Riemann Surfaces: Intersection Numbers and String Scattering Amplitudes

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To my family and friends
without whom my achievements were not possible

Abstract

In this thesis we explore mathematical foundations of scattering amplitudes both in string theory and quantum field theories. Scattering amplitudes are the contact point between theoretical and experimental physics. They are used to check theoretical results in experiments for example Standard Model predictions are tested in LHC measurements of cross sections. Furthermore, they can be used to understand the underlying theoretical structure as well.

There are two main ways to discuss tree-level scattering amplitudes: First, in the bottom up approach one can study amplitudes in different quantum field theories with various matter contents and symmetries. Second, one can use the top down approach and employ string theory as the high energy quantum gravity and study scattering processes in string theory. Then, one can relate them to different field theories in the low energy limit. Utilising these methods one can not only construct more efficient methods to calculate scattering processes (e.g. spinor-helicity formalism) but also establish structural relations among different theories (e.g. gauge/gravity duality).

Our discussion in this thesis stands in the middle of the two aforementioned methods. We discuss Riemann surfaces and define advanced topological structures on top of them namely the twisted cohomology. In particular, we explain the recent development regarding twisted forms/cycles that allows us to construct different tree-level scattering amplitudes [5, 6, 7] both in string theory and quantum field theories. Here, we use the relationship between string theory and quantum field theories (i.e. the low energy limit of string theory) to introduce an algorithm by which we are able to produce new twisted forms. The intersection numbers of these new twisted forms can be used to calculate scattering amplitudes of different theories more efficiently.

Furthermore, we take advantage of this new mathematical method and study the structure of scattering amplitudes. In particular, we explore the double copy construction [8] in two separate avenues. First we construct the first ever double copy for the massive spin-2 field through string theory. We show that this massive double copy can be compared to bimetric gravity [9]. Second, we discuss the role of the twisted cohomology in double copy and put forward a novel method to understand the double copy construction [8] in terms of twisted differentials as well as producing (and suggesting) new double copy theories.

Zusammenfassung

In dieser Arbeit untersuchen wir die mathematischen Grundlagen von Streuamplituden sowohl in Stringtheorie als auch in Quantenfeldtheorie. Streuamplituden sind die Schnittstelle zwischen theorerischer und experimenteller Physik. Sie können dafür verwendet werden, die Vorhersagen des Standardmodells der Teilchenphysik für Streuamplituden von Gluonen und Quarks mit den Versuchsdaten zu vergleichen. Eine weitere Anwendung von Streuamplituden ist der Vergleich der zugrundeliegenden Struktur von verschiedenen theoretischen Modellen.

Man kann "Baumniveau" Streuamplituden mit zwei unterschiedlichen Herangehensweise untersuchen: Bei der ersten Methode, dem "Bottom-up" Ansatz, brechnet und analysiert man Streuamplituden von verschiedenen Quantenfeldtheorien, die unterschiedliche Felder und Symmetrien enthalten. Beim zweiten (Top-down) Ansatz benutzt man Streuamplituden, die in Stringtheorie brechnet wurden, welche eine Vervollständigung von Quantenfeldtheorien im ultravioletten Breich darstellt, um im Grenzwert für niedrige Energie eine Verbindung zu Streuamplituden, die man aus Quantenfeldtheorien erhält, herzustellen. Diese beide Methoden werden in diese Arbeit verwendet, um zum einen ein neues Berechnungsverfahren für Streuamplituden zu entwickeln (z.B. Spinor-Helicity Formalismus) und zum andren eine Verbindung zwischen ansonsten vollständig verschiedene Theorien herzustellen (z.B. die Dualität zwischen Eich- und Gravitationstheorien).

Diese Arbeit stellt eine Verbindung zwischen den beiden genannten Methoden her. Dafür betrachten wir Riemannfläche und definieren topologische Strukturen nämlich "Twisted Kohomologie". Insbesondere, untersuchen wir die neulich entwickelten "Twisted Forms/Cycles", welche die Berechnung von verschiedenen Baumniveau Streuamplituden von Stringtheorie und Quantenfeldtheorien ermöglichen [5, 6, 7]. Wir benutzen die Beziehung zwischen Streuamplituden in Stringtheorie unf Quantenfeldtheorie, welche durch Grenzwert für niedrige Energien gegeben ist, um einen Algorithmus zu entwickeln, mit dem wir neue "twisted forms" herstellen können. Die Schnittzahl dieser neuen "Twisted forms" können wir zur effiziente Berechnung von Amplituden in Quantenfeldtheorien verwenden.

Außerdem benutzen wir die "twisted Kohomologie", um Strukturen in Streuamplituden zu finden und zu untersuchen. Insbesondere betrachten wir dabei Baumniveau Streuamplituden in der Doppelkopie sowohl für massive als auch masselos Zustände. Für Massive spin-2 konstruieren wir die Doppelkopie einer Streuamplituden in Stringtheorie . Darüber hinaus stellen wir fest, dass diese Doppelkopie einer Streuamplituden vergleichbar mit einer bimetrischen Gravitationstheorie (eine Theorie mit massive und masslose Felder mit spin-2 [9]) bis zur kubischen Ordnung ist. Des Weiteren diskutieren wir die Rolle der "twisted Kohomologie" in der Doppelkopie und argumentieren, wie man dieser neuentwickelten Methode die Doppelkopie-Konstruktion [8] in Form von "twisted differential" verstehen und neuer Doppelkopie-Theorien konstruieren kann.

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In any scientific work there are many people who can influence ones work. I had the great fortune to have possibly the best environment to study and work. So, it is my honor and pleasure to be able to thank them and show my gratitude.

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Last but not least, I should thank my parents and sister for their love and support. They provided me with the possibility to pursuit my goals. The privilege I know is rare and I am forever grateful to them.

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Seyed Pouria Mazloumi

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Wovon man nicht sprechen kann darüber muss man schweigen.
Tractatus logico-philosophicus,
Ludwig Wittgenstein

Chapter 1

Introduction

Quantum field theory (QFT) is one of the main tools that we have invented in theoretical physics to model the universe. It employs sophisticated mathematical structures from functional analysis to algebraic geometry to produce a vivid picture of the observable universe to date. One of the main shortcomings of the current formulation of quantum field theory (as is used in the Standard Model of particle physics (SM)) is the accommodation of gravity into the formalism. Attempts to produce a consistent notion of the quantum field theory of gravity, also known as quantum gravity, started at the same time as the advent of quantum mechanics. As of the time of writing this thesis, the most mathematically consistent formalism of quantum gravity is *string theory*. It provides the construction that, among other states, includes a graviton (massless spin-2) state. More importantly, this graviton (massless spin-2) state interacts with other states in the theory in a ghost free and mathematically consistent way. In addition, due to the existence of the conformally symmetric two-dimensional world sheet the usual UV divergence issues do not appear in string theory [10],[11],[12]. All of these good features come at the expense of introducing higher mathematical structures in string theory such as algebraic topology, number theory, K-theory, etc [13]. String theory like the Standard Model is a S-matrix theory and one of the main objects in any quantum field theories with the S-matrix formulations is the scattering amplitude (i.e. elements of the S-matrix). The scattering amplitude gives the probability of transitioning from one state to another within the asymptotic Hilbert space of the theory [1]. These probabilities become complicated when the theory has nontrivial interactions and the only practical method to calculate such interactions is weak coupling perturbation theory[2]. Understanding scattering amplitudes has far-reaching uses than just the transition probability between states one of the more notable programs is the *amplitudhedron* [15] which establishes that scattering amplitudes can be defined (regardless of Lagrangian) as volumes of mathematical object called amplitudhedron described by Polygons. In case of string theory since it can be considered as high energy completion of Standard Model one can take the amplitude and construct effective actions by taking the string-length to zero (particle limit) and expanding the

[1]By asymptotic we mean the Hilbert space corresponding to the free part of the theory.

[2]for a review on non-perturbative methods cf. [14]

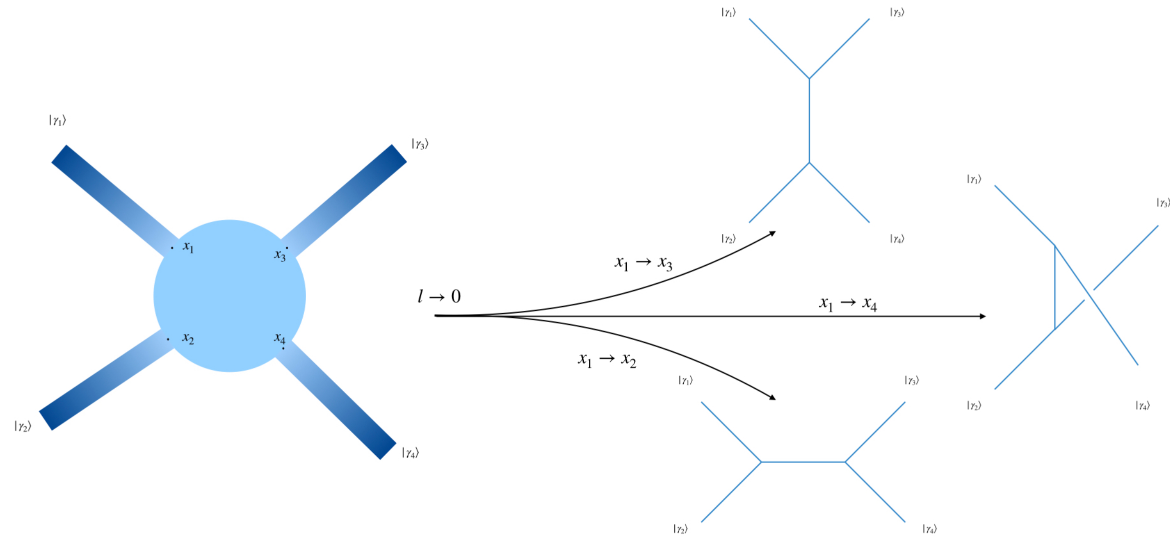

Figure 1.1: Effective amplitudes upon taking length of string $l \to 0$

amplitude in the length of the string. In this limit string scattering amplitudes will have different pole structures which correspond to different field theoretic scattering channels. For example, as we depicted in figure 1.1, for the 4-point scattering upon taking the effective limit the pole structure correspond to s, t, and u channels.

Effective actions

Generic scattering amplitude involves Lorentz invariant terms of polarizations and momenta of the states

$$\mathcal{A}(\varepsilon_I, p_J) = F(\alpha', \varepsilon_i \cdot p_j, p_i \cdot p_j, \varepsilon_i \cdot \varepsilon_j)\,. \tag{1.0.1}$$

One can construct an effective Lagrangian, corresponding to a set of matter field ϕ_i, with the replacement:

$$\begin{aligned} &\varepsilon_i \longrightarrow \phi_i \qquad p_i \longrightarrow i\partial_i\,, \\ &F(\alpha', \varepsilon_i \cdot p_j, p_i \cdot p_j, \varepsilon_i \cdot \varepsilon_j) \mapsto \sum_n \alpha'^n \mathcal{L}_n^{eff}(\phi_i, \partial_i \phi)\,, \end{aligned} \tag{1.0.2}$$

truncated in orders of α'.

This means that associated with every scattering amplitude of string theory (or any UV completion of SM) there is an effective quantum field theory in the low energy limit, determined by the expansion in α'. Many different theories have been constructed through the effective action method such as general relativity (GR) and Yang-Mills (YM) [11, 16]. In earlier times this construction of effective actions was done to show that indeed string theory is indeed a good candidate for quantum gravity since it is inclusive of already established theories as low energy limits. In contrast, we can use this method in reverse, meaning we accept that string theory is a quantum gravity theory and construct the effective actions for theories whose formulations are cumbersome using established quantum field theory methods. Specifically, we looked at theories involving massive spin-2 fields [3, 4]. It has taken a lot of effort and discussions by different groups to produce a ghost free formulation of massive spin-2 theory known as dRGT-gravity [17]. Additionally, coupling the massive spin-2 to matter fields, in particular, to graviton (massless spin-2) required the same amount of attention (as the formulation of the ghost free pure massive spin-2) which was done by Hassan and Rosen [18] and it is known as *bimetric gravity*. However, in both of these cases, it is still an open problem how one can dynamically generate the mass of the spin-2 field. That is where our discussion of string theory comes into play. Since string theory already includes gravity and many other theories as low energy limits, we can try to establish the massive spin-2 as an effective action of string theory and subsequently define the connection of massive spin-2 to other theories including massless spin-2 i.e. gravity. Having this in mind we looked at two candidates one open and one closed string state and found the following structure:

Bimetric as an effective action

Expanding the massive spin-2 bimetric potential and comparing it with string effective Lagrangian we obtain:

- Massive spin two closed string: Full match up to cubic order.
- Massive spin two open string: Same Lagrangian with different coefficients.

Effective field theory is not the only avenue through which one can study string scattering amplitudes. As scattering amplitudes are maps between different states, understanding their algebraic topology/geometry implications can help us to understand dualities (symmetries of Hilbert spaces) such as *color-kinematic duality* [19]. One of the latest methods in the study of scattering amplitudes is *intersection of twisted forms*. It has been shown [5, 7, 6] that many scattering amplitudes including string scattering amplitudes can be written as intersection numbers of twisted forms. These are differential forms φ which can be defined over the moduli space of Riemann surfaces with local structure associated with differential operator $\nabla_\omega := d + \omega\wedge$ which defines the following cohomology:

Twisted cohomology

$$H^m_{\pm\omega}(X, \nabla_{\pm\omega}) = \frac{\nabla_\omega\, m - \textit{closed forms}}{\nabla_\omega\, m - \textit{exact forms}} \,. \tag{1.0.3}$$

The dual space $H^m_{-\omega}$ can be obtained from $H^m_{+\omega}$ by sending $\omega \to -\omega$. The intersection number on the twisted cohomology groups is the invariant pairing between two forms $\varphi_\pm \in H^m_{\pm\omega}$ and defined by the integral

$$\langle \varphi_+, \varphi_- \rangle_\omega := \int_{\mathcal{M}_{0,n}} \iota_\omega(\varphi_+) \wedge \varphi_- \, . \tag{1.0.4}$$

String amplitudes are also defined over the moduli space of Riemann surfaces with genus g and n punctures $\mathcal{M}_{g,n}$ and one can show that associated to any string scattering amplitude $\mathcal{A}(n)$ we can find twisted forms $\varphi_{\pm,n}$[7]. Hence by using the relation between string theory and other theories (as low energy limits of string theory) we can define twisted forms for those effective field theory amplitudes. This is the second part of our study [2, 1]. We managed to construct two new twisted forms with the use of string theory:

New twisted forms

Using the superstring mixed open-closed amplitude we have:

$$\widetilde{\varphi}^{EYM}_{\pm,n;r} = \bigwedge_{i=1}^{n+r-3} dz_i \; F\left(\frac{\varepsilon_i \cdot p_j}{z_i - z_j}, \frac{p_i \cdot p_j}{z_i - z_j}, \frac{\varepsilon_i \cdot \varepsilon_j}{z_i - z_j} \right) . \tag{1.0.5}$$

Having done the same for the bosonic string we have:

$$\widetilde{\varphi}^{Bosonic}_{\pm,n;r} = \bigwedge_{i=1}^{n+r-3} dz_i \; G\left(\frac{\varepsilon_i \cdot p_j}{z_i - z_j}, \frac{p_i \cdot p_j}{z_i - z_j}, \frac{\varepsilon_i \cdot \varepsilon_j}{z_i - z_j} \right) . \tag{1.0.6}$$

We derive to explain and discuss F and G functions in the coming chapter 5.

Since the intersection numbers of twisted forms are defined over Riemann surfaces without boundary, in order to obtain the above results we had to introduce an *embedding* of the disk onto the sphere.

Furthermore, we managed to use the CHY formalism of scattering amplitudes to show that these two twisted forms indeed correspond to Einstein Yang-Mills and Weyl Yang-Mills respectively. The CHY formalism [20] is a formulation of scattering amplitude as an integral *localized over solutions of scattering equations* and it has its roots in *ambitwistor string theory* [21] and is defined as:

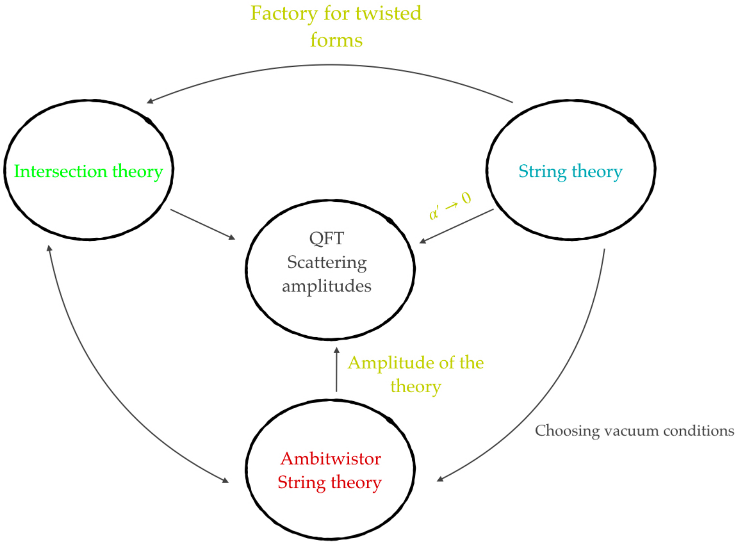

Figure 1.2: Relation among different fields

CHY integral representation

$$\mathcal{A}_{CHY}(n) = \int_{\mathcal{M}_{0,n}} d\mu_n \prod_{a=1}^{n}{}' \delta(f_a)\, \mathcal{I}_L(p,\varepsilon,z)\mathcal{I}_R(p,\varepsilon,z)\,,$$
$$f_a \equiv \sum_{\substack{b=1 \\ b\neq a}} \frac{p_a \cdot p_b}{z_a - z_b} \quad , \quad a = 1,\ldots,n\,. \tag{1.0.7}$$

Where f_a is a set of scattering equations. Both formulations CHY and ambitwistor include only massless states, however, one can try to use the intersection numbers to describe massive states [7].

Additionally, we use the algebraic structure of twisted cohomology to establish relations among different amplitudes as well as describe the color kinematic (CK) duality. CK duality states that in a scattering amplitude of gauge theories, one can find a set of kinematic variables (functions of polarization and momenta) that satisfy the same Jacobi identity of the gauge algebra. These kinematic variables can replace the gauge factors in the amplitude. This replacement leads to a description of double copy known as *BCJ double copy* [8, 19, 22]. The double copy in scattering amplitude processes refers to the relation between gauge and gravity theories schematically this can be written as:

$$gauge \otimes gauge \sim gravity\,.$$

In this work, we propose a new description of this duality in the language of intersection theory.

Double copy in Intersection theory

For two theories T_1 and T_2 given by the intersection numbers

$$T_1 = \langle \varphi^1_+, \varphi^{color} \rangle \,, \qquad T_2 = \langle \varphi^2_-, \varphi^{color} \rangle \,, \tag{1.0.8}$$

respectively, one schematically obtains for their double copy:

$$T_1 \otimes T_2 = \langle \varphi^1_+, \varphi^2_- \rangle \ . \tag{1.0.9}$$

This thesis is dedicated to explaining and establishing the results which we have introduced above. The structure of the thesis is as follows:

I. Mathematical preliminaries: This part includes the first two chapters of this work. In chapter 2 we discuss the basics of Riemann surfaces since, we need them for both intersection numbers as well as string scattering amplitudes. In this chapter, we are also introducing the standard de Rahm cohomology over Riemann surfaces as well as defining the *moduli space* of (punctured) Riemann surfaces. The moduli space is the domain over which our amplitude integrations are defined. In chapter 3 we introduce the twist structure over the Riemann surfaces and extend it to the homology/cohomology formalism and construct the twisted de Rahm cohomology. Having defined the twisted forms (and twisted cycles) we then define the tools that we employ to construct numbers out of them specifically, the twist period and twisted intersection number. Both are defined as integrals over the Riemann surfaces. Generally, these integrals will be related to elliptic functions [23, 24]. We finish chapter 3 by giving the saddle point approximation of the intersection number which we are going to use to validate and test our results.

II. Physics preliminaries: In part II, we will give a short but concise discussion of two main physical topics of this work. First, from the quantum field theory perspective, in chapter 4 we explain the CHY formalism and how it can be used to produce scattering amplitudes of different quantum field theories. Then, we will go through the list of theories that we are using in this work from Yang-Mills and general relativity to bimetric gravity. We give their matter content as well as their CHY formulation. In chapter 5, we delve into string theory and discuss its origins and its features. Our goal is to introduce the matter content of string theory (string spectrum) together with details of calculation of string scattering amplitude. We finish this chapter by discussing the KLT double copy [25]. This is the relation between closed and open string amplitudes schematically defined as:

$$\mathcal{A}^{open} \otimes \mathcal{A}^{open} \sim \mathcal{A}^{closed} \,.$$

III. Scattering amplitudes: In the final part we will present our results and explain how we produce them. In chapter 6 we evident how different types of amplitudes (QFT and string amplitudes) can be constructed using the intersection of twisted forms that we introduce in earlier chapters. We discuss both string and quantum field theory amplitudes and explain how the saddle point approximation and CHY formalism can be used to create an algorithm by which one can construct new twisted forms using solely string amplitudes. We explain our results namely, the two new twisted forms and their construction in detail. In addition, we show that these new twisted forms can be used to describe amplitudes of several different quantum field theories such as Einstein Yang-Mills and Weyl Yang-Mills. We finish our results in chapter 7, where we address double copy constructions both in string theory and intersection theory. First, we use pure string scattering amplitude to produce the bimetric gravity. We use open and closed string states as candidates for the massive spin-2 field. Looking at closed string candidate gives us the opportunity to construct the first-ever massive double copy of string amplitudes to this date. We conclude the chapter by giving a new description of double copy with the use of intersection numbers. We explain how the double copy and color-kinematic duality can arise in a theory whose intersection number amplitude description includes the φ^{color} twisted form. We use this to conjecture new double copies, such as higher derivative gravity.

IV. We finish this thesis with remarks on open questions and future directions of the study of Riemann surfaces, scattering amplitudes, and elliptic integrals.

Part I

Mathematical Preliminaries

Chapter 2

Riemann Surfaces

2.1 Preface

Riemann surfaces are the centerpiece of many studies in theoretical physics as well as mathematics. In physics, Riemann surfaces appear in different fields from condensed matter physics to string theory. Our interest in them starts (but is not limited) with string theory applications. In string theory, Riemann surfaces are used to describe string worldsheets and they are prominent in the string scattering amplitudes since any n-point string amplitude involves an integration over the moduli space of the associated Riemann surface. Therefore, given the relation between string and QFT amplitudes, studying these surfaces and their moduli is going to help us better understand scattering amplitudes both at string theory and QFT level. Furthermore, we are going to define and explore, in the coming chapters, the intersection number of twisted forms over Riemann surfaces. These intersection numbers will produce a class of Feynman integrals that are related to amplitudes, independent of string theory or any other QFT. Hence, having a general overview of these objects is going to be beneficial for our purposes. As summary, this chapter includes the following topics:

1. Basics of Riemann surfaces

2. Monodromy

3. De Rahm and singular cohomology

4. Moduli space of Riemann surfaces

Needless to say, the topic of Riemann surfaces is a vast field of research and we are not even going to attempt to be exhaustive in the topics that we discuss here. Therefore, we invite the reader to the following references for further reading on the topic [26, 27, 28, 29].

2.2 Basics of Riemann surfaces

Definition 2.2.1. A **Riemann surface** X is a *Hausdorff* topological space together with a collection of open sets $U_\alpha \in X$ where U_αs cover X (i.e. $X = \bigcup_\alpha U_\alpha$). For each α we have the homeomorphism

$$\psi_\alpha : U_\alpha \to \tilde{U}_\alpha \,,$$

where $\tilde{U}_\alpha$ is an open set in $\mathbf{C}$ with the property that the transition map $\psi_\alpha \circ \psi_\beta^{-1}$ is **Holomorphic** on its domain of definition.

A reader who is familiar with manifolds can see that a Riemann surface is a connected complex manifold with the complex dimension one and the atlas $(U_\alpha, \tilde{U}_\alpha, \psi_\alpha)$. If one identifies $\mathbf{C}$ as $\mathbf{R}^2$ then the Riemann surface is a manifold of real dimension two. Note that the converse is not always true i.e. not all 2 real-dimensional manifolds are Riemann surfaces. This is due to the holomorphicity requirement on the transition maps.

In particular, we can also describe a Riemann surface as a Riemannian manifold M mod Weyl (conformal) transformations. Therefore, the Riemann surface is invariant under the following symmetries:

- Diffeomorphism transformation: This is the group of all diffeomorphism transformations from manifold X to itself:

$$\begin{aligned} f : & M \to M \,, \\ & z^i \mapsto f^i(z^i) \,, \end{aligned} \tag{2.2.1}$$

  where f and its inverse f^{-1} are differentiable and bijection.

- Weyl (conformal) transformations: The set of local transformations that rescale the metric by a positive number namely:

$$\begin{aligned} \varphi : & M \to M \,, \quad \varphi \in C^\infty \,, \\ & z \mapsto \varphi(z) \,, \\ & \varphi^* g = \Lambda g \,, \qquad \text{for} \quad \Lambda > 0 \,, \end{aligned} \tag{2.2.2}$$

  where $\varphi^* g$ is the standard *pullback* of the metric g under $\varphi(z)$.

So we have two equivalent description of a Riemann surface either a complex manifold with holomorphic transformations or a $2d$ real manifold which is also invariant under the Weyl scaling.

Euler character and Genus

Two of the topological invariants that are mainly used in the study of Riemann surfaces are *Euler character* and *genus*. They characterize some of topological properties of the surface. We have the following definitions:

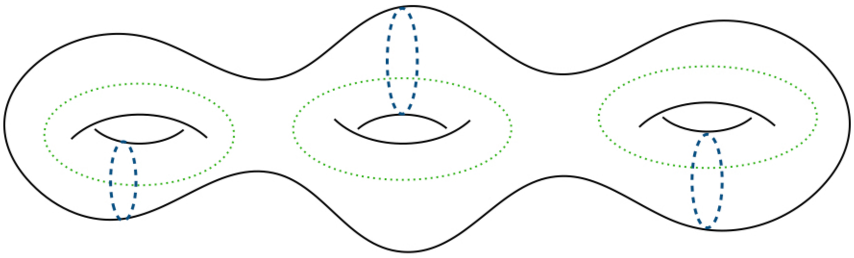

Figure 2.1: A generic Riemann surface with $g = 3$ and $b = 0$ meaning it has three holes and no boundary

Euler character

- *Genus* is an integer number counting the number of *holes* in a surface.
- *Euler character* of a surface X with genus g and b boundary components is another integer defines as:

$$\chi(X) = 2 - 2g - b\,.$$

In particular, for a 2 dimensional Riemannian manifold (X, g) we can define the Euler character with the Einstein-Hilbert action:

$$\int_X d^2x\sqrt{-g} = 4\pi\chi(X)\,. \qquad (2.2.3)$$

Now we can look at two simple yet important examples

Example (2.1): Riemann surfaces

1. **Any open set in C** is a Riemann surface. In particular, as we are going to see later the upper half plane $H = \{z \in \mathbf{C} : Im(z) > 0\}$ is a Riemann surface. Through the following Mobius transformation H can be mapped to the Riemann disk $D_2 = \{w : |w|, < 1\}$:

$$w = \frac{z - i}{z + i}\,.$$

2. **Riemann Sphere** S_2. The sphere is the one point compactification of the complex plane $\mathbf{C}$ by adding the point ∞ i.e.

$$S_2 = \mathbf{C} \cup \{\infty\}\,.$$

3. The **two torus** T^2 is a genus one ($g = 1$) Riemann surface defined as:

$$T^2 = S_1 \times S_1\,.$$

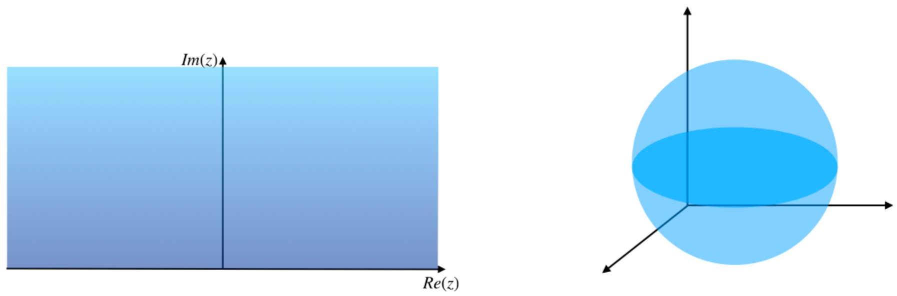

Figure 2.2: The upper half plane and Sphere as Riemann surfaces

The first two examples are going to be used extensively in the following chapters as they correspond to open and closed string worldsheets.

Next we define the Möbius map. This map is going to be important since it corresponds to the coordinate transformation under $PSL(2,\mathbf{R})$ and $PSL(2,\mathbf{C})$, two important groups in our future discussions.

Definition 2.2.2. Given a Riemann surfaces X, one can define the Möbius transformation between over an open subset $U \subset X$ with complex coordinate $(z, \bar{z})$:

$$z \mapsto \frac{az+b}{cz+d}\,, \qquad \text{where, } a,b,c,d \in \ \mathbf{R}\,\textit{or}\ \ \mathbf{C} \text{ with } ad-bc=1\,. \tag{2.2.4}$$

This map describes the action of $PSL(2,\mathbf{R})$ or $PSL(2,\mathbf{C})$ on a Riemann surface X with coordinates z_i. The former corresponds to the automorphisms of the upper half plane H (i.e. $Aut(H) = PSL(2,\mathbf{R})$) and the latter corresponds to the automorphisms of the Riemann sphere S_2 (i.e. $Aut(S_2) = PSL(2,\mathbf{C})$).

2.2.1 Riemann surfaces with marked points (punctured)

One other possible way to define Riemann surfaces is to define them as algebraic curves. To avoid confusion we should make the terminology clear. From the point of view of real coordinates, Riemann surfaces are indeed a two-dimensional *surface*. However, from the complex analysis point of view, a Riemann surface is a one-dimensional algebraic curve. Therefore, we can define a Riemann surface as a *set of solutions* in complex projective plane $\mathbf{CP}^n$. This method is going to be particularly useful when we are going to look at string scattering amplitudes in the coming chapters. Given set of zeros of function $P(z,w) \in \mathbf{C}$ as:

$$\begin{aligned} &P : \mathbf{C}^2 \to \mathbf{C}\,, \\ &X = \{(z,w) \in \mathbf{C}^2 : P(z,w) = 0\}\,. \end{aligned} \tag{2.2.5}$$

Then X can be made into a Riemann surface if for every point $(x_0, y_0) \in \mathbf{C}^2$ one of the two derivatives P_w or P_z are non-zero (for the details of the proof cf. [26]).

Example (2.2): Riemann surfaces with marked points

Take a Riemann surface defined by the following equations :

$$w^2 = f(z) = (z - z_1)(z - z_2)(z - z_3)...(z - z_n)\,.$$

There are branch points at each of the points $\{z_1, z_2, z_3, ..., z_n\}$.

Using this notion we can see that generic polynomials will produce multi-sheet Riemann surfaces. Meaning, upon projecting the Riemann surface $S \subset X$ to $\mathbf{C}$ there are at least two points of S mapped to each point of $\mathbf{C}$.

2.3 Monodromy

The Monodromy of a universal covering on Riemann surfaces has great importance to our discussion. Since we are going to discuss Riemann surfaces with punctures and integration over them then, understanding the monodromy of these particular Riemann surfaces is necessary in the present work. Therefore, we are going to give the definition and examples of these types of monodromies. In the following, we are going to have physical applications in mind for more mathematical-oriented examples and discussion we invite the reader to the following references [27, 26]. First, we need to define the fundamental group. Generically speaking, this group corresponds to loops on a given topological space:

Definition 2.3.1. Given a topological space Y a loop based at y_0 is defined to be continuous map:

$$\gamma(t) : [0,1] \to Y$$

such that $\gamma(0) = \gamma(1) = y_0$. A **homotopy** is a continuous deformation between two given loops with the same base. Namely two loops associated to γ_1 and γ_2 both based at y_0 are *homotopic* to each other iff there exists a continuous map $h(r,t)$ such that

$$h : [0,1] \times [0,1] \to Y\,.$$

- $h(0,t) = h(1,t) = y_0$ for all t.
- $h(r,0) = \gamma_1(r)$ and $h(r,1) = \gamma_2(r)$ for $r \in [0,1]$.

The hotompic relation is an equivalence relation between two given loops. So the first fundamental group is given by:

$$\pi_1(Y, y_0) = \frac{\{\textit{all loops based at } y_0\}}{\textit{homotopy}}\,.$$

This definition is very general and can be applied to topological spaces. However, we will focus on Riemann surfaces. We want to describe how going around loops on a Riemann surface X affects the holomorphic functions over that surface. Since these functions take values in the covering of a Riemann surface, we need to understand the homotopy group within the covering map.

Definition 2.3.2. A map $p : X \to Y$ is called a **local homeomorphism** (and X is called a *covering space* of Y) if around each point of $x \in X$,there is an open neighbourhood U such that $p|_U$ is a homeomorphism to its image in Y

Definition 2.3.3. A *proper* local homeomorphism is a *covering map.*

The number of points in the inverse image of a point, under a covering map, is locally constant (since the base space is connected). This number is called the *number of sheets* of the covering. To see this better we can look at some simple examples:

Example (2.3): Examples of covering maps and their sheet number

1. The map from
$$\begin{aligned} &\mathbf{R} \to S^1\,, \\ &t \mapsto e^{2\pi\, i\, t}\,, \end{aligned} \tag{2.3.6}$$
is a covering map with infinitely many sheets
2. The map
$$\begin{aligned} &S_2 \to S_2\,, \\ &z \mapsto z^n\,, \end{aligned} \tag{2.3.7}$$
for a fixed positive integer n, is a covering with n sheets.
3. Let $f(z)$ be a complex polynomial considered as a map $\mathbf{C} \to \mathbf{C}$, and let F be the set of critical points of $f(z)$. Then the induced map $\mathbf{C} - f^{-1}(F) \to C - F$ is a covering map and has $deg(f(z))$ sheets.

Putting the homotopy group and the covering together we can now define the monodromy.

Definition 2.3.4. Given two Riemann surfaces X and Y with the covering map $p : X \to Y$. Let $y_0 \in Y$ be a base point we define

$$J = \pi_1(Y, y_0)\,, \qquad \text{and} \qquad F = p^{-1}(y_0)\,,$$

where F is called the *fiber* of p. The action of the group J on F is a group permutation and is called *monodromy.*

Let $x \in F$ and $\alpha \in J$ then take α to be a loop $C : [0, 1] \to Y$. Lift C to get a path γ in X with $\gamma(0) = x$, then we can define the action of the group J map as:

$$\begin{aligned} &F \times J \to F\,, \\ &x \times \alpha = \gamma(1)\,. \end{aligned} \tag{2.3.8}$$

It is illuminating if we take a look at a very simple example of the z^2 function:

Example (2.4): Some covering maps and their sheets number

Take the function in the example 2.2.1 for $n = 2$:

$$\begin{aligned} p : S_2 &\to S_2 \,, \\ z &\mapsto z^2 \,. \end{aligned} \tag{2.3.9}$$

Given a point z_0 and then the map $p^{-1}(z_0)$ is the square root of z_0 which has two elements we label them by $+$ and $-$. we can order them without loss of generality:

$$F = p^{-1}(z_o) = \sqrt{z_0} \sim \{z_+, z_-\} \,. \tag{2.3.10}$$

Now, the loop $C : [0, 1] \to S_2$ centered at z_0 can be defined as:

$$\gamma(t)_{z_0} := z_0 e^{i2\pi t} \,. \tag{2.3.11}$$

We can see that going around the loop multiplies the roots by a minus sign:

$$F' = p^{-1}(e^{i2\pi t} z_o) = \sqrt{z_0 e^{i2\pi t}} = \sqrt{z_0} e^{i\pi t} \overset{t=1}{=} -\sqrt{z_0} \,. \tag{2.3.12}$$

The set of roots is still the same (since they differ only by an overall sign) but their order has reversed. We have:

$$F' = p^{-1}(e^{i2\pi t} z_o) \sim \{z_-, z_+\} \,. \tag{2.3.13}$$

This permutation is the monodromy of p.

Looking back at the definition of Riemann surfaces with marked points (2.2.5). We can see that since Riemann surfaces defined in this way does not include the branch point $\mathbf{C} - \{\textit{cirital points of } f\}$ (it is "punctured") then integration over contours (loops) on the Riemann surface X which is defined over the covering space of the surfaces $\hat{X}$ will include the monodromy of the cover around the punctures.

2.4 Calculus on Riemann surfaces

It is clear that in order to perform any local computation (i.e. differentiation and integration) over a Riemann surface we require to construct the calculus. We are going to see that while discussing the contour integration of holomorphic functions over Riemann surfaces ,the monodromies of cover of the Riemann surface becomes relevant. We assume that the reader is familiar with the standard concepts of vector spaces and manifolds and tensor products (cf. standard references [27, 30]). We start with a set of definitions:

1. The tangent space T_pX of the complex manifold X over the point p is the set of all possible vectors at the point p given by the differential of the of all possible curves C

going through p. The tangent space is a vector space which is spanned by $\{\partial_{z_i}, \bar{\partial}_{\bar{z}_i}\}$ for a n-dimensional complex manifold.

2. The cotangent space $T_p^\star X$ is the dual space of the $T_p X$ and is spanned by $\{dz_i, d\bar{z}_i\}$ in case of the n dimensional complex manifold.

Differential forms

- All holomorphic functions over a Riemann surface belong to the cotangent space and are called *zero forms*. The space of all holomorphic functions is denoted by $\Omega^0(X)$.
$$g(z) \in \Omega^0(X)\,.$$
- An elements of $T_p^\star X$ are called *one forms*. The space of one forms can be decomposed into holomorphic and anti-holomorphic parts and is denoted by $\Omega^1_C(X) = \Omega(X)^{1,0} \oplus \Omega^{0,1}(X)$. It can be written as:
$$\begin{aligned} &\alpha = \sum_i g_i(z) dz_i \otimes \sum_i \tilde{g}_i(\bar{z}) d\bar{z}_i\,, \qquad \alpha \in \Omega^1_C(X)\,,\\ &\textit{holomorphic form: } \beta = \sum_i g_i(z) dz_i \qquad \bar{\partial} g_i = 0\,,\\ &\textit{anti-holomorphic form: } \bar{\beta} = \sum_i \tilde{g}_i(\bar{z}) d\bar{z}_i \qquad \partial \tilde{g}_i = 0\,, \end{aligned} \tag{2.4.14}$$
where $g_i(z)$ are zero forms
- Elements of $T_p X$ are called *vectors* (as they are vectors tangent to the point p). A vector V can be written as:
$$\begin{aligned} &V = \sum_i f_i(z) \partial_{z_i} \otimes \sum_i \tilde{f}_i(\bar{z}) \partial_{\bar{z}_i}\,,\\ &\text{holomorphic vector: } W = \sum_i f_i(z) \partial_{z_i}\,, \qquad \bar{\partial} f_i = 0\,,\\ &\text{anti-holomorphic vector: } \bar{W} = \sum_i \tilde{f}_i(\bar{z}) \partial_{\bar{z}_i}\,, \qquad \partial \tilde{f}_i = 0\,. \end{aligned} \tag{2.4.15}$$
We can check the duality between the vectors and forms and see that by acting on each other we obtain complex functions:
$$V(\alpha) = \sum_i \left(f_i(z) g_i(z) + \tilde{f}_i(\bar{z}) \tilde{g}_i(\bar{z}) \right) \in \Omega^0(X)\,.$$

Using the tensor product " $\otimes$ " and tensoring the elements of the tangent bundle one will arrive at the general notion of *tensors* for a discussion on tensors cf. [27, 30]. The objects that are relevant to the current work are a special class of forms that are known as holomorphic/anti-holomorphic *differential forms*. Differential forms are defined by using the *exterior algebra* equipped with a product called *wedge product*:

Differential forms

First, we define the anti-summarized tensor product known as *wedge product*:

$$\alpha \wedge \beta := \alpha \otimes \beta - \beta \otimes \alpha \,. \tag{2.4.16}$$

A holomorphic basis element for n forms is defined as:

$$\bigwedge_i^n dz_i = dz_1 \wedge dz_2 \wedge dz_3 \wedge ... \wedge dz_n \,. \tag{2.4.17}$$

The elements of the space of n forms are denoted by $\Omega^n(X)$. Now we can define the differential d acting on forms as:

$$\begin{aligned} &d : \Omega^{n,m}(X) \to \Omega^{n+1,m}(X) \,, \\ &\bar{d} : \Omega^{n,m}(X) \to \Omega^{n,m+1}(X) \,, \\ &df(z) = \partial_z f(z) dz \qquad f(z) \in \Omega^0(X) \,, \\ &(dd)f = \bar{d}\bar{d}f = 0 \,, \\ &d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta \qquad \alpha \in \Omega^k(X) \,. \end{aligned} \tag{2.4.18}$$

We invite the reader to check for the second example in 2.4 that indeed $d(\alpha)$ is in $\Omega^2_C(X)$. Given the differential operator d, we can categorize forms into different cases:

- **Exact forms**: A form that is differential of a lower order form is called *exact*. Meaning $\alpha \in \Omega^i(X)$ is an exact from iff
$$\exists\, \beta \in \Omega^{i-1}(X) : \alpha = d\beta \,.$$
- **Closed forms**: A forms whose differential is zero is called *closed*. Meaning, $\alpha \in \Omega^i(X)$ is a closed from iff
$$d\alpha = 0 \,.$$

Since $d^2 = 0$ all exact forms are closed but not all closed forms are exact. Further, using the definition of the differential we can see that on a complex manifold with dimension n we have a particular form:

$$\begin{aligned} &\alpha = \sum_i \alpha_i(z, \bar{z}) \Big(\bigwedge_{k=1}^n dz_k \Big) \wedge \Big(\bigwedge_{k=1}^n d\bar{z}_k \Big) \in \Omega^{n,n}(X) \,, \\ &d\alpha = \bar{d}\alpha = 0 \to \Omega^m(X) \equiv 0, \qquad m > n \,, \end{aligned} \tag{2.4.19}$$

and α is known as the top form. In addition, we can see that there are no forms higher than the top form. Having defined the differential, we are ready to define the integration over a Riemann surface:

Integral of forms

Suppose X is an oriented[a] Riemann surface and β is a $(1,1)-$from with compact support and supported on the domain of a coordinate chart on X. Meaning, in these local coordinates we have:

$$\beta = f(z,\bar{z})dz \wedge d\bar{z}\,,$$

then we define the integral of β over X by:

$$\int_X \beta = \int\limits_{\mathbf{C}} f(z,\bar{z})dzd\bar{z} = \frac{i}{2}\int\limits_{\mathbf{R}^2} f(x+iy, x-iy)dxdy\,. \tag{2.4.20}$$

Similarly for a holomorphic (anti-holomorphic) differential form $\alpha \in \Omega^{1,0}(X)$ (or $\Omega^{0,1}(X)$) the integration will be over a path $\gamma \subset X$. Therefore, we have:

$$\int_\gamma \alpha = \int\limits_{z_1}^{z_2} f(z)dz = \int\limits_{\mathbf{R}} f(x+iy)(dx+idy)\,. \tag{2.4.21}$$

[a] All Riemann surfaces that we are considering are oriented

We want to consider is contour integration over holomorphic functions that have branch points, which is the accumulation point of monodromy, differential forms and integration.

2.4.1 Contour integration of punctured Riemann surfaces

For a function $f(z,\bar{z})$ with a set of poles and branch points F to be well defined we have to remove the points from the Riemann surface and obtain the Riemann surface with punctures $(X\backslash F)$. However, in doing so the integration becomes ill defines since there is a nontrivial isomorphism between loops in $X\backslash F$ and loops in the covering $\mathbf{C}$. In particular, this affects contour integration since a contour C is a loop on a Riemann surface X, and when we want to perform the integration of the function $f(z,\bar{z})$ over the covering of X i.e. $\mathbf{C}$ we have to define the monodromy of the curve going around the points. In order to do this we define the following functional:

Monodromy functional

Given a contour integration C we define the monodromy of the holomorphic (anti-holomorphic) form $I(z)$ $(\bar{I}(\bar{z}))$ as:

$$Mon_C\{I(z)\} := \text{Mondromy of the function going around the contour } C\,. \tag{2.4.22}$$

Let us explain this in an example:

Example (2.6): Monodromy functional

We have the holomorphic function $I(z) = z^{2/5}$ over the disk and we take the loop $\gamma \subset D_2$ going along the boundary which corresponds to the loop in $C \subset \mathbf{C}$ with the following monodromy:

$$\oint_{\gamma} I(z) = \oint_{C} Mon_C\{I(z)\} := \int_{C_1} dxI(x) + e^{i\pi\frac{2}{5}} \int_{C_2} dxI(x) = 0\,. \tag{2.4.23}$$

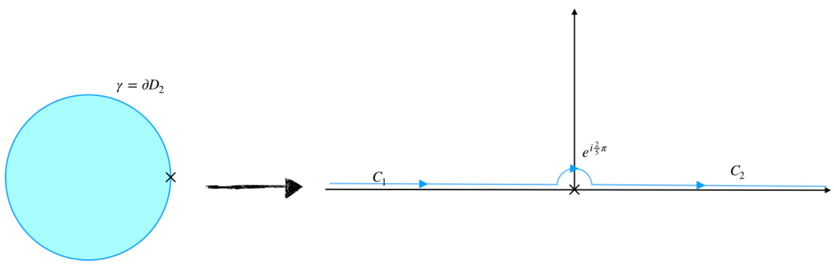

Figure 2.3: Loop γ and contour C

Here, we have taken the radius of the semi-circle C_2 contour to zero and therefore it is not appearing in the integral

We finish this section by looking at a special function that we are going to encounter a lot in this work namely the *Koba-Nielsen* factor.

2.4.2 Koba-Nielsen monodromies

The Koba-Nielsen factor was first constructed in the meason amplitudes [31, 32] but in string theory it can be seen as the contraction of the plane wave vertex operators we are going to discuss this in detail in chapter 5. For the discussion of this section, we take this factor as a holomorphic function and calculate its monodromy over a given contour integral. We start by giving the expression of the Koba-Nielsen (KN) factor:

Koba-Nielesn factor

Given a set of complex points z in the complex plane and non integer real variables ζ_i associated to each point i.e.

$$z := \{z_i : z_i \in \mathbf{C}\}\,,$$
$$\zeta := \{\zeta_i : \zeta_i \in \mathbf{R}\backslash\mathbf{Z}\}\,. \tag{2.4.24}$$

We can define the KN factor as:

$$KN := \prod_{i>j}^{n} |z_i - z_j|^{\zeta_i \cdot \zeta_j} \, . \tag{2.4.25}$$

We can see that the KN factor is a holomorphic function with branch points at $z_i - z_j = 0$. Looking at the third example we gave for the monodromy in example 2.3 we can take the KN factor to be the function $f(z)$, Therefore, the KN defines a map from Riemann surface to the complex numbers so it defines a is a covering and we can calculate the monodromy around a loop. We take the Riemann surface to be the disk D_2 and the loop γ to be a contour integration going around the boundary. By using the map in example 2.2 the disk is mapped to the upper half-plane and the boundary is mapped to the real line. Therefore, the variables $z_i = x_i$ become real:

$$\oint_{\gamma} dz_1 KN = \oint_{x_k} dx_1 \prod_{i>j}^{n} |x_i - x_j|^{\zeta_i \cdot \zeta_j} \, . \tag{2.4.26}$$

Now we can define the monodromy concerning a contour integration as:

Monodromy of KN factor

Given a contour integration γ we can w.l.g. order the branch points along the direction of the contour noted by $\sigma : \{x_2 < x_3 < x_4 < ..., x_n\}$ we have:

$$\oint_{\gamma} dz_1 KN = \oint_{\mathbf{R}} dx_1 \prod_{i>j}^{n} |x_i - x_j|^{\zeta_i \cdot \zeta_j} \, . \tag{2.4.27}$$

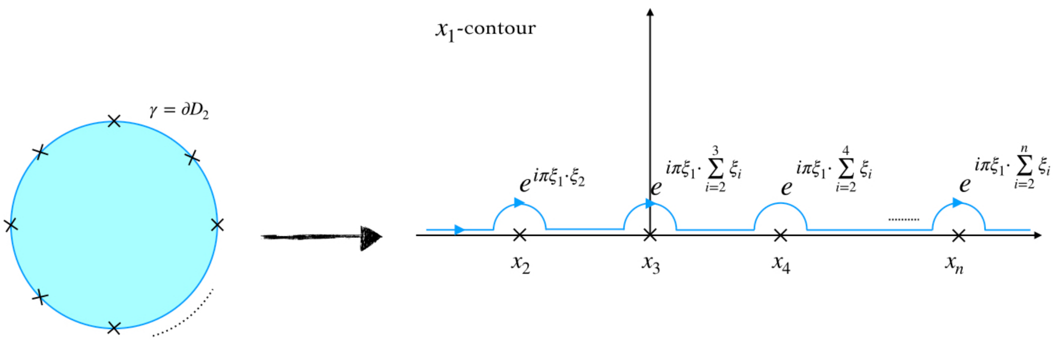

Figure 2.4: The monodromy for variable x_1 of the Koba-Nielsen factor.

We obtain the monodromy of the KN factor:

$$Mon_{C_1}\{KN\} = e^{i\pi F(\sigma)}\ KN$$

$$F(\sigma) = \sum_{i>j} f(\zeta_1 \cdot \zeta_j; x_1, x_j)), \qquad f(\zeta_i \cdot \zeta_j; x_1, x_j) = \begin{cases} \zeta_1 \cdot \zeta_j & x_1 > x_j\,, \\ 0 & x_1 < x_j\,. \end{cases} \tag{2.4.28}$$

We should note here that our choice fixed the positions of the variables in the Koba-Nilesen to be on the real line. Generically, this needs not to be true we can have some of the variables in the bulk of the disk. However, this case is irrelevant to our discussion.[1]

2.5 Singular homology and De Rahm cohomology

So far we have discussed the differential forms and integration over Riemann surfaces. We now want to introduce two important structures over the differential forms that we are going to use throughout this work, namely: singular homology and de Rahm cohomology. Geometrically, singular homology studies and classifies loops on a given manifold. In order to formalize that we need to define the *chain complex* and the associated *boundary* operator:

Definition 2.5.1. A chain complex is a collection of abelian groups G_i indexed by integers together with a sequence of homomorphism $\partial : G_i \to G_{i-1}$ such that $\partial^2 : G_i \to G_{i-2}$ is *zero*. The operator ∂ is called the *boundary operator* or differential.

Singular Homology

If $G_\star = (\{G_i\}, \partial)$ is a chain complex then we define its homology to be the graded group:

$$H_i(G_\star) = \frac{ker\ \partial : C_i \to C_{i-1}}{im\ \partial : C_{i+1} \to C_i}\,. \tag{2.5.29}$$

Simply stated, homology puts all possible cycles with vanishing boundary, that can be deformed to each other by a boundary of another cycle, in the same class. There is a dual space to the homology with the dual operator to the boundary operator δ known as the *singular cohomology* denoted by:

$$H^i(G_\star) = \frac{ker\ \delta : C_i \to C_{i+1}}{im\ \delta : C_{i-1} \to C_i}\,. \tag{2.5.30}$$

Given the properties of the exterior differential that we discussed in (2.4.18), we can see that the differential can be used to define chain complex and is dual to the homology (since ∂_{z_i} and dz_i are dual). This is known as the *de Rahm cohomology* and is defined as:

[1] We are going to see in the chapter 5 that this case will correspond to EYM amplitudes.

De Rahm cohomology

The ith de Rahm cohomology group pf X is the complex vector space:

$$H^i_\Omega(X) = \frac{ker\ d : \Omega_i \to \Omega^{i+1}}{im\ d : \Omega_{i-1} \to \Omega_i} = \frac{closed\ i\text{-}froms}{exact\ i\text{-}froms}\,. \tag{2.5.31}$$

Given the fact that the differential of a top form on a manifold is zero i.e. $d\alpha^{top} = 0$ and all other forms above the top forms are identically zero. The cohomology group is also capped at the n cohomology group meaning:

$$H^i(X) = 0\,, \qquad i > n\,. \tag{2.5.32}$$

The *de Rahm theorem* relates the singular cohomology and the de Rahm cohomology.It states that there is an isomorphism

$$H^i(G_\star) \to H^i_\Omega(X)$$

.

2.5.1 Poincare duality

The Poincare duality gives us the relation between homology and the cohomology groups namely:

Theorem 1. *Let M be a complex manifold of complex dimension n. Then there is the following isomorphism:*

$$H^i_\Omega(X) \cong H_{n-i\,\Omega}(X)\,. \tag{2.5.33}$$

Meaning, to any differential α we can associate a cycle c.

2.5.2 Intersection number

There are different instances in that we use the space of forms with compact support ($\Omega^i_c(X)$). For example, we defined the integration over the space of compactly supported forms. The other one would be the intersection number. The definition of the intersection number goes through the Poincare duality which has the following implication.

Intersecion number

Let γ be loop over the Riemann surface X then integration over γ yields a linear map:

$$\begin{aligned} I_\gamma &: H^1(S) \to \mathbf{R}\,, \\ I_\gamma(\omega) &: \omega \mapsto \int_\gamma \omega\,. \end{aligned} \tag{2.5.34}$$

Then by the use of the Poincare duality, we have that for *any* loop γ there is a *compactly supported* form θ over X such that:

$$I_\gamma(\omega) = \int_X \theta \wedge \omega\,. \tag{2.5.35}$$

Now, the integral only depends on the class of θ (due to Stokes' theorem). Therefore we can define the intersection number as the bilinear paring:

$$\begin{aligned} H^i_c(X) \times H^i(X) &\to \mathbf{C}\,, \\ \langle \theta, \omega \rangle &\mapsto \int_X \theta \wedge \omega\,, \end{aligned} \tag{2.5.36}$$

where $\theta \in H^i_c(X)$.

We are going to use this definition of intersection number for the rest of this work.

2.6 Moduli space of Riemann surfaces

As we have seen so far there are a lot of topological and analytical features associated with the Riemann surfaces from monodromy to cohomology. Now we want to discuss the classification of Riemann surfaces. Meaning, we want to define the class of all Riemann surfaces that can be continuously deformed to each other using a continuous variable. Therefore, we can make a bigger space (space of all possible Riemann surfaces) from these variables. In this bigger space, each point describes a Riemann surface and it is known as *Moduli space* of Riemann surface. Here we are mostly following [11, 26].
First, we can see the simple topological classification of Riemann surfaces, namely the *genus* classification. We know that all surfaces with the same genus are homeomorphic to each other. For example, all genus zero Riemann surfaces (without boundary) are homeomorphic to sphere S_2 or all genus one Riemann surfaces are homeomorphic to torus T^2 (a Doughnut is the same as a cup with handle). However, this topological information is not enough we need the information about the local structure to be added to this equivalency between Riemann surfaces. Therefore, we define the classes as:

Equivalent classes of Riemann surfaces

Two Riemann surfaces are equivalent if there is a homeomorphism between them that preserves geometry (in other words they are holomorphically isomorphic). This map takes holomorphic functions to holomorphic functions.

Since we are still using the subset of homeomorphisms between Riemann surfaces we still have the genus classification i.e. we cannot map surfaces with different genus to each other.

The main difference is that not all surfaces with a given genus are equivalent[2] and we now need to take into account the local geometries i.e. the metric on the surface. So, we can classify surfaces with their metrics. We define the space of all possible metrics of the surface X with genus g as $\mathcal{G}_g(X)$. However, as we discussed the Riemann surface is invariant under diffeomorphisms and Weyl transformations. Therefore any two metrics g_1 and g_2 in $\mathcal{G}_g(X)$ that are related by diffeomorphism and/or Weyl transformation will correspond to the same surface. Therefore, we define the moduli space of a Riemann surface X as:

Moduli space of Riemann surface X_g

$$\mathcal{M}(X)_g := \frac{\mathcal{G}_g(X)}{(\mathit{diff} \times \mathit{Weyl})_g} . \tag{2.6.37}$$

A few simple examples are in order:

Example (2.7): Example of moduli spaces

1. Riemann sphere: The moduli space of the Riemann sphere is trivial "one point" meaning, all possible metrics on a Riemann sphere are equivalent up to *Weyl* $\times$ *diff* transformations.
2. Torus: The moduli space of the T^2 is the upper half plane mod $PSL(2, \mathbf{Z})$

One of the main tools that we are going to use is the *uniformization theorem* which states that any simply connected Riemann surface is holomorphically isomorphic to one of the following three:

- The Riemann sphere S_2.
- The complex plane C.
- The upper half-plane H or equivalently the unit disc D_2.

Therefore, using the uniformization theorem we can classify all genus zero Riemann surfaces. For example, using the theorem we can see that all compact genus zero Riemann surfaces are isomorphic to the Riemann sphere S_2 and hence have trivial moduli (see example 2.6).

2.6.1 Punctured Riemann surfaces

All of the discussion in this work will be over Riemann surfaces with marked (punctured) points. Hence, we are going to take a closer look at the moduli space of Riemann surfaces with n punctures $X_{g,n}$. We take the positions of n punctures as a configuration of

[2]since, as mentioned, we do not have the full homeomorphisms only the geometry preserving homeomorphisms

$X_{g,n}$ therefore, for every possible configuration/position of punctures we have a different Riemann surface. Therefore, we can add the positions to the moduli space coordinates. Therefore we modify the definition (2.6.37) as:

Moduli space of punctured Riemann surfaces $X_{g,n}$

$$\mathcal{M}(X)_{g,n} := \frac{\mathcal{G}_g \times X^n}{(\textit{diff} \times \textit{Weyl})_g} . \tag{2.6.38}$$

Since each puncture can be at any position in X we have n copies of X i.e. X^n. Then the coordinate of the $\mathcal{M}(X)_{g,n}$ will be $(\underbrace{\tau_1, \tau_2, ..., \tau_k}_{\textit{metric moduli}}, \underbrace{z_1, z_2, ..., z_n}_{\textit{puncture positions}})$

For the genus zero case, we have the following:

$$\mathcal{M}(X)_{0,n} := \frac{\mathcal{G}_0 \times X^n}{(\textit{diff} \times \textit{Weyl})_g} = \frac{X^n}{V_{CKG}(X)} , \tag{2.6.39}$$

where the $V_{CKG}(X)$ is the volume of the *conformal killing group* of the surface X. This is the subgroup of the $(\textit{diff} \times \textit{Weyl})_g$ that keeps the metric invariant and therefore remains after the quotient of $\mathcal{G}_0$. The conformal killing group for the sphere is $PSL(2, \mathbf{C})$, and for the disk is $PSL(2, \mathbf{R})$. One can easily implement the conformal killing group by (*gauge*) fixing the positions of some of the punctures. As an important set of examples (to our work) we take a look at the moduli space of n-punctured Riemann sphere and disk.

Example (2.8): Moduli space of S_2 and D_2

1. $\mathcal{M}(S_2)_{0,n} = \{(z_1, z_2, z_3, ..., z_n) \in (S_2)^n / PSL(2, \mathbf{C})\} = (S_2)^{n-3} = (\mathbf{CP}^1)^{n-3}$.

2. $\mathcal{M}(D_2)_{0,n} = \{(x_1, x_2, x_3, ..., x_n) \in (\mathbf{R})^n / PSL(2, \mathbf{R})\} = (\mathbf{R})^{n-3}$.

Here we have fixed three positions to remove the conformal killing volume.

Chapter 3

Intersection number of twisted forms

3.1 Preface

In recent years thanks to the work by Mizera [7] twisted cohomology was brought into the scene of theoretical physics. In particular, in two very close fields of scattering amplitudes:

1. Defining twisted differential forms that describe tree level amplitudes of specific theory[1, 2, 5].
2. Calculation of elliptic integrals that appear in loop field theory amplitudes [23, 33].

The main difference between the two topics is that in the first the properties of the associated field theory (mass, interactions, color kinematic duality, etc) are reflected in the twisted form. However, in the second case, one wants to calculate only the elliptic integral regardless of the theory that gave rise to it. In this chapter, we are going to give an overview of the Intersection theory of twisted forms. We will discuss the following:

1. First, we are going to define twisted cohomology/homology. We are going to explain what is a twisted form/cycle.
2. Second we will introduce the intersection number associated with twisted forms. Then we present different methods to calculate them.
3. Finally, we will use these methods in practice and explain in detail how these intersection numbers are calculated with different examples. We finish by looking at some limits and commenting on their relevance to the next chapters.

3.2 Basics of twisted cohomology

In this section, we are going to give the basics of twisted de Rahm cohomology. We are going to be following mathematical literature [34] with the physicist twist on them. We are going to summarize the important practical results that will use in the last section. However, We believe it is important to understand the mathematical foundations of this topic. So we start with the basic definitions. Like the de Rahm cohomology on any manifold, we have the analogous theorem to de Rahm theorem but before giving the theorem we need to define the main ingredient i.e. *locally constant sheaf* or *local system*.

Definition 3.2.1. A *Sheaf* S on a space X is an assignment of a set $S(U)$ to each open set $U \subset X$ with the "restriction map"

$$\rho_{UV} : S(U) \to S(V)\,, \qquad \text{when} \qquad V \subset U\,.$$

Definition 3.2.2. A *local constant sheaf* or *local system* on a space X is a sheaf S such that for any point $x \in X$ there is an open neighborhood U containing x such that $S|_U$ is constant sheaf.

As an example we have the *local constant sheaf* or *local system* $\mathcal{L}_\omega$ as:

$$\begin{aligned} \mathcal{L}_\omega :& \pi^1(X) \to C^\times \\ & [\gamma] \mapsto \exp\big(\int_\gamma \omega\big), \qquad \omega : \text{closed}. \end{aligned} \tag{3.2.1}$$

Here $\mathcal{L}_\omega$ is the locally constant sheaf of solutions $\nabla_\omega h = 0$, $h \in \Omega^0_\omega(M)$ which is the rank one complex local system on X. In fact, we have more: There is a bijection between the representation of the first fundamental group $\pi^1(X)$ and local systems on X [35]. Now we modify the definition of homology:

Definition 3.2.3. Let G be an Abelian group and Δ_p be the standard complex on X. Then the tensor product $\Delta_p \otimes G$ is a chain complex with the differential $\partial \otimes \mathbf{1}$. We define *homology group with coefficient in* G by:

$$H_p(X; G) = H_p(\Delta_p \otimes G)\,, \tag{3.2.2}$$

Where $H_p(\Delta_p \otimes G)$ is the homology with respect to the "new" boundary operator.

We can take the local sheaf $\mathcal{L}_\omega$, which is an Abelian representation of the first fundamental group $\pi_1(X)$. Hence, we have the homology group with coefficient in $\mathcal{L}_\omega$ as:

$$H_p(X; \mathcal{L}_\omega) = H_p(\Delta_p \otimes \mathcal{L}_\omega)\,. \tag{3.2.3}$$

An element of the twisted homology is called a twisted *cycle* and is defined as:

$$C_{\gamma,\omega} = C_\gamma \otimes \exp\big(\int_\gamma \omega\big)\,, \tag{3.2.4}$$

where C_γ is standard cycle over manifold X.

Twisted de Rahm cohomology

Let ω be a closed holomorphic one form on M. The covariant derivative with respect to ω is defined by:

$$\begin{aligned} &\nabla_\omega := d \pm \omega\wedge\,, \\ &\Omega_\omega(X)^p := \Omega(X)^p \otimes \exp\Big(\int_\gamma \omega\Big)\,. \end{aligned} \tag{3.2.5}$$

Then, the (smooth) twisted de Rahm cohomology complex of X is $(\Omega_\omega(X)^p, \nabla_\omega)$. We denote the cohomology of $(\Omega_\omega(X)^p, \nabla_\omega)$ by H^p_ω we have

$$H^p_\omega(M, \nabla_{\pm\omega}) = \frac{\{\varphi \in \Omega^p_\omega(X) \mid \nabla_{\pm\omega}\varphi = 0\}}{\nabla_{\pm\omega}\Omega^{p-1}_\omega(M)}\,. \tag{3.2.6}$$

Theorem 2. Given the twisted singular cohomology $H^p_\omega(X, \mathcal{L}_\omega)$ the map:

$$\begin{aligned} \Lambda :=& H^p_\omega(X, \nabla_\omega) \to H^p(X, \mathcal{L}_\omega)\,, \\ &[\varphi] \mapsto \left([\xi] \otimes exp\Big(\int_\gamma \omega\Big) \mapsto [c] \otimes\, exp\Big(\int_\gamma \omega\Big) \mapsto \int_c \xi\ exp\Big(\int_\gamma \omega\Big)\right)\,, \end{aligned} \tag{3.2.7}$$

to the de Rahm cohomology is a vector space isomorphism.

An important choice for ω which we are going to use throughout this work is

$$\omega = \sum_{1\le i<j\le n} \lambda_i \cdot \lambda_j\ d\log(z_i - z_j)\,,$$

for this choice we have the following cycle:

$$C_{\gamma,\omega} = C_\gamma \otimes e^{i\pi\phi(\gamma)} \prod_{i=1}^{n-3} |z_i - z_j|^{\lambda_i\cdot\lambda_j} = C_\gamma \otimes KN\,, \tag{3.2.8}$$

where KN is the Koba-Nielsen factor we introduced in (2.4.25). In simpler words, this means that for each cycle that goes around branch points, we associate a complex number to that part of the loop that goes around branch points. We are going to discuss this twist further in the next subsection.

3.2.1 Hyperplane twisted cohomology

So far we have introduced the twisted de Rahm cohomology as an abstract construction. Now we will explicitly choose a twist ω which enables us to give examples and perform calculations. Looking at the definition (3.2.6) any covariant holomorphic one-formcan be a candidate for the twist ω. One of the more important cases of twisted de Rahm cohomologies is the case of complements of *hyperplane* as general hypergeometric functions [36, 37]. An *l-arrangement* of set $\mathcal{K}$ of hyperplanes is a finite set of distinct hyperplanes in the l dimensional complex (projective) space $\mathbf{C}^l$ (note that l is *not* the number of the hyperplanes). We define the complement of $\mathcal{K}$ by $M(\mathcal{K}) = \mathbf{C}^l \backslash \cup_{H\in\mathcal{K}} H$. Then, we define the logarithmic one-form ω_H of a hyperplane H, with the defining linear polynomial, α_H as:

$$\omega_H := d\log\alpha_H\,, \qquad \text{where: } H = ker(\alpha_H)\,. \tag{3.2.9}$$

A *weight* λ of $\mathcal{K}$ is defined by:

$$\lambda = (\lambda_H; H \in \mathcal{K})\,, \qquad \lambda_H \in \mathbf{C}\,. \tag{3.2.10}$$

We define the logarithmic one-form by:

$$\omega(\mathcal{K},\lambda) = \sum_{H\in\mathcal{K}} \lambda_H\omega_H\,, \qquad \sum_H \lambda_H = 0\,. \tag{3.2.11}$$

Then, we have the twisted de Rahm cohomology with respect to this one-form:

$$H(M,\nabla_\omega) \sim H(M,\mathcal{L}_\omega), \qquad \nabla_\omega = d + \omega(\mathcal{K},\lambda)\wedge\,, \tag{3.2.12}$$

where $\mathcal{L}_\omega$ is the rank one local system on $M(\mathcal{K})$ whose monodromy around a hyperplane $H \in \mathcal{K}$ is $exp\{-i2\pi\lambda_H\}$ we can also define the multivalued analytic function $U(\mathcal{K},\lambda)$ on $M(\mathcal{K})$ as:

$$\begin{aligned} U(\mathcal{K},\lambda) &:= \prod_{H\in\mathcal{K}} \alpha_H^{\lambda_H}\,, \\ \omega(\mathcal{K},\lambda) &= d\,\log\,U(\mathcal{K},\lambda)\,. \end{aligned} \tag{3.2.13}$$

All of these definitions are going to be given physical meaning when we get to discuss string amplitudes in the coming sections.

Example (3.1): Dimension one case

The twisted de Rahm cohomology in one dimension ($l = 1$) is the complex line minus some points. Let $\mathcal{K}$ be the set of points in $\mathbf{C}$ and $\lambda = (\lambda_i, i \in \mathcal{K})$ be its weights. Taking the coordinate z of $\mathbf{C}$ we have the following:

$$\begin{aligned} U(\mathcal{K},\lambda) &= \prod_{p\in\mathcal{K}} (z-p)^{\lambda_p}\,, \\ \omega(\mathcal{K},\lambda) &= \sum_{p\in\mathcal{K}} \lambda_p \frac{dz}{z-p}\,. \end{aligned} \tag{3.2.14}$$

We finish this section with a discussion on a particular version of the hyperplane twisted cohomology. Given a l arrangement hyperplane twisted cohomology setup, we choose the set of polynomial hyperplanes in $\mathbf{C}^{k+1}$ to be the following:

$$\begin{aligned} \mathcal{K} &:= \{\alpha_{ij} = 0\}\,, \\ \alpha_{ij} &= z_i - z_j\,. \end{aligned} \tag{3.2.15}$$

Therefore, the twist ω will be for a given set of weights λ_{ij}:

$$\omega = \sum_{ij} \lambda_{ij} d \log (z_i - z_j), \qquad \sum_{i,j} \lambda_{ij} = 0 \tag{3.2.16}$$

The complement space of $\mathcal{K}$ has the following form:

$$M(\mathcal{K}) = \mathbf{C}^l \backslash \bigcup_{ij} \{(z_i - z_j) = 0\} . \tag{3.2.17}$$

This setup will be used in the amplitude discussion.

3.2.2 Dual spaces

In order to construct an intersection number we need the dual space to the $H^p(M, \nabla_\omega)$. One can see that the local constant sheaf $\mathcal{L}_\omega$ has the natural dual space $\mathcal{L}_{-\omega}$ which is given by:

$$\mathcal{L}_\omega \otimes \mathcal{L}_{-\omega} = e^{\int_\gamma \omega} \times e^{-\int_\gamma \omega} = \mathbf{1} . \tag{3.2.18}$$

Therefore, we have the homology with the local coefficients in $\mathcal{L}_{-\omega}$ i.e. $H(M, \nabla_{-\omega})$ as the dual space of the $H(M, \nabla_\omega)$. However, there is another possibility. We can take the complex conjugation of the homology $H(M, \nabla_{-\omega})$ and use the complex dual i.e. $H(M, \nabla_{\overline{\omega}})$ as the dual space. This space also has its own dual local sheaf $H(M, \nabla_{\overline{-\omega}})$. There is a nice isomorphism between the dual twisted cohomologies thanks to Hanamura and Yoshida [38]:

$$\begin{aligned} H^n_\omega &\simeq H^n_{-\overline{\omega}} , \\ H^n_{-\omega} &\simeq H^n_{\overline{\omega}} . \end{aligned} \tag{3.2.19}$$

So, we can use these spaces interchangeably. However, after choosing the elements one has to be careful to not mix them in the middle of calculations since the parings of elements will be inverse matrices of each other. Lastly, thanks to work by Amato it was shown that the only nontrivial cohomology/homology group is for $k = dim^{complex} M$.

3.3 Numbers from twisted homology: Intersections and periods

So far, we have discussed the basics of twisted homology and we have defined cohomologies and their duals. Now we use this machinery to obtain complex numbers from these objects. As we mentioned in section 2.5.2 We have two main tools:

- Twisted periods
- Intersection numbers

3.3.1 Twisted periods

The twisted period is defined as the paring of a cycle in the homology $H_k(M, \nabla_\omega)$ and a twisted differential form in the cohomology $H^k(M, \nabla_\omega)$. The simplest way to produce numbers out of these two objects is by integration numbers. We have the following:

twisted period

Let $\varphi_+ \in H^k(M, \nabla_\omega)$ be a twisted k-form and $C_\gamma \in H_k(M, \nabla_{-\omega})$ be a twisted $k-$cycle. Then, we define their paring by:

$$\langle C_{\gamma,\omega}|\varphi_+\rangle_\omega := \int_{C_\gamma} KN\varphi_+ \,. \tag{3.3.20}$$

This gives rise to period integrals on moduli space. The Euler-Mellin integral representations for different hypergeometric functions can be interpreted as this pairing [23, 34]. A similar construction applies for dual cycles $\tilde{C}_\delta$ of the twisted homology group $H_k^{-\omega}(M, \mathcal{L}_{-\omega})$ with the cycle

$$\tilde{C}_\delta \otimes KN^{-1}, \tag{3.3.21}$$

which in turn gives rise to the period integrals:

$$\langle \tilde{C}_\delta \otimes KN^{-1}|\varphi_-\rangle := \int_{\tilde{C}_b} KN^{-1}\ \varphi_-\ . \tag{3.3.22}$$

3.3.2 Intersection number

Now, we build up a very important tool in our work namely the intersection number of twisted de Rahm cohomology. As mentioned before we will look at the twisted cohomology $H^k(M, \nabla_\omega)$ where twisted forms are associated with the positions of hyperplanes and M is a k dimensional complex projective space. In particular, we consider the intersection of *holomorphic top forms*[1] on M. First, we need to take care of the fact that space of twisted differential $k-$forms on M, (i.e. $\mathcal{D}^k(M)$) is not compactly supported. In order to use the definition of the intersection number (2.5.36) over these forms we need to compactify this space. This is done via the isomorphism ι_ω^k [7, 34]:

$$\iota_\omega^k : H^k(M, \nabla_\omega) \to H_c^k(M, \nabla_\omega)\,, \tag{3.3.23}$$

where $H_c^k(M, \nabla_\omega)$ is the cohomology of the twisted $k-$forms with compact support. The map ι_ω implicitly solves also another problem. As we mentioned we take the holomorphic top forms. However, any wedge of holomorphic top forms will be *zero*. The isomorphism ι_ω introduces anti holomorphic direction and we avoid the zero wedge[2]. Our main task throughout this section will be to realize the map ι_ω^k for twisted forms. Having this setup we can define the intersection number of two twisted top forms:

[1]This means that for a top form φ we have $d\varphi = \nabla_\omega\varphi = 0$

[2]we see this in detail whilst giving the proof of the intersection number.

Intersection number

Let $\varphi_- \in H^k(M, \nabla_\omega)$ and $\varphi_+ \in H^k(M, \nabla_{-\omega})$ be two twisted $k-$forms (top forms). Then, we define their intersection number by:

$$\langle \varphi_-, \varphi_+ \rangle_\omega{}^a := \int_M \iota^k_\omega(\varphi_-) \wedge \varphi_+ \,. \tag{3.3.24}$$

[a]Notice that we are using $\langle | \rangle$ for periods and $\langle , \rangle$ for intersection numbers

In order to construct the intersection number we chose the second twisted form from the dual cohomology $H^k(M, \nabla_{-\omega})$ so that the result of the intersection number (3.3.24) will be $\mathbf{C}$ number not an element of the group G as in (3.2.2). One can choose the twisted form to be in the complex conjugate dual space, i.e. $H^k(M, \nabla_{\overline{\omega}})$. In the subsequent chapters we are going discuss different uses of this intersection number. For now, we employ the twisted dual forms in $H^k(M, \nabla_{-\omega})$. First, we give the result for the intersection number and then we give a sample of the proof [34]. The intersection number (3.3.24) is given by:

Intersection number

Let $\varphi_- \in H^k(M, \nabla_\omega)$ and $\varphi_+ \in H^k(M, \nabla_{-\omega})$ be two twisted $k-$forms. Then, we define their intersection number by:

$$\langle \varphi_-, \varphi_+ \rangle_\omega := \int_M \iota^k_\omega(\varphi_-) \wedge \varphi_+ = (2\pi i)^k \sum_p \widetilde{Res}_{z_p}(\varphi_+ \nabla^{-1}_\omega \varphi_-)\,, \tag{3.3.25}$$

where $z_p = (z_1, ..., z_k) \in \mathbf{C}^k$ are the poles of $(\varphi_+ \nabla^{-1}_\omega \varphi_-)$(in the local coordinates of complex manifold M). We take the multidimensional residue $\widetilde{Res}_{z_p}$ as the following:

$$\widetilde{Res}_{z_p}(\varphi_+ \nabla^{-1}_\omega \varphi_-) := Res_{z=z_k}(Res_{z=z_{k-1}}(Res_{z=z_{k-2}}(...(Res_{z=z_1} \varphi_+ \nabla^{-1}_\omega \varphi_-)...)))\,. \tag{3.3.26}$$

Several points are in order:

- First, we have implicitly assumed that both forms φ_- and φ_+ are dependent on all the coordinate $(z_1, z_2, ..., z_k)$. This is not necessary: For example, one of the twisted forms can be dependent on (z_1, z_4, z_5) and the other (z_2, z_3, z_4) in that case we have to look at the common coordinate (cf. [34]). This will not be relevant for our discussions we always assume the top form. Therefore, we always have all coordinates of the space.

- The equation we take the residue layer by layer (from z_1 to z_k). Of course, the result is not dependent on the ordering of the layer. However, it can be easier to do the calculation in one particular order.

Now is a good time to look at some examples of intersection numbers:

Example (3.2): Example of intersection number of twisted forms

We look at the simple example in which the manifold is $\mathbf{C}^2$ with the local coordinates (z, w). In order to be able to picture the arrangement of the hyperplanes we project the coordinates to their real parts. We introduce the following hyperplanes:

$$\begin{aligned} f_1 &= Re(z) - 2\,, \\ f_2 &= Re(z) + Re(w) - 2\,, \\ f_3 &= Re(w) - 5\,. \end{aligned} \tag{3.3.27}$$

Therefore, the domain of integration is given by: $M(\mathcal{K}) = \mathbf{C}^2 \setminus \bigcup_{i=1}^{3} \{f_i = 0\}$. Now we have the twisted one form and the twisted forms as:

$$\begin{aligned} \omega &= \sum_{i=1}^{3} \lambda_i d\log f_i = \lambda_1 d\log f_1 + \lambda_2 d\log f_2 + \lambda_3 d\log f_3 \\ \varphi_- &= d\log \frac{f_1}{f_2} =, \qquad \varphi_+ = d\log \frac{f_3}{f_2}\,. \end{aligned} \tag{3.3.28}$$

The intersection number of φ_- and φ_+ then is given by:

$$\langle \varphi_-, \varphi_+ \rangle_\omega = \int_{M(\mathcal{K})} \iota_\omega(\varphi_-) \wedge \varphi_+ = \frac{1}{\lambda_2}\,. \tag{3.3.29}$$

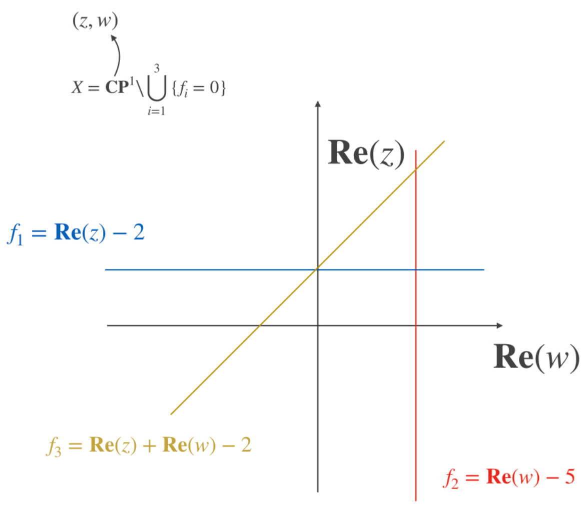

Figure 3.1: Example of intersection number of logarithmic forms in hyperplane geometry

3.3.3 Proof for the $k = 1$ case

As discussed, in order to calculate the integral in (3.3.24) we need to construct a twisted form with compact support as an element of the de Rahm twisted cohomology. The main idea behind the proof is as follows:

1. First, given a twisted form φ with poles at z_i. We remove from its support open neighborhoods U_i centered at given pole z_i.

2. Since we want to preserve the cohomology we amend the removed patches with a function ψ_i for each z_i with the following property:

$$\begin{aligned} &\psi_i\text{: } \textit{is regular at } z_i\,, \\ &\varphi|_{U_i} = \nabla_\omega \psi_i\,. \end{aligned} \tag{3.3.30}$$

3. Now we glue this new function and replace it with the removed patch.

Let us build this structure concretely. Given the twisted form φ with poles at $z = \{z_1, z_2, z_3, ..., z_k\}$ and Riemann surface X. we can always find an open neighborhood U_i centered at each z_i that does not intersect any other neighborhood. Meaning:

$$U_i \subset X, \qquad z_i \in U_i \qquad U_i \cap U_j = \emptyset \quad \forall i \neq j\,. \tag{3.3.31}$$

Then, take $V_i \subset U_i$ and $D_i : U_i \backslash V_i$ and define the following function over the X:

$$h_{z_i}(t) = \begin{cases} h_{z_i}(t) = 1, & t \in V_i\,, \\ 0 \leq h_{z_i}(t), \leq 1 & t \in D_i\,, \\ h_{z_i}(t) = 0, & t \in U_i^c\,, \end{cases} \tag{3.3.32}$$

where U_i^c is the complement of U_i. One can see clearly that function the $h_{z_i}(t)$ patches any function defined over z_i by multiplication. Further, one can also see that function $h_{z_i}(t)$ cannot be pure holomorphic (since a holomorphic function over a bounded disk must be constant). This is the main point, as we mentioned before, the anti-holomorphic parts are added to $\iota_\omega(\varphi)$ through compactifying isomorphism.

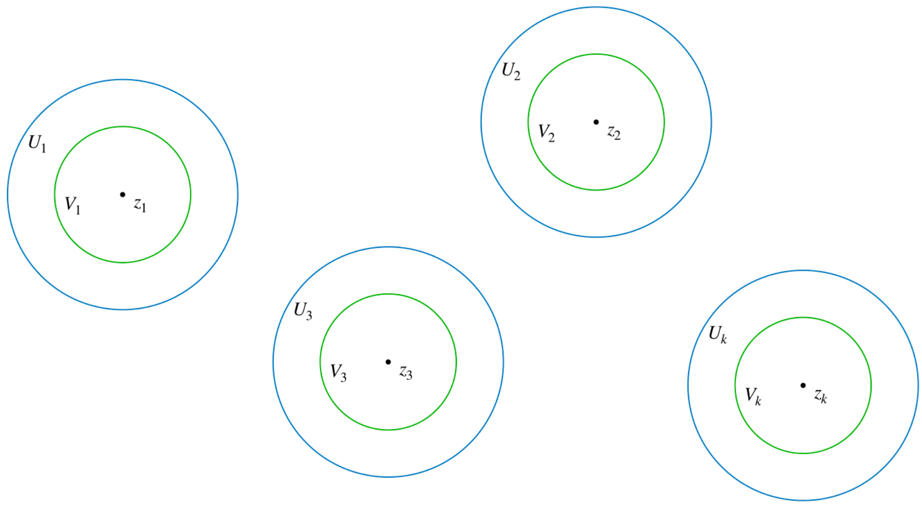

Figure 3.2: The set U_i and V_i around punctures z_i

So we define the compactified version of φ as the following:

$$\iota_\omega(\varphi) = \varphi - \sum_i \nabla_\omega(\psi_i h_{z_i}(t)) \qquad \varphi|_{U_i} = \nabla_\omega \psi_i \,. \tag{3.3.33}$$

First of all, we can see that $\iota_\omega(\varphi)$ is in the same cohomology class as φ since their difference is a total derivative. We can check that indeed $\iota_\omega(\varphi)$ has compact support over X:

$$\begin{aligned} \iota_\omega(\varphi) &= \varphi - \sum_i \nabla_\omega(\psi_i h_{z_i}(t)) = \varphi + \sum_i \Bigg(\nabla_\omega(\psi_i) h_{z_i}(t) + \psi_i \nabla_\omega (h_{z_i}(t)) \Bigg) \\ &= \varphi + \sum_i \Bigg(\varphi h_{z_i}(t) + \psi_i d(h_{z_i}(t)) \Bigg) \\ \iota_\omega(\varphi) &= \underbrace{\varphi + \sum_i \varphi h_{z_i}(t)) +}_{\text{zero on the punctures}} \underbrace{\sum_i \psi_i d(h_{z_i}(t))}_{\text{regular on the punctures}} \quad . \end{aligned} \tag{3.3.34}$$

The last part, which is computational, is to find the functions ψ_i this means that we have to solve the following equation for every patch U_i:

$$\varphi|_{U_i} = \nabla_\omega \psi_i = d\psi_i + \omega \wedge \psi_i \,. \tag{3.3.35}$$

However, the ∇_ω^{-1} is not invertible so we solve this equation by series expansion. We expand the twisted form φ and the function ψ in the local coordinates around z_i:

$$\begin{aligned}\varphi(z) &= \left(\frac{a_{-1}}{z_i} + a_0 + a_1 z_i + ...\right)dz\,,\\ \psi(z) &= \sum_{m=0}^{\infty} c_m z_i^m\,,\\ \omega(z) &= \left(\frac{b_{-1}}{z_i} + b_0 + b_1 z_i + ...\right)dz\,.\end{aligned} \tag{3.3.36}$$

Now, we can plug this back into the equation (3.3.35) and we have:

$$\begin{aligned}&\varphi = d\psi + \omega\wedge\psi\,,\\ &\left(\frac{a_{-1}}{z_i} + a_0 + a_1 z_i + ...\right)dz = \sum_{m=0}^{\infty} m c_m z_i^{m-1}dz + \left(\frac{b_{-1}}{z_i} + b_0 + b_1 z_i + ...\right)\wedge\sum_{m=0}^{\infty} c_m z_i^m dz\\ &\sum_{m=0}^{\infty}\left(m c_m + \sum_{q=-1}^{m-1} a_q c_{m-q-1}\right) z_i^{m-1} = \sum_{m=0}^{\infty} b_{m-1} z_i^{m-1}\,.\end{aligned} \tag{3.3.37}$$

This equation can be solved for c_m and the series will converge in the neighborhood U_i [34]. Therefore, we have the function ψ in terms of the Laurent expansion. We can calculate the intersection number (3.3.24) with the use of Stokes' and the residue theorem:

$$\begin{aligned}\langle\varphi_-,\varphi_+\rangle &= \int_M \iota_\omega(\varphi_-)\wedge\varphi_+ = \int_M\left(\varphi_- - \sum_i \nabla_\omega(\psi_i h_{z_i}(t))\right)\wedge\varphi_+\\ &= -\int_M\left(\sum_i \psi_i d(h_{z_i}(t))\right)\wedge\varphi_+ = -\sum_i\int_{D_i}\left(\psi_i d(h_{z_i}(t))\right)\wedge\varphi_+\\ &= -\sum_i\int_{\partial D_i}\left(\psi_i h_{z_i}(t)\right)\varphi_+ = 2\pi i\sum_i Res_{z=z_i}(\psi_i\varphi_+)\\ &= 2\pi i\sum_i Res_{z=z_i}(\nabla_\omega^{-1}\varphi_-\varphi_+)\,.\end{aligned} \tag{3.3.38}$$

Let us make an important contrast here: In the generic case (3.3.25) we have a k dimensional space. So each z_i corresponds to a complex dimension. The twisted form can have many poles along each dimension and therefore we had a sequence of residues. In the example above we put $k = 1$ (so only one complex dimension) and the last sum runs over residues of the poles in z.

3.3.4 Twisted period relations

We finish this section by discussing the last paring among the homology/cohomology elements that we have omitted so far. In the twisted period case we discussed the pairing between twisted cycles and twisted differentials. Then, for the intersection numbers we used the Poincare duality and paired two twisted differentials. Now, we want to see what is the pairing of two cycles and explain the relationship between all these pairings.

Saddle point approximation

Given a k-twisted cycle $C_{\gamma,\omega} \in H_k(M, \nabla_\omega)$ and a dual twisted cycle $\tilde{C}_{\delta,-\omega} \in H_k(M, \nabla_{-\omega})$ the paring between them is given by $(dim\, H_k) \times (dim\, H_k)$ period matrix:

$$[C_{\gamma,\omega}, \tilde{C}_{\delta,-\omega}]_{ij} = S_{ij}\,. \tag{3.3.39}$$

Using the fact that the only nontrivial cohomology group is given by the complex dimension of M we can use the Poincare polynomial to write the dimension (rank) of $H^k(M, \nabla_\omega)$ in terms of the Euler character for a D complex dimensional space [39]:

$$(-1)^D dim H^D(M, \nabla_\omega) = \chi(M)\,. \tag{3.3.40}$$

One can construct a basis of $\chi(M)^2$ period integrals Π^+_{ab} cf. [40]. Conversely, we have the inverse matrix S^{-1} with two bases $\{C_\gamma\}_{\gamma=1}^{\chi(M)}$ and $\{\tilde{C}_\delta\}_{\delta=1}^{\chi(M)}$ of twisted cycles as the following the intersection matrix:

$$\langle C_\gamma \otimes KN \mid \tilde{C}_\delta \otimes KN^{-1}\rangle = S^{-1}_{ab}\,. \tag{3.3.41}$$

This gives rise to the twisted period relations [34]. Using this matrix element we introduce the last proposition in this section thanks to Mizera [7] which is very useful to expand the intersection numbers.

Saddle point approximation

Let $U \cong V \cong W \cong X$ be four isomorphic complex vector spaces with non-degenerate bilinear pairings denoted by $\langle u, v\rangle$, $\langle u, x\rangle$, $\langle w, x\rangle$, $\langle w, v\rangle$ for $u \in U$, $v \in V$, $w \in W$ and $x \in X$ which are normalized. Then the bilinears between basis vectors $\{u_a\}_{a=1}^{dim\, U} \in U$, $\{v_a\}_{a=1}^{dim\, v} \in V$, $\{w_a\}_{a=1}^{dim\, W} \in W$, $\{x_a\}_{a=1}^{dim\, X} \in X$, are related by:

$$\langle u_a | x_d\rangle = \sum_{b,c=1}^{dim\, U} \langle u_a | v_b\rangle S_{bc} \langle w_c | x_d\rangle\,, \tag{3.3.42}$$

Where S is a $(dim\, U) \times (dim, U)$ matrix with entries $S_{ab} := \langle w_a^\vee | v_b^\vee\rangle$ where the orthonormal basis vectors are denoted by $\{v_a^\vee\}_{a=1}^{dim\, v} \in V$ and the inverse S^{-1} as $S^{-1}_{ab} := \langle w_a | v_b\rangle$.

