Emergent Quasi-Bosonicity in Interacting Fermi Gases

Dissertation von Martin Ravn Christiansen

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Emergent Quasi-Bosonicity in Interacting Fermi Gases

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Declaration

This thesis is based on the work of the papers [11, 12, 13], to which I contributed significantly.

Work of [11] features in the chapters 2, 3, 4, 6, 7, 8, 10, A and B.

Work of [12] features in chapter 11.

Work of [13] features in the chapters 2, 4, 5, 6, 7 and 8.

The contents of the chapters 9 and C are original.

München, 06.02.2023 Martin Ravn Christiansen

Eidesstattliche Versicherung

(Siehe Promotionsordnung vom 12.07.11, § 8, Abs. 2 Pkt. .5.)

Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

München, 06.02.2023 Martin Ravn Christiansen

Contents

Abstract

English

This thesis concerns the correlation structure of interacting Fermi gases on a torus in the mean-field regime. A bosonization method in the spirit of Sawada[6] is developed to analyze the system, and is applied to obtain an upper bound for the correlation energy of the system for a wide class of repulsive interaction potentials, including the Coulomb potential.

This upper bound includes both a bosonic contribution, as found in the bosonic model of Sawada, and an exchange contribution, as was found by Gell-Mann and Brueckner[5] but which was missed by Sawada's model.

An extension to weakly attractive potentials is also presented, as is an outline of the derivation of an effective Hamiltonian for regular interaction potentials, and the construction of plasmon states for this outside of the mean-field setting.

This thesis is based on the papers [11, 12, 13].

Deutsch

Diese Dissertation betrifft die Korrelationsstruktur der wechselwirkenden Fermi Gase auf einem Torus im Mittelfeldregime. Es wird eine Bosonisierungsmethode im Geist von Sawada[6] entwickelt, um das System zu analysieren und zur Herleitung einer oberen Schranke der Korrelationsenergie des Systems für eine breite Klasse von abstoßenden Wechselwirkungspotenziale, einschließlich des Coulomb-Potenzials.

Diese obere Schranke beinhaltet sowohl einen bosonisches Beitrag, wie in dem bosonischen Modell von Sawada, als auch einen Vertauschungsbeitrag, wie er von Gell-Mann und Brueckner[5] entdeckt wurde, der aber von Sawadas Modell nicht erfasst wurde.

Eine Erweiterung zu schwach attraktiven Potenzialen wird ebenfalls vorgestellt, ebenso wie ein Umriss der Herleitung eines effektiven Hamiltonoperators für reguläre Wechselwirkungspotenziale, und die Konstruktion von Plasmonzuständen für diesen außerhalb des Mittelfeldrahmens.

Diese Dissertation basiert auf den Fachartikeln [11, 12, 13].

Chapter 1

Introduction

A Fermi gas is a quantum system described by a Hamiltonian of the form

$$H = -\sum_{i=1}^{N} \Delta_i + \sum_{1 \leq i < j \leq N} V(x_i - x_j)$$

on a fermionic N-particle space. Here the first term represents the kinetic energy of the fermions (in units where $\frac{\hbar^2}{2m} = 1$) while the second term represents pair interactions through a potential V.

The potential of greatest physical interest is the (background-subtracted) Coulomb potential, in which case the system is referred to as *jellium*. Jellium is the simplest model of electrons in a metal which still includes all electron-electron interactions.

In the 1930-40's, theoretical calculations based on applying the Hartree-Fock approximation to the jellium model exhibited a large discrepancy when compared to experimental values. Furthermore, pertubative methods broke down already at second-order, presenting the physicists of the time with the puzzle of how to model an interacting many-body system without being able to apply perturbative methods.

As the Hartree-Fock approximation amounts to neglecting particle correlations, the question was how to include these in the computation. The first steps toward this was taken in the early 1950's by Bohm and Pines[1, 2, 3, 4], who argued that the correlations at play were of an essentially bosonic nature, which would manifest itself as quantized collective electron oscillations, which they dubbed *plasmons*.

Adding plasmon modes to the jellium model by hand, they argued that these served to regularize the electron-electron interaction to the point that second-order perturbation could be applied - provided that certain terms appearing in their analysis could be neglected, the assumption of which was referred to as the "Random Phase Approximation" (RPA).

The validity of the RPA and the manner in which the plasmons were introduced was a somewhat controversial issue, but they were effectively justified by two later works: The first was by Gell-Mann and Brueckner[5], who were able to derive the *correlation energy* - the difference between the ground state and Fermi state energies - of the jellium model directly, by performing a formal resummation of the divergent perturbation series for this, and finding agreement with Pines' calculation.

The second work was by Sawada[6] (and expanded on by Sawada-Brueckner-Fukuda-Brout[7]). He observed that certain terms of the Hamiltonian could, when expressed in the second-quantized picture, be interpreted as quadratic operators with respect to almost-bosonic operators. By studying this corresponding bosonic Hamiltonian, he was also able to derive the correlation energy - with the exception of one term, which was explicitly fermionic in nature.

With these works, the correlation energy was thought to be well-understood by the physics community, but presenting a mathematically rigorous derivation of this remains a major open problem in mathematical physics to this day. Recently there has however been much progress on the corresponding mean-field problem, in which the potential is scaled by a factor proportional to $N^{-\frac{1}{3}}$.

The first results on this problem were by Benedikter-Nam-Porta-Schlein-Seiringer[8, 9] (see also [10]), who were able to prove an asymptotic formula for the correlation energy for highly regular potentials V by employing a bosonization method, albeit in a manner different from Sawada's original observation, to define an analog of a bosonic Bogolubov transformation which could be applied to analyze the system.

Subsequently I and my Ph.D. advisors extended this result significantly in [11], in which we both proved an asymptotic formula for the correlation energy for more general potentials and additionally derived an effective quasi-bosonic Hamiltonian governing the low-lying eigenstates of the Fermi gas. We accomplished this by developing a bosonization method different from that of [8, 9] and more in the spirit of Sawada.

The aim of this thesis is to present this method and the results we have obtained by it.

1.1 Main Results

Before stating the main results, let us introduce the setting properly and define some notation: We consider for a given Fermi momentum $k_F > 0$ the mean-field Hamiltonian

$$H_N = -\sum_{i=1}^{N} \Delta_i + k_F^{-1} \sum_{1 \le i < j \le N} V(x_i - x_j) \tag{1.1.1}$$

on $\mathcal{H}_N = \bigwedge^N L^2(\mathbb{T}^3; \mathbb{C}^s)$, where $\mathbb{T}^3$ is the 3-torus of sidelength 2π and $s \in \mathbb{N}$ is the number of spin states of the system. The number of particles, N, is determined by k_F through the relation $N = s\left|\overline{B}(0, k_F) \cap \mathbb{Z}^3\right|$.

We take the interaction potential V to admit the Fourier decomposition

$$V(x) = \frac{1}{(2\pi)^3} \sum_{k \in \mathbb{Z}^3} \hat{V}_k e^{ik\cdot x} \tag{1.1.2}$$

and assume that the Fourier coefficients obey (with $\mathbb{Z}^3_* = \mathbb{Z}^3 \backslash \{0\}$)

$$\hat{V}_k = \hat{V}_{-k} \quad \text{and} \quad \hat{V}_k \ge 0, \quad k \in \mathbb{Z}^3_*, \tag{1.1.3}$$

in other words we consider a symmetric and repulsive interaction potential.

We define for $k \in \mathbb{Z}^3_*$ the *lune* L_k by

$$L_k = \left\{ p \in \mathbb{Z}^3 \mid |p - k| \leq k_F < |p| \right\} \tag{1.1.4}$$

and let further $\lambda_{k,p} = \frac{1}{2}\left(|p|^2 - |p-k|^2\right)$ for $p \in L_k$.

The main focus of the thesis is the derivation of the following:

Theorem 1.1.1. *Let $\sum_{k\in\mathbb{Z}^3} \hat{V}_k^2 < \infty$. Then it holds that*

$$\inf(\sigma(H_N)) \leq E_F + E_{\text{corr,bos}} + E_{\text{corr,ex}} + C\sqrt{\sum_{k\in\mathbb{Z}^3_*} \hat{V}_k^2 \min\{|k|, k_F\}}, \quad k_F \to \infty,$$

where $E_F = \langle \psi_F, H_N \psi_F \rangle$ is the energy of the Fermi state,

$$E_{\text{corr,bos}} = \frac{1}{\pi} \sum_{k\in\mathbb{Z}^3_*} \int_0^\infty F\left(\frac{s\hat{V}_k k_F^{-1}}{(2\pi)^3} \sum_{p\in L_k} \frac{\lambda_{k,p}}{\lambda_{k,p}^2 + t^2} \right) dt, \quad F(x) = \log(1+x) - x,$$

is the bosonic contribution (to the correlation energy) and

$$E_{\text{corr,ex}} = \frac{s k_F^{-2}}{4(2\pi)^6} \sum_{k,l\in\mathbb{Z}^3_*} \hat{V}_k \hat{V}_l \sum_{p,q\in L_k\cap L_l} \frac{\delta_{p+q,k+l}}{\lambda_{k,p} + \lambda_{k,q}}$$

is the exchange contribution, for a constant $C > 0$ depending only on $\sum_{k\in\mathbb{Z}^3_*} \hat{V}_k^2$ and s.

This result was originally presented in [13] (for $s = 1$). Although we have so far only been able to prove this asymptotic statement as an upper bound, it constitutes a major improvement over the corresponding one of [11]: Not only does it apply to singular potentials (including the Coulomb potential), it also includes the "exchange contribution" $E_{\text{corr,ex}}$, which is the term that was missing from Sawada's purely bosonic model, and which was also lacking in the previously proved results for non-singular potentials (for which $E_{\text{corr,ex}}$ is of much lower order than the rest).

In the case of the Coulomb potential, i.e. $\hat{V}_k \sim |k|^{-2}$, $E_{\text{corr,bos}}$ is of order $k_F \log(k_F)$ and $E_{\text{corr,ex}}$ is of order k_F, while the error term of the theorem is of order $\sqrt{\log(k_F)}$. The precision of the result is thus almost an entire order of magnitude. Furthermore, we may observe that for any potential with $\sum_{k\in\mathbb{Z}^3} \hat{V}_k^2 < \infty$, the error term is at most of order $\sqrt{k_F}$ whereas $E_{\text{corr,bos}}$ is at least order k_F, so there is always a sharp distinction between the correlation energy and the error term.

After concluding this theorem we will make the observation that our proof in fact allows us to generalize this result to slightly *attractive* potentials, proving the following:

Theorem 1.1.2. *Assuming the weaker condition that $\hat{V}_k \geq -(1-\epsilon)\frac{4\pi^2}{s}$ for some $\epsilon > 0$ and all $k \in \mathbb{Z}^3_*$, it continues to hold that*

$$\inf(\sigma(H_N)) \leq E_F + E_{\text{corr,bos}} + E_{\text{corr,ex}} + C\sqrt{\sum_{k\in\mathbb{Z}^3_*} \hat{V}_k^2 \min\{|k|, k_F\}}, \quad k_F \to \infty,$$

where now $C > 0$ depends on $\sum_{k\in\mathbb{Z}^3_*} \hat{V}_k^2$, s and ϵ.

This result has not been presented before. We remark that the condition on $\hat{V}_k$ is nearly optimal, in the sense that if $\hat{V}_k < -\frac{4\pi^2}{s}$ for some $k \in \mathbb{Z}^3_*$ then the corresponding term of $E_{\text{corr,bos}}$ is not even well-defined, as the argument of the logarithm of the integrand is then strictly negative near $t = 0$.

These results only concern upper bounds for the ground state energy of H_N. In [11] we also proved the following stronger operator-level result regarding H_N, albeit only under high regularity assumptions on V:

Theorem 1.1.3. *Let $\sum_{k\in\mathbb{Z}^3_*} \hat{V}_k \, |k| < \infty$. Then there exists a unitary transformation $\mathcal{U} : \mathcal{H}_N \to \mathcal{H}_N$, depending implicitly upon k_F, such that*

$$\mathcal{U} H_N \mathcal{U}^* = E_F + E_{\text{corr,bos}} + H_{\text{eff}} + \mathcal{E}$$

where

$$H_{\text{eff}} = H'_{\text{kin}} + 2 \sum_{k\in\mathbb{Z}^3_*} \sum_{p,q\in L_k} \langle e_p, (\tilde{E}_k - h_k) e_q \rangle b^*_{k,p} b_{k,q}$$

for $\tilde{E}_k = \left(h_k^{\frac{1}{2}} (h_k + 2P_k) h_k^{\frac{1}{2}} \right)^{\frac{1}{2}}$.

Furthermore, it holds for every normalized eigenstate Ψ of H_N with $\langle \Psi, H_N \Psi \rangle \leq E_F + \kappa k_F$, $\kappa > 0$, that the error operator $\mathcal{E}$ obeys

$$|\langle \Psi, \mathcal{E}\Psi \rangle| \, , \, |\langle \mathcal{U}\Psi, \mathcal{E}\mathcal{U}\Psi \rangle| \leq C k_F^{1-\frac{1}{94}+\varepsilon}, \quad k_F \to \infty,$$

for any $\varepsilon > 0$, the constant $C > 0$ depending only on V, κ and ε.

In words, the theorem states that the Hamiltonian H_N is, with respect to the *low-lying* eigenstates (as demarked by the condition $\langle \Psi, H_N \Psi \rangle \leq E_F + \kappa k_F$), up to the constant terms $E_F + E_{\text{corr,bos}}$ unitarily equivalent with the effective Hamiltonian H_{eff}, to leading order in k_F.

Here the effective Hamiltonian consists of two parts: The *localized kinetic operator* H'_{kin}, which appears naturally during the extraction of E_F, and a *quasi-bosonic* term involving the *excitation operators* (for $s = 1$)

$$b_{k,p} = c^*_{p-k} c_p, \quad b^*_{k,p} = c^*_p c_{p-k}, \quad k \in \mathbb{Z}^3_*, \, p \in L_k, \tag{1.1.5}$$

where $(c^*_p)_{p\in\mathbb{Z}^3_*}$ and $(c_p)_{p\in\mathbb{Z}^3_*}$ denote the fermionic creation and annihilation operators associated with the plane-wave states. In the definition of the quasi-bosonic term also appears certain "one-body operators" $h_k, P_k : \ell^2(L_k) \to \ell^2(L_k)$ which naturally appear during the diagonalization process which extracts $E_{\text{corr,bos}}$.

From the fact that $\tilde{E}_k \geq h_k$ it follows that $H_{\text{eff}} \geq 0$, so (as the ground state certainly is low-lying) the theorem in particular implies that

$$\inf(\sigma(H_N)) = E_F + E_{\text{corr,bos}} + O\left(k_F^{1-\frac{1}{94}+\varepsilon} \right), \quad k_F \to \infty, \tag{1.1.6}$$

i.e. the ground-state energy is indeed $E_F + E_{\text{corr,bos}}$ to leading order, provided $\sum_{k\in\mathbb{Z}^3_*} \hat{V}_k |k| < \infty$. Note that $E_{\text{corr,ex}}$ is absent, which is a consequence of the assumed regularity - even just assuming boundedness of V, i.e. that $\sum_{k\in\mathbb{Z}^3_*} \hat{V}_k < \infty$, it holds that $E_{\text{corr,ex}} \le C k_F^{-1}$.

We will not give a full proof of Theorem 1.1.3 in this thesis, but in Section 10 we present the main ideas and techniques that lead to its conclusion.

What is particularly noteworthy about Theorem 1.1.3 is that it not only yields a lower bound on the correlation energy, but also identifies the operator which should govern the low-lying excitations of the system - in the physical case this would include the plasmon states. Unfortunately the mean-field scaling suppresses these states, making it difficult to say much about H_{eff} in this setting.

Given the physical importance of plasmons it is however interesting to extrapolate this result and consider H_{eff} by itself without imposing the mean-field scaling or strict regularity assumptions on the potential, which is what we did in [12], obtaining a result of the following form:

Theorem 1.1.4. *In the non-mean-field scaled setting the following holds: Let $\hat{V}_k = g|k|^{-2}$, $k \in \mathbb{Z}^3_*$, for some $g > 0$. Then for any $\delta \in \left(0, \frac{1}{2}\right)$ and $\varepsilon \in (0,2)$ there exists for all $k \in \overline{B}\left(0, k_F^\delta\right) \cap \mathbb{Z}^3_*$ and $M \le k_F^\varepsilon$ a normalized state $\Psi \in \mathcal{H}_N$ such that*

$$\|(H_{\text{eff}} - M\epsilon_k)\Psi\| \le C|k|^{-1}\sqrt{k_F} M^{\frac{5}{2}}, \quad k_F \to \infty,$$

where ϵ_k denotes the greatest eigenvalue of $2\tilde{E}_k$. ϵ_k obeys $\epsilon_k \ge c k_F^{\frac{3}{2}}$ and

$$0 \le \epsilon_k - 2\sqrt{\frac{s}{(2\pi)^3}\frac{g}{|k|^2}\sum_{p\in L_k}\lambda_{k,p} + \frac{\sum_{p\in L_k}\lambda_{k,p}^3}{\sum_{p\in L_k}\lambda_{k,p}}} \le C k_F^{-\frac{1}{2}}|k|^4, \quad k_F \to \infty,$$

for constants $c, C > 0$ depending only on g.

We present a proof of this in Section 11.

The theorem states that for k and M in certain ranges, there exists an "approximate eigenvector" Ψ for H_{eff} with approximate eigenvalue $M\epsilon_k$ - in fact Ψ is explicitly given as the normalization of

$$b^*(\phi)^M \psi_F, \quad b_k^*(\phi) = \sum_{p\in L_k} \langle e_p, \phi\rangle b_{k,p}^*, \tag{1.1.7}$$

where ϕ is the normalized eigenstate of $2\tilde{E}$ with eigenvalue ϵ_k, which mimics the definition of a bosonic state with M "ϕ excitations".

Calling Ψ an approximate eigenvector is justified by Markov's inequality in the operator form $1_{\mathbb{R}\setminus[E-\delta,E+\delta]}(H) \le \delta^{-1}|H-E|$, as it implies that

$$\left\|1_{\mathbb{R}\setminus[M\varepsilon_k-\delta, M\varepsilon_k+\delta]}(H_{\text{eff}})\Psi\right\| \ll 1, \quad |k|^{-1}\sqrt{k_F}M^{\frac{5}{2}} \ll \delta, \tag{1.1.8}$$

i.e. Ψ is spectrally localized at $E = M\varepsilon_k$ on the scale $|k|^{-1}\sqrt{k_F}M^{\frac{5}{2}}$. As $M\varepsilon_k \sim M k_F^{\frac{3}{2}}$ this is a nontrivial statement for $M \ll (k_F|k|)^{\frac{2}{3}}$. One can also view this in a dynamical setting: By the time evolution estimate $\left\|\left(e^{-itH}-e^{-itE}\right)\psi\right\| \leq \|(H-E)\psi\|\, t$ the theorem implies that

$$\left\|\left(e^{-itH_{\text{eff}}}-e^{iM\epsilon_k t}\right)\Psi\right\| \ll 1, \quad M\epsilon_k t \ll k_F\,|k|\,M^{-\frac{3}{2}}; \tag{1.1.9}$$

as $(M\epsilon_k)^{-1}$ is the characteristic timescale of oscillation of Ψ this is again non-trivial for $M \ll (k_F\,|k|)^{\frac{2}{3}}$.

The formula for ϵ_k is also interesting - if one formally replaces the Riemann sums by their corresponding integrals, one finds that (to leading order)

$$\epsilon_k \sim \sqrt{2gn + \frac{12}{5}k_F^2\,|k|^2} \tag{1.1.10}$$

where $n = \frac{N}{(2\pi)^3} = \frac{s|B_F|}{(2\pi)^3} \sim \frac{1}{(2\pi)^3}\frac{4\pi s}{3}k_F^3$ is the number density of the system. In the physical case $g = 4\pi e^2$ (e being the elementary charge), so recalling that $\frac{\hbar^2}{2m} = 1$ we find

$$\epsilon_k \sim \sqrt{8\pi e^2 n\frac{\hbar^2}{2m} + \frac{12}{5}k_F^2\,|k|^2} = \hbar\sqrt{\frac{4\pi n e^2}{m} + \frac{12}{5}\frac{k_F^2\,|k|^2}{\hbar^2}} = \hbar\sqrt{\omega_0^2 + \frac{3}{5}v_F^2\,|k|^2} \tag{1.1.11}$$

where $\omega_0 = \sqrt{\frac{4\pi n e^2}{m}}$ is the famous plasmon frequency (in CGS units) and $v_F = m^{-1}\hbar k_F = 2\hbar^{-1}k_F$ is the Fermi velocity, corresponding to the well-known plasmon frequency dispersion relation

$$\omega_k^2 \approx \omega_0^2 + \frac{3}{5}v_F^2\,|k|^2\,. \tag{1.1.12}$$

This shows that if Theorem 1.1.3 could be generalized to the full physical setting, it would not only account for the correlation energy but also for the plasmons predicted by Bohm and Pines in the 1950's.

Although proving such a result would be an extremely challenging task, it is our hope that the work covered by this thesis will be useful in this endeavor.

1.2 Outline of the Thesis

We begin our analysis of the Hamiltonian H_N in Section 2 by extracting the leading order contribution to the ground state energy of H_N, which is the energy of the Fermi state ψ_F. We do this by normal-ordering H_N (in its second-quantized form) "with respect to ψ_F". After doing so we observe that the resulting terms which violate the separation between states inside and outside the Fermi ball are quasi-bosonic, in that they obey commutation relations reminiscent of the canonical commutation relations of a bosonic system.

In Section 3 we review the theory of bosonic Bogolubov transformations, originally introduced in [14] to explain the phenomenon of superfluidity, to prepare for the analysis of the quasi-bosonic operators. In particular we describe how one may explicitly define a Bogolubov transformation which diagonalizes a given positive-definite quadratic Hamiltonian.

We then apply the bosonic theory to our study of the Fermi gas in Section 4 wherein we implement the diagonalization procedure in the quasi-bosonic setting. This is done by mimicking the bosonic case to define a quasi-bosonic Bogolubov transformation $e^{\mathcal{K}}$ which diagonalizes the bosonizable terms of H_N up to *exchange terms* - terms which arise due to the deviation from the exact CCR.

In Section 5 we justify that the transformation $e^{\mathcal{K}}$ is well-defined by establishing that the generating kernel $\mathcal{K}$ is in fact a bounded operator under the condition $\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2 < \infty$. More generally we establish a bound on $\mathcal{K}$ in terms of the *excitation number operator* $\mathcal{N}_E$ which will also allow us to control error terms later on by a Gronwall-type argument.

In order to analyze the exchange terms which appeared during the diagonalization procedure we require detailed information on the one-body operators of the corresponding bosonic problem. We analyze these in section 6, obtaining asymptotically optimal elementwise estimates of the main operators.

We then turn to the exchange terms themselves in Section 7. By performing a detailed analysis of all of the possible kinds of terms which emerge from these upon normal-ordering with respect to ψ_F, we extract the exchange contribution $E_{\text{corr,ex}}$ and bound the remaining terms using $\mathcal{N}_E$.

In Section 8 we bring all our work together. After deriving bounds on the *non-bosonizable terms* - the terms of H_N which do not fit into the quasi-bosonic setting - we apply our prior results to estimate the energy of the trial state $e^{\mathcal{K}}\psi_F$, which results in the proof of Theorem 1.1.1.

This is followed by Section 9 wherein we describe the modifications necessary to extend Theorem 1.1.1 to weakly attractive potentials in order to conclude Theorem 1.1.2.

In Section 10 we first present a general outline of the approach that leads to Theorem 1.1.3, followed by a more detalied examination of the key ideas which leads to its conclusion.

Finally, in Section 11, we consider plasmon states for the effective operator of Theorem 1.1.3 in the non-mean-field setting, proving a generalization of Theorem 1.1.4 valid for arbitrary repulsive potentials V.

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Chapter 2

Localization of the Hamiltonian at the Fermi State

In this section we begin our study of the interacting Fermi gas by extracting the energy of the Fermi state ψ_F from the Hamiltonian operator H_N. We do this by normal-ordering H_N "with respect to ψ_F", a procedure which we refer to as localization since it serves to fix ψ_F as our point of reference, making it analogous to the vacuum state of a field theory.

The result of this procedure is summarized in the following:

Proposition 2.0.1. *It holds that*

$$H_N = E_F + H'_{\mathrm{kin}} + \sum_{k\in\mathbb{Z}^3_*} \frac{\hat{V}_k k_F^{-1}}{2\,(2\pi)^3}\Big(2B_k^* B_k + B_k B_{-k} + B_{-k}^* B_k^*\Big) + \mathcal{C} + \mathcal{Q}$$

where $E_F = \langle \psi_F, H_N \psi_F\rangle$ *is the energy of the Fermi state,*

$$H'_{\mathrm{kin}} = \sum_{p\in B_F^c}^{\sigma} |p|^2\, c_{p,\sigma}^* c_{p,\sigma} - \sum_{p\in B_F}^{\sigma} |p|^2\, c_{p,\sigma} c_{p,\sigma}^*, \quad B_k = \sum_{p\in L_k}^{\sigma} c_{p-k,\sigma}^* c_{p,\sigma},$$

and

$$\mathcal{C} = \frac{k_F^{-1}}{(2\pi)^3} \sum_{k\in\mathbb{Z}^3_*} \hat{V}_k \,\mathrm{Re}\Big(\big(B_k + B_{-k}^*\big)^* D_k\Big)$$

$$\mathcal{Q} = \frac{k_F^{-1}}{2\,(2\pi)^3} \sum_{k\in\mathbb{Z}^3_*} \hat{V}_k \left(D_k^* D_k - \sum_{p\in L_k}^{\sigma}\Big(c_{p,\sigma}^* c_{p,\sigma} + c_{p-k,\sigma} c_{p-k,\sigma}^*\Big)\right)$$

for $D_k = \mathrm{d}\Gamma\Big(P_{B_F} e^{-ik\cdot x} P_{B_F}\Big) + \mathrm{d}\Gamma\Big(P_{B_F^c} e^{-ik\cdot x} P_{B_F^c}\Big)$.

After carrying out this procedure we will see how the concept of quasi-bosonicity emerges: The operators B_k of the above representation obey commutation relations which are analogous to the canonical commutation relations of a bosonic system. We end the section by exploring this phenomenon, in particular showing how the kinetic operator H'_{kin} can be made to fit into such a bosonic picture by considering the *excitation operators*

$$b_{k,p} = \frac{1}{\sqrt{s}} \sum_{\sigma=1}^{s} c^*_{p-k,\sigma} c_{p,\sigma}, \quad b^*_{k,p} = \frac{1}{\sqrt{s}} \sum_{\sigma=1}^{s} c^*_{p,\sigma} c_{p-k,\sigma}. \tag{2.0.1}$$

2.1 Notation and Conventions

Before we begin the analysis proper we review the notation which we will use throughout the paper.

We consider the one-particle space $\mathfrak{h} = L^2(\mathbb{T}^3; \mathbb{C}^s)$, where $\mathbb{T}^3 = [0, 2\pi]^3$ with periodic boundary conditions and $s \in \mathbb{N}$ is the number of spin states of the system. We denote by $\mathcal{H}_N = \bigwedge^N \mathfrak{h}$ the associated fermionic N-particle space.

$\mathfrak{h}$ is spanned by the orthonormal basis of plane wave states $(u_{p,\sigma})_{p\in\mathbb{Z}^3}^{1\leq\sigma\leq s}$, given by

$$u_{p,\sigma}(x) = (2\pi)^{-\frac{3}{2}} e^{ip\cdot x} v_\sigma, \quad p \in \mathbb{Z}^3, \tag{2.1.1}$$

where v_σ denotes the σ-th standard basis vector of $\mathbb{C}^s$.

We denote by $c^*_{p,\sigma}$, $c_{p,\sigma}$ the creation and annihilation operators associated to the plane wave states, which obey the canonical anticommutation relations (CAR)

$$\left\{c_{p,\sigma}, c^*_{q,\tau}\right\} = \delta_{p,q}\delta_{\sigma,\tau}, \quad \{c_{p,\sigma}, c_{q,\tau}\} = 0 = \left\{c^*_{p,\sigma}, c^*_{q,\tau}\right\}, \tag{2.1.2}$$

for all $p, q \in \mathbb{Z}^3$ and $1 \leq \sigma, \tau \leq s$.

Sums involving the creation and annihilation operators will generally run over all spin states. To reduce clutter we will denote this by writing the summed indices over the sum signs, leaving the summation range implicit, e.g. for the number operators $\mathcal{N}$ we simply write

$$\mathcal{N} = \sum_{p\in\mathbb{Z}^3} \sum_{\sigma=1}^{s} c^*_{p,\sigma} c_{p,\sigma} = \sum_{p\in\mathbb{Z}^3}^{\sigma} c^*_{p,\sigma} c_{p,\sigma}. \tag{2.1.3}$$

For a given Fermi momentum $k_F > 0$ we denote by B_F the (closed) Fermi ball

$$B_F = \overline{B}(0, k_F) \cap \mathbb{Z}^3 \tag{2.1.4}$$

and write B_F^c for the complement of B_F with respect to $\mathbb{Z}^3$. We define ψ_F to be the Fermi state

$$\psi_F = \bigwedge_{p\in B_F}^{\sigma} u_{p,\sigma} \in \mathcal{H}_N, \quad N = s\,|B_F|\,. \tag{2.1.5}$$

For the sake of brevity we define $\mathbb{Z}^3_* = \mathbb{Z}^3 \backslash \{0\}$ and for $k \in \mathbb{Z}^3_*$ define the lune L_k by

$$L_k = (B_F + k)\backslash B_F = \left\{p \in \mathbb{Z}^3 \mid |p - k| \leq k_F < |p|\right\}. \tag{2.1.6}$$

The Hamiltonian Operator H_N

We consider for given $k_F > 0$ the mean-field Hamiltonian

$$H_N = H_{\text{kin}} + k_F^{-1} H_{\text{int}}, \quad D(H_N) = D(H_{\text{kin}}), \tag{2.1.7}$$

on $\mathcal{H}_N$, where $N = s\,|B_F|$. H_{kin} is the standard kinetic operator

$$H_{\text{kin}} = \mathrm{d}\Gamma(-\Delta) = -\sum_{i=1}^{N} \Delta_i, \quad D(H_{\text{kin}}) = \bigwedge^{N} H^2\left(\mathbb{T}^3; \mathbb{C}^s\right), \tag{2.1.8}$$

and H_{int} describes the pairwise interaction between N particles through a potential $V : \mathbb{T}^3 \to \mathbb{R}$,

$$H_{\text{int}} = \sum_{1 \leq i < j \leq N} V(x_i - x_j). \tag{2.1.9}$$

We will take $V \in L^2(\mathbb{T}^3)$, in which case H_N is a self-adjoint operator on $\mathcal{H}_N$. Letting the Fourier decomposition of V be given by

$$V(x) = \frac{1}{(2\pi)^3} \sum_{k \in \mathbb{Z}^3} \hat{V}_k e^{ik\cdot x} \tag{2.1.10}$$

we furthermore assume that $\hat{V}_k = \hat{V}_{-k}$ and $\hat{V}_k \geq 0$ for all $k \in \mathbb{Z}^3_*$, i.e. that V is *repulsive.*

For the remainder of the thesis we will work in the second-quantized picture, in which it is well-known that H_{kin} and H_{int} can be expressed as

$$H_{\text{kin}} = \sum_{p \in \mathbb{Z}^3}^{\sigma} |p|^2 \, c_{p,\sigma}^* c_{p,\sigma}, \quad H_{\text{int}} = \frac{1}{2\,(2\pi)^3} \sum_{k \in \mathbb{Z}^3} \hat{V}_k \sum_{p,q \in \mathbb{Z}^3}^{\sigma,\tau} c_{p+k,\sigma}^* c_{q-k,\tau}^* c_{q,\tau} c_{p,\sigma}. \tag{2.1.11}$$

2.2 Extraction of the Fermi State Energy

It is well-known that the Fermi state ψ_F is characterized by the conditions

$$c_p \psi_F = 0 = c_q^* \psi_F, \quad p \in B_F^c, \, q \in B_F, \tag{2.2.1}$$

and so the Fermi state energy $E_F = \langle \psi_F, H_N \psi_F \rangle$ can be extracted from H_N by normal-ordering this "with respect to ψ_F", in the sense that the creation and annihilation operators of equation (2.1.11) are normal-ordered as if $c_{p,\sigma}^*$ were an annihilation operator for $p \in B_F$.

Consider first the kinetic operator: By the CAR we can write H_{kin} in the form

$$\begin{aligned} H_{\text{kin}} &= \sum_{p \in B_F^c}^{\sigma} |p|^2 \, c_{p,\sigma}^* c_{p,\sigma} + \sum_{p \in B_F}^{\sigma} |p|^2 \, c_{p,\sigma}^* c_{p,\sigma} = \sum_{p \in B_F^c}^{\sigma} |p|^2 \, c_{p,\sigma}^* c_{p,\sigma} + \sum_{p \in B_F}^{\sigma} |p|^2 - \sum_{p \in B_F}^{\sigma} |p|^2 \, c_{p,\sigma} c_{p,\sigma}^* \\ &= s \sum_{p \in B_F} |p|^2 + H_{\text{kin}}' \end{aligned} \tag{2.2.2}$$

where we define the *localized kinetic operator* $H'_{\mathrm{kin}} : D(H_{\mathrm{kin}}) \subset \mathcal{H}_N \to \mathcal{H}_N$ by

$$H'_{\mathrm{kin}} = \sum_{p \in B_F^c}^{\sigma} |p|^2 \, c^*_{p,\sigma} c_{p,\sigma} - \sum_{p \in B_F}^{\sigma} |p|^2 \, c_{p,\sigma} c^*_{p,\sigma}. \tag{2.2.3}$$

H'_{kin} is clearly normal-ordered with respect to ψ_F, and so the quantity $s \sum_{p \in B_F} |p|^2$ is simply the kinetic energy of ψ_F, whence we can write the relation between H_{kin} and H'_{kin} as

$$H_{\mathrm{kin}} = \langle \psi_F, H_{\mathrm{kin}} \psi_F \rangle + H'_{\mathrm{kin}}. \tag{2.2.4}$$

To normal-order H_{int} we first rewrite this in a factorized form: By the CAR we can write

$$\begin{aligned}
H_{\mathrm{int}} &= \frac{1}{2\,(2\pi)^3} \sum_{k \in \mathbb{Z}^3} \hat{V}_k \sum_{p,q \in \mathbb{Z}^3}^{\sigma,\tau} c^*_{p+k,\sigma} \Big(c_{p,\sigma} c^*_{q-k,\tau} - \delta_{p,q-k} \delta_{\sigma,\tau} \Big) c_{q,\tau} \\
&= \frac{1}{2\,(2\pi)^3} \sum_{k \in \mathbb{Z}^3} \hat{V}_k \left(\left(\sum_{p \in \mathbb{Z}^3}^{\sigma} c^*_{p+k,\sigma} c_{p,\sigma} \right) \left(\sum_{q \in \mathbb{Z}^3}^{\tau} c^*_{q-k,\tau} c_{q,\tau} \right) - \sum_{q \in \mathbb{Z}^3}^{\sigma} c^*_{q,\sigma} c_{q,\sigma} \right) \\
&= \frac{1}{2\,(2\pi)^3} \sum_{k \in \mathbb{Z}^3} \hat{V}_k \Big(\mathrm{d}\Gamma\Big(e^{-ik\cdot x}\Big)^* \mathrm{d}\Gamma\Big(e^{-ik\cdot x}\Big) - \mathcal{N} \Big) \\
&= \frac{N(N-1)}{2\,(2\pi)^3} \hat{V}_0 + \frac{1}{2\,(2\pi)^3} \sum_{k \in \mathbb{Z}^3_*} \hat{V}_k \Big(\mathrm{d}\Gamma\Big(e^{-ik\cdot x}\Big)^* \mathrm{d}\Gamma\Big(e^{-ik\cdot x}\Big) - N \Big)
\end{aligned} \tag{2.2.5}$$

where we recognized the operator $\mathrm{d}\Gamma\Big(e^{-ik\cdot x}\Big)$ as

$$\mathrm{d}\Gamma\Big(e^{-ik\cdot x}\Big) = \sum_{p,q \in \mathbb{Z}^3}^{\sigma,\tau} \Big\langle u_{p,\sigma}, e^{-ik\cdot x} u_{q,\tau} \Big\rangle \, c^*_{p,\sigma} c_{q,\tau} = \sum_{p,q \in \mathbb{Z}^3}^{\sigma,\tau} \delta_{p,q-k} \delta_{\sigma,\tau} c^*_{p,\sigma} c_{q,\tau} = \sum_{q \in \mathbb{Z}^3}^{\tau} c^*_{q-k,\tau} c_{q,\tau} \tag{2.2.6}$$

and used that $\mathrm{d}\Gamma\Big(e^{-i(0\cdot x)}\Big) = \mathrm{d}\Gamma(1) = \mathcal{N} = N$ on $\mathcal{H}_N$. Now, with $P_{B_F} : \mathfrak{h} \to \mathfrak{h}$ denoting the orthogonal projection onto $\mathrm{span}(u_{p,\sigma})_{p \in \mathbb{Z}^3}^{1 \leq \sigma \leq s}$ and $P_{B_F^c} = 1 - P_{B_F}$ denoting its complement, we can decompose $\mathrm{d}\Gamma\Big(e^{-ik\cdot x}\Big)$ as

$$\mathrm{d}\Gamma\Big(e^{-ik\cdot x}\Big) = \mathrm{d}\Gamma\Big(\Big(P_{B_F} + P_{B_F^c}\Big) e^{-ik\cdot x} \Big(P_{B_F} + P_{B_F^c}\Big)\Big) = B_k + B^*_{-k} + D_k \tag{2.2.7}$$

where the operator B_k is given by

$$B_k = \mathrm{d}\Gamma\Big(P_{B_F} e^{-ik\cdot x} P_{B_F^c}\Big) = \sum_{p \in B_F}^{\sigma} \sum_{q \in B_F^c}^{\tau} \delta_{p,q-k} \delta_{\sigma,\tau} c^*_{p,\sigma} c_{q,\tau} = \sum_{q \in L_k}^{\tau} c^*_{q-k,\tau} c_{q,\tau} \tag{2.2.8}$$

as the Kronecker delta $\delta_{p,q-k}$ precisely restrict the summation to $q \in L_k$, and the operator D_k is simply

$$D_k = \mathrm{d}\Gamma\Big(P_{B_F} e^{-ik\cdot x} P_{B_F}\Big) + \mathrm{d}\Gamma\Big(P_{B_F^c} e^{-ik\cdot x} P_{B_F^c}\Big). \tag{2.2.9}$$

We can thus write H_{int} as

$$H_{\mathrm{int}} = \frac{N(N-1)}{2\,(2\pi)^3}\hat{V}_0 + \frac{1}{2\,(2\pi)^3}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k\Big(\Big(\Big(B_k + B^*_{-k}\Big)^*\Big(B_k + B^*_{-k}\Big) - N\Big) \qquad (2.2.10)$$
$$+2\,\mathrm{Re}\Big(\Big(B_k + B^*_{-k}\Big)^* D_k\Big) + D^*_k D_k\Big).$$

Now, it is easily verified that $B_k\psi_F = D_k\psi_F = D^*_k\psi_F = 0$ for any $k \in \mathbb{Z}^3_*$, and so the terms on the last line are effectively normal-ordered, and it only remains to normal-order the terms of the first sum. For this we calculate the commutator $[B_k, B^*_k]$: By the CAR and basic commutator identities, we find that

$$\begin{aligned}
[B_k, B^*_k] &= \sum_{p\in L_k}^{\sigma}\sum_{q\in L_k}^{\tau}\Big[c^*_{p-k,\sigma}c_{p,\sigma}, c^*_{q,\tau}c_{q-k,\tau}\Big] \\
&= \sum_{p\in L_k}^{\sigma}\sum_{q\in L_k}^{\tau}\Big(c^*_{p-k,\sigma}\Big[c_{p,\sigma}, c^*_{q,\tau}c_{q-k,\tau}\Big] + \Big[c^*_{p-k,\sigma}, c^*_{q,\tau}c_{q-k,\tau}\Big]c_{p,\sigma}\Big) \\
&= \sum_{p\in L_k}^{\sigma}\sum_{q\in L_k}^{\tau}c^*_{p-k,\sigma}\Big(\Big\{c_{p,\sigma}, c^*_{q,\tau}\Big\}c_{q-k,\tau} - c^*_{q,\tau}\{c_{p,\sigma}, c_{q-k,\tau}\}\Big) \qquad (2.2.11)\\
&+ \sum_{p\in L_k}^{\sigma}\sum_{q\in L_k}^{\tau}\Big(\Big\{c^*_{p-k,\sigma}, c^*_{q,\tau}\Big\}c_{q-k,\tau} - c^*_{q,\tau}\Big\{c^*_{p-k,\sigma}, c_{q-k,\tau}\Big\}\Big)c_{p,\sigma} \\
&= \sum_{p\in L_k}^{\sigma}\sum_{q\in L_k}^{\tau}\delta_{p,q}\delta_{\sigma,\tau}c^*_{p-k,\sigma}c_{q-k,\tau} - \sum_{p\in L_k}^{\sigma}\sum_{q\in L_k}^{\tau}\delta_{p-k,q-k}\delta_{\sigma,\tau}c^*_{q,\tau}c_{p,\sigma} \\
&= \sum_{p\in L_k}^{\sigma}c^*_{p-k,\sigma}c_{p-k,\sigma} - \sum_{p\in L_k}^{\sigma}c^*_{p,\sigma}c_{p,\sigma} = s\,|L_k| - \sum_{p\in L_k}^{\sigma}\Big(c^*_{p,\sigma}c_{p,\sigma} + c_{p-k,\sigma}c^*_{p-k,\sigma}\Big)
\end{aligned}$$

and using also that $\hat{V}_k = \hat{V}_{-k}$ we may then write H_{int} as

$$H_{\mathrm{int}} = \frac{N(N-1)}{2\,(2\pi)^3}\hat{V}_0 - \frac{1}{2\,(2\pi)^3}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k(N - s\,|L_k|) + \frac{1}{2\,(2\pi)^3}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k\Big(2B^*_kB_k + B^*_kB^*_{-k} + B_{-k}B_k\Big)$$
$$+ \frac{1}{2\,(2\pi)^3}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k\left(2\,\mathrm{Re}\Big(\Big(B_k + B^*_{-k}\Big)^*D_k\Big) + D^*_kD_k - \sum_{p\in L_k}^{\sigma}\Big(c^*_{p,\sigma}c_{p,\sigma} + c_{p-k,\sigma}c^*_{p-k,\sigma}\Big)\right). \qquad (2.2.12)$$

Note that the sum $\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k(N - s\,|L_k|)$ is actually finite, as $s\,|L_k| = s\,|B_F| = N$ when $|k| > 2k_F$.

The terms on the right-hand side of this equation are now normal-ordered, and in particular we see that

$$\langle\psi_F, H_{\mathrm{int}}\psi_F\rangle = \frac{N(N-1)}{2\,(2\pi)^3}\hat{V}_0 - \frac{1}{2\,(2\pi)^3}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k(N - s\,|L_k|) \qquad (2.2.13)$$

whence we can write

$$k_F^{-1} H_{\text{int}} = \left\langle \psi_F, k_F^{-1} H_{\text{int}} \psi_F \right\rangle + \sum_{k \in \mathbb{Z}^3_*} \frac{\hat{V}_k k_F^{-1}}{2\,(2\pi)^3} \Big(2B_k^* B_k + B_k^* B_{-k}^* + B_{-k} B_k \Big) + \mathcal{C} + \mathcal{Q} \quad (2.2.14)$$

where the *cubic* and *quartic* terms $\mathcal{C}$ and $\mathcal{Q}$ are defined by

$$\mathcal{C} = \frac{k_F^{-1}}{(2\pi)^3} \sum_{k \in \mathbb{Z}^3_*} \hat{V}_k \operatorname{Re}\Big(\big(B_k + B_{-k}^* \big)^* D_k \Big) \quad (2.2.15)$$

$$\mathcal{Q} = \frac{k_F^{-1}}{2\,(2\pi)^3} \sum_{k \in \mathbb{Z}^3_*} \hat{V}_k \left(D_k^* D_k - \sum_{p \in L_k}^{\sigma} \Big(c_{p,\sigma}^* c_{p,\sigma} + c_{p-k,\sigma} c_{p-k,\sigma}^* \Big) \right).$$

The terms $\mathcal{C}$ and $\mathcal{Q}$ constitute the *non-bosonizable terms*: They fall outside the quasi-bosonic approach we will introduce below, and so we consider them as error terms to be analyzed separately at the end.

Combining the equations (2.2.4) and (2.2.14) now yields Proposition 2.0.1.

2.3 Remarks on the Localization Procedure

Before continuing with our analysis we must comment on some subtle details of the localization procedure.

Consider the localized kinetic operator H'_{kin}, which we defined by

$$H'_{\text{kin}} = \sum_{p \in B_F^c}^{\sigma} |p|^2 \, c_{p,\sigma}^* c_{p,\sigma} - \sum_{p \in B_F}^{\sigma} |p|^2 \, c_{p,\sigma} c_{p,\sigma}^*. \quad (2.3.1)$$

This expression is a sum of two terms, one manifestly positive and one manifestly negative. As the creation and annihilation operator for orthogonal states are (algebraically) independent, one would therefore not expect H'_{kin} to have a definite sign. But this is not the case, as we can argue that

$$H'_{\text{kin}} = H_{\text{kin}} - \langle \psi_F, H_{\text{kin}} \psi_F \rangle \geq 0 \quad (2.3.2)$$

since $\langle \psi_F, H_{\text{kin}} \psi_F \rangle$ is the ground state energy of H_{kin}.

The resolution of this apparent paradox lies in the domains of definition: The argument for non-definiteness of H'_{kin} is valid *when viewed as an operator on the full Fock space* $\mathcal{F}^-(\mathfrak{h})$, where the assertion that $\langle \psi_F, H_{\text{kin}} \psi_F \rangle$ is the ground state energy of H_{kin} is wrong.

That $H'_{\text{kin}} \geq 0$ is nonetheless correct when viewed as an operator on $\mathcal{H}_N$, precisely by the second observation. The first argument fails in this case because the creation and annihilation operators (or more precisely, the products $c_{p,\sigma}^* c_{p,\sigma}$) are *not* independent on $\mathcal{H}_N$: Normal-ordering $\mathcal{N}$ with respect to ψ_F, we see that

$$N = \mathcal{N} = \sum_{p \in \mathbb{Z}^3}^{\sigma} c_{p,\sigma}^* c_{p,\sigma} = \sum_{p \in B_F^c}^{\sigma} c_{p,\sigma}^* c_{p,\sigma} + \sum_{p \in B_F}^{\sigma} 1 - \sum_{p \in B_F}^{\sigma} c_{p,\sigma} c_{p,\sigma}^* \quad (2.3.3)$$

$$= s\,|B_F| + \sum_{p\in B_F^c}^{\sigma} c_{p,\sigma}^* c_{p,\sigma} - \sum_{p\in B_F}^{\sigma} c_{p,\sigma} c_{p,\sigma}^*,$$

so as $N = s\,|B_F|$ we conclude the identity

$$\sum_{p\in B_F^c}^{\sigma} c_{p,\sigma}^* c_{p,\sigma} = \sum_{p\in B_F}^{\sigma} c_{p,\sigma} c_{p,\sigma}^* \quad \text{on}\,\mathcal{H}_N. \tag{2.3.4}$$

This is the statement of *particle-hole symmetry*: The expression on the left-hand side is appropriately labeled the *excitation number operator* $\mathcal{N}_E$, since just as $\mathcal{N}$ "counts" the number of particles in a state of the full Fock space, $\mathcal{N}_E$ "counts" the number of states lying outside B_F in a state on $\mathcal{H}_N$, which is to say the number of excitations relative to ψ_F.

The expression on the right-hand side may be similarly thought of as a "hole number operator", as it similarly counts the number of states lying inside B_F that a given state is *lacking*. Equation (2.3.4) thus makes explicit the observation that any excitation relative to ψ_F must be accompanied by a "hole".

This also explains why H'_{kin}, despite being the difference of two positive operators, remains positive: To take advantage of the negative part, one must create a hole in the Fermi ball. But particle number conservation then demands that one must create an excitation outside this, and as $|p| > k_F \geq |q|$ for all $p \in B_F^c$, $q \in B_F$, this procedure will always lead to an increase in energy.

In fact we can use equation (2.3.4) to make this argument precise, since it implies that

$$H'_{\text{kin}} = H'_{\text{kin}} - k_F^2\mathcal{N}_E + k_F^2\mathcal{N}_E = \sum_{p\in B_F^c}^{\sigma} \left(|p|^2 - k_F^2\right) c_{p,\sigma}^* c_{p,\sigma} + \sum_{p\in B_F}^{\sigma} \left(k_F^2 - |p|^2\right) c_{p,\sigma} c_{p,\sigma}^* \tag{2.3.5}$$

and now both of the sums on the right-hand side are manifestly non-negative.

2.4 The Quasi-Bosonic Excitation Operators

Now we consider the structure of the terms

$$\sum_{k\in\mathbb{Z}_*^3} \frac{\hat{V}_k k_F^{-1}}{2\,(2\pi)^3} \left(2B_k^* B_k + B_k^* B_{-k}^* + B_{-k} B_k\right), \tag{2.4.1}$$

which appear in the decomposition of H_N of Proposition 2.0.1, further. Consider the operators $B_k = \sum_{p\in L_k}^{\sigma} c_{p-k,\sigma}^* c_{p,\sigma}$: It is easily seen that for any $k, l \in \mathbb{Z}_*^3$ it holds that $[B_k, B_l] = [B_k^*, B_l^*] = 0$, while a slight modification of the calculation of equation (2.2.11) shows that

$$[B_k, B_l^*] = s\,|L_k|\,\delta_{k,l} - \sum_{p\in L_k}^{\sigma} \sum_{q\in L_l} \left(\delta_{p-k,q-l} c_{q,\sigma}^* c_{p,\sigma} + \delta_{p,q} c_{q-l,\sigma} c_{p-k,\sigma}^*\right). \tag{2.4.2}$$

Consider the sum on the right: By the Cauchy-Schwarz and triangle inequalities we can bound the first part as

$$\left|\sum_{p\in L_k}^{\sigma}\sum_{q\in L_l}\left\langle\Psi,\delta_{p-k,q-l}c_{q,\sigma}^{*}c_{p,\sigma}\Psi\right\rangle\right|\leq\sum_{p\in L_k}^{\sigma}\sum_{q\in L_l}\delta_{p-k,q-l}\left\|c_{q,\sigma}\Psi\right\|\left\|c_{p,\sigma}\Psi\right\| \tag{2.4.3}$$
$$\leq\sqrt{\sum_{q\in L_l}^{\sigma}\left\|c_{q,\sigma}\Psi\right\|^2}\sqrt{\sum_{p\in L_k}^{\sigma}\left\|c_{p,\sigma}\Psi\right\|^2}\leq\langle\Psi,\mathcal{N}_E\Psi\rangle$$

for any $\Psi\in\mathcal{H}_N$, and likewise for the second part of the sum. If one now defines the rescaled operators $B_k'=(s\,|L_k|)^{-\frac{1}{2}}B_k$, one sees that these obey commutation relations of the form

$$\left[B_k',(B_l')^*\right]=\delta_{k,l}+O\Big(k_F^{-2}\mathcal{N}_E\Big),\quad[B_k',B_l']=0=\left[(B_k')^*,(B_l')^*\right], \tag{2.4.4}$$

since (as we will see) $|L_k|\geq ck_F^2$. With respect to states for which $\langle\Psi,\mathcal{N}_E\Psi\rangle$ is small, these relations approximate the canonical commutation relations for *bosonic* creation and annihilation operators a_k^*, a_k, which are

$$[a_k,a_l^*]=\delta_{k,l},\quad[a_k,a_l]=0=[a_k^*,a_l^*]\,. \tag{2.4.5}$$

This motivates describing the B_k as being *quasi-bosonic* operators. In view of this, it is tempting to view the terms

$$\sum_{k\in\mathbb{Z}_*^3}\frac{\hat{V}_k k_F^{-1}}{2\,(2\pi)^3}\Big(2B_k^*B_k+B_k^*B_{-k}^*+B_{-k}B_k\Big) \tag{2.4.6}$$

as analogous to a quadratic Hamiltonian in the bosonic setting, to which the theory of Bogolubov transformations applies. This is the spirit of what we will do, but there is a catch: The kinetic operator H_{kin}' is not of a similar form, and the operators B_k do not behave bosonically with respect to it.

The solution to this problem is to further decompose the operators B_k: We define for $k\in\mathbb{Z}_*^3$, $p\in L_k$, the *excitation operators* $b_{k,p}^*$, $b_{k,p}$ by

$$b_{k,p}=\frac{1}{\sqrt{s}}\sum_{\sigma=1}^{s}c_{p-k,\sigma}^*c_{p,\sigma},\quad b_{k,p}^*=\frac{1}{\sqrt{s}}\sum_{\sigma=1}^{s}c_{p,\sigma}^*c_{p-k,\sigma}. \tag{2.4.7}$$

The name is due to the fact that the action of $b_{k,p}^*$ is to annihilate a state at momentum $p-k\in B_F$ and create a state at momentum $p\in B_F^c$ (irrespective of spin), which is to say excite the state $p-k$ to p.

Note that the $b_{k,p}$ and B_k operators are simply related as $B_k=\sqrt{s}\sum_{p\in L_k}b_{k,p}$. Furthermore, the excitation operators also obey quasi-bosonic commutation relations:

Lemma 2.4.1. *For any $k,l\in\mathbb{Z}_*^3$, $p\in L_k$ and $q\in L_l$ it holds that*

$$\left[b_{k,p},b_{l,q}^*\right]=\delta_{k,l}\delta_{p,q}+\varepsilon_{k,l}(p;q),\quad[b_{k,p},b_{l,q}]=0=\left[b_{k,p}^*,b_{l,q}^*\right],$$

where $\varepsilon_{k,l}(p;q)=-s^{-1}\sum_{\sigma=1}^{s}\Big(\delta_{p,q}c_{q-l,\sigma}c_{p-k,\sigma}^*+\delta_{p-k,q-l}c_{q,\sigma}^*c_{p,\sigma}\Big)$.

Proof: By the CAR and commutator identities we calculate that

$$[b_{k,p}, b_{l,q}] = \frac{1}{s}\sum_{\sigma,\tau=1}^{s}\left[c_{p-k,\sigma}^{*}c_{p,\sigma}, c_{q-l,\tau}^{*}c_{q,\tau}\right] = \frac{1}{s}\sum_{\sigma,\tau=1}^{s}\left(c_{p-k,\sigma}^{*}\left[c_{p,\sigma}, c_{q-l,\tau}^{*}c_{q,\tau}\right] + \left[c_{p-k,\sigma}^{*}, c_{q-l,\tau}^{*}c_{q,\tau}\right]c_{p,\sigma}\right)$$
$$= \frac{1}{s}\sum_{\sigma,\tau=1}^{s}\left(c_{p-k,\sigma}^{*}\left\{c_{p,\sigma}, c_{q-l,\tau}^{*}\right\}c_{q,\tau} - c_{q-l,\tau}^{*}\left\{c_{p-k,\sigma}^{*}, c_{q,\tau}\right\}c_{p,\sigma}\right) = 0 \tag{2.4.8}$$

as the anticommutators vanish by disjointness of B_F and B_F^c. $[b_{k,p}^{*}, b_{l,q}^{*}]$ then likewise vanishes, while for $[b_{k,p}, b_{l,q}^{*}]$

$$\left[b_{k,p}, b_{l,q}^{*}\right] = \frac{1}{s}\sum_{\sigma,\tau=1}^{s}\left[c_{p-k,\sigma}^{*}c_{p,\sigma}, c_{q,\tau}^{*}c_{q-l,\tau}\right] = \frac{1}{s}\sum_{\sigma,\tau=1}^{s}\left(c_{p-k,\sigma}^{*}\left\{c_{p,\sigma}, c_{q,\tau}^{*}\right\}c_{q-l,\tau} - c_{q,\tau}^{*}\left\{c_{p-k,\sigma}^{*}, c_{q-l,\tau}\right\}c_{p,\sigma}\right)$$
$$= \frac{1}{s}\sum_{\sigma=1}^{s}\left(\delta_{p,q}c_{p-k,\sigma}^{*}c_{q-l,\sigma} - \delta_{p-k,q-l}c_{q,\sigma}^{*}c_{p,\sigma}\right) = \delta_{k,l}\delta_{p,q} + \varepsilon_{k,l}(p;q). \tag{2.4.9}$$

□

Again these commutation relations are similar to those of bosonic operators, now indexed by $k \in \mathbb{Z}_*^3$ and $p \in L_k$, but differing by the appearance of the *exchange correction* $\varepsilon_{k,l}(p;q)$, which evidently acts by exchanging the hole states with momenta $p-k$ and $q-l$ if $p = q$, i.e. if the excited states match, or swaps the states with momenta p and q if $p-k = q-l$, i.e. if the hole states match.

The presence of $\varepsilon_{k,l}(p;q)$ can be considered a consequence of the fact that holes and excited states are not uniquely associated with one another - indeed, for any $p \in B_F^c$, *every* hole state can be excited into this state, so there is a kind of "overlap" between the excitation operators, which the exchange correction accounts for.

Unlike what was the case for the B_k' operators, these correction terms can however not be expected to be "small" individually. They can however still be considered small "on average", as the sum $\sum_{p\in L_k}\sum_{q\in L_l}\varepsilon_{k,l}(p;q)$ simply reproduces the correction term of equation (2.4.2) (up to a spin factor).

This is generally an unavoidable point: As we will see in Section 7, the exchange contribution of Theorem 1.1.1 in fact originates from these exchange corrections, so an attempt at treating these as simple error terms (as was done in the works [8, 9, 10]) is bound to miss this.

Now, the reason that the excitation operators are preferable to the B_k operators is that these do in fact behave bosonically with respect to H_{kin}':

Lemma 2.4.2. *For any $k \in \mathbb{Z}_*^3$ and $p \in L_k$ it holds that*

$$\left[H_{\text{kin}}', b_{k,p}^{*}\right] = \left(|p|^2 - |p-k|^2\right)b_{k,p}^{*}.$$

Proof: As $H_{\text{kin}}' = \sum_{q\in B_F^c}^{\tau}|q|^2 c_{q,\tau}^{*}c_{q,\tau} - \sum_{q\in B_F}^{\tau}|q|^2 c_{q,\tau}c_{q,\tau}^{*}$ we calculate the commutator with each sum: First is

$$\sum_{q\in B_F^c}^{\tau}\left[|q|^2 c_{q,\tau}^{*}c_{q,\tau}, b_{k,p}^{*}\right] = \frac{1}{\sqrt{s}}\sum_{q\in B_F^c}^{\sigma,\tau}|q|^2\left[c_{q,\tau}^{*}c_{q,\tau}, c_{p,\sigma}^{*}c_{p-k,\sigma}\right] \tag{2.4.10}$$

$$= \frac{1}{\sqrt{s}} \sum_{q \in B_F^c}^{\sigma,\tau} |q|^2 \left(c_{q,\tau}^* \left\{ c_{q,\tau}, c_{p,\sigma}^* \right\} c_{p-k,\sigma} - c_{p,\sigma}^* \left\{ c_{q,\tau}^*, c_{p-k,\sigma} \right\} c_{q,\tau} \right)$$
$$= \frac{1}{\sqrt{s}} \sum_{q \in B_F^c}^{\sigma,\tau} \delta_{p,q} \delta_{\sigma,\tau} |q|^2 c_{q,\tau}^* c_{p-k,\sigma} = \frac{|p|^2}{\sqrt{s}} \sum_{\sigma=1}^{s} c_{p,\sigma}^* c_{p-k,\sigma} = |p|^2 b_{k,p}^*$$

as the second anticommutator vanishes by disjointness of B_F and B_F^c. Similarly, for the second sum

$$\sum_{q \in B_F}^{\tau} \left[|q|^2 c_{q,\tau} c_{q,\tau}^*, b_{k,p}^* \right] = \frac{1}{\sqrt{s}} \sum_{q \in B_F}^{\sigma,\tau} |q|^2 \left[c_{q,\tau} c_{q,\tau}^*, c_{p,\sigma}^* c_{p-k,\sigma} \right] \tag{2.4.11}$$
$$= \frac{1}{\sqrt{s}} \sum_{q \in B_F}^{\sigma,\tau} |q|^2 \left(-c_{q,\tau} c_{p,\sigma}^* \left\{ c_{q,\tau}^*, c_{p-k,\sigma} \right\} + \left\{ c_{q,\tau}, c_{p,\sigma}^* \right\} c_{p-k,\sigma} c_{q,\tau}^* \right)$$
$$= -\frac{1}{\sqrt{s}} \sum_{q \in B_F}^{\sigma,\tau} |q|^2 \delta_{q,p-k} \delta_{\sigma,\tau} c_{q,\tau} c_{p,\sigma}^* = \frac{1}{\sqrt{s}} \sum_{\sigma=1}^{s} |p-k|^2 c_{p,\sigma}^* c_{p-k,\sigma} = |p-k|^2 b_{k,p}^*$$

and the claim follows.

□

This commutation relation mimicks that of a diagonal bosonic quadratic operator, which is

$$\left[\sum \epsilon_{l,q} a_{l,q}^* a_{l,q}, a_{k,p}^* \right] = \epsilon_{k,p} a_{k,p}^* \tag{2.4.12}$$

whence we may informally think of H'_{kin} as

$$H'_{\text{kin}} \sim \sum_{k \in \mathbb{Z}^3} \sum_{p \in L_k} \left(|p|^2 - |p-k|^2 \right) b_{k,p}^* b_{k,p}. \tag{2.4.13}$$

In fact the lemma tells us that H'_{kin} is much better behaved than the expression on the right-hand side: Unlike that, the commutator $[H'_{\text{kin}}, b_{k,p}^*]$ behaves *exactly* bosonically, without any additional error terms. In the subsequent sections we will see that it is precisely through such commutators that H'_{kin} will enter our analysis. For this reason, working with the excitation operators $b_{k,p}$ will prove to be extremely advantageous.

Chapter 3

Overview of Bosonic Bogolubov Transformations

In this section we review some of the general theory of Bogolubov transformations in the bosonic setting. Although the object of study of this thesis is a fermionic system, our approach to this will be through a quasi-bosonic analysis of the fermionic Hamiltonian, and while this of course differs from the exact bosonic case, we will carry out the quasi-bosonic analysis by imitating the exact bosonic setting. For this reason we find it best to review this first so that we may focus on the implementation of the analysis and the discrepancies arising from the quasi-bosonicity in the remainder of the thesis.

This is particularly important as our treatment of Bogolubov transformations will differ from the "usual" one, in that we will view *quadratic operators*, formed by pairs of creation and annihilation operators, as the fundamental object of study, rather than the creation and annihilation operators themselves.

Before we begin the review we must remark on the level of rigor of this section: Bosonic creation and annihilation operators are inherently unbounded operators, and so a full account of this subject would necessitate discussing domains of definition and other subtle details. As the purpose of this section is only to motivate our approach to the fermionic problem later on we will however not address these here.

We will employ the following notation: V denotes a *real* n-dimensional Hilbert space, to which is associated the bosonic Fock space $\mathcal{F}^+(V) = \bigoplus_{N=0}^{\infty} \bigotimes_{\mathrm{sym}}^{N} V$. To any element $\varphi \in V$ there corresponds the creation and annihilation operators $a^*(\varphi)$ and $a(\varphi)$, which act on $\mathcal{F}^+(V)$. These are (formal) adjoints of one another and obey the canonical commutation relations (CCR): For any $\varphi, \psi \in V$ it holds that

$$[a(\varphi), a^*(\psi)] = \langle \varphi, \psi \rangle \,, \quad [a(\varphi), a(\psi)] = 0 = [a^*(\varphi), a^*(\psi)] \,. \tag{3.0.1}$$

Furthermore, the mappings $\varphi \mapsto a(\varphi), a^*(\varphi)$ are linear.

3.1 Quadratic Hamiltonians and Bogolubov Transformations

Similarly to how we can to any $\varphi \in V$ associate the two operators $a(\varphi)$ and $a^*(\varphi)$, we can to any symmetric operators $A, B : V \to V$ associate two kinds of *quadratic operators* acting on $\mathcal{F}^+(V)$: The first kind is the usual second-quantization, given by

$$\mathrm{d}\Gamma(A) = \sum_{i,j=1}^{n} \langle e_i, Ae_j \rangle \, a^*(e_i)a(e_j) = \sum_{i=1}^{n} a^*(Ae_i)a(e_i) \tag{3.1.1}$$

where $(e_i)_{i=1}^n$ denotes any orthonormal basis of V (the operator is independent of this choice, as guaranteed by Lemma 3.2.1 below). The second kind is of the form

$$\begin{aligned} Q(B) &= \sum_{i,j=1}^{n} \langle e_i, Be_j \rangle \, (a(e_i)a(e_j) + a^*(e_j)a^*(e_i)) \\ &= \sum_{i=1}^{n} (a(Be_i)a(e_i) + a^*(e_i)a^*(Be_i)). \end{aligned} \tag{3.1.2}$$

We define a *quadratic Hamiltonian* to be an operator H, acting on $\mathcal{F}^+(V)$, of the form

$$H = 2\,\mathrm{d}\Gamma(A) + Q(B). \tag{3.1.3}$$

(The factor of 2 will be convenient below.)

