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Chow motives of projective, homogeneous E_7-varieties

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Abstract

This thesis compiles four main results concerning adjoint semisimple linear algebraic groups G of exceptional type E_7 over an abstract field k with $\mathrm{char}(k) = 0$.

The first one is the decomposition of most of the projective, homogeneous E_7-varieties X, which are twisted forms of G_0/P_Θ for G_0 denoting the split adjoint E_7, into Chow motives with $\mathbb{F}_2$ coefficients. The motivic decompositions depend on several invariants of the given group G, such as the Tits index, the motivic J-invariant and the Tits algebras of G. We also use the coaction map on the Chow ring of $\overline{X}$, which was recently defined by Petrov and Semenov, for this, giving more insight into its behavior on non rational algebraic cycles.

The second main result is the provision of a table containing the possible combinations of the mentioned invariants. We also mostly settle the question how these parameters change under extension to function fields $k(X)/k$, for X being a twisted form of G_0/P_Θ or the Severi-Brauer variety of the Tits algebra of G. This extends the well known index reduction formulas proven by Merkurjev, Panin and Wadsworth.

As a third main result we examine groups of type E_7, which are obtained from a construction by Tits and use an Albert algebra and a Quaternion algebra as input. We then relate the invariants of the input to the invariants of the output and calculate some of the motivic decompositions of the projective, homogeneous G-varieties of the output.

The last main result is the unexpected discovery of a Galois cohomological degree five invariant for any semisimple linear algebraic group of exceptional type E_7, which splits over the function field of the Severi-Brauer variety of its Tits algebra. It is trivial if and only if the twisted form of the respective variety G_0/P_1 of maximal parabolic subgroups of type 1 has a zero cycle of odd degree. Such anisotropic cases are obtained by the construction of Tits, for example. The construction of the invariant involves some of the afore mentioned results, along the same techniques used to prove them.

Zusammenfassung

Diese Dissertation setzt sich aus vier Hauptergebnissen über adjungierte halbeinfache algebraische Gruppen G vom Ausnahmetyp E_7 über einem abstrakten Körper k mit $\operatorname{char}(k) = 0$ zusammen.

Dabei stellt das erste Ergebnis die Zerlegung der meisten projektiven, homogenen E_7-Varietäten in Chow Motive mit $\mathbb{F}_2$-Koeffizienten dar. Die betrachteten Varietäten sind getwistete Formen von G_0/P_Θ, wobei G_0 die zerfallene adjungierte Gruppe von Typ E_7 sei. Ihre motivischen Zerlegungen hängen von verschiedenen Invarianten der entsprechenden Gruppe, wie dem Tits Index, der motivischen J-Invariante und den Tits Algebren von G ab. Wir verwenden dafür auch die Kowirkung auf dem Chow ring von $\overline{X}$, welche kürzlich von Petrov und Semenov eingeführt wurde. Einige der erzielten Ergebnisse helfen ihr Verhalten auf nicht rationalen algebraischen Zykeln besser zu verstehen.

Das zweite Hauptresultat ist eine Übersicht über die möglichen Kombinationen der drei genannten Invarianten. Wir beantworten auch die Frage, wie sich diese Invarianten über Funktionenkörpern $k(X)$ ändern fast vollständig. Dabei ist X entweder eine getwistete Form von G_0/P_Θ oder aber die Severi-Brauer Varietät einer Tits Algebra von G. Unser Ergebnis erweitert die wohlbekannten Indexreduktionsformeln von Merkurjev, Panin und Wadsworth.

Für unser drittes Hauptresultat untersuchen wir Gruppen vom Typ E_7, welche aus einer Konstruktion von Tits stammen und die als Input eine Albert Algebra und eine Quaternion Algebra verwendet. Wir stellen einen Zusammenhang zwischen den Invarianten des Inputs und des Outputs her und berechnen einige der motivischen Zerlegungen der projektiven, homogenen G-Varietäten des Outputs.

Das letzte Hauptresultat ist die Entdeckung einer Galois-kohomologischen Grad fünf Invariante, für halbeinfache lineare algebraische Gruppen vom Typ E_7, welche über dem Funktionenkörper der Severi-Brauer Varietät ihrer Tits Algebra zerfallen. Diese Invariante ist genau dann nicht trivial, wenn die getwistete Form der Varietät G_0/P_1 der maximalen parabolischen Untergruppen vom Typ 1 kein Nullzykel vom ungeraden Grad hat. Solche anisotropen Gruppen können zum Beispiel aus der Konstruktion von Tits entstehen. Die Konstruktion der neuen Invariante benutzt einige der zuvor erwähnten Resultate und auch deren Beweistechniken.

Main results

Theorem. *Let G be an anisotropic, adjoint algebraic group of type E_7 over k with $\mathrm{char}(k) = 0$, which splits over the generic point of the Severi-Brauer variety of its Tits algebra. Then there is a functorial invariant $h_5 \in H^5(k, \mu_2)$, such that for any field extension L/k one has $\mathrm{res}(h_5)_{L/k} = 0 \in H^5(L, \mu_2)$ if and only if X_1 has a zero cycle of odd degree over L.*

Theorem. *Let G be an adjoint algebraic group of type E_7, with motivic J_2-invariant $J_2(G)$. Let $\mathcal{R}_J$ denote the upper motive of the Borel variety of G. When $J_2 = (0,1,1,1)$ holds, the Chow motives of the G-varieties X_1, X_7 decompose as*

$$M(X_1) = \mathcal{U}(X_1) \oplus \textstyle\bigoplus_{i\in I} \mathcal{R}_J(i),$$

$$M(X_7) = \mathcal{U}(X_7) \oplus \mathcal{U}(X_7)(1),$$

$$\text{with } P(I,t) = t^2(t^{13}-1)/(t-1).$$

When $J_2 = (1,1,1,1)$ holds, $M(X_1)$ and $M(X_7)$ are indecomposable.

Theorem. *Let G be an anisotropic, adjoint algebraic group of type E_7 with a non split Tits algebra A. Then $J_2(G) = (1,1,0,0)$ and $\mathrm{ind}(A) = 2$ hold over k if and only if G has semisimple anisotropic kernel D_4 over $k(\mathrm{SB}(A))$.*

Theorem. *Given an adjoint algebraic group G of type E_7 over k, with Tits algebra A of index 2, motivic J_2-invariant $J_2(G)$ and semisimple anisotropic kernel G_{an}, the following holds for $\mathfrak{p} = [G_{an}, J_2(G), \mathrm{ind}(A)]$ over the generic point of the G-variety X_1.*

$\mathfrak{p}$	$\mathrm{res}_{k(X_1)/k}(\mathfrak{p})$
$[E_7, (1,1,1,*), 2]$	$[D_6, (1,1,1,0), 2]$
$[E_7, (1,1,0,0), 2]$	$[D_4 \times A_1, (1,1,0,0), 2]$
$[E_7, (1,0,0,0), 2]$	$[A_1^3, (1,0,0,0), 2]$

Theorem. *Let G be the output of the $F_4 \times A_1 \to E_7$ construction, with input $(\mathcal{J}, Q)$. Then depending on the number of common slots of $f_3(\mathcal{J}), Q$ the table below holds.*

Q	$f_3(\mathcal{J})$	$f_5(\mathcal{J})$	Slots	If G is isotropic	If G is isotropic
0	$f_3(\mathcal{J})$	0	0	–	$[D_4, (0,1,0,0), 1]$
0	$f_3(\mathcal{J})$	$f_5(\mathcal{J})$	0	–	$[D_4, (0,1,0,0), 1]$
Q	$f_3(\mathcal{J})$	0	0	–	$[D_5 \times A_1, (1,1,0,0), 2]$
Q	$f_3(\mathcal{J})$	0	1	–	$[D_4 \times A_1, (1,1,0,0), 2]$
Q	$f_3(\mathcal{J})$	0	2	–	$[A_1^3, (1,0,0,0), 2]$
Q	$f_3(\mathcal{J})$	$f_5(\mathcal{J})$	0	$[E_7, (1,1,0,0), 2]$	$[D_5 \times A_1, (1,1,0,0), 2]$
Q	$f_3(\mathcal{J})$	$f_5(\mathcal{J})$	1	$[E_7, (1,1,0,0), 2]$	$[D_4 \times A_1, (1,1,0,0), 2]$
Q	$f_3(\mathcal{J})$	$f_5(\mathcal{J})$	2	$[E_7, (1,0,0,0), 2]$	$[A_1^3, (1,0,0,0), 2]$

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Contents

Chapter 1

Introduction

The main topic of this thesis are linear algebraic groups G of exceptional type E_7, over a field of characteristic 0 and the Chow motives of the projective, homogeneous G-varieties. These are varieties which become isomorphic to varieties of type $\overline{G}/\overline{P_\Theta}$ over $\overline{k}$, with $\overline{P_\Theta}$ being a parabolic subgroup of $\overline{G}$. We will calculate many of the motivic decompositions of the twisted forms of these varieties.

Although the concept of motives as an universal cohomology theory was invented by Grothendieck, the very first publication on motives is due to Manin in [Ma68]. The problem of calculating the motive of the Severi-Brauer variety of a central simple algebra over a field was approached by Nikita Karpenko in his work [Kar95]. Such varieties arise as certain projective, homogeneous G-varieties of algebraic groups G of type A_n. In particular, Karpenko found that the motive of an anisotropic Severi-Brauer variety of a division algebra is indecomposable.

Shortly after Vishik (in [Vis98]) and also Rost (in [Ro98]) calculated motives of some quadrics. These varieties are isomorphic to some projective, homogeneous G-varieties of some algebraic groups G of type B_n and D_n. The fact that the motivic summands in the motivic decompositions in general do not arise as the whole motive of a variety, but just as a piece of the motive of some variety, plays an important role Voevodsky's proof of the Milnor conjecture (see [Voe96]) and also the more general Bloch–Kato conjecture.

While generally the motive of a quadric can consist of several non isomorphic motivic summands, the motive of an anisotropic Pfister quadric turned out to contain only copies of one motivic summand up to isomorphism, just like Severi-Brauer varieties. Projective, homogeneous G-varieties with this property arise in the framework of *generically split varieties* and were later systematically considered by Petrov, Semenov, and Zainoulline in [PSZ], [GSV] and [GSV2]. These are basically all G-varieties which motivically resemble the Borel variety of a given G. One consequence of this work is that the motive of the Borel variety of any algebraic group is totally understood. Besides, Yagita has calculated the motivic cohomology of the Borel variety of a simply connected algebraic group in some cases in [Yag].

As for motives of arbitrary projective, homogeneous G-varieties, Bonnet treated the case of G being of Killing-Cartan type G_2 in [Bo03]. These are also generically split varieties, even though this term was not yet invented back then. His result is the first on motives of algebraic groups of exceptional type. Also it turned out that the motivic decompositions of the G_2-varieties encode information on the only Galois cohomological invariant f_3 of the torsors of these exceptional groups, the *Rost invariant.*

Most of the Galois cohomological machinery was developed by Serre in [Serre], who conjectured that the Rost invariant exists for any simple simply connected algebraic group. A general construction of the Rost invariant can be found in [GMS] by Garibaldi, Merkurjev and Serre.

Recently Merkurjev has determined the structure of the group of normalized cohomological invariants of degree three for most types of *adjoint* algebraic groups in [Mer16]. His work also extends some of the C_n cases which have been established by Garibaldi, Parimala, Tignol in [GPT] before.

In [NSZ] and [McD09] the motivic decompositions for F_4 (mod 3 and mod 2 respectively) were calculated by Nikolenko, Semenov, Zainoulline and MacDonald in the second case. Again the already known Galois cohomological F_4-invariants f_3, f_5 and g_3, turned out to be reflected in the calculated motivic decompositions. In fact, given the values of these invariants, one can exactly determine the structure of the motivic summands occurring in the motivic decomposition of every projective, homogeneous F_4-variety. This included the Borel variety as well, whose motive depends only on f_3 and g_3 (i.e. the mod 2 or mod 3 case).

Meanwhile there has been some progress made on the motives of generalized Severi-Brauer varieties by Zhykhovich in [Zhy] and Junkins, Krashen, Lemire in [JKL].

Then in [Shells] Garibaldi, Petrov, and Semenov calculated the motivic decompositions for adjoint groups of type E_6 mod 3. For this they used a refined concept of *shells*, a technique originally developed by Vishik in [Vis04] for quadrics. This E_6 case is different from the other exceptional ones so far, since with the introduction of the motivic J-invariant for arbitrary algebraic groups by Petrov, Semenov, Zainoulline in [PSZ] (see [Vis05] for the original construction for quadrics) it came apparent that for anisotropic groups G of type E_6 mod 3 there is not just one specific motivic decomposition of the projective, homogeneous G-varieties, but several (see [Shells], Table 8.A).

The decomposition depends on the respective torsor one twists a split algebraic group G_0 with to obtain the studied G and the respective G-varieties. The J-invariant allows to distinguish between these cases. It translates rationality of algebraic cycles in the Chow ring of the Borel variety of this G into a numerical information. It was also used in the classification of generically split varieties in [GSV] and [GSV2] before the work [Shells].

As the J-invariant can sometimes differentiate several anisotropic algebraic groups, it can be thought of to be finer than the famous *Tits index*, originally invented by Tits in [Tits66]. The Tits index of an algebraic group is also utilized in this work as an input for the Chernousov-Gille-Merkurjev-Brosnan algorithm (see [Shells, Chapter 6] for its functionality). This method was already used in [Shells] for solving the E_6 case. It is a combination of the results made by Chernousov, Stefan Gille and Merkurjev in [CGM] and Brosnan in [Bro05]. We benefit from its *Maple* implementation, which is due to Nikolenko, Petrov, and Semenov (see e.g. [NS06]), when performing calculations in the Chow ring of some $\overline{G}/\overline{P_\Theta}$.

Since algebraic groups of type E_7 have many more possible Tits indexes and values for $J_2(G)$ in comparison to E_6, our undertaking has a much higher level of complexity. Our first main task is to determine all possible combinations of these invariants for adjoint algebraic groups of type E_7. This takes eight chapters, as it involves many techniques like shells, general Chow theory, index reduction, Galois cohomological invariants and partly constructions of certain algebraic groups.

In the second step we then calculate motivic decompositions for these cases.

Problematically it is a highly non trivial task to predict the behavior of a motivic summand and even the Tits index of a given algebraic group under a field extension in general. This is why we are considering another invariant called the *Tits algebras* introduced by Tits in [Tits71].

Once one knows how all these invariants change under certain field extensions, we can apply some going up techniques to lift specific algebraic cycles to a base field. Additionally involving combinatorial arguments enables us to prove or sometimes disprove that some motivic decomposition holds.

The possible index of the Tits algebras depends on the Tits index (see [DG, Table 8.]) of an algebraic group. Therefore it has been of general interest to know how the index of the Tits algebra changes under certain field extensions long before. These questions are related to the *index reduction formulas* treated in a series of papers by Merkurjev, Panin and Wadsworth in [MPW], [MPW2].

We will call the triples of our considered invariants *phases* (Tits index, J-invariant, Tits algebra).

Our many results on the possible transitions between phases after field extension generalize the index reduction formulas in a broad sense. We also consider several constructions of groups of type E_7 and decode how the input parameters of the famous $F_4 \times A_1$ Tits construction affect the phase of the outcome. This construction was already completely understood by Garibaldi in his work [Gar01] for a real closed base field. We will allow an arbitrary base field of characteristic zero.

The topics of phases and motives mix in a complementary way, meaning that in order to establish some motivic decompositions, we use certain phases and transitions, while other transitions and phases arise from considering motivic decompositions.

In contrast to motives mod 3 of the E_6-varieties, which gave no further insight into Galois cohomological invariants so far, we will find a phase with associated motivic decompositions, from which we derive the existence of a Galois cohomological degree five invariant for algebraic groups of type E_7, which split over the generic point of the Severi-Brauer variety of their Tits algebra, as final result. Such groups exist over $\mathbb{R}$ for example. A similar result for algebraic groups of type E_8 with trivial Rost invariant was obtained by Semenov in [S16]. However, obtaining the motivic decomposition needed for applying this construction is much harder in our E_7 case.

1.1 Outline of the thesis

The twelve chapters of this work can be divided into a recital part, spanning Chapters 1. through 7. and an original part mostly starting in Chapter 8, except for some small lemmas and examples provided in the chapters before.

The recital part starts with the obligatory basics in notation in Chapter 1. The Chapter 2. covers basics on split linear algebraic groups, such as root systems, the Tits index, Borel varieties and some classification results. The case of arbitrary linear algebraic groups over general fields (i.e. twisted forms of split groups) is considered in the Chapter 3. It introduces basics on central simple algebras, Galois cohomology and cohomological invariants of algebraic groups as well. One being the famous Tits algebras.

As exceptional algebraic groups of type E_7 mod 2 can be thought of a general case of some algebraic groups of type D_n, we treat quadratic forms and central simple algebras with orthogonal involution in Chapter 4. Considering quadrics provides many known examples of motivic decompositions of varieties into Chow motives, which are introduced in Chapter 5. This chapter also serves as an overview of several of the aspects of our motivic techniques used (uniqueness of decompositions, lifting idempotents, Tate and Rost motives etc.).

The probably most important ingredient of this whole thesis, Karpenko's theorem, is also discussed there. It restricts the considerations of possible motivic summands in the motivic decomposition of any G-variety to a few basic cases, which still have to be determined of course.

These basic cases are not just altered by the Tits index, but also by the motivic J-invariant, treated in Chapter 6. It allows to differentiate between anisotropic algebraic groups for example. Also it heavily influences the motivic decomposition type, at least in case of the Borel variety of a given algebraic group.

Chapter 7 deals with generically split varieties. By [GSV] knowing the motivic J-invariant of an algebraic group G, one can decide which projective, homogeneous G-varieties are generically split. More recently it has been found out in [PS22] that in some cases one can reversely conclude the motivic J-invariant from knowing whether Rost motives occur in the motivic decomposition of any projective, homogeneous G-variety X, by using a certain coaction ρ on $\mathrm{Ch}(\overline{X})$. We also treat ρ and some of its features in Chapter 7. For example we find that its enough to know $\rho(pt)$ for $pt \in \mathrm{Ch}(X)$ to conclude if X is a generically splitting variety or not.

The original part of the thesis starts in Chapter 8. In this chapter we link the themes from all previous chapters and introduce triples, called *phases*, of the Tits index, the J_2-invariant and the index of the Tits algebras of G, to form some kind of super invariant. We determine which phases are possible to occur at most, and prove that except for a few cases all of them are admissible indeed.

As the motivic decompositions of the projective, homogeneous G-varieties depend on the phase of G, we hence have determined a coarse classification of motivic decomposition types for E_7 by doing so. Then in Chapter 9. and Chapter 10. the motivic decompositions are concretely calculated phase by phase. The Chapter. 9 only treats the cases where all Tits algebras of G are split. This chapter is not totally original, as some cases are already known. Still it features true originality, when $M(X_1)$ is calculated.

The contents of Chapter 10. are much more complicated. Often we reduce a case to one of the cases from Chapter 9. We start with the cases, when G contains a torus of rather big rank, and then slowly work our way up to the anisotropic case, where $J_2 = (1, 1, 1, 1)$ holds. The case of G having anisotropic kernel D_6 is also of great interest, as it deals with motives of involution varieties of a $\mathbf{HSpin}_{12}$.

Then in Chapter 11., we consider an $F_4 \times A_1$ construction of E_7 and sketch a proof for showing how one can construct anisotropic algebraic groups of type E_7, with $J_2 = (1, 1, 0, 0)$. In order to prove this generally, one needs to know an exact formula for the Killing-Form of G, which takes very much effort and time constraints did not allow us to do. The anisotropic $J_2 = (1, 0, 0, 0)$ case is also considered, but it is not a new result that such groups exist. In the last Chapter 12., we then calculate the motivic decomposition for the anisotropic $(1, 0, 0, 0)$ case and also some motives in the $(1, 1, 0, 0)$ case.

The very last section of Chapter 12. deals with the construction of a Galois cohomological invariant for E_7 with $J_2 = (1, 0, 0, 0)$ via the decompositions obtained in the first section of the chapter.

1.2 Generalities and notation

In this short section we set conventions on the notation, we will use throughout this thesis.

1.2.1. We will generally work over an abstract field k with characteristic zero. But usually it is enough to demand $\mathrm{char}(k) \neq 2$. Sometimes we assume cohomological invariants mod $p \neq 2$ to be zero, without k being algebraically closed. Such cases are provided by 2-special fields, which are fields k such that every finite field extension of k is of degree 2^n for some non-negative integer n. Such fields exist by [EKM, Proposition 101.16].

1.2.2. By a *scheme*, we mean separated scheme of finite type over a field. By a *variety*, we mean an integral scheme. Usually a variety over a field k is denoted by X. If we want to emphasize that is considered over k (i.e. $\mathrm{Spec}(k)$ is the base), we write X/k.

1.2.3. When mentioning the set of natural numbers $\mathbb{N}$, the number 0 is not included. We write $\mathbb{N}_0$, in case it is included.

1.2.4. When we express motivic decompositions we use an indexation, which sometimes relies on multisets. Unlike a usual set, a multiset can contain several copies of the same element.

Chapter 2

Algebraic groups

This chapter serves as basic introduction to the terms and concepts we encounter the most often. Canonical references for the theory of algebraic groups are [Inv], [Hum] and [Hum2].

2.1 Basics of algebraic groups

In this section we introduce the most basic facts about algebraic groups. Our main references are [Inv] and [Hum].

2.1.1 Definition. An *algebraic group over k* is a variety G over k, endowed with the structure of a group given by morphisms

$$m : G \times G \longrightarrow G,\ (x, y) \longmapsto xy \text{ (multiplication)}$$

$$i : G \longrightarrow G,\ x \longmapsto x^{-1} \text{ (inverse)}$$

of varieties and an identity element $e \in G$. If G is a subgroup of the general linear group $\mathbf{GL}_n$ of invertible matrices of rank n, then it is called a *linear algebraic group* over k. A *closed subgroup H* of G is a subgroup, which is closed in the Zariski topology.

2.1.2. The set of k-rational points of G carries a canonical group structure. We often just call G a group or an algebraic group, even though we always mean a linear algebraic group when the symbol G or H is used. We write G/k in case we want to emphasize that the base field of G is k. The only exception is that we write $\overline{G}$ for $G/\overline{k}$. Here are some examples.

2.1.3 Example. The most basic example may be the group p-th roots of unity μ_p. Another example is the multiplicative group $\mathbb{G}_m$ of invertible elements. If G and H are algebraic groups, the product variety $G \times H$ is an algebraic group as well, by considering the product morphisms $m_G \times m_H$, $i_G \times i_H$ and the identity element $e_G \times e_H$.

2.1.4 Example. If G is an algebraic group, the group theoretic concepts of the center $Z(G)$, the centralizer, the normalizer or the commutator subgroup $[G, G]$ of G extend to algebraic groups as well.

2.1.5 Example. Fix an algebraic closed field k with $\mathrm{char}(k) \neq 2$. Consider the following n-th *orthogonal group* $\mathbf{O}_n := \{M \in \mathbf{GL}_n \mid MM^T = e\}$ for $n \geq 2$. The determinant map det: $\mathbf{O}_n \to \mathbb{G}_m$ is a well known homomorphism, also satisfying $\det(M) = \det(M^T)$. Thus we have $\det(e) = \det(M)^2 = 1$ for all elements M of $\mathbf{O}_n$. We can restrict the determinant to μ_2, without loss of generality and obtain a surjective homomorphism of algebraic groups det: $\mathbf{O}_n \twoheadrightarrow \mu_2$. Since the determinant map is continuous and μ_2 consists of two irreducible components, we see that generally $\mathbf{O}_n$ has two connected components for every $n \geq 2$.

2.1.6 Definition. A linear algebraic group G is called *connected*, if it is irreducible as a variety. If G is not trivial, it is called *semisimple*, if it is connected and $\overline{G}$ has no nontrivial solvable, connected, normal subgroups. The n-fold product $\mathbb{G}_m \times \ldots \times \mathbb{G}_m$ is called a *split torus of rank n*. We call a group T a *torus of rank n*, if it becomes isomorphic a split torus of rank n over $\overline{k}$. A torus of G is called *maximal*, if it is not strictly contained in another torus of G. We call G *split*, if it contains a split maximal torus. If G contains a split maximal torus which has rank n over k, we say that *G has k-rank n*.

2.1.7. For a list of the *concrete types* (i.e. for example $\mathbf{SO}_n$, $\mathbf{Spin}_n$, $\mathbf{SL}_n$, etc.) of split semisimple linear algebraic groups, including their definitions, see [Inv, §25]. It is well known that for every such concrete type there is exactly one algebraic group over an algebraically closed field. However, over an arbitrary field there are usually several kinds of the same concrete type, called *twisted forms* (of $\mathbf{SO}_n$ for example). We dedicate the Chapter 3 to discussing these issues. Speaking of types of groups, there is another meaningful way of clustering algebraic groups. It takes a different approach than writing down (matrix) equations, but relies on *root systems*.

2.2 Classification by Root systems

Our main references for the topic of root systems are [Inv, §24], from which we copy most of our content.

2.2.1 Definition. Let V be a finite dimensional $\mathbb{R}$-vector space and let V^* denote its dual space. An endomorphism $s \in \mathrm{End}(V)$ is called a *reflection with respect to* $\alpha \in V$ for $\alpha \neq 0$, if

1. $s(\alpha) = -\alpha$,

2. there is a hyperplane $W \subset V$ such that $s_{|W} = \mathrm{Id}$.

We denote the reflection s by s_α in that case. Consider the natural pairing

$$V^* \otimes V \to \mathbb{R},\ \chi \otimes v \mapsto \langle \chi, v \rangle = \chi(v).$$

A reflection s with respect to α is then given by the formula $s(v) = v - \langle \chi, v \rangle \alpha$ for a unique element $\chi \in V^*$, with $\chi_{|W} = 0$ and $\langle \chi, \alpha \rangle = 2$.

A finite subset $\Phi \subset V \neq 0$ is called a *(reduced) root system* (Φ, V) if

1. $0 \neq \Phi$ spans V.

2. If $\alpha \in \Phi$ and $x\alpha \in \Phi$ for $x \in \mathbb{R}$, then $x = \pm 1$.

3. For each $\alpha \in \Phi$ there is a reflection s_α such that $s_\alpha(\Phi) = \Phi$.

4. For each $\alpha, \beta \in \Phi, s_\alpha(\beta) = \beta$ is an integral multiple of α.

The elements $\alpha \in \Phi \subset V$ are called *roots*. For $\alpha \in \Phi$, we define $\alpha^* \in V^*$ by

$$s_\alpha(v) = v - \langle \alpha^*, v \rangle \cdot \alpha.$$

These α^* are called *coroots* and generate the dual root system $\Phi^* = \{\alpha^* \in V^*\}$.

Two root systems $(\Phi_1, V_1), (\Phi_2, V_2)$ are called *isomorphic* if there is an isomorphism of vector spaces $\phi : V_1 \to V_2$, with $f(\Phi_1) = \Phi_2$.

For a family of root systems (Φ_i, V_i) for $i \in I$, consider $V = \oplus_{i \in I} V_i$ and the union $\Phi = \bigcup_{i \in I} \Phi_i$. Then the root system (Φ, V) is called the *sum* of the Φ_i. A root system Φ of V is called *irreducible* if it is not the sum of some root systems Φ_1, Φ_2.

2.2.2 Definition. Let Φ be a root system in V. We denote by Λ_r the additive subgroup of V, which is additively generated by all $\alpha \in \Phi$. It is a lattice, called the *root lattice*. A vector $v \in V$ is called a *weight*, if $\alpha^*(v) \in \mathbb{Z}$ for all $\alpha \in \Phi$. We obtain another lattice

$$\Lambda := \{v \in V \mid \langle \alpha^*, v \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi\},$$

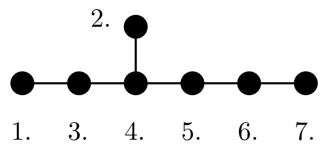
called the *weight lattice*.

2.2.3. Since any $\alpha \in \Phi$ is contained in V as well, and $\langle \alpha^*, \alpha \rangle \in \mathbb{Z}$ holds, we have that $\Lambda_r \subset \Lambda$. Also by the properties of roots, the quotient Λ/Λ_r is finite.

2.2.4 Definition. A subset $\Delta \subset \Phi$ of a root system Φ in V is called a *system of simple roots* or a *base* of Φ, if for any $\alpha \in \Phi$ there are unique $n_\beta \in \mathbb{Z}$, such that $\alpha = \sum_{\beta \in \Delta} n_\beta \cdot \beta$ and either all $n_\beta \geq 0$ or all $n_\beta \leq 0$ holds. The number of elements of Δ is called its *rank*.

2.2.5. If $\Delta \subset \Phi$ is a base of Φ in V, its rank naturally equals the dimension of V. We now outline how irreducible root systems can be classified. The classification of reducible root systems then follows from decomposing a root system into irreducible components and then applying the following classification method by Dynkin diagrams.

2.2.6. (Dynkin diagrams) In [Inv, §24] it is explained that based on a system of simple roots Δ, following certain rules, one can assign a diagram (which we also denote by Δ) in a unique way to it, which is called *Dynkin diagram*. Fundamental results on Dynkin diagrams include that a root system is irreducible if and only if its Dynkin diagram consists of one component. The notion of the rank of Δ transitions also to Dynkin diagrams. The root system of E_7 for example has rank 7, which translates into its Dynkin diagram below having seven nodes.



Two root systems are isomorphic, if and only if their Dynkin diagrams coincide. Thus in order to classify (irreducible) root systems, one just needs to classify (connected) Dynkin diagrams. This complete classification, including the parameters n_β for all roots of any system of simple roots (and thus Dynkin diagrams) is enlisted in [Inv, §24.A]. The surprising thing about it is that, apart from four so called *classic* infinite families of Dynkin diagrams, denoted A_n, B_n, C_n, D_n (the n denotes the rank), there are five unexpected ones G_2, F_4, E_6, E_7 and E_8, called *exceptional*.

2.2.7 Remark. It is rather obvious that for example the E_6 root system is contained in E_7, since the respective Dynkin diagrams are contained in each other. However, there are also much less obvious inclusions, such as $D_8 \subset E_8$ (see [BdS]).

2.2.8 Definition. Assume $\Delta \subset \Phi$ is a base of Φ in V. For Δ and the weight lattice Λ, we define the *cone of dominant weights* in Λ as

$$\Lambda_+ := \{\chi \in \Lambda \mid \langle \alpha^*, \chi \rangle \geq 0 \text{ for all } \alpha \in \Delta\}.$$

It is well known that one can introduce a partial ordering on Λ, by setting $\chi > \chi'$ if $\chi' - \chi$ is a sum of simple roots. For any $\lambda \in \Lambda/\Lambda_r$ there exists a unique *minimal dominant weight* $\chi(\lambda) \in \Lambda_+$ in the coset λ.

2.2.9. So far this section has nothing to do with algebraic groups at first sight. Before we establish this connection, note that the theory of the Lie algebra Lie(G) of an algebraic group is discussed in [Inv, §21]. Also it should already be clear, that once one knows that there is some correspondence between algebraic groups and Dynkin diagrams, the classification above transits to algebraic groups. This is the other method of classification mentioned afore.

2.2.10 Definition. (From algebraic groups to Dynkin diagrams) Let G be a split semisimple algebraic group and let Lie(G) denote its Lie algebra. We fix a split maximal torus $T \subset G$. We define $\widehat{T} := \mathrm{Hom}(T, \mathbb{G}_m)$ and call it the *character group* of T. Consider the adjoint representation of G introduced in [Inv, Exm. 22.19]

$$\mathbf{ad}\colon G \longrightarrow \mathbf{GL}(\mathrm{Lie}(G)).$$

Restricting the adjoint representation to T, we obtain a subgroup $\mathrm{ad}(T)$ of $\mathbf{GL}(\mathrm{Lie}(G))$. As G is split, T is diagonalizable in the usual sense, thus by [Inv, 22.20] we have a direct sum decomposition of Lie(G) into some vector spaces V_α for each $0 \neq \alpha \in \widehat{T}$, called the *weights* of **ad**. The weights are known to be uniquely defined for $V_\alpha \neq 0$. In this concrete setup, the weights are also called the *roots of* G and denoted by $\Phi(G)$.

2.2.11 Theorem. ([Inv, Thrm. 25.1]) *The set of all roots $\Phi(G)$ of G is a root system in $\widehat{T} \otimes \mathbb{R}$.*

□

2.2.12. Because all split maximal tori in a split group are conjugate, the choice of $T \subset G$ does not effect $\Phi(G)$. Therefore we have a unique assignment of a Dynkin diagram to a given split semisimple linear algebraic group, as announced.

2.2.13 Definition. Assume G is an algebraic group over k, which is not necessarily split. Let $\overline{G}$ have Dynkin diagram $\Delta(\overline{G})$. We then say that G has the same *Killing-Cartan type*, i.e. $\Delta(G) = \Delta(\overline{G})$.

2.2.14. There are further isomorphism results on groups and root systems in [Inv, §25], which are all intuitive. Now that we have sketched the proof of the classification of simple (see [Inv, 25.A]) split algebraic groups by Killing-Cartan types, we can deduce further properties of split groups from their root system.

2.2.15 Proposition. ([Inv, Thrm. 25.2]) *For any $\alpha \in \Phi(G)$ and $\chi \in \widehat{T}$ one has $\langle \alpha, \chi \rangle \in \mathbb{Z}$. In particular $\Lambda_r \subset \widehat{T} \subset \Lambda$ holds.*

□

2.2.16 Definition. Consider a split semisimple algebraic group G. We fix a split maximal torus $T \subset G$. Then G is often denoted by G^{sc} and called *simply connected*, if $\Lambda/\widehat{T}$ is trivial. In case $\widehat{T}/\Lambda_r$ is trivial, G is called *adjoint* and often written as G^{ad}. Let G be a not necessarily split semisimple algebraic group. We then call G simply connected or adjoint if $\overline{G}$ is simply connected or adjoint respectively.

2.2.17. By the proof of [Inv, Thrm. 26.7], G^{sc} is in fact the unique (up to isomorphism) cover for any algebraic group G of the same Killing-Cartan type as G^{sc}. This includes groups which are neither simply connected nor adjoint. By the results in the reference there are surjective group homomorphisms $G^{sc} \to G \to G^{ad}$ with finite kernels for any such group G. Thus the term *simply connected* is not just a random name, as G^{sc} does remind one to a universal covering space from topology, which is characterized by being simply connected in the topological sense.

2.2.18. A similar kind of naming holds for the term of a group being *adjoint*. It comes from the fact that the image of the adjoint representation of G has a trivial center by [Inv, §25 Central isogenies]. By the reference, any adjoint group G is isomorphic the factor group of the simply connected group of the same Killing-Cartan type G^{sc} by its center $Z(G^{sc})$.

2.2.19. (Enumeration of simple roots) For enumerating the nodes of the Dynkin diagram $\Delta(G)$ of G (and thus the simple roots), we use what is known as the *Bourbaki enumeration* (see [Bou]). Also in case we cite results involving any possibly different enumeration of the simple roots, we will translate it to this enumeration. The only exception from this enumeration are the calculations done with the *Chow Maple package* from [NS06], since its inputs rely on an enumeration which is due to *Stembridge* and are used in the Maple package [St04], on which the one from [NS06] is based on.

2.2.20. (Translation of enumeration) To help comprehending our results by the *Chow Maple package*, we give the translation from Bourbaki to Stembridge. For groups of type D_n the i-th Bourbaki root is the $(n+1-i)$-th Stembridge root. For groups of type F_4 the i-th Bourbaki root is the $(5-i)$-th Stembridge root. For groups of types A_n, E_6, E_7, E_8 the enumerations coincide. The roots of groups of type B_n and C_n will not be considered.

2.3 Parabolic subgroups and projective, homogeneous G-varieties

This section roughly explains how certain projective varieties can virtually be thought of as being attached to a given algebraic group (see [Hum2, §30]). We assume first that the base field k is algebraically closed.

2.3.1 Definition. Let G be an algebraic group over an algebraically closed field k. A subgroup $B \subset G$ is called a *Borel subgroup* of G, if it is a closed, connected, solvable group, which is not properly included in any bigger group satisfying these properties. The collection of all such groups is an actual variety by [Hum2, 23.3] and called *Borel variety* and denoted by $\mathfrak{X}$.

2.3.2 Definition. Let G be an algebraic group. A closed subgroup $P \subset G$ is called *parabolic subgroup*, if it includes a Borel subgroup of G. This also covers the case of P being a Borel subgroup of G itself. The quotient G/P is projective and is called a *projective, homogeneous* G-variety in this case. The term *homogeneous* means that G operates transitively on G/P.

2.3.3 Definition. (Notation) Consider a semisimple algebraic group G over an algebraically closed field k. Let Θ be a subset of $\Delta(G)$. The simple roots in Θ correspond to parabolic subgroups of G (see also [Shells, Chapter 2]). We normalize the notation in the same way as in the reference. This means that the group generated by the set theoretic complement of Θ is denoted by P_Θ. Thus the projective, homogeneous G-variety $X_\Theta := G/P_\Theta$ has all elements generated by the roots of G modded out that are not in Θ. We write $X_{1,3}$, when $\Theta = \{1, 3\}$ holds for example. In case Θ contains one element, P_Θ is generated by all simple roots but one and is also called *maximal*.

2.3.4. The extreme cases are $\Theta = \emptyset$, which means $X_\Theta \simeq \mathrm{Spec}(k)$ and $\Theta = \Delta(G)$, in which case the projective, homogeneous variety $X_{\Delta(G)}$ is just the *Borel variety*.

2.3.5 Definition. Let G be an algebraic group over some field k. If G contains no split torus of at least k-rank one, it is called *anisotropic*. If the k-rank of G is at least one, we say that G is *isotropic*. If G contains a Borel subgroup over k, it is called *quasi split over k*.

2.3.6. In the next chapter we will revisit the definition of the varieties X_Θ in the case when G is anisotropic. For this one needs some Galois cohomological machinery.

2.3.7 Definition. We say a field extension L/k is *Galois*, if it is finite, separable and normal. We denote the respective *Galois group* by $\mathrm{Gal}(L/k)$. We define the so called *absolute Galois group* of k by

$$\Gamma := \mathrm{Gal}(\overline{k}/k) = \varprojlim \mathrm{Gal}(L/k), \text{ for } L/k \text{ Galois.}$$

2.3.8 Remark. It is well known that $\mathrm{Gal}(\overline{k}/k)$ acts on the Dynkin diagram of an algebraic group as well. A group with a non trivial $\mathrm{Gal}(\overline{k}/k)$-action on its Dynkin diagram, is called *outer* and otherwise *inner*. For this action to be non trivial, the Dynkin diagram needs to admit some symmetry. One can also define being quasi split using the $\mathrm{Gal}(\overline{k}/k)$-action (see [Inv, 27.C]). But we do only consider groups of inner type in this thesis. Note that any group of inner type is quasi split if and only if it is split by [Inv, 27.C the comment after Prop. 27.8].

Chapter 3

Torsors and cohomological invariants

So far it did not necessarily become clear from our definitions, that there may be anisotropic algebraic groups of the same concrete Killing-Cartan type over the same field which are not isomorphic. Also we have considered algebraic groups basically just as abstract objects arising from root systems. We would like to consider algebraic groups which are not split, but are so called *twisted forms*. Most properties of twisted forms depend on some kind of underlying object. These objects are called *torsors*. The theory of torsors for algebraic groups is strongly connected to Galois cohomology, which in turn allows to introduce Galois cohomological invariants of algebraic groups.

3.1 Galois Cohomology

This section contains a few basics on Galois cohomology. We copy all of it from [Inv] and the fundamental work [Serre].

3.1.1 Definition. A discrete topological space M with continuous Γ-left action, is called a Γ-*set*. If Γ acts by group homomorphisms on a Γ-set M, i.e.

$$\sigma(a_1 \otimes a_2) = \sigma(a_1) \otimes \sigma(a_2)$$

holds, and if M is also a commutative group, we call it a Γ-*module*. Let M be a Γ-module (resp. a Γ-set). If M is just a Γ-set, assume that $n \leq 1$. We denote by $H^n(k, M) := H^n(\Gamma, M)$ the n-th *Galois cohomology group* of k (resp. the Galois cohomology set) with values in M as defined in [Inv, §28.A].

If M, N are Γ-modules, one can consider the *cup product*

$$\cup : H^n(k, M) \times H^m(k, N) \longrightarrow H^{n+m}(k, M \otimes_{\mathbb{Z}} N),\ (\alpha, \beta) \longmapsto \alpha \cup \beta.$$

It is inherited from the tensor product $M \otimes_{\mathbb{Z}} N$, which naturally arises from the Γ-left action by group homomorphisms on M, N.

By [EKM, 99.C] there is a map $\mathrm{res}_{L/k} : H^n(k, M) \longrightarrow H^n(L, M)$, for an arbitrary field extension L/k, which is called the *restriction* from k to L. This includes the case, where L is the function field $k(X)$ for some smooth projective variety X.

3.1.2 Definition. Let μ_2 denote the group of the second roots of unity and assume that $\mathrm{char}(k) \neq 2$ for a field k. Then by [Inv, §30] one can identify $\mu_2 \otimes \mu_2$ with μ_2, so $H^n(k, \mu_2^{\otimes n}) = H^n(k, \mu_2)$ holds for $n > 0$. We define the *Galois cohomology ring* mod 2

$$H(k, \mu_2) := \bigoplus_{i=0}^{\infty} H^i(k, \mu_2).$$

An element $(a_1) \cup \ldots \cup (a_n) \in H^n(k, \mu_2)$ is called a *pure symbol*. It is a consequence of the *norm residue theorem* mod 2, formerly known as the *Milnor conjecture*, that each element $\alpha \in H(k, \mu_2)$ is a sum of pure symbols. The elements $(a_i) \in H^1(k, \mu_2)$ making up the pure symbols, α is a sum of, are called its *slots*.

Two elements $\alpha \in H(k, \mu_2)$ and $\beta \in H(k, \mu_2)$ are said to have n *common slots*, if there are n not necessarily distinct $(a_i) \in H^1(k, \mu_2)$ occurring in every of their summands. Assume all summands of $\alpha \in H(k, \mu_2)$ have n common slots (a_j) for $j \in [1 : n]$. Let $\beta = (a_1) \cup \ldots \cup (a_n)$ hold. Then we say that β *divides* α.

3.1.3. The first Galois cohomology $H^1(k, M)$ is of particular interest, especially when M is a linear algebraic group. In this case $H^1(k, M)$ is known to be just a set. We come back to this in the next section.

3.2 Torsors

We shortly explain to concept of torsors of an algebraic group and point out their connection to Galois cohomology in this section (see [Inv, §28]).

3.2.1 Definition. ([Inv, Chapter VII]) Let G be a linear algebraic group over a field k. A *G-torsor* or a *principal homogeneous space* over k is a non-empty algebraic variety ξ over k equipped with an action of G such that G acts on ξ simply transitive.

Two G-torsors ξ and ξ' are called *isomorphic*, if there exists a G-equivariant isomorphism $m\colon \xi \to \xi'$ over k.

3.2.2. Considering the right action of G on itself, the definition makes any algebraic group G into its own torsor, called the *trivial torsor*. Note that any two G-torsors over k become eventually trivial and thus isomorphic over $\overline{k}$.

3.2.3 Definition. A G-torsor which can specialize to any given G-torsor is called a *versal torsor*.

3.2.4. The mathematical term *versal* can conceptually be understood as the idea of something being *universal*. The existence of versal torsors is proven in [GMS, Chapter I 5.3]. As this idea suggests, proving or disproving something about a versal torsor is often enough to cover all other cases.

3.2.5 Example. The torsors for groups of type G_2 are known to correspond to the so called *Octonion algebras* $\mathcal{O}$ (see [Inv, Thrm. 25.14 and §39]). The torsors for groups of type F_4 are known to correspond to *Albert algebras* (see [Inv, Thrm. 25.13 and §40]) and will be denoted by $\mathcal{J}$.

3.2.6. For Albert algebras it is known, that some of them are division, while others are not. It also known that the property of $\mathcal{J}$ being division, does not solely determine whether $\mathrm{Aut}(\mathcal{J})$ is isotropic or not. This makes the consideration of the *Tits index* and later on *Galois cohomological invariants* necessary, which are introduced in Section 3.4 and Section 3.6. There is a connection between the torsors of an algebraic group and Galois cohomology.

3.2.7 Theorem. ([Inv, Prop. 28.14]) *For a linear algebraic group G over an arbitrary field k, there exists a functorial bijection from the set of isomorphism classes of G-torsors over k to $H^1(k, G)$.*

□

3.2.8 Remark. Note that while $H^i(k, \mu_p)$ is an abelian group for all $i \in \mathbb{N}_0$ and all primes p, the situation is very different for $H^1(k, G_0)$, with G_0 being some split algebraic group. In that case we can only be sure about $H^1(k, G_0)$ being a pointed set with the trivial torsor as the distinguished element. For example we can consider the reduced norm map Nrd: $\mathbf{GL}_1(A) \longrightarrow \mathbb{G}_m$ (see [Inv, §20]) for some central simple algebra A (see section 3.4). Using a Hilbert 90 type argument (see [Inv, Corollary 29.4]), one can show that $H^1(k, \mathbf{SL}_1(A)) \simeq k^*/\mathrm{Nrd}(A^*)$ holds . For groups of type E_7 instead there is not any such description, as the E_7-torsors are only known rather abstractly (see [Gar01]). The E_8 case marks the least understood one.

3.3 Twisted forms

In this section we take on twisted forms. Note that in [Inv, §28.C.] the notion of the actual twisting is explained.

3.3.1 Definition. Let G_0 be a split semisimple algebraic group over k. Consider two torsors $\xi, \xi' \in H^1(k, G_0)$ and the groups $G \simeq {}_\xi G_0$ and $G' \simeq {}_{\xi'} G_0$. Then G and G' are called (inner) *twisted forms* of G_0 and also of each other. Twisting a group G by a versal G_0-torsor (defined over a larger field) results in a so called *versal form* of G. We call any group G *simply connected* or *adjoint*, if the split $\overline{G}$ is simply connected or adjoint respectively.

3.3.2. In case one twists a split group G_0 with a versal G_0-torsor ξ, the invariants of ${}_\xi G_0$ take in some sense the highest possible or most abstract value. For example, twisting G_0 with a versal torsor, will result in a form of G_0, which is anisotropic. In contrast to this, twisting with the trivial torsor results in the split form G_0 itself.

3.3.3 Definition. (**Twisted G-varieties**) So far, we have introduced the G-varieties G/P_Θ only in case G is split. Let G_0 be a split group over k and let ξ be a G_0-torsor. Let P_Θ is be a parabolic subgroup of G_0. Note that P_Θ is also necessarily split. Let $G \simeq {}_\xi G_0$ hold. We reset $X_\Theta := {}_\xi(G_0/P_\Theta)$, to denote the twist of G_0/P_Θ by ξ. Consider some field extension L/k, we then set $X_\Theta/L := {}_{res(\xi)_{L/k}}(G_0/P_\Theta)$ or simply say that we *consider* X_Θ over L.

3.3.4. Equivalently we can define the varieties X_Θ as the varieties of parabolic subgroups of $G = {}_\xi G_0$ of type Θ. Observe that by [SGAIII, Cor. XXVI.3.6] these varieties are defined over k. This also includes the case of the Borel variety. We obtain the following very well known corollary.

3.3.5 Corollary. *Let $\mathfrak{X}$ be the Borel variety of an algebraic group G of inner type over k. Then G is split if and only if $\mathfrak{X}$ has a k-rational point.*

Proof: By the definition of quasi split groups, G is quasi split exactly if it has a Borel group defined over k. This is measured by $\mathfrak{X}$ having a rational point over k. As by [Inv, 27.C the comment after Prop. 27.8] any group of inner type is split (i.e. contains a split maximal torus) if it is quasi split, the claim follows.

□

3.3.6. Twisting a split group G_0 into G, does not just determine whether its anisotropic or not, but also alters other features as we will see. Attempting to describe and classify these changes is the actual reason we are interested in the motivic decompositions of the respective projective, homogeneous G-varieties.

3.4 The Tits index

In this section we introduce the Tits index. This invariant of algebraic groups was introduced in [Tits66] and is one of the most important ones in algebraic group theory. Note that since we only consider groups of inner type in this thesis, we omit the notion of the $\mathrm{Gal}(\overline{k}/k)$-action in conjunction with the definition of Tits index (see [Inv, §26]).

