_0\,. \tag{7.55}$$

that may cancel the contribution (7.52). We conclude that the behavior of the peculiar expression (7.51) at the core depends on the value of an integration constant. Without specifying this parameter, the expression is ambiguous. For the specific value $\hat{\phi}_0 = -\hat{\alpha}/16$, the term (7.55) cancels against (7.52) and (7.51) vanishes everywhere. In this case, the only physical δ-source term on the left-hand side of (7.50) arises from the contribution $\tilde{B}$, which can be reproduced by a 7-brane localized at $\rho = 0$ with Einstein-frame action

$$S_7 = -T_7 \int d^{10}x\,\sqrt{-g}\,\frac{\delta(\rho)}{2\pi\rho}\,. \tag{7.56}$$

Since there is no contribution in the dilaton equation, we need to set $a = 0$ and choose $\hat{\lambda} = \kappa_{10}^2 T_7 = 2\pi$. Note that for a 7-brane $\hat{\lambda}$ is dimensionless.

7.4.3 Comparison to Solution II$^+$

Let us now verify whether this object behaves in the desired way close to the core $\rho = 0$, i.e., in the same way as the singular geometry (7.24) of Solution II$^+$. The parameter capturing the local behavior close to the ETW-wall in the string frame is the critical exponent $\delta_s = 2\sqrt{2}$ in the scaling relation (7.2). If we have identified the end-of-the-world defect, it should scale in the same way.

The ρ-dependent functions (7.46) close to $\rho = 0$ read

$$\begin{aligned} \hat{A}(\rho) &\simeq -\hat{K}\log\rho\,, \quad \hat{\chi}(\rho) \simeq -\hat{K}\hat{\alpha}\log\rho\,, \\ \hat{B}(\rho) &\simeq -\hat{K}\left(\tfrac{\hat{\alpha}^2}{32} - \tfrac{7}{2} + \tfrac{1}{\hat{K}}\right)\log\rho\,. \end{aligned} \tag{7.57}$$

The string-frame distance to the core is

$$\Delta \sim \int_0^\rho d\rho'\,e^{\hat{B}(\rho') + \frac{1}{4}\hat{\chi}(\rho')} \sim \rho^{-\hat{K}\left(\frac{\hat{\alpha}^2}{32} + \frac{\hat{\alpha}}{4} - \frac{7}{2}\right)}, \tag{7.58}$$

and for $\hat{K} > 0$ it is finite only if $\frac{\hat{\alpha}^2}{32} + \frac{\hat{\alpha}}{4} - \frac{7}{2} < 0$. This distance can be expressed in terms of the canonically normalized field $D = \hat{\chi}/\sqrt{2}$ as

$$\Delta \sim \exp\left(\tfrac{\sqrt{2}}{\hat{\alpha}}\left(\tfrac{\hat{\alpha}^2}{32} + \tfrac{\hat{\alpha}}{4} - \tfrac{7}{2}\right) D\right), \tag{7.59}$$

By comparison to the scaling relation (7.2), we get $\delta_s = 2\sqrt{2}$ (with Δ finite) precisely for

$$\hat{\alpha} = \alpha^+ = 4(4\sqrt{2} - 5)\,. \tag{7.60}$$

As for the scalar curvature, we find close to $\rho = 0$, and in the string frame

$$|\mathcal{R}| \sim \frac{1}{\rho^2} e^{-2\left(\hat{B}(\rho) + \frac{1}{4}\hat{\chi}(\rho)\right)} \sim \rho^{2\hat{K}\left(\frac{\hat{\alpha}^2}{32} + \frac{\hat{\alpha}}{4} - \frac{7}{2}\right)} \sim e^{2\sqrt{2}D}, \tag{7.61}$$

in accordance with the scaling behavior (7.2), again for $\delta_s = 2\sqrt{2}$. In the Einstein-frame, we similarly found $\delta_E = \frac{5}{2}\sqrt{2}$, consistently leading to $\hat{\alpha} = \alpha^+$, as well. Thus, close to the core, the coordinate ρ in the 7-brane solution corresponds to the coordinate y (or better Δ) in the original domain wall solution, for $y \to \pm\infty$. Similarly, the φ-coordinate for the ETW 7-brane solution (7.47) is related to the periodic r-coordinate of the domain wall (7.18) via $8\hat{K}\varphi = 8K(|r| - 3R/2) = \frac{\pi}{2R}|r| - \frac{3\pi}{4}$. Therefore, depending on $\hat{K}$, the former domain wall coordinate r parametrizes a segment of the φ circular coordinate. Choosing the value $\hat{K} = 1/8$ gives 2π-periodicity to φ.

The main result from this scaling analysis is that indeed the non-isotropic ETW 7-brane solution has the right properties close to the core to close off the singularity found in the neutral domain wall solution.

Let us review the parameters entering these two solutions and their physical meaning. The bulk solution for neutral domain wall the admitted five parameters,

$$\alpha,\, K,\, R,\, e^{\phi_0},\, \lambda\,. \tag{7.62}$$

The jump conditions at the core and requiring a finite size for the spontaneously compactified longitudinal direction, such that the dynamical cobordism framework was applicable, fixed $\alpha = 4(4\sqrt{2} - 5)$ and led to the two conditions

$$K = \frac{\pi}{16R}\,, \qquad e^{\frac{5}{4}\phi_0} \sim \frac{1}{\lambda R}\,. \tag{7.63}$$

We were left with two unconstrained parameters like e.g. the radius R and the overall scale of the string coupling constant $g_s = e^{\phi_0}$. In a concrete string theory setting, the tension would also be fixed. Note that for large radius R the string coupling becomes small so that one expects to have good control over the employed low-energy effective action, which is just dilaton-gravity in this case.

For the 7-brane, we started with five parameters in the bulk

$$\hat{\alpha},\, \hat{K},\, \rho_0,\, e^{\hat{\phi}_0},\, \hat{\lambda}\,, \tag{7.64}$$

but ensuring the appropriate δ-function was contributing fixed the tension $\hat{\lambda} = 2\pi$ and gave rise to one relation

$$\phi_0 \sim \hat{\alpha}, \tag{7.65}$$

leaving us with three free parameters. Requiring that the 7-brane closes off the singularities present in the neutral domain wall solution (i.e. that it is really the ETW-brane) fixed the parameter $\hat{\alpha} = \alpha^+$. While we are left again with two free parameters, $\hat{K}$ and the radial scale ρ_0, the string coupling constant is fixed at $g_s = \exp(-\hat{\alpha}/16) \approx 0.85$, which is at the boundary of controllability over the utilized low-energy effective action. However, the dilaton-gravity solution per se is valid for arbitrary values of $\hat{\alpha}$ and consequently g_s. Therefore, we can retain perturbative control and are thus confident that the solution captures some physical features of the ETW 7-brane.

7.4.4 The geometry of the ETW 7-brane

The geometry around the ETW $7^{\pm}$-branes is particularly interesting, matching our expectations for a cobordism defect that caps off spacetime. We set from now on $\hat{K} = 1/8$, so that φ is 2π-periodic and can be expressed intuitively in conventional polar plots. First, we display the ρ-dependent functions (7.49) with (7.53) in figure 7.3 below.

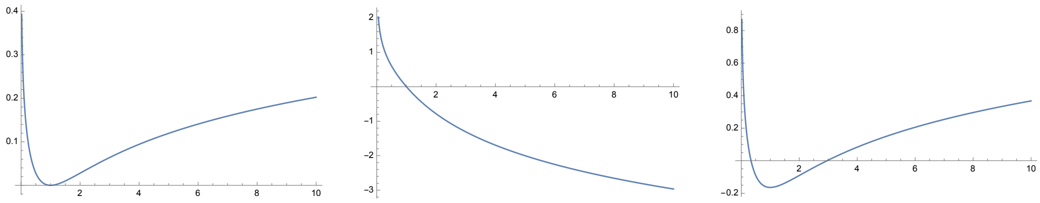

Figure 7.3: The three functions $\hat{A}(\rho)$, $\hat{B}(\rho)$ and $\hat{\chi}(\rho)$, displayed from left to right. We have chosen $\rho_0 = 1$ and $\phi_0 = -\hat{\alpha}/16$.

The figure reflects the discussed-log ρ behavior close to the core and no further singularities appear. The finite proper length of the radial direction in the string frame can be computed as

$$L_\rho = \int_0^\infty d\rho\, e^{\hat{B}(\rho)+\frac{1}{4}\hat{\chi}(\rho)} = \int_0^\infty d\rho\, \frac{\rho_0}{\rho} \left(\cosh\left[\log\left(\frac{\rho}{\rho_0}\right)\right]\right)^{\frac{1}{8}\left(\frac{\hat{\alpha}^2}{32}+\frac{\hat{\alpha}}{4}-\frac{7}{2}\right)} \approx 7.38\,\rho_0\,. \quad (7.66)$$

The angular dependence for the ETW 7^--brane is displayed in figure 7.4. The solution is clearly non-isotropic, with a singular behavior at the two angles $\varphi_1 = \pi/2$ and $\varphi_2 = 3\pi/2$. At φ_1 the string coupling goes to zero, whereas at φ_2 it diverges.