The importance of quadratic Hamiltonians lies in the fact that they can (under suitable assumptions) be *diagonalized*, in the sense that there exists a unitary transformation $\mathcal{U} : \mathcal{F}^+(V) \to \mathcal{F}^+(V)$ such that

$$\mathcal{U} H \mathcal{U}^* = 2\,\mathrm{d}\Gamma(E) + E_0 \tag{3.1.4}$$

for a symmetric operator $E : V \to V$ and $E_0 \in \mathbb{R}$, i.e. a quadratic Hamiltonian is unitarily equivalent to a second-quantized one-body operator plus a constant. As second-quantized operators are simple objects, the properties of quadratic Hamiltonians are thus in principle also simple, provided one can describe $\mathcal{U}$ explicitly enough to relate the operators A and B to E.

In this section we review the explicit construction of such *Bogolubov transformations* $\mathcal{U}$. More precisely, we will consider the Bogolubov transformations which can be written as $\mathcal{U} = e^{\mathcal{K}}$ where $\mathcal{K}$ is of the form

$$\begin{aligned} \mathcal{K} &= \frac{1}{2} \sum_{i,j=1}^{n} \langle e_i, Ke_j \rangle \, (a(e_i)a(e_j) - a^*(e_j)a^*(e_i)) \\ &= \frac{1}{2} \sum_{i=1}^{n} (a(Ke_i)a(e_i) - a^*(e_i)a^*(Ke_i)) \end{aligned} \tag{3.1.5}$$

for a symmetric operator $K : V \to V$ (the *transformation kernel*). Note that from the second line it is clear that $\mathcal{K}^* = -\mathcal{K}$, so such a $\mathcal{K}$ will indeed generate a unitary transformation.

The action of $e^{\mathcal{K}}$ on creation and annihilation operators can be determined as follows: By the CCR we compute that

$$\begin{aligned}[\mathcal{K}, a(\varphi)] &= \frac{1}{2}\sum_{i=1}^{n}\left[a(Ke_i)a(e_i) - a^*(e_i)a^*(Ke_i), a(\varphi)\right] = \frac{1}{2}\sum_{i=1}^{n}\left[a(\varphi), a^*(e_i)a^*(Ke_i)\right] \\ &= \frac{1}{2}\sum_{i=1}^{n}\left(a^*(e_i)\left[a(\varphi), a^*(Ke_i)\right] + \left[a(\varphi), a^*(e_i)\right]a^*(Ke_i)\right) \\ &= \frac{1}{2}\sum_{i=1}^{n}\left(a^*(e_i)\left\langle \varphi, Ke_i\right\rangle + \left\langle \varphi, e_i\right\rangle a^*(Ke_i)\right) \\ &= \frac{1}{2}\left(a^*\left(\sum_{i=1}^{n}\left\langle e_i, K\varphi\right\rangle e_i\right) + a^*\left(K\sum_{i=1}^{n}\left\langle e_i, \varphi\right\rangle e_i\right)\right) = a^*(K\varphi)\end{aligned} \tag{3.1.6}$$

and taking the adjoint likewise shows that $[\mathcal{K}, a^*(\varphi)] = a(K\varphi)$, so

$$\begin{aligned}[\mathcal{K}, a(\varphi)] &= a^*(K\varphi) \\ [\mathcal{K}, a^*(\varphi)] &= a(K\varphi).\end{aligned} \tag{3.1.7}$$

$[\mathcal{K}, \cdot]$ thus acts on creation and annihilation operators by "swapping" each type into the other and applying the operator K to their arguments. From this one can now deduce that

$$\begin{aligned}e^{\mathcal{K}}a(\varphi)e^{-\mathcal{K}} &= a(\cosh(K)) + a^*(\sinh(K)) \\ e^{\mathcal{K}}a^*(\varphi)e^{-\mathcal{K}} &= a^*(\cosh(K)) + a(\sinh(K))\end{aligned} \tag{3.1.8}$$

since by the Baker-Campbell-Hausdorff formula

$$\begin{aligned}e^{\mathcal{K}}a(\varphi)e^{-\mathcal{K}} &= a(\varphi) + [\mathcal{K}, a(\varphi)] + \frac{1}{2!}[\mathcal{K}, [\mathcal{K}, a(\varphi)]] + \frac{1}{3!}[\mathcal{K}, [\mathcal{K}, [\mathcal{K}, a(\varphi)]]] + \cdots \\ &= a(\varphi) + a^*(K\varphi) + \frac{1}{2!}a\left(K^2\varphi\right) + \frac{1}{3!}a^*\left(K^3\varphi\right) + \cdots \\ &= a\left(\left(1 + \frac{1}{2!}K^2 + \cdots\right)\varphi\right) + a^*\left(\left(K + \frac{1}{3!}K^3 + \cdots\right)\varphi\right) = a(\cosh(K)) + a^*(\sinh(K))\end{aligned} \tag{3.1.9}$$

and likewise for $e^{\mathcal{K}}a^*(\varphi)e^{-\mathcal{K}}$.

3.2 The Action of $e^{\mathcal{K}}$ on Quadratic Operators

As our interest in Bogolubov transformations lie in their diagonalization of quadratic Hamiltonians it is however not the transformation of $a(\cdot)$ and $a^*(\cdot)$ that will interest us, but rather the transformation of $\mathrm{d}\Gamma(\cdot)$ and $Q(\cdot)$. The latter can of course be deduced from the former, but this approach is disadvantageous in the quasi-bosonic setting, which is why we will proceed differently.

First, let us make an observation on the structure of the quadratic operators which will simplify calculation significantly: The operators

$$\mathrm{d}\Gamma(A) = \sum_{i=1}^{n} a^*(Ae_i)a(e_i) \tag{3.2.1}$$

$$Q(B) = \sum_{i=1}^{n} (a(Be_i)a(e_i) + a^*(e_i)a^*(Be_i))$$

are both of a "trace-form", in the sense that we can write $\mathrm{d}\Gamma(A)$ (say) in the form $\mathrm{d}\Gamma(A) = \sum_{i=1}^{n} q(e_i, e_i)$ where

$$q(x, y) = a^*(Ax)a(y), \quad x, y \in V, \tag{3.2.2}$$

defines a bilinear mapping from $V \times V$ into the space of operators on $\mathcal{F}^+(V)$, similar to how $\mathrm{tr}(T) = \sum_{i=1}^{n} q(e_i, e_i)$ for $q(x, y) = \langle x, Ty \rangle$. This is worth noting since all such expressions are both basis-independent and obey an additional property, which for the trace is the familiar cyclicity property.

As we will encounter such trace-form sums repeatedly throughout this paper, we state this property in full generality:

Lemma 3.2.1. *Let $\langle V, \langle \cdot, \cdot \rangle \rangle$ be an n-dimensional Hilbert space and let $q : V \times V \to W$ be a sesquilinear mapping into a vector space W. Let $(e_i)_{i=1}^{n}$ be an orthonormal basis for V. Then for any linear operators $S, T : V \to V$ it holds that*

$$\sum_{i=1}^{n} q(Se_i, Te_i) = \sum_{i=1}^{n} q(ST^*e_i, e_i).$$

As a particular consequence, the expression $\sum_{i=1}^{n} q(e_i, e_i)$ is independent of the basis chosen.

Proof: By orthonormal expansion we find that

$$\begin{aligned}\sum_{i=1}^{n} q(Se_i, Te_i) &= \sum_{i=1}^{n} q\left(Se_i, \sum_{j=1}^{n} \langle e_j, Te_i \rangle e_j\right) = \sum_{i,j=1}^{n} \langle T^*e_j, e_i \rangle q(Se_i, e_j) \\ &= \sum_{j=1}^{n} q\left(S \sum_{i=1}^{n} \langle e_i, T^*e_j \rangle e_i, e_j\right) = \sum_{i=1}^{n} q(ST^*e_i, e_i).\end{aligned} \tag{3.2.3}$$

The basis-independence follows from this by noting that if $(e_i')_{i=1}^{n}$ is any other orthonormal basis, then with $U : V \to V$ denoting the unitary transformation defined by $Ue_i = e_i'$, $1 \leq i \leq n$, we see that

$$\sum_{i=1}^{n} q(e_i', e_i') = \sum_{i=1}^{n} q(Ue_i, Ue_i) = \sum_{i=1}^{n} q(UU^*e_i, e_i) = \sum_{i=1}^{n} q(e_i, e_i). \tag{3.2.4}$$

□

(In the present real case sesquilinearity is of course just bilinearity.)

The lemma thus allows us to move operators from one argument to the other when under a sum, which will be immensely useful when simplifying expressions. This can indeed be seen as a generalization of the cyclicity property of the trace, since the lemma can be applied to see that

$$\mathrm{tr}(ST) = \sum_{i=1}^{n} \langle e_i, STe_i \rangle = \sum_{i=1}^{n} \langle S^*e_i, Te_i \rangle = \sum_{i=1}^{n} \langle S^*T^*e_i, e_i \rangle = \sum_{i=1}^{n} \langle e_i, TSe_i \rangle = \mathrm{tr}(TS), \tag{3.2.5}$$

but it should be noted that cyclicity in this sense is not a general property of trace-form sums.

With this lemma we can easily calculate the commutators of $\mathcal{K}$ with $\mathrm{d}\Gamma(\cdot)$ and $Q(\cdot)$:

Proposition 3.2.2. *For any symmetric operators $A, B : V \to V$ it holds that*

$$[\mathcal{K}, 2\,\mathrm{d}\Gamma(A)] = Q(\{K, A\})$$
$$[\mathcal{K}, Q(B)] = 2\,\mathrm{d}\Gamma(\{K, B\}) + \mathrm{tr}(\{K, B\}).$$

Proof: Using equation (3.1.7) and the lemma we compute

$$\begin{aligned}[\mathcal{K}, \mathrm{d}\Gamma(A)] &= \sum_{i=1}^{n} [\mathcal{K}, a^*(Ae_i)a(e_i)] = \sum_{i=1}^{n} (a^*(Ae_i)\,[\mathcal{K}, a(e_i)] + [\mathcal{K}, a^*(Ae_i)]\,a(e_i)) \\ &= \sum_{i=1}^{n} (a^*(Ae_i)a^*(Ke_i) + a(KAe_i)a(e_i)) = \sum_{i=1}^{n} (a(KAe_i)a(e_i) + a^*(e_i)a^*(KAe_i)),\end{aligned} \tag{3.2.6}$$

and since the annihilation operators commute there holds the identity

$$\sum_{i=1}^{n} a(KAe_i)a(e_i) = \sum_{i=1}^{n} a(e_i)a((KA)^*e_i) = \sum_{i=1}^{n} a(AKe_i)a(e_i) \tag{3.2.7}$$

and likewise for the second term, so including a factor of 2 we can write

$$[\mathcal{K}, 2\,\mathrm{d}\Gamma(A)] = \sum_{i=1}^{n} (a(\{K, A\}\,e_i)a(e_i) + a^*(e_i)a^*(\{K, A\}\,e_i)) = Q(\{K, B\}) \tag{3.2.8}$$

as claimed. For $Q(B)$ we note that $Q(B) = 2\,\mathrm{Re}(\sum_{i=1}^{n} a(Be_i)a(e_i))$ and calculate as above that

$$\begin{aligned}[\mathcal{K}, Q(B)] &= 2\,\mathrm{Re}\left(\sum_{i=1}^{n} [\mathcal{K}, a(Be_i)a(e_i)]\right) = 2\,\mathrm{Re}\left(\sum_{i=1}^{n} (a(Be_i)\,[\mathcal{K}, a(e_i)] + [\mathcal{K}, a(Be_i)]\,a(e_i))\right) \\ &= 2\,\mathrm{Re}\left(\sum_{i=1}^{n} (a(Be_i)a^*(Ke_i) + a^*(KBe_i)a(e_i))\right) \\ &= 2\,\mathrm{Re}\left(\sum_{i=1}^{n} (a(BKe_i)a^*(e_i) + a^*(KBe_i)a(e_i))\right).\end{aligned} \tag{3.2.9}$$

By the lemma we see that

$$\begin{aligned}2\,\mathrm{Re}\left(\sum_{i=1}^{n} a(BKe_i)a^*(e_i)\right) &= \sum_{i=1}^{n} (a(BKe_i)a^*(e_i) + a(e_i)a^*(BKe_i)) \\ &= \sum_{i=1}^{n} (a(e_i)a^*((BK)^*e_i) + a(e_i)a^*(BKe_i)) \\ &= \sum_{i=1}^{n} a(e_i)a^*(\{K, B\}\,e_i)\end{aligned} \tag{3.2.10}$$

and likewise

$$2\,\mathrm{Re}\left(\sum_{i=1}^{n} a^*(KBe_i)a(e_i)\right) = \sum_{i=1}^{n} a^*(\{K,B\}\,e_i)a(e_i), \tag{3.2.11}$$

so by the CCR

$$\begin{aligned}[\mathcal{K},Q(B)] &= \sum_{i=1}^{n}(a(e_i)a^*(\{K,B\}\,e_i) + a^*(\{K,B\}\,e_i)a(e_i)) \\ &= 2\sum_{i=1}^{n} a^*(\{K,B\}\,e_i)a(e_i) + \sum_{i=1}^{n}\langle e_i,\{K,B\}\,e_i\rangle \\ &= 2\,\mathrm{d}\Gamma(\{K,B\}) + \mathrm{tr}(\{K,B\}).\end{aligned} \tag{3.2.12}$$

□

Note the similarity between these commutators and those of equation (3.1.7) - again $[\mathcal{K},\cdot]$ acts by "swapping the types and applying K to the argument", although now the types are those of the quadratic operators and the application of K is taking the anticommutator.

Although the action of $e^{\mathcal{K}}$ on the quadratic operators can again be deduced from the Baker-Campbell-Hausdorff formula, we now derive this by an "ODE-style" argument, as this will generalize better to the quasi-bosonic setting of the next section:

Proposition 3.2.3. *For any symmetric operator $T : V \to V$ it holds that*

$$\begin{aligned} e^{\mathcal{K}}(2\,\mathrm{d}\Gamma(T))e^{-\mathcal{K}} &= 2\,\mathrm{d}\Gamma(T_1) + Q(T_2) + \mathrm{tr}(T_1 - T) \\ e^{\mathcal{K}}Q(T)e^{-\mathcal{K}} &= 2\,\mathrm{d}\Gamma(T_2) + Q(T_1) + \mathrm{tr}(T_2)\end{aligned}$$

where $T_1, T_2 : V \to V$ are given by

$$T_1 = \frac{1}{2}\Big(e^{K}Te^{K} + e^{-K}Te^{-K}\Big), \quad T_2 = \frac{1}{2}\Big(e^{K}Te^{K} - e^{-K}Te^{-K}\Big).$$

Proof: We prove the first identity, the second following similarly.

Consider an expression of the form $e^{-t\mathcal{K}}(2\,\mathrm{d}\Gamma(A(t)) + Q(B(t)))e^{t\mathcal{K}}$ where $A(t), B(t) : V \to V$ are any symmetric operators with $t \mapsto A(t), B(t)$ differentiable. Taking the derivative, we find by Proposition 3.2.2 that

$$\begin{aligned} &\frac{d}{dt}e^{-t\mathcal{K}}(2\,\mathrm{d}\Gamma(A(t)) + Q(B(t)))e^{t\mathcal{K}} \\ &= e^{-t\mathcal{K}}(2\,\mathrm{d}\Gamma(A'(t)) + Q(B'(t)) - [\mathcal{K}, 2\,\mathrm{d}\Gamma(A(t)) + Q(B(t))])e^{t\mathcal{K}} \\ &= e^{-t\mathcal{K}}(2\,\mathrm{d}\Gamma(A'(t) - \{K,B(t)\}))e^{t\mathcal{K}} + e^{-t\mathcal{K}}Q(B'(t) - \{K,A(t)\})e^{t\mathcal{K}} - \mathrm{tr}(\{K,B(t)\}).\end{aligned} \tag{3.2.13}$$

Consequently, if $A(t)$ and $B(t)$ are solutions of the system

$$A'(t) = \{K,B(t)\}\,, \quad B'(t) = \{K,A(t)\}\,, \tag{3.2.14}$$

then the first two terms vanish, i.e.

$$\frac{d}{dt}e^{-t\mathcal{K}}(2\,\mathrm{d}\Gamma(A(t)) + Q(B(t)))e^{t\mathcal{K}} = -\,\mathrm{tr}(\{K,B(t)\}). \tag{3.2.15}$$

The fundamental theorem of calculus thus implies that

$$\begin{aligned} e^{-\mathcal{K}}(2\,\mathrm{d}\Gamma(A(1)) + Q(B(1)))e^{\mathcal{K}} &= 2\,\mathrm{d}\Gamma(A(0)) + Q(B(0)) - \int_0^1 \operatorname{tr}(\{K, B(t)\})dt \\ &= 2\,\mathrm{d}\Gamma(A(0)) + Q(B(0)) - \operatorname{tr}(A(1) - A(0)), \end{aligned} \tag{3.2.16}$$

and imposing also the initial conditions

$$A(0) = T, \quad B(0) = 0, \tag{3.2.17}$$

this can be rearranged to

$$e^{\mathcal{K}}(2\,\mathrm{d}\Gamma(T))e^{-\mathcal{K}} = 2\,\mathrm{d}\Gamma(A(1)) + Q(B(1)) + \operatorname{tr}(A(1) - T). \tag{3.2.18}$$

The claim now follows by the observation that

$$\begin{aligned} A(t) &= \frac{1}{2}\Big(e^{tK}Te^{tK} + e^{-tK}Te^{-tK}\Big) \\ B(t) &= \frac{1}{2}\Big(e^{tK}Te^{tK} - e^{-tK}Te^{-tK}\Big) \end{aligned} \tag{3.2.19}$$

are precisely the solutions of this system: The initial conditions are clearly satisfied, as is the ODE since

$$\begin{aligned} \frac{d}{dt}\Big(e^{tK}Te^{tK} \pm e^{-tK}Te^{-tK}\Big) &= e^{tK}\{K, T\}\,e^{tK} \pm e^{-tK}\{-K, T\}\,e^{-tK} \\ &= \Big\{K, e^{tK}Te^{tK} \mp e^{-tK}Te^{-tK}\Big\}. \end{aligned} \tag{3.2.20}$$

□

Diagonalization of Quadratic Hamiltonians

Having derived the transformation laws we can now describe how to diagonalize the quadratic Hamiltonian $H = 2\,\mathrm{d}\Gamma(A) + Q(B)$: By Proposition 3.2.3, this transforms as

$$\begin{aligned} e^{\mathcal{K}}He^{-\mathcal{K}} &= 2\,\mathrm{d}\Gamma\Big(\frac{1}{2}\Big(e^{K}Ae^{K} + e^{-K}Ae^{-K}\Big)\Big) + Q\Big(\frac{1}{2}\Big(e^{K}Ae^{K} - e^{-K}Ae^{-K}\Big)\Big) \\ &\quad + 2\,\mathrm{d}\Gamma\Big(\frac{1}{2}\Big(e^{K}Be^{K} - e^{-K}Be^{-K}\Big)\Big) + Q\Big(\frac{1}{2}\Big(e^{K}Be^{K} + e^{-K}Be^{-K}\Big)\Big) \\ &\quad + \operatorname{tr}\Big(\frac{1}{2}\Big(e^{K}Ae^{K} + e^{-K}Ae^{-K}\Big) - A\Big) + \operatorname{tr}\Big(\frac{1}{2}\Big(e^{K}Be^{K} - e^{-K}Be^{-K}\Big)\Big) \\ &= 2\,\mathrm{d}\Gamma\Big(\frac{1}{2}\Big(e^{K}(A+B)e^{K} + e^{-K}(A-B)e^{-K}\Big)\Big) \\ &\quad + Q\Big(\frac{1}{2}\Big(e^{K}(A+B)e^{K} - e^{-K}(A-B)e^{-K}\Big)\Big) \\ &\quad + \operatorname{tr}\Big(\frac{1}{2}\Big(e^{K}(A+B)e^{K} + e^{-K}(A-B)e^{-K}\Big) - A\Big). \end{aligned} \tag{3.2.21}$$

As the diagonalization of H is the statement that the $Q(\cdot)$ term vanishes, we see that the *diagonalization condition* is that K obeys

$$e^{K}(A+B)e^{K} = e^{-K}(A-B)e^{-K}. \tag{3.2.22}$$

Indeed, if this holds then we evidently have that

$$e^{\mathcal{K}}He^{-\mathcal{K}} = 2\,\mathrm{d}\Gamma(E) + \operatorname{tr}(E-A) \tag{3.2.23}$$

for $E = e^{K}(A+B)e^{K} = e^{-K}(A-B)e^{-K}$.

There remains the question of when such a kernel K exists. For this it holds that the condition $A \pm B > 0$ not only suffices, but in this case a diagonalizing K can be explicitly defined, which is furthermore unique (the following is a generalization and simplification of the arguments used in [8, 15]):

Proposition 3.2.4. *Let $A, B : V \to V$ be symmetric operators such that $A \pm B > 0$. Then*

$$K = -\frac{1}{2}\log\left((A-B)^{-\frac{1}{2}}\left((A-B)^{\frac{1}{2}}(A+B)(A-B)^{\frac{1}{2}}\right)^{\frac{1}{2}}(A-B)^{-\frac{1}{2}}\right)$$

is the unique symmetric solution of

$$e^{K}(A+B)e^{K} = e^{-K}(A-B)e^{-K}.$$

Proof: Write $A_{\pm} = A \pm B$ for brevity. Then we can write the diagonalization condition as

$$A_{+} = e^{-2K}A_{-}e^{-2K}. \tag{3.2.24}$$

Multiplying by $A_{-}^{-\frac{1}{2}}$ on both sides yields

$$A_{-}^{\frac{1}{2}}A_{+}A_{-}^{\frac{1}{2}} = A_{-}^{\frac{1}{2}}e^{-2K}A_{-}e^{-2K}A_{-}^{\frac{1}{2}} = \left(A_{-}^{\frac{1}{2}}e^{-2K}A_{-}^{\frac{1}{2}}\right)^{2}, \tag{3.2.25}$$

so as both $A_{-}^{\frac{1}{2}}A_{+}A_{-}^{\frac{1}{2}}$ and $A_{-}^{\frac{1}{2}}e^{-2K}A_{-}^{\frac{1}{2}}$ are positive operators it must be the case that

$$A_{-}^{\frac{1}{2}}e^{-2K}A_{-}^{\frac{1}{2}} = \left(A_{-}^{\frac{1}{2}}A_{+}A_{-}^{\frac{1}{2}}\right)^{\frac{1}{2}} \tag{3.2.26}$$

whence

$$-2K = \log\left(A_{-}^{-\frac{1}{2}}\left(A_{-}^{\frac{1}{2}}A_{+}A_{-}^{\frac{1}{2}}\right)^{\frac{1}{2}}A_{-}^{-\frac{1}{2}}\right) \tag{3.2.27}$$

which is the claim.

□

Chapter 4

Diagonalization of the Bosonizable Terms

In this section we diagonalize the *bosonizable terms*, which is to say the expression

$$H'_{\mathrm{kin}} + \sum_{k \in \mathbb{Z}^3_*} \frac{\hat{V}_k k_F^{-1}}{2\,(2\pi)^3} \Big(2B_k^* B_k + B_k B_{-k} + B_{-k}^* B_k^* \Big). \tag{4.0.1}$$

In Section 2 we saw that these behave in a quasi-bosonic fashion, and this "diagonalization" is indeed in the sense of Bogolubov transformations. To this end we start by casting the bosonizable terms into a form which more closely mirrors that of the quadratic operators which we considered in the previous section.

Once this is done it will be clear how to define a quasi-bosonic Bogolubov transformation $e^{\mathcal{K}}$ which emulates the properties of the transformation in the exact bosonic setting. We can then repeat the calculations of the previous section - keeping also in mind the additional terms which arise from the exchange correction - to determine the action of this transformation on the bosonizable terms.

With this established we then specify a particular generator $\mathcal{K}$ which will diagonalize these terms, and in the process extract the bosonic contribution to the correlation energy. The main result of this section is summarized in the following (in notation defined below):

Theorem 4.0.1. *Let $\sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2 < \infty$. Then there exists a unitary transformation $e^{\mathcal{K}} : \mathcal{H}_N \to \mathcal{H}_N$ such that*

$$\begin{aligned}
&e^{\mathcal{K}} \left(H'_{\mathrm{kin}} + \sum_{k \in \mathbb{Z}^3_*} \frac{\hat{V}_k k_F^{-1}}{2\,(2\pi)^3} \Big(2B_k^* B_k + B_k B_{-k} + B_{-k}^* B_k^* \Big) \right) e^{-\mathcal{K}} \\
&= \sum_{k \in \mathbb{Z}^3_*} \mathrm{tr}\Big(e^{-K_k} h_k e^{-K_k} - h_k - P_k \Big) + H'_{\mathrm{kin}} + 2 \sum_{k \in \mathbb{Z}^3_*} Q_1^k\Big(e^{-K_k} h_k e^{-K_k} - h_k \Big) \\
&+ \sum_{k \in \mathbb{Z}^3_*} \int_0^1 e^{(1-t)\mathcal{K}} \Big(\varepsilon_k(\{K_k, B_k(t)\}) + 2\,\mathrm{Re}\Big(\mathcal{E}_k^1(A_k(t)) \Big) + 2\,\mathrm{Re}\Big(\mathcal{E}_k^2(B_k(t)) \Big) \Big) e^{-(1-t)\mathcal{K}} dt
\end{aligned}$$

where for any $k \in \mathbb{Z}_*^3$ *the operators* $h_k, P_k : \ell^2(L_k) \to \ell^2(L_k)$ *are defined by*

$$\begin{aligned} h_k e_p &= \lambda_{k,p} e_p & \lambda_{k,p} &= \tfrac{1}{2}\left(|p|^2 - |p-k|^2\right) \\ P_k(\cdot) &= \langle v_k, \cdot \rangle v_k & v_k &= \sqrt{\tfrac{s\hat{V}_k k_F^{-1}}{2\,(2\pi)^3}} \textstyle\sum_{p\in L_k} e_p, \end{aligned}$$

the operator $K_k : \ell^2(L_k) \to \ell^2(L_k)$ *is defined by*

$$K_k = -\frac{1}{2}\log\left(h_k^{-\frac{1}{2}}\left(h_k^{\frac{1}{2}}(h_k + 2P_k)h_k^{\frac{1}{2}}\right)^{\frac{1}{2}} h_k^{-\frac{1}{2}}\right)$$

and for $t \in [0,1]$ *the operators* $A_k(t), B_k(t) : \ell^2(L_k) \to \ell^2(L_k)$ *are given by*

$$\begin{aligned} A_k(t) &= \frac{1}{2}\left(e^{tK_k}(h_k + 2P_k)e^{tK_k} + e^{-tK_k}h_k e^{-tK_k}\right) - h_k \\ B_k(t) &= \frac{1}{2}\left(e^{tK_k}(h_k + 2P_k)e^{tK_k} - e^{-tK_k}h_k e^{-tK_k}\right). \end{aligned}$$

The condition that $\sum_{k\in\mathbb{Z}_*^3} \hat{V}_k^2 < \infty$ arises to ensure that the diagonalizing generator $\mathcal{K}$ is a well-defined (and even bounded) operator. We will however postpone the proof of this until the next section, to focus on the diagonalization procedure first.

(Even though $\mathcal{K}$ is bounded, there are still some subleties to address due to the unboundedness of the transformed operators. We have included these considerations in appendix section C for the interested reader.)

4.1 Formalizing the Bosonic Analogy

Recall that we defined the quasi-bosonic excitation operators by

$$b_{k,p} = \frac{1}{\sqrt{s}}\sum_{\sigma=1}^{s} c_{p-k,\sigma}^* c_{p,\sigma}, \quad b_{k,p}^* = \frac{1}{\sqrt{s}}\sum_{\sigma=1}^{s} c_{p,\sigma}^* c_{p-k,\sigma}, \quad k \in \mathbb{Z}_*^3,\ p \in L_k, \tag{4.1.1}$$

which obey the commutation relations

$$\left[b_{k,p}, b_{l,q}^*\right] = \delta_{k,l}\delta_{p,q} + \varepsilon_{k,l}(p;q), \quad [b_{k,p}, b_{l,q}] = 0 = \left[b_{k,p}^*, b_{l,q}^*\right], \tag{4.1.2}$$

for $\varepsilon_{k,l} = -s^{-1}\sum_{\sigma=1}^{s}\left(\delta_{p,q} c_{q-l,\sigma} c_{p-k,\sigma}^* + \delta_{p-k,q-l} c_{q,\sigma}^* c_{p,\sigma}\right)$. The relation between these and the B_k operators is simply $B_k = \sqrt{s}\sum_{p\in L_k} b_{k,p}$, so we can express the non-kinetic part of the bosonizable terms as

$$\begin{aligned} &\sum_{k\in\mathbb{Z}_*^3} \frac{\hat{V}_k k_F^{-1}}{2\,(2\pi)^3}\left(2B_k^* B_k + B_k B_{-k} + B_{-k}^* B_k^*\right) \\ &= \sum_{k\in\mathbb{Z}_*^3}\left(2\sum_{p,q\in L_k} \frac{s\hat{V}_k k_F^{-1}}{2\,(2\pi)^3} b_{k,p}^* b_{k,p} + \sum_{p,q\in L_k} \frac{s\hat{V}_k k_F^{-1}}{2\,(2\pi)^3}\left(b_{k,p} b_{-k,-p} + b_{-k,-p}^* b_{k,p}^*\right)\right). \end{aligned} \tag{4.1.3}$$

The expressions inside the parenthesis are similar to the quadratic operators we considered in the previous section, and to exploit this similarly we define for any operators $A, B : \ell^2(L_k) \to \ell^2(L_k)$ quasi-bosonic quadratic operators $Q_1^k(A)$ and $Q_2^k(B)$ by

$$\begin{aligned} Q_1^k(A) &= \sum_{p,q\in L_k} \langle e_p, A e_q\rangle \, b_{k,p}^* b_{k,q} \\ Q_2^k(B) &= \sum_{p,q\in L_k} \langle e_p, B e_q\rangle \left(b_{k,p} b_{-k,-q} + b_{-k,-q}^* b_{k,p}^* \right), \end{aligned} \tag{4.1.4}$$

where $(e_p)_{p\in L_k}$ is the standard orthonormal basis of $\ell^2(L_k)$.

Note that the spaces $\ell^2(L_k)$ play the role of the one-body space V of the previous section[1], and that $Q_1^k(A)$ and $Q_2^k(B)$ are analogous to $\mathrm{d}\Gamma(A)$ and $Q(B)$ of the equations (3.1.1) and (3.1.2) (since we already use $\mathrm{d}\Gamma(\cdot)$ to denote the fermionic second-quantization on $\mathcal{H}_N$, we deviate slightly from that notation for the quasi-bosonic operators).

Note also that the $Q_2^k(\cdot)$ terms involve excitation operators of both momentum k and $-k$. For this reason we will have to treat operators corresponding to the lunes L_k and L_{-k} simultaneously when deriving the transformation identities below.

To write the right-hand side of equation (4.1.3) in this notation, define a vector $v_k \in \ell^2(L_k)$ by

$$v_k = \sqrt{\frac{s\hat{V}_k k_F^{-1}}{2\,(2\pi)^3}} \sum_{p\in L_k} e_p \tag{4.1.5}$$

and consider the operator $P_k : \ell^2(L_k) \to \ell^2(L_k)$ which acts according to $P_k(\cdot) = \langle v_k, \cdot\rangle\, v_k$. Then

$$\langle e_p, P_k e_q\rangle = \langle e_p, v_k\rangle \langle v_k, e_q\rangle = \frac{s\hat{V}_k k_F^{-1}}{2\,(2\pi)^3}, \quad p, q \in L_k, \tag{4.1.6}$$

so we simply have

$$\sum_{k\in\mathbb{Z}_*^3} \frac{\hat{V}_k k_F^{-1}}{2\,(2\pi)^3} \left(2B_k^* B_k + B_k B_{-k} + B_{-k}^* B_k^* \right) = \sum_{k\in\mathbb{Z}_*^3} \left(2\, Q_1^k(P_k) + Q_2^k(P_k) \right). \tag{4.1.7}$$

Generalized Excitation Operators

For the purpose of computation (in particular so that we can exploit Lemma 3.2.1 to the fullest) it is convenient to also introduce a basis-independent notation for the quasi-bosonic operators. We thus define, for any $k \in \mathbb{Z}_*^3$ and $\varphi \in \ell^2(L_k)$, the *generalized excitation operators* $b_k(\varphi)$ and $b_k^*(\varphi)$ by

$$b_k(\varphi) = \sum_{p\in L_k} \langle \varphi, e_p\rangle\, b_{k,p}, \quad b_k^*(\varphi) = \sum_{p\in L_k} \langle e_p, \varphi\rangle\, b_{k,p}^*. \tag{4.1.8}$$

[1] As in that case we will only consider $\ell^2(L_k)$ as a *real* vector space.

The assignments $\varphi \mapsto b_k(\varphi), b_k^*(\varphi)$ are then linear, and so it follows from equation (4.1.2) that the generalized excitation operators obey the commutation relations

$$[b_k(\varphi), b_l(\psi)] = [b_k^*(\varphi), b_l^*(\psi)] = 0 \qquad (4.1.9)$$
$$[b_k(\varphi), b_l^*(\psi)] = \delta_{k,l} \langle \varphi, \psi \rangle + \varepsilon_{k,l}(\varphi; \psi)$$

for all $k, l \in \mathbb{Z}^3$, $\varphi \in \ell^2(L_k)$ and $\psi \in \ell^2(L_l)$, where the exchange correction $\varepsilon_{k,l}(\varphi;\psi)$ is given by

$$\varepsilon_{k,l}(\varphi;\psi) = -\frac{1}{s} \sum_{p \in L_k}^{\sigma} \sum_{q \in L_l} \langle \varphi, e_p \rangle \langle e_q, \psi \rangle \left(\delta_{p,q} c_{q-l,\sigma} c_{p-k,\sigma}^* + \delta_{p-k,q-l} c_{q,\sigma}^* c_{p,\sigma} \right). \qquad (4.1.10)$$

In terms of these the quadratic operators $Q_1^k(A)$ and $Q_2^k(B)$ are expressed as

$$Q_1^k(A) = \sum_{p \in L_k} b_k^*(Ae_p) b_{k,p} \qquad (4.1.11)$$
$$Q_2^k(B) = \sum_{p \in L_k} \left(b_k(Be_p) b_{-k,-p} + b_{-k,-p}^* b_k^*(Be_p) \right).$$

It will also be useful to express the relation

$$\left[H'_{\mathrm{kin}}, b_{k,p}^* \right] = \left(|p|^2 - |p-k|^2 \right) b_{k,p}^* \qquad (4.1.12)$$

of Lemma 2.4.2 in a basis-independent way: Defining operators $h_k : \ell^2(L_k) \to \ell^2(L_k)$ by

$$h_k e_p = \lambda_{k,p} e_p, \quad \lambda_{k,p} = \frac{1}{2} \left(|p|^2 - |p-k|^2 \right), \qquad (4.1.13)$$

linearity yields the general commutator

$$[H'_{\mathrm{kin}}, b_k^*(\varphi)] = \sum_{p \in L_k} \left(|p|^2 - |p-k|^2 \right) \langle e_p, \varphi \rangle \, b_{k,p}^* = 2\, b_k^*(h_k \varphi). \qquad (4.1.14)$$

4.2 The Quasi-Bosonic Bogolubov Transformation

Let a collection of symmetric operators $K_l : \ell^2(L_l) \to \ell^2(L_l)$, $l \in \mathbb{Z}_*^3$, be given. Then we define the associated quasi-bosonic Bogolubov kernel $\mathcal{K} : \mathcal{H}_N \to \mathcal{H}_N$ by

$$\mathcal{K} = \frac{1}{2} \sum_{l \in \mathbb{Z}_*^3} \sum_{p,q \in L_l} \langle e_p, K_l e_q \rangle \left(b_{l,p} b_{-l,-q} - b_{-l,-q}^* b_{l,p}^* \right) \qquad (4.2.1)$$
$$= \frac{1}{2} \sum_{l \in \mathbb{Z}_*^3} \sum_{q \in L_l} \left(b_l(K_l e_q) b_{-l,-q} - b_{-l,-q}^* b_l^*(K_l e_q) \right),$$

in analogy with equation (3.1.5). It is clear from the second equation that $\mathcal{K}^* = -\mathcal{K}$, and so $\mathcal{K}$ generates a unitary transformation $e^{\mathcal{K}}$.

This of course depends on $\mathcal{K}$ being well-defined - as it is an infinite sum, this is not obvious. As mentioned at the beginning of this section, we will consider this issue in the next section, in which we establish that $\mathcal{K}$ is in fact a bounded operator provided $\sum_{l\in\mathbb{Z}^3_*}\|K_l\|^2_{\mathrm{HS}}<\infty$.

We will make the additional assumption about the operators K_k that they are symmetric under the negation $k\to -k$, in the sense that

$$\langle e_p, K_k e_q\rangle = \langle e_{-p}, K_{-k} e_{-q}\rangle\,,\quad k\in\mathbb{Z}^3_*,\ p,q\in L_k. \tag{4.2.2}$$

Letting $I_k:\ell^2(L_k)\to\ell^2(L_{-k})$ denote the unitary mapping acting according to $I_k e_p = e_{-p}$, $p\in L_k$, this condition is expressed in terms of operators as

$$I_k K_k = K_{-k} I_k. \tag{4.2.3}$$

It is easily seen that the operators h_k and P_k defined above also satisfy this relation. The reason for imposing this condition is to ensure that Lemma 3.2.1 allows us to move operators between arguments also for $Q_2^k(\cdot)$-type terms, since e.g.

$$\begin{aligned}\sum_{q\in L_l} b_l(K_l e_q) b_{-l,-q} &= \sum_{q\in L_l} b_l(K_l e_q) b_{-l}(e_{-q}) = \sum_{q\in L_l} b_l(K_l e_q) b_{-l}(I_l e_q) \\ &= \sum_{q\in L_l} b_l(e_q) b_{-l}(I_l K_l^* e_q) = \sum_{q\in L_l} b_{l,q} b_{-l}(I_l K_l e_q) = \sum_{q\in L_l} b_{l,q} b_{-l}(K_{-l} e_{-q}).\end{aligned} \tag{4.2.4}$$

$\mathcal{K}$ Commutators

As in the previous section we must calculate several commutators involving $\mathcal{K}$ before we can determine the action of $e^{\mathcal{K}}$ on the bosonizable terms. We start by computing the commutator of $\mathcal{K}$ with an excitation operator:

Proposition 4.2.1. *For any $k\in\mathbb{Z}^3_*$ and $\varphi\in\ell^2(L_k)$ it holds that*

$$\begin{aligned}[\mathcal{K}, b_k(\varphi)] &= b^*_{-k}(I_k K_k\varphi) + \mathcal{E}_k(\varphi)\\ [\mathcal{K}, b^*_k(\varphi)] &= b_{-k}(I_k K_k\varphi) + \mathcal{E}_k(\varphi)^*\end{aligned}$$

where

$$\mathcal{E}_k(\varphi) = \frac{1}{2}\sum_{l\in\mathbb{Z}^3_*}\sum_{q\in L_l}\left\{\varepsilon_{k,l}(\varphi; e_q), b^*_{-l}(K_{-l} e_{-q})\right\}.$$

Proof: It suffices to determine $[\mathcal{K}, b_k(\varphi)]$. Using Lemma 3.2.1 we calculate that

$$\begin{aligned}[\mathcal{K}, b_k(\varphi)] &= \frac{1}{2}\sum_{l\in\mathbb{Z}^3_*}\sum_{q\in L_l}\left(\left[b_l(K_l e_q) b_{-l}(e_{-q}) - b^*_{-l}(e_{-q}) b^*_l(K_l e_q), b_k(\varphi)\right]\right)\\ &= \frac{1}{2}\sum_{l\in\mathbb{Z}^3_*}\sum_{q\in L_l}\left(b^*_{-l}(e_{-q})\left[b_k(\varphi), b^*_l(K_l e_q)\right] + \left[b_k(\varphi), b^*_{-l}(e_{-q})\right] b^*_l(K_l e_q)\right)\end{aligned} \tag{4.2.5}$$

$$= \frac{1}{2}\sum_{l\in\mathbb{Z}^3_*}\sum_{q\in L_l}\Big(b^*_{-l}(K_{-l}e_{-q})\,[b_k(\varphi),b^*_l(e_q)] + \Big[b_k(\varphi),b^*_{-l}(e_{-q})\Big]\,b^*_l(K_l e_q)\Big)$$
$$= \frac{1}{2}\sum_{l\in\mathbb{Z}^3_*}\sum_{q\in L_l}\Big\{\Big[b_k(\varphi),b^*_{-l}(e_{-q})\Big],b^*_l(K_l e_q)\Big\}$$

where we lastly substituted $l\to -l$, $q\to -q$ in the first term. Using the commutation relations of equation (4.1.9) we then find that

$$\begin{aligned}
[\mathcal{K},b_k(\varphi)] &= \frac{1}{2}\sum_{l\in\mathbb{Z}^3_*}\sum_{q\in L_l}\{\delta_{k,-l}\,\langle\varphi,e_{-q}\rangle + \varepsilon_{k,-l}(\varphi;e_{-q}),b^*_l(K_l e_q)\}\\
&= \sum_{q\in L_{-k}}\langle\varphi,e_{-q}\rangle\,b^*_{-k}(K_{-k}e_q) + \frac{1}{2}\sum_{l\in\mathbb{Z}^3_*}\sum_{q\in L_l}\{\varepsilon_{k,-l}(\varphi;e_{-q}),b^*_l(K_l e_q)\} \qquad (4.2.6)\\
&= b^*_{-k}\left(K_{-k}I_k\sum_{q\in L_k}\langle\varphi,e_q\rangle\,e_q\right) + \frac{1}{2}\sum_{l\in\mathbb{Z}^3_*}\sum_{q\in L_l}\Big\{\varepsilon_{k,l}(\varphi;e_q),b^*_{-l}(K_{-l}e_{-q})\Big\}\\
&= b^*_{-k}(I_kK_k\varphi) + \mathcal{E}_k(\varphi).
\end{aligned}$$

□

Note how these commutators compare to those of equation (3.1.7) - again $\mathcal{K}$ “swaps the type and applies K”, but now there is also a reflection from L_k to L_{-k}, as well as an additional term involving the exchange correction.

Using this relation we can now determine the commutator with Q^k_1 terms:

Proposition 4.2.2. *For any $k\in\mathbb{Z}^3_*$ and symmetric operators $A_{\pm k}:\ell^2(L_{\pm k})\to\ell^2(L_{\pm k})$ such that $I_kA_k = A_{-k}I_k$, it holds that*

$$\Big[\mathcal{K},2\,Q^k_1(A_k)+2\,Q^{-k}_1(A_{-k})\Big] = Q^k_2(\{K_k,A_k\}) + 2\,\mathrm{Re}\Big(\mathcal{E}^1_k(A_k)\Big) + (k\to -k)$$

where

$$\mathcal{E}^1_k(A_k) = \sum_{l\in\mathbb{Z}^3_*}\sum_{p\in L_k}\sum_{q\in L_l} b^*_k(A_ke_p)\Big\{\varepsilon_{k,l}(e_p;e_q),b^*_{-l}(K_{-l}e_{-q})\Big\}.$$

Proof: Using Proposition 4.2.1 (and Lemma 3.2.1 together with symmetry of A_k) we find that

$$\begin{aligned}
\Big[\mathcal{K},Q^k_1(A_k)\Big] &= \sum_{p\in L_k}[\mathcal{K},b^*_k(A_ke_p)b_k(e_p)] = \sum_{p\in L_k}(b^*_k(A_ke_p)\,[\mathcal{K},b_k(e_p)] + [\mathcal{K},b^*_k(A_ke_p)]\,b_k(e_p))\\
&= \sum_{p\in L_k}\Big(b^*_k(A_ke_p)b^*_{-k}(I_kK_ke_p) + b_{-k}(I_kK_kA_ke_p)b_k(e_p)\Big)\\
&+ \sum_{p\in L_k}(b^*_k(A_ke_p)\mathcal{E}_k(e_p) + \mathcal{E}_k(A_ke_p)^*b_k(e_p)) \qquad (4.2.7)\\
&= \sum_{p\in L_k}\Big(b^*_k(A_kK_ke_p)b^*_{-k,-p} + b_{-k,-p}b_k(A_kK_ke_p)\Big) + 2\,\mathrm{Re}\left(\sum_{p\in L_k}b^*_k(A_ke_p)\mathcal{E}_k(e_p)\right)
\end{aligned}$$

$$= Q_2^k(A_k K_k) + 2\,\mathrm{Re}\left(\sum_{p\in L_k} b_k^*(A_k e_p)\mathcal{E}_k(e_p)\right).$$

Now, the assumption that $I_k A_k = A_{-k} I_k$ yields

$$\begin{aligned}\sum_{p\in L_k} b_k(A_k K_k e_p) b_{-k,-p} &= \sum_{p\in L_k} b_k(I_k A_{-k} K_{-k} e_{-p}) b_{-k}(e_{-p}) = \sum_{p\in L_k} b_k(e_p) b_{-k}(K_{-k} A_{-k} e_{-p}) \\ &= \sum_{p\in L_{-k}} b_{-k}(K_{-k} A_{-k} e_p) b_{k,-p} \end{aligned} \tag{4.2.8}$$

and likewise $\sum_{p\in L_k} b_{-k,-p}^* b_k^*(A_k K_k e_p) = \sum_{p\in L_{-k}} b_{k,-p}^* b_{-k}^*(K_{-k} A_{-k} e_p)$, whence

$$Q_2^k(A_k K_k) = Q_2^{-k}(K_{-k} A_{-k}). \tag{4.2.9}$$

Summing over both k and $-k$, and introducing a factor of 2, we thus find

$$\begin{aligned}\left[\mathcal{K}, 2\,Q_1^k(A_k) + 2\,Q_1^{-k}(A_{-k})\right] &= 2\,Q_2^k(A_k K_k) + 2\,\mathrm{Re}\left(2\sum_{p\in L_k} b_k^*(A_k e_p)\mathcal{E}_k(e_p)\right) \\ &\quad + 2\,Q_2^{-k}(A_{-k}K_{-k}) + 2\,\mathrm{Re}\left(2\sum_{p\in L_{-k}} b_{-k}^*(A_{-k} e_p)\mathcal{E}_{-k}(e_p)\right) \\ &= Q_2^k(\{K_k, A_k\}) + 2\,\mathrm{Re}\Big(\mathcal{E}_k^1(A_k)\Big) + (k \to -k)\end{aligned} \tag{4.2.10}$$

where $\mathcal{E}_k^1(A_k) = 2\sum_{p\in L_k} b_k^*(A_k e_p)\mathcal{E}_k(e_p)$ follows simply by expansion.

□

To state the commutator of $\mathcal{K}$ with Q_2^k-type terms, we first note the identity

$$\begin{aligned}\sum_{p\in L_k} b_k(e_p) b_k^*(A_k e_p) &= \sum_{p\in L_k} b_k^*(A_k e_p) b_k(e_p) + \sum_{p\in L_k} [b_k(e_p), b_k^*(A_k e_p)] \\ &= \sum_{p\in L_k} b_k^*(A_k e_p) b_k(e_p) + \sum_{p\in L_k} \langle e_p, A_k e_p\rangle + \sum_{p\in L_k} \varepsilon_{k,k}(e_p; A_k e_p) \\ &= Q_1^k(A_k) + \mathrm{tr}(A_k) + \varepsilon_k(A_k)\end{aligned} \tag{4.2.11}$$

where we introduced the convenient notation

$$\begin{aligned}\varepsilon_k(A_k) = \sum_{p\in L_k} \varepsilon_{k,k}(e_p; A_k e_p) &= -\frac{1}{s}\sum_{p,q\in L_k}^{\sigma} \langle e_q, A_k e_p\rangle \left(\delta_{p,q} c_{q-k,\sigma} c_{p-k,\sigma}^* + \delta_{p-k,q-k} c_{q,\sigma}^* c_{p,\sigma}\right) \\ = -\frac{1}{s}\sum_{p\in L_k}^{\sigma} \langle e_p, A_k e_p\rangle &\left(c_{p,\sigma}^* c_{p,\sigma} + c_{p-k,\sigma} c_{p-k,\sigma}^*\right).\end{aligned} \tag{4.2.12}$$

The commutator is then given by the following:

Proposition 4.2.3. *For any $k \in \mathbb{Z}_*^3$ and symmetric operators $B_{\pm k} : \ell^2(L_{\pm k}) \to \ell^2(L_{\pm k})$ such that $I_k B_k = B_{-k} I_k$, it holds that*

$$\left[\mathcal{K}, Q_2^k(B_k) + Q_2^{-k}(B_{-k})\right] = 2\,Q_1^k(\{K_k, B_k\}) + \mathrm{tr}(\{K_k, B_k\}) + \varepsilon_k(\{K_k, B_k\})$$

$$+\,2\,\mathrm{Re}\Big(\mathcal{E}_k^2(B_k)\Big) + (k \to -k)$$

where

$$\mathcal{E}_k^2(B_k) = \frac{1}{2}\sum_{l\in\mathbb{Z}_*^3}\sum_{p\in L_k}\sum_{q\in L_l}\left\{b_k(B_k e_p), \left\{\varepsilon_{-k,-l}(e_{-p};e_{-q}), b_l^*(K_l e_q)\right\}\right\}.$$

Proof: Writing $Q_2^k(B_k)$ as $Q_2^k(B_k) = 2\,\mathrm{Re}\Big(\sum_{p\in L_k} b_k(B_k e_p) b_{-k}(e_{-p})\Big)$, we calculate

$$\begin{aligned}
\left[\mathcal{K}, Q_2^k(B_k)\right] &= 2\,\mathrm{Re}\left(\sum_{p\in L_k}\left(b_k(B_k e_p)\left[\mathcal{K}, b_{-k}(e_{-p})\right] + \left[\mathcal{K}, b_k(B_k e_p)\right] b_{-k}(e_{-p})\right)\right)\\
&= 2\,\mathrm{Re}\left(\sum_{p\in L_k}\left(b_k(B_k e_p)\left[\mathcal{K}, b_{-k}(e_{-p})\right] + \left[\mathcal{K}, b_k(e_p)\right] b_{-k}(B_{-k} e_{-p})\right)\right)\\
&= 2\,\mathrm{Re}\left(\sum_{p\in L_k}\Big(b_k(B_k e_p) b_k^*(I_{-k}K_{-k}e_{-p}) + b_{-k}^*(I_k K_k e_p) b_{-k}(B_{-k}e_{-p})\Big)\right) \qquad (4.2.13)\\
&\quad+ 2\,\mathrm{Re}\left(\sum_{p\in L_k}\left(b_k(B_k e_p)\mathcal{E}_{-k}(e_{-p}) + \mathcal{E}_k(e_p) b_{-k}(B_{-k}e_{-p})\right)\right)\\
&= 2\,\mathrm{Re}\left(\sum_{p\in L_k}\Big(b_{k,p} b_k^*(K_k B_k e_p) + b_{-k}^*(K_{-k}B_{-k}e_{-p}) b_{-k,-p}\Big)\right)\\
&\quad+ 2\,\mathrm{Re}\left(\sum_{p\in L_k}\left(b_k(B_k e_p)\mathcal{E}_{-k}(e_{-p}) + \mathcal{E}_k(e_p) b_{-k}(B_{-k}e_{-p})\right)\right).
\end{aligned}$$

Now

$$\begin{aligned}
2\,\mathrm{Re}\left(\sum_{p\in L_k} b_{k,p} b_k^*(K_k B_k e_p)\right) &= \sum_{p\in L_k} b_{k,p} b_k^*(K_k B_k e_p) + \sum_{p\in L_k} b_k(K_k B_k e_p) b_{k,p}^*\\
&= \sum_{p\in L_k} b_{k,p} b_k^*(K_k B_k e_p) + \sum_{p\in L_k} b_{k,p} b_k^*(B_k K_k e_p) \qquad (4.2.14)\\
&= \sum_{p\in L_k} b_{k,p} b_k^*(\left\{K_k, B_k\right\} e_p)
\end{aligned}$$

and likewise $2\,\mathrm{Re}\Big(\sum_{p\in L_k} b_{-k}^*(K_{-k}B_{-k}e_{-p}) b_{-k,-p}\Big) = \sum_{p\in L_k} b_{-k}^*(\left\{K_{-k}, B_{-k}\right\} e_{-p})$, so

$$\begin{aligned}
&2\,\mathrm{Re}\left(\sum_{p\in L_k}\Big(b_{k,p} b_k^*(K_k B_k e_p) + b_{-k}^*(K_{-k}B_{-k}e_{-p}) b_{-k,-p}\Big)\right)\\
&= \sum_{p\in L_k} b_{k,p} b_k^*(\left\{K_k, B_k\right\} e_p) + \sum_{p\in L_k} b_{-k}^*(\left\{K_{-k}, B_{-k}\right\} e_{-p}) b_{-k,-p} \qquad (4.2.15)\\
&= Q_1^k(\left\{K_k, B_k\right\}) + \mathrm{tr}(\left\{K_k, B_k\right\}) + \varepsilon_k(\left\{K_k, B_k\right\}) + Q_1^{-k}(\left\{K_{-k}, B_{-k}\right\}),
\end{aligned}$$

whence summing over k and $-k$ yields

$$\left[\mathcal{K}, Q_2^k(B_k) + Q_2^{-k}(B_{-k})\right] = 2\,Q_1^k(\{K_k, B_k\}) + \operatorname{tr}(\{K_k, B_k\}) + \varepsilon_k(\{K_k, B_k\}) + 2\operatorname{Re}\left(\sum_{p \in L_k} \{b_k(B_k e_p), \mathcal{E}_{-k}(e_{-p})\}\right) + (k \to -k) \tag{4.2.16}$$

and $\mathcal{E}_k^2(B_k) = \sum_{p \in L_k} \{b_k(B_k e_p), \mathcal{E}_{-k}(e_{-p})\}$ follows by expansion, yielding the claim.

□

Finally, for the transformation of H'_{kin}, we also calculate the commutator $[\mathcal{K}, H'_{\text{kin}}]$:

Proposition 4.2.4. *It holds that*

$$[\mathcal{K}, H'_{\text{kin}}] = \sum_{k \in \mathbb{Z}^3_*} Q_2^k(\{K_k, h_k\}).$$

Proof: By equation (4.1.14) we have

$$[H'_{\text{kin}}, b_k(\varphi)] = -2\,b_k(h_k\varphi), \quad [H'_{\text{kin}}, b_k^*(\varphi)] = 2\,b_k^*(h_k\varphi), \tag{4.2.17}$$

so using that $I_k h_k = h_{-k} I_k$ we find

$$\begin{aligned}
[\mathcal{K}, H'_{\text{kin}}] &= \frac{1}{2}\sum_{k \in \mathbb{Z}^3_*}\sum_{q \in L_k}\left([b_k(K_k e_q) b_{-k}(e_{-q}), H'_{\text{kin}}] - \left[b_{-k}^*(e_{-q}) b_k^*(K_k e_q), H'_{\text{kin}}\right]\right) \\
&= -\frac{1}{2}\sum_{k \in \mathbb{Z}^3_*}\sum_{q \in L_k}\left(b_k(K_k e_q)\,[H'_{\text{kin}}, b_{-k}(e_{-q})] + [H'_{\text{kin}}, b_k(K_k e_q)]\,b_{-k}(e_{-q})\right) \\
&+ \frac{1}{2}\sum_{k \in \mathbb{Z}^3_*}\sum_{q \in L_k}\left(b_{-k}^*(e_{-q})\,[H'_{\text{kin}}, b_k^*(K_k e_q)] + \left[H'_{\text{kin}}, b_{-k}^*(e_{-q})\right] b_k^*(K_k e_q)\right) \\
&= \sum_{k \in \mathbb{Z}^3_*}\sum_{q \in L_k}\left(b_k(K_k e_q) b_{-k}(h_{-k} e_{-q}) + b_k(h_k K_k e_q) b_{-k}(e_{-q})\right) \\
&+ \sum_{k \in \mathbb{Z}^3_*}\sum_{q \in L_k}\left(b_{-k}^*(e_{-q}) b_k^*(h_k K_k e_q) + b_{-k}^*(h_{-k} e_{-q}) b_k^*(K_k e_q)\right) \\
&= \sum_{k \in \mathbb{Z}^3_*}\sum_{q \in L_k}\left(b_k(\{K_k, h_k\}\, e_q) b_{-k}(e_{-q}) + b_{-k}^*(e_{-q}) b_k^*(\{K_k, h_k\}\, e_q)\right) \\
&= \sum_{k \in \mathbb{Z}^3_*} Q_2^k(\{K_k, h_k\}).
\end{aligned} \tag{4.2.18}$$

□

4.3 Transformation of the Bosonizable Terms

With all the commutators calculated we can now determine the action of $e^{\mathcal{K}}$ on quadratic operators:

Proposition 4.3.1. *For any $k \in \mathbb{Z}^3_*$ and symmetric operators $T_{\pm k} : \ell^2(L_{\pm k}) \to \ell^2(L_{\pm k})$ such that $I_k T_k = T_{-k} I_k$ it holds that*

$$e^{\mathcal{K}}\Big(2\,Q_1^k(T_k) + 2\,Q_1^{-k}(T_{-k})\Big)e^{-\mathcal{K}} = \mathrm{tr}\Big(T_k^1(1) - T_k\Big) + 2\,Q_1^k\Big(T_k^1(1)\Big) + Q_2^k\Big(T_k^2(1)\Big)$$
$$+ \int_0^1 e^{(1-t)\mathcal{K}}\Big(\varepsilon_k\Big(\Big\{K_k, T_k^2(t)\Big\}\Big) + 2\,\mathrm{Re}\Big(\mathcal{E}_k^1\Big(T_k^1(t)\Big)\Big) + 2\,\mathrm{Re}\Big(\mathcal{E}_k^2\Big(T_k^2(t)\Big)\Big)\Big)e^{-(1-t)\mathcal{K}}dt + (k \to -k)$$

and

$$e^{\mathcal{K}}\Big(Q_2^k(T_k) + Q_2^{-k}(T_{-k})\Big)e^{-\mathcal{K}} = \mathrm{tr}\Big(T_k^2(1)\Big) + 2\,Q_1^k\Big(T_k^2(1)\Big) + Q_2^k\Big(T_k^1(1)\Big)$$
$$+ \int_0^1 e^{(1-t)\mathcal{K}}\Big(\varepsilon_k\Big(\Big\{K_k, T_k^1(t)\Big\}\Big) + 2\,\mathrm{Re}\Big(\mathcal{E}_k^1\Big(T_k^2(t)\Big)\Big) + 2\,\mathrm{Re}\Big(\mathcal{E}_k^2\Big(T_k^1(t)\Big)\Big)\Big)e^{-(1-t)\mathcal{K}}dt + (k \to -k)$$

where for $t \in [0,1]$

$$T_k^1(t) = \frac{1}{2}\Big(e^{tK_k}T_k e^{tK_k} + e^{-tK_k}T_k e^{-tK_k}\Big)$$
$$T_k^2(t) = \frac{1}{2}\Big(e^{tK_k}T_k e^{tK_k} - e^{-tK_k}T_k e^{-tK_k}\Big).$$

Proof: We prove the first identity, the second following by a similar argument.

As in the proof of Proposition 3.2.3 we consider the expression $e^{-t\mathcal{K}}\Big(2\,Q_1^k(T_k^1(t)) + Q_2^k(T_k^2(t))\Big)e^{t\mathcal{K}}$, where $T_k^1(t)$ and $T_k^2(t)$ are the solutions of the system

$$\Big(T_k^1\Big)'(t) = \Big\{K_k, T_k^2(t)\Big\}, \quad \Big(T_k^2\Big)'(t) = \Big\{K_k, T_k^1(t)\Big\}, \tag{4.3.1}$$

with initial conditions $T_k^1(0) = T_k$, $T_k^2(0) = 0$.

By the Propositions 4.2.2 and 4.2.3 the derivative of such an expression satisfies

$$e^{t\mathcal{K}}\left(\frac{d}{dt}e^{-t\mathcal{K}}\Big(2\,Q_1^k\Big(T_k^1(t)\Big) + Q_2^k\Big(T_k^2(t)\Big)\Big)e^{t\mathcal{K}}\right)e^{-t\mathcal{K}} + (k \to -k)$$
$$= 2\,Q_1^k\Big(\Big(T_k^1\Big)'(t)\Big) + Q_2^k\Big(\Big(T_k^2\Big)'(t)\Big) - \Big[\mathcal{K}, 2\,Q_1^k\Big(T_k^1(t)\Big) + Q_2^k\Big(T_k^2(t)\Big)\Big] + (k \to -k) \tag{4.3.2}$$
$$= 2\,Q_1^k\Big(\Big(T_k^1\Big)'(t)\Big) - 2\,Q_1^k\Big(\Big\{K_k, T_k^2(t)\Big\}\Big) - \mathrm{tr}\Big(\Big\{K_k, T_k^2(t)\Big\}\Big) - \varepsilon_k\Big(\Big\{K_k, T_k^2(t)\Big\}\Big) - 2\,\mathrm{Re}\Big(\mathcal{E}_k^2\Big(T_k^2(t)\Big)\Big)$$
$$+ Q_2^k\Big(\Big(T_k^2\Big)'(t)\Big) - Q_2^k\Big(\Big\{K_k, T_k^1(t)\Big\}\Big) - 2\,\mathrm{Re}\Big(\mathcal{E}_k^1\Big(T_k^1(t)\Big)\Big) + (k \to -k)$$
$$= -\mathrm{tr}\Big(\Big(T_k^1\Big)'(t)\Big) - \varepsilon_k\Big(\Big\{K_k, T_k^2(t)\Big\}\Big) + 2\,\mathrm{Re}\Big(\mathcal{E}_k^1\Big(T_k^1(t)\Big)\Big) + 2\,\mathrm{Re}\Big(\mathcal{E}_k^2\Big(T_k^2(t)\Big)\Big) + (k \to -k),$$

so by the fundamental theorem of calculus

$$e^{-\mathcal{K}}\Big(2\,Q_1^k\Big(T_k^1(1)\Big) + Q_2^k\Big(T_k^2(1)\Big)\Big)e^{\mathcal{K}} + (k \to -k) = 2\,Q_1^k(T_k) - \mathrm{tr}\Big(T_k^1(1) - T_k\Big) \tag{4.3.3}$$
$$- \int_0^1 e^{-t\mathcal{K}}\Big(\varepsilon_k\Big(\Big\{K_k, T_k^2(t)\Big\}\Big) + 2\,\mathrm{Re}\Big(\mathcal{E}_k^1\Big(T_k^1(t)\Big)\Big) + 2\,\mathrm{Re}\Big(\mathcal{E}_k^2\Big(T_k^2(t)\Big)\Big)\Big)e^{t\mathcal{K}}dt + (k \to -k)$$

whence conjugation by $e^{\mathcal{K}}$ and rearrangement yields

$$e^{\mathcal{K}}\Big(2\,Q_1^k(T_k)+2\,Q_1^{-k}(T_{-k})\Big)e^{-\mathcal{K}}=\mathrm{tr}\Big(T_k^1(1)-T_k\Big)+2\,Q_1^k\Big(T_k^1(1)\Big)+Q_2^k\Big(T_k^2(1)\Big) \tag{4.3.4}$$
$$+\int_0^1 e^{(1-t)\mathcal{K}}\Big(\varepsilon_k\Big(\Big\{K_k,T_k^2(t)\Big\}\Big)+2\,\mathrm{Re}\Big(\mathcal{E}_k^1\Big(T_k^1(t)\Big)\Big)+2\,\mathrm{Re}\Big(\mathcal{E}_k^2\Big(T_k^2(t)\Big)\Big)\Big)e^{-(1-t)\mathcal{K}}dt+(k\to -k)$$

which is the claim.

□

With the transformation of quadratic operators determined we can also derive the transformation of H'_{kin}:

Proposition 4.3.2. *It holds that*

$$e^{\mathcal{K}}H'_{\mathrm{kin}}e^{-\mathcal{K}}=\sum_{k\in\mathbb{Z}_*^3}\mathrm{tr}\Big(h_k^1(1)-h_k\Big)+H'_{\mathrm{kin}}+\sum_{k\in\mathbb{Z}_*^3}\Big(2\,Q_1^k\Big(h_k^1(1)-h_k\Big)+Q_2^k\Big(h_k^2(1)\Big)\Big)$$
$$+\sum_{k\in\mathbb{Z}_*^3}\int_0^1 e^{(1-t)\mathcal{K}}\Big(\varepsilon_k\Big(\Big\{K_k,h_k^2(t)\Big\}\Big)+2\,\mathrm{Re}\Big(\mathcal{E}_k^1\Big(h_k^1(t)-h_k\Big)\Big)+2\,\mathrm{Re}\Big(\mathcal{E}_k^2\Big(h_k^2(t)\Big)\Big)\Big)e^{-(1-t)\mathcal{K}}dt$$

where for $t\in[0,1]$

$$h_k^1(t)=\frac{1}{2}\Big(e^{tK_k}h_ke^{tK_k}+e^{-tK_k}h_ke^{-tK_k}\Big)$$
$$h_k^2(t)=\frac{1}{2}\Big(e^{tK_k}h_ke^{tK_k}-e^{-tK_k}h_ke^{-tK_k}\Big).$$

Proof: By the Propositions 4.2.2 and 4.2.4 we see that

$$\left[\mathcal{K},H'_{\mathrm{kin}}-\sum_{k\in\mathbb{Z}_*^3}2\,Q_1^k(h_k)\right]=-\sum_{k\in\mathbb{Z}_*^3}2\,\mathrm{Re}\Big(\mathcal{E}_k^1(h_k)\Big) \tag{4.3.5}$$

whence by the fundamental theorem of calculus

$$e^{\mathcal{K}}\left(H'_{\mathrm{kin}}-\sum_{k\in\mathbb{Z}_*^3}2\,Q_1^k(h_k)\right)e^{-\mathcal{K}}=H'_{\mathrm{kin}}-\sum_{k\in\mathbb{Z}_*^3}2\,Q_1^k(h_k)-\sum_{k\in\mathbb{Z}_*^3}\int_0^1 e^{t\mathcal{K}}\Big(2\,\mathrm{Re}\Big(\mathcal{E}_k^1(h_k)\Big)\Big)e^{-t\mathcal{K}}dt \tag{4.3.6}$$

or

$$e^{\mathcal{K}}H'_{\mathrm{kin}}e^{-\mathcal{K}}=H'_{\mathrm{kin}}+\sum_{k\in\mathbb{Z}_*^3}\Big(e^{\mathcal{K}}\Big(2\,Q_1^k(h_k)\Big)e^{-\mathcal{K}}-2\,Q_1^k(h_k)\Big) \tag{4.3.7}$$
$$-\sum_{k\in\mathbb{Z}_*^3}\int_0^1 e^{(1-t)\mathcal{K}}\Big(2\,\mathrm{Re}\Big(\mathcal{E}_k^1(h_k)\Big)\Big)e^{-(1-t)\mathcal{K}}dt.$$

Applying Proposition 4.3.1 now yields the claim.