So we can summarize the parings in the following:

Space	$H^k_{-\omega}$	$H_k^{-\omega}$
H^k_ω	$\langle\varphi_-,\varphi_+\rangle$ *Intersecion number*	$\langle C_{\gamma_i,\omega}\vert\varphi_+\rangle$ *Twisted period*
H_k^ω	$\langle\varphi_-\vert\tilde{C}_{\gamma_i,-\omega}\rangle$ *Twisted period*	$[C_{\gamma,\omega},\tilde{C}_{\delta,-\omega}]_{ij}=S_{ij}$ *Period matrix*

Table 3.1: Different parings of a twisted differential form: First with a dual twisted form and second with a twisted cycle

3.4 Saddle point approximation

The intersection number that we introduced in (3.3.24) is an integral over the space $M(\mathcal{K})$. We further discussed, the case of logarithmic forms, where we can use the *residue* method to calculate the value of the intersection number. However, it might not be always the case that we have the tools to calculate their residues or that the twisted forms are not logarithmic. Therefore, the result that we discussed so far is not going to help us to obtain intersection numbers. It is fortunate that under proper conditions there are extensive mathematical theories to approximate integrals. We can use these methods to calculate intersection numbers without actually calculating the integrals. The method that we are going to discuss in this section is the saddle point approximation. First, we can look at the definition and property of the saddle point approximation:

Saddle point approximation

For a given integral of the form:

$$\int_C I(z)exp\{\lambda g(z)\}dz \sim \left(\frac{2\pi}{\lambda}\right)^{n/2}\frac{1}{\sqrt{\det g''(z_0)_{ij}}}e^{\lambda g(z_0)}\left(I(z_0)+\mathcal{O}(\lambda^{-1})\right), \quad (3.4.43)$$

where C is a fixed contour in the complex z plane and $I(z)$ and $g(z)$ are analytic function in some region $D\subset\mathbf{C}^n$. The second derivative g''_{ij} is the Hessian matrix of the function $g(z)$ and z_0 is the non degenerate saddle point:

$$\begin{aligned} z_0:\ \partial_z g(z) &= 0\,,\\ g''_{ij} &:= \frac{\partial^2 g(z)}{\partial_{z_i}\partial_{z_j}}\,. \end{aligned} \quad (3.4.44)$$

The saddle point approximation is based on the method of steepest decent [41]. More importantly, in order to be able to perform the above expansion we had to make use of the *complex Morse lemma* which states the following:

Complex Morse lemma

Let $g(z)$ be a holomorphic function and z_0 be a *non-degenerate saddle point.* Then near z_0, there exist a local coordinate such that $g(z)$ is exactly quadratic:

$$\begin{aligned} &z_0 : \partial g(z) = 0\,, \qquad \text{and} \ \ g''(z_0) \neq 0\,, \\ &\exists (w_1, w_2, ..., w_n)\,, \qquad \text{such that } g(w) = g(z_0) + \frac{1}{2}\sum_{i=1}^{n} \mu_i w_i^2\,. \end{aligned} \tag{3.4.45}$$

Using the saddle point approximation we can approximate the intersection number given in (3.3.24). The first, thing to notice is that the form of the integral in the intersection number is not of the form given in (3.4.43). In order to bring the integral of the intersection number into the format that we need for our discussion we make use of proposition 3.3.4. We set $U = H^k_{-\omega}$, $X = U = H^k_{\omega}$, $W = H^{\omega}_k$ and $V^{-\omega}_k$ and since we are going to use this approximation for the moduli space of punctured Riemann surface we can use the fact that the Euler character is given by [42]:

$$\chi(\mathcal{M}_{0,n}) = (-1)^{n-3}(n-3)!\,,$$

and set the dimension of the top cohomology to $(n-3)$. Then, we have for the two twisted top forms $\varphi_- = \hat{\varphi}_-(z)dz^{n-3}$ and $\varphi_+ = \hat{\varphi}_+(z)dz^{n-3}$ the following expansion:

$$\langle \varphi_-, \varphi_+ \rangle_\omega = \sum_{i,j} \langle \varphi_- | \tilde{C}_{\gamma_i,-\omega} \rangle S_{ij} \langle C_{\gamma_j,\omega} | \varphi_+ \rangle_\omega\,. \tag{3.4.46}$$

Using the integral formula of the periods in (3.3.20), we have:

$$\begin{aligned} \langle \varphi_-, \varphi_+ \rangle_\omega &= \sum_{i,j} \langle \varphi_- | \tilde{C}_{\gamma_i,-\omega} \rangle S_{ij} \langle C_{\gamma_j,\omega} | \varphi_+ \rangle_\omega \\ &= \sum_i \int_{C_{\gamma_i}} \int_{C_{\gamma_j}} exp\Big\{ \int_{\gamma_i} -\omega \Big\}\ \hat{\varphi}_-(z)\ \ S_{ij}\ \ exp\Big\{ \int_{\gamma_j} \omega \Big\}\ \hat{\varphi}_+(z)\,, \\ \langle \varphi_-, \varphi_+ \rangle_\omega &= \sum_i \left(\int_{C_{\gamma_i}} exp\Big\{ \int_{\gamma_i} \omega \Big\}\ \hat{\varphi}_-(z) \right)\ S_{ij}\ \left(\int_{C_{\gamma_j}} exp\Big\{ - \int_{\gamma_j} \omega \Big\}\ \hat{\varphi}_+(z) \right), \end{aligned} \tag{3.4.47}$$

which is equivalent to the twisted Riemann period relations by Cho and Matsumoto [43]. For an orthonormal basis of the spaces $H^k_{\pm\omega}$ and $H^k_{\pm\overline{\omega}}$ (for example Park-Taylor basis) we can set $S_{ij} = \delta_{ij}$. We are now ready to use the saddle point approximation for the two integrals in the last expression. We have for the twist

$$\omega = \sum_{1 \leq i < j \leq n} \lambda_i \cdot \lambda_j\ d\log(z_i - z_j)\,,$$

the following:

$$
\begin{aligned}
\langle\varphi_-,\varphi_+\rangle_\omega &= \sum_i \left(\int_{C_{\gamma_i}} exp\Big\{\int_{\gamma_i} -\omega\Big\}\ \hat{\varphi}_-(z)\right)\ \left(\int_{C_{\gamma_i}} exp\Big\{\int_{\gamma_i} \omega\Big\}\ \hat{\varphi}_+(z)\right) \\
\lim_{\alpha'\to\infty}\langle\varphi_-,\varphi_+\rangle_\omega &= \lim_{\alpha'\to\infty}\left[\sum_i \left(\int_{C_{\gamma_i}} exp\Big\{\int_{\gamma_i} -\omega\Big\}\ \hat{\varphi}_-(z)\right)\ \left(\int_{C_{\gamma_i}} exp\Big\{\int_{\gamma_i} \omega\Big\}\ \hat{\varphi}_+(z)\right)\right] \\
&= \lim_{\alpha'\to\infty}\sum_i\left[\frac{1}{\sqrt{\det H_{ij}}}\ exp\Big\{\int_{\gamma_i} -\omega\Big\}\ \hat{\varphi}_-(z) + \mathcal{O}(\alpha'^{-1})\right)_{z=z_i} \\
&\quad \left(\frac{1}{\sqrt{\det(-H_{ij})}}\ exp\Big\{\int_{\gamma_i} \omega\Big\}\ \hat{\varphi}_+(z) + \mathcal{O}(\alpha'^{-1})\right)_{z=z_i}\Bigg], \\
\langle\varphi_-,\varphi_+\rangle_\omega &= \sum_i\left(\frac{1}{\sqrt{\det H_{ij}}}\ exp\Big\{\int_{\gamma_i} -\omega\Big\}\ \lim_{\alpha'\to\infty}\hat{\varphi}_-(z)\right)_{z=z_i} \\
&\quad \left(\frac{1}{\sqrt{\det(-H_{ij})}}\ exp\Big\{\int_{\gamma_i} \omega\Big\}\ \lim_{\alpha'\to\infty}\hat{\varphi}_+(z)\right)_{z=z_i}.
\end{aligned} \tag{3.4.48}
$$

Above we have used the following definitions:

$$
\begin{aligned}
H_{ij} &= \frac{\partial^2\int_\gamma\omega}{\partial_{z_i}\partial_{z_j}} \overset{\substack{Morse\\lemma}}{=} \prod_{i=2}^{n-2}\omega_i\,, \\
\omega &= \sum_i \omega_i dz_i\,, \qquad z_i : \Big\{z\in\mathbf{C}^{n-3}\Big|\frac{\partial\int_\gamma\omega}{\partial z_i} = \omega_i = 0\Big\}.
\end{aligned} \tag{3.4.49}
$$

Using this expansion we can formulate the saddle point approximation of the intersection number of twisted top forms:

Saddle point approximation of the intersection number

Using the definition of the Koba-Nielsen factor in terms of the local coefficient we have:

$$
\begin{aligned}
\lim_{\alpha'\to\infty}\langle\varphi_-,\varphi_+\rangle_\omega &= \sum_i\left(\frac{1}{\det H_{ij}}\ KN\ \lim_{\alpha'\to\infty}\hat{\varphi}_-(z)\ KN^{-1}\ \lim_{\alpha'\to\infty}\hat{\varphi}_+(z)\right)_{z=z_i} \\
&= \sum_i\left(\frac{\lim_{\alpha'\to\infty}\hat{\varphi}_-(z)\ \ \lim_{\alpha'\to\infty}\hat{\varphi}_+(z)}{\prod_{i=1}^{n-2}\omega_i}\right)_{z=z_i},
\end{aligned} \tag{3.4.50}
$$

where z_is are solutions to $w_i = 0$. We can make use of the delta distribution and write the sum over saddle points $\omega_i = 0$ as complex integral:

$$\lim_{\alpha' \to \infty} \langle \varphi_-, \varphi_+ \rangle_\omega = \int_M \left(\bigwedge_{i=1}^{n} dz \right) \delta(w) \left(\lim_{\alpha' \to \infty} \hat{\varphi}_-(z) \quad \lim_{\alpha' \to \infty} \hat{\varphi}_+(z) \right). \tag{3.4.51}$$

In the subsequent chapters, we are going to see why the integral formulation of the intersection number's saddle point approximation is useful.

Part II

Physical Preliminaries

Chapter 4

Quantum field theories and amplitudes

4.1 Preface

In this chapter, we are going to give a short introduction to some specific quantum field theories and their amplitudes. Throughout this work we make contact with all of these theories and therefore, we want to explain their matter content and Lagrangian (if available). The amplitude formulation that we are going to use for these theories is not the normal Feynman integral (diagram) amplitudes instead we use the CHY integral representations. Hence, we discuss the CHY integral formula for a given quantum field theory first. Then, one by one we go through different theories and provide their matter content Lagrangian and CHY amplitudes. In the coming chapters, we will try to produce bimetric gravity both through pure string theory amplitudes and intersection numbers. Therefore at the end of this chapter we discuss bimetric gravity in more detail.

4.2 CHY representation

The CHY formulation of the tree level scattering amplitudes is one of the new structural methods to formulate scattering amplitudes in terms of integrals over Riemann surfaces [20]. It has its origin in ambitwistor strings. The ambitwistor string has been defined as a chiral string theory. By construction, it only contains left–moving world–sheet fields, has no massive states and its tree level amplitudes reproduce the CHY formulae for massless scattering [21]. Furthermore, it has been shown that its action can be constructed using the tensionless string limit ($\alpha' \to \infty$) [44]. This correspondence works at both classical and quantum levels. The CHY formalism provides an integral formula for scattering amplitudes localized over solutions of the scattering equations. These equations are given by:

$$f_a = \sum_{\substack{b=1 \\ b\neq a}} \frac{p_a \cdot p_b}{\sigma_a - \sigma_b}\,, \tag{4.2.1}$$

Here p_as are *on-shell, massless* momenta associated to a given scattering [20, 45, 46]. Therefore, a tree level scattering amplitude of massless particles can be written as an integral over moduli space of punctured Riemann surface with a theory-dependent integrand which can always be factorized into left and right functions. It has the following form known as *CHY representation*:

CHY integral representation

$$\begin{aligned}
\mathcal{A}_{CHY}(n) &= \int\limits_{\mathcal{M}_{0,n}} d\mu_n \prod_{a=1}^{n}{}' \delta(f_a)\ \mathcal{I}_L(p,\varepsilon,\sigma)\mathcal{I}_R(p,\varepsilon,\sigma)\,, \\
f_a &\equiv \sum_{\substack{b=1\\ b\neq a}} \frac{p_a\cdot p_b}{\sigma_a-\sigma_b}\quad , \quad a=1,\ldots,n\,, \\
d\mu_n &= \frac{d^n\sigma_a}{SL(2,\mathbf{C})}\,.
\end{aligned} \tag{4.2.2}$$

The prime above the product means that it should be taken after quotienting the $SL(2,\mathbf{C})$ and fixing three positions (cf. [20]). We should also note that in order to distinguish the string world sheet coordinates and the CHY formulation we are using σ_as instead of zs for the complex variables.

The integrands $\mathcal{I}_L(p,\varepsilon,\sigma)$ and $\mathcal{I}_R(p,\varepsilon,\sigma)$ are unique to the underlying theory. For example, for the two basic theories of general relativity and Yang-Mills, we have:

CHY for GR and YM

The CHY integral for the amplitude of n pure gravitons in GR

$$\mathcal{A}^{GR}_{CHY}(n) = \int\limits_{\mathcal{M}_{0,n}} d\mu_n \prod_{a=1}^{n}{}' \delta(f_a)\mathrm{Pf}'\psi_n\mathrm{Pf}'\psi_n\,, \tag{4.2.3}$$

and pure n gluons in YM is given by

$$\mathcal{A}^{YM}_{CHY}(n) = \int\limits_{\mathcal{M}_{0,n}} d\mu_n \prod_{a=1}^{n}{}' \delta(f_a)\mathcal{C}_n\mathrm{Pf}'\psi_n\,. \tag{4.2.4}$$

We are going to describe the matrix ψ_n in the following section.

We gather the known CHY formulae that we are using in the following two tables. The first one represents all the spin-0 and spin-1 theories and the second will be dedicated to spin-2 theories.

4.2.1 Spin-one and zero theories

First, for the spin-1/scalar fields, we have:

Theory	CHY representation	Amplitude
bi-adjoint scalar	$\mathcal{C}_n\mathcal{C}_n$	n color scalar
Einstein	$\mathrm{Pf}'\psi_n\mathrm{Pf}'\psi_n$	n gravitons
Yang-Mills	$\mathcal{C}_n\mathrm{Pf}'\psi_n$	n gluons
Einstein Yang-Mills	$\mathrm{Pf}'\Psi_{S_r}\mathrm{Pf}'\psi_n$	r graviton n gluon
special Galilean (sGal)	$(\mathrm{Pf}'A_n)^4$	n higher derivative scalars
NLSM	$\mathcal{C}_n(\mathrm{Pf}'A_n)^2$	n scalars
Born–Infeld (BI)	$(\mathrm{Pf}'A_n)^2\,\mathrm{Pf}'\psi_n$	n spin 1
Einstein–Maxwell (EM)	$\mathrm{Pf}X_n\,\mathrm{Pf}'\Psi_{S_{r:n}}\,\mathrm{Pf}'\psi_{n+r}$	r gravitons n photons
Dirac Born–Infeld (DBI)	$\mathrm{Pf}X_n\,\mathrm{Pf}'\Psi_{S_{r:n}}\,(\mathrm{Pf}'A_{n+r})^2$	r gluons n color scalars
Yang-Mills scalar (YMS)	$\mathrm{Pf}X_n\,\mathrm{Pf}'\Psi_{S_{r:n}}\,\mathcal{C}_{n+r}$	r gluons n color scalars
Generalized Yang-Mills Scalar (gen.YMS)	$\mathcal{C}_n\mathrm{Pf}\Psi_{S_r}\mathcal{C}_{n+r}$	r gluons n color scalars
Extended Dirac Born-Infeld (ext.DBI)	$\mathcal{C}_n\mathrm{Pf}\Psi_{S_r}(\mathrm{Pf}'A_{n+r})^2$	r gluons n higher derivative scalars
$(DF)^2$	$\mathcal{C}_n\,W_{\underbrace{11...1}_{n}}$	n gluons (higher derivative)
$(DF)^2$–Photon	$(\mathrm{Pf}'A_n)^2\,W_{\underbrace{11...1}_{n}}$	n higher derivative photons

Table 4.1: Known spin-zero (scalar field) and spin-1 (gauge field) theories and their CHY representations.

4.2.2 Spin-2 theories

for the spin 2 theories we have the following:

Theory	CHY representation	Amplitude
GR	$\mathrm{Pf}'\psi_n \mathrm{Pf}'\psi_n$	n gravitons
Conformal Gravity (CG)	$\mathrm{Pf}'\psi_n \, W_{\underbrace{11\ldots1}_{n}}$	n gravitons (Weyl)
$(Weyl)^3$ *or* R^3	$\left(W_{\underbrace{11\ldots1}_{n}}\right)^2$	n gravitons (higher derivative)

Table 4.2: Known CHY integrands for spin 2 theories

In table 4.1 the $\mathrm{Pf}L$ denotes the pfaffian of matrix L and the $\mathrm{Pf}'L$ denotes the reduced pfaffian of L meaning removals of two rows and two columns (the result of the integrals does not depend on the choice of these rows and columns) as for the matrix L_n we have:

$$\mathrm{Pf}'L_n \equiv \frac{(-1)^{k+l}}{z_k - z_l} \, \mathrm{Pf}L_{[kl]} \,, \tag{4.2.5}$$

with the index k, l denoting removals of rows k, l and columns k, l. The two complex factors $\mathcal{C}_n$, i.e. Park-Taylor form, and X are given by:

$$\begin{aligned} \mathcal{C}_n &= \frac{1}{(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3)....(\sigma_n - \sigma_1)} \,, \\ X_{ab} &= \begin{cases} \frac{1}{\sigma_a - \sigma_b} & a \neq b \,, \\ 0 & a = b \,. \end{cases} \end{aligned} \tag{4.2.6}$$

The matrices in the CHY representation are defined in the same way as [20]:

$$\psi_n = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix} \,, \tag{4.2.7}$$

with the three $n \times n$ matrices:

$$A_{ij} = \begin{cases} 0, & i = j \\ \frac{p_i p_j}{z_i - z_j}, & i \neq j \end{cases} \,, \quad C_{ij} = \begin{cases} -\sum\limits_{k \neq i}^{n} \frac{\varepsilon_i p_k}{z_i - z_k}, & i = j \\ \frac{\varepsilon_i p_j}{z_i - z_j}, & i \neq j \end{cases} \,, \quad B_{ij} = \begin{cases} 0, & i = j \,, \\ \frac{\varepsilon_i \varepsilon_j}{z_i - z_j}, & i \neq j \,, \end{cases} \tag{4.2.8}$$

Furthermore, by using $\mathrm{Pf}M^2 = \det M$ we have the following relation:

$$\frac{\det A_{[kl]}}{z_{kl}^2} = (\mathrm{Pf}' A_n)^2 \,. \tag{4.2.9}$$

where A_n is the submatrix of ψ_n The two extended $\Psi_{S_{r:n}}$ and Ψ_{S_r} are given by:

$$\Psi_{S_{r:n}} = \begin{pmatrix} A_{ab} & A_{aj} & (-C^t)_{ab} \\ A_{ib} & A_{ij} & (-C^t)_{aj} \\ C_{ab} & C_{aj} & B_{ab} \end{pmatrix} . \tag{4.2.10}$$

The submatrices have the same definition as (4.2.7) with the index (ab) and (ij) are related to gravitons and photon legs respectively. For the Ψ_{S_r} we have:

$$\Psi_{S_r} = \begin{pmatrix} A_r & -C^T_{n+r} \\ C_{n+r} & B_r \end{pmatrix} , \tag{4.2.11}$$

with the three $r \times r$–submatrices:

$$A_{ij} = \begin{cases} 0 \, , & i = j \\ \frac{p_i p_j}{\sigma_i - \sigma_j} \, , & i \neq j \end{cases} , \quad C_{ij} = \begin{cases} -\sum\limits_{k\neq i}^{n+r} \frac{\varepsilon_i \cdot p_k}{\sigma_i - \sigma_k} \, , & i = j \\ \frac{\varepsilon_i p_j}{\sigma_i - \sigma_j} \, , & i \neq j \end{cases} , \quad B_{ij} = \begin{cases} 0 \, , & i = j \\ \frac{\varepsilon_i \cdot \varepsilon_j}{\sigma_i - \sigma_j} \, , & i \neq j \end{cases} \tag{4.2.12}$$

The function W_L is used e.g. in [47] and the set L of indices of the function W_L refer to a product of Lam–Yao cycles:

$$W_{\underbrace{11\ldots1}_{n}} = \prod_{i=1}^{n} C_{ii} = \prod_{i=1}^{n} \left(\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\varepsilon_i \cdot p_j}{z_{ij}} \right) . \tag{4.2.13}$$

In the next section, we are going to give a summary of these theories.

4.3 Effective actions and their CHY amplitudes

In tables 4.1 and 4.2 we introduce many different theories and their CHY amplitude representation. Now want to explore their Lagrangians, matter content, and amplitudes. In fact, we need to emphasize that for some of these theories, the exact form of their Lagrangian is not known or different names are used by various authors. Note that for the theories that do not include gravity we set the flat metric and therefore the invariant measure is $d^d x$ in d-dimension.

4.3.1 Einstein Yang-Mills (EYM)

As the name appropriately suggests Einstein Yang-Mills theory is the coupling of GR to a non-abelian $SU(N)$ Yang-Mills. Therefore it has the following Lagrangian:

$$\frac{1}{\sqrt{-g}} \mathcal{L}_{EYM} = -\frac{1}{4} F^{\mu\nu\, a} F_{\mu\nu\, a} + R \, . \tag{4.3.14}$$

The amplitude of n gluons and r gravitons in EYM theory is given by the CHY integral formulation:

$$\mathcal{A}_{EYM}(n;r) = \int_{\mathcal{M}_{0,n+r}} d\mu_{n+r} \prod_{k=2}^{n+r-2} \delta(f_k)\, \mathcal{C}_n(\mathrm{Pf}'\psi_{S_r})\mathrm{Pf}\psi_{n+r}\,. \tag{4.3.15}$$

We can see upon putting $n = 0$ we have pure GR (graviton) amplitude (4.2.3) and by putting $r = 0$ we have pure Yang-Mills (gluon) amplitude (4.2.4)

4.3.2 Special Galilean (sGal)

Galilean theories in d dimension are defined as scalar effective field theories involving higher derivative interactions in the potential [48, 49] namely

$$\mathcal{L}_{Gal} = \frac{1}{2}\,\partial^\mu\phi\partial_\mu\phi + \sum_{n=3}^{d-1} g_n\,(\partial\phi)^2(\partial_\mu\partial^\mu\phi)^{n-2}\,, \tag{4.3.16}$$

or variants thereof [50]. The Galilean theory is called special if it features a $\mathbf{Z}_2$ symmetry. The amplitude of n scalars in this theory [50] can be written in terms of CHY formulation in the following way:

$$\mathcal{A}_{sGal}(n) = \int_{\mathcal{M}_{0,n}} d\mu_n \prod_{k=2}^{n-2} \delta(f_k)\; (\mathrm{Pf}'A_n)^4\;. \tag{4.3.17}$$

4.3.3 Non–linear sigma model (NLSM)

The non–linear sigma model (NLSM) is defined as a theory with scalars Φ together with an embedding onto a manifold M with a non–linear interaction potential $V(\Phi)$. The theory for N scalars fields Φ satisfying a $U(N)$ symmetry can be written in terms of Cayley parametrization [50]:

$$\mathcal{L}_{NLSM} = \frac{1}{8\lambda^2}\mathrm{Tr}\left\{\, \partial_\mu U(\Phi)\partial^\mu U(\Phi)\,\right\}\,, \tag{4.3.18}$$

with

$$U(\Phi) := (\mathbf{1}+\lambda\Phi)\,(\mathbf{1}-\lambda\Phi)^{-1}\,. \tag{4.3.19}$$

The field Φ may be written in the adjoint representation as $\Phi = \phi^I T^I$ with T^I being the generators of $U(N)$. Upon expanding $U(\Phi)$ in terms of Φ yields the usual scalar kinetic term. The scattering amplitude of this theory involving n scalars [50] can be given in terms of the CHY formulation as:

$$\mathcal{A}_{NLSM}(n) = \int_{\mathcal{M}_{0,n}} d\mu_n \prod_{k=2}^{n-2} \delta(f_k)\; \mathcal{C}_n\,(\mathrm{Pf}'A_n)^2\,. \tag{4.3.20}$$

4.3.4 Born–Infeld theory (BI)

The Born–Infeld theory is the non–linear generalization of Maxwell theory [51]. The Lagrangian for this theory in $d = 4$ is given by the non–linear interaction

$$\mathcal{L}_{BI} = \ell^{-2}\left(\sqrt{-\det_4(\eta_{\mu\nu} + \ell F_{\mu\nu})} - 1\right), \tag{4.3.21}$$

where the scale ℓ can be related to the inverse string tension α' as $\ell = 2\pi\alpha'$. The n–point amplitude of this theory [50] can be expressed as CHY representation in terms of the following localization integral:

$$\mathcal{A}_{BI}(n) = \int_{\mathcal{M}_{0,n}} d\mu_n \prod_{k=2}^{n-2} \delta(f_k)\ \mathrm{Pf}'\psi_n\ (\mathrm{Pf}'A_n)^2 . \tag{4.3.22}$$

4.3.5 Generalized Yang–Mills Scalar (gen.YMS)

Generalized Yang-Mills Scalar (gen.YMS) is described by YM gauge theory coupled to scalars $\phi^{a\tilde{a}}$ with both one color a and one flavor index $\tilde{a}$ [50] i.e.:

$$\mathcal{L}_{gen.YMS} = -\frac{1}{4}F_a^{\mu\nu}F^{a\mu\nu} - \frac{1}{2}(D_\mu\phi^{a\tilde{a}})^2 + \lambda\ f^{abc}f^{\tilde{a}\tilde{b}\tilde{c}}\ \phi^{a\tilde{a}}\phi^{b\tilde{b}}\phi^{c\tilde{c}} . \tag{4.3.23}$$

If the second set of labels $\tilde{a}$ also represents color indices we have two color groups and obtain YM plus a cubic bi-adjoint scalar theory $YM + \phi^3$. Furthermore, without the three-point interaction $\lambda \to 0$ in the Lagrangian (4.3.23) we are obtaining Yang-Mills Scalar (YMS) theory. As in the definition of (4.3.23) the group structure of the theory at hand is extended by a flavor group for multiple scalars $\phi^{a\tilde{a}}$. However, here we shall consider only one scalar in the adjoint representation of the gauge group with a single trace for the flavor group. As proposed above, we can construct the CHY representation of the gen.YMS amplitude involving r gluons and n scalars:

$$\mathcal{A}_{gen.YMS}(n;r) = \int_{\mathcal{M}_{0,n+r}} d\mu_{n+r} \prod_{k=2}^{n+r-2} \delta(f_k)\ \mathcal{C}_n\ \mathrm{Pf}\Psi_{S_r}\ \mathcal{C}_{n+r} . \tag{4.3.24}$$

4.3.6 Extended Dirac Born–Infeld

There are two extensions of the Born–Infeld (BI) theory, which we will discuss throughout this work. Firstly, there is the Dirac Born–Infeld (DBI) theory. This theory is defined as a scalar extension of the BI theory (4.3.21) with n scalars and the Lagrangian is given by [50]:

$$\mathcal{L}_{DBI} = \ell^{-2}\left(\sqrt{-\det_4(\eta_{\mu\nu} - \ell^2\partial_\mu\phi^I\partial_\nu\phi^I + \ell F_{\mu\nu})} - 1\right). \tag{4.3.25}$$

Furthermore, one can extend this theory by generalizing the scalar kinetic term by the function $U(\Phi)$ given in (4.3.19) leading to the extended DBI theory (ext.DBI) with the corresponding Lagrangian

$$\mathcal{L}_{ext.DBI} = \ell^{-2}\left(\sqrt{-\det_4(\eta_{\mu\nu} - \frac{\ell^2}{4\lambda^2}\mathrm{Tr}(\partial_\mu U^\dagger \partial_\nu U) - \ell^2 W_{\mu\nu} + \ell F_{\mu\nu})} - 1\right), \tag{4.3.26}$$

and:

$$W_{\mu\nu} = \sum_{m=1}^{\infty}\sum_{k=0}^{m-1} \frac{2(m-k)}{2m-1}\,\lambda^{2m+1}\,\mathrm{Tr}(\partial_{[\mu}\Phi\;\Phi^{2k}\partial_{\nu]}\Phi\;\Phi^{2(m-k)-1})\,. \tag{4.3.27}$$

For $\lambda \to 0$ the expression (4.3.26) yields the DBI action (4.3.25), while we recover NSLM for $\ell \to 0$, respectively. The amplitude for ext.DBI involving r gluons and n scalars can be constructed as

$$\mathcal{A}_{ext.DBI}(n;r) = \int\limits_{\mathcal{M}_{0,n+r}} d\mu_{n+r}\prod_{k=2}^{n+r-2}\delta(f_k)\;\mathcal{C}_n\;\mathrm{Pf}\Psi_{S_r}\;(\mathrm{Pf}'A_{n+r})^2\,, \tag{4.3.28}$$

which is the CHY amplitude given in [50]. The double copy structure of BI, DBI, and ext.DBI theories will be discussed in chapter 7. Finally, let us review some of the basics of the vector theories appearing in Table 4.1.

4.3.7 Higher derivative gauge theory $(DF)^2$

In the literature, there is a variety of higher derivative gauge theories denoted by $(DF)^2$. Here we use the following definition from [52]:

$$\mathcal{L}_{(DF)^2} = \frac{1}{2}(D_\mu F^{a\,\mu\nu})^2 - \frac{g}{3}\,F^3 + \frac{1}{2}(D_\mu\varphi^a)^2 + \frac{g}{2}\,C^{\alpha\,ab}\varphi^\alpha F^a_{\mu\nu}F^{b\,\mu\nu} + \frac{g}{3!}\,d^{\alpha\beta\gamma}\varphi^\alpha\varphi^\beta\varphi^\gamma\,. \tag{4.3.29}$$

In particular, we have the higher order cubic gauge interaction and gauge covariant derivative

$$\begin{aligned} F^3 &= f^{abc}F^{a\,\mu}_{\nu}F^{b\,\nu}_{\gamma}F^{c\,\gamma}_{\mu}\,,\\ D_\mu\varphi^\alpha &= \partial_\mu\varphi^\alpha - ig(T^a_R)^{\alpha\beta}A^a_\mu\,\varphi^\beta\,, \end{aligned} \tag{4.3.30}$$

respectively, with φ in a real representation R and real generators of the gauge group $(T^a_R)^{\alpha\beta}$. This theory has six propagating degrees of freedom. One accounting for the "auxiliary" scalar field φ, which is not considered in the scattering amplitude under consideration. In addition, we have five degrees of freedom from the higher derivative gauge theory which includes a negative norm state (i.e. ghost). The amplitude of this theory, which describes the scattering of n higher derivative spin–one vectors, is given by the CHY formulation:

$$\mathcal{A}_{(DF)^2}(n) = \int\limits_{\mathcal{M}_{0,n}} d\mu_n\prod_{k=2}^{n-2}\,\delta(f_k)\;\mathcal{C}_n\;W_{\underbrace{11\ldots1}_{n}}\,. \tag{4.3.31}$$

4.3.8 $(DF)^2$ – Photon

Next, the $(DF)^2$ – Photon theory involves a higher derivative $U(1)$ gauge theory in the Einstein gravity background. The relevant Lagrangian is given by:

$$\begin{aligned}\frac{1}{\sqrt{-g}}\mathcal{L}_{(DF)^2-\text{Photon}} =& \frac{1}{2\kappa^2}R + \frac{1}{4}(D_\mu F^{\alpha\beta})^2 + \frac{1}{8}RF^2 - \frac{1}{6}\kappa^2 D_\mu F_{\alpha\beta}D^\alpha F^{\mu\gamma}F_{\gamma\delta}F^{\delta\beta} \\ &+ \frac{1}{48}\kappa^2(D_\mu F^{\alpha\beta})^2F^2 + \mathcal{O}(\kappa^4)\,.\end{aligned} \tag{4.3.32}$$

The amplitude $\mathcal{A}_{(DF)^2-\text{Photon}}(n)$ involving n photons can be computed by [53]:

$$\mathcal{A}_{(DF)^2-\text{Photon}}(n) = \int\limits_{\mathcal{M}_{0,n}} d\mu_n \prod_{k=2}^{n-2}\delta(f_k)\; W_{\underbrace{11\ldots1}_{n}}\,(\text{Pf}'A_n)^2\,, \tag{4.3.33}$$

In (4.3.33) we have the submatrix A_n defined in (4.2.8) and the function $W_{\underbrace{11\ldots1}_{n}}$ is given in equation (6.3.39). We can now turn into spin 2 theories in table 4.2 and provide a brief review of their properties and features.

4.3.9 Conformal Gravity

We start with conformal gravity (CG). This name is associated with different theories. The simplest definition corresponds to the (Weyl)^2 action (pure (Weyl)^2 conformal gravity), i.e.

$$\mathcal{L}_{CG} = \kappa_W^{-2}\,\sqrt{-g}\,\left(W_{\mu\nu\alpha\beta}\right)^2\,, \tag{4.3.34}$$

with the coupling constant κ_W and $W_{\mu\nu\alpha\beta}$ the Weyl tensor

$$W_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{1}{2}\left(R_{\alpha\gamma}g_{\beta\delta} - R_{\alpha\delta}g_{\beta\gamma} + R_{\beta\delta}g_{\alpha\gamma} - R_{\beta\gamma}g_{\alpha\delta}\right) + \frac{1}{6}\,R(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma})\,, \tag{4.3.35}$$

which has the same symmetry properties (w.r.t.its indices) as the Riemann tensor $R_{\alpha\beta\gamma\delta}$. In addition to the square of the Riemann tensor the square of the Weyl tensor is the second independent quadratic curvature invariant. We have the following relation:

$$W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} = R_{GB} + 2\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}\,R^2\right)\,, \tag{4.3.36}$$

with the Gauss-Bonnet term

$$R_{GB} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4\,R_{\mu\nu}R^{\mu\nu} + R^2\,, \tag{4.3.37}$$

which for $d = 4$ reduces to a topological surface term. The latter can be added to the action without changing the (classical) theory in a spacetime which is asymptotically

Minkowski. Therefore, in $d=4$ the Lagrangian (4.3.34) can be written in terms of the Riemann scalar R and Ricci tensor $R_{\mu\nu}$ as:

$$\mathcal{L}_{CG} = \kappa_W^{-2}\,\sqrt{-g}\,\left(\frac{1}{3}R^2 - R^{\mu\nu}R_{\mu\nu}\right)\,. \tag{4.3.38}$$

As stated in the table 4.2 the scattering amplitude $\mathcal{A}_{CG}(n)$ of this theory involving n spin-2 particles can be represented as the following CHY representation [53]

$$\mathcal{A}_{CG}(n) = \int\limits_{\mathcal{M}_{0,n}} d\mu_n \prod_{k=2}^{n-2}\delta(f_k)\ \mathrm{Pf}'\psi_n\ W_{\underbrace{11\ldots1}_{n}}\,, \tag{4.3.39}$$

Conformal gravity theory[1] propagates six degrees of freedom (packaged into the metric $g_{\mu\nu}$) which contain a massless spin–two (two degrees of freedom), a massless spin–one (two dof.), and a massless ghost spin–two fields (two degrees) [55]. Conformal gravity exhibits the double copy structure [52]:

$$\text{Conformal Gravity} = (DF)^2 \otimes YM\,. \tag{4.3.40}$$

Note that both factors $(DF)^2$ and YM have vectors $\tilde{A}_\mu$ and A_ν, respectively, which tensor according to

$$\tilde{A}_\mu \otimes A_\nu = g_{\mu\nu} \oplus B_{\mu\nu} \otimes \phi\,, \tag{4.3.41}$$

and giving rise to a graviton $g_{\mu\nu}$, an antisymmetric two–form $B_{\mu\nu}$ and a dilaton field ϕ. Looking at the double copy (4.3.39) we evidence the origin of the massless spin-1 degrees of freedom as descending from the YM theory and the massless spin-2 and ghost spin–two fields are stemming from $(DF)^2$. In chapter 7 we will discuss this double copy in terms of intersection numbers

4.3.10 Einstein–Weyl gravity

Einstein-Weyl gravity is a modification of GR by adding the square of the Weyl tensor [55, 56]:

$$\mathcal{L}_{EW} = \sqrt{-g}\,\left(m^2\,R + \kappa_W^{-2}\,W^2_{\mu\nu\alpha\beta}\right)\,. \tag{4.3.42}$$

Above, the parameter m relates to the string scale $m^2 \sim \alpha'^{-1}$ (likewise to the Planck scale $m \sim M_{\text{Planck}}$). In fact, the mass parameter m interpolates between two-derivative and four-derivative theories. In this way, by taking the limit $\alpha' \to \infty$ we reproduce pure $(\text{Weyl})^2$ conformal gravity. Similar to equation (6.3.38) we have:

$$\lim_{\alpha'\to\infty}\mathcal{L}_{EW}(\alpha') \simeq \mathcal{L}_W\,, \tag{4.3.43}$$

[1] We should emphasize here that although this therory has been hinted to be related to conformal gravity (the naming is clear evidence) it has not been shown to exactly correspond to the conformal gravity in the form of (4.3.38). What has been shown for 4-point amplitude is that it corresponds to Berkovits-Witten super conformal gravity [54]. However, looking at the bosonic part of the theory it will correspond to Weyl+axion thoery[52].

On the other hand, for $\alpha' \to 0$ we end up at (non–pure) Einstein gravity. It can be shown, that the EW action (4.3.42) describes both massless and massive spin-2 states up to total derivatives. EW gravity has seven degrees of freedom accounting for two degrees of freedom from the standard massless spin–two graviton and an additional five (ghost) dof. for the massive spin–two field all packaged inside $g_{\mu\nu}$. In fact, starting from an action for a particular bimetric gravity with the two spin–two fields $g_{\mu\nu}$ and $f_{\mu\nu}$ with a mass term for the latter and eliminating $f_{\mu\nu}$ through its equations of motion yields (4.3.42) [57]. We shall return to these properties in subsection 7.4.3.

4.3.11 $Weyl^3$ or R^3 gravity

Another spin-2 higher derivative theory can be constructed through the six derivative terms like $Weyl^3$ or R^3. Some speculation about the existence of $Weyl^3$ is made in [58], where arguments on dimensional analysis are presented leading to the following CHY representation:

$$\mathcal{A}_{Weyl^3}(n) = \int\limits_{\mathcal{M}_{0,n}} d\mu_n \prod_{k=2}^{n-2} \delta(f_k)\ (W_{\underbrace{11\ldots1}_{n}})^2\,. \tag{4.3.44}$$

4.4 Bimetric and higher derivative gravity

4.4.1 Bimetric theory

Introduction

Here we are going to give a quick overview of bimetric theory, we are going to start by defining the theory and going through the construction of Lagrangian and then we are going to discuss different limits of this theory to other known theories (GR and massive gravity). Finally, we are going to look at the perturbative expansion of the bimetric action up to cubic order which we will use to compare with effective actions obtained from string amplitudes.

The field content of the bimetric theory consists of two spin-2 tensor fields i.e. a metric $g_{\mu\nu}$ and $f_{\mu\nu}$ field, both are dynamical [9, 59]. The task at hand is to construct a Lagrangian out of these two fields such that it satisfies the symmetries i.e. two sets of Lorenz symmetry of each field (which we treat as gauge symmetries) and also the diagonal subgroup of two diffeomorphisms associated with the two metrics. So schematically we have

$$\mathcal{L}_{BG} = \mathcal{L}^g_{kinetic} + \mathcal{L}^f_{kinetic} + V(f, g)\,. \tag{4.4.45}$$

To write the correct action we have to be aware of several issues: The introduction of the second tensor field $f_{\mu\nu}$ and its interactions with $g_{\mu\nu}$ can cause the breaking of the diffeomorphism invariance and results in an increased number of propagating degrees of freedom from 2 to $2+4 = 6$. Further, these interaction terms without proper constraint will create a ghost-like scalar degree of freedom known as Boulware-Deser ghosts. Therefore, we need careful analysis of the Hamiltonian and imposing Dirac constraints that would give us the necessary conditions on the potential $V(f,g)$ in order to have a healthy ghost free theory for the detail of these constraints and calculations can be found in [18, 59].

4.4.2 Bimetric action

Considering all that mentioned before we can now write the action for the massive bimetric and discuss different aspects of it. The ghost-free action for Hassan-Rosen bimetric theory is given by [9]

$$S_{HR} = m_g^2 \int \sqrt{g} dx^4 R(g) + m_f^2 \int \sqrt{f} dx^4 R(f) - 2m^4 \int \sqrt{g} \sum_{n=0}^{4} \beta_n e_n(g^{-1}f) \,, \quad (4.4.46)$$

where m_g and m_f are Planck masses associated with each of the two metrics and $m = \sqrt{m_g m_f}$. The $e_n(S)$ is defined as:

$$e_n(S) = \frac{1}{n!(4-n)!} \epsilon^{\mu_1\mu_2\ldots\mu_n\lambda_{n+1}\ldots\lambda_4} \epsilon_{\nu_1\nu_2\ldots\nu_n\lambda_{n+1}\ldots\lambda_4} S^{\nu_1}_{\mu_1} \ldots S^{\nu_n}_{\mu_n} \,. \quad (4.4.47)$$

Here we defined $\beta_n = b_n \frac{n!(4-n)!}{2}$. b_ns are parameters of the potential which measure interaction strengths. However, out of five of them (i.e. 0 to 4), two are not contributing since e_0 and e_4 are proportional to the cosmological constants of the $g_{\mu\nu}$ and $f_{\mu\nu}$ prospectively. The equations of motion for both fields are Einstein-type equations. We have:

$$\begin{aligned} R_{\mu\nu}(g) - \frac{1}{2} R(g) g^{\mu\nu} + \frac{m^4}{m_g^2} V^g_{\mu\nu}(g,f,\beta_n) = 0 \,, \\ R_{\mu\nu}(f) - \frac{1}{2} R(f) f^{\mu\nu} + \frac{m^4}{m_f^2} V^f_{\mu\nu}(g,f,\beta_n) = 0 \,, \end{aligned} \quad (4.4.48)$$

where V^g and V^f are defined as:

$$\begin{aligned} V^g_{\mu\nu}(g,f,\beta_n) &= g_{\mu\rho} \sum_{n=0}^{3} (-1)^n \beta_n (Y_{(n)})^\rho_\mu(S) \,, \\ V^f_{\mu\nu}(g,f,\beta_n) &= f_{\mu\rho} \sum_{n=0}^{3} (-1)^n \beta_{4-n} (Y_{(n)})^\rho_\mu(S^{-1}) \,, \\ (Y_{(n)})^\rho_\mu(S) &= \sum_{n=0}^{4} e_n(S)(S^{n-k})^\rho_\nu \,. \end{aligned} \quad (4.4.49)$$

The potential term in the Lagrangian (4.4.46) breaks the two diffeomorphisms associated with each of the spin 2 fields (although each Ricci scalar in (4.4.46) is diffeo-invariant). However, the action is invariant under the diagonal subgroup of the two diffeomorphisms (coordinate) transformations which can be characterized by

$$\begin{aligned} \delta X^\mu &= \xi^\mu \,, \\ \delta_\xi g_{\mu\nu} &= -2 g_{\rho(\mu} \nabla_{\nu)} \xi^\rho \,, \\ \delta_\xi f_{\mu\nu} &= -2 f_{\rho(\mu} \tilde{\nabla}_{\nu)} \xi^\rho \,. \end{aligned} \quad (4.4.50)$$

Here the $\tilde{\nabla}$ is defined as the covariant derivative compatible with the $f_{\mu\nu}$ tensor. Therefore, we can count degrees of freedom of 2 symmetric spin 2 fields which each have 10 degrees of freedom. By gauge fixing we can remove 4 degrees of freedom from each of them and end up with 12 dof in total. Next, we should take into account the Bianchi identity which can take the form $\nabla^\mu V^g_\mu = 0$ or $\tilde{\nabla}^\mu V^f_\mu = 0$. Using this we can remove 4 more from the remaining degrees of freedom which brings our counting to $12 - 4 = 8$ dof. This is consistent with the expectation of having 2 propagating degrees of freedom from massless and 5 from massive spin 2 fields. However, here we have an extra degree of freedom, which gives rise to a Boulware-Deser ghost. This means that we need more constraints to eliminate this ghost. The corresponding constraint which removes the canonical momentum of the ghost mode is shown to exist [59].

4.4.3 Coupling to matter and GR limit

The only known matter fields which can be coupled to the bimetric Lagrangian without introducing ghosts are:

$$S_m = \int \sqrt{g} dx^4 \mathcal{L}_m(g, \Phi_g) + \int \sqrt{f} \tilde{\mathcal{L}}_m(f, \Phi_f)\,, \tag{4.4.51}$$

where $\mathcal{L}_m$ and $\tilde{\mathcal{L}}_m$ are standard minimally coupled matter Lagrangians as in GR and stand for sets of matter fields of any kind. Also, it has been shown [60, 61, 62] that it is not possible to couple the same (dynamical) matter field to both metrics using minimal couplings since it will reintroduce the Boulware-Deser ghost.

In the equations of motion, the matter couplings enter in the form of stress-energy tensors

$$T^g_{\mu\nu} = -\frac{1}{\sqrt{g}} \frac{\delta(\sqrt{g}\mathcal{L}_m(g, \Phi_g))}{\delta g^{\mu\nu}}, \quad T^f_{\mu\nu} = -\frac{1}{\sqrt{f}} \frac{\delta(\sqrt{f}\tilde{\mathcal{L}}_m(f, \Phi_f))}{\delta f^{\mu\nu}}\,. \tag{4.4.52}$$

So we get the modified bimetric equations in presence of matter couplings as

$$\begin{aligned} G_{\mu\nu}(g) + \frac{m^4}{m_g^2} V^g_{\mu\nu}(g, f, \beta_n) &= \frac{1}{m_g^2} T^g_{\mu\nu}\,, \\ G_{\mu\nu}(f) + \frac{m^4}{m_f^2} V^f_{\mu\nu}(g, f, \beta_n) &= \frac{1}{m_f^2} T^f_{\mu\nu}\,. \end{aligned} \tag{4.4.53}$$

Now, we can have a look at the two important limits of bimetric gravity to GR and massive gravity.

GR limits

What we want to show here is that, unlike massive gravity, bimetric gravity has a well-defined GR limit. Also, one can note that since the bimetric theory, that we have introduced so far, is symmetric with respect to $g_{\mu\nu}$ and $f_{\mu\nu}$ one can choose either of them to be the metric associated with the resulting GR theory after taking the limit. Here we take $g_{\mu\nu}$ but one can choose $f_{\mu\nu}$ and just follow the same steps.

The starting point is the observation that the bimetric potential satisfies the following relation,

$$\sqrt{g} g^{\rho\mu} V^g_{\rho\nu} + \sqrt{f} f^{\rho\mu} V^f_{\rho\nu} - \sqrt{g} V \delta^\mu_\nu = 0 \,. \tag{4.4.54}$$

With the use of this relation, one can combine the equations of motion (4.4.63) and obtain

$$g^{\mu\rho}\mathcal{G}_{\rho\nu}(g) + \alpha^2 det(\sqrt{g^{-1}f}) f^{\mu\rho}\mathcal{G}_{\rho\nu}(f) + \frac{m^4}{m_g^2} V\delta^\mu_\nu = \frac{1}{m_g^2}(g^{\mu\rho}T^g_{\rho\nu} + f^{\mu\rho}T^f_{\rho\nu}) \,. \tag{4.4.55}$$

Now, we can look at the parameter $\alpha \equiv \frac{m_f}{m_g}$ and we can see that in the limit $\alpha \to 0$ the last equation reduces to

$$g^{\mu\rho}\mathcal{G}_{\rho\nu}(g) + \frac{m^4}{m_g^2} V\delta^\mu_\nu = \frac{1}{m_g^2}(g^{\mu\rho}T^g_{\rho\nu} + f^{\mu\rho}T^f_{\rho\nu}) \,. \tag{4.4.56}$$

Under the condition that there are no matter couplings to the second metric $f_{\mu\nu}$ we have $T^f_{\mu\nu} = 0$ and therefore we can take the divergence of the equation with the covariant derivative that is compatible with the metric $g_{\mu\nu}$. This divergence for covariantly conserved sources would give $V =$ constant on-shell. Finally, the equation will reduce to the Einstein equation for a single physical metric $g_{\mu\nu}$ with Planck mass m_g and cosmological constant $\frac{m^4}{m_g^2}V$

$$\mathcal{G}_{\mu\nu} + \frac{m^4}{m_g^2} V g_{\mu\nu} = \frac{1}{m_g^2} T^g_{\mu\nu} \,. \tag{4.4.57}$$

So we can define the GR limit of bimetric theory as

$$\alpha \to 0, \quad m_g = const, \quad T^f_{\mu\nu} = 0 \,. \tag{4.4.58}$$

It is worth noting that these set of conditions will deform the equation of motion of the $f_{\mu\nu}$ to a pure algebraic equation

$$V^f_{\mu\nu} = 0 \,. \tag{4.4.59}$$

The generic solutions to this equation are proportional backgrounds $f_{\mu\nu} = c^2 g_{\mu\nu}$ with c determined by the condition $\Lambda_f(c) = 0$ which we are going to discuss shortly.

4.4.4 Massive gravity limits

Having looked at the $\alpha \to 0$ limit we can look at the other possible $\alpha \to \infty$ limit. We are going to show that this limit will give us massive gravity from the bimetric gravity. This limit is defined as

$$\begin{aligned} &\alpha \equiv \infty, \quad m_g = const. \quad \frac{1}{M^2} T_{\mu\nu} = \frac{1}{m_f^2} \tilde{T}^f_{\mu\nu} = \text{const.} \\ &\beta'_4 \equiv \frac{m_g^2}{m_f^4}, \quad \beta_4 = const. \quad \beta_n = \text{const}, \quad for \;\; n < 3 \,. \end{aligned} \tag{4.4.60}$$

Here the limits have been defined in such a way that we can reconstruct the massive gravity by decoupling $f_{\mu\nu}$ from $g_{\mu\nu}$. Meaning we have introduced a new mass scale M and stress-energy tensor $\tilde{T}$ here so that we have a mass scaling of massive metric $f_{\mu\nu}$. Also, the scaling of β_4 is required to keep a cosmological constant term for $f_{\mu\nu}$. Now, the $f_{\mu\nu}$ is given by

$$\mathcal{G}_{\mu\nu}(f) + \frac{m^4}{m_g^2}\beta_4' f_{\mu\nu} = \frac{1}{M^2}T_{\mu\nu}\,, \tag{4.4.61}$$

which is an Einstein equation with cosmological constant $\frac{m^4}{m_g^2}\beta_4'$ and Planck mass M.

Thus, the limit does the decoupling of the two metrics as mentioned above. Now, we can use equation (4.4.61) to obtain a solution for $f_{\mu\nu}$ and replace it in the equation of motion for $g_{\mu\nu}$.

The resulting equation for $g_{\mu\nu}$ will be the same as the one obtained by massive gravity action with the fixed reference metric $f_{\mu\nu}$ that solves the Einstein equations. For example by putting $f_{\mu\nu} = \eta_{\mu\nu}$ one can reconstruct dRGT [17] massive gravity from bimetric gravity. However, this not only covers all massive gravity theories but also includes more possible theories that do not have a massive gravity counterpart (i.e. they become singular for $\alpha \to \infty$).

4.4.5 Proportional Background and mass Eigenstates

Given the theory at hand, we need to discuss some solutions of this theory in which one can compute the interaction terms and compare them to the effective action calculated from the string amplitudes [63].

The simplest and yet remarkably important class of solutions to the bimetric equations of motion in vacuum is obtained by making an ansatz that conformally relates the two metrics $\bar{f}_{\mu\nu} = c(x)^2\bar{g}_{\mu\nu}$, where $c(x)$ is a space-time dependent function. Putting this back into one of the forms of the Bianchi identity (e.g. $\nabla^\mu V^g_\mu = 0$) will can see that $c(x)$ is a constant. So we have:

$$\bar{f}_{\mu\nu} = c^2\bar{g}_{\mu\nu} \quad c = const\,. \tag{4.4.62}$$

Therefore, the set of equations of motions for $f_{\mu\nu}$ and $g_{\mu\nu}$ will take the familiar form:

$$\begin{aligned}
&\mathcal{G}_{\mu\nu}(\bar{g}) + \Lambda_g\bar{g}_{\mu\nu} = 0, \quad \Lambda_g = \frac{m^4}{m_g^2}(\beta_0 + 3c\beta_1 + 3c^2\beta_2 + c^3\beta_3)\,,\\
&\mathcal{G}_{\mu\nu}(\bar{g}) + \Lambda_f\bar{g}_{\mu\nu} = 0, \quad \Lambda_f = \frac{m^4}{m_g^2c^2}(c\beta_1 + 3c^2\beta_2 + 3c^3\beta_3 + c^4\beta_4)\,.
\end{aligned} \tag{4.4.63}$$

Given the fact that the Einstein tensor is scale-invariant (i.e. $\mathcal{G}(c^2g) = \mathcal{G}(g)$). The proportional background includes all solutions of GR. The type of the solution we get is dependent on Λs which in turn fixes the proportionality constant c of our ansatz in terms of the parameters of the theory and hence it specifies the solution completely. Further, one can look deeper and observe that by taking the difference between the two equations in (4.4.63) we end up with the condition:

$$\Lambda_g(c) = \Lambda_f(c)\,. \tag{4.4.64}$$

This relation can be used to determine βs.

Perturbation and mass eingensates

Now, we are going to look at the usual setup to discuss the perturbation around the vacuum created by the proportional Background solution introduced above. Here, we have the expansion of the two fields:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \quad f_{\mu\nu} = c^2 \bar{g}_{\mu\nu} + \delta f_{\mu\nu}\,. \tag{4.4.65}$$

The variation of the square-root matrix is given by

$$\bar{g}_{\rho\mu}\delta S^{\rho}_{\nu} = \frac{1}{2c}(\delta f_{\mu\nu} - c^2 \delta g_{\mu\nu})\,. \tag{4.4.66}$$

In this expansion the equations of motion (4.4.63) will be

$$\begin{aligned} &\tilde{\mathcal{E}}^{\rho\sigma}_{\mu\nu}\delta g_{\rho\sigma} - \bar{\Lambda}_g(\delta g_{\mu\nu} - \frac{1}{2}\bar{G}_{\mu\nu}\bar{G}^{\rho\sigma}\delta g_{\rho\sigma}) - N\bar{G}_{\mu\rho}(\delta S^{\rho}_{\nu} - \delta^{\rho}_{\nu}\delta S^{\alpha}_{\alpha}) = 0 \\ ,\ &\tilde{\mathcal{E}}^{\rho\sigma}_{\mu\nu}\delta f_{\rho\sigma} - \bar{\Lambda}_f(\delta f_{\mu\nu} - \frac{1}{2}\bar{G}_{\mu\nu}\bar{G}^{\rho\sigma}\delta f_{\rho\sigma}) - \alpha^{-2}N\bar{G}_{\mu\rho}(\delta S^{\rho}_{\nu} - \delta^{\rho}_{\nu}\delta S^{\alpha}_{\alpha}) = 0 \\ . \end{aligned} \tag{4.4.67}$$

Here N is dependent on c, α, and β_n and can be read off explicitly from the Fierz-Pauli mass. Also, we have used of rescale invariance of the metric to rewrite:

$$\begin{aligned} &\bar{G}_{\mu\nu} \equiv (1+\alpha^2 c^2)\bar{g}_{\mu\nu}\,, \\ &\bar{\Lambda}_g \equiv (1+\alpha^2 c^2)^{-1}\bar{\Lambda}_g\,, \\ &R(\bar{G})_{\mu\nu} = \bar{\Lambda}_g \bar{G}_{\mu\nu}\,. \end{aligned} \tag{4.4.68}$$

The kinematic operator $\tilde{\mathcal{E}}^{\rho\sigma}_{\mu\nu}$ around the background $\bar{G}_{\mu\nu}$ is well known and given by

$$\tilde{\mathcal{E}}^{\rho\sigma}_{\mu\nu} = -\frac{1}{2}\Big[\delta^{\rho}_{\mu}\delta\sigma_{\nu}\bar{\nabla}^2 + \bar{G}^{\rho\sigma}\bar{\nabla}_{\mu}\bar{\nabla}_{\nu} - \delta^{\rho}_{\mu}\bar{\nabla}^{\sigma}\bar{\nabla}_{\mu} - \delta^{\rho}_{\nu}\bar{\nabla}^{\sigma}\bar{\nabla}_{\mu} - \bar{G}^{\rho\sigma}\bar{G}_{\mu\nu}\bar{\nabla}^2 + \bar{G}_{\mu\nu}\bar{\nabla}^{\rho}\bar{\nabla}^{\sigma}\Big]\,, \tag{4.4.69}$$

In which the $\bar{\nabla}$ is the covariant derivative with respect to background metric $\bar{G}_{\mu\nu}$. One can diagonalize the above equations and write them in terms of a massless and a massive perturbation. It is clear that these diagonal fields are the mass eigenstates of bimetric gravity [64]. To check this we can use the diagonalized form of the fields as

$$\delta G_{\mu\nu} \equiv \delta g_{\mu\nu} + \alpha^2 \delta f_{\mu\nu}, \quad \delta M_{\mu\nu} = \frac{1}{2c}(\delta f_{\mu\nu} - c^2 \delta g_{\mu\nu})\,. \tag{4.4.70}$$

Plugging these definitions back into (4.4.67) one will get,

$$\begin{aligned} &\tilde{\mathcal{E}}^{\rho\sigma}_{\mu\nu}\delta G_{\rho\sigma} - \bar{\Lambda}_g(\delta G_{\mu\nu} - \frac{1}{2}\bar{G}_{\mu\nu}\delta G) = 0\,, \\ &\tilde{\mathcal{E}}^{\rho\sigma}_{\mu\nu}\delta M_{\rho\sigma} - \bar{\Lambda}_f(\delta M_{\mu\nu} - \frac{1}{2}\bar{G}_{\mu\nu}\delta M) + \frac{\bar{m}^2_{FP}}{2}(\delta M_{\mu\nu} - \bar{g}_{\mu\nu}\delta M) = 0\,, \end{aligned} \tag{4.4.71}$$

which are perturbative equations of motion for massless and massive spin-2 fields, respectively. The trace δG and δM are defined with respect to background metric $\bar{G}_{\mu\nu}$ (i.e. $\delta G = \bar{G}_{\mu\nu}\delta G^{\mu\nu}$) and the Fierz-Pauli mass is defined as

$$\bar{m}_{FP}^2 = \frac{(1+\alpha^2c^2)^2}{\alpha^2 c}N = \frac{m^4}{m_g^2}\frac{1}{\alpha^2c^2}(c\beta_1 + 2c^2\beta_2 + c^3\beta_3)\,, \tag{4.4.72}$$

where c can be regarded as a function of the Planck mass. Here βs can be determined with the condition (4.4.64). We can associate two propagating degrees of freedom to $\delta G_{\mu\nu}$ and the remaining five to $\delta M_{\mu\nu}$.

The linearised action up to the quadratic order in terms of the mass eigenstate is

$$\begin{aligned} S[\delta G, \delta M] = \int d^4x\sqrt{|g|}\Bigg[&\mathcal{L}^2_{GR}(\delta G) + \frac{1}{m_{PI}}\mathcal{L}^3_{GR}(\delta G) + \mathcal{L}^2_{GR}(\delta M) + \frac{1-\alpha^2}{\alpha m_{PI}}\mathcal{L}^3_{GR}(\delta M) \\ &+ \mathcal{L}^2_{FP}(\delta M) + \frac{1}{m_{PI}}\mathcal{L}^3_{GR}(\delta G, \delta M)\Bigg]\,. \end{aligned} \tag{4.4.73}$$

Here we have collected terms with respect to the number of fields that appear in the interactions we have

$$\begin{aligned} \mathcal{L}^2_{GR}(\delta G) =& \frac{1}{4}\Bigg[\nabla^\rho\delta G\nabla_\rho\delta G - \nabla^\rho\delta G_{\mu\nu}\nabla_\rho\delta G^{\mu\nu} - 2\nabla_\rho\nabla_\mu\delta G^{\mu\rho} + 2\nabla_\nu\delta G_{\mu\rho}\nabla^\rho\delta G^{\mu\nu} \\ &+ 2\Lambda(\delta G^{\mu\nu}\delta G_{\mu\nu} - \frac{1}{2}\delta G^2)\Bigg]\,, \\ \mathcal{L}^2_{FP}(\delta M) =& -\frac{m^2_{FP}}{4}(\delta M_{\mu\nu}\delta M^{\mu\nu} - \delta M^2)\,, \\ \mathcal{L}^3_{GR}(\delta G) =& \frac{1}{4}\Bigg[\delta G^{\mu\nu}(\nabla_\mu\delta G_{\rho\sigma}\nabla_\nu\delta G^{\rho\sigma} - \nabla_\mu\delta G\nabla_\nu\delta G + 2\nabla_\nu\delta G\nabla^\rho h_{\mu\rho} + 2\nabla_\nu\delta G_{\mu\rho}\nabla^\rho\delta G - 2\nabla_\rho\delta G\nabla^\rho h_{\mu\nu} \\ &+ 2\nabla_\rho\delta G_{\mu\nu}\nabla_\nu\delta G^{\rho\sigma} - 4\nabla_\nu\delta G_{\rho\sigma}\nabla^\sigma\delta G^\rho_\mu - 2\nabla^\rho\delta G_{\nu\sigma}\nabla^\sigma h_{\mu\rho} + 2\nabla_\sigma\delta G_{\nu\rho}\nabla^\sigma\delta G^\rho_\mu) \\ &\frac{1}{2}\delta G(\delta G^{\mu\nu}(\nabla_\mu\delta G_{\rho\sigma}\nabla_\nu\delta G^{\rho\sigma} - \nabla_\mu\delta G\nabla_\nu\delta G + 2\nabla_\nu\delta G\nabla^\rho\delta G_{\mu\rho} + 2\nabla_\nu\delta G_{\mu\rho}\nabla^\rho\delta G \\ &- 2\nabla_\rho\delta G\nabla^\rho\delta G_{\mu\nu}) - \frac{\Lambda}{3}(\delta G^3 - 6\delta G\delta G_{\mu\nu}\delta G^{\mu\nu} + 8\delta G^\mu_\rho\delta G^\rho_\nu\delta G^\nu_\mu)\Bigg]\,. \end{aligned} \tag{4.4.74}$$

$$
\begin{aligned}
\mathcal{L}^3_{GR}(\delta G,\delta M) = &-\frac{m^2_{\mathrm{FP}}(1+\alpha^2)(\beta_1+\beta_2)}{4\alpha(\beta_1+\beta_2+\beta_3)}e_3(M)-\frac{m^2_{\mathrm{FP}}}{24\alpha}\Big[-2[M]^3+9[M][M^2]-7[M^3]+\alpha(-3[G][M]^2\\
&+12[M][GM]+3[G][M^2]-12[GM^2])+\alpha^2([M]^3-6[M][M^2]+5[M^3])\Big]\\
&-\frac{\Lambda}{4}\Big[[G][M]^2-4[M][GM]-2[G][M^2]+8[GM^2]\Big]\\
&+\frac{1}{4}\Big[G^{\mu\nu}\big(\partial_\mu M_{\rho\sigma}\partial_\nu M^{\rho\sigma}-\partial_\mu M\partial_\nu M+2\partial_\nu M\partial_\rho M^\rho_\mu+2\partial_\nu M^\rho_\mu\partial_\rho M-2\partial_\rho M\partial^\rho M_{\mu\nu}\\
&+2\partial_\rho M_{\mu\nu}\partial_\sigma M^{\rho\sigma}-4\partial_\nu M_{\rho\sigma}\partial^\sigma M^\rho_\mu-2\partial_\rho M_{\nu\sigma}\partial^\sigma M^\rho_\mu+2\partial_\sigma M_{\nu\rho}\partial^\sigma M^\rho_\mu\big)\\
&+\frac{1}{2}G\big(\partial_\rho M\partial^\rho M-\partial_\rho M_{\mu\nu}\partial^\rho M^{\mu\nu}-2\partial_\rho M\partial_\mu M^{\mu\rho}+2\partial_\rho M_{\mu\nu}\partial^\nu M^{\mu\rho}\big)\Big]\\
&+\frac{1}{2}\Big[M^{\mu\nu}\big(\partial_\mu G_{\rho\sigma}\partial_\nu M^{\rho\sigma}-\partial_\mu G\partial_\nu M+\partial^\rho G_{\rho\mu}\partial_\nu M+\partial_\nu G_{\mu\rho}\partial^\rho M-\partial_\rho G_{\mu\nu}\partial^\rho M\\
&+\partial_\rho G^{\rho\sigma}\partial_\sigma M_{\mu\nu}-2\partial_\mu G^{\rho\sigma}\partial_\sigma M_{\nu\rho}+\partial_\mu G\partial^\rho M_{\rho\nu}+\partial^\rho G_{\mu\nu}\partial^\sigma M_{\rho\sigma}-2\partial_\rho G_{\mu\sigma}\partial_\nu M^{\rho\sigma}\\
&-2\partial^\rho G_{\mu\sigma}\partial^\sigma M_{\nu\rho}+2\partial^\rho G_{\mu\sigma}\partial_\rho M^\sigma_\nu+\partial^\rho G\partial_\nu M_{\mu\rho}-\partial^\rho G\partial_\rho M_{\mu\nu}\big)\\
&+\frac{1}{2}M\big(\partial_\rho G\partial 50^\rho M-\partial_\rho G_{\mu\nu}\partial^\rho M^{\mu\nu}-\partial_\rho G\partial_\sigma M^{\rho\sigma}-\partial_\rho G^{\rho\sigma}\partial_\sigma M+2\partial_\rho G_{\mu\nu}\partial^\nu M^{\mu\rho}\big)\Big]\,.
\end{aligned}
\tag{4.4.75}
$$

In the last term, we gathered all cubic interactions originating from the bimetric potential. Meaning, all possible terms that one can write in a ghost free theory consist of a massive and a massless spin-2 field. There are no terms of the form $\delta G\delta G\delta M$ present, meaning that there is no decay of massive state into massless gravitons at tree level. Also, note that there are no $\delta G\delta G\delta G$ terms present and thus all the self-interaction of massless gravitons come from the Einstein-Hilbert term. The construction of the double copy (7.4.152) may be related to some limit of bimetric gravity. The starting point of this map is the connection between *higher derivative* gravity and bimetric gravity. We are going to discuss higher derivative gravity and the relation to the bimetric gravity in chapter 6.

Chapter 5

String perturbation theory

5.1 Preface

In this chapter, we are going to give a review of string perturbation theory. By the end of this chapter we will have a solid foundation for the following topics:

1. String theory and its symmetries

2. Scattering amplitudes in string theory

In the first section, we are going to discuss string theory. This famous topic has been the subject of so many great books and reviews throughout the past 50 years. However, in order to be consistent and self-sufficient we are going to give a recap of the topic from the scattering amplitude point of view. Therefore, we will look at the Lagrangian formalism of string theory as a sigma model. Then we are going to expand the discussion to conformal field theory (CFT). We are going to explain the framework of CFTs and how amplitudes are calculated in a generic CFT. We invite the reader to see the following (noninclusive) references for more details of the basics of string theory [10, 11, 12, 65]. Then, having looked at CFT structures we will use them in string theory to discuss the following:

- String quantization and spectrum.

- Gauge fixing and BRST quantization.

- Vertex operators for different mass levels.

- Calculation of string S matrix elements (n-point functions) .

5.2 Superstring theory

String theory is a natural extension of the classical and quantum mechanical notion of particles. Specifically, in contrast to the zero-dimensional object (i.e. a particle) which is moving in spacetime and creating a world line, one can extend it to any m dimensional object which is swiping a world-volume (Σ) in spacetime. This topic was first addressed by Dirac back in 1962 discussing the motion of an extended object in a gravitational field. The action for this m dimensional object in a d dimension background with a metric $g_{\mu\nu}$ is given by the Nambo-Goto type Lagrangian:

$$S^{m-extedned} = -T \int_\Sigma dV = -T \int_\Sigma dx^{m+1} \sqrt{-\det_{\alpha\beta}\left(g^{\mu\nu}\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}\right)}\,, \qquad (5.2.1)$$

where T is the tension of the world-volume and $X^\mu(\sigma_1, \sigma_2, ..., \sigma_{m+1})$ is the map from the $m+1$ dimensional world-volume to the embedding spacetime (cf. figure 5.1). The action

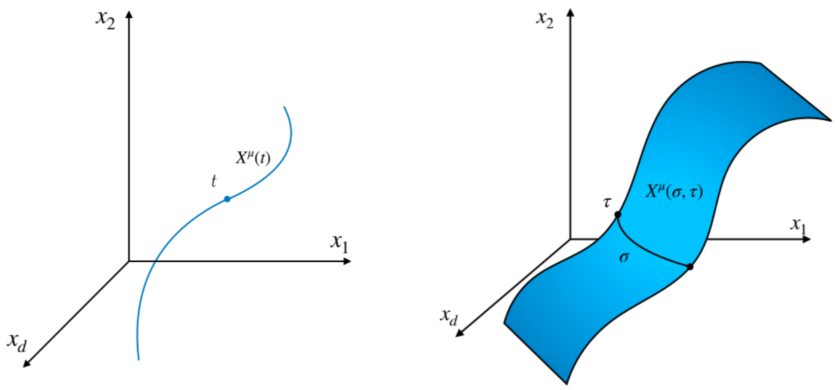

Figure 5.1: Particle's world-line vs string's world-sheet

(5.2.1) is the volume of the world-volume and the equation of motion naturally extremises this area. Although this action has an understandable geometrical interpretation it is not suitable for quantization due to the square root of the field $X^\mu(\sigma_1, \sigma_2, ..., \sigma_{m+1})$. By quantization we mean either old covariant quantization (OCQ) using the creation/annihilation operators (which requires mode expansion around a solution of classical equations of motion) or the path-integral quantization which involves BRST symmetry and associated ghost systems. Therefore, we need another action describing the field $X^\mu(\sigma_1, \sigma_2, ..., \sigma_{m+1})$ with the same classical equations of motion as the action in (5.2.1) and use it for quantization. This requirement is the definition of a sigma model which is a theory describing fields that swipe an embedded volume.

For the purposes of the current work, we are going to restrict ourselves to a one-dimensional extended object with a two-dimensional world-sheet (Σ). Having set the world-sheet to be two-dimensional one can write the corresponding (Polyakov) action to (5.2.1) in the following way as a sigma model:

$$\begin{aligned} S_1^{bosonic} &= -\frac{1}{4\pi\alpha'}\int_\Sigma d\sigma d\tau\sqrt{-h}\, g_{\mu\nu}\, h^{\alpha\beta}\partial_\alpha X^\mu(\sigma,\tau)\partial_\beta X^\nu(\sigma,\tau)\,, \\ S_2^{fermionic} &= -\frac{1}{4\pi}\int_\Sigma d\sigma d\tau\sqrt{-h}\, g_{\mu\nu}\, h^{\alpha\beta}\, i\,\overline{\psi}^\mu \rho_\alpha \partial_\beta \psi^\nu\,, \\ S^{superstring} &= S_1 + S_2 + S_{Aux}\,, \end{aligned} \qquad (5.2.2)$$

where (σ, τ) are the coordinates on the world-sheet and indices (μ, ν) and (α, β) run over spacetime and world-sheet coordinates, respectively. Here we have written two types of actions S_1 for a bosonic spacetime field and S_2 for a fermionic spacetime field. We require both fields in order to have bosons and fermions in the matter content of the resulting theory. One of the more important results of including bosons and fermions is *supersymmetry* (SUSY) which is the symmetry that relates bosons and fermions to each other. We are going to discuss the details of supersymmetry in the coming sections. We should point out that to have local supersymmetry we need to add additional action with some *auxiliary fields* that we labeled as S_{aux} [10, 11, 12].

5.2.1 Symmetries and gauge invariance

We take the action in (5.2.1) as the "string action" and from now on we are going to use this action and its associated S-matrix for the current work. The first thing to study about this action is its symmetries. They can be used as constraining conditions on scattering amplitudes. There are two type of symmetries *local* and *global* [12]:

1. Global symmetry: Poincare invariance of the target space. This symmetry acts on the fields as:
$$\begin{aligned} &\delta X^\mu = \Lambda^\mu{}_\nu X^\nu + a^\mu, \qquad \Lambda_{\mu\nu} = -\Lambda_{\nu\mu}\,, \\ &\delta h^{\alpha\beta} = 0\,. \end{aligned} \tag{5.2.3}$$
Where the matrix $\Lambda_{\mu\nu}$ is constant.

2. Local symmetries:

   - Supersymmetry and Super Weyl rescaling
$$Q(boson) \to (fermion).$$
   - Diffeomorphism invariance of the string world–sheet
$$\begin{aligned} \delta X^\mu &= -\xi^\alpha \partial_\alpha X^\mu\,, \\ \delta h_{\alpha\beta} &= -(\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha)\,. \end{aligned} \tag{5.2.4}$$
   - Weyl rescaling
$$\begin{aligned} \delta X^\mu &= 0, \\ \delta h^{\alpha\beta} &= 2\lambda h^{\alpha\beta}. \end{aligned} \tag{5.2.5}$$

The $\Lambda_{\mu\nu}$ and a^μ are the generators of the Lorentz transformations and translations, respectively. Using these symmetries one can fix the world-sheet metric to conformal gauge. Meaning, set the metric to be flat Minkowski metric locally: $h_{\alpha\beta} = \eta_{\alpha\beta}$. In addition, one can fix the superconformal gauge which removes S_3 from (5.2.1) and give us the action that we are going to use from now on as superstring action:

$$S^{Superstring} = -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \Big(\partial_\alpha X^\mu(\sigma,\tau) \partial^\alpha X_\mu(\sigma,\tau) + i\alpha' \overline{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \Big) + \lambda\chi. \tag{5.2.6}$$

The last term $\lambda\chi$ is the coupling λ to Euler character χ of the surface. This term, as we discussed in 2.2, rises when one takes into account the Ricci (i.e. gravity) term on the two-dimensional Riemann surface. As expected it is topological and counts the genus g of the surface. Later in this chapter, we see that this term is related to the string coupling g_s which, as we will show, controls the string interactions (i.e. the coupling of interactions). The energy-momentum tensor of this action is then given by variation with respect to the metric. However, as this theory is supersymmetric there is a fermionic counterpart to the energy-momentum tensor which is computed through variation of the action with respect to gravitino χ^α. A careful reader might notice that in action (5.2.6) there is no gravitino. The reason for that is after the superconformal gauge fixing it disappears from the action. Therefore, in order to calculate the fermionic energy-momentum tensor one has to look at the full action before gauge fixing. Doing so we have the two energy-momentum tensors as follows:

$$\begin{aligned} T_B^{\alpha\beta} &= -\frac{1}{\alpha'}\left(\partial^\alpha X^\mu \partial^\beta X_\mu - \frac{1}{2}h^{\alpha\beta}\partial^\gamma X^\mu \partial_\gamma X_\mu\right) - \frac{i}{4}\left(\overline{\psi}^\mu \rho^\alpha \partial^\beta \psi_\mu + \overline{\psi}^\mu \rho^\beta \partial^\alpha \psi_\mu\right), \\ T_F^\alpha &= -\frac{1}{4}\sqrt{\frac{2}{\alpha'}}\rho^\alpha \rho^\beta \psi^\mu \partial_\beta X_\mu. \end{aligned} \tag{5.2.7}$$

Here, we have fixed the tension of the surface as $T = \frac{1}{2\pi\alpha'}$ and therefore the length of the string is given by $l = 2\pi\sqrt{\alpha'}$.

We should point out the importance of the energy-momentum tensor. It generates the aforementioned symmetries of the theory. The mode expansion of this tensor will give raise to the operators generating symmetries by acting on the states. They form symmetry algebras of the theory which we are going to discuss later.

The full equations of motion are given by the variation of the action with respect to the fields X^μ, $h^{\alpha\beta}$, ψ^μ and χ^α and we have:

$$\begin{aligned} &\frac{\delta S}{\delta X_\mu} = 0 \to \frac{1}{\sqrt{-h}}\partial_\alpha(\sqrt{-h}h^{\alpha\beta}\partial_\beta X^\mu) = \Box X^\mu = 0, \\ &\frac{\delta S}{\delta \psi_\mu} = 0 \to \rho^\alpha \partial_\alpha \psi^\mu = 0, \\ &\frac{\delta S}{\delta h_{\alpha\beta}} = 0 \to T^{\alpha\beta} = 0, \\ &\frac{\delta S}{\delta \overline{\chi}_\alpha} = 0 \to T_F^\alpha = 0. \end{aligned} \tag{5.2.8}$$

While computing the variations of the action for the differential equations of motion one has to impose different boundary conditions. These conditions will give rise to different types of strings. For the fields X^μ we obtain:

- *non-periodic* boundary conditions:

  1. *Neumann* boundary condition:

$$\partial^\sigma X^\mu\Big|_{\sigma=0} = 0\,, \quad \partial^\sigma X^\mu\Big|_{\sigma=l} = 0\,.$$

where l is the length of the string.

2. *Drichlet* boundary condition:

$$\delta X^{\mu}\Big|_{\sigma=0} = 0, \quad \delta X^{\mu}\Big|_{\sigma=l} = 0.$$

These two boundary conditions will give rise to four different possibilities (NN)-(DN)-(ND)-(DD) all of which are known as **open** String.

- *periodic* boundary condition:

$$X^{\mu}(\tau, 0) = X^{\mu}(\tau, l), \qquad \partial^{\sigma} X^{\mu}(\tau, 0) = \partial^{\sigma} X^{\mu}(\tau, l), \qquad h_{\alpha\beta}(\tau, 0) = h_{\alpha\beta}(\tau, l).$$

This condition will correspond to *loops* which is known as **closed** strings.

Similarly, one has boundary conditions for field ψ^{μ}. However, since the field ψ^{μ} is fermion on the world–sheet there are more possible boundary conditions known as *Ramond* and *Neveu-Schwarz* we have the following:

- *non-periodic* boundary conditions (for **open** strings):

$$\psi^{\mu}(\tau, 0) = \pm\overline{\psi}^{\mu}(\tau, 0), \qquad \psi^{\mu}(\tau, l) = \pm\overline{\psi}^{\mu}(\tau, l), \tag{5.2.9}$$

The overall relative sign between ψ and $\overline{\psi}$ is a matter of convention. We follow here the standard notation setting:

$$\psi^{\mu}(\tau, 0) = \overline{\psi}^{\mu}(\tau, 0), \qquad \psi^{\mu}(\tau, l) = \pm\overline{\psi}^{\mu}(\tau, l). \tag{5.2.10}$$

The plus sign is for *Ramond* Sector and the minus sign is for *Neveu-Schwarz* sector.

- *periodic* boundary condition (for **closed** strings):

$$\begin{aligned} &\psi^{\mu}(\sigma + l) = +\psi^{\mu}(\sigma), \qquad \overline{\psi}^{\mu}(\sigma + l) = +\overline{\psi}^{\mu}(\sigma) \qquad \text{Ramond (R) sector,} \\ &\psi^{\mu}(\sigma + l) = -\psi^{\mu}(\sigma), \qquad \overline{\psi}^{\mu}(\sigma + l) = -\overline{\psi}^{\mu}(\sigma) \qquad \text{Neveu-Schwarz (NS) sector.} \end{aligned} \tag{5.2.11}$$

Since the choice for the boundary of the left and right movers (closed string) are independent one can have (NS-NS), (R-R), (NS-R), and (R-NS) sectors. The strings in the R sectors are spacetime fermions and the strings in the NS sectors are spacetime bosons. Therefore, (NS-NS), (R-R) sectors of closed strings are spacetime boson and (NS-R) and (R-NS) are spacetime fermions.

D-branes and Chan-Paton factor

One of the more important consequences of the Dirichlet boundary condition is the existence of D-branes. It was shown at the start of the second string revolution that the endpoints of the open string with the Dirichlet boundary condition can be considered as extended dynamical objects [66]. The inclusion of these objects in spacetime will break the Poincare symmetry. For D brane with dimension r in d dimension we have the following decomposition[1]:

$$SO(1,d-1) \to SO(1,r-1) \times SO(d-r). \tag{5.2.12}$$

This decomposition will affect tensors in spacetime, in particular, the momenta of the fields. We define the D matrix associated with the brane in the following

D Matrix

In the flat background, we have the diagonal D matrix:

$$D^{\mu}{}_{\nu} = \begin{cases} \delta^{\alpha}{}_{\beta} & (\alpha,\beta) \in \text{parallel to brane}, \\ -\delta^{i}{}_{j} & (i,j) \in \text{perpendicular to brane}. \end{cases} \tag{5.2.13}$$

Then for a generic vector V, we have:

$$V^{\parallel} = \frac{1}{2}\left(V + DV\right), \quad V^{\perp} = \frac{1}{2}\left(V - DV\right), \tag{5.2.14}$$

and the momentum conservation will be:

$$\sum_i V_i^{\parallel} = 0. \tag{5.2.15}$$

This is due to the remaining Poincare group $SO(1,r-1)$.

In addition, at the end points of each open string (the geometrical position of the brane), we find degrees of freedom associated with the parallel directions to the brane. These extra degrees of freedom can be gathered in representations of a $U(n)$ Lie group. All open strings transform in $n \times n$ adjoint representation under this $U(n)$ symmetry. So we can write the open string state as:

$$|open\rangle = |m,k;a\rangle = |m,k\rangle \otimes T^a, \tag{5.2.16}$$

where a is the label of the $U(n)$ adjoint representation (associated Gell-Mann matrices T^a representing the generators of Lie algebra). In the construction of gauge theories as effective actions this Chan-Paton symmetry is related to the gauge group of the field theory.

[1] Usually the time direction is always parallel to the brane.

5.3 Conformal field theory description

In this section, we give a lightning introduction to $2d$ conformal field theory by using string theory as an example (for further detail cf. [67, 68]). We also give the important results that we are going to use in the following sections and chapters. In the first step, we are going to change coordinates and map the world-sheet coordinates (σ, τ) to complex coordinate $(z, \overline{z})$ via:

$$z = \tau + i\sigma, \qquad \overline{z} = \tau - i\sigma. \tag{5.3.17}$$

From now on we are going to use the complex coordinates since they let us the tools of complex analysis. In these coordinates the action (5.2.1) will be:

$$S^{superstring} = \frac{1}{4\pi\alpha'} \int dz d\overline{z} \left(2\partial X^\mu(z, \overline{z}) \overline{\partial} X_\mu(z, \overline{z}) + \alpha'(\psi^\mu \overline{\partial} \psi_\mu + \overline{\psi}^\mu \partial \overline{\psi}_\mu) \right). \tag{5.3.18}$$

The equations of motion are given by:

$$\begin{aligned} &\partial\overline{\partial} X^\mu = \overline{\partial}\partial X^\mu = 0, \\ &\overline{\partial}\psi = \partial\overline{\psi} = 0. \end{aligned} \tag{5.3.19}$$

Using the definition of *holomorphic* and *anti–holomorphic* functions over complex plane we can see that the on-shell fields $\partial X^\mu(z)$ and $\psi^\mu(z)$ are holomorphic and fields $\overline{\partial} X^\mu(\overline{z})$ and $\overline{\psi}^\mu(\overline{z})$ are anti–holomorphic. This is going to be useful in the conformal field theory discussion.

Conformal Symmetry

We start the analysis by looking at the conformal symmetry group $Conf(3, 1)$. It includes three types of transformations:

- Poincare

$$\begin{aligned} &x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu; \qquad \Lambda_{\mu\nu} = -\Lambda_{\nu\mu} \longleftrightarrow M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu), \\ &P_\mu = -i\partial_\mu. \end{aligned} \tag{5.3.20}$$

- Dilatation

$$x'^\mu = \alpha x^\mu \longleftrightarrow D = -i x_\mu \partial^\mu. \tag{5.3.21}$$

- Special conformal transformations:

$$x'^\mu = \frac{x^\mu - a^\mu x^2}{1 - 2a \cdot x + a^2 x^2} \longleftrightarrow K_\mu = -i(x^2\partial_\mu - 2x_\mu x_\nu \partial^\nu). \tag{5.3.22}$$

Above we gave the transformations of the conformal group and their infinitesimal representations. Geometrically speaking conformal group maps infinitesimal squares to infinitesimal squares but rescales them by a position-dependent factor. Since the action

(5.2.1) (or (5.3.18)) is invariant under these transformations we can consider string theory as defined in the action (5.3.18) as a free conformal field theory. The conformal symmetry can be also seen on the level of the energy-momentum tensor. In particular, all conformal field theories have a traceless energy-momentum tensor meaning:

$$\textit{Conformal field theory} \rightarrow T^{\mu}{}_{\mu} = 0. \tag{5.3.23}$$

This tracelessness should hold on both classical and quantum levels. Requiring the energy-momentum tensor to be traceless on the quantum level (which is known as the Weyl anomaly) forces us to fix the dimension of the embedding spacetime. This dimension is known as the *critical* dimension. For superstring theory it is $d = 10$ and for bosonic string it is $d = 26$.

Critial dimension

For superstring theory the Weyl anomaly will be proportional to:

$$\langle T^{\mu}{}_{\mu} \rangle \sim \frac{3}{2} D - 15 \stackrel{!}{=} 0 \rightarrow D = 10. \tag{5.3.24}$$

A similar analysis for the pure bosonic string will give $D = 26$

Primary fields

Now we are going to look at the building blocks of a conformal field theory: *Primary fields*. In order to do that let us start with the Lorentz group $SO(3,1)$ as an example. Following special relativity one wants to impose that physical fields transform covariantly under the Lorentz group. This requires physical fields to be in a representations of the Lorentz group which are known generally as tensor of the group (e.g. scalar ϕ, vector A_μ, etc). Consequently, Lorentz invariant actions can be written as scalar functions which are the inner product of tensors.

The same logic applies here to the conformal group. Since we want to write theories that exhibit conformal symmetries we need the field content to be covariant under the conformal transformations. These fields are known as *primary fields*. Formally, one can define them as follows:

Primary field

If $z \mapsto f(z)$ a conformal transformation and field $\psi(z, \overline{z})$ a primary field of conformal weight $(h, \overline{h})$ then:

$$\psi(z, \overline{z}) \mapsto \psi'(z, \overline{z}) = \left(\frac{\partial f}{\partial z} \right)^{h} \left(\frac{\partial \overline{f}}{\partial \overline{z}} \right)^{\overline{h}} \psi(f(z), \overline{f}(\overline{z})), \tag{5.3.25}$$

if $f \in \mathrm{PSL}(2, \mathbf{C})$ then the field is called quasi-primary.

Not all fields in a CFT need to be (quasi) primary there can also be secondary fields. We can see that the fields ∂X^μ amd ψ^μ are conformal primary fields with weights $h_X = 1$ and $h_\psi = \frac{1}{2}$.

5.3.1 Radial ordering and OPE

One of the important tasks in calculating n point functions in canonically quantized quantum field theories is to *time order* the products of creation and annihilation operators. This is due to the inherent ambiguity of the order of multiplication of the quantum operators.

Time ordering for quantum field ϕ

$$\langle 0|\prod_{i=1}^{n}\phi(x_i)|0\rangle \mapsto \langle 0|T\left(\prod_{i=1}^{n}\phi(x_i)\right)|0\rangle. \tag{5.3.26}$$

The notion of *time ordering* is defined as:

$$T\Bigg(\phi(x_1)\phi(x_2)\Bigg) = \begin{cases} \phi(x_1)\phi(x_2) & x_1^0 > x_2^0, \\ \phi(x_2)\phi(x_1) & x_2^0 > x_1^0. \end{cases} \tag{5.3.27}$$

In 2d CFTs, given the properties of conformal symmetry, one can always map the time direction to the radial distance to the origin of the complex plane (which is the transformation we introduced in (5.3.17)). Therefore, the notion of time ordering is replaced with *radial ordering* and one can define it similarly

Radial ordering for conformal field ψ

$$\langle 0|\prod_{i=1}^{n}\psi(z_i,\overline{z}_i)|0\rangle \mapsto \langle 0|R\left(\prod_{i=1}^{n}\psi(z_i,\overline{z}_i)\right)|0\rangle. \tag{5.3.28}$$

Then the *radial ordering* defined as:

$$R\Bigg(\psi(z_1,\overline{z}_1)\psi(z_2,\overline{z}_2)\Bigg) = \begin{cases} \psi(z_1,\overline{z}_1)\psi(z_2,\overline{z}_2) & |z_1| > |z_2|, \\ \psi(z_2,\overline{z}_2)\psi(z_1,\overline{z}_1) & |z_2| > |z_1|, \end{cases} \tag{5.3.29}$$

where $|z_i|$ is the distance of field $\psi(z_1,\overline{z}_1)$ to the origin.

Wick's theorem

We finish this section with *Wick's theorem.* This is the most important tool to calculate any quantum field theory (including conformal field theory) n-point amplitude. We are going to use this theorem extensively while calculating any string amplitude.