The proper length of the angular direction in the string frame is

$$\begin{aligned} L_\varphi &= \int_0^{2\pi} d\varphi\, e^{V(\varphi)+\frac{1}{4}\psi(\varphi)} \\ &= \int_0^{2\pi} d\varphi \left|\cos\varphi\right|^{\frac{1}{8}\left(\frac{\hat{\alpha}^2}{32}+\frac{\hat{\alpha}}{4}+\frac{9}{2}\right)} \left|\tan\left(\frac{\varphi}{2}+\frac{\pi}{4}\right)\right|^{\pm\frac{1}{8}(\hat{\alpha}+4)} \approx 1.07\,(2\pi)\,. \end{aligned} \quad (7.67)$$

In fact, while both L_ρ and L_φ are finite, the area $A = \tilde{L}_\rho L_\varphi$ diverges, since it involves

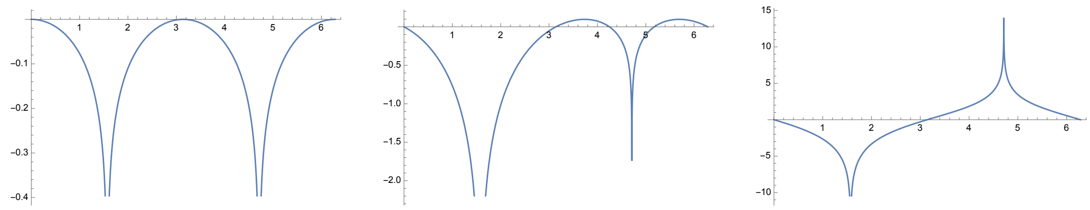

Figure 7.4: The three functions $\hat{U}(\varphi)$, $\hat{V}(\phi)$, $\hat{\psi}(\varphi)$ for the ETW 7^- brane. For the 7^+ brane the plots are just shifted by π.

the divergent integral

$$\tilde{L}_\rho = \int_0^\infty d\rho\, \rho\, e^{\hat{B}(\rho)+\frac{1}{4}\hat{\chi}(\rho)} \to \infty\,. \tag{7.68}$$

Hence, the ETW $7^\pm$-brane solution is non-compact. We find the contour plot of the warp factor $\exp(V(\varphi) + \frac{1}{4}\psi(\varphi))$ in figure 7.5 particularly instructive.

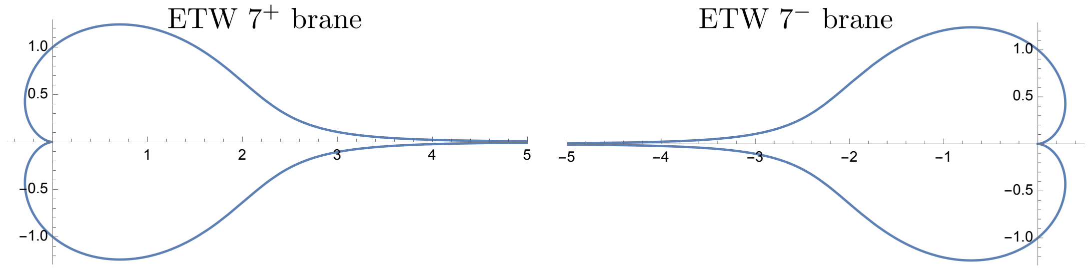

Figure 7.5: Contour plot of the φ-dependent warp factor $\exp(V(\varphi) + \frac{1}{4}\psi(\varphi))$ in string frame. For a more intuitive depiction, we have chosen $\varphi = 0$ along the negative vertical axis.

This figure makes the structure of the ETW-branes clear: at the weak coupling direction, $\varphi = \mp\pi/2$ for the $7^\pm$ brane, the length scale shrinks to zero and geometry disappears into nothing. At the opposite, strong coupling direction, $\varphi = \pm\pi/2$ for the $7^\pm$ brane, the length scale goes to infinity. This reflects the expected divergent behavior (7.31). Note that for each $\varphi \neq \mp\pi/2$ (for the $7^\pm$ brane), the string coupling diverges as one radially approaches the core at $\rho = 0$.

The picture in the Einstein frame is slightly different, as can already be expected from the middle plot in figure 7.4. The radial length scale goes to zero for both $\varphi = \pi/2$ and $\varphi = -\pi/2$. This is still consistent with equation (7.32), just lacks the immediate visual confirmation that spacetime ends "away from the two branes".

7.5 Summary and recent developments

In this chapter, we discussed the Dynamical Cobordism framework as a way to probe the cobordism defects present in setups with Dynamical Tadpoles. While there are many settings in the literature where the Local Dynamical Cobordism description, with its postulated scaling, has proven to be very successful, we tried to go a step beyond identifying the cobordism singularities. In particular, after studying the backreaction of a non-supersymmetric 9-dimensional gauge-neutral domain wall and describing the induced spacetime singularities in the dynamical cobordism framework, we tried to provide an explicit description for the accompanying cobordism defect. We have identified a promising candidate: we found an explicit solution to the leading order string equations of motion, describing a defect $7^{\pm}$-brane that can close off the singularities. These ETW-branes have a non-isotropic geometry around them, thus breaking the rotational symmetry in the transversal directions. They have positive tension $\hat{\lambda} = \kappa_{10}^2 T_{7^{\pm}} = 2\pi$ and, even more interestingly, in the string frame are governed by an action

$$S = -T_{7^{\pm}} \int d^{10}x \sqrt{-g}\, e^{-2\Phi}\, \delta^2(\vec{r})\,. \tag{7.69}$$

This does not coincide with any other 8-dimensional object in the string literature, so it is a promising candidate for a new object in string theory, predicted and detected through cobordism.

Before closing off this chapter, we must mention the recent results of Blumenhagen-Kneissl-Wang [196], which effectively is an extensive generalization of our setup: This newer paper extended our analysis to a generalized Dudas-Mourad and a generalized Blumenhagen-Font model in arbitrary dimensions. An impressive analysis identified codimension-1 and codimension-2 ETW-branes as solutions to the supergravity equations of motion, some charged and some uncharged. Interestingly, apart from new defects with novel couplings, just like in our example, regular BPS branes could also be detected in this framework. This result lends much credence to this full line of research since it points towards both the dynamical cobordism framework being correct and signals that the analysis is sensible since it can describe the D-branes.

Of course, there are many open directions one could still explore. The physical interpretation of the solution I, II^- is still unclear since the non-BPS brane proposal was mostly speculative. For the ETW-branes of our work and of [196], a stability analysis would be interesting, especially given that we have to do with a non-supersymmetric theory, to begin with. Moreover, it would be interesting to find whether this behavior persists in higher codimensions, both in the case that the original domain wall is of higher codimension but also in the case that at the singularity, multiple scalar fields go to infinity.

Chapter 8

Closing words

Now that we have reached the end, it is time to return to the very start of this thesis - the title. By now, it should be clear what the significance of each term is.

First and foremost, the *Swampland Program* has been the framework on which this work was based. It provides a clear guideline on how to make contact between Gravity and Quantum Field Theory by delineating the set of theories that can be UV-completed to Quantum Gravity from those who cannot, through a set of quantitative, predictive statements, the Swampland Conjectures. While the Swampland Conjectures can be used to make predictions about low-energy theories, much of their value is inherently linked with their formulation procedure. A convincing Swampland Conjecture is well supported, with arguments in its favor stemming from string theoretical models, black hole physics, and/or holography, or even partially proven. Hence one can learn a lot trying to formulate a conjecture, testing it, or attempting to violate it.

The largest part of this thesis treated *Cobordism* through the lens of the Cobordism Conjecture. In particular, we studied both types of charge trivialization: gauging and breaking. In both cases, we used auxiliary frameworks that intertwined the Cobordism Conjecture with *Tadpoles* - yet the type and physical meaning of these tadpoles is radically different.

On the gauging side, discussed in chapter 6, the relevant tadpoles appear in tadpole cancellation conditions, which are inescapable consistency conditions inherently related to the quantum nature of gravity. A solid mathematical background establishes a correspondence between cobordism and K-theory groups, which are known to be gauged. The proposal of [35], that cobordism and K-theory global charges can be jointly gauged leads to tadpole cancellation conditions involving both K-theory and cobordism invariants. We tested this proposal under dimensional reduction and uncovered that it reproduces all expected patterns expected from the usual dimensional reduction in cohomology, while simultaneously taking care of quantum mechanical effects. The tadpole cancellation conditions arising from this bottom-up procedure are compatible with known tadpole conditions in string theory, supporting the K-theory/cobordism interplay even further.

On the breaking side, our analysis in chapter 7 relates to dynamical tadpoles, which do not signal a pathology of the theory, rather than the lack of maximally symmetric vacua. The induced spacetime-dependent solutions often feature singularities, interpreted in the Dynamical Cobordism [36, 37] framework as cobordism end-of-the-world defects. We study the backreaction of a non-supersymmetric, positive tension domain wall and verify that the induced singularities behave in accordance with the Dynamical Cobordism picture. We provide an explicit description of the cobordism-predicted defects as novel highly anisotropic solutions to the dilaton-gravity equations of motion.

Both works related to cobordism exhibit how powerful it is as a framework. We uncovered tadpole cancellation conditions valid off-shell and gathered evidence for a previously unknown object in string theory. One can only imagine that the Cobordism Conjecture will further improve our knowledge of string theory.

Another part of the thesis concerned the *Dark Dimension* and its string theoretical realization. The small positive observed value of the cosmological constant can be viewed as an indication that our universe is realized at an asymptotic limit of the moduli space [33], leading to a tower of light states with masses scaling like $m \sim \Lambda^{1/4}$, which signals the decompactification of one extra mesoscopic dimension. As became clear from our discussion in chapter 5, the identification of a tower of states exhibiting the $\Lambda^{1/4}$ scaling, such as the highly redshifted KK modes at the tip of a Klebanov-Strassler throat, can be viewed as a “proof of concept” for the feasibility of the proposal. For a concrete realization, however, it is necessary to adopt a more holistic approach, emphasizing both the consistency of the full string model and its compatibility with experimental and observational constraints.