□

With the transformation formulas derived we can now conclude the main part of Theorem 4.0.1: By the two previous propositions, we see that

$$\begin{aligned} e^{\mathcal{K}}\left(H'_{\mathrm{kin}} + \sum_{k\in\mathbb{Z}^3_*}\Big(2\,Q^k_1(P_k) + Q^k_2(P_k)\Big)\right)e^{-\mathcal{K}} &= \sum_{k\in\mathbb{Z}^3_*}\mathrm{tr}\Big(h^1_k(1) - h_k + P^1_k(1) - P_k + P^2_k(1)\Big) \\ + H'_{\mathrm{kin}} + \sum_{k\in\mathbb{Z}^3_*}\Big(2\,Q^k_1\Big(h^1_k(1) - h_k + P^1_k(1) + P^2_k(1)\Big) &+ Q^k_2\Big(h^2_k(1) + P^2_k(1) + P^1_k(1)\Big)\Big) \quad (4.3.8) \\ + \sum_{k\in\mathbb{Z}^3_*}\int_0^1 e^{(1-t)\mathcal{K}}\Big(\varepsilon_k\Big(\Big\{K_k, h^2_k(t) + P^2_k(t) + P^1_k(t)\Big\}\Big) &+ 2\,\mathrm{Re}\Big(\mathcal{E}^1_k\Big(h^1_k(t) - h_k + P^1_k(t) + P^2_k(t)\Big)\Big) \\ &+ 2\,\mathrm{Re}\Big(\mathcal{E}^2_k\Big(h^2_k(t) + P^2_k(t) + P^1_k(t)\Big)\Big)\Big)e^{-(1-t)\mathcal{K}}dt, \end{aligned}$$

which is to say

$$\begin{aligned} & e^{\mathcal{K}}\left(H'_{\mathrm{kin}} + \sum_{k\in\mathbb{Z}^3_*}\Big(2\,Q^k_1(P_k) + Q^k_2(P_k)\Big)\right)e^{-\mathcal{K}} \\ &= \sum_{k\in\mathbb{Z}^3_*}\mathrm{tr}(A_k(1) - P_k) + H'_{\mathrm{kin}} + \sum_{k\in\mathbb{Z}^3_*}\Big(2\,Q^k_1(A_k(1)) + Q^k_2(B_k(1))\Big) \quad (4.3.9) \\ &+ \sum_{k\in\mathbb{Z}^3_*}\int_0^1 e^{(1-t)\mathcal{K}}\Big(\varepsilon_k(\{K_k, B_k(t)\}) + 2\,\mathrm{Re}\Big(\mathcal{E}^1_k(A_k(t))\Big) + 2\,\mathrm{Re}\Big(\mathcal{E}^2_k(B_k(t))\Big)\Big)e^{-(1-t)\mathcal{K}}dt \end{aligned}$$

where the operators $A_k(t), B_k(t) : \ell^2(L_k) \to \ell^2(L_k)$ are given by

$$\begin{aligned} A_k(t) &= \frac{1}{2}\Big(e^{tK_k}(h_k + 2P_k)e^{tK_k} + e^{-tK_k}h_k e^{-tK_k}\Big) - h_k \quad (4.3.10) \\ B_k(t) &= \frac{1}{2}\Big(e^{tK_k}(h_k + 2P_k)e^{tK_k} - e^{-tK_k}h_k e^{-tK_k}\Big). \end{aligned}$$

We can now choose the kernels K_k such that this expression is diagonalized, i.e. such that the $Q^k_2(\cdot)$ terms vanish. Evidently this is saying that $B_k(1) = 0$, so we arrive at the diagonalization condition

$$e^{K_k}(h_k + 2P_k)e^{K_k} = e^{-K_k}h_k e^{-K_k}. \quad (4.3.11)$$

Note that this is really the condition of equation (3.2.22) of the previous section, with $A = h_k + P_k$ and $B_k = P_k$. As such we see by Proposition 3.2.4 that we must choose

$$K_k = -\frac{1}{2}\log\left(h_k^{-\frac{1}{2}}\left(h_k^{\frac{1}{2}}(h_k + 2P_k)h_k^{\frac{1}{2}}\right)^{\frac{1}{2}}h_k^{-\frac{1}{2}}\right). \quad (4.3.12)$$

Since the diagonalization condition is then fulfilled, it follows that also

$$A_k(1) = e^{-K_k}h_k e^{-K_k} - h_k \quad (4.3.13)$$

and the formula of Theorem 4.0.1 is proved.

Chapter 5

Controlling the Transformation Kernel

In this section we prove that under the condition $\sum_{l\in\mathbb{Z}^3_*}\|K_l\|_{\mathrm{HS}}^2<\infty$, the operator defined by

$$\mathcal{K}=\frac{1}{2}\sum_{l\in\mathbb{Z}^3_*}\sum_{p,q\in L_l}\langle e_p,K_le_q\rangle\left(b_{l,p}b_{-l,-q}-b^*_{-l,-q}b^*_{l,p}\right) \tag{5.0.1}$$

is bounded. More precisely, we prove the following estimate:

Proposition 5.0.1. *For all $\Phi,\Psi\in\mathcal{H}_N$ it holds that*

$$|\langle\Phi,\mathcal{K}\Psi\rangle|\leq\sqrt{5}\sqrt{\sum_{l\in\mathbb{Z}^3_*}\|K_l\|_{\mathrm{HS}}^2}\sqrt{\langle\Phi,(\mathcal{N}_E+1)\Phi\rangle\langle\Psi,(\mathcal{N}_E+1)\Psi\rangle}.$$

Recalling that

$$\mathcal{N}_E=\sum_{p\in B_F^c}^{\sigma}c^*_{p,\sigma}c_{p,\sigma}=\sum_{p\in B_F}^{\sigma}c_{p,\sigma}c^*_{p,\sigma} \tag{5.0.2}$$

we have the trivial bound $\mathcal{N}_E=\sum_{p\in B_F}^{\sigma}c_{p,\sigma}c^*_{p,\sigma}\leq s\,|B_F|=N$, whence the proposition indeed implies boundedness, as an estimate of the form $|\langle\Phi,\mathcal{K}\Psi\rangle|\leq C\,\|\Phi\|\,\|\Psi\|$ follows. Additionally, we will see in the next section that the kernels of equation (4.3.12) obey

$$\|K_k\|_{\mathrm{HS}}\leq C\hat{V}_k,\quad k\in\mathbb{Z}^3_*, \tag{5.0.3}$$

for a constant $C>0$ independent of k, so the criterion $\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2<\infty$ does indeed imply boundedness of our diagonalizing kernel $\mathcal{K}$ hence existence of the unitary transformation $e^{\mathcal{K}}$ asserted by Theorem 4.0.1.

Preliminary Analysis

Define

$$\tilde{\mathcal{K}}=\frac{1}{2}\sum_{l\in\mathbb{Z}^3_*}\sum_{p,q\in L_l}\langle e_p,K_le_q\rangle\,b_{l,p}b_{-l,-q} \tag{5.0.4}$$

so that $\mathcal{K} = \tilde{\mathcal{K}} - \tilde{\mathcal{K}}^*$. Then for any $\Phi, \Psi \in \mathcal{H}_N$

$$|\langle \Phi, \mathcal{K}\Psi\rangle| \leq |\langle \Phi, \tilde{\mathcal{K}}\Psi\rangle| + |\langle \Psi, \tilde{\mathcal{K}}\Phi\rangle| \tag{5.0.5}$$

so we need only bound a quantity of the form $|\langle \Phi, \tilde{\mathcal{K}}\Psi\rangle|$.

Note that by expanding $b_{-l,-q} = s^{-\frac{1}{2}} \sum_{\sigma=1}^{s} c^*_{-q+l,\sigma} c_{-q,\sigma}$ we can write $\tilde{\mathcal{K}}$ as

$$\tilde{\mathcal{K}} = \frac{1}{2\sqrt{s}} \sum_{l \in \mathbb{Z}^3_*} \sum_{p,q \in L_l}^{\sigma} \langle e_p, K_l e_q\rangle\, b_{l,p} c^*_{-q+l,\sigma} c_{-q,\sigma} = \frac{1}{2\sqrt{s}} \sum_{q \in B_F^c}^{\sigma} \left(\sum_{l \in \mathbb{Z}^3_*} \sum_{p \in L_l} 1_{L_l}(q) \langle e_p, K_l e_q\rangle\, b_{l,p} c^*_{-q+l,\sigma} \right) c_{-q,\sigma} \tag{5.0.6}$$

whence we may estimate

$$\begin{aligned} |\langle \Phi, \tilde{\mathcal{K}}\Psi\rangle| &= \frac{1}{2\sqrt{s}} \left| \sum_{q \in B_F^c}^{\sigma} \left\langle \sum_{l \in \mathbb{Z}^3_*} \sum_{p \in L_l} 1_{L_l}(q) \langle K_l e_q, e_p\rangle\, c_{-q+l,\sigma} b^*_{l,p} \Phi, c_{-q,\sigma}\Psi \right\rangle \right| \\ &\leq \frac{1}{2\sqrt{s}} \sum_{q \in B_F^c}^{\sigma} \left\| \sum_{l \in \mathbb{Z}^3_*} \sum_{p \in L_l} 1_{L_l}(q) \langle K_l e_q, e_p\rangle\, c_{-q+l,\sigma} b^*_{l,p} \Phi \right\| \|c_{-q,\sigma}\Psi\| \\ &\leq \frac{1}{2\sqrt{s}} \sqrt{\sum_{q \in B_F^c}^{\sigma} \left\| \sum_{l \in \mathbb{Z}^3_*} \sum_{p \in L_l} 1_{L_l}(q) \langle K_l e_q, e_p\rangle\, c_{-q+l,\sigma} b^*_{l,p} \Phi \right\|^2} \sqrt{\sum_{q \in B_F^c}^{\sigma} \|c_{-q,\sigma}\Psi\|^2} \\ &= \frac{1}{2} \sqrt{\frac{1}{s} \sum_{q \in B_F^c}^{\sigma} \left\| \sum_{l \in \mathbb{Z}^3_*} \sum_{p \in L_l} 1_{L_l}(q) \langle K_l e_q, e_p\rangle\, c_{-q+l,\sigma} b^*_{l,p} \Phi \right\|^2} \sqrt{\langle \Psi, \mathcal{N}_E \Psi\rangle}. \end{aligned} \tag{5.0.7}$$

Now, the operator appearing under the root can be written as

$$\begin{aligned} &\sum_{l \in \mathbb{Z}^3_*} \sum_{p \in L_l} 1_{L_l}(q) \langle K_l e_q, e_p\rangle\, c_{-q+l,\sigma} b^*_{l,p} = \frac{1}{\sqrt{s}} \sum_{l \in \mathbb{Z}^3_*} \sum_{p \in L_l}^{\tau} 1_{L_l}(q) \langle K_l e_q, e_p\rangle\, c^*_{p,\tau} c_{p-l,\tau} c_{-q+l,\sigma} \\ &= \frac{1}{\sqrt{s}} \sum_{p' \in B_F^c}^{\tau} \sum_{q', r' \in B_F} \left(\sum_{l \in \mathbb{Z}^3_*} \sum_{p \in L_l} \delta_{p',p} \delta_{q',p-l} \delta_{r',-q+l} 1_{L_l}(q) \langle K_l e_q, e_p\rangle \right) c^*_{p',\tau} c_{q',\tau} c_{r',\sigma}. \end{aligned} \tag{5.0.8}$$

The introduction of these Kronecker δ's has no effect by itself, but it highlights that this operator can be written simply in the form

$$\frac{1}{\sqrt{s}} \sum_{p \in B_F^c}^{\tau} \sum_{q,r \in B_F} A_{p,q,r} c^*_{p,\tau} c_{q,\tau} c_{r,\sigma} \tag{5.0.9}$$

for some coefficients $A_{p,q,r}$. We will now derive a general estimate for such an expression.

5.1 A Higher Order Fermionic Estimate

Recall that the "standard fermionic estimate" can be stated as

$$\left\| \sum A_k c_k \Psi \right\|, \left\| \sum A_k c_k^* \Psi \right\| \leq \sqrt{\sum |A_k|^2}\, \|\Psi\|, \tag{5.1.1}$$

which can be proved by appealing to the CAR as follows: Trivially

$$\begin{aligned}\left\|\sum A_k c_k \Psi\right\|^2 &= \left\langle \sum A_k c_k \Psi, \sum A_l c_l \Psi \right\rangle = \left\langle \Psi, \left(\sum A_k c_k\right)^* \left(\sum A_l c_l\right) \Psi \right\rangle \\ &\leq \left\langle \Psi, \left(\sum A_k c_k\right)^* \left(\sum A_l c_l\right) \Psi \right\rangle + \left\langle \Psi, \left(\sum A_l c_l\right) \left(\sum A_k c_k\right)^* \Psi \right\rangle \\ &= \left\langle \Psi, \left\{ \left(\sum A_k c_k\right)^*, \left(\sum A_l c_l\right) \right\} \Psi \right\rangle\end{aligned} \tag{5.1.2}$$

since all that was done was the addition of a non-negative term. By the CAR, however,

$$\left\{ \left(\sum A_k c_k\right)^*, \left(\sum A_l c_l\right) \right\} = \sum \overline{A_k} A_l \left\{ c_k^*, c_l \right\} = \sum \overline{A_k} A_l \delta_{k,l} = \sum |A_k|^2 \tag{5.1.3}$$

whence the bound immediately follows. This establishes the uniquely fermionic property that sums of creation and annihilation operators can be estimated independently of the number operator, unlike in the bosonic case.

One can imagine generalizing this to quadratic expressions of the form $\sum_{k,l} A_{k,l} c_k c_l$, but this fails: The issue is that the CAR only yields a commutation relation for such expressions, and not an anticommutation relation, whence the argument above can not be applied.

We may however make the observation that for cubic expressions, such as $\sum_{k,l,m} A_{k,l,m} c_k^* c_l c_m$, the CAR does yield an anticommutation relation, allowing the trick to be applied. The anticommutator is of course not constant, but rather a combination of quadratic, linear and constant expressions, but this still yields a reduction in “number operator order”, which will be crucial for our estimation of $e^{\mathcal{K}} \mathcal{N}_E^m e^{-\mathcal{K}}$ later on.

To derive such an estimate we first calculate the following basic anticommutator:

Lemma 5.1.1. *For any $p, p' \in B_F^c$, $q, q', r, r' \in B_F$ and $1 \leq \sigma, \tau, \tau' \leq s$ it holds that*

$$\begin{aligned}\left\{ \left(c_{p,\tau}^* c_{q,\tau} c_{r,\sigma}\right)^*, c_{p',\tau'}^* c_{q',\tau'} c_{r',\sigma} \right\} &= \delta_{p,p'}^{\tau,\tau'} c_{q',\tau'} c_{r',\sigma} c_{r,\sigma}^* c_{q,\tau}^* + \delta_{q,q'}^{\tau,\tau'} c_{p',\tau'}^* c_{r',\sigma} c_{r,\sigma}^* c_{p,\tau} + \delta_{r,r'} c_{p',\tau'}^* c_{q',\tau'} c_{q,\tau}^* c_{p,\tau} \\ &- \delta_{r,q'}^{\sigma,\tau'} c_{p',\tau'}^* c_{r',\sigma} c_{q,\tau}^* c_{p,\tau} - \delta_{q,r'}^{\tau,\sigma} c_{p',\tau'}^* c_{q',\tau'} c_{r,\sigma}^* c_{p,\tau} \\ &- \delta_{q,q'}^{\tau,\tau'} \delta_{r,r'} c_{p',\tau'}^* c_{p,\tau} - \delta_{p,p'}^{\tau,\tau'} \delta_{q,q'}^{\tau,\tau'} c_{r',\sigma} c_{r,\sigma}^* - \delta_{p,p'}^{\tau,\tau'} \delta_{r,r'} c_{q',\tau'} c_{q,\tau}^* \\ &+ \delta_{q,r'}^{\tau,\sigma} \delta_{r,q'}^{\sigma,\tau'} c_{p',\tau'}^* c_{p,\tau} + \delta_{p,p'}^{\tau,\tau'} \delta_{r,q'}^{\sigma,\tau'} c_{r',\sigma} c_{q,\tau}^* + \delta_{p,p'}^{\tau,\tau'} \delta_{q,r'}^{\tau,\sigma} c_{q',\tau'} c_{r,\sigma}^* \\ &+ \delta_{p,p'}^{\tau,\tau'} \delta_{q,q'}^{\tau,\tau'} \delta_{r,r'} - \delta_{p,p'}^{\tau,\tau'} \delta_{q,r'}^{\tau,\sigma} \delta_{r,q'}^{\sigma,\tau'}.\end{aligned}$$

Proof: The proof is a straightforward but lengthy calculation using the CAR: First we note

$$\begin{aligned}\left(c_{p,\tau}^* c_{q,\tau} c_{r,\sigma}\right)^* c_{p',\tau'}^* c_{q',\tau'} c_{r',\sigma} &= c_{r,\sigma}^* c_{q,\tau}^* c_{p,\tau} c_{p',\tau'}^* c_{q',\tau'} c_{r',\sigma} \\ &= -c_{r,\sigma}^* c_{q,\tau}^* c_{p',\tau'}^* c_{p,\tau} c_{q',\tau'} c_{r',\sigma} + \delta_{p,p'}^{\tau,\tau'} c_{r,\sigma}^* c_{q,\tau}^* c_{q',\tau'} c_{r',\sigma} \\ &= -c_{p',\tau'}^* c_{r,\sigma}^* c_{q,\tau}^* c_{q',\tau'} c_{r',\sigma} c_{p,\tau} + \delta_{p,p'}^{\tau,\tau'} c_{r,\sigma}^* c_{q,\tau}^* c_{q',\tau'} c_{r',\sigma}\end{aligned} \tag{5.1.4}$$

and

$$c_{r,\sigma}^* c_{q,\tau}^* c_{q',\tau'} c_{r',\sigma} = -c_{r,\sigma}^* c_{q',\tau'} c_{q,\tau}^* c_{r',\sigma} + \delta_{q,q'}^{\tau,\tau'} c_{r,\sigma}^* c_{r',\sigma} = c_{r,\sigma}^* c_{q',\tau'} c_{r',\sigma} c_{q,\tau}^* - \delta_{q,r'}^{\tau,\sigma} c_{r,\sigma}^* c_{q',\tau'} + \delta_{q,q'}^{\tau,\tau'} c_{r,\sigma}^* c_{r',\sigma}$$

$$\begin{aligned}
&= -c_{q',\tau'}c^*_{r,\sigma}c_{r',\sigma}c^*_{q,\tau} + \delta^{\sigma,\tau'}_{r,q'}c_{r',\sigma}c^*_{q,\tau} - \delta^{\tau,\sigma}_{q,r'}c^*_{r,\sigma}c_{q',\tau'} + \delta^{\tau,\tau'}_{q,q'}c^*_{r,\sigma}c_{r',\sigma} \qquad (5.1.5)\\
&= c_{q',\tau'}c_{r',\sigma}c^*_{r,\sigma}c^*_{q,\tau} - \delta_{r,r'}c_{q',\tau'}c^*_{q,\tau} + \delta^{\sigma,\tau'}_{r,q'}c_{r',\sigma}c^*_{q,\tau} - \delta^{\tau,\sigma}_{q,r'}c^*_{r,\sigma}c_{q',\tau'} + \delta^{\tau,\tau'}_{q,q'}c^*_{r,\sigma}c_{r',\sigma}\\
&= c_{q',\tau'}c_{r',\sigma}c^*_{r,\sigma}c^*_{q,\tau} - \delta^{\tau,\tau'}_{q,q'}c_{r',\sigma}c^*_{r,\sigma} - \delta_{r,r'}c_{q',\tau'}c^*_{q,\tau} + \delta^{\sigma,\tau'}_{r,q'}c_{r',\sigma}c^*_{q,\tau} + \delta^{\tau,\sigma}_{q,r'}c_{q',\tau'}c^*_{r,\sigma}\\
&+ \delta^{\tau,\tau'}_{q,q'}\delta_{r,r'} - \delta^{\tau,\sigma}_{q,r'}\delta^{\sigma,\tau'}_{r,q'}.
\end{aligned}$$

Consequently

$$\begin{aligned}
-c^*_{p',\tau'}c^*_{r,\sigma}c^*_{q,\tau}c_{q',\tau'}c_{r',\sigma}c_{p,\tau} &= -c^*_{p',\tau'}c_{q',\tau'}c_{r',\sigma}c^*_{r,\sigma}c^*_{q,\tau}c_{p,\tau} + c^*_{p',\tau'}\left(\delta^{\tau,\tau'}_{q,q'}c_{r',\sigma}c^*_{r,\sigma} + \delta_{r,r'}c_{q',\tau'}c^*_{q,\tau}\right)c_{p,\tau}\\
&- c^*_{p',\tau'}\left(\delta^{\sigma,\tau'}_{r,q'}c_{r',\sigma}c^*_{q,\tau} + \delta^{\tau,\sigma}_{q,r'}c_{q',\tau'}c^*_{r,\sigma}\right)c_{p,\tau} - c^*_{p',\tau'}\left(\delta^{\tau,\tau'}_{q,q'}\delta_{r,r'} - \delta^{\tau,\sigma}_{q,r'}\delta^{\sigma,\tau'}_{r,q'}\right)c_{p,\tau}\\
&= -c^*_{p',\tau'}c_{q',\tau'}c_{r',\sigma}\left(c^*_{p,\tau}c_{q,\tau}c_{r,\sigma}\right)^* + \delta^{\tau,\tau'}_{q,q'}c^*_{p',\tau'}c_{r',\sigma}c^*_{r,\sigma}c_{p,\tau} + \delta_{r,r'}c^*_{p',\tau'}c_{q',\tau'}c^*_{q,\tau}c_{p,\tau}\\
&- \delta^{\sigma,\tau'}_{r,q'}c^*_{p',\tau'}c_{r',\sigma}c^*_{q,\tau}c_{p,\tau} - \delta^{\tau,\sigma}_{q,r'}c^*_{p',\tau'}c_{q',\tau'}c^*_{r,\sigma}c_{p,\tau} \qquad (5.1.6)\\
&- \delta^{\tau,\tau'}_{q,q'}\delta_{r,r'}c^*_{p',\tau'}c_{p,\tau} + \delta^{\tau,\sigma}_{q,r'}\delta^{\sigma,\tau'}_{r,q'}c^*_{p',\tau'}c_{p,\tau}
\end{aligned}$$

and

$$\begin{aligned}
\delta^{\tau,\tau'}_{p,p'}c^*_{r,\sigma}c^*_{q,\tau}c_{q',\tau'}c_{r',\sigma} &= \delta^{\tau,\tau'}_{p,p'}c_{q',\tau'}c_{r',\sigma}c^*_{r,\sigma}c^*_{q,\tau} - \delta^{\tau,\tau'}_{p,p'}\left(\delta^{\tau,\tau'}_{q,q'}c_{r',\sigma}c^*_{r,\sigma} + \delta_{r,r'}c_{q',\tau'}c^*_{q,\tau}\right)\\
&+ \delta^{\tau,\tau'}_{p,p'}\left(\delta^{\sigma,\tau'}_{r,q'}c_{r',\sigma}c^*_{q,\tau} + \delta^{\tau,\sigma}_{q,r'}c_{q',\tau'}c^*_{r,\sigma}\right) + \delta^{\tau,\tau'}_{p,p'}\left(\delta^{\tau,\tau'}_{q,q'}\delta_{r,r'} - \delta^{\tau,\sigma}_{q,r'}\delta^{\sigma,\tau'}_{r,q'}\right) \quad (5.1.7)\\
&= \delta^{\tau,\tau'}_{p,p'}c_{q',\tau'}c_{r',\sigma}c^*_{r,\sigma}c^*_{q,\tau} - \delta^{\tau,\tau'}_{p,p'}\delta^{\tau,\tau'}_{q,q'}c_{r',\sigma}c^*_{r,\sigma} - \delta^{\tau,\tau'}_{p,p'}\delta_{r,r'}c_{q',\tau'}c^*_{q,\tau}\\
&+ \delta^{\tau,\tau'}_{p,p'}\delta^{\sigma,\tau'}_{r,q'}c_{r',\sigma}c^*_{q,\tau} + \delta^{\tau,\tau'}_{p,p'}\delta^{\tau,\sigma}_{q,r'}c_{q',\tau'}c^*_{r,\sigma} + \delta^{\tau,\tau'}_{p,p'}\delta^{\tau,\tau'}_{q,q'}\delta_{r,r'} - \delta^{\tau,\tau'}_{p,p'}\delta^{\tau,\sigma}_{q,r'}\delta^{\sigma,\tau'}_{r,q'}.
\end{aligned}$$

Insertion of these two identities into equation (5.1.4) yields the claim.

□

We can now conclude the desired bound:

Proposition 5.1.2. *Let $A_{p,q,r} \in \mathbb{C}$ for $p \in B_F^c$ and $q, r \in B_F$ with $\sum_{p\in B_F^c}\sum_{q,r\in B_F}|A_{p,q,r}|^2 < \infty$ be given. Then for any $\Psi \in \mathcal{H}_N$*

$$\frac{1}{s}\sum_{\sigma=1}^{s}\left\|\sum_{p\in B_F^c}^{\tau}\sum_{q,r\in B_F}A_{p,q,r}c^*_{p,\tau}c_{q,\tau}c_{r,\sigma}\Psi\right\|^2 \le 5s\sum_{p\in B_F^c}\sum_{q,r\in B_F}|A_{p,q,r}|^2\left\langle\Psi,(\mathcal{N}_E+1)\Psi\right\rangle.$$

Proof: As in the proof of the standard fermionic estimate, we have

$$\begin{aligned}
&\left\|\sum_{p\in B_F^c}^{\tau}\sum_{q,r\in B_F}A_{p,q,r}c^*_{p,\tau}c_{q,\tau}c_{r,\sigma}\Psi\right\|^2\\
&= \left\langle\sum_{p\in B_F^c}^{\tau}\sum_{q,r\in B_F}A_{p,q,r}c^*_{p,\tau}c_{q,\tau}c_{r,\sigma}\Psi, \sum_{p'\in B_F^c}^{\tau'}\sum_{q',r'\in B_F}A_{p',q',r'}c^*_{p',\tau'}c_{q',\tau'}c_{r',\sigma}\Psi\right\rangle\\
&\le \sum_{p,p'\in B_F^c}^{\tau,\tau'}\sum_{q,q',r,r'\in B_F}\overline{A_{p,q,r}}A_{p',q',r'}\left\langle\Psi,\left\{\left(c^*_{p,\tau}c_{q,\tau}c_{r,\sigma}\right)^*, c^*_{p',\tau'}c_{q',\tau'}c_{r',\sigma}\right\}\Psi\right\rangle
\end{aligned}$$

so by the identity of the preceding lemma

$$
\begin{aligned}
&\sum_{\sigma=1}^{s}\left\|\sum_{p\in B_F^c}\sum_{q,r\in B_F}A_{p,q,r}c_p^*c_qc_r\Psi\right\|^2\\
\leq &\sum_{\sigma,\tau,\tau'=1}^{s}\sum_{p,p'\in B_F^c}\sum_{q,q',r,r'\in B_F}\overline{A_{p,q,r}}A_{p',q',r'}\left\langle\Psi,\left(\delta_{p,p'}^{\tau,\tau'}c_{q',\tau'}c_{r',\sigma}c_{r,\sigma}^*c_{q,\tau}^*+\delta_{q,q'}^{\tau,\tau'}c_{p',\tau'}^*c_{r',\sigma}c_{r,\sigma}^*c_{p,\tau}\right.\right.\\
&\qquad\qquad\left.\left.+\delta_{r,r'}c_{p',\tau'}^*c_{q',\tau'}c_{q,\tau}^*c_{p,\tau}\right)\Psi\right\rangle \qquad (5.1.8)\\
-&\sum_{\sigma,\tau,\tau'=1}^{s}\sum_{p,p'\in B_F^c}\sum_{q,q',r,r'\in B_F}\overline{A_{p,q,r}}A_{p',q',r'}\left\langle\Psi,\left(\delta_{r,q'}^{\sigma,\tau'}c_{p',\tau'}^*c_{r',\sigma}c_{q,\tau}^*c_{p,\tau}+\delta_{q,r'}^{\tau,\sigma}c_{p',\tau'}^*c_{q',\tau'}c_{r,\sigma}^*c_{p,\tau}\right)\Psi\right\rangle\\
-&\sum_{\sigma,\tau,\tau'=1}^{s}\sum_{p,p'\in B_F^c}\sum_{q,q',r,r'\in B_F}\overline{A_{p,q,r}}A_{p',q',r'}\left\langle\Psi,\left(\delta_{q,q'}^{\tau,\tau'}\delta_{r,r'}c_{p',\tau'}^*c_{p,\tau}+\delta_{p,p'}^{\tau,\tau'}\delta_{q,q'}^{\tau,\tau'}c_{r',\sigma}c_{r,\sigma}^*+\delta_{p,p'}^{\tau,\tau'}\delta_{r,r'}c_{q',\tau'}c_{q,\tau}^*\right)\Psi\right\rangle\\
+&\sum_{\sigma,\tau,\tau'=1}^{s}\sum_{p,p'\in B_F^c}\sum_{q,q',r,r'\in B_F}\overline{A_{p,q,r}}A_{p',q',r'}\left\langle\Psi,\left(\delta_{q,r'}^{\tau,\sigma}\delta_{r,q'}^{\sigma,\tau'}c_{p',\tau'}^*c_{p,\tau}+\delta_{p,p'}^{\tau,\tau'}\delta_{r,q'}^{\sigma,\tau'}c_{r',\sigma}c_{q,\tau}^*+\delta_{p,p'}^{\tau,\tau'}\delta_{q,r'}^{\tau,\sigma}c_{q',\tau'}c_{r,\sigma}^*\right)\Psi\right\rangle\\
+&\sum_{\sigma,\tau,\tau'=1}^{s}\sum_{p,p'\in B_F^c}\sum_{q,q',r,r'\in B_F}\overline{A_{p,q,r}}A_{p',q',r'}\left\langle\Psi,\left(\delta_{p,p'}^{\tau,\tau'}\delta_{q,q'}^{\tau,\tau'}\delta_{r,r'}-\delta_{p,p'}^{\tau,\tau'}\delta_{q,r'}^{\tau,\sigma}\delta_{r,q'}^{\sigma,\tau'}\right)\Psi\right\rangle.
\end{aligned}
$$

We estimate the different types of expressions appearing above. Firstly, by the standard fermionic estimate,

$$
\begin{aligned}
&\sum_{\sigma,\tau,\tau'=1}^{s}\sum_{p,p'\in B_F^c}\sum_{q,q',r,r'\in B_F}\overline{A_{p,q,r}}A_{p',q',r'}\left\langle\Psi,\delta_{p,p'}^{\tau,\tau'}c_{q',\tau'}c_{r',\sigma}c_{r,\sigma}^*c_{q,\tau}^*\Psi\right\rangle\\
&=\sum_{p\in B_F^c}^{\sigma,\tau}\left\langle\sum_{q',r'\in B_F}\overline{A_{p,q',r'}}c_{r',\sigma}^*c_{q',\tau}^*\Psi,\sum_{q,r\in B_F}\overline{A_{p,q,r}}c_{r,\sigma}^*c_{q,\tau}^*\Psi\right\rangle=\sum_{p\in B_F^c}^{\sigma,\tau}\left\|\sum_{q,r\in B_F}\overline{A_{p,q,r}}c_{r,\sigma}^*c_{q,\tau}^*\Psi\right\|^2\\
&\leq\sum_{p\in B_F^c}^{\sigma,\tau}\left(\sum_{q\in B_F}\left\|\sum_{r\in B_F}\overline{A_{p,q,r}}c_{r,\sigma}^*c_{q,\tau}^*\Psi\right\|\right)^2\leq\sum_{p\in B_F^c}^{\sigma,\tau}\left(\sum_{q\in B_F}\sqrt{\sum_{r\in B_F}|A_{p,q,r}|^2}\left\|c_{q,\tau}^*\Psi\right\|\right)^2 \qquad (5.1.9)\\
&\leq\sum_{p\in B_F^c}^{\sigma,\tau}\left(\sum_{q,r\in B_F}|A_{p,q,r}|^2\right)\left(\sum_{q\in B_F}\left\|c_{q,\tau}^*\Psi\right\|^2\right)=s\sum_{p\in B_F^c}\sum_{q,r\in B_F}|A_{p,q,r}|^2\langle\Psi,\mathcal{N}_E\Psi\rangle
\end{aligned}
$$

and likewise for the other two terms on the first line of equation (5.1.8). For the terms on the second line we similarly estimate

$$
\begin{aligned}
&\left|\sum_{\sigma,\tau,\tau'=1}^{s}\sum_{p,p'\in B_F^c}\sum_{q,q',r,r'\in B_F}\overline{A_{p,q,r}}A_{p',q',r'}\left\langle\Psi,\delta_{r,q'}^{\sigma,\tau'}c_{p',\tau'}^*c_{r',\sigma}c_{q,\tau}^*c_{p,\tau}\Psi\right\rangle\right|\\
&=\left|\sum_{r\in B_F}^{\sigma,\tau}\left\langle\sum_{p'\in B_F^c}\sum_{r'\in B_F}\overline{A_{p',r,r'}}c_{r',\sigma}^*c_{p',\sigma}\Psi,\sum_{p\in B_F^c}\sum_{q\in B_F}\overline{A_{p,q,r}}c_{q,\tau}^*c_{p,\tau}\Psi\right\rangle\right|
\end{aligned}
$$

$$\begin{aligned}
&\leq \sum_{r\in B_F}^{\sigma,\tau} \left\| \sum_{p'\in B_F^c}\sum_{r'\in B_F} \overline{A_{p',r,r'}} c_{r',\sigma}^* c_{p',\sigma}\Psi \right\| \left\| \sum_{p\in B_F^c}\sum_{q\in B_F} \overline{A_{p,q,r}} c_{q,\tau}^* c_{p,\tau}\Psi \right\| \qquad (5.1.10)\\
&\leq \sum_{r\in B_F}^{\sigma,\tau} \sum_{p,p'\in B_F^c} \sqrt{\sum_{r'\in B_F} |A_{p',r,r'}|^2} \, \|c_{p',\sigma}\Psi\| \sqrt{\sum_{q\in B_F} |A_{p,q,r}|^2} \, \|c_{p,\tau}\Psi\| \\
&\leq \sum_{r\in B_F} \sqrt{\sum_{p'\in B_F^c}^{\sigma} \sum_{r'\in B_F} |A_{p',r,r'}|^2} \sqrt{\sum_{p'\in B_F^c}^{\sigma} \|c_{p',\sigma}\Psi\|^2} \sqrt{\sum_{p\in B_F^c}^{\tau}\sum_{q\in B_F} |A_{p,q,r}|^2} \sqrt{\sum_{p\in B_F^c}^{\tau} \|c_{p,\tau}\Psi\|^2} \\
&\leq s \sum_{p\in B_F^c}\sum_{q,r\in B_F} |A_{p,q,r}|^2 \langle \Psi, \mathcal{N}_E \Psi\rangle .
\end{aligned}$$

The terms on the third line of equation (5.1.8) all factorize in a manifestly non-positive fashion, and so can be dropped, while for the fourth line

$$\left| \sum_{\sigma,\tau,\tau'=1}^{s} \sum_{p,p'\in B_F^c} \sum_{q,q',r,r'\in B_F} \overline{A_{p,q,r}} A_{p',q',r'} \left\langle \Psi, \delta_{q,r'}^{\tau,\sigma}\delta_{r,q'}^{\sigma,\tau'} c_{p',\tau'}^* c_{p,\tau}\Psi \right\rangle \right| \qquad (5.1.11)$$

$$\begin{aligned}
&= \left| \sum_{q,r\in B_F}^{\sigma} \left\langle \sum_{p'\in B_F^c} \overline{A_{p',r,q}} c_{p',\sigma}\Psi, \sum_{p\in B_F^c} \overline{A_{p,q,r}} c_{p,\sigma}\Psi \right\rangle \right| \leq \sum_{q,r\in B_F}^{\sigma} \left\| \sum_{p'\in B_F^c} \overline{A_{p',r,q}} c_{p',\sigma}\Psi \right\| \left\| \sum_{p\in B_F^c} \overline{A_{p,q,r}} c_{p,\sigma}\Psi \right\| \\
&\leq \sum_{q,r\in B_F}^{\sigma} \sqrt{\sum_{p\in B_F^c} |A_{p',r,q}|^2} \sqrt{\sum_{p\in B_F^c} |A_{p,q,r}|^2} \, \|\Psi\|^2 \leq s \sum_{p\in B_F^c}\sum_{q,r\in B_F} |A_{p,q,r}|^2 \|\Psi\|^2 .
\end{aligned}$$

Lastly, the terms on the fifth line are seen to simply be constant and easily bounded by $s^2 \sum_{p\in B_F^c}\sum_{q,r\in B_F} |A_{p,q,r}|^2 \|\Psi\|^2$, whence the proposition follows.

□

We can now conclude the following bound for $\tilde{\mathcal{K}}$:

Proposition 5.1.3. *For any $\Phi, \Psi \in \mathcal{H}_N$ it holds that*

$$|\langle \Phi, \tilde{\mathcal{K}}\Psi\rangle| \leq \frac{\sqrt{5}}{2} \sqrt{\sum_{l\in\mathbb{Z}_*^3} \|K_l\|_{\mathrm{HS}}^2} \sqrt{\langle \Phi, (\mathcal{N}_E + 1)\Phi\rangle \langle \Psi, \mathcal{N}_E \Psi\rangle}.$$

Proof: By the equations (5.0.7) and (5.0.8), combined with the estimate of the previous proposition, we can estimate

$$|\langle \Phi, \tilde{\mathcal{K}}\Psi\rangle| \leq \frac{\sqrt{5}}{2} \sqrt{\sum_{q\in B_F^c}\sum_{p'\in B_F^c}\sum_{q',r'\in B_F} \left| \sum_{l\in\mathbb{Z}_*^3}\sum_{p\in L_l} \delta_{p',p}\delta_{q',p-l}\delta_{r',-q+l} 1_{L_l}(q) \langle K_l e_q, e_p\rangle \right|^2} \qquad (5.1.12)$$
$$\cdot \sqrt{\langle \Phi, (\mathcal{N}_E + 1)\Phi\rangle \langle \Psi, \mathcal{N}_E \Psi\rangle},$$

and by repeated elimination of the Kronecker δ's the sum reduces to

$$\sum_{q\in B_F^c}\sum_{p'\in B_F^c}\sum_{q',r'\in B_F} \left| \sum_{l\in\mathbb{Z}_*^3}\sum_{p\in L_l} \delta_{p',p}\delta_{q',p-l}\delta_{r',-q+l} 1_{L_l}(q) \langle K_l e_q, e_p\rangle \right|^2$$

$$= \sum_{q \in B_F^c} \sum_{p' \in B_F^c} \sum_{q' \in B_F} \sum_{l \in \mathbb{Z}^3_*} \left| \sum_{p \in L_l} \delta_{p',p} \delta_{q',p-l} 1_{L_l}(q) \langle K_l e_q, e_p \rangle \right|^2 \tag{5.1.13}$$
$$= \sum_{q \in B_F^c} \sum_{q' \in B_F} \sum_{l \in \mathbb{Z}^3_*} \sum_{p \in L_l} |\delta_{q',p-l} 1_{L_l}(q) \langle K_l e_q, e_p \rangle|^2 = \sum_{l \in \mathbb{Z}^3_*} \sum_{p,q \in L_l} |\langle K_l e_q, e_p \rangle|^2 = \sum_{l \in \mathbb{Z}^3_*} \|K_l\|_{\mathrm{HS}}^2 .$$

□

The bound of Proposition 5.0.1 now follows by the observation that $|\langle \Phi, \mathcal{K} \Psi \rangle| \leq |\langle \Phi, \tilde{\mathcal{K}} \Psi \rangle| + |\langle \Psi, \tilde{\mathcal{K}} \Phi \rangle|$.

Chapter 6

Analysis of One-Body Operators

In this section we analyze the operators K_k, $A_k(t)$ and $B_k(t)$ which appeared during the diagonalization process of Section 4.

We first consider operators of the form e^{-2K_k} and e^{2K_k} in detail, obtaining asymptotically optimal matrix element estimates for these. We then extend these estimates to K_k itself, as well as $\sinh(-tK_k)$ and $\cosh(-tK_k)$ for any $t \in [0,1]$. With these we then turn to $A_k(t)$ and $B_k(t)$.

We end the analysis with the integral $\int_0^1 B_k(t)\,dt$, which will appear in the next section during our extraction of the exchange contribution.

In all, we prove the following:

Theorem 6.0.1. *It holds for any $k \in \mathbb{Z}^3_*$ that*

$$\operatorname{tr}\Big(e^{-K_k} h_k e^{-K_k} - h_k - P_k\Big) = \frac{1}{\pi}\int_0^\infty F\left(\frac{s\hat{V}_k k_F^{-1}}{(2\pi)^3}\sum_{p\in L_k}\frac{\lambda_{k,p}}{\lambda_{k,p}^2 + t^2}\right)dt,$$

where $F(x) = \log(1+x) - x$. Furthermore, as $k_F \to \infty$,

$$\|K_k\|_{\mathrm{HS}} \leq C\hat{V}_k \min\left\{1, k_F^2\,|k|^{-2}\right\}$$

and for all $p, q \in L_k$ and $t \in [0,1]$

$$\begin{aligned}
\left|\langle e_p, K_k e_q\rangle\right| &\leq C\frac{\hat{V}_k k_F^{-1}}{\lambda_{k,p} + \lambda_{k,q}} \\
\left|\langle e_p, (-K_k) e_q\rangle - \frac{s\hat{V}_k k_F^{-1}}{2\,(2\pi)^3}\frac{1}{\lambda_{k,p} + \lambda_{k,q}}\right| &\leq C\frac{\hat{V}_k^2 k_F^{-1}}{\lambda_{k,p} + \lambda_{k,q}} \\
\left|\langle e_p, A_k(t) e_q\rangle\right|, \left|\langle e_p, B_k(t) e_q\rangle\right| &\leq C\Big(1 + \hat{V}_k^2\Big)\hat{V}_k k_F^{-1} \\
\left|\left\langle e_p, \Big(\int_0^1 B_k(t)\,dt\Big) e_q\right\rangle - \frac{s\hat{V}_k k_F^{-1}}{4\,(2\pi)^3}\right| &\leq C\Big(1 + \hat{V}_k\Big)\hat{V}_k^2 k_F^{-1} \\
\left|\langle e_p, \{K_k, B_k(t)\}\, e_q\rangle\right| &\leq C\Big(1 + \hat{V}_k^2\Big)\hat{V}_k^2 k_F^{-1}
\end{aligned}$$

for a constant $C > 0$ depending only on s.

6.1 Matrix Element Estimates for K-Quantities

To ease the notation we will abstract the problem slightly: Instead of $\ell^2(L_k)$ we consider a general n-dimensional Hilbert space $(V, \langle\cdot,\cdot\rangle)$, let $h : V \to V$ be a positive self-adjoint operator on V with eigenbasis $(x_i)_{i=1}^n$ and eigenvalues $(\lambda_i)_{i=1}^n$, and let $v \in V$ be any vector such that $\langle x_i, v\rangle \geq 0$ for all $1 \leq i \leq n$. Theorem 6.0.1 will then be obtained at the end by insertion of the particular operators h_k and P_k.

Throughout this section we will also write $P_w : V \to V$, $w \in V$, to denote the operator

$$P_w(\cdot) = \langle w, \cdot\rangle\, w. \tag{6.1.1}$$

We define $K : V \to V$ by

$$K = -\frac{1}{2}\log\Big(h^{-\frac{1}{2}}\Big(h^{\frac{1}{2}}(h+2P_v)h^{\frac{1}{2}}\Big)^{\frac{1}{2}}h^{-\frac{1}{2}}\Big) = -\frac{1}{2}\log\Big(h^{-\frac{1}{2}}\Big(h^2+2P_{h^{\frac{1}{2}}v}\Big)^{\frac{1}{2}}h^{-\frac{1}{2}}\Big). \tag{6.1.2}$$

Then e^{-2K} is given by

$$e^{-2K} = h^{-\frac{1}{2}}\Big(h^2+2P_{h^{\frac{1}{2}}v}\Big)^{\frac{1}{2}}h^{-\frac{1}{2}} \tag{6.1.3}$$

while e^{2K} takes the form

$$e^{2K} = h^{\frac{1}{2}}\Big(h^2+2P_{h^{\frac{1}{2}}v}\Big)^{-\frac{1}{2}}h^{\frac{1}{2}} = h^{\frac{1}{2}}\bigg(\Big(h^2+2P_{h^{\frac{1}{2}}v}\Big)^{-1}\bigg)^{\frac{1}{2}}h^{\frac{1}{2}}. \tag{6.1.4}$$

We can rewrite the inverse of $h^2 + 2P_{h^{\frac{1}{2}}v}$ using the Sherman-Morrison formula:

Lemma 6.1.1 (The Sherman-Morrison Formula)**.** *Let $A : V \to V$ be an invertible self-adjoint operator. Then for any $w \in V$ and $g \in \mathbb{C}$, the operator $A + gP_w$ is invertible if and only if $\langle w, A^{-1}w\rangle \neq -g^{-1}$, with inverse*

$$(A+gP_w)^{-1} = A^{-1} - \frac{g}{1+g\,\langle w, A^{-1}w\rangle}P_{A^{-1}w}.$$

Applying the Sherman-Morrison formula with $A = h^2$, $w = h^{\frac{1}{2}}v$ and $g = 2$ we obtain

$$\Big(h^2+2P_{h^{\frac{1}{2}}v}\Big)^{-1} = h^{-2} - \frac{2}{1+2\,\langle v, h^{-1}v\rangle}P_{h^{-\frac{3}{2}}v} \tag{6.1.5}$$

so e^{-2K} and e^{2K} are given by

$$\begin{aligned} e^{-2K} &= h^{-\frac{1}{2}}\Big(h^2+2P_{h^{\frac{1}{2}}v}\Big)^{\frac{1}{2}}h^{-\frac{1}{2}} \\ e^{2K} &= h^{\frac{1}{2}}\left(h^{-2} - \frac{2}{1+2\,\langle v, h^{-1}v\rangle}P_{h^{-\frac{3}{2}}v}\right)^{\frac{1}{2}}h^{\frac{1}{2}}. \end{aligned} \tag{6.1.6}$$

To proceed further we apply the following integral representation of the square root of a one-dimensional perturbation, first presented in [8]:

Proposition 6.1.2. *Let $A : V \to V$ be a positive self-adjoint operator. Then for any $w \in V$ and $g \in \mathbb{R}$ such that $A + gP_w > 0$ it holds that*

$$(A+gP_w)^{\frac{1}{2}} = A^{\frac{1}{2}} + \frac{2g}{\pi}\int_0^\infty \frac{t^2}{1+g\left\langle w,(A+t^2)^{-1}w\right\rangle} P_{(A+t^2)^{-1}w}\,dt$$

and

$$\operatorname{tr}\Big((A+gP_w)^{\frac{1}{2}}\Big) = \operatorname{tr}\Big(A^{\frac{1}{2}}\Big) + \frac{1}{\pi}\int_0^\infty \log\Big(1+g\left\langle w,\left(A+t^2\right)^{-1}w\right\rangle\Big)dt.$$

We have included a proof of this in appendix section A.1.

By the trace formula we can immediately deduce the following identity:

Proposition 6.1.3. *It holds that*

$$\operatorname{tr}\Big(e^{-K}he^{-K} - h - P_v\Big) = \frac{1}{\pi}\int_0^\infty F\Big(2\left\langle v, h\left(h^2+t^2\right)^{-1}v\right\rangle\Big)dt$$

where $F(x) = \log(1+x) - x$.

Proof: By cyclicity of the trace and the previous proposition

$$\begin{aligned}\operatorname{tr}\Big(e^{-K}he^{-K} - h - P_v\Big) &= \operatorname{tr}\Big(h^{\frac{1}{2}}e^{-2K}h^{\frac{1}{2}}\Big) - \operatorname{tr}(h) - \operatorname{tr}(P_v) = \operatorname{tr}\Big(\Big(h^2 + 2P_{h^{\frac{1}{2}}v}\Big)^{\frac{1}{2}}\Big) - \operatorname{tr}(h) - \|v\|^2 \\ &= \frac{1}{\pi}\int_0^\infty \log\Big(1+2\left\langle v, h\left(h^2+t^2\right)^{-1}v\right\rangle\Big)dt - \|v\|^2, \end{aligned} \tag{6.1.7}$$

so noting that the integral identity $\int_0^\infty \frac{a}{a^2+t^2}dt = \frac{\pi}{2}$, $a > 0$, implies that

$$\frac{1}{\pi}\int_0^\infty 2\left\langle v, h\left(h^2+t^2\right)^{-1}v\right\rangle dt = \frac{2}{\pi}\sum_{i=1}^n |\langle e_i, v\rangle|^2 \int_0^\infty \frac{\lambda_i}{\lambda_i^2+t^2}dt = \sum_{i=1}^n |\langle e_i, v\rangle|^2 = \|v\|^2 \tag{6.1.8}$$

we can absorb the term $-\|v\|^2$ into the integral for the claim.

□

Estimation of e^{-2K} and e^{2K}

Using the square-root formula we now derive elementwise estimates for e^{-2K} and e^{2K}:

Proposition 6.1.4. *For all $1 \le i, j \le n$ it holds that*

$$\frac{2}{1+2\langle v, h^{-1}v\rangle}\frac{\langle x_i, v\rangle\langle v, x_j\rangle}{\lambda_i+\lambda_j} \le \left\langle x_i, \left(e^{-2K}-1\right)x_j\right\rangle, \left\langle x_i, \left(1-e^{2K}\right)x_j\right\rangle \le 2\frac{\langle x_i, v\rangle\langle v, x_j\rangle}{\lambda_i+\lambda_j}.$$

Proof: From the first equality of equation (6.1.6) we can apply the identity of Proposition 6.1.2 with $A = h^2$, $w = h^{\frac{1}{2}}v$ and $g = 2$ to see that

$$e^{-2K} = h^{-\frac{1}{2}}\left(h + \frac{4}{\pi}\int_0^\infty \frac{t^2}{1+2\left\langle h^{\frac{1}{2}}v, (h^2+t^2)^{-1}h^{\frac{1}{2}}v\right\rangle} P_{(h^2+t^2)^{-1}h^{\frac{1}{2}}v}\,dt\right)h^{-\frac{1}{2}} \tag{6.1.9}$$

$$= 1 + \frac{4}{\pi}\int_0^\infty \frac{t^2}{1+2\left\langle v, h(h^2+t^2)^{-1}v\right\rangle} P_{(h^2+t^2)^{-1}v} dt$$

whence for any $1 \leq i, j \leq n$

$$\left\langle x_i, \left(e^{-2K} - 1\right)x_j\right\rangle = \frac{4}{\pi}\int_0^\infty \frac{t^2}{1+2\left\langle v, h(h^2+t^2)^{-1}v\right\rangle} \frac{\langle x_i, v\rangle}{\lambda_i^2+t^2}\frac{\langle v, x_j\rangle}{\lambda_j^2+t^2} dt \qquad (6.1.10)$$
$$= \frac{4}{\pi}\langle x_i, v\rangle \langle v, x_j\rangle \int_0^\infty \frac{1}{1+2\left\langle v, h(h^2+t^2)^{-1}v\right\rangle}\frac{t}{\lambda_i^2+t^2}\frac{t}{\lambda_j^2+t^2} dt.$$

Noting that

$$\frac{1}{1+2\langle v, h^{-1}v\rangle} \leq \frac{1}{1+2\left\langle v, h(h^2+t^2)^{-1}v\right\rangle} \leq 1, \quad t \geq 0, \qquad (6.1.11)$$

and recalling that $\langle x_i, v\rangle \geq 0$ by assumption, we conclude

$$\left\langle x_i, \left(e^{-2K} - 1\right)x_j\right\rangle \geq \frac{4}{\pi}\frac{\langle x_i, v\rangle \langle v, x_j\rangle}{1+2\langle v, h^{-1}v\rangle}\int_0^\infty \frac{t}{\lambda_i^2+t^2}\frac{t}{\lambda_j^2+t^2} dt \qquad (6.1.12)$$
$$\left\langle x_i, \left(e^{-2K} - 1\right)x_j\right\rangle \leq \frac{4}{\pi}\langle x_i, v\rangle \langle v, x_j\rangle \int_0^\infty \frac{t}{\lambda_i^2+t^2}\frac{t}{\lambda_j^2+t^2} dt$$

from which the claim follows by an application of the integral identity

$$\int_0^\infty \frac{t}{a^2+t^2}\frac{t}{b^2+t^2}\, dt = \frac{\pi}{2}\frac{1}{a+b}, \quad a, b > 0. \qquad (6.1.13)$$

Similarly, for e^{2K}, we have by equation (6.1.6) that applying Proposition 6.1.2 with $A = h^{-2}$, $w = h^{-\frac{3}{2}}v$ and $g = -2(1 + 2\langle v, h^{-1}v\rangle)^{-1}$ yields

$$e^{2K} = h^{\frac{1}{2}}\left(h^{-1} - \frac{4}{\pi}\int_0^\infty \frac{t^2}{1+2\langle v, h^{-1}v\rangle - 2\left\langle h^{-\frac{3}{2}}v, (h^{-2}+t^2)^{-1}h^{-\frac{3}{2}}v\right\rangle} P_{(h^{-2}+t^2)^{-1}h^{-\frac{3}{2}}v} dt\right)h^{\frac{1}{2}}$$
$$= 1 - \frac{4}{\pi}\int_0^\infty \frac{t^2}{1+2\left\langle v, h^{-1}(h^{-2}+t^2)^{-1}v\right\rangle t^2} P_{(h^{-2}+t^2)^{-1}h^{-1}v} dt \qquad (6.1.14)$$

from which the claimed inequality follows as before by the observation that

$$\frac{1}{1+2\langle v, h^{-1}v\rangle} \leq \frac{1}{1+2\left\langle v, h^{-1}(h^{-2}+t^2)^{-1}v\right\rangle t^2} \leq 1, \quad t \geq 0, \qquad (6.1.15)$$

as well as the integral identity

$$\frac{1}{ab}\int_0^\infty \frac{t}{a^{-2}+t^2}\frac{t}{b^{-2}+t^2}\, dt = \frac{\pi}{2}\frac{1}{a+b}, \quad a, b > 0. \qquad (6.1.16)$$

□

Note that these estimates are asymptotically optimal, in the sense that the left-hand side reduces to the right-hand side as $\langle v, h^{-1}v\rangle \to 0$. In our case we will see that $\left\langle v_k, h_k^{-1}v_k\right\rangle \sim \hat{V}_k$, so this amounts to optimal estimates for "small" $\hat{V}_k$.

Below it will be more convenient to consider the hyperbolic functions $\sinh(-2K)$ and $\cosh(-2K)$ rather than e^{-2K} and e^{2K}. The previous proposition implies the following for these operators:

Corollary 6.1.5. *For any $1 \leq i, j \leq n$ it holds that*

$$\begin{aligned}\langle x_i, \sinh(-2K)x_j\rangle &\leq 2\frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i + \lambda_j}\\ \langle x_i, (\cosh(-2K) - 1)x_j\rangle &\leq \frac{2\langle v, h^{-1}v\rangle}{1 + 2\langle v, h^{-1}v\rangle}\frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i + \lambda_j}.\end{aligned}$$

Proof: As $\sinh(-2K) = \frac{1}{2}\left(\left(e^{-2K} - 1\right) + \left(1 - e^{2K}\right)\right)$ we can bound

$$\langle x_i, \sinh(-2K)x_j\rangle = \frac{1}{2}\left(\left\langle x_i, \left(e^{-2K} - 1\right)x_j\right\rangle + \left\langle x_i, \left(1 - e^{2K}\right)x_j\right\rangle\right) \leq 2\frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i + \lambda_j} \tag{6.1.17}$$

and as similarly $\cosh(-2K) - 1 = \frac{1}{2}\left(\left(e^{-2K} - 1\right) - \left(1 - e^{2K}\right)\right)$ also

$$\begin{aligned}\langle x_i, (\cosh(-2K) - 1)x_j\rangle &= \frac{1}{2}\left(\left\langle x_i, \left(e^{-2K} - 1\right)x_j\right\rangle - \left\langle x_i, \left(1 - e^{2K}\right)x_j\right\rangle\right)\\ &\leq \frac{1}{2}\left(2\frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i + \lambda_j} - \frac{2}{1 + 2\langle v, h^{-1}v\rangle}\frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i + \lambda_j}\right) \qquad (6.1.18)\\ &= \frac{2\langle v, h^{-1}v\rangle}{1 + 2\langle v, h^{-1}v\rangle}\frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i + \lambda_j}.\end{aligned}$$

□

General Estimates

Now we extend our elementwise estimates to more general operators. First we consider K itself:

Proposition 6.1.6. *For any $1 \leq i, j \leq n$ it holds that*

$$\frac{1}{1 + 2\langle v, h^{-1}v\rangle}\frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i + \lambda_j} \leq \langle x_i, (-K)x_j\rangle \leq \frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i + \lambda_j}.$$

Proof: As $K = -\frac{1}{2}\log\left(h^{-\frac{1}{2}}\left(h^2 + 2P_{h^{\frac{1}{2}}v}\right)^{\frac{1}{2}}h^{-\frac{1}{2}}\right)$ and

$$h^{-\frac{1}{2}}\left(h^2 + 2P_{h^{\frac{1}{2}}v}\right)^{\frac{1}{2}}h^{-\frac{1}{2}} \geq h^{-\frac{1}{2}}hh^{-\frac{1}{2}} = 1 \tag{6.1.19}$$

we see that $K \leq 0$. From the identity

$$-x = \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} \Big(1 - e^{2x}\Big)^m, \quad x \leq 0, \tag{6.1.20}$$

which follows by the Mercator series, we thus have that $-K = \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} \Big(1 - e^{2K}\Big)^m$. Noting that Proposition 6.1.4 in particular implies that $\left\langle x_i, \Big(1 - e^{2K}\Big) x_j \right\rangle \geq 0$ for all $1 \leq i, j \leq n$, whence also $\left\langle x_i, \Big(1 - e^{2K}\Big)^m x_j \right\rangle \geq 0$ for any $m \in \mathbb{N}$, we may estimate

$$\begin{aligned} \langle x_i, (-K) x_j \rangle &= \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} \left\langle x_i, \Big(1 - e^{2K}\Big)^m x_j \right\rangle \geq \frac{1}{2} \left\langle x_i, \Big(1 - e^{2K}\Big) x_j \right\rangle \\ &\geq \frac{1}{1 + 2\langle v, h^{-1} v\rangle} \frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i + \lambda_j} \end{aligned} \tag{6.1.21}$$

which is the lower bound. This similarly implies that $\langle x_i, (-K)^m x_j \rangle \geq 0$ for all $1 \leq i, j \leq n$, $m \in \mathbb{N}$, so the upper bound now also follows from Proposition 6.1.4 by noting that

$$\frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i + \lambda_j} \geq \frac{1}{2} \left\langle x_i, \Big(e^{-2K} - 1\Big) x_j \right\rangle = \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m!} \langle x_i, (-2K)^m x_j \rangle \geq \langle x_i, (-K) x_j \rangle . \tag{6.1.22}$$

□

The fact that $\langle x_i, (-K)^m x_j \rangle \geq 0$ for all $1 \leq i, j \leq n$, $m \in \mathbb{N}$, has the important consequence that for any such i and j, the functions

$$t \mapsto \langle x_i, \sinh(-tK) x_j \rangle , \ \langle x_i, (\sinh(-tK) + tK) x_j \rangle , \ \langle x_i, (\cosh(-tK) - 1) x_j \rangle \tag{6.1.23}$$

are non-negative and convex for $t \in [0, \infty)$, as follows by considering the Taylor expansions of the operators involved. This allows us to extend the bounds of Corollary 6.1.5 to arbitrary $t \in [0, 1]$:

Proposition 6.1.7. *For all* $1 \leq i, j \leq n$ *and* $t \in [0, 1]$ *it holds that*

$$\begin{aligned} \frac{1}{1 + 2\langle v, h^{-1} v\rangle} \frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i + \lambda_j} t \leq \langle x_i, \sinh(-tK) x_j \rangle &\leq \frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i + \lambda_j} t \\ 0 \leq \langle x_i, (\cosh(-tK) - 1) x_j \rangle &\leq \frac{\langle v, h^{-1} v\rangle}{1 + 2\langle v, h^{-1} v\rangle} \frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i + \lambda_j} \\ \left| \left\langle x_i, \Big(e^{tK} - 1\Big) x_j \right\rangle \right| &\leq \frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i + \lambda_j}. \end{aligned}$$

Proof: By the noted convexity we immediately conclude the upper bounds

$$\langle x_i, \sinh(-tK) x_j \rangle \leq \frac{t}{2} \langle x_i, \sinh(-2K) x_j \rangle \leq \frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i + \lambda_j} t \tag{6.1.24}$$

$$\langle x_i, (\cosh(-tK)-1)x_j\rangle \le \frac{t}{2}\langle x_i, (\cosh(-2K)-1)x_j\rangle \le \frac{\langle v, h^{-1}v\rangle}{1+2\langle v, h^{-1}v\rangle}\frac{\langle x_i, v\rangle\langle v, x_j\rangle}{\lambda_i+\lambda_j}$$

and by non-negativity of $\langle x_i, (\sinh(-tK)+tK)x_j\rangle$ and Proposition 6.1.6, the lower bound

$$\langle x_i, \sinh(-tK)x_j\rangle \ge \langle x_i, (-tK)x_j\rangle \ge \frac{1}{1+2\langle v, h^{-1}v\rangle}\frac{\langle x_i, v\rangle\langle v, x_j\rangle}{\lambda_i+\lambda_j}t. \tag{6.1.25}$$

Lastly we can apply the non-negativity of the hyperbolic operators to conclude the bound for $e^{tK}-1$ as

$$\begin{aligned}\left|\left\langle x_i, \left(e^{tK}-1\right)x_j\right\rangle\right| &= |\langle x_i, ((\cosh(-tK)-1)-\sinh(-tK))x_j\rangle| \qquad (6.1.26)\\ &\le \max\{\langle x_i, (\cosh(-tK)-1)x_j\rangle, \langle x_i, \sinh(-tK)x_j\rangle\} \le \frac{\langle x_i, v\rangle\langle v, x_j\rangle}{\lambda_i+\lambda_j}.\end{aligned}$$

□

6.2 Matrix Element Estimates for $A(t)$ and $B(t)$

We now consider operators $A(t), B(t) : V \to V$ defined by

$$\begin{aligned}A(t) &= \frac{1}{2}\Big(e^{tK}(h+2P_v)e^{tK} + e^{-tK}he^{-tK}\Big) - h \qquad (6.2.1)\\ B(t) &= \frac{1}{2}\Big(e^{tK}(h+2P_v)e^{tK} - e^{-tK}he^{-tK}\Big)\end{aligned}$$

for $t \in [0,1]$. We decompose these as

$$\begin{aligned}A(t) &= A_h(t) + e^{tK}P_ve^{tK} \qquad (6.2.2)\\ B(t) &= (1-t)P_v + B_h(t) + e^{tK}P_ve^{tK} - P_v\end{aligned}$$

where, with

$$C_K(t) = \cosh(-tK)-1 \quad \text{and} \quad S_K(t) = \sinh(-tK), \tag{6.2.3}$$

the operators $A_h(t)$ and $B_h(t)$ are given by

$$\begin{aligned}A_h(t) &= \cosh(-tK)\,h\cosh(-tK) + \sinh(-tK)\,h\sinh(-tK) - h \qquad (6.2.4)\\ &= \{h, C_K(t)\} + S_K(t)\,h\,S_K(t) + C_K(t)\,h\,C_K(t)\end{aligned}$$

and

$$\begin{aligned}B_h(t) &= -\sinh(-tK)\,h\cosh(-tK) - \cosh(-tK)\,h\sinh(-tK) + tP_v \qquad (6.2.5)\\ &= tP_v - \{h, S_K(t)\} - S_K(t)\,h\,C_K(t) - C_K(t)\,h\,S_K(t).\end{aligned}$$

We begin by estimating the $e^{tK}P_ve^{tK}$ terms:

Proposition 6.2.1. *For all* $1 \leq i, j \leq n$ *and* $t \in [0,1]$ *it holds that*

$$\left|\left\langle x_i, \left(e^{tK} P_v e^{tK} - P_v\right) x_j \right\rangle\right| \leq \left(2 + \left\langle v, h^{-1} v\right\rangle\right) \left\langle v, h^{-1} v\right\rangle \langle x_i, v\rangle \langle v, x_j\rangle .$$

Proof: Writing

$$e^{tK} P_v e^{tK} - P_v = \left\{P_v, e^{tK} - 1\right\} + \left(e^{tK} - 1\right) P_v \left(e^{tK} - 1\right) \tag{6.2.6}$$

we see that

$$\begin{aligned}\left\langle x_i, \left(e^{tK} P_v e^{tK} - P_v\right) x_j \right\rangle &= \langle x_i, v\rangle \left\langle \left(e^{tK} - 1\right) v, x_j \right\rangle + \left\langle x_i, \left(e^{tK} - 1\right) v\right\rangle \langle v, x_j\rangle \\ &\quad + \left\langle x_i, \left(e^{tK} - 1\right) v\right\rangle \left\langle \left(e^{tK} - 1\right) v, x_j \right\rangle .\end{aligned} \tag{6.2.7}$$

Now, by Proposition 6.1.7 we can for any $1 \leq i \leq n$ estimate

$$\begin{aligned}\left|\left\langle x_i, \left(e^{tK} - 1\right) v\right\rangle\right| &= \left|\sum_{j=1}^{n} \left\langle x_i, \left(e^{tK} - 1\right) x_j \right\rangle \langle x_j, v\rangle\right| \leq \sum_{j=1}^{n} \frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i + \lambda_j} \langle x_j, v\rangle \\ &\leq \langle x_i, v\rangle \sum_{j=1}^{n} \frac{\left|\langle x_j, v\rangle\right|^2}{\lambda_j} = \langle x_i, v\rangle \left\langle v, h^{-1} v\right\rangle\end{aligned} \tag{6.2.8}$$

whence the claim follows.