Theorem (3.1): Wick theorem

Theorem 3. Any time (radial) ordered product of operators can be written as a sum over *normal ordered* products of these operators where distinct pairs of the operator are contracted in all possible ways. Explicitly:

$$R\left(\prod_{i=1}^{n}\psi(z_i,\bar{z}_i)\right) = \underbrace{\left(\prod_{i=1}^{n}\psi(z_i,\bar{z}_i)\right)}_{contraction} + \underbrace{:\left(\prod_{i=1}^{n}\psi(z_i,\bar{z}_i)\right):}_{Normal\ ordered} + \text{regular terms}. \quad (5.3.30)$$

A product of operators $\mathcal{P}$ is normal ordered (Wick ordered) when all creation operators are to the left of all annihilation operators and it is denoted by the ": $\mathcal{P}$:" sign.

Therefore, in the CFT the functional dependence of operator products, when they *contract* (i.e. get close to each other $z_1 \to z_2$), is important. This requires the notion of the *operator product expansion* (OPE) which states that two local operators close together can be approximated to arbitrary accuracy by a sum of local operators [12, 67, 69]. Formally, we have:

$$\mathcal{O}_i(z_1,\bar{z}_1)\mathcal{O}_j(z_2,\bar{z}_2) = \sum_k C_{ij}{}^k((z_1-z_2))\mathcal{O}_k(z_2,\bar{z}_2)\,, \quad (5.3.31)$$

where $C_{ij}{}^k((z_1-z_2))$ is a meromorphic function of $(z-w)$. Using the OPEs of the

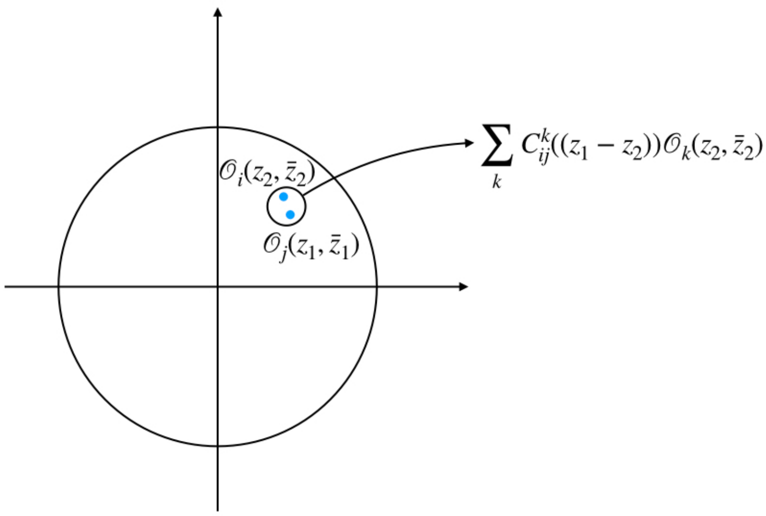

Figure 5.2: OPE of two operators at $(z_1,\bar{z}_1)$ and $(z_2,\bar{z}_2)$

conformal fields one can perform the contractions of Wick's theorem. Using the definition of a primary field $\psi(w,\overline{w})$ in (5.3.25) one can show that in a given CFT with energy-momentum tensor $T(z)$ the OPE of the $T(z)\psi(w,\overline{w})$ is given by:

$$\begin{aligned} T(z)\psi(w,\overline{w}) &\sim \frac{h}{(z-w)^2}\psi(w,\overline{w}) + \frac{1}{(z-w)}\partial\psi(w,\overline{w}) + \dots, \\ \overline{T}(\overline{z})\psi(w,\overline{w}) &\sim \frac{\overline{h}}{(\overline{z}-\overline{w})^2}\psi(w,\overline{w}) + \frac{1}{(\overline{z}-\overline{w})}\overline{\partial}\psi(w,\overline{w}) + \dots . \end{aligned} \tag{5.3.32}$$

above the "..." means terms that are regular in $z-w$. We use this notation for regular terms throughout this work.

n-point function of primary fields

Having a conformal (superconformal) symmetry for a given theory constrains the type of functions that can appear in the OPE of the primary fields. Since the primary fields behave as in equation (5.3.25) then the resulting OPE should also behave the same way. Therefore, the form of two and three-point functions are fixed up to constant coefficients.

Primary fields correlation

Taking $\psi_i(z)$ to be a holomorphic primary field with conformal dimension h_i then we have for the two and three-point functions (the anti–holomorphic case is similar with conformal weight $\overline{h}_i$):

$$\begin{aligned} \langle\psi_i(z)\psi_j(w)\rangle &= \frac{d_{ij}\delta_{h_i,h_j}}{(z-w)^{h_i+h_j}}, \\ \langle\psi_1(z_1)\psi_2(z_2)\psi_3(z_3)\rangle &= \frac{C_{123}}{(z_{12})^{h_1+h_2-h_3}(z_{13})^{h_1+h_3-h_2}(z_{23})^{h_2+h_3-h_1}}, \end{aligned} \tag{5.3.33}$$

where $z_{ij} = z_i - z_j$.

As a very important example, we can take the superstring action in (5.2.6) and write down the OPEs of the primary fields ∂X^μ and ψ^μ in the following as:

Disk	Sphere
$X^\mu(z)X^\nu(w) \sim -\alpha' \ln(z-w) + ...$	$X^\mu(z)X^\nu(w) \sim -\frac{1}{2}\alpha' \ln(z-w) + ...$
$X^\mu(z)X^\nu(\overline{w}) \sim -\alpha' \ln(z-\overline{w}) + ...$	$X^\mu(z)X^\nu(\overline{w}) \sim 0$
$\partial X^\mu(z)\partial X^\nu(w) \sim -2\alpha \frac{\eta^{\mu\nu}}{(z-w)^2} + ...$	$\partial X^\mu(z)\partial X^\nu(w) \sim -\frac{1}{2}\alpha' \frac{\eta^{\mu\nu}}{(z-w)^2} + ...$
$\partial X^\mu(z)\partial X^\nu(\overline{w}) \sim -2\alpha \frac{\eta^{\mu\nu}}{(z-\overline{w})^2} + ...$	$\partial X^\mu(z)\partial X^\nu(\overline{w}) \sim 0$
$\psi^\mu(z)\psi^\nu(w) \sim \frac{\eta^{\mu\nu}}{(z-w)} + ...$	$\psi^\mu(z)\psi^\nu(w) \sim \frac{\eta^{\mu\nu}}{(z-w)} + ...$
$\psi^\mu(z)\overline{\psi}^\nu(\overline{w}) \sim \frac{\eta^{\mu\nu}}{(z-\overline{w})} + ...$	$\psi^\mu(z)\overline{\psi}^\nu(\overline{w}) \sim 0$

Table 5.1: Two point functions of different fields over sphere and disk

First, we note that the field X^μ is not primary but since we need to use the OPE to derive other OPEs we included it in the list. Second, the contractions between holomorphic and anti-holomorphic fields are dependent on the Riemann surface of the world–sheet for example over the sphere there are no mixed contractions. Whereas for the disk there are contractions between the holomorphic and anti-holomorphic fields. In order to study tree level amplitudes we do not need to look for more Riemann surfaces and these OPEs will be sufficient to calculate all amplitudes required for the present work.

5.3.2 Mode expansion and Virasoro algebra

To construct the physical spectrum in canonical quantization of superstring theory we require two ingredients:

- Creation and annihilation operators
- Symmetry generators to constrain physical state conditions.

For the first ingredient, we should (Fourier) mode expand the primary fields ∂X^μ and ψ^μ on shell (i.e. on the solution of equations of motion). Using the equations of motion given in (5.3.19) we have the following expansion for the field content of superstring theory [11]:

$$\begin{aligned} \partial X^\mu(z) &= -i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{m=-\infty}^{\infty} \frac{\alpha_m^\mu}{z^{m+1}}; \qquad \overline{\partial} X^\mu(\overline{z}) = -i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{m=-\infty}^{\infty} \frac{\tilde{\alpha}_m^\mu}{\overline{z}^{m+1}}\,, \\ \psi^\mu &= \sum_{r\in\mathbf{Z}+\nu} \frac{b_r^\mu}{z^{r+1/2}}; \qquad \overline{\psi}^\mu(\overline{z}) = \sum_{r\in\mathbf{Z}+\nu} \frac{\tilde{b}_r^\mu}{\overline{z}^{r+1/2}}\,, \end{aligned} \tag{5.3.34}$$

where for $\nu = 0$ we have the Ramond and for $\nu = \frac{1}{2}$ we have Neveu-Schwarz sectors. The energy-momentum tensors (5.2.7) in complex coordinates are given by:

$$
\begin{aligned}
&T_B(z) = -\frac{1}{\alpha'} : \partial X^\mu \partial X_\mu - \frac{1}{2}\psi^\mu \overline{\partial}\psi_\mu; \qquad \tilde{T}_B(\overline{z}) = -\frac{1}{\alpha'} : \overline{\partial} X^\mu \overline{\partial} X_\mu - \frac{1}{2}\overline{\psi}^\mu \partial \overline{\psi}_\mu \, , \\
&T_F(z) = i\left(\frac{2}{\alpha'}\right)^{1/2} \psi^\mu \partial X_\mu; \qquad \tilde{T}_F(\overline{z}) = i\left(\frac{2}{\alpha'}\right)^{1/2} \overline{\psi}^\mu \overline{\partial} X_\mu \, .
\end{aligned}
\tag{5.3.35}
$$

Therefore, for the energy-momentum tensors, we have the following expansion:

$$
\begin{aligned}
T_B(z) &= \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}}; \qquad \tilde{T}_B(\overline{z}) = \sum_{m=-\infty}^{\infty} \frac{\tilde{L}_m}{\overline{z}^{m+2}} \, , \\
T_F(z) &= \sum_{r \in \mathbf{Z}+\nu} \frac{G_r}{z^{r+3/2}}; \qquad \tilde{T}_F(\overline{z}) = \sum_{r \in \mathbf{Z}+\nu} \frac{\tilde{G}_r}{\overline{z}^{r+3/2}} \, ,
\end{aligned}
\tag{5.3.36}
$$

where ν defines the RNS sectors as above. Upon using the canonical quantization condition one obtains the following algebra for the modes [10, 11, 12, 65]:

Mode Algebra for RNS superstring

First, we have the canonical quantization condition:

$$
\begin{aligned}
[\alpha^\mu_m, \alpha^\nu_n] &= [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\eta^{\mu\nu}\delta_{m+n,0} \, . \\
\{b^\mu_r, b^\nu_s\} &= \{\tilde{b}^\mu_r, \tilde{b}^\nu_s\} = \eta^{\mu\nu}\delta_{r+s,0} \, .
\end{aligned}
\tag{5.3.37}
$$

Using these algebras and the definition (5.2.7) of T_B and T_F on obtain the well known supervirasoro algebra:

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{c}{12}\delta_{r+s,0}. \\
[L_m, G_r] &= \frac{m-2r}{2}G_{m+r}.
\end{aligned}
\tag{5.3.38}
$$

where $r \in \mathbf{Z}$ for the *Ramond* sector and $r \in \mathbf{Z} + \frac{1}{2}$ for the *Neveu-Schwarz* sector of superstring theory.

Central charge

In the supervirasoro algebra, we have been using the factor c without explanation. Here we are going to define c which is known as *central charge*. We should start by explaining that one of the main features of CFTs in physics is that they do not require a Lagrangian description. Meaning, to have a CFT one does not need an action as in (5.2.6). As we discussed the singular part of the OPEs are fixed through the conformal symmetry. So the only necessary objects are the primary fields and the algebra of the modes. In order to fix the algebra one has to take into account that the Virasoro algebra admits extensions. Meaning, one can add a term to the algebra without breaking it.

This is known as *central extension* and can be parameterized by a *central charge c*. We have noted this extension in the algebra (5.3.2) in the commutation of L_ms. The central charge for a given CFT is associated with the "soft" breaking of the conformal symmetry. Physically speaking, it means how a given CFT reacts to the introduction of *macroscopic length* into the theory, for example a boundary [67]. Further, given the central charge, the form of the OPE of the energy-momentum tenor is fixed:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \dots . \tag{5.3.39}$$

Therefore, giving the primary fields together with the c will be sufficient to define a CFT.

5.4 Supersymmetry

So far, in our discussion we have used the concept of *supersymmetry* sporadically. Here we are going to give a very brief introduction to the topic [70]. As we mentioned in the beginning supersymmetry is the symmetry that relates (maps) bosons to fermions. Quantum field theories originally had two types of symmetries:

- Internal symmetries like gauge symmetries. Mathematically speaking these are defined as extra structures (bundles) on top of the spacetime manifold. Normally they are compact Lie groups like $U(1)$, $SU(N)$, etc (with some generator $\mathbf{G}$).
- On the other hand, there are external symmetries that are associated with the (isometries of) spacetime manifold itself, i.e. Poincare symmetry. The Poincare symmetry is generated through rotations $M_{\mu\nu}$ and translations P_μ.

All fields in quantum field theories are in a *representation* of the spacetime symmetry (e.g. they are scalar, vector, tensor, etc). In contrast, this is not the case for internal symmetries, for example, not all fields are eclectically charged. These two types of symmetries commute with each other meaning we we can write the full symmetry as a product:

$$\begin{aligned} &[G, P_\mu] = [G, M_{\mu\nu}] = 0, \\ &\text{Symmetries: Poincare } \otimes \text{ internal symmetry group.} \end{aligned} \tag{5.4.40}$$

Further, it was shown that any attempt to enlarge the above structure will result in a trivial theory (i.e. $S-$matrix= $\mathbf{1}$). This is known as the Coleman-Mandula *no go theorem* [71]. Like any other no go theorem it was circumvented by avoiding one or more of the assumptions. In this case, Coleman-Mandula assumed that the additional group is a Lie group with a commuting Lie algebra with bosonic generators (one can see that P_μ and $M_{\mu\nu}$ are in the tensor representation of the Poincare group). Therefore, it was circumvented by assuming that the "new" symmetry does not have a commuting but an anti-commuting algebra with fermionic generators [72, 73]. This is a generalization of the Lie algebra known as graded Lie algebra schematically we have:

$$\begin{aligned}&\text{Lie algebra:} \quad [A,B] = -[B,A],\\&\text{graded Lie algebra:} \quad [A,B] = (-1)^k[B,A].\end{aligned} \tag{5.4.41}$$

This change in the assumption to the Coleman-Mandula no-go-theorem allowed for a symmetry with fermionic generators Q_α where $\alpha = 1,2$ index is the $SU(2)$ representation index. The algebra of supersymmetry generators are given by:

$$\begin{aligned}&[Q_\alpha, P_\mu] = 0, \qquad [Q_\alpha, G^a] = 0,\\&[Q_\beta, M_{\mu\nu}] = \frac{1}{2}(\sigma^{\mu\nu})^\alpha{}_\beta Q_\alpha, \qquad \{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta}P_\mu,\\&Q^2 = 0.\end{aligned} \tag{5.4.42}$$

This type of symmetry is called *Supersymmetry* (SUSY) and a theory that exhibits this symmetry is supersymmetric like the superstring theory we have introduced so far. Looking at the above algebra one can readily see that supersymmetry leaves the internal symmetry and the Hamiltonian (which is the conserved charge of the translation current P_μ) intact and just affects the Lorentz generators. In other words, while acting on a physical fields by a supersymmetry generator (charge) preserves the mass and the charge of the fields, it will change its representation under the Lorentz group. Since the generator is fermionic it will map bosons to fermions and fermions to bosons. In short, for a state with mass m, spin s and charge q of the internal group we have[2]:

$$Q|m,q,s\rangle = |m,q,s \pm \frac{1}{2}\rangle. \tag{5.4.43}$$

In a supersymmetric theory the fields are going to be in representations of SUSY which means that states form multiplets through the action of the supersymmetry operator Q as demonstrated above. All of the states inside a multiplet have the same mass and charge but different spins. Looking at the algebra we see that the representation of the supersymmery depends on the mass of the states for the massive states we have:

$$\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2m\delta_{\alpha\dot\beta},$$

and for the massless states we have

$$\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2E\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}.$$

Therefore, for the massive case we have states with spins $(s, s\pm\frac{1}{2}, s)$ and for the massless case we have states with *helicities* $(\lambda, \lambda+\frac{1}{2})$ and $(-\lambda, -\lambda-\frac{1}{2})$ in order to preserve CPT symmetry.

[2]The plus-minus in the spin depends on the choice of the representation of the Q is not important since whatever the sign of Q, $\overline{Q}$ will be the reverse and both are always present so its a matter of choice

The fact that Q is nilpotent will insure that the multiplets are finite (i.e. the number of states inside the multiplet is finite). Looking back at the above example we have the massive states (with choice ot plus sign for the Q):

$$\begin{aligned} Q|m,q,s\rangle &\to |m,q,s+\frac{1}{2}\rangle, \\ \overline{Q}|m,q,s\rangle &\to |m,q,s-\frac{1}{2}\rangle, \\ \overline{Q}|m,q,s+\frac{1}{2}\rangle &= \overline{Q}(Q|m,q,s\rangle) = |m,q,s\rangle, \end{aligned} \tag{5.4.44}$$

and the multiplet will be:

$$\begin{pmatrix} |m,q,s\rangle \\ |m,q,s+\frac{1}{2}\rangle \end{pmatrix} \oplus \begin{pmatrix} |m,q,s\rangle \\ |m,q,s-\frac{1}{2}\rangle \end{pmatrix}. \tag{5.4.45}$$

So far, we have only mentioned theories with one supersymmetry i.e. there is only one Q. However, it is possible to have more than one Q^I with $I = 1, 2, 3, ..., \mathcal{N}$ this is known as *extended* SUSY. The value of $\mathcal{N}$ will determine the extended SUSY of a theory.

Since multiplets are invariant under all of the extended SUSY, having more than one SUSY generator creates a rotation symmetry among the SUSY generators. In case of more than one Q we can always rotate the Q^Is meaning supersymmetric theories (and multiplet) are invariant under:

$$Q^I = \lambda_{IJ} Q^J. \tag{5.4.46}$$

Since Q^I are complex the λ_{IJ} are going to be elements of $SU(\mathcal{N})$. Therefore, a theory with extended $\mathcal{N}$ supersymmetry has an extra $SU(\mathcal{N})$ symmetry known as $R-$symmetry. We finish this section by mentioning the two types of supersymmetry that we encounter in studying superstrings

- World-sheet SUSY: This is the supersymmetry associated with the action (5.2.6). It is seen as the map between bosonic field ∂X^μ to fermionic field ψ^μ. As we are going to see in the next section there can be different numbers of supersymmetries on the world-sheet.

- Spacetime SUSY: This type of supersymmetry is associated with the spacetime fields and produced through the content of the superstrings. It is realized in the effective actions constructed through string theory amplitudes like supergravities.

We should emphasize being bosons and fermions on the world–sheet and spacetime are completely different subjects. For example, we have the spin-2 massless state in the superstring that contains both ∂X and ψ fields, as well as its superpartner gravitino, which also contains both ∂X and ψ fields. The number of supersymmetry in the spacetime SUSY actions are $\mathcal{N} = 1$ for open strings and $\mathcal{N} = 2$ for closed strings. Through various compactifications the number of SUSY can be enhanced or broken for example compacitfying over a six-torus will enhance spacetime SUSY to $\mathcal{N} = 8$ (cf. [12, 11, 65, 10]).

5.5 String spectrum

This section is one of the more important parts of this work. Here, we introduce the spectrum of the closed and open superstring theory[3]. We are going to do so in three steps: First, we construct the states using the covariant method of creation and annihilation operators. Second, we will give the little group and spacetime Lorentz representation of each state and finally we introduce the notion of *operator state correspondence* and give the vertex operator associated with each state.

We start by introducing different types of superstring theories. There are five consistent superstrings known so far:

- Type I: String theory involve both open and closed strings with $\mathcal{N} = 1$ supersymmetry with the gauge group $SO(32)$ or $E_8 \times E_8$.
- Type IIA: Closed string theory with with $\mathcal{N} = (1, 1)$ world–sheet supersymmetry. The left and right movers of this theory are of *opposite* chirality. So, it is a non-chiral theory.
- Type IIB: Closed string theory with with $\mathcal{N} = (1, 1)$ world–sheet supersymmetry. The left and right movers of this theory are of *same* chirality. So, it is a chiral theory.
- $E_8 \times E_8$ and $SO(32)$ Heterotic: Closed string theory whose left movers are 26 dimensional bosonic string and 16 dimensional of these are compactified on torus and the right movers are 10 superstring modes with $\mathcal{N} = (0, 1)$ world–sheet supersymmetry

For the current work, we are only going to look at the (universal) (NS,NS) part of Type II superstrings in the background of D-branes interacting with NS open strings.

5.5.1 Construction of States

The most intuitive method to construct the states of the superstring is to do it in the old covariant method. In this method every expansion coefficient introduced in section 5.3.2 will be a quantum operator satisfying the canonical commutation or anti-commutation relations for bosons and fermions, respectively. In the superstring case, we have two primary fields with the corresponding creation and annihilation operators namely:

$$\begin{aligned} &\partial X^\mu(z) \leftrightarrow (\alpha_n^\mu, \alpha_n^{\dagger\mu}), \qquad \overline{\partial} X^\mu(\overline{z}) \leftrightarrow (\tilde{\alpha}_n^\mu, \tilde{\alpha}_n^{\dagger\mu}), \\ &\psi^\mu(z) \leftrightarrow (b_r^\mu, b_r^{\dagger\mu}), \qquad \overline{\psi}(\overline{z}) \leftrightarrow (\tilde{b}_r^\mu, \tilde{b}_r^{\dagger\mu}), \end{aligned} \tag{5.5.47}$$

where m is an integer and r is a half integer numbers. The complex conjugate operators are related to each other by the change of positive and negative modes (i.e. the integer m):

$$(\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu, \qquad (\tilde{\alpha}_n^\mu)^\dagger = \tilde{\alpha}_{-n}^\mu\,.$$

[3]We mention the results for bosonic string theory whenever it is necessary

To evaluate the mass of different string states it is easier follow the recipe of canonical quantization in the light cone coordinates given by:

$$x^\mu = (x^+, x^-, x^i), \qquad i = 2, ..., D-1; \qquad x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)\,.$$

Therefore, the index μ of the oscilators (α^μ_n, b^μ_r) decomposes the same way to $\mu = (\pm, i)$. One finds the Hamiltonian of the light cone action (5.2.6) by using the number operator N which is defined for the RNS superstring as:

$$\begin{aligned} N &= N_B + N_F = \sum_{n>0} \alpha^i_{-n}\alpha^i_n + \sum_{r>0} r b^i_{-r} b^i_r\,, \\ H &= \begin{cases} \alpha' p^2 + N - \frac{1}{2} & NS\,, \\ \alpha' p^2 + N & R\,. \end{cases} \end{aligned} \tag{5.5.48}$$

Using $p^2 = -m^2$ the mass operator will be given by:

$$\alpha' m^2 = \begin{cases} N - \frac{1}{2} & NS\,, \\ N & R\,. \end{cases} \tag{5.5.49}$$

In this mass operator we have avoided the contribution from the D-branes[4].

To construct a state we need to define the vacuum and act by creation and annihilation operators (5.5.47) on it. We define the vacuum as the state annihilated by positive modes:

String state

$$\begin{aligned} \alpha^\mu_n|0\rangle &= 0 \qquad \text{For all } n > 0, \\ b^\mu_r|0\rangle &= 0 \qquad \text{For all } r > 0. \end{aligned} \tag{5.5.50}$$

One can make other choices for the vacuum in particular one can choose positive modes for holomorphic fields and negative norms for anti-holomorphic fields this will give rise to *ambitwistor* constructions in string theory [74]. Then, a generic string state can be given by[a]:

$$|\gamma\rangle = \prod_{k,l} b^{\mu_l}_{-r_l}\alpha^{\nu_k}_{-n_k}|0\rangle. \tag{5.5.51}$$

Using the mass operator in (5.5.49) will give us the following mass for the state:

$$\alpha' m^2(|\gamma\rangle) = \begin{cases} \sum_k n_k + \sum_l r_l - \frac{1}{2} & \text{NS-sector.} \\ \sum_k n_k + \sum_l r_l & \text{R-sector,} \end{cases} \tag{5.5.52}$$

[4]The D-brane contribution is given by $\frac{1}{4\pi^2\alpha'}(\Delta X)^2$ where ΔX is the distance between two D-branes that open string starts and end between them.

[a]Since we want to use BRST symmetry in the next section and we do not need to use the transverse coordinates in BRST quantization to fix the gauge we also avoided the light cone coordinates here.

5.5.2 BRST symmetry and ghost system

There are many different symmetries involved in the superstring theory from superconformal symmetry to Chan-Paton gauge symmetries. Any physical state should be constrained with all the symmetries of the theory. Therefore, not every possible combination of the creation operator as written in (5.5.51) are part of the spectrum and we have to constrain them with *physical conditions*. Here, we are going to provide the physical state conditions using the supervirasoro algebra modes acting on the states:

Physical conditions

There are two types of conditions that we are going to put on the states:

- Symmetry constraints
- Positive norm constraints

As we discussed before, the symmetries of superstring are generated through currents that are given by the energy-momentum tensor. A state will be invariant (physical) if its variation is zero and since the mode expansion of T_B and T_F (5.3.36) are the generators of superconformal symmetry one can define the following conditions for open strings:

$$\begin{aligned} G_r|\gamma_{phys}\rangle &= 0 \\ L_m|\gamma_{phys}\rangle &= 0 \\ L_0|\gamma_{phys}\rangle &= 0 \end{aligned} \tag{5.5.53}$$

Here all modes are positive (i.e. $r, m > 0$). For the closed string, we should add the anti-holomorphic partner of these relations, meaning:

$$\begin{aligned} \overline{G}_r|\gamma_{phys}\rangle &= 0, \\ \overline{L}_m|\gamma_{phys}\rangle &= 0, \\ \overline{L}_0|\gamma_{phys}\rangle &= 0, \end{aligned} \tag{5.5.54}$$

as well as the level matching condition:

$$L_0 - \overline{L}_0|\gamma_{phys}\rangle = 0. \tag{5.5.55}$$

This condition insures that the masses and conformal weight of the left and right mover fields are the same.

The second type of constraint is the positive norm constraint. Meaning, we want the physical states to have positive norms:

$$\langle\gamma|\gamma\rangle > 0.$$

This is implemented through the normal ordering constant inside L_0 which has been fixed for the NS sector. This constraint is satisfied when $a^{NS} = -\frac{1}{2}$ for NS sector and $a^R = 0$ for R sector . This condition has a more important consequence since the eigenvalue of the operator L_0 is the conformal weight of a given CFT state. Then this condition means that all physical states have conformal dimension zero:

$$L_0|\gamma_{phys}\rangle = 0 \Rightarrow L_0|\gamma_{phys}\rangle = (0)|\gamma_{phys}\rangle \to h_\gamma = 0. \tag{5.5.56}$$

It is clear that checking every single state, for every constraint given above, is an exhaustive task. Further, if one wants to use the path integral formulation (which we are going to do in the next section) this method is not applicable. The solution is to use the BRST symmetry. Here, we are going to provide a brief review of this symmetry and how it works in practice. As mentioned the BRST symmetry is most useful in the path integral formulation where one's goal is to fix gauge symmetries and avoid integrating over an infinite measure. In order to achieve that in the path integral formulation one uses the method by *Faddeev-popov* [75]. To implement the system we need to introduce two more conformal field theories:

- The $b-c$ ghost system of *anti-commuting bosons* that is given by the following action:

$$S^{b-c} = \frac{1}{2\pi}\int d^2z(b\overline{\partial}c + \overline{b}\partial\overline{c}). \tag{5.5.57}$$

  with the equations of motion:

$$\overline{\partial}c = \overline{\partial}b = 0, \qquad \partial\overline{c} = \partial\overline{b} = 0. \tag{5.5.58}$$

  Here, as it can be seen by equations of motion and dimensional analyses of the action $b(z)$ is a holomorphic ($\overline{b}(\overline{z})$ is an anti-holomorphic) function with conformal dimension $(h,\overline{h}) = (2,0)$ ($(h,\overline{h}) = (0,2)$) and $c(z)$ is a holomorphic ($\overline{c}(\overline{z})$ anti-holomorphic) function with conformal dimension $(h,\overline{h}) = (-1,0)$ ($(h,\overline{h}) = (0,-1)$).

  The $b-c$ ghost system is responsible for fixing the gauge for the bosonic part (X^μ) of the action (5.2.6). In addition, we need the energy-momentum tensor[5]:

$$T_B^{b,c} = 2 : \partial c(z) b(z) : + : c(z)\partial b(z) : . \tag{5.5.59}$$

- The $\beta-\gamma$ system of *commuting fermions* is the superpartner of $b-c$ system. It is given by the action:

$$S^{\beta-\gamma} = \frac{1}{2\pi}\int d^2z(\beta\overline{\partial}\gamma + \overline{\beta}\partial\overline{\gamma}). \tag{5.5.60}$$

[5] The anti-holomorphic part is obtained by complex conjugation

This system also has a holomorphic field $\beta(z)$ (anti-holomorphic field $\overline{\beta}(\overline{z})$) with conformal dimension

$$(h,\overline{h}) = (\frac{3}{2},0), \qquad (h,\overline{h}) = (0,\frac{3}{2})\,,$$

and $\gamma(z)$ is a holomorphic ($\overline{\gamma}(\overline{z})$ anti-holomorphic) function with conformal dimension

$$(h,\overline{h}) = (-\frac{1}{2},0), \qquad (h,\overline{h}) = (0,\frac{3}{2})\,.$$

Similarly, one can define the following energy-momentum tensor[6]:

$$T_B^{\beta,\gamma} = -\frac{3}{2} :\partial\gamma(z)\beta(z): -\frac{1}{2} :\gamma(z)\partial\beta(z): \,. \tag{5.5.61}$$

An easier way of working with the theory of $\beta-\gamma$ system is by bosnization which assigns boson fields (ϕ,χ) to (β,γ) as:

$$\beta = e^{-\phi}e^{\chi}\partial\chi, \qquad \gamma = e^{\phi}e^{-\chi}\,. \tag{5.5.62}$$

together with the following OPEs:

$$\phi(z)\phi(w) \sim \ln(z-w), \qquad \chi(z)\chi(w) \sim \ln(z-w)\,. \tag{5.5.63}$$

Finally, as we mentioned in the case of the matter fields X^μ and ψ^μ (cf. (5.2.7)) we noted there was the superpartner of the matter energy-momentum tensor corresponding to the gravitino. Similarly, there is a superpartner to the energy-momentum tensors (5.5.61) and (5.5.59) coming from the variation of the action with respect to the gravitino. We denote it by $T_F^{b,c,\beta,\gamma}$ and it is given by:

$$T_F^{b,c,\beta,\gamma} = \frac{1}{2} :\gamma(z)b(z): - :c(z)\partial\beta(z): -\frac{3}{2} :\partial c(z)\beta(z): \,. \tag{5.5.64}$$

We can summarise the mass and conformal weight of the fields in the full theory in the following tables. We have the mass dimensions:

Field	α'	X^m, c	γ	Q	ψ^m	β, T_β	b, T_b
Mass dimension	-2	-1	$-1/2$	0	$1/2$	$3/2$	2

and similarly for the conformal weight:

Operator	$\partial X^m(z)$	$\psi^m(z)$	$e^{q\phi(z)}$	$e^{i\alpha' pX(z)}$
Weight	$(1,0)$	$(1/2,0)$	$(-\frac{1}{2}q(q+2),0)$	$(\frac{\alpha' p^2}{4},0)$

[6]This is the holomorphic part of the energy-momentum tensor the anti-holomorphic has the same form with the anti-holomorphic fields.

for the ghost system:

Operator	b	c	β	γ
Weight	$(-1,0)$	$(2,0)$	$(\frac{3}{2},0)$	$(-\frac{1}{2},0)$

Therefore, taking all these parts together we can write the full *gauge fixed* string action:

$$\begin{aligned} S^{superstring}_{gaugefixed} =& S^X + S^\psi + S^{b-c} + S^{\beta-\gamma}, \\ S^{superstring}_{gaugefixed} =& \frac{1}{4\pi}\int dz d\overline{z}\ \Bigg[\frac{2}{\alpha'}\partial X^\mu(z,\overline{z})\overline{\partial}X_\mu(z,\overline{z}) + \psi^\mu\overline{\partial}\psi_\mu + \overline{\psi}^\mu\partial\overline{\psi}_\mu \\ & + 2\Bigg(b\overline{\partial}c^z + \overline{b}\partial\overline{c} + \beta\overline{\partial}\gamma + \overline{\beta}\partial\overline{\gamma}\Bigg)\Bigg] + \lambda\chi. \end{aligned} \tag{5.5.65}$$

Now we can construct the nilpotent BRST charge in terms of the energy-momentum tensors (5.5.59),(5.5.61),(5.5.64) we get[7]:

$$\begin{aligned} Q_{BRST} =& \oint_{C_0}\frac{dz}{2\pi i} j^F_{BRST} \\ =& \oint_{C_0}\frac{dz}{2\pi i}\Bigg\{ :c(z)\Big[T_B^{X,\psi}(z) + \frac{1}{2}(T_B^{b,c} + T_B^{\beta,\gamma})(z)\Big]: + \\ & :\gamma(z)\Big[T_F^{X,\psi}(z) + \frac{1}{2}T_F^{b,c,\beta,\gamma}(z)\Big]: \Bigg\}. \end{aligned} \tag{5.5.66}$$

Here, we have used the notation: $T_B^{X,\psi}(z) = T_B^X(z) + T_B^\psi(z)$ and $T_F^{X,\psi}(z) = T_F^X(z) + T_F^\psi(z)$. One can check that this charge is indeed nilpotent $Q^2_{BRST} = 0$ and therefore one can define the BRST cohomology with the action of Q_{BRST} on the states. We can use this cohomology to define the physical states. A state is said to be BRST *invariant* if it is annihilated by Q_{BRST}:

$$Q_{BRST}|\gamma\rangle = 0 \tag{5.5.67}$$

We discussed the generic definition of cohomology in chapter 2 (cf. 2.5) and mentioned that a state that is BRST invariant as in (5.5.67) is either of the form $|\alpha\rangle = Q_{BRST}|\delta\rangle$ (i.e. exact state) or not. The exact state α has zero norm due to the nilpotency and hermiticity of the BRST charge operator ($\langle\alpha|QQ|\alpha\rangle = 0$). Since the BRST charge commutes with the Hamiltonian these exact states will decouple from the S-matrix of the theory. Therefore, one can define the physical state in the Hilbert state of the string theory as:

[7] The $\overline{Q}$ is defined in the same way with anti-holomorphic energy-momentum tensors.

Physical state

A state $|\gamma\rangle$ is physical if it is closed under the BRST cohomology but not exact. Meaning:

$$\begin{aligned} &|\gamma\rangle \in \mathcal{H}_{full}, \\ &Q_{BRST}|\gamma\rangle = 0, \qquad |\gamma\rangle \neq Q|\alpha\rangle, \qquad \forall|\alpha\rangle. \end{aligned} \tag{5.5.68}$$

Therefore, a generic state in the Hilbert space can be decomposed as the following product:

$$|\gamma\rangle \in \mathcal{H}_{full}, \Rightarrow |\gamma\rangle = \underbrace{|X;\psi\rangle}_{matter\ fields} \otimes \underbrace{|\downarrow_{b,c}; q_{\beta,\gamma}\rangle}_{ghost\ field}. \tag{5.5.69}$$

One can show by direct computation that the condition (5.5.68) is equivalent to the conditions in (5.5.53). The ghost part of the state (5.5.69) is the permissible vacuum of the $b-c$ and $\beta-\gamma$ system. These two theories similar as the matter fields are CFTs. They have oscillation modes and therefore a vacuum state that is annihilated by all non-negative modes and contains the zero modes. These zero modes commute with the Hamiltonian of the respective systems [8]. This results in a degeneracy of the vacuum. Under the $b-c$ system is easy to consider since the $b-c$ fields are anti-commuting i.e. *Graßmann* the zero modes b_0 and c_0 will have only two eigenstates denoted by $|\downarrow\rangle$ and $|\uparrow\rangle$. The two degenerate states in terms of field modes of the $b-c$ system are given by:

$$\begin{aligned} |\downarrow\rangle &= c_1|0\rangle_{b-c}, \\ |\uparrow\rangle &= c_0c_1|0\rangle_{b-c}. \end{aligned} \tag{5.5.70}$$

In order to satisfy all conditions in (5.5.53) and (5.5.54) we had to choose $|\downarrow\rangle$ for the vacuum string state. Furthermore, the ghosts always cancel the the light cone coordinates $x^{\pm}$ therefore we do not need to use the transverse indices (i,j) for the creation annihilation operators while constructing the states.

The situation for $\beta-\gamma$ is more complicated. Since we cannot use the Grassmann properties (they are commuting fields) we will have infinite equivalent possibilities which are known as *ghost picture* we will get back to this in the next section when we construct the vertex operators.

5.5.3 Operator state correspondence and Vertex operator

So far our discussion on string theory as a $2d$ CFT has been done in two parallel paths. First, we have the operator formulation. We introduced the full action of the gauge fixed superstring and gave the primary fields which upon quantization become local operators. Second, in order to construct the states of superstring theory we followed covariant

[8] b_0 (this is not the zero mode of he matter field ψ, it is the mode expansion of the field b) and c_0 commute with L_0^{b-c} similarly for β_0 and γ_0

quantization meaning we had the mode expansion of the matter fields acting on the vacuum and created a generic state in (5.5.69). Now, we want to point out the relationship between local operator and a given state in the CFT. This is known as *operator-state correspondence* with the following statement:

Operator-state correspondence

There is an isomorphism between the local operators on the complex plane **C** and the initial state on a cylinder.
It is not difficult to check that the isomorphism map is given by:

$$\begin{aligned} &S^1 \times \mathbf{R} \to \mathbf{C}, \\ &z \mapsto e^{-iw}, \qquad w = \tau + i\sigma. \end{aligned} \tag{5.5.71}$$

One can clearly see that the circle at $\tau = -\infty$ is mapped to the origin and the circle $\tau = \infty$ is mapped to the infinity.

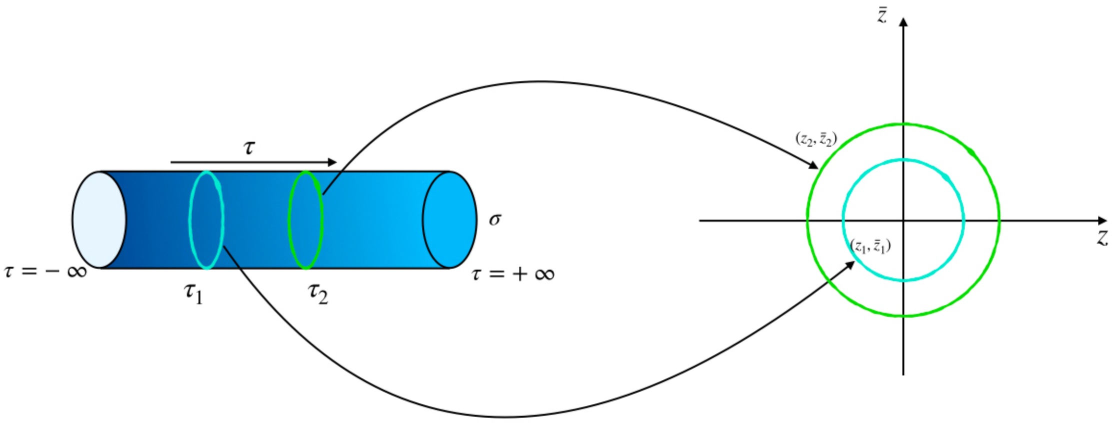

Figure 5.3: Map from cylinder to complex plane

This map gives us the power to map the propagation of closed string[9] to operators on the complex plane with the *initial state* given as the local operator at the origin which is known as *vertex operator*. We are going to discuss extensively the implication and application of this operator and map in the next section. For now, we give the general recipe to construct vertex operator associated with any state of the form (5.5.69) in the NS sector. We have the following construction:

[9] The same map exists for open string mapping the ribbon to the upper half plane

Vertex operator for open string

The generic state $|\gamma\rangle = \underbrace{|X;\psi;a\rangle}_{\textit{matter fields}} \otimes \underbrace{|\downarrow_{b,c};q_{\beta,\gamma}\rangle}_{\textit{ghost fields}}$ has two parts: a matter part and a ghost $(b-c/\beta-\gamma)$ part. For the matter part we have:

$$|X;\psi;a\rangle_{\textit{matter fields}} = T^a \prod_{k,l} b^{\mu_l}_{-r_l}\alpha^{\nu_k}_{-n_k}|0\rangle. \tag{5.5.72}$$

where T^a is the generator of the Chan-Paton symmetry. Then, we introduce the *Fourier back* transformation map of the creation modes α^{μ}_{-n} and b^{ν}_{-r}. Taking $n \in \mathbf{Z}^+$ and r is the positive half integer $\mathbf{Z}+\frac{1}{2}$ (NS sector):

$$\begin{aligned} |0\rangle &\to \int \frac{dz}{2\pi} e^{ik\cdot X}, \\ \alpha^{\mu}_{-n} &= \left(\frac{2}{\alpha'}\right)^{1/2} \oint \frac{dz}{2\pi} z^n \partial X^{\mu}(z) \to \left(\frac{2}{\alpha'}\right)^{1/2} \frac{i}{(n-1)!}\partial^n X^{\mu}(0), \\ b^{\mu}_{-r} &= \oint \frac{dz}{2\pi} z^{r+\frac{1}{2}}\psi^{\mu}(z) \to \frac{i}{(r-\frac{1}{2})!}\partial^{r-\frac{1}{2}}\psi^{\mu}(0). \end{aligned} \tag{5.5.73}$$

Therefore, the matter part of the vertex operator will be:

$$|X;\psi;a\rangle_{\textit{matter fields}} = T^a \prod_{k,l} b^{\mu_l}_{-r_l}\alpha^{\nu_k}_{-n_k}|0\rangle \Rightarrow T^a : \mathcal{F}(\partial^{r-\frac{1}{2}}\psi^{\mu};\partial^n X^{\nu}) : , \tag{5.5.74}$$

above we have used normal ordering for the vertex operator. In the ghost part we have the following state $|\downarrow_{b,c}\rangle \otimes |q_{\beta,\gamma}\rangle_{\textit{ghost charges}}$. Functionally, the choice of the state $|\downarrow_{b,c}\rangle$ amounts to adding $c(z)$ to the matter vertex operator or using the integrated vertex operator. The contribution of the the $\beta-\gamma$ system enters through the bosonized field ϕ. Putting all together we have:

$$V(\partial^{r-\frac{1}{2}}\psi^{\mu};\partial^n X^{\nu},\phi) \otimes |\downarrow_{b,c}\rangle = c(z)\; T^a : \mathcal{V}(\partial^{r-\frac{1}{2}}\psi^{\mu};\partial^n X^{\nu},\phi) : . \tag{5.5.75}$$

We will address the ϕ dependence (i.e. $|q_{\beta,\gamma}\rangle_{\textit{ghost charge}}$) in the picture changing section.

Finally, the last thing is to impose the BRST condition (5.5.68) on the local vertex operator. From the operator-state correspondence we know that the acting on states is mapped to commutation of operators. Meaning, the vertex operator should commute with Q_{BRST} up to a total derivative:

$$[Q_{BRST}, V] = \text{total derivative}. \tag{5.5.76}$$

Integrated vertex operator

We should emphasize some small but important subtlety. The operator state correspondence, as we introduced it in the form of the equation (5.5.73), results in the following map:

Integrated vertex operator

$$\begin{aligned} |\gamma_{open}\rangle &\leftrightarrow c(z)\mathcal{V} \sim \int dz V(z), \\ |\gamma_{closed}\rangle &\leftrightarrow c(z)\tilde{c}(\overline{z})\mathcal{V} \sim \int dz d\overline{z} V(z,\overline{z}). \end{aligned} \tag{5.5.77}$$

The $c(z)\mathcal{V}$ is known as *unintegrated* vertex operator and $\int dz V(z)$ as *integrated* vertex operator.

In this work, we construct the integrated vertex operators by building the integrand $V(z)$ and do the integration while calculating the scattering amplitude. Therefore, we are not going to add $c(z)$ while constructing the vertex operator later we are going to take care of the $b-c$ system in the path integral formulation.

String coupling

Before we construct the spectrum we need to take into account the topological term associated with the Euler character. As we mentioned in the beginning this term arises due to the (non-dynamical) gravity on the two-dimensional world-sheet. This factor in the full amplitude only affects the relative weight of terms with different world–sheet topologies to each other. Meaning, that we have a different factor for genus $g = 0$ (e.g. sphere) surface compared to the genus one $g = 1$ (e.g, torus). Therefore, the factor $e^{S_{top}} = e^{-\lambda\chi}$ in (5.5.65) can be seen as the coupling controlling the string interactions g_s. Looking at the definition of the Euler character for the emission and absorption of the open and closed strings we have disk and sphere worldsheets, respectively. So we can define the open string and closed string couplings as:

$$\begin{gathered} \textit{open string:Disk with } n \textit{ punctures} \rightarrow \chi = 2-2g-b+\frac{n_o}{2}\,, \\ \textit{closed string:Sphere with } n \textit{ punctures} \rightarrow \chi = 2-2g-b+n_c\,, \\ \textit{Mixed open and closed string:Disk with } (n_o,n_c) \textit{ punctures} \rightarrow \chi = 2-2g-b+\frac{n_o}{2}+n_c\,, \end{gathered} \tag{5.5.78}$$

where g is the genus and b is the number of the boundaries of the surface. Now, if we look at the generic n point puncture case for disk and sphere topologies we have:

$$\begin{aligned} &n_o \textit{ open string over disk: } (g=0,\, b=1) \Rightarrow e^{\lambda(2-1)+\frac{n_o}{2}\lambda} = e^{\lambda}(e^{\frac{1}{2}\lambda})^{n_o}\,, \\ &n_c \textit{ closed string over sphere: } (g=0,\, b=0) \Rightarrow e^{\lambda(2-0)+n_c\lambda} = e^{2\lambda}(e^{\lambda})^{n_c}\,. \end{aligned} \tag{5.5.79}$$

Since adding a puncture changes the Euler character by one, in order to systematically take this increase into account for different amplitudes (with different number of punctures) we first factor out the common (puncture independent) part $2-2g-b$ and then define in the remaining:

$$\begin{aligned}&\textit{open string} \rightarrow e^{\frac{1}{2}\lambda} \rightarrow g_o,\\&\textit{closed string} \rightarrow e^{\lambda} \Rightarrow g_c \rightarrow g_c \sim g_o^2.\end{aligned} \tag{5.5.80}$$

and add a factor of g_o and g_c to the vertex operator of the open and closed string, respectively:

$$\begin{aligned}&\textit{open string:} \quad V_q(x) \mapsto g_o V(x),\\&\textit{closed string:} \quad V_{q,\bar{q}}(z,\bar{z}) \mapsto g_c V_{q,\bar{q}}(z,\bar{z}).\end{aligned} \tag{5.5.81}$$

The overall factors $e^{2\lambda}$ and e^{λ} will remain in the action and will be added in the path integral.

Picture changing

As we mentioned the construction of the state $|q_{\beta,\gamma}\rangle_{ghost\ charges}$ requires taking into account the infinite degeneracy associated with the zero modes of the $\beta-\gamma$ system. This infinite degeneracy cannot be resolved. Therefore, all vertex operators associated with string state will have infinitely many versions each associated with a degenerate state. One can label the degenerate state with a charge q known as the *picture number*. In order to construct the picture number we are going to use the bosonized version of the $\beta-\gamma$ system that we introduced before in (5.5.62). In the bosonized $\beta-\gamma$ system the ϕ dependence of the vertex operator (5.5.75) is of the following form:

$$V(\partial^k\psi^\mu;\partial^m X^\nu,\phi)\Big|_{|q_{\beta,\gamma}\rangle} \sim e^{q\phi(z)}. \tag{5.5.82}$$

where q is the picture number. For the purpose of our work, we are going to discuss the canonical picture number (picture number $q=-1$) of vertex operators and explain for a given vertex operator with a chosen picture number how one can change the picture number with the use of the picture changing operator.

The constraint that we use to fix the picture number of a vertex operator for a given matter state $|X;\psi\rangle$ is the conformal weight condition we set in (5.5.56). There we noticed that using the physical state conditions the full physical state must have conformal dimension *zero*. Hence we have the following condition:

$$\begin{aligned}&h(|\gamma\rangle) = h(\underbrace{|X;\psi\rangle}_{\textit{matter fields}} \otimes \underbrace{|\downarrow_{b,c};q_{\beta,\gamma}\rangle}_{\textit{ghost charges}}) = 0\\&\Rightarrow h(\underbrace{|X;\psi\rangle}_{\textit{matter fields}}) + h(\underbrace{|\downarrow_{b,c};q_{\beta,\gamma}\rangle}_{\textit{ghost charges}}) = 0\\&\Rightarrow h(\underbrace{|X;\psi\rangle}_{\textit{matter fields}}) + h(\underbrace{|\downarrow_{b,c}\rangle + h(|q_{\beta,\gamma}\rangle}_{\textit{ghost charges}})) = h_m + h_c + h(|q_{\beta,\gamma}\rangle)) = 0\\&\Rightarrow h_m - 1 + h_q = 1 \Rightarrow h_q = 1 - h_m.\end{aligned} \tag{5.5.83}$$

This will constrain the value of the q for a given matter state. As an important example let us look at the NS massless state:

Example (5.1): NS massless vector boson

The NS vacuum from the matter section is given by $|0\rangle$ and the ground state is given by $T^a\, b^\mu_{-1/2}|0\rangle$. Adding the ghost part we obtain :

$$\begin{aligned} &T^a\, b^\mu_{-1/2}|0\rangle \otimes |\downarrow_{b-c}; q_{\beta-\gamma}\rangle = g_o\, T^a\; e^{q\phi}\; b^\mu_{-1/2}|0\rangle \otimes |\downarrow_{b-c}\rangle, \\ &h(e^{q\phi}\;\; b^\mu_{-1/2}|0\rangle|\downarrow_{b-c}\rangle) = (-\frac{1}{2}q^2 - q) + (-1) + (\frac{1}{2}) = 0 \Rightarrow q = -1. \end{aligned} \tag{5.5.84}$$

Then, the vertex operator with picture number $q = -1$ is given by:

$$\begin{aligned} &V(\psi, X, \phi) \otimes |\downarrow_{b-c}\rangle = g_o\, T^a\; e^{-\phi}\;\; \psi^\mu e^{ik\cdot X} \otimes |\downarrow_{b-c}\rangle, \\ &V(\psi, X, \phi) = g_o\, T^a\; e^{-\phi}\;\; \psi^\mu e^{ik\cdot X}, \qquad k^2 = 0. \end{aligned} \tag{5.5.85}$$

This is the canonical vertex operator for the massless vector boson in type I NS superstring theory.

The final piece of the puzzle of picture number is the picture-changing operator. We need to define an operator whose action on the state does not change the physical state condition while changing the picture number. Clearly, since we want to keep the state physical, we are going to use the BRST charge. Looking back at the definition of the Q_{BRST} we see that the part associated with the superpartner of the matter energy-momentum tensor is given by:

$$Q_F^{X,\psi} = -\oint \frac{dz}{2\pi i}\gamma(z)\; T_F^{X,\psi} = -\frac{i}{2}\sqrt{\frac{2}{\alpha'}}\oint \frac{dz}{2\pi i} e^{-\chi} e^{\phi}\psi_\mu \partial X^\mu. \tag{5.5.86}$$

Here we have to use the bosonization of γ in (5.5.62) and the superpartner energy-momentum tensor in (5.2.7). One can see, given the exponential of ϕ, this part of the BRST charge has picture number $q = +1$ and therefore it can increase the picture number by one while acting on a state. All physical states commute up to a total derivative with all parts of the BRST operator including $Q_F^{X,\psi}$. Hence, one can define the picture-changing operator as the following:

Picture changing operator

$$P_{+1} = Q_F^{X,\psi} \cdot e^{\chi}, \tag{5.5.87}$$

with the action on the vertex operator with picture charge q, V^q defined as[a]

$$V^{q+1} = \lim_{w\to z} P_{+1}(w) V^q = \lim_{w\to z} e^{\phi(w)} T_F^{X,\psi}(w)\; V^q(z). \tag{5.5.88}$$

The limit will pick the terms proportional to z^0 in the OPEs of the product fields.

[a]For the anti-holomorphic picture number $\tilde{q}$ we use $\overline{P}_{+1}$

Let us demonstrate how this picture changing works in an example:

Example (5.2):Picture changing

We start by looking at the vertex operator of the massless boson we defined in the canonical ghost picture in (5.5.85) up to a numerical normalization we had:

$$V^{\mu}_{-1} = g_o\, T^a\, e^{-\phi}\, \psi^{\mu}\, e^{ik\cdot X}, \qquad k^2 = 0\,. \tag{5.5.89}$$

The picture changing will be:

$$\begin{aligned}
V^{\nu}_0 &= g_o\, T^a \lim_{w\to z} P_{+1}(w) V^{\nu}_{-1} = \lim_{w\to z} e^{\phi(w)} T^{X,\psi}_F(w)\left(e^{-\phi} c\psi^{\nu} e^{ik\cdot X}\right)\\
&= g_o\, T^a \lim_{w\to z} e^{\phi(w)} \left(\sqrt{\frac{2}{\alpha'}}\psi_{\mu}\partial X^{\mu}\right)(w)\left(e^{-\phi} c\psi^{\nu} e^{ik\cdot X}\right)(z)\\
&= g_o\, T^a \sqrt{\frac{2}{\alpha'}} c(z) \lim_{w\to z}\left[\left(e^{\phi(w)} e^{-\phi(z)}\right)\left(\psi_{\mu}(w)\psi^{\nu}(z)\right)\left(\partial X^{\mu}(w) e^{ik\cdot X(z)}\right)\right]\\
&= g_o\, T^a \sqrt{\frac{2}{\alpha'}} c(z) \lim_{w\to z}\left[\left((z-w) + \mathcal{O}(z-w)^2\right)\left(\frac{\eta^{\mu\nu}}{(z-w)} + \psi_{\mu}(z)\psi^{\nu}(z)\mathcal{O}(z-w)\right)\right.\\
&\times\left.\left(\frac{\alpha'}{2}\frac{ik^{\nu}}{(z-w)} e^{ik\cdot X(z)} + \partial X^{\mu}(z) e^{ik\cdot X(z)}\mathcal{O}(z-w)\right)\right].
\end{aligned} \tag{5.5.90}$$

Multiplying the above expression and taking the limit $w \to z$ we obtain the vertex operator for the massless opens string boson in the $q = 0$ picture number as:

$$V^{\mu}_0 = g_o\, T^a \sqrt{\frac{2}{\alpha'}}\left(\partial X^{\mu} - 2i\alpha'(k\cdot\psi)\psi^{\mu}\right)(z) e^{ik\cdot X(z)}, \qquad k^2 = 0. \tag{5.5.91}$$

5.5.4 String vertex-spectrum

Now we are ready to write down the spectrum of closed and open string states. Throughout this work we are interested in amplitudes and theories involving bosons (e.g. gauge fields or gravitons). Therefore, we are going to look at the NS sector of open strings and (NS,NS) sector of closed strings (which is the common part of both type IIA and IIB)[10]. We start with the vacuum and then act on it with creation operator. After imposing the BRST symmetry we obtain the physical state. All vertex operators in this section are the integrand of the integrated vertex operator and they are in the canonical picture number (−1). For the open string, we have the following states ordered by their masses:

1. The vacuum of the NS string is denoted by $|0\rangle$ and the vertex operator is given by a plane wave:
$$|0\rangle \leftrightarrow g_o\, e^{ik\cdot X(z)}, \qquad k^2 = \frac{1}{2}.$$

[10]The (R,R) sector is also bosonic however it is not useful for the current work

This state is a tachyon. This is a systemic problem of the string Hilbert space which is cured by projecting out the problematic states. This is known as GSO projection[11] which truncates the spectrum so that it is tachyon free while the space-time supersymmetry is preserved. In fact, the different superstring theories that we introduced at the beginning of this section were constructed after the GSO projection. From now on we only list states that are in the Hilbert space after the GSO projection.

2. The first excited state is given by the action of the fermionic creation operator:

$$b^{\mu}_{-1/2}|0;a\rangle \leftrightarrow \varepsilon_\mu g_o\, e^{-\phi}\, T^a\ \psi^\mu e^{ik\cdot X}, \qquad k^2=0. \tag{5.5.92}$$

3. The next excited state can be made with both bosonic α^μ_m and fermionic b^μ_r operators. We have the following:

$$\begin{aligned}&\left(b^{\mu}_{-3/2}+b^{\mu}_{-1/2}b^{\nu}_{-1/2}b^{\lambda}_{-1/2}+\alpha^{\mu}_{-1}b^{\nu}_{-1/2}\right)|0;a\rangle \leftrightarrow \\ &g_o\, e^{-\phi(z)}\ T^a\ \left(H_\mu\partial\psi^\mu+E_{\mu\nu\lambda}\psi^\mu\psi^\nu\psi^\lambda+iB_{\mu\nu}\partial X^\mu\psi^\nu\right)(z)e^{ik\cdot X(z)}.\end{aligned} \tag{5.5.93}$$

The polarization vectors associated with each term in this vertex operator will include *unphysical* degrees of freedom. Therefore, we have to enforce the BRST constraint on the vertex operator. This is done by commuting the vertex operator with the BRST charge and requiring that it vanishes up to a total derivative, which constraints the polarization vectors as [76, 77]:

$$[Q_{BRST},V]=\text{total derivative}\Rightarrow\begin{cases}k^\mu E_{\mu\nu\lambda}=0,\\ 2\alpha' k^\mu B_{\mu\nu}+H_\nu=0,\\ B^\mu{}_\mu+k^\mu H_\mu=0.\end{cases} \tag{5.5.94}$$

A solution to this system of equations that accounts for all 128 degrees of freedom can be given by setting $H_\mu=0$. Therefore, after invoking the BRST symmetry on this generic vertex operator we end up with the following *physical* states:

- One massive spin-2 state that has propagating 44 degrees of freedom in 10 dimension with the vertex operator:

$$\begin{aligned}&V_{-1}=g_o\ T^a\ B_{\mu\nu}e^{-\phi(z)}\left(i\partial X^\mu\psi^\nu\right)(z)e^{ik\cdot X(z)},\\ &k^2=-\frac{1}{\alpha'}\qquad B_{[\mu\nu]}=0\qquad B^\mu{}_\mu=0\qquad B_{\mu\nu}k^\nu=0.\end{aligned} \tag{5.5.95}$$

[11] In GSO projection states are labed by projecting operator $(-1)^F$ where F is the fermion number $F=\sum_{n>0}b^i_{-n}b^i_n$. One can chose different projection by taking positive or negative sign for holomorphic and anti-holomorphic sectors.

- The 3-from field propagating 84 degrees of freedom in 10 dimension as the following:

$$
\begin{aligned}
&V_{-1} = g_o\ T^a\ E_{\mu\nu\lambda} e^{-\phi(z)} \Bigg(\psi^\mu \psi^\nu \psi^\lambda \Bigg)(z) e^{ik\cdot X(z)}, \\
&k^2 = -\frac{1}{\alpha'} \qquad E_{\{\mu\nu\lambda\}} = 0 \qquad\qquad E_{\mu\nu\lambda} k^\lambda = 0.
\end{aligned}
\tag{5.5.96}
$$

- The remaining part associated to the $\partial\psi^\mu$ will decouple after imposing the BRST constraint $H_\mu = 0$.

For the discussion of our current work, we do not need more than the first massive level. We summarize the details of these states in the following table:

State	M^2	Vertex operator	Little group	Representation	
$\vert 0\rangle$	$-\frac{1}{\alpha'}$	$e^{-\phi} e^{ikX(z)}$	SO(9)	1	$\bullet$
$b^\mu_{-1/2}\vert 0\rangle$	0	$g_o e^{-\phi} \psi^\mu e^{ikX(z)}$	SO(8)	$(8)_v$	$\square$
$b^\mu_{-1/2} b^\nu_{-1/2} b^\lambda_{-1/2}\vert 0\rangle$	$\frac{1}{\alpha'}$	$g_o\ T^a\ E_{\mu\nu\lambda} e^{-\phi(z)} \Bigg(\psi^\mu \psi^\nu \psi^\lambda \Bigg)(z) e^{ik\cdot X(z)}$	SO(9)	84	$\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$
$\alpha^\mu_{-1} b^\nu_{-1/2}\vert 0\rangle$	$\frac{1}{\alpha'}$	$g_o\ T^a\ B_{\mu\nu} e^{-\phi(z)} (i\partial X^\mu \psi^\nu)(z) e^{ik\cdot X(z)}$	SO(9)	44	$\square\square$

Table 5.2: Open string spectrum in the NS sector.

We finish this section by constructing the closed string NS-NS sector of the superstring. To do that we used the well-known fact that the Hilbert space of closed string is the tensor product of two open string Hilbert spaces:

$$
\begin{aligned}
&H_{closed} = H_{open} \otimes H_{open}, \\
&V^{closed}(k_c, \overline{k_c})_{q,\overline{q}} = V^{open}_q(k_o) \times \tilde{V}^{open}_{\tilde{q}}(\overline{k}_o), \qquad k_c^2 = 4k_o^2; \quad \overline{k}_c^2 = 4\overline{k}_o^2.
\end{aligned}
\tag{5.5.97}
$$

The level matching condition (5.5.55) forces the conformal weight of anti-holomorphic sector to be equal to the holomorphic sector of the closed string:

$$
h = \tilde{h}\,.
$$

This can also be seen as the representation of *double copy* on the string Hilbert space level. Further, the process of double copying geometrically corresponds to gluing both ends of the open string together. Therefore, the Chan-Paton factors will be traced out and give the identity matrix. Using this double copy construction one can construct the following table of states for the (NS-NS) closed string:

State	M^2	Vertex operator	Little group	Representation	
$\|0\rangle$	$-\frac{4}{\alpha'}$	$e^{-\phi(z)-\overline{\phi}(\overline{z})}e^{ikX}$	SO(9)	1	$\bullet$
$\tilde{b}^{\nu}_{-1/2}b^{\mu}_{-1/2}\|0\rangle$	0	$g_c e^{-\phi(z)-\overline{\phi}(\overline{z})}\psi^{\mu}(z)\overline{\psi}^{\nu}(\overline{z})e^{ikX(z,\overline{z})}$	SO(8)	$(8)_v \otimes (8)_v$	$\square \otimes \square$

Table 5.3: Spectrum of NS-NS closed string

For the first massive level, we have the following possibilities:

1. First, we can multiply the two "spin-2" states as
$$\alpha^{\mu}_{-1}b^{\nu}_{-1/2}|0\rangle \otimes \tilde{\alpha}^{\lambda}_{-1}\tilde{b}^{\delta}_{-1/2}|0\rangle.$$
The vertex operator will be:
$$V_{-1,-1} = \varepsilon_{\mu\nu\lambda\delta}\; g_c\; e^{-\phi(z)-\overline{\phi}(\overline{z})}\; \partial X^{\mu}(z,\overline{z})\; \psi^{\nu}(z)\overline{\partial}X^{\lambda}(z,\overline{z})\overline{\psi}^{\delta}(\overline{z})e^{ikX(z,\overline{z})}. \tag{5.5.98}$$
The $SO(9)$ little group representation is given by the product:
$$\square\!\square \otimes \square\!\square = 44 \times 44 = 910 \otimes 495 \otimes 450 \otimes 44 \otimes 36 \otimes 1 \tag{5.5.99}$$

2. Second, we can multiply the two 3-forms states:
$$b^{\mu}_{-1/2}b^{\nu}_{-1/2}b^{\alpha}_{-1/2}|0\rangle \otimes \tilde{b}^{\delta}_{-1/2}\tilde{b}^{\lambda}_{-1/2}\tilde{b}^{\gamma}_{-1/2}|0\rangle$$
The vertex operator will be:
$$V_{-1,-1} = \varepsilon_{\mu\nu\alpha\lambda\delta\gamma}\; g_c\; e^{-\phi(z)-\overline{\phi}(\overline{z})}\; \psi^{\mu}(z)\psi^{\nu}(z)\psi^{\alpha}(z)\; \overline{\psi}^{\delta}(\overline{z})\overline{\psi}^{\lambda}(\overline{z})\overline{\psi}^{\gamma}(\overline{z})e^{ikX(z,\overline{z})}. \tag{5.5.100}$$
The $SO(9)$ little group representation is given by the product:
$$\begin{array}{c}\square\\\square\\\square\end{array} \otimes \begin{array}{c}\square\\\square\\\square\end{array} = 84 \times 84 = 2772 \otimes 1980 \otimes 924 \otimes 594 \otimes 126 \otimes 84 \otimes 44 \otimes 36 \otimes 1. \tag{5.5.101}$$

3. Third, we can multiply the two 3-forms and the massive spin-2 states:
$$b^{\mu}_{-1/2}b^{\nu}_{-1/2}b^{\alpha}_{-1/2}|0\rangle \otimes \tilde{\alpha}^{\lambda}_{-1}\tilde{b}^{\delta}_{-1/2}|0\rangle.$$
The vertex operator will be:
$$V_{-1,-1} = \varepsilon_{\mu\nu\alpha\lambda\delta}\; g_c\; e^{-\phi(z)-\overline{\phi}(\overline{z})}\; \psi^{\mu}(z)\psi^{\nu}(z)\psi^{\alpha}(z)\overline{\partial}X^{\lambda}(z,\overline{z})\overline{\psi}^{\delta}(\overline{z})e^{ikX(z,\overline{z})}, \tag{5.5.102}$$
with the group representation in SO(9) as:
$$\begin{array}{c}\square\\\square\\\square\end{array} \otimes \square\!\square = 84 \times 44 = 2457 \otimes 924 \otimes 231 \otimes 84. \tag{5.5.103}$$

The degrees of freedom in each vertex operator are coded inside the polarization vectors and by constraining the polarization one can pick the intended tensor representation under the little group decomposition.

5.6 String interactions

In this section, we are going to discuss superstring interactions. Our dissuasion is designed to be self-consistent and sufficient to understand this topic from the beginning to the calculation of massive tree level amplitudes in superstring theory. In contrast to the previous chapter we are not going to use light cone quantization instead, we are using the path-integral quantization of string theory.

5.6.1 Geometrical picture

First, we give some geometrical intuition on string interactions. As discussed before superstring theory is defined as the two-dimensional world–sheet embedded in 10 dimensional space[12]. Therefore, propagation and interaction involving string states also include $2d$ surfaces. In figure 5.4 we have depicted the free propagation open and closed strings.

Figure 5.4: Open and closes string propagating as Ribbon and Cylinder respectively

The interaction for string theories also is defined over a surface (as depicted in figure 5.5) this is of one the most consequential differences between string theory and quantum field theories. Since in field theory, the interaction is localized at a point (figure 5.5) in spacetime, it can be defined in Lorentz invariant formulations. Therefore it will happen at the same point for all Lorentz frames. However, given the surface interaction in string theory (figure 5.5) different reference frames will see the interaction happen at different points in their respective time. The most important consequence of this feature is that at any local neighborhood of a string interaction, it looks like a free propagation. This fact also manifest itself in the fact that string theory as defined in the action (5.2.6) is a free theory in terms of the world–sheet matter fields (i.e. ∂X^μ and ψ^μ). We are going to see the exact computational ramification of this in the following sections.

The next order of business is to understand the scattering diagrams in terms of Riemann surfaces. As we have advertised in the previous chapters the string scattering (for example the two scatterings depicted in the figure 5.5) correspond to a Riemann surface. To define a scattering amplitude (i.e. S-matrix element) we need to define asymptotic Hilbert spaces H_{in} and H_{out}. Then, the scattering amplitude would be the probability of

[12] 26 dimensions for bosonic string

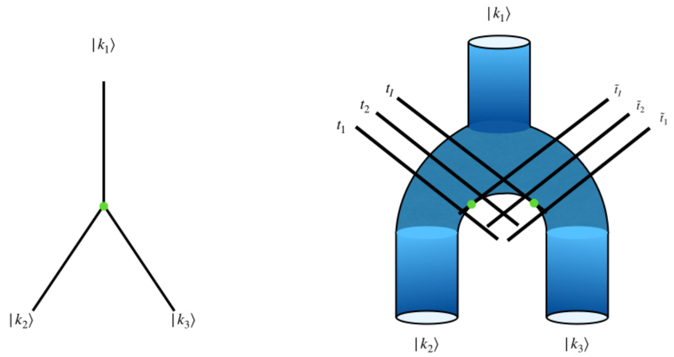

Figure 5.5: String theory interactions vs field theory interactions. In the string case, we have depicted three different time slices following two Lorenz reference frames t and $\tilde{t}$.

transitioning from the initial state $|I\rangle \in H_{in}$ to the final state $|F\rangle \in H_{out}$. Namely :

$$S(I \to F) := \langle I|F\rangle; \text{ for } |I\rangle \in H_{in}, \quad |F\rangle \in H_{out}. \tag{5.6.104}$$

Note that generically the states $|I\rangle$ and $|F\rangle$ can be multi-string (analog to multi-particle) states. To look at string scattering amplitudes we take two *in* and *out* Hilbert spaces as defined in the previous section for the open and closed string spectrum meaning:

$$\begin{aligned} \textit{n-open string: } H^n_{open} &= \bigoplus_{i=1}^{n} H^{(i)}_{open}\,, \\ \textit{m-closed string: } H^m_{closed} &= \bigoplus_{i=1}^{m} H^{(i)}_{closed}\,, \\ \textit{Mixed n-open and m-closed strings: } H^{n+m}_{mixed} &= H^n_{open} \oplus H^m_{closed}\,, \end{aligned} \tag{5.6.105}$$

where all the $H^{(i)}$s are the same single string Hilbert spaces (for closed and open strings, respectively) and we used the (i) notation for the direct sum whereas the H^n denotes the n string (closed or open) Hilbert space.

The associated Riemann surface of a given string scattering can be given by using the *operator-state* correspondence that we introduced in 5.5.3. To do so let us present the picture of a string scattering in terms of S-Matrix perturbation theory.

- First, we chose the initial state from the asymptotic Hilbert space associated with the scattering i.e. H_{in}:
$$|I\rangle \in H_{in}.$$
- Second, this state propagates from the infinite past freely until it interacts locally
- Third, they interact locally according to the interaction rules of superstring theory.

$$S(I, F) : |I\rangle \to |F\rangle.$$

- Finally, after the interaction, the resulting states propagate, again freely, to the infinite future and they are elements of the final Hilbert space H_{out}

$$|F\rangle \in H_{out}.$$

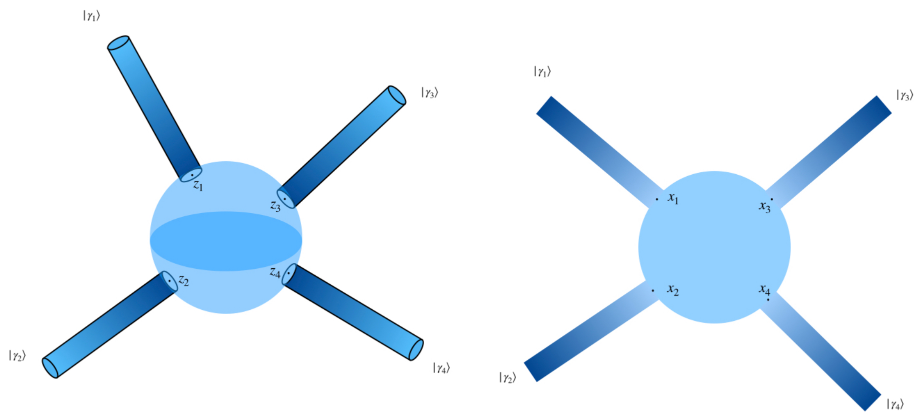

Figure 5.6: Scattering amplitude of closed and open string from infinite past/future to/from the interaction surface

Our task is to introduce the method to calculate all four steps that we sketched above. For the first and last steps the map 5.5.3, which we introduced before, is very useful. Since the string state propagates from infinity to the interaction surface and from the interaction surface back to infinity, the interaction point is the half cylinder and so we can use the map 5.5.3 to isomorphe the semi-infinite cylinder to a local disk glued on top of the interaction surface (see figure 5.7).

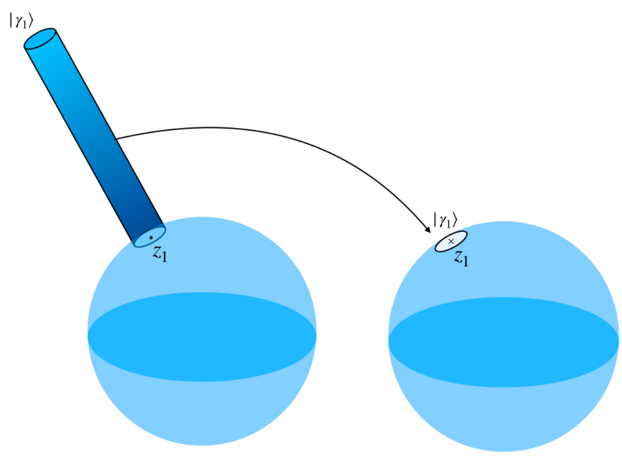

Figure 5.7: Mapping the half cylinder to the disk with puncture

Since we are dealing with the half cylinder (i.e. τ coordinate in figure (5.4) goes from infinity to the τ_0 reaching interaction surface) after mapping it to the complex plane according to the (5.5.71) we obtain a disk centered at z_1 (as in figure 5.7) that has a boundary and can be deformed. Therefore, we can shrink it to the point of the z_1. This construction then makes the surface of scattering amplitude a Riemann sphere with the states inserted as a local operator at the point of a puncture (e.g. at z_1). We can repeat this for all states involved in the scattering. Similarly, for the open string, we can map the ribbon to the real line with a marked point, and upon shrinking the line we get a puncture at x_1. Therefore, we can define the surface of string scattering amplitude as the following:

String scattering amplitude

Any string scattering amplitude involving open, closed, or mixed string states can be mapped to a punctured Riemann surface, where the states are inserted as local operators (vertex operators) into the point of the punctures. For pure open and mixed open-closed amplitude the surface will have a boundary (e.g. disk, annulus, etc) and for the pure closed string the Riemann surface will not have a boundary (e.g. sphere, torus, etc).

Therefore, all of the notions that we introduced in the mathematical preliminaries can be used in the calculation of the string scattering amplitudes.

5.6.2 Scattering amplitude

So far we have introduced the calculational prerequisites and the geometrical setup of a generic scattering amplitude namely:

- Construction of the physical string vertex operators.
- Wick's theorem as the main method of calculating the radial-ordered $n-$point functions.
- The punctured Riemann surface as the world–sheet of the interaction surface.

Now, we need to put all these three pillars of our setup into action. The natural way to see the use of all we have done so far is the path integral formulation of superstring theory.

Path integral

We have for the path integral the following for the superstring action (5.3.18):

$$\mathcal{P} = \int \frac{dX d\psi dg}{V_{diff\times Weyl}} \exp\{-S^{superstring}\}\,. \tag{5.6.106}$$

Here we have to use the $b-c$ and $\beta-\gamma$ ghost systems to gauge fix by adding the following (Feddeev-Popov like terms):

$$\Delta_{FP} = \int dc\ db \exp\{-S^{b-c}\} + \int d\beta\ d\gamma \exp\{-S^{\beta-\gamma}\}\,. \tag{5.6.107}$$

Therefore, what remains from integration over all possible metrics is the sum of all possible compact topologies.

Using the above definition of the path integral and the operator state correspondence we can write an element of the $S-$matrix as:

$S-$matrix element

We have s generic formula for the expectation value of an operator $F(z,\overline{z})$:

$$\langle F(z,\overline{z})\rangle = \sum_{\substack{compact\\topologies}} \int \frac{dX d\psi dg}{V_{diff\times Weyl}} F(z,\overline{z})\ e^{-n_X\lambda} \exp\{-S^{superstring}\}, \tag{5.6.108}$$

above, the factor $e^{-n_X\lambda}$ is the overall factor we took out in (5.5.79). Using operator state correspondence an element of $S-$matrix involving n states going to m states $\langle n|m\rangle$ can be written as the vacuum expectation value of the vertex operators:

$$\begin{aligned}\langle I|F\rangle &= \langle 0|\prod_{o,p} b^{\mu'_p}_{r'_p}\alpha^{\nu'_o}_{n'_o}\prod_{k,l} b^{\mu_l}_{-r_l}\alpha^{\nu_k}_{-n_k}|0\rangle = \\ &= \Big\langle \prod_{i=1}^{n}\int d^2z_i\sqrt{-h} : V_i^{q_i}(z_i,\overline{z}_i) : \prod_{j=1}^{m}\int d^2z_j\sqrt{-h} : V_j^{q_i}(z_j,\overline{z}_j) : \Big\rangle \\ &= \Big\langle \prod_{l=1}^{n+m}\int d^2z_l\sqrt{-h} : V_l^{q_l}(z_l,\overline{z}) : \Big\rangle.\end{aligned} \tag{5.6.109}$$

The vertex operator have the picture number q_l. we should point out here that we have not yet fixed the gauge to the superconformal gauge (i.e. setting the world–sheet metric $h_{\alpha\beta} = \eta_{\alpha\beta}$). This is the task of the ghost part of the integral which we will discuss shortly. Now, we have the following path integral definition for this element:

$$\begin{aligned}\langle I|F\rangle &= \Big\langle \prod_{l=1}^{n+m}\int d^2z_l\sqrt{-h} : V_l^{q_l}(z_l,\overline{z}_l) : \Big\rangle = \\ &\sum_{\substack{compact\\topologies}} \int \frac{dX d\psi dg}{V_{diff\times Weyl}} \left(\prod_{l=1}^{n+m}\int d^2z_l\sqrt{-h} : V_l(z_l,\overline{z}_l) :\right) e^{-n_X\lambda}\ \exp\{-S^{superstring}\}.\end{aligned} \tag{5.6.110}$$

First, we have to take care of the measure. As mentioned we use the Faddeev-Popov ghost system to implement the superconformal gauge. This part of the path integral calculation is involved. However, the results that are relevant to our discussion, i.e. the tree level amplitudes, are rather simple to obtain. Therefore, we leave the detailed discussion on the ghost path integral to references [10, 11, 12] and only give the relevant results. Fortunately, in the tree level scattering amplitudes there is a minimal contribution from the ghost systems which fixes the superconformal gauge:

- The $b-c$ system: Only the zero modes of the c fields are relevant. These modes are related to the conformal killing group (CKV) of the world-sheet. Further, there are no b moduli that we have to take care at the genus zero level.
- The $\beta-\gamma$ system: The only relevant part for the tree level amplitude is that in the vertex operator insertions, the picture number should be overall -2 for holomorphic and anti-holomorphic sectors independently. Meaning, for open string $q_{total} = -2$ and for closes string $q_{total} = (-2, -2)$. We implement this by adding a delta function $\delta(\sum_i q_i + 2)$.

We arrive now at the following formula for the scattering amplitude:

Tree-level gauge fixed Path integral

$$S(I, F) = \int dX\, d\psi \left[\left(\prod_{l=1}^{n+m} \int \frac{d^2 z_l}{V_{CKV}} : V_l^{q_l}(z_l, \overline{z}_l) : \delta(\sum_k q_k + 2) \right) \times \delta_{ghost}\, e^{-n_X \lambda} \exp\{-S^{superstring}\} \right], \tag{5.6.111}$$

$$\delta_{ghost} = \det C^a_{0j}.$$

For the dimension of the matrix C^a_{0j} and its determinant one needs to make use of the Riemann-Roch theorem.

Using this formula for the tree level amplitude we find a simple result for a given Riemann surface X. For the ghost part, in the case of a $g = 0$ orientable surface, the choices are either disk or sphere:

$$\delta_{ghost} = \begin{cases} \textit{Disk:}\ \det C^a_{0j} = C^g_{D_2} \langle c(z_1)c(z_2)c(z_3)\rangle_{D_2} = C^g_{D_2} z_{12}z_{23}z_{31}, \\ \textit{Sphere:}\ \det C^a_{0j} = C^g_{S_2} \langle c(z_1)c(z_2)c(z_3)\tilde{c}(\overline{z}_1)\tilde{c}(\overline{z}_2)\tilde{c}(\overline{z}_3)\rangle_{S_2} \\ \qquad\qquad = C^g_{S_2} |z_{12}|^2 |z_{23}|^2 |z_{31}|^2. \end{cases} \tag{5.6.112}$$

Now, we can take the matter part. We have already taken into account the string coupling in the definition of the vertex operators (5.5.81) and from now on we look at only tree level amplitudes so $g = 0$. We have the following:

$$S(I; F) = \int dX\, d\psi \left(\prod_{l=1}^{n+m} \frac{d^2 z_l}{V_{CKV}} : V_l^{q_l}(z_l, \overline{z}_l) : \delta\Big(\sum_k q_k + 2\Big) \right) \delta_{ghost}\, e^{-n_X \lambda} \times\ exp\left\{ -\frac{1}{4\pi\alpha'} \int dz d\overline{z} \left(2\partial X^\mu(z,\overline{z})\overline{\partial} X_\mu(z,\overline{z}) + \alpha'(\psi^\mu \overline{\partial}\psi_\mu + \overline{\psi}^\mu \partial \overline{\psi}_\mu) \right) \right\}. \tag{5.6.113}$$

Next, we exchange the order of the integrals: The dz integral is now over the moduli space of the punctured Riemann surface $\mathcal{M}_{g,n}$:

$$S(I;F) = \int_{\mathcal{M}_{0,n+m}} \prod_{l=1}^{n+m} \frac{d^2 z_l}{V_{CKG}} \int dX\, d\psi \left(\prod_{l=1}^{n+m} : V_l^{q_l}(z_l,\overline{z}_l) : \delta(\sum_k q_k + 2) \right) \delta_{ghost}\; e^{-n_X \lambda}$$
$$\times\; exp\left\{ -\frac{1}{4\pi\alpha'} \int dz d\overline{z} \left(2\partial X^\mu(z,\overline{z}) \overline{\partial} X_\mu(z,\overline{z}) + \alpha'(\psi^\mu \overline{\partial}\psi_\mu + \overline{\psi}^\mu \partial \overline{\psi}_\mu) \right) \right\},$$
$$\text{where } \mathcal{M}_{0,n+m} = \frac{\mathbf{g}_0 \times \mathcal{M}_{n+r}}{\mathit{diff} \times Weyl}. \tag{5.6.114}$$

Here we had to quotient the part of the world-sheet symmetries associated with the conformal killing group. This symmetry is a remnant of the diff $\times$ Weyl symmetry after gauge fixing. For the disk, it is $SL(2,\mathbf{R})$ and sphere, it is $SL(2,\mathbf{C})$. In practice, the quotient can be done by:

$$\prod_{l=1}^{n} \frac{d^2 z_l}{SL(2,\mathbf{R})} \to \text{Fix three real positions in } z_l.$$
$$\prod_{l=1}^{n} \frac{d^2 z_l}{SL(2,\mathbf{C})} \to \text{Fix three complex positions in } z_l. \tag{5.6.115}$$

Further, $\mathbf{g}_0$ is the space of all metrics on a genus zero Riemann surface which for the tree level amplitudes that we are going to consider is one element set i.e. flat metric. The $\mathcal{M}_{n+r}$ is the moduli space of $n + r$-punctured Riemann surface (see our discussion in section 2.6). Now, the inner integral will give the following vacuum expectation value:

$$S(I;F) = \int_{\mathcal{M}_{0,n+m}} \prod_{l=1}^{n+m} \frac{d^2 z_l}{V_{CKG}}\; \delta_{ghost}\; e^{-n_X \lambda} \left\langle \prod_{k=1}^{n+m} : V_k^{q_k}(z_k,\overline{z}_k) : \delta\Big(\sum_k q_k + 2\Big) \right\rangle_X , \tag{5.6.116}$$

where X is the associated Riemann surface (e.g. disk, sphere,...). Since the action is a free action (no interaction term only kinetic terms) we can use the standard techniques of Wick's theorem that we introduced before to calculate the scattering amplitude:

$$S(I;F) = \int_{\mathcal{M}_{0,n+m}} \prod_{l=1}^{n+m} \frac{d^2 z_l}{V_{CKG}}\; \delta_{ghost}\; e^{-n_X \lambda}\; C_X^{matter}\; \delta\Big(\sum_k q_k + 2\Big)\, \mathbf{Con}_X\{ \prod_{k=1}^{n+m} V_k^{q_k}(z_k,\overline{z}_k)\} , \tag{5.6.117}$$

where $\mathbf{Con}_X\{F(z,\overline{z})\}$ means all possible *pair* contractions of the functional F given the Riemann surface X. It is worth reminding the reader of the reason that we only take the pair contractions. In principle, Wick's theorem requires *all* possible contractions. However, as we explained in the previous section, locally, all string interactions are free string propagations and therefore we only have pair contractions. This also can be seen in the functional form of the path integral as it is a free theory and only has free propagation. In addition, we have the constant C_X^{matter} coming out of the expectation value together with the ghost counterpart C^g_X, that we introduced in (5.6.112), and $e^{-\lambda}$. They can be fixed for the disk and sphere as:

$$\begin{aligned} e^{-\lambda} C^g_{D_2} C^{matter}_{D_2} &= C_{D_2} = \frac{1}{\alpha' g_o^2}, \\ e^{-2\lambda} C^g_{S_2} C^{matter}_{S_2} &= C_{S_2} = \frac{8\pi}{\alpha' g_c^2}. \end{aligned} \tag{5.6.118}$$

Furthermore, when performing the contractions the holomorphic condition on the matter fields will enforce the spacetime momentum conservation [12, 69] (this can also be seen in the zero modes of the matter path integral). Therefore, in all amplitudes the momentum conservation as

$$\sum_i k_i = 0$$

is implied.

Before looking at examples of scattering amplitude it is very useful to look at another method to repackage the contractions in (5.6.117) in terms of Grassmann integral [10]. Looking closer at the Wick's theorem we see that after each contraction the associated fields are removed and replaced with the OPE, therefore, if we associate Graßmann variables $(\theta, \overline{\theta})$ to each matter field as $(\partial X^\mu, \psi^\mu, \overline{\psi}^\mu) \to (\theta\overline{\theta}\partial X^\mu, \theta\psi^\mu, \overline{\theta}\,\overline{\psi}^\mu)$ we can use the properties of Graßmann integrals and rewrite the contractions. For example for n open superstrings spin-1 states we have the following:

$$\mathcal{A}^{open}(p_1, p_2, p_3, ..., p_n) = \\ \int_{C_\gamma} d\mu_n \int \prod_{i=1}^{n} d\theta_i d\overline{\theta_i} \frac{\theta_k \theta_l}{z_k - z_l} \exp\left\{ -\alpha'^2 \sum_{i \neq j} \frac{\theta_i \theta_j p_i \cdot p_j + \overline{\theta}_i \overline{\theta}_j \varepsilon_i \cdot \varepsilon_j + 2(\theta_i - \theta_j)\overline{\theta_i} \varepsilon_i \cdot p_j}{z_i - z_j \mp \alpha'^{-1} \theta_i \theta_j} \right\} \times KN. \tag{5.6.119}$$

In the case of closed strings one has to add the anti-holomorphic part with the same integrand to the amplitude. For the detail of how to write the vertex operator and the contractions in terms of Grassmann variables see Appendix A.

Example of string scattering amplitudes

We finish this section by giving some examples of string amplitudes to see how equation (5.6.117) works in practice. The very first and important example is the n-point tachyonic amplitude. As we mentioned before the vacuum of the NS sector is a tachyon and is projected out from the spectrum through the GSO projection. Therefore, it is not present in any consistent string amplitude. However, the plane wave $e^{ik\cdot X(z)}$ contractions, which are associated with this amplitude is always present in all string amplitudes (with different values of k^2 depending on the mass of the state) since all vertex operators have a plane wave factor. This contraction is so famous that it has been named the *Koba-Nielsen* factor [31] and we introduced it already in chapter 2. This factor plays a central role in the KLT relations and intersection discussions we do in the coming sections. So let us take a look at this factor and calculate its form from string amplitudes.

Example (5.3): n-point tachyon (Koba-Nielsen)

The vertex operator is given by the plane wave as[a]:

$$V(x,k) = e^{ik\cdot X(x)}. \tag{5.6.120}$$

The Riemann surface is the disk that is punctured over the boundary with n injection of this vertex operator. We can fix three positions (x_1, x_2, x_n) and change the measure from a product over n to $n-3$ dx_ls. Then, the amplitude will be:

$$\begin{aligned} A(k_1,k_2,k_3,...,k_n) &= \int \prod_{l=1}^{n} \frac{dx_l}{SL(2,\mathbf{R})} \left\langle \prod_{k=1}^{n} : V_k(x_l) : \right\rangle_{D_2} \\ &= \int \prod_{l=2}^{n-1} dx_l \left\langle : V(x_1) :: V(x_2) :: V(x_3) : ... : V(x_n) : \right\rangle_{D_2} \\ &= \int \prod_{l=2}^{n-1} dx_l \left\langle : e^{ik_1\cdot X(x_1)} :: e^{ik_2\cdot X(x_2)} :: e^{ik_3\cdot X(x_3)} : ... : e^{ik_n\cdot X(x_m)} : \right\rangle_{D_2}. \end{aligned} \tag{5.6.121}$$

Doing the contraction using the OPEs in (5.1) we have:

$$A(k_1,k_2,k_3,...,k_n) = \int \prod_{l=2}^{n-1} dx_l \prod_{1\le i<j}^{n} |x_i - x_j|^{k_i\cdot k_j}. \tag{5.6.122}$$

The integrand in the above amplitude is the famous Koba-Nielsen factor for the open string.

$$KN^{open}(x,k) := \prod_{1\le i<j}^{n} |x_i - x_j|^{k_i\cdot k_j}. \tag{5.6.123}$$

We have a similar calculation for closed strings with just anti-holomorphic fields added to the vertex operators. For closed strings we have:

$$KN^{closed}(z,\overline{z},k,\overline{k}) := \prod_{1\le i<j}^{n} |z_i - z_j|^{k_i\cdot k_j} |\overline{z}_i - \overline{z}_j|^{\overline{k}_i\cdot\overline{k}_j} \tag{5.6.124}$$

[a] We do not add the picture number and ghost factor since it is not in the spectrum from the beginning.