There are many open promising directions for each of these aforementioned projects, already discussed in the respective chapters. Here, we want to close off this thesis by focusing on the bigger picture: *The Swampland program is a promising avenue for connecting the real world and string theory and improving our formal understanding of quantum gravity. Exciting times are ahead!*

Appendix A

Basic mathematical prerequisites

This appendix includes supplementary material to the main part of the thesis, mostly concerning mathematical background and results. In section A we summarize basic facts regarding Kähler and Calabi-Yau manifolds, in section B we review some invariant polynomials and characteristic classes which become relevant both in our cobordism discussion and also for Chern-Simons actions and anomaly cancellation. For the aforementioned sections, we will mostly follow [43, 324, 325].

A.1 Complex manifolds

Definition:
A **complex n-dimensional manifold** is a differentiable manifold, with an open covering $\{U_{a\in A}\}$ and coordinate functions $z_i : U_i \to \mathbb{C}^n$ such that the transition functions $z_a \circ z_b^{-1}$ between any U_a, U_b are holomorphic wherever defined.

In practice, a complex manifold is a space that locally looks like $\mathbb{C}^n$. One can easily see that any n-dimensional complex manifold is a real manifold of dimension 2n, while the opposite is not always true. In terms of local coordinates, one can write:

$$z^i = x^{2i-1} + ix^{2i} \ , \quad \bar{z}^i = x^{2i-1} - ix^{2i} \ , \ i = 1, \ldots, n. \tag{A.1}$$

Then a real one form ω can be decomposed as:

$$\omega = \sum_{\mu=1}^{2n} \omega_\mu dx^\mu \equiv \omega^{(1,0)} + \omega^{(0,1)}, \tag{A.2}$$

where the superscript (1,0) denotes a holomorphic 1-form, while (0,1) corresponds to an antiholomorphic 1-form. In general, a form of degree (p,0) is a *holomorphic* p-form and similarly for the anti-holomorphic case. Denoting by $\Omega^{(p,q)}$ the set of (p+q)-forms of type (p,q), forms of higher degree r decompose as:

$$\Omega^r = \bigoplus_{p+q=r} \Omega^{(p,q)}. \tag{A.3}$$

Moreover, there exists the differential operator d, which is nilpotent $d^2 = 0$, and decomposes as $d = \partial + \bar{\partial}$, where $\partial : \Omega^{(p,q)} \to \Omega^{(p+1,q)}$ and $\bar{\partial} : \Omega^{(p,q)} \to \Omega^{(p,q+1)}$. The nilpotency of d results in $\partial^2 = \bar{\partial}^2 = 0$.

A.2 Kähler manifolds

Let us now introduce a *Hermitian metric* on a complex manifold. This is something that all complex manifolds admit.

Definition:
A **hermitian metric** is a tensor field of the form $ds^2 = 2\sum_{i,j=1}^n g_{i\bar{j}} dz^1 \otimes d\bar{z}^j$, such that $g_{j\bar{i}} = g_{i\bar{j}}^*$, with $g_{i\bar{j}}$ positive-definite.

To any hermitian metric one can naturally associate a real (1,1)-form, as:

$$\omega = i \sum_{i,j=1}^n g_{i\bar{j}} dz^i \wedge d\bar{z}^j. \tag{A.4}$$

The volume form can be expressed as $dvol = \omega^n$. If now the additional requirement $d\omega = 0$ is met, i.e., the associated two-form to the Hermitian metric is closed, we are dealing with a *Kähler* metric. In this case ω is also called the *kähler form.* In the main text of the thesis, we will mostly denote this form with J. We are now ready to give the definition for a Kähler manifold.

Definition:
A **Kähler manifold** is a complex manifold together with a Kähler metric, i.e., a Hermitian metric with a closed associated (1,1)-form, the Kähler form.

The fact that ω is closed has a consequence of great importance: One can show that there exists a real function $\mathcal{K}$, the *Kähler potential*, so that one can always locally express the metric in terms of derivatives of the Kähler potential. More specifically, locally it holds:

$$g_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} \mathcal{K}. \tag{A.5}$$

Kähler manifolds exhibit one additional important property. The fact that their metric is of the special form specified above results in the vanishing of many of the Christoffel symbols. More specifically, the (0,1) and (1,0) tangent spaces do not mix. This way, one can see that for the holonomy group of a Kähler manifold we have $\mathcal{H} \subseteq U(n) \subset SO(n)$.

Cohomology of Kähler manifolds

Since we are dealing with complex manifolds, instead of the usual de Rham cohomology, the relevant cohomology classification will be the Dolbeault cohomology. Roughly speaking, it is defined in a similar fashion to the de Rham case, replacing the differential d with the operator $\bar{\partial} : \Omega^{(p,q)} \to \Omega^{(p,q+1)}$. The Dolbeault cohomology groups over a manifold M are defined schematically as:

$$H^{(p,q)}_{\bar{\partial}}(M) = \frac{\{\omega_{p,q} : \bar{\partial}\omega_{p,q} = 0\}}{\{\omega_{p,q} : \omega_{p,q} = \bar{\partial}\alpha_{p,q-1}\}} \tag{A.6}$$

Let us introduce a few more bits of terminology. The dimensions of the Dolbeault cohomology groups are denoted by $h^{p,q}$ and are called *Hodge numbers*. For compact Kähler manifolds they are finite. For manifolds of low dimension, they are usually arranged in the so-called Hodge diamond. For a manifold of (complex) dimension 3 the Hodge diamond has the following form:

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array}$$

One can show that for n-dimensional Kähler manifolds the following hold:

- $h^{p,q} = h^{q,p}$,
- $h^{p,q} = h^{n-p,n-q}$,
- $h^{p,p} > 0, \quad p = 1, \ldots, n.$

Finally, both the Euler number χ and the Betti numbers b_r of a manifold can be written in terms of the Hodge numbers:

$$\sum_{p+q=r} h^{p,q} = b_r, \quad \chi(M) = \sum_{p,q} (-1)^{p+q} h^{p,q} = \sum_r (-1)^r b_r. \tag{A.7}$$

A.3 Calabi-Yau manifolds

As we have already seen in the main part of the thesis, a Calabi-Yau is a Ricci-flat compact Kähler manifold. Since we are interested in compactifying down to 4d, the relevant Calabi-Yau's are the threefolds, which have Hodge diamonds of the form shown above. The properties listed above allow us to significantly reduce the number of unknown Hodge numbers, down to only two: $h^{1,1}, h^{2,1}$. Let us sketch how this happens. To start with, there exists a unique holomorphic (3,0)-form, hence $h^{3,0} = h^{0,3} = 1$. Since CY manifolds have reduced holonomy, $h^{p,0} = 0$ for $p \neq 0, n$. Hence $h^{1,0} = h^{2,0} = 0$, and by the properties above also $h^{0,1} = h^{0,2} = 0$ and $h^{1,3} = h^{3,1} = h^{2,3} = h^{3,2} = 0$. Moreover, since we only have 1 connected component, $h^{0,0} = h^{3,3} = 1$. So the only a priori undetermined Hodge numbers are $h^{1,1} = h^{2,2}$ and $h^{2,1} = h^{1,2}$. The Hodge diamond for a Calabi-Yau threefold CY_3 hence becomes:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & \mathrm{h}^{1,1} & & 0 & \\ 1 & & \mathrm{h}^{2,1} & & \mathrm{h}^{2,1} & & 1 \\ & 0 & & \mathrm{h}^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

The definition above is not unique. One can equivalently define the Calabi-Yau $n-$fold as a compact, complex manifold that admits a (unique) Ricci-Flat metric $g_{a\bar{b}}$. The equivalence of these two statements is not trivial. Calabi initially conjectured [326] that a vanishing c_1 implies the existence of a unique Ricci-flat Kähler metric and provided the proof [327] for compact Kähler manifolds.

The following equivalent properties follow directly from the definition:

– A unique covariantly constant $(n, 0)$-form Ω exists.

– A unique holomorphic $(n, 0)$-form Ω exists.

– The holonomy group of the manifold is $\mathcal{H} \subseteq SU(n)$.

Appendix B

Characteristic Classes

In this section we attempt to gather some basic facts and properties of several characteristic classes that are relevant in the main part of the thesis. Our presentation closely follows [324]. A less physics-oriented introductory reference can be found in [69], while [304] is one of the classic references in the topic.

B.1 Chern classes

Consider a complex vector bundle $E \xrightarrow{\pi} M$ with fiber $F = \mathbb{C}^k$, and with the base manifold M being $m-$dimensional. The structure group G is a subgroup of $GL(k, \mathbb{C})$, and we denote the $\mathfrak{g}$-valued gauge potential and field strength by $\mathcal{A}$ and $\mathcal{F}$ respectively.

Definition:
The **total Chern class** $c(\mathcal{F})$ is defined by:

$$c(\mathcal{F}) \equiv det\left(I + \frac{i\mathcal{F}}{2\pi}\right). \tag{B.1}$$

It is a direct sum of even-degree forms:

$$c(\mathcal{F}) = 1 + c_1(\mathcal{F}) + c_2(\mathcal{F}) + \dots, \tag{B.2}$$

with $c_j(\mathcal{F}) \in \Omega^{2j}(M)$ being the **jth Chern class**.