3.4.1 Definition. Let G be a semisimple algebraic group over k. Let $\mathcal{T}(G) \subset \Delta(G)$ be the set consisting of the simple roots α_i, for which the G-varieties X_i have a rational point over k. Then $\mathcal{T}(G)$ is called the *Tits index* of G over k. Choose a maximal split torus T in G. Then one considers its centralizer denoted by $Z_G(T)$. The derived subgroup $[Z_G(T), Z_G(T)]$ of $Z_G(T)$ is then defined to be the *semisimple anisotropic kernel* of G over k. We denote it by G_{an}. Sometimes we just call it the *anisotropic kernel* of G.

3.4.2 Example. If for some algebraic group G/k, the variety X_Θ has a rational point over k only for $\Theta = \{1\}$, we have $\mathcal{T}(G) = \{1\}$ for example. Note that the Tits index of an anisotropic group is by definition equal to the empty set. Finally note that the disjoint union of $\Delta(G_{an})$ and $\mathcal{T}(G)$ equals $\Delta(G)$.

3.4.3. All theoretically possible Tits indexes are enlisted in [Tits66]. Interestingly only a few of those that are combinatorially possible exist. An extended version of the table in [Tits66] was provided in [DG]. It does also contain the information of whether a Tits index can occur over a p-special field or not. The refined version of the Tits index is called a *Tits p-index*. The table also contains information on the Tits algebras, which we introduce later. We will from now on refer to these tables as the *Tits classification*. We sometimes call a Tits p-index, a *Tits index occurring* mod p. As can be seen in the reference, one can visualize the Tits index by circling the respective nodes in the Dynkin diagram.

3.4.4 Remark. Any torus in G/k of k-rank n extends to a torus in G/L of at least L-rank n over a field extension L/k. Therefore $\mathcal{T}(G/k) \subset \mathcal{T}(G/L)$ holds in general. A group G is quasi split if and only if $\mathcal{T}(G) = \Delta(G)$ holds. Note that if G is adjoint and $\Delta(G_{an})$ is not connected, then G_{an} it is not a direct product in general, but it is often known to be a *central product*.

3.4.5. If one considers the anisotropic Tits index of some group considered over k, then the other Tits indexes theoretically arise over some field extensions of L/k and L'/k. But even in case they do, this does not mean that the isotropic X_Θ over LL' are exactly those which are either isotropic over L or L'. Consider the following example.

3.4.6 Example. Take an isotropic group of type A_5. By the Tits classification, it is possible for such groups to have $\{3\}$ or $\{2,4\}$ as Tits index. Extending scalars does in any case either split the respective group or does not alter the Tits index. Thus one can not obtain the Tits index $\{2,3,4\} = \{3\} \cup \{2,4\}$.

3.5 Central simple algebras and Brauer groups

For a deep treatment of the topic of central, simple algebras and their relation to Galois cohomology, see [GSz], which is our main reference for this section. The goal of this subsection is just to introduce the Brauer group, its elements and a well known isomorphism in Galois cohomology. We just cite several lemmas to make it a bit comprehensible how the Brauer group was even invented. This section probably marks the least innovative one.

3.5.1 Definition. A finite dimensional k-algebra A is called *central* if its center is isomorphic to k. When every two sided ideal of A is trivial or A itself, then A is called *simple*. In case A is as central and simple k-algebra, we call it a *CSA over k* and often write A/k.

3.5.2 Lemma. ([GSz, Lemma 2.2.2]) *Let A be a finite dimensional k-algebra and let L/k be a finite field extension. Then A/L is a CSA if and only A/k is a CSA.*

□

3.5.3 Lemma. ([GSz, Corollary 2.2.3.]) *Let A be a CSA over k. Then the dimension of A as a k-vector space is a square.*

□

3.5.4 Definition. Let A be a CSA over k. The integer $\sqrt{\dim_k(A)}$ is called the *degree* of A. If $\mathrm{Mat}_{n\times n}(L) \simeq A/L$ holds for a field extension L/k and a suitable n, we say that L *splits* A or A *splits* over L.

3.5.5 Lemma. ([GSz, Corollary 2.2.6]) *Let A be a CSA over k. Then there exists a finite, separable field extension L/k over which A splits.*

□

3.5.6 Wedderburn's Theorem. ([GSz, Theorem 2.1.3.]) *Let A be a CSA of degree n over k. Then there exists a unique division algebra D, such that $A \simeq \mathrm{Mat}_{m\times m}(D)$ for a suitable m.*

□

3.5.7 Definition. Let A be a CSA over k, with $A \simeq \mathrm{Mat}_{m\times m}(D)$ for a division algebra D. The degree of D is called the *index* of A. We write $\mathrm{ind}(A)$ for it.

3.5.8 Definition. (The Brauer group of a field) Given two CSAs A, B over k, we say that A is *Brauer equivalent* to B, if there is a division algebra D over k and positive integers m, n, such that $A \simeq \mathrm{Mat}_{m\times m}(D)$ and $B \simeq \mathrm{Mat}_{n\times n}(D)$. If $A \simeq \mathrm{Mat}_{m\times m}(D)$ holds, then D is Brauer equivalent to A. The set of CSAs over k mod Brauer equivalence and equipped with the tensor product $\otimes$ as an operation is called the *Brauer group* of k and denoted by $\mathrm{Br}(k)$ (see [GSz 2.4]).

3.5.9. That $\mathrm{Br}(k)$ is actually a group can be seen by considering the opposite algebra A^{op} of A (see [GSz, Proposition 2.4.8.]). The fact that $\mathrm{Br}(k)$ is also abelian, follows naturally from the tensor product of CSAs being a commutative operation.

It also known that the Brauer group of a field has torsion. The subgroup of p-torsion elements of $\mathrm{Br}(k)$ is usually denoted by ${}_p\mathrm{Br}(k)$. Often the Brauer group is considered as an additive group and thus $+$ is used for denoting the group operation. The following connection between the Brauer group and Galois cohomology groups is well known and important.

3.5.10 Theorem. ([Inv, p. 397 and §30]) *Let $\mathbb{G}_m$ denote the multiplicative group* over k *viewed as an algebraic group. There are isomorphisms* $\mathrm{Br}(k) \simeq H^2(k, \mathbb{G}_m)$ *and* ${}_p\mathrm{Br}(k) \simeq H^2(k, \mu_p)$.

□

3.6 Cohomological invariants

We now introduce the notion of cohomological invariants for algebraic groups very briefly. On the one hand this is necessary for being able to properly introduce Tits algebras in the next chapter. On the other hand we will prove the existence of a cohomological invariant for certain groups of type E_7 in the final chapter. This involves the invariants of F_4, which requires them to be introduced priorly. Our references for this topic, originally introduced by Serre, are [GMS] or [Inv].

3.6.1 Definition. Let G be a split algebraic group over a field k and let M be a Γ-*module.* A map

$$m : H^1(k, G) \longrightarrow H^n(k, M),$$

which is functorial in k and with $m(0) = 0$ is called a *normalized degree n cohomological invariant* of G. If $M \simeq \mu_p^{\otimes(n-1)}$ holds, we say that m is an *invariant* mod p.

We write $\mathrm{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(n-1))_{\mathrm{norm}}$ to denote the *group of normalized degree n invariants of G* (see [GMS, Appendix A]).

A normalized invariant f_n of degree n is called *decomposable*, if there is another invariant f_m and some $\alpha \in H(k, \mathbb{Q}/\mathbb{Z}(n))$, such that $f_n = f_m \cup \alpha$. Otherwise it is called *indecomposable*. The factor group

$$\mathrm{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(n-1))_{\mathrm{ind}} := \mathrm{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(n-1))_{\mathrm{norm}} / \mathrm{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(n-1))_{\mathrm{dec}}$$

is named the group of *indecomposable invariants* of degree n, while the denominator on the right hand side is the group of *decomposable invariants* of degree n of G. It follows from our definition, that both of these groups contain only normalized invariants.

3.6.2. See [GMS, Part 2] for further details. Let us assume that the characteristic of k is not p. We have that $\mu_p^{\otimes(n-1)}$ is contained in $\mathbb{Q}/\mathbb{Z}(n-1)$, so all mod p invariants of degree n are of course contained in $\mathrm{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(n-1))_{\mathrm{norm}}$. It is well known that for each group type there are so called *torsion primes* p (see [GMS]) and in case p is not a torsion prime for G, then $\mathrm{Inv}^n(G, \mu_p^{\otimes(n-1)})$ is trivial for all n.

3.6.3. For algebraic groups of type B_n, D_n, G_2 the only torsion primes are 2. Groups of type F_4, E_6 and E_7 also have 3 torsion. The most exceptional cases are constituted by E_8, which also haves 5 torsion, and A_n, for which every prime p dividing $n+1$ is known to occur as torsion prime. A famous cohomological invariant of degree 3 is the *Rost invariant.*

3.6.4 Theorem. ([GMS, Part 2. Thrm. 9.11]) *For an absolutely simple simply connected algebraic group G, the group $\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ is finite cyclic and with canonical generator R_G.*

□

3.6.5. The generator R_G from the theorem is the mentioned Rost invariant. It is of great interest, since for the group of invariants of degree 2, one has $\mathrm{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}} \simeq \mathrm{Pic}(G)$ by [Inv, Proposition 31.19]. Combining with the fact that for every semisimple simply connected group $\mathrm{Pic}(G) = 0$ holds (see [San, Lemme 6.9]), one sees that there is no non trivial degree 2 invariant for simply connected groups.

Thus the degree three invariants are naturally the next biggest invariants for simply connected groups to consider in terms of degree. For groups G of type F_4 a lot is known about the cohomological invariants of $H^1(k, G)$.

3.6.6 Theorem. ([Inv, §40]) *Let F_4 denote a split group of the same type over a field k of characteristic unequal to $2, 3$. There are the following invariants defined on $H^1(k, F_4)$, which distinguish the Albert algebras $\mathcal{J} \in H^1(k, F_4)$.*

$$f_3 : H^1(k, F_4) \longrightarrow H^3(k, \mu_2)$$

$$g_3 : H^1(k, F_4) \longrightarrow H^3(k, \mu_3^{\otimes 2})$$

$$f_5 : H^1(k, F_4) \longrightarrow H^5(k, \mu_2)$$

Further consider $G \simeq \mathrm{Aut}(\mathcal{J})$. The Albert algebra $\mathcal{J}$ is division if and only if $g_3(\mathcal{J}) \neq 0$ holds. Assume it is not division. Then the group G is isotropic if and only if $f_5(\mathcal{J}) = 0$ holds. Generally G is split if and only if $f_3(\mathcal{J}) = g_3(\mathcal{J}) = 0$ holds. Additionally, $f_3(\mathcal{J})$ always divides $f_5(\mathcal{J})$. Lastly, when $\mathcal{J}$ is division, G is anisotropic.

□

3.6.7 Remark. Even though Albert algebras $\mathcal{J}$ are not the F_4-torsors, but just correspond to them, we often write $\mathcal{J} \in H^1(k, F_4)$. We proceed the same with Octonion algebras.

3.6.8. The f_3 invariant is often referred to as the *even part* of the Rost invariant R_{F_4}. It defines an Octonion algebra $\mathcal{O}$ over k. We can think of $\mathcal{O}$ as lying under $\mathcal{J}$, just like a division algebra lies under a CSA by Wedderburn's theorem. Except for the striking difference that it is possible for $\mathcal{O}$ to be split (i.e. $f_3(\mathcal{J}) = 0$) without

$\mathcal{J}$ being split as well (i.e. $g_3(\mathcal{J}) \neq 0$) as the statements from [Inv, §40] suggest. In case $g_3(\mathcal{J})$ is zero, $\mathcal{J}$ is also called *reduced*.

3.6.9. The Rost invariant was first discovered for Octonion algebras by Hurwitz. It yields an invariant $f_3 : H^1(k, G_2) \to H^3(k, \mu_2)$. Assume we assign to a given Albert algebra its underlying Octonion algebra. Since this assignment is surjective by the construction methods of Albert algebras introduced in [Inv, §39], combining it with the f_3 invariant of G_2 one obtains the f_3 from the theorem.

3.6.10 Remark. A question naturally arising in conjunction with the Rost invariant is, whether any simply connected algebraic group with zero Rost invariant is split, like in the case of groups of type G_2 and F_4 (see [Inv, §39] again). For groups of type E_8 it is not always the case by a result of Jacobson (see [Jac]). This was exploited in [S16] to construct an indecomposable degree five invariant for anisotropic groups of type E_8 with zero Rost invariant, out of the motivic decompositions of the projective, homogeneous E_8-varieties. This inspired our final chapter, even though we consider adjoint groups of type E_7, for which the Rost invariant is not defined.

3.6.11. In fact very little is known about cohomological invariants of torsors of adjoint groups of type E_7. Above degree two, the only result so far is given by Merkurjev, who has calculated the group of indecomposable degree 3 invariants in [Mer16]. It turns out, that for torsors of simple adjoint E_7s all indecomposable mod 2 invariants of degree 3 are trivial (see [Mer16, Theorem 4.9]). The example below is an interesting consequence from this result, which is seemingly unnoticed by the experts so far.

3.6.12 Example. (Invariants of E_7 and an application) In [Gar01], objects corresponding to the E_7-torsors, called *gifts* were determined. Take a CSA of degree 56 named A and a symplectic involution τ on A (see [Inv §2] for involutions). Then one needs a map $\pi : A \to A$ fulfilling five special requirements in relation to τ (see [Gar01, Definition 3.2]). The triple (A, τ, π) then forms a gift.

From the definition of gifts, it is not clear whether for a given pair (A, τ) there is such a π, to make (A, τ, π) a gift. But it is well known that the pairs (A, τ) correspond to C_{28}-torsors. Consider a group of type C_{28} given by $\mathbf{PGSp}(A, \tau)$. Simply deleting π from the gift (A, τ, π), gives a map $m : H^1(k, E_7^{ad}) \to H^1(k, C_{28})$.

In [Mer16, Theorem 4.6], we see that that $\mathrm{Inv}^3(\mathbf{PGSp}(A, \tau), \mu_2)_{\mathrm{ind}}$ is cyclic of order two. We denote its generator by f_3. Composing f_3 with m, yields a normalized invariant

$$f_3 \circ m : H^1(k, E_7^{ad}) \longrightarrow H^3(k, \mu_2).$$

This composed invariant $f_3 \circ m$ must be indecomposable, since f_3 is indecomposable (in fact the only invariant of lower degree is the Tits algebra introduced next chapter and which is $A \in \mathrm{Br}(k)$ and also coincides for both groups involved). But as $\mathrm{Inv}^3(E_7^{ad}, \mu_2)_{\mathrm{ind}} = 0$, as proven by Merkurjev, it follows that the composition $f_3 \circ m$ is zero in general. It follows that any C_{28}-torsor (A, τ), for which f_3 is not zero, can not lie in the image of m. Thus we see that f_3 detects (at least some) pairs (A, τ), which do not admit a map π, such that (A, τ, π) is a gift.

3.6.13. (Tits constructions) Another application of the Galois cohomology functor in the realm of algebraic groups is the construction of groups out of others. This often works as follows. Consider an embedding of split groups $H \hookrightarrow G$. It is not required that H is simple, i.e. one could for example choose H to be of type $A_2 \times A_2$ and G to be of type E_6. Then apply $H^1(k, -)$ and consider the induced pushforward map. The fiddly part is then to prove statements on the output in terms of the input.

3.6.14. The original idea of this procedure is due to Tits and bears his name, although there are also some concrete constructions of certain groups referred to as Tits constructions as well. Adjoint groups of type E_7 mod 2, for example are proven to completely arise from a $D_6 \times A_1$ construction by Petrov in [P13]. This construction however is far from being deciphered in terms of their input versus their output yet. In the second last chapter, we take a look at a $F_4 \times A_1$ construction also originally due to Tits and present the outputs in terms of the cohomological invariants of the input. We need the notion of Tits algebras for such considerations, which we present in the next section.

3.7 Tits algebras

We now introduce the first ever discovered *general construction* of a cohomological invariant of algebraic groups, the *Tits algebras* (see [Tits71]). There are many possibilities to construct Tits algebras. For a compact overview on several of these see [S15, Chapter 3 and 4]. We loosely copy the methods of construction via boundary morphism and representations from this source (see [Inv, §27] for a deeper treatment). We need a bit more of representation theory, as Tits algebras actually measure to which degree a certain representation is defined over the base, if one wants to call it that.

3.7.1 Definition. Let G_0 be an split semisimple algebraic group over k. We fix a split maximal torus $T \subset G_0$ and consider an irreducible representation
$\rho : G_0 \to \mathbf{GL}(V)$. Restricting ρ to T, we obtain some weights in Λ (analogously to the case of the adjoint representation), since T is diagonalizable. Using the partial ordering (see [Inv, §24]) on these weights, we can pick a biggest element, called the *highest weight of* ρ.

3.7.2 Definition. (Tits algebras via representation theory) Consider a split semisimple algebraic group G_0. We fix a split maximal torus $T \subset G_0$. Now consider an (inner) twist G of G_0 by an $\xi \in H^1(k, G_0)$, and fix an $\omega \in \Lambda_+ \cap \widehat{T}$. The CSA denoted A_ω is called a *Tits algebra* of G corresponding to ω, if there is a group homomorphism
$\rho : G \to \mathrm{GL}_1(A_\omega)$, such that the representation $\rho \otimes k_{\mathrm{sep}} : G/k_{\mathrm{sep}} \to \mathrm{GL}_1(A_\omega \otimes_k k_{\mathrm{sep}})$ of the split group G/k_{sep} is the representation with the highest weight ω.

Let Λ_r be the root lattice of G. There is the *Tits homomorphism* (see [Inv, 27.7])

$$\beta \colon \Lambda/\Lambda_r \longrightarrow \mathrm{Br}(k),\ \omega \longmapsto A_\omega$$

where A_ω is the Tits algebra of G of the weight $\omega \in \Lambda_+ \cap \widehat{T}$, which is the unique representative of ω in Λ/Λ_r. If all Tits algebras of G are split, G is called *strongly inner.*

3.7.3 Remark. By [Inv, Thrm. 27.1] there is in fact a bijection between $\Lambda_+ \cap \widehat{T}$ and the set of isomorphism classes of irreducible representations ρ of G_0, mapping the class of ρ to its highest weight ω. So the number of possibly non Brauer equivalent Tits algebras is for example bounded by the rank of G_0.

3.7.4 Definition. (Tits algebras via boundary morphism) Consider a split semisimple algebraic group G over a field k. We consider the following exact sequence

$$1 \to Z(G) \to G^{sc} \to G \to 1,$$

with $Z(G)$ denoting the center of G and G^{sc} the split simply connected group with the same type as G. Note that $G \simeq G^{sc}/Z(G)$ is just the split adjoint group of the same type as G. The sequence induces a long exact sequence in Galois cohomology (see [Serre]), from which we can cut out the following piece

$$H^1(k, G^{sc}) \to H^1(k, G) \xrightarrow{\partial} H^2(k, Z(G)).$$

Now we consider an irreducible representation $\rho : G \to \mathrm{GL}_n$. We restrict it to the center and obtain $\lambda_\rho : Z(G) \to \mathbb{G}_m$. This map does induce a map $(\lambda_\rho)_*$ on the Galois cohomology level, which gives us the following composition.

$$H^1(k, G) \xrightarrow{\partial} H^2(k, Z(G)) \xrightarrow{(\lambda_\rho)_*} H^2(k, \mathbb{G}_m) \simeq \mathrm{Br}(k).$$

The image of any $\xi \in H^1(k, G)$ under this composition is called the Brauer class of a *Tits algebra* of the twist ${}_\xi G$. Since ξ is fixed, it depends only on ρ and is denoted by A_ρ. The number of elements $\lambda_\rho \in \mathrm{Hom}(Z(G), \mathbb{G}_m) = \widehat{T}/\Lambda_r$ is finite. Thus the number of Brauer classes of Tits algebras of ${}_\xi G$ is finite, too.

3.7.5 Remark. ([GSV, p. 11 Example 4)]) For algebraic groups of type E_7, the Tits algebras A_{ω_i} for $i = 1, 3, 4, 6$ are always split, while the Tits algebras A_{ω_i} for $i = 2, 5, 7$ are Brauer equivalent. Thus for a group G of type E_7, we can speak of *the* Tits algebra A of G, although we usually mean A_{ω_7}. Also the Tits algebras mod 3 are split for all groups of type E_7. The dependence of the possible indexes of A, which are $\{1, 2, 4, 8\}$, from the Tits index is also included in the Tits classification in [DG].

3.7.6 Example. Every group of type G_2, F_4 or E_8 is simply connected and adjoint, since for each of these types the root lattice and the weight lattice coincide (see [Inv, 24.A]). Thus all Tits algebras for groups of these types are split.

3.8 Severi-Brauer varieties

3.8.1 Definition. Let A denote a CSA with $\deg(A) = n$. Its well known that the dimension of any right ideal I in A, is divisible by $\deg(A)$. We call the quotient $\mathrm{rdim}(I) := \dim_k(I)/\deg(A)$ the *reduced dimension* of I. Then the variety

$$\mathrm{SB}_i(A) := \{I \in A \mid I \text{ is a right ideal of } A, \text{ with } \mathrm{rdim}_k(I) = i\}$$

is called, the i-th *Severi-Brauer variety* of A, for i being an integer in $[1 : n-1]$. For $i = 1$ we simply write $\mathrm{SB}(A)$ and call it *the Severi-Brauer variety* of A.

3.8.2. It is well known (see [GSz, 5.1]) that Severi-Brauer varieties $\mathrm{SB}(A)$ are twisted forms of projective spaces. Thus the Chow rings mod p of $\mathrm{SB}(A)/\overline{k}$ are simply isomorphic to $\mathbb{F}_p[h]/\langle h^m \rangle$, while m is known to be equal to $\deg(A)$. Another well known fact is that A with $\mathrm{ind}(A) = p^j$, reduces its index over $k(\mathrm{SB}_{p^m}(A))$ to p^m for $m \leq j$ (see [SvB]). If one passes to $k(\mathrm{SB}_l(A))$, such that $\gcd(p^j, l) = 1$ holds, $\mathrm{SB}(A)$ splits and the class of A becomes split by the following remark.

3.8.3 Remark. ([GSz, Remark 5.3.7]) A direct consequence from the results on [GSz, section 5.3] is that if A is a CSA over k, its class in $\mathrm{Br}(k)$ is trivial if and only if $\mathrm{SB}(A)$ has a rational point over k. This naturally applies to any field extension L/k and thus $A \otimes_k L$ is trivial in $\mathrm{Br}(L)$ if and only $\mathrm{SB}(A)/L = \mathrm{SB}(A \otimes_k L)$ has a rational point.

3.8.4. As every variety X has rational points over $k(X)$, it is clear that A is split over $k(\mathrm{SB}(A))$ by the lemma above. In the case an algebraic group G has non trivial Tits algebras A_i for $i \in [1 : n]$ over k, this raises the question whether G splits over $k(\mathrm{SB}(A_i))$ for some i. For groups of type E_7 we describe the general criteria for this to happen in Chapter 7.

3.8.5 Example. By checking the Tits classification of the Tits indexes of E_7, we can already find such a case as special case. Every group G of type E_7 with anisotropic, semisimple kernel of type A_1^3 (these always have a Tits algebra A of index two) is split over $k(\mathrm{SB}(A))$. This follows because there is only one Tits index with at least the same nodes circled like in the case of G having anisotropic kernel of type A_1^3 and additionally having split Tits algebra, namely the split one.

Chapter 4

Quadrics and involution varieties

4.1 Quadratic forms and quadrics

In this section we introduce some of the most basic definitions and fundamental results about quadratic forms. Everything can be found in [EKM]. The content is just intended for the sake of completeness, it is very well known to everyone working in the field.

4.1.1 Definition. Let k be a field with $\mathrm{char}(k) \neq 2$. A *quadratic form* or just *form* q over k of $\mathrm{rank}(q) = n$ is a homogeneous polynomial over k of degree 2 in n indeterminate variables X_i.

It is well known that one can diagonalize every quadratic form, i.e., bring it to the form $q = \sum_{i=1}^{n} a_i X_i^2$, with $a_i \in k$, and q is called non degenerate, if all $a_i \in k^*$. In this thesis we consider only non degenerate quadratic forms. The smooth projective variety

$$X_q := \{a \in \mathbb{P}^{n-1} \mid q(a) = 0\} \subset \mathbb{P}^{n-1}$$

of dimension $n-2$ is named a *quadric*. The form $X^2 - Y^2$ is called a *hyperbolic plane*. A quadric X_q or the form q are called *isotropic* over k if X_q has a k-rational point. Otherwise it is called *anisotropic*.

4.1.2. For the definition of the addition $\perp$ and multiplication $\otimes$ of quadratic forms, see [EKM, p. 44] and [EKM, p. 51]. It is well known that X_q is isotropic if and only if q contains a hyperbolic plane as a subform. In that case one can cancel the hyperbolic plane or planes, to obtain an anisotropic subform. It is also a fundamental result by Witt, that if two quadratic forms are isometric after one has canceled at least one hyperbolic plane, they are already isometric before cancellation (see [EKM, Thrm. 8.4]).

4.1.3 Definition. For quadratic forms q, p, we write $q \perp p$ for their *orthogonal sum* and $q \otimes p$ for their *tensor product*. If a form q is isotropic, we say that it *splits off* one (or several) hyperbolic planes. The isometry class of a maximal anisotropic subform of q is called its *anisotropic kernel*. If q consists only of hyperbolic planes, it is called *hyperbolic* and X_q is called *split*. The number of hyperbolic planes contained in q over k is called the *Witt index* of q over k and denoted $w(q/k)$ or just $w(q)$.

4.1.4 Definition. (Splitting pattern) Assume q is anisotropic over k. Since passing to $k_1 := k(X_q)$ necessarily makes q isotropic, its Witt index will become positive. The Witt index of q over $k(X_q)$ is called the *first Witt index* of q, denoted by $w_1(q)$.

Assume q_1 is the anisotropic kernel of q over k_1. Then one can pass to $k_1(X_{q_1})$ and repeat the procedure. The Witt index of q over k_i, is named the *i-th Witt index* and denoted by $w_i(q)$. One eventually finds a field k_i over which q is split, i.e., its Witt index equals $[\dim q/2]$. The sequence of Witt indexes one obtains, carries important information on q. We define the so called *splitting pattern* of q by

$$[w_1(q), w_2(q) - w_1(q), \ldots, w_m(q) - w_{m-1}(q)].$$

In this definition $w_m(q)$ marks the last Witt index in the described process until q is split. Thus the splitting pattern has length m. This way of writing down the splitting pattern uses the so called *relative* Witt indexes. Some authors use the *absolute splitting pattern* $[w_1(q), w_2(q), \ldots, w_m(q)]$ sometimes. A table containing the relative splitting patterns for quadratic forms up to rank 12 can be found at the end of [Vis04] and will be referred to as the *splitting pattern table*.

4.1.5 Example. For a quadratic form q of $\mathrm{rank}(q) > 4$ consider the algebraic group $G \simeq \mathbf{SO}(q)$ (see [Inv, §23]). It is known that the G-variety X_1 is isomorphic to X_q. Thus X_1 is isotropic if and only if X_q is. But much more is true. The Witt index of q is reflected in the Tits index of G and vice versa. If for example q is a rank $2n$ form and has Witt index $i \leq n$, the anisotropic kernel of $D_n \simeq \mathbf{SO}(q)$ is $D_{n-i} \simeq \mathbf{SO}(q')$ with q' denoting the anisotropic kernel of q.

4.1.6 Definition. Let us consider the set of isomorphism classes of all quadratic forms over k denoted by $\widehat{W(k)}$. With $\perp$ and $\otimes$ it is a ring. It contains an ideal generated by the hyperbolic plane $X^2 - Y^2$. The *Witt ring* is defined as $W(k) := \widehat{W(k)}/\langle X^2 - Y^2\rangle$. Often when we deal with a quadratic form, we actually mean the class of q in the Witt ring. One invariant defined on $W(k)$ for example is the (signed) *discriminant*

$$\text{disc: } W(k) \longrightarrow k^*/k^{*2},\ \textstyle\sum_{i=1}^n a_i X_i^2 \longmapsto (-1)^{n(n-1)/2} \prod_{i=1}^n a_i.$$

4.1.7. On the Witt ring one can define a map assigning to q its rank, considered mod 2. The kernel of this map I, which is also called the *fundamental ideal*, and its powers define a filtration on the Witt ring, which is then key to the development of the modern quadratic form theory.

A part of it are Milnor's famous conjecture (which now is a theorem). Thanks to this highly non trivial result, one can identify I^n/I^{n+1} with $H^n(k, \mu_2)$ via an isomorphism r_n for any $n \in \mathbb{N}_0$ (see [EKM, §16] for more on Voevodsky and the Milnor conjecture). For example, r_1 is induced by disc$(-)$, as $H^1(k, \mu_2) \simeq k^*/k^{*2}$ holds.

4.1.8 Definition. Consider the quadratic form $X^2 - aY^2$ for some $a \in k^*$. We denote it by $\langle\langle a\rangle\rangle$. We write $\varphi_n = \langle\langle a_1, \ldots, a_n\rangle\rangle$ for the tensor product $\bigotimes_{i=1}^n \langle\langle a_i\rangle\rangle$. It is called a *n-Pfister form*. Its class in $H^n(k, \mu_2)$ under r_n is known to be the pure symbol $(a_1) \cup \ldots \cup (a_n)$.

4.1.9. It is very well known that any Pfister form is either anisotropic or hyperbolic and that among quadratic forms of even rank, Pfister forms uniquely satisfy this property up to scaling. Further it is well known that any proper subform φ' of an anisotropic Pfister form φ over k which is also a Pfister form, stays anisotropic over $k(X_\varphi)$ (this follows from [EKM, Thrm. 26.5]).

If for example one passes to the generic point of X_φ for $\varphi = \langle\langle 1,1,1,1,1\rangle\rangle$ over $\mathbb{R}$, the form $\langle\langle 1,1\rangle\rangle$ stays anisotropic. If however the rank of two Pfister forms φ, φ' coincides and let us say $X_{\varphi'}$ is hyperbolic over $k(X_\varphi)$, then φ and φ' are isometric over k by [EKM, Corollary 23.6]. Lastly, it is also known that the n-Pfister forms over k generate I^n additively.

4.2 Involution varieties

In this section we introduce varieties naturally assigned to pairs (A, σ), for a CSA A with an orthogonal involution σ. The original involution varieties (see [Tao92]) are known to be twisted forms of quadrics. For the theory of involutions, along with the adjoint form q_σ and the discriminant of orthogonal involutions and a classification of group types like $\mathbf{SO}(A, \sigma)$ or $\mathbf{HSpin}(A, \sigma)$, see [Inv, §26].

4.2.1 Definition. Let A be a CSA of degree $\deg(A) = 2n$ over k and let σ be an orthogonal involution on A with trivial discriminant. Then

$$\mathcal{I}(A,\sigma)_i := \{I \subset A \mid I \text{ is a right ideal of } A, \text{ with } \mathrm{rdim}_k(I) = i \text{ and } \sigma(I)I = 0\}$$

is called the i-th *involution variety* of (A,σ), for i being an integer in $[1 : n]$. We just write $\mathcal{I}(A,\sigma)$ for the first involution variety. Involution varieties for $i > 1$ are also often called *generalized involution varieties.*

4.2.2. One can define an analogue for symplectic involutions τ and extend the definition to groups of type C_n. But we do not use or prove anything making these considerations necessary. The interesting part is, that the G-variety X_i for a group G of inner type D_n, is known to be isomorphic to $\mathcal{I}(A,\sigma)_i$ or $\mathcal{I}(A,\tau)_i$ for $i < n-1$. The cases of $i = n-1$ or n constitute a special case by [MT95, Examples 2.4.5], as $\mathcal{I}(A,\sigma)_{n-1}$ is in fact $X_{n-1,n}$, while $\mathcal{I}(A,\sigma)_n$ is not a homogeneous variety and consists of two connected components, which are X_{n-1} and X_n.

If one splits A by passing to $k(\mathrm{SB}(A))$, the involution variety $\mathcal{I}(A,\sigma)$ becomes isomorphic to a quadric X_{q_σ}. The quadratic form q_σ over $k(\mathrm{SB}(A))$ is solely defined by σ (see [Inv §1]) and is said to be *adjoint* to σ.

4.2.3. It is well known, that $\mathcal{I}(A,\sigma)$ is a closed subvariety of $\mathrm{SB}(A)$ of codimension 1. This is not true for $i > 1$, as comparing dimensions shows. One may wonder what the Tits index of G over $k(\mathrm{SB}(A))$ is. Despite all efforts this is still a widely unsolved issue if A is not division. Some insight is given by a result of Karpenko.

4.2.4 Theorem. ([Kar09, Thrm 3.3]) *Let (A,σ) be a* CSA *with* $\deg(A) = 2n$, *Brauer class D, an orthogonal involution σ and* $\mathrm{ind}(A/k) = a$. *Let q_σ be the quadratic form adjoint to σ over $k(\mathrm{SB}(D))$. Then the Witt index of q_σ is divisible by a.*

□

4.2.5 Remark. In [Inv, 8.B] the definition of the Clifford algebra (see [EKM, §11]) is extended from quadratic forms q to (A,σ). Also, some of the Tits algebras ω_i for a group of type D_n defined by (A,σ), may be trivial over $k(\mathrm{SB}(A))$ where q_σ is defined, but not over k.

Concretely the Tits algebras A_{ω_i} over k are trivial, when $i \leq n-2$ is even by [MT95, the part before Corollary 2.11] and isomorphic to A, when i is odd. We see that if D_n is defined by a quadratic form (i.e. A is split), the only ω_i for which the Tits algebras are possibly not trivial are $i = n-1, n$.

Chapter 5

Chow motives

In this chapter we introduce the motivic category $\mathcal{M}_k$ of Chow motives with $\mathbb{F}_p$ coefficients of smooth projective varieties over k, alongside some of its features. As a preparation we start with Chow rings, before we can properly define $\mathcal{M}_k$. Good resources are [Ful] and [EKM, Chapter X]. For motives exclusively one can use [EKM, Chapter XII]. We then introduce the notion of upper motives and explain how the motivic decompositions of projective, homogeneous G-varieties can be restricted to the study of their upper motive. This constitutes a major simplification to the task of calculating the motivic decompositions of all projective, homogeneous E_7-varieties.

We then introduce the theory of shells, which has been used in [Shells] to establish the mod 3 motivic decompositions of the projective, homogeneous E_6-varieties and was fundamental to a lot of proofs. Also we give many examples of known decompositions, including proofs, to display how to apply the techniques introduced previously. The examples and their proofs are also established to be referenced to later on, when we calculate the much more complicated decompositions for E_7. We close the chapter by mentioning a well known algorithm, which calculates motivic decompositions of projective, homogeneous G-varieties, provided G is ***isotropic***.

5.1 The Chow functor and algebraic cycles

The category of Chow motives arises from a construction incorporating the Chow functor and thus the notion of Chow groups and algebraic cycles. These fundamentals of intersection theory can be found in [Ful]. We assume the reader knows about algebraic cycles and rational equivalence. A survey on the concept of an adequate equivalence relation for defining Chow groups (also of different kind than the ones we use) can be found in [Sam].

5.1.1 Definition. Let k be a field. For some smooth variety X, we denote the i-th *Chow group* of algebraic cycles of dimension i on X up to *rational equivalence*, with $\mathbb{Z}$ coefficients by $\mathrm{CH}_i(X)$. The i-th Chow group of algebraic cycles of codimension i on X up to rational equivalence, with $\mathbb{Z}$ coefficients is denoted by $\mathrm{CH}^i(X)$.

We define the *Chow ring* of X by $\mathrm{CH}^*(X) := \bigoplus_{i=0}^{\dim(X)} \mathrm{CH}^i(X)$. Additionally we set $\mathrm{Ch}^i(X) := \mathrm{CH}^i(X) \otimes \mathbb{F}_p$ and $\mathrm{Ch}^*(X) := \mathrm{CH}^*(X) \otimes \mathbb{F}_p$. The definition of $\mathrm{CH}_*(X)$ and $\mathrm{Ch}_*(X)$ are analogues.

If U is a closed subvariety of X, we denote its class in $\mathrm{CH}^*(X)$ also by U and usually refer to it as a cycle, too. A cycle $\alpha \in \mathrm{CH}^i(X)$, which solely represents the class of a closed subvariety of X, is called a *prime cycle*.

5.1.2 Remark. Most of the time we do not include the grading when writing $\mathrm{CH}(X)$ or $\mathrm{Ch}(X)$, except when want to emphasize that the grading is respected. We mostly use $\mathbb{F}_2$ coefficients for $\mathrm{Ch}(X)$ throughout this whole thesis. Especially when we calculate motives of the projective, homogeneous E_7-varieties. We point out if other $\mathbb{F}_p$ coefficients are used, by referring to it as the *mod p case*.

5.1.3. (Ring structure on $\mathrm{CH}(X)$) By definition $\mathrm{CH}(X)$ is $\mathbb{Z}$-module. But it also carries the structure of commutative ring, which one obtains by considering the intersection product of classes of algebraic cycles (see [Ful, 8.0 and 8.3]) and extending it from prime cycles to general sums of prime cycles. Consider $\alpha \in \mathrm{CH}^i(X)$ and $\beta \in \mathrm{CH}^j(X)$. We then write $\alpha\beta$ for their product in $\mathrm{CH}^{i+j}(X)$.

The structure of the Chow rings mod p of split algebraic groups considered as varieties is well known. The Chow rings of the G-varieties G/P are not completely known in terms of generators and relations yet due to their complexity. Even when G is split. But they can in theory be obtained by an algorithmic approach.

5.1.4. (Chow maple package) The *Chow maple package*, we use for many calculations, allows us to calculate arbitrary products in the Chow ring of $X_\Theta \simeq G/P_\Theta$ for any split simple algebraic group G. It uses methods, which are explained in [Shells, 5. and 6.] and expresses each generator of a Chow group $\mathrm{Ch}^i(G/P_\Theta)$ in terms of Weyl coordinates (see [Shells, 5.] or [Hum, 10.3] for more on Weyl groups).

Assume for example, that the routine *chow generators* outputs the elements $z[3,4,3,6,7,4], z[4,2,3,6,5,4], z[7,4,6,3,5,4]$ (in this exact order) as generators of $\mathrm{Ch}^6(X_\Theta)$. We will then refer to them as $\gamma_{6,1}, \gamma_{6,2}, \gamma_{6,3}$.

If P_Θ is a maximal parabolic subgroup of G, then there is only one generator of $\mathrm{Ch}^1(X_\Theta)$, which we usually denote by h.

5.1.5 Definition. (Poincaré polynomials of Chow rings) Let X be a smooth projective, homogeneous variety. We define the Poincaré polynomial of $\mathrm{Ch}(X)$ via

$$P(X,t) := \textstyle\sum_{i=0}^{\infty} \mathrm{rank}_{\mathbb{F}_p}(\mathrm{Ch}^i(\overline{X}))t^i \in \mathbb{N}_0[t],$$

where $\overline{X}$ denotes X over $\overline{k}$.

In case X is the G-variety X_Θ for $\Theta \subset \Delta(G)$, one can calculate $P(X_\Theta, t)$ by dividing some concrete polynomials by each other (see [GSV, Def. 2.5] for how to obtain these). A polynomial $s(t) = \sum_{i=0}^{n} a_i t^i \in \mathbb{N}_0[t]$ with $\deg(s(t)) = n$ is called *symmetric*, if $a_i = a_{n-i}$ for all $i \in [0:n]$. By Poincaré duality, we have that $P(X,t)$ is symmetric.

5.1.6 Example. Let A denote a CSA. Then $P(\mathrm{SB}(A),t) = (t^{\deg(A)} - 1)/(t-1)$ holds. If q is a non degenerate quadratic form of odd rank n, then one has $P(X_q,t) = (t^{n-1} - 1)/(t-1)$ for the quadric X_q. If n is even, one has $P(X_q,t) = (t^{n-1} - 1)/(t-1) + t^{(n-2)/2}$, i.e. $\mathrm{Ch}^{(n-2)/2}(X_q)$ has $\mathbb{F}_2$-rank 2.

5.1.7 Definition. Let $f : X \to Y$ be a proper morphism for smooth varieties varieties X, Y. We then obtain a *pushforward* map $f_* : \mathrm{CH}(X) \to \mathrm{CH}(Y)$. If $Y \simeq \mathrm{Spec}(k)$ holds, the respective pushforward map is denoted by deg: $\mathrm{CH}(X) \to \mathbb{Z}$ and is called the *degree map*.

5.1.8 Definition. Let X be a smooth variety over k and L/k be a field extension of k. Then there is the *restriction map* $\mathrm{res}_{L/k}\colon \mathrm{CH}(X/k) \to \mathrm{CH}(X/L)$. Elements which lie in the image of the restriction map are called *rational over* k.

5.1.9. The two definitions above hold analogously for $\mathrm{Ch}(-)$. The k-rational elements form a subring of $\mathrm{Ch}(X/L)$ for an extension L/k, since *res* is a homomorphism of rings. Calculations of rational cycles in $\mathrm{Ch}(X/k)$ turn out to be notoriously hard. One tool to calculate rational cycles is the Steenrod map defined below.

5.1.10 Definition. (Steenrod operations) By [Voe01], [Br03] and [Pr] there is a map $S^l : \mathrm{Ch}^i(X) \to \mathrm{Ch}^{i+l(p-1)}(X)$, called *$l$-th Steenrod operation mod p*, for Chow groups mod any prime p. It satisfies

1. S^0 is the identity on $\mathrm{Ch}^l(X)$.
2. $S^l(\alpha)$ is zero, if α is in $\mathrm{Ch}^i(X)$ for $i < l$.
3. $S^l(\alpha) = \alpha^p$, if α is in $\mathrm{Ch}^l(X)$.
4. $S^l(\alpha)$ is rational, if α is rational.

One can also consider $S^\bullet : \mathrm{Ch}(X) \to \mathrm{Ch}(X)$, called the *total Steenrod operation mod p*. It is defined as $S^\bullet = \sum_{l=0}^{\dim(X)} S^l$.

5.1.11 Remark. Using the *Chow maple package*, we are able to calculate the l-th Steenrod operation. Still, this only helps to calculate rational cycles out of others. To instead fundamentally say something about rational cycles in $\mathrm{Ch}(X)$ (for X being an inner twist of G/P_Θ), we consider the following well known exact sequence (see [MT95, 2.3 proof of Thrm B])

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(\overline{X}) \xrightarrow{\beta} \mathrm{Br}(k) \to \mathrm{Br}(k(X)),$$

obtained from the Hochschild-Serre spectral sequence. The map β is basically the Tits homomorphism. Identifying $\mathrm{Ch}^1(X)$ with $\mathrm{Pic}(X)$, we can conclude that if $\overline{X} = \overline{X_i}$ holds, the unique generator h (which is just the cycle class corresponding to $\omega_i \in \Lambda$) is rational if and only if the respective Tits algebra $\beta(h)$ is trivial. This reveals a connection between algebraic cycles on specific projective, homogeneous varieties and the Tits algebras.

5.2 The category of Chow motives

In this section we introduce Chow motives mod p. Many categories of motives bear the same spirit of this construction, but incorporate a different equivalence relation or not even the Chow functor. The different ingredients often alter the features of the constructed category. Our main reference for the construction of Chow motives is [EKM, §62 to §64].

5.2.1 Definition. (The category of correspondences) Let k be a field. We denote the category of smooth projective varieties over k by $\mathcal{V}ar_k$. Let X, Y be varieties in $\mathcal{V}ar_k$. Let X_l for $l = 1, \ldots, n$ be the irreducible components of X and set $d_l = \dim(X_l)$. We define the *correspondences from X to Y* as

$$\mathrm{Corr}(X, Y) := \coprod_{l=1}^{n} \mathrm{Ch}_{d_l}(X_l \times Y).$$

with a correspondence product described below.

An element $f \in \mathrm{Mor}_{\mathcal{V}ar_k}(X, Y)$, induces an element $[\Gamma_f] \in \mathrm{Corr}(X, Y)$, via its graph Γ_f. We obtain a functor from $\mathcal{V}ar_k$ to $\mathcal{C}orr_k$, the just defined category of *correspondences* over k. It satisfies

$$\mathrm{Obj}(\mathcal{C}orr_k) = \{X \mid X \in \mathrm{Obj}(\mathcal{V}ar_k)\},$$

$$\mathrm{Mor}_{\mathcal{C}orr_k}(X, Y) = \mathrm{Corr}(X, Y).$$

The *diagonal morphism* $\Delta_X \colon X \to X \times X$ and its graph

$$\Gamma_{\Delta_X} \in \mathrm{Corr}(X, X) = \mathrm{End}(X)$$

are of special interest.

5.2.2. (The correspondence product) In addition to the obvious product structure on $\mathrm{Ch}(X \times X)$, coming from the intersection product on $\mathrm{Ch}(X \times X)$, there is another product called *correspondence product*, denoted by $\circ$. To define it properly, we consider smooth projective varieties X, Y, Z and their product $X \times Y \times Z$. Let $pr_{12}, pr_{13}, pr_{23}$ be the projection to $X \times Y$, $X \times Z$ and $Y \times Z$ respectively. Then the product of $\alpha \in \mathrm{Corr}(X, Y)$ and $\beta \in \mathrm{Corr}(Y, Z)$ is given by

$$\alpha \circ \beta = pr_{13*}(pr_{12}^*(\alpha) pr_{23}^*(\beta)) \in \mathrm{Corr}(X, Z).$$

The *identity element* in $\mathrm{End}(X)$ is given by the class of Δ_X. An element $\rho \in \mathrm{End}(X)$ with $\rho \circ \rho = \rho$, is called a *projector* or an *idempotent*.

5.2.3 Definition. (The category of Chow motives) The category $\mathcal{M}_k$ of Chow motives over k is obtained by taking the idempotent completion (also known as the Karoubi envelope) of $\mathcal{C}orr_k$. It satisfies the following

$$\mathrm{Obj}(\mathcal{M}_k) = \{(X, \rho) \mid X \in \mathrm{Obj}(\mathcal{C}orr_k), \rho \in \mathrm{End}(X), \rho \circ \rho = \rho\}$$

$$\mathrm{Mor}_{\mathcal{M}_k}((X, \rho), (Y, \pi)) = \pi \circ \mathrm{Mor}_{\mathcal{C}orr_k}(X, Y) \circ \rho \subset \mathrm{Mor}_{\mathcal{C}orr_k}(X, Y).$$

The motive (X, Δ_X) of $X \in \mathrm{Obj}(\mathcal{V}ar_k)$ is called *the motive of* X and will be denoted by $M(X)$.

5.2.4 Definition. If $\rho \in \mathrm{End}(X)$ is a projector, then $\Delta_X - \rho$ is a projector too and we call (X, ρ) a *motivic summand* of $M(X)$. If (X, ρ) is a motivic summand of $M(X)$ and there are no non trivial projectors $\pi_1, \pi_2 \in \mathrm{End}(X)$, such that $\rho = \pi_1 + \pi_2$, while $\pi_1 \circ \pi_2 = \pi_2 \circ \pi_1 = 0$ holds, we say that $(X, \rho) \simeq N$ is an *indecomposable* motivic summand of $M(X)$. We then write $N \in M(X)$.

There is a realization functor $r : \mathcal{M}_k \longrightarrow \mathbb{Z}\text{-Mod}$, given via

$$r((X, \rho)) := \mathrm{Im}(\mathrm{Ch}^*(X) \xrightarrow{pr_1^*} \mathrm{Ch}^*(X \times X) \xrightarrow{\rho} \mathrm{Ch}^*(X \times X) \xrightarrow{pr_{2*}} \mathrm{Ch}^*(X)).$$

The middle arrow denotes taking the *intersection product* with ρ. The maps pr_1^*, pr_{2*} denote the pullback and pushforward of the projections to the first and second component. Because the realization functor r naturally commutes with the Chow functor Ch: $\mathcal{V}ar_k \to \mathbb{Z}\text{-Mod}$, it is also common to write $\mathrm{Ch}(N)$ for $r(N)$.