Using the result of this example we can build a decomposition of the contractions given in (5.6.117). We saw in the construction of the vertex operator that all of them include a plane wave factor $e^{ik\cdot X}$ and in the previous example we calculated the contractions of plane wave factors. Therefore, we can decompose the contraction functional in (5.6.117) for open strings (i.e. only holomorphic fields) as:

$$\begin{aligned}\mathbf{Con}_{D_2}\{\prod_{k=1}^{n} V_k(z_k)\} &= \mathbf{Con}_{D_2}\{\prod_{l=1}^{n} e^{ik_l\cdot X_l(z)}\} \times \mathbf{Con}_{D_2}\{rest\},\\ \mathbf{Con}_{D_2}\{\prod_{k=1}^{n} V_k(z_k)\} &= KN \times I(z),\end{aligned} \tag{5.6.125}$$

where we name all leftover contractions $I(z)$ which is a holomorphic function. Then, the amplitude will take the form:

$$\mathcal{A}^{open}(k_1, k_2, k_3, ..., k_n) = \frac{1}{\alpha' g_o^2} \int_{\mathcal{M}_{0,n}} \prod_{l=1}^{n} \frac{dz_l}{SL(2,\mathbf{R})}\ KN\ I'(z). \tag{5.6.126}$$

The prime means that we added the z dependence inside $\delta_{D_2}^{ghost}$ to the function $I(z)$. Similarly, for the closed string, we have:

$$\mathcal{A}^{closed}(k_1, k_2, k_3, ..., k_n) = \frac{1}{\alpha' g_c^2} \int_{\mathcal{M}_{0,n}} \prod_{l=1}^{n} \frac{d^2 z_l}{SL(2,\mathbf{C})}\ KN \times \overline{KN}\ I'(z) \times \overline{I}'(\overline{z}). \tag{5.6.127}$$

These formulae are going to be useful when we compare the string amplitudes to parings in twisted homology.

The next set of examples will include different three point tree level amplitudes i.e. 3-open strings, 3-closed strings, and $(2,1)$ mixed open-closed strings.

Example (5.4): 3 open string amplitude

The next important example is the simplest: three massless open strings amplitude which has vertex operators in two pictures:

$$\begin{aligned}&V_{-1}^{\mu}(x,k,\varepsilon) = \varepsilon_\mu g_o T^a e^{-\phi}\psi^\mu e^{ik\cdot X} \qquad k^2=0 \quad \varepsilon\cdot k = 0,\\ &V_0^{\mu}(x,k,\varepsilon) = g_o\ T^a\ \sqrt{\frac{2}{\alpha'}}\left(\partial X^\mu - 2i\alpha'(k\cdot\psi)\psi^\mu\right)(z) e^{ik\cdot X(z)} \qquad k^2=0; \quad \varepsilon\cdot k = 0.\end{aligned} \tag{5.6.128}$$

The Riemann surface for the pure open string is the disk with punctures on the boundary which is mapped to the upper half-plane with the open strings on the real line. In this case we have three punctures. So we have the following scattering amplitude:

$$\begin{aligned}A(k_1,k_2,k_3) &= \int_{\mathcal{M}_{0,3}} \prod_{l=1}^{3} \frac{dx_l}{SL(2,\mathbf{R})}\ \delta_{ghost}\ \ e^{-\lambda} C_{D_2}^{matter} \left\langle \prod_{l=1}^{3} : V_l^{q_l}(k_l,x_l) : \delta\Big(\sum_k q_k + 2\Big)\right\rangle_{D_2}\\ &= \int_{\mathcal{M}_{0,3}} \prod_{l=1}^{3} \frac{dx_l}{SL(2,\mathbf{R})}\ e^{-\lambda} C_{D_2}^{g} C_{D_2}^{matter} \langle c(x_1)c(x_2)c(x_3)\rangle_{D_2} \left\langle : V_{-1}^{\mu}(x_1) :: V_{-1}^{\mu}(x_2) :: V_0^{\mu}(x_3) : \right\rangle_{D_2}.\end{aligned} \tag{5.6.129}$$

Now, we can quotient the $SL(2,\mathbf{R})$ by fixing all three positions of the open strings to $(x_1,x_2,x_3) \mapsto (0,1,\infty)$. This is the standard choice that we are going to use repeatedly in this work. So by fixing these positions the integral disappears. However, we are going to do this at the end. Here we want to show that constructions makes the integrand x_i independent without the gauge fixing implemented:

$$A(k_1,k_2,k_3) = \frac{g_o^3}{\alpha'\sqrt{\alpha'}g_o^2}\int_{\mathcal{M}_{0,3}}\prod_{l=1}^{3}\frac{dx_l}{SL(2,\mathbf{R})}\ x_{12}x_{23}x_{31}\ \ Tr(T^aT^bT^c)\varepsilon_\mu\varepsilon_\nu\varepsilon_\alpha$$
$$\times\left\langle :e^{-\phi(x_1)}\psi(x_1)^\mu e^{ik_1X(x_1)} :: e^{-\phi(x_2)}\psi(x_2)^\mu e^{ik_1X(x_2)} :: \left(\partial X^\alpha - 2i\alpha'(k\cdot\psi)\psi^\alpha\right)(x_3)e^{ik\cdot X(x_3)}:\right\rangle_{D_2} \tag{5.6.130}$$

Using the OPEs of the X^μ and ψ^μ fields we can do the contractions:

$$A(k_1,k_2,k_3) = \frac{g_o}{\sqrt{\alpha'}}\int_{\mathcal{M}_{0,3}}\prod_{l=1}^{3}\frac{dx_l}{SL(2,\mathbf{R})}\ \ Tr(T^aT^bT^c)$$
$$\times\ \varepsilon^1_\mu\varepsilon^2_\nu\varepsilon^3_\alpha\left[x_{12}x_{23}x_{31}\left(-\frac{\eta^{\mu\nu}k_1^\alpha}{x_{12}x_{13}}-\frac{\eta^{\mu\nu}k_2^\alpha}{x_{12}x_{23}}+\frac{\eta^{\mu\alpha}k_3^\nu-\eta^{\nu\alpha}k_3^\mu}{x_{13}x_{23}}\right)\right],$$
$$A(k_1,k_2,k_3) = \frac{g_o}{\sqrt{\alpha'}}\left(\int_{\mathcal{M}_{0,3}}\prod_{l=1}^{3}\frac{dx_l}{SL(2,\mathbf{R})}\right)\ Tr(T^aT^bT^c)\ \times\varepsilon^1_\mu\varepsilon^2_\nu\varepsilon^3_\alpha\left[\eta^{\mu\nu}k_1^\alpha+\eta^{\nu\alpha}k_2^\mu+\eta^{\mu\alpha}k_3^\nu\right]. \tag{5.6.131}$$

Now, we can see that the last "integrand" is not dependent on any of positions x_i and in fact the quotienting CKV is necessary to avoid infinities coming out of the integral. So we have the following result for three massless open strings:

$$A(k_1,k_2,k_3) = \frac{g_o}{\sqrt{\alpha'}}\ Tr(T^aT^bT^c)\ \times\varepsilon^1_\mu\varepsilon^2_\nu\varepsilon^3_\alpha\left[\eta^{\mu\nu}k_1^\alpha+\eta^{\nu\alpha}k_2^\mu+\eta^{\mu\alpha}k_3^\nu\right]. \tag{5.6.132}$$

One can readily check that this amplitude (which is exact α') is the exact same form of the three point Yang-Mills amplitude [11]. The next example is the three closed string amplitude. The important part is that not only it will show how to work out sphere amplitude but also we can demonstrate the amplitude double copy feature between open and closed strings:

Example (5.5): 3 closed string amplitude

For the three massless closed string amplitude, we have the vertex operators in two pictures:

$$V_c^{(0,0)}(z,\bar z,\varepsilon,k) = \frac{g_c}{\alpha'}\ \varepsilon_{\mu\nu}\left[i\bar\partial X^\mu+\frac{\alpha'}{2}(k\cdot\bar\psi)\bar\psi^\mu(\bar z)\right]\left[i\partial X^\nu+\frac{\alpha'}{2}(k\cdot\psi)\psi^\nu(z)\right]e^{ikX(z,\bar z)},$$
$$V_c^{(-1,-1)}(z,\bar z,\varepsilon,k) = g_c\ \varepsilon_{\mu\nu}\ e^{-\bar\phi(\bar z)}\bar\psi^\mu(\bar z)\ e^{-\phi(z)}\psi^\nu(z)\ e^{ik\cdot X(z,\bar z)}, \tag{5.6.133}$$
$$k^2=0,\qquad k^\mu\cdot\varepsilon_{\mu\nu}=0\ ,\qquad \varepsilon^\mu{}_\mu=0\qquad .$$

The Riemann surface for the pure closed string is the punctured sphere S_2. In this case three punctures. So we have the following scattering amplitude:

$$A(k_1,k_2,k_3)=\int_{\mathcal{M}_{0,3}}\prod_{l=1}^{3}\frac{d^2z_l}{SL(2,\mathbf{C})}\;\delta_{ghost}\;\;e^{-2\lambda}C^{matter}_{S_2}\left\langle\prod_{l=1}^{3}:V_l^{q_l}(z_l,\overline{z}_l,k_l):\delta\Big(\sum_i q_i+2\Big)\right\rangle_{S_2}$$

$$=\int_{\mathcal{M}_{0,3}}\prod_{l=1}^{3}\frac{d^2z_l}{SL(2,\mathbf{C})}\;e^{-\lambda}C^{g}_{S_2}C^{matter}_{S_2}\delta^{S_2}_{ghost}\left\langle:V^{\mu}_{-1}(z_1,\overline{z}_1,k_1)::V^{\mu}_{-1}(z_2,\overline{z}_2,k_2)::V^{\mu}_{0}(z_3,\overline{z}_3,k_3):\right\rangle_{S_2}\tag{5.6.134}$$

Now, we can quotient the measure by the volume of $SL(2,\mathbf{C})$ by fixing all three positions of closed strings to $(z_1,z_2,z_3)\mapsto(0,1,\infty)$. As before the integral cancels out and we get from the pair contractions:

$$\begin{aligned}A(k_1,k_2,k_3)&=\frac{g_c^3}{\alpha'^2g_c^2}\Bigg[|z_{12}|^2|z_{23}|^2|z_{31}|^2\;\varepsilon^1_{\mu_1\nu_1}\varepsilon^2_{\mu_2\nu_2}\varepsilon^3_{\mu_3\nu_3}\\&\times\Big\langle:e^{-\phi(z_1)}e^{-\overline{\phi}(\overline{z}_1)}\psi(z_1)^{\mu_1}\overline{\psi}(\overline{z}_1)^{\nu_1}e^{ik_1X(z_1,\overline{z}_1)}::e^{-\phi(z_2)}e^{-\overline{\phi}(\overline{z}_2)}\psi(z_2)^{\mu_2}\overline{\psi}(\overline{z}_2)^{\nu_2}e^{ik_2X(z_2,\overline{z}_2)}:\\&:\Big[i\overline{\partial}X^{\mu_3}+\frac{\alpha'}{2}(k_3\overline{\psi})\overline{\psi}^{\mu_3}(\overline{z}_3)\Big]\Big[i\partial X^{\nu_3}+\frac{\alpha'}{2}(k_3\psi)\psi^{\nu_3}(z_3)\Big]e^{ik_3X(z_3,\overline{z}_3)}:\Big\rangle_{S_2}\Bigg]\end{aligned}\tag{5.6.135}$$

Using the OPEs of the X^μ and ψ^μ fields (5.1) over the sphere we see that the contractions will decompose into holomorphic and anti-holomorphic sectors:

$$\begin{aligned}A(k_1,k_2,k_3)&=\frac{g_c}{\alpha'}\varepsilon^1_{\mu_1\nu_1}\varepsilon^2_{\mu_2\nu_2}\varepsilon^3_{\mu_3\nu_3}|z_{12}|^2|z_{23}|^2|z_{31}|^2\\&\times\Bigg[\left(\frac{\eta^{\mu_1\mu_2}k_1^{\mu_3}}{z_{12}z_{13}}+\frac{\eta^{\mu_1\mu_2}k_2^{\mu_3}}{z_{12}z_{23}}\frac{\eta^{\mu_1\mu_3}k_3^{\mu_2}-\eta^{\mu_2\mu_3}k_3^{\mu_1}}{z_{13}z_{23}}\right)\left(\frac{\eta^{\nu_1\nu_2}k_1^{\nu_3}}{\overline{z}_{12}\overline{z}_{13}}+\frac{\eta^{\nu_1\nu_2}k_2^{\nu_3}}{\overline{z}_{12}\overline{z}_{23}}-\frac{\eta^{\nu_1\nu_3}k_3^{\nu_2}-\eta^{\nu_2\nu_3}k_3^{\nu_1}}{\overline{z}_{13}\overline{z}_{23}}\right)\Bigg].\end{aligned}\tag{5.6.136}$$

So the result for the amplitude of three massless closed strings is:

$$\begin{aligned}&A(k_1,k_2,k_3)=\frac{g_c}{\alpha'}\varepsilon^1_{\mu_1\nu_1}\varepsilon^2_{\mu_2\nu_2}\varepsilon^3_{\mu_3\nu_3}\\&\left(\eta^{\mu_1\mu_2}k_1^{\mu_3}+\eta^{\mu_2\mu_3}k_2^{\mu_1}+\eta^{\mu_1\mu_3}k_3^{\mu_2}\right)\left(\eta^{\nu_1\nu_2}k_1^{\nu_3}+\eta^{\nu_2\nu_3}k_2^{\nu_1}+\eta^{\nu_1\nu_3}k_3^{\nu_2}\right).\end{aligned}\tag{5.6.137}$$

Looking at the result of the closed string amplitude (5.6.137) we can exhibit the double copy structure of the amplitude. Given the on-shell conditions one can always decomposes the polarization tensor of the spin-2 field as the tensor product of two spin-1 fields:

$$\varepsilon_{\mu_1\nu_1}=\varepsilon_{\mu_1}\otimes\varepsilon_{\nu_1}.$$

Using this decomposition the amplitude of the three closed strings can be regrouped as:

$$\begin{aligned}A(k_1,k_2,k_3)&=\left(\frac{g_c}{\alpha'}\right)^{1/2}\varepsilon^1_{\mu_1}\varepsilon^2_{\mu_2}\varepsilon^3_{\mu_3}\left(\eta^{\mu_1\mu_2}k_1^{\mu_3}+\eta^{\mu_2\mu_3}k_2^{\mu_1}+\eta^{\mu_1\mu_3}k_3^{\mu_2}\right)\\&\otimes\left(\frac{g_c}{\alpha'}\right)^{1/2}\varepsilon^1_{\nu_1}\varepsilon^2_{\nu_2}\varepsilon^3_{\nu_3}\left(\eta^{\nu_1\nu_2}k_1^{\nu_3}+\eta^{\nu_2\nu_3}k_2^{\nu_1}+\eta^{\nu_1\nu_3}k_3^{\nu_2}\right),\end{aligned}\tag{5.6.138}$$

comparing this with our three point open string amplitude (5.6.132) we can see the double copy:

$$A^{closed}(p_1,p_2,p_3) \sim A^{open}(k_1,k_2,k_3) \otimes A^{open}(k_1,k_2,k_3). \tag{5.6.139}$$

where a tracing out of the two sets of Chan-Paton generators is understood. Next, we are going to look at the two open and one closed string amplitude. We have:

Example (5.6): 2 open and 1 closed string amplitude

We have an amplitude of two massless open strings and a massless closed string. For the open strings, we have the vertex operators in two pictures:

$$\begin{aligned} &V^{\mu}_{-1}(x,k,\varepsilon) = \varepsilon_\mu g_o T^a e^{-\phi}\psi^\mu e^{ik\cdot X} \qquad k^2=0 \quad \varepsilon\cdot k = 0\,, \\ &V^{\mu}_{0}(x,k,\varepsilon) = g_o\, T^a \sqrt{\frac{2}{\alpha'}}\left(\partial X^\mu - 2i\alpha'(k\cdot\psi)\psi^\mu\right)(z)e^{ik\cdot X(z)} \qquad k^2=0; \quad \varepsilon\cdot k=0\,, \end{aligned} \tag{5.6.140}$$

For the closed string, we have the following possibilities:

$$\begin{aligned} &V_c^{(0,0)}(z,\bar{z},\varepsilon,q) = \frac{g_c}{\alpha'}\,\varepsilon_{\mu\nu}\left[i\bar{\partial}X^\mu + \frac{\alpha'}{2}(\tilde{q}\overline{\psi})\overline{\psi}^\mu(\bar{z})\right]\left[i\partial X^\nu + \frac{\alpha'}{2}(q\psi)\psi^\nu(z)\right]e^{iqX(z,\bar{z})}, \\ &V_c^{(-1,-1)}(z,\bar{z},\varepsilon,q) = g_c\,\varepsilon_{\mu\nu}\,e^{-\overline{\phi}(\bar{z})}\overline{\psi}^\mu(\bar{z})\,e^{-\phi(z)}\psi^\nu(z)\,e^{iq\cdot X(z,\bar{z})}\,, \\ &q^2=0, \qquad \varepsilon^\mu{}_\mu = 0, \end{aligned} \tag{5.6.141}$$

This amplitude can be considered as an closed string scattered off of a D-brane (cf. 5.2.1). Therefore, the momentum conservation will be along the brane and we have the following:

$$\begin{aligned} &q = \frac{1}{2}(q+Dq) + \frac{1}{2}(q-Dq) = q_{||} + q_\perp \to \tilde{q} = Dq \\ &k_1 + k_2 + \frac{1}{2}(q+Dq) = 0 \end{aligned} \tag{5.6.142}$$

The Riemann surface for the mixed open-closed strings is the disk with punctures on the boundary for open string states and punctures on the interior of the disk for the closed string. We map the disk to the upper half-plane with the open strings on the real line and the closed string on the upper half-plane. Therefore, we have three punctures: Two real and one complex. So we have the following scattering process:

$$\begin{aligned} \mathcal{A}(2;1) &= \int \frac{dz_1 dz_2 d^2z_3}{SL(2,\mathbf{R})}\,\delta_{ghost}\,e^{-\lambda}\,C^g_{D_2}\left\langle V_o^{(-1)}(\varepsilon_1,k_1,z_1)V_o^{(-1)}(\varepsilon_2,k_2,z_2)V_c^{(0,0)}(\varepsilon_q,q,z_3,\bar{z}_3)\right\rangle_{D_2} \\ &= \frac{g_o^2 g_c}{\alpha'}C_{D_2}\varepsilon_\mu\varepsilon_\nu\varepsilon_{\alpha\beta}\int\frac{dz_1dz_2d^2z_3}{SL(2,\mathbf{R})}\Bigg\langle e^{-\phi(z_1)}\psi^\mu(z_1)e^{ik_1\cdot X(z_1)}\;e^{-\phi(z_2)}\psi^\nu(z_2)e^{ik_2\cdot X(z_2)} \\ &\quad\times\left[i\bar{\partial}X^\alpha(\bar{z}_3) + \frac{\alpha'}{2}(\tilde{q}\overline{\psi})\overline{\psi}^\alpha(\bar{z}_3)\right]\left[i\partial X^\beta(z_3) + \frac{\alpha'}{2}(q\psi)\psi^\beta(z_3)\right]e^{iqX(z_3,\bar{z}_3)}\Bigg\rangle_{D_2}\,. \end{aligned} \tag{5.6.143}$$

We have the following on–shell constraints (A.2.7) for the polarization and momenta of the amplitude:

$$\begin{aligned} &\varepsilon_{\alpha\beta} = \varepsilon_\alpha\otimes\varepsilon_\beta \\ &k^\mu\varepsilon_\mu = 0,\;\; q^\alpha\varepsilon_{\alpha\beta} = \tilde{q}^\alpha\varepsilon_{\alpha\beta} = 0 \\ &k_1\cdot k_2 = k_1\cdot q = k_2\cdot q = 0 \\ &k_1\cdot k_2 = k_1\cdot\tilde{q} = k_2\cdot\tilde{q} = 0\,. \end{aligned} \tag{5.6.144}$$

Here, we have used z for the position of the open strings but it should be clear that they are real. Doing the contractions we have (up to overall numerical normalization):

$$\begin{aligned}\mathcal{A}(2;1) = \frac{g_c}{\alpha'^2}\int_{\mathbf{R}}\int_{\mathbf{C}} \frac{dx_1 dx_2 d^2 z_3}{SL(2,\mathbf{R})}\delta_{ghost}\ \frac{\varepsilon_\mu\varepsilon_\nu\varepsilon_\alpha\varepsilon_\beta}{z_1-z_2} \\ \times\Bigg\{\frac{g^{\mu\nu}}{z_1-z_2}\left(\frac{g^{\alpha\beta}}{(z_3-\overline{z}_3)^2}+\frac{k_1^\alpha z_{12}}{(z_1-\overline{z}_3)(z_2-\overline{z}_3)}\frac{k_2^\beta z_{12}}{(z_1-z_3)(z_2-z_3)}\right) \\ +\frac{1}{2}\frac{k_1^\alpha z_{12}}{(z_1-\overline{z}_3)(z_2-\overline{z}_3)}\left(-\frac{q^\mu}{z_1-z_3}\frac{g^{\nu\beta}}{z_2-z_3}+\frac{q^\nu}{z_1-z_3}\frac{g^{\mu\beta}}{z_2-z_3}\right) \\ +\frac{1}{2}\frac{k_1^\beta z_{12}}{(z_1-z_3)(z_2-z_3)}\left(-\frac{\tilde{q}^\mu}{z_1-\overline{z}_3}\frac{g^{\nu\alpha}}{z_2-\overline{z}_3}+\frac{\tilde{q}^\nu}{z_1-\overline{z}_3}\frac{g^{\mu\alpha}}{z_2-\overline{z}_3}\right)\Bigg\}\end{aligned} \tag{5.6.145}$$

which with the choice of spacetime-filling brane (i.e. $(Dq)^\mu = q^\mu$) can be simplified to

$$\begin{aligned}&\mathcal{A}(2;1) = \\ &\frac{g_c}{\alpha'^2}\int\frac{dz_1 dz_2 d^2 z_3}{SL(2,\mathbf{R})}\ \delta_{ghost}\Bigg\{\frac{(\varepsilon_3 k_1)}{(z_1-z_3)(z_1-\overline{z}_3)(z_2-z_3)(z_2-\overline{z}_3)}\left[-\frac{1}{2}(\varepsilon_2\varepsilon_4)(\varepsilon_1 q)+\frac{1}{2}(\varepsilon_1\varepsilon_4)(\varepsilon_2 q)\right] \\ &+\frac{(\varepsilon_4 k_1)}{(z_1-z_3)(z_1-\overline{z}_3)(z_2-z_3)(z_2-\overline{z}_3)}\left[-\frac{1}{2}(\varepsilon_2\varepsilon_3)(\varepsilon_1 q)+\frac{1}{2}(\varepsilon_1\varepsilon_3)(\varepsilon_2\ q)+(\varepsilon_1\varepsilon_2)(\varepsilon_3\cdot k_2)\right]\Bigg\},\end{aligned}$$

We have the result for two massess open, and one massless closed strings:

$$\begin{aligned}\mathcal{A}(2;1) = \frac{g_c}{\alpha'^2}\int\frac{dz_1 dz_2 d^2 z_3}{SL(2,\mathbf{R})}\delta_{ghost}\ \frac{(\varepsilon_3\cdot k_1)}{|(z_1-z_3)|^2|(z_2-z_3)|^2} \\ \times\Bigg\{-(\varepsilon_2\varepsilon_3)(\varepsilon_1 q)+(\varepsilon_1\varepsilon_3)(\varepsilon_2 q)+(\varepsilon_1\varepsilon_2)(\varepsilon_3 k_2)\Bigg\}.\end{aligned} \tag{5.6.146}$$

Now, we can quotient the volume of $SL(2,\mathbf{R})$ by three real positions of the two open strings and real (or imaginary part) of the closed string to $(z_1, z_2, \Re z_3) \mapsto (x, -x, 0)$.

5.7 Amplitude double copy: KLT relations

So far, we have encountered various notions of *double copy* in string theory.

- First, we saw (cf. 5.5.3) that at the level of the coupling, we have:

$$g_o^2 \sim g_c. \tag{5.7.147}$$

- Second, while constructing the Hilbert space we used the fact that the closed string Hilbert space is the tensor product of two open string Hilbert spaces.

$$H_{closed} = H_{open}\otimes H_{open}. \tag{5.7.148}$$

The natural continuation to this line of different double copies is to ask whether this double copy relation can be extended to $S-$matrix elements i.e. scattering amplitudes.

This question for pure closed string amplitudes was answered by the Kawai-Lewellen-Tye in 1986 [25]. In their landmark work they showed that the amplitude of pure closed strings can be written as the product of two open strings. Schematically we have:

$$\mathcal{A}_n^{closed}(q) = F(k,\overline{k})\ \mathcal{A}_n^{open}(k) \times \overline{\mathcal{A}}_n^{open}(\overline{k}). \tag{5.7.149}$$

In this section, we are going to go through this equation. We are going to show the proof of this relation, discuss the function $F(k,\overline{k})$ and show how one can use this to calculate closed string amplitudes with only calculating open string amplitudes.

5.7.1 KLT relations

We start by giving the derivation for the equation (5.7.149). First, let us motivate the intuition behind this formula by discussing the geometrical description of the double copy. We showed in the previous sections that the open string amplitude corresponds to a punctured disk. This can be viewed as a punctured hemisphere. Therefore, by gluing two hemispheres (i.e. amplitudes) together at their boundaries one will get a punctured sphere (see figure 5.8).

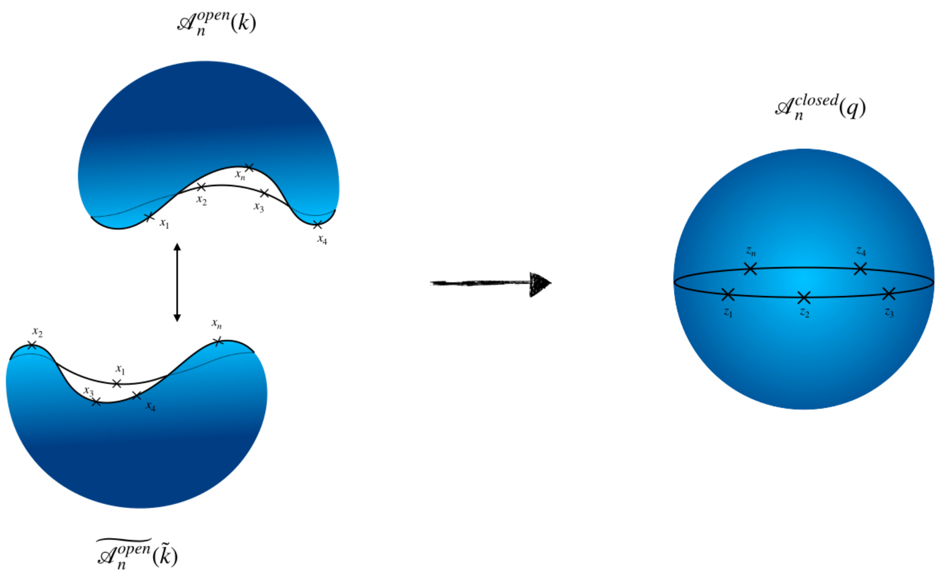

Figure 5.8: Gluing two open strings disk amplitudes will give a (sphere) closed string amplitude

The main issue with this gluing is the map from two integrals over punctured real lines $\int_{\mathbf{R}_n} \otimes \int_{\mathbf{R}_n}$ to an integral over the punctured complex plane $\int_{\mathbf{C}_n}$. This is the point where the function $F(k,\tilde{k})$ comes into play. It makes sure that the this map $(\int_{\mathbf{R}_n} \otimes \int_{\mathbf{R}_n} \to \int_{\mathbf{C}_n})$ is single valued. This is the main result of the KLT amplitude relations. However, they do it in the reverse order. In order words, they take the closed string integral which is integrated over the punctured complex plane and decompose it into two real integrals. Schematically the map is given by:

$$\int_{\mathbf{C}_n} I(z,\overline{z},q) \mapsto F(k(q),\tilde{k}(q))\left(\int_{\mathbf{R}_n} I(z,k)\otimes\int_{\mathbf{R}_n}\tilde{I}(\overline{z},\tilde{k})\right), \tag{5.7.150}$$

Here, we calculate the precise form of the KLT relation for pure closed string amplitude. Looking at the definition of the amplitude we gave in (5.6.117) we have the following integral:

$$\begin{aligned}\mathcal{A}^{closed}(q_1,q_2,q_3,...,q_n) &= \int_{\mathbf{C}^n}\prod_{l=1}^{n}\frac{d^2z_l}{V_{CKG}}\quad I(q,\tilde{q},z,\overline{z}),\\ I(z,\overline{z}) &:= \delta_{ghost}\ e^{-n_{S_2}\lambda}\ C^{matter}_{S_2}\ \mathbf{Con}_{S_2}\{\prod_{m=1}^{n}V_m(q_m,\tilde{q}_m,z_m,\overline{z}_m)\},\end{aligned} \tag{5.7.151}$$

We drop the picture number since this construction works for both bosonic and superstring amplitudes and it will not affect calculations. First, we look at the vertex operators $V_k(z_k,\overline{z}_k)$ using the fact that the Hilbert space of closed string is the double copy of open strings as well as the operator state correspondence will result in the decomposition of the closed string vertex operator into two open string vertex operators one holomorphic and one anti-holomorphic:

$$\begin{aligned}&V^{closed}_m(z_m,\overline{z}_m,q_m,\tilde{q}_m) = V^{open}_m(z_m,q_m)\otimes\overline{V}^{open}_m(\overline{z}_m,\tilde{q}_m),\\ &\delta^{ghost}_{S_2}\ e^{-n_{S_2}\lambda}\ C^{matter}_{S_2} = \delta^{ghost}_{D_2}\ e^{-n_{D_2}\lambda}\ C^{matter}_{D_2}\otimes\overline{\delta}^{ghost}_{D_2}\ e^{-n_{D_2}\lambda}\ C^{matter}_{D_2},\end{aligned} \tag{5.7.152}$$

Looking at the OPEs given in the table 5.1 we can see that for the case of the sphere there are no cross contractions between holomorphic $(\partial X^\mu,\psi^\mu)$ and anti-holomorphic $(\overline{\partial}X^\mu,\overline{\psi}^\mu)$ fields. Therefore, the contraction function $\mathbf{Con}_{S_2}$ will be decomposed into holomorphic and anti-holomorphic parts, Namely:

$$\begin{aligned}&\mathbf{Con}_{S_2}\{\prod_{m=1}^{n}V^c_m(z_m,\overline{z}_m,q_m,\tilde{q}_m)\}\\ &= \mathbf{Con}_{S_2}\{\prod_{m=1}^{n}V^o_m(z_m,q_m)\}\otimes\mathbf{Con}_{S_2}\{\prod_{m=1}^{n}\overline{V}^o_m(\overline{z}_m,\tilde{q}_m)\}\\ &= \mathbf{Con}_{D_2}\{\prod_{m=1}^{n}V^o_m(z_m,\frac{1}{2}q_m)\}\otimes\mathbf{Con}_{D_2}\{\prod_{m=1}^{n}\overline{V}^o_m(\overline{z}_m,\frac{1}{2}\tilde{q}_m).\end{aligned} \tag{5.7.153}$$

In the last line, we have changed the contraction surface from the sphere to the disk (to be able to make contact with the open string amplitude). However, the OPEs on the sphere are the same as on the disk only up to numerical factors which are taken into account by rescaling the momentum (the factor $\frac{1}{2}$ in the vertex operators)[13].

$$\frac{1}{2}q = k^o,\qquad \frac{1}{2}\tilde{q} = \overline{k^o},$$

[13]This is also known as the doubling trick.

Plugging this back into (5.7.151) we have:

$$
\begin{aligned}
&\mathcal{A}^{closed}(q_1, q_2, q_3, ..., q_n) = \int_{\mathbf{C}^n} \prod_{l=1}^{n} \frac{d^2 z_l}{V_{CKG}} \\
&\times \left(\delta_{ghost}\, e^{-n_{D_2}\lambda}\, \mathbf{Con}_{D_2}\{ \prod_{m=1}^{n} V_m(z_m, k_m^o)\} \otimes \overline{\delta}_{ghost}\, e^{-n_{D_2}\lambda}\, \mathbf{Con}_{D_2}\{ \prod_{m=1}^{n} \overline{V}_m(\overline{z}_m, \overline{k^o}) \right), \\
&\mathcal{A}^{closed}(k_1, k_2, k_3, ..., k_n) = \int_{\mathbf{C}^n} \prod_{l=1}^{n} \frac{d^2 z_l}{V_{CKG}} \left(I_n^{open}(z, k^o) \otimes \overline{I}_n^{open}(\overline{z}, \overline{k^o}) \right).
\end{aligned}
\tag{5.7.154}
$$

Above, we have used the definition of the open string amplitude integrand given in (5.6.117). This *nice* form of the closed string amplitude is very close to a double copy of open string amplitudes. The last but most important step left is to change the integration from the punctured sphere to the boundary of the punctured disk. So start at the measure, without loss of generality we use the CKV and fix three positions. After removing them from the product we obtain the following:

$$
\begin{aligned}
&\int \prod_{l=2}^{n-1} d^2 z_l = \int \prod_{l=2}^{n-1} dz\; d\overline{z}\,, \\
&z_l = x_l + i y_l; \qquad \overline{z} = x_l - i y_l, \\
&\int \prod_{l=2}^{n-1} dz\; d\overline{z} = \left(\frac{i}{2}\right)^{n-3} \prod_{l=2}^{n-1} \int_{\mathbf{R}} dx_l \int_{\mathbf{R}} dy_l\,.
\end{aligned}
\tag{5.7.155}
$$

Naively, one would think that by using the last equation our task is finished. We have two real integrals and the integrands decomposed as in (5.7.154). However, this does not work if we plug the last line back into the (5.7.154) we see that we cannot regroup each integral as open string amplitudes, because the variables in the integrand (5.7.154) are complex (z or $\overline{z}$) and not real (x or y). Therefore, we need two real variables to replace z and $\overline{z}$ in the integrand. This is the main idea behind the KLT paper which can be summarized in the following steps:

1. First, take the imaginary part of each complex variable z_l (i.e. y_l) and consider it as an independent variable.

2. Second, do an analytic continuation on the real variable y_l into the complex plain. Meaning, now y_l^c is complex:
$$y_l^c = y_l + i\omega_l,$$
and the complex variables will be:
$$
\begin{aligned}
z_l &= x_l + i(y_l + i\omega_l) = x_l - \omega_l + i y_l, \\
\overline{z}_l &= x_l - i(y_l + i\omega_l) = x_l + \omega_l - i y_l,
\end{aligned}
\tag{5.7.156}
$$

Therefore, in this continuation (x, y_l^c), the original integration $(dz\,,\,d\bar{z})$ can be represented as a complex integral over the contour of $C_o := \omega_l = 0$ for y_l^c. Meaning:

$$\prod_{l=2}^{n-1}\int_{\mathbf{C}} dz_l\, d\bar{z}_l = \left(\frac{i}{2}\right)^{n-3}\prod_{l=2}^{n-1}\int_{\mathbf{R}} dx_l \int_{C_o} dy_l^c\,, \tag{5.7.157}$$

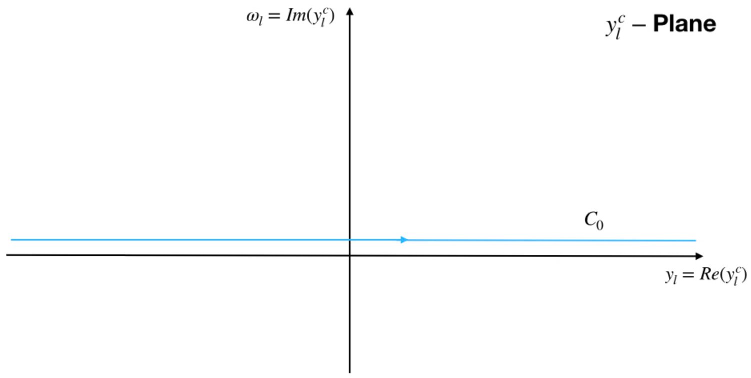

Figure 5.9: The contour of the disk integral for the analytically continued y_l^c. C_0 corresponds to the real integral of original y_l.

3. The most *involved* step is that we now rotate the C_o contour around the complex plane from the real axis to the imaginary axis and call it C_1 as in figure 5.10. Therefore, the y_l^c goes from being pure real (on the contour C_0) to pure imaginary (on the contour C_1). This is possible due to the fact that there are no obstructions (poles, genus, etc) along the rotation. The only important factor, that we should take care of, is the monodromy of the integrand while rotating the contour. We can use the Cauchy's theorem as the following:

$$\begin{aligned}
&\oint_C dy_l^c\ \mathcal{I}(y_l^c) = \int_{C_o} dy_l^c\ \mathcal{I}(y_l^c) + \int_{C_1} dy_l^c\ \mathcal{I}(y_l^c) + \int_{C_\infty} dy_l^c\ \mathcal{I}(y_l^c) = 0.\\
&\int_{C_o} dy_l^c \mathcal{I}(y_l^c) = -\int_{C_1} dy_l^c \mathcal{I}(y_l^c) = -\int_{\infty}^{-\infty} dy_l^c\ Mon_{C_1}\{\mathcal{I}(y_l^c)\}\\
&= \int_{\mathbf{R}} d\omega_l\ Mon_{C_1}\{\mathcal{I}(y_l^c)\}\,,
\end{aligned} \tag{5.7.158}$$

where the function $Mon_{C_1}\{\mathcal{I}\}$ gives the monodromy around each critical point of $\mathcal{I}$ given the contour C_1 as we defined in (2.4.22).

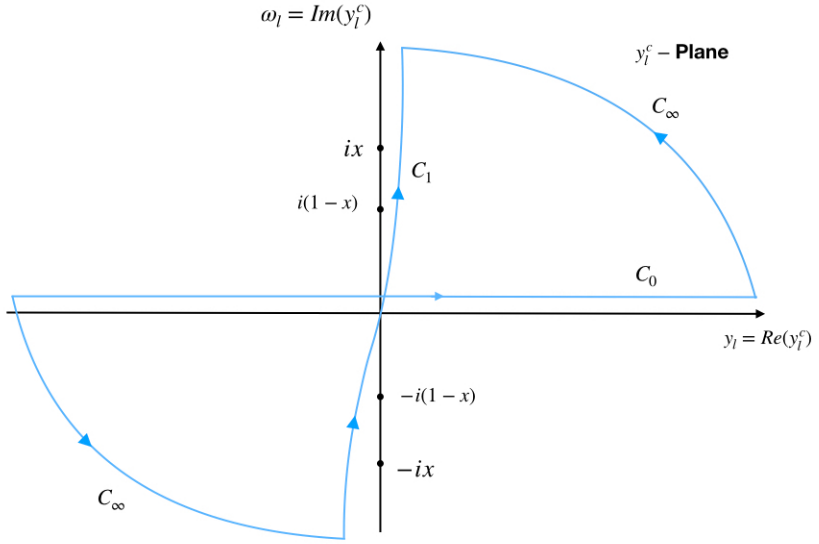

Figure 5.10: Rotating the contour C_0 to C_1 in the complex y_l^c variable. The poles associated with y_l^c are on the imaginary line. We have depicted them for the four point case and they are given by $\pm ix$ and $\pm i(1-x)$.

4. Now we go back to the closed string amplitude integral:

$$\int \prod_{l=2}^{n-1} dz\, d\overline{z}\, \mathcal{I}(z,\overline{z}) = \left(\frac{i}{2}\right)^{n-3} \prod_{l=2}^{n-1} \int_{\mathbf{R}} dx_l \int_{C_o} dy_l^c\, \mathcal{I}(x_l - \omega_l + iy_l, x_l + \omega_l - iy_l)$$
$$= \left(\frac{i}{2}\right)^{n-3} \prod_{l=2}^{n-1} \int_{\mathbf{R}} dx_l \int_{\mathbf{R}} d\omega_l\; Mon_{C_1}\{\mathcal{I}(x_l - \omega_l, x_l + \omega_l)\}\,. \tag{5.7.159}$$

The main difference to the real variable case we had in (5.7.155) is that since over the contour C_1, the real part of y_l^c is zero, the variables $(z_l, \overline{z}_l)$ inside the integral are real and given by:

$$z_l = \xi_l := x_l - \omega_l, \qquad \overline{z}_l = \eta_l := x_l + \omega_l, \tag{5.7.160}$$

5. Finally, for convenience we change the variables from (x_l, ω_l) to (ξ_l, η_l) and we have the following integral

$$\int \prod_{l=2}^{n-1} dz\, d\overline{z}\, \mathcal{I}(z,\overline{z}) = (i)^{n-3} \prod_{l=2}^{n-1} \int_{\mathbf{R}} d\xi_l \int_{\mathbf{R}} d\eta_l\; Mon_{C_1}\{\mathcal{I}(\xi_l, \eta_l)\}\,, \tag{5.7.161}$$

Now we can look back at the pure closed string amplitude in (5.7.154) and implement equation (5.7.161). We can see that the holomorphic functions $I(z) \to I(\xi)$ and anti-holomorphic functions $\overline{I}(\overline{z}) \to I(\eta)$. Using this we have:

$$\begin{aligned}\mathcal{A}^{closed}(k_1,k_2,k_3,...,k_n) &= \int_{S_n} \prod_{l=1}^{n} \frac{d^2 z_l}{V_{CKG}} \left(I_n^{open}(z,k^o) \otimes \overline{I}_n^{open}(\overline{z},\overline{k^o}) \right) \\ &= (i)^{n-3} \prod_{l=2}^{n-1} \int_{\mathbf{R}} d\xi_l \int_{\mathbf{R}} d\eta_l \; Mon_{C_1} \left\{ I_n^{open}(\xi,k^o) \otimes \overline{I}_n^{open}(\eta,\overline{k^o}) \right\},\end{aligned} \tag{5.7.162}$$

Looking at the form of the open string integrand in (5.6.125) we can see that the branch points correspond to the Koba-Nielsen factor (since the field contractions will give integer powers of $z_i - z_j$). Therefore, the branch points of $\mathcal{I}$ are at $z_i - z_j = 0$ and $\overline{z}_i - \overline{z}_j = 0$. We can easily read the position of the branch points for each y_l. They are all on the imaginary axis and given by:

$$y_l^c = i(x_l - \xi_l)\,, \qquad y_l^c = -i(x_l - \eta_l)\,, \tag{5.7.163}$$

Using the position of the branch points we can use the monodromy discussion we did in the first chapter (cf. 2.4.2) and calculate the monodromy of the integrand. For a given ordering of the variables ξ as σ and η as σ', over the real line the monodromy of the integrand (as given by the decomposition (5.6.125)) corresponds to the mondromy of the KN since the other contractions give integer contribution to the phase F. We obtain:

$$\begin{aligned}&Mon_{C_1} \left\{ I_n^{open}(\xi_\sigma,k^o) \otimes \overline{I}_n^{open}(\eta_{\sigma'},\overline{k^o}) \right\} = e^{i\pi F(\sigma,\sigma')} \; I_n^{open}(\xi,k^o) \otimes \overline{I}_n^{open}(\eta,\overline{k^o})\,, \\ &F(\sigma,\sigma') = \sum_{i>j} f(k_i \cdot k_j; (\xi_i - \xi_j), (\eta_i - \eta_j)), \\ &f(k_i \cdot k_j; \xi, \eta) = \begin{cases} k_i \cdot k_j & \xi\eta < 0 \\ 0 & \xi\eta > 0 \end{cases},\end{aligned} \tag{5.7.164}$$

We plug this back into (5.7.162) using the fact that the two real line integrations for ξ and η correspond to all possible relative permutations we obtain:

$$\begin{aligned}\mathcal{A}^{closed}(q_1,q_2,q_3,...,q_n) &= (i)^{n-3} \prod_{l=2}^{n-1} \int_{\mathbf{R}} d\xi_l \int_{\mathbf{R}} d\eta_l \, e^{i\pi F(\sigma,\sigma')} \; I_n^{open}(\xi,k) \otimes \overline{I}_n^{open}(\eta,\overline{k^o}) \\ &= (i)^{n-3} \int_{\mathbf{R}} \prod_{l=2}^{n-1} d\xi_l \; I_n^{open}(\xi,k) \otimes \int_{\mathbf{R}} \prod_{l=2}^{n-1} d\eta_l \; \overline{I}_n^{open}(\eta,\overline{k^o}) \; e^{i\pi F(\sigma,\sigma')} \\ \mathcal{A}^{closed}(q_1,q_2,q_3,...,q_n) &= (i)^{n-3} \sum_{\sigma,\sigma'} e^{i\pi F(\sigma,\sigma')} \mathcal{A}_n^{open}(\sigma) \otimes \overline{\mathcal{A}}_n^{open}(\sigma').\end{aligned} \tag{5.7.165}$$

This is the famous KLT formula and it shows the double copy relation as closed string amplitudes can be written as the product of open string amplitudes. We saw this relation explicitly in the amplitude examples. For the three point amplitude there are no integrals and therefore the function $F(\sigma, \sigma') = 0$. Meaning:

$$\mathcal{A}^{closed}(q_1, q_2, q_3) = \mathcal{A}_3^{open}(\sigma) \otimes \overline{\mathcal{A}}_3^{open}(\sigma')\,, \tag{5.7.166}$$

The computational use of this relation is present even in the simple three point case, because the KLT formalism only requires open string amplitudes therefore the number of contractions is going to be half of the closed string counterpart. This comes at the expense of computing the function $F(\sigma, \sigma')$ for different orderings of external legs (σ, σ').

Part III

Scattering Amplitudes and Double Copies

Chapter 6

Amplitudes from intersection numbers

6.1 Preface

In this chapter, we will describe how all the topics that we have discussed, in both mathematical and physical preliminaries, go together and produce our results. The intersection number of twisted forms has been used in recent years to construct a new method to describe amplitudes as well as amplitude relations [5, 6, 7]. The goal of this chapter is to introduce the algorithm we use in order to produce new twisted forms, as well as, give a new understanding of double copy construction in terms of twisted cohomology [1, 2].

First, we discuss the relationship between twisted homology/cohomology and string scattering amplitudes. In short, we are going to show that all string amplitudes can be written in terms of twisted cohomology. The most important implication of this relation for us is that we can look at string amplitudes and bring them to the form that is comparable with intersection numbers of twisted forms and then look at the equivalency between the two formulations and construct new twisted forms. The motivation behind our procedure is straightforward: in the intersection theory formalism (as we have seen so far) there are no notions of a physical system. A twisted form, originally, does not carry any information about physical properties such as the mass or spin of a state. Therefore, constructing amplitudes of physical states becomes a matter of guessing. In contrast, while using string theory we can choose the states that we are scattering, meaning we know their masses, spins, etc. Hence, going through string scattering amplitudes enables us to look for twisted forms of specific amplitudes involving desired physical states.

6.2 Setup

We can now gather ingredients which we will take from previous chapters to construct our algorithm to build new twisted forms as depicted in figure 6.1. We start with twisted cohomology. We are going to use full mathematical setup we introduced in chapters 2

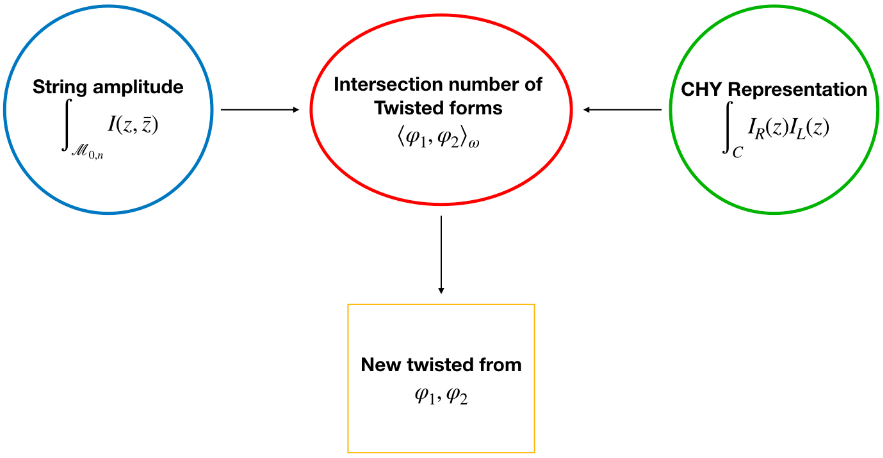

Figure 6.1: The method we use to construct new twisted forms

and 3. In particular, we are going to use the intersection number formula and its saddle point approximation. Let us remind ourselves of those two formulae:

Ingredient I: Intersection theory

$$\langle \varphi_-, \varphi_+ \rangle_\omega := \int_M \iota^k_\omega(\varphi_-) \wedge \varphi_+$$
$$\lim_{\alpha' \to \infty} \langle \varphi_-, \varphi_+ \rangle_\omega = \int_M \left(\bigwedge_{i=1}^{n-3} dz \right) \delta(w) \left(\lim_{\alpha' \to \infty} \hat{\varphi}_-(z) \quad \lim_{\alpha' \to \infty} \hat{\varphi}_+(z) \right) \tag{6.2.1}$$

From our discussions in chapter 5, we are taking the spectrum of the string which we introduced in section 5.5.4 as well as the integral formula of the generic string scattering amplitude:

Ingredient II: String scattering amplitude

$$\mathcal{A}(k_1, k_2, ..., k_n) = \int_{\mathcal{M}_{0,n}} \prod_{l=1}^{n} \frac{d^2 z_l}{V_{CKG}} \; \delta_{ghost} \; e^{-n_X \lambda} \left\langle \prod_{k=1}^{n} : V_k(z_k, \bar{z}_k) : \right\rangle_X$$
$$\mathcal{A}(k_1, k_2, ..., k_n) = \int_{\mathcal{M}_{0,n}} \prod_{l=1}^{n} \frac{d^2 z_l}{V_{CKG}} \; \delta_{ghost} \; e^{-n_X \lambda} \; C_X^{matter} \; \mathbf{Con}_X \{ \prod_{k=1}^{n} V_k(z_k, \bar{z}_k) \} \tag{6.2.2}$$

Finally, from the CHY representation of quantum field theory amplitudes, that we discussed in chapter 4, we take the CHY integral representation of a given scattering amplitude:

Ingredient III: CHY integral representation

$$\begin{aligned}\mathcal{A}_{CHY}(n) &= \int\limits_{\mathcal{M}_{0,n}} d\mu_n \prod_{a=1}^{n}{}' \delta(f_a)\ \mathcal{I}_L^n(p,\varepsilon,\sigma)\mathcal{I}_R^n(p,\varepsilon,\sigma) \\ f_a &\equiv \sum_{\substack{b=1\\ b\neq a}} \frac{p_a\cdot p_b}{\sigma_a-\sigma_b} \quad , \quad a=1,\ldots,n\ ,\end{aligned} \tag{6.2.3}$$

One can already see the apparent similarities between the three formulations. In the next section, we are going to make this relation concrete.

6.3 Construction of amplitudes in twisted cohomology

The very first step, in our construction, is the domain of integration. We have three different integrals: The intersection number, the string scattering amplitude, and CHY integral formula. We are not going to touch the latter since we use it as evidence that our results indeed produce the amplitude that we claim. Therefore, we have two ingredients left, i.e. the string amplitude which is integrated over the moduli space of the punctured Riemann surfaces and the intersection number of twisted forms over Riemann surfaces. In order to construct a twisted cohomology which matches the string amplitudes we take the space of the twisted cohomology to be the moduli space of the punctured Riemann surface i.e. $M=\mathcal{M}_{0,n}$. Therefore, the cohomology in the local sheaf (see theorem 2) will be:

$$\begin{aligned}&\psi \in \Omega^k(\mathcal{M}_{0,n}) \\ &\varphi = \psi \otimes e^{\int_\gamma \omega} \in H(\mathcal{M}_{0,n},\mathcal{L}_\omega)\end{aligned} \tag{6.3.4}$$

The fact that $\mathcal{M}_{0,n}$ is invariant under $SL(2,\mathbf{C})$ (see section 2.6) induces the same invariance over the twisted cohomology (6.3.4). Meaning, the two parts ψ and $e^{\int_\gamma \omega}$ should have the opposite *Möbius charge*[1]. Now we can choose the following hyperplane twist:

$$\begin{aligned}f_i &:= \sum_{j\neq i}^{n} \ln\left(z_i-z_j\right)^{\alpha' p_i\cdot p_j} \\ \omega &= \alpha' \sum_{1\leq i,j\leq n} 2p_ip_j\ d\ln(z_i-z_j), \qquad \sum_{ij} p_i\cdot p_j = 0\end{aligned} \tag{6.3.5}$$

with n on–shell momenta p_i. The factor α' is chosen such that ω is dimensionless.

Now, with this ω we construct the twisted cohomology and the twisted forms on the moduli space of punctured sphere $\mathcal{M}_{0,n} := \mathbf{CP}^{n-3}$. We consider only the top twisted

[1] This is another way of saying that under a $SL(2,\mathbf{C})$ transformation (2.2.4) each term in the tensor product should transform opposite to the other

forms, therefore, we consider $(n-3)$-twisted forms φ as elements in cohomology equivalence classes

$$\varphi \simeq \varphi + \nabla_{\pm\omega}\xi \ , \tag{6.3.6}$$

with a rational $n-4$ form ξ and the Gauss–Manin connection $\nabla_{\pm\omega} = d \pm \omega\wedge$ with d the exterior derivative and closed holohmorphic one–form (twist) ω (6.3.5). As we discussed, we have the $(n-3)$–th twisted cohomology group (see the discussion in chapter 3):

$$H^{n-3}_{\pm\omega}(\mathcal{M}_{0,n}, \nabla_{\pm\omega}) = \frac{\{\varphi \in \Omega^{n-3}(\mathcal{M}_{0,n}) \mid \nabla_{\pm\omega}\varphi = 0\}}{\nabla_{\pm\omega}\Omega^{n-4}(\mathcal{M}_{0,n})} \ . \tag{6.3.7}$$

The dual space $H^{n-3}_{-\omega}$ can be obtained from $H^{n-3}_{+\omega}$ by sending $\omega \to -\omega$. The intersection number on the twisted cohomology groups is the invariant pairing between two forms $\varphi_\pm \in H^{n-3}_{\pm\omega}$ and defined by the integral (3.3.24)

$$\langle \varphi_+, \varphi_- \rangle_\omega := \left(-\frac{\alpha'}{2\pi i}\right)^{n-3} \int_{\mathcal{M}_{0,n}} \iota_\omega(\varphi_+) \wedge \varphi_- \ , \tag{6.3.8}$$

over the space $\mathcal{M}_{0,n}$. The map $\iota_\omega(\varphi_+) \in H^{n-3}_{+\omega}$ is the restriction of the twisted form over the compact support $H^{n-3}_{\omega,c}(\mathcal{M}_{0,n}, \nabla_\omega)$ of $H^{n-3}_{\omega}(\mathcal{M}_{0,n}, \nabla_\omega)$. Otherwise, the integral over the moduli space $\mathcal{M}_{0,n}$ would not be well–defined since the latter is non–compact. Now, we can have our first example of a twisted form with this twist.

Example (6.1):Examples of twisted forms

The $(n-3)$ (Parke–Taylor) form which is also known as *color form* is given by:

$$PT(\sigma) = \frac{d\mu_n}{(z_{\sigma(1)} - z_{\sigma(2)}) \ldots (z_{\sigma(n-1)} - z_{\sigma(n)})} = \mathcal{C}_n(\sigma)\ d\mu_n \ \in H^{n-3}_{\pm\omega} \quad , \quad \sigma \in S_n \ . \tag{6.3.9}$$

Above we have the measure

$$d\mu_n = z_{jk} z_{jl} z_{kl} \prod_{\substack{i=1 \\ i \notin \{j,k,l\}}}^{n} dz_i \ , \tag{6.3.10}$$

which is a degree $n-3$ holomorphic form on $\mathcal{M}_{0,n}$, with z_j, z_k, z_l being three arbitrary marked points fixed by $SL(2, \mathbf{C})$ invariance.

Further, one can see that the Koba–Nielsen factor can be constructed in terms of ω as

$$KN \equiv \prod_{1 \le i,j \le n} |z_i - z_j|^{2\alpha' p_i \cdot p_j} = e^{\int_\gamma \omega} \ , \tag{6.3.11}$$

for some path γ.

Intersection numbers (6.3.8) are always rational functions of kinematic invariants with simple poles in the kinematic invariants. The following concrete computation of intersection numbers (6.3.8) for a particular choice of color form will be very illuminating.

Example (6.2): Intersection number of Park-Taylor factors

we start by the $PT(1,2,3,4)$ for both $\varphi_\pm$. In this case, we have $n=4$, therefore, the top form on $\mathcal{M}_{0,4}$ is a *one* form. we assume the variables (z_2,z_3,z_4) are fixed at (w_2,w_3,w_4) and $z_1=z$ then we have four cases: First we have $z\to w_2$:

$$\begin{aligned}
\varphi_+ &= \frac{dz}{(z-w_2)(w_2-w_3)(w_3-w_4)(w_4-z)} = \frac{PT(2,3,4)}{z-w_2} + \frac{PT(2,3,4)}{w_4-w_2} + \ldots\,, \\
\omega &= \alpha'\left(\frac{s_{12}}{z-w_2} + \frac{s_{13}}{z-w_3} + \frac{s_{14}}{z-w_2}\right) = \frac{\alpha' s_{12}}{z-w_2} + \left(\frac{\alpha' s_{13}}{w_2-w_3} + \frac{\alpha' s_{14}}{w_2-w_4}\right) + \ldots\,,
\end{aligned} \tag{6.3.12}$$

and therefore we can directly calculate the compactifing functions $\psi = \nabla_\omega^{-1} = \sum_{m=0}^{\infty} c_m z^m$ through the expansion (3.3.37):

$$\begin{aligned}
c_0 &= \frac{b_{-1}}{a_{-1}} = PT(2,3,4)\frac{1}{\alpha' s_{12}}\,, \\
c_1 &= \frac{b_0 - a_0 c_0}{(1+a_{-1})} = PT(2,3,4)\frac{\frac{1}{w_4-w_2} - \left(\frac{\alpha' s_{13}}{w_2-w_3} + \frac{\alpha' s_{14}}{w_2-w_4}\right)\frac{1}{\alpha' s_{12}}}{(1+\alpha' s_{12})} \\
&= PT(2,3,4)\frac{s_{13}}{s_{12}(1+\alpha' s_{12})}\frac{(w_4-w_3)}{(w_2-w_4)(w_2-w_3)}\,,
\end{aligned} \tag{6.3.13}$$

and similarly we have for the ψ:

$$\psi = \sum_{m=0}^{\infty} c_m z^m = PT(2,3,4)\left(\frac{1}{\alpha' s_{12}} + \frac{s_{13}}{s_{12}(1+\alpha' s_{12})}\frac{(z-w_2)(w_4-w_3)}{(w_2-w_4)(w_2-w_3)} + \ldots\right)\,, \tag{6.3.14}$$

Similarly, for $z\to w_4$ we have:

$$\begin{aligned}
c_0 &= \frac{b_{-1}}{a_{-1}} = PT(2,3,4)\frac{1}{\alpha' s_{14}}\,, \\
c_1 &= \frac{b_0 - a_0 c_0}{(1+a_{-1})} = PT(2,3,4)\frac{\frac{1}{w_2-w_4} - \left(\frac{\alpha' s_{12}}{w_4-w_2} + \frac{\alpha' s_{13}}{w_4-w_3}\right)\frac{1}{\alpha' s_{14}}}{(1+\alpha' s_{14})} \\
&= PT(2,3,4)\left(\frac{s_{13}}{s_{14}(1+\alpha' s_{14})}\frac{(w_2-w_3)}{(w_2-w_4)(w_3-w_4)}\right)\,, \\
\psi &= \sum_{m=0}^{\infty} c_m z^m = PT(2,3,4)\left(\frac{1}{\alpha' s_{14}} + \frac{s_{13}}{s_{14}(1+\alpha' s_{14})}\frac{(z-w_4)(w_2-w_3)}{(w_2-w_4)(w_3-w_4)}\right)\,,
\end{aligned} \tag{6.3.15}$$

and finally for $z\to w_3$ we obtain:

$$\begin{aligned}
\varphi_+ &= \frac{dz}{(z-w_2)(w_2-w_3)(w_3-w_4)(w_4-z)} = \frac{1}{(w_2-w_3)^2(w_3-w_4)^2} + \ldots\,, \\
\omega &= \alpha'\left(\frac{s_{12}}{z-w_2} + \frac{s_{13}}{z-w_3} + \frac{s_{14}}{z-w_2}\right) = \frac{\alpha' s_{13}}{z-w_3} + \left(\frac{\alpha' s_{12}}{w_2-w_3} + \frac{\alpha' s_{14}}{w_2-w_4}\right) + \ldots\,,
\end{aligned} \tag{6.3.16}$$

note that the there are no poles in $PT(1,2,3,4)$ for $z \to w_3$. We have for the expansions:

$$\begin{aligned} c_0 &= \frac{b_{-1}}{a_{-1}} = 0\,, \\ c_1 &= \frac{b_0 - a_0 c_0}{(1+a_{-1})} = \frac{1}{(w_2-w_3)^2(w_3-w_4)^2}\frac{1}{(1+\alpha' s_{13})}\,, \\ \psi &= \sum_{m=0}^{\infty} c_m z^m = PT(2,3,4)\frac{(w_4-w_2)}{(w_2-w_3)(w_3-w_4)}\frac{(z-w_3)}{(1+\alpha' s_{13})} + \ldots\,, \end{aligned} \tag{6.3.17}$$

and so we can calculate the intersection number:

$$\langle PT(1,2,3,4), PT(1,2,3,4)\rangle_\omega = 2\pi i \sum_{i=2}^{4} Res_{z=z_i}(\psi_i \varphi_+) = \frac{1}{\alpha' s_{12}} + \frac{1}{\alpha' s_{14}}\,, \tag{6.3.18}$$

Here we evidence that, despite the definition (6.3.8) and the twist (6.3.5) involve higher orders in α' the localization procedure entering the computation of the intersection number (6.3.8) provides pure field–theory results. In fact, the intersection numbers (6.3.8) localize near the boundary $\partial\mathcal{M}_{0,n}$ of the moduli space where two or more points z_i coalesce. While the space (3.2.6) of ordinary $(n-3)$–forms φ is $(n-2)!$ dimensional the space of twisted $(n-3)$–forms (i.e. the $(n-3)$ Betti number) is given, by using of the Poincare polynomial by the Euler character.

$$\dim(H^{n-3}_\omega) = \chi(\mathcal{M}_{0,n}) = (n-3)!\,,$$

Also the twisted homology group $H^\omega_{n-3}(\mathcal{M}_{0,n}, e^{\int_\gamma \omega}) = H^\omega_{n-3}(\mathcal{M}_{0,n}, KN)$, which we now associate with the (multivalued) Koba-Nielsen function KN, is $(n-3)!$ dimensional. As we discussed in (3.2.8) elements of the latter are specified by a cycle C_γ and local coefficients (6.3.11) as:

$$\begin{aligned} &C_\gamma := \{(x_1, x_2, x_3, \ldots, x_{n-3} \in \mathbf{R}^{n-3} | x_{a(1)} < x_{a(2)} < x_{a(3)} < \ldots < x_{a(n-3)}\} \\ &C_{\gamma,\omega} = C_\gamma \otimes KN\ . \end{aligned} \tag{6.3.19}$$

The twisted homology cycles (6.3.19) are Poincare dual to the twisted cohomology H^{n-3}_ω and one can consider the following pairing as (cf. 3.3.20):

$$\langle C_\gamma \otimes KN | \varphi_+\rangle := \int_{C_\gamma} KN\ \varphi_+\ , \tag{6.3.20}$$

The first thing to notice is that the period (6.3.20) has taken the form of an open string amplitude. Looking back at the open string amplitude formula (5.6.126) we see that up to constants we have:

$$
\begin{aligned}
&\mathcal{A}^{open}(k_1, k_2, k_3, ..., k_n) = \int_{\mathcal{M}_{0,n}} \prod_{l=1}^{n} \frac{dz_l}{SL(2,\mathbf{R})} \; KN \; \tilde{I}(z) = \sum_{\gamma} \int_{C_\gamma} \prod_{l=1}^{n-3} dz_l \; KN \; \tilde{I}(z) \\
&\langle C_\gamma \otimes KN | \varphi_+ \rangle := \int_{C_\gamma} KN \; \varphi_+ \; , \\
&\Rightarrow \mathcal{A}^{open}(k_1, k_2, k_3, ..., k_n) = \sum_{\gamma} \langle C_\gamma \otimes KN | \varphi_+ \rangle \, .
\end{aligned} \tag{6.3.21}
$$

As we mentioned in equation (3.2.19) one can choose for the dual twisted form to be $H^k(\mathcal{M}(A), \nabla_{\overline{\omega}})$. Then, the dual homology objects (3.3.21) and (3.3.22) can be related by replacing KN^{-1} and $\overline{KN}$. The pairing H_{ω}^{n-3} and $H_{\overline{\omega}}^{n-3}$ is more suited to describe closed string world–sheet integrals. Specifically, we can compare a paring of two forms $\overline{\varphi}_+$ and φ_+ and closed string amplitudes. Using the formula for closed strings (5.6.127) we can see that the closed string amplitudes up to constants will be:

$$
\begin{aligned}
&\mathcal{A}^{closed}(k_1, k_2, k_3, ..., k_n) = \int_{\mathcal{M}_{0,n}} \prod_{l=1}^{n-3} d^2 z_l \; |KN|^2 \; \tilde{I}(z) \times \bar{\tilde{I}}(\bar{z}) \;\; , \\
&\langle \bar{\varphi}_+, \varphi_+ \rangle = \int_{\mathcal{M}_{0,n}} \prod_{l=1}^{n-3} d^2 z_l \; |KN|^2 \; \bar{\varphi}_+ \wedge \varphi_+ \qquad \bar{\varphi}_+ \in H_{\overline{\omega}}^{n-3}, \quad \varphi_+ \in H_{\omega}^{n-3}.
\end{aligned} \tag{6.3.22}
$$

In particular, in this language the twisted period relations (3.4.47) can be interpreted as KLT relations [78]. We shall make use of this isomorphism while constructing our twisted forms for EYM amplitude. To make contact with amplitudes, let us present examples of twisted forms. Worldsheet string correlators are borrowed to construct these twisted forms for field theories. Plugging the latter into the intersection number (6.3.8) yields a (CHY) field theory amplitudes.

Looking now at the integral itself we can use the saddle point approximation that we constructed in section 3.4 with an arbitrary orthonormal basis and obtain the saddle point approximation of the intersection number of ω introduced in (6.3.5):

$$
\begin{aligned}
&\lim_{\alpha' \to \infty} \langle \varphi_+, \varphi_- \rangle_\omega = \int_{\mathbf{CP}^{n-3}} \left(\bigwedge_{i=1}^{n-3} dz \right) \delta(w) \left(\lim_{\alpha' \to \infty} \hat{\varphi}_+(z) \quad \lim_{\alpha' \to \infty} \hat{\varphi}_-(z) \right) , \\
&\omega = \sum_{i=1}^{n-3} \omega_i dz_i ; \qquad \omega_i = \sum_{\substack{j=1 \\ j \neq i}} \frac{p_i \cdot p_j}{z_i - z_j} \, .
\end{aligned} \tag{6.3.23}
$$

A keen-eyed reader will immediately see that this is the exact CHY formula with the scattering equations given by ω_is.

6.3.1 CHY Amplitudes from twisted cohomology pairing

As we mentioned, there is a direct relation between saddle point approximation of the intersection number of twisted forms (6.3.23) and the CHY integral representation of amplitudes. Here, we are going to make this relation concrete and provide a list of already-known examples. We discussed the CHY integrals in chapter 4. There, we mentioned that the amplitude for a given theory is written in terms of an integral, localized over the solutions of the scattering equations:

$$\begin{aligned} \mathcal{A}_{CHY}(n) &= \int\limits_{\mathcal{M}_{0,n}} d\mu_n \prod_{a=1}^{n}{}' \delta(f_a)\ \mathcal{I}_L^n(p,\varepsilon,\sigma)\mathcal{I}_R^n(p,\varepsilon,\sigma) \\ f_a &\equiv \sum_{\substack{b=1\\b\neq a}} \frac{p_a\cdot p_b}{\sigma_a-\sigma_b} \quad , \quad a=1,\ldots,n\ , \end{aligned} \tag{6.3.24}$$

Furthermore, we discussed the saddle point approximation of intersection numbers and we showed for the twist given in (6.3.5) that we have the following:

$$\begin{aligned} \lim_{\alpha'\to\infty}\langle\varphi_-,\varphi_+\rangle_\omega &= \int_{\mathbf{CP}^{n-3}}\left(\bigwedge_{i=1}^{n-3} dz\right)\delta(w)\left(\lim_{\alpha'\to\infty}\hat{\varphi}_-(z)\quad\lim_{\alpha'\to\infty}\hat{\varphi}_+(z)\right) \\ \omega = \sum_{i=1}^{n-3}\omega_i dz_i;&\qquad \omega_i=\sum_{\substack{j=1\\j\neq i}}\frac{p_i\cdot p_j}{z_i-z_j}, \end{aligned} \tag{6.3.25}$$

By comparing the two formulations we arrive at the following relation:

Intersection number vs CHY

$$\begin{aligned} \text{limit of the intersection numbers } \lim_{\alpha'\to\infty}\langle\varphi_+,\varphi_-\rangle_\omega &\sim \text{CHY integral} \\ \lim_{\alpha'\to\infty}\hat{\varphi}_\pm(p,\varepsilon,z) &\leftrightarrow I(p,\varepsilon,\sigma)_{L,R} \end{aligned} \tag{6.3.26}$$

The last relation is the important relation in our discussions and we are going to use it extensively. In both directions, this relation is practical. Meaning, we take an intersection number of two twisted forms then take its limit and construct new CHY integrands. Conversely, we can look at known CHY integrands and construct *limits* of twisted forms. In addition to these two, we make further use of this relation: we will, in the next sections, construct twisted forms based on the string theory spectrum. Then with the use of this relation we *check* that our *stringy motivated* twisted form is producing the desired amplitude. The main difference between reverse engineering twisted forms and string theory-motivated ones is that with string theory we obtain the full-functional structure of the twisted form. In contrast, reverse engineering only gives us the limit of twisted form functions. We use the following algorithm to construct twisted forms:

Intersection number vs CHY

1. We chose the state in string theory that is describing the desired features (spin, mass, etc).
2. Then we calculate the amplitude of that state.
3. Using the relations (6.3.21) and (6.3.22) we read off *candidate twisted forms* from the integrands.
4. Finally we plug the intersection number of candidate twisted forms into the limit in (6.3.26) and calculate the CHY integral.

We accept a twisted form if it produces the correct CHY integrand.

The first example, that we use to showcase the above discussion, is the twisted form of Yang-Mills (YM) amplitude. We start by searching for the string state associated with vector gauge bosons i.e. we need a spin-1 state that is massless, and charged under the gauge group $SU(N)$. Looking at the spectrum of the open NS string, that we gave in table 5.2, we can see that the first excited state has all of the desired properties[2]. The amplitude of n such state in the Grassmann notation is given by [10]:

$$\mathcal{A}^{open}(p_1, p_2, p_3, ..., p_n) = \int_{C_\gamma} d\mu_n \int \prod_{i=1}^{n} d\theta_i d\bar{\theta}_i \frac{\theta_k\theta_l}{z_k - z_l} \exp\left\{ -\alpha'^2 \sum_{i\neq j} \frac{\theta_i\theta_j p_i\cdot p_j + \bar{\theta}_i\bar{\theta}_j \varepsilon_i\cdot\varepsilon_j + 2(\theta_i - \theta_j)\bar{\theta}_i\varepsilon_i\cdot p_j}{z_i - z_j \mp \alpha'^{-1}\theta_i\theta_j} \right\} \times KN\,, \tag{6.3.27}$$

Using the relation (6.3.21) we have the following twisted form:

$$\varphi^{gauge}_{\pm,n} = d\mu_n \int \prod_{i=1}^{n} d\theta_i d\bar{\theta}_i \frac{\theta_k\theta_l}{z_k - z_l} \exp\left\{ -\alpha'^2 \sum_{i\neq j} \frac{\theta_i\theta_j p_i\cdot p_j + \bar{\theta}_i\bar{\theta}_j \varepsilon_i\cdot\varepsilon_j + 2(\theta_i - \theta_j)\bar{\theta}_i\varepsilon_i\cdot p_j}{z_i - z_j \mp \alpha'^{-1}\theta_i\theta_j} \right\}. \tag{6.3.28}$$

Now, we are ready to plug this back in (6.3.26) and obtain the CHY integrand associated with this twisted form:

$$\begin{aligned}
\mathcal{I}_{L,R} &:= \lim_{\alpha'\to\infty} \hat{\varphi}^{gauge}_{\pm,n} = \int \prod_{i=1}^{n} d\theta_i d\bar{\theta}_i \frac{\theta_k\theta_l}{z_k - z_l} \exp\left\{ -\sum_{i\neq j} \frac{\theta_i\theta_j p_i\cdot p_j + \bar{\theta}_i\bar{\theta}_j \varepsilon_i\cdot\varepsilon_j + 2(\theta_i - \theta_j)\bar{\theta}_i\varepsilon_i\cdot p_j}{z_i - z_j} \right\}, \\
\mathcal{I}_{L,R} &:= \lim_{\alpha'\to\infty} \hat{\varphi}^{gauge}_{\pm,n} = \int \prod_{i=1}^{n} d\theta_i d\bar{\theta}_i \frac{\theta_k\theta_l}{z_k - z_l} \exp\left\{ -\sum_{i\neq j} (\theta_i\ \bar{\theta}_i)\Psi_{ij} \begin{pmatrix}\theta_i \\ \bar{\theta}_j\end{pmatrix} \right\},
\end{aligned} \tag{6.3.29}$$

In the last line, we have regrouped the terms in the exponential in a matrix notation where rows and columns are associated to $(\theta_i\ \bar{\theta}_i)$. The matrix Ψ has the exact form that of (4.2.7). Using the properties of Grassmannian integrals (cf. appendix A) we can

[2] Here we are pretending that we do not know that this state gives Yang-Mills theory as effective action.

compute integral in terms of Pfaffian of the matrix Ψ:

$$\mathcal{I}_{L,R} := \lim_{\alpha'\to\infty} \varphi^{gauge}_{\pm,n} = \int \prod_{i=1}^{n} d\theta_i d\bar{\theta}_i \frac{\theta_k\theta_l}{z_k - z_l} \exp\left\{ -\sum_{i\neq j} (\theta_i \ \bar{\theta}_i)\Psi_{ij} \begin{pmatrix}\theta_i \\ \bar{\theta}_j\end{pmatrix} \right\} = \text{Pf}'\Psi_n \,, \tag{6.3.30}$$

Now, reminding ourselves of the CHY integral formula (4.2.4) for Yang-Mills amplitudes we see that this is one of the integrands. Indeed if we calculate the saddle point limit of the intersection number of this twisted form with the Park-Taylor factor we have:

$$\begin{aligned} \lim_{\alpha'\to\infty} \langle \varphi^{color}_{-,n}, \varphi^{gauge}_{+,n} \rangle &= \int_{\mathbf{CP}^{n-3}} \left(\bigwedge_{i=1}^{n-3} dz \right) \delta(w) \left(\lim_{\alpha'\to\infty} \hat{\varphi}^{color}_{-,n}(z) \quad \lim_{\alpha'\to\infty} \hat{\varphi}^{gauge}_{+,n}(z) \right) \\ &= \int_{\mathbf{CP}^{n-3}} \left(\bigwedge_{i=1}^{n-3} dz \right) \delta(w) \mathcal{C}_n \ \text{Pf}'\Psi_n = \mathcal{A}^{YM}(p_1, p_2, ..., p_n) \,, \end{aligned} \tag{6.3.31}$$

So, we have successfully constructed a twisted form arising from string theory. This twisted form and many other similar twisted forms were constructed by Mizera [6]. We gather them in the following table:

Theory	$\varphi_{-,n}$	$\varphi_{+,n}$	CHY representation	Amplitude
bi-adjoint scalar	$\varphi^{color}_{-,n}$	$\varphi^{color}_{+,n}$	$\mathcal{C}_n\mathcal{C}_n$	n color scalar
Einstein	$\varphi^{gauge}_{-,n}$	$\varphi^{gauge}_{+,n}$	$\text{Pf}'\psi_n\text{Pf}'\psi_n$	n gravitons
Yang-Mills	$\varphi^{color}_{-,n}$	$\varphi^{gauge}_{+,n}$	$\mathcal{C}_n\text{Pf}'\psi_n$	n gluons
YM+$(DF)^2$	$\varphi^{color}_{-,n}$	$\varphi^{bosonic}_{+,n}$	??	n higher derivative gluons
Einstein–Weyl	$\varphi^{gauge}_{-,n}$	$\varphi^{bosonic}_{+,n}$	??	n spin 2
special Galilean (sGal)	$\varphi^{scalar}_{-,n}$	$\varphi^{scalar}_{+,n}$	$(\text{Pf}'A_n)^4$	n higher derivative scalars
NLSM	$\varphi^{color}_{-,n}$	$\varphi^{scalar}_{+,n}$	$\mathcal{C}_n(\text{Pf}'A_n)^2$	n scalars
Born–Infeld (BI)	$\varphi^{scalar}_{-,n}$	$\varphi^{gauge}_{+,n}$	$(\text{Pf}'A_n)^2\,\text{Pf}'\psi_n$	n spin 1

Table 6.1: Known theories, their pairs of twisted forms and their CHY representations.

The twisted forms in table 6.1 assume the following form:

$$\varphi^{color}_{\pm,n} = d\mu_n \, \frac{\text{Tr}(T^{c_1}T^{c_2}\dots T^{c_n})}{(z_1 - z_2)(z_2 - z_3)\dots(z_n - z_1)} \equiv \text{Tr}(T^{c_1}T^{c_2}\dots T^{c_n}) \ PT(1, 2, \dots, n) \,,$$

$$
\begin{aligned}
\varphi^{gauge}_{\pm,n} &= d\mu_n \int \prod_{i=1}^{n} d\theta_i d\bar{\theta}_i \frac{\theta_k\theta_l}{z_k - z_l} \exp\left\{ -\alpha'^2 \sum_{i\neq j} \frac{\theta_i\theta_j p_i\cdot p_j + \bar{\theta}_i\bar{\theta}_j \varepsilon_i\cdot\varepsilon_j + 2(\theta_i-\theta_j)\bar{\theta}_i\varepsilon_i\cdot p_j}{z_i - z_j \mp \alpha'^{-1}\theta_i\theta_j} \right\} , \\
\varphi^{scalar}_{\pm,n} &= d\mu_n\, (\mathrm{Pf}' A_n)^2 = d\mu_n\, \frac{\det A_{[kl]}}{(z_k-z_l)^2} , \\
\varphi^{bosonic}_{\pm,n} &= d\mu_n \left(\pm\frac{1}{\alpha'}\right)^{\lfloor \frac{n-2}{2} \rfloor} \int \prod_{i=1}^{n} d\theta_i d\bar{\theta}_i \exp\left\{ \sum_{i\neq j}^{n} \left(\pm\sqrt{\alpha'} \frac{\theta_i\bar{\theta}_i \varepsilon_i\cdot p_j}{z_i - z_j} + \frac{\theta_i\bar{\theta}_i\theta_j\bar{\theta}_j\varepsilon_i\cdot\varepsilon_j}{(z_i-z_j)^2} \right) \right\} \\
&\overset{\alpha'\to\infty}{\longrightarrow} \quad W_{\underbrace{11\ldots1}_{n}}\, d\mu_n \ .
\end{aligned}
\tag{6.3.32}
$$

Actually, all twisted forms in table 6.1 are derived from string theory, which in turn also motivated the corresponding CHY descriptions. For instance, φ^{gauge}, as we showed, refers to superstring theory and is relevant to describe both SYM and GR amplitudes. Therefore, it is natural to discuss the twisted form $\varphi^{bosonic}$ stemming from bosonic string theory. While the partial gluon subamplitudes $A_{YM}(1,\ldots,n)$ are the building blocks for the open superstring amplitudes [79, 80, 81], i.e.

$$
\begin{aligned}
&\mathcal{A}_{\substack{open\\ superstring}}(1,\pi(2,\ldots,n-2),n-1,n) \\
&\qquad = \sum_{\rho,\tau\in S_{n-3}} Z(\pi|\tau)\ S[\rho|\tau]_1\ A_{YM}(1,\rho(2,\ldots,n-2),n-1,n) \ ,
\end{aligned}
\tag{6.3.33}
$$

with the iterated disk integrals $Z(\pi|\tau) = Z(1,\pi(2,\ldots,n-2),n-1,n|1,\tau(2,\ldots,n-2),n,n-1)$, the same role is played by the subamplitudes $B(1,\ldots,n)$ for $(DF)^2+YM$ gauge theory for the open bosonic string [82], i.e.:

$$
\begin{aligned}
&\mathcal{A}_{\substack{open\\ bosonic\ string}}(1,\pi(2,\ldots,n-2),n-1,n) \\
&\qquad = \sum_{\rho,\tau\in S_{n-3}} Z(\pi|\tau)\ S[\rho|\tau]_1\ B(1,\rho(2,\ldots,n-2),n-1,n) \ .
\end{aligned}
\tag{6.3.34}
$$

Above the KLT kernel $S[\rho|\tau]_1$ is given by the $k!\times k!$–matrix [83, 84]

$$
S[\sigma|\rho]_\ell := S^{(0)}[\,\sigma(1,\ldots,k)\,|\,\rho(1,\ldots,k)\,]_1 = \prod_{t=1}^{k}\left(p_1 p_{t_\sigma} + \sum_{r<t} p_{r_\sigma} p_{t_\sigma} \theta(r_\sigma,t_\sigma) \right) \ , \tag{6.3.35}
$$

with $j_\sigma = \sigma(j)$ and $\theta(r_\sigma,t_\sigma) = 1$ if the ordering of the legs r_σ, t_σ is the same in both $\sigma(1,\ldots,k)$ and $\rho(1,\ldots,k)$, and zero otherwise. By comparing the two expressions (6.3.33) and (6.3.34) it is natural to construct from the latter the twisted form $\varphi^{bosonic}$ relevant to a $(DF)^2+YM$ theory. The latter involves couplings of YM and $(DF)^2$ theory. The bosonic field theory of the latter is defined by the following Lagrangian [52]

$$
\begin{aligned}
\mathcal{L}_{YM+(DF)^2} &= \frac{1}{2}\,(D_\mu F^{a\,\mu\nu})^2 - \frac{g}{3}\,F^3 + \frac{1}{2}\,(D_\mu\varphi^a)^2 + \frac{g}{2}\,C^{\alpha\,ab}\varphi^\alpha F^a_{\mu\nu}F^{b\,\mu\nu} \\
&\quad + \frac{g}{3!}\,d^{\alpha\beta\gamma}\varphi^\alpha\varphi^\beta\varphi^\gamma - \frac{1}{2}\,m^2(\varphi^\alpha)^2 - \frac{1}{4}\,m^2F^2 \ ,
\end{aligned}
\tag{6.3.36}
$$

with the field content comprising a massless gluon, a massive gluon and a massive scalar. The Lagrangian of $(DF)^2$–theory was displayed in eq. (4.3.29). To the latter, the YM part is added as a mass deformation, which in total gives rise to the Lagrangian (6.3.36). As a consequence, there are mass terms proportional to m^2 for the fields φ^α and $F^a_{\mu\nu}$. Within bosonic string theory this mass is related to the string tension as

$$m^2 = -\alpha'^{-1} , \tag{6.3.37}$$

accounting for the string tachyonic modes representing both the massive gluon and the massive scalar. In fact, for $\alpha' \to 0$ we recover pure YM theory (after multiplying the Lagrangian by an overall factor of $m^{-2} = \alpha'$), while for $\alpha' \to \infty$ we obtain $(DF)^2$ theory. Hence, after taking the limit $\alpha' \to \infty$ the mass (6.3.37) goes to zero and the Lagrangian boils down to the $(DF)^2$–theory, i.e.

$$\lim_{\alpha'\to\infty} \mathcal{L}_{YM+(DF)^2}(\alpha') \simeq \mathcal{L}_{DF^2} , \tag{6.3.38}$$

with the latter referring to (4.3.29). This is consistent with the CHY construction, which works only for massless theories. Likewise, the $\alpha' \to \infty$ limit of the intersection number (6.3.8) discards the mass term and reproduces the massless $(DF)^2$ theory. In (6.3.32) for $\varphi^{bosonic}$ we have introduced the function W^3, which arises in its $\alpha' \to \infty$ limit. Notice, that $W_{11\ldots1}$ is produced from the twisted form $\varphi^{bosonic}$ in the limit $\alpha' \to \infty$. In this limit the twisted form $\varphi^{bosonic}_{\pm,n}$ can be used for describing the $(DF)^2$ theory. The limit $\alpha' \to \infty$ removes in $\varphi^{bosonic}_{\pm,n}$ all the contractions $\epsilon_i\epsilon_j$ in agreement with the absence of those terms in the $(DF)^2$ theory. Therefore, we define the following form:

$$\varphi^{bosonic}_{\pm,n} = d\mu_n \left(\pm\frac{1}{\alpha'}\right)^{\lfloor\frac{n-2}{2}\rfloor} \int \prod_{i=1}^{n} d\theta_i d\bar\theta_i \, \exp\left\{ \sum_{i\neq j}^{n} \left(\pm\sqrt{\alpha'} \frac{\theta_i\bar\theta_i\varepsilon_i \cdot p_j}{z_i - z_j} \right) \right\} = W_{\underbrace{11\ldots1}_{n}} \, d\mu_n \, . \tag{6.3.39}$$

The last equality is up to an overall sign. On the other hand, for the CHY representation of the $YM + (DF)^2$ theory, we cannot use the limit $\alpha' \to \infty$ since this theory is massive with the mass given by (6.3.37). As a consequence, taking the limit $\alpha' \to \infty$ removes the mass term in (6.3.36) referring to the YM part of this theory. As discussed before, in this limit, only the CHY representation of the $(DF)^2$–theory is reproduced.

A similar situation applies to Einstein–Weyl theory (cf. subsection 4.3.10). In this case, the mass term refers to the Einstein term and one obtains only the conformal gravity (i.e. Weyl) amplitude in the limit $\alpha' \to \infty$. Additionally, this also relates to the notion of massive CHY amplitudes. To conclude while we do have representations in twisted intersection theory for amplitudes of both $YM + (DF)^2$ and $(DF)^2$ theories we only have CHY representations for the latter. The same is true for conformal gravity and Einstein–Weyl theories, cf. Section 7.4.

[3] We defined the function W first in (4.2.13).

6.4 Construction of new twisted forms

One of the main results of this work is to use the method that we described above to construct new twisted forms that correspond to amplitudes of different field theories. As mentioned in our algorithm we are going to use both bosonic and superstring amplitudes as building blocks. Our new results are the following:

- $\tilde{\varphi}^{EYM}$ for *Einstein Yang-Mills* amplitude.
- $\tilde{\varphi}^{bosonic}$ for *Weyl Yang-Mills* and *higher derivative gravity.*

For both of these new twisted forms we are going to use an *embedding* of the disk onto the sphere which we are going to motivate and explain in subsequent sections.
We start with the Einstein Yang-Mils (EYM) theory. We take the conventional superstring theory to provide twisted forms which are suitable to describe EYM amplitudes in their $\alpha' \to \infty$ limit i.e. massless spin-2 and spin-1 states in superstrings. While the selection of the relevant twisted form from (6.3.27) to describe pure gauge and gravity amplitudes follows from an analog expression in open superstring theory (A.1.2) the description of the EYM amplitudes is different due to the presence of both gluons and gravitons. In fact, in string theory such an amplitude is described by a superstring disk amplitude involving both open and closed strings [85]. Due to the boundary of the disk world–sheet interactions between holomorphic and anti–holomorphic closed string fields appear. This mixing has to be taken into account when building the twisted forms relevant to EYM amplitudes. In this section, we prepare the necessary steps for constructing the latter. In particular, we shall promote the disk correlator onto a larger space by some field extension of the open string vertex operator. This map promotes the string theory on the disk to a holomorphic theory on the sphere.