The Chern classes $c_j(\mathcal{F})$ with $2j > m$ vanish trivially for dimensional reasons, while the series terminates at $c_k(\mathcal{F}) = det\left(\frac{i\mathcal{F}}{2\pi}\right)$, and for $j > k$ $c_j(\mathcal{F}) = 0$. For a general $\mathcal{F}$ one can express the individual Chern classes as follows:

$$c_0(\mathcal{F}) = 1, \tag{B.3a}$$
$$c_1(\mathcal{F}) = \frac{i}{2\pi} tr\mathcal{F}, \tag{B.3b}$$
$$c_2(\mathcal{F}) = \frac{1}{2}\left(\frac{i}{2\pi}\right)^2 [tr\mathcal{F} \wedge tr\mathcal{F} - tr(\mathcal{F} \wedge \mathcal{F})], \tag{B.3c}$$
$$\ldots$$
$$c_k(\mathcal{F}) = \left(\frac{i}{2\pi}\right)^k det\mathcal{F}. \tag{B.3d}$$

For convenience, we will often just refer to the Chern class using only the vector bundle as the argument, i.e. $ch(E)$, instead of specifying the curvature $\mathcal{F}_E$. While there are many important properties of the Chern classes, let us here simply state that the total Chern class of a Whitney sum bundle $E \oplus F$, where $F = \mathbb{C}^l \xrightarrow{\pi'} M$ is another bundle, is

$$c(E \oplus F) = c(E) \wedge c(F). \tag{B.4}$$

For further properties of Chern classes, we point the interested reader to [324], which we very closely follow throughout this appendix.

Moreover, since it will be of use for the later definition of Todd classes, let us define the total Chern class of a complex line bundle L as

$$c(L) = 1 + c_1(L) \equiv 1 + x. \tag{B.5}$$

Obviously, the series terminates at c_1 since for the line bundle any higher-rank forms are vanishing trivially. The so-called *splitting principle* states that the Chern class of an n-dimensional bundle E is the same as the Chern class of a Whitney sum of line bundles, $c(E) = \prod_{i=1}^{n}(1 + x_i)$, where now x_i is the first Chern class of the line bundle L_i.

B.2 Chern characters

Consider again the complex vector bundle $F \to E \xrightarrow{\pi} M$ with fibre $F = \mathbb{C}^k$, M $m-$dimensional, structure group G a subgroup of $GL(k, \mathbb{C})$, and $\mathfrak{g}$-valued gauge potential $\mathcal{A}$ and field strength $\mathcal{F}$.

Definition:
The **total Chern character** $ch(\mathcal{F})$ is defined by:

$$ch(\mathcal{F}) \equiv tr \exp\left(\frac{i\mathcal{F}}{2\pi}\right) = \sum_{j=1} \frac{1}{j!} tr\left(\frac{i\mathcal{F}}{2\pi}\right)^j. \tag{B.6}$$

The **jth Chern character** is defined by:

$$ch_j(\mathcal{F}) \equiv \frac{1}{j!} tr\left(\frac{i\mathcal{F}}{2\pi}\right)^j. \tag{B.7}$$

The series terminates since $c_j(\mathcal{F})$ vanishes for $2j > m$. For a general $\mathcal{F}$ one can write the Chern characters in terms of Chern classes as follows:

$$ch_0(\mathcal{F}) = k, \tag{B.8a}$$
$$ch_1(\mathcal{F}) = c_1(\mathcal{F}), \tag{B.8b}$$
$$ch_2(\mathcal{F}) = \frac{1}{2}\big(c_1(\mathcal{F})^2 - 2c_2(\mathcal{F})\big)^2, \tag{B.8c}$$
$$\ldots$$

where we remind the reader that k is the fiber dimension of the bundle. Again, for simplicity, we will often just use the bundle as the argument. Suppose again that we have two vector bundles E,F over M. The Chern characters for the tensor product and the Whitney sum are given in terms of the Chern characters of the individual bundles as:

$$ch(E \otimes F) = ch(E) \wedge ch(F), \tag{B.9a}$$
$$ch(E \oplus F) = ch(E) \oplus ch(F). \tag{B.9b}$$

B.3 Todd classes

As before in this appendix, we consider the complex vector bundle $E \to M$, where per the splitting principle $c(E) = \prod_{i=1}^{m}(1 + x_i)$.

Definition:
The **Todd class** $Td(\mathcal{F})$ is defined by:

$$Td(\mathcal{F}) = \prod_j \frac{x_j}{1 - e^{-x_j}}, \tag{B.10}$$

and it can be expanded in terms of the Chern classes of $\mathcal{F}$ as

$$Td(\mathcal{F}) = 1 + \frac{1}{2}c_1(\mathcal{F}) + \frac{1}{12}\big(c_1(\mathcal{F})^2 + c_2(\mathcal{F})\big) + \dots \tag{B.11}$$

Let us provide the explicit expressions for the first four terms of (B.11), where for ease of notation we suppress the argument $\mathcal{F}$ of all Chern classes.

$$Td_0(\mathcal{F}) = 1, \tag{B.12a}$$
$$Td_1(\mathcal{F}) = \frac{1}{2}c_1, \tag{B.12b}$$
$$Td_2(\mathcal{F}) = \frac{1}{12}(c_1^2 + c_2), \tag{B.12c}$$
$$Td_3(\mathcal{F}) = \frac{1}{24}(c_1 c_2), \tag{B.12d}$$
$$Td_4(\mathcal{F}) = \frac{1}{720}(-c_1^4 + 4c_1^2 c_2 + 3c_2^2 + c_1 c_3 - c_4). \tag{B.12e}$$

Finally, for the Whitney sum of two complex vector bundles E,F over M we have

$$Td(E \oplus F) = Td(E) \wedge Td(F). \tag{B.13}$$

As expected, in the same fashion as the Chern classes, the Todd class of the Whitney sum is the product of the individual classes.

B.4 Pontrjagin classes

The setup for the definition of Pontrjagin classes is somewhat different than before since now we consider a *real* vector bundle E, of real dimension $dim_{\mathbb{R}}E = k$, over an m-dimensional manifold M. The structure group G is now (a subgroup of) $O(k)$ and the field strength $\mathcal{F}$ is skew-symmetric.

Definition:
The **total Pontrjagin class** $p(\mathcal{F})$ is defined by:

$$p(\mathcal{F}) \equiv det\left(I + \frac{\mathcal{F}}{2\pi}\right) = \left(I - \frac{\mathcal{F}}{2\pi}\right), \tag{B.14}$$

where the second equality is due to the skew-symmetry of $\mathcal{F}$. It is an even function in $\mathcal{F}$ expanded as

$$p(\mathcal{F}) = 1 + p_1(\mathcal{F}) + p_2(\mathcal{F}) + \ldots, \tag{B.15}$$

with $p_j(\mathcal{F}) \in H^{4j}(M;\mathbb{R})$ being the **jth Pontrjagin class** .

In terms of $\mathcal{F}$ the first few Pontrjagin classes are

$$p_1(\mathcal{F}) = -\frac{1}{2}\left(\frac{1}{2\pi}\right)^2 tr\mathcal{F}^2, \tag{B.16a}$$

$$p_2(\mathcal{F}) = \frac{1}{8}\left(\frac{1}{2\pi}\right)^4 \big((tr\mathcal{F}^2)^2 - 2tr\mathcal{F}^4\big), \tag{B.16b}$$

$$p_3(\mathcal{F}) = \frac{1}{48}\left(\frac{1}{2\pi}\right)^6 \big(-(tr\mathcal{F}^2)^3 + 6tr\mathcal{F}^2 tr\mathcal{F}^4 - 8tr\mathcal{F}^6\big), \tag{B.16c}$$

$$p_4(\mathcal{F}) = \frac{1}{384}\left(\frac{1}{2\pi}\right)^8 \big((tr\mathcal{F}^2)^4 - 12(tr\mathcal{F}^2)^2 tr\mathcal{F}^4 \tag{B.16d}$$

$$+ 32tr\mathcal{F}^2 tr\mathcal{F}^6 + 12(tr\mathcal{F}^4)^4 - 48tr\mathcal{F}^8\big), \tag{B.16e}$$

$$\ldots$$

$$p_{[k/2]}(\mathcal{F}) = \left(\frac{1}{2\pi}\right)^k det\mathcal{F}. \tag{B.16f}$$

It is also possible to express the Pontrjagin classes in terms of Chern classes. To this end, one first should address the discrepancy between the type of fibers in the definitions of the two classes - this happens by complexifying the real fiber E, which results in the complex fiber $E^{\mathbb{C}}$. Then we have

$$p_j(E) = (-1)^j c_{2j}(E^{\mathbb{C}}). \tag{B.17}$$

Finally, let us mention that for Whitney sums it holds that

$$p(E \oplus F) = p(E) \wedge p(F). \tag{B.18}$$

B.5 Hirzebruch L-polynomial

Considering the same setup as for the Pontrjagin classes, we can define a different invariant polynomial, which becomes important in the context of the Hirzebruch signature theorem.

Definition:
The **Hirzebruch L-polynomial** $L(x)$ is defined by:

$$L(x) = \prod_{j=1}^{k} \frac{x_j}{tanh x_j} = \prod_{j=1}^{k} \left(1 + \sum_{n\geq 1} (-1)^{n-1} \frac{2^{2n}}{(2n)!} B_n x_j^{2n}\right), \tag{B.19}$$

where B_n are the Bernoulli numbers.

The L-polynomial can be expressed in terms of Pontrjagin classes, and we present the first few terms

$$L_1(\mathcal{F}) = \frac{1}{3} p_1, \tag{B.20a}$$
$$L_2(\mathcal{F}) = \frac{1}{45}(-p_1^2 + 7p_2), \tag{B.20b}$$
$$L_3(\mathcal{F}) = \frac{1}{945}(2p_1^3 - 13p_1p_2 + 62p_3), \tag{B.20c}$$
$$\dots$$

where we have suppressed the argument $\mathcal{F}$ in the Pontrjagin classes for notational clarity. Additionally, as for the Pontrjagin classes, it holds

$$L(E \oplus F) = L(E) \wedge L(F). \tag{B.21}$$

B.6 $\hat{A}$ genus

There is another polynomial of great importance for physics that directly relates to the Pontrjagin classes and is an even function of x_j.