Let l be the smallest number, for which $\mathrm{Ch}^l((X, \rho)) \neq 0$ holds and j be the biggest number, for which $\mathrm{Ch}^j((X, \rho)) \neq 0$ holds. We say that (X, ρ) *starts* in l and *ends* in j. If a motive (X, ρ) starts in l, we call any nonzero cycle in $\mathrm{Ch}^l(X)$ a *generic point* of (X, ρ). Typically abuse the language refer to it as *the* generic point. The number $j - l$ is called the *dimension* of (X, ρ). We write $\dim((X, \rho))$ for it.

5.2.5 Definition. The category $\mathcal{M}_k$ is known to be tensor additive, meaning one can add and multiply motives. For two motives $(X, \rho), (Y, \pi)$ we have

$$(X, \rho) \oplus (Y, \pi) = (X \sqcup Y, \rho + \pi),$$
$$(X, \rho) \otimes (Y, \pi) = (X \times Y, \rho \times \pi).$$

A *motivic decomposition* of X is a finite collection of nontrivial projectors $\rho_i \in \mathrm{End}(X)$, such that $\Delta_X = \sum \rho_i$ and $\rho_i \circ \rho_j = \rho_j \circ \rho_i = \delta_{i,j} \rho_i$. The last property is also called *mutually orthogonal*. The motivic decomposition is then given by $M(X) = (X, \Delta_X) = \bigoplus_i (X, \rho_i)$. The realization map naturally respects decompositions, i.e. we have that $r(M(X)) = \bigoplus_i r((X, \rho_i))$.

5.2.6. While it is certainly of general interest to calculate the Chow ring when considering motives, the way $M(X)$ decomposes provides additional information. The decomposition reflects the rationality of algebraic cycles in $\mathrm{Ch}(X)$ over a certain base field. So the base field does play an essential role for determining the motivic decomposition type of X.

5.2.7 Definition. Consider field extension L/k. The restriction map on $\mathrm{Ch}(X)$ extends naturally to $\mathrm{Ch}(X \times X)$. Thus we can consider a restriction for motives. Assume that an **indecomposable** motivic summand $N \in M(X/k)$ becomes decomposable into some motivic summands N_i over L. We then write $\mathrm{res}_{L/k}(N) = \oplus N_i$ and say that N splits off the (or a specific) N_i over L. Also, we say that the N_i *are glued* over k. We might also call them *glued to* N over k. Consider an indecomposable motivic summand $(X/L, \pi) \in \mathrm{res}_{L/k}(M(X)) = M(X/L)$. If the projector π is (not) rational over k, we say $(X/L, \pi)$ is *(not) visible* over k. If a motive N is visible over k, we sometimes write N/k to indicate that we consider it over k.

5.2.8. The diagonal $\Delta_{X/L}$ is always defined over the base field k. But other projectors in $\mathrm{End}(X/L)$ are often not defined over k. This is what constitutes the main problem in calculating motivic decompositions. We will encounter cases where $\Delta_{X/L}$ is also the only rational projector in $\mathrm{End}(X/L)$. The smallest possible motivic summand is introduced below.

5.2.9 Definition. (Tate motives) Consider the projective line $\mathbb{P}^1$. The algebraic cycle $pt \times \mathbb{P}^1 \in \mathrm{Ch}(\mathbb{P}^1 \times \mathbb{P}^1)$ defines a projector in $\mathrm{End}(\mathbb{P}^1)$. Thus we obtain a motive $\mathbb{F}_p(1) := (\mathbb{P}^1, pt \times \mathbb{P}^1)$. Also we set $\mathbb{F}_p := M(\mathrm{Spec}(k)) = (\mathrm{Spec}(k), pt \times pt)$. The first one is *the Tate motive*, while the second one is the *trivial Tate motive*. The Tate motive defines the endofunctor

$$(-) \otimes \mathbb{F}_p(1) : \mathcal{M}_k \longrightarrow \mathcal{M}_k.$$

We call $(-) \otimes \mathbb{F}_p(1)$ the *Tate shift*. The motive $N(i) := N \otimes \mathbb{F}_p(1)^{\otimes i}$ is called the i-th *Tate shift* of N. This includes $\mathbb{F}_p(i) = \mathbb{F}_p(1)^{\otimes i}$ as well, which is simply called *a Tate motive*. Tate shifting some motive N by i, increases the starting point of N

by i. A motive is called *split* over k, if it completely decomposes into a sum of Tate motives (this includes shifts and the trivial Tate motive as well) over k.

5.2.10 Definition. Two motives $(X, \rho), (Y, \pi)$ are *isomorphic* if there are elements $\alpha \in \mathrm{Mor}_{\mathcal{M}_k}((X, \rho), (Y, \pi))$ and $\beta \in \mathrm{Mor}_{\mathcal{M}_k}((Y, \pi), (X, \rho))$, such that $\alpha \circ \beta = \pi$ and $\beta \circ \alpha = \rho$ holds.

5.2.11 Definition. (Poincaré polynomials of motives) Let N be a motive in $\mathcal{M}_k$. We assume that $\mathrm{rank}_{\mathbb{F}_p}(\mathrm{Ch}^i(N/\overline{k}))$ is finite for all $i \in \mathbb{N}_0$. We define

$$P(N, t) := \textstyle\sum_{i=0}^{\infty} \mathrm{rank}_{\mathbb{F}_p}(\mathrm{Ch}^i(N/\overline{k}))t^i \in \mathbb{N}_0[t]$$

as the *Poincaré polynomial* of N.

5.2.12 Example. If N equals $\mathbb{F}_2(i)$, we have $P(N, t) = t^i$ for example. Thus it makes even more sense to think about Tate motives as the most basic material. We generally have that $P(M(X), t) = P(X, t)$ holds, no matter if X is motivically decomposable or not.

5.2.13 Definition. (Shift polynomials) Let us assume we are given a motivic decomposition $M(X) = \bigoplus N_i$ into indecomposable summands. We pick one summand N_i. Now consider a motive N, which is isomorphic to N_i up to a Tate shift, but has starting point zero (i.e. we basically ignore the Tate shift of N_i). If a polynomial $O(N, t) = \sum a_i t^i$ in $\mathbb{N}_0[t]$ has the property that each a_i equals the number of Tate shifts of N in codimension i, which occur in $M(X)$, we say that $O(N, t)$ *codes the shifts* of N in $M(X)$.

5.2.14 Example. For example we consider $M(X) = A \oplus B(2) \oplus A(4)$. In this case we have $O(A, t) = 1 + t^4$ and $O(B, t) = t^2$. Observe that $P(X, t) - P(B, t)t^2$ is necessarily the product of $O(A, t) = 1 + t^4$ and $P(A, t)$.

In other words, the polynomial $P(N, t)$ tells us how N looks, while the polynomial $O(N, t)$ tells us which shifts of N are contained in $M(X)$.

5.2.15 Definition. (Tate polynomials) If in the definition of shift polynimals, N is the trivial Tate motive $\mathbb{F}_p$, we write $T(X/k, t)$ instead of $O(N, t)$ and call it the *Tate polynomial* of X over k. Sometimes we omit the field k, if there is no danger of confusion.

5.2.16 Definition. (Subpolynomials) Two **symmetric** polynomials $s(t)$, $f(t) \in \mathbb{N}_0[t]$ are called *subpolynomials* of a symmetric polynomial

$$P(t) = \sum_{i=0}^{n} a_i t^i \in \mathbb{N}_0[t],$$

if $P(t) - f(t)s(t)$ is **symmetric** and **contained** in $\mathbb{N}_0[t]$.

5.2.17 Example. Our definition of subpolynomials may look a bit strange at first sight. Take for example $O(t) = 1 + t^2 + t^3 + t^5$. It has the obvious subpolynomials $1, t^2, t^3, t^5$ in the usual sense. However by the definition, $s(t) = 1 + t$ is also a subpolynomial of $O(t)$ in our sense, as $s(t)t^2 = t^2 + t^3$. No subpolynomial of $O(t)$ in our sense would be $1+t^2+t^3$, since it is not symmetric. But of course $r(t) = 1+t^2$ is a subpolynomial of $O(t)$ in our sense, since $r(t)1 = 1+t^2$ is symmetric and contained in $O(t)$.

Now consider $P(X,t) = P(M(X),t) = 1+t+t^2+t^3+2t^4+t^5+t^6+t^7+t^8$ for some projective variety X, and some motive N, with $P(N,t) = 1+t$. Then $M(X)$ cannot be decomposed completely into shifts of N, since $P(X,t)/P(N,t)$ is not in $\mathbb{N}_0[t]$. There is simply no subpolynomial $s(t)$ of $P(X,t)$, such that $s(t)P(N,t) = P(X,t)$. We can think of $M(X)$ as a bread board. The indecomposable motivic summands of $M(X)$ are like patch cables. But they are not limited to connecting only two ports (i.e. generators in $\mathrm{Ch}(X)$) in general. More examples follow in Section 5.5.

5.2.18 Remark. Note that for any indecomposable motivic summand $N \simeq (X,\rho)$, the Poincaré polynomial of N is symmetric if X is a projective, homogeneous G-variety for some semisimple algebraic group G. By modifying [Zhy, Prop. 2.3.1] (simply change $M(X)(i)$ to $N \simeq (X,\rho)$) this holds for any motivic decomposition of $M(X)$ into indecomposable motivic summands itself, meaning a motivic summand N of $M(X)$ starting in codimension i either has an isomorphic counterpart N^* in $M(X)$, which starts in codimension $\dim(X) - i - \dim(N)$ or it is sitting in the very middle of the decomposition. By sitting in the middle, we mean that the rational numbers $i + \frac{\dim(N)}{2}$ and $\frac{\dim(X)}{2}$ coincide. An indecomposable motivic summand N of $M(X)$ can only sit in the middle, if the relation $\dim(N) \equiv \dim(X) \bmod 2$ holds.

5.2.19. In general many questions in conjunction with motives are unanswered. One is whether there are so called *phantom summands*. A phantom summand is a motive over k, which vanishes over some field extension L/k. For Chow motives of projective, homogeneous varieties this can not happen, since the Rost nilpotence theorem (in short RNT) below is known to hold in the case.

Also by the definition of Poincaré polynomials, it follows that for projective, homogeneous varieties two motives over k become isomorphic over $\overline{k}$ if and only if their Poincaré polynomials coincide. Since for any Poincaré polynomial $P(N,t)$ we consider, $P(N,1)$ is finite, this allows the following conclusion. In any possibly infinite tower of field extensions of k, there are only finitely many steps, such that a motive decomposes further than in the step before, until it is split. This motivates the Section 5.4 on shells.

5.2.20 Rost Nilpotence Theorem. ([CGM, Section 8]) *Let X be a projective, homogeneous variety over k. Then for every field extension L/k, the kernel of the natural ring homomorphism* $\ker(\mathrm{End}(M(X/k)) \to \mathrm{End}(M(X/L)))$ *consists of nilpotent correspondences.*

□

5.2.21 Remark. (Uniqueness of a decomposition) In [S06, Corollary 5.6] an example was established, showing that at least for Chow motives with *integer coefficients* a motivic decomposition is not unique in general. Also no canceling rule holds for motives with integer coefficients in general, i.e. the relation $M \oplus N \simeq M \oplus N'$ does not imply that $N \simeq N'$ holds (see [CPSZ, Remark 2.8]).

The property of unique decompositions, which naturally implies that the canceling rule holds, is also called the *Krull-Schmidt property.*

Luckily the Krull-Schmidt property is known to hold for all Chow motives mod p of projective, homogeneous varieties by [CM]. Taking also the RNT into account, we see that the motives of all varieties considered by us behave in a good way. By that we mean that every indecomposable motivic summand N over k will map to one motivic summand over L/k (i.e. it will stay indecomposable) or it will decompose into smaller indecomposable motivic summands, whose Poincaré polynomials exactly add up to the Poincaré polynomial of N over k (i.e. we can virtually lift several motivic summands at once to the base k, where they are glued) and on top this all happens in a unique way.

5.2.22. Another mystery about motives, is their possible structure (i.e. Poincaré polynomials). Besides the symmetry the Poincaré polynomial of a motive, not much is known. Consider the easiest imaginable case of a motive which splits into exactly two Tate motives over some field extension. Such motives are called *binary.*

5.2.23 Binary summand Theorem. ([Shells, Corollary 7.6]) *Let k be a field with* $\mathrm{char}(k) = 0$ *and let N be an indecomposable direct summand of the motive of a smooth projective variety X such that over $k(X)$ the motive N splits as $\mathbb{F}_2 \oplus \mathbb{F}_2(a)$. Then $a = \dim(N) = 2^n - 1$ for some $n \in \mathbb{N}_0$.*

□

5.3 Upper motives

In this section, we present facts on upper motives, which have been introduced by Karpenko. He has proven that for the motives of projective, homogeneous G-varieties the possible indecomposable motivic summands are very limited and depend on G. Finding them boils down to determining the so called upper motives of the projective, homogeneous G-varieties.

5.3.1 Definition. Let G be a semisimple algebraic group and let X be a projective, homogeneous G-variety over a field k. An indecomposable motivic summand of $M(X)$, which has its starting point in codimension 0, is called the *upper motive* of X. We denote the upper motive of X by $\mathcal{U}(X)$.

5.3.2. We emphasize that by [Kar13, Remark 2.13] the upper motive is unique up to motivic isomorphism. Using the notion of the upper motive, one can express motivic indecomposability briefly by stating that $M(X) \simeq \mathcal{U}(X)$ holds.

5.3.3 Remark. There are two very well known facts concerning the zero cycles of any smooth projective variety X, which we use several times. The first one is that the upper motive of X is isomorphic to $\mathbb{F}_p$ if and only if X has a zero cycle γ of degree d coprime to p. Unfortunately it does not follow that X is isotropic then, as an example for certain G-varieties of unitary groups G shows (see [Pa]). The second fact is that in case X has a zero cycle of degree coprime to p, it then becomes isotropic over a field extension L/k of degree coprime to p. For quadrics the following statement holds.

5.3.4 Springer's Theorem. ([EKM, Corollary 18.5]) *Let L/k be a finite field extension of odd degree. Suppose that q/k is an anisotropic quadratic form over k. Then q/L is anisotropic.*

□

5.3.5 Remark. Combining this result with the basic knowledge that a variety which is anisotropic and has a zero cycle of odd degree becomes isotropic over a field extension of odd degree, we conclude that any anisotropic quadric does **not** have a zero cycle of odd degree. We will refer to this conclusion as *Springer's theorem*, too.

5.3.6 Definition. Let G_0 denote a split algebraic group over k. Consider the group $G \simeq {}_{\xi}(G_0)$ for some fixed $\xi \in H^1(k, G_0)$ and let X be the projective, homogeneous G-variety ${}_{\xi}(G_0/P_\Theta)$. We define

$$\#M(X/k) := \{N \in \mathrm{Obj}(\mathcal{M}_k) \mid N \in M(X/k) \text{ indecomposable}\} \ \mathbf{mod}\ \mathbb{F}_p(1).$$

By the $\mathbf{mod}\ \mathbb{F}_p(1)$ expression we mean that we ignore the Tate shift of N. In case there is no danger of confusion, we will suppress the base field k in the notation. Also we define

$$\#G/k := \bigcup\nolimits_{\Theta \subset \Delta(G)} \#M(X).$$

5.3.7. Often we omit the base field k in the notation and simply write $\#G$ for $\#G/k$. As $\#G$ depends solely on ξ, we write $\#G/L$ for a field extension L/k, when $\mathrm{res}(\xi)_{L/k}$ is considered for twisting. The class of $\mathcal{U}(X)$ is always contained in $\#M(X)$. In case G is split, every projective, homogeneous G-variety X has $\#M(X) = \{\mathbb{F}_p\}$.

5.3.8. Unfortunately $\#M(-)$ behaves counter intuitive in the sense, that $\#M(X/k)$ can contain more, as many as, or less elements than $\#M(X/L)$. For example if $M(X/k)$ is indecomposable, but splits into at least two motivic summands over L, which are not isomorphic up to Tate shift. Overall $\#M(-)$ does measure the motivic diversity of X depending on ξ. Karpenko proved this result about $\#G$, which is of highest importance to us.

5.3.9 Theorem. ([Kar13, Theorem 3.5]) (Karpenko's theorem) *Let G be a semisimple algebraic group of inner type over k and let X be a projective, homogeneous G-variety. Then each indecomposable motivic summand in $M(X)$ is isomorphic to a shift $\mathcal{U}(Y)(i)$ of the upper motive of some projective, homogeneous G-variety Y, such that the Tits index of G over $k(X)$ is contained in the Tits index of G over $k(Y)$.*

□

5.3.10. Karpenko's theorem heavily restricts the number of possible elements of $\#G$. Although it is loosely bounded by $2^{\mathrm{rank}(G_0)}$, as this is the maximal number of G-varieties of the form G/P_Θ, we will find out in the last chapters that there are never a lot of elements in $\#G$, for G being of type E_7. Also we see that the theorem does not require Y to be different from X, i.e. it may happen that an element $\mathcal{U}(X) \in \#G$ does only have one representative X. Establishing the structure of the upper motives of all projective, homogeneous G-varieties is the key step in calculating the whole motivic decompositions of all projective, homogeneous G-varieties.

5.3.11. The decompositions of the motives of the projective, homogeneous E_6-varieties mod 3, established in [Shells] show that knowing the Tits index of a group is not enough to calculate $\#G$ (see [Shells, Table 8.A along with Table 10.A]). In fact another invariant J_p, which we introduce in Chapter 6., is needed to distinguish between the anisotropic cases.

5.3.12. Lastly we cite a key result concerning upper motives, which interacts very well with Karpenko's theorem. In order to prove it, one needs some more definitions which we have not introduced, since we do not need them. We are content with naming a source, where more background is given. Note, that the theorem holds for Chow groups with $\mathbb{F}_p$ coefficients.

5.3.13 Theorem. ([Zhy, Corollary 3.3.7.]) *Let G be a semisimple algebraic group of inner type over k and let X, Y be projective, homogeneous G-varieties. Then X has a zero cycle of degree one over $k(Y)$ and vice versa if and only if $\mathcal{U}(X) \simeq \mathcal{U}(Y)$ holds over k.*

□

5.4 Shells

This section very briefly introduces the the concept of shells. It was originally introduced by Vishik during the research on Chow motives of quadrics. Later it was further developed in [Shells] and successfully used to provide a complete motivic decomposition of the projective, homogeneous E_6-varieties mod 3. The work [Shells] is the main inspiration for this whole thesis. Interestingly we will barely use shells explicitly, except for our final main result proven in the last chapter.

5.4.1 Definition. (Shells) Let Θ be a subset of vertices of the Dynkin diagram $\Delta(G)$ of a semisimple algebraic group G of inner type over k. In analogy to [Shells, 4.] define the *big shell* $\mathrm{SH}_{\leq\Theta}(X)$ of a projective, homogeneous G-variety X as the union for all i of the cycles $b \in \mathrm{Ch}^i(\overline{X})$ such that

1. b is rational over $k(X_\Theta)$ and
2. there is an $a \in \mathrm{Ch}_i(\overline{X})$ rational over $k(X_\Theta)$ such that $\deg(ab) = 1 \in \mathbb{F}_p$

We call $\mathrm{SH}_{\leq\Theta}(X_\Theta)$ the *first shell*. Further we define the *small shell* $\mathrm{SH}_\Theta(X)$ as the union of all i for all cycles $b \in \mathrm{Ch}^i(\overline{X})$ such that b is the starting point of an indecomposable direct summand in $M(X)$ isomorphic to a Tate shift of $\mathcal{U}(X_\Theta)$.

5.4.2. This definition may become a bit more understandable, if one recalls that it is inspired from Vishik's research on the motives of quadrics. Consider some group of type B_n or D_n defined by a anisotropic quadratic form q. Passing from k to the generic point of the anisotropic quadric X_q makes it isotropic and thus some Tate motives occur in $M(X_q)$ over the function field of X_q. All Tate motives come from copies of the upper motive of X_q. This is because firstly $M(X_q/k)$ contains no Tate motives by Springer's theorem. But secondly, by the definition of the generic point, there is no other projective, homogeneous G-variety Y, such that X_q is anisotropic over $k(Y)$. On the other hand, all other projective, homogeneous G-varieties Y, which become isotropic over $k(X_q)$ have the same upper motive as X_q by Karpenko's theorem.

Assume there are n Tate motives in $M(X_q)$ over $k(X_q)$. Then n must be divisible by the number of shifts of $\mathcal{U}(X_q/k)$ contained in $M(X_q/k)$. In fact it was shown by Vishik, that there are exactly $n/2$ shifts of $\mathcal{U}(X_q/k)$ contained in $M(X_q/k)$. The first shell describes the starting points of these shifts.

5.4.3. Then one can consider the anisotropic kernel q' of q over $k(X_q)$ and pass to $k(X_{q'})$, to calculate the next shell. More Tate motives in $M(X)$ occur and so on. In other words, the Tits index, which extends the concept of the Witt index, is reflected in the algebraic cycles in the shells. The definition of shells extends further to the case of varieties of type G/P_Θ, not isomorphic to quadrics and thus from groups of type B_n and D_n corresponding to quadratic forms to all Killing-Cartan types.

5.5 Examples of known motivic decompositions

The following examples give an overview on the progress of motivic decompositions made during the last decades. We will also use these well known results later in many of our arguments. So this services as basic material for some calculations done later on. We start with Severi-Brauer varieties also proven by Karpenko. From a motivic viewpoint Severi-Brauer Varieties turn out to be among the least complicated varieties. For basic results about quadrics see [EKM]. For some decompositions we provide proofs, even though they are kind of well known, but may not have been written down properly anywhere. In the fundamental work [Kar95] some results were obtained, which can be summarized as follows.

5.5.1 Theorem. ([Kar95]) *Let D be the division algebra Brauer equivalent to A. Then the unique motivic decomposition of $M(\mathrm{SB}(A))$ is given by*

$$M(\mathrm{SB}(A)) = \bigoplus_{i=0}^{m} \mathrm{SB}(D)(c_i),$$

with $P(\mathrm{SB}(A), t)/P(\mathrm{SB}(D), t) = \sum_{i=0}^{m} c_i t^i \in \mathbb{N}_0[t]$ and $m = \deg(A)/\mathrm{ind}(A)$. The motive of $\mathrm{SB}(A)$ is indecomposable if and only if A is division.

□

5.5.2. Note that this result does also hold for Chow motives with $\mathbb{Z}$ coefficients by some lifting argument proven in [PSZ]. In any case one has $\#M(\mathrm{SB}(A)) = \#M(\mathrm{SB}(D))$. The theorem does also reveal the structure of the upper motive of $\mathrm{SB}(A)$. It is simply given by $P(\mathcal{U}(\mathrm{SB}(A)),t) = P(\mathcal{U}(\mathrm{SB}(D)),t) = (t^{\mathrm{ind}(A)}-1)/(t-1)$. For generalized Severi-Brauer varieties, similar results are known about their Chow-motives (see [Zhy]).

5.5.3. (Pfister quadrics) Consider an anisotropic n-Pfister form φ over k and a field extension L/k. Since φ/L is hyperbolic if and only if it is isotropic, we have $\#M(X_\varphi) = \{\mathcal{U}(X_\varphi)\}$. The surprising result that $M(X_\varphi)$ is actually decomposable when X is anisotropic, was established by Rost.

5.5.4 Theorem. ([Ro98, Prop. 19]) *Let φ be an anisotropic n-Pfister form. Then there is an indecomposable motivic summand $\mathcal{R}_n$, such that the motive of the quadric X_φ decomposes as*

$$M(X_\varphi) = \bigoplus_{i=0}^{2^{n-1}-1} \mathcal{R}_n(i),$$

with $P(\mathcal{R}_n,t) = 1 + t^{2^{n-1}-1}$.

□

5.5.5 Remark. (Rost motives) The motives $\mathcal{R}_n$ are widely known as (original) *Rost motives.* For $n = 1$ this motive equals the motive of a quadratic extension of the base field k and for $n = 2$ it is known that $\mathcal{R}_2$ is isomorphic to $M(X_q)$ for X_q being a hyperplane section of a 2-Pfister quadric (see [Vis04]).

5.5.6 Example. (Motivic decompositions of F_4) Let us consider $F_4 \simeq G = \mathrm{Aut}(\mathcal{J})$ for some Albert algebra $\mathcal{J}$. The mod 3 case of the motivic decompositions of the projective, homogeneous F_4-varieties was solved in [NSZ]. It turns out that it is similar to the decomposition of Pfister quadrics, in the sense that every $M(X_\Theta)$ for $\Theta \subset \Delta(F_4)$ is decomposable and $\#M(X_\Theta) = \{\mathcal{R}_J\}$ for some motive $\mathcal{R}_J$ with $P(\mathcal{R}_J,t) = 1 + t^4 + t^8$, when F_4 is anisotropic. As the only other Tits 3-index is the split one, there is nothing left to prove for the mod 3 case.

For the mod 2 case there are three Tits 2-indexes. All $M(X_\Theta)$ are decomposable, with $\#M(X_\Theta) = \{\mathcal{R}_3\}$ and $4 \notin \Theta$, when G is not split. The decomposition of $M(X_4)$, when G is anisotropic was calculated in [McD09]. It is given by

$$M(X_4) = \mathcal{R}_5 \bigoplus \oplus_{i\in I}\mathcal{R}_3(i)$$

for some multiset I. Interestingly the $\mathcal{R}_J$ mod 3 splits into Tate motives if and only if $g_3(\mathcal{J}) = 0$ holds. An analogous relation holds for the mod 2 Rost motives $\mathcal{R}_5$ and $f_5(\mathcal{J})$ and $\mathcal{R}_3$ and $f_3(\mathcal{J})$.

5.5.7. (Generalized Rost motives) The motive $\mathcal{R}_J$ is *called generalized Rost motive.* It was originally introduced by Voevodsky (see [Voe03, Chapter 5.]) and is one of the key ingredients in the proof to the Bloch–Kato conjecture for general n, p, as it allows for the motive of any so called *norm variety* X to decompose.

By (the general) definition of these varieties there exists a non zero pure symbol $\alpha \in H^n(k, \mu_p^{\otimes n})$, which becomes zero over $k(X)$. This marks our first point of contact with a connection between motives and Galois cohomology. The concept of generalized Rost motives of Voevodsky generalizes even further in Chapter 7, when we introduce generically split varieties.

5.5.8. (Motives of quadrics with splitting pattern $[1, 2]$) Consider two 2-Pfister forms φ, φ' with no common slot. We set $q := \varphi \perp -\varphi'$ in $W(k)$. Quadratic forms of this type are known as *Albert forms* and exclusively have splitting pattern $[1, 2]$ by the splitting pattern table. We need to know the motive of an anisotropic X_q for later use. We provide a proof of its well known structure, to give an outline on the techniques used in later proofs and to the make this thesis a bit more self contained. Also this simple proof may give an idea why proving decomposability is usually much harder, than proving that the motive of a projective variety is indecomposable.

5.5.9 Lemma. *Let q be an anisotropic quadratic form of rank 6 with splitting pattern $[1, 2]$. Then the motive of X_q is indecomposable.*

Proof: Passing to $k(X_q)$, one hyperbolic plane is split off from q, because of the splitting pattern of q. This is the same as to say there are exactly two Tate motives in $M(X_q)$ over $k(X_q)$ by [EKM, Proposition 70.1]. As $\dim(X_q) = 4$ holds, the motivic decomposition of $M(X_q)$ is given by $\mathbb{F}_2 \oplus M(X_{q'})(1) \oplus \mathbb{F}_2(4)$ over $k(X_q)$, with q' denoting the anisotropic kernel of q. The remaining splitting pattern $[2]$ belonging to q' indicates that q' is a 2-Pfister form. Using Rost's result on motives of Pfister forms, we obtain the decomposition $M(X_q) = \mathbb{F}_2 \oplus \mathcal{R}_2(1) \oplus \mathcal{R}_2(2) \oplus \mathbb{F}_2(4)$ over $k(X_q)$, with $P(\mathcal{R}_2, t) = 1 + t$. Now we show that none of the summands in the decomposition can be lifted to k.

For the Tate motives this is clear, as by Springer's theorem from [EKM, Corollary 18.5] X_q is anisotropic if and only if it has no zero cycle of odd degree over k. The two Tate motives are glued to each other over k, because of the symmetry of the decomposition, so $\mathcal{U}(X_q)$ is 4-dimensional over k. By the binary summand theorem from [Shells, Corollary 7.6] there are no 4-dimensional binary motives. Thus, using the symmetry of the decomposition again, both of the two summands $\mathcal{R}_2(1) \oplus \mathcal{R}_2(2)$ in $M(X_q/k(X_q))$ are glued to $\mathcal{U}(X_q)$ over k. □

5.5.10 Lemma. *Let q' be an anisotropic quadratic form of rank 12 with splitting pattern $[2, 4]$. Then the unique motivic decomposition of $M(X_{q'})$ into indecomposable motivic summands is given by*

$$M(X_{q'}) = \mathcal{U}(X_{q'}) \oplus \mathcal{U}(X_{q'})(1),$$

with $P(\mathcal{U}(X_{q'}), t) = 1 + t^2 + t^4 + t^5 + t^7 + t^9$.

Proof: Note that Pfister classified anisotropic quadratic forms of rank 12 with splitting pattern $[2, 4]$. Namely, such forms are exactly of the type $q' = \varphi \perp -\varphi'$ in $W(k)$, with φ, φ' being 3-Pfister forms with exactly one common slot.

By [Vis98, Theorem 4.1] the motive of $X_{q'}$ is isomorphic to $N \oplus N(1)$ for some motive N. Moreover, the motive N is indecomposable. Indeed, otherwise we would have a direct summand in the motive of our quadric starting in the second shell. But then all its shifts within the second shell would be also direct summands of the motive of the quadric q'. Then the upper motive of q' would be binary of dimension 9, which contradicts the binary summand theorem.

□

5.5.11 Lemma. (Vishik) *Let q be an anisotropic quadratic form of rank 10. Assume its splitting pattern is equal to $[1, 2, 2]$. Then the motive of X_q is indecomposable.*

In case q is an anisotropic quadratic form of rank 8 *with splitting pattern* $[2,2]$, *the motive of X_q decomposes into indecomposable motivic summands as*

$$M(X_q) = \mathcal{U}(X_q) \oplus \mathcal{U}(X_q)(1),$$

with $P(\mathcal{U}(X_q), t) = 1 + t^2 + t^3 + t^5$.

Proof: By a result of Pfister, in the rank 8 case the form q is proportional to a difference of a 3-Pfister form and a 2-Pfister form having exactly one common slot.

Exactly as in the previous lemma we obtain a decomposition

$$M(X_q) = N \oplus N(1)$$

(see [Vis04, first part of the proof of Prop. 5.10]), and the motive N is indecomposable, since otherwise we would get a binary motive of dimension 5.

Assume now that the form q has rank 10 and splitting pattern $[1,2,2]$. In this case the form q corresponds to a difference of a 3-Pfister form and a 2-Pfister form having no common slots.

By the result on the splitting pattern $[2,2]$, the motive of X_q decomposes as $\mathbb{F}_2 \oplus N(1) \oplus N(2) \oplus \mathbb{F}_2(8)$ over $k(X_q)$, with $P(N,t) = 1 + t^2 + t^3 + t^5$ as above. It follows from the symmetry argument and from the binary summand theorem, that over k the motive of $M(X_q)$ is indecomposable. $\square$

5.5.12. (Motives of involution varieties) There are very few known results about motives of involution varieties in the literature up to this day. One is due to Karpenko, roughly stating that all Tate motives in $M(\mathcal{I}(A,\sigma))$ over $k(\mathcal{I}(A,\sigma))$ come from shifts of $\mathcal{U}(\mathrm{SB}(A))$, visible over k (see ([Kar09, Prop. 4.1])). Another known result was established in [Nes, Remark 7.2.1]. It provides an example of an involution variety with indecomposable motive. The result holds for a twist with a versal $\mathbf{HSpin}_8$-torsor.

5.5.13 Example. The result [Kar09, Prop. 4.1] of Karpenko raises the question, how $M(\mathcal{I}(A,\sigma))$ looks over k, if $\mathcal{I}(A,\sigma)$ remains anisotropic over the generic point of $\mathrm{SB}(A)$. This is also unknown in general. However, it is known that the group $\mathbf{Spin}(A,\sigma)$, $\mathbf{SO}(A,\sigma)$, $\mathbf{PGO}^+(A,\sigma)$ or $\mathbf{HSpin}(A,\sigma)$ is anisotropic over $k(\mathrm{SB}(A))$ (for $\mathrm{char}(k) \neq 2$), when A is division of degree 2^n.

To see this (it is originally proven in [Kar11, Thrm 5.3] by Karpenko), first remember that $\mathcal{I}(A,\sigma)$ is a closed subvariety of $\mathrm{SB}(A)$ with codimension 1. It is therefore clear that $\mathrm{SB}(A)$ is isotropic over $k(\mathcal{I}(A,\sigma))$. But when A is division, $M(\mathrm{SB}(A))$ is indecomposable by Theorem 5.5.1 and thus isomorphic to $\mathcal{U}(\mathrm{SB}(A))$. Therefore $\mathcal{U}(\mathrm{SB}(A))$ and $\mathcal{U}(\mathcal{I}(A,\sigma))$ can not be isomorphic, as $\mathcal{U}(\mathrm{SB}(A))$ has dimension $d = 2^n - 1$ and $\mathcal{U}(\mathcal{I}(A,\sigma))$ has at most dimension $d - 1 = 2^n - 2$.

5.5.14. In the case of generalized involution varieties, even less is known about the motives than for generalized Severi-Brauer varieties. This has mainly to do with the mysterious isotropy behavior of G over $k(\mathrm{SB}(A))$. In Theorem 10.4.9, we provide a complete motivic decomposition of the projective, homogeneous $\mathbf{HSpin}_{12}$-varieties corresponding to maximal parabolic subgroups, for the case when the motivic J-invariant $J_2(G)$ (defined in Chapter 6) is maximal. This includes the 10-dimensional involution variety, which turns out to have an indecomposable motive.

5.6 The Chernousov-Gille-Merkurjev-Brosnan algorithm

Most of the motivic decompositions introduced in the previous section are obtained, by passing to a field extension L/k such that a certain anisotropic variety X becomes isotropic over L. Then, using the Rost nilpotence theorem, a lifting procedure involving combinatorial arguments is performed. So calculating the decomposition of $M(X/L)$ is usually the first step in calculating $M(X/k)$. There is an algorithmic procedure, which *kind of* achieves this. In its full generality it is the result of the work by Chernousov-Gille-Merkurjev (see [CGM]) and Brosnan (see [Br05]). We refer to it the *CGMB algorithm* from now on. For more background on the math involved see [Shells, Chapter 6].

5.6.1 Theorem. ([CGM, Thrm 7.4], [Br05, Thrm 7.5]) *Let G be an isotropic semisimple algebraic group, with semisimple anisotropic kernel H. Let X be a projective, homogeneous G-variety. Then there are projective, homogeneous H-varieties Y_i such that*

$$M(X) = \bigoplus_{i\in I} M(Y_i)^{\oplus c_i}(s_i) \oplus \bigoplus_{j\in J} \mathbb{F}_p^{\oplus d_i}(t_j)$$

for some multisets I, J and some numbers $c_i, s_i, d_j, t_j \in \mathbb{N}_0$.

□

5.6.2. In the decomposition above, it is possible that there are no Tate motives at all of course. The algorithm is implemented in the *Chow maple package* (see [NS06]) and can be performed by executing the command *prodbases(H, P, G)* to calculate $M(X)$, when X is a twisted form of G/P and H is the semisimple anisotropic kernel of G. The unsatisfying fact concerning the algorithm is that it only establishes the decomposition in the following sense. For example, it understands that a shift of $M(Y)$ occurs in $M(X)$ for some projective, homogeneous H-variety Y. But it does not compute an actual decomposition of $M(Y)$ because H is anisotropic and thus the algorithm can not be applied to $M(Y)$.

Chapter 6

The motivic J-invariant

In this chapter we introduce the so called motivic J-invariant. In the first section we introduce general results about the J-invariant, after defining it. In the second section we focus on particular cases of of the value of J in conjunction with algebraic groups of type E_7. We provide many examples in both sections, which will be needed during the course of many proofs later on. Originally the concept of the

J-invariant was developed by Vishik in [Vis05] in order to define a new invariant for quadratic forms (and thus some groups of type B and D), by measuring rational cycles in the Chow groups of quadratic Grassmannians. The generalization to arbitrary algebraic groups was established in [PSZ], which is our main reference for this chapter. Recently a refined version of the concept for a broad class of generalized oriented cohomology theories in the sense of Levine–Morel has been defined in [PS22].

6.1 Construction of the J-Invariant

6.1.1. We consider a split semisimple linear algebraic group G_0 over k, with a split maximal torus T and a Borel subgroup B of G_0 containing T. Let $G := {}_\xi G_0$ be a twist of G_0 by $\xi \in H^1(k, G_0)$. Note that G is of inner type since it is a twisted form of a split group. We consider a twisted form of the Borel variety $\mathfrak{X} := {}_\xi(G_0/B)$. We need a result by Grothendieck for defining of the J-invariant the same way as in [PSZ]. In [Gr58, p. 21, Remark 2] it is shown that the pullback $\pi^*\colon \mathrm{Ch}(G_0/B) \to \mathrm{Ch}(G_0)$ of the quotient map $\pi\colon G_0 \to G_0/B$ is surjective. Moreover,

$$\mathrm{Ch}^*(G_0) \simeq \mathbb{F}_p[e_1, \ldots, e_r]/(e_1^{p^{k_1}}, \ldots, e_r^{p^{k_r}})$$

for some integers r, k_i and with $\operatorname{codim} e_i =: d_i$. We assume that the sequence of d_i is non-decreasing.

6.1.2 Definition. (J-invariant) Let $\xi \in H^1(k, G_0)$ and let G and $\mathfrak{X}$ be as above. We identify $\mathrm{Ch}(\overline{\mathfrak{X}})$ with $\mathrm{Ch}(G_0/B)$ and consider the image of the following composition of maps

$$\mathfrak{J} : \mathrm{Ch}(\mathfrak{X}) \xrightarrow{res} \mathrm{Ch}(\overline{\mathfrak{X}}) \xrightarrow{\pi^*} \mathrm{Ch}(G_0)$$

Since both maps are ring homomorphisms, $\mathrm{im}(\mathfrak{J})$ is a subring of $\mathrm{Ch}(G_0)$. For each $1 \le i \le r$ we set j_i to be the smallest non negative integer such that $\mathrm{im}(\mathfrak{J})$ contains an element a with the greatest monomial $e_i^{p^{j_i}}$ in respect to the *wdegrevlex* order. Thus it is of the form

$$a = e_i^{p^{j_i}} + \sum_{x^M \lneq x_i^{p^{j_i}}} c_M e^M,\ c_M \in \mathbb{F}_p.$$

The r-tuple of integers $(j_1, \ldots, j_n)$ is then called the *J-invariant of ξ modulo p* and denoted by $J_p(\xi)$. Assume that G_0 is simple and not of type D_4. If G is a twist of G_0 by ξ, one can show that the J-invariant of ξ depends only on G and we denote the J-invariant of ξ by $J_p(G)$.

6.1.3. (Notation and remarks) If the Dynkin diagram of a split semisimple group admits a symmetry it is possible that twisting with different torsors ξ, ξ' results in isomorphic groups, but $J_p(\xi) \neq J_p(\xi')$ holds. Otherwise this can not happen. The issue arises, e.g., for groups G of type $\mathbf{PGO}_8^+$ (see [QSZ, 2.]). But we mostly do not consider such groups and keep writing $J_p(G)$.

As any group G of inner type is split over k if and only if $\mathfrak{X}$ has a rational point over k, the J_p-invariant is normalized in the sense that it is zero, when G is split. Often we will suppress ξ or G in the notation and just write J_p. If all entries of J_p are zero, we write $J_p = 0$.

The values of $J_p(G)$, which can potentially occur for any type of G, are summarized in a table in [PSZ, at end of section 4.]. We will refer to it as the *J_p-table.* The primes p for which $J_p(G)$ is not always zero, are in fact the torsion primes of the respective type. Note the differences of potential values of $J_2(G)$ between simply connected and adjoint groups of type E_7. The dependencies among the j_i, which were established in [PSZ] using Steenrod operations, are very helpful to us. In general it is unknown, if all theoretically possible values presented in the J_p-table do actually occur for some of the group types. We list the following essential properties of the J-invariant, which to prove takes too much effort to perform here.

6.1.4 Theorem. *Let G be a semisimple algebraic group of inner type over a base field k, p a prime integer and $J_p(G) = (j_1, \ldots, j_r)$. Then the following properties hold*

1. (transfer argument) *If L/k is a field extension of degree coprime to p then $J_p(G/k) = J_p(G/L)$ holds.*

2. (cut off) *Let H be the semisimple anisotropic kernel of G. Then $J_p(G) = J_p(H)$ holds. Also in case $J_p(H)$ has only $s < r$ entries, the $r - s$ entries unique to $J_p(G)$ are zero.*

3. (decrease) *If L/k is a field extension then $(j_i)_L \leq (j_i)_k$ holds for all $1 \leq i \leq r$.*

4. (triviality 1.) *$J_p(G)$ is zero if and only if G splits over a field extension of k of degree coprime to p.*

5. (triviality 2.) *Assume that G does not have simple components of type E_8 and that $J_p(G)$ is zero for all p. Then G is split.*

Proof: For the first statement see [PSZ, Proposition 5.18 (ii)]. The second statement is [PSZ, Corollary 5.19]. In the reference it is formulated in terms of the generalized Rost motive $\mathcal{R}_J$ from [PSZ, Theorem 5.17] (it is denoted by $\mathcal{R}_p(G)$ there). For the third and fifth statement see [GSV, Proposition 3.9, 1. and 2.]. The fourth statement is essentially [PSZ, Corollary 6.7]. $\square$

6.1.5. From the properties above, we see that the J-invariant behaves intuitive in any aspect relating field extensions. Remembering the definition of Tits p-indexes, it becomes clear that considering p-special fields is a potentially good way of simplification of many calculations involving J_p. Also the properties **1**. and **4.** reveal a connection between zero cycles of coprime to p degree in the Chow groups of the projective, homogeneous G-varieties and the value of $J_p(G)$. This will come into play in some proofs later on and in the next chapter.

6.1.6 Example. (J-invariant of G_2) Let G be an algebraic group of type G_2. From the J_p-table we see, that $p = 2$ is the only prime, for which J_p is not always zero.

Also there are only two possibilities for $J_2(G)$. Considering that there are only two possible Tits indexes for G and taking into account the property **5.** above, we see that the Tits index of G does exactly correspond to the value of $J_2(G)$. We obtain $J_2(G) = (1)$ in case G is anisotropic and $J_2(G) = (0)$ in case G is split.

6.1.7 Example. (J-invariant of F_4) Now let us consider an algebraic group G of type F_4. From the J_p-table we see, that there are two possibilities for the value of $J_2(G)$ and also $J_3(G)$. There are three Tits 2-indexes and two Tits 3-indexes. One can deduce whether G is split or not just by looking at the values of $J_2(G)$ and $J_3(G)$ by the property **5.** In case G is not split and $J_3(G) = (0)$, one necessarily has that $J_2(G) = (1)$. The condition $J_3(G) = (0)$ is fulfilled if k is a 2-special field for example by the property **4.** of the J-invariant. The exact Tits index however can not be deduced from knowing that $J_2(G) = (1)$ and $J_3(G) = (0)$ holds, but of course vice versa. In case $J_2(G) = (0)$ holds, the value of $J_3(G)$ indicates whether G is split or not as there is only one Tits index corresponding to each value, similar to groups of type G_2.

6.1.8 Example. (J-invariant of E_6) For groups G of type E_6 the J_p-table reveals that $p = 3$ is the prime for which things are the most complicated. Let us assume that $J_2(G)$ is zero. In [Shells, Table 10.A] we see that, in contrast to the F_4 mod 2 case, the Tits index of G *can* in fact be deduced from knowing $J_3(G)$ but not vice versa.

6.1.9 Remark. Let G be a split semisimple algebraic group, Q some parabolic subgroup of G. Let C denote the commutator $[L, L]$ of the Levi subgroup of Q (see [Hum2, 30.2]). By [GSV2, Lemma 2.3] we can identify $\mathrm{Ch}^*(C)$ with $\mathrm{Ch}^*(Q)$. If we also use the result of [PS22, Lemma 6.2], then there is a right exact sequence of graded rings

$$\mathrm{Ch}^*(G/Q) \to \mathrm{Ch}^*(G) \twoheadrightarrow \mathrm{Ch}^*(C) \to 0.$$

6.2 The J-Invariant of E7

In this small section we discuss some observations and cite results we need later in Chapter 8., to calculate the values of the J-invariant in conjunction with the Tits index for adjoint groups of type E_7.

6.2.1. (Values of $J_p(E_7)$) We reproduce the E_7 part from the J_p-table for the readers convenience. The numbers k_i are the maximal value for each component j_i of $J_p(G)$, while the d_i are the exceptional p-degrees (and thus the codimensions of the generators of $\mathrm{Ch}(G)$).

G	p	k_i	d_i	Restrictions
E_7^{sc}	2	$1,1,1$	$3,5,9$	$j_1 \geq j_2 \geq j_3$
E_7^{ad}	2	$1,1,1,1$	$1,3,5,9$	$j_2 \geq j_3 \geq j_4$
E_7	3	1	4	

We see that there are **eight** theoretically possible values for $J_2(G)$ in case G has type E_7^{ad}, including the value zero. For $J_3(G)$ there are only two possibilities, including the value zero. Assuming $J_2(G)$ is zero and taking into account the Tits classification this shows that $J_3(G) = (1)$ in case G has anisotropic kernel of type E_6 and $J_3(G) = 0$ if G is split, by property **5.** and because other Tits 3-indexes do not exist.

We see that for groups of type E_7 the modulo 2 case is the case of actual interest. Also because there are only **seven** Tits 2-indexes. For later we need a theorem on the relation of the index of the Tits algebra of an E_7^{ad} and the values of its J_2-invariant.

6.2.2 Theorem. ([GSV, Proposition 4.2]) *Let G be an adjoint semisimple algebraic group of inner type over k. Let p be a prime integer and $J_2(G) = (j_1, \dots, j_r)$. Then $j_i = 0$ holds for all i with $d_i = 1$ if and only if the indexes of all Tits algebras of G are coprime to p.*

□

6.2.3 Corollary. *Let G be an adjoint semisimple algebraic group of type E_7 with $J_p(G) = (j_1, j_2, j_3, j_4)$. Then $j_1 = 0$ holds if and only if the Tits algebra of G is split.*

Proof: Checking the E_7^{ad} mod 2 row in the J_p-table, shows that the only d_i with $d_i = 1$ is d_1. Also every group of type E_7 has only one possibly non trivial Tits algebra up to Brauer equivalence by Remark 3.7.5. The claim follows by applying the theorem above. □

6.2.4 Remark. Let H be the semisimple anisotropic kernel of an adjoint algebraic group of type E_7. If H is of type D_6 it is a halfspin group $\mathbf{HSpin}_{12}$. This can be derived by a careful combinatorial analysis of the root data. In fact, it is easy to exclude the cases $\mathbf{SO}_{12}$ and $\mathbf{PGO}_{12}^{+}$ using the J_p-table, as groups of type $\mathbf{SO}_{12}$ (resp. $\mathbf{PGO}_{12}^{+}$) have k_1 (resp. k_2) parameter equal to 3, while groups of type E_7 in general have each k_i equal to 1, and by Remark 6.1.9 the canonical homomorphism $\mathrm{Ch}^*(E_7) \to \mathrm{Ch}^*(D_6)$ must be surjective.

6.2.5 Example. Let q be an anisotropic quadratic form of rank 8. Assume that q has trivial discriminant and Clifford invariant (see [EKM, §14]). It is then necessarily isometric to a 3-Pfister form φ by the Arason–Pfister Hauptsatz (see [EKM, Thrm. 6.18]).

We consider the group $G \simeq \mathbf{SO}(q)$. By consulting the J_p-table and because of the [GSV, Proposition 4.2], we see that $J_2(G) = (0, j_2)$ holds, as the Tits algebras of G are trivial. As j_2 is either 1 or 0 and G is anisotropic by our assumption, j_2 can not be zero, since this would mean that G is split by property **5.** of the J-invariant. Thus $J_2(G) = (0, 1)$ holds.