□

Note that for $\left\langle x_i, e^{tK} P_v e^{tK} x_j \right\rangle$ this in particular implies the bound

$$\left|\left\langle x_i, e^{tK} P_v e^{tK} x_j \right\rangle\right| \leq \left(1 + \left\langle v, h^{-1} v\right\rangle\right)^2 \langle x_i, v\rangle \langle v, x_j\rangle . \tag{6.2.9}$$

We now consider $A_h(t)$ and $B_h(t)$:

Proposition 6.2.2. *For all* $1 \leq i, j \leq n$ *and* $t \in [0,1]$ *it holds that*

$$\left|\langle x_i, A_h(t) x_j\rangle\right|, \left|\langle x_i, B_h(t) x_j\rangle\right| \leq 4 \left\langle v, h^{-1} v\right\rangle \langle x_i, v\rangle \langle v, x_j\rangle .$$

Proof: The estimates of Proposition 6.1.7 imply that

$$\begin{aligned}\left|\langle x_i, \{h, C_K(t)\} x_j\rangle\right| &= (\lambda_i + \lambda_j) \left|\langle x_i, C_K(t) x_j\rangle\right| \leq (\lambda_i + \lambda_j) \frac{\langle v, h^{-1} v\rangle}{1 + 2\langle v, h^{-1} v\rangle} \frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i + \lambda_j} \\ &\leq \left\langle v, h^{-1} v\right\rangle \langle x_i, v\rangle \langle v, x_j\rangle\end{aligned} \tag{6.2.10}$$

and

$$\begin{aligned}\left|\langle x_i, S_K(t)\, h\, S_K(t) x_j\rangle\right| &= \left|\sum_{k=1}^{n} \lambda_k \langle x_i, S_K(t) x_k\rangle \langle x_k, S_K(t) x_j\rangle\right| \leq \sum_{k=1}^{n} \lambda_k \frac{\langle x_i, v\rangle \langle v, x_k\rangle}{\lambda_i + \lambda_k} \frac{\langle x_k, v\rangle \langle v, x_j\rangle}{\lambda_k + \lambda_j} \\ &\leq \langle x_i, v\rangle \langle v, x_j\rangle \sum_{k=1}^{n} \frac{\left|\langle x_k, v\rangle\right|^2}{\lambda_k} = \left\langle v, h^{-1} v\right\rangle \langle x_i, v\rangle \langle v, x_j\rangle .\end{aligned} \tag{6.2.11}$$

The latter estimate only relied on the inequality

$$|\langle x_i, S_K(t)x_j\rangle| \leq \frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i + \lambda_j}, \tag{6.2.12}$$

which is also true for $C_K(t)$, so the terms

$$C_K(t)\, h\, C_K(t), \quad C_K(t)\, h\, S_K(t) \quad \text{and} \quad S_K(t)\, h\, C_K(t) \tag{6.2.13}$$

also obey this estimate. It thus only remains to bound $tP_v - \{h, S_K(t)\}$. From Proposition 6.1.7 we see that

$$\frac{\langle x_i, v\rangle \langle v, x_j\rangle}{1 + 2\langle v, h^{-1}v\rangle} t \leq \langle x_i, \{h, S_K(t)\}\, x_j\rangle \leq \langle x_i, v\rangle \langle v, x_j\rangle\, t \tag{6.2.14}$$

whence

$$\begin{aligned} &|\langle x_i, (tP_v - \{h, S_K(t)\})x_j\rangle| = \langle x_i, P_v x_j\rangle\, t - \langle x_i, \{h, S_K(t)\}\, x_j\rangle \\ \leq &\left(1 - \frac{1}{1 + 2\langle v, h^{-1}v\rangle}\right) \langle x_i, v\rangle \langle v, x_j\rangle\, t = \frac{2\langle v, h^{-1}v\rangle}{1 + 2\langle v, h^{-1}v\rangle} \langle x_i, v\rangle \langle v, x_j\rangle\, t \\ \leq &\, 2\left\langle v, h^{-1}v\right\rangle \langle x_i, v\rangle \langle v, x_j\rangle\, . \end{aligned} \tag{6.2.15}$$

□

Combining equation (6.2.9) and Proposition 6.2.2 we conclude the following:

Proposition 6.2.3. *For all* $1 \leq i, j \leq n$ *and* $t \in [0, 1]$ *it holds that*

$$|\langle x_i, A(t)x_j\rangle|\, , \, |\langle x_i, B(t)x_j\rangle| \leq 3\Big(1 + \left\langle v, h^{-1}v\right\rangle\Big)^2 \langle x_i, v\rangle \langle v, x_j\rangle\, .$$

Analysis of $\{K, B(t)\}$ and $\int_0^1 B(t)\, dt$

We end by estimating $\{K, B(t)\}$ and $\int_0^1 B(t)\, dt$, the latter of which will be needed for the analysis of the exchange contribution in the next section.

First is $\{K, B(t)\}$:

Proposition 6.2.4. *For all* $1 \leq i, j \leq n$ *and* $t \in [0, 1]$ *it holds that*

$$|\langle x_i, \{K, B(t)\}\, x_j\rangle| \leq 6\Big(1 + \left\langle v, h^{-1}v\right\rangle\Big)^2 \left\langle v, h^{-1}v\right\rangle \langle x_i, v\rangle \langle v, x_j\rangle\, .$$

Proof: Using the Propositions 6.1.6 and 6.2.3 we see that

$$\begin{aligned} |\langle x_i, KB(t)x_j\rangle| &= \left|\sum_{k=1}^n \langle x_i, Kx_k\rangle \langle x_k, B(t)x_j\rangle\right| \leq 3\Big(1 + \left\langle v, h^{-1}v\right\rangle\Big)^2 \sum_{k=1}^n \frac{\langle x_i, v\rangle \langle v, x_k\rangle}{\lambda_i + \lambda_k} \langle x_k, v\rangle \langle v, x_j\rangle \\ &\leq 3\Big(1 + \left\langle v, h^{-1}v\right\rangle\Big)^2 \sum_{k=1}^n \frac{|\langle x_k, v\rangle|^2}{\lambda_k} \langle x_i, v\rangle \langle v, x_j\rangle \end{aligned} \tag{6.2.16}$$

$$= 3\Big(1 + \left\langle v, h^{-1}v \right\rangle\Big)^2 \left\langle v, h^{-1}v \right\rangle \langle x_i, v\rangle \langle v, x_j\rangle .$$

This estimate is also valid for $|\langle x_i, B(t)Kx_j\rangle|$ so the claim follows.

□

Finally is $\int_0^1 B(t)\,dt$:

Proposition 6.2.5. *For all* $1 \leq i, j \leq n$ *it holds that*

$$\left|\left\langle x_i, \left(\int_0^1 B(t)\,dt\right)x_j\right\rangle - \frac{1}{2}\langle x_i, v\rangle \langle v, x_j\rangle\right| \leq \left(6 + \left\langle v, h^{-1}v\right\rangle\right)\left\langle v, h^{-1}v\right\rangle \langle x_i, v\rangle \langle v, x_j\rangle .$$

Proof: Noting that $\frac{1}{2}\langle x_i, v\rangle \langle v, x_j\rangle = \frac{1}{2}\langle x_i, P_v x_j\rangle$ and that

$$\begin{aligned}\int_0^1 B(t)\,dt - \frac{1}{2}P_v &= \int_0^1 \Big((1-t)P_v + B_h(t) + e^{tK}P_v e^{tK} - P_v\Big)\,dt - \frac{1}{2}P_v \\ &= \int_0^1 \Big(B_h(t) + e^{tK}P_v e^{tK} - P_v\Big)\,dt\end{aligned} \tag{6.2.17}$$

we can estimate using the Propositions 6.2.1 and 6.2.2 that

$$\begin{aligned}\left|\left\langle x_i, \left(\int_0^1 B(t)\,dt - \frac{1}{2}P_v\right)x_j\right\rangle\right| &\leq \int_0^1 |\langle x_i, B_h(t)x_j\rangle|\,dt + \int_0^1 \left|\left\langle x_i, \left(e^{tK}P_v e^{tK} - P_v\right)x_j\right\rangle\right|dt \\ &\leq \left(6 + \left\langle v, h^{-1}v\right\rangle\right)\left\langle v, h^{-1}v\right\rangle \langle x_i, v\rangle \langle v, x_j\rangle .\end{aligned} \tag{6.2.18}$$

□

Insertion of the Particular Operators h_k and P_k

Recall that the particular operators we must consider are $h_k, P_k : \ell^2(L_k) \to \ell^2(L_k)$ defined by

$$\begin{array}{rclcrcl} h_k e_p &=& \lambda_{k,p} e_p & & \lambda_{k,p} &=& \frac{1}{2}\left(|p|^2 - |p-k|^2\right) \\ P_k(\cdot) &=& \langle v_k, \cdot\rangle\, v_k & & v_k &=& \sqrt{\frac{s\hat{V}_k k_F^{-1}}{2\,(2\pi)^3}}\sum_{p\in L_k} e_p. \end{array} \tag{6.2.19}$$

For these we have that

$$\left\langle v_k, h_k^{-1}v_k\right\rangle = \frac{s\hat{V}_k k_F^{-1}}{2\,(2\pi)^3}\sum_{p\in L_k}\lambda_{k,p}^{-1}. \tag{6.2.20}$$

In appendix section B we obtain the following estimates for sums of the form $\sum_{p\in L_k}\lambda_{k,p}^{\beta}$:

Proposition 6.2.6. *For any* $k \in \mathbb{Z}^3_*$ *and* $\beta \in [-1, 0]$ *it holds that*

$$\sum_{p\in L_k}\lambda_{k,p}^{\beta} \leq C_\beta \begin{cases} k_F^{2+\beta}\,|k|^{1+\beta} & |k| \leq 2k_F \\ k_F^3\,|k|^{2\beta} & |k| > 2k_F \end{cases}$$

for a constant $C_\beta > 0$ *independent of* k *and* k_F*.*

In particular, it holds that $\sum_{p\in L_k}\lambda_{k,p}^{-1}\leq Ck_F\min\{1,k_F^2\,|k|^{-2}\}$, so

$$\left\langle v_k,h_k^{-1}v_k\right\rangle\leq C\hat{V}_k \tag{6.2.21}$$

for a constant $C>0$ depending only on s. Additionally, independently of p and q it holds that

$$\langle e_p,v_k\rangle\langle v_k,e_q\rangle=\frac{s\hat{V}_k k_F^{-1}}{2\,(2\pi)^3} \tag{6.2.22}$$

and for any $t\geq 0$

$$\left\langle v,h\left(h^2+t^2\right)^{-1}v\right\rangle=\frac{s\hat{V}_k k_F^{-1}}{2\,(2\pi)^3}\sum_{p\in L_k}\frac{\lambda_{k,p}}{\lambda_{k,p}^2+t^2}. \tag{6.2.23}$$

Inserting these quantities into the statements of the Propositions 6.1.3, 6.1.6, 6.2.3 and 6.2.4 yields Theorem 6.0.1, noting also that by Proposition 6.1.6

$$\begin{aligned}\|K_k\|_{\mathrm{HS}}&=\sqrt{\sum_{p,q\in L_k}|\langle e_p,K_k e_q\rangle|^2}\leq\frac{s\hat{V}_k k_F^{-1}}{2\,(2\pi)^3}\sqrt{\sum_{p,q\in L_k}\frac{1}{(\lambda_{k,p}+\lambda_{k,q})^2}}\leq\frac{s\hat{V}_k k_F^{-1}}{2\,(2\pi)^3}\sum_{p\in L_k}\lambda_{k,p}^{-1}\\&\leq C\hat{V}_k\min\{1,k_F^2\,|k|^{-2}\}.\end{aligned} \tag{6.2.24}$$

Chapter 7

Analysis of Exchange Terms

In this section we analyze the *exchange terms*, by which we mean the quantities of the expression

$$\sum_{k\in\mathbb{Z}_*^3}\int_0^1 e^{(1-t)\mathcal{K}}\Big(\varepsilon_k(\{K_k,B_k(t)\})+2\,\mathrm{Re}\Big(\mathcal{E}_k^1(A_k(t))\Big)+2\,\mathrm{Re}\Big(\mathcal{E}_k^2(B_k(t))\Big)\Big)e^{-(1-t)\mathcal{K}}dt \quad (7.0.1)$$

which appears in Theorem 4.0.1 - the name is apt, as these enter our calculations due to the presence of the exchange correction $\varepsilon_{k,l}(p;q)$ of the quasi-bosonic commutation relations.

To be more precise, what we consider in this section are the operators $\varepsilon_k(\{K_k,B_k(t)\})$, $\mathcal{E}_k^1(A_k(t))$ and $\mathcal{E}_k^2(B_k(t))$ - the effect of the integration will be handled by Gronwall estimates in the next section.

The exchange terms are primarily to be regarded as error terms, and the main result of this section is the following estimates for them:

Theorem 7.0.1. *For any $\Psi\in\mathcal{H}_N$ and $t\in[0,1]$ it holds that*

$$\left|\sum_{k\in\mathbb{Z}_*^3}\langle\Psi,\varepsilon_k(\{K_k,B_k(t)\})\Psi\rangle\right|\leq Ck_F^{-1}\langle\Psi,\mathcal{N}_E\Psi\rangle$$

$$\sum_{k\in\mathbb{Z}_*^3}\left|\left\langle\Psi,\mathcal{E}_k^1(A_k(t))\Psi\right\rangle\right|\leq C\sqrt{\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2\min\{|k|,k_F\}}\left\langle\Psi,\left(\mathcal{N}_E^3+1\right)\Psi\right\rangle$$

$$\sum_{k\in\mathbb{Z}_*^3}\left|\left\langle\Psi,\left(\mathcal{E}_k^2(B_k(t))-\left\langle\psi_F,\mathcal{E}_k^2(B_k(t))\psi_F\right\rangle\right)\Psi\right\rangle\right|\leq C\sqrt{\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2\min\{|k|,k_F\}}\left\langle\Psi,\mathcal{N}_E^3\Psi\right\rangle$$

for a constant $C>0$ depending only on $\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2$ and s.

Note the presence of the constant terms $\langle\psi_F,\mathcal{E}_k^2(B_k(t))\psi_F\rangle$ in the final estimate of the theorem. By adding and subtracting these, we see that

$$\text{Exchange Terms}=\sum_{k\in\mathbb{Z}_*^3}\int_0^1\left\langle\psi_F,2\,\mathrm{Re}\Big(\mathcal{E}_k^2(B_k(t))\Big)\psi_F\right\rangle dt+\text{Error Terms.} \quad (7.0.2)$$

The quantity $\sum_{k\in\mathbb{Z}^3_*}\int_0^1 \langle \psi_F, 2\,\mathrm{Re}(\mathcal{E}_k^2(B_k(t)))\psi_F\rangle\,dt$ is the *exchange contribution* (to the correlation energy), which is not generally negligible for singular potentials V. We end the section by determining the leading behavior of these:

Proposition 7.0.2. *It holds that*

$$\left|\sum_{k\in\mathbb{Z}^3_*}\int_0^1 \left\langle \psi_F, 2\,\mathrm{Re}\Big(\mathcal{E}_k^2(B_k(t))\Big)\psi_F\right\rangle dt - E_{\mathrm{corr,ex}}\right| \le C\sqrt{\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2\min\{|k|,k_F\}}$$

for a constant $C>0$ depending only on $\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2$ and s, where

$$E_{\mathrm{corr,ex}} = \frac{sk_F^{-2}}{4\,(2\pi)^6}\sum_{k,l\in\mathbb{Z}^3_*}\hat{V}_k\hat{V}_l\sum_{p,q\in L_k\cap L_l}\frac{\delta_{p+q,k+l}}{\lambda_{k,p}+\lambda_{k,q}}.$$

Analysis of ε_k Terms

Let us first consider terms of the form $\sum_{k\in\mathbb{Z}^3_*}\varepsilon_k(A_k)$, where we recall that $\varepsilon_k(A_k)$ is given by

$$\varepsilon_k(A_k) = -\frac{1}{s}\sum_{p\in L_k}^{\sigma}\langle e_p, A_k e_p\rangle\left(c_{p,\sigma}^*c_{p,\sigma} + c_{p-k,\sigma}c_{p-k,\sigma}^*\right). \tag{7.0.3}$$

When summing over $k\in\mathbb{Z}^3_*$, we can split the sum into two parts and interchange the summations as follows:

$$\begin{aligned}
-\sum_{k\in\mathbb{Z}^3_*}\varepsilon_k(A_k) &= \frac{1}{s}\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in L_k}^{\sigma}\langle e_p, A_k e_p\rangle c_{p,\sigma}^*c_{p,\sigma} + \frac{1}{s}\sum_{k\in\mathbb{Z}^3_*}\sum_{q\in(L_k-k)}^{\sigma}\langle e_{q+k}, A_k e_{q+k}\rangle c_{q,\sigma}c_{q,\sigma}^* \\
&= \frac{1}{s}\sum_{p\in B_F^c}^{\sigma}\left(\sum_{k\in\mathbb{Z}^3_*}1_{L_k}(p)\langle e_p, A_k e_p\rangle\right)c_{p,\sigma}^*c_{p,\sigma} \\
&\quad + \frac{1}{s}\sum_{q\in B_F}^{\sigma}\left(\sum_{k\in\mathbb{Z}^3_*}1_{L_k}(q+k)\langle e_{q+k}, A_k e_{q+k}\rangle\right)c_{q,\sigma}c_{q,\sigma}^*.
\end{aligned} \tag{7.0.4}$$

Recalling that the excitation number operator is given by

$$\mathcal{N}_E = \sum_{p\in B_F^c}^{\sigma}c_{p,\sigma}^*c_{p,\sigma} = \sum_{q\in B_F}^{\sigma}c_{q,\sigma}c_{q,\sigma}^* \tag{7.0.5}$$

on $\mathcal{H}_N$, we can then immediately conclude that

$$\begin{aligned}
\pm\sum_{k\in\mathbb{Z}^3_*}\varepsilon_k(A_k) &\le \frac{1}{s}\left(\sup_{p\in B_F^c}\sum_{k\in\mathbb{Z}^3_*}1_{L_k}(p)\,|\langle e_p, A_k e_p\rangle| + \sup_{q\in B_F}\sum_{k\in\mathbb{Z}^3_*}1_{L_k}(q+k)\,|\langle e_{q+k}, A_k e_{q+k}\rangle|\right)\mathcal{N}_E \\
&\le \frac{2}{s}\left(\sum_{k\in\mathbb{Z}^3_*}\sup_{p\in L_k}|\langle e_p, A_k e_p\rangle|\right)\mathcal{N}_E.
\end{aligned} \tag{7.0.6}$$

By the estimates of the previous section we thus obtain the first estimate of Theorem 7.0.1:

Proposition 7.0.3. *For any $\Psi \in \mathcal{H}_N$ and $t \in [0,1]$ it holds that*

$$\left|\sum_{k\in\mathbb{Z}_*^3} \langle \Psi, \varepsilon_k(\{K_k, B_k(t)\})\Psi\rangle\right| \le Ck_F^{-1} \langle \Psi, \mathcal{N}_E\Psi\rangle$$

for a constant $C > 0$ depending only on $\sum_{k\in\mathbb{Z}_*^3} \hat{V}_k^2$ and s.

Proof: By Theorem 6.0.1 we have that

$$|\langle e_p, \{K_k, B_k(t)\}\, e_q\rangle| \le C\Big(1+\hat{V}_k^2\Big)\hat{V}_k^2 k_F^{-1}, \quad k\in\mathbb{Z}_*^3,\ p,q\in L_k, \tag{7.0.7}$$

for a constant $C > 0$ depending only on s, so

$$\left|\sum_{k\in\mathbb{Z}_*^3} \langle \Psi, \varepsilon_k(\{K_k, B_k(t)\})\Psi\rangle\right| \le \frac{2}{s}\left(\sum_{k\in\mathbb{Z}_*^3}\sup_{p\in L_k} |\langle e_p, \{K_k, B_k(t)\}\, e_p\rangle|\right)\langle \Psi, \mathcal{N}_E\Psi\rangle \tag{7.0.8}$$
$$\le Ck_F^{-1}\sum_{k\in\mathbb{Z}_*^3}\Big(1+\hat{V}_k^2\Big)\hat{V}_k^2 \langle \Psi, \mathcal{N}_E\Psi\rangle \le Ck_F^{-1}\Big(1+\|\hat{V}\|_\infty^2\Big)\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2 \langle \Psi, \mathcal{N}_E\Psi\rangle .$$

As $\|\hat{V}\|_\infty^2 \le \|\hat{V}\|_2^2 = \sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2$ the claim follows.

□

7.1 Analysis of $\mathcal{E}_k^1$ Terms

We consider terms of the form

$$\mathcal{E}_k^1(A_k) = \sum_{l\in\mathbb{Z}_*^3}\sum_{p\in L_k}\sum_{q\in L_l} b_k^*(A_k e_p)\left\{\varepsilon_{k,l}(e_p;e_q), b_{-l}^*(K_{-l}e_{-q})\right\}. \tag{7.1.1}$$

Recalling that $\varepsilon_{k,l}(e_p;e_q)$ is given by

$$\varepsilon_{k,l}(e_p;e_q) = -\frac{1}{s}\sum_{\sigma=1}^{s}\Big(\delta_{p,q}c_{q-l,\sigma}c_{p-k,\sigma}^* + \delta_{p-k,q-l}c_{q,\sigma}^*c_{p,\sigma}\Big) \tag{7.1.2}$$

we see that $\mathcal{E}_k^1(A_k)$ splits into two sums as

$$\begin{aligned}
-s\,\mathcal{E}_k^1(A_k) &= \sum_{l\in\mathbb{Z}_*^3}\sum_{p\in L_k}^{\sigma}\sum_{q\in L_l} b_k^*(A_k e_p)\left\{\delta_{p,q}c_{q-l,\sigma}c_{p-k,\sigma}^*, b_{-l}^*(K_{-l}e_{-q})\right\}\\
&+\sum_{l\in\mathbb{Z}_*^3}\sum_{p\in(L_k-k)}^{\sigma}\sum_{q\in(L_l-l)} b_k^*(A_k e_{p+k})\left\{\delta_{p,q}c_{q+l,\sigma}^*c_{p+k,\sigma}, b_{-l}^*(K_{-l}e_{-q-l})\right\} \qquad (7.1.3)\\
&= \sum_{l\in\mathbb{Z}_*^3}\sum_{p\in L_k\cap L_l}^{\sigma} b_k^*(A_k e_p)\left\{c_{p-l,\sigma}c_{p-k,\sigma}^*, b_{-l}^*(K_{-l}e_{-p})\right\}\\
&+\sum_{l\in\mathbb{Z}_*^3}\sum_{p\in(L_k-k)\cap(L_l-l)}^{\sigma} b_k^*(A_k e_{p+k})\left\{c_{p+l,\sigma}^*c_{p+k,\sigma}, b_{-l}^*(K_{-l}e_{-p-l})\right\}.
\end{aligned}$$

The two sums on the right-hand side have the same "schematic form": They can both be written as

$$\sum_{l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}^{\sigma} b_k^*(A_k e_{p_1})\left\{\tilde{c}^*_{p_2,\sigma}\tilde{c}_{p_3,\sigma}, b^*_{-l}(K_{-l}e_{p_4})\right\},\quad \tilde{c}_p = \begin{cases} c_p & p\in B_F^c \\ c_p^* & p\in B_F\end{cases}, \tag{7.1.4}$$

where the index set is either the lune $S_k = L_k$ or the set of corresponding hole states $S_k = L_k - k$, and depending on this index set the variables p_1, p_2, p_3, p_4 are given by

$$(p_1,p_2,p_3,p_4) = \begin{cases} (p, p-l, p-k, -p) & S_k = L_k \\ (p+k, p+l, p+k, -p-l) & S_k = L_k - k\end{cases}. \tag{7.1.5}$$

Note that in either case p_1, p_3 depend only on p and k, while p_2, p_4 depend only on p and l. Additionally, p_1 is always an element of L_k and p_4 is always an element of L_{-l}.

Since $b_{k,p} = s^{-\frac{1}{2}}\sum_{\sigma=1}^s c^*_{p-k,\sigma}c_{p,\sigma} = s^{-\frac{1}{2}}\sum_{\sigma=1}^s \tilde{c}_{p-k,\sigma}\tilde{c}_{p,\sigma}$ it is easily seen that $[b,\tilde{c}] = 0$, so in normal-ordering (with respect to ψ_F) the summand of equation (7.1.4) we find

$$\begin{aligned} & b_k^*(A_k e_{p_1})\left\{\tilde{c}^*_{p_2,\sigma}\tilde{c}_{p_3,\sigma}, b^*_{-l}(K_{-l}e_{p_4})\right\} \\ &= b_k^*(A_k e_{p_1})\tilde{c}^*_{p_2,\sigma}\tilde{c}_{p_3,\sigma}b^*_{-l}(K_{-l}e_{p_4}) + b_k^*(A_k e_{p_1})b^*_{-l}(K_{-l}e_{p_4})\tilde{c}^*_{p_2,\sigma}\tilde{c}_{p_3,\sigma} \\ &= 2\,\tilde{c}^*_{p_2,\sigma}b_k^*(A_k e_{p_1})b^*_{-l}(K_{-l}e_{p_4})\tilde{c}_{p_3,\sigma} + \tilde{c}^*_{p_2,\sigma}b_k^*(A_k e_{p_1})\left[\tilde{c}_{p_3,\sigma}, b^*_{-l}(K_{-l}e_{p_4})\right]. \end{aligned} \tag{7.1.6}$$

To bound a sum of the form $\sum_{k\in\mathbb{Z}^3_*}\mathcal{E}_1^k(A_k)$ it thus suffices to estimate the two schematic forms

$$\begin{gathered} \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}^{\sigma} \tilde{c}^*_{p_2,\sigma}b_k^*(A_k e_{p_1})b^*_{-l}(K_{-l}e_{p_4})\tilde{c}_{p_3,\sigma} \\ \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}^{\sigma} \tilde{c}^*_{p_2,\sigma}b_k^*(A_k e_{p_1})\left[b_{-l}(K_{-l}e_{p_4}), \tilde{c}^*_{p_3,\sigma}\right]^*. \end{gathered} \tag{7.1.7}$$

Preliminary Estimates

We prepare for the estimation of these schematic forms by deriving some auxilliary bounds for the operators involved.

Recall that for any $k\in\mathbb{Z}^3_*$ and $\varphi\in\ell^2(L_k)$, the excitation operator $b_k(\varphi)$ is given by

$$b_k(\varphi) = \sum_{p\in L_k}\langle\varphi, e_p\rangle\, b_{k,p} = \frac{1}{\sqrt{s}}\sum_{p\in L_k}^{\sigma}\langle\varphi, e_p\rangle\, c^*_{p-k,\sigma}c_{p,\sigma}. \tag{7.1.8}$$

We observe that the exchange correction $\varepsilon_{k,k}(\varphi;\varphi)$ arising from the commutator $[b_k(\varphi), b_k^*(\varphi)]$ is non-positive: Indeed, this is given by

$$\varepsilon_{k,k}(\varphi;\varphi) = -\frac{1}{s}\sum_{p,q\in L_k}^{\sigma}\langle\varphi, e_p\rangle\langle e_q,\varphi\rangle\left(\delta_{p,q}c_{q-k,\sigma}c^*_{p-k,\sigma} + \delta_{p-k,q-k}c^*_{q,\sigma}c_{p,\sigma}\right) \tag{7.1.9}$$

$$= -\frac{1}{s} \sum_{p \in L_k}^{\sigma} |\langle e_p, \varphi \rangle|^2 \left(c_{p-k,\sigma} c_{p-k,\sigma}^* + c_{p,\sigma}^* c_{p,\sigma} \right) \leq 0.$$

Using this we can bound both $b_k(\varphi)$ and $b_k^*(\varphi)$ as follows:

Proposition 7.1.1. *For any $k \in \mathbb{Z}_*^3$, $\varphi \in \ell^2(L_k)$ and $\Psi \in \mathcal{H}_N$ it holds that*

$$\|b_k(\varphi)\Psi\| \leq \|\varphi\| \, \|\mathcal{N}_k^{\frac{1}{2}} \Psi\|, \quad \|b_k^*(\varphi)\Psi\| \leq \|\varphi\| \, \|(\mathcal{N}_k + 1)^{\frac{1}{2}} \Psi\|,$$

where $\mathcal{N}_k = \sum_{p \in L_k} b_{k,p}^* b_{k,p}$.

Proof: By the triangle and Cauchy-Schwarz inequalities we immediately obtain

$$\|b_k(\varphi)\Psi\| \leq \sum_{p \in L_k} |\langle \varphi, e_p \rangle| \, \|b_{k,p}\Psi\| \leq \|\varphi\| \sqrt{\sum_{p \in L_k} \|b_{k,p}\Psi\|^2} = \|\varphi\| \, \|\mathcal{N}_k^{\frac{1}{2}} \Psi\| \tag{7.1.10}$$

and the bound for $\|b_k^*(\varphi)\Psi\|$ now follows from this, since the above observation implies that

$$b_k(\varphi) b_k^*(\varphi) = b_k^*(\varphi) b_k(\varphi) + \langle \varphi, \varphi \rangle + \varepsilon_{k,k}(\varphi; \varphi) \leq \|\varphi\|^2 (\mathcal{N}_k + 1). \tag{7.1.11}$$

□

Note that the operator

$$\mathcal{N}_k = \sum_{p \in L_k} b_{k,p}^* b_{k,p} = \frac{1}{s} \sum_{p \in L_k}^{\sigma,\tau} c_{p,\sigma}^* c_{p-k,\sigma} c_{p-k,\tau}^* c_{p,\tau} \tag{7.1.12}$$

can be estimated directly in terms of $\mathcal{N}_E$ as $\mathcal{N}_k \leq \mathcal{N}_E$, since for any $\Psi \in \mathcal{H}_N$

$$\begin{aligned} \langle \Psi, \mathcal{N}_k \Psi \rangle &= \sum_{p \in L_k} \|b_{k,p}\Psi\|^2 = \sum_{p \in L_k} \left\| \frac{1}{\sqrt{s}} \sum_{\sigma=1}^{s} c_{p-k,\sigma}^* c_{p,\sigma} \Psi \right\|^2 \leq \sum_{p \in L_k} \left(\frac{1}{\sqrt{s}} \sum_{\sigma=1}^{s} \left\| c_{p-k,\sigma}^* c_{p,\sigma} \Psi \right\| \right)^2 \\ &\leq \sum_{p \in L_k}^{\sigma} \left\| c_{p-k,\sigma}^* c_{p,\sigma} \Psi \right\|^2 \leq \sum_{p \in L_k}^{\sigma} \|c_{p,\sigma}\Psi\|^2 \leq \langle \Psi, \mathcal{N}_E \Psi \rangle \end{aligned} \tag{7.1.13}$$

by the usual fermionic estimate. Below we will generally only use this cruder estimate, but $\mathcal{N}_k$ is useful for some bounds since it can be summed over $k \in \mathbb{Z}_*^3$: By rearranging the summations one concludes that

$$\begin{aligned} \sum_{k \in \mathbb{Z}_*^3} \langle \Psi, \mathcal{N}_k \Psi \rangle &\leq \sum_{k \in \mathbb{Z}_*^3} \sum_{p \in L_k}^{\sigma} \left\| c_{p-k,\sigma}^* c_{p,\sigma} \Psi \right\|^2 = \sum_{k \in \mathbb{Z}_*^3} \sum_{p \in L_k}^{\sigma} \left\langle \Psi, c_{p,\sigma}^* c_{p,\sigma} c_{p-k,\sigma} c_{p-k,\sigma}^* \Psi \right\rangle \\ &= \left\langle \Psi, \sum_{p \in B_F^c}^{\sigma} c_{p,\sigma}^* c_{p,\sigma} \sum_{k \in \mathbb{Z}_*^3} 1_{L_k}(p) c_{p-k,\sigma} c_{p-k,\sigma}^* \Psi \right\rangle \\ &= \left\langle \Psi, \sum_{p \in B_F^c}^{\sigma} c_{p,\sigma}^* c_{p,\sigma} \sum_{k \in (B_F + p)} c_{p-k,\sigma} c_{p-k,\sigma}^* \Psi \right\rangle = \left\langle \Psi, \sum_{p \in B_F^c}^{\sigma} c_{p,\sigma}^* c_{p,\sigma} \sum_{q \in B_F} c_{q,\sigma} c_{q,\sigma}^* \Psi \right\rangle, \end{aligned} \tag{7.1.14}$$

so noting that $\sum_{q\in B_F} c_{q,\sigma}c_{q,\sigma}^* = \mathcal{N}_E - \sum_{q\in B_F}^{\tau\neq\sigma} c_{q,\tau}c_{q,\tau}^*$ we can estimate

$$\sum_{k\in\mathbb{Z}_*^3} \langle\Psi, \mathcal{N}_k\Psi\rangle \leq \left\langle\Psi, \sum_{p\in B_F^c}^{\sigma} c_{p,\sigma}^* c_{p,\sigma}\mathcal{N}_E\Psi\right\rangle - \left\langle\Psi, \sum_{p\in B_F^c}^{\sigma} c_{p,\sigma}^* c_{p,\sigma} \sum_{q\in B_F}^{\tau\neq\sigma} c_{q,\tau}c_{q,\tau}^*\Psi\right\rangle \tag{7.1.15}$$
$$= \left\langle\Psi, \mathcal{N}_E^2\Psi\right\rangle - \sum_{p\in B_F^c}^{\sigma}\left\langle\Psi, c_{p,\sigma}^*\left(\sum_{q\in B_F}^{\tau\neq\sigma} c_{q,\tau}c_{q,\tau}^*\right)c_{p,\sigma}\Psi\right\rangle \leq \left\langle\Psi, \mathcal{N}_E^2\Psi\right\rangle$$

i.e. $\sum_{k\in\mathbb{Z}_*^3}\mathcal{N}_k \leq \mathcal{N}_E^2$. (Equality even holds for $s = 1$.)

We also note that for any $\Psi\in\mathcal{H}_N$ and $p\in\mathbb{Z}^3$

$$\sum_{\sigma=1}^{s}\|\mathcal{N}_k^{\frac{1}{2}}\tilde{c}_{p,\sigma}\Psi\|^2 \leq \sum_{\sigma=1}^{s}\|\tilde{c}_{p,\sigma}\mathcal{N}_k^{\frac{1}{2}}\Psi\|^2 \leq \sum_{\sigma=1}^{s}\|\tilde{c}_{p,\sigma}\mathcal{N}_E^{\frac{1}{2}}\Psi\|^2 \tag{7.1.16}$$
$$\sum_{\sigma=1}^{s}\|(\mathcal{N}_k+1)^{\frac{1}{2}}\tilde{c}_{p,\sigma}\Psi\|^2 \leq \sum_{\sigma=1}^{s}\|\tilde{c}_{p,\sigma}(\mathcal{N}_k+1)^{\frac{1}{2}}\Psi\|^2 \leq \sum_{\sigma=1}^{s}\|\tilde{c}_{p,\sigma}(\mathcal{N}_E+1)^{\frac{1}{2}}\Psi\|^2,$$

as follows by the inequality (considering $p\in B_F^c$ for definiteness)

$$\sum_{\sigma=1}^{s}\tilde{c}_{p,\sigma}^*\mathcal{N}_k\tilde{c}_{p,\sigma} = \frac{1}{s}\sum_{q\in L_k}^{\sigma,\tau,\rho} c_{p,\sigma}^*c_{q,\tau}^*c_{q-k,\tau}c_{q-k,\rho}^*c_{q,\rho}c_{p,\sigma} = \frac{1}{s}\sum_{q\in L_k}^{\sigma,\tau,\rho} c_{q,\tau}^*c_{q-k,\tau}c_{q-k,\rho}^*\left(c_{q,\rho}c_{p,\sigma}^* - \delta_{p,q}\delta_{\sigma,\tau}\right)c_{p,\sigma}$$
$$= \mathcal{N}_k\sum_{\sigma=1}^{s}c_{p,\sigma}^*c_{p,\sigma} - \frac{1}{s}\sum_{\sigma,\tau=1}^{s}1_{L_k}(p)c_{p,\tau}^*c_{p-k,\tau}c_{p-k,\sigma}^*c_{p,\sigma} \tag{7.1.17}$$
$$= \mathcal{N}_k\sum_{\sigma=1}^{s}c_{p,\sigma}^*c_{p,\sigma} - 1_{L_k}(p)b_{k,p}^*b_{k,p} \leq \mathcal{N}_k\sum_{\sigma=1}^{s}c_{p,\sigma}^*c_{p,\sigma}$$

and the fact that $\sum_{\sigma=1}^{s}\left[\tilde{c}_{p,\sigma}^*\tilde{c}_{p,\sigma}, \mathcal{N}_k\right] = 0 = \sum_{\sigma=1}^{s}\left[\tilde{c}_{p,\sigma}^*\tilde{c}_{p,\sigma}, \mathcal{N}_E\right]$. Similarly[1]

$$\sum_{\sigma=1}^{s}\|\mathcal{N}_E^{\frac{1}{2}}\tilde{c}_{p,\sigma}\Psi\|^2 \leq \sum_{\sigma=1}^{s}\|\tilde{c}_{p,\sigma}\mathcal{N}_E^{\frac{1}{2}}\Psi\|^2, \quad \sum_{\sigma=1}^{s}\|(\mathcal{N}_E+1)^{\frac{1}{2}}\tilde{c}_{p,\sigma}\Psi\|^2 \leq \sum_{\sigma=1}^{s}\|\tilde{c}_{p,\sigma}(\mathcal{N}_E+1)^{\frac{1}{2}}\Psi\|^2. \tag{7.1.18}$$

To analyze the commutator term $\left[b_{-l}(K_{-l}e_{p_4}), \tilde{c}_{p_3,\sigma}^*\right]$ we calculate a general identity: For any $l\in\mathbb{Z}_*^3$, $\psi\in\ell^2(L_l)$ and $p\in\mathbb{Z}^3$

$$\left[b_l(\psi), \tilde{c}_{p,\sigma}^*\right] = \frac{1}{\sqrt{s}}\sum_{q\in L_l}^{\tau}\langle\psi, e_q\rangle\begin{cases}\left[c_{q-l,\tau}^*c_{q,\tau}, c_{p,\sigma}\right] & p\in B_F\\ \left[c_{q-l,\tau}^*c_{q,\tau}, c_{p,\sigma}^*\right] & p\in B_F^c\end{cases}$$
$$= \frac{1}{\sqrt{s}}\sum_{q\in L_l}^{\tau}\langle\psi, e_q\rangle\begin{cases}-c_{q,\tau}\left\{c_{q-l,\tau}^*, c_{p,\sigma}\right\} & p\in B_F\\ c_{q-l,\tau}^*\left\{c_{q,\tau}, c_{p,\sigma}^*\right\} & p\in B_F^c\end{cases} \tag{7.1.19}$$

[1] There is a slight ambiguity here: $\mathcal{N}_E = \sum_{p\in B_F^c}^{\sigma} c_{p,\sigma}^*c_{p,\sigma} = \sum_{q\in B_F}^{\sigma} c_{q,\sigma}c_{q,\sigma}^*$ holds on $\mathcal{H}_N$, but an element such as $\tilde{c}_{p,\sigma}\Psi$ belongs to $\mathcal{H}_{N\pm1}$. This is of no importance, however, since these inequalities hold no matter if $\mathcal{N}_E$ is understood as $\sum_{p\in B_F^c}^{\sigma} c_{p,\sigma}^*c_{p,\sigma}$ or $\sum_{q\in B_F}^{\sigma} c_{q,\sigma}c_{q,\sigma}^*$. On the same note, the estimate of equation (7.1.13) is valid for either case even if $\Psi\in\mathcal{H}_{N\pm1}$.

$$= \begin{cases} -1_{L_l}(p+l)s^{-\frac{1}{2}}\left\langle \psi, e_{p+l}\right\rangle \tilde{c}_{p+l,\sigma} & p \in B_F \\ 1_{L_l}(p)s^{-\frac{1}{2}}\left\langle \psi, e_p\right\rangle \tilde{c}_{p-l,\sigma} & p \in B_F^c \end{cases},$$

so for our particular commutator we obtain

$$\left[b_{-l}(K_{-l}e_{p_4}), \tilde{c}^*_{p_3,\sigma}\right] = \begin{cases} -1_{L_{-l}}(p_3-l)s^{-\frac{1}{2}}\left\langle K_{-l}e_{p_4}, e_{p_3-l}\right\rangle \tilde{c}_{p_3-l,\sigma} & S_k = L_k \\ 1_{L_{-l}}(p_3)s^{-\frac{1}{2}}\left\langle K_{-l}e_{p_4}, e_{p_3}\right\rangle \tilde{c}_{p_3+l,\sigma} & S_k = L_k - k \end{cases}. \quad (7.1.20)$$

It will be crucial to our estimates that the prefactors obey the following:

Proposition 7.1.2. *For any $k, l \in \mathbb{Z}^3_*$ and $p \in S_k \cap S_l$ it holds that*

$$\left|1_{L_{-l}}(p_3-l)s^{-\frac{1}{2}}\left\langle K_{-l}e_{p_4}, e_{p_3-l}\right\rangle\right| \leq C\hat{V}_{-l}k_F^{-1}\frac{1_{L_{-k}}(p_2-k)1_{L_{-l}}(p_3-l)}{\sqrt{\lambda_{k,p_1}+\lambda_{-k,p_2-k}}\sqrt{\lambda_{-l,p_3-l}+\lambda_{-l,p_4}}}, \quad S_k = L_k,$$

and

$$\left|1_{L_{-l}}(p_3)s^{-\frac{1}{2}}\left\langle K_{-l}e_{p_4}, e_{p_3}\right\rangle\right| \leq C\hat{V}_{-l}k_F^{-1}\frac{1_{L_{-k}}(p_2)1_{L_{-l}}(p_3)}{\sqrt{\lambda_{k,p_1}+\lambda_{-k,p_2}}\sqrt{\lambda_{-l,p_3}+\lambda_{-l,p_4}}}, \quad S_k = L_k - k,$$

for a constant $C > 0$ depending only on s.

Proof: Recall that p_1, p_2, p_3, p_4 are given by

$$(p_1, p_2, p_3, p_4) = \begin{cases} (p, p-l, p-k, -p) & S_k = L_k \\ (p+k, p+l, p+k, -p-l) & S_k = L_k - k \end{cases}. \quad (7.1.21)$$

From this we see that for any $p \in S_k \cap S_l$

$$\begin{aligned} \begin{cases} 1_{L_{-l}}(p_3-l) & S_k = L_k \\ 1_{L_{-l}}(p_3) & S_k = L_k - k \end{cases} &= \begin{cases} 1_{B_F^c}(p-k-l)1_{B_F}(p-k) & S_k = L_k \\ 1_{B_F^c}(p+k)1_{B_F}(p+k+l) & S_k = L_k - k \end{cases} \\ &= \begin{cases} 1_{B_F^c}(p-l-k)1_{B_F}(p-l) & S_k = L_k \\ 1_{B_F^c}(p+l)1_{B_F}(p+l+k) & S_k = L_k - k \end{cases} \quad (7.1.22) \\ &= \begin{cases} 1_{L_{-k}}(p_2-k) & S_k = L_k \\ 1_{L_{-k}}(p_2) & S_k = L_k - k \end{cases} \end{aligned}$$

where the assumption that $p \in S_k \cap S_l$ enters to ensure that $1_{B_F}(p-k) = 1 = 1_{B_F}(p-l)$ or $1_{B_F^c}(p+k) = 1 = 1_{B_F^c}(p+l)$, respectively. Importantly this also implies that, when combined with such an indicator function, we also have the identity

$$\begin{cases} \lambda_{-l,p_3-l} + \lambda_{-l,p_4} & S_k = L_k \\ \lambda_{-l,p_3} + \lambda_{-l,p_4} & S_k = L_k - k \end{cases}$$

$$= \frac{1}{2}\begin{cases} |p-k-l|^2 - |p-k-l+l|^2 + |-p|^2 - |-p+l|^2 & S_k = L_k \\ |p+k|^2 - |p+k+l|^2 + |-p-l|^2 - |-p-l+l|^2 & S_k = L_k - k \end{cases} \tag{7.1.23}$$

$$= \frac{1}{2}\begin{cases} |p|^2 - |p-k|^2 + |p-l-k|^2 - |p-l-k+k|^2 & S_k = L_k \\ |p+k|^2 - |p+k-k|^2 + |p+l|^2 - |p+l+k|^2 & S_k = L_k - k \end{cases}$$

$$= \begin{cases} \lambda_{k,p_1} + \lambda_{-k,p_2-k} & S_k = L_k \\ \lambda_{k,p_1} + \lambda_{-k,p_2} & S_k = L_k - k \end{cases}.$$

The claim now follows by applying these identities to the estimates

$$\left|1_{L_{-l}}(p_3-l)s^{-\frac{1}{2}}\left\langle K_{-l}e_{p_4}, e_{p_3-l}\right\rangle\right| \leq C\frac{1_{L_{-l}}(p_3-l)\hat{V}_{-l}k_F^{-1}}{\lambda_{-l,p_3-l}+\lambda_{-l,p_4}}, \qquad S_k = L_k, \tag{7.1.24}$$

$$\left|1_{L_{-l}}(p_3)s^{-\frac{1}{2}}\left\langle K_{-l}e_{p_4}, e_{p_3}\right\rangle\right| \leq C\frac{1_{L_{-l}}(p_3)\hat{V}_{-l}k_F^{-1}}{\lambda_{-l,p_3}+\lambda_{-l,p_4}}, \qquad S_k = L_k - k,$$

which are given by Theorem 6.0.1.

□

Below we will only use the simpler bound

$$\begin{cases} \left|1_{L_{-l}}(p_3-l)s^{-\frac{1}{2}}\left\langle K_{-l}e_{p_4}, e_{p_3-l}\right\rangle\right| & S_k = L_k \\ \left|1_{L_{-l}}(p_3)s^{-\frac{1}{2}}\left\langle K_{-l}e_{p_4}, e_{p_3}\right\rangle\right| & S_k = L_k - k \end{cases} \leq C\frac{\hat{V}_{-l}k_F^{-1}}{\sqrt{\lambda_{k,p_1}\lambda_{-l,p_4}}} \tag{7.1.25}$$

but for the $\mathcal{E}_k^2$ terms the more general ones will be needed.

Estimation of $\sum_{k\in\mathbb{Z}^3_*}\mathcal{E}_k^1(A_k(t))$

Now the main estimate of this subsection:

Proposition 7.1.3. *For any collection of symmetric operators (A_k) and $\Psi\in\mathcal{H}_N$ it holds that*

$$\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}^{\sigma}\left|\left\langle\Psi, \tilde{c}^*_{p_2,\sigma}b_k^*(A_ke_{p_1})b_{-l}^*(K_{-l}e_{p_4})\tilde{c}_{p_3,\sigma}\Psi\right\rangle\right| \leq C\sqrt{\sum_{k\in\mathbb{Z}^3_*}\max_{p\in L_k}\|A_ke_p\|^2}\|(\mathcal{N}_E+1)^{\frac{3}{2}}\Psi\|^2$$

$$\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}^{\sigma}\left|\left\langle\Psi, \tilde{c}^*_{p_2,\sigma}b_k^*(A_ke_{p_1})\left[b_{-l}(K_{-l}e_{p_4}), \tilde{c}^*_{p_3,\sigma}\right]^*\Psi\right\rangle\right| \leq Ck_F^{-\frac{1}{2}}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\|A_kh_k^{-\frac{1}{2}}\|^2_{\mathrm{HS}}}\,\|(\mathcal{N}_E+1)\Psi\|^2$$

for a constant $C>0$ depending only on $\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2$ and s.

Proof: Using the triangle and Cauchy-Schwarz inequalities and Proposition 7.1.1 we estimate

$$\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}^{\sigma}\left|\left\langle\Psi, \tilde{c}^*_{p_2,\sigma}b_k^*(A_ke_{p_1})b_{-l}^*(K_{-l}e_{p_4})\tilde{c}_{p_3,\sigma}\Psi\right\rangle\right|$$

$$
\begin{aligned}
&\leq \sum_{k,l\in\mathbb{Z}_*^3}\sum_{p\in S_k\cap S_l}^{\sigma}\|b_k(A_ke_{p_1})\tilde{c}_{p_2,\sigma}\Psi\|\left\|b_{-l}^*(K_{-l}e_{p_4})\tilde{c}_{p_3,\sigma}\Psi\right\| \\
&\leq \sum_{k\in\mathbb{Z}_*^3}\sum_{p\in S_k}^{\sigma}\sum_{l\in\mathbb{Z}_*^3}1_{S_l}(p)\,\|A_ke_{p_1}\|\,\|K_{-l}e_{p_4}\|\,\|\mathcal{N}_k^{\frac{1}{2}}\tilde{c}_{p_2,\sigma}\Psi\|\|(\mathcal{N}_{-l}+1)^{\frac{1}{2}}\tilde{c}_{p_3,\sigma}\Psi\| \\
&\leq \sum_{k\in\mathbb{Z}_*^3}\left(\max_{p\in L_k}\|A_ke_p\|\right)\sum_{p\in S_k}\sqrt{\sum_{\sigma=1}^{s}\|\tilde{c}_{p_3,\sigma}(\mathcal{N}_E+1)^{\frac{1}{2}}\Psi\|^2}\sqrt{\sum_{l\in\mathbb{Z}_*^3}1_{S_l}(p)\,\|K_{-l}e_{p_4}\|^2} \qquad (7.1.26)\\
&\qquad\cdot\sqrt{\sum_{l\in\mathbb{Z}_*^3}^{\sigma}1_{S_l}(p)\|\tilde{c}_{p_2,\sigma}\mathcal{N}_k^{\frac{1}{2}}\Psi\|^2} \\
&\leq \sum_{k\in\mathbb{Z}_*^3}\left(\max_{p\in L_k}\|A_ke_p\|\right)\|\mathcal{N}_E^{\frac{1}{2}}\mathcal{N}_k^{\frac{1}{2}}\Psi\|\sqrt{\sum_{p\in S_k}^{\sigma}\|\tilde{c}_{p_3,\sigma}(\mathcal{N}_E+1)^{\frac{1}{2}}\Psi\|^2}\sqrt{\sum_{p\in S_k}\sum_{l\in\mathbb{Z}_*^3}1_{S_l}(p)\,\|K_{-l}e_{p_4}\|^2} \\
&\leq \sqrt{\sum_{k\in\mathbb{Z}_*^3}\max_{p\in L_k}\|A_ke_p\|^2}\sqrt{\sum_{l\in\mathbb{Z}_*^3}\|K_l\|_{\mathrm{HS}}^2}\,\|(\mathcal{N}_E+1)\Psi\|\,\sqrt{\sum_{k\in\mathbb{Z}_*^3}\|\mathcal{N}_E^{\frac{1}{2}}\mathcal{N}_k^{\frac{1}{2}}\Psi\|^2} \\
&= \sqrt{\sum_{k\in\mathbb{Z}_*^3}\max_{p\in L_k}\|A_ke_p\|^2}\sqrt{\sum_{l\in\mathbb{Z}_*^3}\|K_l\|_{\mathrm{HS}}^2}\,\|(\mathcal{N}_E+1)\Psi\|\,\|\mathcal{N}_E^{\frac{3}{2}}\Psi\|
\end{aligned}
$$

and the first bound now follows by recalling that $\|K_l\|_{\mathrm{HS}}^2 \leq C\hat{V}_l$. For the second we have by the equations (7.1.20) and (7.1.25) that

$$
\begin{aligned}
&\sum_{k,l\in\mathbb{Z}_*^3}\sum_{p\in S_k\cap S_l}^{\sigma}\left|\left\langle\Psi,\tilde{c}_{p_2,\sigma}^*b_k^*(A_ke_{p_1})\left[b_{-l}(K_{-l}e_{p_4}),\tilde{c}_{p_3,\sigma}^*\right]^*\Psi\right\rangle\right| \\
&\leq \sum_{k,l\in\mathbb{Z}_*^3}\sum_{p\in S_k\cap S_l}^{\sigma}\left\|\left[b_{-l}(K_{-l}e_{p_4}),\tilde{c}_{p_3,\sigma}^*\right]\tilde{c}_{p_2,\sigma}\Psi\right\|\|b_k^*(A_ke_{p_1})\Psi\| \\
&\leq C\sum_{l\in\mathbb{Z}_*^3}\sum_{p\in S_l}^{\sigma}\sum_{k\in\mathbb{Z}_*^3}1_{S_k}(p)\,\|A_ke_{p_1}\|\,\frac{\hat{V}_{-l}k_F^{-1}}{\sqrt{\lambda_{k,p_1}\lambda_{-l,p_4}}}\,\|\tilde{c}_{p_3\mp l,\sigma}\tilde{c}_{p_2,\sigma}\Psi\|\,\|(\mathcal{N}_k+1)^{\frac{1}{2}}\Psi\| \\
&\leq Ck_F^{-1}\|(\mathcal{N}_E+1)^{\frac{1}{2}}\Psi\|\sum_{p}\sum_{l\in\mathbb{Z}_*^3}\frac{1_{S_l}(p)\hat{V}_{-l}}{\sqrt{\lambda_{-l,p_4}}}\sqrt{\sum_{k\in\mathbb{Z}_*^3}1_{S_k}(p)\|A_kh_k^{-\frac{1}{2}}e_{p_1}\|^2} \qquad (7.1.27)\\
&\qquad\cdot\sqrt{\sum_{k\in\mathbb{Z}_*^3}1_{S_k}(p)\left(\sum_{\sigma=1}^{s}\|\tilde{c}_{p_3\mp l,\sigma}\tilde{c}_{p_2,\sigma}\Psi\|\right)^2} \\
&\leq Ck_F^{-1}\|(\mathcal{N}_E+1)^{\frac{1}{2}}\Psi\|\sum_{p}\sqrt{\sum_{k\in\mathbb{Z}_*^3}1_{S_k}(p)\|A_kh_k^{-\frac{1}{2}}e_{p_1}\|^2}\sqrt{\sum_{l\in\mathbb{Z}_*^3}1_{S_l}(p)\frac{\hat{V}_{-l}^2}{\lambda_{-l,p_4}}} \\
&\qquad\cdot\sqrt{\sum_{l\in\mathbb{Z}_*^3}^{\sigma}1_{S_l}(p)\|\tilde{c}_{p_2,\sigma}\mathcal{N}_E^{\frac{1}{2}}\Psi\|^2}
\end{aligned}
$$

$$\leq Ck_F^{-1}\|(\mathcal{N}_E+1)^{\frac{1}{2}}\Psi\|\,\|\mathcal{N}_E\Psi\|\sqrt{\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in S_k}\|A_kh_k^{-\frac{1}{2}}e_{p_1}\|^2}\sqrt{\sum_{l\in\mathbb{Z}^3_*}\hat{V}^2_{-l}\sum_{p\in S_l}\frac{1}{\lambda_{-l,p_4}}}$$
$$\leq Ck_F^{-1}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\|A_kh_k^{-\frac{1}{2}}\|^2_{\mathrm{HS}}}\sqrt{\sum_{l\in\mathbb{Z}^3_*}\hat{V}_l^2\sum_{p\in L_l}\frac{1}{\lambda_{l,p}}}\|(\mathcal{N}_E+1)^{\frac{1}{2}}\Psi\|\,\|\mathcal{N}_E\Psi\|$$

where we noted that $\|A_ke_{p_1}\|\,\lambda_{k,p_1}^{-\frac{1}{2}}=\|A_kh_k^{-\frac{1}{2}}e_{p_1}\|$ and also estimated

$$\sum_{k\in\mathbb{Z}^3_*}1_{S_k}(p)\left(\sum_{\sigma=1}^{s}\|\tilde{c}_{p_3\mp l,\sigma}\tilde{c}_{p_2,\sigma}\Psi\|\right)^2\leq s\sum_{k\in\mathbb{Z}^3_*}^{\sigma}1_{S_k}(p)\,\|\tilde{c}_{p_3\mp l,\sigma}\tilde{c}_{p_2,\sigma}\Psi\|^2\leq C\sum_{k\in\mathbb{Z}^3_*}^{\sigma,\tau}1_{S_k}(p)\,\|\tilde{c}_{p_3\mp l,\tau}\tilde{c}_{p_2,\sigma}\Psi\|^2$$
$$\leq C\sum_{\sigma=1}^{s}\|\mathcal{N}_E^{\frac{1}{2}}\tilde{c}_{p_2,\sigma}\Psi\|^2. \tag{7.1.28}$$

The claim follows as $\sum_{p\in L_l}\lambda_{l,p}^{-1}\leq Ck_F$.

□

The bound on $\sum_{k\in\mathbb{Z}^3_*}\mathcal{E}_k^1(A_k(t))$ of Theorem 7.0.1 now follows by our matrix element estimates:

Proposition 7.1.4. *For any $\Psi\in\mathcal{H}_N$ and $t\in[0,1]$ it holds that*

$$\sum_{k\in\mathbb{Z}^3_*}\left|\left\langle\Psi,\mathcal{E}_k^1(A_k(t))\Psi\right\rangle\right|\leq C\sqrt{\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2\min\{|k|,k_F\}}\left\langle\Psi,\left(\mathcal{N}_E^3+1\right)\Psi\right\rangle$$

for a constant $C>0$ depending only on $\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2$ and s.

Proof: By Theorem 6.0.1 we have

$$|\langle e_p,A_k(t)e_q\rangle|\leq C\left(1+\hat{V}_k^2\right)\hat{V}_kk_F^{-1},\quad k\in\mathbb{Z}^3_*,\ p,q\in L_k, \tag{7.1.29}$$

so

$$\sum_{k\in\mathbb{Z}^3_*}\max_{p\in L_k}\|A_k(t)e_p\|^2=\sum_{k\in\mathbb{Z}^3_*}\max_{p\in L_k}\sum_{q\in L_k}|\langle e_q,A_k(t)e_p\rangle|^2\leq Ck_F^{-2}\sum_{k\in\mathbb{Z}^3_*}\left(1+\hat{V}_k^2\right)^2\hat{V}_k^2\,|L_k|$$
$$\leq Ck_F^{-2}\sum_{k\in\mathbb{Z}^3_*}\left(\hat{V}_k^2+\hat{V}_k^6\right)\min\left\{k_F^2\,|k|\,,k_F^3\right\} \tag{7.1.30}$$
$$\leq C\left(1+\|\hat{V}\|_\infty^4\right)\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2\min\{|k|\,,k_F\}$$

where we used that $|L_k|\leq C\min\{k_F^2\,|k|\,,k_F^3\}$. Likewise

$$\sum_{k\in\mathbb{Z}^3_*}\|A_k(t)h_k^{-\frac{1}{2}}\|^2_{\mathrm{HS}}=\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k}\left|\left\langle e_p,A_k(t)h_k^{-\frac{1}{2}}e_q\right\rangle\right|^2\leq Ck_F^{-2}\sum_{k\in\mathbb{Z}^3_*}\left(1+\hat{V}_k^2\right)^2\hat{V}_k^2\,|L_k|\sum_{q\in L_k}\frac{1}{\lambda_{k,q}}$$
$$\leq Ck_F\left(1+\|\hat{V}\|_\infty^4\right)\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2\min\{|k|\,,k_F\} \tag{7.1.31}$$

since $\sum_{q\in L_k}\lambda_{k,q}^{-1}\leq Ck_F$. Inserting these estimates into Proposition 7.1.3 yields the claim.

□

7.2 Analysis of $\mathcal{E}_k^2$ Terms

Now we come to the terms

$$\mathcal{E}_k^2(B_k) = \frac{1}{2}\sum_{l\in\mathbb{Z}_*^3}\sum_{p\in L_k}\sum_{q\in L_l}\{b_k(B_k e_p),\{\varepsilon_{-k,-l}(e_{-p};e_{-q}),b_l^*(K_l e_q)\}\}. \tag{7.2.1}$$

We will analyze these similarly to the $\mathcal{E}_k^1(A_k)$ terms. Noting that

$$\varepsilon_{-k,-l}(e_{-p};e_{-q}) = -\frac{1}{s}\sum_{\sigma=1}^{s}\Big(\delta_{p,q}c_{-q+l,\sigma}c^*_{-p+k,\sigma} + \delta_{p-k,q-l}c^*_{-q,\sigma}c_{-p,\sigma}\Big) \tag{7.2.2}$$

we find that $\mathcal{E}_k^2(B_k)$ splits into two sums as

$$\begin{aligned} -2s\,\mathcal{E}_k^2(B_k) &= \sum_{l\in\mathbb{Z}_*^3}\sum_{p\in L_k}^{\sigma}\sum_{q\in L_l}\Big\{b_k(B_k e_p),\Big\{\delta_{p,q}c_{-q+l,\sigma}c^*_{-p+k,\sigma},b_l^*(K_l e_q)\Big\}\Big\} \\ &+\sum_{l\in\mathbb{Z}_*^3}\sum_{p\in(L_k-k)}^{\sigma}\sum_{q\in(L_l-l)}\Big\{b_k(B_k e_{p+k}),\Big\{\delta_{p,q}c^*_{-q-l,\sigma}c_{-p-k,\sigma},b_l^*(K_l e_{q+l})\Big\}\Big\} \\ &= \sum_{l\in\mathbb{Z}_*^3}\sum_{p\in L_k\cap L_l}^{\sigma}\Big\{b_k(B_k e_p),\Big\{c_{-p+l,\sigma}c^*_{-p+k,\sigma},b_l^*(K_l e_p)\Big\}\Big\} \\ &+\sum_{l\in\mathbb{Z}_*^3}\sum_{p\in(L_k-k)\cap(L_l-l)}^{\sigma}\Big\{b_k(B_k e_{p+k}),\Big\{c^*_{-p-l,\sigma}c_{-p-k,\sigma},b_l^*(K_l e_{p+l})\Big\}\Big\} \end{aligned} \tag{7.2.3}$$

and again these share a common schematic form, namely

$$\sum_{l\in\mathbb{Z}_*^3}\sum_{p\in S_k\cap S_l}^{\sigma}\Big\{b_k(B_k e_{p_1}),\Big\{\tilde{c}^*_{p_2,\sigma}\tilde{c}_{p_3,\sigma},b_l^*(K_l e_{p_4})\Big\}\Big\} \tag{7.2.4}$$

where the momenta are now

$$(p_1,p_2,p_3,p_4) = \begin{cases}(p,-p+l,-p+k,p) & S_k = L_k \\ (p+k,-p-l,-p-k,p+l) & S_k = L_k - k\end{cases}. \tag{7.2.5}$$

Again p_1, p_3 only depend on p and k while p_2, p_4 only depend on p and l.