Definition:
The $\hat{A}$ **(A-roof) genus** $\hat{A}(\mathcal{F})$ or **Dirac genus** is defined by:

$$\hat{A}(\mathcal{F}) = \prod_{j=1}^{k} \frac{x_j/2}{sinh(x_j/2)} = \prod_{j=1}^{k} \left(1 + \sum_{n\geq 1} (-1)^{n} \frac{2^{2n}-2}{(2n)!} B_n x_j^{2n}\right), \tag{B.22}$$

B_n being the Bernoulli numbers.

The first few terms of the series expansion are

$$\hat{A}_1(\mathcal{F}) = -\frac{1}{24}p_1, \tag{B.23a}$$

$$\hat{A}_2(\mathcal{F}) = \frac{1}{5760}(7p_1^2 - 4p_2), \tag{B.23b}$$

$$\hat{A}_3(\mathcal{F}) = \frac{1}{967680}(-31p_1^3 + 44p_1p_2 - 16p_3), \tag{B.23c}$$

$$\ldots .$$

Finally, for the Whitney sum we have

$$\hat{A}(E \oplus F) = \hat{A}(E) \wedge \hat{A}(F). \tag{B.24}$$

B.7 Stiefel-Whitney classes

The last classes we will introduce are the so-called Stiefel-Whitney classes. These are classes of great importance for physics, as they relate to whether spinors are admissible in certain manifolds, and they are defined using real bundles. We present here the axiomatic formulation of [69].

Consider the real vector bundle $E \to B$. There is a unique sequence of functions $w_1, w_2, \ldots$ assigning to the vector bundle a class $w_i(E) \in H^i(B;\mathbb{Z}_2)$, satisfying the properties:

- $w_i(f^*(E)) = f^*(w_i(E))$ for a pullback $f^*(E)$.
- $w(E_1 \oplus E_2) = w(E_1) \smile w(E_2)$, where $w = 1 + w_1 + w_2 + \ldots \in H^*(B;\mathbb{Z}_2)$.
- $w_i(E) = 0$ if $i > \mathrm{dim}E$.
- For the canonical line bundle $E \to \mathbb{R}P^\infty$, $w_1(E)$ is a generator of $H^1(\mathbb{R}P^\infty;\mathbb{Z}_2)$

The sum $w(E) = 1 + w_1(E) + w_2(E) + \ldots$ is the **total Stiefel-Whitney class, and w_i is the ith Stiefel Whitney class.**

Appendix C

Elements of differential topology

C.1 Short exact sequences, extensions, and Ext

Consider the abelian groups A, B, and C. A short sequence

$$0 \longrightarrow B \xrightarrow{\beta} C \xrightarrow{\alpha} A \longrightarrow 0 \tag{C.1}$$

is exact, if the map β is injective and the map α surjective, i.e. if $\ker(\alpha) = \mathrm{Im}(\beta)$. Then C is called an *extension of A by B*, denoted as $C = e(A, B)$. According to the Splitting Lemma for abelian groups, the extension is trivial, i.e., $C = A \oplus B$, iff there is a left inverse to β iff there is a right inverse to α. Then the short exact sequence is split. In general, the extension is not necessarily unique and there can be more extensions besides the trivial one. Equivalence classes of extensions of A by B are in one-to-one correspondence with elements of the group $\mathrm{Ext}^1(A, B)$, with the trivial extension corresponding to 0.

The definition and main properties of the groups $\mathrm{Ext}^n(A, B)$ can be found in e.g., [328], chapter 3. As stated in Lemma 3.3.1, if A and B are abelian $\mathrm{Ext}^n(A, B) = 0$ for $n \geq 2$. It is $\mathrm{Ext}^0(A, B) = \mathrm{Hom}(A, B)$, while $\mathrm{Ext}^1(A, B)$ classifies extensions of A by B, as anticipated above. Two useful properties of these groups are

$$\mathrm{Ext}^n(\oplus_i A_i, B) = \Pi_i \mathrm{Ext}^n(A_i, B), \tag{C.2}$$

$$\mathrm{Ext}^n(A, \Pi_i B_i) = \Pi_i \mathrm{Ext}^n(A, B_i), \tag{C.3}$$

with the direct product and direct sum coinciding for abelian groups. For cyclic groups, we recall the results

$$\mathrm{Ext}^1(\mathbb{Z}, \mathbb{Z}) = 0, \tag{C.4}$$

$$\mathrm{Ext}^1(\mathbb{Z}, \mathbb{Z}_n) = 0, \tag{C.5}$$

$$\mathrm{Ext}^1(\mathbb{Z}_n, \mathbb{Z}) = \mathbb{Z}_n, \tag{C.6}$$

$$\mathrm{Ext}^1(\mathbb{Z}_m, \mathbb{Z}_n) = \mathbb{Z}_k, \tag{C.7}$$

where $k = \mathrm{GCD}(m, n)$. All of this is used in the calculations of section 6.4

Let us give two simple examples to illustrate how everything works in a combined way. Let us consider the short exact sequence

$$0 \to \mathbb{Z}_2 \to e(\mathbb{Z}_2, \mathbb{Z}_2) \to \mathbb{Z}_2 \to 0. \tag{C.8}$$

Since $\mathrm{Ext}^1(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$, $e(\mathbb{Z}_2, \mathbb{Z}_2)$ is not split, instead we have two possible extensions. Indeed, it is well-known that there are two short exact sequences

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0, \tag{C.9}$$

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0. \tag{C.10}$$

Instead, the short exact sequence

$$0 \to \mathbb{Z}_3 \to \mathbb{Z}_6 \to \mathbb{Z}_2 \to 0, \tag{C.11}$$

is split, since $\mathrm{Ext}^1(\mathbb{Z}_2, \mathbb{Z}_3) = 0$.

C.2 Universal Coefficient Theorem

The universal coefficient theorem (see e.g. [328]) can be used to express (co)homology groups of a topological space X with coefficients in a left $\mathbb{Z}$-module A in terms of (co)homology groups with coefficients in $\mathbb{Z}$. It can be formulated both for homology and cohomology.

The version for homology groups states that there is a short (noncanonically) split exact sequence

$$0 \to H_n(X) \otimes A \to H_n(X; A) \to \mathrm{Tor}_1(H_{n-1}(X), A) \to 0. \tag{C.12}$$

The definition of the groups $\mathrm{Tor}_n(A, B)$ can be found e.g. in [328], chapter 3. As stated in Proposition 3.1.2 and 3.1.4, if A and B are abelian, $\mathrm{Tor}_n(A, B)$ are torsion abelian groups and they vanish for $n \geq 2$; if A is also torsion free, $\mathrm{Tor}_1(A, B) = 0$. The version for cohomology groups states that there is a short (noncanonically) split exact sequence

$$0 \to \mathrm{Ext}^1 H_{n-1}(X; A) \to H_n(X; A) \to \mathrm{Hom}(H_{n-1}(X), A) \to 0. \tag{C.13}$$

C.3 Properties of Steenrod squares

In this appendix, we collect some useful facts about Steenrod squares. For a nice, pedagogical review and for more information, we refer the reader to [329] and references therein. A standard textbook is [304]. We will work at prime 2, but it is possible to generalize the discussion to any prime p.

A cohomology operation of degree i is a map

$$H^n(X;\mathbb{Z}_2) \to H^{n+i}(X;\mathbb{Z}_2). \tag{C.14}$$

It is said to be stable if it commutes with the suspension isomorphism. Steenrod squares, Sq^i, are stable cohomology operations of degree i satisfying the following defining properties, for any $i \geq 0$:

a) $Sq^0 = \mathrm{Id}$;

b) $Sq^i(x) = x \cup x$, for $x \in H^i(X;\mathbb{Z}_2)$;

c) $Sq^i(x) = 0$, for $x \in H^j(X;\mathbb{Z}_2)$ and $j < i$;

d) $Sq^i(x \cup y) = \sum_{m+n=i} Sq^m(x) \cup Sq^n(y)$ (Cartan formula).

e) $Sq^i \circ Sq^j = \sum_{k=0}^{\lfloor i/2 \rfloor} \begin{pmatrix} j-k-1 \\ i-2k \end{pmatrix}_{\mathrm{mod}\ 2} Sq^{i+j-k} \circ Sq^k$, for $0 < i < 2j$. (Adem relation).