If we look at this example from the viewpoint of a Spin group, i.e., if we take G of type $\mathbf{Spin}(q)$, then by the same considerations things are easier and $J_2(G) = (1)$ holds.

Finally, from the $\mathbf{PGO}^{+}(q)$ viewpoint, i.e., in the case of $G = \mathbf{PGO}^{+}(q)$, one has $J_2(G) = (0, 0, 1)$, as there are two generators in $\mathrm{Ch}^1(\overline{G})$ in this case (i.e. $d_1 = d_2 = 1$).

The key take away of this example is, that for any D_4 defined by a anisotropic 3-Pfister form the parameter j_i of the J-invariant which corresponds to the d_i with $d_i = 3$, has the value 1, while all others are zero.

6.2.6 Example. (Quadratic forms of even rank with splitting pattern [1, 2, 2] or [2, 2]). Consider an anisotropic quadratic form $q := \varphi_3 \perp -\varphi_2$ in $W(k)$, with φ_3, φ_2 being 3-Pfister resp. 2-Pfister forms having one or none common slots. We consider $G \simeq \mathbf{SO}(q)$, as well as the quadrics X_q and X_{φ_2}. We have seen in the proof of Lemma 5.5.11 that q has splitting pattern $[1,2,2]$ or $[2,2]$, depending on the number of common slots of φ_3, φ_2.

When we pass to $k(X_{\varphi_2})$, the class of φ_2 becomes trivial in $\mathrm{Br}(k)$, because φ_2 becomes hyperbolic. Since φ_2 is supposed to not divide φ_3, the Witt class of q becomes isomorphic to φ_3. Thus $H := G_{an}/k(X_{\varphi_2})$ is given by $\mathbf{SO}(\varphi_3)$ and $J_2(H) = (0,1)$ holds as in the example before. By property **3.** of the J-invariant, we have $J_2(G/k) = (*,1)$. The Brauer class of φ_2 is isomorphic to the Tits algebra of ω_5 and ω_6, thus by Theorem 6.2.2 we have $J_2(G/k) = (1,1)$ in both cases.

6.2.7 Example. (Quadratic forms with splitting pattern [1, 2, 2] or [2, 2] in E_7) Consider an adjoint group G of type E_7 with semisimple anisotropic kernel $D_5 \times A_1$ (compare the Tits classification). In this case one can associate with this group an anisotropic quadratic form q of rank 10 with splitting pattern $[1,2,2]$ (see [Tits90] for this result). Thus, using property **2.** of the J-invariant and the previous example, we obtain in this case that $J_2(G) = (1,1,0,0)$ holds.

If we pass to $k(X_q)$, then q becomes isotropic and its anisotropic kernel q' has splitting pattern $[2,2]$. In this case the semisimple anisotropic kernel of G is of type $D_4 \times A_1$, and we can repeat the argument and see that $J_2(G/k(X_q)) = (1,1,0,0)$ holds. It follows that for the enveloping E_7, we have $j_1 = j_2 = 1$, and $j_3 = j_4 = 0$ in both cases.

6.2.8 Example. (D_6 in E_7) Let us consider an adjoint group of type E_7. By the Tits classification one of the possibilities for its semisimple anisotropic kernel is to be of type D_6. Consulting the J_p-table, we see that for any group of type D_6, the J_2-invariant has at most three non zero entries. Using the cut off property, we can deduce that any adjoint E_7 with anisotropic kernel D_6 has $J_2 = (j_1, j_2, j_3, 0)$. In fact the same considerations for the other possible Tits indexes, show that every isotropic E_7 has $j_4 = 0$. Also each entry of J_2 is bounded by the restrictions shown in the J_p-table for adjoint groups of type E_7.

6.2.9. In general it is unknown how $J_p(G)$ changes under field extension of degree divisible by p, for any algebraic group G. In the Example 6.2.6 the first value j_1 of $J_2(G)$ changed by passing to $k(\mathrm{SB}(A)) \simeq k(X_{\varphi_2})$ as a consequence of [GSV, Proposition 4.2]. The fact that only j_1 (i.e. the value j_i for which $d_i = 1$ holds) changed, is no coincidence. It turns out that this is a general phenomenon as recently proven by Zhykhovich.

6.2.10 Theorem. ([Zhy22, Theorem 4.1]) *Let G be an algebraic group of inner type over k and let A be a Tits algebra of G. Assume that $J_p(G/k) = (j_1, \ldots, j_n)$. We denote the value of j_i over $k(\mathrm{SB}(A))$ by j_i'. Then $j_i = j_i'$ if $d_i > 1$ holds for the exceptional p-degree d_i.*

□

6.2.11 Corollary. *Let G be an adjoint algebraic group of type E_7 with Tits algebra A and $J_p(G) = (j_1, j_2, j_3, j_4)$. Then over $k(\mathrm{SB}(A))$ one has $J_p(G) = (0, j_2, j_3, j_4)$.*

□

Chapter 7

Generically split varieties

In this chapter we introduce the so called *generically split varieties.* These varieties are completely understood in terms of their motivic decompositions. They mark the most basic case in terms of motivic decompositions. It turns out (see [PSZ]) that for the generically split varieties of an algebraic group G, one can completely deduce the motivic decomposition just from knowing the value of $J_p(G)$. This allows us to tell which projective, homogeneous E_7-varieties may have a somehow surprising motive and which ones do not. In the second section we briefly discuss the so called *coaction* ρ, originally introduced in [PS22]. It recently has been discovered that there is a deep connection between shifts of the upper motive of $\mathfrak{X}$ occurring in a motivic decomposition of a projective, homogeneous G-variety X, the value of $J_p(G)$ and the coaction ρ on $\mathrm{Ch}(G_0)$.

7.1 Definitions and properties

7.1.1 Definition. Let G be an algebraic group over k and let X be a projective, homogeneous G-variety. We say that X is a *generically split variety* for G, if G splits over $k(X)$. We usually use the abbreviation *GSV*.

7.1.2. The first GSV which may come to mind is the Borel variety $\mathfrak{X}$ of any algebraic group G (of inner type). This illustrates, that for every G there is a natural GSV attached to it. We have already encountered other examples of GSVs, such as the Severi-Brauer variety $\mathrm{SB}(A)$ for A being a CSA. The property which makes $\mathrm{SB}(A)$ a GSV for $G \simeq \mathbf{PGL}_1(A)$ or $G \simeq \mathbf{SL}_1(A)$, is the statement of [GSz, Remark 5.3.7], as $\mathbf{PGL}_1(A)$ is split if and only if A is trivial in $\mathrm{Br}(k)$.

7.1.3. Considering the whole theory of splitting patterns of quadratic forms, it comes apparent that for groups G of type B_n and D_n, the G-variety X_1 is not always a GSV. If G is however given by a n-Pfister form φ with $n \geq 2$, it follows from the defining property of Pfister forms of only being anisotropic or hyperbolic, that $X_\varphi \simeq X_1$ splits G.

7.1.4. In Example 6.2.5 and Example 6.2.6, we have seen that for groups of type $D_n \simeq \mathbf{SO}(q)$ for example, the value of $J_2(G)$ is reflected in the splitting pattern of q. This suggests a connection between the motivic J-invariant of an algebraic group G of type D_n and the GSV property of the G-variety X_1. It turns out that in fact the GSV property of *any* variety X_Θ is linked to the motivic J-invariant of G via the motivic decomposition of $M(X_\Theta)$, for all inner types of algebraic groups G.

7.1.5 Theorem. ([PSZ, Theorem 5.17]) *Let G be a semisimple linear algebraic group of inner type over a field k, let p be a prime integer and let $J = (j_1, \dots, j_r)$ denote the motivic J-invariant of G modulo p. Let X be an anisotropic generically split, projective, homogeneous G-variety. Then the motive of X decomposes uniquely in indecomposable motivic summands as*

$$M(X) = \bigoplus_{i \geq 0} \mathcal{R}_J(i)^{\oplus c_i},$$

$$\textit{with } P(\mathcal{R}_J, t) = \prod_{i=1}^{r} \frac{1 - t^{d_i p^{j_i}}}{1 - t^{d_i}},$$

and the integers c_i are the coefficients of the quotient

$$\sum_{i \geq 0} c_i t^i = P(\mathrm{Ch}(X), t)/P(\mathcal{R}_J, t).$$

□

7.1.6. The numbers d_i appearing in the statement above, are the exceptional p-degrees, which can also be found in the J_p-table for each group type. As all other parameters in the statement are also well known and contained in the J_p-table, the question for the motivic decomposition is completely settled not just in theory, but specifically. As for the motivic decomposition of any GSV, we are done.

One may ask which values of $J_p(G)$ make a projective, homogeneous G-variety a GSV, depending on the type of G. These questions, including the E_7 case, where completely answered in [GSV2, Theorem 3.3]. We refer to these results as the *GSV-table*. The next question to answer is, how many distinguishable cases (i.e. twists) of a variety exist, which are not GSV? We widely settle this in the chapter about *phases*.

7.1.7 Remark. We would like to point out that the striking theorem above is a corollary of many results, which we will not discuss for the sake of brevity. One noteworthy thing however to keep in mind is, that the theorem describes many generic points of motivic summands and thus rational cycles in $\mathrm{Ch}(X)$. The result [GSV, Theorem 5.5] does show that these generic points are in fact the **only** rational cycles in $\mathrm{Ch}(X)$ for any GSV X over k. This may shed a bit more light on the motivation behind proving Theorem 6.2.2.

If $d_i = 1$ and $j_i = 0$ holds, the monomial t is not part of $P(\mathcal{R}_J, t)$ and thus it is necessarily the generic point of $\mathcal{R}_J(1)$ in $M(\mathfrak{X})$ and therefore rational. The Theorem 6.2.2 shows, that the exact opposite is also true, i.e. h (represented by t) is not rational if $j_i = 1$ holds for the respective i.

7.1.8. (Generalized Rost motives again) Let us assume that $\sum_{i=1}^r j_i = 1$ holds for $J_p(G) = (j_1, \dots, j_r)$ and $p = 2$. Then the generalized Rost motive $\mathcal{R}_J$ coincides with the original, binary Rost motive, which appears in the decomposition of any anisotropic Pfister quadric as in Example 5.5.4. For odd primes p (and the same sum relation of the j_i), one obtains the generalized Rost motive of Voevodsky. If $\sum_{i=1}^{r} j_i > 1$, we are confronted with even more general generalized Rost motives.

We will usually also refer to $\mathcal{R}_J$ just as *Rost motive* and emphasize if it is binary. Note also that the theorem implies that $\#M(X) = \{\mathcal{R}_J\}$ holds, if X is a GSV. Remember the example of most of the projective, homogeneous F_4-varieties. As $\mathfrak{X}$ naturally is always a GSV for groups G of inner type, it also follows that the class of $\mathcal{R}_J$ is always contained in $\#G$.

7.1.9 Lemma. *Let G be a non split algebraic group of type E_7, with J-invariant $J_2(G)$ and the Tits algebra A. Then G splits over the function field of* $\mathrm{SB}(A)$ *if and only if $J_2(G) = (1,0,0,0)$ holds.*

Proof: If $J_2(G)$ has the desired value then by Theorem 6.2.2, G splits over $k(\mathrm{SB}(A))$. If on the other hand one has that $\mathrm{SB}(A)$ splits G, it follows from the proof of [Shells, Proposition 10.7] that $\mathrm{ind}(A) = 2$ holds and $J_2(G)$ has the desired value. □

7.1.10 Example. (E_7 mod 3) Checking the Tits classification, it becomes clear that every group G of type E_7 mod 3 is either split or has anisotropic kernel of type E_6. As also shown in the GSV-table, it follows that every anisotropic projective, homogeneous G-variety mod 3 is necessarily a GSV. Therefore, concerning motivic decompositions there are no more open questions about E_7 mod 3.

7.1.11 Example. (E_6 mod 3) Here is an illustration on how the property of being a GSV is often used in proofs. The [Shells, Table 8.A], shows the possible values of $J_3(G)$ for any adjoint group G of inner type E_6. The Poincaré polynomials of $\mathcal{R}_J$ depending on $J_3(G)$ are contained in [Shells, Table 8.B]. Assume G is versal with Tits algebra A. Over k we automatically have $J_3(G) = (2,1)$. Let us pass to $k(\mathrm{SB}(A))$. We obtain that $j_1 = 0$ by Theorem 6.2.2. But $\mathrm{SB}(A)$ can not split G, since its Poincaré polynomial does not coincide with $P(\mathcal{R}_{(2,1)}, t)$. By Theorem 6.2.2, we obtain that $J_3(G) = (0,1)$ holds over $k(\mathrm{SB}(A))$. If G would be isotropic but not split, then by the Tits classification its anisotropic kernel would be of type A_2^2. But in this case the Tits algebra of G would not be trivial. Thus it is anisotropic, which shows that the respective line in the [Shells, Table 8.A] can be obtained fairly easy.

7.2 The coaction of $\mathrm{Ch}(G/P)$

In this small section we give a short definition of the coaction ρ of algebraic groups G from [PS22]. We use it later on for calculating some Rost motives in certain motivic decompositions, as it is connected to the value of $J_p(G)$, by a main result (namely [PS22, Theorem 6.4]) established by Petrov and Semenov. We explain this connection and introduce some techniques around calculating the J-invariant out of the coaction and vice versa, which are all related to this main result.

7.2.1 Definition. Let G_0 be a split semisimple group over k and let E be a G_0-torsor. We write J for the bi-ideal in $\mathrm{Ch}^*(G_0)$ generated by $\mathrm{Im}(\mathrm{Ch}^{>0}(E) \xrightarrow{res} \mathrm{Ch}^{>0}(G_0))$ (here we identify $\mathrm{Ch}^*(\overline{E})$ and $\mathrm{Ch}^*(G_0)$, see [PS22, Section 4]). We define the bialgebra $H^* := \mathrm{Ch}^*(G_0)/J$.

Let X be a smooth projective cellular variety with a G_0-left action. We define the structure of a right H^*-comodule on $\mathrm{Ch}^*(\overline{X})$ as the composition

$$\rho : \mathrm{Ch}^*(\overline{X}) \to \mathrm{Ch}^*(\overline{G}_0 \times \overline{X}) \to \mathrm{Ch}^*(G_0) \otimes_{\mathrm{Ch}(pt)} \mathrm{Ch}^*(\overline{X}) \to H^* \otimes_{\mathrm{Ch}(pt)} \mathrm{Ch}^*(\overline{X}),$$

where the first map is the pullback of the action of $\overline{G}_0$ on $\overline{X}$. We will omit the grading on the Chow ring in the future to simplify the notation. For a cycle β in $\mathrm{Ch}(\overline{X})$ we write $\rho(\beta) > a \otimes b$ to indicate that the cycle $a \otimes b$ is a summand of $\rho(\beta)$.

7.2.2. This definition is adopted from [PS22, Definition 4.6] and [PS22, Definition 4.10]. The definitions given in the reference are valid for any oriented cohomology theory A^* in the sense of Levine-Morel (compare [LM, Remark 2.4.14(2)]), like $\mathrm{Ch}^*(-)$ for example. This explains why ours looks a bit different, while it is actually less general. Note that the coaction is a multiplicative map, since all intermediate maps in its definition are ring homomorphisms. It follows from the definition of J above, that H^* and $\mathrm{Ch}^*(G_0)$ coincide when the considered torsor is versal, because in this case its Chow ring is just $\mathbb{F}_p$.

This means if the values of $\rho(x_i)$ for each generator x_i of $\mathrm{Ch}^*(\overline{X})$ are known when one considers versal torsor, one can easily deduce the behavior of ρ in the other cases, in which and $J_p(G)$ is usually not maximal. We give an example how this works after the following lemma.

7.2.3 Lemma. ([PS22, Lemma 7.1 and 7.2]) *Consider the split form of* $\mathbf{SO}_n$ *for* $n = 2m+2$ *or* $n = 2m+1$. *We have* $\mathrm{Ch}^*(\mathbf{SO}_n) \simeq \mathbb{F}_2[e_1, \dots, e_m]/(e_i^2 = e_{2i})$ *with* $\operatorname{codim} e_i = i$ *if* $i \leq m$ *and* $e_i = 0$ *if* $i > m$.

Let h, l *be generators of* $\mathrm{Ch}^*(\overline{X_1})$, *where* h *is the class of a hyperplane section of the quadric* $\overline{X}_1$ *and* l *is the class of a maximal totally isotropic subspace in* $\overline{X}_1$. *Consider the coaction* ρ of $\mathrm{Ch}^*(\mathbf{SO}_n)$ *on* $\mathrm{Ch}^*(\overline{X_1})$. *Then the following holds.*

$$\rho(l) = \textstyle\sum_{i=1}^{m} e_i \otimes h^{m-i} + 1 \otimes l.$$

□

7.2.4 Example. We consider a split group G_0 of type $\mathbf{SO}_{14}$ over k. The Chow ring of G_0 has three generators e_1, e_3, e_5 by the J_p-table. We focus on $G_0/P_1 =: X_1$. It is well known that $\mathrm{Ch}(\overline{X_1})$ is generated by h, l, satisfying $h^7 = 0, l^2 = pt \in \mathrm{Ch}^{12}(\overline{X_1})$ (see [EKM]). By the J_p-table the maximal value of $J_2(G)$ is given by $(3, 2, 1)$. Using the lemma above, we see that in this case

$$\rho(l) = e_1 \otimes h^5 + e_1^2 \otimes h^4 + e_3 \otimes h^3 + e_1^4 \otimes h^2 + e_5 \otimes h + e_3^2 \otimes 1 + 1 \otimes l$$

holds. Assume now that we are given some inner twist G of G_0 over k, such that $J_2(G) = (0, 2, 1)$ for example. Then all powers of e_1 are modded out of $\mathrm{Ch}(G_0)$ by J, when considering $H^* = \mathrm{Ch}(G_0)/J$. Concretely, e_1 is now zero in H^*, which means that the summands of the form $e_1^i \otimes b$ become zero, too. Thus

$$\rho(l) = e_3 \otimes h^3 + e_5 \otimes h + e_3^2 \otimes 1 + 1 \otimes l$$

holds in this case.

7.2.5 Lemma. ([PS22, Lemma 4.12]) *Let β be a rational cycle in* $\mathrm{Ch}(\overline{X})$. *Then* $\rho(\beta) = 1 \otimes \beta \in H^* \otimes_{\mathrm{Ch}(pt)} \mathrm{Ch}(\overline{X})$ *holds.*

□

7.2.6. One may ask whether the converse statement of the lemma above is true in general. In Lemma 10.3.7 we show, that at least for the variety E_7/P_1 there is a case when a cycle β is not rational, but gets mapped onto $1 \otimes \beta$ by ρ. The lemma above will be needed a lot later. Further below we prove some lemmas as examples for the application of the coaction. One of the most important results of [PS22] is the discovery of the following theorem, which is crucial for our future calculations.

7.2.7 Theorem. ([PS22, Theorem 6.4]) *Let P be a parabolic subgroup of a split semisimple algebraic group G_0 over a field k and let ξ be a G_0-torsor over k. Let ρ denote the coaction of H^* on* $\mathrm{Ch}^*(G_0/P)$ *and let $J = (j_1, \dots, j_r)$ denote the motivic J-invariant mod p. Every summand of the Chow motive $M({}_\xi(G_0/P))$ with coefficients $\mathbb{F}_p$ which is isomorphic to a Tate shift of $\mathcal{R}_J$ has a generic point $\alpha \in$* $\mathrm{Ch}^*(G_0/P)$ *such that for some $\beta \in$* $\mathrm{Ch}^*(G_0/P)$ *we have*

$$\rho(\beta) = E_J \otimes \alpha + \sum a_i \otimes b_i$$

for some a_i, b_i with $\mathrm{codim}(a_i) < \mathrm{codim}(E_J)$, *where $E_J = e_1^{p^{j_1-1}} \cdots e_r^{p^{j_r-1}}$. Conversely, for every β of this form there is a summand of the Chow motive $M({}_\xi(G_0/P))$ with coefficients $\mathbb{F}_p$ which is isomorphic to a Tate shift of $\mathcal{R}_J$ and whose generic point is α.*

□

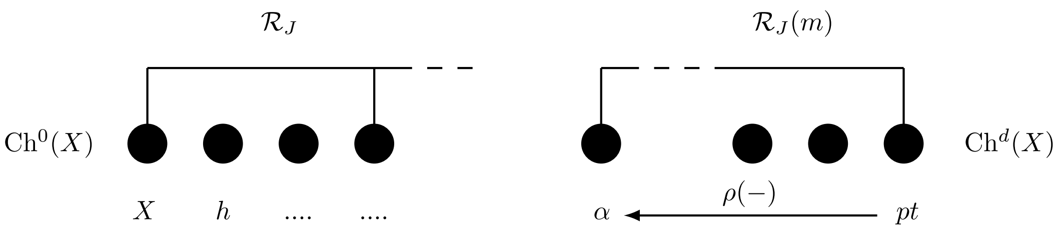
7.2.8. The cycle E_J in the theorem above reflects the value of $J_p(G)$ in a one to one manner. So, if we completely know the coaction on $\mathrm{Ch}(\overline{X})$, we can tell whether there are Rost motives in $M(X)$ just by looking at $J_p(G)$. Also, for excluding the possibility of Rost motives in $M(X)$, it is enough to know whether $\mathrm{Ch}(\overline{X})$ has less generators than $J_p(\overline{G})$ has entries **and** that ρ can not send a generator x_i of $\mathrm{Ch}(\overline{X})$ to a cycle containing a summand of the form $e_ie_j \otimes \alpha$ for $i \neq j$. This can sometimes be concluded from the codimensions of the generators of $\mathrm{Ch}(\overline{X})$ and $\mathrm{Ch}(\overline{G})$, because ρ naturally preserves codimensions. To make the theorem above more comprehensible, see the following example on how $\rho(\beta)$ defining a Rost motive (via the theorem above) behaves under field extensions.

7.2.9 Example. Let us consider a non split adjoint group G of type E_7 over k. The G-variety X_3 is a GSV in case the Tits algebra of G is trivial. In the chapter about phases we will see that $J_2(G)$ equals $(0,1,1,1), (0,1,1,0)$ or $(0,1,0,0)$, corresponding to the anisotropic kernels E_7, D_6 and D_4 respectively.

Since X_3 is a GSV, the quotient $P(X_3,t)/P(\mathcal{R}_J,t)$ codes the shifts of $\mathcal{R}_J$ in $M(X_3)$ in these cases. The copy $\mathcal{R}_J(m)$ in $M(X)$ with the highest shift m is necessarily unique, because the ending point of $\mathcal{R}_J(m)$ is the cycle $pt \in \mathrm{Ch}^d(\overline{X}_3)$ with $d = \dim(X_3)$.

We can use [PS22, Theorem 6.4] to conclude that there is a cycle $\beta \in \mathrm{Ch}(\overline{X}_3)$, with $\rho(\beta) = E_J \otimes \alpha_m + \sum a \otimes b$, with α_m denoting the starting point of $\mathcal{R}_J(m)$.

It is easy to see that the cycle β is pt, as the codimension of E_J generally equals the dimension of $\mathcal{R}_J$. Here is an illustration of the situation for some imaginative GSV X and a imaginative Rost motive $\mathcal{R}_J$.



Thus $m = 30$ for $J_2(G) = (0,1,1,1)$ and $\mathcal{R}_J(30)$ with generic point α_{30} is contained in $M(X_3/k)$ and defined by pt via ρ, since $\mathrm{codim}(e_3e_5e_9) = \dim(\mathcal{R}_{(0,1,1,1)}) = 17$ and $\dim(X_3) = 47$ holds.

Now comes the surprise. When we pass to L/k, such that $J_2(G/L) = (0,1,1,0)$ holds (see Lemma 8.2.2 for the existence) and $\mathrm{codim}(e_3e_5) = \dim(\mathcal{R}_{(0,1,1,0)}) = 8$, then we can show by the very same considerations as above that $\mathcal{R}_{(0,1,1,0)}(39)$ is contained $M(X_3/L)$ and that its generic point α_{39} is in fact also defined by $\beta = pt$ via $\rho(pt)$.

Generally for any GSV, if pt defines a Rost motive $\mathcal{R}_J(m)$ over k, it defines a Rost motive $\mathcal{R}_{J'}(l)$ over L/k and $l = m + \mathrm{codim}(E_J) - \mathrm{codim}(E_{J'})$. So if the J-invariant decreases over L, the same cycle pt defines a Rost motive of lower dimension and generic point of higher codimension over L.

7.2.10. Another conclusion from the example is that for any X (not necessarily a GSV), $\rho(pt)$ contains all summands of the form $E_J \otimes \alpha_i$, such that one can derive from the E_J exactly for which values of the J-invariant, the GSV property applies to X (provided E_J strictly has the biggest codimension in the sense of [PS22, Theorem 6.4]). Saying this in a different way, if we know an exact formula for $\rho(pt)$ for a projective, homogeneous G-variety X, then we can tell for which values of the J-invariant the variety X is a GSV for a group G.

7.2.11 Lemma. *Let G be an algebraic group of type E_8, with $J_2(G) = (1,1,1,1)$. Then the Chow motive of the projective, homogeneous G-variety X_8 contains no shifts of Rost motives.*

Proof: By [DuZ10, Theorem 7] the Chow ring of X_8 is generated by four cycles h, x_6, x_{10}, x_{15}. The subscript marks their codimension, while h is the generator of $\mathrm{Ch}^1(X_8)$ as usual. In order for $M(X_8)$ to contain Rost motives, we need to find a cycle $\beta \in \mathrm{Ch}(\overline{X_8})$, such that $\rho(\beta) > e_3e_5e_9e_{15} \otimes \alpha$ holds for some α in $\mathrm{Ch}(\overline{X_8})$ by [PS22, Theorem. 6.4]. Note that h is rational over k since the Tits algebras of any E_8 are all split and thus by Lemma 7.2.5 we have $\rho(h^i) = 1 \otimes h^i$. As ρ is codimension preserving, the e_{15} portion comes from x_{15}. Also e_9 can not be contributed by x_6 but only by x_{10}. This means that the e_3e_5 portion, which is 8-codimensional, has to be contributed by x_6, which is impossible. Note that since we use $\mathbb{F}_2$ coefficients, considering any even power of the generators can not give an odd power of any e_i

under ρ. A product β consisting of even and odd powers of the generators, will therefore always contain an even power of some e_i in any summand of $\rho(\beta)$, too.

□

7.2.12 Lemma. *Consider $G \simeq \mathbf{SO}(q)$ of inner type D_{m+1}, $m \geq 2$, with $J_2(G) = (j_1, \ldots, j_r)$. Then the Chow motive of the G-variety X_1 contains no shifts of Rost motives over k, in case at least two entries j_i in $J_2(G)$ are nonzero.*

Proof: The Chow ring of $\overline{X}_1$ is known to be generated by $h \in \mathrm{Ch}^1(\overline{X}_1)$ and $l \in \mathrm{Ch}^m(\overline{X}_1)$. Every cycle $\beta \in \mathrm{Ch}^i(\overline{X}_1)$ is either a power of h, a power of h multiplied with l, or $h^m + l$.

The cycle h is definitely rational. Thus we have that $\rho(h) = 1 \otimes h$ holds by Lemma 7.2.5. Using the formula for $\rho(l)$ from Lemma 7.2.3, it follows that $\rho(lh^a) = \sum_{i=1}^{m} e_i \otimes h^{m-i+a} + 1 \otimes lh^a$ holds. By Theorem 7.2.7, it follows that $M(X_1)$ contains no Rost motives, if more than one entry in $J_2(G)$ is unequal to zero. □

7.2.13 Lemma. *For a projective, homogeneous G-variety X let $h \in \mathrm{Ch}(\overline{X})$ be rational homogeneous and assume that there are homogeneous cycles $\alpha \in \mathrm{Ch}^i(\overline{X})$, $\beta \in \mathrm{Ch}(\overline{X})$ such that the conditions of* [PS22, Theorem 6.4] *are satisfied for a given value of $J_p(G)$. Then for all natural numbers l with $\beta h^l \neq 0, \alpha h^l \neq 0$ the conditions of the theorem are also satisfied for the cycles $\alpha h^l, \beta h^l$. In particular it follows that, if there is a Rost motive $R_J(i) \in M(X)$ with generic point α, then there is a Rost motive $R_J(i+l) \in M(X)$ with generic point αh^l.*

Proof: Since we assume that h is rational, we have $\rho(h) = 1 \otimes h$ by [PS22, Lemma 4.12]. In case α, β satisfy the equation and initial requirements from [PS22, Theorem 6.4] for our value of J_p, then the cycle $0 \neq \beta h^l$ maps to $\rho(\beta h^l) = E_J \otimes \alpha h^l + \sum a_i \otimes b_i h^l$, since ρ is a homomorphism of rings. Provided αh^l is not zero, the summand $E_J \otimes \alpha h$ is also not zero. The other summands $a_i \otimes b_i$, for which initially $\mathrm{codim}(a_i) < \mathrm{codim}(E_J)$ is supposed to hold, become $a_i \otimes b_i h^l$. Thus the condition on the codimensions of the a_i is still satisfied, as E_J does not change either. □

7.2.14. The theorem above in conjunction with [PS22, Theorem 6.4] can be thought of as a generalization of [Shells, Theorem 4.10]. But instead of demanding b, which is h in our case, to be from the first shell, rationality alone is enough for the theorem to hold, provided the motive M is a Rost motive.

Chapter 8

Phases of algebraic groups of type E7

In this section we introduce collections of invariants of an algebraic group, which we call a *phase*. We calculate many possible phases for twisted forms of a split adjoint group of type E_7 mod 2.

8.1 Definitions and properties

We have already introduced well known examples, which show that one can not derive the motivic decomposition of the projective, homogeneous F_4-varieties mod 2 or the projective, homogeneous E_6-varieties mod 3 solely from the Tits index or the J-invariant of the respective group. Additionally for some types of groups it is possible that two groups having the same Tits index have Tits algebras with different index (compare the Tits classification for E_7). This can become an issue when one wants to know, how the semisimple anisotropic kernel of a group changes under a field extension, possibly of the kind $k(X)$ for some projective variety X. Such questions where treated in [SvB] and [MPW], [MPW2] and spawned the *index reduction formulas*. The idea of index reduction was further developed in [PS07], where Tits automata were introduced.

The results in [Shells, Table 8.A] and [Shells, Table 10.A] show how motivic decompositions depend on the three invariants Tits index, Tits algebra and motivic J-invariant. As groups of type E_7 are even more complicated than F_4 or E_6 in many aspects, this suggests that one should also consider these three invariants, when calculating motives of projective, homogeneous E_7-varieties.

8.1.1 Definition. Consider a split adjoint group G_0 of type E_7 and let ξ be a G_0-torsor. Let G be the twist of G_0 by ξ with semisimple anisotropic kernel G_{an}, Tits algebra A and motivic J_2-invariant $J_2(G)$. We call the triple

$$\mathfrak{p}(G) := [G_{an}, J_2(G), \operatorname{ind}(A)]$$

the *phase* of G. We just write $\mathfrak{p}$ sometimes. A phase is called *admissible* if it does occur over some field k. Given two phases $\mathfrak{p}$ and $\mathfrak{p}'$, with $\mathfrak{p}$ occurring over k, we say that there is a *transition* from $\mathfrak{p}$ to $\mathfrak{p}'$ if there is a field extension L/k such that $\mathfrak{p}'$ occurs over L. When L is a field extension of the form $L \simeq k(X)$ for some smooth projective variety X, we say that X *induces* a transition to $\mathfrak{p}'$. The phase $[\emptyset, 0, 1]$ is called *split*.

8.1.2. Since over a field extension all three invariants contained in a phase can only stay the same or decrease, a phase does also behave that way. Eventually a phase will transit to the split phase. One can extend the definition to other groups and torsion primes, but we are only interested in E_7^{ad} mod 2. It is known that when twisting a split group with a versal torsor, all invariants G_{an}, $J_2(G)$, $\operatorname{ind}(A)$, will have take their maximal value, which gives us some kind of versal phase $[E_7, (1, 1, 1, 1), 8]$. It follows from the definition of a versal torsor, that the phase of a versal form can specialize to any other admissible phase. Every GSV does induce a transition to the split phase, for example. Figuring out how transitions to other phases are induced, is part of our research during the coming chapters.

8.1.3 Remark. When some projective, homogeneous G-varieties X_i and X_j induce the same transition, then it follows that the upper motive of X_i and X_j are isomorphic, simply because then both varieties become isotropic of the generic point of the other and therefore each have a zero cycle of odd degree then. So determining patterns of how phases change over a tower of fields means to determine whether certain upper motives in a chain of transitions become isomorphic or not.

It surely would be desirable for a one to one correspondence between $\mathfrak{p}(G)$ and $\#G$ or $\#G \cup \mathcal{U}(\mathrm{SB}(A))$ to exist, but this is not necessarily the case, as G_{an} is not a motivic invariant. Take two anisotropic quadratic forms of the same rank for example. The motivic decompositions of their quadrics depend on their splitting pattern and may be totally different. Also, we can not guarantee that two quadratic forms with the same J_2-invariants have the same splitting pattern. On top of that, we do not know such a thing as an intrinsic splitting pattern depending on ξ for groups of type E_7. Indeed when a group G of type E_7 is *isotropic*, we will see from the individual calculations of the motives of the projective, homogeneous G-varieties, that every admissible phase determines $\#G$ uniquely.

8.2 Phases of strongly inner E7s

This section contains the admissible phases in case G has trivial Tits algebra. It is basically a warm up, since these phases can easily be copied from the results in [Shells], where the simply connected case was treated. We start with the most isotropic case D_4 and work our way up to the anisotropic case.

8.2.1 Lemma. *The only phase with Tits index D_4 is $[D_4, (0,1,0,0), 1]$.*

Proof: It is known that the anisotropic kernel D_4 inside a group G of type E_7 is defined by a 3-Pfister form φ. So if it is anisotropic, we can use Example 6.2.5 to conclude that we have $J_2(D_4) = (0,1)$. Thus by the cut off property of the J-invariant, $J_2(G) = (0,1,0,0)$ holds. □

8.2.2 Lemma. *The only phase with Tits index D_6 and* $\mathrm{ind}(A) = 1$, *is* $[D_6, (0,1,1,0), 1]$.

Proof: By Theorem 6.2.2, we necessarily have $j_1 = 0$, since the considered G of type E_7 is strongly inner. Any strongly inner group H of type D_6 inside a E_7 is defined by a rank 12 quadratic form q, such that all Tits algebras of H are split. Otherwise the enveloping E_7 would have non trivial Tits algebras, which violates our initial requirement. By the Tits classification, there are only two Tits indexes more isotropic and having split Tits algebras than a D_6 with split Tits algebras, for such a group H. One is to have anisotropic kernel D_4, the other one is the split one.

Since there is no splitting pattern [6] by the splitting pattern table for a quadratic form q of rank 12, the admissible splitting pattern is $[2,4]$ by the same table. Such forms are given by $q = \langle\langle a,b,c\rangle\rangle \perp -\langle\langle a,d,e\rangle\rangle$ in $W(k)$, which have exactly one common slot $\langle\langle a\rangle\rangle$ (see [Vis04, p.79]).

In [Shells, Table 10.B], we see that $J_2(\mathbf{Spin}(q)) = (1,1)$ holds. From the parameters in the J_p-table and Theorem 6.2.2 it follows that we have $J_2(G) = (0,1,1,0)$ by the cut off property. □

8.2.3 Lemma. *The phase $[E_7, (0,1,1,1), 1]$ is admissible.*

Proof: By the comment in Example 6.2.8 any adjoint E_7 is anisotropic, when it has $J_2 = (0,1,1,1)$. Since a simply connected E_7 with $J_2 = (1,1,1)$ does exist by the results in [Shells], it follows that an adjoint E_7 with $J_2 = (0,1,1,1)$ exists.

Let G_0 be a split simply connected E_7 and G be a twisted form of G_0 by $\xi \in H^1(k, G_0)$, such that $J_2(G) = (1,1,1)$ holds. Consider its image $\xi' \in H^1(k, G_0')$ under the map induced from the covering map $G_0 \to G_0'$, for G_0' being the split adjoint E_7. Twisting G_0' with ξ', then gives the respective value for J_2 and thus an anisotropic adjoint E_7.

Alternatively, take a versal form of an adjoint E_7. It has $J_2 = (1,1,1,1)$, as this is the maximal value for J_2. Pass to $k(\mathrm{SB}(A))$. By Zhykhovich's theorem for example, J_2 has the desired value. □

8.2.4. The proofs of the lemmas above show that for the isotropic inner E_7s there is a one to one correspondence between the semisimple anisotropic kernel of the respective group and its J_2-invariant. For the anisotropic cases, note that $[E_7, (0,1,0,0), 1]$ and $[E_7, (0,1,1,0), 1]$ are impossible by the results compiled in [Shells, Table 10.B].

8.3 Phases of general E7s

Now we prove the admissibility of all of the isotropic and some of the anisotropic phases.

8.3.1 Lemma. *Adjoint groups G of type E_7 with Tits index A_1^3 have $J_2(G) = (1,0,0,0)$.*

Proof: From the Tits classification it is known that the anisotropic kernel A_1^3 occurs and the Tits algebra A of G has $\mathrm{ind}(A) = 2$ in such cases. Also X_7 is necessarily a GSV in this case, as by the index reduction formula in [MPW2] one has $\mathrm{ind}(A \otimes_k k(X_7)) = 1$ and thus the Tits classification leaves no other possibility for the Tits index of G than the split one. By the GSV-table X_7 is a GSV if and only if $j_2 = 0$ holds. Since G/k is not strongly inner, we have $j_1 \neq 0$ by Theorem 6.2.2. Considering the restrictions holding for J_2 by the J_p-table, this leaves $(1,0,0,0)$ as only possibility for $J_2(G)$. □

8.3.2 Lemma. *Adjoint groups G of type E_7 with Tits index $D_4 \times A_1$ or $D_5 \times A_1$ have $J_2(G) = (1,1,0,0)$.*

Proof: In [Tits90] it is mentioned that groups G of type E_7 with anisotropic kernel $D_5 \times A_1$ or $D_4 \times A_1$ exist and are classified by quadratic forms $q = \varphi_3 \perp -\varphi_2$ in $W(k)$, with φ_3, φ_2 being a 2- or 3-Pfister form respectively and having none (then one has D_5) or one (this gives D_4) common slot. We have treated such quadratic forms in Example 6.2.6 and Example 6.2.7. The statement of this lemma is basically the second example. Using the cut off property, it follows that the entries j_3, j_4 of $J_2(G)$ are zero. □

8.3.3 Remark. The construction of groups of type E_7 with anisotropic kernel $D_5 \times A_1$ and other types can be deeply understood on the level of so called *structurable algebras*. In the case above these are simply the Octonion and Quaternion algebras. The procedure can be found in [Allison] for example. The quadratic form q defining such groups is in fact the norm form of the difference of the norm form f_3 of the Octonion algebra $\mathcal{O}$ and φ_2 of the Quaternion algebra Q used to construct the group G.

Note that the resulting G has Q as as its Tits algebra. This holds necessarily, as passing to $k(X_{\varphi_2})$ makes G have anisotropic kernel D_4 being defined by φ_3 by the Example 6.2.6. At the same time, splitting the Tits algebra of G makes G strongly inner, so there can not be another non zero mod 2 invariant like φ_2 for G, since every group of type E_7 has at most one none split Tits algebra up to Brauer equivalence by Remark 3.7.5 and φ_3 (now defining G) lies in I^3. As X_3 is not a GSV when $j_1 = 1$, this gives the following sequence of phases and transitions

$$[D_5 \times A_1, (1,1,0,0), 2] \quad \xrightarrow{X_1} \quad [D_4 \times A_1, (1,1,0,0), 2] \quad \xrightarrow{X_3} \quad [A_1^3, (1,0,0,0), 2].$$

8.3.4. We now deal with the most complicated isotropic case, being groups of type E_7 with anisotropic kernel D_6 and non split Tits algebras. Such groups can have $\operatorname{ind}(A) = 2$ or 4 for their Tits algebra A by the Tits classification. There is a construction for each of these two cases. We thank Skip Garibaldi for explaining the constructions, which are also shortly discussed in [DG, 4.5.1.].

8.3.5 Theorem. *Adjoint algebraic groups G of type E_7 with anisotropic kernel D_6, Tits algebra A and* $\operatorname{ind}(A) = 2$ *or* 4 *exist and satisfy* $J_2(G) = (1,1,1,0)$.

Proof: By [GQ09, Rmrk. 3.3], we can use [GQ09, Theorem 3.1] to produce the groups in question. By Remark 6.2.4 their anisotropic kernel is given by $\mathbf{HSpin}(A, \sigma)$, for A being a degree 12 CSA and σ an orthogonal involution with trivial discriminant and Clifford invariant (see [Inv, §8]).

Case of $\operatorname{ind}(A) = 4$. For the existence of the $\operatorname{ind}(A) = 4$ case, one can check [Inv, p.148 Exercise 13]. These groups arise from degree 6 algebras B with unitary involution τ, which induces an orthogonal involution σ on the discriminant algebra $D(B)$ (see [Inv, §10]). The quadratic form q_σ adjoint to σ, is the difference of two 3-Pfister forms having exactly one common slot. Such forms have splitting pattern $[2,4]$ by [Vis04]. They classify strongly inner groups of type E_7 with anisotropic kernel D_6. Their Witt index is therefore $0, 2$ or 6.

By [Kar09, Thrm 3.3], $\operatorname{ind}(A/k)$ divides the Witt index of q_σ over $k(\mathrm{SB}(A))$ in the sense that if it does not divide any of these numbers, q_σ is anisotropic over $k(\mathrm{SB}(A))$. This is obviously the case here. Thus we have that the anisotropic kernel of G stays D_6 over $k(\mathrm{SB}(A))$. By Lemma 8.2.2 the value of $J_2(G)$ over $k(\mathrm{SB}(A))$ equals $(0,1,1,0)]$. The value of J_2 over k is therefore at least $(0,1,1,0)$ by the decreasing property of the J-invariant. Since A/k is not split, it is in fact at least $(1,1,1,*)$ by Theorem 6.2.2. By the conclusion of Example 6.2.8, $j_4 = 0$ holds over k.

Case of $\operatorname{ind}(A) = 2$. For an existence proof in the the index 2 case, we can consider [GQ9, Example 2.3]. In this case the Tits algebra A is Brauer equivalent to a

Quaternion algebra Q. We can not use Karpenko's result here. But the crucial part in the existence proof is that the anisotropic kernel of G over $k(\mathrm{SB}(Q))$ is also of type D_6. Applying the same argument as above (i.e reducing to the strongly inner case) does prove the statement on $J_2(G)$. $\square$

8.3.6 Theorem. *Adjoint groups G of type E_7, with $J_2(G) = (1, 1, 1, 1)$ have a Tits algebra A with* $\mathrm{ind}(A) \in \{2, 4, 8\}$. *Every of these three values is admissible.*

Proof: Let D be the division algebra lying under A. If G is versal, then $\mathrm{ind}(A) = 8$, which is known to be the maximal possible value. We can use Zhykovich's theorem and pass to $k(\mathrm{SB}(D))$, to obtain $[E_7, (0, 1, 1, 1), 1]$. When we pass from k to $k(\mathrm{SB}_4(D))$ or $k(\mathrm{SB}_2(D))$, the index of A changes to 4 or 2 by the index reduction formula in [SvB]. The field extension $k(\mathrm{SB}(D))/k$ factors through $k(\mathrm{SB}_4(D))$ or $k(\mathrm{SB}_2(D))$, amounting to the following transitions (the $*$ symbolizes 2 or 4)

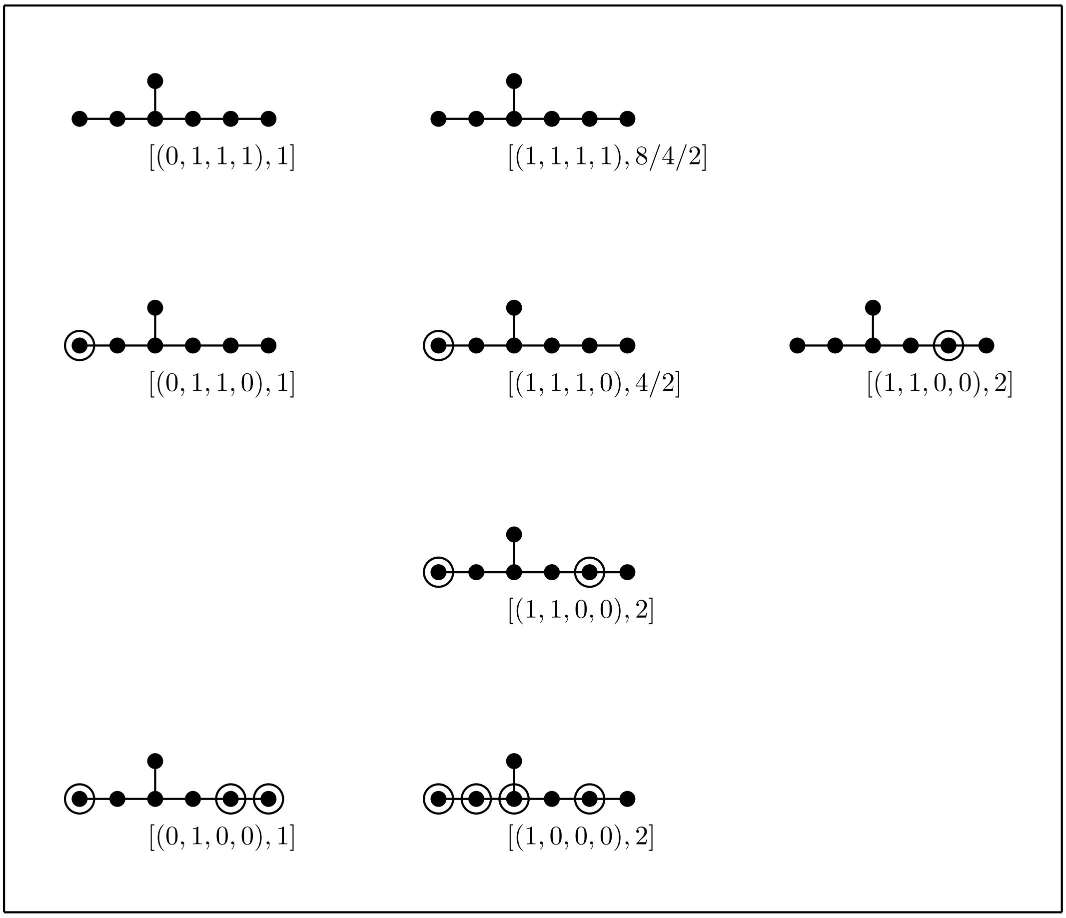
$$[E_7, (1, 1, 1, 1), 8] \xrightarrow{\mathrm{SB}_*(D)} [E_7, (1, 1, 1, 1), *] \xrightarrow{\mathrm{SB}(D)} [E_7, (0, 1, 1, 1), 1].$$

One may stress that G could become isotropic over $L := k(\mathrm{SB}_*(D))$. But this is impossible, as in that case passing from L to $L(\mathrm{SB}(D))$ can not yield $[E_7, (0, 1, 1, 1), 1]$ then. This would contradict the general commutativity of the restriction map.

$\square$

8.3.7. In the corollary below, we summarize the results of this section in a big diagram. It will come in handy for anyone who enjoys a visual support for the many proofs to come. We often show that a phase can (not) transition into another one, when certain upper motives are (not) isomorphic. This overview should speed up comprehending the arguments. The arrangement of the phases is carefully chosen. We will later on show that there are some more anisotropic phases. But this does not interfere with the representation below.

8.3.8 Corollary. (Phase classification) *Let G be an adjoint algebraic group of type E_7. Then the following phases of G are admissible. There are no other phases with isotropic G than those shown below.*



Proof: This table is just a summary of the results from the lemmas in the two sections before.

□

8.3.9 Corollary. *Let G be an anisotropic group of type E_7 with $J_2 = (1,1,0,0)$ or $(1,0,0,0)$ and Tits algebra A. Then* $\operatorname{ind}(A) = 2$.