6.4.1 Embedding of the disk onto the sphere

In the setup for ambitwistor strings as well as intersection theory all results are derived on the sphere [6, 45, 21]. Therefore, to construct new twisted forms within our construction we should establish an embedding of the amplitude on the disk onto the sphere. In order to do so, we may analytically continue the positions of the open string fields x_i and treat them as complex coordinates $z_i \in \mathbf{C}$ living on an auxiliary sphere world–sheet. This continuation also requires promoting the fields of the open string vertex operators (A.2.3) and (A.2.5), which are defined over the real line (boundary of the upper half plane), to the full complex plane and analyzing their holomorphic and anti-holomorphic properties. In order to accomplish this we shall look at their equations of motion:

$$\begin{aligned} \partial\overline{\partial}\widetilde{X}^{\mu} &= 0 \ , \\ \overline{\partial}\psi^{\mu} &= 0 \ , \\ \partial\overline{\psi}^{\mu} &= 0 \ . \end{aligned} \tag{6.4.40}$$

Preserving these equations of motion we have:

Open string sphere embedding

We can define an embedding of the n open string vertex positions $x_i \in \mathbf{R}$ located on the boundary of the disk onto the sphere $z_i \in \mathbf{C}$ as:

$$x_i \longmapsto z_i \quad , \quad i = 1, \ldots, n \ . \tag{6.4.41}$$

Similarly, we define an embedding of the open string fields on the disk onto the sphere compatible with the equations of motion as:

$$\begin{aligned} X^\mu(x) &\longmapsto X^\mu(z) + \widetilde{X}^\mu(\overline{z}) \ \Rightarrow \begin{cases} \partial X & \mapsto \partial X \\ k \cdot X & \mapsto k \cdot (X + \widetilde{X}) \ , \end{cases} \\ \psi^\mu(x) &\longmapsto \psi^\mu(z) \ . \end{aligned} \tag{6.4.42}$$

In addition, to adopt a given color ordering from the disk an anti–holomorphic gauge current $\tilde{J}^c(\bar{z})$ is appended to each open string vertex operator. In total, the string vertex operators on the sphere will be defined as:

$$\begin{aligned} V_o(\varepsilon_i, k_i, x_i) &\longmapsto V_o(\varepsilon_i, k_i, z_i)\ e^{ik_i \widetilde{X}(\overline{z}_i)}\ \tilde{J}^{c_i}(\bar{z}_i) \ , & i = 1, \ldots, n \ , \\ V_c(\varepsilon_s, q_s, z_{n+s}, \overline{z}_{n+s}) &\longmapsto V_c = V_o(\varepsilon_s, q_s, z_{n+s})\ V_o(\widetilde{\varepsilon}_s, \widetilde{q}_s, \overline{z}_{n+s}) \ , & s = 1, \ldots, r \ . \end{aligned} \tag{6.4.43}$$

Thus, the open string vertex operators acquire an additional anti-holomorphic field dependence, while the closed string vertex operators remain untouched. In this embedding we also promote the conformal Killing group from $SL(2, \mathbf{R})$ to $SL(2, \mathbf{C})$ and the full disk correlator $\langle \ldots \rangle_{D_2}$ has to be the treated as if it was defined on the sphere $\langle \ldots \rangle_{S_2}$. Meaning, for the generic disk correlator of n open and r closed strings:

$$\Big\langle \prod_{i=1}^n V_o(\varepsilon_i, k_i, z_i) \ \prod_{s=1}^r V_c(\varepsilon_s, q_s, z_{n+s}, \bar{z}_{n+s}) \Big\rangle_{D_2} \ , \tag{6.4.44}$$

we shall now consider the product of correlators on the double cover S_2:

$$\begin{aligned} &\Big\langle \prod_{i=1}^n V_o(\varepsilon_i, k_i, z_i) \ \prod_{s=1}^r V_c(\varepsilon_s, q_s, z_{n+s}, \bar{z}_{n+s}) \Big\rangle_{D_2} \\ &\longmapsto \ \Big\langle \prod_{i=1}^n V_o(\varepsilon_i, k_i, z_i)\ e^{ik_i \widetilde{X}(\overline{z}_i)} \ \prod_{s=1}^r V_c(\varepsilon_s, q_s, z_{n+s}, \bar{z}_{n+s}) \Big\rangle_{S_2} \times \Big\langle \tilde{J}^{c_1}(\bar{z}_1) \ldots \tilde{J}^{c_n}(\bar{z}_n) \Big\rangle_{S_2} \ . \end{aligned} \tag{6.4.45}$$

In particular, the map (6.4.43) changes the Koba–Nielsen factor to a correlator on the double cover S_2

$$KN_{D_2} \longmapsto \ KN_{S_2} \ , \tag{6.4.46}$$

with

$$KN_{S_2} = \prod_{i<j}^{n} |z_j - z_i|^{2\alpha' k_i k_j} \prod_{a<b}^{r} |z_{n+b} - z_{a+n}|^{2\alpha' q_a q_b} \prod_{i=1}^{n}\prod_{a=1}^{r} |z_{a+n} - z_i|^{2\alpha' k_i q_a} \equiv KN \cdot \overline{KN} \ , \tag{6.4.47}$$

which in turn can be split into a pair of pure holomorphic and anti–holomorphic factors

$$\begin{aligned} KN &= \prod_{i<j}^{n} (z_j - z_i)^{\alpha' k_i k_j} \prod_{a<b}^{r} (z_{n+b} - z_{a+n})^{\alpha' q_a q_b} \prod_{i=1}^{n}\prod_{a=1}^{r} (z_{a+n} - z_i)^{\alpha' k_i q_a} \\ &= \prod_{i<j}^{n+r} (z_j - z_i)^{\alpha' p_i p_j} =: e^{\int_\gamma \omega} \ , \\ \overline{KN} &= \prod_{i<j}^{n} (\overline{z}_j - \overline{z}_i)^{\alpha' k_i k_j} \prod_{a<b}^{r} (\overline{z}_{n+b} - \overline{z}_{a+n})^{\alpha' q_a q_b} \prod_{i=1}^{n}\prod_{a=1}^{r} (\overline{z}_{a+n} - \overline{z}_i)^{\alpha' k_i q_a} \\ &= \prod_{i<j}^{n+r} (\bar{z}_j - \bar{z}_i)^{\alpha' p_i p_j} =: e^{\int_{\bar\gamma} \bar\omega} \ , \end{aligned} \tag{6.4.48}$$

respectively. Similar to (6.3.11) in (6.4.48) we have defined the one–forms ω and $\bar{\omega}$ specifying the twisted cohomologies H^{n+r-3}_{ω} and $H^{n+r-3}_{\overline{\omega}}$, respectively.

It is worth pointing out the resemblance of our embedding to the construction of heterotic string theory in which one has a superstring sector for the right movers and a bosonic string sector for the left movers. However, one crucial difference is that we are dealing with a left and right supersymmetric closed string sector and only extend the real variables of the open string. It would be interesting to find connections between our construction and the heterotic ambitwistor string.

6.4.2 Sphere integrand from the superstring disk embedding

In this secton we summarize our procedure to extract twisted forms from superstring disk amplitudes amplitudes $\mathcal{A}(n;r)$ which results in the amplitudes for EYM theory. The superstring amplitude involves n open and r closed strings. The closed string spectrum, as we discussed in chapter 5, includes the graviton state and the open string describes the gluon (massless charged vector state) cf. 5.5.4. These states are described by vertex operators given in subsection A.2. Since we are interested in the construction of twisted forms we suppress the complex integral over the vertex operator positions and consider only the disk correlator (6.4.44), which computes the integrand $\mathcal{I}(n;r)$ for the superstring disk amplitude $\mathcal{A}(n;r)$. Now, we reformulate the disk correlator (6.4.44) as a sphere correlator (6.4.45) by extending the open string vertex operators onto the complex plane using (6.4.43). The specific color ordering of the open strings along the disk boundary is adopted by the gauge current correlator. The gauge correlator $\langle \tilde{J}^{c_1}(\bar{z}_1)\ldots\tilde{J}^{c_n}(\bar{z}_n)\rangle_{S_2}$ decomposes into a sum over various gauge group structures. Since we are only concerned with a single trace color structure of the form $\mathrm{Tr}(T^{c_1}\ldots T^{c_n})$ we shall project the gauge current correlator onto the relevant color form (6.3.9):

$$\left\langle \tilde{J}^{c_1}(\bar{z}_1)\dots\tilde{J}^{c_n}(\bar{z}_n)\right\rangle_{S_2} \longrightarrow \mathrm{Tr}(T^{c_1}\dots T^{c_n})\ \mathcal{C}(1,2,\dots,n)\ , \tag{6.4.49}$$

with:

$$\mathcal{C}(1,2,\dots,n) = \frac{1}{(\bar{z}_1-\bar{z}_2)(\bar{z}_2-\bar{z}_3)\cdot\ldots\cdot(\bar{z}_n-\bar{z}_1)}\ . \tag{6.4.50}$$

As the next step, we analyze the field contractions of the correlator (6.4.45) and express the field interactions in terms of the Graßmann formalism (A.1.2). This leads to the following integrand:

$$\begin{aligned}
\mathcal{I}(n;r) = \mathcal{C}(1,2,\dots,n)\ \times \Big\langle &\int (\prod_{i=1}^{n+2r} d\theta_i d\bar{\theta}_i)\ \frac{\theta_1\theta_2}{z_1-z_2} \\
\times \exp&\Big\{ik_1\cdot(X+\widetilde{X}) + \theta_1\bar{\theta}_1\varepsilon_1\cdot\partial X + \theta_1\sqrt{\alpha'}k_1\cdot\psi + \bar{\theta}_1\sqrt{\alpha'}\varepsilon_1\cdot\psi\Big\} \\
&\vdots \\
\times \exp&\Big\{ik_n\cdot(X+\widetilde{X}) + \theta_n\bar{\theta}_n\varepsilon_n\cdot\partial X + \theta_n\sqrt{\alpha'}k_n\cdot\psi + \bar{\theta}_n\sqrt{\alpha'}\varepsilon_n\cdot\psi\Big\} \\
\times \exp&\Big\{iq_1\cdot X + i\widetilde{q}_1\cdot\widetilde{X} + \theta_{n+1}\bar{\theta}_{n+1}\varepsilon_{n+1}\cdot\partial X + \theta_{n+1}\sqrt{\alpha'}q_1\cdot\psi + \bar{\theta}_{n+1}\sqrt{\alpha'}\varepsilon_{n+1}\cdot\psi \\
&+\theta_{n+2}\bar{\theta}_{n+2}\widetilde{\varepsilon}_{n+1}\cdot\overline{\partial}\widetilde{X} + \theta_{n+2}\sqrt{\alpha'}\widetilde{q}_1\cdot\overline{\psi} + \bar{\theta}_{n+2}\sqrt{\alpha'}\widetilde{\varepsilon}_{n+1}\cdot\overline{\psi}\Big\} \\
&\vdots \\
\times \exp&\Big\{iq_r\cdot X + i\widetilde{q}_r\cdot\widetilde{X} + \theta_{2r+n-1}\bar{\theta}_{2r+n-1}\varepsilon_{n+r}\cdot\partial X + \theta_{2r+n-1}\sqrt{\alpha'}q_r\cdot\psi + \bar{\theta}_{2r+n-1}\sqrt{\alpha'}\varepsilon_{n+r}\cdot\psi \\
&+\ \theta_{2r+n}\bar{\theta}_{2r+n}\widetilde{\varepsilon}_{n+r}\cdot\overline{\partial}\widetilde{X} + \theta_{2r+n}\sqrt{\alpha'}\widetilde{q}_r\cdot\overline{\psi} + \bar{\theta}_{2r+n}\sqrt{\alpha'}\widetilde{\varepsilon}_{n+r}\cdot\overline{\psi}\Big\}\Big\rangle_{S_2}\ .
\end{aligned} \tag{6.4.51}$$

The background ghost charge is taken into account by the term $\frac{\theta_1\theta_2}{z_1-z_2}$ and we have labeled the external momenta as:

$$\begin{aligned}
&\text{open strings: } \{k_1,k_2...,k_n\}\ , \\
&\text{Closed strings: } \{q_1,\widetilde{q}_1,q_2,\widetilde{q}_2,...,q_r,\widetilde{q}_r\}\ ,
\end{aligned} \tag{6.4.52}$$

The integrand (6.4.51) can be further simplified such that the fermionic variables $\theta_i\bar{\theta}_i$ resemble the pattern dictated by the map (6.4.43). To simplify we introduce the (ordered) generalized coordinates as:

$$\{\zeta_1,...,\zeta_{n+2r}\} := \{z_1,z_2,...,z_n,z_{n+1},z_{n+1},z_{n+2},z_{n+2},...,,z_{n+r},z_{n+r}\}\ , \tag{6.4.53}$$
$$\{\xi_1,...,\xi_{n+2r}\} := \{\varepsilon_1,\varepsilon_2,...,\varepsilon_n,\varepsilon_{n+1},\widetilde{\varepsilon}_{n+1},\varepsilon_{n+2},\widetilde{\varepsilon}_{n+2},...,\varepsilon_{n+r},\widetilde{\varepsilon}_{n+r}\}\ , \tag{6.4.54}$$

Note that we deliberately chose the first set $\{\zeta\}$ to *not* be minimal. We are going to take the complex conjugation whenever necessary and it is important to keep this ordering of the elements in each set. Having these new variables we can use the matrix notation in Grßmann variables and regroup the integrand in the following way:

$$
\begin{aligned}
\mathcal{I}(n;r) = \int \prod_{i=1}^{n+2r} d\theta_i d\bar{\theta}_i \ \frac{\theta_1\theta_2}{\zeta_1-\zeta_2} \ \exp\left\{ \frac{1}{2}\alpha'^2 \sum_{i,j\in\mathcal{S}} \begin{pmatrix}\theta_i\\ \bar{\theta}_i\end{pmatrix}^t \psi_{n+r} \begin{pmatrix}\theta_j\\ \bar{\theta}_j\end{pmatrix} \pm \frac{1}{2}\alpha' \sum_{i,j\in\mathcal{S}} \frac{\theta_i\theta_j\bar{\theta}_i\bar{\theta}_j(\xi_i\cdot\xi_j)}{(\zeta_i-\zeta_j)^2} \right\} \\
\times \exp\left\{ \frac{1}{2}\alpha'^2 \sum_{i,j\in\mathcal{S}_r} \begin{pmatrix}\theta_i\\ \bar{\theta}_i\end{pmatrix}^t \psi_r \ \begin{pmatrix}\theta_j\\ \bar{\theta}_j\end{pmatrix} \pm \frac{1}{2}\alpha' \sum_{i,j\in\mathcal{S}_r} \frac{\theta_i\theta_j\bar{\theta}_i\bar{\theta}_j(\xi_i\cdot\xi_j)}{(\bar{\zeta}_i-\bar{\zeta}_j)^2} \right\} \\
\times \mathcal{C}(1,2,\ldots,n) \times KN\cdot\overline{KN} \ .
\end{aligned}
\tag{6.4.55}
$$

where we have used the matrix definitions (4.2.7) and introduced the two sets $\mathcal{S}$ and $\mathcal{S}_r$, given by:

$$
\begin{aligned}
\mathcal{S} &:= \{1,2,3,...,n,n+1,n+3,...,n+2r-1\} \ , \\
\mathcal{S}_r &:= \{n+2,n+4,...,n+2r\} \ .
\end{aligned}
\tag{6.4.56}
$$

Here, $\mathcal{S}_r$ represents the set of indices accounting for the anti–holomorphic parts of the r graviton vertex operators (A.2.4), while $\mathcal{S}$ is the set of indices labeling the holomorphic parts of both gluons and gravitons. By applying the relation (6.3.22) we can read two twisted forms[4]:

$$
\begin{aligned}
\varphi^{EYM}_{\pm,n;r} &= d\mu_{n+r} \ \int \prod_{i\in\mathcal{S}} \frac{\theta_1\theta_2}{\zeta_1-\zeta_2} \ d\theta_i d\bar{\theta}_i \\
&\quad \times \exp\left\{ \frac{1}{2}\alpha'^2 \sum_{i,j\in\mathcal{S}} \begin{pmatrix}\theta_i\\ \bar{\theta}_i\end{pmatrix}^t \psi_{n+r} \ \begin{pmatrix}\theta_j\\ \bar{\theta}_j\end{pmatrix} \right\} \exp\left\{ \pm \frac{1}{2}\alpha' \sum_{i,j\in\mathcal{S}} \frac{\theta_i\theta_j\bar{\theta}_i\bar{\theta}_j(\xi_i\cdot\xi_j)}{(\zeta_i-\zeta_j)^2} \right\} , \\
\widetilde{\varphi}^{EYM}_{\pm,n;r} &= d\mu_{n+r} \ \mathcal{C}(1,2,\ldots,n) \ \int \prod_{i\in\mathcal{S}_r} d\theta_i d\bar{\theta}_i \ \exp\left\{ \frac{1}{2}\alpha'^2 \sum_{i,j\in\mathcal{S}_r} \begin{pmatrix}\theta_i\\ \bar{\theta}_i\end{pmatrix}^t \psi_r \ \begin{pmatrix}\theta_j\\ \bar{\theta}_j\end{pmatrix} \right\} \Bigg|_{\bar{\zeta}_l\to\zeta_l} \\
&\quad \times \exp\left\{ \pm \frac{1}{2}\alpha' \sum_{i,j\in\mathcal{S}_r} \frac{\theta_i\theta_j\bar{\theta}_i\bar{\theta}_j(\xi_i\cdot\xi_j)}{(\zeta_i-\zeta_j)^2} \right\} ,
\end{aligned}
$$

The first twisted form is not new it is already known as φ^{gauge} through the identification:

$$
\varphi^{EYM}_{\pm,n;r} \equiv \varphi^{gauge}_{\pm,n+r} \Bigg|_{\substack{z_l=\zeta_l, \quad l=1,\ldots,n \\ z_{n+k}=\zeta_{n+2k-1}, \ k=1,\ldots,r \\ \\ \theta_{n+k}=\theta_{n+2k-1} \\ \bar{\theta}_{n+k}=\bar{\theta}_{n+2k-1}, \ k=1,\ldots,r}} \ .
\tag{6.4.57}
$$

In fact, it was expected that we obtain the twisted form associated with the superstring amplitudes since we are extending the superstring disk amplitudes which is the origin of the φ^{gauge}. In contrast, the second twisted form that is labeled as $\widetilde{\varphi}^{EYM}_{\pm,n;r}$ is new. We claim that the intersection number of this twisted form together with φ^{gauge} corresponds

[4]Note that in (6.3.22) we have one of the forms to be complex conjugate and an element of $H_{\overline{\omega}}$. By taking the complex conjugation in our procedure we construct the counterpart of this form in H_{ω}.

to the EYM amplitude. Again, following our algorithm we can check this by comparing the tensionless limit ($\alpha' \to \infty$) of the intersection number to the CHY formula for this amplitude (4.3.15):

$$\lim_{\alpha'\to\infty} \langle \varphi^{gague}_{-,n+r}, \widetilde{\varphi}^{EYM}_{+,n;r} \rangle = \int\limits_{\mathcal{M}_{0,n+r}} d\mu_{n+r} \prod_{k=2}^{n+r-2} \delta(f_k)\ \mathcal{C}_n(\mathrm{Pf}'\psi_{S_r})\mathrm{Pf}\psi_{n+r}\ . \tag{6.4.58}$$

which is indeed the CHY representation of the EYM amplitude.

Example 2 gluon and 1 graviton EYM amplitudes

In this subsection, we are going to explicitly compute the string amplitude involving two gluons and one graviton and show how our algorithm works to construct twisted forms. The corresponding amplitude is described by a disk world–sheet with two open and one closed string states. We calculated this amplitude in example 5.6.2 with the standard contraction method. Note that the result (5.6.146) is exact to all orders in α'. In the following, we shall only be interested in the integrand of (5.6.146) and treat the latter as a complex function depending on the three vertex positions $z_i \in \mathbf{C}$. Following the steps in (A.1.1) we shall now introduce fermionic variables $\theta_i, \bar{\theta}_i$ to describe the integrand of the amplitude (5.6.143). We can write the integrand as:

$$\begin{aligned}\mathcal{I}(2;1) = \Bigg\langle \int d\theta_1 d\bar{\theta}_1 d\theta_2 d\bar{\theta}_2 d\theta_3 d\bar{\theta}_3 d\theta_4 d\bar{\theta}_4 \frac{\theta_1\theta_2}{z_1 - z_2} \\ &\times \exp[ik_1\cdot X + \theta_1\bar{\theta}_1\varepsilon_1\cdot\partial X + \theta_1\sqrt{\alpha'}k_1\cdot\psi + \bar{\theta}_1\sqrt{\alpha'}\varepsilon_1\cdot\psi] \\ &\times \exp[ik_2\cdot X + \theta_2\bar{\theta}_2\varepsilon_2\cdot\partial X + \theta_2\sqrt{\alpha'}k_2\cdot\psi + \bar{\theta}_2\sqrt{\alpha'}\varepsilon_2\cdot\psi] \\ &\times \exp[iq\cdot X + i\widetilde{q}\cdot\widetilde{X} + \theta_3\bar{\theta}_3\varepsilon_3\cdot\partial X + \theta_3\sqrt{\alpha'}q\cdot\psi + \bar{\theta}_3\sqrt{\alpha'}\varepsilon_3\cdot\psi \\ &\quad + \theta_4\bar{\theta}_4\widetilde{\varepsilon}_3\cdot\bar{\partial}\widetilde{X} + \theta_4\sqrt{\alpha'}\widetilde{q}\cdot\overline{\psi} + \bar{\theta}_4\sqrt{\alpha'}\widetilde{\varepsilon}_3\cdot\overline{\psi}]\Bigg\rangle_{D_2=S_2} ,\end{aligned} \tag{6.4.59}$$

which in turn can be expressed in terms of the Graßmann integral (A.1.2). We should note here that due to the kinematics (5.6.144) of this amplitude, the disk and sphere correlators result in the same terms. We open up the sums above and expand the exponential up to the quadratic order in the fermionic variables $\theta_i, \bar{\theta}_i$ which leads to a non-vanishing Graßmann integral:

$$\begin{aligned}\mathcal{I}(2;1) = \alpha'^2 \int \prod_{i=1}^{4} \frac{\theta_1\theta_2}{z_1 - z_2} d\theta_i d\bar{\theta}_i \Bigg\{ \frac{(\bar{\theta}_1\theta_3\varepsilon_1\cdot q)(\bar{\theta}_3\bar{\theta}_2\varepsilon_3\cdot\varepsilon_2)}{(z_1 - z_3)(z_3 - z_2)} \theta_4\bar{\theta}_4 \left(\frac{(\widetilde{\varepsilon}_3\cdot k_1)}{z_1 - \overline{z}_3} + \frac{(\widetilde{\varepsilon}_3\cdot k_2)}{z_2 - \overline{z}_3} \right) \\ + \frac{(\bar{\theta}_2\theta_3\varepsilon_2\cdot q)(\bar{\theta}_3\bar{\theta}_1\varepsilon_3\cdot\varepsilon_1)}{(z_3 - z_1)(z_2 - z_3)} \theta_4\bar{\theta}_4 \left(\frac{(\widetilde{\varepsilon}_3\cdot k_1)}{z_1 - \overline{z}_3} + \frac{(\widetilde{\varepsilon}_3\cdot k_2)}{z_2 - \overline{z}_3} \right) \\ + \frac{(\bar{\theta}_2\bar{\theta}_1\varepsilon_2\cdot\varepsilon_1)}{(z_2 - z_1)} \theta_3\bar{\theta}_3 \left(\frac{(\varepsilon_3\cdot k_1)}{z_1 - z_3} + \frac{(\varepsilon_3\cdot k_2)}{z_2 - z_3} \right) \theta_4\bar{\theta}_4 \left(\frac{(\widetilde{\varepsilon}_3\cdot k_1)}{z_1 - \overline{z}_3} + \frac{(\widetilde{\varepsilon}_3\cdot k_2)}{z_2 - \overline{z}_3} \right) \Bigg\} .\end{aligned} \tag{6.4.60}$$

Many of the possible terms vanish due to the on-shell conditions (5.6.144). In fact, using momentum conservation and performing the Graßmann integrals we arrive at

$$\mathcal{I}(2;1) = C\ \frac{(\widetilde{\varepsilon}_3 \cdot k_1)}{|(z_1-z_3)|^2|(z_2-z_3)|^2}\left\{(\varepsilon_2\cdot\varepsilon_3)(\varepsilon_1\cdot q)-(\varepsilon_1\cdot\varepsilon_3)(\varepsilon_2\cdot q)-(\varepsilon_1\cdot\varepsilon_2)(\varepsilon_3\cdot k_2)\right\}\ , \tag{6.4.61}$$

which agrees with (5.6.146). The result (6.4.61) may be compared with the four open superstring amplitude following from (A.1.2) for $n=4$. However, the main differences in the case at hand originate from applying the on-shell condition (5.6.144) which discards all the terms proportional to $p_i \cdot p_j$ and leads to (6.4.61). Furthermore, due to $\varepsilon^\mu{}_\mu = 0$ in (6.4.61) there is no contribution from the bosonic contraction $\langle \partial X \partial X\rangle$.

We now rewrite (6.4.61) in terms of matrix notation and Graßmann integration. For this, we introduce the 8×8 matrix with a block structure

$$\Psi^{(2,1)} = \Psi_3 \otimes \Psi_1\ , \tag{6.4.62}$$

which splits into the 6×6 block matrix $\Psi_3 =$

$$\begin{pmatrix} 0 & 0 & 0 & \frac{-\varepsilon_1\cdot q}{z_1-z_3}-\frac{\varepsilon_1\cdot k_2}{z_1-z_2} & \frac{-\varepsilon_2\cdot k_1}{z_2-z_1} & \frac{-\varepsilon_3\cdot k_1}{z_3-z_1} \\ 0 & 0 & 0 & \frac{-\varepsilon_1\cdot k_2}{z_1-z_2} & \frac{-\varepsilon_2\cdot k_1}{z_2-z_1}-\frac{\varepsilon_2\cdot q}{z_2-z_3} & \frac{-\varepsilon_3\cdot k_2}{z_2-z_3} \\ 0 & 0 & 0 & \frac{-\varepsilon_1\cdot q}{z_1-z_3} & \frac{-\varepsilon_2\cdot q}{z_2-z_3} & \frac{-\varepsilon_3\cdot k_1}{z_3-z_1}-\frac{\varepsilon_3\cdot k_2}{z_3-z_2} \\ \frac{\varepsilon_1\cdot q}{z_1-z_3}+\frac{\varepsilon_1\cdot k_2}{z_1-z_2} & \frac{\varepsilon_1\cdot k_2}{z_1-z_2} & \frac{\varepsilon_1\cdot q}{z_1-z_3} & 0 & \frac{\varepsilon_1\cdot\varepsilon_2}{z_1-z_2} & \frac{\varepsilon_1\cdot\varepsilon_3}{z_1-z_3} \\ \frac{\varepsilon_2\cdot k_1}{z_2-z_1} & \frac{\varepsilon_2\cdot k_1}{z_2-z_1}+\frac{\varepsilon_2\cdot q}{z_2-z_3} & \frac{\varepsilon_2\cdot q}{z_2-z_3} & -\frac{\varepsilon_1\cdot\varepsilon_2}{z_1-z_2} & 0 & \frac{\varepsilon_2\cdot\varepsilon_3}{z_2-z_3} \\ \frac{\varepsilon_3\cdot k_1}{z_3-z_1} & \frac{\varepsilon_3\cdot k_2}{z_2-z_3} & \frac{\varepsilon_3\cdot k_1}{z_3-z_1}+\frac{\varepsilon_3\cdot k_2}{z_3-z_2} & -\frac{\varepsilon_1\cdot\varepsilon_3}{z_1-z_3} & -\frac{\varepsilon_2\cdot\varepsilon_3}{z_2-z_3} & 0 \end{pmatrix}, \tag{6.4.63}$$

and the following 2×2 block matrix:

$$\Psi_1 = \begin{pmatrix} 0 & \frac{\widetilde{\varepsilon}_3\cdot k_1 \overline{z}_{21}}{\overline{z}_{13}\overline{z}_{23}} \\ -\frac{\widetilde{\varepsilon}_3\cdot k_1 \overline{z}_{21}}{\overline{z}_{13}\overline{z}_{23}} & 0 \end{pmatrix}\ . \tag{6.4.64}$$

We have defined the matrix (6.4.62) as concatenation in order to properly describe the action of the Graßmann matrix notation. More precisely, we define:

$$\sum_{i,j=1}^{4}(\theta_j\ \bar\theta_j)\ \Psi^{(2;1)}\begin{pmatrix}\theta_i\\ \bar\theta_i\end{pmatrix} := \sum_{i,j=1}^{3}(\theta_j\ \bar\theta_j)\Psi_3\begin{pmatrix}\theta_i\\ \bar\theta_i\end{pmatrix} + (\theta_4\ \bar\theta_4)\Psi_1\begin{pmatrix}\theta_4\\ \bar\theta_4\end{pmatrix}\ . \tag{6.4.65}$$

With these preparations and applying the well-known formula for the Graßmann integrals we obtain

$$\int\prod_{i=1}^{m} d\theta_i d\bar\theta_i\ \exp\left\{\sum_{i,j=1}^{m}(\theta_j\ \bar\theta_j)M\begin{pmatrix}\theta_i\\ \bar\theta_i\end{pmatrix}\right\} = \text{Pf}\ M\ , \tag{6.4.66}$$

with M being a $2m\times 2m$ matrix. We can express (6.4.61) as:

$$\mathcal{I}(2;1) = \int\prod_{i=1}^{4}\frac{\theta_1\theta_2}{z_1-z_2}d\theta_i d\bar\theta_i \exp\left\{\alpha'^2\sum_{i,j=1}^{4}(\theta_j\ \bar\theta_j)\Psi^{(2;1)}\begin{pmatrix}\theta_i\\ \bar\theta_i\end{pmatrix}\right\}$$

$$= \int \prod_{i=1}^{4} \frac{\theta_1\theta_2}{z_1-z_2} d\theta_i d\bar{\theta}_i \exp\left\{ \alpha'^2\ (\theta_3\ \bar{\theta}_1\ \bar{\theta}_2\ \bar{\theta}_3)\Psi_3^{12} \begin{pmatrix} \theta_3 \\ \bar{\theta}_1 \\ \bar{\theta}_2 \\ \bar{\theta}_3 \end{pmatrix} + \alpha'^2\ \theta_4\bar{\theta}_4\ \frac{\widetilde{\varepsilon}_3\cdot k_1\overline{z}_{21}}{\overline{z}_{13}\overline{z}_{23}} \right\}$$

$$= \mathrm{Pf}\Psi_1\ \frac{\mathrm{Pf}\Psi_3^{12}}{z_1-z_2} = \mathrm{Pf}\Psi_1\ \mathrm{Pf}'\Psi_3\ . \tag{6.4.67}$$

According to the definition (4.2.5) the prime at the Pfaffian accounts for the additional factor $\frac{1}{z_1-z_2}$ in the integral and we have used the definition of $\mathrm{Pf}\Psi_1$ as shown in 7.3.76:

$$\mathrm{Pf}\Psi_1 = (\varepsilon_3\cdot k_1)\ \frac{\overline{z}_{12}}{\overline{z}_{13}\overline{z}_{23}}\ . \tag{6.4.68}$$

Above, we again used that we may set $\widetilde{\varepsilon}_3 = \varepsilon_3$. Note, that the anti–holomorphic part of the graviton vertex (A.2.4) and hence half of its fermionic variables $(\theta_4, \overline{\theta}_4)$ are nested into the block diagonal matrix Ψ_1, which is concatenated with Ψ_3, cf. also (6.4.65). As a consequence, this procedure yields a different Graßmannian structure than in the pure open superstring case (A.1.2).

Let us make some comments. Firstly, the expression (6.4.67) is identical to the integrand of the disk amplitude (5.6.143). This is due to $KN_{D_2} = 1$ and our comment after eq. (5.6.146) that we are dealing with an all-order exact expression in α'. All these properties are special due to the three–particle kinematics and the on-shell conditions (5.6.144). As a consequence of exactness in α' the three-point amplitude (5.6.143) behaves identically in the $\alpha' \to 0$ and $\alpha' \to \infty$ limits, i.e. the integrand (6.4.67) is uniform in α'.

Equipped with the results from the previous sections here we want to check our newly constructed twisted form $\widetilde{\varphi}^{EYM}$ to express the EYM amplitude involving two gluons and one graviton by an appropriate intersection number (6.3.8). Then, the CHY amplitude (4.3.15) is found in the leading $\alpha' \to \infty$ limit of the latter as:

$$\mathcal{A}_{CHY}(2;1) = \lim_{\alpha'\to\infty} \langle \varphi_+, \varphi_- \rangle_\omega\ . \tag{6.4.69}$$

In the following, we shall motivate the construction of our two twisted forms by calculating the string scattering contractions (6.4.51), which furnishes the following direct product structure of two Graßmann integrals

$$\begin{aligned} \mathcal{I}(2;1) = &\int \prod_{i=1}^{3} \frac{\theta_1\theta_2}{z_1-z_2}\ d\theta_i d\bar{\theta}_i\ \exp\left\{ \alpha'^2 \sum_{i,j=1}^{3} (\theta_j\ \bar{\theta}_j)\Psi_3 \begin{pmatrix} \theta_i \\ \bar{\theta}_i \end{pmatrix} \right\} \\ &\times \int d\theta_4 d\bar{\theta}_4\ \exp\left\{ \alpha'^2\ (\theta_4\ \bar{\theta}_4)\Psi_1 \begin{pmatrix} \theta_4 \\ \bar{\theta}_4 \end{pmatrix} \right\}\ KN\cdot\overline{KN}\ , \end{aligned} \tag{6.4.70}$$

with the matrices Ψ_3 and Ψ_1 given in (4.2.7). Above we have appended Koba–Nielsen factors, which of course are trivial $KN_{D_2} = |KN|^2 = 1$ due to kinematical structure of this amplitude (5.6.144). The expression (6.4.70) reminds us of a KLT-like product, which connects a holomorphic and anti-holomorphic sector yet without any color ordering. We will associate the first holomorphic factor of (6.4.70) to be an element of $H^{r+n-3}_{+\omega}$. On the other hand, by using the isomorphism (3.2.19) between dual twisted cohomologies we may associate the second anti–holomorphic factor of (6.4.70) with a twisted form from $H^{r+n-3}_{-\omega}$ by imposing the map:

$$\begin{aligned} \overline{z_i} &\longrightarrow z_i \quad , \quad i=1,\ldots,n+r \ , \\ \overline{KN} &\longrightarrow KN^{-1} \ . \end{aligned} \tag{6.4.71}$$

With these preparations, as our first twisted form, we can choose:

$$\varphi^{EYM}_{\pm,2;1} = d\mu_3 \int \prod_{i=1}^{3} \frac{\theta_1\theta_2}{z_1-z_2}\ d\theta_i d\bar{\theta}_i\ \exp\left\{\alpha'^2 \sum_{i,j=1}^{3} (\theta_j\ \bar{\theta}_j)\Psi_3 \begin{pmatrix}\theta_i\\ \bar{\theta}_i\end{pmatrix}\right\} , \tag{6.4.72}$$

which as we mentioned before is the same as $\varphi^{gauge}_{\pm,3}$. For the second twisted form, we now choose

$$\widetilde{\varphi}^{EYM}_{\pm,2;1} = d\mu_3\ \mathcal{C}(1,2) \int d\theta_4 d\bar{\theta}_4\ \exp\left\{\alpha'^2 (\theta_4\ \bar{\theta}_4)\Psi_1 \begin{pmatrix}\theta_4\\ \bar{\theta}_4\end{pmatrix}\right\}\Bigg|_{\overline{z_l}\to z_l} , \tag{6.4.73}$$

subject to (6.4.71), which entails:

$$\Psi_1\big|_{\overline{z_l}\to z_l} = \begin{pmatrix} 0 & \frac{\varepsilon_3\cdot k_1 z_{21}}{z_{13}z_{23}} \\ -\frac{\varepsilon_3\cdot k_1 z_{21}}{z_{13}z_{23}} & 0 \end{pmatrix} , \qquad \mathrm{Pf}\Psi_1\big|_{\overline{z_l}\to z_l} = (\varepsilon_3\cdot k_1)\ \frac{z_{12}}{z_{13}z_{23}} \ . \tag{6.4.74}$$

In order to properly describe the color ordering of (6.4.76) we have augmented (6.4.73) with the following Park–Taylor factor:

$$\mathcal{C}(1,2) = \frac{1}{(z_1-z_2)(z_2-z_1)} \ . \tag{6.4.75}$$

Eventually, computing the intersection number (6.3.8) of our two twisted forms (6.4.72) and (6.4.73) yields

$$\begin{aligned} \mathcal{A}_{CHY}(2;1) = \langle \widetilde{\varphi}^{EYM}_{2;1}, \varphi^{EYM}_{2;1}\rangle_\omega = &\int \frac{dz_1 dz_2 dz_3}{SL(2,\mathbf{C})}\ \frac{(\varepsilon_3\cdot k_1)}{(z_1-z_2)^2(z_1-z_3)^2(z_2-z_3)^2} \\ &\times \Big\{ (\varepsilon_2\cdot\varepsilon_3)(\varepsilon_1\cdot q) - (\varepsilon_1\cdot\varepsilon_3)(\varepsilon_2\cdot q) + (\varepsilon_1\cdot\varepsilon_2)(\varepsilon_3\cdot k_1) \Big\} , \end{aligned} \tag{6.4.76}$$

which is the CHY amplitude given in (4.3.15). Actually, in the case at hand the $\alpha'\to\infty$ limit (6.4.76) is exact in α', i.e.:

$$\mathcal{A}_{CHY}(2;1) = \lim_{\alpha'\to\infty} \langle \widetilde{\varphi}^{EYM}_{2;1}, \varphi^{EYM}_{2;1}\rangle_\omega \ . \tag{6.4.77}$$

No lower orders in α' show up due to the limited number of contractions for this case.

6.4.3 Sphere integrand from the bosonic string disk embedding

Similar to the superstring we can now apply the embedding to the bosonic string amplitude. While the twisted form $\varphi^{bosonic}_{\pm,n}$ may directly be derived from the bosonic string disk amplitude for the construction of the form $\widetilde{\varphi}^{bosonic}_{\pm,n;r}$ we apply the embedding of disk fields onto the sphere (6.4.42). We chose the massless states of both closed and open strings. The structure of the amplitude is given similar to the superstring as a correlator over the disk:

$$\Big\langle \prod_{i=1}^{n} V_o^{bosonic}(\varepsilon_i, k_i, z_i) \ \prod_{s=1}^{r} V_c^{bosonic}(\varepsilon_s, q_s, z_{n+s}, \bar{z}_{n+s}) \Big\rangle_{D_2} \tag{6.4.78}$$

Following the embedding (6.4.42) we then obtain the following integrand:

$$\begin{aligned}
\mathcal{I}^{bosonic}(n;r) = \mathcal{C}(1,2,\ldots,n) \ \times \Big\langle \ &\int (\prod_{i=1}^{n+2r} d\theta_i d\bar{\theta}_i) \ \frac{\theta_1\theta_2}{z_1 - z_2} \\
\times \exp\Big\{ ik_1 \cdot (X + \widetilde{X}) + \theta_1\bar{\theta}_1\varepsilon_1 \cdot \partial X \Big\} \\
&\vdots \\
\times \exp\Big\{ ik_n \cdot (X + \widetilde{X}) + \theta_n\bar{\theta}_n\varepsilon_n \cdot \partial X \Big\} \\
\times \exp\Big\{ iq_1 \cdot X + i\widetilde{q}_1 \cdot \widetilde{X} + \theta_{n+1}\bar{\theta}_{n+1}\varepsilon_{n+1} \cdot \partial X \ + \theta_{n+2}\bar{\theta}_{n+2}\widetilde{\varepsilon}_{n+1} \cdot \bar{\partial}\widetilde{X} \Big\} \\
&\vdots \\
\times \exp\Big\{ iq_r \cdot X + i\widetilde{q}_r \cdot \widetilde{X} + \theta_{2r+n-1}\bar{\theta}_{2r+n-1}\varepsilon_{n+r} \cdot \partial X + \theta_{2r+n}\bar{\theta}_{2r+n}\widetilde{\varepsilon}_{n+r} \cdot \bar{\partial}\widetilde{X} \Big\} \ \Big\rangle_{S_2} .
\end{aligned} \tag{6.4.79}$$

In this integrand we again used the same labeling (6.4.52) for the momenta. One can see the difference between the two integrands (6.4.79) and (6.4.51) is the absence of the fermionic string field ψ in the contractions. We can regroup this integrand with the use of the function

$$\widetilde{W}_{\underbrace{11\ldots1}_{r}} = \prod_{i\in\mathcal{R}} \left(\sum_{j\in\mathcal{N}} \frac{\varepsilon_i \cdot p_j}{z_{ij}} + \sum_{\substack{j\in\mathcal{R}\\ j\neq i}} \frac{\varepsilon_i \cdot p_j}{z_{ij}} \right) , \tag{6.4.80}$$

which has been introduced in [82] and generalizes (4.2.13) and also describes the Weyl–YM theory. The set $\mathcal{N}$ encompasses all open string labels while the set $\mathcal{R}$ comprises all closed string labels. We have the integrand (6.4.79) as:

$$\mathcal{I}^{bosonic}(n;r) = \mathcal{C}(1,2,\ldots,n)\widetilde{W}_{\underbrace{11\ldots1}_{r}} \times \exp\left\{ \pm \frac{1}{2}\alpha' \sum_{i>j\in} \frac{\theta_i\theta_j\bar{\theta}_i\bar{\theta}_j(\xi_i\cdot\xi_j)}{(\zeta_i-\zeta_j)^2} \right\} , \tag{6.4.81}$$

In the function (6.4.80) the diagonal elements of the C–block (4.2.8) entering the matrix ψ_n in (4.2.7) appear. A special limit of the function (6.4.80) is exhibited in (6.3.32), which gives rise to (4.2.13) and stems from the twisted form $\varphi^{bosonic}_{\pm,n}$ originating from open bosonic string theory. Here, we want to construct the new twisted form $\widetilde{\varphi}^{bosonic}_{\pm,n;r}$ associated with (6.4.80). Following the structure in (6.3.22) we can write the twisted form $\widetilde{\varphi}^{bosonic}_{\pm,n;r}$ as:

$$\widetilde{\varphi}^{bosonic}_{\pm,n;r} = d\mu_{n+r} \left(\pm\frac{1}{\alpha'}\right)^{\lfloor\frac{n+r-2}{2}\rfloor} \mathcal{C}_n\{z_i\} \int \prod_{i=1}^{r} d\theta_i d\bar{\theta}_i$$
$$\times \exp\left\{ \pm\sqrt{\alpha'} \sum_{i\in\mathcal{R}} \theta_i\bar{\theta}_i \left(\sum_{j\in\mathcal{N}} \frac{\varepsilon_i\cdot p_j}{z_i - z_j} + \sum_{\substack{j\in\mathcal{R}\\ j\neq i}} \frac{\varepsilon_i\cdot p_j}{z_i - z_j} \right) + \sum_{\substack{j\neq i\\ j,i\in\mathcal{R}}} \frac{\theta_i\bar{\theta}_i\theta_j\bar{\theta}_j\varepsilon_i\cdot\varepsilon_j}{(z_i - z_j)^2} \right\} .$$

Finally, we take the $\alpha' \to \infty$ limit as

$$\lim_{\alpha'\to\infty} \widetilde{\varphi}^{bosonic}_{\pm,n;r} = \mathcal{C}_n \prod_{i\in\mathcal{R}} \left(\sum_{j\in\mathcal{N}} \frac{\varepsilon_i\cdot p_j}{z_{ij}} + \sum_{\substack{j\in\mathcal{R}\\ j\neq i}} \frac{\varepsilon_i\cdot p_j}{z_{ij}} \right) d\mu_{n+r} \equiv \mathcal{C}_n\, \widetilde{W}_{\underbrace{11...1}_{r}}\, d\mu_{n+r} \ , \tag{6.4.82}$$

which comprises the color form $\mathcal{C}_n$ (over the set of legs $\mathcal{N}$) and the function $\widetilde{W}_{\underbrace{11...1}_{r}}$ referring to the set of legs $\mathcal{R}$ introduced in (6.4.80). We are going to discuss and explore, in the coming section, theories that can be described through this newly constrcuted twsited form.

6.5 Theories from the Einstein Yang-Mills form $\tilde{\varphi}^{EYM}_{\pm,n;r}$

We built the first extension of table 6.1 in the previous section. There, we have shown that one can extend the twisted intersection description (6.3.8) to Einstein Yang-Mills (EYM) amplitudes by introducing an embedding formalism. Concretely, we introduced the twisted form:

$$\widetilde{\varphi}^{EYM}_{\pm,n;r} = d\mu_{n+r}\, \mathcal{C}(1,2,3,...,n) \int \prod_{i\in\mathcal{S}_r} d\theta_i d\bar{\theta}_i\ \exp\left\{ \frac{1}{2}\alpha'^2 \sum_{i,j\in\mathcal{S}_r} \begin{pmatrix}\theta_i\\ \bar{\theta}_i\end{pmatrix}^t \Psi_r \begin{pmatrix}\theta_j\\ \bar{\theta}_j\end{pmatrix} \right\}\Bigg|_{\overline{\zeta_l}\to\zeta_l}$$
$$\times \exp\left\{ \pm\frac{1}{2}\alpha' \sum_{i,j\in\mathcal{S}_r} \frac{\theta_i\theta_j\overline{\theta}_i\overline{\theta}_j(\xi_i\cdot\xi_j)}{(\overline{\zeta}_i - \overline{\zeta}_j)^2} \right\} . \tag{6.5.83}$$

With (6.5.83) we have the following addition to table 6.1

Theory	φ_-	φ_+	CHY representation	Amplitude
EYM	$\tilde{\varphi}^{EYM}_{-,n;r}$	$\varphi^{gauge}_{+,n+r}$	$\mathcal{C}_n\ \mathrm{Pf}\Psi_{S_r}\ \mathrm{Pf}'\psi_{n+r}$	r gravitons, n gluons

Table 6.2: Twisted forms for EYM amplitude and its CHY representation.

to construct EYM amplitudes involving n gluons and r gravitons as:

$$\mathcal{A}_{EYM}(n;r) = \lim_{\alpha'\to\infty} \langle \tilde{\varphi}^{EYM}_{-,n;r}, \varphi^{gauge}_{+,n+r} \rangle_\omega = \int_{\mathcal{M}_{0,n+r}} d\mu_{n+r} \prod_{k=2}^{n+r-2} \delta(f_k) \lim_{\alpha'\to\infty} \hat{\tilde{\varphi}}^{EYM}_{-,n;r} \hat{\varphi}^{gauge}_{+,n+r}$$
$$= \int_{\mathcal{M}_{0,n+r}} d\mu_{n+r} \prod_{k=2}^{n+r-2} \delta(f_k)\, \mathcal{C}_n \,\mathrm{Pf}\Psi_{S_r}\, \mathrm{Pf}'\psi_{n+r} \ . \tag{6.5.84}$$

Actually, we can go further by looking at other theories, which have similar CHY structures to EYM amplitude. The first extension originates from paring $\tilde{\varphi}^{EYM}_{-,n;r}$ with other twisted forms from (6.3.32) and comparing (6.3.23) with the corresponding CHY representation [50]. This way it is straightforward to construct amplitudes for generalized Yang-Mills scalar (gen.YMS) and extended Dirac Born–Infeld theory (ext.DBI) supplementing our table 6.1 by the following content:

Theory	φ_-	φ_+	CHY representation	Amplitude
Generalized Yang-Mills Scalar (gen.YMS)	$\tilde{\varphi}^{EYM}_{-,n;r}$	$\varphi^{color}_{+,n+r}$	$\mathcal{C}_n \mathrm{Pf}\Psi_{S_r} \mathcal{C}_{n+r}$	r gluons n color scalars
Extended Dirac Born-Infeld (ext.DBI)	$\tilde{\varphi}^{EYM}_{-,n;r}$	$\varphi^{scalar}_{+,n+r}$	$\mathcal{C}_n \mathrm{Pf}\Psi_{S_r} (\mathrm{Pf}' A_{n+r})^2$	r gluons n higher derivative scalars

Table 6.3: Additional theories that can be described through the new twisted form $\tilde{\varphi}^{EYM}$.

In order to verify table 6.3 we compute the intersection numbers (6.3.8) in the limit $\alpha' \to \infty$ given by using (6.3.23) involving the pairs of twisted forms for gen.YMS and ext.DBI and compare the results with their corresponding CHY representations. We discussed the CHY integral of the gen.YMS in (4.3.23) we had

$$\mathcal{A}_{gen.YMS}(n;r) = \lim_{\alpha'\to\infty} \langle \varphi^{color}_{-,n+r}, \tilde{\varphi}^{EYM}_{+,n;r} \rangle_\omega = \int_{\mathcal{M}_{0,n+r}} d\mu_{n+r} \prod_{k=2}^{n+r-2} \delta(f_k) \lim_{\alpha'\to\infty} \hat{\varphi}^{color}_{-,n+r} \hat{\tilde{\varphi}}^{EYM}_{+,n;r}$$
$$= \int_{\mathcal{M}_{0,n+r}} d\mu_{n+r} \prod_{k=2}^{n+r-2} \delta(f_k)\, \mathcal{C}_n \,\mathrm{Pf}\Psi_{S_r}\, \mathcal{C}_{n+r} \ . \tag{6.5.85}$$

Similarly, we had for the Extended Driac Born-Infeld theory, that we introduced in chapter 4, the CHY integrand given in(4.3.26). The amplitude for ext.DBI involving r gluons and

n scalars can be constructed through the following intersection number

$$\mathcal{A}_{ext.DBI}(n;r) = \lim_{\alpha'\to\infty} \langle \varphi^{scalar}_{-,n+r}, \tilde{\varphi}^{EYM}_{+,n;r} \rangle_\omega = \int_{\mathcal{M}_{0,n+r}} d\mu_{n+r} \prod_{k=2}^{n+r-2} \delta(f_k) \lim_{\alpha'\to\infty} \hat{\varphi}^{scalar}_{-,n+r} \hat{\tilde{\varphi}}^{EYM}_{+,n;r} ,$$
$$= \int_{\mathcal{M}_{0,n+r}} d\mu_{n+r} \prod_{k=2}^{n+r-2} \delta(f_k)\, \mathcal{C}_n \, \mathrm{Pf}\Psi_{S_r} \, (\mathrm{Pf}' A_{n+r})^2 , \tag{6.5.86}$$

which is the CHY representation given in [50]. The double copy structure of BI, DBI, and ext.DBI theories will be discussed in section 7.4 (resulting in the representations (7.4.130), (7.4.124), and (7.4.133), respectively).

6.6 Theories with the Einstein–Maxwell form $\tilde{\varphi}^{EM}_{\pm,n;r}$

Above we have used the twisted form (6.5.83) of EYM amplitude to compute the intersection number (6.3.23) for the amplitudes of EYM, gen.YMS and ext.DBI theories which are given in (6.5.84), (4.3.24), and (4.3.28), respectively. While for the latter set of theories the CHY representations are constructed in [50], their corresponding twisted forms had not been constructed. Similarly, the twisted form for Einstein-Maxwell (EM) theory, i.e. Einstein gravity with an $U(1)^m$ gauge group, as it arises from compactification of m dimensions, can be used to construct other amplitudes for theories that interact with Maxwell theory. One would expect these theories to be defined in some limit of the Yang-Mills theory. We organize them in the following table 6.4 based on [50].

Theory	CHY representation	Amplitude
Einstein–Maxwell (EM)	$\mathrm{Pf}X_n \, \mathrm{Pf}'\Psi_{S_{r:n}} \, \mathrm{Pf}'\psi_{n+r}$	r gravitons n photons
Dirac Born–Infeld (DBI)	$\mathrm{Pf}X_n \, \mathrm{Pf}'\Psi_{S_{r:n}} \, (\mathrm{Pf}'A_{n+r})^2$	r gluons n color scalars
Yang-Mills scalar (YMS)	$\mathrm{Pf}X_n \, \mathrm{Pf}'\Psi_{S_{r:n}} \, \mathcal{C}_{n+r}$	r gluons n color scalars

Table 6.4: Known theories yet without twisted form description.

The matrix $\Psi_{S_{r:n}}$ is similar to the object Ψ_{S_r} introduced in [2]. The latter is a $(2r)\times(2r)$–matrix with only those indices included, which refer to the highest spin particles of the theory under consideration. On the other hand, the object $\Psi_{S_{r:n}}$ is a $(2r+n)\times(2r+n)$–matrix with an additional sector contributing to the lower spin particles. More concretely, for EM amplitude we have [50]

$$\Psi_{S_{r:n}} = \begin{pmatrix} A_{ab} & A_{aj} & (-C^t)_{ab} \\ A_{ib} & A_{ij} & (-C^t)_{aj} \\ C_{ab} & C_{aj} & B_{ab} \end{pmatrix} , \tag{6.6.87}$$

described by the sets of graviton indices $a, b \in \{1, \ldots, r\}$ and the photon (gluon) indices $i, j \in \{1, \ldots, n\}$. In cases of DBI and YMS amplitudes, we have the sets of gluon indices $a, b \in \{1, \ldots, r\}$ and the scalar indices $i, j \in \{1, \ldots, n\}$. The only common part in the CHY representation of all three theories is comprised of the Pfaffian $\mathrm{Pf}X_n$ of the $n \times n$ matrix X with matrix elements:

$$X_{ab} = \begin{cases} \frac{1}{z_a - z_b} & a \neq b \ , \\ 0 & a = b \ . \end{cases} \tag{6.6.88}$$

This Pfaffian describes a correlator involving an even number of n fermions, i.e.:

$$\langle \psi(z_1) \ldots \psi(z_n) \rangle = \mathrm{Pf}X_n \ . \tag{6.6.89}$$

Therefore, the number n of photons must be even. The twisted form of EM amplitude can be represented by the following $(n+r)$–form:

$$\tilde{\varphi}^{EM}_{\pm,n;r} = d\mu_{n+r} \int \prod_{i \in \mathcal{S}_r} d\theta_i d\bar{\theta}_i \, \frac{\theta_k \theta_l}{z_k - z_l} \, \exp\left\{ \frac{1}{2} \alpha'^2 \sum_{i,j \in \{r,n\}} \begin{pmatrix} \theta_i \\ \bar{\theta}_i \end{pmatrix}^t \Psi_{S_{r:n}} \begin{pmatrix} \theta_j \\ \bar{\theta}_j \end{pmatrix} \right\} \times \mathrm{Pf}X_n \ , \tag{6.6.90}$$

where we have used the Pfaffian in equation (6.6.89). We should emphasize that we did not construct this twisted form directly from string theory and so it does not contain any α' corrections (it is exact). One can attempt to produce this twisted form by taking the limit $SU(N) \to U(1)$ of $\tilde{\varphi}^{EYM}_{\pm,n;r}$. However, this limit is not possible since the interactions of the gauge boson are governed by the $SU(N)$ algebra (in particular the structure constant) and taking this limit is ill-defined. Further, open strings with just $U(1)$ charges will decouple from the rest of the spectrum [11] and therefore there are no good candidates within critical string theory. Further investigation on different compactification-spectrums might give us better candidates to build the twisted form from Maxwell theory which includes string corrections.

Taking this twisted form $\tilde{\varphi}^{EM}_{\pm,n;r}$ together with the twisted form for Yang-Mills amplitude, $\varphi^{gauge}_{-,n+r}$, we can construct the Einstein-Maxwell amplitude

$$\begin{aligned} \mathcal{A}_{EM}(n;r) &= \lim_{\alpha' \to \infty} \langle \varphi^{gauge}_{-,n+r}, \tilde{\varphi}^{EM}_{+,n;r} \rangle_\omega = \int\limits_{\mathcal{M}_{0,n+r}} d\mu_{n+r} \prod_{k=2}^{n+r-2} \delta(f_k) \lim_{\alpha' \to \infty} \hat{\varphi}^{gauge}_{-,n+r} \, \hat{\tilde{\varphi}}^{EM}_{+,n;r} \ , \\ &= \int\limits_{\mathcal{M}_{0,n+r}} d\mu_{n+r} \prod_{k=2}^{n+r-2} \delta(f_k) \, \mathrm{Pf}X_n \, \mathrm{Pf}'\Psi_{S_{r:n}} \, \mathrm{Pf}'\psi_{n+r} \ , \end{aligned} \tag{6.6.91}$$

involving r gravitons and n photons. Similar constructions can be established for the two other theories displayed in table 6.4. In total, we have the following pairings of twisted forms:

Theory	φ_-	φ_+	Amplitude
EM	$\varphi^{gauge}_{-,n+r}$	$\tilde{\varphi}^{EM}_{+,n;r}$	r graviton n photon
DBI	$\varphi^{scalar}_{-,n+r}$	$\tilde{\varphi}^{EM}_{+,n;r}$	r gluons n color scalars
YMS	$\varphi^{color}_{-,n+r}$	$\tilde{\varphi}^{EM}_{+,n;r}$	r gluons n color scalars

Table 6.5: Theories which can be described by $\tilde{\varphi}^{EM}_{\pm,n;r}$.

6.7 New theories involving $\tilde{\varphi}^{bosonic}_{\pm,r;n}$

So far, we have discussed the theories involving the new twisted form $\tilde{\varphi}^{EYM}_{\pm,r;n}$. In this section we take a look at $\tilde{\varphi}^{bosonic}_{\pm,r;n}$. We should point out that all these theories are new theories from amplitude considerations and hence we did not put them in the preliminary chapter 4.

6.7.1 Generalized Weyl scalar

We introduce the so-called generalized Weyl scalar similar to the generalized YM scalar amplitude (4.3.24). Meaning, we have DF^2 (cf. section 4.3.7) coupled to a scalar theory in the adjoint representation of the $SU(N)$ gauge group. We set the following Lagrangian for the theory:

$$\mathcal{L}_{gen.DFS} = \sqrt{-g}\ \left(m^2\ R + \kappa_W^{-2}\ W^2_{\mu\nu\alpha\beta}\right) - \frac{1}{2}(D_\mu\phi^{a\tilde{a}})^2 + \lambda\ f^{abc} f^{\tilde{a}\tilde{b}\tilde{c}}\ \phi^{a\tilde{a}}\phi^{b\tilde{b}}\phi^{c\tilde{c}} \tag{6.7.92}$$

Since we know that the Weyl and adjoint scalar theories are associated with functions $W_{\underbrace{11...1}_r}$ and $\mathcal{C}_n$, respectively. We can pair them by extending the $W_{\underbrace{11...1}_r}$ to $\widetilde{W}_{\underbrace{11...1}_r}$ by our embedding[5]. This means with the paring of our newly constructed twisted form in (6.4.82) and the color form $\varphi^{color}_{\pm,n+r}$ we can describe this theory. In the $\alpha' \to \infty$ limit we conjecture the corresponding CHY integrand from the twisted intersection (6.3.8):

$$\lim_{\alpha'\to\infty} \langle \tilde{\varphi}^{bosonic}_{-,n;r}, \varphi^{color}_{+,n+r}\rangle_\omega \simeq \mathcal{C}_n\ \widetilde{W}_{\underbrace{11...1}_r}\ \mathcal{C}_{n+r}\ . \tag{6.7.93}$$

[5]As we say this is a conjecture. However, it is an educated guess since in all previous cases (e.g. gen.YMS, EYM, etc.) coupling the higher spin theory to a lower spin required the embedded twisted form of the higher spin theory.

6.7.2 Weyl-YM

Similarly, to the EYM amplitude we can construct the Weyl-YM theory. Meaning, we can replace the "Einstein" part of the theory with *Weyl* gravity (see section 4.3.10). We conjecture the following Lagrangian for this theory:

$$\mathcal{L}_{WYM} = \sqrt{-g}\, \left(m^2\, R + \kappa_W^{-2}\, W^2_{\mu\nu\alpha\beta}\right) - \frac{1}{4} F^{\mu\nu\, a} F_{\mu\nu, a}\,, \tag{6.7.94}$$

For the EYM amplitude we used the open and closed superstring together with the underlying embedding formalism to construct $\tilde{\varphi}^{EYM}_{-,n;r}$, while $\tilde{\varphi}^{bosonic}_{-,n;r}$ is derived from (6.4.43) involving the open and closed bosonic string. Therefore, we can construct this amplitude using the limit of the intersection number of $\tilde{\varphi}^{bosonic}_{-,n;r}$ together with $\varphi^{gauge}_{-,n+r}$ we have:

$$\lim_{\alpha'\to\infty} \langle \tilde{\varphi}^{bosonic}_{-,n;r}, \varphi^{gauge}_{+,n+r} \rangle_\omega \simeq \mathcal{C}_n\, \widetilde{W}_{\underbrace{11...1}_{r}}\, \psi_{n+r}\ . \tag{6.7.95}$$

6.7.3 $Weyl^3$-DF^2

Finally, we finish this chapter with the $Weyl^3$-DF^2 theory. The CHY amplitude of the $Weyl^3$ theory is given by the $W_{11...1}$ function (cf. section 4.3.11). As we discussed it is known from previous work by Mizera [6] that this function can be obtained through the tensionless ($\alpha' \to \infty$) limit of $\varphi^{bosonic}_{-,n;r}$. In addition, we know that similar to the $Weyl^3$, the DF^2 amplitude it is given by the same function. Therefore, we plan to construct an extension of the $Weyl^3$ theory that can be paired with the DF^2 theory. We are going to use the EYM theory as our guiding tool. In that case, one has the function $\mathrm{Pf}\psi_n$ in the same role as $W_{11...1}$. If we square the function we obtain gravity (spin-2) theory otherwise it corresponds to the gauge theory (YM). In comparison, the extension of matrix ψ_n was constructed in [20] and was the function $\mathrm{Pf}\psi_{S_r}$. Paring it with $\mathrm{Pf}\psi_{n+r}$ produced the EYM amplitude. This extension in the CHY formulation corresponds to the embedding of the disk, that we introduced in section 6.4.1, in the twisted form language. Therefore, we follow the same steps but now with $W_{11...1} \leftrightarrow \varphi^{bosonic}_{-,n;r}$. We build the embedded version of the twisted form $\varphi^{bosonic}_{-,n;r}$ in section 6.4.3 and denote it by $\tilde{\varphi}^{bosonic}_{-,n;r}$. By putting together the two twisted form $\varphi^{bosonic}_{-,n;r}$ and $\tilde{\varphi}^{bosonic}_{-,n+r}$ we can conjecture the following amplitude for the $Weyl^3$-DF^2 theory:

$$\lim_{\alpha'\to\infty} \langle \tilde{\varphi}^{bosonic}_{-,n;r}, \varphi^{bosonic}_{+,n+r} \rangle_\omega \simeq \mathcal{C}_n\, \widetilde{W}_{\underbrace{11...1}_{r}}\, W_{\underbrace{11...1}_{n+r}}\ . \tag{6.7.96}$$

We refrain to give a specific Lagrangian for this theory since the $Weyl^3$ in the context of CHY formulation given in [21] does not have a Lagrangian formulation yet.

Chapter 7

Double copy and amplitude relations

7.1 Preface

In this chapter, we are going to discuss double copy and amplitude relations both in string theory and intersection numbers. We first look at the connection between intersection theory and BCJ-KK amplitude relations [7, 8, 22, 86]. We give a proof for the BCJ-KK relations from the equivalent classes in twisted cohomology theory. Then, we use intersection formulation of the amplitude relations to expand the EYM amplitude (that we found in the previous chapter) in terms of pure Yang-Mills amplitudes [40, 87].

Next, we turn to the double copy of massive states in string theory [3, 4]. Historically, string theory is the origin of the concept of the double copy construction as we discussed in section 5.7. The KLT double copy [25] was one of the first works that introduced the notion $(gauge)^2 \sim gravity$. However, their work and the subsequent discussions on this topic were primarily done for massless states (except for attempts involving massive scalars)[88, 89, 90, 91]. Here, we attempt to produce the massive spin-2 state as described in bimetric gravity (which we introduced in chapter 4) from string theory, using both the double copy and non-double copy methods.

We finish the section by making an observation regarding the relation between BCJ double copy [22, 92, 93, 94, 95, 96] and twisted cohomology. In short, we are going to claim that the theories, which can be BCJ double copied, are the ones that have $\varphi_{\pm}^{color}$ in their twisted representation. We are going to support this claim by looking at different examples as well as showing that for any two such theories, the resulting double copy has a KLT matrix representation.

7.2 Amplitude relations from intersection theory

7.2.1 BCJ-KK amplitude relations

The BCJ-KK amplitude relations are the set of relations among gauge amplitudes that relate different color ordered subamplitudes to each other [8, 19, 86]. The starting point is the fact that all tree level gauge amplitudes can be written as the following sum of color ordered subamplitudes:

$$\mathbf{A}^{YM}(n) = \sum_{\tau} \frac{n_\tau c_\tau}{\prod_t p_t^2} = \sum_{\tau} c_\tau \mathcal{A}(\sigma_\tau)\,, \tag{7.2.1}$$

where τ is the set of all tree graphs that have only *cubic vertices* with n external legs (there are $(2n-5)!!$ such graphs) with the set of internal legs t. The $\mathcal{A}(\sigma_\tau)$s are the colored order amplitudes with respect to the ordering of the graph σ_τ. The c_τs are color factors associated with each cubic vertex operator (which is given by the structure constant of the gauge group) and n_τs are the corresponding kinematic numerators of the amplitude. The very first non-trivial relation (aside from cyclic symmetry and reversal) was the photon decoupling which was generalized by *Kleiss and Kuijf* (*KK* relations) as [86]:

- Photon decoupling:

$$\sum_{\sigma \in cyclic} \mathcal{A}(1, \sigma(2,3,4,...,n)) = 0\,. \tag{7.2.2}$$

- KK relations:

$$\mathcal{A}(1,\{\alpha\},n,\{\beta\}) = (-1)^{|\beta|} \sum_{\sigma_i \in OP(\{\alpha\},\{\beta\}^T)} \mathcal{A}(1,\{\sigma_i\},n,)\,. \tag{7.2.3}$$

where $OP(\{\alpha\},\{\beta\}^T)$ stands for ordered permutation of the set α and the transposed of the set β.

The next set of relations among the color ordered amplitudes came as the consequence of color algebra. The important structure among the c_τs is that they can be grouped into triplets (i,j,k), which are associated to the s,t and u channels leg orderings, that satisfy the Jacobi identity:

$$c_i + c_j + c_k = 0. \tag{7.2.4}$$

This gave rise to the so-called *color kinematic duality.* It states that for every triplet (i,j,k) graphs of tree level amplitudes whose color factors satisfy the Jacobi relations there exists a set of numerators n_i,n_j and n_k of those amplitudes (up to gauge transformation) that satisfy the same Jacobi relations meaning:

$$c_i + c_j + c_k = 0 \;\leftrightarrow\; n_i + n_j + n_k = 0\,. \tag{7.2.5}$$

Since the color ordered amplitudes $\mathcal{A}(\sigma_\tau)$ are functions of the n_is the above identity imposes relations among associated color ordered amplitudes. These are known as the *BCJ* relations and can have the following representation:

$$\sum_{i=1}^{n-1} p_1 \cdot (p_2 + p_3 + ... + p_i)\mathcal{A}(2,,3,...,i,1,i+1,...,n) = 0\,. \tag{7.2.6}$$

These sets of relations will exhaust a basis for the color ordered amplitudes. Starting with $n!$ possible permutations, after implementing the BCJ-KK relations we obtain the

$(n-3)!$ basis. Although every theory that exhibits the color kinematic duality (7.2.5) will have a BCJ-KK basis (i.e. the color ordered subamplitudes will obey the BCJ-KK relations) [22, 19] the converse has not yet been proven but highly motivated. Meaning a gauge theory whose color ordered subamplitudes will obey the BCJ-KK relations would exhibit color kinematic duality. These relations were proven in string theory in [97] and [98] through string monodromy relations. There also works discussing these relation on the loop level (i.e. world-sheets with genus bigger than zero [99],[100],[101])

7.2.2 Amplitude relations in intersection theory

It has been discussed in [5] that the fundamental KK and BCJ relations, in the language of twisted cohomology, may be understood as the equivalence class relation associated with the derivative (exact twisted form) of the color form $\varphi^{color}_{\pm,n-1}$ i.e.:

$$[\varphi_n + \nabla_\omega \varphi^{color}_{n-1}] = [\varphi_n] \Rightarrow (d \pm d\omega\wedge)\, \varphi^{color}_{\pm;n-1} \sim 0 \in H^{n-3}_\omega \ . \tag{7.2.7}$$

Recall, that we are using both notations φ^{color} and $PT(1,2,\ldots,n)$ for the color form interchangeably. To derive amplitude relations in intersection theory we first note that $d\omega$ corresponds to the scattering equations (4.2.1), i.e.:

$$d\omega = \sum_{i=2}^{n-2} S_i\, dz_i \ , \qquad \text{with:} \qquad S_i = \sum_{j\neq i}^{n-2} \frac{s_{ij}}{z_{ij}} \ . \tag{7.2.8}$$

Secondly, to simplify the calculation we introduce the insertion function $\text{Ins}(i)_{jk}$s:

$$\text{Ins}(i)_{j,k} := \frac{z_{jk}}{z_{ji} z_{ik}} \ , \tag{7.2.9}$$

which acts on the color form $PT(1,\ldots,n)$ as operator:

$$\text{Ins}(i)_{j,k}\ PT(1,2,3,\ldots,j,k,...,n) = PT(1,2,3,\ldots,j,i,k,...,n) \ . \tag{7.2.10}$$

Therefore, by using momentum conservation and performing some rearrangements we can write S_i in terms of $\text{Ins}(i)_{j,k}$ as

$$S_i = \sum_{j\neq i}^{n-2} x(i)_j\ \text{Ins}(i)_{j,j+1} \ , \tag{7.2.11}$$

where $x(i)_j$ is defined as:

$$x(i)_j = p_i \cdot \sum_{k=2}^{j} p_k \ . \tag{7.2.12}$$

Now, by using $\varphi^{color}_{\pm;n-1} = PT(1,2,\ldots,n-1)$ and $d\varphi^{color}_{\pm;n-1} = 0$ we rewrite (7.2.7) in the following way:

$$\begin{aligned} \Phi &:= (d \pm d\omega\wedge)\, \varphi^{color}_{\pm;n-1} = \pm(d\omega\wedge)\, \varphi^{color}_{\pm;n-1} \\ &= (d\omega\big|_{dz_n}\wedge)\varphi^{color}_{\pm;n-1} = (S_n dz_n\wedge)\varphi^{color}_{\pm;n-1} \simeq 0\ \in H^{n-3}_\omega \ . \end{aligned} \tag{7.2.13}$$

Above $d\omega|_{dz_n}$ projects onto the one–form part dz_n of $d\omega$. Eventually, from (7.2.13) we deduce the following relation

$$\sum_{j\neq n}^{n-2} x(n)_j \ \text{Ins}(n)_{j,j+1} PT(1,2,3,...,n-1) = \sum_{j\neq n}^{n-2} x(n)_j \ PT(1,2,3,..,j,n,j+1...,n-1) = 0, \tag{7.2.14}$$

which assumes the form of a BCJ relation. To produce the BCJ relations for YM amplitudes we shall calculate the fowling intersection number

$$\begin{aligned} \langle \varphi^{gauge}_{-,n}, \Phi\rangle_\omega &= \langle \varphi^{gauge}_{-,n}, (S_n dz_n \wedge) \varphi^{color}_{\pm;n-1}\rangle_\omega \\ &= \langle \varphi^{gauge}_{-,n}, \sum_{j\neq n}^{n-2} x(n)_j PT(1,2,3,..,j,n,j+1...,n-1)\rangle_\omega \\ &= \sum_{j\neq n}^{n-2} x(n)_j \ \underbrace{\langle \varphi^{gauge}_{-,n}, PT(1,2,3,..,j,n,j+1...,n-1)\rangle_\omega}_{\mathcal{A}_{YM}(1,2,3,..,j,n,j+1...,n-1)} = 0 \ , \end{aligned} \tag{7.2.15}$$

from which the BCJ relation for YM amplitudes follows:

$$\sum_{j\neq n}^{n-2} x(n)_j \ \mathcal{A}_{YM}(1,2,3,..,j,n,j+1...,n-1) = 0 \ . \tag{7.2.16}$$

On the other hand, for the KK relations we start at the $n+1$–form

$$\varphi^{color}_{\pm,n+1}(\sigma_l) := PT(1,2,...,l,p,l+1,....,n) \ ,$$

with $n+1$ legs. One additional leg denoted by $p \equiv n+1$ is appended such that there are in total $n+1$ legs. Furthermore σ_l denotes the particular ordering of those $n+1$ legs as: $\sigma_l \leftrightarrow (1,2,...,l,p,l+1,....,n)$. As a consequence of the KK relations for PT factors, we have the identity:

$$\sum_{l=1}^{n} \varphi^{color}_{\pm,n+1}(\sigma_l) = \sum_{l=1}^{n} PT(1,2,\ldots,l,p,l+1,....,n) = 0 \ . \tag{7.2.17}$$

Inserting (7.2.17) into the intersection number (6.3.8) appended by $\varphi^{gauge}_{-,n+1}$ yields

$$\langle \varphi^{gauge}_{-,n+1}, \sum_{l=1}^{n} \varphi^{color}_{+,n+1}(\sigma_l)\rangle_\omega = \sum_{l=1}^{n} \langle \varphi^{gauge}_{-,n+1}, \varphi^{color}_{+,n+1}(\sigma_l)\rangle_\omega = 0 \ , \tag{7.2.18}$$

from which, the KK relation for YM amplitudes of generic helicity configurations follows:

$$\sum_{l=1}^{n} \mathcal{A}_{YM}(1,2,\ldots,l,p,l+1,....,n) = 0 \ . \tag{7.2.19}$$

7.2.3 Expansion of EYM amplitude

From chapter 3, we know that any intersection number (6.3.8) may be expanded w.r.t.,a orthonormal basis of n–forms $\bigcup_{a=1}^{(n-3)!} \{\Phi_{+,a}\} \in H^{n-3}_{+\omega}$. Generally, together with the dual basis $\bigcup_{b=1}^{(n-3)!} \{\Phi^\vee_{-,b}\} \in H^{n-3}_{-\omega}$ with the intersection matrix as: (see proposition (3.3.4))

$$\langle \Phi_{+,a}, \Phi^\vee_{-,b} \rangle_\omega = \delta_{ab} \ . \tag{7.2.20}$$

We have the following expansion of twisted n–forms φ^1_+, φ^2_-

$$\varphi^1_+ = \sum_{a=1}^{(n-3)!} \langle \Phi^\vee_{-,a}, \varphi_+ \rangle_\omega \ \Phi_{+,a} \ , \tag{7.2.21}$$

$$\varphi^2_- = \sum_{b=1}^{(n-3)!} \langle \varphi_-, \Phi_{+,b} \rangle_\omega \ \Phi^\vee_{-,b} \ , \tag{7.2.22}$$

respectively, for the case of $\Phi_{+,a} = PT(a)$ leading to the following expansion of the intersection number:

$$\langle \varphi^1_+, \varphi^2_- \rangle = \sum_{a=1}^{(m-3)!} \langle PT^\vee(a), \varphi^1_+ \rangle_\omega \ \langle \varphi^2_-, PT(a) \rangle_\omega \ . \tag{7.2.23}$$

Choosing $\varphi^1_+ = \tilde{\varphi}^{EYM}_{+,n;r}$, $\varphi^2_- = \varphi^{EYM}_{-,n;r} = \varphi^{gauge}_{-,n+r}$ the orthogonal decomposition (7.2.23) can be used to express EYM amplitudes in terms of a linear combination of (a basis) $m := n + r$–point YM subamplitudes [2]:

$$\mathcal{A}_{EYM}(n;r) = \lim_{\alpha' \to \infty} \sum_{a=1}^{(m-3)!} \langle PT^\vee(a), \widetilde{\varphi}^{EYM}_{+,n;r} \rangle_\omega \ \ \mathcal{A}_{YM}(a) \ . \tag{7.2.24}$$

In addition, we may use BCJ–KK relations for further simplifications of $\mathcal{A}_{YM}(a)$.

To determine the expansion coefficients $\langle PT^\vee(a), \widetilde{\varphi}^{EYM}_{+,n;r} \rangle_\omega$ in (7.2.24) we exemplify the one graviton case $r = 1$. We label the momentum of this one graviton by $p \equiv n+1$. From [2] we have for the EYM twisted form $\tilde{\varphi}^{EYM}_{n;1}$

$$\begin{aligned} \lim_{\alpha' \to \infty} \tilde{\varphi}^{EYM}_{n;1} &= d\mu_{n+1} \ PT(1, \ldots, n) \ \sum_{l=1}^{n-1} (\varepsilon_p \cdot x_l) \frac{z_{l,l+1}}{z_{l,p} z_{p,l+1}} \\ &= d\mu_{n+1} \ \sum_{l=1}^{n-1} (\varepsilon_p \cdot x_l) \ PT(1, 2, ..., l, p, l+1, ..., n) \ , \end{aligned} \tag{7.2.25}$$

with the definition $x_l^\mu = \sum_{j=1}^{l} k_j^\mu$. Inserting (7.2.25) into (7.2.24) yields:

$$\begin{aligned}\mathcal{A}_{EYM}(n;1) &= \lim_{\alpha'\to\infty} \sum_{\alpha\in S_{n-3}} \langle PT^\vee(\alpha), \tilde{\varphi}^{EYM}_{n;1}\rangle\ \mathcal{A}_{YM}(\alpha) \\ &= \sum_{\alpha\in S_{n-3}} \sum_{l=1}^{n-1} (\varepsilon_p\cdot x_l)\ \langle PT^\vee(\alpha), PT(1,2,...,l,p,l+1,...,n)\rangle_\omega\ \mathcal{A}_{YM}(\alpha)\ .\end{aligned} \tag{7.2.26}$$

Notice, that above the α'–dependence (i.e. string scale not to be mistaken with the ordering) has dropped. Now we can use the orthonormality condition (7.2.20), i.e.

$$\langle PT^\vee(\alpha), PT(\beta)\rangle_\omega = \delta_{\alpha,\beta}\ , \tag{7.2.27}$$

to label the ordering of $PT(1,2,...,l,p,l+1,...,n)$ by σ_l and (7.2.26) becomes

$$\begin{aligned}\mathcal{A}_{EYM}(n;1) &= \sum_{\alpha\in S_{n-3}} \sum_{l=1}^{n-1} (\varepsilon_p\cdot x_l)\ \langle PT^\vee(\alpha), PT(\sigma_l)\rangle_\omega\ \mathcal{A}_{YM}(\alpha)\ . \\ &= \sum_{\alpha\in S_{n-3}} \sum_{l=1}^{n-1} (\varepsilon_p\cdot x_l)\ \delta_{\alpha,\sigma_l}\ \mathcal{A}_{YM}(\alpha) \\ &= \sum_{l=1}^{n-1} (\varepsilon_p\cdot x_l)\ \mathcal{A}_{YM}(1,2,...,l,p,l+1,...,n)\ ,\end{aligned} \tag{7.2.28}$$

This is the expansion of the EYM amplitude involving 1 graviton and n gluons in terms of color ordered subamplitudes of $n+1$ pure gluons, which was first calculated in [102]. Next, let us also evaluate the coefficients $\langle PT^\vee(a), \widetilde{\varphi}^{EYM}_{+,n;r}\rangle_\omega$ in (7.2.24) for the two graviton case $r=2$. Again, the starting point is the twisted form $\tilde{\varphi}^{EYM}_{n;2}$ for the EYM amplitude given in [2]

$$\begin{aligned}\lim_{\alpha'\to\infty} \widetilde{\varphi}^{EYM}_{\pm,n;2} &= d\mu_{n+2}\ PT(1,2,3,...,n) \int \prod_{i\in\mathcal{S}_2} d\theta_i d\bar{\theta}_i\ \exp\left\{\frac{1}{2}\alpha'^2 \sum_{i,j\in\mathcal{S}_2} \begin{pmatrix}\theta_i\\ \bar{\theta}_i\end{pmatrix}^t \Psi_{\mathcal{S}_2}\ \begin{pmatrix}\theta_j\\ \bar{\theta}_j\end{pmatrix}\right\}\Bigg|_{\bar{\zeta}_l\to\zeta_l} \\ &= d\mu_{n+2}\ PT(1,2,3,...,n)\ \mathrm{Pf}'\Psi_2\ ,\end{aligned} \tag{7.2.29}$$

with $\mathcal{S}_2 = \{n+2, n+4\}$. To compute the coefficient $\langle PT_a^\vee, \tilde{\varphi}^{EYM}_{n;2}\rangle$ of the decomposition (7.2.24) we expand the twisted form (7.2.29) in terms of a PT basis (for a derivation we refer to 7.3.78). We have:

$$
\begin{aligned}
&\lim_{\alpha'\to\infty} \tilde{\varphi}^{EYM}_{n;2} = \\
&d\mu_{n+2}\Bigg\{ \sum_{k\geq l=1}^{n-1} (\varepsilon_q\cdot x_l)(\varepsilon_p\cdot x_k) PT(1,2,...,l,p,l+1,...k,q,k+1,...,n) + (p\leftrightarrow q) \\
&-\sum_{l=1}^{n-1}(\varepsilon_p\cdot q)(\varepsilon_q\cdot x_l)\sum_{j=2}^{l-1} PT(1,j-1,q,j,...l-1,p,l,...,n) + (p\leftrightarrow q) \\
&-\frac{1}{2}\sum_{l=1}^{n-1}(\varepsilon_p\cdot\varepsilon_q)s_{p,l}\sum_{j=2}^{l} PT(1,j-1,q,p,j,...,l,...,n) + (p\leftrightarrow q)\Bigg\} .
\end{aligned}
\tag{7.2.30}
$$

Now we proceed similar to the one graviton case and insert (7.2.30) into (7.2.24) and apply the orthonormality condition (7.2.27) to get:

$$
\begin{aligned}
\mathcal{A}_{EYM}(n;2) &= \lim_{\alpha'\to\infty}\sum_{a=1}^{(n-3)!}\langle PT^\vee(a), \tilde{\varphi}^{EYM}_{n;2}\rangle\, \mathcal{A}_{YM}(a) \\
&= \sum_{k\geq l=1}^{n-1} (\varepsilon_q\cdot x_l)(\varepsilon_p\cdot x_k)\mathcal{A}_{YM}(1,2,...,l,p,l+1,...k,q,k+1,...,n) + (p\leftrightarrow q) \\
&-\sum_{l=1}^{n-1}(\varepsilon_p\cdot q)(\varepsilon_q\cdot x_l)\sum_{j=2}^{l-1}\mathcal{A}_{YM}(1,j-1,q,j,...l-1,p,l,...,n) + (p\leftrightarrow q) \\
&\quad -\frac{1}{2}\sum_{l=1}^{n-1}(\varepsilon_p\cdot\varepsilon_q)s_{p,l}\sum_{j=2}^{l}\mathcal{A}_{YM}(1,j-1,q,p,j,...,l,...,n) + (p\leftrightarrow q)\ .
\end{aligned}
\tag{7.2.31}
$$

This result provides the expansion of the EYM amplitude involving n gluons and two gravitons in terms of pure YM $n+2$–point subamplitudes. The expression (7.2.31) has been already given in [103] using the expansion of CHY integrand, while here we applied twisted intersection theory to derive it.

7.3 Massive spin-2 double copy from string theory

For the past fifty years string scattering amplitudes have been used for various purposes. We have discussed in the previous chapters the two main ones. The effective actions through the low energy limit of string theory and double copy structure between massless gauge and gravity theories (KLT). In this section, we are going to make use of both. We try to answer whether or not the full bimetric gravity (cf. subsection 7.4.3) can be produced as an effective action of a string amplitudes. In addition, using the string amplitudes, we try to construct a double copy for this theory. The latter is more nuanced since it would involve a double copy description for a massive spin-2 state. Recently, there have been attempts to construct such a double copy through field theory ansatz [90, 104, 105] mainly using the massive vector field (Proca Yang-Mills) and double copy of it's amplitudes. However, these either failed or broke down at the level of higher point tree level amplitude.

Our aim is now to use string theory states to: First, produce the bimetric action in the mass eigenstate (or a limit of it) through string states. Second, using the double copy of string amplitudes we try to construct bimetric gravity as a double copy. Looking at the spectrum of the NS sector that we constructed in the section 5.5.4 we have two possible candidates:

- Open string: It is the first massive level of the open string state given in the last row of table 5.2. We refer to it as the boundary state
- Closed string: The closed string state is constructed as the tensor product of two open strings in the first mass level.

So far, we have avoided string compactifications. However, in order to make contact with the results of bimetric gravity we need to reduce (i.e. compactify) the dimension of spacetime from 10 (or 26 for bosonic string) to 4-dimensions. This is a very vast and deep topic and here we are going to state the results and explain the consequence of the compactification procedure. There are many good references such as [10, 11, 12, 106, 107] we refer the reader to them for details.

7.3.1 Compactification to four dimensions

The goal of compactification in string theory is to reduce the number of dimensions from 10 (or 26 in the bosonic string) to four dimensions. So that the resulting effective actions are comparable with our visible universe. The geometrical picture is to take the 10 dimensional spacetime manifold and wrap six dimensions around a *compactifying* manifold and then take the size of that manifold to be small. The means that the spacetime and its isometries break into a product manifold structure. We have the following structure:

$$\begin{aligned} &\mathbf{R}^{1,9} \to \mathbf{R}^{1,3} \otimes \mathbf{M}_6 \,, \\ &SO(1,9) \mapsto SO(1,3) \otimes G \,, \end{aligned} \tag{7.3.32}$$

where $\mathbf{M}_6$ is the internal manifold and the group G is the symmetry associated with it. Having established this structure the spacetime fields of our theory $(\partial X^\alpha, \psi^\alpha)$ will decompose into internal and external parts as:

$$\begin{aligned} &\partial X^\alpha = (\partial X^\mu, \partial Z^M)\,, \\ &\psi^\alpha = (\psi^\mu, \Psi^M)\,, \\ &\alpha = 0,1,2,...,9\,, \qquad \mu = 0,1,2,3\,, \qquad M = 4,5,...,9. \end{aligned} \tag{7.3.33}$$

In the above decompositions the fields $(\partial Z^M, \Psi^M)$ and $(\partial X^\mu, \psi^\mu)$ are the internal and external fields, respectively. Given the fact that we have a product space, the correlators will take the following form:

$$\begin{aligned}
\langle X^\mu(z_i)X^\nu(z_j)\rangle &= -2\alpha'\eta^{\mu\nu}\ln(z_i-z_j), & \langle\psi^\mu(z_i)\psi^\nu(z_j)\rangle &= \frac{\eta^{\mu\nu}}{(z_i-z_j)}\,,\\
\langle X^\mu(z_i)Z^N(z_j)\rangle &= 0, & \langle\psi^\mu(z_i)\Psi^N(z_j)\rangle &= 0\,,\\
\langle Z^M(z_i)Z^N(z_j)\rangle &= -2\alpha'\delta^{MN}\ln(z_i-z_j), & \langle\Psi^M(z_i)\Psi^N(z_j)\rangle &= \frac{\delta^{MN}}{(z_i-z_j)}\,.
\end{aligned} \tag{7.3.34}$$

We can see that, because of the product space structure (7.3.32), there are no contractions between internal and external fields. Now we turn to the CFT operators and how they are modified under compactification. First, we have to note that the field content of the compactify theory which in this case is a superconformal field theory, contains more than than just internal and external matter fields. Because, depending on the internal manifold $\mathbf{M}_6$, by compactifying, the SUSY supercurrent will also separate into external and internal spin fields Σ^I which in turn might break or enhance the spacetime supersymmetry. We will discuss the following cases with different supersymmetry:

- $\mathcal{N}=4$: This is the case for compactifying on a six-dimensional torus $\mathbf{M}_6 = T^6$ which is the maximally supersymmetric case (enhancement). The internal symmetry group in this case will be $G = SO(6) = SU(4)$ with the Lie algebra: $\mathfrak{g} = \mathfrak{so}(6)\times\mathfrak{u}(1)^6$.
- $\mathcal{N}=2$: This amount of SUSY can be achieved through internal manifolds like $\mathbf{M}_6 = K3\times T^2$ orientifold with $D5/D9$ branes. The internal symmetry group is $G = SU(2)$ and the Lie algebra $\mathfrak{g} = \mathfrak{su}(2)\times\mathfrak{u}(1)$.
- $\mathcal{N}=1$: This theory can be constructed with the *Calabi-Yau* orientfold $D3/D7$ or $D5/D9$. The internal symmetry group is $G = U(1)$ and the Lie algebra $\mathfrak{g} = \mathfrak{u}(1)$.

From the point of view of the four dimensional physics the internal supercurrents Σ^I produce *Kac-Moody currents* $\mathcal{J}^{IJ}$ (I and J are the supersymmetry indices) with conformal dimension one which are now physical. The OPE of these currents for extended SUSY (i.e. $\mathcal{N}=2,4$) are:

$$\begin{aligned}
\Sigma^I(z)\overline{\Sigma}^J(w) &= \tfrac{\delta^{IJ}}{(z-w)^{3/4}}\,\mathbf{I} + (z-w)^{1/4}\,\mathcal{J}^{IJ}(w)+\dots\,,\\
\Sigma^I(z)\Sigma^J(w) &\sim (z-w)^{1/4}\,\psi^{IJ}(w)\,,
\end{aligned} \tag{7.3.35}$$

where $\mathbf{I}$ is the identity matrix and $\psi^{IJ}(w)$ conformal weight $\frac{1}{2}$ field. For $\mathcal{N}=1$ we have enhancement to $\mathcal{N}=2$ due to the current $\mathcal{J}$ and the two corresponding spin fields $\Sigma^\pm(z)$ are associated to the opposite $U(1)$ charges defined as:

$$\Sigma^\pm(z)\overline{\Sigma}^\mp(w) = \tfrac{1}{(z-w)^{3/4}}\,\mathbf{I} \pm \tfrac{\sqrt{3}}{2}(z-w)^{1/4}\,\mathcal{J}(w)+\dots\,. \tag{7.3.36}$$

We should point out that in the case of compactification, it is not always possible to express the current $\mathcal{J}^{IJ}$ as a function of (internal or external) world sheet fields. This is not an issue, since for a field in a conformal field theory we only require the conformal weight and

OPEs with other fields, which we do have for $\mathcal{J}^{IJ}$ currents. Each of these currents $\mathcal{J}^\star$ is going to be charged under the internal symmetry associated to the compactification and their contractions are governed by the algebra of the Lie group of the internal symmetry. Hence, we can separate the abelian and non-abelian cases:

- Abelian $U(1)$ case: This is the case for $\mathcal{N}=1$ SUSY where the R-symmetry is $U(1)$. So the algebra of the Kac-Moody current $\mathfrak{g}=\mathfrak{u}(1)$. The contraction then will be:

$$\langle \mathcal{J}(z_i)\mathcal{J}(z_j)\rangle = \frac{1}{(z_i-z_j)^2}, \qquad \langle \mathcal{J}(z_i)\mathcal{J}(z_j)\mathcal{J}(z_k)\rangle \sim 0\,. \tag{7.3.37}$$

The current $\mathcal{J}$ can be written in terms of a free boson CFT $H(z)$ as:

$$\begin{aligned} &\mathcal{J}(z) = i\partial H(z), \qquad \Sigma(z) = e^{iqH(z)}, \qquad \mathcal{O}(z) = e^{i\sqrt{3}H}\,, \\ &\langle H(z)H(w)\rangle = \ln(z-w)\,. \end{aligned} \tag{7.3.38}$$

- Non-abelian $G=SU(2)$: This is the case for extended $\mathcal{N}=2$ SUSY where the R-symmetry is $SU(2)$. So the algebra of the Kac-Moody currents is the following

$$\mathcal{N}=2 \Rightarrow \mathfrak{g} = \mathfrak{su}(2)\times\mathfrak{u}(1)\,. \tag{7.3.39}$$

There are two currents one $\mathcal{J}$ associated to the $U(1)$ and a $SU(2)$ in the triplet representation $\mathcal{J}^A$ with the contractions:

$$\begin{aligned} \mathcal{J}^A(z_1)\mathcal{J}^B(z_2) &\sim \frac{\delta^{AB}}{z_{12}^2} + \frac{i\sqrt{2}\,\varepsilon^{ABC}}{z_{12}}\,\mathcal{J}^C(z_2)\,, \\ \langle \mathcal{J}^A(z_1)\mathcal{J}^B(z_2)\mathcal{J}^C(z_3)\rangle &= \frac{\varepsilon^{ABC}}{z_{12}z_{13}z_{23}}\,. \end{aligned} \tag{7.3.40}$$

the ε^{ABC} is the structure constant of the $\mathfrak{su}(2)$ Lie algebra. In the same fashion as the previous case, we can construct both currents $\mathcal{J}$ and $\mathcal{J}^3$ as:

$$\begin{aligned} &\mathcal{J}(z) = i\partial H_s(z), \qquad \mathcal{J}^3 = i\partial H_3(z)\,, \\ &\langle H(z)H(w)\rangle = \ln(z-w)\,. \end{aligned} \tag{7.3.41}$$

where $H_s(z)$ and $H_3(z)$ are two decoupled free scalar CFTs with the same OPE as $H(z)$.