The map $Sq^1 \equiv \tilde{\beta}$ is an example of a Bockstein homomorphism. It is associated with the short exact sequence

$$0 \to \mathbb{Z}_2 \stackrel{\times 2}{\to} \mathbb{Z}_4 \stackrel{\rho}{\to} \mathbb{Z}_2 \to 0, \tag{C.15}$$

where the first map is multiplication by 2 and the second (ρ) is the reduction modulo 2, which induces the long exact sequence

$$\ldots \stackrel{\tilde{\beta}}{\to} H^n(X;\mathbb{Z}_2) \stackrel{\times 2}{\to} H^n(X;\mathbb{Z}_4) \stackrel{\rho}{\to} H^n(X;\mathbb{Z}_2) \stackrel{\tilde{\beta}}{\to} H^{n+1}(X;\mathbb{Z}_2) \to \ldots. \tag{C.16}$$

Here, $\tilde{\beta}$ is the connecting homomorphism between cohomology groups of different degree. Another Bockstein homomorphism, called β in the main text, can be constructed in association with the short exact sequence

$$0 \to \mathbb{Z} \stackrel{\times 2}{\to} \mathbb{Z} \stackrel{\rho}{\to} \mathbb{Z}_2 \to 0, \tag{C.17}$$

inducing in turn the long exact sequence

$$\ldots \stackrel{\beta}{\to} H^n(X;\mathbb{Z}) \stackrel{\times 2}{\to} H^n(X;\mathbb{Z}) \stackrel{\rho}{\to} H^n(X,\mathbb{Z}_2) \stackrel{\beta}{\to} H^{n+1}(X,\mathbb{Z}) \to \ldots. \tag{C.18}$$

The two Bocksteins are related by

$$\tilde{\beta} = \rho \circ \beta. \tag{C.19}$$

At odd degree $i = 2k+1$, one can define an integral lift of the Steenrod squares,

$$Sq_{\mathbb{Z}}^{2m+1} = \beta \circ Sq^{2m}, \tag{C.20}$$

which is such that $\rho \circ Sq^{2m+1}_{\mathbb{Z}} = Sq^{2m+1}$ and maps $H^n(X;\mathbb{Z}_2) \to H^{n+i}(X;\mathbb{Z})$. One further gets a map between integral cohomology by first reducing modulo 2 and then acting with $Sq^i_{\mathbb{Z}}$,

$$Sq^i_{\mathbb{Z}} \circ \rho : H^n(X;\mathbb{Z}) \to H^{n+i}(X;\mathbb{Z}). \tag{C.21}$$

An integral lift of Sq^i for even $i = 2m$ does not exist.[1]

Given an element $x \in H^{k-i}(X;\mathbb{Z}_2)$, with $k = \dim(X)$, the action of the Steenrod squares can be defined as

$$Sq^i(x) = \nu_i \cup x, \tag{C.22}$$

where $\nu_i \in H^i(X;\mathbb{Z}_2)$ is the i-th Wu class of X (more precisely, of a real vector bundle over X of rank k, which we generically take to be the tangent bundle), such that $\nu_i = 0$ if $i > k - i$. Since the total Wu class is the Steenrod square of the total Stiefel-Whitney class, one can express each of the single Wu classes in terms of Stiefel-Whitney classes. At lower degree, one has

$$\begin{aligned} \nu_1 &= w_1, \\ \nu_2 &= w_2 + w_1 \cup w_1, \\ \nu_3 &= w_1 \cup w_2. \end{aligned} \tag{C.23}$$

In certain cases, one can give an alternative action of Sq^i, namely (see e.g. [65, 67])

$$Sq^i(y) = \iota_*(w_i(N)) \cup y, \tag{C.24}$$

where $y \in H^n(X;\mathbb{Z}_2)$, N is the normal bundle of the submanifold $Y \subset X$ Poincaré dual to y and $\iota : Y \to X$ is the inclusion.[2] This is most convenient for physical purposes, such as checking the absence of Freed–Witten anomalies for branes wrapping Y, on which we comment in section 6.4.3 (there, following [65, 67], we directly employ the integral lift $W_3(N)$ of $w_3(N)$ and omit the pushforward ι_*).

C.4 Wedge sum, smash product and reduced suspension

Consider two pointed topological spaces (X, x_0) and (Y, y_0). The wedge sum, $X \vee Y$, is defined as

$$X \vee Y = X \sqcup Y/\sim, \tag{C.25}$$

[1] This can be proven as follows. Suppose it exists an integral lift for the even case, $Sq^{2m} = \rho \circ Sq^{2m}_{\mathbb{Z}}$. Exactness of the sequence (C.18) means that $\ker \beta = \mathrm{Im}\rho$, implying in turn $\beta \circ Sq^{2m} = \beta(\rho(Sq^{2m}_{\mathbb{Z}})) = 0$. However, this is contradiction with the Adem relation $Sq^1 \circ Sq^{2m} = Sq^{2m+1} \neq 0$ (recall $Sq^1 = \rho \circ \beta$). Thus, such an integral lift $Sq^{2m}_{\mathbb{Z}}$ cannot exist.

[2] More in general [304], one can define an action $Sq^i(u) = \pi^*(w_i(\xi)) \cup u$, with $u \in H^k(E;\mathbb{Z}_2)$ and $w_i(\xi) \in H^i(B;\mathbb{Z}_2)$, for any k-plane bundle $\xi : F \to E \xrightarrow{\pi} B$ of which the normal bundle $N(B)$ is a particular case.

where the equivalence relation identifies the two base points x_0 and y_0. The smash product, $X \wedge Y$, is defined as the quotient of the cartesian product by the wedge sum

$$X \wedge Y = \frac{X \times Y}{X \vee Y} \tag{C.26}$$

It satisfies the properties

$$X \wedge Y \cong Y \wedge X, \tag{C.27}$$
$$(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge X), \tag{C.28}$$

where the symbol $\cong$ means homeomorphic as topological spaces.

Consider then the n-sphere S^n. The reduced suspension of X is defined as

$$\Sigma X \cong S^1 \wedge X. \tag{C.29}$$

The construction can be iterated

$$\Sigma^n X \cong S^n \wedge X. \tag{C.30}$$

An important case is when $X = S^k$, thus giving

$$\Sigma^n S^k \cong S^{n+k}. \tag{C.31}$$

We also recall that

$$\Sigma^0 \wedge X \cong S^0 \wedge X \cong X, \tag{C.32}$$

where $S^0 \cong \mathrm{pt} \sqcup \mathrm{pt}$. Another useful formula is

$$\Sigma(X \times Y) \cong \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y). \tag{C.33}$$

C.5 Cobordism groups of spheres and tori

It can be proved by induction that for a generic structure ξ the cobordism groups of spheres S^k and tori T^k have a simple decompositions in terms of the respective cobordism groups of the point, namely

$$\Omega^\xi_n(S^k) = \Omega^\xi_n(\mathrm{pt}) \oplus \Omega^\xi_{n-k}(\mathrm{pt}), \tag{C.34}$$
$$\Omega^\xi_n(T^k) = \bigoplus_{i=0}^{k} \binom{k}{i} \Omega^\xi_{n-i}(\mathrm{pt}), \tag{C.35}$$

where groups with negative index are assumed to be vanishing.
We start from the cobordism groups of spheres, S^k. For S^1, we have

$$\begin{aligned}\Omega^\xi_n(S^1) &= \Omega^\xi_n(\mathrm{pt}) \oplus \tilde{\Omega}^\xi_n(S^1) = \Omega^\xi_n(\mathrm{pt}) \oplus \tilde{\Omega}^\xi_n(\Sigma(S^0)) \\ &= \Omega^\xi_n(\mathrm{pt}) \oplus \tilde{\Omega}^\xi_{n-1}(S^0) = \Omega^\xi_n(\mathrm{pt}) \oplus \Omega^\xi_{n-1}(\mathrm{pt}).\end{aligned} \tag{C.36}$$

The suspension axiom $\tilde{\Omega}^{\xi}_n(\Sigma X) = \tilde{\Omega}^{\xi}_{n-1}(X)$ [171] justifies the step from the first to the second line, while in the last step we used that $\tilde{\Omega}^{\xi}_n(S^0) = \Omega^{\xi}_n(\mathrm{pt})$, which follows from

$$\Omega^{\xi}_n(S^0) = \Omega^{\xi}_n(\mathrm{pt} \sqcup \mathrm{pt}) = \Omega^{\xi}_n(\mathrm{pt}) \oplus \Omega^{*}_n(pt) = \Omega^{\xi}_n(\mathrm{pt}) \oplus \tilde{\Omega}^{\xi}_n(S^0). \tag{C.37}$$

Assuming the formula to hold for S^k, we prove it for S^{k+1}. Using again the Splitting Lemma (4.5) and the suspension axiom, we have

$$\begin{aligned}\Omega^{\xi}_n(S^{k+1}) &= \Omega^{\xi}_n(\mathrm{pt}) \oplus \tilde{\Omega}^{\xi}_n(S^{k+1}) = \Omega^{\xi}_n(\mathrm{pt}) \oplus \tilde{\Omega}^{\xi}_n(\Sigma(S^k)) \\ &= \Omega^{\xi}_n(\mathrm{pt}) \oplus \tilde{\Omega}^{\xi}_{n-1}(S^k) = \Omega^{\xi}_n(\mathrm{pt}) \oplus \Omega^{\xi}_{n-k-1}(\mathrm{pt}).\end{aligned} \tag{C.38}$$

This proves (C.34) by induction.