Proof: Surely $\operatorname{ind}(A)$ can not be 1, as by Lemma 6.2.2 one would have $j_1 = 0$. Assume that $\operatorname{ind}(A) > 2$. Then over $k(X_1)$ we have $\operatorname{ind}(A) = 4$, by the index reduction formula [MPW2]. By the Tits classification the only possible anisotropic kernel for such an isotropic G is of type D_6. But for such a G one has $J_2 = (1,1,1,0)$ by the phase classification. This violates the fact, that the J-invariant can not become bigger over field extension. All other possibilities do not violate the requirements, as the other isotropic cases have the desired values for J_2 and $\operatorname{ind}(A)$ simultaneously. □

8.3.10. Unlike in the case for E_6 mod 3, as pointed out in the motivation subsection, we have now found cases (the ones above) where $\operatorname{ind}(A)$ can be derived solely from J_p. Note that we have not proven the existence of such groups as in the corollary yet. We consider the constructing of such groups in the Chapter 11 by sketching a proof. The existence of anisotropic E_7s with $J_2 = (1, 1, 1, 0)$ remains a mysterious case though. We manage to derive some restrictions on it and calculate some motivic decompositions in Chapter 10. But we can not prove its existence.

Chapter 9

Motivic decompositions for strongly inner E7s

In this chapter we establish the motivic decomposition of all projective, homogeneous G-varieties for an adjoint group G of type E_7 with trivial Tits algebra into Chow motives with $\mathbb{F}_2$ coefficients. Except for the case where G is anisotropic, these results are already known technically as they follow rather easily by applying the CGMB algorithm, considering Tate motives and certain polynomials and looking into [Shells]. We solve all cases phase by phase, starting with the most isotropic case. We include these for the sake of completion and as a reference for the calculations in later chapters.

The results are mostly presented in tables containing many decompositions at once. Note that in the logic which holds in the tables, the Tate polynomial $T(X, t)$ of a variety X is zero in case X is anisotropic. The tables consist of three pieces. One showing the decomposition, one containing the structure of the summands and one containing the shifting information of the summands. Sometimes the shifting information are only given implicit, because the polynomials become too big or the information are not of further interest to us. In case we recite them in later results, they are stated explicitly.

When we are dealing with different values of the J-invariant in tables or proofs, we explicitly write $\mathcal{R}_{(1,1,0,0)}$ for example, instead of $\mathcal{R}_J$ when there is no danger of confusion.

By the GSV-table the varieties of X_2, X_3, X_4 and X_5 are GSVs. We focus only on the cases where P_Θ is a maximal parabolic subgroup, except for $P_{1,6}$. The other non maximal cases do not provide any added value and can be derived from these base cases easily.

9.1 The phase [D4,(0,1,0,0),1]

This first case is the most simple one and only needed for proofs of the other cases where G is less isotropic.

9.1.1 Theorem. *Let G have phase $[D_4,(0,1,0,0),1]$. Then the following unique decompositions of the Chow motives of projective, homogeneous G-varieties into indecomposable motivic summands hold*

Θ	$M(X_\Theta)$
$\Theta \subset \{1,6,7\}$	$\bigoplus_{t\in T_\Theta} \mathbb{F}_2(t) \oplus \bigoplus_{i\in I_\Theta} \mathcal{R}_J(i)$
Any other	$\bigoplus_{i\in I_\Theta} \mathcal{R}_J(i)$

Index	**Poincaré Polynomial**
$\mathcal{R}_J$	$(1+t^3)$

Index	**Shift/Tate Polynomial**
T_1	$1+t^8+t^{16}+t^{17}+t^{25}+t^{33}$
T_6	$(1+t^8+t^{16})(1+t^9)(1+t^{17})$
T_7	$(1+t)(1+t^9)(1+t^{17})$
$T_{1,6}$	$(1+t^8)T_6$
I_1	$t(1+t+t^2)(1+t^6)(1+t^3+t^5+t^8+t^9+t^{11}+t^{12}+t^{15}+t^{17}+t^{20})$
I_6	$t(1+t)(1+t^6)(1+t+t^2+2t^3+t^4+4t^5+5t^7+8t^9+9t^{11}+10t^{13}+$ $9t^{15}+10t^{17}+9t^{19}+8t^{21}+5t^{23}+4t^{25}+t^{26}+2t^{27}+t^{28}+t^{29}+t^{30})$
I_Θ	$[P(X_\Theta,t)-T_\Theta]/\mathcal{R}_J$

Proof: If G has anisotropic kernel D_4 over k, then by the Tits classification the only possibility for G to become more isotropic over a field extension of k, is to become split. Applying Karpenko's theorem and the fact that the J_2-invariant equals 0 if and only if G is split (provided $J_3 = 0$ holds), we see that $\#G/k = \{\mathbb{F}_2, \mathcal{R}_{(0,1,0,0)}\}$ does hold. So all anisotropic projective, homogeneous G-varieties are GSVs in this situation. The actual computations are straightforward. First, using the CGMB algorithm, we calculate the Tate motives in the motive of each isotropic projective, homogeneous G-variety X_Θ. This gives us the Tate polynomials $P(T_\Theta,t)$. Secondly, we obtain the shift polynomial $P(I_\Theta,t)$ describing the Rost motives contained in $M(X)$, by subtracting the Tate polynomial $P(T_\Theta,t)$ from the Poincaré polynomial $P(X_\Theta,t)$ of X_Θ and dividing the difference by $P(\mathcal{R}_{(0,1,0,0)},t)$. $\square$

9.2 The phase [D6,(0,1,1,0),1]

9.2.1 Theorem. *Let G have phase $[D_6,(0,1,1,0),1]$. Then the following unique decompositions of the Chow motive of projective, homogeneous G-varieties into indecomposable motivic summands hold*

Θ	$M(X_\Theta)$
$\{1\}$	$\mathbb{F}_2 \oplus \mathbb{F}_2(33) \oplus \mathcal{U}(X_7)(8) \oplus \mathcal{U}(X_7)(16) \oplus \bigoplus_{i\in I_1} \mathcal{R}_J(i)$
$\{2,3,4,5\} \cap \Theta \neq 0$	$\bigoplus_{i\in I_\Theta} \mathcal{R}_J(i)$
Any other	$\bigoplus_{u\in O_\Theta} \mathcal{U}(X_7)(u) \oplus \bigoplus_{i\in I_\Theta} \mathcal{R}_J(i)$

Index	**Poincaré Polynomial**
$\mathcal{U}(X_7)$	$(1+t^5)(1+t^2+t^4)$
$\mathcal{R}_J$	$(1+t^3)(1+t^5)$

Index	**Shift/Tate Polynomial**
O_6	$(1+t^8+t^{16})(1+t^{17})$
O_7	$(1+t)(1+t^{17})$
$O_{1,6}$	$(1+t^8)O_6$
$O_\Theta : \{2,3,4,5\} \cap \Theta \neq 0$	0
I_1	$t(t^{24}-1)/(t-1)+t^{11}(1+t+t^2+t^3)$
I_7	$t^6(1+t)(1+t^2)(1+t^4)$
I_Θ	$[P(X_\Theta,t)-O_\Theta P(\mathcal{U}(X_7),t)]/\mathcal{R}_J$

Proof:

Calculating $\#G$: First note that by the GSV-table X_7/k and X_6/k are not GSVs. Since a phase can only decrease under field extension, passing to the generic point of X_7 induces a transition to $[D_4,(0,1,0,0),1]$, as there is no other phase possible by the phase classification. We have seen in Lemma 8.2.2 that the anisotropic kernel of G/k is given by $\mathbf{HSpin}(q)$, with q having splitting pattern $[2,4]$. The upper motives of X_q/k and X_7/k are isomorphic (see [Shells, Lemma 10.15], where this was already considered). The motive of X_q/k decomposes as $\mathcal{U}(X_q/k)\oplus\mathcal{U}(X_q/k)(1)$, as shown in Lemma 5.5.10, from which the structure of $\mathcal{U}(X_7/k) \simeq \mathcal{U}(X_q/k)$ follows. As X_1/k is already isotropic and the other projective, homogeneous G-varieties are either GSVs or induce transitions to $[D_4,(0,1,0,0),1]$ as well and thus have an upper motive isomorphic to $\mathcal{U}(X_q/k)$, we obtain that $\#G/k = \{\mathbb{F}_2, \mathcal{U}(X_7), \mathcal{R}_{(0,1,1,0)}\}$.

Calculating $M(X_\Theta)$: For establishing the motivic decompositions of the G-varieties X_Θ which are not GSVs, we can consider their Tate polynomials $T(X_\Theta,t)$ over $k(X_7)$. As the Tate motives come from copies of $\mathcal{U}(X_7)$, which splits off Tate motives in codimension 0 and 9, dividing the Tate polynomials $T(X,t)$ by $(1+t^9)$ gives us the shift polynomials $O(X_\Theta,t)$ coding the shifting information about $\mathcal{U}(X_7)$ in $M(X_\Theta)$. We use the Tate motives as some kind of skeleton of the $\mathcal{U}(X_7)$s here. Finally we need to subtract the product of $P(\mathcal{U}(X_7),t)$ and $O(X_\Theta,t)$ from $P(X_\Theta,t)$ and divide these differences by $P(\mathcal{R}_{(0,1,1,0)},t)$.

The case of $M(X_1)$: The only case for which this does not work is X_1, since X_1/k

is already isotropic. This means $M(X_1/k)$ contains Tate motives. Therefore we need to subtract these Tate motives that are visible over k and which are coded by $(1+t^{33})$ (input: prodbases([2, 3, 4, 5, 6, 7], [2, 3, 4, 5, 6, 7], E7) to see this), before we proceed as in the case of the other X_Θs. □

9.3 The phase [E7,(0,1,1,1),1]

9.3.1. For this case the projective, homogeneous G-varieties which are not GSVs over k, are X_Θ for $\Theta \subset \{1,6,7\}$ by the GSV-table. Also taking the Tits classification into account the varieties $X_6, X_7, X_{6,7}, X_{1,6,7}$ induce a transition to $[D_4,(0,1,0,0),1]$ and thus have isomorphic upper motive over k. Just from looking at the possible phases it is hypothetically possible that over $k(X_1)$ one does also obtain $[D_4,(0,1,0,0),1]$. To prove that this is wrong, checking the shifts of the Tate motive in $M(X_\Theta)$ over $k(X_\Theta)$ is enough. We incorporate a result from [Shells] to prove the following.

9.3.2 Lemma. *Let G have phase $[E_7,(0,1,1,1),1]$. Then the upper motives of the projective, homogeneous G-varieties X_1 and X_7 are not isomorphic over k.*

Proof: We first prove that X_1/k has no zero cycle of odd degree.

No odd degree zero cycle on X_1: Assume the opposite holds, i.e. X_1/k is anisotropic and has the desired zero cycle. Then there is a field extension L/k of odd degree, such that X_1 becomes isotropic over L. As the J_p-invariant does not change over field extensions of degree coprime to p, the value of $J_2(G/L))$ is still equal to $(0,1,1,1)$. By the Tits classification an isotropic E_7 mod 2 has D_6 as biggest possible anisotropic kernel (i.e. it has a k-torus of rank 1). By Example 6.2.8 every isotropic adjoint E_7 has at most $j_4=0$. It follows that X_1/k can not have a zero cycle of odd degree.

Main statement: By the GSV-table, X_7/k is not a GSV and thus induces a transition to $[D_4,(0,1,0,0),1]$. Assume the upper motives of X_1/k and X_7/k are isomorphic. Then X_7/k does not have a zero cycle of odd degree, as otherwise X_1/k would also have one.

We now can apply [Shells, Lemma 10.8], from which it follows that $\mathbb{F}_2(9)$ in $M(X_7)$ over $k(X_7)$ is glued with $\mathcal{U}(X_7/k)$. But $\mathbb{F}_2(9)$ is not contained in $M(X_1)$ over $k(X_7)$ by Theorem 9.1.1.

□

9.3.3. Our considerations show that $\#G/k = \{\mathcal{U}(X_1), \mathcal{U}(X_7), \mathcal{R}_{(0,1,1,1)}\}$ holds. Also we see that X_1 induces a transition from to $[E_7,(0,1,1,1),1]$ to $[D_6,(0,1,1,0),1]$. This was already concluded in [Shells] for E_7^{sc}. We now calculate the motivic decomposition for X_7, which is surprisingly easy obtained. The structure of $\mathcal{U}(X_7)$ will also be used when treating the harder case of X_1.

9.3.4 Theorem. *Let G have phase $[E_7,(0,1,1,1),1]$. Then the unique decomposition of the Chow motives of the projective, homogeneous G-variety X_7 into indecomposable motivic summands is given by*

$$M(X_7) = \mathcal{U}(X_7) \oplus \mathcal{U}(X_7)(1),$$

with $P(\mathcal{U}(X_7), t) = (1 + t^2 + t^4 + t^6 + t^8 + t^{10} + t^{12})(1 + t^5)(1 + t^9)$.

Proof: Recall that we have $\#G/k = \{\mathcal{U}(X_1), \mathcal{U}(X_7), \mathcal{R}_{(0,1,1,1)}\}$, so ruling out the possibility of $\mathcal{U}(X_1/k)$ or $\mathcal{R}_{(0,1,1,1)}$ occurring in $M(X_7/K)$ is our first step. By the lemma above G has anisotropic kernel D_6 over $k(X_1)$. Since the upper motive of X_7 becomes isomorphic to $\mathcal{U}(X_q)$ over $k(X_1)$ by Theorem 9.2.1, X_7/k does not have a zero cycle of odd degree by Springer's theorem. Thus no copy of $\mathcal{U}(X_1)$ can be contained in $M(X_7)$ over k. Now we show that there are no Rost motives in $M(X_7/k)$.

No Rost motives: By [Kac, Table II] $P(\mathcal{R}_{(0,1,1,1)}, t) = (1+t^9)P(\mathcal{R}_{(0,1,1,0)}, t)$ holds. This means that if there is a Rost motive $\mathcal{R}_{(0,1,1,1)}(i)$ in $M(X_7/k)$, then there is a Rost motive $\mathcal{R}_{(0,1,1,0)}(i + 9)$ in $M(X_7)$ over $k(X_1)$. By Theorem 9.2.1 we know that in the decomposition of $M(X_7)$ no pair of Rost motives with shifts $i, i+9$ exists over $k(X_1)$ as $P(I_7, t) = t^6(1 + t + t^2 + t^3 + t^4 + t^5 + t^6 + t^7)$ holds in this case. Therefore $M(X_7/k)$ consists only of copies of $\mathcal{U}(X_7/k)$.

Structure of $\mathcal{U}(X_7)$: In the decomposition of $M(X_7)$, in case G has anisotropic kernel D_6, there appear several copies of $\mathcal{U}(X_q)$, whose generic points are given by $O(X_7, t) = (1+t)(1+t^{17})$ by Theorem 9.2.1. Using this, Karpenko's theorem and the symmetry of the decomposition, there are only three hypothetical possibilities for the structure of $\mathcal{U}(X_7/k)$. Either it splits over $k(X_1)$ as $\mathcal{U}(X_q) \oplus \mathcal{U}(X_q)(1)$ plus some Rost motives, or $\mathcal{U}(X_q) \oplus \mathcal{U}(X_q)(17)$ plus some Rost motives, or it is isomorphic to the whole motive of X_7 and splits as shown in Theorem 9.2.1. However, passing to $k(X_7)$ and checking the Tate motives calculated in Theorem 9.1.1, we see that $M(X_7)$ contains $\mathbb{F}_2(1)$, while $M(X_6)$ does not. Since by the phase classification $\mathcal{U}(X_6/k)$ and $\mathcal{U}(X_7/k)$ are isomorphic over, this rules out the possibilities one and three. $\square$

9.3.5 Theorem. *Let G have phase $[E_7, (0,1,1,1), 1]$. Then the following unique decompositions of the Chow motives of projective, homogeneous G-varieties into indecomposable motivic summands hold*

Θ	$M(X_\Theta)$
$\{2,3,4,5\} \cap \Theta \neq 0$	$\bigoplus_{i \in I_\Theta} \mathcal{R}_J(i)$
$\{7\}$	$\mathcal{U}(X_7) \oplus \mathcal{U}(X_7)(1)$
Any other but $\{1\}$	$\bigoplus_{u \in O_\Theta} \mathcal{U}(X_7)(u) \oplus \bigoplus_{i \in I_\Theta} \mathcal{R}_J(i)$

Index	**Poincaré Polynomial**
$\mathcal{U}(X_7)$	$(1+t^2+t^4+t^6+t^8+t^{10}+t^{12})(1+t^5)(1+t^9)$
$\mathcal{R}_J$	$(1+t^3)(1+t^5)(1+t^9)$

Index	**Shift/Tate Polynomial**
O_6	$(1+t^8+t^{16})$
$O_{1,6}$	$(1+t^8)O_6$
$O_\Theta : \{2,3,4,5\} \cap \Theta \neq 0$	0
I_Θ	$[P(X_\Theta, t) - O_\Theta P(\mathcal{U}(X_7), t)]/\mathcal{R}_J$

Proof: The motivic decompositions for the varieties X_Θ with $\{1\} \neq \Theta \subset \{1,6,7\}$ follow easily from the decomposition of $M(X_7)$, as their upper motives are all isomorphic to $\mathcal{U}(X_7)$ by the Tits classification. Also over $k(X_1)$ these varieties remain anisotropic by Lemma 9.3.2 (and also have no zero cycle of odd degree). Therefore no copy of $\mathcal{U}(X_1)$ can be contained in any of their motives over k. The Tate motives split off by $\mathcal{U}(X_7)$ over $k(X_7)$ are encoded by

$$T(X_7, t) := 1 + t^9 + t^{17} + t^{26}$$

and are obtained by encoding the Tate motives calculated in Theorem 9.2.1 as a polynomial and then dividing by $1+t$ as the theorem above suggests. So passing to $k(X_7)$ and applying the CGMB algorithm to identify Tate motives, writing them as a polynomial $T(X_\Theta, t)$ and dividing by $T(X_7, t)$, one obtains a shift polynomial $O(X_\Theta, t)$ with the shifting information about the copies of $\mathcal{U}(X_7/k)$ in $M(X_\Theta/k)$. Then subtracting the product $O(X_\Theta, t)P(\mathcal{U}(X_7/k), t)$ from $P(X_\Theta, t)$ and dividing by $P(\mathcal{R}_{(0,1,1,1)}, t)$, gives $P(I_\Theta, t)$, which encodes the shifting information about the Rost motives in $M(X_\Theta/k)$. □

9.3.6. Interestingly the decomposition of $M(X_7)$ was also recently obtained in [PS22, Proposition 8.8] by using the coaction map ρ. We explicitly listed its decomposition in the table above to emphasize that it contains no Rost motives. Knowing the structure of $\mathcal{U}(X_7)$, we can finally calculate the decomposition of $M(X_1/k)$. Also we will heavily incorporate the notion of ρ for this. First we need to know about the generators of $\mathrm{Ch}(X_1)$.

9.3.7 Lemma. *For a split adjoint algebraic group G of type E_7 and the projective, homogeneous G-variety X_1 the following holds.*
$\mathrm{Ch}(G) \simeq \mathbb{F}_2[e_1, e_3, e_5, e_9]/\langle e_1^2, e_3^2, e_5^2, e_9^2\rangle$ *and* $\mathrm{Ch}(X_1) \simeq \mathbb{F}_2[h, x_4, x_6, x_9]/\langle r_1, r_2, r_3, r_4\rangle$ *for some $r_i \in \mathbb{F}_2[h, x_4, x_6, x_9]$. Let $S^n(-)$ be the n-th Steenrod operation. Then $S^2(e_3) = e_5$, $S^4(e_5) = e_9$ holds.*

Proof: The statements on the Chow rings can be found in [DuZ10, Theorem 6] and [Kac85, Table II]. The first reference does also contain an explicit description of the cycles. The cycle x_6 for example is $\gamma_{6,2}$, the second generator of $\mathrm{Ch}^6(X_1)$ one obtains when executing the *chow generators* command from the *Chow maple package*. For the Steenrod algebra action see [IKT76, Proposition 5.1]. $\square$

9.3.8 Lemma. *Let G be an adjoint algebraic group of type E_7. Consider the projective, homogeneous G-variety X_1. Then the following holds for the coaction map ρ on* $\mathrm{Ch}(\overline{X_1})$

1. $\rho(h) = 1 \otimes h$

2. $\rho(x_4) = e_3 \otimes h + e_1 \otimes h^3 + 1 \otimes x_4$

3. $\rho(x_6) = e_5 \otimes h + e_1 \otimes h^5 + 1 \otimes x_6$

4. $\rho(x_9) > e_9 \otimes 1 + 1 \otimes x_9$

Proof:

1. By Remark 3.7.5 and Remark 5.1.11 $h \in \mathrm{Ch}(X_1)$ is always rational. Therefore we have $\rho(h) = 1 \otimes h$ by [PS22, Lemma 4.12].

2. We prove this statement for each of the two summands $e_1 \otimes h^3$ and $e_3 \otimes h$ separately. The claim on $1 \otimes x_4$ follows from [PS22, Lemma 4.12].

For the $e_1 \otimes h^3$ case, consider Theorem 10.1.1 in the next section. There is a Rost motive $\mathcal{R}_{(1,0,0,0)}(3)$ in $M(X_1)$, when G has phase $[A_1^3, (1,0,0,0), 2]$. By [PS22, Theorem 6.4] it follows that there is some cycle $\beta \in \mathrm{Ch}^4(X_1)$ for which $\rho(\beta)$ contains $e_1 \otimes h^3$. Since h^4 is rational, because the Tits algebra of ω_1 is always split, $\rho(h^4) = 1 \otimes h^4$ holds by [PS22, Lemma 4.12]. Thus adding h^4 to β does not change the statement on $e_1 \otimes h^3$. We conclude that $\beta = x_4$ holds.

For the $e_3 \otimes h$ case, simply consider Theorem 9.1.1. There is a Rost motive $\mathcal{R}_{(0,1,0,0)}(1)$ in $M(X_1)$, when G has phase $[D_4, (0,1,0,0), 1]$. By [PS22, Theorem 6.4] it follows that there is some cycle $\beta \in \mathrm{Ch}^4(X_1)$ for which $\rho(\beta)$ contains $e_3 \otimes h$. Since h^4 is rational, because the Tits algebra of ω_1 is always split, $\rho(h^4) = 1 \otimes h^4$ holds by [PS22, Lemma 4.12]. Thus adding h^4 to β does not change the statement on $e_3 \otimes h$. We conclude that $\beta = x_4$ holds.

3. The third statement follows by applying the total Steenrod operation to the second statement by using the lemma above. Note that the total Steenrod operation commutes with the coaction. It is worth noting that $S^2(x_4) = x_6$, while $S^3(x_6) \neq x_9$ holds.

4. We consider the pullback $\pi^* : \mathrm{Ch}(G/P) \to \mathrm{Ch}(G)$ of the natural projection $\pi : G \to G/P$. By [Xr20, Corollary 2.] we need to show, that $\pi^*(x_9) = e_9$ holds. For this we apply the technique from Remark 6.1.9. Using the same nomenclature as in [PS22, Lemma 6.2], we consider the right exact sequence of Chow rings of split algebraic groups

$$\mathrm{Ch}(G/Q) \to \mathrm{Ch}(G) \twoheadrightarrow \mathrm{Ch}(C) \to 0.$$

In our situation Q is P_1, while G equals E_7^{ad} and C is of type $\mathbf{HSpin}_{12}$, by Remark 6.2.4. By the right exactness of the sequence above, each generator e_1, e_3, e_5 of $\mathrm{Ch}(C)$ has a preimage in $\mathrm{Ch}(E_7^{ad})$, which is generated by e_1, e_3, e_5, e_9. For codimensional reasons and because of the relation $e_i^2 = 0$ for $i = 1, 3, 5$ holding in $\mathrm{Ch}(C)$, the generators $e_1, e_3, e_5 \in \mathrm{Ch}(E_7^{ad})$ map to their counterparts in terms of codimension in $\mathrm{Ch}(C)$.

Thus e_9 either maps to zero, or it maps to $e_1e_3e_5$. In the first case we are done by [Xr20, Corollary 2.] and because $\rho(x_9)$ can not contain $e_1e_3e_5 \otimes 1$. Otherwise X_1 would be a GSV by [PS22, Theorem 6.4] when $J_2 = (1, 1, 1, 0)$ holds. But X_1 is never a GSV, by the GSV-table.

We are left with showing that e_9 does not map to $e_1e_3e_5$. Assume it does. Then the cycle $e_1e_3e_5 + e_9 \in \mathrm{Ch}(E_7^{ad})$ is mapped to $2e_9 = 0 \in \mathrm{Ch}(C)$. Its preimage in $\mathrm{Ch}(X_1)$ has codimension 9. Such a cycle can only be a linear combination of x_4h^5, x_6h^3, x_9, h^9. Now remember that $e_1 \in \mathrm{Ch}(E_7^{ad})$ maps to $e_1 \in \mathrm{Ch}(C)$ and therefore h maps to zero under π^* (this can also be seen from the fact that h is rational and how the coaction treats such cycles), because $e_1^2 = 0$ in $\mathrm{Ch}(G)$ and $\mathrm{Ch}(C)$ (see J_p-table). Thus the preimage of $e_1e_3e_5 + e_9 \in \mathrm{Ch}(E_7^{ad})$ is x_9. Again, this means when $J_2 = (1, 1, 1, 0)$ holds, then X_1 is a GSV. As this is a contradiction, we have that e_9 maps to zero and $\pi^*(x_9) = e_9$ holds. □

9.3.9 Lemma. *Let G have phase $[E_7, (0, 1, 1, 1), 1]$. Then the Chow motive of the projective, homogeneous G-variety X_1 contains exactly one Rost motive in each of the codimensions $l \in [2 : 14]$, each having as generic point h^l for $h \in \mathrm{Ch}^1(X_1)$. These are the only Rost motives in $M(X_1)$.*

Proof: Let us define $\beta := x_4x_6x_9$. The coaction is a ring homomorphism and thus $\rho(x_4)\rho(x_6)\rho(x_9) = \rho(\beta)$ yields. With the result from the lemma above in mind, we set $\rho(x_9) = e_9 \otimes 1 + 1 \otimes x_9 + \sum a_i \otimes b_i$. As ρ preserves codimensions, all of the a_i have strictly smaller codimension than $\mathrm{codim}(e_9)$, except in case one of the a_i is of the form $e_1e_3e_5$. In this case the respective b_i is 1. However if $e_1e_3e_5 \otimes 1 < \rho(x_9)$ would hold, we could use [PS22, Thrm 6.4] to show that the upper motive of X_1 is a Rost motive when $J_2 = (1, 1, 1, 0)$. But by the GSV-table X_1 does never have a Rost motive as its upper motive (i.e. is never a GSV). So we see that the summand of the form $a_i \otimes b_i$ in $\rho(\beta)$, for which a_i has the biggest codimension, is uniquely $e_3e_5e_9 \otimes h^2$. Now the requirements for [PS22, Thrm 6.4] are fulfilled.

Using Lemma 7.2.13, we see that ρ maps $h^l\beta$ to $e_3e_5e_9 \otimes h^{2+l} + \sum a_i' \otimes b_i'$. Considering the fact that $h^{12}\beta \neq 0 \in \mathrm{Ch}(\overline{X_1})$ holds and that h^{14} is the biggest power of h, which is not zero, the first statement follows.

To prove that these are the only Rost motives in $M(X_1)$, observe that for fulfilling the requirements of [PS22, Thrm 6.4] one needs to find a cycle $\beta' \in \mathrm{Ch}(\overline{X_1})$, which gets mapped to $\rho(\beta') = e_3e_5e_9 \otimes \gamma + \sum a_i \otimes b_i$, for some cycle $\gamma \in \mathrm{Ch}(\overline{X_1})$. By the equations of Lemma 9.3.8 it follows that β' has the form $\delta\beta$ for some appropriate cycle $\delta \in \mathrm{Ch}(\overline{X_1})$. By the identities holding for ρ on the generators of $\mathrm{Ch}(\overline{X_1})$, we see that β' can only be non zero in case δ has 1 or some positive power of h as a summand. As $h^{15} = 0$, this makes fifteen possibilities for such a summand in β' if it is a monomial. But since ρ maps $h^l\beta$ to $e_3e_5e_9 \otimes h^{2+l} + \sum a_i \otimes b_i$, the cycles $h^{13}\beta$ and $h^{14}\beta$ are not suitable for β'. One could argue that maybe β multiplied by an even power of x_4, x_6, x_9 is also a cycle defining a Rost motive, because $\rho(x_i^2) = 1 \otimes x_i^2$ holds for $i = 4, 6, 9$. But checking all such combinations, we see that either $x_i^2\beta = 0$ or $h^2 x_i^2 = 0$ holds.

$\square$

9.3.10 Theorem. *Let G have phase $[E_7, (0, 1, 1, 1), 1]$. Then the unique decomposition of the Chow motive of the projective, homogeneous G-variety X_1 into indecomposable motivic summands is given by*

$$M(X_1) = \mathcal{U}(X_1) \oplus \bigoplus_{i \in I} \mathcal{R}_J(i),$$

with $P(I,t) = t^2(t^{13} - 1)/(t - 1)$ and

$$P(\mathcal{U}(X_1), t) = (1 + t^9)(1 + t + t^4 + t^6 + t^8 + t^{12} + t^{16} + t^{18} + t^{20} + t^{23} + t^{24}).$$

Proof: The claim on the Rost motives is simply the lemma above. We have already seen in Lemma 9.3.2, that the upper motives of X_1/k and X_7/k are not isomorphic.

Calculating $\#M(X_1)$: We now show that no shift of $\mathcal{U}(X_7)$ is contained in $M(X_1)$. In the proof of Theorem 9.3.5, we see that $\mathcal{U}(X_7)$ splits off Tate motives given by $T(\mathcal{U}(X_7), t) = 1 + t^9 + t^{17} + t^{26}$ over $k(X_7)$. However by Theorem 9.1.1 we see that $T(X_1, t) = 1 + t^8 + t^{16} + t^{17} + t^{25} + t^{33}$ holds over $k(X_7)$. To prove the claim it is enough to check that there is no polynomial $f(t) \in \mathbb{N}_0[t]$, such that $T(\mathcal{U}(X_7), t)f(t)$ is a subpolynomial of $T(X_1, t)$. This proves that $\#M(X_1/k) = \{\mathcal{U}(X_1), \mathcal{R}_{(0,1,1,1)}\}$.

There is only one copy of $\mathcal{U}(X_1)$: To show that there is only one copy of $\mathcal{U}(X_1/k)$ in $M(X_1/k)$, we pass to $k(X_1)$. Since the upper motives of X_1/k and X_7/k are not isomorphic, G will have anisotropic kernel D_6 and by Theorem 9.2.1 there are only the two Tate motives $\mathbb{F}_2, \mathbb{F}_2(33)$ in $M(X_1)$ over $k(X_1)$. Since X_1/k has no zero cycle of odd degree by the proof of Lemma 9.3.2, this shows that all Tate motives in $M(X_1)$ over $k(X_7)$ come from the one and only $\mathcal{U}(X_1/k)$. $\square$

9.3.11 Remark. The results of this section can be used to calculate all motivic decompositions of the G-varieties of an isotropic group G of type E_8, which has anisotropic kernel of type E_7.

Chapter 10

Motivic decompositions for general E7s

This chapter deals with the motivic decomposition of the projective, homogeneous G-varieties for G being an adjoint algebraic group of type E_7 with non trivial Tits algebra. Recall that by Corollary 6.2.3 this means that $j_1 = 1$ does hold for $J_2(G) = (j_1, j_2, j_3, j_4)$ over k. A highlight of this chapter is the case where G

has anisotropic kernel D_6 and is defined by $\mathbf{HSpin}(A,\sigma)$. In this case, we provide complete motivic decompositions of the respective (generalized) involution varieties $\mathcal{I}_i(A,\sigma)$ in Theorem 10.4.9 (for $i = 5,6$ we consider the $\mathbf{HSpin}(A,\sigma)$-varieties Y_5, Y_6).

The cases of $[E_7,(1,1,0,0),2]$ and $[E_7,(1,0,0,0),2]$ are treated in a separate chapter, because we also take in account how (some of) such groups are constructed and strive for making that chapter a bit more self contained. Also we have not yet proved that these phases are admissible. As a main result, we show that $M(X_1)$ and $M(X_7)$ are indecomposable in the case of $[E_7,(1,1,1,1),*]$. The decompositions of the other projective, homogeneous G-varieties for this phase are not completely established. We only provide the partial result of showing that $M(X_3), M(X_4)$ and $M(X_6)$ contain Rost motives. Finally we provide a motivic decompositions for $M(X_1)$ in the hypothetically existing case of an anisotropic E_7 with $J_2 = (1,1,1,0)$.

By the GSV-table the varieties X_Θ which are not always GSV and for which P_Θ is a maximal parabolic subgroup are X_1, X_3, X_4, X_6, X_7. From the Tits classification it is clear that the upper motives of these varieties, $\mathfrak{X}$ and of $X_{1,6}$ are the only possibly distinct elements in $\#G$. Thus the decompositions of all other varieties X_Θ for other Θs can be obtained from these decompositions by Karpenko's theorem and the use of phase transitions and checking Tate motives. We do not consider these other cases concretely.

The tables containing the motivic decompositions are organized in the same way as in the last chapter.

10.1 The phase [A1 x A1 x A1,(1,0,0,0),2]

Establishing these decompositions is merely a triviality thanks to the advancements made during the last decades. We only provide it for further referencing.

10.1.1 Theorem. *Let G have phase $[A_1^3,(1,0,0,0),2]$. Then the following unique decompositions of the Chow motives of projective, homogeneous G-varieties into indecomposable motivic summands hold*

Θ	$M(X_\Theta)$
$\Theta \subset \{1,3,4,6\}$	$\bigoplus_{t\in T_\Theta} \mathbb{F}_2(t) \oplus \bigoplus_{i\in I_\Theta} \mathcal{R}_J(i)$
Any other	$\bigoplus_{i\in I_\Theta} \mathcal{R}_J(i)$

Index	**Poincaré Polynomial**
$\mathcal{R}_J$	$(1+t)$

Index	**Shift/Tate Polynomial**
T_1	$(1+t+t^2)(1+t^6)(1+t^{10})(1+t^{15})$
T_3	$(1+t^5)(1+t^9)P(T_1,t)$
T_4	$(1+t^5)(1+t^6)P(T_6,t)$
T_6	$(1+t^4+t^8)(1+t^9)(1+t^{10})(1+t^{15})$
$T_{1,6}$	$(1+t+t^2+t^4+t^5+t^6+t^8+t^9+t^{10})(1+t^6)(1+t^9)(1+t^{10})(1+t^{15})$
I_1	$t^3(1+t+t^2)(1+t^3+t^5+t^6+t^8+2t^9+t^{11}+t^{12}+t^{13}+2t^{15}+t^{16}+t^{18}+t^{19}+t^{21}+t^{24})$
I_3	$t^2(1+t+t^2)(1+t^2)(1+t+3t^4+2t^5+2t^6+2t^7+3t^8+5t^9+6t^{10}+3t^{11}+4t^{12}+8t^{13}+7t^{14}+7t^{15}+7t^{16}+7t^{17}+7t^{18}+10t^{19}+7t^{20}+7t^{21}+7t^{22}+7t^{23}+7t^{24}+8t^{25}+4t^{26}+3t^{27}+6t^{28}+5t^{29}+3t^{30}+2t^{31}+2t^{32}+2t^{33}+3t^{34}+t^{37}+t^{38})$
I_6	$t(1+t^4+t^8)(1+t)(1+t^2+t^3+t^4+2t^5+t^6+3t^7+5t^9+6t^{11}+6t^{13}+7t^{15}+6t^{17}+6t^{19}+5t^{21}+3t^{23}+t^{24}+2t^{25}+t^{26}+t^{27}+t^{28}+t^{30})$
I_Θ	$[P(X_\Theta,t)-P(T_\Theta,t)]/P(\mathcal{R}_J,t)$

Proof: By the Tits classification there is no possibility for G/k to become more isotropic over some field extension L/k without splitting. Especially every anisotropic projective, homogeneous G-variety is a GSV over k. Thus $\#G/k = \{\mathbb{F}_2, \mathcal{R}_{(1,0,0,0)}\}$ holds by Karpenko's theorem, as G/k is isotropic. Note that since the Tits algebra A of G/k is not split by Corollary 6.2.3, one has that $\mathcal{U}(\mathrm{SB}(A/k)) \simeq \mathcal{R}_{(1,0,0,0)}$ holds, because $\mathrm{SB}(A/k)$ is a GSV.

The overall situation resembles the $[D_4,(0,1,0,0),1]$ case. The decompositions of the isotropic X_Θ are therefore obtained analogously. First one uses the CGMB algorithm to determine the Tate motives in each $M(X_\Theta)$ and subtracts the polynomial encoding them from $P(X_\Theta,t)$. Then all one needs to do is to divide this difference by $P(\mathcal{R}_{(1,0,0,0)},t)$ to obtain the shifts of the Rost motives contained in $M(X_\Theta)$. $\square$

10.2 The phase [D4 x A1,(1,1,0,0),2]

The decompositions for this case are as nearly as straightforward and easily obtained as in the $(A_1^3, (1,0,0,0), 2)$ case. The results of this chapter will be needed very often later on. We start with calculating $\#G$ and then simply use the CGMB algorithm for providing the actual decompositions of the varieties of our interest.

10.2.1 Lemma. *Let G be an adjoint algebraic group of type E_7 with phase $[D_4 \times A_1, (1,1,0,0), 2]$. Then the upper motives of X_7 and* $\mathrm{SB}(A)$ *are isomorphic and $\#G = \{\mathbb{F}_2, \mathcal{U}(X_3), \mathcal{U}(X_7), \mathcal{R}_{(1,1,0,0)}\}$ holds, with*

$$P(\mathcal{U}(X_3), t) = (1 + t^2 + t^3 + t^5)$$

$$\textit{and } P(\mathcal{U}(X_7), t) = (1 + t).$$

Proof: The existence of the Tate motive in $\#G$ follows from the fact that G is isotropic. Also the upper motives of X_3 and X_4 are isomorphic by the Tits classification, and by the GSV-table X_Θ is a GSV if and only if 2 or 5 are contained in Θ. This means we only need to show that the upper motives of X_7 and X_3 are not isomorphic and then calculate their Poincaré polynomial.

Showing that $\mathcal{U}(X_3) \neq \mathcal{U}(X_7)$: Since $\mathrm{SB}(A)$ is a GSV only if $J_2 = (1,0,0,0)$ holds by Lemma 7.1.9, passing to $k(\mathrm{SB}(A))$ yields the phase $[D_4, (0,1,0,0), 1]$ by the phase classification and as j_1 becomes zero over $k(\mathrm{SB}(A))$ by Theorem 6.2.2. Now X_7 is isotropic, while X_3 is an anisotropic GSV by the GSV-table and therefore has no zero cycle of odd degree over $k(\mathrm{SB}(A))$.

Structure of $\mathcal{U}(X_7)$: By the GSV-table X_7/k is not a GSV, so passing to $k(X_7)$ also induces a transition to $[D_4, (0,1,0,0), 1]$ by the Tits classification. Thus the upper motives of X_7/k and $\mathrm{SB}(A/k)$ are isomorphic. As $\mathrm{SB}(A)$ has a zero cycle of odd degree if and only if $\mathrm{ind}(A) = 1$ holds, $\mathcal{U}(X_7)$ has the desired Poincaré polynomial by the main result of [Kar95] and because $\mathrm{ind}(A/k) = 2$.

Structure of $\mathcal{U}(X_3)$: The statement on the structure of $\mathcal{U}(X_3)$ follows easily by noting that the D_4 part of the semisimple anisotropic kernel of G is defined by a quadratic form q having splitting pattern $[2,2]$ by Lemma 8.3.2. Therefore $\mathcal{U}(X_3) \simeq \mathcal{U}(X_q)$ holds. We can use Lemma 5.5.11, which states that the structure of $\mathcal{U}(X_q)$ is as claimed.

□

10.2.2 Theorem. *Let G have phase $[D_4 \times A_1, (1,1,0,0), 2]$. Then the following unique decompositions of the Chow motives of projective, homogeneous G-varieties into indecomposable motivic summands hold*

Θ	$M(X_\Theta)$
$\{1\}$	$\bigoplus_{t=0,8,25,33} \mathbb{F}_2(t) \oplus \bigoplus_{u\in O_1} \mathcal{U}(X_3)(u) \oplus \mathrm{SB}(D)(16) \oplus \bigoplus_{i\in I_1} \mathcal{R}_J(i)$
$\{3\}$	$\bigoplus_{u\in O_3} \mathcal{U}(X_3)(u) \oplus \bigoplus_{i\in I_3} \mathcal{R}_J(i)$
$\{4\}$	$\bigoplus_{u\in O_4} \mathcal{U}(X_3)(u) \oplus \bigoplus_{i\in I_4} \mathcal{R}_J(i)$
$\{6\}$	$\bigoplus_{t=0,17,25,42} \mathbb{F}_2(t) \oplus \bigoplus_{u\in O_6} \mathcal{U}(X_3)(u) \oplus \bigoplus_{s=8,16,25,33}$ $\mathrm{SB}(D)(s) \oplus \bigoplus_{i\in I_6} \mathcal{R}_J(i)$
$\{7\}$	$\bigoplus_{s=0,9,17,26} \mathrm{SB}(D)(s) \oplus \bigoplus_{i\in I_7} \mathcal{R}_J(i)$
$\{1,6\}$	$\bigoplus_{t=0,8,17,25,25,33,42,50} \mathbb{F}_2(t) \oplus \bigoplus_{u\in O_{1,6}} \mathcal{U}(X_3)(u) \oplus \bigoplus_{s\in S}$ $\mathrm{SB}(D)(s) \oplus \bigoplus_{i\in I_{1,6}} \mathcal{R}_J(i)$

Index	**Poincaré Polynomial**
$\mathcal{U}(X_3)$	$(1+t^2+t^3+t^5)$
$\mathrm{SB}(D)$	$(1+t)$
$\mathcal{R}_J$	$(1+t)(1+t^3)$

Index	**Shift/Tate Polynomial**
O_1	$t+t^2+t^{10}+t^{11}+t^{12}+t^{16}+t^{17}+t^{18}+t^{26}+t^{27}$
O_3	$(1+t+t^2)(1+t^6)(1+t^9)(1+t^{10})(1+t^{15})$
O_4	$(1+t^4+t^8)(1+t^6)(1+t^9)(1+t^{10})(1+t^{15})$
O_6	$t^4(1+t^4+t^6+t^{10}+t^{14}+t^{15}+t^{19}+t^{23}+t^{25}+t^{29})$
$O_{1,6}$	$[T_{1,6}-(1+t^8)(1+t^{17})(1+t^{25})]/(1+t^5)$, *for* $T_{1,6}$ *from* Theorem 10.1.1
S	$t^8(1+t^8)^2(1+t^{17})$
I_1	$t^5(t^{20}-1)/(t-1)$
I_3	$t^3(1+t)(1+t^2)(1+t+t^2+t^3+t^4+3t^5+t^6+4t^7+t^8+7t^9+t^{10}+$ $8t^{11}+10t^{13}+11t^{15}+12t^{17}+11t^{19}+10t^{21}+8t^{23}+t^{24}+7t^{25}+t^{26}+$ $4t^{27}+t^{28}+3t^{29}+t^{30}+t^{31}+t^{32}+t^{33}+t^{34})$
I_4	$[P(X_4,t)-O_iP(\mathcal{U}(X_3),t)]/P(\mathcal{R}_J,t)$
I_6	$[P(X_6,t)-(1+t^{17}+t^{25}+t^{42})-O_6P(\mathcal{U}(X_3),t)-t^8(1+t^8+t^{17}+$ $t^{25})P(\mathrm{SB}(D),t)]/P(\mathcal{R}_J,t)$
I_7	$t^2+t^4+t^6+t^8+t^{10}+t^{11}+t^{12}+t^{13}+t^{15}+t^{17}+t^{19}+t^{21}$
$I_{1,6}$	$[P(X_{1,6},t)-(1+t^8)(1+t^{17})(1+$ $t^{25})-O_{1,6}P(\mathcal{U}(X_3),t)-P(S,t)P(\mathrm{SB}(D),t)]/P(\mathcal{R}_J,t)$

Proof: The decompositions are all obtained in a similar way. For the Tate motives over k one uses the CGMB method. By the lemma above (i.e. by considering $\#G$) there are only two non split phases one can have a transition to. Namely $[D_4, (0,1,0,0), 1]$ by passing to $k(\mathrm{SB}(A))$ because of Lemma 6.2.2 and $[A_1^3, (1,0,0,0), 2]$ by passing to $k(X_3)$ because of the GSV-table and the phase classification.

Determining the copies of $\mathcal{U}(\mathrm{SB}(A))$: Note that $\mathcal{U}(X_3) \simeq \mathcal{U}(X_q)$ decomposes as $\mathbb{F}_2 \oplus \mathcal{R}_{(1,0,0,0)}(2) \oplus \mathbb{F}_2(5)$ over $k(X_3)$ since q has splitting pattern $[2,2]$. Thus the shifts of $\mathcal{U}(\mathrm{SB}(A))$ over k in each $M(X_\Theta)$ can be computed by passing to $k(\mathrm{SB}(A))$, checking for the Tate motives *not visible over* k and dividing the respective polynomial by $1+t$. This works because the D_4 part of the anisotropic kernel of G does not change over $k(\mathrm{SB}(A))$ by the phase classification and Lemma 7.1.9 and thus X_3 still has no zero cycle of odd degree as it turns into a GSV over $k(\mathrm{SB}(A))$.

Determining the copies of $\mathcal{U}(X_3)$: The shifts of $\mathcal{U}(X_3)$ over k are computed by passing to $k(X_3)$, checking for the Tate motives *not visible over* k and dividing the respective polynomial by $1+t^5$. This works because $\mathrm{SB}(A)$ does not split over $k(X_3)$ by the index reduction formula in [MPW2]. The Tate motives over the mentioned field extensions were already calculated in Theorem 9.1.1 and Theorem 10.1.1.

Determining the copies of $\mathcal{R}_J$: Subtracting the polynomials coding Tate motives, shifts of $\mathcal{U}(X_3)$ and the shifts of $\mathcal{U}(\mathrm{SB}(A))$ from the Poincaré polynomials of each X_Θ and dividing by $P(\mathcal{R}_{(1,1,0,0)},t)$ yields the polynomials encoding the shifts of $\mathcal{R}_J$ in $M(X_\Theta)$. This works because by the lemma above there are no other elements in $\#G$ over k.

□

10.3 The phase [D5 x A1,(1,1,0,0),2]

We start with a lemma which will be used very often as it allows to shorten many proofs.

10.3.1 Lemma. *Let G be an adjoint algebraic group of type E_7 over k, with $J_2(G) = (1,1,*,*)$. Then none of the upper motives of $X_1, X_6, X_{1,6}$ are isomorphic to the upper motive of X_3.*

Proof: By Theorem 6.2.2, the Tits algebra of G/k is not split. On top of that $\mathrm{SB}(A)$ is not a GSV for G/k by Lemma 7.1.9. Thus passing to $k(\mathrm{SB}(A))$ yields one of the phases $[E_7,(0,1,1,1),1], [D_6,(0,1,1,0),1], [D_4,(0,1,0,0),1]$, by the phase classification. Note that even if $[E_7,(0,1,1,0),1]$ existed, this does not change the proof.

If the anisotropic kernel of G is D_4, the statement is easy to prove as X_3 is an anisotropic GSV and thus it has no zero cycle of odd degree, while the other varieties in question are isotropic. If the anisotropic kernel of G is E_7 or D_6, we pass to the generic point of X_7 to also obtain anisotropic kernel D_4. This works since X_7 is never a GSV when G has split Tits algebras (i.e. $j_1 = 0$) by the GSV-table.

□

10.3.2 Lemma. *Let G be an adjoint algebraic group of type E_7 with phase $[D_5 \times A_1,(1,1,0,0),2]$. Then the upper motives of X_7 and* $\mathrm{SB}(A)$ *are isomorphic and $\#G = \{\mathbb{F}_2, \mathcal{U}(X_1), \mathcal{U}(X_3), \mathcal{U}(X_7), \mathcal{R}_{(1,1,0,0)}\}$ holds, with*

$$P(\mathcal{U}(X_1),t) = (1+t^4)(1+t+t^2+t^3+t^4),$$

$$P(\mathcal{U}(X_3),t) = (1+t^6)(1+t^2+t^3+t^5),$$

$$\textit{and } P(\mathcal{U}(X_7),t) = (1+t).$$

Proof: The existence of the Tate motive in $\#G$ follows from the fact that G is isotropic. Also by the lemma above the upper motives of X_1 and X_3 are not isomorphic. Note further that the upper motives of X_3 and X_4 are isomorphic by the Tits classification. By the GSV-table G/P_Θ is a GSV if and only if 2 or 5 are contained in Θ.