- Non-abelian $G=SU(4)$: This is the case for extended $\mathcal{N}=4$ SUSY where the R-symmetry is $SU(4)$. We can write the current $\mathcal{J}^{MN}(z)$ in terms of the world sheet fields as:

$$\mathcal{J}^{MN}(z) = \frac{1}{\sqrt{2}}\,\Psi^M\Psi^N(z)\,, \qquad \mathcal{J}^M(z) = \partial Z^M(z)\,. \tag{7.3.42}$$

where Ψ^M and $\partial Z^M(z)$ are the internal fields of $\psi^\alpha(z)$ and $\partial X^\mu(z)$ respectively. So the algebra of the Kac-Moody currents will be:

$$\mathcal{N} = 4 \Rightarrow \mathfrak{g} = \mathfrak{su}(4) \times \mathfrak{u}(1) = \mathfrak{su}(6) \times \mathfrak{u}(1)^6 \,, \tag{7.3.43}$$

and for the current $\mathcal{J}^{MN}(z)$ the OPEs are given by:

$$\begin{aligned} \mathcal{J}_{MN}(z_1)\mathcal{J}^{KL}(z_2) &\sim \frac{\delta_M^{[K}\delta_N^{L]}}{z_{12}^2} + \frac{2\sqrt{2}}{z_{12}}\,\delta^{[K}_{(M}\mathcal{J}^{L]}_{N)}(z_2)\,, \\ \langle \mathcal{J}^{MN}(z_1)\mathcal{J}_{KL}(z_2)\mathcal{J}^{PR}(z_3)\rangle &= \frac{\sqrt{2}}{z_{12}z_{13}z_{23}}\left[\delta^N_{[K}\delta^{[P}_{L]}\delta^{R]M} - \delta^{N[P}\delta^{R]}_{[K}\delta^M_{L]}\right]. \end{aligned} \tag{7.3.44}$$

So, for each case of supersymmetry we have the current and the algebra needed to calculate the contractions.

Internal energy-momentum tensor

We need one more ingredient to be able to calculate vertex operators in a compactified string. That is the internal fermionic energy-momentum tensor. As we showed in (5.5.88) the picture changing operator is constructed out of the fermionic energy-momentum tensor T_F. So in order to be able to change the picture of the vertex operators, for the calculation of amplitudes, we need to know the structure of T_F in different compactifications as well as its OPE with internal fields[1]. We have for our different cases the following:

- $\mathcal{N} = 1$: The internal spin fields of SUSY charges are Σ^+ and Σ^- associated to the current $\mathcal{J}(z)$ of the $U(1)$ R-symmetry. Therefore, the internal fermionic energy-momentum tensor is constructed separately in the same way with the opposite $U(1)$ charges:

$$T_{F,int} = \frac{1}{2}(T^+_{F,int} + T^-_{F,int})\,, \tag{7.3.45}$$

  and together with the OPE:

$$\mathcal{J}(z)T^{\pm}_{F,int}(w) = \pm\frac{1}{\sqrt{3}}\frac{T^{\pm}_{F,int}(w)}{(z-w)} + \mathcal{J}(w)T^{\pm}_{F,int}(w) + \dots\,. \tag{7.3.46}$$

- $\mathcal{N} = 2$: Given the two currents of the $\mathcal{N} = 2$ case namely $\mathcal{J}$ and $\mathcal{J}^A$, which are decoupled, the internal energy-momentum tensor of the $\mathcal{N} = 2$ is also a sum of two decoupled energy-momentum tensors associated with each internal CFT:

$$T_{F,int} = T^{c=3}_{F,int} + T^{c=6}_{F,int}\,, \tag{7.3.47}$$

[1]There are no mixed contraction with internal and external fields therefore we only need OPEs with internal fields

$T^{c=3}_{F,int}$ is associated to the CFT of the $U(1)$ current (the $H_s(z)$ CFT) and it can be written in terms of the bosonic field H_s and the internal complex boson Z^M field as:

$$T^{c=3}_{F,int} = \frac{1}{2\sqrt{2\alpha'}}(i\partial Z e^{-H_s} + i\partial \bar{Z} e^{H_s}) . \tag{7.3.48}$$

The other energy-momentum tensor $T^{c=6}_{F,int}$ is a doublet under $SU(2)$ and has two components of opposite charge $\pm\frac{1}{\sqrt{2}}$ under $\mathcal{J}^3$ and can be written as:

$$T^{c=6}_{F,int} = \frac{1}{\sqrt{2}}\sum_{i=1}^{2} \lambda^i(z) g_i(z) , \tag{7.3.49}$$

where λ^i and g_i are two conformal fields with conformal weights $\frac{1}{4}$ and $\frac{5}{4}$, respectively. We have the following OPEs:

$$\begin{aligned} &T^{c=3}_{F,int}(z) T^{c=6}_{F,int}(w) \sim \text{regular} , \\ &g_i(z)\mathcal{J}^A(w) \sim \text{regular} , \\ &g_i(z)\lambda^i(w) \sim \text{regular} , \\ &\lambda^i(z)\mathcal{J}^A(w) = \frac{(\tau^A)^i{}_j}{\sqrt{2}(z-w)}\,\lambda^j(w) - \frac{1}{\sqrt{2}}(\tau^A)^i{}_j\,\partial\lambda^j(w) + \dots . \end{aligned} \tag{7.3.50}$$

In the last line, we have the Pauli matrices τ^As.

- $\mathcal{N}=4$: For this case we could write the current $\mathcal{J}_{MN}$ and $\mathcal{J}_M$ in terms of internal fields Ψ^M and ∂Z^M (with no need of bosonization). The internal energy-momentum tensor then is given by:

$$T_{F,int} = \frac{i}{2\sqrt{2\alpha'}} \sum_{M=1}^{6} \Psi^M \partial Z^M , \tag{7.3.51}$$

and the OPE:

$$\begin{aligned} \mathcal{J}^{MN}(z) T_{F,int}(w) &= \frac{1}{2\sqrt{\alpha'}}\frac{1}{z-w}\Psi^{[M}\partial Z^{N]}(w) + \dots , \\ \mathcal{J}^{M}(z) T_{F,int}(w) &= \frac{2\alpha'}{(z-w)^2}\Psi^M + \dots . \end{aligned} \tag{7.3.52}$$

7.3.2 Spectrum of compactified string

Now, that we have the internal energy-momentum tensors we need to look at the spectrum of the superstring after compactification and choose our candidates for bimetric gravity. As we discussed in chapter 4 the bimetric gravity in the mass eigenstate is the interacting theory of massive and massless spin-2 fields. Therefore, we need a massless spin-2 (i.e. graviton) and a massive spin-2 states in the compactified superstring spectrum. We do not need to look further than the first massive level of NS open string which includes, after compactification, a massive spin-1 and a massive spin-2 states. For the massive spin-2 candidate both massive spin-1 and spin-2 states are useful. We can use the former and implement the KLT double copy to create a massive spin-2 closed string state and the latter as the direct candidate for the bimetric massive spin-2. We should point out that there are *infinitely many* massive spin-1 and spin-2 states in the string spectrum [10, 11, 12]. However, we chose the lightest levels to probe and so we have the following spectrum for the compactified superstring [76, 77]:

- level 0: It always contains a single massless vector multiplet, whose structure is as follows

$$\mathcal{N}=1: \quad (1,1/2) \quad , \quad \mathbf{2}_{\mathrm{B}}+\mathbf{2}_{\mathrm{F}}\,, \tag{7.3.53a}$$
$$\mathcal{N}=2: \quad \big(1,2(1/2),2(0)\big) \quad , \quad \mathbf{4}_{\mathrm{B}}+\mathbf{4}_{\mathrm{F}}\,, \tag{7.3.53b}$$
$$\mathcal{N}=4: \quad \big(1,4(1/2),6(0)\big) \quad , \quad \mathbf{8}_{\mathrm{B}}+\mathbf{8}_{\mathrm{F}}\,, \tag{7.3.53c}$$

  So the massless candidate is straightforward: We have only one choice for each SUSY (the first vector state in the multiplet). It has the same universal vertex operator for all compactifications as:

$$\begin{aligned} &V_o^0(z,\varepsilon,q) = g_o T^a\sqrt{\frac{2}{\alpha'}}\,\varepsilon_\mu\left(i\partial X^\mu - 2\alpha'(q\cdot\psi)\psi^\mu(z)\right)e^{iqX(z)}\,,\\ &V_c^{(-1)}(z,\varepsilon,q) = g_o\;T^a\varepsilon_\mu\,e^{-\phi(z)}\psi^\mu(z)\;e^{iq\cdot X(z,\bar z)}\,,\\ &q^2=0, \qquad q^\mu\cdot\varepsilon_\mu = 0\;. \end{aligned} \tag{7.3.54}$$

  Since the picture zero vertex operator does not depend on the internal fields and currents, it only contracts with the external T_F and hence picture (-1) vertex operator has the same structure as the non-compact case.

- level 1: It always contains one massive spin-2 multiplet, whose structure is per case as follows

$$\mathcal{N}=1: \quad \big(2,2(3/2),1\big) \quad , \quad \mathbf{8}_{\mathrm{B}}+\mathbf{8}_{\mathrm{F}}\,, \tag{7.3.55a}$$
$$\mathcal{N}=2: \quad \big(2,4(3/2),6(1),4(1/2),0\big) \quad , \quad \mathbf{24}_{\mathrm{B}}+\mathbf{24}_{\mathrm{F}}\,, \tag{7.3.55b}$$
$$\mathcal{N}=4: \quad \big(2,8(3/2),27(1),48(1/2),42(0)\big) \quad , \quad \mathbf{128}_{\mathrm{B}}+\mathbf{128}_{\mathrm{F}}\,, \tag{7.3.55c}$$

Therefore, for the massive candidate we have two choices: First, we have the spin-2 states in the multiplet (the spin-2 state at top of the each multiplet) which like graviton state has the following universal, independent of internal fields, form:

$$\begin{aligned} V_B^{(-1)}(z,B,p) &= \frac{g_o}{(2\alpha')^{1/2}}\;T^a\;e^{-\phi(z)}\,B_{mn}\,i\partial X^m(z)\psi^n(z)\;e^{ipX(z)}\,,\\ V_B^{(0)}(z,\alpha,k) &= \frac{g_o}{(2\alpha')}\;T^a\;B_{\mu\nu}\big[i\partial X^\mu(z)\partial X^\nu(z) - 2i\alpha'\partial\psi^\mu(z)\psi^\nu(z)\,,\\ &\qquad\qquad +2\alpha'\,(k\psi)(z)\,\psi^\nu(z)\partial X^\mu(z)\big]\;e^{ikX(z)}\,,\\ p^2 = -\frac{1}{\alpha'} \quad &, \quad B_{[mn]}=0 \quad , \quad p^m B_{mn}=0 \quad , \quad B^m{}_m = 0\,. \end{aligned} \tag{7.3.56}$$

Second, we have the vector spin-1 states in the multiplets that can be KLT double copied into massive closed string spin-2 states. They have the following vertex operators:

For $\mathcal{N}=1$:

$$V_{\boldsymbol{A}}^{(-1)}(z,a,p) = g_o\sqrt{\frac{\alpha'}{6}}\,T^a\,e^{-\phi(z)}\,a_\mu\,\psi^\mu(z)\mathcal{J}(z)\,e^{ipX(z)}\,, \tag{7.3.57a}$$

$$V_{\boldsymbol{A}}^{(0)}(z,a,p) = g_{\rm o}\,T^a a_\mu\left\{\frac{i}{2\sqrt{\alpha'}}\left[i\partial X^\mu(z)+2\alpha'\,(p\cdot\psi)\psi^\mu(z)\right]\mathcal{J}(z)\,,\right. \tag{7.3.57b}$$

$$\left. +\,\psi^\mu\left[T^+_{\rm F,int}-T^-_{\rm F,int}\right]\right\}e^{ipX(z)}\,.$$

$$p^2=-\frac{1}{\alpha'}\quad,\quad p\cdot a=0\,. \tag{7.3.57c}$$

For $\mathcal{N}=2$:

$$V_{\boldsymbol{A}}^{(-1)}(z,a,p) = g_o\sqrt{\frac{\alpha'}{6}}\,T^a\,e^{-\phi(z)}\,a^A_\mu\,\psi^\mu(z)\mathcal{J}^A(z)\,e^{ipX(z)} \tag{7.3.58a}$$

$$V_{\boldsymbol{A}}^{(0)}(z,a,p) = g_{\rm o}\,T^a a^A_\mu\left\{\frac{i}{\sqrt{2\alpha'}}\left[i\partial X^\mu(z)+2\alpha'\,(p\cdot\psi)\psi^\mu(z)\right]\mathcal{J}^A(z)\right. \tag{7.3.58b}$$

$$\left. +\,\psi^\mu\,g_i\,(\tau^A)^i{}_j\,\lambda^j\right\}e^{ipX(z)}\,.$$

$$p^2=-\frac{1}{\alpha'}\quad,\quad p\cdot a^A=0\,. \tag{7.3.58c}$$

For $\mathcal{N}=4$:

$$V_{\boldsymbol{A}}^{(-1)}(z,a,p) = g_o\sqrt{\frac{\alpha'}{6}}\,T^a\,e^{-\phi(z)}\,a^{MN}_\mu\,\psi^\mu(z)\mathcal{J}^{MN}(z)\,e^{ipX(z)}\,, \tag{7.3.59a}$$

$$V_{\boldsymbol{A}}^{(0)}(z,a,p) = g_{\rm o}\,T^a a^{MN}_\mu\left\{\frac{i}{\sqrt{2}}\left[i\partial X^\mu(z)+2\alpha'\,(p\cdot\psi)\psi^\mu(z)\right]\mathcal{J}^{MN}(z)\,,\right. \tag{7.3.59b}$$

$$\left. +\,\psi^\mu\,\Psi^{[M}\partial Z^{N]}\right\}e^{ipX(z)}\,.$$

$$p^2=-\frac{1}{\alpha'}\quad,\quad p\cdot a^{MN}=0\,. \tag{7.3.59c}$$

The calculation of picture changing is given in appendix C.3, where we look at the open spin states. For closed strings, we are going to explain later, we use the double copy of the open string vertex operator.

7.3.3 Open string spin-2 state

Our first candidate is the massive spin-2 from the open superstring, together with the massless graviton which is the standard NS-NS closed string graviton. For the massive and the massless states, we have the vertex operators given in (7.3.56) and (7.3.54), respectively. We use these states and calculate the following amplitudes:

- three point scattering of massive spin-2 $\mathcal{A}(MMM)$.
- two massive and one massless spin-2 scattering $\mathcal{A}(MMG)$.

Since these two amplitudes involve both open and closed strings, we need to define the D-brane configuration of the open strings. We set a spacetime-filling brane with the D matrix defined as:

$$D^{\mu}_{\nu} = \delta^{\mu}_{\nu} \quad , \quad D^{\mu\nu} = D^{\mu}_{\lambda} g^{\lambda\nu} = g^{\mu\nu} \quad , \quad (Dq)^{\mu} = q^{\mu} \quad , \quad \mu, \nu = 0, \ldots, 3 \, , \tag{7.3.60}$$

and the following kinematics (cf. 5.2.1):

$$\begin{aligned} &\text{open string: } (k_1, k_2) \, , \\ &\text{Closes string: } (\frac{1}{2}q, \frac{1}{2}\bar{q}) = (\frac{1}{2}q, \frac{1}{2}Dq) \, , \\ &\text{Momentum conservation } (k_1 + k_2 + q_{||}) = 0 \, . \end{aligned} \tag{7.3.61}$$

The first amplitude is the scattering of three massive spin-2 states. The Riemann surface is the disk with the three open string states inserted on the boundary of it.

$$\begin{aligned} A^{open}(3,0) &= \int_{\mathcal{M}_{0,3}} \prod_{l=1}^{3} \frac{dx_l}{SL(2,\mathbf{R})} \, \delta_{ghost} \;\; e^{-\lambda} C^{matter}_{D_2} \left\langle \prod_{k=1}^{3} : V_k(x_l, k_l) : \right\rangle_{D_2} \\ &= \int_{\mathcal{M}_{0,3}} \prod_{l=1}^{3} \frac{dx_l}{SL(2,\mathbf{R})} \; e^{-\lambda} C^{g}_{D_2} C^{matter}_{D_2} \langle c(x_1)c(x_2)c(x_3)\rangle_{D_2} \\ &\qquad \left\langle : V_{-1}(x_1, k_1) :: V_{-1}(x_2, k_2) :: V_0(x_3, k_3) : \right\rangle_{D_2} \, , \\ &= \frac{1}{\alpha' g_o} \langle c(x_1)c(x_2)c(x_3)\rangle_{D_2} \left\langle : V_{-1}(x_1, k_1) :: V_{-1}(x_2, k_2) :: V_0(x_3, k_3) : \right\rangle_{D_2} \, , \end{aligned} \tag{7.3.62}$$

with the momentum conservation and massive on-shell condition:

$$k_1 + k_2 + k_3 = 0, \qquad\qquad k_i^2 = -\frac{1}{\alpha'} \, . \tag{7.3.63}$$

Plugging vertex operators back in:

$$
\begin{aligned}
A^{open}(3,0) = \frac{g_o}{\sqrt{\alpha'}}(x_{12}x_{23}x_{31}) \Big\langle &: T^a\ e^{-\phi(x_1)}\ \alpha^1_{\mu\nu}\ i\partial X^\mu(x_1)\psi^\nu(x_2)\ e^{ik_1X(x_1)} : \\
: T^b\ e^{-\phi(x_2)}\ \alpha^2_{\mu\nu}\ i\partial X^\mu(x_2)\psi^\nu(x_2)\ e^{ik_2X(x_2)} &:: T^c\alpha^3_{\mu\nu}\big[i\partial X^\mu(x_3)\partial X^\nu(x_3) \\
-2i\alpha'\partial\psi^\mu(x_3)\psi^\nu(x_3) + 2\alpha'\,(k_3\psi)(x_3)\,\psi^\nu(x_3)\partial X^\mu(x_3)\big]\ e^{ik_3X(x_3)} &\Big\rangle_{D_2} .
\end{aligned}
\tag{7.3.64}
$$

We can remove the integration by fixing the three points with the conformal killing group and after performing the Wick contractions we see that all the coordinate dependence drops out. We obtain:

$$
\begin{aligned}
\mathcal{A}(3,0) \;=\; & \tfrac{g_o}{4\alpha'^3}\mathrm{Tr}(T^{a_1}\{T^{a_2},T^{a_3}\})\Big\{3\,(2\alpha')^2\mathrm{Tr}(\alpha^1\cdot\alpha^2\cdot\alpha^3) + (2\alpha')^3\times \\
& \Big[(k_1\cdot\alpha^2\cdot k_1)(\alpha^3\cdot\alpha^1) + (k_2\cdot\alpha^3\cdot k_2)(\alpha^2\cdot\alpha^1) + (k_3\cdot\alpha^1\cdot k_3)(\alpha^2\cdot\alpha^3) \\
& +3\,k_1\cdot\alpha^2\cdot\alpha^1\cdot\alpha^3\cdot k_2 + 3\,k_2\cdot\alpha^3\cdot\alpha^2\cdot\alpha^1\cdot k_3 + 3\,k_3\cdot\alpha^1\cdot\alpha^3\cdot\alpha^2\cdot k_1\Big] \\
& +(2\alpha')^4\Big[(k_1\cdot\alpha^2\cdot k_1)(k_2\cdot\alpha^3\cdot\alpha^1\cdot k_3) + (k_2\cdot\alpha^3\cdot k_2)(k_3\cdot\alpha^1\cdot\alpha^2\cdot k_1) \\
& +(k_3\cdot\alpha^1\cdot k_3)(k_1\cdot\alpha^2\cdot\alpha^3\cdot k_2)\Big]\Big\} .
\end{aligned}
\tag{7.3.65}
$$

As we can observe this amplitude is exact in the orders of α'. The next amplitude will be the mixed amplitude of two massive and one massles spin-2 i.e. two open and one closed strings. Therefore, the interaction world–sheet is a punctured disk with open strings on the boundary and the closed string in the bulk (figure 7.1):

$$
\begin{aligned}
\mathcal{A}(2;1) &= \int \frac{dx_1dx_2d^2z_3}{SL(2,\mathbf{R})}\ \delta_{ghost}\ e^{-\lambda}\ C^g_{D_2}\Big\langle V_o^{(-1)}(\alpha_1,k_1,x_1)V_o^{(-1)}(\alpha_2,k_2,x_2)V_c^{(0,0)}(\varepsilon_q,q,z_3,\bar{z}_3)\Big\rangle_{D_2} \\
&= \frac{g_o^2 g_c}{\alpha'}C_{D_2}\alpha_{\mu_1\nu_1}\alpha_{\mu_2\nu_2}\varepsilon_{\alpha\beta}\ \int \frac{dx_1dx_2d^2z_3}{SL(2,\mathbf{R})}(x_1-z)(x_1-\overline{z})(z-\overline{z}) \\
&\times \Big\langle T^a\ e^{-\phi(x_1)}\ i\partial X^{\mu_1}(x_1)\psi^{\nu_1}(x_1)\ e^{ik_1X(x_1)}\,T^b\ e^{-\phi(x_2)}\ i\partial X^{\mu_2}(x_2)\psi^{\nu_2}(x_2)\ e^{ik_2X(x_2)} \\
&\times \Big[i\overline{\partial}X^\alpha(\bar{z}_3) + \frac{\alpha'}{2}(Dq\overline{\psi})\overline{\psi}^\alpha(\overline{z}_3)\Big]\Big[i\partial X^\beta(z_3) + \frac{\alpha'}{2}(q\psi)\psi^\beta(z_3)\Big]\ e^{iqX(z_3,\bar{z}_3)}\Big\rangle_{D_2} .
\end{aligned}
\tag{7.3.66}
$$

We can see that fixing the conformal killing group by quotienting the volume $SL(2,\mathbf{R})$ is not going cancel the integral completely. In particular, we can do the following fixing:

$$
(x_1,x_2,z,\bar{z}) \mapsto (x,-x,i,-i)\,, \tag{7.3.67}
$$

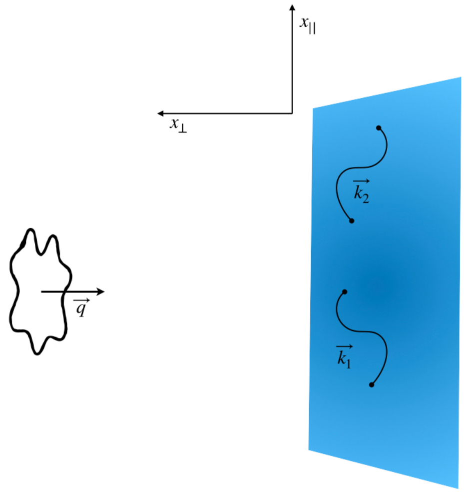

Figure 7.1: Scattering of one closed string from two open strings on a brane

where we have fixed the three real values consisting of two real numbers in z (real and imaginary part of z) and the sum of the position of two open strings $x_2 = -x_1$. With this fixing, the ghost factor δ_{ghost} and the Koba-Nielsen will have the following form:

$$\begin{aligned} \langle c(x_1)c(z)\tilde{c}(\bar{z})\rangle &= (x_1 - z)(x_1 - \bar{z})(z - \bar{z}) = 2i(x-i)(x+i)\,. \\ KN_{(2,1)} &= 4^{s+1}\,|x|^{s+2}(x^2+1)^{-s}\,. \end{aligned} \tag{7.3.68}$$

Although it is a three point amplitude the closed string acts as *two* open strings with left and write momenta $(q, \tilde{q})$ acting as two momenta. Therefore, we can define the 4-point-like the Mandelstam variables:

$$\begin{aligned} s &\equiv \alpha'(k_1+k_2)^2 = -2 + 2\alpha' k_1 k_2 = -2\alpha'\, k_1\cdot q\,, \\ t &\equiv \alpha'(k_1+k_3)^2 = -1 + \alpha' k_1 q\,, \\ u &\equiv \alpha'(k_1+k_4)^2 = -1 + \alpha' k_1 D q\,, \end{aligned} \tag{7.3.69}$$

where we have defined:

$$k_3^\mu \equiv \frac{q^\mu}{2} \quad , \quad k_4^\mu \equiv \frac{(Dq)^\mu}{2}\,. \tag{7.3.70}$$

Now we can do the tedious task of performing the Wick contractions. For the details see C.1.1. We can regroup the amplitude in the following way:

$$\mathcal{A}(2,1) = \frac{g_c}{\alpha'^2}\,\mathrm{Tr}\big(T^a T^b\big)\sum_{i=1}^{4}\mathbf{A}_i\,, \tag{7.3.71}$$

where

$$\begin{aligned}\mathbf{A}_1 &= 4^s\,\alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu}\int\limits_{-\infty}^{\infty}dx\,|x|^{s+2}(x^2+1)^{-s}\tfrac{(x-i)(x+i)}{2x}\Big\{\tfrac{\Theta^{\mu\nu\kappa\lambda\rho\sigma}}{(x-i)^4(x+i)}+\tfrac{\Lambda^{\mu\nu\kappa\lambda\rho\sigma}}{(x-i)(x+i)^4}\\ &+\tfrac{\Xi^{\mu\nu\kappa\lambda\rho\sigma}}{(x-i)^3(x+i)^2}+\tfrac{\Sigma^{\mu\nu\kappa\lambda\rho\sigma}}{(x-i)^2(x+i)^3}-\tfrac{i}{2}\tfrac{\Gamma^{\mu\nu\kappa\lambda\rho\sigma}}{(x-i)^4}-\tfrac{i}{2}\tfrac{\Delta^{\mu\nu\kappa\lambda\rho\sigma}}{(x+i)^4}-\tfrac{i}{2}\tfrac{\Phi^{\mu\nu\kappa\lambda\rho\sigma}}{(x-i)^3(x+i)}\\ &-\tfrac{i}{2}\tfrac{\Psi^{\mu\nu\kappa\lambda\rho\sigma}}{(x-i)(x+i)^3}-\tfrac{i}{2}\tfrac{\Omega^{\mu\nu\kappa\lambda\rho\sigma}}{(x-i)^2(x+i)^2}\Big\},\end{aligned}\tag{7.3.72}$$

$$\begin{aligned}\mathbf{A}_2 &= 4^s\,\alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu}\int\limits_{-\infty}^{\infty}dx\,|x|^{s+2}(x^2+1)^{-s}\tfrac{(x-i)(x+i)}{(2x)^2}\Big\{\tfrac{P^{\mu\nu\kappa\rho}g^{\lambda\sigma}}{(x-i)^4}+\tfrac{\widetilde{P}^{\mu\nu\kappa\rho}g^{\lambda\sigma}}{(x+i)^4}\\ &+\tfrac{Q^{\mu\nu\kappa\rho\lambda\sigma}}{(x-i)^3(x+i)}+\tfrac{\widetilde{Q}^{\mu\nu\kappa\rho\lambda\sigma}}{(x-i)(x+i)^3}+\tfrac{R^{\mu\nu\kappa\rho\lambda\sigma}}{(x-i)^2(x+i)^2}+\tfrac{i}{2}\tfrac{S^{\mu\nu\kappa\rho\lambda\sigma}}{(x-i)^3}-\tfrac{i}{2}\tfrac{\widetilde{S}^{\mu\nu\kappa\rho\lambda\sigma}}{(x+i)^3}\\ &+\tfrac{i}{2}\tfrac{T^{\mu\nu\kappa\rho\lambda\sigma}}{(x-i)^2(x+i)}-\tfrac{i}{2}\tfrac{\widetilde{T}^{\mu\nu\kappa\rho\lambda\sigma}}{(x-i)(x+i)^2}-\tfrac{1}{4}\tfrac{U^{\mu\nu\kappa\rho}g^{\lambda\sigma}}{(x-i)^2}-\tfrac{1}{4}\tfrac{\widetilde{U}^{\mu\nu\kappa\rho}g^{\lambda\sigma}}{(x+i)^2}-\tfrac{1}{4}\tfrac{W^{\mu\nu\kappa\rho}g^{\lambda\sigma}}{(x-i)(x+i)}\Big\},\end{aligned}\tag{7.3.73}$$

$$\begin{aligned}\mathbf{A}_3 &= 4^s\,\alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu}\int\limits_{-\infty}^{\infty}dx\,|x|^{s+2}(x^2+1)^{-s}\tfrac{(x-i)(x+i)}{(2x)^3}\Big\{\tfrac{G^{\mu\nu\kappa\rho}g^{\lambda\sigma}}{(x-i)^3}+\tfrac{H^{\mu\nu\kappa\rho}g^{\lambda\sigma}}{(x+i)^3}\\ &+\tfrac{I^{\mu\nu\kappa\rho\lambda\sigma}}{(x-i)^2(x+i)}+\tfrac{J^{\mu\nu\kappa\rho\lambda\sigma}}{(x-i)(x+i)^2}-i\tfrac{K^{\mu\nu\kappa\rho\lambda\sigma}}{(x-i)^2}-i\tfrac{L^{\mu\nu\kappa\rho\lambda\sigma}}{(x+i)^2}-i\tfrac{M^{\mu\nu\kappa\rho\lambda\sigma}}{(x-i)(x+i)}\\ &+\tfrac{N^{\mu\nu\kappa\rho}g^{\lambda\sigma}}{x+i}+\tfrac{O^{\mu\nu\kappa\rho}g^{\lambda\sigma}}{x-i}\Big\},\end{aligned}\tag{7.3.74}$$

$$\begin{aligned}\mathbf{A}_4 &= 4^s\,\alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu}g^{\lambda\sigma}\int\limits_{-\infty}^{\infty}dx\,|x|^{s+2}(x^2+1)^{-s}\tfrac{(x-i)(x+i)}{(2x)^4}\Big\{A^{\mu\nu\kappa\rho}+\tfrac{B^{\mu\nu\kappa\rho}}{(x-i)(x+i)}\\ &+\tfrac{C^{\mu\nu\kappa\rho}}{(x+i)^2}+\tfrac{\widetilde{\Delta}^{\mu\nu\kappa\rho}}{(x-i)^2}+i\tfrac{E^{\mu\nu\kappa\rho}}{x+i}+i\tfrac{F^{\mu\nu\kappa\rho}}{x-i}\Big\}.\end{aligned}\tag{7.3.75}$$

Above, we have ordered the $\mathbf{A}_i$s in such a way that the index i corresponds to the power $(x_1-x_2)^i$. We can solve[2] these integrals and find the following results:

$$\begin{aligned}\mathbf{A}_1 &= \tfrac{1}{2}4^s\Big\{(\Gamma-\Delta)\,\tfrac{2\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+3}{2}\right)}{\Gamma(s+3)}+\tfrac{1}{2}(\Theta+\Lambda)\,\tfrac{\pi^{3/2}2^{-s-2}(s-3)\sec\left(\frac{\pi s}{2}\right)}{\Gamma\left(\frac{3}{2}-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+1\right)}\\ &-\tfrac{1}{4}(\Xi+\Sigma)\,\tfrac{\Gamma\left(\frac{s-1}{2}\right)\Gamma\left(\frac{s+3}{2}\right)}{\Gamma(s+1)}+\tfrac{1}{16}(\Omega_+-\Omega_-)\,\tfrac{\pi\sec\left(\frac{\pi s}{2}\right)\Gamma\left(\frac{s+3}{2}\right)}{\Gamma\left(\frac{5}{2}-\frac{s}{2}\right)\Gamma(s)}\Big\},\end{aligned}\tag{7.3.76}$$

$$\begin{aligned}\mathbf{A}_2 &= \tfrac{1}{4}4^s\Big\{-(P+\widetilde{P})\tfrac{\sqrt{\pi}2^{-s-1}s\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}+2\right)}-(S+\widetilde{S})\tfrac{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+3}{2}\right)}{\Gamma(s+2)}\\ &-\tfrac{1}{4}(U+\widetilde{U})\tfrac{\Gamma\left(\frac{s-1}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma(s+1)}-\tfrac{1}{4}(W+\widetilde{W})\tfrac{\sqrt{\pi}2^{-s}\Gamma\left(\frac{s-1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\Big\},\end{aligned}\tag{7.3.77}$$

$$\begin{aligned}\mathbf{A}_3 &= \tfrac{1}{8}4^s\Big\{-(G+H)\,\tfrac{\pi^{3/2}2^{-s}\sec\left(\frac{\pi s}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+1\right)}+(K-L)\,\tfrac{\sqrt{\pi}2^{-s+1}\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)}\\ &+\tfrac{1}{4}(N+O)\,\tfrac{\Gamma\left(\frac{s-1}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma(s)}\Big\},\end{aligned}\tag{7.3.78}$$

[2]We used Mathematica in this case.

$$\mathbf{A}_4 = \tfrac{1}{16}\,4^s\left\{2A\,\frac{\sqrt{\pi}2^{-s}s\,\Gamma\left(\frac{s-1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)} - (C+\widetilde{\Delta})\,\frac{\sqrt{\pi}2^{-s}\Gamma\left(\frac{s-1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)} + (E-F)\,\frac{(s-1)\left[\Gamma\left(\frac{s-1}{2}\right)\right]^2}{4\Gamma(s)}\right\}, \tag{7.3.79}$$

We should emphasize that the result (7.3.71) is the *exact*[3] result of the two massive open strings and one massless closed string amplitude. Our next task is to take the low energy limit of this amplitude and produce the effective Lagrangian for the both amplitudes $\mathcal{A}(2;1)$ and $\mathcal{A}(3;0)$.

Amplitude expansion

We are facing a very fundamental problem of finding low energy effective actions for massive string states. As we described before in section 5.5.4, states in string spectrum have the mass:

$$M^2 = \frac{n}{\alpha'}\,, \qquad \text{where} \quad n \in \mathbf{W}$$

. This mass formula in the low energy limit $\alpha' \to 0$ creates two classes of states.

- The massless states which remain massless in this limit.
- The massive levels all of which upon taking this limit will have infinite mass.

This is known as the *decoupling* of the massive modes in the low energy string actions. By taking this limit one would only keep the massless states (the rest will be infinitely heavy). This is one of the main reasons that before [3] there were not many discussions on massive string effective actions.

Having this in mind, let us look at our amplitude (7.3.71). We have gamma functions (the solutions of integrals) that depend on the kinematic Mandelstam variables. We want to expand these functions in the limit $\alpha' \to 0$. This limit is where we face the *decoupling* problem. If the states were massless we did not have any issues. We would have plugged on-shell conditions back in gamma functions and expand them in $\alpha' \to 0$ limit. However, we have massive states involved and this means that their momenta scale with α', in particular:

$$k_i \sim \sqrt{\frac{1}{\alpha'}}, \qquad \text{whereas} \qquad q \sim 1\,.$$

Therefore, we need to be really careful while taking the low energy limit. Taking this scaling into account we see that

$$\alpha'\,k_1 \cdot k_2 \overset{\alpha'\to 0}{\longrightarrow} 1\,, \tag{7.3.80}$$

while

$$\alpha'\,k_{1,2} \cdot q \overset{\alpha'\to 0}{\longrightarrow} 0\,. \tag{7.3.81}$$

Using (7.3.69), this yields

$$s \overset{\alpha'\to 0}{\longrightarrow} 0 \quad \text{or} \quad t \overset{\alpha'\to 0}{\longrightarrow} -1\,, \tag{7.3.82}$$

[3] There are no higher order α'. All corrections terms are included in the gamma functions.

instead of the naive limit value $s \stackrel{\alpha'\to 0}{\longrightarrow} -2$. Now, the issue is to justify which of the two formulations of s in (7.3.69) to choose and use for our expansion. In order to do that we go back to the construction of the *low energy* effective string action. We can clearly see that only taking $\alpha' \to 0$ leads to two different choices. So we need an additional condition for the limit. The new scale that we take into account is $\alpha' k^2$. Since different poles in the gamma functions correspond to exchange of internal strings with the mass equal to the value of the pole [11], in order to obtain the leading order contribution (associated with the massless internal exchange) we should expand the gamma function at pole zero i.e. the value of the Mandelstam must go to zero. Therefore we amend the *low energy* effective string condition:

Low energy limit

We rewrite all functions in terms of formulation of s that goes to zero as $\alpha' \to 0$ we have:

$$\begin{aligned} &\alpha' \to 0 \\ &s = -2\alpha'\, k_1 \cdot q \to 0 \end{aligned} \tag{7.3.83}$$

Doing so, we rewrite s dependence in the gamma functions in eq. (7.3.76),(7.3.77),(7.3.78) and (7.3.79) in terms of the vanishing s and expand in the limit $\alpha' \to 0$. Therefore, using the method we described above we have the following *consistent* low energy limit of the two massive open and one massless closes string amplitude:

Low energy α' expansion

$$\begin{aligned} \mathcal{A}(2,1) \;=\; g_c \Big\{ & -\mathrm{Tr}(\alpha^1 \cdot \alpha^2)\, \varepsilon_{\mu\nu} k_1^\mu k_2^\nu + (\varepsilon \cdot \alpha^2 \cdot \alpha^1)_{\mu\nu} k_1^\mu k_2^\nu \\ & + (\varepsilon \cdot \alpha^1 \cdot \alpha^2)_{\mu\nu} k_1^\nu k_2^\mu + (\varepsilon \cdot \alpha^2 \cdot \alpha^1)_{\mu\nu}\, k_1^\mu q^\nu + (\varepsilon \cdot \alpha^1 \cdot \alpha^2)_{\mu\nu}\, k_2^\mu q^\nu \\ & + \mathrm{Tr}(\varepsilon \cdot \alpha^1 \cdot \alpha^2)\, (k_1 \cdot q) + \tfrac{1}{2}\big[\mathrm{Tr}(\varepsilon \cdot \alpha^2)\alpha^1_{\mu\nu} - 2(\alpha^1 \cdot \varepsilon \cdot \alpha^2)_{\mu\nu} \\ & + \mathrm{Tr}(\varepsilon \cdot \alpha^1)\alpha^2_{\mu\nu}\big]\, q^\mu q^\nu \Big\} + \mathcal{O}\left(\alpha'^3\right) . \end{aligned} \tag{7.3.84}$$

Due to the truncation we can see that this expansion is not exact in α' since we have the exchange of the string state between the open and closed strings which as we mentioned corresponds to the poles of the gamma function. This means we have higher-order string corrections in this expansion which are out of the scope of our current work.

Effective Lagrangian

In order to produce an effective Lagrangian out of these terms we follow the standard replacement for amplitudes:

$$\alpha_{\mu\nu} \to M_{\mu\nu}, \quad \varepsilon_{\mu\nu} \to G_{\mu\nu}, \quad (k,q)^\mu \to i\partial^\mu, \tag{7.3.85}$$

Using these replacement rules on the M^3 amplitude that we found in (7.3.65) we obtain a effective cubic order Lagrangian for $M_{\mu\nu}$ as:

Effective Lagrangian M^3

$$\begin{aligned}\mathcal{L}^{\text{eff}}_{\text{M}^3} &= \tfrac{g_o}{\alpha'}\Big\{\big[M^3\big] + 2\alpha' M^{\mu\nu}\big[\partial_\mu M_{\rho\sigma}\partial_\nu M^{\rho\sigma} - 3\partial_\nu M_{\rho\sigma}\partial^\sigma M^\rho_\mu\big] \\ &\qquad + 4\alpha'^2\,\partial^\mu\partial^\nu M_{\rho\sigma}\partial^\rho M^\kappa_\nu\partial^\sigma M_{\mu\kappa}\Big\}.\end{aligned} \tag{7.3.86}$$

Similarly, for the amplitude of two massive and one massless states (7.3.84) we arrive at the following:

Effective Lagrangian GM^2

$$\begin{aligned}\mathcal{L}^{\text{eff}}_{\text{GM}^2} &= g_c\Big[G^{\mu\nu}\big(\partial_\mu M_{\rho\sigma}\partial_\nu M^{\rho\sigma} - 4\partial_\nu M_{\rho\sigma}\partial^\sigma M^\rho_\mu\big) \\ &\qquad + M^{\mu\nu}\big(\partial_\mu G_{\rho\sigma}\partial_\nu M^{\rho\sigma} - \partial_\rho G_{\mu\sigma}\partial_\nu M^{\rho\sigma}\big)\Big].\end{aligned} \tag{7.3.87}$$

We will compare and contrast these results with the bimetric gravity at the end of this section. But first we take a look at the closed string candidate as well.

7.3.4 Closed string spin-2 state

Now, we turn to the closed string candidate for the massive spin-2 field of the bimetric theory. Our method is as follows: We take the massive *spin*-1 open string states that we identified in (7.3.57) together with the massless vector we discussed in subsection 5.5.4. Then, we calculate three-point tree level amplitudes for different cases:

- Scattering $\mathcal{A}(MMM)$ of three massive spin-1.
- Scattering $\mathcal{A}(MMG)$ of two massive and one massless spin1.
- Scattering $\mathcal{A}(MGG)$ of two massless and one massive spin-1.

These amplitudes will involve massive and massless spin-1 *vectors*. So, using the KLT double copy (cf. section 5.7) we produce closed string amplitude. Since we have the massive/massless spin-1 fields the double copied amplitudes will correspond to the massive/massless spin-2 states. This is a nontrivial but easy to see, since the polarization of

the double copied amplitude will be the tensor product of the polarization of two spin-1 states (i.e. $\alpha_{\mu\nu} = \varepsilon_\mu \otimes \varepsilon_\nu$), it will be transverse and traceless. Therefore, it will correspond to the correct propagating degrees of freedom for massive/massless spin-2 fields (two and five degrees of freedom for massless and massive spin-2 fields, respectively).
Furthermore, since we are looking at the three point amplitude the KLT factor in (5.7.165) is going to be trivial (no kinematic dependence). Therefore, we only need to functionally multiply the amplitudes, meaning:

$$\mathcal{A}_3^{closed} = \mathcal{A}_3^{open} \otimes \mathcal{A}_3^{open} \tag{7.3.88}$$

We have three different possible vertex operators each associated with a different number of SUSY. Looking at the vertex operators in (7.3.57). We can see that the main differences are in the internal current $\mathcal{J}^\star$ and the picture zero of the vertex operators (as a result of different internal energy-momentum tensors). Since we are using the type II string theoey we must to take both amplitudes from the same supersymmetry. Therefore, we are going to have the following double copies:

$$\begin{aligned} \mathcal{N}^{open} = 1 \otimes \mathcal{N}^{open} = 1 \Rightarrow \mathcal{N}^{closed} = 2\,, \\ \mathcal{N}^{open} = 2 \otimes \mathcal{N}^{open} = 2 \Rightarrow \mathcal{N}^{closed} = 4\,, \\ \mathcal{N}^{open} = 4 \otimes \mathcal{N}^{open} = 4 \Rightarrow \mathcal{N}^{closed} = 8\,. \end{aligned} \tag{7.3.89}$$

All of the amplitudes are three-point open string amplitudes with the following structure:

$$\begin{aligned} A^{open}(k_1,k_2,k_3) &= \int_{\mathcal{M}_{0,3}} \prod_{l=1}^{3} \frac{dx_l}{SL(2,\mathbf{R})}\, \delta_{ghost}\;\; e^{-\lambda} C_{D_2}^{matter} \left\langle \prod_{k=1}^{3} : V_k(x_l,k_l) : \right\rangle_{D_2} \\ &= \int_{\mathcal{M}_{0,3}} \prod_{l=1}^{3} \frac{dx_l}{SL(2,\mathbf{R})}\; e^{-\lambda} C_{D_2}^{g} C_{D_2}^{matter} \langle c(x_1)c(x_2)c(x_3)\rangle_{D_2} \\ &\quad\times \left\langle : V_{-1}(x_1,k_1) :: V_{-1}(x_2,k_2) :: V_0(x_3,k_3) : \right\rangle_{D_2} \\ &= \frac{1}{\alpha' g_o} \langle c(x_1)c(x_2)c(x_3)\rangle_{D_2} \left\langle : V_{-1}(x_1,k_1) :: V_{-1}(x_2,k_2) :: V_0(x_3,k_3) : \right\rangle_{D_2}\,. \end{aligned} \tag{7.3.90}$$

In the last line, we have used the fact that the $SL(2,\mathbf{R})$ invariance will fix the positions and remove the world–sheet integral. Now we can plug back vertex operators from (7.3.57) and calculate the contractions[4],

$$\begin{aligned} \mathcal{A}_{AAA} &= \tfrac{g_o}{\sqrt{\alpha'}}\, \epsilon_{1\mu}\, \epsilon_{2\nu}\, \epsilon_{3\lambda}\, \mathrm{Tr}([T^a,T^b]T^c)\, K N_{AAA}\, \mathcal{B}^{\mu\nu\lambda}\,, \\ \mathcal{A}_{AA\boldsymbol{A}} &= \tfrac{g_o}{\sqrt{\alpha'}}\, \epsilon_{1\mu}\, \epsilon_{2\nu}\, a_{3\lambda}\, \mathrm{Tr}([T^a,T^b]T^c)\, K N_{AA\boldsymbol{A}}\, \mathcal{B}^{\mu\nu\lambda} \langle \mathcal{J}_3^\star\rangle\,, \\ \mathcal{A}_{\boldsymbol{AA}A} &= \tfrac{g_o}{\sqrt{\alpha'}}\, a_{1\mu}\, a_{2\nu}\, \epsilon_{3\lambda}\, \mathrm{Tr}([T^a,T^b]T^c)\, K N_{\boldsymbol{AA}A}\, \mathcal{B}^{\mu\nu\lambda} \langle \mathcal{J}_1^\star\, \mathcal{J}_2^\star\rangle\,, \\ \mathcal{A}_{\boldsymbol{AAA}} &= \tfrac{g_o}{\sqrt{\alpha'}}\, a_{1\mu}\, a_{2\nu}\, a_{3\lambda}\, \mathrm{Tr}([T^a,T^b]T^c)\, K N_{\boldsymbol{AAA}}\, \mathcal{B}^{\mu\nu\lambda} \langle \mathcal{J}_1^\star\, \mathcal{J}_2^\star\, \mathcal{J}_3^\star\rangle\,, \end{aligned} \tag{7.3.91}$$

[4] As we also showed in the example 5.6.2 the contractions will also be position independent

where $a_{i\,\mu}$ is to be understood as $a_{i\,\mu}$, $a^{A}_{i\,\mu}$ and $a^{MN}_{i\,\mu}$ and $\mathcal{J}^{\star}_i$ as $\mathcal{J}_i$, $\mathcal{J}^{A}_i$ and $\mathcal{J}^{MN}_i$ for $\mathcal{N} = 1, 2, 4$, respectively and the bold $\boldsymbol{A}$ is the notation for massive legs. The $\mathcal{B}^{\mu\nu\lambda}$ contraction function is given by:

$$\mathcal{B}^{\mu\nu\lambda} \equiv \langle c_1 c_2 c_3\rangle\langle e^{-\phi_1} e^{-\phi_2}\rangle\Big\{\langle\psi_1^{\mu}\psi_2^{\nu}\rangle\Big[\langle ip_1 X_1\, i\partial X_3^{\lambda}\rangle + \langle ip_2 X_2\, i\partial X_3^{\lambda}\rangle\Big] \\ -2\alpha'\Big[\langle\psi_1^{\mu}\left(p_3\cdot\psi_3\right)\rangle\langle\psi_2^{\nu}\psi_3^{\lambda}\rangle - \langle\psi_1^{\mu}\psi_3^{\lambda}\rangle\langle\psi_2^{\nu}\left(p_3\cdot\psi_3\right)\rangle\Big]\Big\}\,. \tag{7.3.92}$$

There are some important points to emphasize:

- First, we have the Chan-Paton factors in the amplitudes these should not be mixed with the Lie algebra of the currents.

- Second, there are extra terms in the vertex operator arising in the picture changing procedure that would have spoiled the structure we have here. However, upon performing the contractions they all vanish and hence they have no impact on the results (see C.3).

- Third, we can see from the second amplitude in (7.3.91) that regardless of the theory (and supersymmetry) we will have a one-point function of the current $\langle\mathcal{J}_i^{\star}\rangle$ in the decay channel of the massive field to two massless fields. This one-point function will force the amplitude to be zero in all cases:

$$\mathcal{A}^{\mathcal{N}=1,2,4}_{AA\boldsymbol{A}} \equiv 0\,. \tag{7.3.93}$$

  This is in agreement with the Landau and Yang theorem that massive particles cannot decay into particles of the same helicity. This result has been extended to higher spins in [108], which we are going to refer to also in the case of spin-2 field after the double copy.

- Looking at the OPE of different currents we see in (7.3.37) that for the $\mathcal{N} = 1$ case the three-point function of $\mathcal{J}_i$s vanishes. Therefore, for this case we also have vanishing amplitude:

$$\mathcal{A}^{\mathcal{N}=1}_{\boldsymbol{AAA}} \equiv 0\,. \tag{7.3.94}$$

- Finally, we calculated the standard massless open string spin-1 amplitude in the example 5.6.2. So we just use the result of the amplitude when necessary.

After multiplying all the functions and polarizations we have the following results for each case:

Massive spin-1 amplitudes

$$\mathcal{A}^{\mathcal{N}=1,2,4}_{\boldsymbol{AA}A} = \frac{g_o}{\sqrt{\alpha'}}\, f^{abc}\left(a_1\cdot a_2\, p_1\cdot\epsilon_3 + a_2\cdot\epsilon_3\, p_2\cdot a_1 + \epsilon_3\cdot a_1\, p_3\cdot a_2\right),$$

$$\begin{aligned}\mathcal{A}^{\mathcal{N}=2}_{\boldsymbol{AAA}} &= \frac{g_o}{\sqrt{\alpha'}}\, f^{abc}\varepsilon^{ABC}\left(a_1^A\cdot a_2^B\, p_1\cdot a_3^C + a_2^B\cdot a_3^C\, p_2\cdot a_1^A + a_3^C\cdot a_1^A\, p_3\cdot a_2^B\right),\\ \mathcal{A}^{\mathcal{N}=4}_{\boldsymbol{AAA}} &= \frac{g_o}{\sqrt{\alpha'}}\, f^{abc}\left(a_1^{MN}\cdot a_2^{ML}\, p_1\cdot a_3^{NL} + a_2^{ML}\cdot a_3^{NL}\, p_2\cdot a_1^{MN} + a_3^{NL}\cdot a_1^{MN}\, p_3\cdot a_2^{ML}\right).\end{aligned} \tag{7.3.95}$$

where in $\mathcal{A}^{\mathcal{N}=2}_{\boldsymbol{AA}A}$ the Kac–Moody $SU(2)$ indices of a_1^A and a_2^B are implicitly contracted with each other, while in $\mathcal{A}^{\mathcal{N}=4}_{\boldsymbol{AA}A}$ the Kac–Moody $SO(6)$ are contracted, namely $a_1\cdot a_2$ stands for $a_1^{MN}\cdot a_2^{MN}$. Recall that a_1^{MN} is anti-symmetric explicitly w.r.t.,the interchange of M and N.

Now we can construct the closed string spin-2 amplitudes $\mathcal{M}$ as the following symmetric double copies:

$$\begin{aligned}\mathcal{M}_{GGG} = \mathcal{A}_{AAA}\otimes\widetilde{\mathcal{A}}_{AAA} \quad &, \quad \mathcal{M}_{GGM} = \mathcal{A}_{AA\boldsymbol{A}}\otimes\widetilde{\mathcal{A}}_{AA\boldsymbol{A}}\\ \mathcal{M}_{MMM} = \mathcal{A}_{\boldsymbol{AAA}}\otimes\widetilde{\mathcal{A}}_{\boldsymbol{AAA}} \quad &, \quad \mathcal{M}_{MMG} = \mathcal{A}_{\boldsymbol{AA}A}\otimes\widetilde{\mathcal{A}}_{\boldsymbol{AA}A}\end{aligned} \tag{7.3.96}$$

where $\widetilde{\mathcal{A}} = \overline{\mathcal{A}}$ is the anti holomorphic left mover of $\mathcal{A}$. In the three point amplitude case, this is not going to affect us since there is no position dependence left in amplitudes. First, after constructing the double copy the massless case we obtain the famous (graviton) result:

$$\begin{aligned}\mathcal{M}^{\mathcal{N}=2,4,8}_{GGG} = g_{\rm c}\Big[&(k_1\cdot\varepsilon_3\cdot k_1)\mathrm{Tr}\big(\varepsilon_1\cdot\varepsilon_2\big) + 2\,k_2\cdot\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3\cdot k_1\Big]\\ &+ \text{ cyclic permutations}\,.\end{aligned} \tag{7.3.97}$$

Second, as mentioned for the case of one massive and two massless amplitude we have vanishing amplitude:

$$\mathcal{M}^{\mathcal{N}=2,4,8}_{GGM} = \ 0\,. \tag{7.3.98}$$

For the other mixture of two massive and one massless case amplitude, we have:

$$\begin{aligned}\mathcal{M}^{\mathcal{N}=2,4,8}_{MMG} \ = \ g_{\rm c}\Big[&(k_1\cdot\varepsilon\cdot k_1)\mathrm{Tr}(\alpha_1\cdot\alpha_2) + 2\,k_2\cdot\alpha_1\cdot\alpha_2\cdot\varepsilon\cdot k_1\Big]\\ &+ \text{cyclic permutations}\,,\end{aligned} \tag{7.3.99}$$

where in $\mathcal{M}^{\mathcal{N}=4}_{MMG}$ and $\mathcal{M}^{\mathcal{N}=8}_{MMG}$ we have the Kac–Moody $SU(2)$ and $SO(6)$ indices, respectively. These indices are implicitly contracted with each other. For example for $\alpha_1^{AA'}$ and

$\alpha_2^{BB'}$ in $SU(2)$ and $\alpha_1^{MNM'N'}$ and $\alpha_2^{OPO'P'}$ in $SO(6)$[5] we have:

$$(k_1 \cdot \alpha_2 \cdot k_1)\mathrm{Tr}(\alpha_1 \cdot \varepsilon) = \begin{cases} (k_1 \cdot \alpha_2^{AA'} \cdot k_1)\mathrm{Tr}(\alpha_1^{AA'} \cdot \varepsilon) \\ (k_1 \cdot \alpha_2^{MNM'N'} \cdot k_1)\mathrm{Tr}(\alpha_1^{MNM'N'} \cdot \varepsilon)\,. \end{cases} \tag{7.3.100}$$

Finally, For the all massive cases, we have:

$$\begin{aligned}
\mathcal{M}_{MMM}^{\mathcal{N}=2} &= 0\,, \\
\mathcal{M}_{MMM}^{\mathcal{N}=4} &= g_{\mathrm{c}}\varepsilon^{ABC}\varepsilon^{A'B'C'}\Big[(k_1 \cdot \alpha_3^{CC'} \cdot k_1)\mathrm{Tr}(\alpha_1^{AA'} \cdot \alpha_2^{BB'}) \\
&\qquad + 2\,k_2 \cdot \alpha_1^{AA'} \cdot \alpha_2^{BB'} \cdot \alpha_3^{CC'} \cdot k_1\Big] + \text{cyclic permutations} \\
\mathcal{M}_{MMM}^{\mathcal{N}=8} &= g_{\mathrm{c}}\Big[(k_1 \cdot \alpha_3^{NLN'L'} \cdot k_1)\mathrm{Tr}(\alpha_1^{MNM'N'} \cdot \alpha_2^{MLM'L'}) \\
&\qquad + 2\,k_2 \cdot \alpha_1^{MNM'N'} \cdot \alpha_2^{MLM'L'} \cdot \alpha_3^{NLN'L'} \cdot k_1\Big] + \text{cyclic permutations}\,.
\end{aligned} \tag{7.3.101}$$

Effective Lagrangian

Having all the spin-2 amplitudes, we are ready to construct the low energy effective Lagrangians. In this case, we have a much easier task since there is no integration left (unlike the previous open string case) and all amplitudes are exact in orders of α'. Therefore, we only need to follow the standard procedure as before and perform the following replacements for both spin-1 and spin-2 amplitudes:

$$\alpha_{\mu\nu} \to M_{\mu\nu}, \quad \varepsilon_{\mu\nu} \to G_{\mu\nu}, \quad a_\mu \to \boldsymbol{A}_\mu^a, \quad \epsilon_\mu \to A_\mu^a, \quad p^\mu \to i\partial^\mu\,. \tag{7.3.102}$$

We have for the spin-1 amplitudes the following effective Lagrangians:

Massive spin-1 effective lagrangian

$$\begin{aligned}
\mathcal{L}_{A^3}^{\mathcal{N}=1,2,4} &= \frac{g_{\mathrm{o}}}{\sqrt{\alpha'}} f^{abc}\left(\partial^\mu A_\nu^a\right) A^{\nu\, b} A_\mu^c\,, \\
\mathcal{L}_{A^2\boldsymbol{A}}^{\mathcal{N}=1,2,4} &= 0\,, \\
\mathcal{L}_{\boldsymbol{A}^2 A}^{\mathcal{N}=1,2,4} &= \frac{g_o}{\sqrt{\alpha'}} f^{abc}\left[\left(\partial^\mu \boldsymbol{A}_\nu^a\right) \boldsymbol{A}^{\nu\, b} A_\mu^c + 2\boldsymbol{A}_\mu^a \left(\partial^\mu \boldsymbol{A}_\nu^b\right) A^{\nu\, c}\right], \\
\mathcal{L}_{\boldsymbol{A}^3} &= \frac{g_o}{\sqrt{\alpha'}} f^{abc} \begin{cases} \varepsilon^{ABC}\left(\partial^\mu \boldsymbol{A}_\nu^{aA}\right) \boldsymbol{A}^{\nu\, bB} \boldsymbol{A}_\mu^{cC}, & \mathcal{N}=2\,, \\ \left(\partial^\mu \boldsymbol{A}_\nu^{aMN}\right) \boldsymbol{A}^{\nu\, bML} \boldsymbol{A}_\mu^{cNL}, & \mathcal{N}=4\,, \\ 0, \quad \mathcal{N}=2,\ . \end{cases}
\end{aligned} \tag{7.3.103}$$

[5]Every index is associated with a different Kac–Moody i.e. in $\alpha_1^{AA'}$ A is index in the one $SU(2)$ and A' is index in another $SU(2)$ current. Similarly for the $SO(6)$.

This result is the *massive spin*-1 effective action from string theory. Although this was not our goal, this is a pleasant side result. One can see the structure of the Proca action which is the Yang-Mills theory deformed by the mass term $m^2 A_\mu A^\mu$ in the effective $\mathcal{L}_{\boldsymbol{A}^3}$ terms. Therefore, this state can also be considered as a string candidate for the Proca field theory. In addition, one can check that the $\boldsymbol{A}^3$ terms are indeed gauge invariant (upon replacement of one polarization with on-shell momenta).

Similarly, for the closed string amplitudes upon replacing (7.3.102) we obtain the following effective Lagrangians:

Massive spin-2 effective lagrangian

$$\begin{aligned}
\mathcal{L}^{\mathcal{N}=2,4,8}_{G^3} &= g_c\, G^{\mu\nu}\big(\partial_\mu G_{\rho\sigma}\partial_\nu G^{\rho\sigma} - 2\partial_\nu G_{\rho\sigma}\partial^\sigma G^\rho_\mu\big)\,,\\
\mathcal{L}^{\mathcal{N}=2,4,8}_{G^2 M} &= 0\,,\\
\mathcal{L}^{\mathcal{N}=2,4,8}_{G M^2} &= g_c\Big[G^{\mu\nu}\big(\partial_\mu M_{\rho\sigma}\partial_\nu M^{\rho\sigma} - 4\partial_\nu M_{\rho\sigma}\partial^\sigma M^\rho_\mu\big)\\
&\qquad + 2M^{\mu\nu}\big(\partial_\mu G_{\rho\sigma}\partial_\nu M^{\rho\sigma} - \partial_\rho G_{\mu\sigma}\partial_\nu M^{\rho\sigma}\big)\Big]\,,\\
\mathcal{L}^{\mathcal{N}=4,8}_{M^3} &= g_c\, M^{\mu\nu}\big(\partial_\mu M_{\rho\sigma}\partial_\nu M^{\rho\sigma} - 2\partial_\nu M_{\rho\sigma}\partial^\sigma M^\rho_\mu\big)\,,\\
\mathcal{L}^{\mathcal{N}=2}_{M^3} &= 0\,.
\end{aligned} \tag{7.3.104}$$

There are several points about these effective actions to note:

- The first observation is the universal form of the massless graviton in all possible supersymmetry cases.
- As expected the decay channel of massive spin-2 state to massless spin-2 state is forbidden by vanishing of $\mathcal{L}_{G^2 M}$.
- For the case of $\mathcal{N}=2$ supersymmetry the $U(1)$ current algebra enforces $\mathcal{L}^{\mathcal{N}=2}_{M^3}=0$ meaning, it is a trivial candidate at the cubic level.

7.3.5 Comparison to bimetric gravity

We can now compare both our effective Lagrangian results for open string (7.3.86),(7.3.87), and for closed string (7.3.104) with bimetric Lagrangian. From the bimetric Lagrangian in the mass eigenbasis we take all cubic interactions. Therefore, looking at (4.4.74),(4.4.75) and changing the notation from $(\delta G_{\mu\nu}, \delta M_{\mu\nu})$ to $(G_{\mu\nu}, M_{\mu\nu})$. We will have the following Lagrangian terms:

Bimetric 3 point interactive Lagrangian

$$\mathcal{L}^{\rm bim}_{\rm G^3} = \frac{1}{m_g\sqrt{1+\alpha^2}}\, G^{\mu\nu}\big(\partial_\mu G_{\rho\sigma}\partial_\nu G^{\rho\sigma} - 2\partial_\nu G_{\rho\sigma}\partial^\sigma G^\rho_\mu\big)\,, \tag{7.3.105a}$$

$$\mathcal{L}^{\text{bim}}_{\text{G}^2\text{M}} = 0\,, \tag{7.3.105b}$$

$$\mathcal{L}^{\text{bim}}_{\text{GM}^2} = \frac{1}{m_g\sqrt{1+\alpha^2}}\Big[G^{\mu\nu}\big(\partial_\mu M_{\rho\sigma}\partial_\nu M^{\rho\sigma} - 4\partial_\nu M_{\rho\sigma}\partial^\sigma M^\rho_\mu\big)\,, \\ + 2M^{\mu\nu}\big(\partial_\mu G_{\rho\sigma}\partial_\nu M^{\rho\sigma} - \partial_\rho G_{\mu\sigma}\partial_\nu M^{\rho\sigma}\big)\Big]\,, \tag{7.3.105c}$$

$$\mathcal{L}^{\text{bim}}_{\text{M}^3} = \frac{(-\beta_1+\beta_3)\,(1+\alpha^2)^{3/2}m_g}{6\,\alpha}\,[M^3] \\ + \frac{(1-\alpha^2)}{m_g\alpha\sqrt{1+\alpha^2}}\,M^{\mu\nu}\big(\partial_\mu M_{\rho\sigma}\partial_\nu M^{\rho\sigma} - 2\partial_\nu M_{\rho\sigma}\partial^\sigma M^\rho_\mu\big)\,. \tag{7.3.105d}$$

Before we proceed we should make a very important point. As a keen reader will notice all the terms in our three Lagrangians i.e. bimetric, open string effective Lagrangian and closed string effective Lagrangian are the same up to relative numerical factors. This is not a coincidence. We are looking at the tree level three-point terms that are Lorenz invariant and ghost free. These terms are the only possible terms. However, this does not mean that our discussion is trivial. In the open string case we manage to define a consistent way to obtain an effective Lagrangian and avoid the issue with the tower of states with an integer gap. Further, in the closed string case, we have produced the first massive double copy for string amplitudes and we are going to check whether or not these massive spin-2 amplitude for open and closed strings will match the bimetric theory.

Open string massive spin-2 state

We can now compare the first amplitude we calculated involving massive spin-2 open string states. As we mentioned all *allowed* terms compatible with symmetries are available in both bimetric theory and our effective Lagrangians. Looking at the bimetric expansion (7.3.105d) and comparing it with the low energy Lagrangian for the massive open string state (7.3.86) and (7.3.87) we can construct the following parametric relation:

Parameter constraints

Comparing the graviton case we matched with the following identification between the string coupling and the bimetric mass

$$\frac{1}{m_g\sqrt{1+\alpha^2}} \stackrel{!}{=} g_{\text{c}}\,, \qquad \text{where:} \qquad \alpha = \frac{m_f}{m_g}\,. \tag{7.3.106}$$

For the terms involving the massive spin-2 state, we a have numerical discrepancy:

$$\begin{aligned} & M^{\mu\nu}\big(\partial_\mu M_{\rho\sigma}\partial_\nu M^{\rho\sigma} - \underbrace{2}_{bimetric}\ \partial_\nu M_{\rho\sigma}\partial^\sigma M^\rho_\mu\big) \\ \text{vs } & M^{\mu\nu}\big(\partial_\mu M_{\rho\sigma}\partial_\nu M^{\rho\sigma} - \underbrace{4}_{open\ string}\ \partial_\nu M_{\rho\sigma}\partial^\sigma M^\rho_\mu\big)\,, \end{aligned} \tag{7.3.107}$$

Given this issue, we refrain from comparing further the couplings. However, let us look at this issue a bit deeper. The cubic term in question is coming from massive spin-2 self-interaction. These interactions as we discussed in section 4.4 are governed by a Einstein-Hilbert type action which is the consequence of having two sets of diffeomorphisms in the bimetric theory. So this comparison tells us that the open string effective action at the cubic level does not respect the diffeomorphisms. Does this mean string theory has ghosts or *bad* degrees of freedom given this discrepancy? The answer is no! String theory is ghost free and this apparent problem will be resolved if one computes all higher derivative (and higher order α') corrections and resum all of them [57]. However, what this means is that this open string state is not a good candidate to represent the massive spin-2 theory as depicted by bimetric and dRGT [17, 18] in the low energy α' truncated level.

Closed string massive spin-2 state

Now, we turn to the result in (7.3.104) for the closed string case. We see that the relative numerical factors are a perfect match (unlike the previous case). However, in string theory, there is only one free parameter α' so to accommodate for that, we have to fix the bimetric parameters β_i. Therefore, we have matched the bimetric Lagrangian for specific points in the parameter space:

Parameter constraints

As before we have the universal graviton coupling:

$$\frac{1}{m_g\sqrt{1+\alpha^2}} \overset{!}{=} g_c\,, \qquad \text{where:} \qquad \alpha = \frac{m_f}{m_g}\,. \tag{7.3.108}$$

For the terms involving the massive spin-2 fields we have the following universal (with respect to different SUSY cases) constraint:

$$m_g^2(1+\alpha^2)\,(\beta_1 + 2\beta_2 + \beta_3) \overset{!}{=} \frac{4}{\alpha'}\,. \tag{7.3.109}$$

For the rest of the terms we have a split between $\mathcal{N}=2$ and $\mathcal{N}=4,8$ cases:

$$\begin{aligned} \mathcal{N} = 2 \Rightarrow \beta_1 \overset{!}{=} \beta_3 \quad , \quad \alpha \overset{!}{=} 1\,, \\ \mathcal{N} = 4,8 \Rightarrow \beta_1 = \beta_3 \quad , \quad \alpha \approx 0.62\,. \end{aligned} \tag{7.3.110}$$

In all three cases $\mathcal{N} = 2, 4, 8$ matching the GM^2 will result in the first constraint $\beta_1 = \beta_3$. For $\mathcal{N} = 2$ we have that the M^3 Lagrangian vanishes and hence we have $\alpha = 1$. In the other two cases matching the M^3 Lagrangian will result in equation $\alpha^2 + \alpha - 1 = 0$ solving it for positive α we give the above result.

Note that for all cases we have that $\mathcal{L}_{G^2M}$ vanishes. This, as we discussed, originates from the spin-1 amplitude (7.3.91). However, it is a very important feature as it has been argued that the absence of the decay channel for the massive spin-2 fields to gravitons is a feature that only ghost free bimetric theory enjoys [64].

7.4 Double copy in intersection theory

One of the most intriguing features of double copy relations is the connection between spin-1 and spin-2 theories and their symmetries. Likewise, the relation between gauge invariance for spin-1 theories and diffeomorphism invariance associated with spin-2 theories. It has been argued in [52] that taking two spin-1 theories that satisfy color-kinematics duality and double copying them results in a spin-2 theory that is invariant under linearized diffeomorphism. This important feature requires further investigation and understanding for the massive case which is notably challenging [89]. In that case, one needs to clarify the role of CK duality and the corresponding KK–BCJ amplitude relations. Then, with this information one may construct massive spin-1 theories which satisfy this duality and check the corresponding double copied theory against diffeomorphism invariance for massive spin-2 theories, e.g. dRGT gravity.

In intersection theory, double copies can simply be constructed by pairing two theories whose description in terms of intersection numbers (6.3.8) comprise a color–form $\varphi^{color} \equiv PT(a)$. The latter constitutes an orthonormal basis, i.e.

$$\langle \Phi_{+,a}, \Phi^\vee_{-,b} \rangle_\omega = \delta_{ab} \quad , \quad \Phi_{+,a} = PT(a) \ , \tag{7.4.111}$$

and this fact allows us to simply "glue" two different theories both containing such a color–form φ^{color}.

Double copy in intersection theory

Concretely, for two theories T_1 and T_2 given by the intersection numbers

$$T_1 = \langle \varphi^1_+, \varphi^{color} \rangle \qquad\qquad T_2 = \langle \varphi^2_-, \varphi^{color} \ , \rangle \tag{7.4.112}$$

respectively. One schematically obtains for their double copy:

$$T_1 \otimes T_2 = \langle \varphi^1_+, \varphi^2_- \rangle \ . \tag{7.4.113}$$

More precisely, with the orthonormal basis (7.4.111) we have

$$\varphi^1_+ = \sum_{a=1}^{(m-3)!} \langle PT^\vee(a), \varphi^1_+ \rangle_\omega \ PT(a) \ ,$$

$$\varphi_-^2 = \sum_{b=1}^{(m-3)!} \langle \varphi_-^2, PT(b)\rangle_\omega \; PT^\vee(b) \; , \tag{7.4.114}$$

and with

$$\langle PT^\vee(a), \varphi_+^1\rangle_\omega = \sum_{b=1}^{(m-3)!} S[a|b] \; \langle PT(b), \varphi_+^1\rangle_\omega \; ,$$

we consider the following manipulations leading to a double copy expression

$$\begin{aligned} \langle \varphi_+^1, \varphi_-^2\rangle &= \sum_{a=1}^{(m-3)!} \langle PT^\vee(a), \varphi_+^1\rangle_\omega \; \langle \varphi_-^2, PT(a)\rangle_\omega \\ &= \sum_{a,b=1}^{(m-3)!} \langle \varphi_-^2, PT(a)\rangle_\omega \; S[a|b] \; \langle PT(b), \varphi_+^1\rangle_\omega \\ &= \sum_{a,b=1}^{(m-3)!} T_1(a) \; S[a|b] \; T_2(b) \; , \end{aligned} \tag{7.4.115}$$

with the intersection form or KLT kernel $S[a|b]$ given in (6.3.35). Therefore, the two theories (7.4.112) involving a color form in their twisted intersection forms give rise to the double copy (7.4.115) denoted by $T_1 \otimes T_2$.

7.4.1 Collections of known double copies

Here, we compile a list of different double copy constructions from pairs of theories discussed before. All of these theories exhibit a color form φ^{color} in their twisted intersection form (6.3.8). We will see how in each case what are the underlying theories and how the double copy of the theory and the amplitude (as $\alpha \to \infty$ limit of the intersection number) is constructed. The main point of this section is to motivate our proposal on the relationship between BCJ double copy and the twisted cohomology.

(i) Special Galilean theory

We start with the special Galilean theory described by gluing two identical NSLM[6] theories

$$\begin{aligned} T_1(a) &= \mathcal{A}_{NLSM}(a) = \lim_{\alpha'\to\infty} \langle PT(a), \varphi_{+,n}^{scalar}\rangle_\omega \; , \\ T_2(b) &= \mathcal{A}_{NLSM}(b) = \lim_{\alpha'\to\infty} \langle \varphi_{-,n}^{scalar}, PT(b)\rangle_\omega \; . \end{aligned} \tag{7.4.116}$$

with (7.4.115) we construct the double copy $T_1 \otimes T_2$ of Galilean theory

$$\mathcal{A}_{sGal}(n) = \lim_{\alpha'\to\infty} \langle \varphi_{-,n}^{scalar}, \varphi_{+,n}^{scalar}\rangle_\omega \; . \tag{7.4.117}$$

[6]Non linear sigma model

describing the scattering of n scalars with higher derivative interaction (4.3.16), i.e.:

$$\mathcal{A}_{sGal}(n) = \sum_{a,b=1}^{(n-3)!} \mathcal{A}_{NLSM}(a)\ S[a|b]\ \mathcal{A}_{NLSM}(b)\ . \tag{7.4.118}$$

(ii) Einstein–Yang–Mills (EYM) Secondly, we consider a double copy from gen.YMS and YM theories

$$\begin{aligned} T_1(a) = \mathcal{A}_{gen.YMS}(a) &= \lim_{\alpha' \to \infty} \langle PT(a), \tilde{\varphi}^{EYM}_{+,n;r} \rangle_\omega\ , \\ T_2(b) = \mathcal{A}_{YM}(b) &= \langle \varphi^{gauge}_{-,n+r}, PT(b) \rangle_\omega\ , \end{aligned} \tag{7.4.119}$$

to construct the double copy $T_1 \otimes T_2$ of EYM amplitudes [2]

$$\mathcal{A}_{EYM}(n;r) = \lim_{\alpha' \to \infty} \langle \varphi^{gauge}_{-,n+r}, \tilde{\varphi}^{EYM}_{+,n;r} \rangle_\omega\ , \tag{7.4.120}$$

describing the scattering of n gluons and r gravitons [50]

$$\mathcal{A}_{EYM}(n;r) = \sum_{a,b=1}^{(m-3)!} \mathcal{A}_{gen.YMS}(a)\ S[a|b]\ \mathcal{A}_{YM}(b)\ , \tag{7.4.121}$$

with $m = n + r$.

(iii) Dirac–Born–Infeld (DBI)

Thirdly, we glue together YMS and NSLM theories

$$\begin{aligned} T_1(a) = \mathcal{A}_{YMS}(a) &= \lim_{\alpha' \to \infty} \langle PT(a), \tilde{\varphi}^{EM}_{+,n;r} \rangle_\omega\ , \\ T_2(b) = \mathcal{A}_{NSLM}(b) &= \lim_{\alpha' \to \infty} \langle \varphi^{scalar}_{-,n+r}, PT(b) \rangle_\omega\ , \end{aligned} \tag{7.4.122}$$

to construct the double copy $T_1 \otimes T_2$ of DBI amplitudes

$$\mathcal{A}_{DBI}(n;r) = \lim_{\alpha' \to \infty} \langle \varphi^{scalar}_{-,n+r}, \tilde{\varphi}^{EM}_{+,n;r} \rangle_\omega\ , \tag{7.4.123}$$

describing the scattering of r gluons and n scalars [50]:

$$\mathcal{A}_{DBI}(n;r) = \sum_{a,b=1}^{(m-3)!} \mathcal{A}_{YMS}(a)\ S[a|b]\ \mathcal{A}_{NLSM}(b)\,. \tag{7.4.124}$$

(iv) Einstein-Maxwell (EM)

Furthermore, EM can be written as a double copy of the following two theories

$$T_1(a) = \mathcal{A}_{YMS}(a) = \lim_{\alpha' \to \infty} \langle PT(a), \tilde{\varphi}^{EM}_{+,n;r} \rangle_\omega\ ,$$

$$T_2(b) = \mathcal{A}_{YM}(b) = \langle \varphi^{gauge}_{-,n+r}, PT(b) \rangle_\omega \ . \tag{7.4.125}$$

giving rise to the double copy $T_1 \otimes T_2$ of EM amplitudes

$$\mathcal{A}_{EM}(n;r) = \lim_{\alpha' \to \infty} \langle \varphi^{gauge}_{-,n+r}, \tilde{\varphi}^{EM}_{+,n;r} \rangle_\omega \ , \tag{7.4.126}$$

involving r gravitons and n photons [50]:

$$\mathcal{A}_{EM}(n;r) = \sum_{a,b=1}^{(m-3)!} \mathcal{A}_{YMS}(a) \ S[a|b] \ \mathcal{A}_{YM}(b) \ . \tag{7.4.127}$$

(v) Born–Infeld (BI) theory

The amplitudes of Born–Infeld theory can be written as a double copy of the following two theories:

$$T_1(a) = \mathcal{A}_{NLSM}(a) = \lim_{\alpha' \to \infty} \langle PT(a), \varphi^{scalar}_{+,n} \rangle_\omega \ ,$$
$$T_2(b) = \mathcal{A}_{YM}(b) = \langle \varphi^{gauge}_{-,n}, PT(b) \rangle_\omega \ . \tag{7.4.128}$$

Then, the double copy $T_1 \otimes T_2$ yields the BI amplitudes

$$\mathcal{A}_{BI}(n) = \lim_{\alpha' \to \infty} \langle \varphi^{gauge}_{-,n}, \varphi^{scalar}_{+,n} \rangle_\omega \ . \tag{7.4.129}$$

accounting for the scattering of n gluons:

$$\mathcal{A}_{BI}(n) = \sum_{a,b=1}^{(n-3)!} \mathcal{A}_{NLSM}(a) \ S[a|b] \ \mathcal{A}_{YM}(b) \ . \tag{7.4.130}$$

(vi) Extended Dirac Born–Infeld (ext.DBI) theory

In addition to BI and DBI the amplitudes of ext.DBI theory can be written as the double copy of the following two theories:

$$T_1(a) = \mathcal{A}_{NLSM}(a) = \lim_{\alpha' \to \infty} \langle PT(a), \varphi^{scalar}_{+,n+r} \rangle_\omega \ ,$$
$$T_2(b) = \mathcal{A}_{gen.YMS}(b) = \langle \tilde{\varphi}^{EYM}_{-,n;r}, PT(b) \rangle_\omega \ . \tag{7.4.131}$$

Then, the double copy $T_1 \otimes T_2$ yields the ext.DBI amplitudes

$$\mathcal{A}_{ext.DBI}(n,r) = \lim_{\alpha' \to \infty} \langle \tilde{\varphi}^{EYM}_{-,n;r}, \varphi^{scalar}_{+,n+r} \rangle_\omega \ , \tag{7.4.132}$$

accounting for the scattering of r gluons and n higher derivative scalars as:

$$\mathcal{A}_{ext.DBI}(n,r) = \sum_{a,b=1}^{(m-3)!} \mathcal{A}_{NLSM}(a) \ S[a|b] \ \mathcal{A}_{gen.YMS}(b) \ . \tag{7.4.133}$$

(vii) $(DF)^2$ – photon theory
Finally, the last spin-1 theory we have is the amplitudes (4.3.33) of $(DF)^2$ – photon theory can be written as a double copy of the following two theories:

$$\begin{aligned} T_1(a) &= \mathcal{A}_{NLSM}(a) = \lim_{\alpha' \to \infty} \langle PT(a), \varphi_{+,n}^{scalar} \rangle_\omega \ , \\ T_2(b) &= \mathcal{A}_{(DF)^2}(b) = \langle \varphi_{-,n}^{Bosonic}, PT(b) \rangle_\omega \ . \end{aligned} \tag{7.4.134}$$

The double copy $T_1 \otimes T_2$ for the amplitudes

$$\mathcal{A}_{(DF)^2-\text{Photon}}(n) = \lim_{\alpha' \to \infty} \langle \varphi_{-,n}^{Bosonic}, \varphi_{+,n}^{scalar} \rangle_\omega \, . \tag{7.4.135}$$

involving n higher derivative photons gives:

$$\mathcal{A}_{(DF)^2-\text{Photon}}(n) = \sum_{a,b=1}^{(n-3)!} \mathcal{A}_{NLSM}(a)\ S[a|b]\ \mathcal{A}_{(DF)^2}(b)\ . \tag{7.4.136}$$

Note, that this double copy involves Einstein gravity and higher–derivative photon terms, cf. equation (4.3.32).

(viii) Conformal gravity
According to table 4.2 the scattering amplitude $\mathcal{A}_{CG}(n)$ of this theory involving n spin-2 particles can be represented as a double copy in agreement with the CHY representation [53]

$$\mathcal{A}_{CG}(n) \sim \lim_{\alpha' \to \infty} \langle \varphi_{-,n}^{gauge}, \varphi_{+,n}^{Bosonic} \rangle_\omega = \int\limits_{\mathcal{M}_{0,n}} d\mu_n \prod_{k=2}^{n-2} \delta(f_k)\ \text{Pf}'\psi_n\ W_{\underbrace{11...1}_{n}}\ , \tag{7.4.137}$$

This intersection number can be written as the following expansion:

$$\begin{aligned} \mathcal{A}_{CG}(n) &= \lim_{\alpha' \to \infty} \langle \varphi_{-,n}^{gauge}, \varphi_{+,n}^{bosonic} \rangle_\omega \\ &= \sum_{a,b=1}^{(n-3)!} \mathcal{A}_{YM}(a)\ S[a|b]\ \mathcal{A}_{(DF)^2}(b)\ \sim YM \otimes (DF)^2 \, , \end{aligned} \tag{7.4.138}$$

matching the already known double copy [52].

(ix) Einstein–Weyl
The EW gravity scattering amplitudes $\mathcal{A}_{EW}(n)$ involving n gravitons can be constructed through bosonic string amplitudes [82]. According to (4.3.43) in the limit $\alpha' \to \infty$ the Einstein–Hilbert part of the EW theory decouples and therefore the CHY representation of this theory cannot be constructed by applying this limit at the intersection number (6.3.26) of twisted forms. On the other hand, by using the bosonic string content one can write the amplitudes of the full theory as [5] (cf. also table 4.1):

$$\mathcal{A}_{EW}(n) = \langle \varphi_{+,n}^{gauge}, \varphi_{-,n}^{bosonic} \rangle_\omega \ . \tag{7.4.139}$$

The double copy structure of this theory is given by:

$$\text{Einstein–Weyl} = [YM + (DF)^2] \otimes YM \sim \langle \varphi^{gauge}_{-,n}, \varphi^{bosonic}_{+,n} \rangle_\omega \ . \tag{7.4.140}$$

Note, that according to (6.3.38) and (4.3.43) in the limit $\alpha' \to \infty$ the double copy structure (7.4.140) will reduce to the double copy of conformal gravity (4.3.40). On the other hand, for $\alpha' \to 0$ we obtain the double copy for non–pure Einstein gravity including an antisymmetric tensor and a dilaton scalar, cf. also (4.3.41).

Actually, the double copy structure (7.4.140) resembles the heterotic string. The amplitudes for the closed heterotic string also referred to as GR+R^2 can be calculated by using the KLT relations implementing the open bosonic string amplitude (6.3.34) together with the open superstring amplitude (6.3.33). The action up to order α'^3 was derived in [16]

$$\begin{aligned} S_{\text{heterotic string}} = -\frac{2}{\kappa^2} \int \mathrm{d}^d x \, \sqrt{g} \, \Big\{ & R - \tfrac{4}{d-2} (\partial_\mu \phi)^2 - \frac{1}{12} H^2 \\ & + \frac{\alpha'}{8} e^{-2\phi} \left(R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right) + \mathcal{O}(\alpha'^3) \Big\}, \end{aligned} \tag{7.4.141}$$

where ϕ represents the dilaton and $H_{\mu\nu\rho} = 3 \left(\partial_{[\mu} B_{\nu\rho]} + \frac{\alpha'}{4} \omega^{ab}_{[\mu} R^{ab}{}_{\nu\rho]} \right)$ is the field strength of the anti–symmetric B–field and the Chern–Simons form ω. Note, that in (7.4.141) the linear order in α' corresponds to the Gauss-Bonnet term (4.3.37). In the limit $\alpha' \to 0$ we recover standard GR including an antisymmetric tensor and a dilaton scalar, cf. (4.3.41). Likewise, in this limit $\alpha' \to 0$ the double copy structure (7.4.140) boils down to $YM \otimes YM$.

(x) $Weyl^3$ or R^3

The amplitude structure of the $Weyl^3$ can be reproduced by the intersection number (6.3.8) involving a pair of the twisted forms (6.3.39)

$$\mathcal{A}_{Weyl^3}(n) = \lim_{\alpha' \to \infty} \langle \varphi^{Bosonic}_{-,n}, \varphi^{Bosonic}_{+,n} \rangle_\omega \ , \tag{7.4.142}$$

leading to the following CHY representation:

$$\mathcal{A}_{Weyl^3}(n) = \int\limits_{\mathcal{M}_{0,n}} d\mu_n \, \prod_{k=2}^{n-2} \delta(f_k) \, (W_{\underbrace{11...1}_{n}})^2 \ . \tag{7.4.143}$$

Looking at equation (4.3.31) it is evident that this integrand is built from two sectors accounting for the $(DF)^2$ theory. Therefore, the twisted form (6.3.39) gives rise to the double copy structure of the amplitude (4.3.44), i.e.:

$$Weyl^3 = (DF)^2 \otimes (DF)^2 \ , \tag{7.4.144}$$

or likewise [52]:

$$(DF)^2 \otimes (DF)^2 \sim (\nabla R)^2 + R^3 \ . \tag{7.4.145}$$

(xi) $Weyl^3$-DF^2

The $Weyl^3$-DF^2 can be considered as double copy of a $(DF)^2$ with $(DF)^2+\phi^3$ theory. More specifically, we shall use the fact that the amplitude of $Weyl^3$–$(DF)^2$ theory may be described by the two twisted forms $\varphi^{bosonic}_{\pm,n+r}$ and $\tilde{\varphi}^{bosonic}_{\pm,n;r}$. The $Weyl^3$–$(DF)^2$ theory can be considered as an exotic cousin of EYM theory. Its CHY representation is assumed to take the following form (cf. subsection 6.7.3):

$$\begin{aligned}\mathcal{A}_{Weyl^3-(DF)^2}(n;r) &= \lim_{\alpha'\to\infty}\langle\varphi^{bosonic}_{-,n+r},\tilde{\varphi}^{bosonic}_{+,n;r}\rangle_\omega \\ &= \int\limits_{\mathcal{M}_{0,n+r}} d\mu_{n+r}\prod_{k=2}^{n+r-2}\delta(f_k)\ \mathcal{C}_n\widetilde{W}_{\underbrace{11\ldots1}_{r}}W_{\underbrace{11\ldots1}_{n+r}}\,.\end{aligned} \tag{7.4.146}$$

The formula (7.4.146) is a conjecture, which presumably may be proven by ambitwistor construction [58]. However, this goes beyond the scope of this work and thus we write the $Weyl^3$-$(DF)^2$ theory in terms of the following double copy of theories:

$$Weyl^3-(DF)^2=(DF)^2\otimes[(DF)^2+\phi^3]=(DF)^2\otimes(DF)^2+(DF)^2. \tag{7.4.147}$$

In table 7.1 we summarized all double copied theories with their originating theories. These are previously known and newly constructed double copies from theories containing a color form φ^{color}. e.g. the double copy of a $(DF)^2$ theory with itself gives rise to the amplitudes of six–derivative gravity R^3 originating from the bosonic ambitwistor string [53]. Note, in contrast double copying F^3 amplitudes leads to the amplitudes produced by Einstein gravity coupled to a dilaton field ϕ and deformed by operators of the form ϕR^2 and R^3 [109]. On the other hand, the conformal supergravity (CSG) amplitudes (of the non–minimal Berkovits–Witten type theory [54]) from table 7.1 follow from double copying $(DF)^2$ and SYM theory [52]. From table 7.1 it can be evidenced that there are new double copies of theories that can be constructed only by using the property that the color form provides CK duality and can be used as an essential part to double copy two different theories. Besides, there is a close relation between string theory and the content of table 7.1 utilizing the inherent double copy of the closed string in terms of open strings (cf. [25, 82, 110]). Moreover, we find a double copy for higher derivative (HD) gravity and bigravity to be discussed in subsections 7.4.3.