Considering the cobordism groups of tori, T^k, the result for $T^1 = S^1$ is already proven in (C.36). Hence, we only assume the formula holds for T^k and we prove it for T^{k+1}. To this end, using (C.33), we can split

$$\Sigma(T^k \times S^1) = \Sigma(T^k) \vee \Sigma(S^1) \vee \Sigma(T^k \wedge S^1) \tag{C.39}$$

and therefore

$$\begin{aligned}\Omega^{\xi}_n(T^{k+1}) &= \Omega^{\xi}_n(\mathrm{pt}) \oplus \tilde{\Omega}^{\xi}_{n+1}(\Sigma(T^{k+1})) = \Omega^{\xi}_n(\mathrm{pt}) \oplus \tilde{\Omega}^{\xi}_{n+1}(\Sigma(T^k \times S^1)) \\ &= \Omega^{\xi}_n(\mathrm{pt}) \oplus \tilde{\Omega}^{\xi}_{n+1}(\Sigma(T^k)) \oplus \tilde{\Omega}^{\xi}_{n+1}(\Sigma(S^1)) \oplus \tilde{\Omega}^{\xi}_{n+1}(\Sigma^2(T^k)) \\ &= \Omega^{\xi}_n(\mathrm{pt}) \oplus \tilde{\Omega}^{\xi}_n(T^k) \oplus \tilde{\Omega}^{\xi}_n(S^1) \oplus \tilde{\Omega}^{\xi}_{n-1}(T^k) \\ &= \Omega^{\xi}_n(T^k) \oplus \Omega^{\xi}_{n-1}(T^k),\end{aligned} \tag{C.40}$$

where we used $\tilde{\Omega}(X \vee Y) = \tilde{\Omega}(X) \oplus \tilde{\Omega}(Y)$, valid for reduced generalized homology theories [171]. We can finally demonstrate that

$$\begin{aligned}\Omega^{\xi}_n(T^{k+1}) &= \Omega^{\xi}_{n-1}(T^k) \oplus \Omega^{\xi}_n(T^k) \\ &= \bigoplus_{i=0}^{k} \binom{k}{i} \Omega^{\xi}_{n-1-i}(\mathrm{pt}) \oplus \bigoplus_{i=0}^{k} \binom{k}{i} \Omega^{\xi}_{n-i}(\mathrm{pt}) \\ &= \bigoplus_{i=1}^{k+1} \binom{k}{i-1} \Omega^{\xi}_{n-i}(\mathrm{pt}) \oplus \bigoplus_{i=0}^{k} \binom{k}{i} \Omega^{\xi}_{n-i}(\mathrm{pt}) \\ &= \bigoplus_{i=0}^{k+1} \binom{k}{i-1} \Omega^{\xi}_{n-i}(\mathrm{pt}) \oplus \bigoplus_{i=0}^{k+1} \binom{k}{i} \Omega^{\xi}_{n-i}(\mathrm{pt}) \\ &= \bigoplus_{i=0}^{k+1} \binom{k+1}{i} \Omega^{\xi}_{n-i}(\mathrm{pt}).\end{aligned} \tag{C.41}$$

In the fourth line, zero was added to both terms, while for the last step Pascal's formula was applied. This concludes the proof of (C.35) by induction.

Appendix D

Tables of Cobordism and K-theory Groups

D.1 Tables of Cobordism Groups

n	0	1	2	3	4	5	6	7	8	9	10
$\Omega_n^{\text{Spin}}(S^1)$	$\mathbb{Z}$	$\mathbb{Z}_2 \oplus \mathbb{Z}$	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	0	0	$2\mathbb{Z}$	$2\mathbb{Z}_2 \oplus 2\mathbb{Z}$	$5\mathbb{Z}_2$
$\Omega_n^{\text{Spin}}(S^2)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}$	0	$\mathbb{Z}$	0	$2\mathbb{Z}$	$2\mathbb{Z}_2$	$3\mathbb{Z}_2 \oplus 2\mathbb{Z}$
$\Omega_n^{\text{Spin}}(S^3)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2 \oplus \mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$2\mathbb{Z}$	$2\mathbb{Z}_2$	$3\mathbb{Z}_2$
$\Omega_n^{\text{Spin}}(S^4)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$2\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$3\mathbb{Z}$	$2\mathbb{Z}_2$	$3\mathbb{Z}_2$
$\Omega_n^{\text{Spin}}(S^5)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$2\mathbb{Z}$	$2\mathbb{Z}_2 \oplus \mathbb{Z}$	$3\mathbb{Z}_2$
$\Omega_n^{\text{Spin}}(S^6)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus 2\mathbb{Z}$	$2\mathbb{Z}_2$	$3\mathbb{Z}_2 \oplus \mathbb{Z}$
$\Omega_n^{\text{Spin}}(S^7)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2 \oplus 2\mathbb{Z}$	$3\mathbb{Z}_2$	$3\mathbb{Z}_2$
$\Omega_n^{\text{Spin}}(S^8)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$3\mathbb{Z}$	$3\mathbb{Z}_2$	$4\mathbb{Z}_2$
$\Omega_n^{\text{Spin}}(S^9)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	$2\mathbb{Z}_2 \oplus \mathbb{Z}$	$4\mathbb{Z}_2$
$\Omega_n^{\text{Spin}}(S^{10})$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	$2\mathbb{Z}_2$	$3\mathbb{Z}_2 \oplus \mathbb{Z}$

Table D.1: Cobordism groups $\Omega_n^{\text{Spin}}(S^k)$, $k = 1, \ldots, 10$.

m	0	1	2	3	4	5	6	7	8	9	10
$\Omega_n^{\text{Spin}}(T^2)$	$\mathbb{Z}$	$2\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus 3\mathbb{Z}_2$	$3\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}$	$2\mathbb{Z}$	$\mathbb{Z}$	0	$2\mathbb{Z}$	$2\mathbb{Z}_2 \oplus 4\mathbb{Z}$	$7\mathbb{Z}_2 \oplus 2\mathbb{Z}$
$\Omega_n^{\text{Spin}^c}(T^2)$	$\mathbb{Z}$	$2\mathbb{Z}$	$2\mathbb{Z}$	$2\mathbb{Z}$	$3\mathbb{Z}$	$4\mathbb{Z}$	$4\mathbb{Z}$	$4\mathbb{Z}$	$6\mathbb{Z}$	$8\mathbb{Z}$	$8\mathbb{Z} \oplus \mathbb{Z}_2$

Table D.2: Cobordism groups of 2-torus $\Omega_n^{\text{Spin}}(T^2)$, $\Omega_n^{\text{Spin}^c}(T^2)$.

n	0	1	2	3	4	5	6	7	8	9	10
$\Omega_n^{\mathrm{Spin^c}}(S^1)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$2\mathbb{Z}$	$2\mathbb{Z}$	$2\mathbb{Z}$	$2\mathbb{Z}$	$4\mathbb{Z}$	$4\mathbb{Z}$	$4\mathbb{Z}\oplus\mathbb{Z}_2$
$\Omega_n^{\mathrm{Spin^c}}(S^2)$	$\mathbb{Z}$	0	$2\mathbb{Z}$	0	$3\mathbb{Z}$	0	$4\mathbb{Z}$	0	$6\mathbb{Z}$	0	$8\mathbb{Z}\oplus\mathbb{Z}_2$
$\Omega_n^{\mathrm{Spin^c}}(S^3)$	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}$	$2\mathbb{Z}$	$\mathbb{Z}$	0	$2\mathbb{Z}$	$4\mathbb{Z}$	$2\mathbb{Z}$	$4\mathbb{Z}\oplus\mathbb{Z}_2$
$\Omega_n^{\mathrm{Spin^c}}(S^4)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$3\mathbb{Z}$	0	$3\mathbb{Z}$	0	$6\mathbb{Z}$	0	$6\mathbb{Z}\oplus\mathbb{Z}_2$
$\Omega_n^{\mathrm{Spin^c}}(S^5)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$2\mathbb{Z}$	$\mathbb{Z}$	$2\mathbb{Z}$	$\mathbb{Z}$	$4\mathbb{Z}$	$2\mathbb{Z}$	$4\mathbb{Z}\oplus\mathbb{Z}_2$
$\Omega_n^{\mathrm{Spin^c}}(S^6)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$2\mathbb{Z}$	0	$3\mathbb{Z}$	0	$5\mathbb{Z}$	0	$6\mathbb{Z}\oplus\mathbb{Z}_2$
$\Omega_n^{\mathrm{Spin^c}}(S^7)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$2\mathbb{Z}$	0	$2\mathbb{Z}$	$\mathbb{Z}$	$4\mathbb{Z}$	$\mathbb{Z}$	$4\mathbb{Z}\oplus\mathbb{Z}_2$
$\Omega_n^{\mathrm{Spin^c}}(S^8)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$2\mathbb{Z}$	0	$2\mathbb{Z}$	0	$5\mathbb{Z}$	0	$5\mathbb{Z}\oplus\mathbb{Z}_2$
$\Omega_n^{\mathrm{Spin^c}}(S^9)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$2\mathbb{Z}$	0	$2\mathbb{Z}$	0	$4\mathbb{Z}$	$\mathbb{Z}$	$4\mathbb{Z}\oplus\mathbb{Z}_2$
$\Omega_n^{\mathrm{Spin^c}}(S^{10})$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$2\mathbb{Z}$	0	$2\mathbb{Z}$	0	$4\mathbb{Z}$	0	$5\mathbb{Z}\oplus\mathbb{Z}_2$

Table D.3: Spin^c cobordism groups of spheres $\Omega_n^{\mathrm{Spin}^c}(S^k)$, $k = 1, \ldots, 10$.

n	0	1	2	3	4	5	6	7	8	9
$\Omega_n^{\mathrm{Spin}^c}(K3)$	$\mathbb{Z}$	0	$23\mathbb{Z}$	0	$25\mathbb{Z}$	0	$47\mathbb{Z}$	0	$50\mathbb{Z}$	0
$\Omega_n^{\mathrm{Spin}^c}(CY_3)$	$\mathbb{Z}$	0	$(b_2+1)\mathbb{Z}$	$b_3\mathbb{Z}$	$(2+2b_2)\mathbb{Z}$	$b_3\mathbb{Z}$	$(3+3b_2)\mathbb{Z}$	$2b_3\mathbb{Z}$	$(5+4b_2)\mathbb{Z}$	$2b_3\mathbb{Z}$

Table D.4: Spin^c cobordism groups of CY manifolds $\Omega_n^{\mathrm{Spin}^c}(K3)$, $\Omega_n^{\mathrm{Spin}^c}(CY_3)$. We omit the $n = 10$ case where the extension problem is not solved.