Structure of $\mathcal{U}(\mathrm{SB}(A))$: Mimicking the part of the proof of Theorem 10.2.2 which deals with the upper motives X_7 and $\mathrm{SB}(A)$, we see that the upper motives of X_7/k and $\mathrm{SB}(A/k)$ are isomorphic and have the desired Poincaré polynomial.

Structure of $\mathcal{U}(X_1)$: The upper motive of X_1 is isomorphic to the upper motive of a quadric X_q with q having splitting pattern $[1, 2, 2]$, because any adjoint E_7 having anisotropic kernel $D_5 \times A_1$ arises as from such a q by the comment in the proof of Lemma 8.3.2. Also $M(X_q)$ is indecomposable by Lemma 5.5.11, which proves the claim on $\mathcal{U}(X_1)$.

Structure of $\mathcal{U}(X_3)$: Let us focus on an anisotropic group H of type D_5 defined by a anisotropic quadratic form q having the splitting pattern $[1, 2, 2]$. Using Karpenko's theorem we have $\#H = \{\mathcal{U}(Y_1), \mathcal{U}(Y_2), \mathcal{R}_{(1,1,0,0)}\}$ with $Y_1/k \simeq X_q/k$. The upper motives of the H-varieties Y_2 and Y_3 are isomorphic to $\mathcal{U}(X_3)$. So we focus on the H-varieties for the rest of the proof.

Passing to $k(Y_2)$ will leave H with anisotropic kernel A_1^2 because of the splitting pattern of q. Performing the CGMB algorithm (input: prodbases([1, 2], [1, 2, 3, 5], D5) for $T(Y_2, t)$ for example), we see that the polynomials encoding the Tate motives in $M(Y_2)$ and $M(Y_3)$ over $k(Y_2)$ are given by

$$T(Y_2, t) := (1 + t + t^2)(1 + t^5)(1 + t^6) \text{ and } T(Y_3, t) := (1 + t^4)(1 + t^5)(1 + t^6).$$

Because of the splitting pattern of q, no copy of $\mathcal{U}(Y_1)$ is contained in neither $M(Y_2)$ nor $M(Y_3)$ over k, as otherwise they would contain a Tate motive over $k(Y_1)$. Since $\mathcal{U}(Y_2/k) \simeq \mathcal{U}(Y_3/k)$ holds, given the few elements of $\#H/k$ and since $\gcd(T(Y_2, t), T(Y_3, t)) = (1+t^5)(1+t^6)$ in $\mathbb{N}_0[t]$, it follows that the Tate motives which $\mathcal{U}(Y_2/k)$ splits off over $k(Y_2)$ are encoded by the one of the following polynomials.

Case 1. $(1 + t^6)$, **Case 2.** $(1 + t^5)$ or **Case 3.** $(1 + t^5)(1 + t^6)$.

Clearly $\#M(Y_2/k) \subset \{\mathcal{U}(Y_2), \mathcal{R}_{(1,1,0,0)}\}$ holds. The value of J_2 does not change over $k(Y_1)$, so the Rost motives $R_{(1,1,0,0)}$ contained in $M(Y_2/k)$ do not change either when passing to $k(Y_1)$, as their occurrence only depends on the value of J_2 and the coaction map ρ by [PS22, Theorem 6.4]. This means that $\mathcal{U}(Y_2/k)$ can not split off Rost motives over $k(Y_1)$.

Now recall from Theorem 10.2.2 that the upper motive of Y_2 over $k(Y_1)$ is isomorphic to the upper motive of a quadric $X_{q'}$ with $P(\mathcal{U}(X_{q'}), t) = 1 + t^2 + t^3 + t^5$. Thus we have that $\mathcal{U}(Y_2/k)$ will exactly split as $\mathcal{U}(X_{q'}) \oplus \mathcal{U}(X_{q'})(6)$ or become isomorphic to $\mathcal{U}(X_{q'})$, when passing to $k(Y_1)$. It follows that **Case 1.** can not be true.

To see that **Case 2.** is also impossible, we show that $\mathcal{U}(X_{q'}) \oplus \mathcal{U}(X_{q'})(6)$ are glued over k. For this we use the [DC, Thrm. 1]. In our situation we set $M := M(Y_1)$

and $N := \mathcal{U}(X_{q'})(1)$. The summand $\mathcal{U}(X_{q'})(1)$ is contained in $M(Y_2)$ over $k(Y_1)$ as can be read off from the structure of $T(Y_2,t)$ above. We also set $X := Y_1, Y := Y_2$.

Because X is a quadric defined by a form q with splitting pattern $[1,2,2]$, we see that over $k(X)$ the indecomposable direct summand N is contained in M. By the theorem of De Clercq the same holds over k. But we have seen in Lemma 5.5.11 that M/k in indecomposable.

□

10.3.3 Theorem. *Let G have phase $[D_5 \times A_1, (1,1,0,0), 2]$. Then the following unique decompositions of the Chow motives of projective, homogeneous G-varieties into indecomposable motivic summands hold*

Θ	$M(X_\Theta)$
$\{1\}$	$\bigoplus_{u=0,25}\mathcal{U}(X_1)(u) \oplus \bigoplus_{q\in O_1}\mathcal{U}(X_3)(q) \oplus \mathrm{SB}(D)(16) \oplus \bigoplus_{i\in I_1}\mathcal{R}_J(i)$
$\{3\}$	$\bigoplus_{u\in O_3}\mathcal{U}(X_3)(u) \oplus \bigoplus_{i\in I_3}\mathcal{R}_J(i)$
$\{4\}$	$\bigoplus_{u\in O_4}\mathcal{U}(X_3)(u) \oplus \bigoplus_{i\in I_4}\mathcal{R}_J(i)$
$\{6\}$	$\bigoplus_{t=0,42}\mathbb{F}_2(t) \oplus \mathcal{U}(X_1)(17) \oplus \bigoplus_{q\in O_6}\mathcal{U}(X_3)(q) \oplus \bigoplus_{s=8,16,25,33}\mathrm{SB}(D)(s) \oplus \bigoplus_{i\in I_6}\mathcal{R}_J(i)$
$\{7\}$	$\bigoplus_{s=0,9,17,26}\mathrm{SB}(D)(s) \oplus \bigoplus_{i\in I_7}\mathcal{R}_J(i)$
$\{1,6\}$	$\bigoplus_{u=0,17,25,42}\mathcal{U}(X_1)(u) \oplus \bigoplus_{q\in O_{1,6}}\mathcal{U}(X_3)(q) \oplus \bigoplus_{s\in S}\mathrm{SB}(D)(s) \oplus \bigoplus_{i\in I_{1,6}}\mathcal{R}_J(i)$

Index	**Poincaré Polynomial**
$\mathcal{U}(X_1)$	$(1+t+t^2+t^3+2t^4+t^5+t^6+t^7+t^8)$
$\mathcal{U}(X_3)$	$(1+t^6)(1+t^2+t^3+t^5)$
$\mathrm{SB}(D)$	$(1+t)$
$\mathcal{R}_J$	$(1+t)(1+t^3)$

Index	**Shift/Tate Polynomial**
O_1	$t^{10}(1+t+t^2)$
O_3	$(1+t+t^2)(1+t^9)(1+t^{10})(1+t^{15})$
O_4	$(1+t^4+t^8)(1+t^9)(1+t^{10})(1+t^{15})$
O_6	$t^4+t^8+t^{23}+t^{27}$
$O_{1,6}$	$(1+t+t^2)[(1+t^4)(1+t^6)(1+t^{19})t^4-(t^{18}+t^{19})]$
S	$t^8(1+t^8)^2(1+t^{17})$
I_1	$t^5(t^{20}-1)/(t-1)$
I_i	$[P(X_i,t){-}O_iP(\mathcal{U}(X_3),t)]/P(\mathcal{R}_J,t)$ *for* $i=3,4$
I_6	$[P(X_6,t){-}(1+t^{42}){-}O_6P(\mathcal{U}(X_3),t){-}t^8(1+t^8+t^{17}+t^{25})P(\mathrm{SB}(D),t)]/P(\mathcal{R}_J,t)$
I_7	$[P(X_7,t){-}(1+t^9+t^{17}+t^{26})P(\mathrm{SB}(D),t)]/P(\mathcal{R}_J,t)$
$I_{1,6}$	$[P(X_{1,6},t){-}(1+t^{17})(1+t^{25})P(\mathcal{U}(X_1),t){-}O_3P(\mathcal{U}(X_3),t){-}P(S,t)P(\mathrm{SB}(D),t)]/P(\mathcal{R}_J,t)$

Proof: The decompositions are obtained analogously to the procedure in Theorem 10.2.2. By the proof of the lemma above, the phases obtained after passing to X_1, X_3 or X_7 are $[D_4 \times A_1, (1,1,0,0), 2], [A_1^3, (1,0,0,0), 2], [D_4, (0,1,0,0), 1]$. The structure of the upper motives was also calculated in the lemma.

To obtain the structure of the decompositions, performing the CGMB algorithm and analyzing the Tate motives over several field extensions proves the claims. The $\mathcal{U}(X'_q)$ and $\mathrm{SB}(A)$ over $k(X_1)$, which are needed for this were calculated in Theorem 10.2.2. Note that $\mathcal{U}(X_3/k)$ splits as $\mathcal{U}(X'_q) \oplus \mathcal{U}(X'_q)(6)$ for q' having splitting pattern $[2, 2]$ over $k(X_1)$.

□

10.3.4. Interestingly the motivic decomposition of $M(X_7)$ is the same for $[D_5 \times A_1, (1, 1, 0, 0), 2]$ and $[D_4 \times A_1, (1, 1, 0, 0), 2]$. This observation motivated the author to prove the following theorem, which is a special case of Zhykhovich's theorem. Coincidentally this was discovered independently by the author in 2018, only a few weeks earlier than Zhykhovich's result.

10.3.5 Theorem. *Let G be an anisotropic, adjoint group of type E_7 over k, with a non split Tits algebra A. Then $J_2(G) = (1, 1, 0, 0)$ and $\mathrm{ind}(A) = 2$ hold over k if and only if G has semisimple, anisotropic kernel D_4 over $k(\mathrm{SB}(A))$.*

Proof: $\Leftarrow$: Since $\mathrm{SB}(A)$ does not split G and A/k is non split, we see that by Lemma 7.1.9 one has $J_2(G/k) = (1, 1, *, *)$. So over $k(X_7)$, the anisotropic kernel of G is D_4 by the phase classification, as X_7 is not a GSV for such a value of J_2 by the GSV-table. Since by assumption it is also D_4 over $k(\mathrm{SB}(A))$, the upper motives of $\mathrm{SB}(A)/k$ and X_7/k are isomorphic. By Theorem 9.1.1 the Tate motives in $M(X_7)$ over $k(X_7)$ are given by

$$T(X_7, t) := 1 + t + t^9 + t^{10} + t^{17} + t^{18} + t^{26} + t^{27}.$$

These Tate motives necessarily come from shifts of $\mathcal{U}(X_7/k)$. Since $\mathcal{U}(X_7/k)$ is isomorphic to $\mathcal{U}(\mathrm{SB}(A/k))$, this leaves as only possibility that $\mathrm{ind}(A/k) = 2$ holds, because in general $P(\mathcal{U}(\mathrm{SB}(A)), t) = (t^{\mathrm{ind}(A)} - 1)/(t - 1)$ holds by Theorem 5.5.1. By Karpenko's theorem the other motives in $M(X_7/k)$ can only be Rost motives $\mathcal{R}_J$ for $J_2 = (1, 1, *, *)$, as the only Tits index more isotropic than D_4 is the split one. Therefore the difference $P(X_7, t) - T(X_7, t)$ has to be divisible by $P(\mathcal{R}_J, t)$. But this is only the case for $J_2 = (1, 0, 0, 0)$ and $J_2 = (1, 1, 0, 0)$. The first case is impossible, because if J_2 was $(1, 0, 0, 0)$ then G would split over $k(\mathrm{SB}(A))$, contradicting our initial assumption.

$\Rightarrow$: Over $k(\mathrm{SB}(A))$ the value of J_2 will be $(0, 1, 0, 0)$, since $\mathrm{SB}(A)$ can only split G for $J_2 = (1, 0, 0, 0)$ and j_1 has to decrease to zero by Theorem 6.2.2, which means that the anisotropic kernel of G is D_4 by the phase classification. □

10.3.6. Another very interesting fact, this time concerning the coaction, can be derived from our previous results. Namely the opposite statement of [PS22, Lemma 4.12] is unfortunately wrong, as the following counter example shows.

10.3.7 Lemma. *Let G be an adjoint algebraic group of type E_7, with the phase $[D_5 \times A_1, (1, 1, 0, 0), 2]$ over k. Consider the coaction ρ on $\mathrm{Ch}(\overline{X_1})$ and the cycle $pt \in \mathrm{Ch}^{33}(\overline{X_1})$. Then $\rho(pt) = 1 \otimes pt$ holds, but pt is* ***not*** *rational over k.*

Proof: Consider the formulas for ρ established in Lemma 9.3.8 and remember that ρ preserves codimensions. In case $J_2(G) = (1, 1, 0, 0)$ holds and because x_9 has codimension 9, $\rho(x_9)$ contains at most the following summands

$$e_1 \otimes h^8, e_1 \otimes x_4h^4, e_1 \otimes x_6h^2, e_3 \otimes h^6, e_3 \otimes x_4h^2, e_3 \otimes x_6, e_1e_3 \otimes h^5, e_1e_3 \otimes x_4h, 1 \otimes x_4h^5, 1 \otimes x_6h^3, 1 \otimes h^9, 1 \otimes x_9.$$

We can only be sure about the last one, but for now we consider the possibility of any of the other summands occurring as well. Multiplying with $\rho(h^{14}) = 1 \otimes h^{14}$ kills every summand $a \otimes bh^i$, with i being positive, because $h^{15} = 0$ holds in $\mathrm{Ch}(\overline{X_1})$. The only cycles which survive this are $e_3 \otimes x_6, 1 \otimes x_9$. Thus $\rho(x_9h^{14}) < e_3 \otimes x_6h^{14} + 1 \otimes x_9h^{14}$ holds. Expanding $\rho(x_4)\rho(x_6) = (e_3 \otimes h + e_1 \otimes h^3 + 1 \otimes x_4)(e_1 \otimes h^5 + 1 \otimes x_6)$, gives

$$e_1e_3 \otimes h^6 + e_1 \otimes x_4h^5 + e_1 \otimes x_6h^3 + e_3 \otimes x_6h + 1 \otimes x_4x_6,$$

by Lemma 9.3.8 and Lemma 9.3.7. Multiplying the product $\rho(x_4)\rho(x_6)$ with $\rho(x_9h^{14})$, yields $\rho(x_4x_6x_9h^{14}) = 1 \otimes x_4x_6x_9h^{14}$. Thus $x_4x_6x_9h^{14} = pt \in \mathrm{Ch}^{33}(\overline{X_1})$ gets mapped to $1 \otimes pt$ by ρ. In case G has phase $[D_5 \times A_1, (1,1,0,0), 2]$, the upper motive of X_1 is isomorphic to the upper motive of an anisotropic quadric. Applying Springer's theorem, it follows that pt can not be rational over k. □

10.3.8 Corollary. *Let G be an anisotropic adjoint algebraic group of type E_7 with motivic J_2-invariant $J_2(G)$. If $J_2(G) = (1,1,0,0)$ holds, then none of the projective, homogeneous G-varieties X_Θ has a zero cycle of odd degree if X_6 does not have one.*

Proof: For the GSVs the statement is clear and by the phase classification and Karpenko's theorem, we can restrict our considerations to X_1, X_3, X_6, X_7 and $X_{1,6}$. An important ingredient is Lemma 10.3.1 along with the index reduction formula from [MPW2] and the phase classification. From this it follows that over the generic points of X_1, X_6 and $X_{1,6}$, the anisotropic kernel of G is either $D_4 \times A_1$ or $D_5 \times A_1$. So keep that in mind, when we pass to the respective $k(X_\Theta)$. Also, let A denote the Tits algebra of G. By Theorem 6.2.2 it is not split over k.

The case X_3: By Lemma 7.1.9 passing to $k(\mathrm{SB}(A))$ does not split G. By the phase classification and the GSV-table, X_3 is an anisotropic GSV over $k(\mathrm{SB}(A))$ and the claim follows.

The case X_7: By the GSV-table, passing to $k(X_3)$ does not split G. We obtain A_1^3 as anisotropic kernel. Now the upper motive of X_7 is isomorphic to the upper motive of $\mathrm{SB}(A)$. As the Brauer class of A is not trivial, $\mathrm{SB}(A)$ has no zero cycle of odd degree over $k(X_3)$.

The case $X_{1,6}$: Passing to $k(X_1)$ gives us an anisotropic kernel $D_4 \times A_1$, while passing to $k(X_6)$ gives us anisotropic kernel $D_5 \times A_1$ or $D_4 \times A_1$. This both follows from the proof of Lemma 10.3.1. In the $D_5 \times A_1$ case the upper motive of $X_{1,6}$ is isomorphic to the upper motive of X_1 but not to $\mathcal{U}(X_6)$, while in the second case it is isomorphic to the upper motive of X_6. Thus we can focus our considerations on X_1 and X_6.

The case X_1: Pass to $k(X_6)$. If G has anisotropic kernel $D_5 \times A_1$, the upper motive of X_1 becomes isomorphic to the upper motive of an anisotropic quadric. Thus by Springer's theorem X_1 can not have the discussed zero cycle over k. If the anisotropic kernel of G is $D_4 \times A_1$ over $k(X_6)$, then $\mathcal{U}(X_1/k)$ is necessarily isomorphic to $\mathcal{U}(X_6/k)$. One can use Lemma 10.3.1 to see this. We have found that it is enough to show that X_6 does not have the zero cycle in question.

□

10.4 The phase [D6,(1,1,1,0),*]

The decompositions of the projective, homogeneous E_7-varieties established in this section can be completely concluded by knowing the upper motives of the respective D_6-varieties. The delicate premise is that the CSA A of the group $H \simeq \mathbf{HSpin}(A,\sigma)$ of type D_6 considered here is not split, which means that the projective, homogeneous H-varieties are mostly involution varieties. For such a D_6 a complete calculation of $\#H$ has never been provided before in the literature. So this section may be of general interest. We derive the motivic decompositions of the projective, homogeneous E_7-varieties from these results. Also the results are used for showing that for a versal form of $\mathbf{HSpin}(A,\sigma) \simeq D_6$ the motives of several projective, homogeneous D_6-varieties are indecomposable.

10.4.1 Remark. (Enumeration of nodes) Before we start, we need to fix an orientation of the fifth and sixth node of $\Delta(D_6)$ to be compliant with the references. We embed $\Delta(D_6)$ in $\Delta(E_7)$ in such a way that the sixth node of $\Delta(D_6)$ equals the second node in $\Delta(E_7)$ (both in Bourbaki enumeration). Let us denote the projective, homogeneous D_6-varieties by Y_Θ and the projective, homogeneous E_7-varieties by X_Θ. Then $\mathcal{U}(X_6) \simeq \mathcal{U}(Y_2)$, $\mathcal{U}(X_3) \simeq \mathcal{U}(Y_5)$ and $\mathcal{U}(X_2) \simeq \mathcal{U}(Y_6)$ holds. This is especially important as only this way Y_6 is always a GSV.

10.4.2. The proof of the lemma below is perfectly suited to be comprehended by using the overview of phases in the phase classification.

10.4.3 Lemma. *Let G be an adjoint algebraic group of type E_7 with semisimple, anisotropic kernel H of type D_6. If $J_2(G) = (1,1,1,0)$ holds, then the motive of the projective, homogeneous H-variety Y_1 is indecomposable.*

Proof: For $H \simeq \mathbf{HSpin}(A,\sigma)$ we consider the H-variety $Y_1 \simeq \mathcal{I}(A,\sigma)$. Passing to $L := k(\mathrm{SB}(A))$ yields the phase $[D_6,(0,1,1,0),1]$ by the proof of Theorem 8.3.5. Now Y_1 becomes isomorphic to a quadric X_q, with q having splitting pattern $[2,4]$. The decomposition of $M(Y_1/L)$ into indecomposable motivic summands was calculated in Lemma 5.5.10 and is given by $M(Y_1/L) = \mathcal{U}(Y_1/L) \oplus \mathcal{U}(Y_1/L)(1)$.

Consider the projective, homogeneous G-variety X_6/k and pass to $k(X_6)$. By the index reduction formula in [MPW2], the index of the Tits algebra of G will be equal to 2 over $k(X_6)$. By the Tits classification this means that the anisotropic kernel of G either reduces to $D_4 \times A_1$ or A_1^3. The upper motive of the G-variety X_7 and thus Y_1 is in any case isomorphic to $\mathcal{U}(\mathrm{SB}(A))$ by Theorem 10.1.1 and Theorem 10.2.2. As $P(\mathcal{U}(\mathrm{SB}(A)),t) = 1+t$ holds in both cases, we are done.

□

10.4.4 Lemma. *Let G be a split algebraic group of type $\mathbf{HSpin}_{12}$. Then the Chow ring of the projective, homogeneous G-variety Y_2 up to codimension 9 is generated in the root enumeration by Stembridge by the four algebraic cycles $h = Z[5]$, $x_2 = Z[6,5]$, $x_4 = Z[1,3,4,5]$, $x_5 = Z[1,2,3,4,5]$.*

Proof: Using the *Chow maple package*, we find a representation of each element in $\mathrm{Ch}^i(Y_2)$ for $i < 10$ by the four generators using trial and error. We do not need the codimensions higher than 9, but with more effort one can show by the same method that these four elements completely generate $\mathrm{Ch}(Y_2)$. For the proof we simply give a table of the generators of $\mathrm{Ch}^i(Y_2)$ in terms of h, x_2, x_4, x_5 for each codimension i.

i	$\gamma_{i,1}$	$\gamma_{i,2}$	$\gamma_{i,3}$	$\gamma_{i,4}$	$\gamma_{i,5}$	$\gamma_{i,6}$
1	h	$-$	$-$	$-$	$-$	$-$
2	$h^2 + x_2$	x_2	$-$	$-$	$-$	$-$
3	h^3	hx_2	$-$	$-$	$-$	$-$
4	x_4	$h^4+h^2x_2+x_2^2 + x_4$	$h^2x_2 + x_2^2$	x_2^2	$-$	$-$
5	x_5	$hx_4 + x_5$	$h^3x_2 + hx_4 + x_5$	h^5	$-$	$-$
6	$hx_5 + x_4x_2$	x_4x_2	$hx_5 + h^2x_4 + x_4x_2$	$hx_5 + h^2x_2^2 + h^2x_4 + x_2^3 + x_4x_2$	x_2^3	$-$
7	$h^2x_5 + hx_4x_2 + x_5x_2$	x_5x_2	$hx_4x_2 + x_5x_2$	$h^3x_4 + h^2x_5 + x_5x_2$	$h^3x_4 + h^2x_5 + x_5x_2 + x_2^3$	$-$
8	$h^3x_5 + h^2x_4x_2 + x_4x_2^2$	$hx_5x_2 + x_4x_2^2$	$x_4x_2^2$	$h^2x_4x_2 + hx_5x_2 + x_4x_2^2$	$h^4x_4 + h^3x_5 + h^2x_4x_2 + x_4x_2^2$	$h^4x_4 + h^3x_5 + h^2x_4x_2 + x_4x_2^2 + x_2^4$
9	$h^4x_5 + h^3x_4x_2 + h^2x_5x_2 + x_5x_2^2$	$h^4x_5 + h^3x_4x_2 + hx_4x_2^2$	$h^4x_5 + h^3x_4x_2 + hx_4x_2^2 + x_5x_2^2$	$h^4x_5 + h^3x_4x_2$	$h^2x_5x_2 + x_5x_4$	$h^4x_5 + x_5x_4 + x_5x_2^2$

□

10.4.5 Lemma. *Let G be an algebraic group of type* $\mathbf{HSpin}_{12}$. *Consider the projective, homogeneous G-variety Y_2. Then the following holds for the coaction map ρ on* $\mathrm{Ch}(\overline{Y_2})$

1. $\rho(h) = 1 \otimes h$

2. $\rho(x_2) = e_1 \otimes h + 1 \otimes x_2$

3. $\rho(x_4) > e_3 \otimes h + 1 \otimes x_4$

4. $\rho(x_5) > e_5 \otimes 1 + 1 \otimes x_5$

Proof:

1. For first identity note that the Tits algebra assigned to $\omega_2 = h$, with $\langle h \rangle = \mathrm{Ch}^1(\overline{Y_2})$ by the Tits homomorphism, is always trivial by [MT95, 2.4.5]. Thus h is always rational and by applying [PS22, Lemma 4.12] the claim follows.

2. Assume we are given an E_7 with phase $[D_6,(1,1,1,0),4)]$. We pass to the the generic point of the D_6-variety Y_5, which is not a GSV by the GSV-table. Note that its upper motive is isomorphic to the one of the E_7-variety X_3. We obtain the phase $[A_1^3,(1,0,0,0),2]$ for the enveloping E_7 over $k(Y_5)$. By the phase classification and Karpenko's theorem, we have $\#G = \{\mathbb{F}_2, \mathcal{R}_{(1,0,0,0)}\}$ over $k(Y_5)$.

Applying the CGMB algorithm to $M(Y_2)$ when the E_7 enveloping G has phase $[A_1^3,(1,0,0,0),2]$ (input: prodbases([2, 4, 6], [1, 2, 3, 4, 6], D6)), we see that neither $\mathbb{F}_2(1)$ nor $\mathbb{F}_2(2)$ is contained in $M(Y_2)$ over $k(Y_5)$, but of course $\mathbb{F}_2$ as Y_2 is isotropic. So h is the starting point of a motive not isomorphic to $\mathbb{F}_2(1)$. Thus the Rost motive $\mathcal{R}_{(1,0,0,0)}(1)$ is contained in $M(Y_2)$ over $k(Y_5)$, because $\#G = \{\mathbb{F}_2, \mathcal{R}_{(1,0,0,0)}\}$ over $k(Y_5)$.

Thus, $\rho(x_2) = e_1 \otimes h + \sum a_i \otimes b_i$, since we just concluded that the generic point of the Rost motive is h. On the other hand, by dimensional reasons and by [PS22, Lemma 4.12] $\sum a_i \otimes b_i$ has to be $1 \otimes x_2$.

3. We assume that the given D_6 occurs as semisimple anisotropic kernel of an adjoint algebraic group of type E_7 denoted by G'. By the phase classification, the second entry of the motivic J_2-invariant of the enveloping E_7 equals 1. Thus the projective, homogeneous E_7-variety X_7 is not a GSV by the GSV-table.

Passing to $L := k(X_7)$, the D_6 becomes isotropic with semisimple anisotropic kernel of type D_4 and $J_2(G') = (0,1,0,0)$. Since there is only one phase which has a smaller anisotropic kernel than D_4, we have that $\#M(Y_2/L)$ contains only Tate motives and Rost motives $\mathcal{R}_{(0,1,0,0)}$ by Karpenko's theorem. We use the CGMB algorithm (input: prodbases([1, 2, 3, 4], [1, 2, 3, 4, 6], D6)) and see that there are only four Tate motives in $M(Y_2/L)$, given by

$$T(Y_2,t) := (1 + t^8 + t^9 + t^{17}).$$

Subtracting $T(Y_2,t)$ from $P(Y_2,t)$ shows that the copy of $\mathcal{R}_{(0,1,0,0)}$ in $M(Y_2)$ with smallest shift is $\mathcal{R}_{(0,1,0,0)}(1)$. Our whole argument is in principle the same as in the case above. We can apply [PS22, Theorem 6.4] to conclude that there is a cycle $\beta \in \mathrm{Ch}^4(\overline{Y_2})$, such that $\rho(\beta)$ contains $e_3 \otimes h$. For codimensional reasons this can only be some sum of x_4, x_2^2, h^4. By the two formulas established above, adding x_2^2 or h^4 to β does not change the fact that $e_3 \otimes h$ is contained in $\rho(x_4)$. Finally, the summand $1 \otimes x_4$ is contained in $\rho(x_4)$ by [PS22, Lemma 4.12].

4. This proof works the same as the prove of the fourth line in Lemma 9.3.8 by using the right exact sequence of split groups from it.

The semisimple part C of the Levi subgroup of the second parabolic subgroup is of the form $(\mathrm{Spin}_8 \times \mathrm{SL}_2)/\mu_2$ and $\mathrm{Ch}(C)$ has two generators e_1, e_3 with $e_1^2 = 0$ and $e_3^2 = 0$.

By the right exactness of sequence from Lemma 9.3.8 and the codimensions of the generators of $\mathrm{Ch}(Y_2)$ the claim follows. □

10.4.6 Lemma. *Let G be an adjoint algebraic group of type E_7 with Tits algebra A and having the phase $[E_7,(1,1,*,0),*]$ or $[D_6,(1,1,*,0),*]$ over k. Then the phase of G over $k(X_6)$ is $[D_4 \times A_1,(1,1,0,0),2]$.*

Proof: First note that by Theorem 6.2.2 $\mathrm{ind}(A) > 1$ holds. Applying the index reduction formula from [MPW2], we see that $\mathrm{ind}(A) = 2$ holds over $k(X_6)$. By the phase classification this leaves only two possibilities of phases that G can have over $k(X_6)$. Either we have $[D_4 \times A_1, (1,1,0,0), 2]$ or $[A_1^3, (1,0,0,0), 2]$. As X_3 is not a GSV by the GSV-table, the second phase is obtained over $k(X_3)$. By Lemma 10.3.1 the upper motives of X_6/k and X_3/k are not isomorphic, so it is impossible for G to have $[A_1^3, (1,0,0,0), 2]$ over $k(X_6)$. □

10.4.7 Lemma. *Let G be an adjoint algebraic group of type E_7 with semisimple, anisotropic kernel H of type D_6. If $J_2(G) = (1,1,1,0)$ holds, then the unique motivic decomposition of the projective, homogeneous H-varieties Y_4 and Y_5 into indecomposable motivic summands is given by*

$$M(Y_4) = \mathcal{U}(Y_5) \oplus \mathcal{U}(Y_5)(4) \oplus \mathcal{U}(Y_5)(8) \oplus \bigoplus_{i \in I_4} \mathcal{R}_{(1,1,1,0)},$$

$$M(Y_5) = \mathcal{U}(Y_5) \oplus \mathcal{U}(Y_5)(1),$$

$$\textit{with } P(\mathcal{U}(Y_5), t) = (1+t^2)(1+t^3)(1+t^4)(1+t^5)$$

$$\textit{and } P(I_4, t) = t(1+t)(1+t^2)(1+t^4)(1+t^2+t^4).$$

Proof: Note that the upper motives of Y_4 and Y_5 are isomorphic. We will only consider Y_5. Also by the Tits classification, Karpenko's theorem and the GSV-table we have $\#M(Y_i) \subset \{\mathcal{U}(Y_5), \mathcal{R}_{(1,1,1,0)}\}$ for $i = 4, 5$.

Tate motives in $\mathcal{U}(Y_5)$: First we pass to the generic point of the G-variety X_3. By the phase classification and the GSV-table the E_7 enveloping H has phase $[A_1^3, (1,0,0,0), 2]$. We calculate the Tate motives in $M(Y_4)$ and $M(Y_5)$ with the CGMB method (input: prodbases([2, 4, 6], [1, 2, 4, 5, 6], D6) for Y_4 for example) and obtain

$$T(Y_4, t) := (1+t^4+t^8)(1+t^5)(1+t^9),$$

$$T(Y_5, t) := (1+t)(1+t^5)(1+t^9).$$

As $\gcd(T(Y_4,t), T(Y_5,t)) = (1+t^5)(1+t^9) \in \mathbb{N}_0[t]$ and $\mathcal{U}(Y_4/k) \simeq \mathcal{U}(Y_5/k)$ holds, $M(Y_5/k)$ contains at least $\mathcal{U}(Y_5/k) \oplus \mathcal{U}(Y_5/k)(1)$. We calculate which copies of $\mathbb{F}_2$ in $T(Y_5,t)$ come from $\mathcal{U}(Y_5/k) \simeq \mathcal{U}(X_3/k)$ and code them by $T(\mathcal{U}(Y_5), t)$. First we pass from k to $k(\mathrm{SB}(A))$ and obtain the phase $[D_6, (0,1,1,0), 1]$ by the proof of Theorem 8.3.5. Now X_3 is a GSV for and thus $\dim(\mathcal{U}(X_3/k)) \geq \dim(R_{(0,1,1,0)}) = 8$ holds. This leaves only $(1+t^9)$ and $(1+t^5)(1+t^9)$ as candidates for $T(\mathcal{U}(Y_5), t)$. Consider the transitions

$$[D_6, (1,1,1,0), *] \quad \xrightarrow{X_6} \quad [D_4 \times A_1, (1,1,0,0), 2] \quad \xrightarrow{X_3} \quad [A_1^3, (1,0,0,0), 2].$$

The second transition holds by the GSV-table and the phase classification. For the first one consider also Lemma 10.3.1. As there are no phantom summands by the RNT, we can directly pass from k to $k(X_3)$ and see that the transition to $[A_1^3, (1,0,0,0), 2]$ virtually factors through $[D_4 \times A_1, (1,1,0,0), 2]$. Surely $\mathcal{U}(Y_5)$ splits completely into copies of $\mathcal{U}(X_q)$ (q has splitting pattern $[2,2]$) and $\mathcal{R}_{(1,1,0,0)}$ over $k(X_6)$ by Theorem 10.2.2. When $T(\mathcal{U}(Y_5), t) = (1+t^9)$ would hold, then over $k(X_6)$ at least $\mathcal{U}(X_q) \oplus \mathcal{U}(X_q)(4)$ would be split off from $\mathcal{U}(Y_5)$, since $\dim(\mathcal{U}(X_q)) = 5$, when

q is anisotropic. But each shift of $\mathcal{U}(X_q)$ splits off two Tate motives over $k(X_3)$, so $T(\mathcal{U}(Y_5),t)$ contains four summands. Also $\mathbb{F}_2(4)$ equating to t^4 is not contained in $T(Y_5,t)$. Therefore $T(\mathcal{U}(Y_5),t)$ equals $(1+t^5)(1+t^9)$.

There are no Rost motives in $M(Y_5/k)$: Since there are only Tate motives and shifts of $\mathcal{R}_{(1,0,0,0)}$ in $M(Y_5)$ over $k(X_3)$ by Theorem 10.1.1, we can calculate the Rost motives in $M(Y_5/k(X_3))$ by subtracting T_6 from $P(Y_5,t)$ and dividing by $P(\mathcal{R}_{(1,0,0,0)},t) = 1+t$. We obtain

$$O_6(t) := t^2+t^3+t^4+t^5+t^6+2t^7+t^8+t^9+t^{10}+t^{11}+t^{12}.$$

Since $M(Y_5/k)$ contains at least $\mathcal{U}(Y_5/k) \oplus \mathcal{U}(Y_5/k)(1)$ and we have just seen that $T(\mathcal{U}(Y_5),t) = (1+t^5)(1+t^9)$ holds, it follows that $\mathcal{U}(Y_5/k)$ splits into at least $\mathcal{U}(X_q) \oplus \mathcal{U}(X_q)(9)$ and thus splits off at least $\mathcal{R}_{(1,0,0,0)}(2) \oplus \mathcal{R}_{(1,0,0,0)}(11)$ and the mentioned Tate motives over $k(X_3)$. Note that $\dim(\mathcal{U}(Y_5/k)) = 14$ is even, while $\dim(\mathcal{R}_{(1,0,0,0)}) = 1$. So no shift of $\mathcal{R}_{(1,0,0,0)}$ is glued in the middle position of $\mathcal{U}(Y_5/k)$ and the number of copies of $\mathcal{R}_{(1,0,0,0)}$ glued to $\mathcal{U}(Y_5/k)$ is even. So, as $O_6(1) = 12$ holds, there are either two, four or six total copies of $\mathcal{R}_{(1,0,0,0)}$ glued to $\mathcal{U}(Y_5/k)$. Twelve is impossible, since we have already concluded that $M(Y_5/k)$ contains $\mathcal{U}(Y_5/k) \oplus \mathcal{U}(Y_5/k)(1)$, i.e. two sifts of $\mathcal{U}(Y_5/k)$. Subtracting the definite copies coded by $t^2(1+t)(1+t^9)$ from $O_6(t)$, it remains

$$O_6(t) - t^2(1+t)(1+t^9) = t^4+t^5+t^6+2t^7+t^8+t^9+t^{10}.$$

Remember that $O_6(t)$ does code shifts of $\mathcal{R}_{(1,0,0,0)}$. So if we want to check whether there can be shifts of $\mathcal{R}_{(1,1,1,0)}$ in $M(Y_5/k)$, we need to check whether $P(\mathcal{R}_{(1,1,1,0)},t)/P(\mathcal{R}_{(1,0,0,0)},t) = (1+t^3)(1+t^5)$ is a subpolynomial of the difference above, which is impossible. Thus $M(Y_5/k) = \mathcal{U}(Y_5/k) \oplus \mathcal{U}(Y_5/k)(1)$ holds.

The decomposition for $M(Y_4)$ now follows analogously by first considering $T(Y_4,t)$ and then subtracting $(1+t^4+t^8)P(\mathcal{U}(Y_5/k,t)$ from $P(M(Y_4/k),t)$ and dividing by $P(\mathcal{R}_{(1,1,1,0)},t)$.

$\square$

10.4.8 Lemma. *Let G be an adjoint algebraic group of type E_7 with semisimple, anisotropic kernel H of type D_6 and Tits algebra A. Assume $J_2(G) = (1,1,1,0)$ holds. Then the motive of the projective, homogeneous H-variety Y_2 decomposes into indecomposable motivic summands as*

$$M(Y_2) \simeq \mathcal{U}(Y_2) \oplus \textstyle\bigoplus_{i\in I} \mathcal{R}_{(1,1,1,0)}(i),$$

with $P(\mathcal{U}(Y_2),t) = (1+t-2t^3+t^5+t^6)(1+t^2+t^3+t^4+t^5+t^6+t^7+t^8+t^9+t^{11})$

$$\text{and } P(I,t) = t^2+t^3+t^4+t^5+t^6.$$

Proof: For the claim on $P(I,t)$ coding the Rost motives, we consider the cycles $x_2x_4x_5h^i$ in $\mathrm{Ch}(\overline{Y_2})$ for $i = 0,1,2,3,4$ and their image under the coaction. By the Lemma 10.4.5 the cycle $\rho(x_2x_4x_5)$ contains $e_1e_3e_5 \otimes h^2$ as biggest summand in the reasoning of the requirements of [PS22, Theorem 6.4]. Since $h^6 \neq 0$ and $h^7 = 0$ holds in $\mathrm{Ch}(\overline{Y_2})$, one obtains only four more shifts of $\mathcal{R}_{(1,1,1,0)}$ in each of the codimensions $2,3,4,5,6$. It also follows that there can not be more than these five

Rost motives contained in $M(Y_2/k)$, because there are no further generators of the needed codimensions in $\mathrm{Ch}(\overline{Y_2})$.

Structure of $\mathcal{U}(Y_2)$: We pass to $k(Y_2)$. By Lemma 10.4.6, the resulting phase of G is $[D_4 \times A_1, (1,1,0,0), 2]$. We use the CGMB algorithm on $M(Y_2)$ over $k(Y_2)$ (input: prodbases([1, 2, 3, 4, 6], [1, 2, 3, 4, 6], D6)), which shows that the following holds

$$M(Y_2) \simeq \mathbb{F}_2 \oplus M(Z_{1,3})(1) \oplus M(Z_4)(4) \oplus M(Z_1)(8) \oplus M(Z_{1,3})(9) \oplus \mathbb{F}_2(17)$$

for some projective, homogeneous varieties Z_Θ (in Bourbaki notation).

Now $\mathcal{U}(Z_4) \simeq \mathcal{U}(Y_4) \simeq \mathcal{U}(X_q)$, with q having splitting pattern $[2,2]$, holds by Lemma 8.3.2. Subtracting all of the Poincaré polynomials $P(Z_\Theta, t)$ except for $P(Z_1, t)$ from $P(Y_2, t)$, shows that $P(Z_1, t) = 1 + t$ holds. So $M(Z_1) \simeq \mathcal{U}(Y_1) \simeq M(\mathrm{SB}(D))$ for D being a division algebra with $\mathrm{ind}(D) = 2$ and $Z_{1,3}$ is a GSV by the GSV-table.

The upper motives of Y_1/k and $Y_4/k, Y_5/k$ have been calculated above and in Lemma 10.4.3, while $Y_3/k, Y_6/k$ are GSVs by the GSV-table. From the proofs of these lemmas we know how these decompose over $k(Y_2)$. We have that $M(Y_1/k) \simeq \mathcal{U}(Y_1/k)$ splits off two shifts of the upper motive of some Severi-Brauer variety over $k(Y_2)$.

Thus no shift of $\mathcal{U}(Y_1/k)$ is contained in $M(Y_2/k)$, provided there are no shifts of upper motives of Severi-Brauer varieties in $M(Z_4)$ over $k(Y_2)$. But this is clear, as passing to the generic point of $\mathrm{SB}(A)$ and using Theorem 10.3.5 we obtain that G has phase $[D_4, (0,1,0,0), 1]$ and thus Y_4 becomes a GSV and can not have such motivic summands over $k(Y_2)$.

The situation is similar for $\mathcal{U}(Y_5/k)$, which splits off $\mathcal{U}(X_q) \oplus \mathcal{U}(X_q)(9)$ by the proof of the lemma above (look at $T(Y_5, t)$) and has dimension 14. If we pass to $k(Y_4)$, we necessarily obtain phase $[A_1^3, (1,0,0,0), 2]$ for the E_7 enveloping H. Using the CGMB method again (input: prodbases([2, 4, 6], [1, 2, 3, 4, 6], D6)), we see that over $k(Y_2)(Y_4)$, $M(Y_2)$ has a Tate polynomial given by

$$1 + t^4 + t^8 + t^9 + t^{13} + t^{17}.$$

The four new Tate motives (i.e. not given by $1 + t^{17}$) can only come from the summand $M(Z_4)(4)$ above, as $\mathrm{SB}(D)$ does not become isotropic over $k(Y_2)(Y_4)$. Since $\mathcal{U}(Y_5/k)$ has dimension 14, we have proven that no shift of $\mathcal{U}(Y_5/k)$ is contained in $M(Y_2/k)$.

Finally, we show that the summand $M(\mathrm{SB}(D))(8)$ can not be seen over k (this is actually only an issue if $\mathrm{ind}(A/k) = 2$). We just pass to $k(\mathrm{SB}(A))$ and remember the proof of Theorem 8.3.5. It follows that H stays anisotropic and thus $M(\mathrm{SB}(D))(8)$ is glued to $\mathcal{U}(Y_2)$ over k. Also Y_2 has no zero cycles of odd degree over $k(\mathrm{SB}(A))$, since by the proof of the theorem its upper motive over $k(\mathrm{SB}(A))$ is isomorphic to the one of an anisotropic quadric.

□

10.4.9 Theorem. *Let G be an adjoint algebraic group of type E_7 with semisimple, anisotropic kernel of type D_6 given by* $\mathbf{HSpin}(A,\sigma)$ *and denoted by H. If $J_2(G) = (1,1,1,0)$ holds, then the unique motivic decomposition of the projective, homogeneous H-varieties Y_i, for $i = [1:6]$ into indecomposable motivic summands is given by*

i	$M(Y_i)$
1	$\mathcal{U}(\mathcal{I}(A,\sigma))$
2	$\mathcal{U}(\mathcal{I}(A,\sigma)_2) \oplus \bigoplus_{l\in I_2} \mathcal{R}_J(l)$
$3,6$	$\bigoplus_{l\in I_i} \mathcal{R}_J(l)$
4	$\mathcal{U}(\mathcal{I}(A,\sigma)_4) \oplus \mathcal{U}(\mathcal{I}(A,\sigma)_4)(4) \oplus \mathcal{U}(\mathcal{I}(A,\sigma)_4)(8) \oplus \bigoplus_{l\in I_4} \mathcal{R}_J(l)$
5	$\mathcal{U}(\mathcal{I}(A,\sigma)_4) \oplus \mathcal{U}(\mathcal{I}(A,\sigma)_4)(1)$

Index	**Poincaré Polynomial**
$\mathcal{U}(\mathcal{I}(A,\sigma))$	$(1+t+t^2+t^3+t^4+2t^5+t^6+t^7+t^8+t^9+t^{10})$
$\mathcal{U}(\mathcal{I}(A,\sigma)_2)$	$(1+t-2t^3+t^5+t^6)(1+t^2+t^3+t^4+t^5+t^6+t^7+t^8+t^9+t^{11})$
$\mathcal{U}(\mathcal{I}(A,\sigma)_4)$	$(1+t^2)(1+t^3)(1+t^4)(1+t^5)$
$\mathcal{R}_J$	$(1+t)(1+t^3)(1+t^5)$

Index	**Shift/Tate Polynomial**
I_2	$t^2+t^3+t^4+t^5+t^6$
I_i	$P(\mathcal{I}(A,\sigma)_i,t)/P(\mathcal{R}_J,t)$ *for* $i=3,6$
I_4	$t(1+t)(1+t^2)(1+t^4)(1+t^2+t^4)$

Proof: The $i = 3,6$ cases follow from the GSV-table. The other results are just Lemma 10.4.3, Lemma 10.4.7 and Lemma 10.4.8 above.

□

10.4.10. It comes as a surprise is that the motive of Y_5 contains no Rost motives. Our results allow the following corollary.

10.4.11 Corollary. *Let G be an adjoint algebraic group of type E_7. Assume its phase is equal to $\mathfrak{p} = [D_6,(1,1,1,0),*]$. Then the following statements on the projective, homogeneous G-varieties X_i and their upper motives hold*

X_i	$\mathrm{res}_{k(X_i)/k}(\mathfrak{p})$	$\mathrm{res}_{k(X_i)/k}(\mathcal{U}(X_i/k))$
X_3, X_4	$[A_1^3,(1,0,0,0),2]$	$\bigoplus_{t=0,5,9,14} \mathbb{F}_2(t) \oplus \bigoplus_{i=2,4,6,7,9,11} \mathcal{R}_{(1,0,0,0)}(i)$
X_6	$[D_4\times A_1,(1,1,0,0),2]$	$\bigoplus_{t=0,17} \mathbb{F}_2(t) \oplus \bigoplus_{l=4,8} \mathcal{U}(X_{q'})(l) \oplus$ $\mathcal{U}(\mathrm{SB}(A))(8) \oplus \bigoplus_{i\in I_6} \mathcal{R}_{(1,1,0,0)}(i)$
X_7	$[D_4,(0,1,0,0),1]$	$\bigoplus_{t=0,1,9,10} \mathbb{F}_2(t) \oplus \bigoplus_{i=2,3,4,5} \mathcal{R}_{(0,1,0,0)}(i)$

Index	**Poincaré Polynomial**
$\mathcal{U}(X_{q'})$	$(1+t^2)(1+t^3)$
$\mathcal{U}(\mathrm{SB}(A))$	$(1+t)$
$\mathcal{R}_{(1,0,0,0)}$	$(1+t)$
$\mathcal{R}_{(1,1,0,0)}$	$(1+t)(1+t^3)$
$\mathcal{R}_{(0,1,0,0)}$	$(1+t^3)$

Index	**Shift Polynomial**
I_6	$t(t^{12}-1)/(t-1)$

Proof: By the GSV-table it is clear that none of the considered varieties is a GSV. The claim on X_3 now follows from the phase classification and the calculations done in Lemma 10.4.7.

The decomposition in the X_6 case is easily derived from the proof of Lemma 10.4.8, where I_6 is named just $O(t)$.