We normal order the summand: As

$$\begin{aligned} & b_k(B_k e_{p_1})\Big\{\tilde{c}^*_{p_2,\sigma}\tilde{c}_{p_3,\sigma},b_l^*(K_l e_{p_4})\Big\} \\ &= \tilde{c}^*_{p_2,\sigma}b_k(B_k e_{p_1})\{\tilde{c}_{p_3,\sigma},b_l^*(K_l e_{p_4})\} + \Big[b_k(B_k e_{p_1}),\tilde{c}^*_{p_2,\sigma}\Big]\{\tilde{c}_{p_3,\sigma},b_l^*(K_l e_{p_4})\} \\ &= 2\,\tilde{c}^*_{p_2,\sigma}b_k(B_k e_{p_1})b_l^*(K_l e_{p_4})\tilde{c}_{p_3,\sigma} + \tilde{c}^*_{p_2,\sigma}b_k(B_k e_{p_1})\Big[b_l(K_l e_{p_4}),\tilde{c}^*_{p_3,\sigma}\Big]^* \\ &+ 2\Big[b_k(B_k e_{p_1}),\tilde{c}^*_{p_2,\sigma}\Big]b_l^*(K_l e_{p_4})\tilde{c}_{p_3,\sigma} + \Big[b_k(B_k e_{p_1}),\tilde{c}^*_{p_2,\sigma}\Big]\Big[b_l(K_l e_{p_4}),\tilde{c}^*_{p_3,\sigma}\Big]^* \\ &= 2\,\tilde{c}^*_{p_2,\sigma}b_l^*(K_l e_{p_4})b_k(B_k e_{p_1})\tilde{c}_{p_3,\sigma} + 2\,\tilde{c}^*_{p_2,\sigma}\left[b_k(B_k e_{p_1}),b_l^*(K_l e_{p_4})\right]\tilde{c}_{p_3,\sigma} \end{aligned} \tag{7.2.6}$$

$$+\,\tilde{c}^*_{p_2,\sigma}\left[b_l(K_le_{p_4}),\tilde{c}^*_{p_3,\sigma}\right]^*b_k(B_ke_{p_1})+\tilde{c}^*_{p_2,\sigma}\left[b_k(B_ke_{p_1}),\left[b_l(K_le_{p_4}),\tilde{c}^*_{p_3,\sigma}\right]^*\right]$$
$$+\,2\,b^*_l(K_le_{p_4})\left[b_k(B_ke_{p_1}),\tilde{c}^*_{p_2,\sigma}\right]\tilde{c}_{p_3,\sigma}+2\left[b_l(K_le_{p_4}),\left[b_k(B_ke_{p_1}),\tilde{c}^*_{p_2,\sigma}\right]^*\right]^*\tilde{c}_{p_3,\sigma}$$
$$-\left[b_l(K_le_{p_4}),\tilde{c}^*_{p_3,\sigma}\right]^*\left[b_k(B_ke_{p_1}),\tilde{c}^*_{p_2,\sigma}\right]+\left\{\left[b_k(B_ke_{p_1}),\tilde{c}^*_{p_2,\sigma}\right],\left[b_l(K_le_{p_4}),\tilde{c}^*_{p_3,\sigma}\right]^*\right\}$$

and simply

$$\begin{aligned}&\left\{\tilde{c}^*_{p_2,\sigma}\tilde{c}_{p_3,\sigma},b^*_l(K_le_{p_4})\right\}b_k(B_ke_{p_1})=\tilde{c}^*_{p_2,\sigma}\left\{\tilde{c}_{p_3,\sigma},b^*_l(K_le_{p_4})\right\}b_k(B_ke_{p_1})\\&=2\,\tilde{c}^*_{p_2,\sigma}b^*_l(K_le_{p_4})b_k(B_ke_{p_1})\tilde{c}_{p_3,\sigma}+\tilde{c}^*_{p_2,\sigma}\left[b_l(K_le_{p_4}),\tilde{c}^*_{p_3,\sigma}\right]^*b_k(B_ke_{p_1})\end{aligned}\tag{7.2.7}$$

the summand decomposes into 8 schematic forms as

$$\begin{aligned}&\left\{b_k(B_ke_{p_1}),\left\{\tilde{c}^*_{p_2,\sigma}\tilde{c}_{p_3,\sigma},b^*_l(K_le_{p_4})\right\}\right\}\\&=4\,\tilde{c}^*_{p_2,\sigma}b^*_l(K_le_{p_4})b_k(B_ke_{p_1})\tilde{c}_{p_3,\sigma}+2\,\tilde{c}^*_{p_2,\sigma}\left[b_k(B_ke_{p_1}),b^*_l(K_le_{p_4})\right]\tilde{c}_{p_3,\sigma}\\&+2\,\tilde{c}^*_{p_2,\sigma}\left[b_l(K_le_{p_4}),\tilde{c}^*_{p_3,\sigma}\right]^*b_k(B_ke_{p_1})+2\,b^*_l(K_le_{p_4})\left[b_k(B_ke_{p_1}),\tilde{c}^*_{p_2,\sigma}\right]\tilde{c}_{p_3,\sigma}\\&+\tilde{c}^*_{p_2,\sigma}\left[b_k(B_ke_{p_1}),\left[b_l(K_le_{p_4}),\tilde{c}^*_{p_3,\sigma}\right]^*\right]+2\left[b_l(K_le_{p_4}),\left[b_k(B_ke_{p_1}),\tilde{c}^*_{p_2,\sigma}\right]^*\right]^*\tilde{c}_{p_3,\sigma}\\&-\left[b_l(K_le_{p_4}),\tilde{c}^*_{p_3,\sigma}\right]^*\left[b_k(B_ke_{p_1}),\tilde{c}^*_{p_2,\sigma}\right]+\left\{\left[b_k(B_ke_{p_1}),\tilde{c}^*_{p_2,\sigma}\right],\left[b_l(K_le_{p_4}),\tilde{c}^*_{p_3,\sigma}\right]^*\right\}.\end{aligned}\tag{7.2.8}$$

Of these it should be noted that only the last one is proportional to a constant (i.e. does not contain any creation or annihilation operators). As the rest annihilate ψ_F, it follows that (when summed) the constant term yields precisely $\langle\psi_F,\mathcal{E}^2_k(B_k)\psi_F\rangle$, whence bounding the other terms amounts to estimating the operator

$$\mathcal{E}^2_k(B_k)-\left\langle\psi_F,\mathcal{E}^2_k(B_k)\psi_F\right\rangle\tag{7.2.9}$$

as in the statement of Theorem 7.0.1.

Estimation of the Top Terms

We begin by bounding the "top" terms

$$\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}^{\sigma}\tilde{c}^*_{p_2,\sigma}b^*_l(K_le_{p_4})b_k(B_ke_{p_1})\tilde{c}_{p_3,\sigma}\quad\text{and}\quad\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}^{\sigma}\tilde{c}^*_{p_2,\sigma}\left[b_k(B_ke_{p_1}),b^*_l(K_le_{p_4})\right]\tilde{c}_{p_3,\sigma}.$$

By the quasi-bosonic commutation relations, the commutator term reduces to

$$\begin{aligned}&\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}^{\sigma}\tilde{c}^*_{p_2,\sigma}\left[b_k(B_ke_{p_1}),b^*_l(K_le_{p_4})\right]\tilde{c}_{p_3,\sigma}\\&=\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in S_k}^{\sigma}\langle B_ke_{p_1},K_ke_{p_1}\rangle\,\tilde{c}^*_{p_3,\sigma}\tilde{c}_{p_3,\sigma}+\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}^{\sigma}\tilde{c}^*_{p_2,\sigma}\varepsilon_{k,l}(B_ke_{p_1};K_le_{p_4})\tilde{c}_{p_3,\sigma}\end{aligned}\tag{7.2.10}$$

where we used that $p_1 = p_4$ and $p_2 = p_3$ when $k = l$. Now, the exchange correction of the second sum splits as

$$\begin{aligned} -s\,\varepsilon_{k,l}(B_k e_{p_1}; K_l e_{p_4}) &= \sum_{q\in L_k}^{\tau} \sum_{q'\in L_l} \langle B_k e_{p_1}, e_q\rangle \langle e_{q'}, K_l e_{p_4}\rangle \left(\delta_{q,q'} c_{q'-l,\tau} c_{q-k,\tau}^* + \delta_{q-k,q'-l} c_{q',\tau}^* c_{q,\tau}\right) \\ &= \sum_{q\in L_k\cap L_l}^{\tau} \langle B_k e_{p_1}, e_q\rangle \langle e_q, K_l e_{p_4}\rangle\, \tilde{c}_{q-l,\tau}^* \tilde{c}_{q-k,\tau} \qquad (7.2.11) \\ &+ \sum_{q\in (L_k-k)\cap(L_l-l)}^{\tau} \langle B_k e_{p_1}, e_{q+k}\rangle \langle e_{q+l}, K_l e_{p_4}\rangle\, \tilde{c}_{q+l,\tau}^* \tilde{c}_{q+k,\tau} \end{aligned}$$

which are both of the schematic form $\sum_{q\in S_k'\cap S_l'}^{\tau} \langle B_k e_{p_1}, e_{q_1}\rangle \langle e_{q_4}, K_l e_{p_4}\rangle\, \tilde{c}_{q_2,\tau}^* \tilde{c}_{q_3,\tau}$.

To estimate $\sum_{k,l\in\mathbb{Z}_*^3} \sum_{p\in S_k\cap S_l}^{\sigma} \tilde{c}_{p_2,\sigma}^* \varepsilon_{k,l}(B_k e_{p_1}; K_l e_{p_4}) \tilde{c}_{p_3,\sigma}$ it thus suffices to consider

$$\sum_{k,l\in\mathbb{Z}_*^3} \sum_{p\in S_k\cap S_l}^{\sigma} \sum_{q\in S_k'\cap S_l'}^{\tau} \langle B_k e_{p_1}, e_{q_1}\rangle \langle e_{q_4}, K_l e_{p_4}\rangle\, \tilde{c}_{p_2,\sigma}^* \tilde{c}_{q_2,\tau}^* \tilde{c}_{q_3,\tau} \tilde{c}_{p_3,\sigma}. \qquad (7.2.12)$$

The estimates for the top terms are as follows:

Proposition 7.2.1. *For any collection of symmetric operators (B_k) and $\Psi \in \mathcal{H}_N$ it holds that*

$$\sum_{k,l\in\mathbb{Z}_*^3} \sum_{p\in S_k\cap S_l}^{\sigma} \left|\left\langle \Psi, \tilde{c}_{p_2,\sigma}^* b_l^*(K_l e_{p_4}) b_k(B_k e_{p_1}) \tilde{c}_{p_3,\sigma}\Psi\right\rangle\right| \leq C\sqrt{\sum_{k\in\mathbb{Z}_*^3} \max_{p\in L_k} \|B_k e_p\|^2} \|\mathcal{N}_E^{\frac{3}{2}}\Psi\|^2$$

$$\sum_{k,l\in\mathbb{Z}_*^3} \sum_{p\in S_k\cap S_l}^{\sigma} \left|\left\langle \Psi, \tilde{c}_{p_2,\sigma}^* \left[b_k(B_k e_{p_1}), b_l^*(K_l e_{p_4})\right] \tilde{c}_{p_3,\sigma}\Psi\right\rangle\right| \leq C\sqrt{\sum_{k\in\mathbb{Z}_*^3} \sum_{p\in L_k} \max_{q\in L_k} |\langle e_p, B_k e_q\rangle|^2}\, \|\mathcal{N}_E\Psi\|^2$$

for a constant $C > 0$ depending only on $\sum_{k\in\mathbb{Z}_*^3} \hat{V}_k^2$ and s.

Proof: The first term we can estimate as in Proposition 7.1.3 by

$$\begin{aligned} &\sum_{k,l\in\mathbb{Z}_*^3} \sum_{p\in S_k\cap S_l}^{\sigma} \left|\left\langle \Psi, \tilde{c}_{p_2,\sigma}^* b_l^*(K_l e_{p_4}) b_k(B_k e_{p_1}) \tilde{c}_{p_3,\sigma}\Psi\right\rangle\right| \\ &\leq \sum_{k,l\in\mathbb{Z}_*^3} \sum_{p\in S_k\cap S_l}^{\sigma} \|b_l(K_l e_{p_4}) \tilde{c}_{p_2,\sigma}\Psi\| \, \|b_k(B_k e_{p_1}) \tilde{c}_{p_3,\sigma}\Psi\| \\ &\leq \sum_{k\in\mathbb{Z}_*^3} \sum_{p\in S_k}^{\sigma} \sum_{l\in\mathbb{Z}_*^3} 1_{S_l}(p)\, \|B_k e_{p_1}\|\, \|K_l e_{p_4}\|\, \|\mathcal{N}_l^{\frac{1}{2}} \tilde{c}_{p_2,\sigma}\Psi\| \|\mathcal{N}_k^{\frac{1}{2}} \tilde{c}_{p_3,\sigma}\Psi\| \qquad (7.2.13) \\ &\leq \sum_{k\in\mathbb{Z}_*^3} \left(\max_{p\in L_k} \|B_k e_p\|\right) \sum_{p\in S_k} \sqrt{\sum_{\sigma=1}^{s} \|\tilde{c}_{p_3,\sigma} \mathcal{N}_k^{\frac{1}{2}}\Psi\|^2} \sqrt{\sum_{l\in\mathbb{Z}_*^3} 1_{S_l}(p)\, \|K_l e_{p_4}\|^2} \sqrt{\sum_{l\in\mathbb{Z}_*^3}^{\sigma} 1_{S_l}(p) \|\tilde{c}_{p_2,\sigma} \mathcal{N}_E^{\frac{1}{2}}\Psi\|^2} \\ &\leq \|\mathcal{N}_E\Psi\| \sum_{k\in\mathbb{Z}_*^3} \left(\max_{p\in L_k} \|B_k e_p\|\right) \sqrt{\sum_{p\in S_k}^{\sigma} \|\tilde{c}_{p_3,\sigma} \mathcal{N}_k^{\frac{1}{2}}\Psi\|^2} \sqrt{\sum_{p\in S_k} \sum_{l\in\mathbb{Z}_*^3} 1_{S_l}(p)\, \|K_l e_{p_4}\|^2} \end{aligned}$$

$$\leq \sqrt{\sum_{l\in\mathbb{Z}^3_*} \|K_l\|^2_{\mathrm{HS}}}\,\|\mathcal{N}_E\Psi\| \sum_{k\in\mathbb{Z}^3_*}\Big(\max_{p\in L_k}\|B_k e_p\|\Big)\|\mathcal{N}_E^{\frac{1}{2}}\mathcal{N}_k^{\frac{1}{2}}\Psi\|$$
$$\leq \sqrt{\sum_{k\in\mathbb{Z}^3_*}\max_{p\in L_k}\|B_k e_p\|^2}\sqrt{\sum_{l\in\mathbb{Z}^3_*}\|K_l\|^2_{\mathrm{HS}}}\,\|\mathcal{N}_E\Psi\|\,\|\mathcal{N}_E^{\frac{3}{2}}\Psi\|.$$

For the commutator term we first consider $\sum_{k\in\mathbb{Z}^3_*}\sum^{\sigma}_{p\in S_k}\langle B_k e_{p_1}, K_k e_{p_1}\rangle\,\tilde{c}^*_{p_3,\sigma}\tilde{c}_{p_3,\sigma}$: This is trivially bounded by

$$\sum_{k\in\mathbb{Z}^3_*}\sum^{\sigma}_{p\in S_k}\left|\langle B_k e_{p_1}, K_k e_{p_1}\rangle\left\langle\Psi,\tilde{c}^*_{p_3,\sigma}\tilde{c}_{p_3,\sigma}\Psi\right\rangle\right| \leq \sum_{k\in\mathbb{Z}^3_*}\max_{p\in L_k}|\langle B_k e_p, K_k e_p\rangle|\sum^{\sigma}_{p\in S_k}\left\langle\Psi,\tilde{c}^*_{p_3,\sigma}\tilde{c}_{p_3,\sigma}\Psi\right\rangle$$
$$\leq \sum_{k\in\mathbb{Z}^3_*}\max_{p\in L_k}|\langle e_p, B_k K_k e_p\rangle|\,\langle\Psi,\mathcal{N}_E\Psi\rangle \qquad (7.2.14)$$

and by the matrix element estimate for K_k of Theorem 6.0.1 we have for any $p\in L_k$ that

$$|\langle B_k e_p, K_k e_p\rangle| \leq \sum_{q\in L_k}|\langle B_k e_p, e_q\rangle|\,|\langle e_q, K_k e_p\rangle| \leq C\sum_{q\in L_k}|\langle e_p, B_k e_q\rangle|\,\frac{\hat{V}_k k_F^{-1}}{\lambda_{k,q}+\lambda_{k,p}} \qquad (7.2.15)$$
$$\leq C\hat{V}_k k_F^{-1}\Big(\max_{q\in L_k}|\langle e_p, B_k e_q\rangle|\Big)\sum_{q\in L_k}\frac{1}{\lambda_{k,q}} \leq C\hat{V}_k\max_{q\in L_k}|\langle e_p, B_k e_q\rangle|$$

since $\sum_{q\in L_k}\lambda_{k,q}^{-1}\leq Ck_F$. Consequently

$$\sum_{k\in\mathbb{Z}^3_*}\sum^{\sigma}_{p\in S_k}\left|\langle B_k e_{p_1}, K_k e_{p_1}\rangle\left\langle\Psi,\tilde{c}^*_{p_3,\sigma}\tilde{c}_{p_3,\sigma}\Psi\right\rangle\right| \leq C\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k\Big(\max_{p,q\in L_k}|\langle e_p, B_k e_q\rangle|\Big)\langle\Psi,\mathcal{N}_E\Psi\rangle$$
$$\leq C\sqrt{\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\max_{p,q\in L_k}|\langle e_p, B_k e_q\rangle|^2}\,\langle\Psi,\mathcal{N}_E\Psi\rangle$$

and clearly $\max_{p,q\in L_k}|\langle e_p, B_k e_q\rangle|^2\leq\sum_{p\in L_k}\max_{q\in L_k}|\langle e_p, B_k e_q\rangle|^2$. Finally

$$\sum_{k,l\in\mathbb{Z}^3_*}\sum^{\sigma}_{p\in S_k\cap S_l}\sum^{\tau}_{q\in S'_k\cap S'_l}\left|\langle B_k e_{p_1}, e_{q_1}\rangle\langle e_{q_4}, K_l e_{p_4}\rangle\left\langle\Psi,\tilde{c}^*_{p_2,\sigma}\tilde{c}^*_{q_2,\tau}\tilde{c}_{q_3,\tau}\tilde{c}_{p_3,\sigma}\Psi\right\rangle\right|$$
$$\leq \sum_{k,l\in\mathbb{Z}^3_*}\sum^{\sigma}_{p\in S_k\cap S_l}\sum^{\tau}_{q\in S'_k\cap S'_l}|\langle B_k e_{p_1}, e_{q_1}\rangle|\,|\langle e_{q_4}, K_l e_{p_4}\rangle|\,\|\tilde{c}_{q_2,\tau}\tilde{c}_{p_2,\sigma}\Psi\|\,\|\tilde{c}_{q_3,\tau}\tilde{c}_{p_3,\sigma}\Psi\|$$
$$\leq \sqrt{\sum_{k,l\in\mathbb{Z}^3_*}\sum^{\sigma}_{p\in S_k\cap S_l}\sum^{\tau}_{q\in S'_k\cap S'_l}|\langle B_k e_{p_1}, e_{q_1}\rangle|^2\,\|\tilde{c}_{q_2,\tau}\tilde{c}_{p_2,\sigma}\Psi\|^2}$$
$$\cdot\sqrt{\sum_{k,l\in\mathbb{Z}^3_*}\sum^{\sigma}_{p\in S_k\cap S_l}\sum^{\tau}_{q\in S'_k\cap S'_l}|\langle e_{q_4}, K_l e_{p_4}\rangle|^2\,\|\tilde{c}_{q_3,\tau}\tilde{c}_{p_3,\sigma}\Psi\|^2} \qquad (7.2.16)$$
$$\leq \sqrt{\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in S_k}\max_{q\in L_k}|\langle e_{p_1}, B_k e_q\rangle|^2\sum^{\sigma}_{l\in\mathbb{Z}^3_*}1_{S_l}(p)\|\tilde{c}_{p_2,\sigma}\mathcal{N}_E^{\frac{1}{2}}\Psi\|^2}$$

$$\cdot \sqrt{\sum_{l\in\mathbb{Z}^3_*}\sum_{p\in S_l}\|K_l e_{p_4}\|^2 \sum_{k\in\mathbb{Z}^3_*}^{\sigma} 1_{S_k}(p)\,\|\tilde{c}_{p_3,\sigma}\Psi\|^2}$$

$$\leq \sqrt{\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in L_k}\max_{q\in L_k}|\langle e_p, B_k e_q\rangle|^2}\sqrt{\sum_{l\in\mathbb{Z}^3_*}\|K_l\|^2_{\mathrm{HS}}}\|\mathcal{N}_E^{\frac{1}{2}}\Psi\|\,\|\mathcal{N}_E\Psi\|$$

whence the claim follows as $\|K_l\|_{\mathrm{HS}} \leq C\hat{V}_l$.

□

Estimation of the Single Commutator Terms

For the single commutator terms

$$\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}^{\sigma}\tilde{c}^*_{p_2,\sigma}\left[b_l(K_l e_{p_4}),\tilde{c}^*_{p_3,\sigma}\right]^* b_k(B_k e_{p_1}) \text{ and } \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}^{\sigma} b^*_l(K_l e_{p_4})\left[b_k(B_k e_{p_1}),\tilde{c}^*_{p_2,\sigma}\right]\tilde{c}_{p_3,\sigma}$$

we note that by equation (7.1.19), the commutator $\left[b_l(K_l e_{p_4}),\tilde{c}^*_{p_3,\sigma}\right]$ is given by

$$\left[b_l(K_l e_{p_4}),\tilde{c}^*_{p_3,\sigma}\right] = \begin{cases} -1_{L_l}(p_3+l)s^{-\frac{1}{2}}\langle K_l e_{p_4}, e_{p_3+l}\rangle\,\tilde{c}_{p_3+l,\sigma} & S_k = L_k \\ 1_{L_l}(p_3)s^{-\frac{1}{2}}\langle K_l e_{p_4}, e_{p_3}\rangle\,\tilde{c}_{p_3-l,\sigma} & S_k = L_k - k \end{cases}. \tag{7.2.17}$$

The prefactors again obey an estimate as in Proposition 7.1.2:

Proposition 7.2.2. *For any $k,l\in\mathbb{Z}^3_*$ and $p\in S_k\cap S_l$ it holds that*

$$\left|1_{L_l}(p_3+l)s^{-\frac{1}{2}}\langle K_l e_{p_4}, e_{p_3+l}\rangle\right| \leq C\hat{V}_l k_F^{-1}\frac{1_{L_k}(p_2+k)1_{L_l}(p_3+l)}{\sqrt{\lambda_{k,p_1}+\lambda_{k,p_2+k}}\sqrt{\lambda_{l,p_3+l}+\lambda_{l,p_4}}},\quad S_k = L_k,$$

and

$$\left|1_{L_l}(p_3)s^{-\frac{1}{2}}\langle K_l e_{p_4}, e_{p_3}\rangle\right| \leq C\hat{V}_l k_F^{-1}\frac{1_{L_k}(p_2)1_{L_l}(p_3)}{\sqrt{\lambda_{k,p_1}+\lambda_{k,p_2}}\sqrt{\lambda_{l,p_3}+\lambda_{l,p_4}}},\quad S_k = L_k - k,$$

for a constant $C>0$ depending only on s.

The proof is essentially the same as that of Proposition 7.1.2 (indeed, this proposition can be obtained directly from the former by appropriate substition, but some care must be used since the p_i's differ in their definition).

For the single commutator terms we again only need the simpler bound

$$\begin{cases} \left|1_{L_l}(p_3+l)s^{-\frac{1}{2}}\langle K_l e_{p_4}, e_{p_3+l}\rangle\right| & S_k = L_k \\ \left|1_{L_l}(p_3)s^{-\frac{1}{2}}\langle K_l e_{p_4}, e_{p_3}\rangle\right| & S_k = L_k - k \end{cases} \leq C\frac{\hat{V}_l k_F^{-1}}{\sqrt{\lambda_{k,p_1}\lambda_{l,p_4}}} \tag{7.2.18}$$

but the full one will be needed for the double commutator terms below. Now the estimate:

Proposition 7.2.3. *For any collection of symmetric operators* (B_k) *and* $\Psi \in \mathcal{H}_N$ *it holds that*

$$\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}^{\sigma}\left|\left\langle \Psi, \tilde{c}^*_{p_2,\sigma}\left[b_l(K_l e_{p_4}), \tilde{c}^*_{p_3,\sigma}\right]^* b_k(B_k e_{p_1})\Psi\right\rangle\right| \leq C k_F^{-\frac{1}{2}}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\|B_k h_k^{-\frac{1}{2}}\|_{\mathrm{HS}}^2\,\|\mathcal{N}_E\Psi\|^2}$$

$$\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}^{\sigma}\left|\left\langle \Psi, b_l^*(K_l e_{p_4})\left[b_k(B_k e_{p_1}), \tilde{c}^*_{p_2,\sigma}\right]\tilde{c}_{p_3,\sigma}\Psi\right\rangle\right| \leq C\sqrt{\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in L_k}\max_{q\in L_k}|\langle e_p, B_k e_q\rangle|^2\,\|\mathcal{N}_E\Psi\|^2}$$

for a constant $C>0$ *depending only on* $\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2$ *and* s.

Proof: As in the second estimate of Proposition 7.1.3 we have

$$\begin{aligned}
&\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}^{\sigma}\left|\left\langle \Psi, \tilde{c}^*_{p_2,\sigma}\left[b_l(K_l e_{p_4}), \tilde{c}^*_{p_3,\sigma}\right]^* b_k(B_k e_{p_1})\Psi\right\rangle\right| \\
&\leq \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}^{\sigma}\left\|\left[b_l(K_l e_{p_4}), \tilde{c}^*_{p_3,\sigma}\right]\tilde{c}_{p_2,\sigma}\Psi\right\|\|b_k(B_k e_{p_1})\Psi\| \\
&\leq C\sum_{l\in\mathbb{Z}^3_*}\sum_{p\in S_l}^{\sigma}\sum_{k\in\mathbb{Z}^3_*}1_{S_k}(p)\,\|B_k e_{p_1}\|\,\frac{\hat{V}_l k_F^{-1}}{\sqrt{\lambda_{k,p_1}\lambda_{l,p_4}}}\,\|\tilde{c}_{p_3\pm l,\sigma}\tilde{c}_{p_2,\sigma}\Psi\|\,\|\mathcal{N}_k^{\frac{1}{2}}\Psi\| && (7.2.19)\\
&\leq C k_F^{-1}\|\mathcal{N}_E^{\frac{1}{2}}\Psi\|\sum_{p}\sum_{l\in\mathbb{Z}^3_*}\frac{1_{S_l}(p)\hat{V}_l}{\sqrt{\lambda_{l,p_4}}}\sqrt{\sum_{k\in\mathbb{Z}^3_*}1_{S_k}(p)\|B_k h_k^{-\frac{1}{2}}e_{p_1}\|^2}\sqrt{\sum_{k\in\mathbb{Z}^3_*}^{\sigma,\tau}1_{S_k}(p)\,\|\tilde{c}_{p_3\pm l,\tau}\tilde{c}_{p_2,\sigma}\Psi\|^2} \\
&\leq C k_F^{-1}\|\mathcal{N}_E^{\frac{1}{2}}\Psi\|\sum_{p}\sqrt{\sum_{k\in\mathbb{Z}^3_*}1_{S_k}(p)\|B_k h_k^{-\frac{1}{2}}e_{p_1}\|^2}\sqrt{\sum_{l\in\mathbb{Z}^3_*}1_{S_l}(p)\frac{\hat{V}_l^2}{\lambda_{l,p_4}}}\sqrt{\sum_{l\in\mathbb{Z}^3_*}^{\sigma}1_{S_l}(p)\|\tilde{c}_{p_2,\sigma}\mathcal{N}_E^{\frac{1}{2}}\Psi\|^2} \\
&\leq C k_F^{-1}\|\mathcal{N}_E^{\frac{1}{2}}\Psi\|\,\|\mathcal{N}_E\Psi\|\sqrt{\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in S_k}\|B_k h_k^{-\frac{1}{2}}e_{p_1}\|^2}\sqrt{\sum_{l\in\mathbb{Z}^3_*}\hat{V}_l^2\sum_{p\in S_l}\frac{1}{\lambda_{l,p_4}}} \\
&\leq C k_F^{-\frac{1}{2}}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\|B_k h_k^{-\frac{1}{2}}\|_{\mathrm{HS}}^2}\sqrt{\sum_{l\in\mathbb{Z}^3_*}\hat{V}_l^2}\|\mathcal{N}_E^{\frac{1}{2}}\Psi\|\,\|\mathcal{N}_E\Psi\|\,.
\end{aligned}$$

By equation (7.1.19) it holds that

$$\left[b_k(B_k e_{p_1}), \tilde{c}^*_{p_2,\sigma}\right] = \begin{cases} -1_{L_k}(p_2+k)s^{-\frac{1}{2}}\left\langle B_k e_{p_1}, e_{p_2+k}\right\rangle \tilde{c}_{p_2+k,\sigma} & p\in B_F \\ 1_{L_k}(p_2)s^{-\frac{1}{2}}\left\langle B_k e_{p_1}, e_{p_2}\right\rangle \tilde{c}_{p_2-k,\sigma} & p\in B_F^c \end{cases} \qquad (7.2.20)$$

so the second term can be bounded as

$$\begin{aligned}
&\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}^{\sigma}\left|\left\langle \Psi, b_l^*(K_l e_{p_4})\left[b_k(B_k e_{p_1}), \tilde{c}^*_{p_2,\sigma}\right]\tilde{c}_{p_3,\sigma}\Psi\right\rangle\right| \\
&\leq \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}^{\sigma}\|b_l(K_l e_{p_4})\Psi\|\left\|\left[b_k(B_k e_{p_1}), \tilde{c}^*_{p_2,\sigma}\right]\tilde{c}_{p_3,\sigma}\Psi\right\|
\end{aligned}$$

$$\leq \sum_{k\in\mathbb{Z}^3_*}\sum_{p\in S_k}^{\sigma}\sum_{l\in\mathbb{Z}^3_*} 1_{S_l}(p)\Big(\max_{q\in L_k}|\langle e_{p_1}, B_k e_q\rangle|\Big)\,\|K_l e_{p_4}\|\,\|\mathcal{N}_l^{\frac{1}{2}}\Psi\|\,\|\tilde{c}_{p_2\pm k,\sigma}\tilde{c}_{p_3,\sigma}\Psi\| \tag{7.2.21}$$

$$\leq \|\mathcal{N}_E^{\frac{1}{2}}\Psi\|\sum_p\sum_{k\in\mathbb{Z}^3_*}1_{S_k}(p)\Big(\max_{q\in L_k}|\langle e_{p_1}, B_k e_q\rangle|\Big)\sqrt{\sum_{l\in\mathbb{Z}^3_*}1_{S_l}(p)\,\|K_l e_{p_4}\|^2}\sqrt{\sum_{l\in\mathbb{Z}^3_*}^{\sigma,\tau}1_{S_l}(p)\,\|\tilde{c}_{p_2\pm k,\tau}\tilde{c}_{p_3,\sigma}\Psi\|^2}$$

$$\leq \|\mathcal{N}_E^{\frac{1}{2}}\Psi\|\sum_p\sqrt{\sum_{l\in\mathbb{Z}^3_*}1_{S_l}(p)\,\|K_l e_{p_4}\|^2}\sqrt{\sum_{k\in\mathbb{Z}^3_*}1_{S_k}(p)\Big(\max_{q\in L_k}|\langle e_{p_1}, B_k e_q\rangle|^2\Big)}\sqrt{\sum_{k\in\mathbb{Z}^3_*}^{\sigma}1_{S_k}(p)\|\tilde{c}_{p_3,\sigma}\mathcal{N}_E^{\frac{1}{2}}\Psi\|^2}$$

$$\leq \|\mathcal{N}_E^{\frac{1}{2}}\Psi\|\,\|\mathcal{N}_E\Psi\|\sqrt{\sum_{l\in\mathbb{Z}^3_*}\sum_{p\in S_l}\|K_l e_{p_4}\|^2}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in S_k}\max_{q\in L_k}|\langle e_{p_1}, B_k e_q\rangle|^2}$$

$$\leq \sqrt{\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in L_k}\max_{q\in L_k}|\langle e_p, B_k e_q\rangle|^2}\sqrt{\sum_{l\in\mathbb{Z}^3_*}\|K_l\|_{\mathrm{HS}}^2}\,\|\mathcal{N}_E^{\frac{1}{2}}\Psi\|\,\|\mathcal{N}_E\Psi\|\,.$$

□

Estimation of the Double Commutator Terms

Finally we have the double commutator terms

$$\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}^{\sigma}\tilde{c}^*_{p_2,\sigma}\left[b_k(B_k e_{p_1}),\left[b_l(K_l e_{p_4}),\tilde{c}^*_{p_3,\sigma}\right]^*\right],\quad \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}^{\sigma}\left[b_l(K_l e_{p_4}),\left[b_k(B_k e_{p_1}),\tilde{c}^*_{p_2,\sigma}\right]^*\right]^*\tilde{c}_{p_3,\sigma}$$

and

$$\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}^{\sigma}\left[b_l(K_l e_{p_4}),\tilde{c}^*_{p_3,\sigma}\right]^*\left[b_k(B_k e_{p_1}),\tilde{c}^*_{p_2,\sigma}\right].$$

An identity for the iterated commutators is obtained by applying the identity of equation (7.1.19) to itself: For any $k,l\in\mathbb{Z}^3_*$, $\varphi\in\ell^2(L_k)$, $\psi\in\ell^2(L_l)$ and $p\in\mathbb{Z}^3_*$

$$\left[b_k(\varphi),\left[b_l(\psi),\tilde{c}^*_{p,\sigma}\right]^*\right] = \begin{cases} -1_{L_l}(p+l)s^{-\frac{1}{2}}\langle e_{p+l},\psi\rangle\left[b_k(\varphi),\tilde{c}^*_{p+l,\sigma}\right] & p\in B_F\\ 1_{L_l}(p)s^{-\frac{1}{2}}\langle e_p,\psi\rangle\left[b_k(\varphi),\tilde{c}^*_{p-l,\sigma}\right] & p\in B_F^c\end{cases} \tag{7.2.22}$$

$$= \begin{cases} -1_{L_k}(p+l)1_{L_l}(p+l)s^{-1}\langle\varphi,e_{p+l}\rangle\langle e_{p+l},\psi\rangle\,\tilde{c}_{p+l-k,\sigma} & p\in B_F\\ -1_{L_k}(p-l+k)1_{L_l}(p)s^{-1}\langle\varphi,e_{p-l+k}\rangle\langle e_p,\psi\rangle\,\tilde{c}_{p-l+k,\sigma} & p\in B_F^c\end{cases}.$$

The estimates are the following:

Proposition 7.2.4. *For any collection of symmetric operators (B_k) and $\Psi\in\mathcal{H}_N$ it holds that*

$$\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}^{\sigma}\left|\left\langle\Psi,\tilde{c}^*_{p_2,\sigma}\left[b_k(B_k e_{p_1}),\left[b_l(K_l e_{p_4}),\tilde{c}^*_{p_3,\sigma}\right]^*\right]\Psi\right\rangle\right|,$$

$$\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}^{\sigma}\left|\left\langle\Psi,\left[b_l(K_l e_{p_4}),\left[b_k(B_k e_{p_1}),\tilde{c}^*_{p_2,\sigma}\right]^*\right]^*\tilde{c}_{p_3,\sigma}\Psi\right\rangle\right|,$$

$$\sum_{k,l\in\mathbb{Z}^3_*}\sum^{\sigma}_{p\in S_k\cap S_l}\left|\left\langle\Psi,\left[b_l(K_le_{p_4}),\tilde{c}^*_{p_3,\sigma}\right]^*\left[b_k(B_ke_{p_1}),\tilde{c}^*_{p_2,\sigma}\right]\Psi\right\rangle\right|,$$

are all bounded by $Ck_F^{-\frac{1}{2}}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\max_{p\in L_k}\|h_k^{-\frac{1}{2}}B_ke_p\|^2}\,\langle\Psi,\mathcal{N}_E\Psi\rangle$ *for a constant* $C>0$ *depending only on* $\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2$ *and* s*.*

Proof: For these estimates we consider only the case $S_k=L_k$ for the sake of clarity, i.e. we let

$$(p_1,p_2,p_3,p_4)=(p,-p+l,-p+k,p);\tag{7.2.23}$$

the case $S_k=L_k-k$ can be handled by similar manipulations.

Using the identity of equation (7.2.22) we start by estimating (by the bound of Proposition 7.2.2)

$$\begin{aligned}
&\sum_{k,l\in\mathbb{Z}^3_*}\sum^{\sigma}_{p\in L_k\cap L_l}\left|\left\langle\Psi,\tilde{c}^*_{p_2,\sigma}\left[b_k(B_ke_{p_1}),\left[b_l(K_le_{p_4}),\tilde{c}^*_{p_3,\sigma}\right]^*\right]\Psi\right\rangle\right|\\
&=\sum_{k,l\in\mathbb{Z}^3_*}\sum^{\sigma}_{p\in L_k\cap L_l}\left|1_{L_k}(p_3+l)1_{L_l}(p_3+l)s^{-1}\langle B_ke_{p_1},e_{p_3+l}\rangle\langle e_{p_3+l},K_le_{p_4}\rangle\left\langle\Psi,\tilde{c}^*_{p_2,\sigma}\tilde{c}_{p_3+l-k,\sigma}\Psi\right\rangle\right|\\
&\leq C\sum_{k,l\in\mathbb{Z}^3_*}\sum^{\sigma}_{p\in L_k\cap L_l}1_{L_k}(p_3+l)\left|\langle B_ke_{p_1},e_{p_3+l}\rangle\right|\frac{\hat{V}_lk_F^{-1}1_{L_k}(p_2+k)1_{L_l}(p_3+l)}{\sqrt{\lambda_{k,p_1}+\lambda_{k,p_2+k}}\sqrt{\lambda_{l,p_3+l}+\lambda_{l,p_4}}}\left\langle\Psi,\tilde{c}^*_{p_2,\sigma}\tilde{c}_{p_2,\sigma}\Psi\right\rangle\\
&\leq Ck_F^{-1}\sum_{l\in\mathbb{Z}^3_*}\hat{V}_l\sum^{\sigma}_{p\in L_l}\sqrt{\sum_{k\in\mathbb{Z}^3_*}1_{L_k}(p)1_{L_k}(p_3+l)\left|\left\langle e_p,h_k^{-\frac{1}{2}}B_ke_{p_3+l}\right\rangle\right|^2}\qquad(7.2.24)\\
&\qquad\cdot\sqrt{\sum_{k\in\mathbb{Z}^3_*}\frac{1_{L_l}(p_3+l)}{\lambda_{l,p_3+l}}}\left\langle\Psi,\tilde{c}^*_{-p+l,\sigma}\tilde{c}_{-p+l,\sigma}\Psi\right\rangle\\
&\leq Ck_F^{-\frac{1}{2}}\sum_{l\in\mathbb{Z}^3_*}\hat{V}_l\sum^{\sigma}_{p\in(L_l-l)}\sqrt{\sum_{k\in\mathbb{Z}^3_*}1_{L_k}(p+l)1_{L_k}(p_3)\left|\left\langle e_{p+l},h_k^{-\frac{1}{2}}B_ke_{p_3}\right\rangle\right|^2}\left\langle\Psi,\tilde{c}^*_{-p,\sigma}\tilde{c}_{-p,\sigma}\Psi\right\rangle\\
&\leq Ck_F^{-\frac{1}{2}}\sum^{\sigma}_{p\in B_F}\sqrt{\sum_{l\in\mathbb{Z}^3_*}\hat{V}_l^2}\sqrt{\sum_{k,l\in\mathbb{Z}^3_*}1_{L_k}(p+l)1_{L_k}(p_3)\left|\left\langle e_{p+l},h_k^{-\frac{1}{2}}B_ke_{p_3}\right\rangle\right|^2}\left\langle\Psi,\tilde{c}^*_{-p,\sigma}\tilde{c}_{-p,\sigma}\Psi\right\rangle\\
&\leq Ck_F^{-\frac{1}{2}}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\max_{p\in L_k}\|h_k^{-\frac{1}{2}}B_ke_p\|^2}\sqrt{\sum_{l\in\mathbb{Z}^3_*}\hat{V}_l^2}\,\langle\Psi,\mathcal{N}_E\Psi\rangle
\end{aligned}$$

where we used that $\sum_{k\in\mathbb{Z}^3_*}1_{L_l}(p_3+l)\lambda_{l,p_3+l}^{-1}\leq\sum_{q\in L_l}\lambda_{l,q}^{-1}\leq Ck_F$.

From equation (7.2.22) we have

$$\begin{aligned}
\left[b_l(K_le_{p_4}),\left[b_k(B_ke_{p_1}),\tilde{c}^*_{p,\sigma}\right]^*\right]&=-1_{L_l}(p_2+k)1_{L_k}(p_2+k)s^{-1}\langle K_le_{p_4},e_{p_2+k}\rangle\langle e_{p_2+k},B_ke_{p_1}\rangle\,\tilde{c}_{p_2+k-l,\sigma}\\
&=-1_{L_k}(p_2+k)1_{L_l}(p_3+l)s^{-1}\langle K_le_{p_4},e_{p_3+l}\rangle\langle e_{p_2+k},B_ke_{p_1}\rangle\,\tilde{c}_{p_3,\sigma}
\end{aligned}$$

so the second term can be similarly estimated as

$$\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}^{\sigma}\left|\left\langle\Psi,\left[b_l(K_l e_{p_4}),\left[b_k(B_k e_{p_1}),\tilde{c}^*_{p_2,\sigma}\right]^*\right]^*\tilde{c}_{p_3,\sigma}\Psi\right\rangle\right|$$
$$\leq C\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in L_k\cap L_l}^{\sigma}\frac{\hat{V}_l k_F^{-1}1_{L_k}(p_2+k)1_{L_l}(p_3+l)}{\sqrt{\lambda_{k,p_1}+\lambda_{k,p_2+k}}\sqrt{\lambda_{l,p_3+l}+\lambda_{l,p_4}}}\left|\langle e_{p_2+k},B_k e_{p_1}\rangle\right|\left\langle\Psi,\tilde{c}^*_{p_3,\sigma}\tilde{c}_{p_3,\sigma}\Psi\right\rangle$$
$$\leq Ck_F^{-1}\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in L_k}^{\sigma}\sqrt{\sum_{l\in\mathbb{Z}^3_*}1_{L_l}(p)\frac{\hat{V}_l^2}{\lambda_{l,p_4}}}\sqrt{\sum_{l\in\mathbb{Z}^3_*}1_{L_k}(p_2+k)\left|\left\langle e_{p_2+k},h_k^{-\frac{1}{2}}B_k e_{p_1}\right\rangle\right|^2}\left\langle\Psi,\tilde{c}^*_{-p+k,\sigma}\tilde{c}_{-p+k,\sigma}\Psi\right\rangle$$
$$\leq Ck_F^{-1}\sum_{p\in B_F}^{\sigma}\sum_{k\in\mathbb{Z}^3_*}1_{L_k-k}(p)\sqrt{\sum_{l\in\mathbb{Z}^3_*}\hat{V}_l^2\frac{1_{L_l}(p+k)}{\lambda_{l,p+k}}}\|h_k^{-\frac{1}{2}}B_k e_{p+k}\|\left\langle\Psi,\tilde{c}^*_{-p,\sigma}\tilde{c}_{-p,\sigma}\Psi\right\rangle \tag{7.2.25}$$
$$\leq Ck_F^{-1}\sum_{p\in B_F}^{\sigma}\sqrt{\sum_{l\in\mathbb{Z}^3_*}\hat{V}_l^2\sum_{k\in\mathbb{Z}^3_*}\frac{1_{L_l}(p+k)}{\lambda_{l,p+k}}}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\|h_k^{-\frac{1}{2}}B_k e_{p+k}\|^2}\left\langle\Psi,\tilde{c}^*_{-p,\sigma}\tilde{c}_{-p,\sigma}\Psi\right\rangle$$
$$\leq Ck_F^{-\frac{1}{2}}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\max_{p\in L_k}\|h_k^{-\frac{1}{2}}B_k e_p\|^2}\sqrt{\sum_{l\in\mathbb{Z}^3_*}\hat{V}_l^2}\left\langle\Psi,\mathcal{N}_E\Psi\right\rangle.$$

Finally, from the equations (7.2.17) and (7.2.20) we see that

$$\begin{aligned}&\left[b_l(K_l e_{p_4}),\tilde{c}^*_{p_3,\sigma}\right]^*\left[b_k(B_k e_{p_1}),\tilde{c}^*_{p_2,\sigma}\right]\\&=1_{L_k}(p_2+k)1_{L_l}(p_3+l)s^{-1}\langle B_k e_{p_1},e_{p_2+k}\rangle\langle e_{p_3+l},K_l e_{p_4}\rangle\tilde{c}^*_{p_3+l,\sigma}\tilde{c}_{p_2+k,\sigma}\end{aligned} \tag{7.2.26}$$

so we estimate

$$\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in S_k\cap S_l}^{\sigma}\left|\left\langle\Psi,\left[b_l(K_l e_{p_4}),\tilde{c}^*_{p_3,\sigma}\right]^*\left[b_k(B_k e_{p_1}),\tilde{c}^*_{p_2,\sigma}\right]\Psi\right\rangle\right|$$
$$\leq C\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in L_k\cap L_l}^{\sigma}\frac{\hat{V}_l k_F^{-1}1_{L_k}(p_2+k)1_{L_l}(p_3+l)}{\sqrt{\lambda_{k,p_1}+\lambda_{k,p_2+k}}\sqrt{\lambda_{l,p_3+l}+\lambda_{l,p_4}}}\left|\langle B_k e_{p_1},e_{p_2+k}\rangle\right|\left\langle\Psi,\tilde{c}^*_{p_3+l,\sigma}\tilde{c}_{p_2+k,\sigma}\Psi\right\rangle$$
$$\leq Ck_F^{-1}\sum_{p\in B_F^c}^{\sigma}\sum_{k,l\in\mathbb{Z}^3_*}1_{L_k\cap L_l}(p)1_{L_k\cap L_l}(-p+k+l)\frac{\hat{V}_l}{\sqrt{\lambda_{l,p}}}\left|\left\langle e_p,h_k^{-\frac{1}{2}}B_k e_{-p+k+l}\right\rangle\right| \tag{7.2.27}$$
$$\cdot\left\langle\Psi,\tilde{c}^*_{-p+k+l,\sigma}\tilde{c}_{-p+k+l,\sigma}\Psi\right\rangle$$
$$=Ck_F^{-1}\sum_{p\in B_F^c}^{\sigma}\sum_{k,l\in\mathbb{Z}^3_*}1_{L_k\cap L_l}(p+k+l)1_{L_k\cap L_l}(-p)\frac{\hat{V}_l}{\sqrt{\lambda_{l,p+k+l}}}\left|\left\langle e_{p+k+l},h_k^{-\frac{1}{2}}B_k e_{-p}\right\rangle\right|\left\langle\Psi,\tilde{c}^*_{-p,\sigma}\tilde{c}_{-p,\sigma}\Psi\right\rangle$$
$$\leq Ck_F^{-1}\sum_{p\in B_F^c}^{\sigma}\sqrt{\sum_{k,l\in\mathbb{Z}^3_*}1_{L_k}(p+k+l)1_{L_k}(-p)\left|\left\langle e_{p+k+l},h_k^{-\frac{1}{2}}B_k e_{-p}\right\rangle\right|^2}$$
$$\cdot\sqrt{\sum_{k,l\in\mathbb{Z}^3_*}\hat{V}_l^2\frac{1_{L_l}(p+k+l)}{\lambda_{l,p+k+l}}}\left\langle\Psi,\tilde{c}^*_{-p,\sigma}\tilde{c}_{-p,\sigma}\Psi\right\rangle\leq Ck_F^{-\frac{1}{2}}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\max_{p\in L_k}\|h_k^{-\frac{1}{2}}B_k e_p\|^2}\sqrt{\sum_{l\in\mathbb{Z}^3_*}\hat{V}_l^2}\left\langle\Psi,\mathcal{N}_E\Psi\right\rangle.$$

□

The $\mathcal{E}_k^2$ bound of Theorem 7.0.1 now follows:

Proposition 7.2.5. *For any $\Psi\in\mathcal{H}_N$ and $t\in[0,1]$ it holds that*

$$\sum_{k\in\mathbb{Z}_*^3}\left|\left\langle\Psi,\left(\mathcal{E}_k^2(B_k(t))-\left\langle\psi_F,\mathcal{E}_k^2(B_k(t))\psi_F\right\rangle\right)\Psi\right\rangle\right|\leq C\sqrt{\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2\min\{|k|,k_F\}}\left\langle\Psi,\mathcal{N}_E^3\Psi\right\rangle$$

for a constant $C>0$ depending only on $\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2$ and s.

Proof: Clearly

$$\max_{p\in L_k}\|B_ke_p\|^2\leq\sum_{p\in L_k}\max_{q\in L_k}|\langle e_p,B_ke_q\rangle|^2,\quad \max_{p\in L_k}\|h_k^{-\frac{1}{2}}B_ke_p\|^2\leq\|B_kh_k^{-\frac{1}{2}}\|_{\mathrm{HS}}^2,\tag{7.2.28}$$

for any B_k, and as our estimate for $B_k(t)$ in Theorem 6.0.1 is the same as that for $A_k(t)$, the bounds

$$\sum_{k\in\mathbb{Z}_*^3}\sum_{p\in L_k}\max_{q\in L_k}|\langle e_p,B_ke_q\rangle|^2,\; k_F^{-1}\sum_{k\in\mathbb{Z}_*^3}\|B_kh_k^{-\frac{1}{2}}\|_{\mathrm{HS}}^2\leq C\left(1+\|\hat{V}\|_\infty^4\right)\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2\min\{|k|,k_F\}\tag{7.2.29}$$

follow exactly as those of Proposition 7.1.4. Insertion into the Propositions 7.2.1, 7.2.3 and 7.2.4 yields the claim.

□

7.3 Analysis of the Exchange Contribution

Finally we determine the leading order of the exchange contribution. To begin we derive a general formula for a quantity of the form $\langle\psi_F,\mathcal{E}_k^2(B_k)\psi_F\rangle$: We can write

$$\begin{aligned}-2\left\langle\psi_F,\mathcal{E}_k^2(B_k)\psi_F\right\rangle&=-\sum_{l\in\mathbb{Z}_*^3}\sum_{p\in L_k}\sum_{q\in L_l}\langle\psi_F,b_k(B_ke_p)\varepsilon_{-k,-l}(e_{-p};e_{-q})b_l^*(K_le_q)\psi_F\rangle\\&=\frac{1}{s}\sum_{l\in\mathbb{Z}_*^3}\sum_{p\in L_k\cap L_l}^{\sigma}\left\langle\psi_F,b_k(B_ke_p)\tilde{c}_{-p+l,\sigma}^*\tilde{c}_{-p+k,\sigma}b_l^*(K_le_p)\psi_F\right\rangle\\&+\frac{1}{s}\sum_{l\in\mathbb{Z}_*^3}\sum_{p\in(L_k-k)\cap(L_l-l)}^{\sigma}\left\langle\psi_F,b_k(B_ke_{p+k})\tilde{c}_{-p-l,\sigma}^*\tilde{c}_{-p-k,\sigma}b_l^*(K_le_{p+l})\psi_F\right\rangle\\&=:A+B\end{aligned}\tag{7.3.1}$$

where, using equation (7.1.19) in the form

$$\left[b_l(\psi),\tilde{c}_{p,\sigma}^*\right]=\begin{cases}-s^{-\frac{1}{2}}\sum_{q\in L_l}\delta_{p,q-l}\left\langle\psi,e_q\right\rangle\tilde{c}_{q,\sigma} & p\in B_F\\ s^{-\frac{1}{2}}\sum_{q\in(L_l-l)}\delta_{p,q+l}\left\langle\psi,e_{q+l}\right\rangle\tilde{c}_{q,\sigma} & p\in B_F^c\end{cases},\tag{7.3.2}$$

the terms A and B are given by

$$A = \frac{1}{s}\sum_{l\in\mathbb{Z}^3_*}\sum_{p\in L_k\cap L_l}^{\sigma}\left\langle\psi_F,\left[b_k(B_ke_p),\tilde{c}^*_{-p+l,\sigma}\right]\left[b_l(K_le_p),\tilde{c}^*_{-p+k,\sigma}\right]^*\psi_F\right\rangle \tag{7.3.3}$$
$$= \frac{1}{s^2}\sum_{l\in\mathbb{Z}^3_*}\sum_{p\in L_k\cap L_l}^{\sigma}\left\langle\psi_F,\left(\sum_{q\in L_k}\delta_{-p+l,q-k}\langle B_ke_p,e_q\rangle\tilde{c}_{q,\sigma}\right)\left(\sum_{q'\in L_l}\delta_{-p+k,q'-l}\langle e_{q'},K_le_p\rangle\tilde{c}^*_{q',\sigma}\right)\psi_F\right\rangle$$
$$= \frac{1}{s}\sum_{l\in\mathbb{Z}^3_*}\sum_{p,q\in L_k\cap L_l}\delta_{p+q,k+l}\langle e_p,B_ke_q\rangle\langle e_q,K_le_p\rangle$$

and

$$B = \frac{1}{s}\sum_{l\in\mathbb{Z}^3_*}\sum_{p\in(L_k-k)\cap(L_l-l)}^{\sigma}\left\langle\psi_F,\left[b_k(B_ke_{p+k}),\tilde{c}^*_{-p-l,\sigma}\right]\left[b_l(K_le_{p+l}),\tilde{c}^*_{-p-k,\sigma}\right]^*\psi_F\right\rangle \tag{7.3.4}$$
$$= \frac{1}{s}\sum_{l\in\mathbb{Z}^3_*}\sum_{p,q\in(L_k-k)\cap(L_l-l)}\delta_{-p-q,k+l}\langle e_{p+k},B_ke_{q+k}\rangle\langle e_{q+l},K_le_{p+l}\rangle.$$

Although it is not obvious, there holds the identity $A = B$. To see this we rewrite both terms: First, for A, note that the presence of the $\delta_{p+q,k+l}$ makes the L_l of the summation $p,q\in L_k\cap L_l$ redundant: For any $p,q\in B_F^c$ there holds the equivalence

$$p,q\in L_{p+q-k}\Leftrightarrow p,q\in L_k \tag{7.3.5}$$

by the trivial identities

$$|p-k| = |q-(p+q-k)|,\quad |q-k| = |p-(p+q-k)|, \tag{7.3.6}$$

so A can be written as

$$A = \frac{1}{s}\sum_{p,q\in L_k}\sum_{l\in\mathbb{Z}^3_*}\delta_{p+q,k+l}\langle e_p,B_ke_q\rangle\langle e_q,K_le_p\rangle = \frac{1}{s}\sum_{p,q\in L_k}\langle e_p,B_ke_q\rangle\langle e_q,K_{p+q-k}e_p\rangle. \tag{7.3.7}$$

A similar observation applies to B: For any $p,q\in B_F$ we likewise have

$$p,q\in(L_{-p-q-k}+p+q+k)\Leftrightarrow p+k,q+k\in L_{p+q+k}\Leftrightarrow p,q\in(L_k-k) \tag{7.3.8}$$

so

$$B = \frac{1}{s}\sum_{p,q\in(L_k-k)}\sum_{l\in\mathbb{Z}^3_*}\delta_{-p-q,k+l}\langle e_{p+k},B_ke_{q+k}\rangle\langle e_{q+l},K_le_{p+l}\rangle \tag{7.3.9}$$
$$= \frac{1}{s}\sum_{p,q\in(L_k-k)}\langle e_{p+k},B_ke_{q+k}\rangle\langle e_{-p-k},K_{-p-q-k}e_{-q-k}\rangle = \frac{1}{s}\sum_{p,q\in L_k}\langle e_p,B_ke_q\rangle\langle e_q,K_{p+q-k}e_p\rangle$$

where we lastly used that the kernels K_k obey

$$\langle e_{-p},K_{-k}e_{-q}\rangle = \langle e_p,K_ke_q\rangle = \langle e_q,K_ke_p\rangle,\quad k\in\mathbb{Z}^3_*,\ p,q\in L_k. \tag{7.3.10}$$

In all we thus have the identity

$$\left\langle \psi_F, \mathcal{E}_k^2(B_k)\psi_F \right\rangle = -\frac{1}{s}\sum_{l\in\mathbb{Z}_*^3}\sum_{p,q\in L_k\cap L_l} \delta_{p+q,k+l} \left\langle e_p, B_k e_q\right\rangle \left\langle e_q, K_l e_p\right\rangle \tag{7.3.11}$$
$$= -\frac{1}{s}\sum_{p,q\in L_k} \left\langle e_p, B_k e_q\right\rangle \left\langle e_q, K_{p+q-k} e_p\right\rangle .$$

Our matrix element estimates of the last section now yield the following:

Proposition (7.0.2)**.** *It holds that*

$$\left| \sum_{k\in\mathbb{Z}_*^3} \int_0^1 \left\langle \psi_F, 2\,\mathrm{Re}\Big(\mathcal{E}_k^2(B_k(t))\Big)\psi_F \right\rangle dt - E_{\mathrm{corr,ex}} \right| \leq C\sqrt{\sum_{k\in\mathbb{Z}_*^3} \hat{V}_k^2 \min\left\{|k|, k_F\right\}}$$

for a constant $C > 0$ depending only on $\sum_{k\in\mathbb{Z}_*^3} \hat{V}_k^2$ and s, where

$$E_{\mathrm{corr,ex}} = \frac{s k_F^{-2}}{4\left(2\pi\right)^6} \sum_{k,l\in\mathbb{Z}_*^3} \hat{V}_k\hat{V}_l \sum_{p,q\in L_k\cap L_l} \frac{\delta_{p+q,k+l}}{\lambda_{k,p}+\lambda_{k,q}}.$$

Proof: Since all the one-body operators are real-valued we can drop the $\mathrm{Re}(\cdot)$ and apply the above identity for

$$\sum_{k\in\mathbb{Z}_*^3} \int_0^1 \left\langle \psi_F, 2\,\mathrm{Re}\Big(\mathcal{E}_k^2(B_k(t))\Big)\psi_F \right\rangle dt = \sum_{k\in\mathbb{Z}_*^3} 2\left\langle \psi_F, \mathcal{E}_k^2\Big(\int_0^1 B_k(t)\,dt\Big)\psi_F \right\rangle \tag{7.3.12}$$
$$= \frac{2}{s}\sum_{k,l\in\mathbb{Z}_*^3}\sum_{p,q\in L_k\cap L_l} \delta_{p+q,k+l} \left\langle e_p, \Big(\int_0^1 B_k(t)\,dt\Big) e_q\right\rangle \left\langle e_q, (-K_l) e_p\right\rangle .$$

Now, note that $E_{\mathrm{corr,ex}}$ can be written as

$$E_{\mathrm{corr,ex}} = \frac{1}{s}\sum_{k,l\in\mathbb{Z}_*^3}\sum_{p,q\in L_k\cap L_l} \delta_{p+q,k+l} \frac{s\hat{V}_k k_F^{-1}}{2\left(2\pi\right)^3} \frac{s\hat{V}_l k_F^{-1}}{2\left(2\pi\right)^3} \frac{1}{\lambda_{l,p}+\lambda_{l,q}} \tag{7.3.13}$$

since, much as in Proposition 7.1.2, the $\delta_{p+q,k+l}$ implies the following identity for the denominators:

$$\lambda_{l,p}+\lambda_{l,q} = \frac{1}{2}\Big(|p|^2 - |p-l|^2\Big) + \frac{1}{2}\Big(|q|^2 - |q-l|^2\Big) \tag{7.3.14}$$
$$= \frac{1}{2}\Big(|p|^2 - |q-k|^2\Big) + \frac{1}{2}\Big(|q|^2 - |p-k|^2\Big) = \lambda_{k,p}+\lambda_{k,q}.$$

We thus see that

$$\sum_{k\in\mathbb{Z}_*^3} \int_0^1 \left\langle \psi_F, 2\,\mathrm{Re}\Big(\mathcal{E}_k^2(B_k(t))\Big)\psi_F \right\rangle dt - E_{\mathrm{corr,ex}}$$

$$= \frac{2}{s} \sum_{k,l \in \mathbb{Z}^3_*} \sum_{p,q \in L_k \cap L_l} \delta_{p+q,k+l} \left(\left\langle e_p, \left(\int_0^1 B_k(t)\, dt \right) e_q \right\rangle - \frac{s \hat{V}_k k_F^{-1}}{4\,(2\pi)^3} \right) \langle e_q, (-K_l) e_p \rangle \tag{7.3.15}$$

$$+ \frac{1}{s} \sum_{k,l \in \mathbb{Z}^3_*} \sum_{p,q \in L_k \cap L_l} \delta_{p+q,k+l} \frac{s \hat{V}_k k_F^{-1}}{2\,(2\pi)^3} \left(\langle e_q, (-K_l) e_p \rangle - \frac{s \hat{V}_l k_F^{-1}}{2\,(2\pi)^3} \frac{1}{\lambda_{l,p} + \lambda_{l,q}} \right) =: A + B.$$

We estimate A and B. By the matrix element estimates of Theorem 6.0.1 we have that (using our freedom to replace $\lambda_{l,p} + \lambda_{l,q}$ by $\lambda_{k,p} + \lambda_{k,q}$)

$$\begin{aligned} |A| &\le C \sum_{k,l \in \mathbb{Z}^3_*} \sum_{p,q \in L_k \cap L_l} \delta_{p+q,k+l} \left(1 + \hat{V}_k \right) \hat{V}_k^2 k_F^{-1} \frac{\hat{V}_l k_F^{-1}}{\lambda_{l,p} + \lambda_{l,q}} \\ &\le C k_F^{-2} \left(1 + \|\hat{V}\|_\infty \right) \sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2 \sum_{p \in L_k} \frac{1}{\sqrt{\lambda_{k,p}}} \sum_{q \in L_k} \frac{\hat{V}_{p+q-k}}{\sqrt{\lambda_{k,q}}} \\ &\le C k_F^{-\frac{3}{2}} \left(1 + \|\hat{V}\|_\infty \right) \sqrt{\sum_{l \in \mathbb{Z}^3_*} \hat{V}_l^2} \sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2 \sum_{p \in L_k} \frac{1}{\sqrt{\lambda_{k,p}}} \\ &\le C \left(1 + \|\hat{V}\|_\infty \right) \sqrt{\sum_{l \in \mathbb{Z}^3_*} \hat{V}_l^2} \sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2 \, |k|^{\frac{1}{2}} \min\{1, k_F^{\frac{3}{2}} |k|^{-\frac{3}{2}}\} \end{aligned} \tag{7.3.16}$$

where we applied the inequality $\sum_{q \in L_k} \lambda_{k,q}^{-1} \le C k_F$ and also used that Proposition 6.2.6 implies that

$$\sum_{p \in L_k} \frac{1}{\sqrt{\lambda_{k,p}}} \le C k_F^{\frac{3}{2}} |k|^{\frac{1}{2}} \min\{1, k_F^{\frac{3}{2}} |k|^{-\frac{3}{2}}\} \tag{7.3.17}$$

for a $C > 0$ independent of all quantities. By Cauchy-Schwarz we can further estimate

$$\begin{aligned} \sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2 \, |k|^{\frac{1}{2}} \min\{1, k_F^{\frac{3}{2}} |k|^{-\frac{3}{2}}\} &\le \sqrt{\sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2} \sqrt{\sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2 \, |k| \min\{1, k_F^3 |k|^{-3}\}} \\ &\le \sqrt{\sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2} \sqrt{\sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2 \min\{|k|, k_F\}} \end{aligned} \tag{7.3.18}$$

for the bound of the statement. By similar estimation also

$$|B| \le C \sum_{k,l \in \mathbb{Z}^3_*} \sum_{p,q \in L_k \cap L_l} \delta_{p+q,k+l} \hat{V}_k k_F^{-1} \frac{\hat{V}_l^2 k_F^{-1}}{\lambda_{l,p} + \lambda_{l,q}} \le C \sqrt{\sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2} \sum_{l \in \mathbb{Z}^3_*} \hat{V}_l^2 \, |l|^{\frac{1}{2}} \min\{1, k_F^{\frac{3}{2}} |l|^{-\frac{3}{2}}\} \tag{7.3.19}$$

and the claim follows.

□

Chapter 8

Estimation of the Non-Bosonizable Terms and Gronwall Estimates

In this section we perform the final work which will allow us to conclude Theorem 1.1.1.

The main content of this section lies in the estimation of the non-bosonizable terms, which we recall are the cubic and quartic terms

$$\mathcal{C} = \frac{k_F^{-1}}{(2\pi)^3} \sum_{k\in\mathbb{Z}_*^3} \hat{V}_k \operatorname{Re}\Big(\big(B_k + B_{-k}^*\big)^* D_k\Big) \tag{8.0.1}$$

$$\mathcal{Q} = \frac{k_F^{-1}}{2\,(2\pi)^3} \sum_{k\in\mathbb{Z}_*^3} \hat{V}_k \left(D_k^* D_k - \sum_{p\in L_k}^{\sigma} \Big(c_{p,\sigma}^* c_{p,\sigma} + c_{p-k,\sigma} c_{p-k,\sigma}^* \Big) \right).$$

The cubic terms $\mathcal{C}$ will not present a big obstacle to us: As was first noted in [8] (in their formulation), the expectation value of these in fact vanish identically with respect to the type of trial state we will consider. The bulk of the work will thus be to estimate the quartic terms. We prove the following bounds:

Theorem 8.0.1. *It holds that* $\mathcal{Q} = G + \mathcal{Q}_{\mathrm{LR}} + \mathcal{Q}_{\mathrm{SR}}$ *where for any* $\Psi \in \mathcal{H}_N$

$$|\langle \Psi, G\Psi \rangle| \leq C \sqrt{\sum_{k\in\mathbb{Z}_*^3} \hat{V}_k^2 \min\{|k|, k_F\}}\, \langle \Psi, \mathcal{N}_E \Psi \rangle$$

$$|\langle \Psi, \mathcal{Q}_{\mathrm{LR}}\Psi \rangle| \leq C \sqrt{\sum_{k\in\mathbb{Z}_*^3} \hat{V}_k^2 \min\{|k|, k_F\}}\, \Big\langle \Psi, \mathcal{N}_E^2 \Psi \Big\rangle$$

and $e^{\mathcal{K}} \mathcal{Q}_{\mathrm{SR}} e^{-\mathcal{K}} = \mathcal{Q}_{\mathrm{SR}} + \int_0^1 e^{t\mathcal{K}} (2\operatorname{Re}(\mathcal{G})) e^{-t\mathcal{K}} dt$ *for an operator* $\mathcal{G}$ *obeying*

$$|\langle \Psi, \mathcal{G}\Psi \rangle| \leq C \sqrt{\sum_{k\in\mathbb{Z}_*^3} \hat{V}_k^2 \min\{|k|, k_F\}}\, \Big\langle \Psi, \Big(\mathcal{N}_E^3 + 1\Big) \Psi \Big\rangle,$$

$C > 0$ *being a constant depending only on* $\sum_{k\in\mathbb{Z}_*^3} \hat{V}_k^2$.

(Again there are some technical questions which arise due to the unboundedness of $\mathcal{Q}_{\mathrm{SR}}$. We consider these in appendix section C.3.)