Theory	T_1	T_2	CHY representation
sGal	NSLM	NSLM	$(\mathrm{Pf}'A_n)^4$
EYM	gen.YMS	YM	$\mathcal{C}_n \mathrm{Pf}\Psi_{S_r}\, \mathrm{Pf}'\psi_{n+r}$
DBI	YMS	NSLM	$\mathrm{Pf}X_n\, \mathrm{Pf}'\Psi_{S_{r:n}}\, (\mathrm{Pf}'A_{n+r})^2$
EM	YMS	YM	$\mathrm{Pf}X_n\, \mathrm{Pf}'\Psi_{S_{r:n}}\, \mathrm{Pf}'\psi_{n+r}$
BI	NSLM	YM	$(\mathrm{Pf}'A_n)^2\, \mathrm{Pf}'\psi_n$
ext.DBI	NSLM	YM	$\mathcal{C}_n\, \mathrm{Pf}\Psi_{S_r}\, (\mathrm{Pf}'A_{n+r})^2$
$(DF)^2$-Photon	NSLM	$(DF)^2$	$(\mathrm{Pf}'A_n)^2\, W_{\underbrace{11...1}_{n}}$
Conformal gravity	YM	$(DF)^2$	$\mathrm{Pf}'\psi_n\, W_{\underbrace{11...1}_{n}}$
Einstein–Weyl	YM	YM+$(DF)^2$	*not known*
$Weyl^3$ or R^3	$(DF)^2$	$(DF)^2$	$\left(W_{\underbrace{11...1}_{n}}\right)^2$
$Weyl^3$-DF^2	$(DF)^2+\phi^3$	$(DF)^2$	$\mathcal{C}_n W_{\underbrace{11...1}_{r}} W_{\underbrace{11...1}_{n+r}}$

Table 7.1: Double copies $T_1 \otimes T_2$ through twisted form description (7.4.112) and (7.4.113).

7.4.2 Formal algebraic double copies

In the previous section, we used the known and newly constructed twisted form description of theories to produce double copies. However, there is an additional way to make double copies with the use of a biadjoint scalar. Since the biadjoint scalar is described through two color forms φ^{color} (cf. table 4.1) it can act as the identity in the space of theories. This feature is already shown in the KLT matrix analysis [49, 111] and here we are going to provide an intersection theory argument for it. Let us take the intersection double copy setup we had in (7.4.115) and take one of the theories to be the biadjoint scalar. We have:

$$T_{biadj} = \langle \varphi_-^{color}, \varphi_+^{color} \rangle, \qquad T_2 = \langle \varphi_-^{T_2}, \varphi_+^{color} \rangle\,,$$

$$\begin{aligned} T_{biadj} \otimes T_2 = \langle \varphi_+^1, \varphi_-^2 \rangle &= \sum_{a=1}^{(m-3)!} \langle PT^\vee(a), \varphi_+^{color} \rangle_\omega \, \langle \varphi_-^2, PT(a) \rangle_\omega \\ &= \sum_{a=1}^{(m-3)!} \underbrace{\langle PT^\vee(a), \varphi_+^{color} \rangle_\omega}_{\delta_{ab}} \langle \varphi_-^2, PT(a) \rangle_\omega = \langle \varphi_-^2, PT(a) \rangle_\omega = T_2\,. \end{aligned} \tag{7.4.148}$$

So we can evident that the biadjoint scalar acts as the *identiy* element in the set of theories. Therefore, we can use the elements of table 7.1 and from a set. Together with the tensor product and the biadjoint as the identity elemen. This set forms a *unital magma*. Using this we can construct new elements of the set of theories by multiplying. For example the double copy construction (7.4.120) of EYM amplitude amounts to:

$$\text{gen.YMS} \otimes \text{YM} = [\text{YM} + \phi^3] \otimes \text{YM} = (\text{YM} \otimes \text{YM}) + \text{YM} = \text{GR} + \text{YM} = \text{EYM}\,. \tag{7.4.149}$$

Furthermore, in Table 7.2 we have:

$$\text{gen.YMS} \otimes \text{YMS} = [\text{YM} + \phi^3] \otimes \text{YMS} = (\text{YM} \otimes \text{YMS}) + \text{YMS} = \text{EM} + \text{YMS}\,. \tag{7.4.150}$$

On the other hand, double copying two copies of gen.YMS amplitudes have been argued to produce amplitudes in Einstein–Yang-Mills–scalar (EYMS) theory. The latter simply follows from the compactification of EYM theory [20, 50, 112]. This result can also be anticipated from the decomposition:

$$\text{gen.YMS} \otimes \text{gen.YMS} = \text{gen.YMS} \otimes [YM + \phi^3] = EYM + \text{gen.YMS}\,. \tag{7.4.151}$$

Putting all of these in the next table 7.2 the four double copies (7.4.121), (7.4.124), (7.4.127) and (7.4.133) are also tabulated among some new constructions like EM amplitude squared.

Theory ⊗	YMS	gen.YMS	NLSM	YM
YMS	EM^2	EM + YMS	DBI	EM
gen.YMS	EM + YMS	EYMS	ext.DBI	EYM

Table 7.2: Table of different double copies

This notion is particularly powerful since it does not require knowledge about the resulting theory after double copies and we can use this formulation to investigate the space of theories.

7.4.3 Double copy of $YM + (DF)^2$ theory and higher derivative gravity

In this section we look at one of the main use of the algebraic double copy method, which we discussed in the last section namely the double copy of $YM + (DF)^2$ theory and show that it can be related to higher derivative (HD) gravity. Looking at table 7.1 we can write the double copy of the $YM + (DF)^2$ theory in terms of known spin-2 theories in the following way:

$$\begin{aligned}&[(DF)^2 + YM] \otimes [(DF)^2 + YM] \\ &\sim \underbrace{(YM \otimes YM)}_{GR} + \underbrace{(DF^2 \otimes DF^2)}_{R^3} + \underbrace{(DF^2 \otimes YM)}_{CG} + \underbrace{(YM \otimes DF^2)}_{CG} \\ &\sim GR + CG_1 + CG_2 + R^3 \sim GR + \widetilde{CG} + \mathcal{O}(R^3) \ .\end{aligned} \tag{7.4.152}$$

Since the amplitude of the $YM + (DF)^2$ theory is described by the intersection number (6.3.8) of $\varphi^{bosonic}_{\pm,n}$ and $\varphi^{color}_{\pm,n}$ we can write the amplitude of the resulting double copied theory as:

$$\mathcal{A}(n) = \langle \varphi^{bosonic}_{-,n}, \varphi^{bosonic}_{+,n} \rangle \ . \tag{7.4.153}$$

The twisted form $\varphi^{bosonic}_{\pm,n}$ stems from the bosonic string and is given in (6.3.32). The amplitude (7.4.153) only involves the metric $g_{\mu\nu}$ described by the higher derivative Lagrangian, which comprises the conformal term+GR+R^3 corrections (7.4.152). Therefore, looking at the structure of the double copy in (7.4.152) we see that this amplitude corresponds to the interaction of n spin-2 fields (i.e. $g_{\mu\nu}$) with higher derivative interactions. Hence, it is worth looking at the theories known as HD gravity. The latter can be defined by the following Lagrangian [113]:

$$\mathcal{L} = m_g^2 \sqrt{g} \left[\lambda_1 R(g) + \lambda_3 \left(\frac{1}{3} R^2 - R^{\mu\nu} R_{\mu\nu} \right) \right] + \mathcal{O}(R^3) \ , \tag{7.4.154}$$

with the following equations of motion

$$B_{\mu\nu} + \frac{\lambda_1}{\lambda_3} G_{\mu\nu} + \mathcal{O}(R^3) = 0 \ , \tag{7.4.155}$$

where $B_{\mu\nu}$ and $G_{\mu\nu}$ are the Bach and Einstein tensors, respectively. Some possible candidates for the $\mathcal{O}(R^3)$ operators in gravity can be found in [114]. The Lagrangian in (7.4.154) describes GR plus Weyl (conformal) interactions together with higher order curvature corrections $\mathcal{O}(R^3)$. Comparing this Lagrangian to the double copy structure in (7.4.152) we can see that the content and the type of interactions are the same.

Actually, the amplitudes for the closed bosonic string also referred to as GR+R^2+R^3 can be calculated by using the KLT relations and the open bosonic string amplitudes (6.3.34). The action up to order α'^3 was derived in [115]

$$S_{\substack{\text{closed} \\ \text{bosonic string}}} = -\frac{2}{\kappa^2} \int \mathrm{d}^d x \sqrt{g} \left\{ R - \tfrac{4}{d-2} (\partial_\mu \phi)^2 - \frac{1}{12} H^2 \right. \tag{7.4.156}$$

$$+ \frac{\alpha'}{4} e^{-2\phi} \left(R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right)$$
$$+ \alpha'^2 e^{-4\phi} \left(\frac{1}{16} R^{\mu\nu}{}_{\alpha\beta} R^{\alpha\beta}{}_{\lambda\rho} R^{\lambda\rho}{}_{\mu\nu} - \frac{1}{12} R^{\mu\nu}{}_{\alpha\beta} R^{\nu\lambda}{}_{\beta\rho} R^{\lambda\mu}{}_{\rho\alpha}\right) + \mathcal{O}(\alpha'^3)\Big\},$$

where ϕ represents the dilaton and $H_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]}$ is the field strength of the anti–symmetric B–field. Note, that in (7.4.156) the linear order in α' corresponds to the Gauss-Bonnet term which for $d = 4$ reduces to a topological surface term. In (7.4.152) there is also the higher order correction R^3, which in the bosonic closed string (7.4.156) originates at quadratic order in $\alpha'^2 R^3$. It should be emphasized that in higher–point gravitational amplitudes the order α'^2 cannot be reproduced from the double copy of the α' order of the corresponding open bosonic string amplitudes (6.3.34) due to additional $\alpha'^2\zeta_2 F^4$ contributions from a single open string sector [109, 116, 117].

It is worth mentioning that HD gravity may be related to bimetric gravity. The latter is constructed through interactions of two spin two fields $g_{\mu\nu}$ and $f_{\mu\nu}$ with a nonlinear interacting potential [18]. It has been shown [118] that by expanding the potential of bimetric gravity around a specific solution of $f_{\mu\nu}$ in terms of $g_{\mu\nu}$ and integrating out $f_{\mu\nu}$ one can match the Lagrangian of bimetric gravity to that of HD gravity. Therefore, the intersection number associated with the amplitude in (7.4.153) can also be related to some particular parameters of the integrated bimetric gravity known as partially massless bimetric gravity, cf. subsection 7.4.3.

Bimetric theory as a double copy

We discussed bimetric gravity in detail in section 4.4 there is a special limit of this theory known as partially massless (PM) bimetric theory, which is defined over the following background solution to eliminate $f_{\mu\nu}$ [118]

$$f_{\mu\nu} = a^2 g_{\mu\nu} + \frac{2\gamma}{m^2} P_{\mu\nu} + \mathcal{O}(m^{-4}) \ , \tag{7.4.157}$$

where $P_{\mu\nu}$ is the Schouten tensor defined as:

$$P_{\mu\nu} = \frac{1}{d-2}\left(R_{\mu\nu} - \frac{R}{2(d-1)} g_{\mu\nu}\right) \tag{7.4.158}$$

Plugging this relation back in the bimetric theory we can integrate out the field $f_{\mu\nu}$. In [118] it was shown that the kinetic term of the second metric $f_{\mu\nu}$ and the bimetric potential assume the following form (in arbitrary dimension):

$$\begin{aligned} m^2\sqrt{g}V(S;\beta_n) = m^2\sqrt{g}\sum_{n=0}^{4}\beta_n e_n(S) &\sim \Lambda_0\ \sqrt{g}[1+\kappa R(g)] \\ &+ \sqrt{g}\frac{\Lambda_1}{m^2}\ \left(\frac{d}{4(d-1)}R^2 - R^{\mu\nu}R_{\mu\nu}\right) + \mathcal{O}(R^3)\ , \\ \sqrt{f}R(f) \sim (\Lambda_2)^{d-2}\sqrt{g}R(g) &- \frac{\Lambda_3}{m^2}\sqrt{g}\ \left(\frac{d}{4(d-1)}R^2 - R^{\mu\nu}R_{\mu\nu}\right) + \mathcal{O}(R^3)\ . \end{aligned} \tag{7.4.159}$$

The parameters $\Lambda_i(\beta_n)$ are dimensionless. For simplicity, we absorbed other dimensionless parameters, e.g. γ in (7.4.157), in the parameters Λ_i. For details of the calculation we refer to [118]. We fix the parameters β_n in such a way that the term including the cosmological constant vanishes i.e. $\Lambda_0 = 0$. For the full fledged PM bimetric gravity, one has to fix all remaining β_n so that also $\Lambda_2(\beta_n) = 0$ [118]. The last two terms of (7.4.159) correspond to the CG Lagrangian (4.3.38). More concretely, one has the following identity, cf. (4.3.36)

$$(W_{\mu\nu\alpha\beta})^2 = R_{GB} + \left(\frac{d}{4(d-1)}R^2 - R^{\mu\nu}R_{\mu\nu}\right) , \tag{7.4.160}$$

with R_{GB} the Gauss-Bonnet term, which is a total derivative in four dimensions and can be discarded in the Lagrangian. After putting for $d = 4$ the relations (7.4.159) back into (4.4.46) we have evidence that there are two copies of CG together with the kinetic term for the first metric $g_{\mu\nu}$ described by GR. Concretely, we obtain:

$$\begin{aligned}\mathcal{L} = m_g^2 \sqrt{g} \Bigg[& R(g) + \alpha^2\Lambda_2^2 R(g) - \alpha^2 \frac{\Lambda_3}{m^2} \left(\frac{1}{3}R^2 - R^{\mu\nu}R_{\mu\nu}\right) \\ & - 2\,\frac{\Lambda_1}{m^2} \left(\frac{1}{3}R^2 - R^{\mu\nu}R_{\mu\nu}\right) \Bigg] + \mathcal{O}(R^3) \ .\end{aligned} \tag{7.4.161}$$

Combining the terms (7.4.161) into the action we reproduce (7.4.154) with:

$$\begin{aligned}\lambda_1 &= (1 + \alpha^2\Lambda_2^2) \ , \\ \lambda_3 &= -\frac{1}{m^2}\,(2\Lambda_1 + \alpha^2\Lambda_3) \ .\end{aligned} \tag{7.4.162}$$

Then, the resulting Lagrangian (7.4.154) describes a HD gravity with higher order corrections due to the bimetric theory. Calculating the equations of motions for the action (7.4.154) in the leading order[7] of m^2 one ends up with the same constraints (7.4.155) as in EW theory (cf. [119]) and the comment below equation (4.3.43). Recall, that we have eliminated $f_{\mu\nu}$ through (7.4.157) and obtained the Lagrangian (7.4.154). Therefore we can relate the intersection number (7.4.153) to the amplitude of the PM bimetric gravity.

Similar to EW theory PM bimetric gravity propagates seven degrees of freedom (packaged into the metric $g_{\mu\nu}$), which contain a massless spin-2 and a massive spin-2 states. In PM bimetric gravity the scattering amplitude $\mathcal{A}(n)$ describes the interaction of n spin-2 fields $g_{\mu\nu}$. The latter is described by the Lagrangian (7.4.154) comprising the conformal term+GR+R^3 corrections. Eventually, the amplitude $\mathcal{A}(n)$ may be computed by using (6.3.8) and is given by the intersection number (7.4.153). Looking at table 7.1 we may anticipate for PM bimetric gravity the double copy structure (7.4.152).

In addition to GR in (7.4.152) there are the two CG theories originating in (7.4.161). The latter can be combined in one single subject to (7.4.162), while the R^3 term is of higher order m^4. A final comment is that choosing the potential in bimetric action (4.4.46) as

$$V(\beta; S) \approx\ m^2\ [(f^\mu_\mu)^2 - f^{\mu\nu}f_{\mu\nu}] \tag{7.4.163}$$

[7]Note, that R^3 is of order m^4.

leads to Einstein–Weyl gravity. This has been pointed out in [56] and also commented below equation (4.3.43). Finally, a detailed discussion on string scattering amplitudes for full bimetric gravity with comments on their double copy structure can be found in [3, 4].

Chapter 8

Conclusions

In this thesis, we have studied Riemann surfaces and their applications to scattering amplitudes. We saw that Riemann surfaces appear as the natural underlying worldsheet of string theory amplitudes, CHY formalism and twisted intersection theory. We discussed the spectrum of the superstring and use the low energy limit ($\alpha' \to 0$) to produce effective actions. In particular, we calculated the string amplitude with massive external legs both with spin-2 states in open and closed string sector. Specifically, for the massive closed string state, we constructed for the first time the double copy of massive states in superstring theory. Furthermore, by taking the low energy limit we managed to produce the effective action of the massive spin-2 field, which we compared up to cubic order, with the massive gravity in the context of bimetric theory.

Cubic Effective Lagrangian

$$\begin{aligned} \mathcal{L}^{\text{eff}}_{\text{M}^3} &= \frac{g_o}{\alpha'}\Big\{\big[M^3\big] + 2\alpha' M^{\mu\nu}\big[\partial_\mu M_{\rho\sigma}\partial_\nu M^{\rho\sigma} - \kappa\,\partial_\nu M_{\rho\sigma}\partial^\sigma M^\rho_\mu\big] \\ &\qquad + 4\alpha'^2\,\partial^\mu\partial^\nu M_{\rho\sigma}\partial^\rho M^\kappa_\nu\partial^\sigma M_{\mu\kappa}\Big\}\,. \end{aligned} \tag{8.0.1}$$

$$\begin{aligned} \mathcal{L}^{\text{eff}}_{\text{GM}^2} &= g_c\Big[G^{\mu\nu}\big(\partial_\mu M_{\rho\sigma}\partial_\nu M^{\rho\sigma} - 4\partial_\nu M_{\rho\sigma}\partial^\sigma M^\rho_\mu\big) \\ &\qquad + M^{\mu\nu}\big(\partial_\mu G_{\rho\sigma}\partial_\nu M^{\rho\sigma} - \partial_\rho G_{\mu\sigma}\partial_\nu M^{\rho\sigma}\big)\Big] \end{aligned} \tag{8.0.2}$$

where the numeric factor κ is the only difference between closed and open strings. We have:

$$\begin{aligned} &\textit{Open string spin-2} \leftrightarrow \kappa = 4 \\ &\textit{Closed string spin-2} \leftrightarrow \kappa = 2 \\ &\textit{Bimetric potnetial} \leftrightarrow \kappa = 2 \end{aligned} \tag{8.0.3}$$

As a result, we observe that for closed string massive spin-2 state as double copy we have a perfect match up to a particular choice of parameters in bimetric theory.

Furthermore, by studying Riemann surfaces, we also observed that adding additional structure namely the differential twist ω on top of the standard de Rahm cohomology creates the twisted cohomology.

Twisted de Rahm cohomology

$$\nabla_\omega := d \pm \omega \wedge \tag{8.0.4}$$

Then, the (smooth) twisted de Rahm cohomology complex of X is $(\Omega_\omega(X)^p, \nabla_\omega)$. Denote the cohomology of $(\Omega_\omega(X)^p, \nabla_\omega)$ by H^p_ω we have

$$H^p_\omega(M, \nabla_{\pm\omega}) = \frac{\{\varphi \in \Omega^p_\omega(X) \mid \nabla_{\pm\omega}\varphi = 0\}}{\nabla_{\pm\omega}\Omega^{p-1}_\omega(M)} \tag{8.0.5}$$

Similar to any other mathematical description more structure gives us more expressive power. In the case of twisted homology/cohomology, we immediately see that one can describe closed and open string amplitudes in terms of intersection numbers of twisted forms.

Amplitudes in twisted cohomology

$$\begin{gathered}\mathcal{A}^{open}(k_1, k_2, k_3, ..., k_n) = \langle C_\gamma \otimes KN | \varphi_+ \rangle := \int_{C_\gamma} KN\ \varphi_+ \\ C_\gamma \otimes KN \in H_{n-3}(\mathcal{M}_{0,n}\nabla_\omega), \quad \varphi_+ \in H^{n-3}(\mathcal{M}_{0,n}, \nabla_\omega)\end{gathered} \tag{8.0.6}$$

$$\begin{gathered}\mathcal{A}^{closed}(k_1, k_2, k_3, ..., k_n) = \langle \bar{\varphi}_+ | \varphi_+ \rangle = \int_{\mathcal{M}_{0,n}} |KN|^2\ \bar{\varphi}_+ \wedge \varphi_+ \\ \bar{\varphi}_+ \in H^{n-3}_{\overline{\omega}}, \quad \varphi_+ \in H^{n-3}_\omega\end{gathered} \tag{8.0.7}$$

Exploiting this relation among string amplitudes and intersection numbers we produced new twisted forms $\tilde{\varphi}^{EYM}_{\pm,n;r}$ and $\tilde{\varphi}^{bosonic}_{\pm,n;r}$, while the former is associated with the superstring mixed open-closed amplitudes the latter with the bosonic mixed open-closed amplitudes. Then, we used the CHY formalism of scattering amplitudes to provide proof that these twisted forms produce a family of Einstein Yang-Mills and Weyl Yang-Mills type amplitudes, respectively.

In addition we used the twisted cohomology description of the amplitude relations and proposed a new method to understand the BCJ double copy. In particular, we evident that all theories with φ^{color} will elevade to a double copy with a KLT matrix.

Double copy in Intersection theory

For two theories T_1 and T_2 given by the intersection numbers

$$T_1 = \langle \varphi^1_+, \varphi^{color} \rangle \qquad T_2 = \langle \varphi^2_-, \varphi^{color} \rangle \tag{8.0.8}$$

we consider the following manipulations leading to a double copy expression

$$\begin{aligned}\langle\varphi_+^1,\varphi_-^2\rangle &= \sum_{a=1}^{(m-3)!}\langle PT^\vee(a),\varphi_+^1\rangle_\omega\ \langle\varphi_-^2,PT(a)\rangle_\omega \\ &= \sum_{a,b=1}^{(m-3)!}\langle\varphi_-^2,PT(a)\rangle_\omega\ S[a|b]\ \langle PT(b),\varphi_+^1\rangle_\omega \\ &= \sum_{a,b=1}^{(m-3)!}T_1(a)\ S[a|b]\ T_2(b) \qquad (8.0.9)\end{aligned}$$

with the intersection form or KLT matrix $S[a|b]$ given in (6.3.35).

Since the origin of the BCJ double copy is in the color-kinematic duality, this formulation can be used in the future to generalize the notion of the color-kinematic duality.

This study opens up more avenues of research in the field of amplitudes and Feynman integrals. For instance, the very first extension would be to ask what twisted forms describe massive string amplitudes. This question has its own challenges since the CHY formulation, that we discussed here, includes exclusively the massless amplitudes (because of its origins in ambitwistor strings). Therefore, one cannot provide a field theory amplitude directly. Conversely, it might open up a path to a formulation of CHY integral for massive amplitudes. One may wonder what would happen if we take the massive superstring amplitude, that we provided in this work for the bimetric effective action, and construct the twisted form associated with it. Furthermore one-loop double copies may be constructed in similar way as we did here starting from [120].

The other area of research involving intersection numbers is Feynman integrals as we mentioned in the chapter 3 intersection numbers can be related to the Euler integral representation of hypergeometric functions as well as the Baikov representation of the Feynman integrals [24]. The intersection numbers do not carry physical information on their own. Meaning, we can use them in different contexts. For example, one can write loop amplitudes integrals as intersection number of twisted forms and use the lower loop (e.g. one loop) twisted forms to describe higher loops [33].

Appendix A

Grassmann formulation of string Scattering amplitude

A.1 Preface

In this appendix we introduce the Grassmann variables, integrals and how to write vertex operators in terms of these variables. We mostly follow [10]. As we discussed, string amplitudes are described by a punctured Riemann surface with each puncture representing a vertex operator position associated to creation or annihilation of a string state. e.g.,the closed superstring tree–level amplitude assumes the form

$$\mathcal{A}(n) = \int\limits_{\mathcal{M}_{0,n}} d\mu_n \, \langle V(\varepsilon_1, p_1, z_1)...V(\varepsilon_n, p_n, z_n)\rangle = \int\limits_{\mathcal{M}_{0,n}} \prod_{i<j}^{n} |z_i - z_j|^{2\alpha' p_i \cdot p_j} \, |F(p, \varepsilon)|^2 \ , \tag{A.1.1}$$

with the measure (6.3.10) and the twisted gauge form (6.3.27):

$$F(p, \varepsilon) = d\mu_n \int \prod_{i=1}^{n} d\theta_i d\bar{\theta}_i \, \frac{\theta_k \theta_l}{z_k - z_l} \exp\left(\sum_{i \neq j} \frac{\theta_i \theta_j p_i \cdot p_j + \bar{\theta}_i \bar{\theta}_j \varepsilon_i \cdot \varepsilon_j + 2(\theta_i - \theta_j)\bar{\theta}_i \varepsilon_i \cdot p_j}{z_i - z_j - \alpha'^{-1} \theta_i \theta_j} \right) . \tag{A.1.2}$$

Here, we deal with the massless states of the closed and open superstring describing a graviton and a gluon, respectively. Scattering amplitudes of Mixed open and closed strings at tree level are described by a disk worldsheet. In the sequel we shall introduce some necessary tools for computing disk amplitudes. Later we shall use the latter to construct twisted forms for EYM amplitudes. For this it is instrumental that we look at the standard vertex operators of superstring theory in terms of Graßmann variables [10].

A.2 Vertex operators

The open string vertex positions x_i are located at the boundary of the disk, while closed string vertex positions z_i are inserted in the bulk of the disk. The momentum and polarization of a massless gluon are given by k_ρ and ϵ_μ, respectively. Similarly, for a massless graviton we have the left– and right–moving momenta $q_\rho, \tilde{q}_\rho$ and the polarization $\epsilon_{\mu\nu}$. To cancel the ghost background charge on the genus zero Riemann surface we define their vertex operators in the zero and (-1)–ghost pictures. We have the following list of vertex operators:

- Open string vertex operator in zero–ghost picture[1]:

$$V_o^{(0)}(x,\varepsilon,k) = \varepsilon_\mu \left[\partial X^\mu + \alpha'(k\cdot\psi)\psi^\mu\right] e^{ik\cdot X} \\ \longrightarrow \int d\theta d\bar{\theta}\ \exp\left\{ik\cdot X + \theta\bar{\theta}\varepsilon\cdot\partial X + \theta\sqrt{\alpha'}k\cdot\psi + \bar{\theta}\sqrt{\alpha'}\varepsilon\cdot\psi\right\} \tag{A.2.3}$$

- Closed string vertex operator in $(0,0)$–ghost picture:

$$V_c^{(0,0)}(z,\overline{z},\varepsilon,q) = \varepsilon_{\mu\nu}\left[i\overline{\partial}\widetilde{X}^\mu + \frac{\alpha'}{2}(\widetilde{q}\widetilde{\psi})\widetilde{\psi}^\mu(\overline{z})\right]\left[i\partial X^\nu + \frac{\alpha'}{2}(q\psi)\psi^\nu(z)\right] e^{iqX(z,\overline{z})} \\ \longrightarrow \int d\theta_1 d\bar{\theta}_1 d\theta_2 d\bar{\theta}_2\ \exp\left\{iq\cdot X + i\widetilde{q}\cdot\widetilde{X} + \theta_1\bar{\theta}_1\varepsilon\cdot\partial X + \theta_1\sqrt{\alpha'}q\cdot\psi \right. \\ \left. + \bar{\theta}_1\sqrt{\alpha'}\varepsilon\cdot\psi + \theta_2\bar{\theta}_2\varepsilon\cdot\overline{\partial}\widetilde{X} + \theta_2\sqrt{\alpha'}\widetilde{q}\cdot\widetilde{\psi} + \bar{\theta}_2\sqrt{\alpha'}\varepsilon\cdot\widetilde{\psi}\right\}\ . \tag{A.2.4}$$

- Open string vertex operator in the (-1)–ghost picture:

$$V_o^{(-1)}(x,\alpha,k) = \varepsilon_\mu\ e^{-\phi(x)}\psi^\mu(x)\ e^{ik\cdot X(x)} \longrightarrow \int d\theta \exp\left\{ik\cdot X + \theta\varepsilon\cdot\psi\right\}\ . \tag{A.2.5}$$

- Closed string vertex operator in the $(-1,-1)$ picture:

$$V_c^{(-1,-1)}(z,\overline{z},\varepsilon,q) = \varepsilon_{\mu\nu}\ e^{-\bar{\phi}(\bar{z})}\widetilde{\psi}^\mu(\bar{z})\ e^{-\phi(z)}\psi^\nu(z)\ e^{iq\cdot X(z,\bar{z})} \\ \longrightarrow \int d\theta d\bar{\theta}\ \exp\left\{iq\cdot X + i\widetilde{q}\cdot\widetilde{X} + \bar{\theta}\widetilde{\varepsilon}\cdot\widetilde{\psi} + \theta\varepsilon\cdot\psi\right\}\ . \tag{A.2.6}$$

Finally, for the string S–matrix the following on–shell conditions must be imposed:

$$\begin{aligned} &k^2 = q^2 = 0\ , \\ &k^\mu\varepsilon_\mu = 0\ , \qquad q^\mu\cdot\varepsilon_{\mu\nu} = 0\ , \qquad \varepsilon^\mu_\mu = 0\ . \end{aligned} \tag{A.2.7}$$

These are the standard massless, transverse and traceless conditions of on–shell string states.

[1]The map of the string vertex operators to their Graßmann representations is also given. For this we have removed the ghost field ϕ by noting that the total ghost charge of (-2) in the amplitude must be contributed by choosing those ghost pictures such that a factor $\langle e^{-\phi(x_i)}e^{-\phi(x_j)}\rangle = (x_i - x_j)^{-1}$ is exhibited.

Appendix B

EYM string amplitude

B.1 The case of one graviton EYM amplitude

In this section, we perform the first steps towards extending our result (6.4.76) for two gluons and one graviton to the generic case by increasing the number of gluons, i.e. open strings in the underlying string correlator (6.4.45). To simplify the Graßmann integral (6.4.51) we perform similar steps, which took us from (6.4.59) to (6.4.67). In addition, following subsection 6.4.2 for the open and closed string states we introduce the unifying notation (subject to $\varepsilon_{n+1} = \widetilde{\varepsilon}_{n+1}$):

$$\begin{aligned} \{\xi_1, ..., \xi_{n+2}\} &:= \{\varepsilon_1, \varepsilon_2, ..., \varepsilon_n, \varepsilon_{n+1}, \widetilde{\varepsilon}_{n+1}\} \ , \\ \{p_1, ..., p_{n+2}\} &:= \{k_1, k_2 ..., k_n, q_1, \widetilde{q}_1\} \ . \end{aligned}$$

With these preparations we can write (6.4.51) as

$$\begin{aligned} \mathcal{I}(n;1) = \mathcal{C}(1,2,...,n) & \int \prod_{i=1}^{n+2} d\theta_i d\bar{\theta}_i \ \frac{\theta_1\theta_2}{z_1-z_2} KN \cdot \overline{KN} \\ & \times \exp\left\{ \alpha'^2 \left(\sum_{\substack{i,j=1 \\ i\neq j}}^{n+1} \frac{(\bar{\theta}_i\theta_j\xi_i\cdot p_j)}{z_i - z_j \mp \alpha'^{-1}\theta_i\theta_j} + \sum_{i>j}^{n+1} \frac{(\bar{\theta}_i\bar{\theta}_j\xi_i\cdot\xi_j)}{z_i - z_j \mp \alpha'^{-1}\theta_i\theta_j} \right.\right. \\ & + \sum_{\substack{i,j=1 \\ i\neq j}}^{n+1} \frac{(\theta_i\bar{\theta}_i\xi_i\cdot p_j)}{z_i - z_j \mp \alpha'^{-1}\theta_i\theta_j} + \sum_{i>j}^{n+1} \frac{(\theta_i\theta_j p_i\cdot p_j)}{z_i - z_j \mp \alpha'^{-1}\theta_i\theta_j} \\ & \left.\left. + \sum_{j=1}^{n} \frac{(\theta_{n+2}\overline{\theta}_{n+2}\xi_{n+2}\cdot p_j)}{\overline{z}_{n+1} - \overline{z}_j \mp \alpha'^{-1}\theta_{n+2}\theta_j} \right) \right\} \ . \end{aligned} \tag{B.1.1}$$

Let us make some comments. Firstly, $KN \cdot \overline{KN}$ is the Koba–Nielsen factor (6.3.11) for $n+1$ closed strings. Secondly, comparing (B.1.1) with (6.4.67) we evidence that the main difference is the additional last term in the exponential of (B.1.1). It accounts for the new type of interactions between the anti–holomorphic field $\bar{\partial}X$ from the single graviton vertex and the anti–holomorphic open string fields $e^{ik_i\tilde{X}(\bar{z}_i)}$ from the gluon vertices subject to the map (6.4.43). On the other hand, the disk amplitude (5.6.145) involves additional interactions between holomorphic and anti–holomorphic fields which are accounted for by the additional terms produced by the map (6.4.43).

At any rate, due to special kinematics of the three–point amplitude (5.6.144) to arrive at (6.4.67) we did not apply the embedding (6.4.43), but simply rewrote the disk integrand (6.4.59) in terms of (A.1.2). Nevertheless, applying the map (6.4.43) at (6.4.59) yields the same result (6.4.67). In order to see this let us look at the equation (B.1.1), which for the case $n=2$ and $KN=1$ becomes:

$$\begin{aligned}\mathcal{I}(2;1)=\mathcal{C}(1,2)\ &\int\prod_{i=1}^{4}d\theta_i d\bar{\theta}_i\ \frac{\theta_1\theta_2}{z_1-z_2}\\ \times\exp&\left\{\alpha'^2\left(\sum_{\substack{i,j=1\\ i\neq j}}^{3}\frac{(\bar{\theta}_i\theta_j\xi_i\cdot p_j)}{z_i-z_j\mp\alpha'^{-1}\theta_i\theta_j}+\sum_{i>j}^{3}\frac{(\bar{\theta}_i\bar{\theta}_j\xi_i\cdot\xi_j)}{z_i-z_j\mp\alpha'^{-1}\theta_i\theta_j}\right.\right.\\ &\left.\left.+\sum_{\substack{i,j=1\\ i\neq j}}^{3}\frac{(\theta_i\bar{\theta}_i\xi_i\cdot p_j)}{z_i-z_j\mp\alpha'^{-1}\theta_i\theta_j}+\sum_{i>j}^{3}\frac{(\theta_i\theta_j p_i\cdot p_j)}{z_i-z_j\mp\alpha'^{-1}\theta_i\theta_j}+\sum_{j=1}^{2}\frac{(\theta_4\overline{\theta}_4\xi_4\cdot p_j)}{\overline{z}_3-\overline{z}_j\mp\alpha'^{-1}\theta_4\theta_j}\right)\right\}\ .\end{aligned}\tag{B.1.2}$$

Expanding the exponential in (B.1.2) w.r.t.,the fermionic variables and using the tracelessness condition of the graviton polarization gives:

$$\begin{aligned}\mathcal{I}(2;1)=\mathcal{C}(1,2)\ &\int\prod_{i=1}^{4}\frac{\theta_1\theta_2}{z_1-z_2}d\theta_i d\bar{\theta}_i\ \left\{\frac{(\bar{\theta}_1\theta_3\varepsilon_1\cdot q)(\overline{\theta}_3\bar{\theta}_2\varepsilon_3\cdot\varepsilon_2)\overline{z}_{21}(\overline{\theta}_4\theta_4\widetilde{\varepsilon}_3\cdot k_1)}{(z_1-z_3)(z_2-z_3)(\overline{z}_1-\overline{z}_3)(\overline{z}_2-\overline{z}_3)}\right.\\ &\left.+\frac{(\bar{\theta}_2\theta_3\varepsilon_2\cdot q)(\overline{\theta}_3\overline{\theta}_1\varepsilon_3\cdot\varepsilon_1)\overline{z}_{21}(\bar{\theta}_4\theta_4\widetilde{\varepsilon}_3\cdot k_1)}{(z_1-z_3)(z_2-z_3)(\overline{z}_1-\overline{z}_3)(\overline{z}_2-\overline{z}_3)}+\frac{(\overline{\theta}_1\overline{\theta}_2\varepsilon_1\cdot\varepsilon_2)(\overline{\theta}_3\theta_3\varepsilon_3\cdot k_1)\overline{z}_{21}(\overline{\theta}_4\theta_4\widetilde{\varepsilon}_3\cdot k_1)}{(z_1-z_3)(z_2-z_3)(\overline{z}_1-\overline{z}_3)(\overline{z}_2-\overline{z}_3)}\right\}\ .\end{aligned}\tag{B.1.3}$$

Up to the color factor (6.4.75) the expression (B.1.3) agrees with (6.4.60) subject to the reality condition $z_1=\bar{z}_1$ and $z_2=\bar{z}_2$ of the two open string positions on the disk.

Now, we proceed similar as in subsection 6.4 by making profit of the KLT like construction of EYM amplitudes established by the map (6.4.43). We shall construct from our integrand (B.1.1) a pair of twisted forms φ_+,φ_-, whose intersection number $\langle\varphi_+,\varphi_-\rangle_\omega$ will give, in its $\alpha'\to\infty$ limit, the EYM amplitude formula of the CHY formalism (4.3.15). For the case of n gluons and one graviton the latter yields the integrand:

$$\mathcal{I}_{n+1}(1,2,\ldots,n;q)=\mathcal{C}(1,2,\ldots,n)\ C_{qq}\ \mathrm{Pf}'\Psi_{n+1}(k_a,q,\varepsilon,\sigma)\ .\tag{B.1.4}$$

Here, $\mathcal{C}(1,2,\ldots,n)$ is the Parke–Taylor factor (6.3.9) and C_{qq} is defined as:

$$C_{qq}=\sum_{l=1}^{n-1}(\varepsilon_q\cdot x_l)\ \frac{\sigma_{l,l+1}}{\sigma_{l,q}\sigma_{l+1,q}}=\mathrm{Pf}\Psi_{S_1}\equiv\mathrm{Pf}\Psi_1\ ,\tag{B.1.5}$$

The latter will be identified with the gauge correlator (6.4.50) of the integrand (B.1.1), which will be associated to the twisted form $\tilde{\varphi}^{EYM}_{\pm,n;1}$ in the same way as proposed in subsection 6.4. To find the correct description of $C_{qq} \equiv \mathrm{Pf}\Psi_1$ and $\mathrm{Pf}'\Psi_{n+1}(k_a, q, \varepsilon, \sigma)$ we proceed as follows.

First, note that the denominators in the exponential proliferate terms subleading in α', in particular:

$$\frac{\bar{\theta}_i\bar{\theta}_j(\xi_i\cdot\xi_j)}{z_i - z_j \mp \alpha'^{-1}\theta_i\theta_j} = \frac{\bar{\theta}_i\bar{\theta}_j(\xi_i\cdot\xi_j)}{z_i - z_j} \pm \alpha'^{-1}\ \frac{\theta_i\theta_j\bar{\theta}_i\bar{\theta}_j(\xi_i\cdot\xi_j)}{(z_i-z_j)^2}\ .$$

With this information the integrand (B.1.1) can be cast into the following form:

$$\begin{aligned}\mathcal{I}(n;1) = \mathcal{C}(1,2,\ldots,n)\ &\int\prod_{i=1}^{n+2} d\theta_i d\bar{\theta}_i\ \frac{\theta_1\theta_2}{z_1-z_2}\quad \exp\left\{\alpha'^2\ \psi_1\theta_{n+2}\overline{\theta}_{n+2}\right\}|KN|^2 \\ \times \exp&\left\{\frac{1}{2}\alpha'^2\sum_{i,j=1}^{n+1}\begin{pmatrix}\theta_i\\ \bar{\theta}_i\end{pmatrix}^t \Psi_{n+1}\begin{pmatrix}\theta_j\\ \bar{\theta}_j\end{pmatrix} \pm \frac{1}{2}\alpha'\ \sum_{\substack{i,j=1\\ i\neq j}}^{n+1}\frac{\theta_i\theta_j\bar{\theta}_i\bar{\theta}_j(\xi_i\cdot\xi_j)}{(z_i-z_j)^2}\right\}\ .\end{aligned} \tag{B.1.6}$$

In (B.1.6) we have introduced $(2n+2)\times(2n+2)$ matrix (4.2.11)

$$\Psi_{n+1} \equiv \Psi_{S_{n+1}} = \begin{pmatrix}A_{ij} & -C_{ji}\\ C_{ij} & B_{ij}\end{pmatrix}\Bigg|_{\sigma_l=z_l}\quad ,\quad S_{n+1} = \{1,\ldots,n,n+1\}\ , \tag{B.1.7}$$

with the $(n+1)\times(n+1)$ matrices A, B and C being the same as (4.2.7) with σ_l replaced by $z_l,\ l{=}1,\ldots,n+1$. Furthermore, we have defined the 2×2 matrix Ψ_1:

$$\Psi_1 = \begin{pmatrix}0 & \psi_1\\ -\psi_1 & 0\end{pmatrix}\quad ,\quad \psi_1 = \sum_{j=1}^{n}\frac{(\xi_{n+2}p_j)}{\overline{z}_{n+1}-\overline{z}_j}\ . \tag{B.1.8}$$

Similar as in (6.4.62) we can construct the $(2n+4)\times(2n+4)$ matrix $\Psi^{n,1}$ of holomorphic and anti holomorphic block structure

$$\Psi^{(n;1)} = \Psi_{n+1}\otimes\Psi_1 \tag{B.1.9}$$

to write the full order α'^2 of the exponential of (B.1.6) in terms of the concatenation:

$$\sum_{i,j=1}^{n+2}(\theta_j\ \bar{\theta}_j)\ \Psi^{(n;1)}\begin{pmatrix}\theta_i\\ \bar{\theta}_i\end{pmatrix} := \sum_{i,j=1}^{n+1}(\theta_j\ \bar{\theta}_j)\Psi_{n+1}\begin{pmatrix}\theta_i\\ \bar{\theta}_i\end{pmatrix} + (\theta_{n+2}\ \bar{\theta}_{n+2})\Psi_1\begin{pmatrix}\theta_{n+2}\\ \bar{\theta}_{n+2}\end{pmatrix}\ . \tag{B.1.10}$$

Eventually, with this block structure we are now able to construct our pair of twisted forms. We take the holomorphic part of (B.1.6) (described by Ψ_{n+1} and the subleading term) supplemented by the Parke–Taylor factor (6.4.50) to amount to the form $\varphi^{EYM}_{\pm,n;1}$:

$$\begin{aligned}\varphi^{EYM}_{\pm,n;1} = d\mu_{n+1}\ &\int\prod_{i=1}^{n+1}\frac{\theta_1\theta_2}{z_1-z_2}d\theta_i d\bar{\theta}_i \exp\left\{\frac{1}{2}\alpha'^2\sum_{i,j=1}^{n+1}(\theta_j\ \bar{\theta}_j)\Psi_{n+1}\begin{pmatrix}\theta_i\\ \bar{\theta}_i\end{pmatrix}\right\} \\ \times \exp&\left\{\pm\frac{1}{2}\alpha'\ \sum_{\substack{i,j=1\\ i\neq j}}^{n+1}\frac{\theta_i\theta_j\bar{\theta}_i\bar{\theta}_j(\xi_i\cdot\xi_j)}{(z_i-z_j)^2}\right\}\ .\end{aligned} \tag{B.1.11}$$

Furthermore, we extract the anti–holomorphic part of (B.1.6) and apply the isomorphism (6.4.71) to arrive at our second twisted form twisted form $\widetilde{\varphi}^{EYM}_{\pm,n;1}$:

$$\widetilde{\varphi}^{EYM}_{\pm,n;\mathbf{1}} = d\mu_{n+1}\ \mathcal{C}(1,2,\ldots,n) \int d\theta_{n+2} d\bar{\theta}_{n+2}\ \exp\left\{ \alpha'^2\ (\theta_{n+2}\ \bar{\theta}_{n+2})\Psi_1 \begin{pmatrix} \theta_{n+2} \\ \bar{\theta}_{n+2} \end{pmatrix} \right\}\Bigg|_{\overline{z_l}\to z_l} . \tag{B.1.12}$$

The two twisted forms (B.1.11) and (B.1.12) ($\varphi_- = \widetilde{\varphi}^{EYM}_{\pm,n;1}$ and $\varphi_+ = \varphi^{EYM}_{-,n;1}$) are our candidates for reproducing the EYM integrand (B.1.4). In fact, with (B.1.11) and (B.1.12) we are now able to derive the EYM amplitude from (6.3.23). The twisted forms in equations (B.1.11) and (B.1.12) are of the appropriate form for taking the $\alpha' \to \infty$ limit:

$$\begin{aligned}
\lim_{\alpha'\to\infty} \hat{\varphi}^{EYM}_{\pm,n;1} &= \int \prod_{i=1}^{n+1} d\theta_i d\bar{\theta}_i\ \frac{\theta_1\theta_2}{z_1 - z_2}\ \exp\left\{ \frac{1}{2}\alpha'^2 \sum_{i,j=1}^{n+1} (\theta_j\ \bar{\theta}_j)\Psi_{n+1}\ \begin{pmatrix} \theta_i \\ \bar{\theta}_i \end{pmatrix} \right\} + \mathcal{O}(\alpha'^{-1}) \\
&= \frac{\mathrm{Pf}\Psi^{12}_{n+1}}{z_1 - z_2} + \mathcal{O}(\alpha'^{-1}) = \mathrm{Pf}'\Psi_{n+1} + \mathcal{O}(\alpha'^{-1})\ , \\
\lim_{\alpha'\to\infty} \hat{\widetilde{\varphi}}^{EYM}_{\pm,n;1} &= \mathcal{C}(1,2,\ldots,n) \int d\theta_{n+2} d\bar{\theta}_{n+2}\ \ \exp\left\{ \alpha'^2\ (\theta_{n+2}\ \bar{\theta}_{n+2})\Psi_1 \begin{pmatrix} \theta_{n+2} \\ \bar{\theta}_{n+2} \end{pmatrix} \right\}\Bigg|_{\overline{z_l}\to z_l} \\
&= \mathcal{C}(1,2\ldots,n)\ \mathrm{Pf}\Psi_1\big|_{\overline{z_l}\to z_l} .
\end{aligned} \tag{B.1.13}$$

Plugging the expressions (6.4.50) and (B.1.13) into (6.3.23) yields the final expression for the EYM amplitude:

$$\begin{aligned}
\mathcal{A}(1,2,\ldots,n;1) &= \lim_{\alpha'\to\infty} \langle \widetilde{\varphi}^{EYM}_{+,n;1}, \varphi^{EYM}_{-,n;1} \rangle_\omega = \int\limits_{\mathcal{M}_{0,n+1}} d\mu_{n+1} \prod_{a=1}^{n+1}{}' \delta(f_a) \lim_{\alpha'\to\infty} \hat{\varphi}^{EYM}_{n;1} \hat{\widetilde{\varphi}}^{EYM}_{n;1} \\
&= \int\limits_{\mathcal{M}_{0,n+1}} d\mu_{n+1} \prod_{a=1}^{n+1}{}' \delta(f_a)\ \frac{\mathrm{Pf}\Psi_1\big|_{\overline{z_l}\to z_l}\ \mathrm{Pf}'\Psi_{n+1}}{(z_1 - z_2)(z_2 - z_3)\ldots(z_n - z_1)} .
\end{aligned} \tag{B.1.14}$$

In the leading order in α' this result agrees with the expression given in (4.3.15) for the Einstein Yang–Mills amplitude in the CHY formalism. The terms $\mathcal{O}(\alpha'^{-1})$ subleading in α' originate from the limit (B.1.13).

B.2 The generic case of (n, r) EYM amplitude

In this section we shall determine the pair of twisted forms $\varphi^{EYM}_{\pm,n;r}$ and $\widetilde{\varphi}^{EYM}_{\pm,n;r}$ for the multi–graviton case. To illuminate the structure and changes compared to the one–graviton case we first derive the amplitude involving n gluons and two gravitons. Equipped with these preparations we then move to the generic case of n gluons and r gravitons.

B.2.1 Twisted form and intersections for amplitudes of n gluons and two gravitons

In this subsection we consider the disk amplitude (6.4.44) involving n gluons and two gravitons. Subject to (6.4.43) we embed the latter onto the sphere and arrive at (6.4.45). As already anticipated above for the construction of the twisted differentials we do not need the position integrals and may focus on the integrand (6.4.51) written in terms of Graßmann variables. For the case at hand we introduce the unifying notation for polarizations ξ_i, momenta p_j and positions ζ_l exhibited in 6.4.2, more precisely (subject to $\tilde{\varepsilon}_{n+i} = \varepsilon_{n+i},\ i = 1,\ldots,r$)

$$\begin{aligned}
&\{\zeta_1,...,\zeta_{n+4}\} = \{z_1, z_2, ..., z_n, z_{n+1}, z_{n+1}, z_{n+2}, z_{n+2}\}\ ,\\
&\{\xi_1,...,\xi_{n+4}\} := \{\varepsilon_1, \varepsilon_2, ..., \varepsilon_n, \varepsilon_{n+1}, \widetilde{\varepsilon}_{n+1}, \varepsilon_{n+2}, \widetilde{\varepsilon}_{n+2}\}\ ,\\
&\{p_1,...,p_{n+4}\} := \{k_1, k_2..., k_n, q_1, \widetilde{q}_1, q_2, \widetilde{q}_2\}\ ,
\end{aligned} \tag{B.2.15}$$

and perform the field contractions using the correlators on the sphere. With these preparations the integrand (6.4.51) assumes the following form:

$$\begin{aligned}
\mathcal{I}(n;2) = \mathcal{C}(1,2,\ldots,n)\ &\int \prod_{i=1}^{n+4} d\theta_i d\bar{\theta}_i\ \frac{\theta_1\theta_2}{\zeta_1-\zeta_2}\\
\times \exp\Bigg\{\alpha'^2\Bigg(&\sum_{\substack{i,j=1\\ i\neq j\\ i,j\neq n+2}}^{n+3} \frac{(\bar{\theta}_i\theta_j\xi_i\cdot p_j)}{\zeta_i-\zeta_j\mp\alpha'^{-1}\theta_i\theta_j} + \sum_{\substack{i,j=1\\ i>j\\ i,j\neq n+2}}^{n+3} \frac{(\bar{\theta}_i\bar{\theta}_j\xi_i\cdot\xi_j)}{\zeta_i-\zeta_j\mp\alpha'^{-1}\theta_i\theta_j}\\
&+ \sum_{\substack{i,j=1\\ i\neq j\\ i,j\neq n+2}}^{n+3} \frac{(\theta_i\bar{\theta}_i\xi_i\cdot p_j)}{\zeta_i-\zeta_j\mp\alpha'^{-1}\theta_i\theta_j} + \sum_{\substack{i,j=1\\ i>j\\ i,j\neq n+2}}^{n+3} \frac{(\theta_i\theta_j p_i\cdot p_j)}{\zeta_i-\zeta_j\mp\alpha'^{-1}\theta_i\theta_j}\\
&+ \sum_{\substack{i,j\in\{n+2,n+4\}\\ i\neq j}} \frac{(\bar{\theta}_i\theta_j\xi_i\cdot p_j)}{\overline{\zeta}_i-\overline{\zeta}_j\mp\alpha'^{-1}\theta_i\theta_j} + \sum_{\substack{j=1\\ j\notin\{n+1,n+3\}\\ i\in\{n+2,n+4\}}}^{n+4} \frac{(\theta_i\bar{\theta}_i\xi_i\cdot p_j)}{\overline{\zeta}_i-\overline{\zeta}_j\mp\alpha'^{-1}\theta_i\theta_j}\\
&+ \sum_{\substack{j=n+2\\ i=n+4}} \frac{(\bar{\theta}_i\bar{\theta}_j\xi_i\cdot\xi_j)}{\overline{\zeta}_i-\overline{\zeta}_j\mp\alpha'^{-1}\theta_i\theta_j} + \sum_{\substack{j=n+2\\ i=n+4}} \frac{(\theta_i\theta_j p_i\cdot p_j)}{\overline{\zeta}_i-\overline{\zeta}_j\mp\alpha'^{-1}\theta_i\theta_j}\Bigg)\Bigg\}\ \times KN\cdot\overline{KN}\ .
\end{aligned} \tag{B.2.16}$$

Note, that in (B.2.16) the first four terms in the exponential account for the holomorphic field contractions, while the last four terms represent the anti–holomorphic field contractions subject to the map (6.4.43). Furthermore, $KN\cdot\overline{KN}$ is the Koba–Nielsen factor (6.3.11) for $n+2$ closed strings. In the following we explicitly outline the steps leading to (B.2.16) and discuss the underlying matrix structure w.r.t.,the fermionic variables. First of all, we discuss the anti–holomorphic sector of the two–graviton case. Looking at α'^2 order terms we encounter two contributions. Firstly, there is the correlator of the bosonic part of one graviton vertex (A.2.4) with both the open string exponentials stemming from the map (6.4.43) and a single exponential from the second graviton (with $z_{l,m} = z_l - z_m$):

$$\begin{aligned}\langle\widetilde{\varepsilon}^{n+1}_\mu\overline{\partial}\widetilde{X}^\mu(\overline{z}_{n+1})\left(\prod_{i=1}^n e^{ik_i\widetilde{X}(\overline{z}_i)}\right)\times e^{iq_2\widetilde{X}(\overline{z}_{n+2})}\rangle &= i\left(\sum_{i=1}^n\frac{\widetilde{\varepsilon}^{n+1}\cdot k_i}{\overline{z}_i-\overline{z}_{n+1}}\right)+i\frac{\widetilde{\varepsilon}^{n+1}\cdot q_2}{\overline{z}_{n+2}-\overline{z}_{n+1}}\\
&= i\left(\sum_{i=1}^{n-1}\frac{\widetilde{\varepsilon}^{n+1}\cdot k_i}{\overline{z}_i-\overline{z}_{n+1}}\right)+i\frac{\widetilde{\varepsilon}^{n+1}\cdot k_n}{\overline{z}_n-\overline{z}_{n+1}}+i\frac{\widetilde{\varepsilon}^{n+1}\cdot q_2}{\overline{z}_{n+2}-\overline{z}_{n+1}}\\
&= i\sum_{i=1}^{n-1}\frac{\widetilde{\varepsilon}^{n+1}\cdot k_i}{\overline{z}_i-\overline{z}_{n+1}}-i\frac{\widetilde{\varepsilon}^{n+1}\cdot\sum_{j=1}^{n-1}k_j+q_2}{\overline{z}_n-\overline{z}_{n+1}}+i\frac{\widetilde{\varepsilon}^{n+1}\cdot q_2}{\overline{z}_{n+2}-\overline{z}_{n+1}}\\
&= i\sum_{i=1}^{n-1}\frac{\widetilde{\varepsilon}^{n+1}\cdot k_i}{\overline{z}_i-\overline{z}_{n+1}}-i\sum_{j=1}^{n-1}\frac{\widetilde{\varepsilon}^{n+1}\cdot k_j}{\overline{z}_n-\overline{z}_{n+1}}+i\frac{(\widetilde{\varepsilon}^{n+1}\cdot q_2)\overline{z}_{n,n+2}}{\overline{z}_{n+2,n+1}\overline{z}_{n,n+1}}\\
&= i\sum_{i=1}^{n-1}(\widetilde{\varepsilon}^{n+1}\cdot k_i)\frac{\overline{z}_{ni}}{(\overline{z}_i-\overline{z}_{n+1})(\overline{z}_n-\overline{z}_{n+1})}+i\frac{(\widetilde{\varepsilon}^{n+1}\cdot q_2)\overline{z}_{n,n+2}}{\overline{z}_{n+2,n+1}\overline{z}_{n,n+1}}\\
&= -i\sum_{i=1}^{n-1}(\widetilde{\varepsilon}^{n+1}\cdot k_i)\sum_{l=i}^{n-1}\frac{\overline{z}_{l,l+1}}{(\overline{z}_l-\overline{z}_{n+1})(\overline{z}_{l+1}-\overline{z}_{n+1})}+i\frac{(\widetilde{\varepsilon}^{n+1}\cdot q_2)\overline{z}_{n,n+2}}{\overline{z}_{n+2,n+1}\overline{z}_{n,n+1}}\\
&= i\left\{\sum_{l=1}^{n-1}(\widetilde{\varepsilon}^{n+1}\cdot x_l)\frac{\overline{z}_{l,l+1}}{\overline{z}_{l,n+1}\overline{z}_{n+1,l+1}}+(\widetilde{\varepsilon}^{n+1}\cdot q_2)\frac{\overline{z}_{n+2,n}}{\overline{z}_{n+2,n+1}\overline{z}_{n+1,n}}\right\}=:\Xi^{n+2}\ .\end{aligned}\tag{B.2.17}$$

The above correlator Ξ^{n+2} amounts to the generalized version of the expression (B.1.5) for the case of more than one graviton. In the last line of (B.2.17) the first term in the sum is exactly (B.1.5) and the second term takes into account the additional interaction with the second graviton. For Ξ^l the upper index l indicates the corresponding pair of Graßmann variables in the anti–holomorphic sector. For the case at hand we have

$$\langle\theta_{n+2}\overline{\theta}_{n+2}\ \widetilde{\varepsilon}^{n+1}_{\mu_1}\ \overline{\partial}\widetilde{X}^{\mu_1}(\overline{z}_{n+1})\left(\prod_{i=1}^n e^{ik_i\widetilde{X}(\overline{z}_i)}\right)\times e^{iq_{2\mu_2}\widetilde{X}^{\mu_2}(\overline{z}_{n+2})}\rangle=\theta_{n+2}\overline{\theta}_{n+2}\ \Xi^{n+2}\ ,\tag{B.2.18}$$

and similarly for the second graviton we encounter:

$$\langle\theta_{n+4}\overline{\theta}_{n+4}\ \widetilde{\varepsilon}^{n+2}_{\mu_2}\ \overline{\partial}\widetilde{X}^{\mu_2}(\overline{z}_{n+2})\left(\prod_{i=1}^n e^{ik_i\widetilde{X}(\overline{z}_i)}\right)\times e^{iq_{1\mu_1}\widetilde{X}^{\mu_1}(\overline{z}_{n+1})}\rangle=\theta_{n+4}\overline{\theta}_{n+4}\ \Xi^{n+4}\ .\tag{B.2.19}$$

Secondly, there is the fermionic piece from the anti–holomorphic parts of the two graviton vertex operators (A.2.4):

$$\begin{aligned}&\theta_{n+2}\overline{\theta}_{n+2}\theta_{n+4}\overline{\theta}_{n+4}\ \langle\widetilde{\varepsilon}^{n+1}_{\mu_1}\widetilde{\psi}^{\mu_1}q_1\cdot\widetilde{\psi}\ \widetilde{\varepsilon}^{n+2}_{\mu_2}\widetilde{\psi}^{\mu_2}q_2\cdot\widetilde{\psi}\rangle_{S_2}\\
&\quad=\theta_{n+2}\overline{\theta}_{n+2}\theta_{n+4}\overline{\theta}_{n+4}\left\{-\frac{\widetilde{\varepsilon}_{n+1}\cdot\widetilde{\varepsilon}_{n+2}}{\overline{z}_{n+1}-\overline{z}_{n+2}}\frac{q_1\cdot q_2}{\overline{z}_{n+1}-\overline{z}_{n+2}}+\frac{\widetilde{\varepsilon}_{n+1}\cdot q_2}{\overline{z}_{n+1}-\overline{z}_{n+2}}\frac{\widetilde{\varepsilon}_{n+2}\cdot q_1}{\overline{z}_{n+1}-\overline{z}_{n+2}}\right\}\\
&\qquad=-\left(\overline{\theta}_{n+2}\overline{\theta}_{n+4}\frac{\widetilde{\varepsilon}_{n+1}\cdot\widetilde{\varepsilon}_{n+2}}{\overline{z}_{n+1}-\overline{z}_{n+2}}\right)\ \left(\theta_{n+2}\theta_{n+4}\frac{q_1\cdot q_2}{\overline{z}_{n+1}-\overline{z}_{n+2}}\right)\end{aligned}$$

$$+\left(\overline{\theta}_{n+2}\theta_{n+4}\frac{\widetilde{\varepsilon}_{n+1}\cdot q_2}{\overline{z}_{n+1}-\overline{z}_{n+2}}\right)\left(\theta_{n+2}\overline{\theta}_{n+4}\frac{\widetilde{\varepsilon}_{n+2}\cdot q_1}{\overline{z}_{n+1}-\overline{z}_{n+2}}\right) . \tag{B.2.20}$$

The expressions (B.2.18), (B.2.19) and (B.2.20) can be extracted from the last four terms of the exponential (B.2.16). Let us now comprise all terms into a matrix. We have the following relation

$$\begin{aligned}&\Bigg\langle\left(\prod_{i=1}^{n}e^{ik_i\widetilde{X}(\overline{z}_i)}\right)\exp\{i\widetilde{q}_1\cdot\widetilde{X}+\theta_{n+2}\bar{\theta}_{n+2}\widetilde{\varepsilon}_{n+1}\cdot\overline{\partial}\widetilde{X}+\theta_{n+2}\widetilde{q}_1\cdot\widetilde{\psi}+\bar{\theta}_{n+2}\widetilde{\varepsilon}_{n+1}\cdot\widetilde{\psi}\}\\&\times\exp\{i\widetilde{q}_2\cdot\widetilde{X}+\theta_{n+4}\bar{\theta}_{n+4}\widetilde{\varepsilon}_{n+2}\cdot\overline{\partial}\widetilde{X}+\theta_{n+4}\widetilde{q}_2\cdot\widetilde{\psi}+\bar{\theta}_{n+4}\widetilde{\varepsilon}_{n+2}\cdot\widetilde{\psi}\}\Bigg\rangle_{S_2}\Bigg|_{\alpha'\to\infty}\\&=\exp\left\{\frac{1}{2}\alpha'^2\sum_{i,j\in\{n+2,n+4\}}\begin{pmatrix}\theta_i\\\bar{\theta}_i\end{pmatrix}^t\Psi_2\begin{pmatrix}\theta_j\\\bar{\theta}_j\end{pmatrix}\right\}+\mathcal{O}(\alpha'^{-1}) ,\end{aligned} \tag{B.2.21}$$

where Ψ_2 is given by:

$$\Psi_2=\begin{pmatrix}0&\frac{q_1\cdot q_2}{\overline{z}_{n+1,n+2}}&\Xi^{n+2}&\frac{\widetilde{\varepsilon}_{n+2}\cdot q_1}{\overline{z}_{n+2,n+1}}\\-\frac{q_1\cdot q_2}{\overline{z}_{n+1,n+2}}&0&\frac{-\widetilde{\varepsilon}_{n+1}\cdot q_2}{\overline{z}_{n+1,n+2}}&\Xi^{n+4}\\-\Xi^{n+2}&\frac{\widetilde{\varepsilon}_{n+1}\cdot q_2}{\overline{z}_{n+1,n+2}}&0&\frac{\widetilde{\varepsilon}_{n+1}\cdot\widetilde{\varepsilon}_{n+2}}{\overline{z}_{n+1,n+2}}\\-\frac{\widetilde{\varepsilon}_{n+2}\cdot q_1}{\overline{z}_{n+2,n+1}}&-\Xi^{n+4}&-\frac{\widetilde{\varepsilon}_{n+1}\cdot\widetilde{\varepsilon}_{n+2}}{\overline{z}_{n+1,n+2}}&0\end{pmatrix} . \tag{B.2.22}$$

The matrix (B.2.22) is the same expression (denoted by Ψ_S) defined in already in equation (4.2.11) of [20] for the two–graviton case. Here, Ξ^{n+2} and Ξ^{n+4} are the objects introduced in (B.2.18) and (B.2.19) and correspond to the two anti–holomorphic graviton fields labelled by $n+2$ and $n+4$, respectively. Putting everything together, for the anti holomorphic part we have:

$$\begin{aligned}&\overline{KN}^{-1}\,\mathcal{I}(n;2)\Big|_{\substack{\text{anti-}\\\text{holomorphic}\\\alpha'\to\infty}}=\mathcal{C}(1,2,\ldots,n)\\&\times\int\prod_{i\in\{n+2,n+4\}}d\theta_i d\bar{\theta}_i\ \exp\left\{\frac{1}{2}\alpha'^2\sum_{i,j\in\{n+2,n+4\}}\begin{pmatrix}\theta_i\\\bar{\theta}_i\end{pmatrix}^t\Psi_2\begin{pmatrix}\theta_j\\\bar{\theta}_j\end{pmatrix}\right\}+\mathcal{O}(\alpha'^{-1})\\&=\mathcal{C}(1,2,\ldots,n)\ \mathrm{Pf}\Psi_2+\mathcal{O}(\alpha'^{-1}) .\end{aligned} \tag{B.2.23}$$

For the holomorphic part of (B.2.17) we are dealing with $n+r=n+2$ open superstring vertex operators which can be described by the pure open superstring disk amplitude, which is related to the Pfaffian of the matrix Ψ_{n+2}:

$$\begin{aligned}KN^{-1}\,\mathcal{I}(n;2)\Big|_{\substack{\text{holomorphic}\\\alpha'\to\infty}}&=\left\langle\prod_{j=1}^{n}V_o(\varepsilon_j,k_j,z_j)\prod_{l=1}^{2}V_o(\varepsilon_l,q_l,z_l)\right\rangle_{S_2}\Bigg|_{\alpha'\to\infty}\\&=\left\langle\prod_{j=1}^{n}V_o(\varepsilon_j,k_j,z_j)\prod_{l=1}^{2}V_o(\varepsilon_l,q_l,z_l)\right\rangle_{D_2}\Bigg|_{\alpha'\to\infty}=\mathrm{Pf}'\Psi_{n+2} .\end{aligned} \tag{B.2.24}$$

It is clear that these contractions give rise to the first four terms of the integrand (B.2.16) as they are identical to contractions of the $n+2$ open strings given in the standard formula (A.1.2). After putting together the anti–holomorphic (B.2.23) and holomorphic (B.2.24) parts we have

$$\lim_{\alpha'\to\infty} \overline{KN}^{-1}\, \mathcal{I}(n;2)\Big|_{\substack{\text{anti-}\\ \text{holomorphic}}} \cdot KN^{-1}\, \mathcal{I}(n;2)\Big|_{\text{holomorphic}} = \mathcal{C}(1,2,\ldots,n) \times \mathrm{Pf}\Psi_2\ \mathrm{Pf}'\Psi_{n+2} + \mathcal{O}(\alpha'^{-1})\ , \tag{B.2.25}$$

with the $(2n+4)\times(2n+4)$ matrix Ψ_{n+2} and the 4×4 matrix Ψ_2 both assuming the form of (4.2.11) with entries (4.2.12)

$$\Psi_{n+2} = \Psi_S = \begin{pmatrix} A_{ij} & -C_{ji} \\ C_{ij} & B_{ij}\end{pmatrix}\Bigg|_{\substack{\sigma_l=\zeta_l,\ l=1,\ldots,n+1\\ \sigma_{n+2}=\zeta_{n+3}}} \quad , \quad S=\{1,\ldots,n,n+1,n+2\}\ , \tag{B.2.26}$$

$$\Psi_2 = \Psi_{S_2} = \begin{pmatrix} A_{ij} & -C_{ji} \\ C_{ij} & B_{ij}\end{pmatrix}\Bigg|_{\substack{\sigma_l=\bar\zeta_l,\ l=1,\ldots,n\\ \sigma_{n+1}=\bar\zeta_{n+2},\ \sigma_{n+2}=\bar\zeta_{n+4}}} \quad , \quad S_2=\{n+1,n+2\}\ , \tag{B.2.27}$$

respectively. Finally, noting the expansion of denominator terms w.r.t.,α'

$$\frac{\overline{\theta}_i\overline{\theta}_j(\xi_i\cdot\xi_j)}{\zeta_i-\zeta_j\mp\alpha'^{-1}\theta_i\theta_j} = \frac{\overline{\theta}_i\overline{\theta}_j(\xi_i\cdot\xi_j)}{\zeta_i-\zeta_j} \pm \alpha'^{-1}\frac{\theta_i\theta_j\overline{\theta}_i\overline{\theta}_j(\xi_i\cdot\xi_j)}{(\zeta_i-\zeta_j)^2}\ , \tag{B.2.28}$$

with (B.2.26) and (B.2.27) we can write the integrand (B.2.16) in terms of the generalized notation (B.2.15) as the following product:

$$\begin{aligned}\mathcal{I}(n;2) = &\int \prod_{i=1}^{n+4} d\theta_i d\bar\theta_i\ \frac{\theta_1\theta_2}{\zeta_1-\zeta_2}\exp\left\{\frac{1}{2}\alpha'^2 \sum_{\substack{i,j=1\\ i,j\neq n+2}}^{n+3} \begin{pmatrix}\theta_i\\ \bar\theta_i\end{pmatrix}^t \Psi_{n+2}\begin{pmatrix}\theta_j\\ \bar\theta_j\end{pmatrix} \pm \frac{1}{2}\alpha' \sum_{\substack{i,j=1\\ i,j\neq n+2}}^{n+3} \frac{\theta_i\theta_j\overline{\theta}_i\overline{\theta}_j(\xi_i\cdot\xi_j)}{(\zeta_i-\zeta_j)^2}\right\}\\ &\times\exp\left\{\frac{1}{2}\alpha'^2 \sum_{i,j\in\{n+2,n+4\}} \begin{pmatrix}\theta_i\\ \bar\theta_i\end{pmatrix}^t \Psi_2\begin{pmatrix}\theta_j\\ \bar\theta_j\end{pmatrix} \pm \frac{1}{2}\alpha' \sum_{i,j\in\{n+2,n+4\}} \frac{\theta_i\theta_j\overline{\theta}_i\overline{\theta}_j(\xi_i\cdot\xi_j)}{(\overline{\zeta}_i-\overline{\zeta}_j)^2}\right\}\\ &\times \mathcal{C}(1,2,\ldots,n)\ \times KN\cdot\overline{KN}.\end{aligned} \tag{B.2.29}$$

Our integrand (B.2.29) furnishes a KLT like structure factorizing holomorphic and anti–holomorphic terms. In addition, to this factorized form the two sets of fermionic variables $\bigcup_{j\in\{1,\ldots,n+1,n+3\}}\{\theta_j,\bar\theta_j\}$ and $\bigcup_{i\in\{n+2,n+4\}}\{\theta_i,\bar\theta_i\}$ can be attributed, respectively.

Eventually, after these preparations we are able to construct the pair of twisted forms $\varphi^{EYM}_{\pm,n;2}$ and $\widetilde\varphi^{EYM}_{\pm,n;2}$. Similarly to (B.1.11) we take the pure holomorphic part of (B.2.29) to constitute the twisted form $\varphi^{EYM}_{\pm,n;2}$

$$\varphi^{EYM}_{\pm,n;2} = d\mu_{n+2}\int \prod_{\substack{i=1\\ i\neq n+2}}^{n+3} d\theta_i d\bar\theta_i\ \frac{\theta_1\theta_2}{\zeta_1-\zeta_2}\ \exp\left\{\frac{1}{2}\alpha'^2 \sum_{\substack{i,j=1\\ i,j\neq n+2}}^{n+3} \begin{pmatrix}\theta_i\\ \bar\theta_i\end{pmatrix}^t \Psi_{n+2}\begin{pmatrix}\theta_j\\ \bar\theta_j\end{pmatrix}\right\}$$

$$\times \exp\left\{ \pm \frac{1}{2}\alpha' \sum_{\substack{i,j=1 \\ i,j\neq n+2}}^{n+3} \frac{\theta_i\theta_j\overline{\theta}_i\overline{\theta}_j(\xi_i\cdot\xi_j)}{(\zeta_i-\zeta_j)^2} \right\} , \tag{B.2.30}$$

with Ψ_{n+2} defined in (B.2.26). To find our second twisted form $\widetilde{\varphi}^{EYM}_{\pm,n;2}$ we take the anti–holomorphic part of (B.2.29) and apply the isomorphism (6.4.71) to arrive at:

$$\widetilde{\varphi}^{EYM}_{\pm,n;2} = d\mu_{n+2}\ \mathcal{C}(1,2,\ldots,n)\int \prod_{i\in\{n+2,n+4\}} d\theta_i d\bar{\theta}_i\ \exp\left\{ \frac{\alpha'^2}{2} \sum_{i,j\in\{n+2,n+4\}} \begin{pmatrix}\theta_i\\ \bar{\theta}_i\end{pmatrix}^t \Psi_2 \begin{pmatrix}\theta_j\\ \bar{\theta}_j\end{pmatrix} \right\}\Bigg|_{\bar{\zeta}_l\to\zeta_l}$$
$$\times \exp\left\{ \pm \frac{1}{2}\alpha' \sum_{i,j\in\{n+2,n+4\}} \frac{\theta_i\theta_j\overline{\theta}_i\overline{\theta}_j(\xi_i\cdot\xi_j)}{(\zeta_i-\zeta_j)^2} \right\} . \tag{B.2.31}$$

Our two twisted forms $\varphi_- = \widetilde{\varphi}^{EYM}_{+,n;2}$ and $\varphi_+ = \varphi^{EYM}_{-,n;2}$ reproduce the EYM integrand in the $\alpha'\to\infty$ limit (6.3.23). These limits can be determined as:

$$\lim_{\alpha'\to\infty} \hat{\varphi}^{EYM}_{\pm,n;2} = \int \prod_{\substack{i=1\\ i\neq n+2}}^{n+3} d\theta_i d\bar{\theta}_i\ \frac{\theta_1\theta_2}{\zeta_1-\zeta_2}\ \exp\left\{ \frac{1}{2}\alpha'^2 \sum_{\substack{i,j=1\\ i,j\neq n+2}}^{n+3} \begin{pmatrix}\theta_i\\ \bar{\theta}_i\end{pmatrix}^t \Psi_{n+2} \begin{pmatrix}\theta_j\\ \bar{\theta}_j\end{pmatrix} \right\} \pm \mathcal{O}(\alpha'^{-1})$$
$$= \frac{\mathrm{Pf}\Psi^{12}_{n+2}}{\zeta_1-\zeta_2} + \mathcal{O}(\alpha'^{-1}) = \mathrm{Pf}'\Psi_{n+2} + \mathcal{O}(\alpha'^{-1}) ,$$

$$\lim_{\alpha'\to\infty} \hat{\widetilde{\varphi}}^{EYM}_{\pm,n;2} = \mathcal{C}(1,2,\ldots,n) \int \prod_{i\in\{n+2,n+4\}} d\theta_i d\bar{\theta}_i \exp\left\{ \frac{\alpha'^2}{2} \sum_{i,j\in\{n+2,n+4\}} \begin{pmatrix}\theta_i\\ \bar{\theta}_i\end{pmatrix}^t \Psi_2 \begin{pmatrix}\theta_j\\ \bar{\theta}_j\end{pmatrix} \right\}\Bigg|_{\bar{\zeta}_l\to\zeta_l}$$
$$\pm \mathcal{O}(\alpha'^{-1}) = \mathcal{C}(1,2,\ldots,n)\ \mathrm{Pf}\Psi_2\big|_{\bar{\zeta}_l\to\zeta_l} + \mathcal{O}(\alpha'^{-1}) \tag{B.2.32}$$

Using these limits in (6.3.23) gives the final result for EYM amplitude in the CHY formalism:

$$\mathcal{A}(n;2) = \lim_{\alpha'\to\infty} \langle \widetilde{\varphi}^{EYM}_{+,n;2}, \varphi^{EYM}_{-,n;2}\rangle_\omega = \int_{\mathcal{M}_{0,n+2}} d\mu_{n+2} \prod_{a=1}^{n+2}{}' \delta(f_a) \lim_{\alpha'\to\infty} \hat{\varphi}^{EYM}_{-,n;2}\ \hat{\widetilde{\varphi}}^{EYM}_{+,n;2}$$
$$= \int_{\mathcal{M}_{0,n+2}} d\mu_{n+2} \prod_{a=1}^{n+2}{}' \delta(f_a)\ \frac{\mathrm{Pf}\Psi_2\big|_{\bar{\zeta}_l\to\zeta_l}\ \mathrm{Pf}'\Psi_{n+2}}{(z_1-z_2)(z_2-z_3)\ldots(z_n-z_1)} . \tag{B.2.33}$$

B.2.2 Twisted form and intersections for amplitudes of n gluons and r gravitons

Finally, in this subsection we extend our results for EYM amplitudes to the all multiplicity case. We shall consider the EYM amplitude involving n gluons and r gravitons. We start with the correlator (6.4.45) on the sphere, which can be expressed in terms of Graßmann variables as given in (6.4.51). Performing the Wick contractions on the sphere we obtain for the integrand:

$$\begin{aligned}\mathcal{I}(n;r) = \int \prod_{i=1}^{n+2r} \frac{\theta_1\theta_2}{\zeta_1-\zeta_2}\, d\theta_i d\bar\theta_i\, \exp\Bigg\{\alpha'^2\Bigg(&\sum_{\substack{i,j,\in\mathcal{S}\\ i\neq j}} \frac{(\bar\theta_i\theta_j\xi_i\cdot p_j)}{\zeta_i-\zeta_j\mp\alpha'^{-1}\theta_i\theta_j} + \sum_{\substack{i,j,\in\mathcal{S}\\ i\neq j}} \frac{(\theta_i\bar\theta_i\xi_i\cdot p_j)}{\zeta_i-\zeta_j\mp\alpha'^{-1}\theta_i\theta_j}\\ &+ \sum_{\substack{i,j,\in\mathcal{S}\\ i>j}} \frac{(\bar\theta_i\bar\theta_j\xi_i\cdot\xi_j)}{\zeta_i-\zeta_j\mp\alpha'^{-1}\theta_i\theta_j} + \sum_{\substack{i,j,\in\mathcal{S}\\ i>j}} \frac{(\theta_i\theta_j p_i\cdot p_j)}{\zeta_i-\zeta_j\mp\alpha'^{-1}\theta_i\theta_j} + \sum_{\substack{j\in\{1,\ldots,n\}\cup\mathcal{S}_r\\ i\in\mathcal{S}_r}} \frac{(\theta_i\bar\theta_i\xi_i\cdot p_j)}{\bar\zeta_i-\bar\zeta_j\mp\alpha'^{-1}\theta_i\theta_j}\\ &+ \sum_{\substack{i,j\in\mathcal{S}_r\\ i\neq j}} \frac{(\bar\theta_i\theta_j\xi_i\cdot p_j)}{\bar\zeta_i-\bar\zeta_j\mp\alpha'^{-1}\theta_i\theta_j} + \sum_{\substack{i,j\in\mathcal{S}_r\\ i>j}} \frac{(\bar\theta_i\bar\theta_j\xi_i\cdot\xi_j)}{\bar\zeta_i-\bar\zeta_j\mp\alpha'^{-1}\theta_i\theta_j} + \sum_{\substack{i,j\in\mathcal{S}_r\\ i>j}} \frac{(\theta_i\theta_j p_i\cdot p_j)}{\bar\zeta_i-\bar\zeta_j\mp\alpha'^{-1}\theta_i\theta_j}\Bigg)\Bigg\}\\ &\times\mathcal{C}(1,2,\ldots,n)\times KN\cdot\overline{KN}\ .\end{aligned} \tag{B.2.34}$$

In (B.2.34) we have introduced the two sets $\mathcal{S}$ and $\mathcal{S}_r$, given by:

$$\begin{aligned}\mathcal{S} &:= \{1,2,3,...,n,n+1,n+3,...,n+2r-1\}\ ,\\ \mathcal{S}_r &:= \{n+2,n+4,...,n+2r\}\ .\end{aligned} \tag{B.2.35}$$

Here, $\mathcal{S}_r$ represents the set of indices accounting for the anti–holomorphic parts of the r graviton vertex operators (A.2.4), while $\mathcal{S}$ is the set of indices labelling the holomorphic parts of both gluons and gravitons. Furthermore, in the exponential of (B.2.34) the sums run over indices denoting fermionic variables $\theta_i, \overline{\theta}_i$, the set of the generalized momenta p_i, polarizations ξ_j, and positions ζ_j defined in (B.2.15). As in (B.2.16) the first four terms in the exponential account for the holomorphic field contractions, while the last four terms represent the anti–holomorphic field contractions subject to the map (6.4.43). Additionally, $KN\cdot\overline{KN}$ is the Koba–Nielsen factor (6.3.11) for $n+r$ closed strings.

We shall follow similar steps as in the previous subsection to extract from (B.2.34) a pair of twisted forms suitable for describing the multi–leg EYM amplitude. Let us first look at the CHY integral introduced in (4.3.15) for this general case

$$\mathcal{I}_{n+r}(n;r) = \mathcal{C}(1,2,\ldots,n)\ \mathrm{Pf}\Psi_r\ \mathrm{Pf}'\Psi_{n+r}(k_a,q_a,\varepsilon,\sigma)\ , \tag{B.2.36}$$

with $\mathcal{C}(1,2,\ldots,n)$ being the Parke–Taylor factor (6.3.9). As in our constructions above the latter is to be identified with the gauge current factor $\mathcal{C}$ in (B.2.34) which enters the definition of $\widetilde{\varphi}^{EYM}_{\pm,n;r}$.

Again, by using (B.2.28) in the exponential of the integrand (B.2.34) we may disentangle quadratic from linear orders in α'. As a consequence the eight sums accounting for the quadratic order α'^2 can compactly be written as

$$\sum_{i,j\in\mathcal{S}}(\theta_i\ \bar\theta_i)\Psi_{n+r}\ \begin{pmatrix}\theta_j\\ \bar\theta_j\end{pmatrix} + \sum_{i,j\in\mathcal{S}_r}(\theta_i\ \bar\theta_i)\Psi_r\ \begin{pmatrix}\theta_j\\ \bar\theta_j\end{pmatrix}\ , \tag{B.2.37}$$

with the $(2n+2r)\times(2n+2r)$ matrix Ψ_{n+r} and the $2r\times 2r$ matrix Ψ_r both assuming the form of (4.2.11) with entries (4.2.12)

$$\Psi_{n+r} = \Psi_S = \begin{pmatrix} A_{ij} & -C_{ji} \\ C_{ij} & B_{ij} \end{pmatrix}\Bigg|_{\substack{\sigma_l=\zeta_l,\ l=1,\ldots,n \\ \sigma_{n+k}=\zeta_{n+2k-1},\ k=1,\ldots,r}} \ , \quad S = \{1,\ldots,n,n+1,\ldots,n+r\}\ , \tag{B.2.38}$$

$$\Psi_r = \Psi_{S_r} = \begin{pmatrix} A_{ij} & -C_{ji} \\ C_{ij} & B_{ij} \end{pmatrix}\Bigg|_{\substack{\sigma_l=\bar\zeta_l,\ l=1,\ldots,n \\ \sigma_{n+k}=\bar\zeta_{n+2k},\ k=1,\ldots,r}} \ , \quad S_r = \{n+2,\ldots,n+2r\}\ , \tag{B.2.39}$$

respectively. After taking into account the linear order in α' originating from the expansion (B.2.28) with the above matrices (B.2.38) and (B.2.39) the integrand (B.2.34) can be cast into:

$$\begin{aligned} \mathcal{I}(n;r) = \int \prod_{i=1}^{n+2r} d\theta_i d\bar\theta_i\ \frac{\theta_1\theta_2}{\zeta_1-\zeta_2}\ \exp\left\{\frac{1}{2}\alpha'^2 \sum_{i,j\in\mathcal{S}} \begin{pmatrix}\theta_i\\ \bar\theta_i\end{pmatrix}^t \Psi_{n+r} \begin{pmatrix}\theta_j\\ \bar\theta_j\end{pmatrix} \pm \frac{1}{2}\alpha' \sum_{i,j\in\mathcal{S}} \frac{\theta_i\theta_j\bar\theta_i\bar\theta_j(\xi_i\cdot\xi_j)}{(\zeta_i-\zeta_j)^2}\right\} \\ \times \exp\left\{\frac{1}{2}\alpha'^2 \sum_{i,j\in\mathcal{S}_r} \begin{pmatrix}\theta_i\\ \bar\theta_i\end{pmatrix}^t \Psi_r\ \begin{pmatrix}\theta_j\\ \bar\theta_j\end{pmatrix} \pm \frac{1}{2}\alpha' \sum_{i,j\in\mathcal{S}_r} \frac{\theta_i\theta_j\bar\theta_i\bar\theta_j(\xi_i\cdot\xi_j)}{(\bar\zeta_i-\bar\zeta_j)^2}\right\} \\ \times\, \mathcal{C}(1,2,\ldots,n) \times KN\cdot\overline{KN}\ . \end{aligned} \tag{B.2.40}$$

Again, our integrand (6.4.55) furnishes a KLT like structure factorizing holomorphic and anti–holomorphic terms. In addition, to this factorized form in lines of (B.2.37) the two sets of fermionic variables $\bigcup_{j\in\mathcal{S}}\{\theta_j,\bar\theta_j\}$ and $\bigcup_{i\in\mathcal{S}_r}\{\theta_i,\bar\theta_i\}$ can be attributed, respectively.

Now, we are prepared to construct the pair of twisted forms $\varphi^{EYM}_{\pm,n;r}$ and $\tilde\varphi^{EYM}_{\pm,n;r}$. As first differential form we define:

$$\begin{aligned} \varphi^{EYM}_{\pm,n;r} = d\mu_{n+r} \int \prod_{i\in\mathcal{S}} \frac{\theta_1\theta_2}{\zeta_1-\zeta_2}\ d\theta_i d\bar\theta_i \\ \times \exp\left\{\frac{1}{2}\alpha'^2 \sum_{i,j\in\mathcal{S}} \begin{pmatrix}\theta_i\\ \bar\theta_i\end{pmatrix}^t \Psi_{n+r}\ \begin{pmatrix}\theta_j\\ \bar\theta_j\end{pmatrix}\right\} \exp\left\{\pm\frac{1}{2}\alpha' \sum_{i,j\in\mathcal{S}} \frac{\theta_i\theta_j\bar\theta_i\bar\theta_j(\xi_i\cdot\xi_j)}{(\zeta_i-\zeta_j)^2}\right\}\ . \end{aligned} \tag{B.2.41}$$

In this form $\varphi^{EYM}_{\pm,n;r}$, as we mentioned in the chapter 5, can be identified with the twisted gauge form (6.3.27), i.e.:

$$\varphi^{EYM}_{\pm,n;r} \equiv \varphi^{gauge}_{\pm,n+r} \tag{B.2.42}$$

For the second twisted form $\tilde\varphi^{EYM}_{\pm,n;r}$ we take the anti–holomorphic part of (B.2.34) and apply the isomorphism (6.4.71):

$$\begin{aligned} \tilde\varphi^{EYM}_{\pm,n;r} = d\mu_{n+r}\ \mathcal{C}(1,2,\ldots,n) \int \prod_{i\in\mathcal{S}_r} d\theta_i d\bar\theta_i\ \exp\left\{\frac{1}{2}\alpha'^2 \sum_{i,j\in\mathcal{S}_r} \begin{pmatrix}\theta_i\\ \bar\theta_i\end{pmatrix}^t \Psi_r\ \begin{pmatrix}\theta_j\\ \bar\theta_j\end{pmatrix}\right\}\Bigg|_{\bar\zeta_l\to\zeta_l} \\ \times \exp\left\{\pm\frac{1}{2}\alpha' \sum_{i,j\in\mathcal{S}_r} \frac{\theta_i\theta_j\bar\theta_i\bar\theta_j(\xi_i\cdot\xi_j)}{(\zeta_i-\zeta_j)^2}\right\}\ . \end{aligned} \tag{B.2.43}$$

With these two twisted forms (6.4.57) and (B.2.43) we can compute the EYM amplitude through the expression for the intersection number (6.3.23) in the $\alpha' \to \infty$ limit. For this we determine the following limits:

$$\lim_{\alpha'\to\infty} \hat{\varphi}^{EYM}_{\pm,n;r} = \int \prod_{i\in\mathcal{S}_r} d\theta_i d\bar{\theta}_i \frac{\theta_1\theta_2}{\zeta_1-\zeta_2} \exp\left\{\frac{1}{2}\alpha'^2 \sum_{i,j\in\mathcal{S}} \begin{pmatrix}\theta_i\\ \bar{\theta}_i\end{pmatrix}^t \Psi_{n+r} \begin{pmatrix}\theta_j\\ \bar{\theta}_j\end{pmatrix}\right\} + \mathcal{O}(\alpha'^{-1})$$
$$= \frac{\mathrm{Pf}\Psi^{12}_{n+r}}{\zeta_1-\zeta_2} + \mathcal{O}(\alpha'^{-1}) = \mathrm{Pf}'\Psi_{n+r} + \mathcal{O}(\alpha'^{-1}) \ ,$$

$$\lim_{\alpha'\to\infty} \hat{\tilde{\varphi}}^{EYM}_{\pm,n;r} = \mathcal{C}(1,2,\ldots,n) \int \prod_{i\in\mathcal{S}_r} d\theta_i d\bar{\theta}_i \ \exp\left\{\frac{1}{2}\alpha'^2 \sum_{i,j\in\mathcal{S}_r} \begin{pmatrix}\theta_i\\ \bar{\theta}_i\end{pmatrix}^t \Psi_r \begin{pmatrix}\theta_j\\ \bar{\theta}_j\end{pmatrix}\right\}\Bigg|_{\bar{\zeta}_l\to\zeta_l} + \mathcal{O}(\alpha'^{-1})$$
$$= \mathcal{C}(1,2,\ldots,n) \ \mathrm{Pf}\Psi_r\big|_{\bar{\zeta}_l\to\zeta_l} + \mathcal{O}(\alpha'^{-1}) \ . \tag{B.2.44}$$

Putting the limits (B.2.44) into (6.3.23) subject to the choice $\varphi_+ = \tilde{\varphi}^{EYM}_{+,n;r}$ and $\varphi_- = \varphi^{EYM}_{-,n;r}$ yields the EYM amplitude for n gluons and r gravitons:

$$\mathcal{A}(n;r) = \lim_{\alpha'\to\infty} \langle \tilde{\varphi}^{EYM}_{+,n;r}, \varphi^{EYM}_{-,n;r}\rangle_\omega = \int_{\mathcal{M}_{0,n+r}} d\mu_{n+r} \prod_{a=1}^{n+r}{}' \delta(f_a) \lim_{\alpha'\to\infty} \hat{\varphi}^{EYM}_{-,n;r} \ \hat{\tilde{\varphi}}^{EYM}_{+,n;r}$$
$$= \int_{\mathcal{M}_{0,n+r}} d\mu_{n+r} \prod_{a=1}^{n+r}{}' \delta(f_a) \ \frac{\mathrm{Pf}\Psi_r\big|_{\bar{\zeta}_l\to\zeta_l} \ \mathrm{Pf}'\Psi_{n+r}}{(z_1-z_2)(z_2-z_3)\ldots(z_n-z_1)} \ . \tag{B.2.45}$$

This is the integral formula for the Einstein Yang-Mills amplitude (4.3.15) formulated in the CHY formalism for the generic case of n gluons and r gravitons.