The statements on X_7 follow from the GSV-table and the phase classification. By the proof of Theorem 8.3.5, the upper motive of X_7 becomes isomorphic to the upper motive of X_q, with q having splitting pattern $[2,4]$, decomposing as $\mathcal{U}(X_q)\oplus\mathcal{U}(X_q)(1)$ over $k(\mathrm{SB}(A))$ by Lemma 5.5.10. Thus passing to $k(\mathrm{SB}(A))$ and then passing to $k(\mathrm{SB}(A))(X_7)$ is the same from the motivic point of view as passing to $k(X_7)$ directly. The claim now follows from considering the decomposition of $\mathcal{U}(X_q)$ after passing to $k(X_q)$, which is also established in the proof of Lemma 5.5.10. □

10.4.12 Theorem. *Let G have phase $[D_6,(1,1,1,0),*]$. Then the following unique decompositions of the Chow motives of projective, homogeneous G-varieties into indecomposable motivic summands hold*

Θ	$M(X_\Theta)$
$\{1\}$	$\bigoplus_{t=0,33}\mathbb{F}_2(t)\oplus\bigoplus_{q=1,2,17,18}\mathcal{U}(X_3)(q)\oplus\mathcal{U}(X_6)(8)\oplus\bigoplus_{i\in I_1}\mathcal{R}_J(i)$
$\{3\}$	$\bigoplus_{u\in O_3}\mathcal{U}(X_3)(u)\oplus\bigoplus_{i\in I_3}\mathcal{R}_J(i)$
$\{4\}$	$\bigoplus_{u\in O_4}\mathcal{U}(X_3)(u)\oplus\bigoplus_{i\in I_4}\mathcal{R}_J(i)$
$\{6\}$	$\bigoplus_{u=0,25}\mathcal{U}(X_6)(u)\oplus\bigoplus_{q=10,14,18}\mathcal{U}(X_3)(q)\oplus\mathcal{U}(X_7)(16)\oplus\bigoplus_{i\in I_6}\mathcal{R}_J(i)$
$\{7\}$	$\mathcal{U}(X_7)\oplus\mathcal{U}(X_7)(17)\oplus\bigoplus_{i=6,8,10,12}\mathcal{R}_J(i)$
$\{1,6\}$	$\bigoplus_{u=0,8,25,33}\mathcal{U}(X_6)(u)\oplus\bigoplus_{q\in O_{1,6}}\mathcal{U}(X_3)(q)\oplus\bigoplus_{s=16,24}\mathcal{U}(X_7)(s)\oplus\bigoplus_{i\in I_{1,6}}\mathcal{R}_J(i)$

Index	**Poincaré Polynomial**
$\mathcal{U}(X_3)$	$(1+t^2)(1+t^3)(1+t^4)(1+t^5)$
$\mathcal{U}(X_6)$	$(1+t-2t^3+t^5+t^6)(1+t^2+t^3+t^4+t^5+t^6+t^7+t^8+t^9+t^{11})$
$\mathcal{U}(X_7)$	$(1+t^5)(1+t+t^2+t^3+t^4+t^5)$
$\mathcal{R}_J$	$(1+t)(1+t^3)(1+t^5)$

Index	**Shift/Tate Polynomial**
O_3	$(1+t+t^2)(1+t^6)(1+t^{10})(1+t^{15})$
O_4	$(1+t^4+t^8)(1+t^6)(1+t^{10})(1+t^{15})$
$O_{1,6}$	$(t+t^5+t^9)(1+t+t^9+t^{10}+t^{11}+t^{15}+t^{16}+t^{17}+t^{25}+t^{26})$
I_1	$t^8(t^2+t^3+t^4+t^5+t^6)$
I_i	$[P(X_i,t)-O_iP(\mathcal{U}(X_3),t)]/P(\mathcal{R}_J,t)$ *for* $i=3,4$
I_6	$[P(X_6,t)-(1+t^{25})P(\mathcal{U}(X_6),t)-(t^{10}+t^{14}+t^{18})P(\mathcal{U}(X_3),t)-t^{16}P(\mathcal{U}(X_7),t)]/P(\mathcal{R}_J,t)$
$I_{1,6}$	$[P(X_{1,6},t)-(1+t^8+t^{25}+t^{33})P(\mathcal{U}(X_6),t)-O_{1,6}P(\mathcal{U}(X_3),t)-(t^{16}+t^{24})P(\mathcal{U}(X_7),t)]/P(\mathcal{R}_J,t)$

Proof: We start with $M(X_1)$, by applying the CGMB algorithm. Executing prodbases([2, 3, 4, 5, 6, 7], [2, 3, 4, 5, 6, 7], E7), we obtain

$$M(X_1)=\mathbb{F}_2\oplus M(Y_5)(1)\oplus M(Y_2)(8)\oplus M(Y_5)(17)\oplus\mathbb{F}_2(33)$$

Now we can simply input the results on the motives of the projective, homogeneous

D_6-varieties Y_2, Y_5 proven in Lemma 10.4.7 and Lemma 10.4.8.

Calculation of $M(X_3)$ *and* $M(X_4)$: For the X_3 and X_4 decompositions, we pass from k to the generic point of X_3 and obtain phase $[A_1^3, (1,0,0,0), 2]$ by the proof of Lemma 10.4.7. As by Corollary 10.4.11, $\mathcal{U}(X_3)$ splits off Tate motives coded by $(1+t^5)(1+t^9)$, the polynomials O_3, O_4 are obtained by dividing the polynomials T_3 and T_4 from Theorem 10.1.1, by $(1+t^5)(1+t^9)$. From the Tits classification and Karpenko's theorem, we obtain $\#M(X_3/k) = \{\mathcal{U}(X_3), R_{(1,1,1,0)}\}$. So I_3, I_4 can be easily derived from O_3, O_4.

Calculation of $M(X_6)$ *and* $M(X_7)$: The decomposition of $M(X_6)$ and $M(X_7)$ are as easily obtained of $M(X_1)$. Executing prodbases([2, 3, 4, 5, 6, 7], [1, 2, 3, 4, 5, 7], E7) and prodbases([2, 3, 4, 5, 6, 7], [1, 2, 3, 4, 5, 6], E7), we obtain

$$M(X_6) = M(Y_2) \oplus M(Y_{1,6})(5) \oplus M(Y_4)(10) \oplus M(Y_1)(16) \oplus M(Y_{1,6})(17) \oplus M(Y_2)(25),$$

$$M(X_7) = M(Y_1) \oplus M(Y_6)(6) \oplus M(Y_1)(17).$$

Again we use the decompositions of Y_1, Y_2, Y_4, established in Lemma 10.4.3, Lemma 10.4.8 and Lemma 10.4.7. The other varieties occurring are GSVs by the GSV-table. Now subtracting the polynomials of the copies of the upper motives of Y_1, Y_2, Y_4 from $P(X_6, t)$ and $P(X_7, t)$ and dividing by the Poincaré polynomial of $\mathcal{R}_{(1,1,0,0)}$, one obtains the shifts of the Rost motives in $M(X_6)$ and $M(X_7)$.

Additionally, the upper motives of X_6/k and $X_{1,6}/k$ are isomorphic, as $X_{1,6}/k$ does not have a zero cycle of odd degree. To see this, pass to $k(\mathrm{SB}(A))$. By the proof of Theorem 8.3.5 $\mathcal{U}(X_{1,6})$ becomes isomorphic to the upper motive of an anisotropic quadric. Now use Springer's theorem.

Calculation of $M(X_{1,6})$: Step one is to pass to $k(X_6)$, to determine the copies of $\mathcal{U}(X_6)$ in $M(X_{1,6})$. The polynomial encoding the $\mathbb{F}_2(i)$ in $M(X_{1,6})$ over $k(X_6)$ is given by $(1+t^8)(1+t^{17})(1+t^{25})$, by Theorem 10.2.2. The $\mathbb{F}_2(i)$ split off by $\mathcal{U}(X_6)$ over $k(X_6)$ are given by $1+t^{17}$, by Corollary 10.4.11. Dividing the first by the second polynomial, shows that the copies of $\mathcal{U}(X_6)$ in $M(X_{1,6})$ are given by $V_{1,6} := (1+t^8+t^{25}+t^{33})$.

Now consider the field extensions $k(X_6)$, $k(X_7)$ and $k(X_6)(X_7)$. The phase of G over $k(X_7)$ and $k(X_6)(X_7)$ coincides, as the transition from k to $k(X_6)(X_7)$ factors through $k(X_6)$, where G has anisotropic kernel $D_4 \times A_1$ by Corollary 10.4.11.

Therefore some of the $\mathbb{F}_2(i)$ in $M(X_{1,6})$ over $k(X_7)$ are not coming from copies of $\mathcal{U}(X_7/k)$, but from copies of $\mathcal{U}(X_6/k)$. In Theorem 10.2.2 we have seen the motivic decomposition of $\mathcal{U}(X_6)$ over $k(X_6)$, is given by

1. $\bigoplus_{t=0,17} \mathbb{F}_2(t) \oplus \bigoplus_{l=4,8} \mathcal{U}(X_{q'})(l) \oplus \mathcal{U}(\mathrm{SB}(A))(8) \oplus \bigoplus_{i\in I_6} \mathcal{R}_{(1,1,0,0)}(i)$.

By Corollary 10.4.11 the upper of X_3 over $k(X_6)$ is isomorphic to $\mathcal{U}(X_{q'})$, for q' having splitting pattern $[2,2]$. However, passing to $k(X_7)$ does not make X_3 isotropic by the same corollary, but splits the Tits algebra A of G. Therefore $\mathcal{U}(\mathrm{SB}(A))$ splits as $\mathbb{F}_2 \oplus \mathbb{F}_2(1)$ over $k(X_7)$. We obtain that the $\mathbb{F}_2(i)$, which are split off from $\mathcal{U}(X_6)$ over $k(X_7)$ are given by $T_7 := (1+t^8+t^9+t^{17})$. The $\mathbb{F}_2(i)$ coming from all the shifts of $\mathcal{U}(X_6)$ contained in $M(X_{1,6}/k)$, are given by the product $V_{1,6}(1+t^8+t^9+t^{17})$.

Subtracting this product from $T_{1,6}$ given in Theorem 9.1.1 leaves

$$T_{1,7} - V_{1,6}(1 + t^8 + t^9 + t^{17}) = (t^{16} + t^{17} + t^{24} + 2t^{25} + t^{26} + t^{33} + t^{34}).$$

Dividing by T_7 shows, that there are two copies of $\mathcal{U}(X_7)$ in $M(X_{1,6}/k)$, given by $t^{16} + t^{24}$. Lastly we need to calculate the copies of $\mathcal{U}(X_3)$ in $M(X_{1,6}/k)$. As X_7 stays anisotropic over $k(X_3)$ by Corollary 10.4.11, we only need to consider the behavior of $\mathcal{U}(X_6)$ over $k(X_3)$. We proceed analogously to the case when passing to $k(X_7)$ and consider *1.* again. The $\mathbb{F}_2(i)$ split off by $\mathcal{U}(X_{q'})$ over $k(X_3)$ are given by $(1+t^5)$. After putting this into *1.*, we see that the $\mathbb{F}_2(i)$ split off by $\mathcal{U}(X_6)$ over $k(X_3)$ are given by

$$1 + t^4(1 + t^5) + t^8(1 + t^5) + t^{17}.$$

Multiplying with $V_{1,6}$ gives the $\mathbb{F}_2(i)$ split off from all copies of $\mathcal{U}(X_6/k)$ in $M(X_{1,6}/k)$ over $k(X_3)$. The $\mathbb{F}_2(i)$ in $M(X_{1,6})$ over $k(X_3)$ are coded by (another) $T_{1,6}$ given in Theorem 10.1.1. Also, the $\mathbb{F}_2(i)$ split off by $\mathcal{U}(X_3)$ over $k(X_3)$ are given by $(1+t^5)(1+t^9)$ by Corollary 10.4.11. Subtracting $V_{1,6}(1+t^4(1+t^5)+t^8(1+t^5)+t^{17})$ from $T_{1,6}$ given in Theorem 10.1.1 and dividing by $(1+t^5)(1+t^9)$, finally proves the claim.

□

10.4.13 Remark. (More on ρ) The theorem above gives us more information on the coaction ρ on $\mathrm{Ch}(\overline{X_1})$. Since there are Rost motives in $M(X_1)$, in the theorem above, we can use the results from Lemma 9.3.8 and [PS22, Theorem 6.4], to conclude that $\rho(x_9)$ does contain $e_1 \otimes h^8$ as summand. Also we see that the cycles $x_4x_6x_9h^l$ are mapped to $e_1e_3e_5 \otimes h^{10+l} + \sum a_i \otimes b_i$ by ρ, for $l \in [0:4]$. As $x_4x_6x_9h^2$ is dual to h^{12}, we found a cycle β for which $\rho(\beta) = E_J \otimes \beta^* + \sum a_i \otimes b_i$ holds, where β^* is some cycle dual to β.

10.5 The phase [E7,(1,1,1,1),*]

In this chapter we culminate many of the previous results into proving the motivic indecomposability of X_1 and X_7, in case G has maximal J_2-invariant. Unfortunately we are unable to provide the decompositions for the other projective, homogeneous E_7-varieties, which are not GSVs. We are limited to point out several restrictions on their motivic decompositions. We first establish a compilation of lemmas, to generalize the index reduction formula for the G-variety X_1 in the sense that we take into account the value of $J_2(G)$ and $\mathrm{ind}(A)$. This completely answers the question on the behavior of G over $k(X_1)$, when its Tits algebra has index 2. From this formula we obtain several results about isomorphisms of some upper motives and use it for our proofs. Similar results can also be obtained without using the many lemmas, provided G is a versal form. This shows that in the treated cases, the index of the Tits algebra of G is irrelevant for the motivic decomposition type. This is not a triviality by [GSV, Theorem 4.2] and [GSV, Remark 4.3], concerning the rationality of some algebraic cycles.

10.5.1 Lemma. *Let G be an anisotropic adjoint algebraic group of type E_7 with $J_2(G) = (1,1,1,0)$. Then the only projective, homogeneous G-variety which could have a zero cycle of odd degree is X_1. If $J_2(G) = (1,1,1,1)$ holds, no projective, homogeneous G-variety has a zero cycle of odd degree.*

Proof: Assume $J_2(G) = (1,1,1,1)$ holds and assume that a certain X_Θ has a zero cycle of odd degree over k. By the property **1.** of the motivic J-invariant, there is a field extension L/k of odd degree such that X_Θ becomes isotropic, while $J_2(G/L) = (1,1,1,1)$ holds. By Example 6.2.8 this is impossible. When $J_2(G) = (1,1,1,0)$ holds, we can repeat the argument and see that there needs to be an isotropic group of type E_7 with anisotropic kernel of type other than D_6, having $J_2(G) = (1,1,1,0)$ for $\Theta \neq \{1\}$ to contradict our statement. This is impossible by the phase classification.

□

10.5.2 Lemma. *Let G be an anisotropic adjoint algebraic group of type E_7 with $J_2(G) = (1,1,*,*)$. Then over $k(X_{1,6})$ the anisotropic kernel of G is of type $D_4 \times A_1$.*

Proof: By the phase classification we have to rule out the possibilities of the anisotropic kernel being D_4, A_1^3 or split. However by the index reduction formula from [MPW2] one has $\mathrm{ind}(A/k(X_{1,6})) = \min(2, \mathrm{ind}(A/k))$, which equals 2 in our case. Thus G does not have anisotropic kernel of type D_4 or is split, as it is strongly inner in that case. If one would obtain anisotropic kernel A_1^3 over $k(X_{1,6})$, the upper motives of $X_{1,6}$ and X_3 would be isomorphic, since X_3 is not a GSV by the GSV-table. Now use Lemma 10.3.1 to finish the proof. □

10.5.3. In sight of the phase classification, the upper lemma can be reformulated as saying that the upper motives of X_3 and $X_{1,6}$ are isomorphic if and only if the value of $J_2(G)$ equals $(1,0,0,0)$. The following lemmas imply that the same holds for the upper motives of X_3 and X_1. The very next lemma shows that even if the value of $J_2(G)$ is not maximal one needs to consider $X_{1,6}$ as well for calculating $\#G$.

10.5.4 Lemma. *Let G be an anisotropic adjoint algebraic group of type* E_7 *with $J_2(G) = (1,1,1,*)$ or* $\mathrm{ind}(A) > 2$. *Then $\mathcal{U}(X_1)$ and $\mathcal{U}(X_{1,6})$ are not isomorphic.*

Proof: Assuming $\mathrm{ind}(A) > 2$, we have that $J_2 = (1,1,1,*)$ holds by the phase classification. So we can ignore $\mathrm{ind}(A)$ for the rest of the proof. Also by the Lemma 10.5.1, it follows that $X_{1,6}$ has no zero cycles of odd degree over k. Over $k(\mathrm{SB}(A))$ the group G does not have anisotropic kernel D_4 by the proof of Theorem 8.3.5. If the anisotropic kernel of G over $k(\mathrm{SB}(A))$ is D_6 the claim becomes obvious, as X_1 is isotropic, but $X_{1,6}$ has no zero cycle of odd degree because its upper motives is now isomorphic to $\mathcal{U}(X_q)$ for a anisotropic quadratic form q by Theorem 9.2.1. If it is E_7, pass to the generic point of X_1, over which the anisotropic kernel will reduce to D_6 by Lemma 9.3.2. Then apply the same argument. □

10.5.5 Lemma. *Let G be an anisotropic adjoint algebraic group of type E_7 with $J_2 = (1,1,1,*)$ and* $\mathrm{ind}(A) = 2$. *Then the semisimple, anisotropic kernel of G is of type D_6 over $k(X_1)$.*

Proof: Since $\mathcal{U}(X_1)$ and $\mathcal{U}(X_{1,6})$ are not isomorphic by the lemma above, the fact that over $k(X_{1,6})$ the anisotropic kernel of G will reduce to $D_4 \times A_1$ and the index reduction formula from [MPW2], stating that $\operatorname{ind}(A) = 2$ over $k(X_1)$, we only need to rule out the possibility of the anisotropic kernel of G becoming A_1^3, by the Tits classification. This is covered by Lemma 10.3.1, as otherwise the upper motives of $\mathcal{U}(X_3)$ and $\mathcal{U}(X_{1,6})$ are isomorphic.

□

10.5.6. This series of lemmas allows an interesting corollary. Finally the question on the anisotropic kernel of G over $k(X_1)$ can be completely answered. Some of the results below are already known, but we include them to provide a complete overview. Remember that it is not known whether the phases $[E_7, (1,1,1,0), *]$ for $*$ denoting any value in $\{2,4,8\}$ are admissible.

10.5.7 Corollary. (**Generalized index reduction formula for** X_1). *Given an adjoint algebraic group G of type E_7 over a field k with characteristic unequal to 2, Tits algebra A and phase $\mathfrak{p}$. Consider the G-variety X_1 and pass to $k(X_1)$. Then the following transitions from $\mathfrak{p}$ to $\operatorname{res}_{k(X_1)/k}(\mathfrak{p})$ hold.*

$\mathfrak{p}$	$\operatorname{res}_{k(X_1)/k}(\mathfrak{p})$
$[E_7, (1,1,1,*), 4/8]$	$[D_6, (1,1,1,0), 4]$
$[E_7, (1,1,1,*), 2]$	$[D_6, (1,1,1,0), 2]$
$[E_7, (1,1,0,0), 2]$	$[D_4 \times A_1, (1,1,0,0), 2]$
$[E_7, (1,0,0,0), 2]$	$[A_1^3, (1,0,0,0), 2]$
$[E_7, (0,1,1,1), 1]$	$[D_6, (0,1,1,0), 1]$

Proof: The first line follows from the the classic index reduction formula from [MPW2] and the phase classification, as there is only one isotropic Tits index for E_7 with $\operatorname{ind}(A) = 4$. The second line is proven by using the lemma above, along with the classic index reduction formula from [MPW2] and the phase classification. The third statement follows from Lemma 10.5.2 and noting that the upper motives of X_1 and $X_{1,6}$ are isomorphic in this case, because otherwise the upper motives of X_1 and X_3 would be isomorphic, contradicting Lemma 10.3.1. The fourth statement follows from the GSV-table and the phase classification. The last statement is basically Lemma 9.3.2, along with the phase classification. □

10.5.8 Theorem. *Let G be an algebraic group of type E_7, with $J_2(G) = (1,1,1,1)$. Then the motive of the projective, homogeneous G-variety X_1 is indecomposable.*

Proof: For the first step let us assume that there are no Rost motives $\mathcal{R}_{(1,1,1,1)}$ in $M(X_1/k)$. Passing to $k(\mathrm{SB}(A))$ yields the G-phase $[E_7, (0,1,1,1), 1]$ by Zhykhovich's theorem and the phase classification. In Theorem 9.3.5 we have seen that in case G has this phase, firstly $\#M(X_1) = \{\mathcal{U}(X_1), \mathcal{R}_{(0,1,1,1)}\}$ holds and that secondly there is only one copy of $\mathcal{U}(X_1)$ contained in $M(X_1)$. Since we assume that there are no Rost motives in $M(X_1/k)$, all Rost motives $\mathcal{R}_{(0,1,1,1)}$ occurring in $M(X_1)$ over $k(\mathrm{SB}(A))$ are glued to $\mathcal{U}(X_1/k)$ or come from the upper motive of X_3. The latter holds since, $\mathcal{U}(X_3/k)$ splits completely into shifts of $\mathcal{R}_{(0,1,1,1)}$ over $k(\mathrm{SB}(A))$, as X_3 is a GSV when $j_1 = 0$ holds by the GSV-table. We first give a proof for showing that there are no Rost motives in $M(X_1/k)$ and then address the issue with $\mathcal{U}(X_3/k)(i)$ possibly occurring in $M(X_1/k)$.

No Rost motives: By Theorem 6.2.2 the Tits algebra A/k of G/k has index at least 2 in case $j_1 = 1$ holds. Using the generalized index reduction formula for X_1, it follows that over $k(X_1)$ the phase of G changes to $[D_6, (1, 1, 1, 0), *]$ with $*$ depending on the index of A/k. We use a lifting argument of the motivic decomposition of X_1 over $k(X_1)$ established in Theorem 10.4.12. The polynomial $P(I_1, t) = t^8(t^2 + t^3 + t^4 + t^5 + t^6)$, which encodes the shifts of the Rost motives $\mathcal{R}_{(1,1,1,0)}$ contained in $M(X_1)$ over $k(X_1)$, has no subpolynomial divisible by $(1 + t^9)$ in $\mathbb{N}_0[t]$. However, the Rost motive $\mathcal{R}_{(1,1,1,1)}$ splits into $\mathcal{R}_{(1,1,1,0)} \oplus \mathcal{R}_{(1,1,1,0)}(9)$ over $k(X_1)$. Thus no shift of the Rost motive $\mathcal{R}_{(1,1,1,1)}$ is contained in $M(X_1/k)$.

There is no $\mathcal{U}(X_3/k)$ in $M(X_1/k)$: Let us assume that at least one shift of $\mathcal{U}(X_3/k)$ occurs in $M(X_1/k)$. We pass to $L := k(X_1)$ again and obtain $[D_6, (1, 1, 1, 0), *]$ as above. Note that $P(\mathcal{U}(X_3/k), t) \neq P(\mathcal{U}(X_3/L), t)$ must hold, because if these polynomials were equal, $P(\mathcal{R}_{(0,1,1,1)}, t)$ would divide $P(\mathcal{U}(X_3/L), t)$ in $\mathbb{N}_0[t]$ since X_3 becomes a GSV over $k(\mathrm{SB}(A))$. But this is not the case, as clearly can be seen by comparing $P(\mathcal{U}(X_3/L), t) = (1 + t^2)(1 + t^3)(1 + t^4)(1 + t^5)$ from Theorem 10.4.12 and $P(\mathcal{R}_{(0,1,1,1)}, t) = (1 + t^3)(1 + t^5)(1 + t^9)$.

Also by Theorem 10.4.12, the following copies of $\mathcal{U}(X_3/L)$ and $\mathcal{U}(X_6/L)$ occur in $M(X_1/L)$

$$\mathcal{U}(X_3)(1) \oplus \mathcal{U}(X_3)(2) \oplus \mathcal{U}(X_6)(8) \oplus \mathcal{U}(X_3)(17) \oplus \mathcal{U}(X_3)(18).$$

We need to check any combination of gluing these and then prove that the respective gluing is impossible. As $M(X_3/L)$ does not contain a shift of $\mathcal{U}(X_6)$, by the same theorem, we only need to check combinations of the $\mathcal{U}(X_3)(i)$. By the symmetry of the decomposition there are the following possibilities for the polynomial $O_1(t)$, coding the shifts of $\mathcal{U}(X_3/L)$ split off by $\mathcal{U}(X_3/k)$ when passing to L.

1. $O_1(t) = (1 + t)$
2. $O_1(t) = (1 + t^{16})$
3. $O_1(t) = (1 + t + t^{16} + t^{17})$

Note that $O_1(t) = (1 + t^{17})$ (as well as $(1 + t^{15})$) is no option, as this means that $\mathcal{U}(X_3)(2) \oplus \mathcal{U}(X_3)(17)$ is also glued and isomorphic to the upper motive of another projective, homogeneous G-variety X_Θ than X_3 over k. But as $\mathcal{U}(X_\Theta)$ becomes isomorphic to $\mathcal{U}(X_3)$ over L, this is impossible by the results presented in Theorem 10.4.12.

We also consider the possibility that some of the Rost motives in $M(X_1/L)$, coded by $r(t) := t^8(t^2 + t^3 + t^5 + t^6)$, come from some $\mathcal{U}(X_3/k)(i)$. This gives us $P(\mathcal{U}(X_3/k)) = O_1(t)P(\mathcal{U}(X_3/L), t) + s(t)P(\mathcal{R}_{(1,1,1,0)}, t)$ as candidate for the Poincaré polynomial of $\mathcal{U}(X_3/k)$, for some subpolynomial $s(t)$ of $r(t)$, including the possibility $s(t) = 0$. But $P(\mathcal{U}(X_3/k), t)$ must be divisible by $P(\mathcal{R}_{(0,1,1,1)}, t)$, because X_3 becomes a GSV when we pass from k to $k(\mathrm{SB}(A))$. By try and error we find that none of these polynomials divide the polynomial in question in $\mathbb{N}_0[t]$ (since one can not establish the factor $(1+t^9)$ which occurs in $P(\mathcal{R}_{(0,1,1,1)},t))$. We are done.

□

10.5.9 Remark. One can also use Theorem 10.3.5 for proving that there is only one copy of $\mathcal{U}(X_1/k)$ in $M(X_1/k)$ instead of using Zhykhovich's theorem. It follows that the semisimple anisotropic kernel of G over $k(\mathrm{SB}(A))$ is either E_7 or D_6. In the first case we are done. In the second case one concludes from the decomposition of $M(X_1)$ in Theorem 9.2.1 that all motivic summands in $M(X_1)$, which are not Rost motives, are glued over k. But this approach demands some deeper combinatorial considerations.

10.5.10 Theorem. *Let G be an algebraic group of type E_7, with $J_2(G)=(1,1,1,1)$. Then the motive of the projective, homogeneous G-variety X_7 is indecomposable.*

Proof: Passing from k to $k(\mathrm{SB}(A))$ yields the anisotropic kernel E_7 by Zhykhovich's theorem and the phase classification. In Theorem 9.3.5 we have seen that $M(X_7)=\mathcal{U}(X_7)\oplus\mathcal{U}(X_7)(1)$ holds over $k(\mathrm{SB}(A))$, with $\mathcal{U}(X_7)$ being no Tate motive. Also, passing from k to $k(X_3)$ yields the phase $[A_1^3,(1,0,0,0),2]$ by the GSV-table and the phase classification. By Theorem 10.1.1, $\mathcal{U}(X_7)\simeq\mathcal{U}(\mathrm{SB}(A))$ with Poincaré polynomial equal to $1+t$ holds over $k(X_3)$. □

10.5.11 Remark. In [Hen], the structure of the Chow rings of X_6 and X_3 in terms of generators and relations has been determined very recently. The first one is generated by five elements, which we denote by h,x_2,x_4,x_5,x_9 using the usual nomenclature. For $\mathrm{Ch}(X_3)$ the generators are h,x_2,x_3,x_4,x_5,x_9. An expression in Weyl coordinates can also be found in the reference.

10.5.12 Lemma. *Let G be an adjoint algebraic group of type E_7. Consider the projective, homogeneous G-variety X_6. Then the following holds for the coaction map ρ* on $\mathrm{Ch}(\overline{X_6})$

1. $\rho(h)=1\otimes h$

2. $\rho(x_2)=e_1\otimes h+1\otimes x_2$

3. $\rho(x_4)>e_3\otimes h+1\otimes x_4$

4. $\rho(x_5)>e_5\otimes 1+1\otimes x_5$

5. $\rho(x_9)>e_9\otimes 1+1\otimes x_9$

Proof:

1. For first identity note that the Tits algebra of ω_6 is always trivial by [MT95, 2.4.5]. Thus h is always rational and by applying [PS22, Lemma 4.12] the claim follows.

2. We consider the motivic decomposition of $M(X_6)$ in Theorem 10.1.1. We see that there is the Rost motive $\mathcal{R}_{(1,0,0,0)}(1)$ contained in it. Using [PS22, Theorem 6.4], it follows that there is a cycle β in $\mathrm{Ch}(\overline{X_6})$, such that $\rho(\beta)>e_1\otimes h$. Because of codimensional reasons, x_2 is an admissible choice for β.

3. We consider the motivic decomposition of $M(X_6)$ in Theorem 9.1.1. We see that there is the Rost motive $\mathcal{R}_{(0,1,0,0)}(1)$ contained in it. Using [PS22, Theorem 6.4], it follows that there is a cycle β in $\mathrm{Ch}(X_6)$, such that $\rho(\beta) > e_3 \otimes h$. Because of codimensional reasons, x_4 is an admissible choice for β.

4. and *5.* We determine the interesting summand of the coaction for each of these cases analogously to the last line in the proof of Lemma 9.3.8. Then C is a central product of $D_5 \times A_1$. Since this C occurs as the semisimple anisotropic kernel of E_7^{ad}, its Chow ring has only two generators e_1, e_3 by the very same arguments of the last proof in Lemma 10.4.5.

Using the same right exact sequence as in the proof of Lemma 9.3.8, we can show that e_5, e_9 in $\mathrm{Ch}(E_7^{ad})$ map to zero in $\mathrm{Ch}(C)$. Then the claim follows analogously.

□

10.5.13 Theorem. *Let G be an adjoint algebraic group of type E_7, with motivic J_2-invariant $J_2(G)$. When $J_2(G)$ equals $(1,1,1,1)$, the motive of the projective, homogeneous G-variety X_6 contains exactly one copy of $\mathcal{U}(X_6)$. The only other possible indecomposable motivic summands in $M(X_6)$ are shifts of the Rost motive $\mathcal{R}_{(1,1,1,1)}$.*

Proof: The claim on the Rost motives follows from the lemma above and [PS22, Theorem 6.4]. We multiply the four generators of codimension bigger than 1 and obtain a cycle β. Now, $\rho(h^i\beta) > E_J \otimes h^{2+i}$ holds and E_J uniquely has the biggest codimension of all a with $a \otimes b < \rho(h^i\beta)$ for $i \in [0,12]$.

For the claim on $\mathcal{U}(X_6)$, consider the Tits algebra A of G. It is not split by Theorem 6.2.2. We pass to $L := k(\mathrm{SB}(A))$. By Zhykhovich's theorem and the phase classification, we obtain $[E_7, (0,1,1,1), 1]$. By Theorem 9.3.5, there are three copies of $\mathcal{U}(X_6/L)$ contained in $M(X_6/L)$ of which the two outer ones (in terms of shifts) are necessarily glued over k. To see this, we pass to $k(X_6)$. By the proof of Lemma 10.4.6 and the Tits classification, the semisimple anisotropic kernel of G is either $D_5 \times A_1$ or $D_4 \times A_1$. In the first case we have seen in Theorem 10.3.3 that there are only two Tate motives in $M(X_6)$. Thus the claim follows. In the second case we check Theorem 10.2.2 to see that the Tate motives in $M(X_6)$ are in this case given by $T_6 := 1 + t^{17} + t^{25} + t^{42}$. But $\mathcal{U}(X_6/L)$ has dimension 26 by Theorem 9.3.5. Thus $\mathcal{U}(X_6/k)$ has the same dimension as X_6.

We need to show, that $\mathcal{U}(X_6/L)(8)$ in the middle of $M(X_6/L)$, is also glued to $\mathcal{U}(X_6/k)$ over k. But this mostly follows from the table in Theorem 9.3.5 and Theorem 9.3.10, which enlist the copies of $\mathcal{U}(X_6/L)$ contained in each projective, homogeneous G-variety over L and and from Karpenko's theorem. The only cases to consider are, whether $\mathcal{U}(X_7/k)$ or $\mathcal{U}(X_{1,6}/k)$ could be contained in $M(X_6/k)$. The motive of X_7 is indecomposable over k by the theorem right before the lemma above and splits into two shifts of $\mathcal{U}(X_6/L)$ over L. So we are left with the $X_{1,6}$ case. Let us pass to $k(X_6)$. By the proof of Lemma 10.4.6 and the Tits classification, the semisimple anisotropic kernel of G is either $D_4 \times A_1$ or $D_5 \times A_1$.

In the first case the upper motives of X_6 and $X_{1,6}$ are isomorphic, as by using the proof of Lemma 10.4.6 and the Tits classification again, one definitely has kernel

$D_4 \times A_1$ over $k(X_{1,6})$. Since we have already established that $\mathcal{U}(X_6/k)$ has the same dimension as X_6, the summand $\mathcal{U}(X_6/L)(8)$ is not glued to a copy of $\mathcal{U}(X_{1,6}/k)$ inside of $M(X_6/k)$, unless it is completely isomorphic $\mathcal{U}(X_{1,6}/k)$.

In the second case the only possibility for $\mathcal{U}(X_{1,6}/L)(8)$ to be visible in $M(X_6/k)$ and not glued to anything else, is also to be isomorphic to $\mathcal{U}(X_{1,6}/k)$. This indicates that the dimension of $\mathcal{U}(X_{1,6}/k)$ is 26. Checking Theorem 10.2.2, where G has semisimple anisotropic kernel $D_4 \times A_1$, we see that there is no such Tate motive contained in $M(X_{1,6})$.

$\square$

10.5.14 Lemma. *Let G be an adjoint algebraic group of type E_7. Consider the projective, homogeneous G-variety X_3. Then the following holds for the coaction map ρ on* $\mathrm{Ch}(\overline{X_3})$

1. $\rho(h) = 1 \otimes h$

2. $\rho(x_2) = 1 \otimes x_2$

3. $\rho(x_3) > e_3 \otimes 1 + 1 \otimes x_3$

4. $\rho(x_4) > e_1 \otimes \alpha + 1 \otimes x_4$, *for some* $0 \neq \alpha \in \mathrm{Ch}^3(X_3)$.

5. $\rho(x_5) > e_5 \otimes 1 + 1 \otimes x_5$

6. $\rho(x_9) > e_9 \otimes 1 + 1 \otimes x_9$

Proof:

1. For first identity note that the Tits algebra of ω_3 is always trivial by [MT95, 2.4.5]. Thus h is always rational and by applying [PS22, Lemma 4.12] the claim follows.

2. In Theorem 10.1.1, we see that $M(X_3)$ does not contain $\mathcal{R}_{(1,0,0,0)}(1)$, when $J_2(G) = (1,0,0,0)$. Thus by [PS22, Theorem 6.4], x_2 can not contain $e_1 \otimes h$.

3. We consider the motivic decomposition of $M(X_3)$ in Theorem 9.1.1. By the GSV-table, it follows that X_3 is a GSV with upper motive $\mathcal{R}_{(0,1,0,0)}$. Now the claim follows from considering [PS22, Theorem 6.4] and the codimensions of the generators of $\mathrm{Ch}(X_3)$.

4. Assume G has the phase $[D_4 \times A_1, (1,1,0,0), 2]$. We can use Theorem 10.2.2 and see that $M(X_3)$ contains a Rost motive $\mathcal{R}_{(1,1,0,0)}(3)$ and that this is the Rost motive with smallest shift in $M(X_3)$. By [PS22, Theorem 6.2] there is an cycle $\beta \in \mathrm{Ch}^7(\overline{X_3})$, such that the $\rho(\beta)$ contains a cycle $e_1e_3 \otimes \alpha$, with $\alpha \in \mathrm{Ch}^3(\overline{X_3})$. By the formula above, the codimensions of the generators of $\mathrm{Ch}(\overline{X_3})$ and the fact that 3 is the smallest shift of $\mathcal{R}_{(1,1,0,0)}$ in $M(X_3)$, we see that the $e_1 \otimes \alpha$ portion comes from $\rho(x_4)$ and x_3x_4 is one choice for β. Adding other cycles of codimension 7 to β does not change this. Note that α is equal to some sum of h^3, x_2h, x_3, as these are the generators of $\mathrm{Ch}^3(\overline{X_3})$. It can not be x_3 by itself, as this would make x_3 rational, when ever $J_2(G) \leq (1,1,0,0)$ holds component wise. This is impossible by the formula above.

5. and *6.* We determine the interesting summand of the coaction for each of these analogously to the last proof in Lemma 9.3.8. The parameter C is a central product of $A_5 \times A_1$. Since groups of type A_n have only one generator e_1, we have that $\mathrm{Ch}(C)$ has also only one or two generators e_1, e_1'. Using the same right exact sequence as in the proof of Lemma 9.3.8, we can show that e_5, e_9 in $\mathrm{Ch}(E_7^{ad})$ map to zero in $\mathrm{Ch}(C)$. Then the claim follows analogously.

□

10.5.15 Theorem. *Let G be an adjoint algebraic group of type E_7, with motivic J_2-invariant $J_2(G)$. When $J_2(G)$ equals $(1,1,1,1)$, the motive of the projective, homogeneous G-variety X_3 contains only shifts of $\mathcal{U}(X_3)$ and of the Rost motive $\mathcal{R}_{(1,1,1,1)}$.*

Proof: The claim on the Rost motives follows from the lemma above and [PS22, Theorem 6.4]. We can put $x_3x_4x_5x_9$ into $\rho(-)$ for example, to conclude that $\mathcal{R}_{(1,1,1,1)}(3)$ is contained in $M(X_3/k)$, if the cycles e_i do not cancel out. Note that even though we have not determined the α in $\rho(x_4)$, every possible combination of the cycles discussed above does not contradict our result, as $\rho(x_3), \rho(x_5), \rho(x_9)$ all contain $e_i \otimes 1$ as summand of the form $a \otimes b$ with a having biggest codimension. Since the only b unequal to zero is alpha, we have that the generic point of the Rost motive $\mathcal{R}_{(1,1,1,1)}(3)$ is α (which is not zero by the lemma above).

For the claim on $\mathcal{U}(X_3)$, consider the Tits algebra A of G. It is not split by Theorem 6.2.2. We pass to $L := k(\mathrm{SB}(A))$. By Zhykhovich's theorem and the phase classification, we obtain $[E_7, (0,1,1,1), 1]$. By the GSV-table, X_3/L is a GSV. Thus any indecomposable motivic summand N in $M(X_3/k)$ splits into shifts of Rost motives over L. By Karpenko's theorem, N is isomorphic to the shift of $\mathcal{U}(Y/k)$ for some appropriate projective, homogeneous variety Y/k. As N becomes isomorphic to $\mathcal{R}_{(0,1,1,1)}$ over L, we have that Y/k is either a GSV it becomes one over L. In the first case $\mathcal{U}(Y/k)$ is isomorphic to $\mathcal{R}_{(1,1,1,1)}$. In the second case $\mathcal{U}(Y/k)$ is isomorphic to $\mathcal{U}(X_3/k)$, since X_3 is a representative of the projective, homogeneous varieties, which become a GSV over L.

□

10.5.16. (Unsolved cases) In both of the cases of $M(X_3)$ and $M(X_6)$ we are unable to completely determine the Rost motives from the coaction. Many of the calculations of products in the respective Chow rings crash the algorithm from the *Chow maple package.* While this generally hinders us from determining the concrete structure of the upper motives of X_3 and X_6, the case of X_3 turns out to be very tenacious.

Passing to all established phases and checking motivic decompositions, we can conclude that $M(X_3/k)$ contains at least three shifts of $\mathcal{U}(X_3)$, coded by the shifting polynomial $(1+t+t^2)O_3$ for some symmetric polynomial $O_3 \in \mathbb{N}_0[t]$ (check the Tate motives in Theorem 10.1.1). But it is unclear whether O_3 equals 1 or maybe $(1+t^6)$ for example. For obtaining clarity, a lot of calculations of rational cycles are necessary, which may also crash executing the algorithm.

Facts on $M(X_4)$: This case works similar. Step one is to show that $\#M(X_4) \subset \#M(X_3)$, which follows from the Tits classification. We then have that the shift polynomial of its upper motive is $(1+t^4+t^8)O_3$, for the same O_3 like in the X_3 case, which follows from comparing the Tate motives in Theorem 10.1.1. Simply pass to $k(X_3)$, which yields phase $[A_1^3,(1,0,0,0),1]$ by the GSV-table and the phase classification. Now check the Tate polynomials of X_3 and X_4 in Theorem 10.1.1 to see it. Even if the structure of the upper motive of X_3 and X_4 would by given by $\mathcal{U}(X_3) = P(X_3,t)/(1+t+t^2)$, which is the biggest possibility in terms of dimension, then subtracting $(1+t^4+t^8)P(\mathcal{U}(X_3),t)$ from $P(X_4,t)$ leaves a difference with only positive or zero coefficients. Thus there are definitely Rost motives in $M(X_4)$. But in order to exactly determine them, we need to know the structure of $\mathcal{U}(X_3)$.

Facts on $M(X_{1,6})$: The only thing we know is that if G is versal, then there are transitions to $[D_6,(1,1,1,0),*]$ and $[D_5 \times A_1,(1,1,0,0),2]$ and thus the upper motives of X_6 and $X_{1,6}$ are *not isomorphic* over k. As the Chow ring of $X_{1,6}$ is huge and should basically have at least the same generators as $\mathrm{Ch}(X_1)$ and $\mathrm{Ch}(X_6)$, it is highly unlikely that there are no Rost motives in $M(X_{1,6})$.

10.6 Conclusions on the phase [E7,(1,1,1,0),*]

In this section we use our results to establish the motivic decomposition of X_1 in the case G has the ***hypothetically admissible*** phase $[E_7,(1,1,1,0),*]$ and satisfies some other property. For starters, here is a simply obtainable result.

10.6.1 Lemma. *If an algebraic group G has phase $[E_7,(1,1,1,0),8]$, the projective, homogeneous G-variety X_1 does not have a zero cycle of odd degree.*

Proof: Let A be the Tits algebra of G. Assume X_1 has a zero cycle of odd degree. By the property **1.** of the motivic J-invariant there is an odd degree extension L/k, such that X_1 becomes isotropic over L without $J_2(G)$ changing. The phase over L is necessarily $[D_6,(1,1,1,0),*]$ (with $*$ now being restricted to 2 or 4) by the phase classification. But as L/k has odd degree, A can not change its index to neither 2 nor 4 over L.

□

10.6.2 Lemma. *If the phase $[E_7,(1,1,1,0),8]$ is admissible, then $[E_7,(1,1,1,0),4]$ is also admissible.*

Proof: Assume that we are given $[E_7,(1,1,1,0),8]$ and that $[E_7,(1,1,1,0),4]$ is not admissible. Let D be the Brauer class of the Tits algebra of G. Passing to $L := k(\mathrm{SB}_4(D))$ reduces the index of D to 4 by the index reduction formula from [SvB]. By the phase classification and since we assume that $[E_7,(1,1,1,0),4]$ is not admissible, G/L has phase $[D_6,(1,1,1,0),4]$.

We have seen in Theorem 10.4.12 that there are exactly two Tate motives in $M(X_1/L)$, which are $\mathbb{F}_2$ and $\mathbb{F}_2(33)$. Passing from k to $k(X_1)$ also yields phase $[D_6,(1,1,1,0),4]$ by the index reduction formula for X_1 from [MPW2]. This implies that the upper motives of $\mathrm{SB}_4(D)$ and X_1 are isomorphic over k, because X_1/k

does not have a zero cycle of odd degree by the lemma above. Thus the motives of both varieties contain $\mathbb{F}_2(33)$ over L and $k(X_1)$. This is impossible, since $\dim(\mathrm{SB}_4(D)) = \deg(P(A_7/P_4, t)) = 16$. It follows that G remains anisotropic over L. To see that the J-invariant does not change over L, remember Corollary 8.3.9 and the fact that $\mathrm{ind}(D) = 4$ holds over L.

□

10.6.3 Theorem. *Let G be an adjoint algebraic group of type E_7 with phase $[E_7, (1, 1, 1, 0), 8]$. Assume that G has semisimple, anisotropic kernel $D_5 \times A_1$ over $k(X_6)$. Then the motive of the projective, homogeneous G-variety X_1 decomposes into indecomposable motivic summands as follows*

$$M(X_1) \simeq \mathcal{U}(X_1) \oplus \textstyle\bigoplus_{i\in I} \mathcal{R}_J(i),$$

$$\textit{with } P(I, t) = t^8(t^2 + t^3 + t^4 + t^5 + t^6)$$

$$\textit{and } P(\mathcal{U}(X_1), t) = P(X_1, t) - P(I, t)P(\mathcal{R}_J, t).$$

Proof: Passing to $k(X_1)$ yields the phase $[D_6, (1, 1, 1, 0), 4]$ by the index reduction formula for X_1 from [MPW2]. Thus the claim on the Rost motives follows by Theorem 10.4.12 and the fact that we can lift the Rost motives to k, because of [PS22, Theorem 6.4] and as the value of J_2 is equal over k and $k(X_1)$. The other motivic summands over $k(X_1)$ are all isomorphic to the upper motives of projective, homogeneous D_6-varieties.

Structure of $\mathcal{U}(X_1)$: By the binary summand theorem and the lemma above, the Tate motives $\mathbb{F}_2, \mathbb{F}_2(33)$ in $M(X_1)$ over $k(X_1)$ are glued to other motivic summands over k. It is clear that there is only one copy of $\mathcal{U}(X_1/k)$ contained in $M(X_1/k)$, since there only two Tate motives in $M(X_1)$ over $k(X_1)$.

By the Theorem 10.4.12 we can now check the possibilities of which indecomposable motivic summands *besides* $\mathbb{F}_2, \mathbb{F}_2(33)$ are split off by $\mathcal{U}(X_1/k)$ over $k(X_1)$ from the summands below (the Y_i denote projective, homogeneous D_6-varieties)

$$\mathcal{U}(Y_5)(1) \oplus \mathcal{U}(Y_5)(2) \oplus \mathcal{U}(Y_2)(8) \oplus \mathcal{U}(Y_5)(17) \oplus \mathcal{U}(Y_5)(18).$$

When we pass from k to $k(X_6)$, we obtain anisotropic kernel $D_5 \times A_1$ by the initial requirement. By Theorem 10.3.3 the upper motive of X_1 over $k(X_6)$ has a Poincaré polynomial starting with $1 + t + t^2 + t^3 + t^4$ (it is a motivically indecomposable quadric of dimension 10). As $\mathrm{Ch}^1(\overline{X_1})$ and $\mathrm{Ch}^2(\overline{X_1})$ have $\mathbb{F}_2-$rank 1, this proves that the shifts of $\mathcal{U}(Y_5)$ above are all glued to $\mathcal{U}(X_1/k)$.

To show that $\mathcal{U}(Y_2)(8)$ is also glued to $\mathcal{U}(X_1/k)$, we use a proof by contradiction. Assume the opposite. Then by Karpenko's theorem there is some other G-variety Z over k, with $\mathcal{U}(Y_2)$ as upper motive. If we compare the structures of the elements in $\#G/k(X_1)$ in Theorem 10.4.12, its clear that only X_6 qualifies for Z, as $\mathcal{U}(X_6) \simeq \mathcal{U}(Y_2)$ holds over $k(X_6)$. So we can assume that $\mathcal{U}(X_6/k)(8)$ is contained in $M(X_1/k)$. But by Karpenko's theorem passing to $k(X_6)$ will give us an semisimple anisotropic kernel smaller than D_6, which one obtains when passing to $k(X_1)$ by the index reduction formula from [MPW2] (i.e. X_1 and X_6 both need to become isotropic over $k(X_6)$). This is clearly a contradiction to our requirement on how the kernel of G looks over $k(X_6)$.