With these all the general bounds are established. As all our error estimates are with respect to $\mathcal{N}_E$ and powers thereof, it then only remains to control the effect which the transformation $e^{\mathcal{K}}$ has on these. By a standard Gronwall-type argument this control will follow from the estimate of Proposition 5.1.3, and we then end the section by concluding Theorem 1.1.1.

Analysis of the Cubic Terms

Expanding the $\mathrm{Re}(\cdot)$, the cubic terms are

$$\mathcal{C} = \frac{k_F^{-1}}{2\,(2\pi)^3}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k\Big((B_k^* + B_{-k})D_k + D_k^*\Big(B_k + B_{-k}^*\Big)\Big). \tag{8.0.2}$$

The operators B_k can be written simply as $B_k = s\sum_{p\in L_k} b_{k,p}$ in terms of the excitation operators $b_{k,p} = s^{-\frac{1}{2}}\sum_{\sigma=1}^{s} c_{p-k,\sigma}^* c_{p,\sigma}$, whence it is easily seen that

$$[\mathcal{N}_E, B_k] = -B_k, \quad [\mathcal{N}_E, B_k^*] = B_k^*. \tag{8.0.3}$$

As a consequence, B_k maps the eigenspace $\{\mathcal{N}_E = M\}$ into $\{\mathcal{N}_E = M-1\}$ and B_k^* maps $\{\mathcal{N}_E = M\}$ into $\{\mathcal{N}_E = M+1\}$. Meanwhile, the operators D_k preserve the eigenspaces: Writing $D_k = D_{1,k} + D_{2,k}$ for

$$D_{1,k} = \mathrm{d}\Gamma\Big(P_{B_F} e^{-ik\cdot x} P_{B_F}\Big) = \sum_{p,q\in B_F}^{\sigma} \delta_{p,q-k} c_{p,\sigma}^* c_{q,\sigma} = -\sum_{q\in B_F\cap(B_F+k)}^{\sigma} \tilde{c}_{q,\sigma}^* \tilde{c}_{q-k,\sigma} \tag{8.0.4}$$

$$D_{2,k} = \mathrm{d}\Gamma\Big(P_{B_F^c} e^{-ik\cdot x} P_{B_F^c}\Big) = \sum_{p,q\in B_F^c}^{\sigma} \delta_{p,q-k} c_{p,\sigma}^* c_{q,\sigma} = \sum_{p\in B_F^c\cap\left(B_F^c-k\right)}^{\sigma} \tilde{c}_{p,\sigma}^* \tilde{c}_{p+k,\sigma}$$

these annihilate and create one hole or excitation, respectively, whence $[\mathcal{N}_E, D_k] = 0 = [\mathcal{N}_E, D_k^*]$.

It follows that $\mathcal{C}$ maps the eigenspace $\{\mathcal{N}_E = M\}$ into $\{\mathcal{N}_E = M-1\}\oplus\{\mathcal{N}_E = M+1\}$. Decomposing $\mathcal{H}_N$ orthogonally as $\mathcal{H}_N = \mathcal{H}_N^{\mathrm{even}} \oplus \mathcal{H}_N^{\mathrm{odd}}$ for

$$\mathcal{H}_N^{\mathrm{even}} = \bigoplus_{m=0}^{\infty}\{\mathcal{N}_E = 2m\}, \quad \mathcal{H}_N^{\mathrm{odd}} = \bigoplus_{m=0}^{\infty}\{\mathcal{N}_E = 2m+1\}, \tag{8.0.5}$$

we thus see that $\mathcal{C}$ maps each subspace into the other. On the other hand, since our transformation kernel $\mathcal{K}$ is of the form

$$\mathcal{K} = \frac{1}{2}\sum_{l\in\mathbb{Z}^3_*}\sum_{p,q\in L_l}\langle e_p, K_l e_q\rangle\left(b_{l,p}b_{-l,-q} - b_{-l,-q}^* b_{l,p}^*\right) \tag{8.0.6}$$

we note that $\mathcal{K}$ maps each $\{\mathcal{N}_E = M\}$ into $\{\mathcal{N}_E = M-2\}\oplus\{\mathcal{N}_E = M+2\}$, hence $\mathcal{K}$ preserves $\mathcal{H}_N^{\mathrm{even}}$ and $\mathcal{H}_N^{\mathrm{odd}}$, and so too does the transformation $e^{-\mathcal{K}}$. As any eigenstate $\Psi\in\mathcal{H}_N$ of $\mathcal{N}_E$ is contained in either $\mathcal{H}_N^{\mathrm{even}}$ or $\mathcal{H}_N^{\mathrm{odd}}$, and these are orthogonal, we conclude the following:

Proposition 8.0.2. *For any eigenstate* Ψ *of* $\mathcal{N}_E$ *it holds that*

$$\left\langle e^{-\mathcal{K}}\Psi, \mathcal{C} e^{-\mathcal{K}}\Psi \right\rangle = 0.$$

8.1 Analysis of the Quartic Terms

Now we consider the quartic terms

$$\mathcal{Q} = \frac{k_F^{-1}}{2\left(2\pi\right)^3} \sum_{k\in\mathbb{Z}_*^3} \hat{V}_k \left(D_k^* D_k - \sum_{p\in L_k}^{\sigma} \left(c_{p,\sigma}^* c_{p,\sigma} + c_{p-k,\sigma} c_{p-k,\sigma}^* \right) \right). \tag{8.1.1}$$

We begin by rewriting these: Recalling the decomposition $D_k = D_{1,k} + D_{2,k}$ above, we calculate

$$\begin{aligned} D_{1,k}^* D_{1,k} &= \sum_{p,q\in B_F\cap(B_F+k)}^{\sigma,\tau} \tilde{c}_{p-k,\sigma}^* \tilde{c}_{p,\sigma} \tilde{c}_{q,\tau}^* \tilde{c}_{q-k,\tau} \\ &= \sum_{p,q\in B_F\cap(B_F+k)}^{\sigma,\tau} \tilde{c}_{p-k,\sigma}^* \tilde{c}_{q,\tau}^* \tilde{c}_{q-k,\tau} \tilde{c}_{p,\sigma} + \sum_{q\in B_F\cap(B_F+k)}^{\sigma} \tilde{c}_{q-k,\sigma}^* \tilde{c}_{q-k,\sigma} \\ &= \sum_{p,q\in B_F\cap(B_F+k)}^{\sigma,\tau} \tilde{c}_{p-k,\sigma}^* \tilde{c}_{q,\tau}^* \tilde{c}_{q-k,\tau} \tilde{c}_{p,\sigma} + \sum_{q\in B_F}^{\sigma} 1_{B_F}(q+k) \tilde{c}_{q,\sigma}^* \tilde{c}_{q,\sigma} \end{aligned} \tag{8.1.2}$$

and similarly

$$\begin{aligned} D_{2,k}^* D_{2,k} &= \sum_{p,q\in B_F^c\cap\left(B_F^c-k\right)}^{\sigma,\tau} \tilde{c}_{p+k,\sigma}^* \tilde{c}_{p,\sigma} \tilde{c}_{q,\tau}^* \tilde{c}_{q+k,\tau} \\ &= \sum_{p,q\in B_F^c\cap\left(B_F^c-k\right)}^{\sigma,\tau} \tilde{c}_{p+k,\sigma}^* \tilde{c}_{q,\tau}^* \tilde{c}_{q+k,\tau} \tilde{c}_{p,\sigma} + \sum_{p\in B_F^c}^{\sigma} 1_{B_F^c}(p-k) \tilde{c}_{p,\sigma}^* \tilde{c}_{p,\sigma} \\ &= \sum_{p,q\in B_F^c\cap\left(B_F^c-k\right)}^{\sigma,\tau} \tilde{c}_{p+k,\sigma}^* \tilde{c}_{q,\tau}^* \tilde{c}_{q+k,\tau} \tilde{c}_{p,\sigma} + \mathcal{N}_E - \sum_{p\in B_F^c}^{\sigma} 1_{B_F}(p-k) \tilde{c}_{p,\sigma}^* \tilde{c}_{p,\sigma}. \end{aligned} \tag{8.1.3}$$

For any $k \in \mathbb{Z}_*^3$ we can likewise write $\sum_{p\in L_k}^{\sigma} \left(c_{p,\sigma}^* c_{p,\sigma} + c_{p-k,\sigma} c_{p-k,\sigma}^* \right)$ in the form

$$\begin{aligned} \sum_{p\in L_k}^{\sigma} \left(c_{p,\sigma}^* c_{p,\sigma} + c_{p-k,\sigma} c_{p-k,\sigma}^* \right) &= \sum_{p\in B_F^c}^{\sigma} 1_{B_F}(p-k) \tilde{c}_{p,\sigma}^* \tilde{c}_{p,\sigma} + \sum_{q\in B_F}^{\sigma} 1_{B_F^c}(q+k) \tilde{c}_{q,\sigma}^* \tilde{c}_{q,\sigma} \\ &= \sum_{p\in B_F^c}^{\sigma} 1_{B_F}(p-k) \tilde{c}_{p,\sigma}^* \tilde{c}_{p,\sigma} + \mathcal{N}_E - \sum_{q\in B_F}^{\sigma} 1_{B_F}(q+k) \tilde{c}_{q,\sigma}^* \tilde{c}_{q,\sigma}. \end{aligned} \tag{8.1.4}$$

Noting that $D_{1,k} = 0$ for $|k| > 2k_F$, as then $B_F \cap (B_F + k) = \emptyset$, we thus obtain the decomposition

$$\mathcal{Q} = G + \mathcal{Q}_{\mathrm{LR}} + \mathcal{Q}_{\mathrm{SR}} \tag{8.1.5}$$

where G is

$$G = \frac{k_F^{-1}}{(2\pi)^3} \sum_{k \in \mathbb{Z}^3_*} \hat{V}_k \left(\sum_{q \in B_F}^{\sigma} 1_{B_F}(q+k) \tilde{c}^*_{q,\sigma} \tilde{c}_{q,\sigma} - \sum_{p \in B_F^c}^{\sigma} 1_{B_F}(p-k) \tilde{c}^*_{p,\sigma} \tilde{c}_{p,\sigma} \right), \tag{8.1.6}$$

the *long-range terms* $\mathcal{Q}_{\mathrm{LR}}$ are given by

$$\mathcal{Q}_{\mathrm{LR}} = \frac{k_F^{-1}}{2\,(2\pi)^3} \sum_{k \in \overline{B}(0,2k_F) \cap \mathbb{Z}^3_*} \hat{V}_k \left(\sum_{p,q \in B_F \cap (B_F + k)}^{\sigma,\tau} \tilde{c}^*_{p-k,\sigma} \tilde{c}^*_{q,\tau} \tilde{c}_{q-k,\tau} \tilde{c}_{p,\sigma} + D^*_{1,k} D_{2,k} + D^*_{2,k} D_{1,k} \right) \tag{8.1.7}$$

and the *short-range terms* $\mathcal{Q}_{\mathrm{SR}}$ are

$$\mathcal{Q}_{\mathrm{SR}} = \frac{k_F^{-1}}{2\,(2\pi)^3} \sum_{k \in \mathbb{Z}^3_*} \hat{V}_k \sum_{p,q \in B_F^c \cap \left(B_F^c - k\right)}^{\sigma,\tau} \tilde{c}^*_{p+k,\sigma} \tilde{c}^*_{q,\tau} \tilde{c}_{q+k,\tau} \tilde{c}_{p,\sigma}. \tag{8.1.8}$$

Estimation of G and $\mathcal{Q}_{\mathrm{LR}}$

G and the long-range terms are easily controlled: First, interchanging the summations we can write G as

$$G = \frac{k_F^{-1}}{(2\pi)^3} \sum_{q \in B_F}^{\sigma} \left(\sum_{k \in (B_F - q) \cap \mathbb{Z}^3_*} \hat{V}_k \right) \tilde{c}^*_{q,\sigma} \tilde{c}_{q,\sigma} - \frac{k_F^{-1}}{(2\pi)^3} \sum_{p \in B_F^c}^{\sigma} \left(\sum_{k \in (B_F + p) \cap \mathbb{Z}^3_*} \hat{V}_k \right) \tilde{c}^*_{p,\sigma} \tilde{c}_{p,\sigma} \tag{8.1.9}$$

from which it is obvious that G obeys

$$\pm G \leq \max_{p \in \mathbb{Z}^3_*} \left(\frac{k_F^{-1}}{(2\pi)^3} \sum_{k \in (B_F + p) \cap \mathbb{Z}^3_*} \hat{V}_k \right) \mathcal{N}_E. \tag{8.1.10}$$

This implies the following:

Proposition 8.1.1. *For any $\Psi \in \mathcal{H}_N$ it holds that*

$$|\langle \Psi, G \Psi \rangle| \leq C \sqrt{\sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2 \min\{|k|, k_F\}} \, \langle \Psi, \mathcal{N}_E \Psi \rangle$$

for a constant $C > 0$ independent of all quantities.

Proof: For any $p \in \mathbb{Z}^3$ we estimate by Cauchy-Schwarz

$$\begin{aligned} \sum_{k \in (B_F + p) \cap \mathbb{Z}^3_*} \hat{V}_k &\leq \sqrt{\sum_{k \in (B_F + p) \cap \mathbb{Z}^3_*} \hat{V}_k^2 \min\{|k|, k_F\}} \sqrt{\sum_{k \in (B_F + p) \cap \mathbb{Z}^3_*} \min\{|k|, k_F\}^{-1}} \\ &\leq \sqrt{\sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2 \min\{|k|, k_F\}} \sqrt{\sum_{k \in B_F \setminus \{0\}} |k|^{-1} + k_F^{-1}} \end{aligned} \tag{8.1.11}$$

where we lastly used that $k \mapsto \min\{|k|, k_F\}^{-1}$ is radially non-increasing and that $(B_F + p) \cap \mathbb{Z}^3_*$ contains at most $|B_F|$ points. As it is well-known that $\sum_{k \in \overline{B}(0,R)\backslash\{0\}} |k|^{-1} \leq CR^2$ as $R \to \infty$ the bound follows.

□

$\mathcal{Q}_{\mathrm{LR}}$ can be handled in a similar manner:

Proposition 8.1.2. *For any* $\Psi \in \mathcal{H}_N$ *it holds that*

$$|\langle \Psi, \mathcal{Q}_{\mathrm{LR}} \Psi \rangle| \leq C \sqrt{\sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2 \min\{|k|, k_F\}} \left\langle \Psi, \mathcal{N}_E^2 \Psi \right\rangle$$

for a constant $C > 0$ *independent of all quantities.*

Proof: Consider the first term in the parenthesis of equation (8.1.7): For any $k \in \mathbb{Z}^3_*$ we can estimate

$$\begin{aligned} \sum_{p,q \in B_F \cap (B_F + k)}^{\sigma,\tau} \left| \left\langle \Psi, \tilde{c}^*_{p-k,\sigma} \tilde{c}^*_{q,\tau} \tilde{c}_{q-k,\tau} \tilde{c}_{p,\sigma} \Psi \right\rangle \right| &\leq \sum_{p,q \in B_F \cap (B_F + k)}^{\sigma,\tau} \|\tilde{c}_{q,\tau} \tilde{c}_{p-k,\sigma} \Psi\| \, \|\tilde{c}_{q-k,\tau} \tilde{c}_{p,\sigma} \Psi\| \\ \leq \sqrt{\sum_{p,q \in B_F \cap (B_F + k)}^{\sigma,\tau} \|\tilde{c}_{q,\tau} \tilde{c}_{p-k,\sigma} \Psi\|^2} & \sqrt{\sum_{p,q \in B_F \cap (B_F + k)}^{\sigma,\tau} \|\tilde{c}_{q-k,\tau} \tilde{c}_{p,\sigma} \Psi\|^2} \leq \left\langle \Psi, \mathcal{N}_E^2 \Psi \right\rangle. \end{aligned} \tag{8.1.12}$$

As e.g.

$$\begin{aligned} D^*_{1,k} D_{2,k} &= \sum_{p \in B_F^c \cap \left(B_F^c - k\right)}^{\sigma} \sum_{q \in B_F \cap (B_F + k)}^{\tau} \tilde{c}^*_{q-k,\tau} \tilde{c}_{q,\tau} \tilde{c}^*_{p,\sigma} \tilde{c}_{p+k,\sigma} \\ &= \sum_{p \in B_F^c \cap \left(B_F^c - k\right)}^{\sigma} \sum_{q \in B_F \cap (B_F + k)}^{\tau} \tilde{c}^*_{p,\sigma} \tilde{c}^*_{q-k,\tau} \tilde{c}_{q,\tau} \tilde{c}_{p+k,\sigma} \end{aligned} \tag{8.1.13}$$

since B_F and B_F^c are disjoint, the terms $D^*_{1,k} D_{2,k}$ and $D^*_{2,k} D_{1,k}$ can be handled similarly, whence

$$|\langle \Psi, \mathcal{Q}_{\mathrm{LR}} \Psi \rangle| \leq \frac{3 k_F^{-1}}{2 (2\pi)^3} \left(\sum_{k \in \overline{B}(0, 2k_F) \cap \mathbb{Z}^3_*} \hat{V}_k \right) \left\langle \Psi, \mathcal{N}_E^2 \Psi \right\rangle \leq C \sqrt{\sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2 \min\{|k|, k_F\}} \left\langle \Psi, \mathcal{N}_E^2 \Psi \right\rangle \tag{8.1.14}$$

where $\sum_{k \in \overline{B}(0, 2k_F) \cap \mathbb{Z}^3_*} \hat{V}_k$ was bounded as in equation (8.1.11).

□

Analysis of $\mathcal{Q}_{\mathrm{SR}}$

Lastly we come to

$$\mathcal{Q}_{\mathrm{SR}} = \frac{k_F^{-1}}{2 (2\pi)^3} \sum_{k \in \mathbb{Z}^3_*} \hat{V}_k \sum_{p,q \in B_F^c \cap \left(B_F^c - k\right)}^{\sigma,\tau} \tilde{c}^*_{p+k,\sigma} \tilde{c}^*_{q,\tau} \tilde{c}_{q+k,\tau} \tilde{c}_{p,\sigma}. \tag{8.1.15}$$

Recall that the transformation kernel $\mathcal{K}$ can be written as $\mathcal{K} = \tilde{\mathcal{K}} - \tilde{\mathcal{K}}^*$ for

$$\tilde{\mathcal{K}} = \frac{1}{2}\sum_{l\in\mathbb{Z}^3_*}\sum_{p,q\in L_l}\langle e_p, K_l e_q\rangle\, b_{l,p} b_{-l,-q} = \frac{1}{2}\sum_{l\in\mathbb{Z}^3_*}\sum_{q\in L_l} b_l(K_l e_q) b_{-l,-q}. \tag{8.1.16}$$

To determine $e^{\mathcal{K}}\mathcal{Q}_{\mathrm{SR}}e^{-\mathcal{K}}$ we will need the commutator $[\mathcal{K}, \mathcal{Q}_{\mathrm{SR}}] = 2\,\mathrm{Re}\Big([\tilde{\mathcal{K}}, \mathcal{Q}_{\mathrm{SR}}]\Big)$. Noting that for any $p \in B_F^c$ and $l \in \mathbb{Z}^3_*$, $q \in L_l$, we have

$$\Big[b_{l,q}, \tilde{c}^*_{p,\sigma}\Big] = \frac{1}{\sqrt{s}}\sum_{\tau=1}^{s}\Big[c^*_{q-l,\tau}c_{q,\tau}, c^*_{p,\sigma}\Big] = \frac{1}{\sqrt{s}}\sum_{\tau=1}^{s}\delta_{p,q}\delta_{\sigma,\tau}c^*_{q-l,\tau} = \frac{1}{\sqrt{s}}\delta_{p,q}\tilde{c}_{q-l,\sigma}, \tag{8.1.17}$$

we deduce (with the help of Lemma 3.2.1) that

$$\begin{aligned}
[\tilde{\mathcal{K}}, \tilde{c}^*_{p,\sigma}] &= \frac{1}{2}\sum_{l\in\mathbb{Z}^3_*}\sum_{q\in L_l}\Big(b_l(K_l e_q)\Big[b_{-l,-q}, \tilde{c}^*_{p,\sigma}\Big] + \Big[b_l(K_l e_q), \tilde{c}^*_{p,\sigma}\Big]\, b_{-l,-q}\Big)\\
&= \frac{1}{2}\sum_{l\in\mathbb{Z}^3_*}\sum_{q\in L_l}\Big(b_l(K_l e_q)\Big[b_{-l,-q}, \tilde{c}^*_{p,\sigma}\Big] + \Big[b_{l,q}, \tilde{c}^*_{p,\sigma}\Big]\, b_{-l}(K_{-l}e_{-q})\Big)\\
&= \frac{1}{2\sqrt{s}}\sum_{l\in\mathbb{Z}^3_*}\sum_{q\in L_l}(b_l(K_l e_q)\delta_{p,-q}\tilde{c}_{-q+l,\sigma} + \delta_{p,q}\tilde{c}_{q-l,\sigma}b_{-l}(K_{-l}e_{-q}))\\
&= \frac{1}{\sqrt{s}}\sum_{l\in\mathbb{Z}^3_*}\sum_{q\in L_l}\delta_{p,-q}b_l(K_l e_q)\tilde{c}_{-q+l,\sigma} = \frac{1}{\sqrt{s}}\sum_{l\in\mathbb{Z}^3_*}1_{L_l}(-p)b_l(K_l e_{-p})\tilde{c}_{p+l,\sigma}.
\end{aligned} \tag{8.1.18}$$

Using this we conclude the following:

Proposition 8.1.3. *It holds that* $e^{\mathcal{K}}\mathcal{Q}_{\mathrm{SR}}e^{-\mathcal{K}} = \mathcal{Q}_{\mathrm{SR}} + \int_0^1 e^{t\mathcal{K}}(2\,\mathrm{Re}(\mathcal{G}))e^{-t\mathcal{K}}dt$ *for*

$$\begin{aligned}
\mathcal{G} = {} & \frac{s^{-\frac{1}{2}}k_F^{-1}}{(2\pi)^3}\sum_{k,l\in\mathbb{Z}^3_*}\hat{V}_k\sum_{p,q\in B_F^c\cap\left(B_F^c+k\right)}^{\sigma,\tau}1_{L_l}(q)\tilde{c}^*_{p,\sigma}b_l(K_l e_q)\tilde{c}_{-q+l,\tau}\tilde{c}_{-q+k,\tau}\tilde{c}_{p-k,\sigma}\\
& + \frac{s^{-1}k_F^{-1}}{2\,(2\pi)^3}\sum_{k,l\in\mathbb{Z}^3_*}\hat{V}_k\sum_{p,q\in B_F^c\cap\left(B_F^c+k\right)}^{\sigma,\tau}1_{L_l}(p)1_{L_l}(q)\,\langle K_l e_q, e_p\rangle\,\tilde{c}_{p-l,\sigma}\tilde{c}_{-q+l,\tau}\tilde{c}_{-q+k,\tau}\tilde{c}_{p-k,\sigma}.
\end{aligned}$$

Proof: By the fundamental theorem of calculus

$$e^{\mathcal{K}}\mathcal{Q}_{\mathrm{SR}}e^{-\mathcal{K}} = \mathcal{Q}_{\mathrm{SR}} + \int_0^1 e^{t\mathcal{K}}\,[\mathcal{K}, \mathcal{Q}_{\mathrm{SR}}]\,e^{-t\mathcal{K}}dt \tag{8.1.19}$$

and as noted $[\mathcal{K}, \mathcal{Q}_{\mathrm{SR}}] = 2\,\mathrm{Re}\Big([\tilde{\mathcal{K}}, \mathcal{Q}_{\mathrm{SR}}]\Big)$. Using equation (8.1.18) we compute that $\mathcal{G} := [\tilde{\mathcal{K}}, \mathcal{Q}_{\mathrm{SR}}]$ is given by

$$\mathcal{G} = \frac{k_F^{-1}}{2\,(2\pi)^3}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k\sum_{p\in B_F^c\cap\left(B_F^c+k\right)}^{\sigma}\sum_{q\in B_F^c\cap\left(B_F^c-k\right)}^{\tau}\Big(\tilde{c}^*_{p,\sigma}[\tilde{\mathcal{K}}, \tilde{c}^*_{q,\tau}] + [\tilde{\mathcal{K}}, \tilde{c}^*_{p,\sigma}]\tilde{c}^*_{q,\tau}\Big)\tilde{c}_{q+k,\tau}\tilde{c}_{p-k,\sigma}$$

$$
\begin{aligned}
&= \frac{s^{-\frac{1}{2}}k_F^{-1}}{2\,(2\pi)^3}\sum_{k,l\in\mathbb{Z}^3_*}\hat{V}_k\sum_{p\in B_F^c\cap\left(B_F^c+k\right)}^{\sigma}\sum_{q\in B_F^c\cap\left(B_F^c-k\right)}^{\tau}1_{L_l}(-q)\tilde{c}^*_{p,\sigma}b_l(K_le_{-q})\tilde{c}_{q+l,\tau}\tilde{c}_{q+k,\tau}\tilde{c}_{p-k,\sigma}\\
&+ \frac{s^{-\frac{1}{2}}k_F^{-1}}{2\,(2\pi)^3}\sum_{k,l\in\mathbb{Z}^3_*}\hat{V}_k\sum_{p\in B_F^c\cap\left(B_F^c+k\right)}^{\sigma}\sum_{q\in B_F^c\cap\left(B_F^c-k\right)}^{\tau}1_{L_l}(-p)b_l(K_le_{-p})\tilde{c}_{p+l,\sigma}\tilde{c}^*_{q,\tau}\tilde{c}_{q+k,\tau}\tilde{c}_{p-k,\sigma}\\
&= \frac{s^{-\frac{1}{2}}k_F^{-1}}{2\,(2\pi)^3}\sum_{k,l\in\mathbb{Z}^3_*}\hat{V}_k\sum_{p\in B_F^c\cap\left(B_F^c+k\right)}^{\sigma}\sum_{q\in B_F^c\cap\left(B_F^c-k\right)}^{\tau}1_{L_l}(-q)\left\{b_l(K_le_{-q}),\tilde{c}^*_{p,\sigma}\right\}\tilde{c}_{q+l,\tau}\tilde{c}_{q+k,\tau}\tilde{c}_{p-k,\sigma}\\
&= \frac{s^{-\frac{1}{2}}k_F^{-1}}{2\,(2\pi)^3}\sum_{k,l\in\mathbb{Z}^3_*}\hat{V}_k\sum_{p,q\in B_F^c\cap\left(B_F^c+k\right)}^{\sigma,\tau}1_{L_l}(q)\left\{b_l(K_le_q),\tilde{c}^*_{p,\sigma}\right\}\tilde{c}_{-q+l,\tau}\tilde{c}_{-q+k,\tau}\tilde{c}_{p-k,\sigma},
\end{aligned}
$$

where we for the third inequality substituted $(p,\sigma)\leftrightarrow(q,\tau)$ and $k\rightarrow -k$ in the second sum, noting that then

$$
1_{L_l}(-q)\tilde{c}_{q+l,\tau}\tilde{c}^*_{p,\sigma}\tilde{c}_{p-k,\sigma}\tilde{c}_{q+k,\tau} = 1_{L_l}(-q)\tilde{c}^*_{p,\sigma}\tilde{c}_{q+l,\tau}\tilde{c}_{q+k,\tau}\tilde{c}_{p-k,\sigma} \tag{8.1.20}
$$

as the indicator function (and summation range) ensures that $q+l\neq p$.

By the identity of equation (7.1.19) the anti-commutator is given by

$$
\left\{b_l(K_le_q),\tilde{c}^*_{p,\sigma}\right\} = 2\,\tilde{c}^*_{p,\sigma}b_l(K_le_q) + 1_{L_l}(p)s^{-\frac{1}{2}}\left\langle K_le_q,e_p\right\rangle\tilde{c}_{p-l,\sigma} \tag{8.1.21}
$$

which is inserted into the previous equation for the claim.

□

We bound the $\mathcal{G}$ operator as follows:

Proposition 8.1.4. *For any $\Psi\in\mathcal{H}_N$ it holds that*

$$
\left|\langle\Psi,\mathcal{G}\Psi\rangle\right| \leq C\sqrt{\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2\min\left\{|k|,k_F\right\}}\left\langle\Psi,\left(\mathcal{N}_E^3+1\right)\Psi\right\rangle
$$

for a constant $C>0$ depending only on $\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2$.

Proof: Using Proposition 7.1.1 we estimate the sum of the first term of $\mathcal{G}$ as

$$
\begin{aligned}
&\sum_{k,l\in\mathbb{Z}^3_*}\hat{V}_k\sum_{p,q\in B_F^c\cap\left(B_F^c+k\right)}^{\sigma,\tau}1_{L_l}(q)\left|\left\langle\Psi,\tilde{c}^*_{p,\sigma}b_l(K_le_q)\tilde{c}_{-q+l,\tau}\tilde{c}_{-q+k,\tau}\tilde{c}_{p-k,\sigma}\Psi\right\rangle\right|\\
&\leq \sum_{k,l\in\mathbb{Z}^3_*}\hat{V}_k\sum_{p,q\in B_F^c\cap\left(B_F^c+k\right)}^{\sigma,\tau}1_{L_l}(q)\left\|b_l^*(K_le_q)\tilde{c}_{p,\sigma}\Psi\right\|\left\|\tilde{c}_{-q+l,\tau}\tilde{c}_{-q+k,\tau}\tilde{c}_{p-k,\sigma}\Psi\right\|\\
&\leq \sum_{k,l\in\mathbb{Z}^3_*}\hat{V}_k\sum_{p,q\in B_F^c\cap\left(B_F^c+k\right)}^{\sigma,\tau}1_{L_l}(q)\left\|K_le_q\right\|\left\|\tilde{c}_{p,\sigma}(\mathcal{N}_E+1)^{\frac{1}{2}}\Psi\right\|\left\|\tilde{c}_{p-k,\sigma}\tilde{c}_{-q+l,\tau}\tilde{c}_{-q+k,\tau}\Psi\right\|\\
&\leq \left\|(\mathcal{N}_E+1)\Psi\right\|\sum_{l\in\mathbb{Z}^3_*}\sum_{q\in L_l}^{\tau}\left\|K_le_q\right\|\sum_{k\in\mathbb{Z}^3_*}1_{B_F^c+k}(q)\hat{V}_k\left\|\tilde{c}_{-q+k,\tau}\tilde{c}_{-q+l,\tau}\mathcal{N}_E^{\frac{1}{2}}\Psi\right\|
\end{aligned} \tag{8.1.22}
$$

$$\leq \sqrt{\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2}\,\|(\mathcal{N}_E+1)\Psi\|\sum_{l\in\mathbb{Z}^3_*}\sum_{q\in L_l}^{\tau}\|K_l e_q\|\,\|\tilde{c}_{-q+l,\tau}\mathcal{N}_E\Psi\|$$
$$\leq \sqrt{s}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2}\left(\sum_{l\in\mathbb{Z}^3_*}\|K_l\|_{\mathrm{HS}}\right)\|(\mathcal{N}_E+1)\Psi\|\,\|\mathcal{N}_E^{\frac{3}{2}}\Psi\|.$$

Now, the $\|K_k\|_{\mathrm{HS}}$ estimate of Theorem 6.0.1 and Cauchy-Schwarz lets us estimate

$$\sum_{k\in\mathbb{Z}^3_*}\|K_k\|_{\mathrm{HS}}\leq C\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k\min\{1,k_F^2\,|k|^{-2}\}\leq C\sqrt{\sum_{k\in\mathbb{Z}^3_*}\frac{\min\{1,k_F^4\,|k|^{-4}\}}{\min\{|k|,k_F\}}}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2\min\{|k|,k_F\}},$$

and

$$\sum_{k\in\mathbb{Z}^3_*}\frac{\min\{1,k_F^4\,|k|^{-4}\}}{\min\{|k|,k_F\}}=\sum_{k\in B_F\backslash\{0\}}\frac{1}{|k|}+k_F^3\sum_{k\in\mathbb{Z}^3_*\backslash B_F}\frac{1}{|k|^4}\leq Ck_F^2 \tag{8.1.23}$$

for a constant $C>0$ independent of all quantities, so in all the first term of $\mathcal{G}$ obeys

$$\frac{s^{-\frac{1}{2}}k_F^{-1}}{2\,(2\pi)^3}\sum_{k,l\in\mathbb{Z}^3_*}\hat{V}_k\sum_{p,q\in B_F^c\cap\left(B_F^c+k\right)}1_{L_l}(q)\left|\left\langle\Psi,\tilde{c}_p^*b_l(K_le_q)\tilde{c}_{-q+l}\tilde{c}_{-q+k}\tilde{c}_{p-k}\Psi\right\rangle\right| \tag{8.1.24}$$
$$\leq C\sqrt{\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2\min\{|k|,k_F\}}\,\|(\mathcal{N}_E+1)\Psi\|\,\|\mathcal{N}_E^{\frac{3}{2}}\Psi\|.$$

Similarly, for the second term (using simply that $\|\tilde{c}_{p-l,\sigma}\|_{\mathrm{Op}}=1$ at the beginning)

$$\sum_{k,l\in\mathbb{Z}^3_*}\hat{V}_k\sum_{p,q\in B_F^c\cap\left(B_F^c+k\right)}^{\sigma,\tau}1_{L_l}(p)1_{L_l}(q)\,|\langle K_le_q,e_p\rangle\,\langle\Psi,\tilde{c}_{p-l,\sigma}\tilde{c}_{-q+l,\tau}\tilde{c}_{-q+k,\tau}\tilde{c}_{p-k,\sigma}\Psi\rangle|$$
$$\leq\|\Psi\|\sum_{k,l\in\mathbb{Z}^3_*}\hat{V}_k\sum_{p,q\in B_F^c\cap\left(B_F^c+k\right)}^{\sigma,\tau}1_{L_l}(p)1_{L_l}(q)\,|\langle K_le_q,e_p\rangle|\,\|\tilde{c}_{p-k,\sigma}\tilde{c}_{-q+l,\tau}\tilde{c}_{-q+k,\tau}\Psi\| \tag{8.1.25}$$
$$\leq\sqrt{s}\,\|\Psi\|\sum_{l\in\mathbb{Z}^3_*}\sum_{q\in L_l}^{\tau}\|K_le_q\|\sum_{k\in\mathbb{Z}^3_*}1_{B_F^c+k}(q)\hat{V}_k\|\tilde{c}_{-q+k,\tau}\tilde{c}_{-q+l,\tau}\mathcal{N}_E^{\frac{1}{2}}\Psi\|$$
$$\leq s\sqrt{\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2}\left(\sum_{l\in\mathbb{Z}^3_*}\|K_l\|_{\mathrm{HS}}\right)\|\Psi\|\,\|\mathcal{N}_E^{\frac{3}{2}}\Psi\|.$$

□

8.2 Gronwall Estimates

We now establish control over the operators $e^{\mathcal{K}}\mathcal{N}_E^m e^{-\mathcal{K}}$ for $m=1,2,3$. Consider first the mapping $t\mapsto e^{t\mathcal{K}}\mathcal{N}_Ee^{-t\mathcal{K}}$: Noting that for any $\Psi\in\mathcal{H}_N$

$$\frac{d}{dt}\left\langle\Psi,e^{t\mathcal{K}}(\mathcal{N}_E+1)e^{-t\mathcal{K}}\Psi\right\rangle=\left\langle\Psi,e^{-t\mathcal{K}}\left[\mathcal{K},\mathcal{N}_E\right]e^{-t\mathcal{K}}\Psi\right\rangle, \tag{8.2.1}$$

Gronwall's lemma implies that to bound $e^{t\mathcal{K}}(\mathcal{N}_E+1)e^{-t\mathcal{K}}$ it suffices to control $[\mathcal{K},\mathcal{N}_E]$ with respect to $\mathcal{N}_E+1$ itself. We determine the commutator: As $\mathcal{K}=\tilde{\mathcal{K}}-\tilde{\mathcal{K}}^*$ for

$$\tilde{\mathcal{K}}=\frac{1}{2}\sum_{l\in\mathbb{Z}^3_*}\sum_{p,q\in L_l}\langle e_p,K_l e_q\rangle\, b_{l,p}b_{-l,-q} \tag{8.2.2}$$

and $[b_{l,p},\mathcal{N}_E]=b_{l,p}$ it holds that $[\tilde{\mathcal{K}},\mathcal{N}_E]=2\,\tilde{\mathcal{K}}$, whence

$$[\mathcal{K},\mathcal{N}_E]=2\,\mathrm{Re}\Big([\tilde{\mathcal{K}},\mathcal{N}_E]\Big)=2\,\tilde{\mathcal{K}}+2\,\tilde{\mathcal{K}}^*. \tag{8.2.3}$$

The estimate of Proposition 5.1.3 immediately yields that

$$\pm\,[\mathcal{K},\mathcal{N}_E]\leq C(\mathcal{N}_E+1) \tag{8.2.4}$$

for a constant $C>0$ depending only on $\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2$ and s, whence by Gronwall's lemma

$$\Big\langle\Psi,e^{t\mathcal{K}}(\mathcal{N}_E+1)e^{-t\mathcal{K}}\Psi\Big\rangle\leq e^{C|t|}\,\langle\Psi,(\mathcal{N}_E+1)\Psi\rangle\leq C'\,\langle\Psi,(\mathcal{N}_E+1)\Psi\rangle\,,\quad |t|\leq 1. \tag{8.2.5}$$

This proves the bound for $\mathcal{N}_E$; for $\mathcal{N}_E^2$ we will as in [11] apply the following lemma:

Lemma 8.2.1. *Let A,B,Z be given with $A>0$, $Z\geq 0$ and $[A,Z]=0$. Then if $\pm\,[A,[A,B]]\leq Z$ it holds that*

$$\pm[A^{\frac{1}{2}},[A^{\frac{1}{2}},B]]\leq\frac{1}{4}A^{-1}Z.$$

We include the proof in appendix section A.2.

The estimates are as follows:

Proposition 8.2.2. *For any $\Psi\in\mathcal{H}_N$ and $|t|\leq 1$ it holds that*

$$\Big\langle e^{-t\mathcal{K}}\Psi,(\mathcal{N}_E^m+1)e^{-t\mathcal{K}}\Psi\Big\rangle\leq C\,\langle\Psi,(\mathcal{N}_E^m+1)\Psi\rangle\,,\quad m=1,2,3,$$

for a constant $C>0$ depending only on $\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2$ and s.

Proof: The case of $m=1$ was proved above. For $m=2$ it suffices to control $[\mathcal{K},\mathcal{N}_E^2]$ in terms of $\mathcal{N}_E^2+1$; by the identity $\{A,B\}=A^{\frac{1}{2}}BA^{\frac{1}{2}}+[A^{\frac{1}{2}},[A^{\frac{1}{2}},B]]$ we can write

$$\begin{aligned}\Big[\mathcal{K},\mathcal{N}_E^2\Big]&=\mathcal{N}_E\,[\mathcal{K},\mathcal{N}_E]+[\mathcal{K},\mathcal{N}_E]\,\mathcal{N}_E=\{\mathcal{N}_E,[\mathcal{K},\mathcal{N}_E]\}\\&=\{\mathcal{N}_E+1,[\mathcal{K},\mathcal{N}_E]\}-2\,[\mathcal{K},\mathcal{N}_E]\\&=(\mathcal{N}_E+1)^{\frac{1}{2}}\,[\mathcal{K},\mathcal{N}_E]\,(\mathcal{N}_E+1)^{\frac{1}{2}}+[(\mathcal{N}_E+1)^{\frac{1}{2}},[(\mathcal{N}_E+1)^{\frac{1}{2}},[\mathcal{K},\mathcal{N}_E]]]-2\,[\mathcal{K},\mathcal{N}_E]\end{aligned} \tag{8.2.6}$$

and note that the commutator $[\tilde{\mathcal{K}},\mathcal{N}_E]=2\,\tilde{\mathcal{K}}$ also implies that

$$[\mathcal{N}_E,[\mathcal{N}_E,[\mathcal{K},\mathcal{N}_E]]]=4\,[\mathcal{K},\mathcal{N}_E]\,, \tag{8.2.7}$$

so by Lemma 8.2.1 and equation (8.2.4)

$$\pm\left[\mathcal{K},\mathcal{N}_E^2\right]\leq C\Big((\mathcal{N}_E+1)^2+1+(\mathcal{N}_E+1)\Big)\leq C'\Big(\mathcal{N}_E^2+1\Big). \tag{8.2.8}$$

Similarly, for $\mathcal{N}_E^3$,

$$\begin{aligned}\left[\mathcal{K},\mathcal{N}_E^3\right] &= \mathcal{N}_E^2\left[\mathcal{K},\mathcal{N}_E\right]+\mathcal{N}_E\left[\mathcal{K},\mathcal{N}_E\right]\mathcal{N}_E+\left[\mathcal{K},\mathcal{N}_E\right]\mathcal{N}_E^2\\ &= 3\,\mathcal{N}_E\left[\mathcal{K},\mathcal{N}_E\right]\mathcal{N}_E+\mathcal{N}_E\left[\mathcal{N}_E,\left[\mathcal{K},\mathcal{N}_E\right]\right]+\left[\left[\mathcal{K},\mathcal{N}_E\right],\mathcal{N}_E\right]\mathcal{N}_E\\ &= 3\,\mathcal{N}_E\left[\mathcal{K},\mathcal{N}_E\right]\mathcal{N}_E+\left[\mathcal{N}_E,\left[\mathcal{N}_E,\left[\mathcal{K},\mathcal{N}_E\right]\right]\right]=3\,\mathcal{N}_E\left[\mathcal{K},\mathcal{N}_E\right]\mathcal{N}_E+4\left[\mathcal{K},\mathcal{N}_E\right]\end{aligned} \tag{8.2.9}$$

implies that

$$\pm\left[\mathcal{K},\mathcal{N}_E^3\right]\leq C(\mathcal{N}_E(\mathcal{N}_E+1)\mathcal{N}_E+(\mathcal{N}_E+1))\leq C'\Big(\mathcal{N}_E^3+1\Big) \tag{8.2.10}$$

hence the $m=3$ bound.

□

Conclusion of Theorem 1.1.1

We can now conclude:

Theorem (1.1.1)**.** *It holds that*

$$\inf\sigma(H_N)\leq E_F+E_{\mathrm{corr,bos}}+E_{\mathrm{corr,ex}}+C\sqrt{\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2\min\{|k|,k_F\}},\quad k_F\to\infty,$$

for a constant $C>0$ depending only on $\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2$ and s.

Proof: By the variational principle applied to the trial state $e^{-\mathcal{K}}\psi_F$ we have by Proposition 2.0.1 and the Theorems 4.0.1, 6.0.1 and 8.0.1 that

$$\begin{aligned}\inf\sigma(H_N)\leq{}& E_F+\left\langle\psi_F,e^{\mathcal{K}}\left(H'_{\mathrm{kin}}+\sum_{k\in\mathbb{Z}_*^3}\frac{\hat{V}_kk_F^{-1}}{2\,(2\pi)^3}\Big(2B_k^*B_k+B_kB_{-k}+B_{-k}^*B_k^*\Big)\right)e^{-\mathcal{K}}\psi_F\right\rangle\\ &+\left\langle\psi_F,e^{\mathcal{K}}\mathcal{C}e^{-\mathcal{K}}\psi_F\right\rangle+\left\langle\psi_F,e^{\mathcal{K}}\mathcal{Q}e^{-\mathcal{K}}\psi_F\right\rangle\\ ={}& E_F+E_{\mathrm{corr,bos}}+\langle\psi_F,H'_{\mathrm{kin}}\psi_F\rangle+2\sum_{k\in\mathbb{Z}_*^3}\left\langle\psi_F,Q_1^k\Big(e^{-K_k}h_ke^{-K_k}-h_k\Big)\psi_F\right\rangle\\ &+\sum_{k\in\mathbb{Z}_*^3}\int_0^1\left\langle e^{-(1-t)\mathcal{K}}\psi_F,\Big(\varepsilon_k(\{K_k,B_k(t)\})+2\,\mathrm{Re}\Big(\mathcal{E}_k^1(A_k(t))\Big)+2\,\mathrm{Re}\Big(\mathcal{E}_k^2(B_k(t))\Big)\Big)e^{-(1-t)\mathcal{K}}\psi_F\right\rangle dt\\ &+\left\langle e^{\mathcal{K}}\psi_F,(G+\mathcal{Q}_{\mathrm{LR}})e^{-\mathcal{K}}\psi_F\right\rangle+\langle\psi_F,\mathcal{Q}_{\mathrm{SR}}\psi_F\rangle+\int_0^1\left\langle e^{-t\mathcal{K}}\psi_F,(2\,\mathrm{Re}(\mathcal{G}))e^{-t\mathcal{K}}\psi_F\right\rangle dt\\ ={}& E_F+E_{\mathrm{corr,bos}}+E_{\mathrm{corr,ex}}+\epsilon_1+\epsilon_2+\epsilon_3,\end{aligned}$$

where we also used that

$$H'_{\text{kin}}\psi_F = Q_1^k(A)\psi_F = \mathcal{Q}_{\text{SR}}\psi_F = 0 \tag{8.2.11}$$

and that $\left\langle \psi_F, e^{\mathcal{K}}\mathcal{C}e^{-\mathcal{K}}\psi_F \right\rangle = 0$ by Proposition 8.0.2. The errors ϵ_1, ϵ_2 and ϵ_3 obey

$$\epsilon_1 = \sum_{k\in\mathbb{Z}_*^3}\int_0^1 \left\langle \psi_F, 2\,\mathrm{Re}\Big(\mathcal{E}_k^2(B_k(t))\Big)\psi_F \right\rangle dt - E_{\text{corr,ex}} \leq C\sum_{k\in\mathbb{Z}_*^3}\sqrt{\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2\min\{|k|, k_F\}} \tag{8.2.12}$$

by Proposition 7.0.2,

$$\begin{aligned}\epsilon_2 &= \sum_{k\in\mathbb{Z}_*^3}\int_0^1 \left\langle e^{-(1-t)\mathcal{K}}\psi_F, \Big(\varepsilon_k(\{K_k, B_k(t)\}) + 2\,\mathrm{Re}\Big(\mathcal{E}_k^1(A_k(t))\Big)\Big)e^{-(1-t)\mathcal{K}}\psi_F \right\rangle dt \\ &+ \sum_{k\in\mathbb{Z}_*^3}\int_0^1 \left\langle e^{-(1-t)\mathcal{K}}\psi_F, \Big(2\,\mathrm{Re}\Big(\mathcal{E}_k^2(B_k(t)) - \left\langle \psi_F, \mathcal{E}_k^2(B_k(t))\psi_F\right\rangle\Big)\Big)e^{-(1-t)\mathcal{K}}\psi_F \right\rangle dt \\ &\leq Ck_F^{-1} + C\sqrt{\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2\min\{|k|, k_F\}} \leq C\sqrt{\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2\min\{|k|, k_F\}}\end{aligned} \tag{8.2.13}$$

by Theorem 7.0.1, and

$$\begin{aligned}\epsilon_3 &= \left\langle e^{-\mathcal{K}}\psi_F, (G + \mathcal{Q}_{\text{LR}})e^{-\mathcal{K}}\psi_F \right\rangle + \int_0^1 \left\langle e^{-t\mathcal{K}}\psi_F, (2\,\mathrm{Re}(\mathcal{G}))e^{-t\mathcal{K}}\psi_F \right\rangle dt \\ &\leq C\sqrt{\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2\min\{|k|, k_F\}}\end{aligned} \tag{8.2.14}$$

by Theorem 8.0.1, where we for the last error terms also used that

$$\left\langle e^{-t\mathcal{K}}\psi_F, (\mathcal{N}_E^m + 1)e^{-t\mathcal{K}}\psi_F \right\rangle \leq C, \quad |t| \leq 1,\ m = 1, 2, 3, \tag{8.2.15}$$

as follows by Proposition 8.2.2.

Chapter 9

Extension to Attractive Potentials

We now make the observation that the result of Theorem 1.1.1 generalizes to weakly attractive potentials.

To determine under what conditions we can do this, let us consider where we applied the assumption $\hat{V}_k \geq 0$. This condition did not enter anywhere in Section 2.0.1, so the conclusion of that section, i.e. the representation

$$H_N = E_F + H'_{\text{kin}} + \sum_{k \in \mathbb{Z}^3_*} \frac{\hat{V}_k k_F^{-1}}{2\,(2\pi)^3} \Big(2B_k^* B_k + B_k B_{-k} + B_{-k}^* B_k^* \Big) + \mathcal{C} + \mathcal{Q}, \tag{9.0.1}$$

continues to hold. The first time we applied the condition was in Section 3, when we wrote the bosonizable interaction terms in the form

$$\sum_{k \in \mathbb{Z}^3_*} \frac{\hat{V}_k k_F^{-1}}{2\,(2\pi)^3} \Big(2B_k^* B_k + B_k B_{-k} + B_{-k}^* B_k^* \Big) = \sum_{k \in \mathbb{Z}^3_*} \Big(2\, Q_1^k(P_k) + Q_2^k(P_k) \Big), \tag{9.0.2}$$

since we defined $P_k : \ell^2(L_k) \to \ell^2(L_k)$ to act as $P_k(\cdot) = \langle v_k, \cdot \rangle\, v_k$ for $v_k = \sqrt{\frac{s\hat{V}_k k_F^{-1}}{2\,(2\pi)^3}} \sum_{p \in L_k} e_p$. This definition was made to ensure that

$$\langle e_p, P_k e_q \rangle = \langle e_p, v_k \rangle \langle v_k, e_q \rangle = \frac{s\hat{V}_k k_F^{-1}}{2\,(2\pi)^3}, \quad p, q \in L_k, \tag{9.0.3}$$

but it is clear that this can still be enforced by a slight modification: If we more generally define P_k and v_k by

$$P_k(\cdot) = \operatorname{sgn}(\hat{V}_k) \langle v_k, \cdot \rangle\, v_k, \quad v_k = \sqrt{\frac{s|\hat{V}_k| k_F^{-1}}{2\,(2\pi)^3}} \sum_{p \in L_k} e_p, \tag{9.0.4}$$

then we recover the previous definition for $\hat{V}_k \geq 0$, but now also have that $\langle e_p, P_k e_q \rangle = \frac{s\hat{V}_k k_F^{-1}}{2\,(2\pi)^3}$ even if $\hat{V}_k < 0$.

As the calculations of Section 3 were purely algebraic, we see that the conclusion, i.e. the existence of a unitary transformation $e^{\mathcal{K}}$ such that

$$\begin{aligned} &e^{\mathcal{K}}\left(H'_{\text{kin}}+\sum_{k\in\mathbb{Z}^3_*}\frac{\hat{V}_k k_F^{-1}}{2\left(2\pi\right)^3}\left(2B_k^*B_k+B_kB_{-k}+B_{-k}^*B_k^*\right)\right)e^{-\mathcal{K}} \\ &=\sum_{k\in\mathbb{Z}^3_*}\operatorname{tr}\left(e^{-K_k}h_ke^{-K_k}-h_k-P_k\right)+H'_{\text{kin}}+2\sum_{k\in\mathbb{Z}^3_*}Q_1^k\left(e^{-K_k}h_ke^{-K_k}-h_k\right) \\ &+\sum_{k\in\mathbb{Z}^3_*}\int_0^1 e^{(1-t)\mathcal{K}}\left(\varepsilon_k(\{K_k,B_k(t)\})+2\operatorname{Re}\left(\mathcal{E}_k^1(A_k(t))\right)+2\operatorname{Re}\left(\mathcal{E}_k^2(B_k(t))\right)\right)e^{-(1-t)\mathcal{K}}dt, \end{aligned} \tag{9.0.5}$$

continues to hold (keeping the new definition of P_k in mind), *provided* the diagonalizing kernels

$$K_k=-\frac{1}{2}\log\left(h_k^{-\frac{1}{2}}\left(h_k^{\frac{1}{2}}(h_k+2P_k)h_k^{\frac{1}{2}}\right)^{\frac{1}{2}}h_k^{-\frac{1}{2}}\right) \tag{9.0.6}$$

are still well-defined when $\hat{V}_k<0$.

This is the condition that $h_k+2P_k=h_k-2P_{v_k}>0$. By the Sherman-Morrison formula (Lemma 6.1.1 - as well as motonotony of $t\mapsto h_k+tP_k$) this is the case if and only if

$$1-2\left\langle v_k,h_k^{-1}v_k\right\rangle>0 \tag{9.0.7}$$

which can be expanded and rearranged to

$$\hat{V}_k>-\frac{\left(2\pi\right)^3}{sk_F^{-1}\sum_{p\in L_k}\lambda_{k,p}^{-1}},\quad k\in\mathbb{Z}^3_*. \tag{9.0.8}$$

In appendix section B we prove the following asymptotic behaviour of the Riemann sum $\sum_{p\in L_k}\lambda_{k,p}^{-1}$:

Proposition 9.0.1. *For any $\gamma\in\left(0,\frac{1}{11}\right)$ and $k\in\overline{B}(0,k_F^{\gamma})$ it holds that*

$$\sum_{p\in L_k}\lambda_{k,p}^{-1}=2\pi k_F+O\left(\log(k_F)^{\frac{5}{3}}k_F^{\frac{1}{3}(2+11\gamma)}\right),\quad k_F\to\infty.$$

The condition of equation (9.0.8) thus asymptotically amounts to

$$\hat{V}_k>-\frac{4\pi^2}{s},\quad k\in\mathbb{Z}^3_*, \tag{9.0.9}$$

but as in the statement of Theorem 1.1.2 we will for the purposes of analysis make the slightly stronger assumption that

$$\hat{V}_k\geq-(1-\epsilon)\frac{4\pi^2}{s},\quad k\in\mathbb{Z}^3_*, \tag{9.0.10}$$

for some $\epsilon>0$. With this we can uniformly bound $1-2\left\langle v_k,h_k^{-1}v_k\right\rangle$ away from 0:

Lemma 9.0.2. *Let* $\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2<\infty$ *and* $\hat{V}_k\geq -(1-\epsilon)\frac{4\pi^2}{s}$ *for all* $k\in\mathbb{Z}^3_*$*. Then*

$$\inf_{\left\{k\in\mathbb{Z}^3_*\mid \hat{V}_k<0\right\}}\left(1-2\left\langle v_k,h_k^{-1}v_k\right\rangle\right)\geq C,\quad k_F\to\infty,$$

for a constant $C>0$ *depending only on* ϵ*.*

Proof: Expanding the definitions and applying Proposition 9.0.1, we have for all $k\in\overline{B}(0,k_F^{\frac{1}{20}})$ (say) with $\hat{V}_k<0$ that

$$\begin{aligned}1-2\left\langle v_k,h_k^{-1}v_k\right\rangle &= 1-\frac{s|\hat{V}_k|k_F^{-1}}{(2\pi)^3}\sum_{p\in L_k}\lambda_{k,p}^{-1}\geq 1-(1-\epsilon)\frac{4\pi^2}{s}\frac{sk_F^{-1}}{(2\pi)^3}\sum_{p\in L_k}\lambda_{k,p}^{-1} \\ &= \epsilon+(1-\epsilon)\frac{k_F^{-1}}{2\pi}\left(2\pi k_F-\sum_{p\in L_k}\lambda_{k,p}^{-1}\right)\geq \epsilon - C\log(k_F)^{\frac{5}{3}}k_F^{-\frac{1}{3}\left(1-\frac{11}{20}\right)} \\ &\geq C'\end{aligned}\tag{9.0.11}$$

as $k_F\to\infty$ for some $C'>0$ depending only on ϵ. If instead $k\in\mathbb{Z}^3_*\backslash\overline{B}(0,k_F^{\frac{1}{20}})$ we may note that by the general bound $\sum_{p\in L_k}\lambda_{k,p}^{-1}\leq Ck_F$, we can always estimate

$$1-2\left\langle v_k,h_k^{-1}v_k\right\rangle\geq 1-Cs|\hat{V}_k|,\tag{9.0.12}$$

so noting that

$$\sup_{k\in\mathbb{Z}^3_*\backslash\overline{B}(0,k_F^{\frac{1}{20}})}|\hat{V}_k|\leq\sqrt{\sum_{k\in\mathbb{Z}^3_*\backslash\overline{B}(0,k_F^{\frac{1}{20}})}\hat{V}_k^2}\to 0,\quad k_F\to\infty,\tag{9.0.13}$$

since $\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2<\infty$ we see that we can for k_F sufficiently large assume that

$$\sup_{k\in\mathbb{Z}^3_*\backslash\overline{B}(0,k_F^{\frac{1}{20}})}\left(1-2\left\langle v_k,h_k^{-1}v_k\right\rangle\right)\geq\frac{1}{2}\tag{9.0.14}$$

(say), so either way the claim holds.

□

We remark that a similar argument shows that our condition on $\hat{V}_k$ is nearly optimal, in the sense that if for some $k\in\mathbb{Z}^3_*$ it holds that $\hat{V}_k<-\frac{4\pi^2}{s}$, then the asymptotic result of Proposition 9.0.1 in fact implies that

$$1-2\left\langle v_k,h_k^{-1}v_k\right\rangle<0\tag{9.0.15}$$

for all sufficiently large k_F, in which case the corresponding term of $E_{\mathrm{corr,bos}}$ is not even well-defined as the integrand involves

$$\log\left(1+\frac{s\hat{V}_kk_F^{-1}}{(2\pi)^3}\sum_{p\in L_k}\frac{\lambda_{k,p}}{\lambda_{k,p}^2+t^2}\right)=\log\left(1-2\left\langle v_k,h_k\left(h_k^2+t^2\right)^{-1}v_k\right\rangle\right).\tag{9.0.16}$$

The condition $\hat{V}_k \geq -(1-\epsilon)\frac{4\pi^2}{s}$ thus ensures that our diagonalization procedure (and $E_{\text{corr,bos}}$) remains well-defined, but it is not immediately clear how the one-body estimates of Section 6 are to be modified for the attractive case.

This is the main information that is needed for the generalization to attractive potentials, but it turns out that Theorem 6.0.1 continues to hold almost exactly as stated before, the only difference being an ϵ-dependence and the substitution $\hat{V}_k \to |\hat{V}_k|$ in the error terms:

Proposition 9.0.3. *It holds for any $k \in \mathbb{Z}^3_*$ that*

$$\operatorname{tr}\left(e^{-K_k} h_k e^{-K_k} - h_k - P_k\right) = \frac{1}{\pi}\int_0^\infty F\left(\frac{s\hat{V}_k k_F^{-1}}{(2\pi)^3}\sum_{p\in L_k}\frac{\lambda_{k,p}}{\lambda_{k,p}^2+t^2}\right)dt,$$

where $F(x) = \log(1+x) - x$. Furthermore, as $k_F \to \infty$,

$$\|K_k\|_{\text{HS}} \leq C|\hat{V}_k| \min\left\{1, k_F^2\,|k|^{-2}\right\}$$

and for all $p, q \in L_k$ and $t \in [0,1]$

$$\begin{aligned}
|\langle e_p, K_k e_q\rangle| &\leq C\frac{|\hat{V}_k| k_F^{-1}}{\lambda_{k,p}+\lambda_{k,q}}\\
\left|\langle e_p, (-K_k) e_q\rangle - \frac{s\hat{V}_k k_F^{-1}}{2\,(2\pi)^3}\frac{1}{\lambda_{k,p}+\lambda_{k,q}}\right| &\leq C\frac{\hat{V}_k^2 k_F^{-1}}{\lambda_{k,p}+\lambda_{k,q}}\\
|\langle e_p, A_k(t) e_q\rangle|\,, |\langle e_p, B_k(t) e_q\rangle| &\leq C\left(1+\hat{V}_k^2\right)|\hat{V}_k| k_F^{-1}\\
\left|\left\langle e_p, \left(\int_0^1 B_k(t)\,dt\right)e_q\right\rangle - \frac{s\hat{V}_k k_F^{-1}}{4\,(2\pi)^3}\right| &\leq C\left(1+|\hat{V}_k|\right)\hat{V}_k^2 k_F^{-1}\\
|\langle e_p, \{K_k, B_k(t)\}\, e_q\rangle| &\leq C\left(1+\hat{V}_k^2\right)\hat{V}_k^2 k_F^{-1}
\end{aligned}$$

for a constant $C > 0$ depending only on s and ϵ.

We momentarily postpone the proof to subsection 9.1 below.

With these estimates we are essentially done, since the computations of the Sections 7 and 8 only relied on these, as well as the triangle and Cauchy-Schwarz inequalities. Whenever the triangle inequality was applied, the only difference that is required for attractive potentials is that $\hat{V}_k$ is substituted with $|\hat{V}_k|$, but since we generally apply the Cauchy-Schwarz inequality to estimate in terms of $\hat{V}_k^2$ this makes no difference in the end.

The only modification to Theorem 1.1.1 that is necessary to generalize to the condition $\hat{V}_k > -(1-\epsilon)\frac{4\pi^2}{s}$ is therefore that the constant in the error term is ϵ-dependent, which is Theorem 1.1.2.

9.1 One-Body Estimates for Attractive Modes

To prove Proposition 9.0.3 we return to the general setting of Section 6, i.e. we consider an n-dimensional Hilbert space $(V, \langle\cdot,\cdot\rangle)$, a positive self-adjoint operator $h : V \to V$ with eigenbasis $(x_i)_{i=1}^n$ and a vector $v \in V$ such that $\langle x_i, v\rangle \geq 0$, $1 \leq i \leq n$.

The calculations of this subsection are very reminiscent of those of Section 6, and for that reason we will adopt a brisk pacing, mainly pointing out the necessary modifications - these will mainly be various sign reversals.