Appendix C

Massive string amplitudes

In this appendix we are going to provide some detail on the results and calculations we discussed in chapter 6 for the string amplitude with massive external legs.

C.1 Calculations for $\mathcal{A}(2,1)$

C.1.1 Sample contractions

Using the disk correlators (5.1) and Wick's theorem we perform the contractions in (7.3.66). The formulae are long and here we only present one of the shorter examples:

$$
\begin{aligned}
&-\tfrac{\alpha'^2}{4}\langle :\partial X^\kappa(x_1)e^{ik_1X(x_1)}::\partial X^\rho(x_2)e^{ik_2X(x_2)}::e^{iq\widetilde{X}(\overline{z})}::e^{iqX(z)}:\rangle \\
&\times\langle :e^{-\phi(x_1)}::e^{-\phi(x_2)}:\rangle \quad \langle \psi^\lambda(x_1)\psi^\sigma(x_2):\big(q\widetilde{\psi}(\overline{z})\big)\widetilde{\psi}^\mu(\overline{z})::\big(q\psi(z)\big)\psi^\nu(z):\rangle = \\
&\tfrac{\alpha'^3}{8}\mathcal{E}\Bigg\{-\frac{\left[2\alpha' k_2^\kappa k_1^\rho g^{\lambda\sigma}-g^{\kappa\rho}g^{\lambda\sigma}\right]l^{\nu\mu}}{(x_1-x_2)^4(z-\overline{z})^2}-\frac{\alpha' k_1^\rho q^\kappa g^{\lambda\sigma}l^{\nu\mu}}{(x_1-x_2)^3(x_1-z)(z-\overline{z})^2}-\frac{\alpha'(Dq)^\kappa k_1^\rho g^{\lambda\sigma}l^{\nu\mu}}{(x_1-x_2)^3(x_1-\overline{z})(z-\overline{z})^2} \\
&+\frac{\alpha' k_2^\kappa q^\rho g^{\lambda\sigma}l^{\nu\mu}}{(x_1-x_2)^3(x_2-z)(z-\overline{z})^2}+\frac{\alpha' k_2^\kappa (Dq)^\rho g^{\lambda\sigma}l^{\nu\mu}}{(x_1-x_2)^3(x_2-\overline{z})(z-\overline{z})^2} \\
&-\frac{\left[2\alpha' k_2^\kappa k_1^\rho-g^{\kappa\rho}\right][q^\lambda\widetilde{f}^{\sigma\nu\mu}+g^{\lambda\nu}\widetilde{h}^{\sigma\mu}]}{(x_1-x_2)^3(x_1-z)(x_2-\overline{z})(z-\overline{z})}-\frac{\left[2\alpha' k_2^\kappa k_1^\rho-g^{\kappa\rho}\right]\left[q^\sigma f^{\lambda\nu\mu}+g^{\sigma\nu}h^{\lambda\mu}\right]}{(x_1-x_2)^3(x_1-\overline{z})(x_2-z)(z-\overline{z})} \\
&+\frac{1}{2}\frac{\alpha' q^\kappa q^\rho g^{\lambda\sigma}l^{\nu\mu}}{(x_1-x_2)^2(x_1-z)(x_2-z)(z-\overline{z})^2}+\frac{1}{2}\frac{\alpha' q^\kappa (Dq)^\rho g^{\lambda\sigma}l^{\nu\mu}}{(x_1-x_2)^2(x_1-z)(x_2-\overline{z})(z-\overline{z})^2} \\
&+\frac{1}{2}\frac{\alpha'(Dq)^\kappa q^\rho g^{\lambda\sigma}l^{\nu\mu}}{(x_1-x_2)^2(x_1-\overline{z})(x_2-z)(z-\overline{z})^2}+\frac{1}{2}\frac{\alpha'(Dq)^\kappa (Dq)^\rho g^{\lambda\sigma}l^{\nu\mu}}{(x_1-x_2)^2(x_1-\overline{z})(x_2-\overline{z})(z-\overline{z})^2} \\
&+\frac{\alpha' k_2^\kappa(Dq)^\rho q^\lambda\widetilde{f}^{\sigma\nu\mu}+\alpha' k_2^\kappa(Dq)^\rho g^{\lambda\nu}\widetilde{h}^{\sigma\mu}}{(x_1-x_2)^2(x_1-z)(x_2-\overline{z})^2(z-\overline{z})}-\frac{\alpha' k_1^\rho q^\kappa q^\lambda\widetilde{f}^{\sigma\nu\mu}+\alpha' k_1^\rho q^\kappa g^{\lambda\nu}\widetilde{h}^{\sigma\mu}}{(x_1-x_2)^2(x_1-z)^2(x_2-\overline{z})(z-\overline{z})} \\
&+\frac{\alpha' k_2^\kappa q^\rho q^\sigma f^{\lambda\nu\mu}+\alpha' k_2^\kappa q^\rho g^{\sigma\nu}h^{\lambda\mu}}{(x_1-x_2)^2(x_1-\overline{z})(x_2-z)^2(z-\overline{z})}-\frac{\alpha'(Dq)^\kappa k_1^\rho q^\sigma f^{\lambda\nu\mu}+\alpha'(Dq)^\kappa k_1^\rho g^{\sigma\nu}h^{\lambda\mu}}{(x_1-x_2)^2(x_1-\overline{z})^2(x_2-z)(z-\overline{z})} \\
&-\frac{\alpha' k_1^\rho q^\kappa q^\sigma f^{\lambda\nu\mu}+\alpha' k_1^\rho q^\kappa g^{\sigma\nu}h^{\lambda\mu}}{(x_1-x_2)^2(x_1-z)(x_1-\overline{z})(x_2-z)(z-\overline{z})}-\frac{\alpha'(Dq)^\kappa k_1^\rho q^\lambda\widetilde{f}^{\sigma\nu\mu}+\alpha'(Dq)^\kappa k_1^\rho g^{\lambda\nu}\widetilde{h}^{\sigma\mu}}{(x_1-x_2)^2(x_1-z)(x_1-\overline{z})(x_2-\overline{z})(z-\overline{z})} \\
&+\frac{\alpha' k_2^\kappa(Dq)^\rho q^\sigma f^{\lambda\nu\mu}+\alpha' k_2^\kappa(Dq)^\rho g^{\sigma\nu}h^{\lambda\mu}}{(x_1-x_2)^2(x_1-\overline{z})(x_2-z)(x_2-\overline{z})(z-\overline{z})}+\frac{\alpha' k_2^\kappa q^\rho q^\lambda\widetilde{f}^{\sigma\nu\mu}+\alpha' k_2^\kappa q^\rho g^{\lambda\nu}\widetilde{h}^{\sigma\mu}}{(x_1-x_2)^2(x_1-z)(x_2-z)(x_2-\overline{z})(z-\overline{z})} \\
&+\frac{1}{2}\frac{\alpha' q^\kappa(Dq)^\rho q^\lambda\widetilde{f}^{\sigma\nu\mu}+\alpha' q^\kappa(Dq)^\rho g^{\lambda\nu}\widetilde{h}^{\sigma\mu}}{(x_1-x_2)(x_1-z)^2(x_2-\overline{z})^2(z-\overline{z})}+\frac{1}{2}\frac{\alpha'(Dq)^\kappa q^\rho q^\sigma f^{\lambda\nu\mu}+\alpha'(Dq)^\kappa q^\rho g^{\sigma\nu}h^{\lambda\mu}}{(x_1-x_2)(x_1-\overline{z})^2(x_2-z)^2(z-\overline{z})} \\
&+\frac{1}{2}\frac{\alpha' q^\kappa q^\rho q^\lambda\widetilde{f}^{\sigma\nu\mu}+\alpha' q^\kappa q^\rho g^{\lambda\nu}\widetilde{h}^{\sigma\mu}}{(x_1-x_2)(x_1-z)^2(x_2-z)(x_2-\overline{z})(z-\overline{z})}+\frac{1}{2}\frac{\alpha'(Dq)^\kappa(Dq)^\rho q^\lambda\widetilde{f}^{\sigma\nu\mu}+\alpha'(Dq)^\kappa(Dq)^\rho g^{\lambda\nu}\widetilde{h}^{\sigma\mu}}{(x_1-x_2)(x_1-z)(x_1-\overline{z})(x_2-\overline{z})^2(z-\overline{z})} \\
&+\frac{1}{2}\frac{\alpha' q^\kappa q^\rho q^\sigma f^{\lambda\nu\mu}+\alpha' q^\kappa q^\rho g^{\sigma\nu}h^{\lambda\mu}}{(x_1-x_2)(x_1-z)(x_1-\overline{z})(x_2-z)^2(z-\overline{z})}+\frac{1}{2}\frac{\alpha'(Dq)^\kappa(Dq)^\rho q^\sigma f^{\lambda\nu\mu}+\alpha'(Dq)^\kappa(Dq)^\rho g^{\sigma\nu}h^{\lambda\mu}}{(x_1-x_2)(x_1-\overline{z})^2(x_2-z)(x_2-\overline{z})(z-\overline{z})} \\
&+\frac{1}{2}\frac{\alpha'(Dq)^\kappa q^\rho q^\lambda\widetilde{f}^{\sigma\nu\mu}+\alpha'(Dq)^\kappa q^\rho g^{\lambda\nu}\widetilde{h}^{\sigma\mu}+\alpha' q^\kappa(Dq)^\rho q^\sigma f^{\lambda\nu\mu}+\alpha' q^\kappa(Dq)^\rho g^{\sigma\nu}h^{\lambda\mu}}{(x_1-x_2)(x_1-z)(x_1-\overline{z})(x_2-z)(x_2-\overline{z})(z-\overline{z})}\Bigg\}.
\end{aligned}
\tag{C.1.1}
$$

In the above contraction we have introduced $l^{\mu\nu}$, $h^{\mu\nu}, \widetilde{h}^{\mu\nu}$ and $f^{\mu\nu\lambda}, \widetilde{f}^{\mu\nu\lambda}$, that we define as follows

$$l^{\mu\nu} \equiv -q\cdot DqD^{\mu\nu} + (Dq)^\nu(Dq)^\mu\,, \tag{C.1.2}$$

$$h^{\mu\nu} \;=-\widetilde{h}^{\mu\nu} \equiv\; (Dq)^\mu(Dq)^\nu - D^{\mu\nu}q\cdot Dq\,, \tag{C.1.3}$$

$$f^{\mu\nu\lambda} \;=-\widetilde{f}^{\mu\nu\lambda} \equiv\; -(Dq)^\mu D^{\nu\lambda} + D^{\mu\lambda}(Dq)^\nu\,. \tag{C.1.4}$$

For convenience, we also define the objects

$$c^{\sigma\nu\lambda} \equiv -g^{\sigma\nu}q^\lambda + g^{\lambda\nu}q^\sigma \quad , \quad d^{\sigma\mu\lambda} \equiv -D^{\sigma\mu}(Dq)^\lambda + D^{\lambda\mu}(Dq)^\sigma\,, \tag{C.1.5}$$

that will appear within certain kinematic packages.

C.1.2 The kinematic packages

Here, we list our resulting expressions for all kinematic packages. They are exact in α' (i.e. no truncation), with their constituent terms ordered from the highest power (α'^3) to the lowest (α'^1).

Packages for $\mathbf{A}_1$

$$\begin{aligned}\Theta^{\mu\nu\kappa\lambda\rho\sigma} &= \tfrac{1}{8}\alpha'^3\Big[-(Dk_2)^\mu q^\kappa (Dq)^\rho g^{\sigma\nu} q^\lambda + (Dk_2)^\mu q^\kappa (Dq)^\rho g^{\lambda\nu} q^\sigma \\ &+k_1^\nu q^\kappa (Dq)^\rho D^{\sigma\mu}(Dq)^\lambda - k_1^\nu q^\kappa (Dq)^\rho D^{\lambda\mu}(Dq)^\sigma\Big] \\ &+\tfrac{1}{8}\alpha'^2\Big[D^{\rho\mu} q^\kappa g^{\sigma\nu} q^\lambda - D^{\rho\mu} q^\kappa g^{\lambda\nu} q^\sigma \\ &-g^{\kappa\nu}(Dq)^\rho D^{\sigma\mu}(Dq)^\lambda + g^{\kappa\nu}(Dq)^\rho D^{\lambda\mu}(Dq)^\sigma\Big]\end{aligned} \tag{C.1.6}$$

$$\begin{aligned}\Lambda^{\mu\nu\kappa\lambda\rho\sigma} &= \tfrac{1}{8}\alpha'^3\Big[-k_2^\nu (Dq)^\kappa q^\rho D^{\sigma\mu}(Dq)^\lambda + k_2^\nu (Dq)^\kappa q^\rho D^{\lambda\mu}(Dq)^\sigma \\ &+(Dk_1)^\mu (Dq)^\kappa q^\rho g^{\sigma\nu} q^\lambda - (Dk_1)^\mu (Dq)^\kappa q^\rho g^{\lambda\nu} q^\sigma\Big] \\ &+\tfrac{1}{8}\alpha'^2\Big[+g^{\rho\nu}(Dq)^\kappa D^{\sigma\mu}(Dq)^\lambda - g^{\rho\nu}(Dq)^\kappa D^{\lambda\mu}(Dq)^\sigma \\ &-D^{\kappa\mu} q^\rho g^{\sigma\nu} q^\lambda + D^{\kappa\mu} q^\rho g^{\lambda\nu} q^\sigma\Big]\end{aligned} \tag{C.1.7}$$

$$\begin{aligned}\Xi^{\mu\nu\kappa\lambda\rho\sigma} &= \tfrac{1}{8}\alpha'^3\Big[-k_2^\nu q^\kappa (Dq)^\rho D^{\sigma\mu}(Dq)^\lambda + k_1^\nu q^\kappa q^\rho D^{\sigma\mu}(Dq)^\lambda - k_1^\nu q^\kappa q^\rho D^{\lambda\mu}(Dq)^\sigma \\ &-(Dk_2)^\mu (Dq)^\kappa (Dq)^\rho g^{\sigma\nu} q^\lambda - (Dk_2)^\mu q^\kappa q^\rho g^{\sigma\nu} q^\lambda + (Dk_1)^\mu q^\kappa (Dq)^\rho g^{\sigma\nu} q^\lambda \\ &+(Dk_2)^\mu (Dq)^\kappa (Dq)^\rho g^{\lambda\nu} q^\sigma + (Dk_2)^\mu q^\kappa q^\rho g^{\lambda\nu} q^\sigma - (Dk_1)^\mu q^\kappa (Dq)^\rho g^{\lambda\nu} q^\sigma \\ &+k_2^\nu q^\kappa (Dq)^\rho D^{\lambda\mu}(Dq)^\sigma + k_1^\nu (Dq)^\kappa (Dq)^\rho D^{\sigma\mu}(Dq)^\lambda \\ &-k_1^\nu (Dq)^\kappa (Dq)^\rho D^{\lambda\mu}(Dq)^\sigma\Big] + \tfrac{1}{8}\alpha'^2\Big[-g^{\kappa\nu} q^\rho D^{\sigma\mu}(Dq)^\lambda + g^{\kappa\nu} q^\rho D^{\lambda\mu}(Dq)^\sigma \\ &+D^{\rho\mu}(Dq)^\kappa g^{\sigma\nu} q^\lambda - D^{\rho\mu}(Dq)^\kappa g^{\lambda\nu} q^\sigma\Big]\end{aligned} \tag{C.1.8}$$

$$\begin{aligned}\Sigma^{\mu\nu\kappa\lambda\rho\sigma} &= \tfrac{1}{8}\alpha'^3\Big[-k_2^\nu (Dq)^\kappa (Dq)^\rho D^{\sigma\mu}(Dq)^\lambda + k_2^\nu q^\kappa q^\rho D^{\lambda\mu}(Dq)^\sigma - k_2^\nu q^\kappa q^\rho D^{\sigma\mu}(Dq)^\lambda \\ &+k_2^\nu (Dq)^\kappa (Dq)^\rho D^{\lambda\mu}(Dq)^\sigma + (Dk_2)^\mu (Dq)^\kappa q^\rho g^{\lambda\nu} q^\sigma - (Dk_1)^\mu q^\kappa q^\rho g^{\lambda\nu} q^\sigma \\ &-(Dk_1)^\mu (Dq)^\kappa (Dq)^\rho g^{\lambda\nu} q^\sigma - k_1^\nu (Dq)^\kappa q^\rho D^{\lambda\mu}(Dq)^\sigma + (Dk_1)^\mu q^\kappa q^\rho g^{\sigma\nu} q^\lambda \\ &-(Dk_2)^\mu (Dq)^\kappa q^\rho g^{\sigma\nu} q^\lambda + k_1^\nu (Dq)^\kappa q^\rho D^{\sigma\mu}(Dq)^\lambda \\ &+(Dk_1)^\mu (Dq)^\kappa (Dq)^\rho g^{\sigma\nu} q^\lambda\Big] + \tfrac{1}{8}\alpha'^2\Big[+g^{\rho\nu} q^\kappa D^{\sigma\mu}(Dq)^\lambda - g^{\rho\nu} q^\kappa D^{\lambda\mu}(Dq)^\sigma \\ &-D^{\kappa\mu}(Dq)^\rho g^{\sigma\nu} q^\lambda + D^{\kappa\mu}(Dq)^\rho g^{\lambda\nu} q^\sigma\Big]\end{aligned} \tag{C.1.9}$$

$$\begin{aligned}\Gamma^{\mu\nu\kappa\lambda\rho\sigma} &= \tfrac{1}{8}\alpha'^3\Big[q^\kappa(Dq)^\rho q^\lambda(Dq)^\sigma D^{\nu\mu} - q^\kappa(Dq)^\rho q^\lambda D^{\sigma\mu}(Dq)^\nu \\ &\quad -q^\kappa(Dq)^\rho g^{\lambda\nu}(Dq)^\sigma(Dq)^\mu + q^\kappa(Dq)^\rho g^{\lambda\nu}D^{\sigma\mu}\,qDq\Big]\end{aligned} \tag{C.1.10}$$

$$\begin{aligned}\Delta^{\mu\nu\kappa\lambda\rho\sigma} &= \tfrac{1}{8}\alpha'^3\Big[-(Dq)^\kappa q^\rho q^\sigma(Dq)^\lambda D^{\nu\mu} + (Dq)^\kappa q^\rho q^\sigma D^{\lambda\mu}(Dq)^\nu \\ &\quad +(Dq)^\kappa q^\rho g^{\sigma\nu}(Dq)^\lambda(Dq)^\mu - (Dq)^\kappa q^\rho g^{\sigma\nu}D^{\lambda\mu}\,qDq\Big]\end{aligned} \tag{C.1.11}$$

$$\begin{aligned}\Phi^{\mu\nu\kappa\lambda\rho\sigma} &= \tfrac{1}{16}\alpha'^3\Big[(Dq)^\mu q^\kappa(Dq)^\rho g^{\sigma\nu}q^\lambda - (Dq)^\mu q^\kappa(Dq)^\rho g^{\lambda\nu}q^\sigma \\ &\quad +(Dq)^\nu q^\kappa(Dq)^\rho D^{\lambda\mu}(Dq)^\sigma + q^\kappa q^\rho q^\lambda(Dq)^\sigma D^{\nu\mu} - q^\kappa q^\rho q^\lambda D^{\sigma\mu}(Dq)^\nu \\ &\quad -q^\kappa q^\rho g^{\lambda\nu}(Dq)^\sigma(Dq)^\mu + q^\kappa q^\rho g^{\lambda\nu}D^{\sigma\mu}\,qDq + (Dq)^\kappa(Dq)^\rho q^\lambda(Dq)^\sigma D^{\nu\mu} \\ &\quad -(Dq)^\kappa(Dq)^\rho q^\lambda D^{\sigma\mu}(Dq)^\nu - (Dq)^\kappa(Dq)^\rho g^{\lambda\nu}(Dq)^\sigma(Dq)^\mu \\ &\quad +(Dq)^\kappa(Dq)^\rho g^{\lambda\nu}D^{\sigma\mu}\,qDq - (Dq)^\nu q^\kappa(Dq)^\rho D^{\sigma\mu}(Dq)^\lambda\Big]\end{aligned} \tag{C.1.12}$$

$$\begin{aligned}\Psi^{\mu\nu\kappa\lambda\rho\sigma} &= \tfrac{1}{16}\alpha'^3\Big[(Dq)^\mu(Dq)^\kappa q^\rho g^{\sigma\nu}q^\lambda - (Dq)^\mu(Dq)^\kappa q^\rho g^{\lambda\nu}q^\sigma \\ &\quad +(Dq)^\nu(Dq)^\kappa q^\rho D^{\lambda\mu}(Dq)^\sigma - q^\kappa q^\rho q^\sigma(Dq)^\lambda D^{\nu\mu} + q^\kappa q^\rho q^\sigma D^{\lambda\mu}(Dq)^\nu \\ &\quad +q^\kappa q^\rho g^{\sigma\nu}(Dq)^\lambda(Dq)^\mu - q^\kappa q^\rho g^{\sigma\nu}D^{\lambda\mu}\,qDq - (Dq)^\kappa(Dq)^\rho q^\sigma(Dq)^\lambda D^{\nu\mu} \\ &\quad +(Dq)^\kappa(Dq)^\rho q_2^\sigma D^{\lambda\mu}(Dq)^\nu + (Dq)^\kappa(Dq)^\rho g^{\sigma\nu}(Dq)^\lambda(Dq)^\mu \\ &\quad -(Dq)^\kappa(Dq)^\rho g^{\sigma\nu}D^{\lambda\mu}\,qDq - (Dq)^\nu(Dq)^\kappa q^\rho D^{\sigma\mu}(Dq)^\lambda\Big]\end{aligned} \tag{C.1.13}$$

$$\begin{aligned}\Omega^{\mu\nu\kappa\lambda\rho\sigma} &= \tfrac{1}{16}\alpha'^3\Big[-(Dq)^\nu q^\kappa q^\rho D^{\sigma\mu}(Dq)^\lambda - (Dq)^\nu(Dq)^\kappa(Dq)^\rho D^{\sigma\mu}(Dq)^\lambda \\ &\quad -(Dq)^\mu q^\kappa q^\rho g^{\lambda\nu}q^\sigma + (Dq)^\nu q^\kappa q^\rho D^{\lambda\mu}(Dq)^\sigma + (Dq)^\mu(Dq)^\kappa(Dq)^\rho g^{\sigma\nu}q^\lambda \\ &\quad -(Dq)^\mu(Dq)^\kappa(Dq)^\rho g^{\lambda\nu}q^\sigma + (Dq)^\kappa q^\rho q^\lambda(Dq)^\sigma D^{\nu\mu} + (Dq)^\mu q^\kappa q^\rho g^{\sigma\nu}q^\lambda \\ &\quad -(Dq)^\kappa q^\rho g^{\lambda\nu}(Dq)^\sigma(Dq)^\mu + (Dq)^\kappa q^\rho g^{\lambda\nu}D^{\sigma\mu}\,qDq \\ &\quad +q^\kappa(Dq)^\rho q^\sigma D^{\lambda\mu}(Dq)^\nu + q^\kappa(Dq)^\rho g^{\sigma\nu}(Dq)^\lambda(Dq)^\mu \\ &\quad +(Dq)^\nu(Dq)^\kappa(Dq)^\rho D^{\lambda\mu}(Dq)^\sigma - q^\kappa(Dq)^\rho g^{\sigma\nu}D^{\lambda\mu}\,qDq \\ &\quad -q^\kappa(Dq)^\rho q^\sigma(Dq)^\lambda D^{\nu\mu} - (Dq)^\kappa q^\rho q^\lambda D^{\sigma\mu}(Dq)^\nu\Big]\end{aligned} \tag{C.1.14}$$

and

$$\Gamma \equiv \alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu}\left[\Gamma^{\mu\nu\kappa\lambda\rho\sigma} + \Theta^{\mu\nu\kappa\lambda\rho\sigma}\right], \qquad \Delta \equiv \alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu}\left[\Delta^{\mu\nu\kappa\lambda\rho\sigma} - \Lambda^{\mu\nu\kappa\lambda\rho\sigma}\right] \tag{C.1.15}$$

$$\Theta \equiv \alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu}\left[\Theta^{\mu\nu\kappa\lambda\rho\sigma} - \Phi^{\mu\nu\kappa\lambda\rho\sigma} - \Xi^{\mu\nu\kappa\lambda\rho\sigma}\right] \tag{C.1.16}$$

$$\Lambda \equiv \alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu}\left[\Lambda^{\mu\nu\kappa\lambda\rho\sigma} + \Psi^{\mu\nu\kappa\lambda\rho\sigma} - \Sigma^{\mu\nu\kappa\lambda\rho\sigma}\right] \tag{C.1.17}$$

$$\Xi \equiv \alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu}\left[\Lambda^{\mu\nu\kappa\lambda\rho\sigma} + \Psi^{\mu\nu\kappa\lambda\rho\sigma} - \Omega^{\mu\nu\kappa\lambda\rho\sigma} + \Xi^{\mu\nu\kappa\lambda\rho\sigma} - 2\Sigma^{\mu\nu\kappa\lambda\rho\sigma}\right] \tag{C.1.18}$$

$$\Sigma \equiv \alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu}\Big[\Theta^{\mu\nu\kappa\lambda\rho\sigma} - \Phi^{\mu\nu\kappa\lambda\rho\sigma} + \Omega^{\mu\nu\kappa\lambda\rho\sigma} - 2\Xi^{\mu\nu\kappa\lambda\rho\sigma} + \Sigma^{\mu\nu\kappa\lambda\rho\sigma}\Big] \tag{C.1.19}$$

$$\begin{aligned}\Omega_+ \equiv\ & \alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu}\Big[- \Lambda^{\mu\nu\kappa\lambda\rho\sigma} + \Theta^{\mu\nu\kappa\lambda\rho\sigma} - \Phi^{\mu\nu\kappa\lambda\rho\sigma} - \Psi^{\mu\nu\kappa\lambda\rho\sigma} + 2\Omega^{\mu\nu\kappa\lambda\rho\sigma} \\ & -3\Xi^{\mu\nu\kappa\lambda\rho\sigma} + 3\Sigma^{\mu\nu\kappa\lambda\rho\sigma}\Big]\end{aligned} \tag{C.1.20}$$

$$\begin{aligned}\Omega_- \equiv\ & \alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu}\Big[- \Lambda^{\mu\nu\kappa\lambda\rho\sigma} + \Theta^{\mu\nu\kappa\lambda\rho\sigma} - \Phi^{\mu\nu\kappa\lambda\rho\sigma} - \Psi^{\mu\nu\kappa\lambda\rho\sigma} + 2\Omega^{\mu\nu\kappa\lambda\rho\sigma} \\ & -3\Xi^{\mu\nu\kappa\lambda\rho\sigma} + 3\Sigma^{\mu\nu\kappa\lambda\rho\sigma}\Big]\end{aligned} \tag{C.1.21}$$

Packages for $\mathbf{A}_2$

$$P^{\mu\nu\kappa\rho} = \tfrac{1}{4}\,\alpha'^3\big[k_1^\nu(Dk_2)^\mu q^\kappa(Dq)^\rho\big] - \tfrac{1}{4}\alpha'^2[D^{\rho\mu}k_1^\nu q^\kappa + g^{\kappa\nu}(Dq)^\rho(Dk_2)^\mu] + \tfrac{1}{4}\alpha' g^{\kappa\nu}D^{\rho\mu} \tag{C.1.22}$$

$$\widetilde{P}^{\mu\nu\kappa\rho} = \tfrac{1}{4}\,\alpha'^3\big[(Dk_1)^\mu k_2^\nu(Dq)^\kappa q^\rho\big] - \tfrac{1}{4}\alpha'^2[(Dk_1)^\mu g^{\rho\nu}(Dq)^\kappa + D^{\kappa\mu}q^\rho k_2^\nu] + \tfrac{1}{4}\alpha' D^{\kappa\mu}g^{\rho\nu} \tag{C.1.23}$$

$$\begin{aligned}Q^{\mu\nu\kappa\rho\lambda\sigma} =\ & \tfrac{1}{4}\alpha'^3\Big[[-k_1^\nu q^\kappa(Dk_1)^\mu(Dq)^\rho + k_1^\nu q^\kappa(Dk_2)^\mu q^\rho + (Dk_2)^\mu(Dq)^\rho k_1^\nu(Dq)^\kappa \\ & -(Dk_2)^\mu(Dq)^\rho k_2^\nu q^\kappa]g^{\lambda\sigma} - k_1^\nu k_2^\kappa(Dq)^\rho D^{\lambda\mu}(Dq)^\sigma + D^{\sigma\mu}(Dq)^\lambda k_1^\nu k_2^\kappa(Dq)^\rho \\ & +(Dk_2)^\mu k_2^\kappa(Dq)^\rho g^{\lambda\nu}q^\sigma + (Dk_2)^\mu q^\kappa k_1^\rho g^{\lambda\nu}q^\sigma + k_1^\nu q^\kappa k_1^\rho D^{\sigma\mu}(Dq)^\lambda \\ & -(Dk_2)^\mu k_2^\kappa(Dq)^\rho g^{\sigma\nu}q^\lambda - k_1^\nu q^\kappa k_1^\rho D^{\lambda\mu}(Dq)^\sigma - g^{\sigma\nu}q^\lambda(Dk_2)^\mu q^\kappa k_1^\rho\Big] \\ & +\tfrac{1}{4}\alpha'^2\Big[[g^{\kappa\nu}(Dq)^\rho(Dk_1)^\mu - g^{\kappa\nu}q^\rho(Dk_2)^\mu + D^{\rho\mu}k_2^\nu q^\kappa - D^{\rho\mu}k_1^\nu(Dq)^\kappa]g^{\lambda\sigma} \\ & +g^{\kappa\nu}k_1^\rho D^{\lambda\mu}(Dq)^\sigma - D^{\sigma\mu}(Dq)^\lambda g^{\kappa\nu}k_1^\rho - D^{\rho\mu}k_2^\kappa g^{\lambda\nu}q^\sigma + g^{\sigma\nu}q^\lambda D^{\rho\mu}k_2^\kappa\Big]\end{aligned} \tag{C.1.24}$$

$$\begin{aligned}\widetilde{Q}^{\mu\nu\kappa\rho\lambda\sigma} =\ & \tfrac{1}{4}\alpha'^3\Big[[-(Dk_1)^\mu(Dq)^\kappa k_1^\nu q^\rho - (Dk_1)^\mu(Dq)^\kappa k_2^\nu q^\rho + (Dk_1)^\mu(Dq)^\kappa k_2^\nu(Dq)^\rho \\ & +(Dk_1)^\mu k_2^\nu q^\kappa q^\rho]g^{\lambda\sigma} + D^{\lambda\mu}(Dq)^\sigma k_2^\nu k_2^\kappa q^\rho + D^{\lambda\mu}(Dq)^\sigma k_2^\nu(Dq)^\kappa k_1^\rho \\ & -D^{\sigma\mu}(Dq)^\lambda k_2^\nu k_2^\kappa q^\rho - D^{\sigma\mu}(Dq)^\lambda k_2^\nu(Dq)^\kappa k_1^\rho - g^{\lambda\nu}q^\sigma(Dk_1)^\mu(Dq)^\kappa k_1^\rho \\ & -g^{\lambda\nu}q^\sigma(Dk_1)^\mu k_2^\kappa q^\rho + g^{\sigma\nu}q^\lambda(Dk_1)^\mu(Dq)^\kappa k_1^\rho + g^{\sigma\nu}q^\lambda(Dk_1)^\mu k_2^\kappa q^\rho\Big] \\ & +\tfrac{1}{4}\alpha'^2\Big[[D^{\kappa\mu}q^\rho k_1^\nu - D^{\kappa\mu}(Dq)^\rho k_2^\nu + g^{\rho\nu}(Dk_2)^\mu(Dq)^\kappa - g^{\rho\nu}(Dk_1)^\mu q^\kappa]g^{\lambda\sigma} \\ & -D^{\lambda\mu}(Dq)^\sigma g^{\rho\nu}k_2^\kappa + D^{\sigma\mu}(Dq)^\lambda g^{\rho\nu}k_2^\kappa + g^{\lambda\nu}q^\sigma D^{\kappa\mu}k_1^\rho - g^{\sigma\nu}q^\lambda D^{\kappa\mu}k_1^\rho\Big]\end{aligned} \tag{C.1.25}$$

$$
\begin{aligned}
R^{\mu\nu\kappa\rho\lambda\sigma} = {} & \tfrac{1}{4}\alpha'^3\Big[[-(Dk_1)^\mu k_1^\nu q^\kappa q^\rho - (Dk_1)^\mu k_1^\nu (Dq)^\kappa (Dq)^\rho + (Dk_1)^\mu k_2^\nu q^\kappa (Dq)^\rho \\
& -(Dk_2)^\mu k_2^\nu q^\kappa q^\rho - (Dk_2)^\mu k_2^\nu (Dq)^\kappa (Dq)^\rho + (Dk_2)^\mu k_1^\nu (Dq)^\kappa q^\rho] g^{\lambda\sigma} \\
& -D^{\lambda\mu}(Dq)^\sigma k_1^\nu (Dq)^\kappa k_1^\rho - D^{\lambda\mu}(Dq)^\sigma k_1^\nu k_2^\kappa q^\rho + D^{\lambda\mu}(Dq)^\sigma k_2^\nu q^\kappa k_1^\rho \\
& +D^{\lambda\mu}(Dq)^\sigma k_2^\nu k_2^\kappa (Dq)^\rho + D^{\sigma\mu}(Dq)^\lambda k_1^\nu (Dq)^\kappa k_1^\rho + D^{\sigma\mu}(Dq)^\lambda k_1^\nu k_2^\kappa q^\rho \\
& -D^{\sigma\mu}(Dq)^\lambda k_2^\nu q^\kappa k_1^\rho - D^{\sigma\mu}(Dq)^\lambda k_2^\nu k_2^\kappa (Dq)^\rho - g^{\lambda\nu} q^\sigma (Dk_1)^\mu q^\kappa k_1^\rho \\
& -g^{\lambda\nu} q^\sigma (Dk_1)^\mu k_2^\kappa (Dq)^\rho + g^{\lambda\nu} q^\sigma (Dk_2)^\mu (Dq)^\kappa k_1^\rho + g^{\lambda\nu} q^\sigma (Dk_2)^\mu k_2^\kappa q^\rho \\
& +g^{\sigma\nu} q^\lambda (Dk_1)^\mu q^\kappa k_1^\rho + g^{\sigma\nu} q^\lambda (Dk_1)^\mu k_2^\kappa (Dq)^\rho - g^{\sigma\nu} q^\lambda (Dk_2)^\mu (Dq)^\kappa k_1^\rho \\
& -g^{\sigma\nu} q^\lambda (Dk_2)^\mu k_2^\kappa q^\rho\Big] \\
& +\tfrac{1}{4}\alpha'^2[+g^{\kappa\nu} q^\rho (Dk_1)^\mu + D^{\kappa\mu}(Dq)^\rho k_1^\nu + g^{\rho\nu}(Dk_2)^\mu q^\kappa + D^{\rho\mu} k_2^\nu (Dq)^\kappa] g^{\lambda\sigma}
\end{aligned}
\tag{C.1.26}
$$

$$
\begin{aligned}
S^{\mu\nu\kappa\rho\lambda\sigma} = {} & -\tfrac{1}{8}\alpha'^3\Big[-[(Dq)^\nu q^\kappa (Dk_2)^\mu (Dq)^\rho + (Dq)^\mu (Dq)^\rho k_1^\nu q^\kappa] g^{\lambda\sigma} \\
& +k_1^\rho q^\kappa q^\lambda (Dq)^\sigma D^{\nu\mu} - k_2^\kappa (Dq)^\rho q^\lambda D^{\sigma\mu}(Dq)^\nu - k_2^\kappa (Dq)^\rho g^{\lambda\nu}(Dq)^\sigma (Dq)^\mu \\
& -k_1^\rho q^\kappa q^\lambda D^{\sigma\mu}(Dq)^\nu + k_2^\kappa (Dq)^\rho q^\lambda (Dq)^\sigma D^{\nu\mu} - k_1^\rho q^\kappa g^{\lambda\nu}(Dq)^\sigma (Dq)^\mu \\
& +k_2^\kappa (Dq)^\rho g^{\lambda\nu} D^{\sigma\mu}\, qDq + k_1^\rho q^\kappa g^{\lambda\nu} D^{\sigma\mu}\, qDq\Big] \\
& -\tfrac{1}{8}\alpha'^2[(Dq)^\nu q^\kappa D^{\rho\mu} + (Dq)^\mu (Dq)^\rho g^{\kappa\nu}] g^{\lambda\sigma}
\end{aligned}
\tag{C.1.27}
$$

$$
\begin{aligned}
\widetilde{S}^{\mu\nu\kappa\rho\lambda\sigma} = {} & \tfrac{1}{8}\alpha'^3\Big[(Dq)^\mu (Dq)^\kappa k_2^\nu q^\rho + (Dq)^\nu q^\rho (Dk_1)^\mu (Dq)^\kappa] g^{\lambda\sigma} \\
& +[k_2^\kappa q^\rho + (Dq)^\kappa k_1^\rho]\big[-q^\sigma (Dq)^\lambda D^{\nu\mu} + q^\sigma D^{\lambda\mu}(Dq)^\nu + g^{\sigma\nu}(Dq)^\lambda (Dq)^\mu \\
& -g^{\sigma\nu} D^{\lambda\mu}\, qDq\big]\Big] - \tfrac{1}{8}\alpha'^2[(Dq)^\mu (Dq)^\kappa g^{\rho\nu} + (Dq)^\nu q^\rho D^{\kappa\mu}] g^{\lambda\sigma}
\end{aligned}
\tag{C.1.28}
$$

$$\begin{aligned}
T^{\mu\nu\kappa\rho\lambda\sigma} &= \tfrac{1}{8}\alpha'^3\Big[[(Dq)^\nu(Dq)^\kappa(Dk_2)^\mu(Dq)^\rho+(Dq)^\mu(Dq)^\rho k_1^\nu(Dq)^\kappa \\
&-(Dq)^\mu(Dq)^\rho k_2^\nu q^\kappa-(Dq)^\nu q^\kappa(Dk_1)^\mu(Dq)^\rho+(Dq)^\nu q^\kappa(Dk_2)^\mu q^\rho \\
&+(Dq)^\mu q^\rho k_1^\nu q^\kappa]g^{\lambda\sigma}-\big[(Dq)^\kappa k_1^\rho+k_2^\kappa q^\rho\big]\Big[q^\lambda\big[(Dq)^\sigma D^{\nu\mu}-D^{\sigma\mu}(Dq)^\nu\big] \\
&+g^{\lambda\nu}\big[-(Dq)^\sigma(Dq)^\mu+D^{\sigma\mu}\,qDq\big]\Big] \\
&-\big[q^\kappa k_1^\rho+k_2^\kappa(Dq)^\rho\big]\Big[(Dq)^\nu\big[-D^{\sigma\mu}(Dq)^\lambda+D^{\lambda\mu}(Dq)^\sigma\big] \\
&+(Dq)^\mu\big[-g^{\sigma\nu}q^\lambda+g^{\lambda\nu}q^\sigma\big]\Big]\Big]-\tfrac{1}{8}\alpha'^2[(Dq)^\nu(Dq)^\kappa D^{\rho\mu}+(Dq)^\mu q^\rho g^{\kappa\nu}]g^{\lambda\sigma}
\end{aligned} \tag{C.1.29}$$

$$\begin{aligned}
\widetilde{T}^{\mu\nu\kappa\rho\lambda\sigma} &= \tfrac{1}{8}\alpha'^3\Big[[(Dq)^\mu q^\kappa k_2^\nu q^\rho+(Dq)^\nu(Dq)^\rho(Dk_1)^\mu(Dq)^\kappa-(Dq)^\mu(Dq)^\kappa k_1^\nu q^\rho \\
&+(Dq)^\mu(Dq)^\kappa k_2^\nu(Dq)^\rho-(Dq)^\nu q^\rho(Dk_1)^\mu q^\kappa+(Dq)^\nu q^\rho(Dk_2)^\mu(Dq)^\kappa]g^{\lambda\sigma} \\
&+\big[k_1^\rho q^\kappa+k_2^\kappa(Dq)^\rho\big]\Big[q^\sigma\big[-(Dq)^\lambda D^{\nu\mu}+D^{\lambda\mu}(Dq)^\nu\big]+g^{\sigma\nu}\big[(Dq)^\lambda(Dq)^\mu \\
&-D^{\lambda\mu}\,qDq\big]\Big]+\big[-(Dq)^\mu q^\kappa k_1^\rho-(Dq)^\mu k_2^\kappa(Dq)^\rho\big]\big[-g^{\sigma\nu}q^\lambda+g^{\lambda\nu}q^\sigma\big] \\
&+\big[(Dq)^\nu k_2^\kappa q^\rho+(Dq)^\nu q^\kappa k_1^\rho\big]\big[-D^{\sigma\mu}(Dq)^\lambda+D^{\lambda\mu}(Dq)^\sigma\big]\Big] \\
&-\tfrac{1}{8}\alpha'^2[(Dq)^\mu q^\kappa g^{\rho\nu}+(Dq)^\nu(Dq)^\rho D^{\kappa\mu}]g^{\lambda\sigma}
\end{aligned} \tag{C.1.30}$$

$$\begin{aligned}
U^{\mu\nu\kappa\rho} &= \tfrac{1}{16}\alpha'^3[q_2^\kappa(Dq)^\rho(Dq)^\mu(Dq)^\nu+q^\kappa(Dq)^\rho qDq\,D^{\nu\mu}-q^\kappa(Dq)^\rho(Dq)^\nu(Dq)^\mu] \\
&-\tfrac{1}{8}\alpha'^2\;q^\kappa(Dq)^\rho D^{\nu\mu}
\end{aligned} \tag{C.1.31}$$

$$\begin{aligned}
\widetilde{U}^{\mu\nu\kappa\rho} &= \tfrac{1}{16}\alpha'^3[(Dq)^\kappa q^\rho(Dq)^\mu(Dq)^\nu+(Dq)^\kappa q^\rho qDq\,D^{\nu\mu}-(Dq)^\kappa q^\rho(Dq)^\nu(qD)^\mu] \\
&-\tfrac{1}{8}\alpha'^2\;(Dq)^\kappa q^\rho D^{\nu\mu}
\end{aligned} \tag{C.1.32}$$

$$\begin{aligned}
W^{\mu\nu\kappa\rho} &= \tfrac{1}{16}\alpha'^3\Big[(Dq)^\mu(Dq)^\nu(Dq)^\kappa(Dq)^\rho+(Dq)^\mu(Dq)^\nu q^\kappa q^\rho \\
&-\big[q^\kappa q^\rho+(Dq)^\kappa(Dq)^\rho\big]\big[-qDq\,D^{\nu\mu}+(Dq)^\nu(Dq)^\mu\big]\Big] \\
&-\tfrac{1}{8}\,\alpha'^2\big[D^{\nu\mu}(Dq)^\rho(Dq)^\kappa-D^{\nu\mu}q^\rho q^\kappa\big]
\end{aligned} \tag{C.1.33}$$

and

$$P\equiv\alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu}g^{\lambda\sigma}\,P^{\mu\nu\kappa\rho},\qquad \widetilde{P}\equiv\alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu}g^{\lambda\sigma}\,\widetilde{P}^{\mu\nu\kappa\rho} \tag{C.1.34}$$

$$S\;\equiv\;\alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu}\big[S^{\mu\nu\kappa\rho\lambda\sigma}-Q^{\mu\nu\kappa\rho\lambda\sigma}\big] \tag{C.1.35}$$

$$\widetilde{S} \equiv \alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu}\left[\widetilde{S}^{\mu\nu\kappa\rho\lambda\sigma} - \widetilde{Q}^{\mu\nu\kappa\rho\lambda\sigma}\right] \tag{C.1.36}$$

$$U \equiv \alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu}\left[U^{\mu\nu\kappa\rho}g^{\lambda\sigma} - Q^{\mu\nu\kappa\rho\lambda\sigma} + R^{\mu\nu\kappa\rho\lambda\sigma} - T^{\mu\nu\kappa\rho\lambda\sigma}\right] \tag{C.1.37}$$

$$\widetilde{U} \equiv \alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu}\left[\widetilde{U}^{\mu\nu\kappa\rho}g^{\lambda\sigma} - \widetilde{Q}^{\mu\nu\kappa\rho\lambda\sigma} + R^{\mu\nu\kappa\rho\lambda\sigma} - \widetilde{T}^{\mu\nu\kappa\rho\lambda\sigma}\right] \tag{C.1.38}$$

$$\begin{aligned} W \equiv\ & \alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu}\left[W^{\mu\nu\kappa\rho}g^{\lambda\sigma} + Q^{\mu\nu\kappa\rho\lambda\sigma} + \widetilde{Q}^{\mu\nu\kappa\rho\lambda\sigma} - 2R^{\mu\nu\kappa\rho\lambda\sigma}\right. \\ & \left. + T^{\mu\nu\kappa\rho\lambda\sigma} + \widetilde{T}^{\mu\nu\kappa\rho\lambda\sigma}\right] \end{aligned} \tag{C.1.39}$$

$$\begin{aligned} \widetilde{W} \equiv\ & \alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu}\left[W^{\mu\nu\kappa\rho}g^{\lambda\sigma} + Q^{\mu\nu\kappa\rho\lambda\sigma} + \widetilde{Q}^{\mu\nu\kappa\rho\lambda\sigma} - 2R^{\mu\nu\kappa\rho\lambda\sigma}\right. \\ & \left. + T^{\mu\nu\kappa\rho\lambda\sigma} + \widetilde{T}^{\mu\nu\kappa\rho\lambda\sigma}\right] \end{aligned} \tag{C.1.40}$$

Packages for $\mathbf{A}_3$

$$G^{\mu\nu\kappa\rho} = \tfrac{\alpha'^3}{2}\left[(Dk_2)^\mu k_1^\nu k_2^\kappa (Dq)^\rho + (Dk_2)^\mu k_1^\nu q^\kappa k_1^\rho\right] - \tfrac{\alpha'^2}{2}\left[D^{\rho\mu}k_1^\nu k_2^\kappa + g^{\kappa\nu}k_1^\rho (Dk_2)^\mu\right] \tag{C.1.41}$$

$$H^{\mu\nu\kappa\rho} = \tfrac{\alpha'^3}{2}\left[(Dk_1)^\mu k_2^\nu k_2^\kappa q^\rho + (Dk_1)^\mu k_2^\nu (Dq)^\kappa k_1^\rho\right] - \tfrac{\alpha'^2}{2}\left[g^{\rho\nu}(Dk_1)^\mu k_2^\kappa + D^{\kappa\mu}k_1^\rho k_2^\nu\right] \tag{C.1.42}$$

$$\begin{aligned} I^{\mu\nu\kappa\rho\lambda\sigma} =\ & \tfrac{1}{2}\alpha'^3\Big\{\big[-(Dk_2)^\mu k_2^\nu k_2^\kappa (Dq)^\rho - (Dk_1)^\mu k_1^\nu q^\kappa k_1^\rho - (Dk_1)^\mu k_1^\nu k_2^\kappa (Dq)^\rho \\ & + (Dk_2)^\mu k_1^\nu (Dq)^\kappa k_1^\rho + (Dk_2)^\mu k_1^\nu k_2^\kappa q^\rho - (Dk_2)^\mu k_2^\nu q^\kappa k_1^\rho\big]\, g^{\lambda\sigma} - k_1^\nu k_2^\kappa k_1^\rho\, d^{\sigma\mu\lambda} \\ & + (Dk_2)^\mu k_2^\kappa k_1^\rho\, c^{\sigma\nu\lambda}\Big\} + \tfrac{1}{2}\alpha'^2\big[D^{\rho\mu}k_2^\nu k_2^\kappa\, g^{\lambda\sigma} \\ & + g^{\kappa\nu}(Dk_1)^\mu k_1^\rho\, g^{\lambda\sigma} + \tfrac{1}{2}g^{\kappa\rho}k_1^\nu\, d^{\sigma\mu\lambda} - \tfrac{1}{2}g^{\kappa\rho}(Dk_2)^\mu\, c^{\sigma\nu\lambda}\big] \end{aligned} \tag{C.1.43}$$

$$\begin{aligned} J^{\mu\nu\kappa\rho\lambda\sigma} =\ & \tfrac{1}{2}\alpha'^3\Big\{\big[-(Dk_1)^\mu k_1^\nu (Dq)^\kappa k_1^\rho - (Dk_1)^\mu k_1^\nu k_2^\kappa q^\rho + (Dk_1)^\mu k_2^\nu q^\kappa k_1^\rho \\ & + (Dk_1)^\mu k_2^\nu k_2^\kappa (Dq)^\rho - (Dk_2)^\mu k_2^\nu (Dq)^\kappa k_1^\rho - (Dk_2)^\mu k_2^\nu k_2^\kappa q^\rho\big]\, g^{\lambda\sigma} \\ & - (Dk_1)^\mu k_2^\kappa k_1^\rho\, c^{\sigma\nu\lambda} + k_2^\nu k_2^\kappa k_1^\rho\, d^{\sigma\mu\lambda}\Big\} + \tfrac{1}{2}\alpha'^2\big[D^{\kappa\mu}k_1^\rho k_1^\nu\, g^{\lambda\sigma} \\ & + g^{\rho\nu}(Dk_2)^\mu k_2^\kappa g^{\lambda\sigma} + \tfrac{1}{2}g^{\kappa\rho}(Dk_1)^\mu\, c^{\sigma\nu\lambda} - \tfrac{1}{2}g^{\kappa\rho}k_2^\nu\, d^{\sigma\mu\lambda}\big] \end{aligned} \tag{C.1.44}$$

$$\begin{aligned} K^{\mu\nu\kappa\rho\lambda\sigma} =\ & \tfrac{1}{8}\alpha'^3\Big\{\big[-(Dk_2)^\mu (Dq)^\nu k_2^\kappa (Dq)^\rho - (Dq)^\mu k_1^\nu q^\kappa k_1^\rho - (Dk_2)^\mu (Dq)^\nu q^\kappa k_1^\rho \\ & - (Dq)^\mu k_1^\nu k_2^\kappa (Dq)^\rho\big] g^{\lambda\sigma} + k_2^\kappa k_1^\rho q^\lambda\, \widetilde{f}^{\sigma\nu\mu} + k_2^\kappa k_1^\rho g^{\lambda\nu}\, \widetilde{h}^{\sigma\mu}\Big\} \\ & + \tfrac{1}{8}\alpha'^2\left[g^{\kappa\nu}k_1^\rho (Dq)^\mu + D^{\rho\mu}(Dq)^\nu k_2^\kappa g^{\lambda\sigma} - \tfrac{1}{2}g^{\kappa\rho}q^\lambda\, \widetilde{f}^{\sigma\nu\mu} - \tfrac{1}{2}g^{\kappa\rho}g^{\lambda\nu}\, \widetilde{h}^{\sigma\mu}\right] \end{aligned} \tag{C.1.45}$$

$$
\begin{aligned}
L^{\mu\nu\kappa\rho\lambda\sigma} &= \tfrac{1}{8}\alpha'^3 \Big\{ \big[(Dq)^\mu k_2^\nu k_2^\kappa q^\rho + (Dk_1)^\mu (Dq)^\nu (Dq)^\kappa k_1^\rho + (Dk_1)^\mu (Dq)^\nu k_2^\kappa q^\rho \\
&\quad + (Dq)^\mu k_2^\nu (Dq)^\kappa k_1^\rho\big] g^{\lambda\sigma} + k_2^\kappa k_1^\rho q^\sigma\, f^{\lambda\nu\mu} + k_2^\kappa k_1^\rho g^{\sigma\nu}\, h^{\lambda\mu}\Big\} \\
&\quad + \tfrac{1}{8}\alpha'^2 \big[- D^{\kappa\mu} k_1^\rho (Dq)^\nu - g^{\rho\nu} (Dq)^\mu k_2^\kappa g^{\lambda\sigma} - \tfrac{1}{2} g^{\kappa\rho} q^\sigma\, f^{\lambda\nu\mu} - \tfrac{1}{2} g^{\kappa\rho} g^{\sigma\nu}\, h^{\lambda\mu}\big]
\end{aligned} \tag{C.1.46}
$$

$$
\begin{aligned}
M^{\mu\nu\kappa\rho\lambda\sigma} &= \tfrac{1}{8}\alpha'^3 \Big\{ \big[- (Dk_2)^\mu (Dq)^\nu (Dq)^\kappa k_1^\rho + (Dk_1)^\mu (Dq)^\nu k_2^\kappa (Dq)^\rho \\
&\quad + (Dq)^\mu k_2^\nu q^\kappa k_1^\rho + (Dk_1)^\mu (Dq)^\nu q^\kappa k_1^\rho - (Dq)^\mu k_1^\nu (Dq)^\kappa k_1^\rho - (Dq)^\mu k_1^\nu k_2^\kappa q^\rho \\
&\quad + (Dq)^\mu k_2^\nu k_2^\kappa (Dq)^\rho - (Dk_2)^\mu (Dq)^\nu k_2^\kappa q^\rho\big] g^{\lambda\sigma} + (Dq)^\nu k_2^\kappa k_1^\rho d^{\sigma\mu\lambda} \\
&\quad - (Dq)^\mu k_2^\kappa k_1^\rho c^{\sigma\nu\lambda}\Big\} + \tfrac{1}{16}\,\alpha'^2 \big[- g^{\kappa\rho} (Dq)^\nu d^{\sigma\mu\lambda} + g^{\kappa\rho} (Dq)^\mu c^{\sigma\nu\lambda}\big]
\end{aligned} \tag{C.1.47}
$$

$$
\begin{aligned}
N^{\mu\nu\kappa\rho} &= \tfrac{1}{32}\alpha'^3 \big[- (Dq)^\mu (Dq)^\nu k_2^\kappa q^\rho - (Dq)^\mu (Dq)^\nu (Dq)^\kappa k_1^\rho + (Dq)^\kappa k_1^\rho l^{\nu\mu} \\
&\quad + k_2^\kappa q^\rho l^{\nu\mu}\big] + \tfrac{1}{16}\alpha'^2 \big[D^{\nu\mu} q^\rho k_2^\kappa + D^{\nu\mu} k_1^\rho (Dq)^\kappa\big]
\end{aligned} \tag{C.1.48}
$$

$$
\begin{aligned}
O^{\mu\nu\kappa\rho} &= \tfrac{1}{32}\alpha'^3 \big[- (Dq)^\mu (Dq)^\nu k_2^\kappa (Dq)^\rho - (Dq)^\mu (Dq)^\nu q^\kappa k_1^\rho \\
&\quad + k_1^\rho q^\kappa l^{\nu\mu} + k_2^\kappa (Dq)^\rho l^{\nu\mu}\big] + \tfrac{1}{16}\alpha'^2 \big[D^{\nu\mu} (Dq)^\rho k_2^\kappa + D^{\nu\mu} k_1^\rho q^\kappa\big]
\end{aligned} \tag{C.1.49}
$$

and

$$
\begin{aligned}
G &= \alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu} g^{\lambda\sigma}\, G^{\mu\nu\kappa\rho} \quad , \quad H = \alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu} g^{\lambda\sigma}\, H^{\mu\nu\kappa\rho} \\
K &\equiv \alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu} \big[K^{\mu\nu\kappa\rho\lambda\sigma} + \tfrac{1}{2} I^{\mu\nu\kappa\rho\lambda\sigma}\big] \quad , \quad L = \alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu} \big[L^{\mu\nu\kappa\rho\lambda\sigma} - \tfrac{1}{2} J^{\mu\nu\kappa\rho\lambda\sigma}\big] \\
N &\equiv \alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu} \big[4N^{\mu\nu\kappa\rho} g^{\lambda\sigma} + 2M^{\mu\nu\kappa\rho\lambda\sigma} - I^{\mu\nu\kappa\rho\lambda\sigma} + J^{\mu\nu\kappa\rho\lambda\sigma}\big] \\
O &\equiv \alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu} \big[4O^{\mu\nu\kappa\rho} g^{\lambda\sigma} - 2M^{\mu\nu\kappa\rho\lambda\sigma} + I^{\mu\nu\kappa\rho\lambda\sigma} - J^{\mu\nu\kappa\rho\lambda\sigma}\big]
\end{aligned} \tag{C.1.50}
$$

Packages for $\mathbf{A}_4$

$$
\begin{aligned}
A^{\mu\nu\kappa\rho} &= \tfrac{1}{16}\alpha'^3 \big[l^{\nu\mu} - (Dq)^\mu (Dq)^\nu\big] k_2^\kappa k_1^\rho \\
&\quad + \tfrac{1}{8}\alpha'^2 \Big[D^{\nu\mu} k_1^\rho k_2^\kappa - \tfrac{1}{4}\big(l^{\nu\mu} - (Dq)^\mu (Dq)^\nu\big) g^{\kappa\rho}\Big] - \tfrac{1}{16}\alpha' D^{\nu\mu} g^{\kappa\rho}
\end{aligned} \tag{C.1.51}
$$

$$
B^{\mu\nu\kappa\rho} = -\alpha'^3 \big[(Dk_1)^\mu k_1^\nu + (Dk_2)^\mu k_2^\nu\big] k_2^\kappa k_1^\rho + \tfrac{1}{2}\alpha'^2 \big[(Dk_1)^\mu k_1^\nu + (Dk_2)^\mu k_2^\nu\big] g^{\kappa\rho} \tag{C.1.52}
$$

$$
C^{\mu\nu\kappa\rho} = \alpha'^3\, (Dk_1)^\mu k_2^\nu k_2^\kappa k_1^\rho - \tfrac{1}{2}\alpha'^2\, (Dk_1)^\mu k_2^\nu g^{\kappa\rho} \tag{C.1.53}
$$

$$
\widetilde{\Delta}^{\mu\nu\kappa\rho} = \alpha'^3\, (Dk_2)^\mu k_1^\nu k_2^\kappa k_1^\rho - \tfrac{1}{2}\alpha'^2\, (Dk_2)^\mu k_1^\nu g^{\kappa\rho} \tag{C.1.54}
$$

$$
E^{\mu\nu\kappa\rho} = -\tfrac{\alpha'^3}{2} \big[(Dk_1)^\mu (Dq)^\nu + (Dq)^\mu k_2^\nu\big] k_2^\kappa k_1^\rho + \tfrac{\alpha'^2}{4} \big[(Dk_1)^\mu (Dq)^\nu + (Dq)^\mu k_2^\nu\big] g^{\kappa\rho} \tag{C.1.55}
$$

$$F^{\mu\nu\kappa\rho} = \frac{\alpha'^3}{2}\left[(Dk_2)^\mu(Dq)^\nu + (Dq)^\mu k_1^\nu\right]k_2^\kappa k_1^\rho - \frac{\alpha'^2}{4}\left[(Dk_2)^\mu(Dq)^\nu + (Dq)^\mu k_1^\nu\right]g^{\kappa\rho} \tag{C.1.56}$$

and

$$\begin{aligned} A &\equiv \alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu}g^{\lambda\sigma}A^{\mu\nu\kappa\rho}\,,\; C = \alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu}g^{\lambda\sigma}C^{\mu\nu\kappa\rho}\,,\; \widetilde{\Delta} = \alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu}g^{\lambda\sigma}\widetilde{\Delta}^{\mu\nu\kappa\rho} \\ E &\equiv \alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu}\,g^{\lambda\sigma}\left[2E^{\mu\nu\kappa\rho} + B^{\mu\nu\kappa\rho}\right] \quad , \quad F = \alpha^1_{\kappa\lambda}\alpha^2_{\rho\sigma}\varepsilon_{\mu\nu}\,g^{\lambda\sigma}\left[2F^{\mu\nu\kappa\rho} - B^{\mu\nu\kappa\rho}\right] \end{aligned} \tag{C.1.57}$$

C.1.3 Computation of the relevant integrals

To calculate the integrals of x that appear in our calculation, we first observe that

$$\begin{aligned} \int_{-\infty}^{+\infty} dx|x|^{s+2}(x^2+1)^{-s}\frac{(x+i)(x-i)}{x^4} &= 2\int_0^{+\infty} dx\, x^{s-2}(x^2+1)^{-s}(x+i)(x-i) \\ &= 2\frac{2^{-s}\sqrt{\pi}\,s\,\Gamma\left(\frac{s-1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)} \quad \text{if} \quad \Re(s) > 1\,. \end{aligned} \tag{C.1.58}$$

Similarly,

$$\int_{-\infty}^{+\infty} dx|x|^{s+2}(x^2+1)^{-s}\frac{(x+i)(x-i)}{x^4(x+i)^2} = \int_{-\infty}^{+\infty} dx|x|^{s+2}(x^2+1)^{-s}\frac{(x+i)(x-i)}{x^4(x-i)^2} = I_C + \overline{I}_C \tag{C.1.59}$$

$$\int_{-\infty}^{+\infty} dx|x|^{s+2}(x^2+1)^{-s}\frac{(x+i)(x-i)}{x^4(x+i)} = -\int_{-\infty}^{+\infty} dx|x|^{s+2}(x^2+1)^{-s}\frac{(x+i)(x-i)}{x^4(x-i)} = I_E - \overline{I}_E \tag{C.1.60}$$

$$\int_{-\infty}^{\infty} dx\,|x|^{s+2}(x^2+1)^{-s}\frac{(x+i)(x-i)}{x^3(x-i)^3} = \int_{-\infty}^{\infty} dx\,|x|^{s+2}(x^2+1)^{-s}\frac{(x+i)(x-i)}{x^3(x+i)^3} = I_G + \overline{I}_G \tag{C.1.61}$$

$$\int_{-\infty}^{\infty} dx\,|x|^{s+2}(x^2+1)^{-s}\frac{(x+i)(x-i)}{x^3(x-i)^2} = -\int_{-\infty}^{\infty} dx\,|x|^{s+2}(x^2+1)^{-s}\frac{(x+i)(x-i)}{x^3(x+i)^2} = I_K - \overline{I}_K \tag{C.1.62}$$

$$\int_{-\infty}^{\infty} dx\,|x|^{s+2}(x^2+1)^{-s}\frac{(x+i)(x-i)}{x^3(x-i)} = \int_{-\infty}^{\infty} dx\,|x|^{s+2}(x^2+1)^{-s}\frac{(x+i)(x-i)}{x^3(x+i)} = I_O + \overline{I}_O \tag{C.1.63}$$

$$\int_{-\infty}^{\infty} dx\,|x|^{s+2}(x^2+1)^{-s}\frac{(x+i)(x-i)}{x^2(x-i)^4} = \int_{-\infty}^{\infty} dx\,|x|^{s+2}(x^2+1)^{-s}\frac{(x+i)(x-i)}{x^2(x+i)^4} = I_P + \overline{I}_P \tag{C.1.64}$$

$$\int_{-\infty}^{\infty} dx\,|x|^{s+2}(x^2+1)^{-s}\frac{(x+i)(x-i)}{x^2(x-i)^3} = -\int_{-\infty}^{\infty} dx\,|x|^{s+2}(x^2+1)^{-s}\frac{(x+i)(x-i)}{x^2(x+i)^3} = I_S - \overline{I}_S \tag{C.1.65}$$

$$\int_{-\infty}^{\infty} dx\,|x|^{s+2}(x^2+1)^{-s}\frac{(x+i)(x-i)}{x^2(x-i)^2} = \int_{-\infty}^{\infty} dx\,|x|^{s+2}(x^2+1)^{-s}\frac{(x+i)(x-i)}{x^2(x+i)^2} = I_U + \overline{I}_U \tag{C.1.66}$$

$$\int_{-\infty}^{\infty} dx\,|x|^{s+2}(x^2+1)^{-s}\frac{(x+i)(x-i)}{x^2(x-i)} = -\int_{-\infty}^{\infty} dx\,|x|^{s+2}(x^2+1)^{-s}\frac{(x+i)(x-i)}{x^2(x+i)} = I_W - \overline{I}_W \tag{C.1.67}$$

$$\int_{-\infty}^{\infty} dx\,|x|^{s+2}(x^2+1)^{-s}\tfrac{(x+i)(x-i)}{x(x-i)^4} = -\int_{-\infty}^{\infty} dx\,|x|^{s+2}(x^2+1)^{-s}\tfrac{(x+i)(x-i)}{x(x+i)^4} = I_\Gamma - \overline{I}_\Gamma \tag{C.1.68}$$

$$\int_{-\infty}^{\infty} dx\,|x|^{s+2}(x^2+1)^{-s}\tfrac{(x+i)(x-i)}{x(x-i)^3} = \int_{-\infty}^{\infty} dx\,|x|^{s+2}(x^2+1)^{-s}\tfrac{(x+i)(x-i)}{x(x+i)^3} = I_\Theta + \overline{I}_\Theta \tag{C.1.69}$$

$$\int_{-\infty}^{\infty} dx\,|x|^{s+2}(x^2+1)^{-s}\tfrac{(x+i)(x-i)}{x(x-i)^2} = -\int_{-\infty}^{\infty} dx\,|x|^{s+2}(x^2+1)^{-s}\tfrac{(x+i)(x-i)}{x(x+i)^2} = I_\Sigma - \overline{I}_\Sigma \tag{C.1.70}$$

$$\int_{-\infty}^{\infty} dx\,|x|^{s+2}(x^2+1)^{-s}\tfrac{(x+i)(x-i)}{x(x-i)} = \int_{-\infty}^{\infty} dx\,|x|^{s+2}(x^2+1)^{-s}\tfrac{(x+i)(x-i)}{x(x+i)} = I_{\Omega_-} + \overline{I}_{\Omega_-} \tag{C.1.71}$$

where

$$I_C \equiv -\tfrac{1}{2}\left[\frac{2^{-s}\sqrt{\pi}\Gamma\left(\frac{s-1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)} + i\frac{\Gamma\left(\frac{s}{2}\right)^2}{\Gamma(s)}\right] \quad \text{if} \quad \Re(s) > 1 \tag{C.1.72}$$

$$I_E \equiv \frac{\Gamma\left(\frac{s}{2}\right)^2}{2\Gamma(s)} - i\frac{(s-1)\Gamma\left(\frac{s-1}{2}\right)^2}{4\Gamma(s)} \quad \text{if} \quad \Re(s) > 1 \tag{C.1.73}$$

$$I_G \equiv -\frac{\pi^{3/2}2^{-s-1}\sec\left(\frac{\pi s}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+1\right)} + \frac{i\pi(s-1)\csc\left(\frac{\pi s}{2}\right)\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(-\frac{s}{2}-1\right)\Gamma(s+3)} \quad \text{if} \quad \Re(s) > 0 \tag{C.1.74}$$

$$I_K \equiv \frac{i\sqrt{\pi}2^{-s}\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)} \quad \text{if} \quad \Re(s) > 0 \tag{C.1.75}$$

$$I_O \equiv \frac{\Gamma\left(\frac{s-1}{2}\right)\Gamma\left(\frac{s+1}{2}\right)+i\Gamma\left(\frac{s}{2}\right)^2}{2\Gamma(s)} \quad \text{if} \quad \Re(s) > 1 \tag{C.1.76}$$

$$I_P \equiv -\frac{2^{-s-2}\sqrt{\pi}s\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}+2\right)} \quad \text{if} \quad \Re(s) > -1 \tag{C.1.77}$$

$$I_S \equiv \frac{(1-s)\Gamma\left(\frac{s}{2}+1\right)\Gamma\left(\frac{s}{2}\right)+2i\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+3}{2}\right)}{2\Gamma(s+2)} \quad \text{if} \quad \Re(s) > 0 \tag{C.1.78}$$

$$I_U \equiv \frac{\Gamma\left(\frac{s-1}{2}\right)\Gamma\left(\frac{s+1}{2}\right)+2i\Gamma\left(\frac{s}{2}+1\right)\Gamma\left(\frac{s}{2}\right)}{2\Gamma(s+1)} \quad \text{if} \quad \Re(s) > 1 \tag{C.1.79}$$

$$I_W \equiv -\frac{\pi\csc\left(\frac{\pi s}{2}\right)\Gamma\left(\frac{s}{2}+1\right)}{2\Gamma\left(2-\frac{s}{2}\right)\Gamma(s)} - \frac{i\pi^{3/2}2^{-s}\sec\left(\frac{\pi s}{2}\right)}{\Gamma\left(\frac{3}{2}-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}\right)} \quad \text{if} \quad \Re(s) > 2 \tag{C.1.80}$$

$$I_\Gamma \equiv \frac{(1-s)\Gamma\left(\frac{s}{2}+2\right)\Gamma\left(\frac{s}{2}\right)+2i\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+3}{2}\right)}{\Gamma(s+3)} \quad \text{if} \quad \Re(s) > 0 \tag{C.1.81}$$

$$I_\Theta \equiv \sqrt{\pi}2^{-s-2}\left(\frac{\pi(s-3)\sec\left(\frac{\pi s}{2}\right)}{\Gamma\left(\frac{3}{2}-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+1\right)} + \frac{i(s+3)\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+3}{2}\right)}\right) \quad \text{if} \quad \Re(s) > 1 \tag{C.1.82}$$

$$I_\Sigma \equiv \frac{\sqrt{\pi}2^{-s}\Gamma\left(\frac{s}{2}-1\right)}{\Gamma\left(\frac{s+1}{2}\right)} + \frac{i\Gamma\left(\frac{s-1}{2}\right)\Gamma\left(\frac{s+3}{2}\right)}{\Gamma(s+1)} \quad \text{if} \quad \Re(s) > 2 \tag{C.1.83}$$

$$I_{\Omega_-} \equiv \frac{\pi\left(\frac{\sec\left(\frac{\pi s}{2}\right)\Gamma\left(\frac{s+3}{2}\right)}{\Gamma\left(\frac{5}{2}-\frac{s}{2}\right)} - \frac{i\csc\left(\frac{\pi s}{2}\right)\Gamma\left(\frac{s}{2}+1\right)}{\Gamma\left(2-\frac{s}{2}\right)}\right)}{2\Gamma(s)} \quad \text{if} \quad \Re(s) > 3 \tag{C.1.84}$$

C.1.4 Expansions

$$\begin{aligned}
\mathbf{A}_1 &= (\Gamma-\Delta)\left[\tfrac{\pi}{4}+\tfrac{1}{2}\pi\,\alpha'\,k_1\cdot k_3+\tfrac{1}{6}\pi\,(6+\pi^2)\;\alpha'^2\,(k_1\cdot k_3)^2+\mathcal{O}\left(\alpha'^3\right)\right]\\
&\quad-(\Theta+\Lambda)\left[\tfrac{3\pi}{8}-\pi\,\alpha'\,k_1\cdot k_3+\tfrac{1}{4}\pi(16+\pi^2)\,\alpha'^2\,(k_1\cdot k_3)^2+\mathcal{O}\left(\alpha'^3\right)\right]\\
&\quad+(\Xi+\Sigma)\left[\tfrac{\pi}{8}-\pi\,\alpha'\,k_1\cdot k_3+\tfrac{1}{12}\pi(48+\pi^2)\,\alpha'^2\,(k_1\cdot k_3)^2+\mathcal{O}\left(\alpha'^3\right)\right]\\
&\quad+(\Omega_+-\Omega_-)\left[-\tfrac{1}{12}\pi\,\alpha'\,k_1\cdot k_3+\tfrac{7}{9}\pi\,\alpha'^2\,(k_1\cdot k_3)^2+\mathcal{O}\left(\alpha'^3\right)\right]\\
&= \tfrac{\pi\,\alpha'^2}{16}\left[\mathrm{Tr}(\varepsilon\cdot\alpha^1)\alpha^2_{\mu\nu}q^\mu q^\nu+\mathrm{Tr}(\varepsilon\cdot\alpha^2)\alpha^1_{\mu\nu}q^\mu q^\nu-2(\alpha^1\cdot\varepsilon\cdot\alpha^2)_{\mu\nu}q^\mu q^\nu\right]+\mathcal{O}\left(\alpha'^3\right)
\end{aligned} \tag{C.1.85}$$

$$\begin{aligned}
\mathbf{A}_2 &= (P+\widetilde{P})\;\left[\tfrac{1}{2}\pi\,\alpha'\,k_1\cdot k_3+\pi\,\alpha'^2\,(k_1\cdot k_3)^2+\mathcal{O}\left(\alpha'^3\right)\right]\\
&\quad+(S+\widetilde{S})\;\left[-\tfrac{\pi}{8}-\tfrac{1}{12}\pi^3\,\alpha'^2\,(k_1\cdot k_3)^2+\mathcal{O}\left(\alpha'^3\right)\right]\\
&\quad+(U+\widetilde{U})\;\left[\tfrac{\pi}{8}-\tfrac{1}{2}\pi\,\alpha'\,k_1\cdot k_3+\tfrac{1}{12}\pi(24+\pi^2)\,\alpha'^2\,(k_1\cdot k_3)^2+\mathcal{O}\left(\alpha'^3\right)\right]\\
&\quad+(W+\widetilde{W})\;\left[-\tfrac{\pi}{4}\,\alpha'\,k_1\cdot k_3+\pi\,\alpha'^2\,(k_1\cdot k_3)^2+\mathcal{O}\left(\alpha'^3\right)\right]\\
&= \tfrac{\pi}{8}\alpha'^2\left[\mathrm{Tr}(\varepsilon\cdot\alpha^1\cdot\alpha^2)\,k_1\cdot q+(\varepsilon\cdot\alpha^1\cdot\alpha^2)_{\mu\nu}k_1^\mu q^\nu+(\varepsilon\cdot\alpha^2\cdot\alpha^1)_{\mu\nu}k_2^\mu q^\nu\right]\\
&\quad+\mathcal{O}(\alpha'^3)
\end{aligned} \tag{C.1.86}$$

$$\begin{aligned}
\mathbf{A}_3 &= (G+H)\left[-\tfrac{\pi}{8}-\tfrac{1}{12}\pi^3\,\alpha'^2\,(k_1\cdot k_3)^2+\mathcal{O}\left(\alpha'^3\right)\right]\\
&\quad+(K-L)\left[\tfrac{\pi}{4}+\tfrac{1}{6}\pi^3\,\alpha'^2\,(k_1\cdot k_3)^2+\mathcal{O}\left(\alpha'^3\right)\right]\\
&\quad+(N+O)\left[\tfrac{1}{4}\pi\,\alpha'\,k_1\cdot k_3-\pi\alpha'^2\,(k_1\cdot k_3)^2+\mathcal{O}\left(\alpha'^3\right)\right]\\
&= \tfrac{\pi}{8}\alpha'^2\left[(\varepsilon\cdot\alpha^2\cdot\alpha^1)_{\mu\nu}k_1^\mu k_2^\nu+(\varepsilon\cdot\alpha^1\cdot\alpha^2)_{\mu\nu}k_1^\nu k_2^\mu\right]+\mathcal{O}(\alpha'^3)
\end{aligned} \tag{C.1.87}$$

$$\begin{aligned}
\mathbf{A}_4 &= A\left[\pi\,\alpha'\,k_1\cdot k_3-4\pi\,\alpha'^2(k_1\cdot k_3)^2\;+\mathcal{O}\left(\alpha'^3\right)\right]\\
&\quad+(C+\widetilde{\Delta})\left[\tfrac{\pi}{8}-\tfrac{1}{2}\pi\,\alpha'\,k_1\cdot k_3+\tfrac{1}{12}\pi(24+\pi^2)\,\alpha'^2\,(k_1\cdot k_3)^2+\mathcal{O}\left(\alpha'^3\right)\right]\\
&\quad+(E-F)\left[\tfrac{1}{4}\pi\,\alpha'\,k_1\cdot k_3-\pi\,\alpha'^2\,(k_1\cdot k_3)^2+\mathcal{O}\left(\alpha'^3\right)\right]\\
&= -\tfrac{\pi}{8}\alpha'^2\,\mathrm{Tr}(\alpha^1\cdot\alpha^2)\,\varepsilon_{\mu\nu}k_1^\mu k_2^\nu+\mathcal{O}(\alpha'^3)\,,
\end{aligned} \tag{C.1.88}$$

where q is related to k_3 and k_4 via the definitions (7.3.70) on the disk and in the special case (7.3.60).

C.2 Calculations for the $\mathcal{A}(3,0)$ amplitude

C.2.1 Contractions for the supersymmetric case

Performing the contractions in (7.3.63) using the disk correlators in table (5.1) we obtain

$$\begin{aligned}\mathcal{X}_1 &= i\widetilde{\mathcal{E}}\frac{1}{x_{12}^2x_{13}^2x_{23}^2}\Big[(2\alpha')^2\big[g^{\nu_1\nu_2}g^{\mu_1\mu_3}g^{\mu_2\nu_3}+g^{\nu_1\nu_2}g^{\mu_1\nu_3}g^{\mu_2\mu_3}\big]\\ &+(2\alpha')^3\big[g^{\nu_1\nu_2}g^{\mu_1\mu_2}k_1^{\mu_3}k_1^{\nu_3}-g^{\nu_1\nu_2}g^{\mu_1\mu_3}k_1^{\nu_3}k_1^{\mu_2}-g^{\nu_1\nu_2}g^{\mu_1\nu_3}k_1^{\mu_3}k_1^{\mu_2}\\ &+g^{\nu_1\nu_2}g^{\mu_2\mu_3}k_2^{\mu_1}k_1^{\nu_3}+g^{\nu_1\nu_2}g^{\mu_2\nu_3}k_2^{\mu_1}k_1^{\mu_3}\big]-(2\alpha')^4\,g^{\nu_1\nu_2}k_1^{\mu_3}k_1^{\nu_3}k_1^{\mu_2}k_2^{\mu_1}\Big]\end{aligned}\tag{C.2.89}$$

$$\mathcal{X}_2 = i\widetilde{\mathcal{E}}\frac{1}{x_{12}^2x_{13}^2x_{23}^2}\Big[(2\alpha')^2\,g^{\nu_1\mu_3}g^{\mu_1\mu_2}g^{\nu_2\nu_3}-(2\alpha')^3g^{\nu_1\mu_3}g^{\nu_2\nu_3}k_1^{\mu_2}k_2^{\mu_1}\Big]\tag{C.2.90}$$

$$\begin{aligned}\mathcal{X}_3 &= i\widetilde{\mathcal{E}}\frac{1}{x_{12}^2x_{13}^2x_{23}^2}\Big\{(2\alpha')^3\big[-g^{\mu_1\mu_2}g^{\nu_2\nu_3}k_1^{\mu_3}k_3^{\nu_1}+g^{\mu_1\mu_2}g^{\nu_1\nu_3}k_1^{\mu_3}k_3^{\nu_2}+g^{\mu_1\mu_3}g^{\nu_2\nu_3}k_1^{\mu_2}k_3^{\nu_1}\\ &-g^{\mu_1\mu_3}g^{\nu_1\nu_3}k_1^{\mu_2}k_3^{\nu_2}-g^{\mu_2\mu_3}g^{\nu_2\nu_3}k_2^{\mu_1}k_3^{\nu_1}+g^{\mu_2\mu_3}g^{\nu_1\nu_3}k_2^{\mu_1}k_3^{\nu_2}\big]\\ &+(2\alpha')^4\big[g^{\nu_2\nu_3}k_3^{\nu_1}k_1^{\mu_3}k_1^{\mu_2}k_2^{\mu_1}-g^{\nu_1\nu_3}k_3^{\nu_2}k_1^{\mu_3}k_1^{\mu_2}k_2^{\mu_1}\big]\Big\}\,,\end{aligned}\tag{C.2.91}$$

where we have used momentum conservation (7.3.63) and the on–shell conditions. For $\mathcal{X}_2$ we have also used the fact that the polarization tensors are symmetric in particular for the case of $\alpha^3_{\mu_3\nu_3}$. This was necessary for both the term with no momenta and the one with two momenta in order to extract the overall x–dependence of $\mathcal{X}_2$.

C.2.2 Contractions for the bosonic case

$$\begin{aligned}&-\langle:\partial X_1^{\mu_1}\partial X_1^{\nu_1}e^{ik_1X_1}::\partial X_2^{\mu_2}\partial X_2^{\nu_2}e^{ik_2X_2}::\partial X_3^{\mu_3}\partial X_3^{\nu_3}e^{ik_3X_3}:\rangle\\ &=\frac{(2\alpha')^3\,\widetilde{\mathcal{E}}}{x_{12}^2x_{13}^2x_{23}^2}\Big\{\big[g^{\mu_1\mu_2}g^{\nu_1\mu_3}g^{\nu_2\nu_3}+g^{\mu_1\mu_2}g^{\nu_1\nu_3}g^{\nu_2\mu_3}+g^{\mu_1\nu_2}g^{\nu_1\mu_3}g^{\mu_2\nu_3}+g^{\mu_1\nu_2}g^{\nu_1\nu_3}g^{\mu_2\mu_3}\\ &+g^{\mu_1\mu_3}g^{\nu_1\mu_2}g^{\nu_2\nu_3}+g^{\mu_1\mu_3}g^{\nu_1\nu_2}g^{\mu_2\nu_3}+g^{\mu_1\nu_3}g^{\nu_1\mu_2}g^{\nu_2\mu_3}+g^{\mu_1\nu_3}g^{\nu_1\nu_2}g^{\mu_2\mu_3}\big]\\ &+4\alpha'\Big[g^{\mu_1\mu_3}g^{\nu_1\nu_3}k_1^{\mu_2}k_1^{\nu_2}+g^{\mu_1\mu_2}g^{\nu_1\nu_2}k_2^{\mu_3}k_2^{\nu_3}+g^{\mu_2\mu_3}g^{\nu_2\nu_3}k_3^{\mu_1}k_3^{\nu_1}\\ &+g^{\mu_1\mu_2}g^{\nu_2\mu_3}k_1^{\nu_3}k_2^{\nu_1}+g^{\mu_1\mu_2}g^{\nu_2\nu_3}k_1^{\mu_3}k_2^{\nu_1}-g^{\mu_1\mu_2}g^{\nu_1\mu_3}k_1^{\nu_3}k_1^{\nu_2}-g^{\mu_1\mu_2}g^{\nu_1\nu_3}k_1^{\mu_3}k_1^{\nu_2}\\ &-g^{\mu_1\nu_2}g^{\nu_1\mu_3}k_1^{\nu_3}k_1^{\mu_2}-g^{\mu_1\nu_2}g^{\nu_1\nu_3}k_1^{\mu_3}k_1^{\mu_2}-g^{\mu_1\mu_3}g^{\mu_2\nu_3}k_1^{\nu_2}k_2^{\nu_1}-g^{\mu_1\mu_3}g^{\nu_2\nu_3}k_1^{\mu_2}k_2^{\nu_1}\\ &+g^{\nu_1\mu_2}g^{\nu_2\mu_3}k_1^{\nu_3}k_2^{\mu_1}+g^{\nu_1\mu_2}g^{\nu_2\nu_3}k_1^{\mu_3}k_2^{\mu_1}-g^{\nu_1\mu_3}g^{\mu_2\nu_3}k_1^{\nu_2}k_2^{\mu_1}-g^{\nu_1\mu_3}g^{\nu_2\nu_3}k_1^{\mu_2}k_2^{\mu_1}\Big]\\ &-16\alpha'^2\Big[g^{\mu_1\mu_2}k_1^{\mu_3}k_1^{\nu_3}k_1^{\nu_2}k_2^{\nu_1}-g^{\mu_1\mu_3}k_1^{\mu_2}k_1^{\nu_2}k_1^{\nu_3}k_2^{\nu_1}+g^{\mu_2\mu_3}k_2^{\mu_1}k_2^{\nu_1}k_1^{\nu_3}k_1^{\nu_2}\Big]\\ &+8\alpha'^3\,k_1^{\mu_2}k_1^{\nu_2}k_2^{\mu_3}k_2^{\nu_3}k_3^{\mu_1}k_3^{\nu_1}\Big\}\,.\end{aligned}\tag{C.2.92}$$

Notice that (C.2.89)–(C.2.92) are valid strictly when contracted with the polarization tensors within the amplitude and that we are able to factorize their x–dependence and write it as the overall prefactor $\frac{1}{x_{12}^2 x_{13}^2 x_{23}^2}$.

C.3 Picture changing operator with T_{int}

In this section we show how to we calculated the picture zero vertex operators in (7.3.57), (7.3.58), and (7.3.59). After the compactification, $T_F(z)$ is given in three different forms (7.3.45), (7.3.47), and (7.3.51) associated to each comapctification and SUSY. For all cases $\mathcal{N} = 1, 2, 4$, since $T_{\mathrm{F,int}}(z)$ is decoupled from matter fields we only need to look at the internal current $\mathcal{J}^\star$ contractions with the internal energy momentum tensor. Consequently, we look at each case individually:

- $\mathcal{N} = 1$ case the picture (-1) is given by:

$$V_{\boldsymbol{A}}^{(-1)}(z, a, p) = g_o \sqrt{\frac{\alpha'}{6}}\, T^a\, e^{-\phi(z)}\, a_\mu\, \psi^\mu(z) \mathcal{J}(z)\, e^{ipX(z)} \tag{C.3.93}$$

  As we discussed in the (5.5.88), we should calculate the following limit:

$$\begin{aligned} &V_{\boldsymbol{A}}^{(0)}(z, a, p) = g_{\mathrm{o}}\, T^a\, a_\mu \\ &\times \lim_{w\to z} e^{\phi(z)} \frac{1}{\sqrt{2}} \left[T_{\mathrm{F,int}}^{+}(z) + T_{\mathrm{F,int}}^{-}(z)\right] \left\{ \frac{i}{2\sqrt{2\alpha'}}\, \psi\partial X\,(z) \times\, e^{-\phi(w)}\, \psi^\mu(w) \mathcal{J}(w)\, e^{ipX(w)} \right\} \end{aligned} \tag{C.3.94}$$

  from which, using the the OPE (7.3.46), we find

$$\begin{aligned} V_{\boldsymbol{A}}^{(0)}(z, a, p) = g_{\mathrm{o}}\, T^a a_\mu \Bigg\{ &\tfrac{i}{2\sqrt{\alpha'}} \left[i\partial X^\mu(z) + 2\alpha'\,(p\cdot\psi)\psi^\mu(z)\right] \mathcal{J}(z) \\ &+\psi^\mu \left[T_{\mathrm{F,int}}^{+} - T_{\mathrm{F,int}}^{-}\right] \Bigg\}\, e^{ipX(z)}\,. \end{aligned} \tag{C.3.95}$$

  Here we can directly observe that the internal contribution to the vertex operator (the last term) involves one space time fermion whose odd-point contraction is always zero and therefore it decouples from out three-point calculations.

- $\mathcal{N} = 2$ case we had the following picture zero vertex operator with the internal current $\mathcal{J}^A$:

$$V_{\boldsymbol{A}}^{(-1)}(z, a, p) = g_o \sqrt{\frac{\alpha'}{6}}\, T^a\, e^{-\phi(z)}\, a_\mu^A\, \psi^\mu(z) \mathcal{J}^A(z)\, e^{ipX(z)} \tag{C.3.96}$$

  Using the $T_{F,int}$ in (7.3.47) and OPEs in (7.3.50) we have the action of the picture changing operator as:

$$\begin{aligned} &V_{\boldsymbol{A}}^{(0)}(z, a, p) = g_{\mathrm{o}}\, T^a\, a_\mu^A \lim_{w\to z} e^{\phi(z)} \left(T_{\mathrm{F,int}}^{c=3}(z) + \frac{1}{\sqrt{2}}\, \lambda^i(z)\, g_i(z) \right) \\ &\times \left\{ \frac{i}{2\sqrt{2\alpha'}}\, \psi\partial X\,(z) \times\, e^{-\phi(w)}\, \psi^\mu(w) \mathcal{J}^A(w)\, e^{ipX(w)} \right\} \end{aligned} \tag{C.3.97}$$

Noticing that $T^{c=3}_{\rm F,int}$ and g_i are *decoupled* from the rest of the fields present in $V^{(-1)}_{\boldsymbol{A}}(z,a,p)$, we find that

$$V^{(0)}_{\boldsymbol{A}}(z,a,p) = g_{\rm o}\, T^a a^A_\mu \left\{ \tfrac{i}{\sqrt{2\alpha'}} \big[i\partial X^\mu(z) + 2\alpha'\,(p\cdot\psi)\psi^\mu(z) \big]\, \mathcal{J}^A(z) \right. \\ \left. +\psi^\mu\, g_i\,(\tau^A)^i_j\, \lambda^j \right\} e^{ipX(z)}\,. \tag{C.3.98}$$

- Finally, for the $\mathcal{N}=4$ case, the picture zero vertex operator was:

$$V^{(-1)}_{\boldsymbol{A}}(z,a,p) = g_o \sqrt{\frac{\alpha'}{6}}\, T^a\, e^{-\phi(z)}\, a^{MN}_\mu\, \psi^\mu(z) \mathcal{J}^{MN}(z)\, e^{ipX(z)} \tag{C.3.99}$$

and therefore we have the limit:

$$V^{(0)}_{\boldsymbol{A}}(w,a,p) = g_{\rm o}\, T^a\, a^{MN}_\mu \\ \times \lim_{z\to w} e^{\phi(z)} T_{\rm F,int}(z) \left[\frac{i}{2\sqrt{2\alpha'}}\, \psi\partial X\,(z) \times\, e^{-\phi(w)}\, \psi^\mu(w) \mathcal{J}^{MN}(w)\, e^{ipX(w)} \right] \tag{C.3.100}$$

Upon using the energy momentum tensor (7.3.51) and the OPEs (7.3.52) we find

$$V^{(0)}_{\boldsymbol{A}}(z,a,p) = g_{\rm o}\, T^a a^{MN}_\mu \left\{ \tfrac{i}{\sqrt{2}} \big[i\partial X^\mu(z) + 2\alpha'\,(p\cdot\psi)\psi^\mu(z) \big]\, \mathcal{J}^{MN}(z) \right. \\ \left. +\psi^\mu\, \Psi^{[M} \partial Z^{N]} \right\} e^{ipX(z)}\,. \tag{C.3.101}$$

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