D.2 Tables of K- and KO-theory groups

n	0	1
$K^{-n}(S^1)$	$\mathbb{Z}$	$\mathbb{Z}$
$K^{-n}(S^2)$	$2\mathbb{Z}$	0
$K^{-n}(S^3)$	$\mathbb{Z}$	$\mathbb{Z}$
$K^{-n}(S^4)$	$2\mathbb{Z}$	0
$K^{-n}(S^5)$	$\mathbb{Z}$	$\mathbb{Z}$
$K^{-n}(S^6)$	$2\mathbb{Z}$	0
$K^{-n}(S^7)$	$\mathbb{Z}$	$\mathbb{Z}$
$K^{-n}(S^8)$	$2\mathbb{Z}$	0

n	0	1
$K^{-n}(T^2)$	$2\mathbb{Z}$	$2\mathbb{Z}$
$K^{-n}(T^3)$	$4\mathbb{Z}$	$4\mathbb{Z}$
$K^{-n}(T^4)$	$8\mathbb{Z}$	$8\mathbb{Z}$
$K^{-n}(T^5)$	$16\mathbb{Z}$	$16\mathbb{Z}$
$K^{-n}(T^6)$	$32\mathbb{Z}$	$32\mathbb{Z}$
$K^{-n}(T^7)$	$64\mathbb{Z}$	$64\mathbb{Z}$
$K^{-n}(T^8)$	$128\mathbb{Z}$	$128\mathbb{Z}$

n	0	1
$K^{-n}(K3)$	$24\mathbb{Z}$	0
$K^{-n}(CY_3)$	$(2+2b_2)\mathbb{Z}$	$b_3\mathbb{Z}$

Table D.5: K-groups of spheres, tori, and Calabi-Yau twofolds and threefolds.

n	0	1	2	3	4	5	6	7	8
$KO^{-n}(S^1)$	$\mathbb{Z}\oplus\mathbb{Z}_2$	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	0	0	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}_2$
$KO^{-n}(S^2)$	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_2$
$KO^{-n}(S^3)$	$\mathbb{Z}$	$\mathbb{Z}_2\oplus\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
$KO^{-n}(S^4)$	$2\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$2\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$2\mathbb{Z}$
$KO^{-n}(S^5)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}$
$KO^{-n}(S^6)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_2$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$KO^{-n}(S^7)$	$\mathbb{Z}$	$\mathbb{Z}_2\oplus\mathbb{Z}$	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	0	0	$\mathbb{Z}$
$KO^{-n}(S^8)$	$2\mathbb{Z}$	$2\mathbb{Z}_2$	$2\mathbb{Z}_2$	0	$2\mathbb{Z}$	0	0	0	$2\mathbb{Z}$

Table D.6: KO-groups of spheres $KO^{-n}(S^k)$, $k=1,\ldots,8$. Note that Bott periodicity is respected in a two-fold way, i.e. $KO^{-n}(S^k) = KO^{-n\pm 8}(S^k) = KO^{-n}(S^{k+8})$.

n	0	1	2	3	4	5	6	7
$KO^{-n}(T^2)$	$\mathbb{Z}\oplus 3\mathbb{Z}_2$	$3\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_2$	$2\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}$	$2\mathbb{Z}\oplus\mathbb{Z}_2$
$KO^{-n}(T^3)$	$\mathbb{Z}\oplus 6\mathbb{Z}_2$	$\mathbb{Z}\oplus 4\mathbb{Z}_2$	$3\mathbb{Z}\oplus\mathbb{Z}_2$	$3\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$3\mathbb{Z}\oplus\mathbb{Z}_2$	$3\mathbb{Z}\oplus 4\mathbb{Z}_2$
$KO^{-n}(T^4)$	$2\mathbb{Z}\oplus 10\mathbb{Z}_2$	$4\mathbb{Z}\oplus 5\mathbb{Z}_2$	$6\mathbb{Z}\oplus\mathbb{Z}_2$	$4\mathbb{Z}$	$2\mathbb{Z}$	$4\mathbb{Z}\oplus\mathbb{Z}_2$	$6\mathbb{Z}\oplus 5\mathbb{Z}_2$	$4\mathbb{Z}\oplus 10\mathbb{Z}_2$
$KO^{-n}(T^5)$	$6\mathbb{Z}\oplus 15\mathbb{Z}_2$	$10\mathbb{Z}\oplus\mathbb{Z}_2$	$10\mathbb{Z}\oplus 6\mathbb{Z}_2$	$6\mathbb{Z}$	$6\mathbb{Z}\oplus\mathbb{Z}_2$	$10\mathbb{Z}\oplus 6\mathbb{Z}_2$	$10\mathbb{Z}\oplus 15\mathbb{Z}_2$	$6\mathbb{Z}\oplus 20\mathbb{Z}_2$
$KO^{-n}(T^6)$	$16\mathbb{Z}\oplus 21\mathbb{Z}_2$	$20\mathbb{Z}\oplus 7\mathbb{Z}_2$	$16\mathbb{Z}\oplus\mathbb{Z}_2$	$12\mathbb{Z}\oplus\mathbb{Z}_2$	$16\mathbb{Z}\oplus 7\mathbb{Z}_2$	$20\mathbb{Z}\oplus 21\mathbb{Z}_2$	$16\mathbb{Z}\oplus 35\mathbb{Z}_2$	$12\mathbb{Z}\oplus 35\mathbb{Z}_2$
$KO^{-n}(T^7)$	$36\mathbb{Z}\oplus 28\mathbb{Z}_2$	$36\mathbb{Z}\oplus 8\mathbb{Z}_2$	$28\mathbb{Z}\oplus 2\mathbb{Z}_2$	$28\mathbb{Z}\oplus 8\mathbb{Z}_2$	$36\mathbb{Z}\oplus 28\mathbb{Z}_2$	$36\mathbb{Z}\oplus 56\mathbb{Z}_2$	$28\mathbb{Z}\oplus 70\mathbb{Z}_2$	$28\mathbb{Z}\oplus 56\mathbb{Z}_2$
$KO^{-n}(T^8)$	$72\mathbb{Z}\oplus 36\mathbb{Z}_2$	$64\mathbb{Z}\oplus 10\mathbb{Z}_2$	$56\mathbb{Z}\oplus 10\mathbb{Z}_2$	$64\mathbb{Z}\oplus 36\mathbb{Z}_2$	$72\mathbb{Z}\oplus 84\mathbb{Z}_2$	$64\mathbb{Z}\oplus 126\mathbb{Z}_2$	$56\mathbb{Z}\oplus 126\mathbb{Z}_2$	$64\mathbb{Z}\oplus 84\mathbb{Z}_2$

Table D.7: KO-groups of tori $KO^{-n}(T^k)$, $k=2,\ldots,8$.

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Acknowledgements

First and foremost, I want to thank my supervisor Ralph Blumenhagen. I have learned a lot from him during these past years, and I am grateful to him for always sharing his scientific expertise and intuition, discussing exciting projects, as well as for always being there as a mentor.

I would like to thank Dieter Lüst for creating such a pleasant and stimulating environment for string theory research at MPP, as well as for his support during various stages of my PhD. I would also like to thank Tasos Petkou for his continuous support since my time in Thessaloniki, as well as Ilka Brunner for her role as a mentor this past year.

To all my collaborators in the various projects I worked on during this PhD, Christian, Niccolò, Max, Rafa, and Lorenz, it has been a pleasure, and I have learned a lot from all of you. Thanks a lot, and I hope we work together again in the future. Special thanks to Niccolò and Christian, working with you was *not bad*.

The whole String Theory group at MPP has been an indispensable part of this journey. I would like to thank all present and past members for their insights, great talks, discussions, and the nice atmosphere. I must mention Christian, Niccolò, Pouria, Aleks, Antonia, Andreas, Davide, Thomas, Marco, Ivano, Yixuan, Chuying, Joaquin, Benjamin, as well as former group members Alessandra, Chrysoula, Saskia, and Vero. I wish you all the best for the future.

To my current and past office mates, Pouria, Christian, David, Max - thank you for the pleasant atmosphere, all the discussions, and, at times, the emotional support.

I want to thank Niccolò, Christian, Aleks, Pouria, Andreas, and Thomas for their comments on this thesis and help with the german translation of the abstract. I must express my deepest gratitude to Pouria for his help regarding handing in the thesis.

Finally, I want to thank Ana for being a great flatmate and friend. I will miss the kitchen discussions.

Στους γονείς μου, θέλω να πω πόσο πολύ εκτιμώ ότι πάντοτε με ενθάρρυναν να ακολουθήσω τα όνειρά μου, και τους ευχαριστώ πολύ για την αδιάκοπη, άνευ όρων αγάπη και στήριξη σε ότι και αν προσπάθησα. Ελπίζω να σας κάνω περήφανους. Στον Χάρη και την Ελένη οφείλω ότι είμαι φυσικός - ευχαριστώ. Υπόσχομαι κάποια μέρα να εξηγήσω τον

κόσμο στα ανίψια μου. Τέλος, θέλω να ευχαριστήσω τον Θοδωρή που είναι δίπλα μου όλα αυτά τα χρόνια, ομορφαίνοντας κάθε πτυχή της ζωής μου. Ξέρω ότι πιστεύεις σε μένα πιο πολύ από ότι πιστεύω εγώ στον εαυτό μου, και σε ευχαριστώ για αυτό.

Σας αγαπώ όλους πάρα πολύ, και μου λείπετε. Ευχαριστώ για όλα.