We let $K : V \to V$ be given by

$$K = -\frac{1}{2}\log\left(h^{-\frac{1}{2}}\left(h^{\frac{1}{2}}(h - 2P_v)h^{\frac{1}{2}}\right)^{\frac{1}{2}}h^{-\frac{1}{2}}\right) = -\frac{1}{2}\log\left(h^{-\frac{1}{2}}\left(h^2 - 2P_{h^{\frac{1}{2}}v}\right)^{\frac{1}{2}}h^{-\frac{1}{2}}\right); \quad (9.1.1)$$

we assume that $1 - 2\langle v, h^{-1}v\rangle > 0$ so that K is well-defined. In this case we have that e^{-2K} and e^{2K} are given by

$$\begin{aligned} e^{-2K} &= h^{-\frac{1}{2}}\left(h^2 - 2P_{h^{\frac{1}{2}}v}\right)^{\frac{1}{2}}h^{-\frac{1}{2}} \\ e^{2K} &= h^{\frac{1}{2}}\left(h^{-2} + \frac{2}{1 - 2\langle v, h^{-1}v\rangle}P_{h^{-\frac{3}{2}}v}\right)^{\frac{1}{2}}h^{\frac{1}{2}} \end{aligned} \quad (9.1.2)$$

and it follows from Proposition 6.1.2 that $\mathrm{tr}\left(e^{-K}he^{-K} - h + P_v\right)$ is given by

$$\mathrm{tr}\left(e^{-K}he^{-K} - h + P_v\right) = \frac{1}{\pi}\int_0^\infty F\left(-2\left\langle v, h\left(h^2 + t^2\right)^{-1}v\right\rangle\right)dt, \quad F(x) = \log(1 + x) - x. \quad (9.1.3)$$

The operators e^{-2K} and e^{2K} obey the following matrix element estimates:

Proposition 9.1.1. *For all $1 \leq i, j \leq n$ it holds that*

$$2\frac{\langle x_i, v\rangle\langle v, x_j\rangle}{\lambda_i + \lambda_j} \leq \left\langle x_i, \left(1 - e^{-2K}\right)x_j\right\rangle, \left\langle x_i, \left(e^{2K} - 1\right)x_j\right\rangle \leq \frac{2}{1 - 2\langle v, h^{-1}v\rangle}\frac{\langle x_i, v\rangle\langle v, x_j\rangle}{\lambda_i + \lambda_j}.$$

Proof: By Proposition 6.1.2 we have that

$$\begin{aligned} 1 - e^{-2K} &= 1 - h^{-\frac{1}{2}}\left(h - \frac{4}{\pi}\int_0^\infty \frac{t^2}{1 - 2\left\langle h^{\frac{1}{2}}v, (h^2 + t^2)^{-1}h^{\frac{1}{2}}v\right\rangle}P_{(h^2+t^2)^{-1}h^{\frac{1}{2}}v}dt\right)h^{-\frac{1}{2}} \quad (9.1.4) \\ &= \frac{4}{\pi}\int_0^\infty \frac{t^2}{1 - 2\left\langle v, h(h^2 + t^2)^{-1}v\right\rangle}P_{(h^2+t^2)^{-1}v}dt \end{aligned}$$

and now it holds that

$$1 \leq \frac{1}{1 - 2\left\langle v, h(h^2 + t^2)^{-1}v\right\rangle} \leq \frac{1}{1 - 2\langle v, h^{-1}v\rangle}, \quad t \geq 0, \quad (9.1.5)$$

whence the element estimate follows as in Proposition 6.1.4. Similarly, for e^{2K},

$$
\begin{aligned}
e^{2K} &= h^{\frac{1}{2}}\left(h^{-1}+\frac{4}{\pi}\int_0^\infty \frac{t^2}{1-2\left\langle v,h^{-1}v\right\rangle+2\left\langle h^{-\frac{3}{2}}v,(h^{-2}+t^2)^{-1}h^{-\frac{3}{2}}v\right\rangle}P_{(h^{-2}+t^2)^{-1}h^{-\frac{3}{2}}v}dt\right)h^{\frac{1}{2}}\\
&=1+\frac{4}{\pi}\int_0^\infty \frac{t^2}{1-2\left\langle v,h^{-1}(h^{-2}+t^2)^{-1}v\right\rangle t^2}P_{(h^{-2}+t^2)^{-1}h^{-1}v}dt
\end{aligned}
\tag{9.1.6}
$$

so the claim follows as

$$
1\leq \frac{1}{1-2\left\langle v,h^{-1}(h^{-2}+t^2)^{-1}v\right\rangle t^2}\leq \frac{1}{1-2\left\langle v,h^{-1}v\right\rangle},\quad t\geq 0. \tag{9.1.7}
$$

□

As in Corollary 6.1.5 we can then conclude the bounds

$$
\begin{aligned}
\left\langle x_i,\sinh(2K)x_j\right\rangle &\leq \frac{2}{1-2\left\langle v,h^{-1}v\right\rangle}\frac{\left\langle x_i,v\right\rangle\left\langle v,x_j\right\rangle}{\lambda_i+\lambda_j} \\
\left\langle x_i,(\cosh(2K)-1)x_j\right\rangle &\leq \frac{2\left\langle v,h^{-1}v\right\rangle}{1-2\left\langle v,h^{-1}v\right\rangle}\frac{\left\langle x_i,v\right\rangle\left\langle v,x_j\right\rangle}{\lambda_i+\lambda_j},
\end{aligned}
\tag{9.1.8}
$$

and we note that $\cosh(2K)$ also obeys

$$
\begin{aligned}
\left\langle x_i,(\cosh(2K)-1)x_j\right\rangle &= \frac{1}{2}\left(\left\langle x_i,\left(e^{2K}-1\right)x_j\right\rangle-\left\langle x_i,\left(1-e^{-2K}\right)x_j\right\rangle\right)\\
&\leq \frac{1}{2}\left\langle x_i,\left(e^{2K}-1\right)x_j\right\rangle\leq \frac{1}{1-2\left\langle v,h^{-1}v\right\rangle}\frac{\left\langle x_i,v\right\rangle\left\langle v,x_j\right\rangle}{\lambda_i+\lambda_j}
\end{aligned}
\tag{9.1.9}
$$

so in fact

$$
\left\langle x_i,(\cosh(2K)-1)x_j\right\rangle \leq \frac{\min\left\{1,2\left\langle v,h^{-1}v\right\rangle\right\}}{1-2\left\langle v,h^{-1}v\right\rangle}\frac{\left\langle x_i,v\right\rangle\left\langle v,x_j\right\rangle}{\lambda_i+\lambda_j}. \tag{9.1.10}
$$

By the same arguments used in Proposition 6.1.6, it follows from Proposition 9.1.1 that K obeys the following elementwise bounds:

Proposition 9.1.2. *For any* $1\leq i,j\leq n$ *it holds that*

$$
\frac{\left\langle x_i,v\right\rangle\left\langle v,x_j\right\rangle}{\lambda_i+\lambda_j}\leq \left\langle x_i,Kx_j\right\rangle\leq \frac{1}{1-2\left\langle v,h^{-1}v\right\rangle}\frac{\left\langle x_i,v\right\rangle\left\langle v,x_j\right\rangle}{\lambda_i+\lambda_j}.
$$

As this in particular implies that $\left\langle x_i,Kx_j\right\rangle\geq 0$ for all $1\leq i,j\leq n$, it follows that the functions

$$
t\mapsto \left\langle x_i,\left(e^{tK}-1\right)x_j\right\rangle,\ \left\langle x_i,\sinh(tK)x_j\right\rangle,\ \left\langle x_i,(\sinh(tK)-tK)x_j\right\rangle,\ \left\langle x_i,(\cosh(tK)-1)x_j\right\rangle, \tag{9.1.11}
$$

are non-negative and convex, whence we obtain the following analogue of Proposition 6.1.7:

Proposition 9.1.3. *For all* $1 \leq i, j \leq n$ *and* $t \in [0,1]$ *it holds that*

$$\frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i + \lambda_j} t \leq \langle x_i, \sinh(tK) x_j\rangle \leq \frac{1}{1 - 2\langle v, h^{-1}v\rangle} \frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i + \lambda_j} t$$
$$0 \leq \langle x_i, (\cosh(tK) - 1) x_j\rangle \leq \frac{\min\{1, \langle v, h^{-1}v\rangle\}}{1 - 2\langle v, h^{-1}v\rangle} \frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i + \lambda_j}$$
$$0 \leq \left\langle x_i, \left(e^{tK} - 1\right) x_j\right\rangle \leq \frac{1}{1 - 2\langle v, h^{-1}v\rangle} \frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i + \lambda_j}.$$

Estimation of $A(t)$ and $B(t)$

We thus come to the estimation of $A(t)$ and $B(t)$, which are now given by

$$\begin{aligned} A(t) &= \frac{1}{2}\left(e^{tK}(h - 2P_v)e^{tK} + e^{-tK} h e^{-tK}\right) - h \\ B(t) &= \frac{1}{2}\left(e^{tK}(h - 2P_v)e^{tK} - e^{-tK} h e^{-tK}\right). \end{aligned} \tag{9.1.12}$$

As in Section 6 we decompose these as

$$\begin{aligned} A(t) &= A_h(t) - e^{tK} P_v e^{tK} \\ B(t) &= -(1-t)P_v + B_h(t) - e^{tK} P_v e^{tK} + P_v \end{aligned} \tag{9.1.13}$$

for

$$\begin{aligned} A_h(t) &= \cosh(tK)\, h \cosh(tK) + \sinh(tK)\, h \sinh(tK) - h \\ &= \{h, C_K(t)\} + S_K(t)\, h\, S_K(t) + C_K(t)\, h\, C_K(t) \end{aligned} \tag{9.1.14}$$

and

$$\begin{aligned} B_h(t) &= \sinh(tK)\, h \cosh(tK) + \cosh(tK)\, h \sinh(tK) - tP_v \\ &= \{h, S_K(t)\} - tP_v + S_K(t)\, h\, C_K(t) + C_K(t)\, h\, S_K(t), \end{aligned} \tag{9.1.15}$$

where $S_K(t)$ and $C_K(t)$ are now given by

$$C_K(t) = \cosh(tK) - 1 \quad \text{and} \quad S_K(t) = \sinh(tK). \tag{9.1.16}$$

Since the only effective difference between the statement of Proposition 9.1.3 and that of Proposition 6.1.7 is a factor of $(1 - 2\langle v, h^{-1}v\rangle)^{-1}$, the bound of Proposition 6.2.1 generalizes as (using also the trivial estimate $1 \leq (1 - 2\langle v, h^{-1}v\rangle)^{-1}$)

$$\left|\left\langle x_i, \left(e^{tK} P_v e^{tK} - P_v\right) x_j\right\rangle\right| \leq \frac{(2 + \langle v, h^{-1}v\rangle)\langle v, h^{-1}v\rangle}{(1 - 2\langle v, h^{-1}v\rangle)^2} \langle x_i, v\rangle \langle v, x_j\rangle. \tag{9.1.17}$$

Consequently also

$$\left|\left\langle x_i, e^{tK} P_v e^{tK} x_j\right\rangle\right| \leq \left(\frac{1 + \langle v, h^{-1}v\rangle}{1 - 2\langle v, h^{-1}v\rangle}\right)^2 \langle x_i, v\rangle \langle v, x_j\rangle \tag{9.1.18}$$

and by the same argument

$$|\langle x_i, \{h, C_K(t)\} x_j\rangle| \leq \frac{\langle v, h^{-1}v\rangle}{1-2\langle v, h^{-1}v\rangle} \langle x_i, v\rangle \langle v, x_j\rangle \tag{9.1.19}$$

and

$$|\langle x_i, S_K(t)\, h\, S_K(t) x_j\rangle| \leq \frac{\langle v, h^{-1}v\rangle}{(1-2\langle v, h^{-1}v\rangle)^2} \langle x_i, v\rangle \langle v, x_j\rangle\,, \tag{9.1.20}$$

the latter extending also to the operators $C_K(t)\, h\, C_K(t)$, $S_K(t)\, h\, C_K(t)$ and $C_K(t)\, h\, S_K(t)$.

Proposition 9.1.3 finally implies that

$$\begin{aligned} |\langle x_i, (\{h, S_K(t)\} - tP_v) x_j\rangle| &= \langle x_i, \{h, S_K(t)\} x_j\rangle - \langle x_i, P_v x_j\rangle\, t \\ &\leq \left(\frac{1}{1-2\langle v, h^{-1}v\rangle} - 1\right) \langle x_i, v\rangle \langle v, x_j\rangle\, t \\ &= \frac{2\langle v, h^{-1}v\rangle}{1-2\langle v, h^{-1}v\rangle} \langle x_i, v\rangle \langle v, x_j\rangle\, t, \end{aligned} \tag{9.1.21}$$

so combining all the estimates we conclude the following analogue of the Propositions 6.2.2 and 6.2.3:

Proposition 9.1.4. *For all* $1 \leq i, j \leq n$ *it holds that*

$$\begin{aligned} |\langle x_i, A_h(t) x_j\rangle|\,, |\langle x_i, B_h(t) x_j\rangle| &\leq \frac{4\langle v, h^{-1}v\rangle}{(1-2\langle v, h^{-1}v\rangle)^2} \langle x_i, v\rangle \langle v, x_j\rangle \\ |\langle x_i, A(t) x_j\rangle|\,, |\langle x_i, B(t) x_j\rangle| &\leq 3\left(\frac{1+\langle v, h^{-1}v\rangle}{1-2\langle v, h^{-1}v\rangle}\right)^2 \langle x_i, v\rangle \langle v, x_j\rangle\,. \end{aligned}$$

These estimates again only differ from those of Section 6 by a factor of $(1-2\langle v, h^{-1}v\rangle)^{-2}$, so the statements of the Propositions 6.2.4 and 6.2.5 likewise generalize as

$$|\langle x_i, \{K, B(t)\} x_j\rangle| \leq \frac{(6+\langle v, h^{-1}v\rangle)\langle v, h^{-1}v\rangle}{(1-2\langle v, h^{-1}v\rangle)^3} \langle x_i, v\rangle \langle v, x_j\rangle \tag{9.1.22}$$

and

$$\left|\left\langle x_i, \left(\int_0^1 B(t)\, dt\right) x_j\right\rangle + \frac{1}{2}\langle x_i, v\rangle \langle v, x_j\rangle\right| \leq \frac{(6+\langle v, h^{-1}v\rangle)\langle v, h^{-1}v\rangle}{(1-2\langle v, h^{-1}v\rangle)^2} \langle x_i, v\rangle \langle v, x_j\rangle\,, \tag{9.1.23}$$

respectively.

Conclusion of Proposition 9.0.3

We have now obtained estimates similar to those of Section 6, with only two differences: First, the left-hand sides differ by a sign whenever v (or rather P_v) appears. This only serves to negative the absolute value of $|\hat{V}_k|$ in our new definition of v_k and P_k, however, which is the reason that $|\hat{V}_k|$ only appears on the right-hand sides of Proposition 9.0.3.

The second difference (apart from the absolute value) is various factors of $(1 - 2\langle v, h^{-1}v\rangle)^{-1}$. By Lemma 9.0.2 we can however estimate

$$1 - 2\left\langle v_k, h_k^{-1} v_k \right\rangle \geq C \tag{9.1.24}$$

uniformly in k for a $C > 0$ depending only on ϵ, whence also $\left(1 - 2\left\langle v_k, h_k^{-1} v_k \right\rangle\right)^{-1} \leq C'$ depending only on ϵ. Absorbing this dependence into the overall constant yields Proposition 9.0.3.

Chapter 10

Overview of the Operator Result

In this section we review the main points which lead to the conclusion of Theorem 1.1.3.

We first present a general outline of the approach, and then consider the main points in greater detail in the rest of the section. As in [11] we will focus on the case $s = 1$ for simplicity, and assume as in the theorem that $\sum_{k\in\mathbb{Z}^3_*} \hat{V}_k \, |k| < \infty$.

First we should note that the statement in [11] is slightly more general than that of Theorem 1.1.3, in that with respect to the decomposition

$$\mathcal{U} H_N \mathcal{U}^* = E_F + E_{\mathrm{corr,bos}} + H_{\mathrm{eff}} + \mathcal{E} \tag{10.0.1}$$

the error operator $\mathcal{E}$ is shown to generally obey

$$\pm\mathcal{E} \leq C k_F^{-\frac{1}{94}+\varepsilon} \Big(k_F + H'_{\mathrm{kin}} + k_F^{-1} \mathcal{N}_E H'_{\mathrm{kin}} \Big) \tag{10.0.2}$$

with respect to $D(H'_{\mathrm{kin}})$, and not just the low-lying eigenstates. The particular statement of Theorem 1.1.3 then follows by *a priori* bounds on such states: Define a normalized state $\Psi \in D(H'_{\mathrm{kin}})$ to be low-lying (with respect to H_N) if

$$\langle \Psi, H_N \Psi \rangle \leq E_F + \kappa k_F \tag{10.0.3}$$

for some fixed $\kappa > 0$. Then the following holds:

Proposition 10.0.1. *For any low-lying eigenstate* $\Psi \in D(H'_{\mathrm{kin}})$ *it holds that*

$$\langle \Psi, \mathcal{N}_E \Psi \rangle \leq \langle \Psi, H'_{\mathrm{kin}} \Psi \rangle \leq C k_F, \quad \langle \Psi, \mathcal{N}_E H'_{\mathrm{kin}} \Psi \rangle \leq C k_F^2,$$

for a constant $C > 0$ *depending only on* $\sum_{k\in\mathbb{Z}^3_*} \hat{V}_k \, |k|$ *and* κ.

Let us comment on the quality of these estimates: That $\langle \Psi, H'_{\mathrm{kin}} \Psi \rangle \leq O(k_F)$ is presumably optimal, since H'_{kin} enters directly in $H_N - E_F$ and we already know that $\inf(\sigma(H_N)) \sim E_F + O(k_F)$. The bound $\langle \Psi, \mathcal{N}_E \Psi \rangle \leq O(k_F)$ is likely far from optimal, however, since the trial state we applied for the upper bound had only $\langle \Psi, \mathcal{N}_E \Psi \rangle \leq O(1)$. (It can also be shown that for this state, $\langle \Psi, \mathcal{N}_E H'_{\mathrm{kin}} \Psi \rangle \leq O(k_F)$.)

This point is important for the estimation of error terms later on, since it means that in order to bound these well, they must be bounded in terms of H'_{kin} to the greatest extent possible, rather than just $\mathcal{N}_E$ and its powers (as we have done for the upper bound).

Decomposition of the Hamiltonian

With these *a priori* bounds at our disposal we can turn to the Hamiltonian proper. Here we must at the outset make a slight modification compared to the decomposition of Theorem 2.0.1: We now write

$$H_N' = H_{\text{kin}}' + \sum_{k\in\overline{B}\left(0,k_F^{\gamma}\right)\cap\mathbb{Z}_*^3} \left(2\,Q_1^k(P_k) + Q_2^k(P_k)\right) + \text{ND} + \mathcal{C} + \mathcal{Q} \tag{10.0.4}$$

for some $\gamma > 0$ to be optimized at the end, where the non-diagonalized terms ND are the tail of the interaction terms,

$$\text{ND} = \frac{k_F^{-1}}{2\,(2\pi)^3} \sum_{k\in\mathbb{Z}_*^3\backslash\overline{B}\left(0,k_F^{\gamma}\right)} \hat{V}_k\left(2B_k^* B_k + B_k B_{-k} + B_{-k}^* B_k^*\right). \tag{10.0.5}$$

We do this as we will later on need to estimate Riemann sums which are more singular than $\sum_{p\in L_k}\lambda_{k,p}^{-1}$, and these we can only establish for $|k|$ sufficiently small compared to k_F. This necessitates a cut-off in the transformation, hence in the number of terms we can diagonalize for a given k_F. As $\overline{B}(0,k_F^{\gamma})\cap\mathbb{Z}_*^3$ exhausts $\mathbb{Z}_*^3$ when $k_F\to\infty$, all terms are "eventually" diagonalized, but the tail terms of ND must be treated as errors rather than included in the transformation.

The non-bosonizable terms $\mathcal{C}$ and $\mathcal{Q}$ are likewise bounded prior to the transformation. This is a difficult task since, as mentioned above, these are to be bounded in terms of the kinetic operator. Nonetheless we obtain the following:

Proposition 10.0.2. *It holds that*

$$\begin{aligned}
\pm\text{ND} &\leq C k_F^{-\frac{\gamma}{2}}(k_F + H_{\text{kin}}') \\
\pm(\mathcal{C}+\mathcal{Q}) &\leq C\log(k_F)^{\frac{1}{9}} k_F^{-\frac{1}{18}}\left(H_{\text{kin}}' + k_F^{-1}\mathcal{N}_E H_{\text{kin}}'\right)
\end{aligned}$$

as $k_F\to\infty$ for a constant $C>0$ depending only on $\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k\,|k|$.

We remark that in the end it will be ND which is the dominant error term of $\mathcal{E}$ - the Riemann sum estimates impose the condition $\gamma < \frac{1}{47}$, whence $\pm\text{ND} \leq C k_F^{-\frac{1}{94}+\varepsilon}(k_F + H_{\text{kin}}')$. This is not surprising since the non-diagonalizable terms *do* contribute to the correlation energy, we simply lack singular Riemann sum estimates which are sufficiently uniform in k to meaningfully extract this.

Analysis of Bosonizable Terms

With these bounds the remaining analysis reduces entirely to the (now cut-off) bosonizable terms. For these, Theorem 4.0.1 continues to hold in the form

$$e^{\mathcal{K}}\left(H_{\text{kin}}' + \sum_{k\in\overline{B}\left(0,k_F^{\gamma}\right)\cap\mathbb{Z}_*^3} \frac{\hat{V}_k k_F^{-1}}{2\,(2\pi)^3}\left(2B_k^* B_k + B_k B_{-k} + B_{-k}^* B_k^*\right)\right)e^{-\mathcal{K}}$$

$$= \sum_{k\in\overline{B}\left(0,k_F^\gamma\right)\cap\mathbb{Z}^3_*} \mathrm{tr}(E_k - h_k - P_k) + H'_{\mathrm{kin}} + 2\sum_{k\in\overline{B}\left(0,k_F^\gamma\right)\cap\mathbb{Z}^3_*} Q_1^k(E_k - h_k) \tag{10.0.6}$$

$$+ \sum_{k\in\overline{B}\left(0,k_F^\gamma\right)\cap\mathbb{Z}^3_*} \int_0^1 e^{(1-t)\mathcal{K}}\Big(\varepsilon_k(\{K_k, B_k(t)\}) + 2\,\mathrm{Re}\Big(\mathcal{E}_k^1(A_k(t))\Big) + 2\,\mathrm{Re}\Big(\mathcal{E}_k^2(B_k(t))\Big)\Big)e^{-(1-t)\mathcal{K}}dt$$

where $E_k = e^{-K_k}h_k e^{-K_k}$. The cut-off means that we only recover part of $E_{\mathrm{corr,bos}}$, but the remainder is of lower order as $k_F \to \infty$. Additionally, the following kinetic estimates of the exchange terms, and Gronwall estimates for the kinetic operators, can be derived:

Proposition 10.0.3. *It holds that*

$$\sum_{k\in\overline{B}\left(0,k_F^\gamma\right)\cap\mathbb{Z}^3_*} \mathrm{tr}(E_k - h_k - P_k) = E_{\mathrm{corr,bos}} + O\Big(k_F^{1-\gamma}\Big)$$

$$\pm\,\mathrm{Exchange\,Terms} \leq C\log(k_F)^{\frac{2}{3}} k_F^{\frac{8}{3}\gamma - \frac{1}{3}}\Big(k_F + H'_{\mathrm{kin}} + k_F^{-1}\mathcal{N}_E H'_{\mathrm{kin}}\Big)$$

and for any $t \in [-1,1]$

$$\begin{aligned} e^{t\mathcal{K}}H'_{\mathrm{kin}}e^{-t\mathcal{K}} &\leq C(H'_{\mathrm{kin}} + k_F) \\ e^{t\mathcal{K}}\mathcal{N}_E H'_{\mathrm{kin}}e^{-t\mathcal{K}} &\leq C(\mathcal{N}_E H'_{\mathrm{kin}} + k_F H'_{\mathrm{kin}} + k_F) \end{aligned}$$

for a constant $C > 0$ *depending only on* $\sum_{k\in\mathbb{Z}^3_*} \hat{V}_k\,|k|$.

This leaves only $H'_{\mathrm{kin}} + 2\sum_{k\in\overline{B}\left(0,k_F^\gamma\right)\cap\mathbb{Z}^3_*} Q_1^k(E_k - h_k)$. Now, if we were only considering a lower bound, it would be tempting to think that we are done, since $E_k = e^{-K_k}h_k e^{-K_k} = e^{-K_k}h_k^{\frac{1}{2}}h_k^{\frac{1}{2}}e^{-K_k}$ is isospectral to

$$\tilde{E}_k = h_k^{\frac{1}{2}}e^{-K_k}e^{-K_k}h_k^{\frac{1}{2}} = h_k^{\frac{1}{2}}e^{-2K_k}h_k^{\frac{1}{2}} = \left(h_k^2 + 2P_{h_k^{\frac{1}{2}}v_k}\right)^{\frac{1}{2}} \tag{10.0.7}$$

and $\tilde{E}_k \geq h_k$, so one might suspect that $E_k \geq h_k$ which would imply that $Q_1^k(E_k - h_k) \geq 0$. This is not so, however - $E_k - h_k$ is not non-negative.

To get around this issue we consider a second transformation $e^{\mathcal{J}} : \mathcal{H}_N \to \mathcal{H}_N$ for $\mathcal{J}$ of the form

$$\mathcal{J} = \sum_{k\in\overline{B}\left(0,k_F^\gamma\right)\cap\mathbb{Z}^3_*}\sum_{p,q\in L_k} \langle e_p, J_k e_q\rangle\, b_{k,p}^* b_{k,q} = \sum_{k\in\overline{B}\left(0,k_F^\gamma\right)\cap\mathbb{Z}^3_*}\sum_{p\in L_k} b_k^*(J_k e_p)b_{k,p} \tag{10.0.8}$$

where we take $J_k : \ell^2(L_k) \to \ell^2(L_k)$, $k \in \mathbb{Z}^3_*$, to be a collection of *skew*-symmetric operators. It follows that $\mathcal{J}$ is also skew-symmetric, as

$$\mathcal{J}^* = \sum_{k\in\overline{B}\left(0,k_F^\gamma\right)\cap\mathbb{Z}^3_*}\sum_{p\in L_k} b_k^*(e_p)b_k(J_k e_p) = \sum_{k\in\overline{B}\left(0,k_F^\gamma\right)\cap\mathbb{Z}^3_*}\sum_{p\in L_k} b_k^*(J_k^* e_p)b_k(e_p) = -\mathcal{J}, \tag{10.0.9}$$

so $e^{\mathcal{J}}$ is a unitary transformation.

In the exact bosonic case, a transformation of such a form obeys $e^{\mathcal{J}}d\Gamma(A)e^{-\mathcal{J}} = d\Gamma\left(e^{J}Ae^{-J}\right)$. We thus take the operators J_k to be the principal logarithms of the operators U_k, given by

$$U_k = \left(h_k^{\frac{1}{2}}e^{-2K_k}h_k^{\frac{1}{2}}\right)^{\frac{1}{2}}h_k^{-\frac{1}{2}}e^{K_k}, \tag{10.0.10}$$

which precisely act by taking E_k to $\tilde{E}_k$:

$$\begin{aligned} U_kE_kU_k^* &= \left(h_k^{\frac{1}{2}}e^{-2K_k}h_k^{\frac{1}{2}}\right)^{\frac{1}{2}}h_k^{-\frac{1}{2}}e^{K_k}e^{-K_k}h_ke^{-K_k}e^{K_k}h_k^{-\frac{1}{2}}\left(h_k^{\frac{1}{2}}e^{-2K_k}h_k^{\frac{1}{2}}\right)^{\frac{1}{2}} \\ &= \left(h_k^{\frac{1}{2}}e^{-2K_k}h_k^{\frac{1}{2}}\right)^{\frac{1}{2}}\left(h_k^{\frac{1}{2}}e^{-2K_k}h_k^{\frac{1}{2}}\right)^{\frac{1}{2}} = h_k^{\frac{1}{2}}e^{-2K_k}h_k^{\frac{1}{2}} = \tilde{E}_k. \end{aligned} \tag{10.0.11}$$

It can then be shown to hold that

$$\begin{aligned} &e^{\mathcal{J}}\left(H'_{\text{kin}} + 2\sum_{k\in\overline{B}\left(0,k_F^{\gamma}\right)\cap\mathbb{Z}_*^3}Q_1^k(E_k - h_k)\right)e^{-\mathcal{J}} \\ &= H'_{\text{kin}} + 2\sum_{k\in\overline{B}\left(0,k_F^{\gamma}\right)\cap\mathbb{Z}_*^3}Q_1^k\left(\tilde{E}_k - h_k\right) + 2\sum_{k\in\overline{B}\left(0,k_F^{\gamma}\right)\cap\mathbb{Z}_*^3}\int_0^1 e^{(1-t)\mathcal{J}}\mathcal{E}_k^3(E_k(t))e^{-(1-t)\mathcal{J}}dt \end{aligned} \tag{10.0.12}$$

where $\mathcal{E}_k^3(\cdot)$ is of a similar form to $\mathcal{E}_k^1(\cdot)$ and $\mathcal{E}_k^2(\cdot)$ of the first transformation, while $E_k(t) : \ell^2(L_k) \to \ell^2(L_k)$ is given by

$$E_k(t) = e^{tJ_k}e^{-K_k}h_ke^{-K_k}e^{-tJ_k} - h_k. \tag{10.0.13}$$

The following estimate for the error term, and Gronwall estimates for the kinetic operators with respect to the second transformation, can then be obtained:

Proposition 10.0.4. *It holds for all* $0 < \gamma < \frac{1}{47}$ *that*

$$\pm\sum_{k\in\overline{B}\left(0,k_F^{\gamma}\right)\cap\mathbb{Z}_*^3}\mathcal{E}_k^3(E_k(t)) \leq C\log(k_F)^{\frac{5}{3}}k_F^{\left(5+\frac{2}{3}\right)\gamma-\frac{1}{3}}\left(H'_{\text{kin}} + k_F^{-1}\mathcal{N}_EH'_{\text{kin}}\right)$$

and for $t \in [-1, 1]$

$$\begin{aligned} e^{t\mathcal{J}}H'_{\text{kin}}e^{-t\mathcal{J}} &\leq CH'_{\text{kin}} \\ e^{t\mathcal{J}}\mathcal{N}_EH'_{\text{kin}}e^{-t\mathcal{J}} &\leq C\mathcal{N}_EH'_{\text{kin}} \end{aligned}$$

for a constant $C > 0$ *depending only on* $\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k|k|$.

As mentioned, the condition $\gamma < \frac{1}{47}$ enters in the estimation of one-body estimates for $E_k(t)$ and the Gronwall argument - the Gronwall argument is particularly sensitive to this, as the exponential prefactor diverges as $k_F \to \infty$ if these are not estimated optimally.

Theorem 1.1.3 now follows by taking $\mathcal{U} = e^{\mathcal{J}} e^{\mathcal{K}}$, for which

$$\begin{aligned}\mathcal{U} H_N \mathcal{U}^* = E_F + E_{\mathrm{corr,bos}} + H'_{\mathrm{kin}} + 2 \sum_{k \in \overline{B}\left(0, k_F^{\gamma}\right) \cap \mathbb{Z}_*^3} Q_1^k\left(\tilde{E}_k - h_k\right) \\ + \sum_{k \in \overline{B}\left(0, k_F^{\gamma}\right) \cap \mathbb{Z}_*^3} \mathrm{tr}(E_k - h_k - P_k) - E_{\mathrm{corr,bos}} + \mathcal{U}(\mathrm{ND} + \mathcal{C} + \mathcal{Q})\mathcal{U}^* \\ + e^{\mathcal{J}} (\mathrm{Exchange\,Terms}) e^{-\mathcal{J}} + 2 \sum_{k \in \overline{B}\left(0, k_F^{\gamma}\right) \cap \mathbb{Z}_*^3} \int_0^1 e^{(1-t)\mathcal{J}} \mathcal{E}_k^3(E_k(t)) e^{-(1-t)\mathcal{J}} dt;\end{aligned} \tag{10.0.14}$$

by the estimates obtained, the terms on the second and third lines are bounded by

$$C\left(k_F^{-\frac{\gamma}{2}} + \log(k_F)^{\frac{5}{3}} k_F^{\left(5+\frac{2}{3}\right)\gamma - \frac{1}{3}} + \log(k_F)^{\frac{1}{9}} k_F^{-\frac{1}{18}}\right)\left(k_F + H'_{\mathrm{kin}} + k_F^{-1} \mathcal{N}_E H'_{\mathrm{kin}}\right) \tag{10.0.15}$$

which is optimized as $\gamma \to \frac{1}{47}$ for the prefactor $k_F^{-\frac{1}{94}+\varepsilon}$, $\varepsilon > 0$. It then only remains to estimate the tail of $\sum_{k \in \mathbb{Z}_*^3} Q_1^k\left(\tilde{E}_k - h_k\right)$, but it is not too difficult to show that these obey

$$\pm \sum_{k \in \mathbb{Z}_*^3 \backslash \overline{B}\left(0, k_F^{\gamma}\right)} Q_1^k\left(\tilde{E}_k - h_k\right) \leq C k_F^{-\gamma} H'_{\mathrm{kin}} \tag{10.0.16}$$

and so are likewise negligible.

10.1 ***A Priori* Bounds**

In this subsection we prove Proposition 10.0.1. For the sake of brevity we will write $H'_N = H_N - E_F$, so the definition of a low-lying state is simply that

$$\langle \Psi, H'_N \Psi \rangle \leq \kappa k_F. \tag{10.1.1}$$

First we obtain an *a priori* bound for H'_N itself. Recall that we in Section 2 found that

$$H_{\mathrm{kin}} = \langle \psi_F, H_{\mathrm{kin}} \psi_F \rangle + H'_{\mathrm{kin}} \tag{10.1.2}$$

and note that it follows from the equations (2.2.5) and (2.2.13) that

$$H_{\mathrm{int}} = \langle \psi_F, H_{\mathrm{int}} \psi_F \rangle + \frac{1}{2\,(2\pi)^3} \sum_{k \in \mathbb{Z}_*^3} \hat{V}_k\left(\mathrm{d}\Gamma\left(e^{-ik\cdot x}\right)^* \mathrm{d}\Gamma\left(e^{-ik\cdot x}\right) - |L_k|\right) \tag{10.1.3}$$

so

$$H'_N = H'_{\mathrm{kin}} + \frac{k_F^{-1}}{2\,(2\pi)^3} \sum_{k \in \mathbb{Z}_*^3} \hat{V}_k\left(\mathrm{d}\Gamma\left(e^{-ik\cdot x}\right)^* \mathrm{d}\Gamma\left(e^{-ik\cdot x}\right) - |L_k|\right). \tag{10.1.4}$$

As trivially $\mathrm{d}\Gamma\left(e^{-ik\cdot x}\right)^* \mathrm{d}\Gamma\left(e^{-ik\cdot x}\right) \geq 0$ we can thus apply the bound $|L_k| \leq C k_F^2 |k|$ to conclude that

$$H'_N \geq H'_{\mathrm{kin}} - \frac{k_F^{-1}}{2\,(2\pi)^3} \sum_{k \in \mathbb{Z}_*^3} \hat{V}_k |L_k| \geq H'_{\mathrm{kin}} - C' k_F \sum_{k \in \mathbb{Z}_*^3} \hat{V}_k |k| = H'_{\mathrm{kin}} - C k_F \tag{10.1.5}$$

for a constant $C > 0$ depending only on $\sum_{k\in\mathbb{Z}^3_*} \hat{V}_k\,|k|$.

This immediately implies that the correlation energy is (at most) of order k_F in this case, but more crucial is the implied bound on H'_{kin}: Equation (10.1.5) implies that any low-lying state must obey

$$\langle \Psi, H'_{\text{kin}}\Psi\rangle \leq \langle \Psi, H'_N\Psi\rangle + Ck_F \leq (C+\kappa)k_F, \tag{10.1.6}$$

which is our first *a priori* bound.

This in turn yields an *a priori* bound for $\langle \Psi, \mathcal{N}_E\Psi\rangle$ as well, for recall that we found that the particle-hole symmetry allowed us to express

$$H'_{\text{kin}} = \sum_{p\in B_F^c} |p|^2\, c_p^* c_p - \sum_{p\in B_F} |p|^2\, c_p c_p^* \tag{10.1.7}$$

in the manifestly positive form

$$H'_{\text{kin}} = \sum_{p\in B_F^c} \Big(|p|^2 - k_F^2\Big) c_p^* c_p + \sum_{p\in B_F} \Big(k_F^2 - |p|^2\Big) c_p c_p^*. \tag{10.1.8}$$

This particular form is not useful, as the prefactors in the sums can be arbitrarily small. The only condition we used to obtain this was however that $|p| \geq k_F \geq |q|$, so the same argument shows that for any $\zeta \in \Big[\sup_{q\in B_F} |q|^2, \inf_{p\in B_F^c} |p|^2\Big]$ it holds that

$$H'_{\text{kin}} = \sum_{p\in B_F^c} \Big(|p|^2 - \zeta\Big) c_p^* c_p + \sum_{p\in B_F} \Big(\zeta - |p|^2\Big) c_p c_p^*, \tag{10.1.9}$$

and choosing $\zeta = \frac{1}{2}\Big(\inf_{p\in B_F^c} |p|^2 + \sup_{q\in B_F} |q|^2\Big)$ we have

$$\inf_{p\in\mathbb{Z}^3} \big|\, |p|^2 - \zeta \big| \geq \frac{1}{2} \tag{10.1.10}$$

since $\inf_{p\in B_F^c} |p|^2 - \sup_{q\in B_F} |q|^2 \geq 1$ as $|p|^2, |q|^2 \in \mathbb{Z}$ but $|q|^2 < |p|^2$ for any $p \in B_F^c$ and $q \in B_F$.

We thus conclude the general operator inequality (first noted in [9])

$$H'_{\text{kin}} \geq \frac{1}{2}\sum_{p\in B_F^c} c_p^* c_p + \frac{1}{2}\sum_{p\in B_F} c_p c_p^* = \mathcal{N}_E \tag{10.1.11}$$

and conclude the following:

Proposition 10.1.1. *For any low-lying state $\Psi \in D(H'_{\text{kin}})$ it holds that*

$$\langle \Psi, \mathcal{N}_E\Psi\rangle \leq \langle \Psi, H'_{\text{kin}}\Psi\rangle \leq (C+\kappa)k_F$$

for a $C > 0$ depending only on $\sum_{k\in\mathbb{Z}^3_*} \hat{V}_k\,|k|$.

Bootstrapped Bounds for Eigenstates

In the particular case that Ψ is additionally an eigenstate we can also obtain an *a priori* bound on $\langle\Psi,\mathcal{N}_E H'_{\mathrm{kin}}\Psi\rangle$ by employing a bootstrapping argument (similar to an idea of [17]). It turns out to be easier to bound $\langle\Psi,\mathcal{N}_E^2 H'_{\mathrm{kin}}\Psi\rangle$ and then obtain $\langle\Psi,\mathcal{N}_E H'_{\mathrm{kin}}\Psi\rangle$ as a corollary, so let us consider this: First, by equation (10.1.5), we have the operator inequality

$$\begin{aligned}\mathcal{N}_E^2 H'_{\mathrm{kin}} &= \mathcal{N}_E H'_{\mathrm{kin}}\mathcal{N}_E \leq \mathcal{N}_E H'_N\mathcal{N}_E + Ck_F\mathcal{N}_E^2 \\ &= \frac{1}{2}\Big(\mathcal{N}_E^2 H'_N + H'_N\mathcal{N}_E^2 - [\mathcal{N}_E,[\mathcal{N}_E,H'_N]]\Big) + Ck_F\mathcal{N}_E^2,\end{aligned} \tag{10.1.12}$$

so if $\Psi\in D(H'_{\mathrm{kin}})$ is an eigenstate of H_N such that $H'_N\Psi = E\Psi$, it holds that

$$\begin{aligned}\left\langle\Psi,\mathcal{N}_E^2 H'_{\mathrm{kin}}\Psi\right\rangle &\leq (E+Ck_F)\left\langle\Psi,\mathcal{N}_E^2\Psi\right\rangle - \frac{1}{2}\left\langle\Psi,[\mathcal{N}_E,[\mathcal{N}_E,H'_N]]\,\Psi\right\rangle \\ &\leq (E+Ck_F)\left\langle\Psi,\mathcal{N}_E H'_{\mathrm{kin}}\Psi\right\rangle - \frac{1}{2}\left\langle\Psi,[\mathcal{N}_E,[\mathcal{N}_E,H'_N]]\,\Psi\right\rangle\end{aligned} \tag{10.1.13}$$

where we also used that $\mathcal{N}_E\leq H'_{\mathrm{kin}}$ for the first term.

We must therefore consider $[\mathcal{N}_E,[\mathcal{N}_E,H'_N]]$. Note that by the decomposition of Proposition 2.0.1, we can write

$$H'_N = H_\Delta + \frac{k_F^{-1}}{2\,(2\pi)^3}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k\Big(B_kB_{-k} + B^*_{-k}B^*_k\Big) + \mathcal{C} \tag{10.1.14}$$

for

$$H_\Delta = H'_{\mathrm{kin}} + \frac{k_F^{-1}}{(2\pi)^3}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_kB^*_kB_k + \mathcal{Q}, \tag{10.1.15}$$

where we recall that the cubic terms $\mathcal{C}$ are given by

$$\mathcal{C} = \frac{k_F^{-1}}{2\,(2\pi)^3}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k\Big((B^*_k + B_{-k})D_k + D^*_k\Big(B_k + B^*_{-k}\Big)\Big). \tag{10.1.16}$$

As remarked at the start of Section 8 there holds the commutators

$$[\mathcal{N}_E,B_k] = -B_k,\quad [\mathcal{N}_E,B^*_k] = B^*_k,\quad [\mathcal{N}_E,D_k] = 0 = [\mathcal{N}_E,D^*_k]\,, \tag{10.1.17}$$

which imply that $[\mathcal{N}_E,H_\Delta]=0$ and thus

$$\begin{aligned}[\mathcal{N}_E,[\mathcal{N}_E,H'_N]] &= \frac{k_F^{-1}}{2\,(2\pi)^3}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k\Big(4B_kB_{-k} + 4B^*_{-k}B^*_k + (B^*_k + B_{-k})D_k + D^*_k\Big(B_k + B^*_{-k}\Big)\Big) \\ &= \frac{k_F^{-1}}{(2\pi)^3}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k\,\mathrm{Re}(4B_kB_{-k} + (B^*_k + B_{-k})D_k).\end{aligned} \tag{10.1.18}$$

We note the following estimates for the B_k and D_k operators (the *kinetic bound* on B_k was first obtained in [16]):

Proposition 10.1.2. *For any $k \in \mathbb{Z}^3_*$ and $\Psi \in D(H'_{\mathrm{kin}})$ it holds that*

$$\begin{aligned}
\|B_k\Psi\|^2 &\leq Ck_F \langle \Psi, H'_{\mathrm{kin}}\Psi\rangle \\
\|B_k^*\Psi\|^2 &\leq C\Big(k_F \langle \Psi, H'_{\mathrm{kin}}\Psi\rangle + k_F^2\,|k|\,\|\Psi\|^2\Big) \\
\|D_k\Psi\|^2 &\leq 8\left\langle \Psi, \mathcal{N}_E^2\Psi\right\rangle
\end{aligned}$$

for a constant $C > 0$ independent of all quantities.

Proof: For B_k we can by Cauchy-Schwarz estimate

$$\|B_k\Psi\| = \left\|\sum_{p\in L_k} b_{k,p}\Psi\right\| \leq \sum_{p\in L_k}\|b_{k,p}\Psi\| \leq \sqrt{\sum_{p\in L_k}\lambda_{k,p}^{-1}}\sqrt{\sum_{p\in L_k}\lambda_{k,p}\|b_{k,p}\Psi\|^2} \leq Ck_F^{\frac{1}{2}}\sqrt{\sum_{p\in L_k}\lambda_{k,p}\|b_{k,p}\Psi\|^2} \tag{10.1.19}$$

where we also used that $\sum_{p\in L_k}\lambda_{k,p}^{-1} \leq Ck_F$. For the remaining factor we expand and bound as

$$\begin{aligned}
\sum_{p\in L_k}\lambda_{k,p}\|b_{k,p}\Psi\|^2 &= \frac{1}{2}\sum_{p\in L_k}\left(|p|^2 - |p-k|^2\right)\left\|c_{p-k}^*c_p\Psi\right\|^2 \qquad (10.1.20)\\
&= \frac{1}{2}\sum_{p\in L_k}\left(|p|^2 - k_F^2\right)\left\|c_{p-k}^*c_p\Psi\right\|^2 + \frac{1}{2}\sum_{p\in L_k}\left(k_F^2 - |p-k|^2\right)\left\|c_{p-k}^*c_p\Psi\right\|^2 \\
&\leq \frac{1}{2}\sum_{p\in L_k}\left(|p|^2 - k_F^2\right)\|c_p\Psi\|^2 + \frac{1}{2}\sum_{p\in L_k}\left(k_F^2 - |p-k|^2\right)\left\|c_{p-k}^*\Psi\right\|^2 \\
&= \frac{1}{2}\langle\Psi, H'_{\mathrm{kin}}\Psi\rangle
\end{aligned}$$

where we applied the representation of H'_{kin} given by equation (10.1.8). This implies the first bound. The second then follows as the commutator of equation (2.2.11) shows that

$$\begin{aligned}
\|B_k^*\Psi\|^2 &= \langle\Psi, B_kB_k^*\Psi\rangle = \langle\Psi, B_k^*B_k\Psi\rangle + \langle\Psi, [B_k, B_k^*]\,\Psi\rangle \qquad (10.1.21)\\
&\leq \|B_k\Psi\|^2 + |L_k|\,\|\Psi\|^2 \leq C\Big(k_F\langle\Psi, H'_{\mathrm{kin}}\Psi\rangle + k_F^2\,|k|\,\|\Psi\|^2\Big).
\end{aligned}$$

For D_k, recall the decomposition $D_k = D_{1,k} + D_{2,k}$ we used in Section 8. As $\|D_k\Psi\|^2 \leq 2\|D_{1,k}\Psi\|^2 + 2\|D_{2,k}\Psi\|^2$ it suffices to bound $D_{1,k}$ and $D_{2,k}$. Equation (8.1.2) says that (with $s = 1$)

$$D_{1,k}^*D_{1,k} = \sum_{p,q\in B_F\cap(B_F+k)} c_{p-k}c_qc_{q-k}^*c_p^* + \sum_{q\in B_F} 1_{B_F}(q+k)c_qc_q^* \tag{10.1.22}$$

and the first term we bounded in equation (8.1.12) as

$$\sum_{p,q\in B_F\cap(B_F+k)}\left\langle\Psi, c_{p-k}c_qc_{q-k}^*c_p^*\Psi\right\rangle \leq \left\langle\Psi, \mathcal{N}_E^2\Psi\right\rangle \tag{10.1.23}$$

while the second term trivially obeys

$$\sum_{q\in B_F} 1_{B_F}(q+k)c_q c_q^* \leq \mathcal{N}_E \leq \mathcal{N}_E^2, \tag{10.1.24}$$

so $\|D_{1,k}\Psi\|^2 \leq 2\langle\Psi,\mathcal{N}_E^2\Psi\rangle = 2\|\mathcal{N}_E\Psi\|^2$. $\|D_{2,k}\Psi\|^2$ can be bounded similarly for the claim. □

A bound on $[\mathcal{N}_E,[\mathcal{N}_E,H_N']]$ immediately follows:

Proposition 10.1.3. *It holds that*

$$\pm[\mathcal{N}_E,[\mathcal{N}_E,H_N']] \leq C\Big(k_F + H_{\text{kin}}' + k_F^{-1}\mathcal{N}_E^2\Big)$$

for a constant $C>0$ depending only on $\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k|k|$.

The main eigenstate bound can then be obtained:

Proposition 10.1.4. *For any normalized eigenstate Ψ of H_N with $H_N'\Psi = E\Psi$ it holds that*

$$\Big\langle\Psi,\mathcal{N}_E^2 H_{\text{kin}}'\Psi\Big\rangle \leq C\max\{E,k_F\}^3$$

for a constant $C>0$ depending only on $\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k|k|$.

Proof: Inserting the previous estimate into equation (10.1.13), we obtain

$$\begin{aligned}\Big\langle\Psi,\mathcal{N}_E^2 H_{\text{kin}}'\Psi\Big\rangle &\leq (E+Ck_F)\langle\Psi,\mathcal{N}_E H_{\text{kin}}'\Psi\rangle + C\Big\langle\Psi,\Big(k_F + H_{\text{kin}}' + k_F^{-1}\mathcal{N}_E^2\Big)\Psi\Big\rangle \\ &\leq Ck_F + C\max\{E,k_F\}\langle\Psi,\mathcal{N}_E H_{\text{kin}}'\Psi\rangle\end{aligned} \tag{10.1.25}$$

where we also used that $H_{\text{kin}}', \mathcal{N}_E^2 \leq \mathcal{N}_E H_{\text{kin}}'$ to simplify the expression. Now, by the Cauchy-Schwarz inequality for H_{kin} we can estimate

$$\langle\Psi,\mathcal{N}_E H_{\text{kin}}'\Psi\rangle \leq \sqrt{\langle\Psi,H_{\text{kin}}'\Psi\rangle\langle\Psi,\mathcal{N}_E H_{\text{kin}}'\mathcal{N}_E\Psi\rangle} \leq \sqrt{C\max\{E,k_F\}}\sqrt{\langle\Psi,\mathcal{N}_E^2 H_{\text{kin}}'\Psi\rangle} \tag{10.1.26}$$

where we also applied the inequality $H_{\text{kin}}' \leq H_N' + Ck_F$. It follows by the Cauchy inequality that

$$\begin{aligned}\max\{E,k_F\}\langle\Psi,\mathcal{N}_E H_{\text{kin}}'\Psi\rangle &\leq C(\max\{E,k_F\})^{\frac{3}{2}}\sqrt{\langle\Psi,\mathcal{N}_E^2 H_{\text{kin}}'\Psi\rangle} \\ &\leq C\max\{E,k_F\}^3 + \frac{1}{2}\Big\langle\Psi,\mathcal{N}_E^2 H_{\text{kin}}'\Psi\Big\rangle\end{aligned} \tag{10.1.27}$$

which upon insertion into equation (10.1.25) upon rearrangement yields

$$\Big\langle\Psi,\mathcal{N}_E^2 H_{\text{kin}}'\Psi\Big\rangle \leq 2\Big(Ck_F + C\max\{E,k_F\}^3\Big) \leq C\max\{E,k_F\}^3. \tag{10.1.28}$$

□

We can now conclude the desired estimate:

Corollary 10.1.5. *For any low-lying eigenstate* $\Psi \in D(H'_{\text{kin}})$ *it holds that*

$$\langle \Psi, \mathcal{N}_E H'_{\text{kin}} \Psi \rangle \leq C k_F^2$$

for a constant $C > 0$ *depending only on* $\sum_{k \in \mathbb{Z}^3_*} \hat{V}_k \, |k|$ *and* κ.

Proof: Estimating as in equation (10.1.26) we have by the proposition that

$$\langle \Psi, \mathcal{N}_E H'_{\text{kin}} \Psi \rangle \leq \sqrt{\langle \Psi, H'_{\text{kin}} \Psi \rangle \langle \Psi, \mathcal{N}_E^2 H'_{\text{kin}} \Psi \rangle} \leq C \sqrt{\max \{\kappa k_F, k_F\}^4} \leq C k_F^2. \tag{10.1.29}$$

□

10.2 Bounding the Non-Diagonalized and Non-Bosonizable Terms

We consider the bounds of Proposition 10.0.2. The non-diagonalized terms

$$\mathrm{ND} = \frac{k_F^{-1}}{2\,(2\pi)^3} \sum_{k \in \mathbb{Z}^3_* \setminus \overline{B}\left(0, k_F^\gamma\right)} \hat{V}_k \Big(2 B_k^* B_k + B_k B_{-k} + B_{-k}^* B_k^* \Big) \tag{10.2.1}$$

can be immediately estimated by Proposition 10.1.2 as

$$\begin{aligned} |\langle \Psi, \mathrm{ND} \Psi \rangle| &\leq \frac{k_F^{-1}}{(2\pi)^3} \sum_{k \in \mathbb{Z}^3_* \setminus \overline{B}\left(0, k_F^\gamma\right)} \hat{V}_k \Big(\|B_k \Psi\|^2 + \|B_k^* \Psi\| \, \|B_{-k} \Psi\| \Big) \\ &\leq C k_F^{-1} \sum_{k \in \mathbb{Z}^3_* \setminus \overline{B}\left(0, k_F^\gamma\right)} \hat{V}_k \sqrt{k_F \langle \Psi, H'_{\text{kin}} \Psi \rangle \Big(k_F \langle \Psi, H'_{\text{kin}} \Psi \rangle + k_F^2 \, |k| \, \|\Psi\|^2 \Big)} \\ &\leq C \left(\sum_{k \in \mathbb{Z}^3_* \setminus \overline{B}\left(0, k_F^\gamma\right)} \hat{V}_k \, |k|^{\frac{1}{2}} \right) \Big(\langle \Psi, H'_{\text{kin}} \Psi \rangle + k_F \, \|\Psi\|^2 \Big) \\ &\leq C k_F^{-\frac{\gamma}{2}} \left(\sum_{k \in \mathbb{Z}^3_*} \hat{V}_k \, |k| \right) \Big(\langle \Psi, H'_{\text{kin}} \Psi \rangle + k_F \, \|\Psi\|^2 \Big) \end{aligned} \tag{10.2.2}$$

for any $\Psi \in D(H'_{\text{kin}})$, i.e.

$$\pm \mathrm{ND} \leq C k_F^{-\frac{\gamma}{2}} (k_F + H'_{\text{kin}}). \tag{10.2.3}$$

We again recall the non-bosonizable terms (for $s = 1$):

$$\begin{aligned} \mathcal{C} &= \frac{k_F^{-1}}{(2\pi)^3} \sum_{k \in \mathbb{Z}^3_*} \hat{V}_k \, \mathrm{Re}\Big(\Big(B_k + B_{-k}^* \Big)^* D_k \Big) \\ \mathcal{Q} &= \frac{k_F^{-1}}{2\,(2\pi)^3} \sum_{k \in \mathbb{Z}^3_*} \hat{V}_k \left(D_k^* D_k - \sum_{p \in L_k} \Big(c_p^* c_p + c_{p-k} c_{p-k}^* \Big) \right). \end{aligned} \tag{10.2.4}$$

When $\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k<\infty$ the second terms of $\mathcal{Q}$ are entirely negligible, as

$$0\leq\frac{k_F^{-1}}{2\left(2\pi\right)^3}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k\sum_{p\in L_k}\left(c_p^*c_p+c_{p-k}c_{p-k}^*\right)\leq\frac{k_F^{-1}}{\left(2\pi\right)^3}\left(\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k\right)\mathcal{N}_E\leq Ck_F^{-1}\mathcal{N}_E \tag{10.2.5}$$

so we may disregard these. For the remaining terms we rewrite $\mathcal{C}$: Straightforward computation shows that $[B_{-k},D_k]=0=\left[B_{-k}^*,D_k^*\right]$ for any $k\in\mathbb{Z}^3_*$, and as furthermore $D_k^*=D_{-k}$ we can write $\mathcal{C}$ as

$$\begin{aligned}\mathcal{C}&=\frac{k_F^{-1}}{2\left(2\pi\right)^3}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k\Big((B_k^*+B_{-k})D_k+D_k^*\Big(B_k+B_{-k}^*\Big)\Big)\\&=\frac{k_F^{-1}}{2\left(2\pi\right)^3}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k\Big(B_k^*D_k+D_kB_{-k}+D_k^*B_k+B_{-k}^*D_k^*\Big)\\&=\frac{k_F^{-1}}{\left(2\pi\right)^3}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k(B_k^*D_k+D_k^*B_k)\end{aligned} \tag{10.2.6}$$

so the terms that we need to control are

$$\mathrm{NB}=\frac{k_F^{-1}}{\left(2\pi\right)^3}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k\left(B_k^*D_k+D_k^*B_k+\frac{1}{2}D_k^*D_k\right). \tag{10.2.7}$$

Dividing the summation range into $k\in\overline{B}\left(0,k_F^\delta\right)\cap\mathbb{Z}^3_*$ and $k\in\mathbb{Z}^3_*\backslash\overline{B}\left(0,k_F^\delta\right)$ for some $\delta>0$, we write $\mathrm{NB}=\mathrm{NB}_1+\mathrm{NB}_2$ and estimate NB_2 using Proposition 10.1.2 as

$$\begin{aligned}\pm\mathrm{NB}_2&\leq Ck_F^{-1}\sum_{k\in\mathbb{Z}^3_*\backslash\overline{B}\left(0,k_F^\delta\right)}\hat{V}_k(B_k^*B_k+D_k^*D_k)\leq Ck_F^{-1}\sum_{k\in\mathbb{Z}^3_*\backslash\overline{B}\left(0,k_F^\delta\right)}\hat{V}_k\left(k_FH'_{\mathrm{kin}}+\mathcal{N}_E^2\right)\\&\leq C\left(\sum_{k\in\mathbb{Z}^3_*\backslash\overline{B}\left(0,k_F^\delta\right)}\hat{V}_k\right)\left(H'_{\mathrm{kin}}+k_F^{-1}\mathcal{N}_EH'_{\mathrm{kin}}\right)\leq Ck_F^{-\delta}\left(H'_{\mathrm{kin}}+k_F^{-1}\mathcal{N}_EH'_{\mathrm{kin}}\right).\end{aligned} \tag{10.2.8}$$

For NB_1 we note that by Cauchy-Schwarz and the B_k estimate of Proposition 10.1.2,

$$\left|\left\langle\Psi,\left(B_k^*D_k+D_k^*B_k+\frac{1}{2}D_k^*D_k\right)\Psi\right\rangle\right|\leq C\left(k_F^{\frac{1}{2}}\sqrt{\langle\Psi,H'_{\mathrm{kin}}\Psi\rangle}+\|D_k\Psi\|\right)\|D_k\Psi\|, \tag{10.2.9}$$

so it suffices to obtain an improved D_k estimate for small k.

Detailed Analysis of D_k

We begin by noting the following:

Proposition 10.2.1. *For all $k\in\mathbb{Z}^3_*$ and any $\lambda>0$ it holds that*

$$D_k^*D_k\leq C\left(1+\left|S_{k,\lambda}^1\right|+\left|S_{k,\lambda}^2\right|\right)\mathcal{N}_E+C\lambda^{-\frac{1}{2}}\mathcal{N}_EH'_{\mathrm{kin}}$$

for a constant $C > 0$ independent of k and k_F, where

$$S^1_{k,\lambda} = \left\{p \in B_F \cap (B_F + k) \mid \max\left\{\left|\,|p|^2 - \zeta\right|, \left|\,|p-k|^2 - \zeta\right|\right\} < \lambda\right\}$$
$$S^2_{k,\lambda} = \left\{p \in B_F^c \cap (B_F^c + k) \mid \max\left\{\left|\,|p|^2 - \zeta\right|, \left|\,|p-k|^2 - \zeta\right|\right\} < \lambda\right\}.$$

Proof: It suffices to consider $D_{1,k}$ and $D_{2,k}$; we focus on $D_{1,k}$. Recall again that

$$D^*_{1,k} D_{1,k} = \sum_{p,q \in B_F \cap (B_F + k)} c_{p-k} c_q c^*_{q-k} c^*_p + \sum_{q \in B_F} 1_{B_F}(q+k) c_q c^*_q \tag{10.2.10}$$

so for any $\Psi \in D(H'_{\text{kin}})$

$$\begin{aligned} \|D_{1,k}\Psi\|^2 &\leq \sum_{p,q \in B_F \cap (B_F + k)} \left\|c^*_q c^*_{p-k}\Psi\right\| \left\|c^*_{q-k} c^*_p \Psi\right\| + \langle \Psi, \mathcal{N}_E \Psi\rangle \\ &\leq \sum_{p \in B_F \cap (B_F + k)} \|c^*_{p-k} \mathcal{N}_E^{\frac{1}{2}} \Psi\| \|c^*_p \mathcal{N}_E^{\frac{1}{2}} \Psi\| + \langle \Psi, \mathcal{N}_E \Psi\rangle\,. \end{aligned} \tag{10.2.11}$$

To estimate the sum, we decompose $B_F \cap (B_F + k) = S^1_{k,\lambda} \cup S^1_{\geq\lambda}$ where $S^1_{k,\lambda}$ is as in the statement of the theorem, and

$$S^1_{\geq\lambda} = \left\{p \in B_F \cap (B_F + k) \mid \max\left\{\left|\,|p|^2 - \zeta\right|, \left|\,|p-k|^2 - \zeta\right|\right\} \geq \lambda\right\}. \tag{10.2.12}$$

By this definition and equation (10.1.10) it holds for all $p \in S^1_{\geq\lambda}$ that

$$\sqrt{\left|\,|p|^2 - \zeta\right|}\sqrt{\left|\,|p-k|^2 - \zeta\right|} \geq 2^{-\frac{1}{2}} \lambda^{\frac{1}{2}} \tag{10.2.13}$$

so we can estimate

$$\begin{aligned} &\sum_{p \in B_F \cap (B_F + k)} \|c^*_{p-k} \mathcal{N}_E^{\frac{1}{2}} \Psi\| \|c^*_p \mathcal{N}_E^{\frac{1}{2}} \Psi\| \\ &\leq \left|S^1_{k,\lambda}\right| \|\mathcal{N}_E^{\frac{1}{2}}\Psi\|^2 + \sqrt{2}\lambda^{-\frac{1}{2}} \sum_{p \in S^1_{\geq\lambda}} \sqrt{\left|\,|p|^2 - \zeta\right|}\sqrt{\left|\,|p-k|^2 - \zeta\right|} \|c^*_{p-k} \mathcal{N}_E^{\frac{1}{2}} \Psi\| \|c^*_p \mathcal{N}_E^{\frac{1}{2}} \Psi\| \\ &\leq \left|S^1_{k,\lambda}\right| \langle \Psi, \mathcal{N}_E \Psi\rangle + \sqrt{2}\lambda^{-\frac{1}{2}} \sqrt{\sum_{p \in S^1_{\geq\lambda}} \left|\,|p|^2 - \zeta\right| \|c^*_p \mathcal{N}_E^{\frac{1}{2}} \Psi\|^2} \sqrt{\sum_{p \in S^1_{\geq\lambda}} \left|\,|p-k|^2 - \zeta\right| \|c^*_{p-k} \mathcal{N}_E^{\frac{1}{2}} \Psi\|^2} \\ &\leq \left|S^1_{k,\lambda}\right| \langle \Psi, \mathcal{N}_E \Psi\rangle + \sqrt{2}\lambda^{-\frac{1}{2}} \langle \Psi, \mathcal{N}_E H'_{\text{kin}} \Psi\rangle \end{aligned} \tag{10.2.14}$$

by equation (10.1.9), whence the claim follows.

□

By employing precise lattice point counting techniques of the same kind used in appendix section B.3, the following was obtained in [11]:

Proposition 10.2.2. *For all $k \in \overline{B}(0, k_F) \cap \mathbb{Z}^3_*$ and $0 < \lambda \leq \frac{1}{6} k_F^2$ (depending on k and k_F) it holds that*

$$\left|S^1_{k,\lambda}\right| + \left|S^2_{k,\lambda}\right| \leq C\left(|k|^{-1}\lambda + |k|^{3+\frac{2}{3}} \log(k_F)^{\frac{2}{3}} k_F^{\frac{2}{3}}\right)(\lambda + |k|), \quad k_F \to \infty,$$

for a constant $C > 0$ independent of all quantities.

From this and Proposition 10.2.1 one can then conclude a stronger D_k bound:

Proposition 10.2.3. *For all* $k \in \overline{B}\left(0, k_F^{\delta}\right) \cap \mathbb{Z}_*^3$, $0 < \delta < \frac{2}{31}$, *it holds that*

$$D_k^* D_k \leq C\,|k|^{\frac{11}{9}} \log(k_F)^{\frac{2}{9}} k_F^{\frac{8}{9}} \left(H'_{\text{kin}} + k_F^{-1} \mathcal{N}_E H'_{\text{kin}}\right), \quad k_F \to \infty,$$

for a constant $C > 0$ *independent of all quantities.*

It now follows from equation (10.2.9) that

$$\left|\left\langle \Psi, \left(B_k^* D_k + D_k^* B_k + \frac{1}{2} D_k^* D_k\right)\Psi\right\rangle\right| \leq C\,|k| \log(k_F)^{\frac{1}{9}} k_F^{\frac{17}{18}} \left\langle \Psi, \left(H'_{\text{kin}} + k_F^{-1} \mathcal{N}_E H'_{\text{kin}}\right)\Psi\right\rangle \tag{10.2.15}$$

for $|k| \leq k_F^{\delta}$, $\delta < \frac{2}{31}$, whence

$$\pm \mathrm{NB}_1 \leq C \log(k_F)^{\frac{1}{9}} k_F^{-\frac{1}{18}} \left(H'_{\text{kin}} + k_F^{-1} \mathcal{N}_E H'_{\text{kin}}\right). \tag{10.2.16}$$

As $\frac{2}{31} > \frac{1}{18}$, δ can be chosen such that the NB_2 bound matches this one, yielding Proposition 10.0.2.

10.3 Controlling the Diagonalization

We begin by considering the tail estimate for $E_{\text{corr,bos}}$. Recall that by Theorem 6.0.1 (with $s = 1$)

$$\operatorname{tr}(E_k - h_k - P_k) = \frac{1}{\pi} \int_0^\infty F\left(\frac{\hat{V}_k k_F^{-1}}{(2\pi)^3} \sum_{p \in L_k} \frac{\lambda_{k,p}}{\lambda_{k,p}^2 + t^2}\right) dt, \quad F(x) = \log(1 + x) - x. \tag{10.3.1}$$

As F obeys $|F(x)| \leq \frac{1}{2} x^2$ for $x \geq 0$, we may estimate

$$\begin{aligned}
|\operatorname{tr}(E_k - h_k - P_k)| &\leq \frac{1}{2\pi} \int_0^\infty \left(\frac{\hat{V}_k k_F^{-1}}{(2\pi)^3} \sum_{p \in L_k} \frac{\lambda_{k,p}}{\lambda_{k,p}^2 + t^2}\right)^2 dt = \frac{\hat{V}_k^2 k_F^{-2}}{(2\pi)^7} \sum_{p,q \in L_k} \int_0^\infty \frac{\lambda_{k,p}}{\lambda_{k,p}^2 + t^2} \frac{\lambda_{k,q}}{\lambda_{k,q}^2 + t^2} dt \\
&= \frac{\hat{V}_k^2 k_F^{-2}}{4\,(2\pi)^6} \sum_{p,q \in L_k} \frac{1}{\lambda_{k,p} + \lambda_{k,q}} \leq \frac{\hat{V}_k^2 k_F^{-2}}{4\,(2\pi)^6} \left(\sum_{p \in L_k} \frac{1}{\sqrt{\lambda_{k,p}}}\right)^2 \\
&\leq \frac{\hat{V}_k^2 k_F^{-2}}{4\,(2\pi)^6} \left(C k_F^{\frac{3}{2}} |k|^{\frac{1}{2}}\right)^2 \leq C k_F \hat{V}_k^2\,|k|
\end{aligned} \tag{10.3.2}$$

where we used the integral identity

$$\int_0^\infty \frac{a}{a^2 + t^2} \frac{b}{b^2 + t^2} dt = \frac{\pi}{2} \frac{1}{a + b}, \quad a, b > 0, \tag{10.3.3}$$

and the estimate $\sum_{p\in L_k}\lambda_{k,p}^{-\frac{1}{2}}\leq Ck_F^{\frac{3}{2}}|k|^{\frac{1}{2}}$. Consequently $\sum_{k\in\overline{B}\left(0,k_F^\gamma\right)\cap\mathbb{Z}_*^3}\mathrm{tr}(E_k-h_k-P_k)-E_{\mathrm{corr,bos}}$ is bounded by

$$\begin{aligned}\sum_{k\in\mathbb{Z}_*^3\backslash\overline{B}\left(0,k_F^\gamma\right)}|\mathrm{tr}(E_k-h_k-P_k)|&\leq Ck_F\sum_{k\in\mathbb{Z}_*^3\backslash\overline{B}\left(0,k_F^\gamma\right)}\hat{V}_k^2|k|\leq Ck_F^{1-\gamma}\sum_{k\in\mathbb{Z}_*^3\backslash\overline{B}\left(0,k_F^\gamma\right)}\hat{V}_k^2|k|^2\\&\leq Ck_F^{1-\gamma}\left(\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k|k|\right)^2\leq Ck_F^{1-\gamma}\end{aligned}\tag{10.3.4}$$

as claimed in Proposition 10.0.3, where we used that $\sum_k|a_k|^2\leq\left(\sum_k|a_k|\right)^2$.