$\square$

Chapter 11

Groups of type E7 constructed from F4 and A1

In this chapter we briefly describe the $F_4 \times A_1$ construction for groups of type E_7. It was already researched in [Gar01] and is completely understood over real closed fields. We take things one step further to arbitrary fields of characteristic zero. We obtain many results on the output G, such as the definite value of $J_2(G)$. For some cases the Tits index of G is determined, too. Our results may be of general interest. For the rest of this thesis, we assume that $g_3(\mathcal{J}) = 0$ holds, for any $F_4 \simeq \mathrm{Aut}(\mathcal{J})$ used in the $F_4 \times A_1$ construction.

11.1 Constructing E7 from F4 and A1

In this section we introduce the $F_4 \times A_1$ construction of groups of type E_7 and point out some basics. Also we manage to determine the maximal value of $J_2(G)$ for any group G, which is a result of the construction.

11.1.1 Definition. Let A_1, F_4 and E_7 denote split adjoint groups of the respective type. By [Gar01] there is an embedding of split groups $F_4 \times A_1 \hookrightarrow E_7$. Applying $H^1(k, -)$, yields a map

$$H^1(k, F_4) \times H^1(k, A_1) \longrightarrow H^1(k, E_7).$$

Let ξ lie in the image of this map and assume G is a twist of E_7 by ξ. Then we say that G *comes from the $F_4 \times A_1$ construction.*

11.1.2. If any of the inputs H of the construction is isotropic, the outcome G is also isotropic, since G contains at least the same split tori of H. The hard thing about such constructions to determine is, whether anisotropic inputs yield an anisotropic output. A complete solution for this problem for the $F_4 \times \mu_2$ construction of outer algebraic groups of type ${}^2E_6$ was provided in [GPet]. The authors manage to determine the Tits index of the outcome of the $F_4 \times \mu_2$ construction, based on the relation of the Galois cohomological invariants of the inputs over k. We use the same approach of focusing on the relation of the mod 2 invariants of the F_4 and the A_1 used to construct G. We refer to those as $f_3(\mathcal{J}), f_5(\mathcal{J})$ for the Albert algebra

$\mathcal{J}$ defining the F_4, and Q for the degree two invariant (i.e. the Tits algebra) of the $\mathbf{PGL}_1(Q)$, which defines the A_1.

11.1.3 Remark. In the reference [Gar01], it is demanded that $f_3(\mathcal{J})$ and Q have a common slot. This property is not necessary for the construction to work in general. It is only demanded, because the author in the reference wants to ensure that G splits over a quadratic field extension. This is just important if one wants to restrict the kinds of groups arising from the construction. In fact the case where $f_3(\mathcal{J})$ and Q do not have any common slots is rather interesting, as we will see in a minute. If we demand k to be 2-special, then Theorem 3.6.6 applies and we can be sure that F_4 is isotropic if and only if $f_5(\mathcal{J})$ is zero, as in this case $g_3(\mathcal{J})$ is zero.

11.1.4 Remark. Note that the Tits algebra of any G coming from the $F_4 \times A_1$ construction is Brauer equivalent to the $Q \in \mathrm{Br}(k)$ making up the input A_1. This is not hard to see, as the Tits algebra of F_4 is split in general by Example 3.7.6. Thus the Tits algebra of G is necessarily contributed by A_1.

As a versal form of an adjoint group of type E_7 has a Tits algebra with index 8 by the Tits classification, the $F_4 \times A_1$ can not produce every group of type E_7 as output. It also turns out that any value of $J_2(G)$ for any G coming from the $F_4 \times A_1$ construction is strictly smaller compared to maximal possible value of $J_2(G)$.

11.1.5 Lemma. (Garibaldi) *Let G be the output of the $F_4 \times A_1$ construction. Assume that G does not split over $k(\mathrm{SB}(Q))$. Then the phase of G over $k(\mathrm{SB}(Q))$ is $[D_4, (0,1,0,0), 1]$.*

Proof: First we consider the $F_4 \times A_1$ construction, with Q being already split over k. We write Q_0 for it. Then any ξ in the image of $H^1(k, F_4) \times H^1(k, A_1) \to H^1(k, E_7)$ has as preimage $\mathcal{J} \times Q_0$. This means that ξ is solely determined by $\mathcal{J}$ and thus we have a construction of E_7 stemming from the embedding of $F_4 \hookrightarrow E_7$ of split groups. This embedding factors through E_6. Any non split group of inner type E_6 is known to have anisotropic kernel D_4 mod 2. Thus twisting the split E_7 with ξ, means that the resulting group G is split or does also have anisotropic kernel of type D_4.

Now we assume that Q is not split over k. Extending scalars to $k(\mathrm{SB}(Q))$ splits Q, but not G by assumption. We are again in the situation where Q is split, but the G coming from the $F_4 \times A_1$ construction is not. Thus its anisotropic kernel is of type D_4 by our observation. The rest of the claim now follows from the phase classification. □

11.1.6 Theorem. *Let G be the output of the $F_4 \times A_1$ construction. Then the maximal value of the J_2-invariant $J_2(G)$ is equal to* $(1,1,0,0)$.

Proof: By the lemma above it follows, that if Q is split, $J_2(G)$ is either $(0,1,0,0)$ or zero. So let us assume that Q is not split. We pass to $k(\mathrm{SB}(Q))$. If G splits, then $J_2(G)$ over k is equal to $(1,0,0,0)$ by Lemma 7.1.9. If G does not split, then by the lemma above its anisotropic kernel reduces to D_4. Now we can apply Theorem 10.3.5 and are done. □

11.1.7. The proof of the theorem indicates, that for non split groups G coming from the $F_4 \times A_1$ construction only the three values $(1,1,0,0), (0,1,0,0), (1,0,0,0)$ are possible for $J_2(G)$. To further decode the outcome of the construction, we now determine the impact of cohomological invariants of the input on $J_2(G)$, before we consider how the Tits index is effected by these. The following example from [Gar01] marks the least complicated case.

11.1.8 Example. ([Gar01, 5.4]) Consider the $F_4 \times A_1$ construction with $\mathcal{J}$ split and Q *not* split. Since G contains the split F_4 and thus a split torus of k-rank 4, its anisotropic kernel is either A_1^3 or is split, by the Tits classification. But since Q is its Tits algebra, it is A_1^3. Using the phase classification, or the fact that G necessarily splits over $k(\mathrm{SB}(A))$, it follows that $J_2(G) = (1,0,0,0)$ holds over k.

11.1.9 Example. Assume $k = \mathbb{Q}_p$ for an arbitrary prime p. It is well known that $H^{2+i}(k, \mu_2) = 0$ holds for any $i > 0$. This follows from the calculation of the so called u invariant for quadratic forms (see [EKM, §VI]), which is known to be equal to 4 for $k = \mathbb{Q}_p$. Applying the Arason–Pfister Hauptsatz (see [EKM, Thrm. 6.18]), shows that $I^3 = 0$ holds in this case. So over $\mathbb{Q}_p$, any F_4 mod 2 is split, as the even part of its Rost invariant is zero. Thus the outcome of $F_4 \times A_1$ construction depends only on the choice of Q by the example above and is never anisotropic as it contains a split torus of at least k-rank 4.

11.1.10 Lemma. *Let G be the output of the $F_4 \times A_1$ construction. Assume $\mathcal{J}$ and Q are both not split. If Q divides $f_3(\mathcal{J})$, then $J_2(G)$ equals* $(1,0,0,0)$.

Proof: Passing to $k(\mathrm{SB}(Q))$ kills both, Q and $f_3(\mathcal{J})$. By Theorem 3.6.6 any Albert algebra $\mathcal{J}$ with $g_3(\mathcal{J}) = 0$ used for the construction, is split if and only if $f_3(\mathcal{J}) = 0$ holds. Thus both F_4 and A_1 split over $k(\mathrm{SB}(Q))$ and G has k-rank of at least 5 over $k(\mathrm{SB}(Q))$. By the Tits classification, G is split by $\mathrm{SB}(Q)$, which makes $\mathrm{SB}(Q)$ a GSV. By Lemma 7.1.9 it follows that over k one has $J_2(G) = (1,0,0,0)$. □

11.1.11 Remark. The upper lemma makes no statement about the Tits index of G. How (at least) some of the isotropic Gs with $J_2(G) = (1,0,0,0)$ are obtained, is covered in the example. But how to construct the anisotropic ones in general? Over a real closed field it is known that one has $H^2(k, \mu_2) \simeq \mathbb{Z}/2\mathbb{Z}$.

Thus Q necessarily divides $f_3(\mathcal{J})$, provided neither Q nor $\mathcal{J}$ is split. The lemma above then applies. This case is treated in [Gar01, 6.1] and indeed in this situation G is anisotropic if and only if $f_5(\mathcal{J})$ is not zero over a real closed field. In any case the example shows, that there are anisotropic groups coming from the $F_4 \times A_1$ construction and having $J_2(G) = (1,0,0,0)$, such that none of the projective, homogeneous G-varieties has a zero cycle of odd degree. This follows since k is real closed and thus no field extension of odd degree bigger than 1 exists.

11.1.12 Lemma. *Let G be the output of the $F_4 \times A_1$ construction. Assume $\mathcal{J}$ and Q are both not split. If Q and $f_3(\mathcal{J})$ have one or none common slots, then $J_2(G)$ equals* $(1,1,0,0)$.

Proof: Passing to $k(\mathrm{SB}(Q))$ kills only Q and leaves $f_3(\mathcal{J})$ non zero (it may change though). So G is not split. Now the same arguments of the proof of Lemma 11.1.5

apply, to show that G has anisotropic kernel of type D_4 over $k(\mathrm{SB}(Q))$. Again using Theorem 10.3.5 finishes the proof. □

11.1.13. The problem of whether each G coming from the $F_4 \times A_1$ construction is anisotropic or not aside, there is another issue to deal with. By the phase classification $J_2(G)$ takes the value $(1, 1, 0, 0)$ in case G has anisotropic kernel $D_5 \times A_1$ or $D_4 \times A_1$. From [Tits90] we know that groups of type E_7 having this anisotropic kernel are given by a quadratic form $q = \varphi_3 \perp -\varphi_2 \in W(k)$, which is the difference of a 3-Pfister and a 2-Pfister form having none or one common slot. It turns out that for any G coming from the $F_4 \times A_1$ construction a similar relation exists for $f_3(\mathcal{J})$ and Q, which unsurprisingly carries over to the anisotropic kernel of G.

11.1.14 Lemma. *Let G be the output of the $F_4 \times A_1$ construction. Assume the value of $J_2(G)$ equals $(1, 1, 0, 0)$ and let G be isotropic. Then the semisimple, anisotropic kernel of G is of type $D_5 \times A_1$ if $f_3(\mathcal{J})$ and Q have no common slot and $D_4 \times A_1$ if they have exactly one common slot.*

Proof: First note that $f_3(\mathcal{J})$ and Q split over the same quadratic field extension L/k if and only if they have a common slot. Thus it is enough tho show that G does not split over any quadratic field extension, in case $f_3(\mathcal{J})$ and Q do not have a common slot.

So let us assume that G splits over the quadratic extension L/k and $f_3(\mathcal{J})$ and Q have no common slot over k. Then Q splits over L too, because it is the Tits algebra of G both over k and L. This means that putting the non split $\mathcal{J}/L$ and the split Q/L into the $F_4 \times A_1$ construction, we obtain the split E_7. By the proof of Garibaldi's lemma, this happens only if $\mathcal{J}/L$ is also split, as otherwise one obtains an output with anisotropic kernel D_4. This means $f_3(\mathcal{J})$ and Q must have a common slot over k, contradicting our assumption.

□

11.1.15. Our last task to is to determine criteria, which control whether G is isotropic or not. There is only a partial answer, which goes beyond the sole consideration of $f_5(\mathcal{J})$. It resulted from a discussion with Victor Petrov about the $F_4 \times A_1$ construction. He suggested the consideration of the Killing form (see [Hum, II 5.]) and pointed out that the essentials to make a proof work are already known. Unfortunately explicitly calculating this Killing-Form is very time-consuming. We did not perform this effortful task. This leaves us only with a remark instead of a lemma. We give a sketch of the proof.

11.1.16 Remark. *Let G be the output of the $F_4 \times A_1$ construction. Assume G is isotropic. Then $f_5(\mathcal{J}) \cup Q = 0 \in H^7(k, \mu_2)$ holds.*

Sketch of proof: Let us consider the Killing form $\mathcal{K}_G$ of the group G of type E_7 coming from the $F_4 \times A_1$ construction and assume that G is anisotropic over k. We can take [Jac, (144) on p.117] as a blueprint for $\mathcal{K}_G$. From the reference it is clear that $\mathcal{K}_G$ will incorporate the Killing forms of A_1 and F_4 and additionally some constant terms or factors (i.e. an expression like $\langle 2, 2\rangle$ for example).

Except for these constants, everything else is known. Concretely $\mathcal{K}_G$ contains the direct sum of $\mathcal{K}_{F_4}$ and $\mathcal{K}_{A_1}$ and the tensor product (compare with [Jac]) of the invariant trace forms on the 3-dimensional representation of A_1 and the 26-dimensional representation of F_4.

Consideration of $\mathcal{K}_{A_1}$: Let Q' be the quadratic form in the decomposition $Q = \langle 1 \rangle \perp Q'$. Its well known that $\mathcal{K}_{A_1}$ is similar to Q'. Thus Q' is also the trace form on the 3-dimensional representation of A_1.

Consideration of $\mathcal{K}_{F_4}$: We consult [Mal, Introduction]. Originally $\mathcal{K}_{F_4}$ was calculated by Serre and is known to be equal to

$$\langle -2 \rangle \otimes (f_5(\mathcal{J}) \perp -f_3(\mathcal{J})) \perp \langle -1, -1, -1, -1 \rangle \otimes (f_3(\mathcal{J}) \perp \langle -1 \rangle).$$

Calculation of of $\mathcal{K}_G$: We conclude that $\mathcal{K}_G$ contains a summand $Q' \otimes f_5(\mathcal{J})$. Assume that G is isotropic. Then $\mathcal{K}_G$ can be calculated via [Mal, Theorem 1]. By the theorem all one needs to know to establish $\mathcal{K}_G$, is the Killing form of the anisotropic kernel of G. The maximal value of $J_2(G)$ for any output of the $F_4 \times A_1$ construction is $(1,1,0,0)$. So, by the phase classification, we only need to know the Killing forms of A_1, D_4, D_5.

By [Mal, proof of Theorem 2 on p. 8], the Killing forms of $D_4, D_5 \simeq \mathbf{SO}(q)$ do only consist of quadratic forms made up of products of the coefficients of q. Since q can not contain $f_5(\mathcal{J})$ in any of these cases in general, because its rank is only ten, it follows that $\mathcal{K}_G$ does not contain $Q' \otimes f_5(\mathcal{J}) \neq 0$, when G is isotropic.

To conclude the proof, we need to know that $Q' \otimes f_5(\mathcal{J})$ does not cancel out with some other summand. For showing this, one needs to compare the general Killing form with the Killing forms of isotropic E_7's in the Witt ring (more precisely, calculating modulo subsequent factors of powers of the fundamental ideal of the Witt ring). But for this one needs to know the precise constants occurring in the Killing forms. This procedure takes a lot of effort. In our concrete case, we considered $f_5 \cup Q$, so one needs to do checks up to I^7/I^8.

Note also that for a generic Tits construction the above argument is working, since $f_5(\mathcal{J})$ does not appear in the isotropic cases arising from the Tits construction at all and can not cancel out.

The fact that the semisimple anisotropic kernel of an adjoint group of type E_7 is sometimes a central product of for example D_5 and A_1 and thus it is not really a $\mathbf{SO}_{10}$, is not a problem because $f_5(\mathcal{J})$ is still independent of q by construction.

$\square$

11.1.17 Remark. (Possibility to construct $[E_7, (1,1,1,0), *]$) The sketched proof of the remark above could potentially be used for constructing groups G having the phase $[E_7, (1,1,1,0), *]$. One could use (A, σ) given as in Theorem 8.3.5 as the input for Petrov's $D_6 \times A_1$ construction from [P13]. For this construction, the Quaternion algebra Q defining the A_1 needs to come from the D_6. If one knew the Killing form $\mathcal{K}_{D_6}$ of the $D_6 \simeq \mathbf{HSpin}(A, \sigma)$ defining the anisotropic kernel of the isotropic E_7 and was able to prove that $\mathcal{K}_{E_7}$ contains $\mathcal{K}_{D_6} \otimes \mathcal{K}_Q$, then a similar result like in the remark could be proven.

Then G is most probably anisotropic when $f_n \cup Q \neq 0$, for some summand f_n of $\mathcal{K}_{D_6}$. There is hope to find such an f_n, because a decomposable degree three invariant f_3 for (A, σ) as in Theorem 8.3.5 has recently been established in [MaT20, Thrm. 2.3], along with a concrete formula in case $\mathrm{ind}(A) = 2$ holds. Passing to $k(\mathrm{SB}(A))$ then makes G having anisotropic kernel D_6 defined by the form q_σ adjoint to σ as in the Theorem 8.3.5. So the claim on $J_2(G/k)$ follows, once one knows that an anisotropic G can be constructed with such an input.

11.1.18 Remark. The Remark 11.1.16 suggests, that even over some fields where -1 is a square, the $F_4 \times A_1$ construction can theoretically produce anisotropic groups G of type E_7 with $J_2(G) = (1, 1, 0, 0)$, for which $\#G$ differs. This is very interesting, as this would mean that the phase of a group alone *probably* does ***not*** determine the motivic decomposition type of an anisotropic group.

If Q and f_3 have exactly one common slot and G is anisotropic, then G splits over a quadratic field extension. Also by the index reduction formulas in [MPW] and the phase classification, it is clear that the upper motives of X_1 and X_6 are isomorphic in this case.

But if Q and f_3 have no common slot and G is anisotropic, this is impossible as G splits if and only if Q and f_3 are both split. If passing to $k(X_6)$ does not change the number of common slots of Q and f_3, one can not obtain anisotropic kernel $D_4 \times A_1$, because these groups split over a quadratic field extension, since they are defined by the quadratic form from Lemma 5.5.11.

Thus one obtains anisotropic kernel $D_5 \times A_1$ over $k(X_6)$, which means that the upper motives of X_1 and X_6 are not isomorphic over k. Sadly, determining the behavior of G over $k(X_6)$ seems out of reach.

11.1.19. If we knew the exact Killing-Form of any E_7 coming from the $F_4 \times A_1$ construction, the proof below would be complete.

11.1.20 Remark. *Let G be an anisotropic group of type E_7 over k with motivic J-invariant $J_2(G) = (1, 0, 0, 0)$, which comes from the $F_4 \times A_1$ construction. Then none of the projective, homogeneous G-varieties X_Θ has a zero cycle of odd degree.*

Sketch of proof: We can limit our consideration to X_1, as by the GSV-table and the phase classification, the upper motives of all X_Θ which are not GSVs are isomorphic. Assume that X_1 has the demanded zero cycle. Then there is a field extension L/k of odd degree, such that G/L is isotropic. If G/L is isotropic, it is either split or has anisotropic kernel A_1^3 by the phase classification. The first case is impossible, as then $J_2(G/L)$ would change to zero, violating property **1.** of the J-invariant. The second case is also impossible, as by [Mal, Thrm. 1], $\mathcal{K}_{G/L}$ can be calculated purely as an orthogonal sum of the Killing form of $A_1 \simeq \mathbf{PGL}_1(Q/L)$. So it surely does not contain $f_3(\mathcal{J})$, unless it is zero over L.

Consider the Killing form $\mathcal{K}_{G/k}$. It contains the Killing form of $F_4 \simeq \mathrm{Aut}(\mathcal{J}/k)$ used as input along with a Quaternion algebra Q/k as orthogonal summand, as in the case over $\mathbb{R}$ in [Jac]. Then one proceeds similarly as in Remark 11.1.16 but looking at $f_3(\mathcal{J})$ instead of $f_5(\mathcal{J})$. Note that since the degree of L over k is odd, $f_3(\mathcal{J})$ is still present in the Killing form over L and thus $\mathcal{K}_{G/L}$ can not only be made up of the Killing form of $\mathbf{PGL}_1(Q/L)$.

□

11.1.21. Below we summarize the results on the $F_4 \times A_1$ construction obtained in this chapter. The table is to read as follows. The first three columns indicate whether one of the respective cohomological invariants is zero or not. In case it is zero, we write a 0. Otherwise we write $Q, f_3(\mathcal{J}), f_5(\mathcal{J})$ to denote that the considered invariant is not zero.

The term *Slots* denotes the number of common slots of $f_3(\mathcal{J})$ and Q. The column to its right contains the phase of the output G, *provided* G is anisotropic. Otherwise we write a $-$.

The last column contains the phase of the output G, *provided* it is isotropic. What exactly controls the isotropy of X_6 is unclear at the moment. Zainoulline has recently shown in [Zai, Exm. 3.4] that the (integral) canonical dimension (see [RY]) of E_7^{ad} is bounded by 42, which interestingly equals the dimension of X_6 and X_2.

11.1.22 Corollary. *Let G be the output of the $F_4 \times A_1$ construction. Then its phase depends on $Q, f_3(\mathcal{J}), f_5(\mathcal{J})$ in the following way*

Q	$f_3(\mathcal{J})$	$f_5(\mathcal{J})$	Slots	If G is anisotropic	If G is isotropic
0	$f_3(\mathcal{J})$	0	0	$-$	$[D_4, (0,1,0,0), 1]$
0	$f_3(\mathcal{J})$	$f_5(\mathcal{J})$	0	$-$	$[D_4, (0,1,0,0), 1]$
Q	$f_3(\mathcal{J})$	0	0	$-$	$[D_5 \times A_1, (1,1,0,0), 2]$
Q	$f_3(\mathcal{J})$	0	1	$-$	$[D_4 \times A_1, (1,1,0,0), 2]$
Q	$f_3(\mathcal{J})$	0	2	$-$	$[A_1^3, (1,0,0,0), 2]$
Q	$f_3(\mathcal{J})$	$f_5(\mathcal{J})$	0	$[E_7, (1,1,0,0), 2]$	$[D_5 \times A_1, (1,1,0,0), 2]$
Q	$f_3(\mathcal{J})$	$f_5(\mathcal{J})$	1	$[E_7, (1,1,0,0), 2]$	$[D_4 \times A_1, (1,1,0,0), 2]$
Q	$f_3(\mathcal{J})$	$f_5(\mathcal{J})$	2	$[E_7, (1,0,0,0), 2]$	$[A_1^3, (1,0,0,0), 2]$

□

Chapter 12

Motivic construction of a degree five invariant for E7

In this final chapter we construct a Galois cohomological degree five invariant mod 2 for adjoint algebraic groups of type E_7, which split over the generic point of the Severi-Brauer variety of their Tits algebra. We do this by first calculating the motivic decompositions for the phase $[E_7, (1,0,0,0), 2]$ over k. It turns out that the decomposition of all projective, homogeneous G-varieties that are not GSV over k have an upper motive which is binary and 15-dimensional, provided they do not have a zero cycle of odd degree. The calculations also incorporate results on groups coming from the $F_4 \times A_1$ construction, but do not require that G is the output of the construction. In the case of $[E_7, (1,1,0,0), 2]$ only $M(X_7)$ is calculated. But it is not needed for the calculation of the invariant. The actual prove of existence of the invariant is established by using several results of Voevodsky and Semenov and can be thought of an analogue to the main result of [S16].

12.1 Motivic decomposition for [E7,(1,0,0,0),2]

In this section we consider the motivic decompositions in case the phase of G is $[E_7,(1,0,0,0),2]$. The proof of it marks the magnum opus of this thesis. The decomposition makes an assumption on the zero cycles of X_1. Remember that this case occurs over $\mathbb{R}$ by [Gar01, 6.1] for example.

12.1.1 Theorem. *Let G be an anisotropic group of type E_7 over a field k of characteristic zero, with motivic J-invariant $J_2(G) = (1,0,0,0)$. Assume that the projective, homogeneous G-variety X_1 has no zero cycle of odd degree. Then the upper motive of X_1 is binary and has dimension* 15.

Proof: We cluster this proof into four steps and set $L := k(X_1)$ throughout the whole proof.

1. By the GSV-table X_1 is not a GSV, when $J_2(G) = (1,0,0,0)$ holds. So passing to L induces a transition to $[A_1^3,(1,0,0,0),2]$, by the phase classification. Also the split Tits index is the only one that has more nodes circled than A_1^3. We are in a situation, where we can use the result of De Clercq (see [DC, Thrm. 1.1]), which asserts that we can lift all Rost motives $\mathcal{R}_{(1,0,0,0)}$ in $M(X_1/L)$ to k. Another way to obtain this, would be to consider the coaction ρ, [PS22, Theorem 6.4] and the fact that $J_2(G/k) = J_2(G/L)$ holds. It follows that $\mathcal{U}(X_1/k)$ splits completely into Tate motives over L, since there are no other motivic summands than Rost motives or Tate motives in the motive of any projective, homogeneous G-variety over L by Theorem 10.1.1.

2. Consulting the GSV-table again, we see that the upper motives of all X_Θ for $\Theta \subset \{1,3,4,6\}$ are isomorphic over k. Thus the arguments in the previous step are also completely valid for X_6 for example. Using Karpenko's theorem, it follows that any motivic summand in $M(X_\Theta/k)$ is either a shift of $\mathcal{U}(X_1/k)$ or of $\mathcal{R}_{(1,0,0,0)}$. The polynomials T_1, T_6 from Theorem 10.1.1, which encode the $\mathbb{F}_2(i)$ in $M(X_1/L)$ and $M(X_6/L)$, thus are definitely divisible by the polynomial which encodes the $\mathbb{F}_2(i)$, split off by $\mathcal{U}(X_1/k)$ over L. As we have seen in the first step, this polynomial coincides with $P(\mathcal{U}(X_1/k),t)$. Conspicuously, $\gcd(T_1,T_6) = (1+t^{10})(1+t^{15})$ holds in $\mathbb{N}_0[t]$. Hypothetically this leaves the possibilities $1, (1+t^{10}), (1+t^{15}), (1+t^{10})(1+t^{15})$ for $P(\mathcal{U}(X_1/k),t)$. The first one is impossible, since X_1 is supposed to have no zero cycle of odd degree and thus has no Tate motive as upper motive by Remark 5.3.3. The second one contradicts the binary summand theorem.

3. When expanded, the polynomial T_1 contains the monomial t^6 symbolizing the only $\mathbb{F}_2(6)$ in $M(X_1/L)$. Considering the two possible structures of $\mathcal{U}(X_1/k)$ in question, this Tate motive necessarily marks the generic point of the summand $\mathcal{U}(X_1/k)(6)$ in $M(X_1/k)$. Thus it is clearly rational. But we do not know for sure which cycle defines this generic point. To find out we now calculate all rational cycles in $\mathrm{Ch}^6(X_1/k)$, which is generated by $\gamma_{6,1} = z[4,2,5,4,3,1]$, $x_6 = \gamma_{6,2} = z[2,6,5,4,3,1]$, $\gamma_{6,3} = z[7,6,5,4,3,1]$. Considering the Rost motives occurring in $M(X_1/L)$, which were also calculated in Theorem 10.1.1, we see that $\mathcal{R}_{(1,0,0,0)}(6)$ is contained one time in $M(X_1/k)$. Thus there are exactly two generic points in $\mathrm{Ch}^6(X_1/k)$. One of $\mathcal{U}(X_1/k)(6)$ and one of $\mathcal{R}_{(1,0,0,0)}(6)$. Using the coaction on the generators h, x_4, x_6, x_9 of $\mathrm{Ch}(\overline{X_1})$ established in Lemma 9.3.8, we obtain $\rho(x_4h^3) = e_1 \otimes h^6 + 1 \otimes x_4h^3$. Because of [PS22, Theorem 6.4], this makes h^6 the generic point of $\mathcal{R}_{(1,0,0,0)}(6)$ in $M(X_1/k)$. Now comes the complicated part. We want to show that the generic point of $\mathcal{U}(X_1/k)(6)$ in $M(X_1/k)$ equals the cycle $\gamma_{6,1}$. We do this by showing that none of the cycles $\gamma_{6,2}, \gamma_{6,3}, \gamma_{6,1}+\gamma_{6,2}, \gamma_{6,1}+\gamma_{6,3}$ is rational over k. Note that $h^6 = \gamma_{6,2}+\gamma_{6,3}$ is rational, since h is always rational by Remark 3.7.5. Also note that one can express the cycle $\gamma_{6,1}$ as $\gamma_{6,2}+x_4h^2$. Then by the Lemma 9.3.8, we have

$$\rho(\gamma_{6,2}) = e_1 \otimes h^5 + 1 \otimes \gamma_{6,2},$$

$$\rho(\gamma_{6,3}) = \rho(\gamma_{6,2}+h^6) = \rho(\gamma_{6,2}) + \rho(h^6) = e_1 \otimes h^5 + 1 \otimes (\gamma_{6,2}+h^6) \text{ and}$$

$$\rho(\gamma_{6,1}) = \rho(\gamma_{6,2}+x_4h^2) = \rho(\gamma_{6,2}) + \rho(x_4h^2) = 1 \otimes \gamma_{6,1}.$$

It follows from Lemma 7.2.5 that the cycles $\gamma_{6,2}, \gamma_{6,3}$ can not be rational over k. From the upper equations it follows by the same arguments, that the sums $\gamma_{6,1}+\gamma_{6,2}$ and $\gamma_{6,1}+\gamma_{6,3}$ can also not be rational. To finish this part of the proof, let us assume that $\alpha := \gamma_{6,1}+\gamma_{6,2}+\gamma_{6,3}$ is rational, but $\gamma_{6,1}$ is not. Using the identity $h^6 = \gamma_{6,2}+\gamma_{6,3}$, we see that $\alpha + h^6 = \gamma_{6,1}$ is rational yet. Thus the subgroup of rational cycles in $\mathrm{Ch}^6(X_1/k)$ is generated by $\gamma_{6,1}$ and h^6.

4. Analysing T_1 again, we find that $\mathbb{F}_2(10)$ is contained in $M(X_1/L)$ exactly one time. If this Tate motive comes from a generic point over k, we are done. For showing this, we calculate all rational cycles in $\mathrm{Ch}^{10}(X_1/k)$. First the generic points of the Rost motives. By Theorem 10.1.1, there are exactly two Rost motives in $M(X_1/k)$ starting in codimension 10. Using the coaction, we find

$$\rho(x_6h^5) = e_1 \otimes h^{10} + 1 \otimes x_6h^5 \text{ and}$$

$$\rho(x_4x_6h) = e_1 \otimes (h^3x_6 + h^5x_4)h + 1 \otimes x_4h^6h.$$

This gives us the generic points h^{10} and $h^4x_6 + h^6x_4$ by [PS22, Theorem 6.4]. Using the maple *Chow maple package*, we find that $h^{10} = \gamma_{10,1} + \gamma_{10,2}$ holds, while the second generic point is equal to $\gamma_{10,3}$.

Over L there are precisely three rational cycles in codimension 10, two of which are generic points of $\mathcal{R}_{(1,0,0,0)}(10)$ and one of them is the generic point of $\mathcal{U}(X_1/L)(10)$. If we manage to find a third rational cycle in $\mathrm{Ch}^{10}(X_1/k)$, which is linearly independent from the rational cycles h^{10} and $h^4x_6 + h^6x_4$ above, then we will automatically have a rational cycle of codimension 10 lying in the first shell, and thus a direct summand $\mathcal{U}(X_1/k)(10)$ by [Shells, Corollary 4.11]. This would immediately imply that the upper motive $\mathcal{U}(X_1/k)$ has Poincaré polynomial $1+t^{15}$.

In step *3.*, we have seen that $\gamma_{6,1}$ is rational over k. Interestingly $S^4(\gamma_{6,1}) = \gamma_{10,5} = [6,5,4,2,7,6,5,4,3,1]$ holds. Since the Steenrod operation conserves rationality of cycles, we have found an additional generic point. To substantiate our result, note that the *prodbases* routine calculates the very same cycle for the only Tate motive in $M(X_1/L)$ starting in codimension 10. We are done. $\square$

12.1.2 Theorem. *Let G have phase $[E_7,(1,0,0,0),2]$ and assume that the projective, homogeneous G-variety X_1 does not a have zero cycle of odd degree. Then the motivic decompositions of the projective, homogeneous G varieties are as follows*

Θ	$M(X_\Theta)$
$\Theta \subset \{1,3,4,6\}$	$\bigoplus_{u\in O_\Theta} \mathcal{U}(X_1)(u) \oplus \bigoplus_{i\in I_\Theta} \mathcal{R}_J(i)$
Any other	$\bigoplus_{i\in I_\Theta} \mathcal{R}_J(i)$

Index	**Poincaré Polynomial**
$\mathcal{U}(X_1)$	$(1+t^{15})$
$\mathcal{R}_J$	$(1+t)$

Index	**Poincaré Polynomial**
O_1	$(1+t+t^2)(1+t^6)(1+t^{10})$
O_3	$O_1(1+t^5)(1+t^9)$
O_4	$O_6(1+t^5)(1+t^6)$
O_6	$(1+t^4+t^8)(1+t^9)(1+t^{10})$
$O_{1,6}$	$(1+t+t^2+t^4+t^5+t^6+t^8+t^9+t^{10})(1+t^6)(1+t^9)(1+t^{10})$
I_1	$t^3(1+t+t^2)(1+t^3+t^5+t^6+t^8+2t^9+t^{11}+t^{12}+t^{13}+2t^{15}+t^{16}+t^{18}+t^{19}+t^{21}+t^{24})$
I_Θ	$[P(X_\Theta,t)-O_\Theta(1+t^{15})]/P(\mathcal{R}_J,t)$

Proof: The proof is a simple consequence of the Theorem above and the motivic decompositions established in Theorem 10.1.1. In the first step of the proof above it is shown, that all Rost motives over $k(X_1)$ lift to k. Also the upper motive $\mathcal{U}(X_1)$ of all projective, homogeneous G-varieties, which are not GSV, splits into $\mathbb{F}_2 \oplus \mathbb{F}_2(15)$. Therefore the polynomials describing the shifts of $\mathcal{U}(X_1)$ in the motives of these varieties in this theorem are simply the Tate polynomials from Theorem 10.1.1 divided by $1+t^{15}$ □

12.2 Motivic decomposition for [E7,(1,1,0,0),2]

We write $[E_7,(1,1,0,0),2]_6$ in case the upper motives of X_1, X_6 are not isomorphic and $[E_7,(1,1,0,0),2]_1$ in case they are. Here is our only result.

12.2.1 Theorem. *Let G have phase* $[E_7,(1,1,0,0),2]_1$ or $[E_7,(1,1,0,0),2]_6$ *over k and let A be the Tits algebra of G. Then the unique motivic decomposition of X_7 into indecomposable motivic summands is given by*

$$M(X_7) = \bigoplus_{s=0,9,17,26} \mathcal{U}(\mathrm{SB}(A))(s) \oplus \bigoplus_{i\in I} \mathcal{R}_J(i),$$

$$\textit{with } P(I,t) = t^{21}+t^{19}+t^{17}+t^{15}+t^{13}+t^{12}+t^{11}+t^{10}+t^8+t^6+t^4+t^2$$

$$\textit{and } P(\mathcal{U}(\mathrm{SB}(A)),t) = 1+t.$$

Proof: By the proof of Lemma 10.3.1, the variety X_1 induces a transition to $[D_4\times A_1,(1,1,0,0),2]$. We see in Theorem 10.2.2 that the decomposition of $M(X_7)$

now coincides with the one from the claim. Since the value of $J_2(G)$ does not change, all Rost motives in the decomposition lift to k by [PS22, Theorem 6.4].

To prove that the upper motive of $M(X_7/k)$ is isomorphic to $\mathcal{U}(\mathrm{SB}(A)/k)$, we only need to look at what happens when passing from k to $k(X_7)$ and to $k(\mathrm{SB}(A))$. In both cases we obtain anisotropic kernel D_4 by the GSV-table (for X_7) and the Lemma 7.1.9 (for $\mathrm{SB}(A)$). By Karpenko's theorem there can not be any other motivic summands in $M(X_7/k)$.

□

12.2.2. We have seen in Chapter 10 that passing to $k(X_6)$ makes the upper motives of some quadrics appear in many motivic decompositions of the X_Θ. If we could be sure that these upper motives can not be seen over the base field, then by our techniques applied so far, one can determine many more motivic decompositions. But looking at groups coming from the the $F_4 \times A_1$ construction, it is a possibility that some of these upper motives can in fact be seen over the base.

12.3 Constructing the invariant

In this final section we use results established in [S16] and several other sources, to show how the motivic decomposition obtained in the last section proves the existence of a cohomological invariant of degree five mod 2 for groups G of type E_7, that have $J_2(G) = (1, 0, 0, 0)$. We discuss a few consequences and the relation to the $F_4 \times A_1$ construction, which we know produces some of the mentioned groups.

12.3.1 Proposition. *Assume k is a field of characteristic zero. Let G be an anisotropic group of type E_7 over k, which splits over the generic point of its Tits algebra. Then there is an element $h_5 \in H^5(k, \mu_2)$, such that for any field extension L/k one has $\mathrm{res}(h_5)_{L/k} = 0 \in H^5(L, \mu_2)$ if and only if X_1 has a zero cycle of odd degree over L.*

Proof: Without loss of generality we can assume that the projective, homogeneous G-variety X_1 has no zero cycle of odd degree.

To a certain extent, we can mimic the proof from [S16]. Our tactic is to show that the requirements of [S16, Lemma 6.1 (b)] are satisfied. For this to achieve, we need to transform our initial starting situation. We remind the reader that the zero cycle condition is also included in the requirements of Theorem 12.1.2.

1. We consider the motivic decomposition of $M(X_1)$ from Theorem 12.1.2. It is valid for Chow motives with $\mathbb{F}_2$ coefficients. If one applies [SZ, Theorem 4.3] it becomes valid for Chow motives with $\mathbb{Z}_2$ coefficients, too.

The referenced theorem requires X_1 to be nilsplit, which means that the RNT needs to hold for Chow motives with $\mathbb{F}_2$ coefficients and that $M(X_1)$ becomes isomorphic to a sum of Tate motives over a finite field extension. The first condition is known to be satisfied. The second condition is even more trivial, since we know that G splits when its Tits algebra splits. Thus we can lift the motivic decomposition to $\mathbb{Z}_2$ coefficients.

2. Our goal is to use [S16, Lemma 8.6]. For this we need to adjust the statement of [S16, Lemma 8.5]. It refers to certain groups of type E_8 and their Borel variety. We replace G by an adjoint group of type E_7 and consider X_1 and its motivic decomposition calculated in Theorem 12.1.2. The upper motive of X_1 is not isomorphic to the upper motive of the Borel variety of G in our case. However checking the proof of the mentioned result by Voevodsky [Voe03b, Theorem 4.4], one finds that it is not specific to E_8, Borel varieties or (as also mentioned by Semenov) to norm quadrics. The essential requirement is that X_1 has a binary upper motive with $\mathbb{F}_2$ or $\mathbb{Z}_2$ coefficients. We can lift the decomposition of $M(X_1)$ to $\mathbb{Z}_2$ coefficients by *1.* and we can apply [S16, Lemma 8.6].

Then we obtain an exact triangle $\mathcal{X}\{15\} \to \mathcal{U}(X_1) \to \mathcal{X} \to \mathcal{X}\{15\}[1]$ in Voevodsky's motivic category $\mathrm{DM}_{-}^{eff}(k)$ of effective motives with $\mathbb{Z}_2$ coefficients (see [MVW06] for a deep treatment of motivic cohomology), with $\mathcal{X}$ denoting the motive of the standard simplicial scheme associated with X_1.

3. Let us consider X_1 again. To use [S16, Theorem 6.1 (b)], we need to show, that there is a morphism $Y \to X_1$ for Y being some ν_4-variety (see [S16, Definition 2.3]). In the proof of [Shells, Lemma 7.5], a closed irreducible subvariety Y'' of X is considered. It satisfies the initial conditions of [Shells, Lemma 7.5]. We replace X by our X_1.

As Y'' is a subvariety of X_1, we have a morphism $Y'' \to X_1$. Now we apply [S16, Lemma 7.1] and obtain a smooth projective irreducible variety Y which is birational to Y'' and admits a morphism $Y \to Y''$. It satisfies the requirements of [Shells, Lemma 7.5], too. Patching with the other morphism, shows that there is a morphism $Y \to X_1$.

4. We have seen that $\mathcal{U}(X_1)$ splits as $\mathbb{F}_2 \oplus \mathbb{F}_2(15)$ over any quadratic field extension that splits $\mathrm{SB}(A)$. Thus we can apply [Shells, Lemma 7.5], to conclude that there is some smooth projective irreducible variety Y of dimension 15, with $\mathcal{U}(Y) \simeq \mathcal{U}(X_1)$. We can choose this new Y to be the Y from *3.* right above, because the Y from above does satisfy the same properties required by [Shells, Lemma 7.5].

5. Now we show that this Y is a ν_4-variety (see [S16, Definition 2.3]). For this it is enough the check that the requirements of [S16, Lemma 6.2] are satisfied by Y. This is rather obvious, since the upper motive of Y is binary and 15-dimensional and coincides with the dimension of Y itself. Also it is isomorphic to the $\mathcal{U}(X_1)$ and thus the projector defining it behaves exactly like the respective projector in $\mathrm{Ch}(X_1 \times X_1)$ behaves over quadratic field extensions.

6. Summarizing all of the above, we have a ν_4-variety Y together with a morphism $Y \to X_1$. On top of that we can lift the motivic decomposition of X_1 to $\mathbb{Z}_2$ coefficients and in Theorem 12.1.2, we have seen that $\mathcal{U}(X_1)(18) \simeq \mathcal{U}(X_1)\{\dim(X_1) - 15\}$ is a summand of $M(X_1)$. Since we have the sequence from *2.* in $\mathrm{DM}_{-}^{eff}(k)$, all requirements of [S16, Theorem 6.1 (b)] are satisfied.

□

12.3.2 Remark. If the characteristic of k is not zero, many ingredients for the proof like [S16, Thrm. 6.1] can not be used.

The requirement that G splits over $k(\mathrm{SB}(A))$, does imply that the Brauer class of A divides h_5 by [OVV07, Thrm. 2.1]. This makes h_5 a decomposable invariant, just like any degree three invariant mod 2 for E_7^{ad} as shown by Merkurjev. By Remark 11.1.16, on the anisotropy of groups coming from the $F_4 \times A_1$ construction, one could argue that $f_5(\mathcal{J}) \cup Q$ is a decomposable degree seven invariant.

But this invariant, if one wants to call it that, is not reflected in the structure of the upper motive of any projective, homogeneous G-variety, as the dimension of the respective Rost motive is 63 and thus is simply too large to occur.

Lastly, it is worth mentioning that for classifying E_7^{ad} torsors mod 2 in general one needs at least *eight* parameters by [RY, Theorem 8.19]).

12.3.3 Example. Let $k = \mathbb{R}$ hold. Then it is well known that $H^5(k, \mu_2)$ is generated by $(-1)^5$. Assume we are given a group G of type E_7 over k, for which the new invariant h_5 is defined. If G is anisotropic, then $h_5(G) = (-1)^5$ holds. If $h_5(G)$ was zero, then X_1 would have a zero cycle of odd degree by the proposition above. Also, it would become isotropic over a field extension L/k of odd degree. This is obviously impossible. Thus over $\mathbb{R}$ we have that $h_5(G) = 0$ if and only if G is isotropic. Lastly, if G comes from $F_4 \times A_1$ construction and $k = \mathbb{R}$ holds, then we have $f_5(\mathcal{J}) = h_5(G)$, for $\mathcal{J}$ denoting the Albert algebra used in the construction.

12.3.4. (Comparison with the Semenov invariant) We have seen that once one knows that the upper motive of a certain variety is binary, one can conclude the existence of the invariant without too many other requirements. Finding cases of algebraic groups, such that a projective, homogeneous G-variety has a binary upper motive is a challenge by itself.

In case of the Semenov invariant from [S16] a classic construction by Tits was considered, but with a specific input. The key idea is to consider the $F_4 \times G_2$ construction for E_8 and choose as input pairs, coming solely from an F_4, which is reduced (i.e. $g_3(\mathcal{J}) = 0$). This is achieved by considering as inputs the outputs of the map

$$t : H^1(k, F_4) \longrightarrow H^1(k, F_4) \times H^1(k, G_2),$$

which sends $\mathcal{J}$ to $\mathcal{J} \times \mathcal{O}_{\mathcal{J}}$, while $\mathcal{O}_{\mathcal{J}}$ denotes the Octonion algebra lying under $\mathcal{J}$. The most analogous thing to our case is to require Q to divide $f_3(\mathcal{J})$, when considering the $F_4 \times A_1$ construction.

The even part of the Rost invariant of any E_8 constructed as above is two times $f_3(\mathcal{J})$ and thus zero in $H^3(k, \mu_2)$. The respective set of torsors is denoted by $H^1(k, E_8)_0$. The $F_4 \times A_1$ construction has no analogue for this, because any F_4 is simply connected and so there is no degree two invariant for $\mathcal{J}$ which could be killed by Q. In the case of E_8 one can use the Killing form (see [Jac]), for showing that anisotropic outputs can even be obtained with the chosen inputs. By Remark 11.1.16 it looks very much like this is also possible for the E_7 case.

Also in both cases one passes to $k(X)$, for some projective, homogeneous G-variety X to make G isotropic. In the E_8 case it is concluded that any projective, homogeneous G-variety is a GSV as the Rost invariant stays zero over L and by the result [Gar01a, Theorem 0.5] on the triviality of the kernel of the Rost invariant for groups of low rank, the claim follows. By the GSV-table, then $J_2(G/k) = (0,0,0,1)$ holds.

In the $F_4 \times A_1$ case however, G does not split over $k(X)$ for every X and curiously $J_2(G/k) = (1,0,0,0)$ holds. The J-invariant in the E_7 case is only used to lift the Rost motives to k, while for E_8 it is used to show that $\mathcal{U}(X)$ is binary and 15-dimensional.

In a certain sense the $F_4 \times A_1$ case is more complicated, as G has one more stage of splitting and additionally requires concrete calculations in the Chow ring of some X. Both constructions are completely understood over $\mathbb{R}$ by the results in [Jac].

For any G, for which the Semenov invariant u is defined and which comes from the $F_4 \times G_2$ construction, the invariant u is the $f_5(\mathcal{J})$ coming from the Albert algebra $\mathcal{J}$ used as input. The same relation holds for our new E_7-invariant $h_5(G)$ and $f_5(\mathcal{J})$, as we will see as the final result.

12.3.5. (A conclusion on the Garibaldi invariant) The existence of our invariant kind of has been predicted. In [GS, Remark 3.10] an invariant g for E_7^{sc} is defined by proving that there is a map $m : H^1(k, E_7^{sc}) \to H^1(k, E_8)_0$, which is not trivial in general and then composing it with the invariant u by Semenov.

If one ignores the fact that in one case one considers a simply connected E_7, while in the other case one deals with an adjoint E_7, our new invariant h_5 looks like a composition of the map by Garibaldi and the invariant of Semenov. As the comment at the end of [GS, Remark 3.10] implies, all these invariants agree for $k = \mathbb{R}$ and if the Tits algebra Q is not split.

12.3.6 Theorem. *Let G come from the $F_4 \times A_1$ construction with the inputs $\mathcal{J}, Q \neq 0$, such that $J_2(G) = (1,0,0,0)$ holds. Let $h_5(G)$ denote the new invariant from above. Then $f_5(\mathcal{J}) = h_5(G)$ holds.*

Proof: To tackle this problem, we start with considering the generic case. Note that it is mandatory that Q divides $f_3(\mathcal{J})$, or otherwise $J_2(G) \neq (1,0,0,0)$ and h_5 is not even defined. So the inputs are not totally generic.

In the generic case the variety X_1 has no 0-cycles of odd degree, since we can specialize to the real numbers, where this is the case. (Alternatively, one can argue using the Killing form as in Chapter 11). Therefore, the invariant h_5 is not zero.

On the other hand, in general, if the invariant f_5 is zero, then the output of the Tits construction gives an isotropic group of type E_7. Therefore, if f_5 is zero over some field extension of the base field, then h_5 is zero over this field extension as well.

The description of the kernel of the restriction homomorphism

$$H^n(k, \mu_2) \to H^n(k(X), \mu_2)$$

given in [OVV07, Theorem 2.1] applied to the case, when $n = 5$ and X is the norm quadric associated to the f_5-invariant, implies that $h_5 = f_5$ in the generic case.

But then specializing the h_5-invariant we obtain the general case of the Tits construction as above. $\square$

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