We will not prove the exchange term bounds of the proposition here, but let us mention the idea behind kinetic estimation: The thing to note is that the idea of the kinetic estimate of Proposition 10.1.2 immediately generalizes as

$$\begin{aligned}\|b_k(\varphi)\Psi\|&\leq\sum_{p\in L_k}|\langle\varphi,e_p\rangle|\left\|c_{p-k}^*c_p\Psi\right\|\leq\sqrt{\sum_{p\in L_k}\lambda_{k,p}^{-1}|\langle\varphi,e_p\rangle|^2}\sqrt{\sum_{p\in L_k}\lambda_{k,p}\left\|c_{p-k}^*c_p\Psi\right\|^2}\\&\leq 2^{-\frac{1}{2}}\sqrt{\left\langle\varphi,h_k^{-1}\varphi\right\rangle\langle\Psi,H'_{\mathrm{kin}}\Psi\rangle}\end{aligned}\tag{10.3.5}$$

and so, as $\varepsilon_{k,k}(\varphi;\varphi)\leq 0$, also

$$\|b_k^*(\varphi)\Psi\|^2\leq\|b_k(\varphi)\Psi\|^2+\|\varphi\|^2\|\Psi\|^2\leq\frac{1}{2}\left\langle\varphi,h_k^{-1}\varphi\right\rangle\langle\Psi,H'_{\mathrm{kin}}\Psi\rangle+\|\varphi\|^2\|\Psi\|^2,\tag{10.3.6}$$

so for any $\Psi\in D(H'_{\mathrm{kin}})$

$$\|b_k(\varphi)\Psi\|\leq\|h_k^{-\frac{1}{2}}\varphi\|\sqrt{\langle\Psi,H'_{\mathrm{kin}}\Psi\rangle},\quad\|b_k^*(\varphi)\Psi\|\leq\|h_k^{-\frac{1}{2}}\varphi\|\sqrt{\langle\Psi,H'_{\mathrm{kin}}\Psi\rangle}+\|\varphi\|\|\Psi\|.\tag{10.3.7}$$

These inequalities allow us to arbitrage between the one-body and many-body kinetic operators. As we have good control on both the one-body quantities and the many-body kinetic energy, this is a significant improvement over pure $\mathcal{N}_E$ estimates given our poor control of this quantity.

To illustrate the application of these bounds, let us derive the Gronwall estimate for $e^{t\mathcal{K}}H'_{\mathrm{kin}}e^{-t\mathcal{K}}$; this amounts to controlling

$$[\mathcal{K},H'_{\mathrm{kin}}]=\sum_{k\in\overline{B}\left(0,k_F^\gamma\right)\cap\mathbb{Z}_*^3}Q_2^k(\{K_k,h_k\})\tag{10.3.8}$$

in terms of $H'_{\mathrm{kin}}+k_F$. We derive a general kinetic bound for a $Q_2^k(B)$ operator: By the kinetic estimate

$$\begin{aligned}\left|\left\langle\Psi,Q_2^k(B)\Psi\right\rangle\right|&\leq 2\sum_{p\in L_k}|\langle\Psi,b_k(Be_p)b_{-k,-p}\Psi\rangle|\leq 2\sum_{p\in L_k}\|b_k^*(Be_p)\Psi\|\|b_{-k,-p}\Psi\|\\&\leq 2\sum_{p\in L_k}\left(\|h_k^{-\frac{1}{2}}Be_p\|\sqrt{\langle\Psi,H'_{\mathrm{kin}}\Psi\rangle}+\|Be_p\|\|\Psi\|\right)\|b_{-k,-p}\Psi\|,\end{aligned}\tag{10.3.9}$$

and by Cauchy-Schwarz we have that

$$\sum_{p\in L_k}\|h_k^{-\frac{1}{2}}Be_p\|\,\|b_{-k,-p}\Psi\| \leq \sqrt{\sum_{p\in L_k}\lambda_{k,p}^{-1}\|h_k^{-\frac{1}{2}}Be_p\|^2}\sqrt{\sum_{p\in L_k}\lambda_{-k,-p}\,\|b_{-k,-p}\Psi\|^2} \tag{10.3.10}$$
$$\leq \sqrt{\sum_{p\in L_k}\|h_k^{-\frac{1}{2}}Bh_k^{-\frac{1}{2}}e_p\|^2}\sqrt{\langle\Psi,H'_{\mathrm{kin}}\Psi\rangle} = \|h_k^{-\frac{1}{2}}Bh_k^{-\frac{1}{2}}\|_{\mathrm{HS}}\sqrt{\langle\Psi,H'_{\mathrm{kin}}\Psi\rangle}$$

and similarly

$$\sum_{p\in L_k}\|Be_p\|\,\|b_{-k,-p}\Psi\| \leq \sqrt{\sum_{p\in L_k}\lambda_{k,p}^{-\frac{1}{2}}\,\|Be_p\|^2}\sqrt{\sum_{p\in L_k}\lambda_{-k,-p}\,\|b_{-k,-p}\Psi\|^2} \tag{10.3.11}$$
$$\leq \|Bh_k^{-\frac{1}{2}}\|_{\mathrm{HS}}\sqrt{\langle\Psi,H'_{\mathrm{kin}}\Psi\rangle},$$

whence

$$\left|\left\langle\Psi,Q_2^k(B)\Psi\right\rangle\right| \leq \|h_k^{-\frac{1}{2}}Bh_k^{-\frac{1}{2}}\|_{\mathrm{HS}}\,\langle\Psi,H'_{\mathrm{kin}}\Psi\rangle + \|Bh_k^{-\frac{1}{2}}\|_{\mathrm{HS}}\,\|\Psi\|\,\sqrt{\langle\Psi,H'_{\mathrm{kin}}\Psi\rangle}. \tag{10.3.12}$$

For $B=\{K_k,h_k\}$, it follows from our one-body operator estimates that

$$\|h_k^{-\frac{1}{2}}\,\{K_k,h_k\}\,h_k^{-\frac{1}{2}}\|_{\mathrm{HS}} \leq C\hat{V}_k, \quad \|\,\{K_k,h_k\}\,h_k^{-\frac{1}{2}}\|_{\mathrm{HS}} \leq Ck_F^{\frac{1}{2}}\hat{V}_k\,|k|^{\frac{1}{2}}\,, \tag{10.3.13}$$

so

$$\left|\left\langle\Psi,Q_2^k(\{K_k,h_k\})\Psi\right\rangle\right| \leq C\hat{V}_k\,\langle\Psi,H'_{\mathrm{kin}}\Psi\rangle + Ck_F^{\frac{1}{2}}\hat{V}_k\,|k|^{\frac{1}{2}}\,\|\Psi\|\,\sqrt{\langle\Psi,H'_{\mathrm{kin}}\Psi\rangle} \tag{10.3.14}$$
$$\leq C\hat{V}_k\,|k|^{\frac{1}{2}}\,\langle\Psi,(H'_{\mathrm{kin}}+k_F)\Psi\rangle$$

i.e. $\pm\,Q_2^k(\{K_k,h_k\}) \leq C\hat{V}_k\,|k|^{\frac{1}{2}}\,(H'_{\mathrm{kin}}+k_F)$, whence

$$\pm\,[\mathcal{K},H'_{\mathrm{kin}}] \leq C\left(\sum_{k\in\overline{B}\left(0,k_F^{\gamma}\right)\cap\mathbb{Z}^3_*}\hat{V}_k\,|k|^{\frac{1}{2}}\right)(H'_{\mathrm{kin}}+k_F) \leq C(H'_{\mathrm{kin}}+k_F) \tag{10.3.15}$$

as desired.

10.4 The Second Transformation

In this last subsection we consider the one-body operator estimates needed to control the second transformation. First note that for $\mathcal{J}$ as defined by equation (10.0.8), computation using the quasi-bosonic commutation relations as in Section 4 establishes that $\mathcal{J}$ obeys

$$[\mathcal{J},b_k(\varphi)] = b_k(J_k\varphi) + \sum_{l\in\overline{B}\left(0,k_F^{\gamma}\right)\cap\mathbb{Z}^3_*}\sum_{q\in L_l}\varepsilon_{k,l}(\varphi;e_q)b_l(J_le_q) \tag{10.4.1}$$

hence

$$\left[\mathcal{J},Q_1^k(A)\right]=Q_1^k([J_k,A])+\mathcal{E}_3^k(A)\tag{10.4.2}$$

for symmetric $A:\ell^2(L_k)\to\ell^2(L_k)$, where $\mathcal{E}_3^k(A)$ is given by

$$\mathcal{E}_3^k(A)=2\sum_{l\in\overline{B}\left(0,k_F^\gamma\right)\cap\mathbb{Z}_*^3}\sum_{p\in L_k}\sum_{q\in L_l}\mathrm{Re}(b_k^*(Ae_p)\varepsilon_{k,l}(e_p;e_q)b_l(J_le_q)).\tag{10.4.3}$$

We estimate a generic term of $\mathcal{E}_3^k(A)$ using the kinetic bound of equation (10.3.7) in the manner of [11]: We have

$$\begin{aligned}&\sum_{l\in\overline{B}\left(0,k_F^\gamma\right)\cap\mathbb{Z}_*^3}\sum_{p\in L_k\cap L_l}\left|\left\langle\Psi,b_k^*(Ae_p)c_{p-l}c_{p-k}^*b_l(J_le_p)\Psi\right\rangle\right|\\
&\leq\sum_{l\in\overline{B}\left(0,k_F^\gamma\right)\cap\mathbb{Z}_*^3}\sum_{p\in L_k\cap L_l}\left\|b_k(Ae_p)c_{p-l}^*\Psi\right\|\left\|b_l(J_le_p)c_{p-k}^*\Psi\right\| \qquad (10.4.4)\\
&\leq\sum_{l\in\overline{B}\left(0,k_F^\gamma\right)\cap\mathbb{Z}_*^3}\sum_{p\in L_k\cap L_l}\|h_k^{-\frac{1}{2}}Ae_p\|\|h_l^{-\frac{1}{2}}J_le_p\|\sqrt{\left\langle\Psi,c_{p-l}H_{\mathrm{kin}}'c_{p-l}^*\Psi\right\rangle}\sqrt{\left\langle\Psi,c_{p-k}H_{\mathrm{kin}}'c_{p-k}^*\Psi\right\rangle}\\
&\leq\left(\max_{p\in L_k}\|h_k^{-\frac{1}{2}}Ae_p\|\right)\sqrt{\langle\Psi,H_{\mathrm{kin}}'\Psi\rangle}\sum_{l\in\overline{B}\left(0,k_F^\gamma\right)\cap\mathbb{Z}_*^3}\sum_{p\in L_k\cap L_l}\|h_l^{-\frac{1}{2}}J_le_p\|\sqrt{\left\langle\Psi,c_{p-l}H_{\mathrm{kin}}'c_{p-l}^*\Psi\right\rangle}\\
&\leq\left(\max_{p\in L_k}\|h_k^{-\frac{1}{2}}Ae_p\|\right)\left(\sum_{l\in\overline{B}\left(0,k_F^\gamma\right)\cap\mathbb{Z}_*^3}\|h_l^{-\frac{1}{2}}J_l\|_{\mathrm{HS}}\right)\sqrt{\langle\Psi,H_{\mathrm{kin}}'\Psi\rangle\left\langle\Psi,\mathcal{N}_EH_{\mathrm{kin}}'\Psi\right\rangle}.\end{aligned}$$

Controlling the error term of the transformation of equation (10.0.12) thus requires us to estimate one-body quantities of the form $\max_{p\in L_k}\|h_k^{-\frac{1}{2}}E_k(t)e_p\|$, where $E_k(t)$ is given by

$$E_k(t)=e^{tJ_k}e^{-K_k}h_ke^{-K_k}e^{-tJ_k}-h_k.\tag{10.4.5}$$

We consider this in the abstract one-body setting of Section 6. In this case, the unitary transformation U is given by

$$U=\left(h^2+2P_{h^{\frac{1}{2}}v}\right)^{\frac{1}{4}}h^{-\frac{1}{2}}e^K,\tag{10.4.6}$$

and by using the integral identity

$$a^{\frac{1}{4}}=\frac{2\sqrt{2}}{\pi}\int_0^\infty\left(1-\frac{t^4}{a+t^4}\right)dt,\quad a>0,\tag{10.4.7}$$

one can derive a representation formula for an operator of the form $(A+gP_w)^{\frac{1}{4}}$ similar to that of Proposition 6.1.2 with the following consequence:

Proposition 10.4.1. *For all $1\leq i,j\leq n$ it holds that*

$$\left|\left\langle x_i,\left(\left(h^2+2P_{h^{\frac{1}{2}}v}\right)^{\frac{1}{4}}-h^{\frac{1}{2}}\right)x_j\right\rangle\right|\leq2\frac{\sqrt{\lambda_i\lambda_j}}{\sqrt{\lambda_i}+\sqrt{\lambda_j}}\frac{\langle x_i,v\rangle\langle v,x_j\rangle}{\lambda_i+\lambda_j}.$$

This implies the following elementwise bounds for U:

Proposition 10.4.2. *For all $1 \leq i, j \leq n$ it holds that*

$$|\langle x_i, (U-1)x_j\rangle|, |\langle x_i, (U^*-1)x_j\rangle| \leq 3\left(1+\left\langle v, h^{-1}v\right\rangle\right)\frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i+\lambda_j}.$$

Proof: It suffices to consider $U-1$. Writing

$$\begin{aligned} U-1 &= \left(\left(h^2+2P_{h^{\frac{1}{2}}v}\right)^{\frac{1}{4}} - h^{\frac{1}{2}}\right)h^{-\frac{1}{2}}e^K + h^{\frac{1}{2}}h^{-\frac{1}{2}}e^K - 1 \\ &= \left(e^K-1\right) + \left(\left(h^2+2P_{h^{\frac{1}{2}}v}\right)^{\frac{1}{4}} - h^{\frac{1}{2}}\right)h^{-\frac{1}{2}} + \left(\left(h^2+2P_{h^{\frac{1}{2}}v}\right)^{\frac{1}{4}} - h^{\frac{1}{2}}\right)h^{-\frac{1}{2}}\left(e^K-1\right) \end{aligned} \tag{10.4.8}$$

we estimate each part in turn. Firstly, we already know that

$$\left|\left\langle x_i, \left(e^K-1\right)x_j\right\rangle\right| \leq \frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i+\lambda_j} \tag{10.4.9}$$

by Proposition 6.1.7. Meanwhile, by the previous proposition

$$\left|\left\langle x_i, \left(\left(h^2+2P_{h^{\frac{1}{2}}v}\right)^{\frac{1}{4}} - h^{\frac{1}{2}}\right)h^{-\frac{1}{2}}x_j\right\rangle\right| \leq \frac{1}{\sqrt{\lambda_j}}\frac{2\sqrt{\lambda_i\lambda_j}}{\sqrt{\lambda_i}+\sqrt{\lambda_j}}\frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i+\lambda_j} \leq 2\frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i+\lambda_j} \tag{10.4.10}$$

and using both of these estimates we also find that

$$\begin{aligned} &\left|\left\langle x_i, \left(\left(h^2+2P_{h^{\frac{1}{2}}v}\right)^{\frac{1}{4}} - h^{\frac{1}{2}}\right)h^{-\frac{1}{2}}\left(e^K-1\right)x_j\right\rangle\right| \\ &\leq \sum_{k=1}^n \left|\left\langle x_i, \left(\left(h^2+2P_{h^{\frac{1}{2}}v}\right)^{\frac{1}{4}} - h^{\frac{1}{2}}\right)h^{-\frac{1}{2}}x_k\right\rangle\right| \left|\left\langle x_k, \left(e^K-1\right)x_j\right\rangle\right| \\ &\leq 2\sum_{k=1}^n \frac{\langle x_i, v\rangle \langle v, x_k\rangle}{\lambda_i+\lambda_k}\frac{\langle x_k, v\rangle \langle v, x_j\rangle}{\lambda_k+\lambda_j} \leq 2\left(\sum_{k=1}^n \frac{|\langle x_k, v\rangle|^2}{\lambda_k}\right)\frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i+\lambda_j} \\ &= 2\left\langle v, h^{-1}v\right\rangle \frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i+\lambda_j} \end{aligned} \tag{10.4.11}$$

where we used that $(a+c)^{-1}(b+c)^{-1} \leq c^{-1}(a+b)^{-1}$ for $a, b, c > 0$ (as $c(a+b) \leq (a+c)(b+c)$ follows by expansion). Combining the estimates yields the claim.

□

Recall that for the particular operators h_k and P_{v_k} it holds that $\left\langle v_k, h_k^{-1}v_k\right\rangle \leq C\hat{V}_k$, so for the purposes of estimation this matrix element estimate for $U = e^J$ is almost as good as that for e^K of Proposition 6.1.7. Unlike that proposition, however, we can not extend this to e^{tJ} for general $t \in [0,1]$, as we now lack the required monotonicity.

We can work around this by finding a way to reduce estimates involving e^{tJ} to ones involving U. To provide a concrete example, let us consider a term we would need to control $\mathcal{E}_3^k(E_k(t))$: In the general setting, we consider $E(t)$ defined by

$$E(t) = e^{tJ}e^{-K}he^{-K}e^{-tJ} - h = \Big(e^{tJ}he^{-tJ} - h\Big) + e^{tJ}\Big(e^{-K}he^{-K} - h\Big)e^{-tJ} =: E_1(t) + E_2(t) \tag{10.4.12}$$

and decompose $E_1(t)$ further as

$$E_1(t) = \Big(e^{tJ} - 1\Big)h + h\Big(e^{-tJ} - 1\Big) + \Big(e^{tJ} - 1\Big)h\Big(e^{-tJ} - 1\Big). \tag{10.4.13}$$

We consider the first term, and so need to estimate $\max_{1\leq i\leq n} \|h^{-\frac{1}{2}}\Big(e^{tJ} - 1\Big)hx_i\|$. As mentioned we are to find a way to replace $e^{tJ} - 1$ by $U - 1$ (and possibly $U^* - 1$). Now, J is the principal logarithm of U, and as U is unitary, hence normal, and we are working on a finite-dimensional space (which we now consider as a complex vector space), there exists an orthonormal basis $(w_j)_{j=1}^n$ and real numbers $(\theta_j)_{j=1}^n \subset [-\pi, \pi)$ such that

$$e^{\pm tJ}w_j = e^{\pm it\theta_j}w_j, \quad 1 \leq j \leq n. \tag{10.4.14}$$

With respect to this basis, our task thus amounts to estimating $e^{it\theta} - 1$ in terms of $e^{i\theta} - 1$ and $e^{-i\theta} - 1$. To that end we note the following: There exists a $C > 0$ such that for all $t \in [-1, 1]$ and $\theta \in [-\pi, \pi]$

$$\left|\Big(e^{it\theta} - 1\Big) - t\Big(e^{i\theta} - 1\Big) + \frac{t(1-t)}{2}\Big(e^{i\theta} + e^{-i\theta} - 2\Big)\right| \leq C\left|e^{i\theta} - 1\right|^3. \tag{10.4.15}$$

(There is a particular reason for why we want a cubic error bound - we will explain this at the end.)

This bound follows by considering the series expansion for e^x and compactness of $[-1, 1] \times [-\pi, \pi]$. Motivated by this, we define the operator F_t for $t \in [0, 1]$ by

$$F_t = t(U - 1) - \frac{t(1-t)}{2}(U + U^* - 2). \tag{10.4.16}$$

We then have the following:

Proposition 10.4.3. *For any $T : V \to V$, $x \in V$, $m \in \{1, 2\}$ and $t \in [0, 1]$ it holds that*

$$\left\|T\Big(e^{tJ} - 1 - F_t\Big)x\right\|, \left\|T\Big(e^{-tJ} - 1 - F_t^*\Big)x\right\| \leq C\,\|T(U-1)^m\|_{\mathrm{HS}}\left\|(U-1)^{3-m}x\right\|$$

and for all $1 \leq i, j \leq n$

$$|\langle x_i, F_t x_j\rangle|, |\langle x_i, F_t^* x_j\rangle| \leq C\Big(1 + \left\langle v, h^{-1}v\right\rangle\Big)\frac{\langle x_i, v\rangle\langle v, x_j\rangle}{\lambda_i + \lambda_j}$$

for a constant $C > 0$ independent of all quantities.

Proof: It suffices to consider $\left\|T\left(e^{tJ}-1-F_t\right)x\right\|$. By orthonormal expansion using the basis $(w_j)_{j=1}^n$, it holds by equation (10.4.15) and Cauchy-Schwarz that

$$\begin{aligned}
&\left\|T\left(e^{tJ}-1-F_t\right)x\right\|^2 = \sum_{j=1}^n \left|\left\langle w_j, T\left(e^{tJ}-1-F_t\right)x\right\rangle\right|^2 \\
&= \sum_{j=1}^n \left|\sum_{k=1}^n \left\langle w_j, T\left(e^{tJ}-1-F_t\right)w_k\right\rangle \langle w_k, x\rangle\right|^2 \\
&= \sum_{j=1}^n \left|\sum_{k=1}^n \left(\left(e^{it\theta_k}-1\right) - t\left(e^{i\theta_k}-1\right) + \frac{t(1-t)}{2}\left(e^{i\theta_k}+e^{-i\theta_k}-2\right)\right) \langle w_j, Tw_k\rangle \langle w_k, x\rangle\right|^2 \\
&\leq C\sum_{j=1}^n \left(\sum_{k=1}^n \left|e^{i\theta_k}-1\right|^3 |\langle w_j, Tw_k\rangle| \, |\langle w_k, x\rangle|\right)^2 \qquad (10.4.17) \\
&\leq C\sum_{j=1}^n \left(\sum_{k=1}^n \left|e^{i\theta_k}-1\right|^{2m} |\langle w_j, Tw_k\rangle|^2\right)\left(\sum_{k=1}^n \left|e^{i\theta_k}-1\right|^{2(3-m)} |\langle w_k, x\rangle|^2\right) \\
&= C\left(\sum_{j,k=1}^n |\langle w_j, T(U-1)^m w_k\rangle|^2\right)\left(\sum_{k=1}^n \left|\left\langle w_k, (U-1)^{3-m}x\right\rangle\right|^2\right) \\
&= C\,\|T(U-1)^m\|_{\mathrm{HS}}^2 \left\|(U-1)^{3-m}x\right\|^2
\end{aligned}$$

which implies the first claim. The elementwise estimates for F_t and F_t^* follow immediately from Proposition 10.4.2.

□

By the proposition we then have that

$$\begin{aligned}
\|h^{-\frac{1}{2}}\left(e^{tJ}-1\right)hx_i\| &\leq \|h^{-\frac{1}{2}}F_t hx_i\| + \|h^{-\frac{1}{2}}\left(e^{tJ}-1-F_t\right)hx_i\| \qquad (10.4.18) \\
&\leq \|h^{-\frac{1}{2}}F_t hx_i\| + C\|h^{-\frac{1}{2}}(U-1)^2\|_{\mathrm{HS}}\,\|(U-1)hx_i\|
\end{aligned}$$

and so have reduced the estimation to operators which we have good control over. We can estimate that

$$\begin{aligned}
&\|h^{-\frac{1}{2}}F_t hx_i\|^2 = \sum_{j=1}^n \left|\left\langle x_j, h^{-\frac{1}{2}}F_t hx_i\right\rangle\right|^2 = \sum_{j=1}^n \frac{\lambda_i^2}{\lambda_j}|\langle x_j, F_t x_i\rangle|^2 \qquad (10.4.19) \\
&\leq C\left(1+\left\langle v, h^{-1}v\right\rangle\right)^2 \sum_{j=1}^n \frac{\lambda_i^2}{\lambda_j}\left|\frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i+\lambda_j}\right|^2 \leq C\left(1+\left\langle v, h^{-1}v\right\rangle\right)^2 \left\langle v, h^{-1}v\right\rangle |\langle x_i, v\rangle|^2
\end{aligned}$$

and likewise

$$\begin{aligned}
\|(U-1)hx_i\|^2 &= \sum_{j=1}^n \lambda_i^2 |\langle x_j, (U-1)x_i\rangle|^2 \leq C\left(1+\left\langle v, h^{-1}v\right\rangle\right)^2 \sum_{j=1}^n \lambda_i^2 \left|\frac{\langle x_i, v\rangle \langle v, x_j\rangle}{\lambda_i+\lambda_j}\right|^2 \\
&\leq C\left(1+\left\langle v, h^{-1}v\right\rangle\right)^2 \|v\|^2 |\langle x_i, v\rangle|^2, \qquad (10.4.20)
\end{aligned}$$

while

$$\|h^{-\frac{1}{2}}(U-1)^2\|_{\mathrm{HS}}^2 = \sum_{i,j=1}^{n}\left|\left\langle x_i, h^{-\frac{1}{2}}(U-1)^2 x_j\right\rangle\right|^2 = \sum_{i,j=1}^{n}\frac{1}{\lambda_i}\left|\sum_{k=1}^{n}\langle x_i,(U-1)x_k\rangle\langle x_k,(U-1)x_j\rangle\right|^2$$
$$\leq C\Big(1+\left\langle v,h^{-1}v\right\rangle\Big)^4\sum_{i,j=1}^{n}\frac{1}{\lambda_i}\left|\sum_{k=1}^{n}\frac{\langle x_i,v\rangle\langle v,x_k\rangle}{\lambda_i+\lambda_k}\frac{\langle x_k,v\rangle\langle v,x_j\rangle}{\lambda_k+\lambda_j}\right|^2 \quad (10.4.21)$$
$$\leq C\Big(1+\left\langle v,h^{-1}v\right\rangle\Big)^4\sum_{i,j=1}^{n}\frac{|\langle x_i,v\rangle|^2}{\lambda_i^{\frac{5}{4}}}\frac{|\langle v,x_j\rangle|^2}{\lambda_j^{\frac{5}{4}}}\left(\sum_{k=1}^{n}\frac{|\langle x_k,v\rangle|^2}{\lambda_k^{\frac{7}{8}}\lambda_k^{\frac{3}{8}}}\right)^2$$
$$= C\Big(1+\left\langle v,h^{-1}v\right\rangle\Big)^4\left\langle v,h^{-\frac{5}{4}}v\right\rangle^4,$$

so in all

$$\|h^{-\frac{1}{2}}\Big(e^{tJ}-1\Big)hx_i\| \leq C\Big(1+\left\langle v,h^{-1}v\right\rangle\Big)^3\left(\sqrt{\langle v,h^{-1}v\rangle}+\|v_k\|\left\langle v,h^{-\frac{5}{4}}v\right\rangle^2\right)|\langle x_i,v\rangle|\,. \quad (10.4.22)$$

For the particular operators h_k and P_{v_k}, this implies that

$$\max_{p\in L_k}\|h_k^{-\frac{1}{2}}\Big(e^{tJ_k}-1\Big)h_k e_p\| \leq Ck_F^{-\frac{1}{2}}\Big(1+\hat{V}_k\Big)^3\hat{V}_k\left(1+k_F^{\frac{1}{2}}|k|^{\frac{1}{2}}\left\langle v_k,h_k^{-\frac{5}{4}}v_k\right\rangle^2\right). \quad (10.4.23)$$

The inner product is

$$\left\langle v_k,h_k^{-\frac{5}{4}}v_k\right\rangle = \frac{\hat{V}_k k_F^{-1}}{2\,(2\pi)^3}\sum_{p\in L_k}\lambda_{k,p}^{-\frac{5}{4}} \quad (10.4.24)$$

and this Riemann sum is more singular than what we consider in appendix section B. Nonetheless, the methods used therein - in particular, the summation formula of Proposition B.3.3 - implies the following:

Proposition 10.4.4. *For all $k\in\overline{B}(0,k_F^{\gamma})\cap\mathbb{Z}_*^3$, $0<\gamma<\frac{1}{47}$, it holds that*

$$\sum_{p\in L_k}\lambda_{k,p}^{-\frac{5}{4}} \leq Ck_F^{\frac{3}{4}}|k|^{-\frac{1}{4}}$$

for a constant $C>0$ depending only on γ.

With this we arrive at

$$\max_{p\in L_k}\|h_k^{-\frac{1}{2}}\Big(e^{tJ_k}-1\Big)h_k e_p\| \leq Ck_F^{-\frac{1}{2}}\Big(1+\hat{V}_k\Big)^5\hat{V}_k \quad (10.4.25)$$

provided $\gamma<\frac{1}{47}$, which is sufficient for the purposes of Proposition 10.0.4.

Finally, regarding the bound of (10.4.15), it likewise holds that

$$\left|\Big(e^{it\theta}-1\Big)-t\Big(e^{i\theta}-1\Big)\right| \leq C\left|e^{i\theta}-1\right|^2 \quad (10.4.26)$$

and so, considering simply $F'_t = t(U-1)$, that e.g.

$$\|h^{-\frac{1}{2}}\left(e^{tJ} - 1 - F'_t\right)hx_i\| \leq C\|h^{-\frac{1}{2}}(U-1)\|_{\mathrm{HS}}\,\|(U-1)hx_i\|\,. \tag{10.4.27}$$

The issue with this lies in the fact that we now have to deal with $\|h^{-\frac{1}{2}}(U-1)\|_{\mathrm{HS}}$ instead of $\|h^{-\frac{1}{2}}(U-1)^2\|_{\mathrm{HS}}$; this can be estimated similarly, but with the result

$$\|h^{-\frac{1}{2}}(U-1)\|_{\mathrm{HS}} \leq C\left(1 + \left\langle v, h^{-1}v\right\rangle\right)^2 \left\langle v, h^{-\frac{3}{2}}v\right\rangle. \tag{10.4.28}$$

Formally - i.e. if one replaces the Riemann sums with integrals - it is true that $\left\langle v_k, h_k^{-\frac{3}{2}} v_k\right\rangle \sim \left\langle v_k, h_k^{-\frac{5}{4}} v_k\right\rangle^2$ with respect to k_F, and so there should not be a difference. The result of appendix section B however only extends (optimally) to Riemann sums of the form $\sum_{p\in L_k} \lambda_{k,p}^{\beta}$ for $\beta > -\frac{4}{3}$, and so $\left\langle v_k, h_k^{-\frac{3}{2}} v_k\right\rangle$ it outside the range which we are able to control, even with a cut-off in k.

Chapter 11

Plasmon Modes of the Effective Hamiltonian

In this final section we consider the effective operator

$$H_{\text{eff}} = H'_{\text{kin}} + 2\sum_{k\in\mathbb{Z}^3_*} Q^1_k\Big(\tilde{E}_k - h_k\Big) = H'_{\text{kin}} + 2\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in L_k} b^*_k\Big(\Big(\tilde{E}_k - h_k\Big)e_p\Big)b_{k,p}, \tag{11.0.1}$$

where $\tilde{E}_k = (h_k^2 + 2P_{h_k^{\frac{1}{2}}v_k})^{\frac{1}{2}}$, in detail. As we will consider H_{eff} in isolation from the proper Hamiltonian, we will now omit the mean-field scaling factor k_F^{-1} - concretely this means that $v_k \in \ell^2(L_k)$ is now given by

$$v_k = \frac{s\hat{V}_k}{2\,(2\pi)^3}\sum_{p\in L_k} e_p. \tag{11.0.2}$$

For this section we will fix a $k \in B_F$, let $\phi \in \ell^2(L_k)$ denote the normalized eigenvector of $2\tilde{E}_k$ corresponding to the greatest eigenvalue ϵ_k, and define $\Psi_M \in \{\mathcal{N}_E = M\}$ by

$$\Psi_M = b^*_k(\phi)^M\psi_F, \quad M \in \mathbb{N}_0. \tag{11.0.3}$$

(For the statements of certain propositions below we will understand $\Psi_{-1}, \Psi_{-2} = 0$.)

The main result of this section is the following bound for $\hat{\Psi}_M$:

Theorem 11.0.1. *There exists constants $c, C > 0$ such that if $\hat{V}_k > ck_F^{-1}$ it holds for all $M \leq Ck_F^2\,|k|$ that $\hat{\Psi}_M = \|\Psi_M\|^{-1}\,\Psi_M$ obeys*

$$\|(H_{\text{eff}} - M\epsilon_k)\hat{\Psi}_M\| \leq C'\sqrt{\sum_{l\in\mathbb{Z}^3_*}\min\Big\{1, k_F\hat{V}_l, k_F^3\hat{V}_l\,|l|^{-2}\Big\}\,\hat{V}_l\,|l|^2}\,\frac{M^{\frac{5}{2}}}{\sqrt{k_F}\,|k|}, \quad k_F \to \infty,$$

where ϵ_k denotes the greatest eigenvalue of $2\tilde{E}_k$, which obeys $\epsilon_k \geq c's^{\frac{1}{2}}k_F^{\frac{3}{2}}\,|k|\,\hat{V}_k^{\frac{1}{2}}$ and

$$0 \leq \epsilon_k - 2\sqrt{2\,\langle v_k, h_k v_k\rangle + \frac{\langle v_k, h_k^3 v_k\rangle}{\langle v_k, h_k v_k\rangle}} \leq C'k_F^{-\frac{1}{2}}\,|k|\,\hat{V}_k^{-\frac{3}{2}}, \quad k_F \to \infty,$$

for constants $c', C' > 0$. The constants c, c', C, C' are independent of all quantities.

Note that Theorem 1.1.4 is an immediate consequence of this result: For $\hat{V}_k = g\,|k|^{-2}$, the condition $\hat{V}_k > c k_F^{-1}$ becomes

$$|k| < \sqrt{\frac{g}{c} k_F} \tag{11.0.4}$$

which is ensured for all $k \in \overline{B}\left(0, k_F^{\delta}\right) \cap \mathbb{Z}_*^3$ for k_F suffiently large provided $\delta \in \left(0, \frac{1}{2}\right)$. That $M \leq k_F^{\varepsilon}$ for $\varepsilon \in (0,2)$ similarly ensures that $M \leq C k_F^2 \,|k|$ for k_F sufficiently large, so the conditions of the theorem hold, and the sum can be estimated as

$$\begin{aligned} \sum_{l \in \mathbb{Z}_*^3} \min\left\{1, k_F \hat{V}_l, k_F^3 \hat{V}_l \,|l|^{-2}\right\} \hat{V}_l \,|l|^2 &\leq \max\{1, g\} \sum_{l \in \mathbb{Z}_*^3} \min\left\{1, k_F \,|l|^{-2}, k_F^3 \,|l|^{-4}\right\} \\ &\leq C\left(\sum_{l \in \overline{B}\left(0, \sqrt{k_F}\right) \cap \mathbb{Z}_*^3} 1 + k_F \sum_{l \in B_F \backslash \overline{B}\left(0, \sqrt{k_F}\right)} |l|^{-2} + k_F^3 \sum_{l \in \mathbb{Z}_*^3 \backslash B_F} |l|^{-4}\right) \\ &\leq C\left(\left(\sqrt{k_F}\right)^3 + k_F^2 + k_F^2\right) \leq C k_F^2. \end{aligned} \tag{11.0.5}$$

The statement regarding ϵ_k follows by expanding the inner products $\left\langle v_k, h_k^{\beta} v_k\right\rangle$ and inserting $\hat{V}_k = g\,|k|^{-2}$.

11.1 Properties of the Plasmon State Ψ_M

Owing to the inequality (which in the exact bosonic case would be an equality)

$$\|\Psi_M\|^2 = \|b_k^*(\phi)\Psi_{M-1}\|^2 \leq \|(\mathcal{N}_E + 1)^{\frac{1}{2}} \Psi_{M-1}\|^2 = M\,\|\Psi_{M-1}\|^2 \tag{11.1.1}$$

we can control the ratio $\|\Psi_M\|^{-1} \|\Psi_{M-1}\|$ well from below, but for the purposes of Theorem 11.0.1 it is an upper bound which will be needed. To that end we begin by noting the following:

Lemma 11.1.1. *For any $p \in B_F^c$, $q \in B_F$, $1 \leq \sigma \leq s$ and $M \in \mathbb{N}$ it holds that*

$$\begin{aligned} c_{p,\sigma} \Psi_M &= 1_{L_k}(p) M s^{-\frac{1}{2}} \langle e_p, \phi\rangle\, c_{p-k,\sigma} \Psi_{M-1} \\ c_{q,\sigma}^* \Psi_M &= -1_{L_k}(q+k) M s^{-\frac{1}{2}} \langle e_{q+k}, \phi\rangle\, c_{q+k,\sigma}^* \Psi_{M-1}. \end{aligned}$$

As a consequence it holds for any $l \in \mathbb{Z}_*^3$ and $p \in L_l$ that

$$b_{l,p} \Psi_M = \delta_{k,l} M \langle e_p, \phi\rangle\, \Psi_{M-1} + 1_{L_k}(p) \frac{M(M-1)}{s^{\frac{3}{2}}} \sum_{q \in L_k}^{\sigma} \delta_{p-l,q-k} \langle e_p, \phi\rangle \langle e_q, \phi\rangle\, c_{q,\sigma}^* c_{p-k,\sigma} \Psi_{M-2}.$$

Proof: By equation (7.1.19) we have

$$[c_{p,\sigma}, b_k^*(\phi)] = 1_{L_k}(p) s^{-\frac{1}{2}} \langle e_p, \phi\rangle\, c_{p-k,\sigma} \tag{11.1.2}$$

$$\left[c_{q,\sigma}^*, b_k^*(\phi)\right] = -1_{L_k}(q+k)s^{-\frac{1}{2}} \langle e_{q+k}, \phi\rangle \, c_{q+k,\sigma}^*,$$

so

$$\begin{aligned} c_{p,\sigma}\Psi_M &= c_{p,\sigma} b_k^*(\phi)^M \psi_F = b_k^*(\phi)^M c_{p,\sigma}\psi_F + \sum_{j=0}^{M-1} b_k^*(\phi)^j \left[c_{p,\sigma}, b_k^*(\phi)\right] b_k^*(\phi)^{M-j-1}\psi_F \\ &= \sum_{j=0}^{M-1} b_k^*(\phi)^j \Big(1_{L_k}(p) s^{-\frac{1}{2}} \langle e_p, \phi\rangle \, c_{p-k,\sigma}\Big) b_k^*(\phi)^{M-j-1}\psi_F \qquad (11.1.3) \\ &= 1_{L_k}(p) M s^{-\frac{1}{2}} \langle e_p, \phi\rangle \, c_{p-k,\sigma} b_k^*(\phi)^{M-1}\psi_F = 1_{L_k}(p) M s^{-\frac{1}{2}} \langle e_p, \phi\rangle \, c_{p-k,\sigma}\Psi_{M-1} \end{aligned}$$

and likewise for $c_{q,\sigma}^*\Psi_M$, $q \in B_F$. The expression for $b_{l,p}\Psi_M$ then follows as

$$\begin{aligned} b_{l,p}\Psi_M &= \frac{1}{\sqrt{s}} \sum_{\sigma=1}^{s} c_{p-l,\sigma}^* c_{p,\sigma}\Psi_M = \frac{M}{s} \sum_{\sigma=1}^{s} 1_{L_k}(p) \langle e_p, \phi\rangle \, c_{p-l,\sigma}^* c_{p-k,\sigma}\Psi_{M-1} \\ &= \delta_{k,l} 1_{L_k}(p) \frac{M}{s} \sum_{\sigma=1}^{s} \langle e_p, \phi\rangle \, \Psi_{M-1} - 1_{L_k}(p) \frac{M}{s} \sum_{\sigma=1}^{s} \langle e_p, \phi\rangle \, c_{p-k,\sigma} c_{p-l,\sigma}^* \Psi_{M-1} \\ &= \delta_{k,l} M \langle e_p, \phi\rangle \, \Psi_{M-1} \qquad (11.1.4) \\ &- 1_{L_k}(p) \frac{M(M-1)}{s^{\frac{3}{2}}} \sum_{\sigma=1}^{s} 1_{L_k}(p-l+k) \langle e_p, \phi\rangle \langle e_{p-l+k}, \phi\rangle \, c_{p-k,\sigma} c_{p-l+k,\sigma}^* \Psi_{M-2} \\ &= \delta_{k,l} M \langle e_p, \phi\rangle \, \Psi_{M-1} + 1_{L_k}(p) \frac{M(M-1)}{s^{\frac{3}{2}}} \sum_{q\in L_k}^{\sigma} \delta_{p-l,q-k} \langle e_p, \phi\rangle \langle e_q, \phi\rangle \, c_{q,\sigma}^* c_{p-k,\sigma}\Psi_{M-2} \end{aligned}$$

where we used the identity $1_{L_k}(p-l+k) f(p-l+k) = \sum_{q\in L_k} \delta_{p-l,q-k} f(q)$ to rewrite the second term.

□

This implies the following bound:

Corollary 11.1.2. *For any $M \in \mathbb{N}$ it holds that*

$$\|\Psi_M\|^2 \geq M\Big(1 - \frac{M-1}{s} \|\phi\|_\infty^2\Big) \|\Psi_{M-1}\|^2$$

where $\|\phi\|_\infty = \sup_{p\in L_k} |\langle e_p, \phi\rangle|$.

Proof: We estimate

$$\begin{aligned} \|\Psi_M\|^2 &= \langle \Psi_{M-1}, b(\phi)\Psi_M\rangle = \frac{1}{\sqrt{s}} \sum_{p\in L_k}^{\sigma} \langle \phi, e_p\rangle \Big\langle \Psi_{M-1}, c_{p-k,\sigma}^* c_{p,\sigma}\Psi_M \Big\rangle \\ &= \frac{M}{s} \sum_{p\in L_k}^{\sigma} |\langle e_p, \phi\rangle|^2 \Big\langle \Psi_{M-1}, c_{p-k,\sigma}^* c_{p-k,\sigma}\Psi_{M-1} \Big\rangle \qquad (11.1.5) \\ &= \frac{M}{s} \sum_{p\in L_k}^{\sigma} |\langle e_p, \phi\rangle|^2 \|\Psi_{M-1}\|^2 - \frac{M}{s} \sum_{p\in L_k}^{\sigma} |\langle e_p, \phi\rangle|^2 \Big\langle \Psi_{M-1}, c_{p-k,\sigma} c_{p-k,\sigma}^* \Psi_{M-1} \Big\rangle \end{aligned}$$

$$\geq M\left\Vert \Psi_{M-1}\right\Vert ^{2}-\frac{M}{s}\left\Vert \phi\right\Vert _{\infty}^{2}\left\langle \Psi_{M-1},\mathcal{N}_{E}\Psi_{M-1}\right\rangle =M\Big(1-\frac{M-1}{s}\left\Vert \phi\right\Vert _{\infty}^{2}\Big)\left\Vert \Psi_{M-1}\right\Vert ^{2}$$

where we used that $\mathcal{N}_E\Psi_{M-1}=(M-1)\Psi_{M-1}$.

□

Note that this bound actually applies to all (normalized) $\varphi\in\ell^2(L_k)$ in the form

$$\|b_k^*(\varphi)^M\psi_F\|^2\geq M\Big(1-\frac{M-1}{s}\left\Vert \varphi\right\Vert _{\infty}^{2}\Big)\|b_k^*(\varphi)^{M-1}\psi_F\|^2 \tag{11.1.6}$$

- this is even optimal, with equality holding for all φ which are uniformly supported on some $S\subset L_k$ in the sense that

$$|\langle e_p,\varphi\rangle|=\begin{cases}|S|^{-\frac{1}{2}} & p\in S\\ 0 & p\in L_k\backslash S\end{cases}. \tag{11.1.7}$$

Although ϕ is not uniformly supported, we will see below that it is "almost completely delocalized" as

$$\left\Vert \phi\right\Vert _{\infty}\leq C\left|L_k\right|^{-\frac{1}{2}}, \tag{11.1.8}$$

so the corollary and the inequality $\left\Vert \Psi_M\right\Vert^2\leq M\left\Vert \Psi_{M-1}\right\Vert^2$ implies that

$$1\leq\frac{M\left\Vert \Psi_{M-1}\right\Vert ^{2}}{\left\Vert \Psi_{M}\right\Vert ^{2}}\leq\frac{1}{1-C\frac{M}{s|L_k|}}\leq1+C'\frac{M}{|L_k|},\quad M\ll\left|L_k\right|, \tag{11.1.9}$$

i.e. $M\left\Vert \Psi_M\right\Vert^{-2}\left\Vert \Psi_{M-1}\right\Vert^2\sim1$ for all $M\ll|L_k|\sim O(k_F^2\left|k\right|)$.

The Action of H_{eff} on Ψ_M

Having established control on the state Ψ_M itself we now turn to the action of H_{eff} upon it:

Proposition 11.1.3. *For all $M\in\mathbb{N}$ it holds that*

$$\|(H_{\mathrm{eff}}-M\epsilon_k)\Psi_M\|=\frac{2M(M-1)}{s^{\frac{3}{2}}}\|\mathcal{E}\Psi_{M-2}\|$$

where $\mathcal{E}:\mathcal{H}_N\to\mathcal{H}_N$ is given by

$$\mathcal{E}=\sum_{p,q\in L_k}^{\sigma}\left\langle e_p,\phi\right\rangle\left\langle e_q,\phi\right\rangle\left(\sum_{l\in\mathbb{Z}_*^3}\delta_{p-l,q-k}1_{L_l}(p)b_l^*\Big(\Big(\tilde{E}_l-h_l\Big)e_p\Big)\right)c_{q,\sigma}^*c_{p-k,\sigma}.$$

Proof: By the commutation relation $[H'_{\mathrm{kin}},b_k^*(\phi)]=2b_k^*(h_k\phi)$ it follows as $H'_{\mathrm{kin}}\psi_F=0$ that

$$H'_{\mathrm{kin}}\Psi_M=Mb_k^*(2h_k\phi)\Psi_{M-1}, \tag{11.1.10}$$

so applying Lemma 11.1.1 we find

$$\begin{aligned}
H_{\text{eff}}\Psi_M &= H'_{\text{kin}}\Psi_M + 2\sum_{l\in\mathbb{Z}^3_*}\sum_{p\in L_l} b^*_l\Big(\big(\tilde{E}_l - h_l\big)e_p\Big)b_{l,p}\Psi_M \\
&= Mb^*_k(2h_k\phi)\Psi_{M-1} + 2M\sum_{l\in\mathbb{Z}^3_*}\sum_{p\in L_l}\delta_{k,l}b^*_l\Big(\big(\tilde{E}_l - h_l\big)e_p\Big)\langle e_p,\phi\rangle\,\Psi_{M-1} \\
&+ \frac{2M(M-1)}{s^{\frac{3}{2}}}\sum_{l\in\mathbb{Z}^3_*}\sum_{p\in L_k\cap L_l}\sum^{\sigma}_{q\in L_k}\delta_{p-l,q-k}\langle e_p,\phi\rangle\langle e_q,\phi\rangle\, b^*_l\Big(\big(\tilde{E}_l - h_l\big)e_p\Big)c^*_{q,\sigma}c_{p-k,\sigma}\Psi_{M-2} \\
&= Mb^*_k(2h_k\phi)\Psi_{M-1} + Mb^*_k\Big(2\big(\tilde{E}_k - h_k\big)\phi\Big)\Psi_{M-1} \qquad (11.1.11) \\
&+ \frac{2M(M-1)}{s^{\frac{3}{2}}}\sum^{\sigma}_{p,q\in L_k}\langle e_p,\phi\rangle\langle e_q,\phi\rangle\left(\sum_{l\in\mathbb{Z}^3_*}\delta_{p-l,q-k}1_{L_l}(p)b^*_l\Big(\big(\tilde{E}_l - h_l\big)e_p\Big)\right)c^*_{q,\sigma}c_{p-k,\sigma}\Psi_{M-2} \\
&= Mb^*_k\Big(2\tilde{E}_k\phi\Big)\Psi_{M-1} + \frac{2M(M-1)}{s^{\frac{3}{2}}}\mathcal{E}\Psi_{M-2}.
\end{aligned}$$

By our choice of ϕ the claim now follows as

$$Mb^*_k\Big(2\tilde{E}_k\phi\Big)\Psi_{M-1} = M\epsilon_k b^*_k(\phi)\Psi_{M-1} = M\epsilon_k\Psi_M. \qquad (11.1.12)$$

□

To bound $\|\mathcal{E}\Psi_{M-2}\|$ we note the following generalization of Proposition 7.1.1:

Proposition 11.1.4. *For any collection of vectors $\varphi_k \in \ell^2(L_k)$, $k\in\mathbb{Z}^3_*$, with $\sum_{k\in\mathbb{Z}^3_*}\|\varphi_k\|^2 < \infty$ it holds for all $\Psi\in\mathcal{H}_N$ that*

$$\Big\|\sum_{k\in\mathbb{Z}^3_*} b_k(\varphi_k)\Psi\Big\| \leq \sqrt{\sum_{k\in\mathbb{Z}^3_*}\|\varphi_k\|^2}\|\mathcal{N}_E^{\frac{1}{2}}\Psi\|, \qquad \Big\|\sum_{k\in\mathbb{Z}^3_*} b^*_k(\varphi_k)\Psi\Big\| \leq \sqrt{\sum_{k\in\mathbb{Z}^3_*}\|\varphi_k\|^2}\|(\mathcal{N}_E+1)^{\frac{1}{2}}\Psi\|.$$

Proof: By the triangle and Cauchy-Schwarz inequalities and the usual fermionic estimate we can bound

$$\begin{aligned}
\Big\|\sum_{k\in\mathbb{Z}^3_*} b_k(\varphi_k)\Psi\Big\| &= \frac{1}{\sqrt{s}}\Big\|\sum_{k\in\mathbb{Z}^3_*}\sum^{\sigma}_{p\in L_k}\langle\varphi,e_p\rangle\, c^*_{p-k,\sigma}c_{p,\sigma}\Psi\Big\| = \frac{1}{\sqrt{s}}\Big\|\sum^{\sigma}_{p\in B_F^c}\left(\sum_{k\in\mathbb{Z}^3_*}1_{L_k}(p)\langle\varphi,e_p\rangle\, c^*_{p-k,\sigma}\right)c_{p,\sigma}\Psi\Big\| \\
&\leq \frac{1}{\sqrt{s}}\sum^{\sigma}_{p\in B_F^c}\Big\|\left(\sum_{k\in\mathbb{Z}^3_*}1_{L_k}(p)\langle\varphi,e_p\rangle\, c^*_{p-k,\sigma}\right)c_{p,\sigma}\Psi\Big\| \qquad (11.1.13) \\
&\leq \frac{1}{\sqrt{s}}\sum^{\sigma}_{p\in B_F^c}\sqrt{\sum_{k\in\mathbb{Z}^3_*}1_{L_k}(p)\,|\langle\varphi,e_p\rangle|^2}\,\|c_{p,\sigma}\Psi\| \\
&\leq \sqrt{\sum_{p\in B_F^c}\sum_{k\in\mathbb{Z}^3_*}1_{L_k}(p)\,|\langle\varphi,e_p\rangle|^2}\sqrt{\sum^{\sigma}_{p\in B_F^c}\|c_{p,\sigma}\Psi\|^2} = \sqrt{\sum_{k\in\mathbb{Z}^3_*}\|\varphi_k\|^2}\|\mathcal{N}_E^{\frac{1}{2}}\Psi\|
\end{aligned}$$

for the first claim. The second follows from this, since

$$\left(\sum_{k\in\mathbb{Z}^3_*} b_k(\varphi_k)\right)\left(\sum_{k\in\mathbb{Z}^3_*} b_k(\varphi_k)\right)^* = \left(\sum_{k\in\mathbb{Z}^3_*} b_k(\varphi_k)\right)^*\left(\sum_{k\in\mathbb{Z}^3_*} b_k(\varphi_k)\right) + \sum_{k\in\mathbb{Z}^3_*}\|\varphi_k\|^2 + \sum_{k,l\in\mathbb{Z}^3_*}\varepsilon_{k,l}(\varphi_k;\varphi_l)$$

$$\leq \left(\sum_{k\in\mathbb{Z}^3_*}\|\varphi_k\|^2\right)(\mathcal{N}_E+1)+\sum_{k,l\in\mathbb{Z}^3_*}\varepsilon_{k,l}(\varphi_k;\varphi_l) \tag{11.1.14}$$

and we claim that $\sum_{k,l\in\mathbb{Z}^3_*}\varepsilon_{k,l}(\varphi_k;\varphi_l)\leq 0$. Indeed, as

$$\varepsilon_{k,l}(\varphi_k;\varphi_l)=-\frac{1}{s}\sum_{p\in L_k}^{\sigma}\sum_{q\in L_l}\langle\varphi_k,e_p\rangle\langle e_q,\varphi_l\rangle\left(\delta_{p,q}c_{q-l,\sigma}c^*_{p-k,\sigma}+\delta_{p-k,q-l}c^*_{q,\sigma}c_{p,\sigma}\right) \tag{11.1.15}$$

we see that for the sum corresponding to the $\delta_{p,q}c_{q-l,\sigma}c^*_{p-k,\sigma}$ terms,

$$\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in L_k}^{\sigma}\sum_{q\in L_l}\langle\varphi_k,e_p\rangle\langle e_q,\varphi_l\rangle\delta_{p,q}c_{q-l,\sigma}c^*_{p-k,\sigma}=\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in L_k\cap L_l}^{\sigma}\langle\varphi_k,e_p\rangle\langle e_p,\varphi_l\rangle c_{p-l,\sigma}c^*_{p-k,\sigma}$$

$$=\sum_{p\in B_F^c}^{\sigma}\left(\sum_{l\in\mathbb{Z}^3_*}1_{L_l}(p)\langle\varphi_l,e_p\rangle c^*_{p-l,\sigma}\right)^*\left(\sum_{k\in\mathbb{Z}^3_*}1_{L_k}(p)\langle\varphi_k,e_p\rangle c^*_{p-k,\sigma}\right)\geq 0, \tag{11.1.16}$$

and a similar observation applies to the $\delta_{p-k,q-l}c^*_{q,\sigma}c_{p,\sigma}$ terms.

□

We can then bound $\|\mathcal{E}\Psi_{M-2}\|$ in the following form:

Proposition 11.1.5. *For all $M\in\mathbb{N}$ it holds that*

$$\|\mathcal{E}\Psi_{M-2}\|\leq M\sqrt{M-1}s^{\frac{1}{2}}\|\phi\|_\infty^2\sqrt{\sum_{l\in\mathbb{Z}^3_*}\|\tilde{E}_l-h_l\|_{\mathrm{HS}}^2}\,\|\Psi_{M-2}\|.$$

Proof: Write $B_{p,q}=\sum_{l\in\mathbb{Z}^3_*}\delta_{p-l,q-k}1_{L_l}(p)b_l\Big(\big(\tilde{E}_l-h_l\big)e_p\Big)$ for brevity, so that

$$\mathcal{E}=\sum_{p,q\in L_k}^{\sigma}\langle e_p,\phi\rangle\langle e_q,\phi\rangle B^*_{p,q}c^*_{q,\sigma}c_{p-k,\sigma}, \tag{11.1.17}$$

and note that by the previous proposition, the operators $B^*_{p,q}$ obey

$$\begin{aligned}\sum_{p,q\in L_k}\left\|B^*_{p,q}\Psi_{M-2}\right\|^2&\leq\sum_{p,q\in L_k}\sum_{l\in\mathbb{Z}^3_*}\delta_{p-l,q-k}1_{L_l}(p)\left\|\big(\tilde{E}_l-h_l\big)e_p\right\|^2\|(\mathcal{N}_E+1)^{\frac{1}{2}}\Psi_{M-2}\|^2\\&=(M-1)\sum_{l\in\mathbb{Z}^3_*}\sum_{p\in L_k\cap L_l}\left\|\big(\tilde{E}_l-h_l\big)e_p\right\|^2\|\Psi_{M-2}\|^2\\&\leq(M-1)\sum_{l\in\mathbb{Z}^3_*}\|\tilde{E}_l-h_l\|_{\mathrm{HS}}^2\|\Psi_{M-2}\|^2.\end{aligned} \tag{11.1.18}$$

Due to the identity

$$\left[c^*_{p-k,\sigma}c_{q,\sigma},c^*_{q',\tau}c_{p'-k,\tau}\right]=\delta^{\sigma,\tau}_{p,p'}\delta^{\sigma,\tau}_{q,q'}-\delta^{\sigma,\tau}_{p,p'}c^*_{q',\tau}c_{q,\sigma}-\delta^{\sigma,\tau}_{q,q'}c_{p'-k,\tau}c^*_{p-k,\sigma} \tag{11.1.19}$$

it follows by a partial normal-ordering of $\mathcal{E}^*\mathcal{E}$ that

$$\|\mathcal{E}\Psi_{M-2}\|^2=\sum_{p,p',q,q'\in L_k}^{\sigma,\tau}\langle\phi,e_p\rangle\langle\phi,e_q\rangle\langle e_{p'},\phi\rangle\langle e_{q'},\phi\rangle\left\langle\Psi_{M-2},c^*_{p-k,\sigma}c_{q,\sigma}B_{p,q}B^*_{p',q'}c^*_{q',\tau}c_{p'-k,\tau}\Psi_{M-2}\right\rangle$$

$$
\begin{aligned}
&= \sum_{p,p',q,q'\in L_k}^{\sigma,\tau} \langle \phi, e_p\rangle \langle \phi, e_q\rangle \langle e_{p'}, \phi\rangle \langle e_{q'}, \phi\rangle \left\langle c^*_{p'-k,\tau} c_{q',\tau} B^*_{p,q}\Psi_{M-2}, c^*_{p-k,\sigma} c_{q,\sigma} B^*_{p',q'}\Psi_{M-2}\right\rangle \\
&- \sum_{p,q,q'\in L_k}^{\sigma} |\langle e_p, \phi\rangle|^2 \langle \phi, e_q\rangle \langle e_{q'}, \phi\rangle \left\langle c_{q',\sigma} B^*_{p,q}\Psi_{M-2}, c_{q,\sigma} B^*_{p,q'}\Psi_{M-2}\right\rangle \qquad (11.1.20)\\
&- \sum_{p,p',q\in L_k}^{\sigma} |\langle e_q, \phi\rangle|^2 \langle \phi, e_p\rangle \langle e_{p'}, \phi\rangle \left\langle c^*_{p'-k,\sigma} B^*_{p,q}\Psi_{M-2}, c^*_{p-k,\sigma} B^*_{p',q}\Psi_{M-2}\right\rangle \\
&+ \sum_{p,q\in L_k}^{\sigma} |\langle e_p, \phi\rangle|^2 |\langle e_q, \phi\rangle|^2 \left\| B^*_{p,q}\Psi_{M-2}\right\|^2 \\
&=: T_1 + T_2 + T_3 + T_4.
\end{aligned}
$$

We bound these terms individually. For T_1 we have by Cauchy-Schwarz that

$$
\begin{aligned}
|T_1| &\leq \|\phi\|_\infty^4 \sum_{p,p',q,q'\in L_k}^{\sigma,\tau} \left\| c^*_{p'-k,\tau} c_{q',\tau} B^*_{p,q}\Psi_{M-2}\right\| \left\| c^*_{p-k,\sigma} c_{q,\sigma} B^*_{p',q'}\Psi_{M-2}\right\| \\
&\leq s\,\|\phi\|_\infty^4 \sum_{p,p',q,q'\in L_k}^{\sigma} \left\| c^*_{p-k,\sigma} c_{q,\sigma} B^*_{p',q'}\Psi_{M-2}\right\|^2 \leq s\,\|\phi\|_\infty^4 \sum_{p',q'\in L_k} \sum_{p,q\in L_k}^{\sigma,\tau} \left\| c^*_{p-k,\sigma} c_{q,\tau} B^*_{p',q'}\Psi_{M-2}\right\|^2 \\
&= s\,\|\phi\|_\infty^4 \sum_{p',q'\in L_k} \left\| \mathcal{N}_E B^*_{p',q'}\Psi_{M-2}\right\|^2 = (M-1)^2 s\,\|\phi\|_\infty^4 \sum_{p',q'\in L_k} \left\| B^*_{p',q'}\Psi_{M-2}\right\|^2 \qquad (11.1.21)\\
&\leq (M-1)^3 s\,\|\phi\|_\infty^4 \sum_{l\in\mathbb{Z}^3_*} \|\tilde{E}_l - h_l\|_{\mathrm{HS}}^2 \|\Psi_{M-2}\|^2
\end{aligned}
$$

and similarly for T_2

$$
\begin{aligned}
|T_2| &\leq \|\phi\|_\infty^4 \sum_{p,q,q'\in L_k}^{\sigma} \left\| c_{q',\sigma} B^*_{p,q}\Psi_{M-2}\right\| \left\| c_{q,\sigma} B^*_{p,q'}\Psi_{M-2}\right\| \leq \|\phi\|_\infty^4 \sum_{p,q,q'\in L_k}^{\sigma} \left\| c_{q',\sigma} B^*_{p,q}\Psi_{M-2}\right\|^2 \\
&\leq \|\phi\|_\infty^4 \sum_{p,q\in L_k} \|\mathcal{N}_E^{\frac{1}{2}} B^*_{p,q}\Psi_{M-2}\|^2 = \|\phi\|_\infty^4 (M-1) \sum_{p,q\in L_k} \|B^*_{p,q}\Psi_{M-2}\|^2 \qquad (11.1.22)\\
&\leq (M-1)^2 \|\phi\|_\infty^4 \sum_{l\in\mathbb{Z}^3_*} \|\tilde{E}_l - h_l\|_{\mathrm{HS}}^2 \|\Psi_{M-2}\|^2 .
\end{aligned}
$$

T_3 obeys the same bound and obviously $|T_4| \leq (M-1)s\,\|\phi\|_\infty^4 \sum_{l\in\mathbb{Z}^3_*} \|\tilde{E}_l - h_l\|_{\mathrm{HS}}^2 \|\Psi_{M-2}\|^2$. The claim follows by combining these estimates.

□

We summarize this subsection in the following:

Proposition 11.1.6. *It holds for all $M \in \mathbb{N}$ with $M < \|\phi\|_\infty^{-2}$ that $\hat{\Psi}_M = \|\Psi_M\|^{-1}\Psi_M$ obeys*

$$
\|(H_{\mathrm{eff}} - M\epsilon_k)\hat{\Psi}_M\| \leq \frac{2}{\|\phi\|_\infty^{-2} - M}\sqrt{s^{-2}\sum_{l\in\mathbb{Z}^3_*} \|\tilde{E}_l - h_l\|_{\mathrm{HS}}^2}\, M^{\frac{5}{2}}.
$$

Proof: By inserting the previous estimate into the statement of Proposition 11.1.3 we obtain

$$\|(H_{\text{eff}} - M\epsilon_k)\hat{\Psi}_M\| \leq 2M^2(M-1)^{\frac{3}{2}} \|\phi\|_\infty^2 \sqrt{s^{-2} \sum_{l\in\mathbb{Z}^3_*} \|\tilde{E}_l - h_l\|_{\text{HS}}^2 \frac{\|\Psi_{M-2}\|}{\|\Psi_M\|}}, \tag{11.1.23}$$

and by the lower bound of Corollary 11.1.2 it holds that

$$\frac{\|\Psi_{M-2}\|}{\|\Psi_M\|} = \frac{\|\Psi_{M-2}\|}{\|\Psi_{M-1}\|}\frac{\|\Psi_{M-1}\|}{\|\Psi_M\|} \leq \frac{1}{\sqrt{M(M-1)}}\frac{1}{1-M\|\phi\|_\infty^2} \tag{11.1.24}$$

for $M < \|\phi\|_\infty^{-2}$.

□

11.2 Estimates of One-Body Quantities

To conclude Theorem 11.0.1 it only remains to control the one-body quantities $\|\phi\|_\infty^2$, $\|\tilde{E}_l - h_l\|_{\text{HS}}^2$ and ϵ_k. To this end we return a final time to the setting of Section 6 and consider $\tilde{E} : V \to V$ given by

$$\tilde{E} = \left(h^2 + 2P_{h^{\frac{1}{2}}v}\right)^{\frac{1}{2}} \tag{11.2.1}$$

with normalized eigenvector $\phi \in V$ (chosen such that $\left\langle h^{\frac{1}{2}}v, \phi\right\rangle \geq 0$) corresponding to the greatest eigenvalue ϵ of $2\tilde{E}$. Below it will be more convenient to work in terms of the greatest eigenvalue ε of $\tilde{E}^2$; the eigenvalues are simply related by $\epsilon = 2\sqrt{\epsilon}$.

The eigenvalue equation for ε is

$$\varepsilon\phi = \tilde{E}^2\phi = \left(h^2 + 2P_{h^{\frac{1}{2}}v}\right)\phi = h^2\phi + 2\left\langle h^{\frac{1}{2}}v, \phi\right\rangle h^{\frac{1}{2}}v \tag{11.2.2}$$

and assuming that $\varepsilon > \max_{1\leq i\leq n} \lambda_i^2 =: \lambda_{\max}^2$ this can be rearranged to

$$\phi = 2\left\langle h^{\frac{1}{2}}v, \phi\right\rangle \left(\varepsilon - h^2\right)^{-1} h^{\frac{1}{2}}v. \tag{11.2.3}$$

As ϕ is by assumption normalized and $\left\langle h^{\frac{1}{2}}v, \phi\right\rangle \geq 0$, this implies that ϕ is determined with ε as the only unknown quantity by the formula

$$\phi = \|\left(\varepsilon - h^2\right)^{-1} h^{\frac{1}{2}}v\|^{-1}\left(\varepsilon - h^2\right)^{-1} h^{\frac{1}{2}}v. \tag{11.2.4}$$

In particular, the components of ϕ with respect to the eigenvectors $(x_i)_{i=1}^n$ of h obey

$$\langle x_i, \phi\rangle = \frac{1}{\|(\varepsilon - h^2)^{-1}h^{\frac{1}{2}}v\|}\frac{\sqrt{\lambda_i}}{\varepsilon - \lambda_i^2}\langle x_i, v\rangle, \quad 1 \leq i \leq n. \tag{11.2.5}$$

To ensure that $\varepsilon > \lambda_{\max}^2$, note that by the variational principle there holds the inequality

$$\varepsilon \geq \frac{\left\langle h^{\frac{1}{2}}v, \left(h^2 + 2P_{h^{\frac{1}{2}}v}\right)h^{\frac{1}{2}}v\right\rangle}{\left\langle h^{\frac{1}{2}}v, h^{\frac{1}{2}}v\right\rangle} = 2\langle v, hv\rangle + \frac{\langle v, h^3 v\rangle}{\langle v, hv\rangle} \tag{11.2.6}$$

so $\varepsilon > \lambda_{\max}^2$ is assured if $2\langle v, hv\rangle > \lambda_{\max}^2$. Under this condition we then have the following bound:

Corollary 11.2.1. *Provided* $2\langle v, hv\rangle > \lambda_{\max}^2$ *it holds that*

$$\|\phi\|_\infty \leq \frac{2\sqrt{\langle v, hv\rangle \lambda_{\max}}}{2\langle v, hv\rangle - \lambda_{\max}^2}\|v\|_\infty .$$

Proof: As $t \mapsto t(t-\lambda_{\max}^2)^{-1}$ is decreasing for $t \in (\lambda_{\max}^2, \infty)$, equation (11.2.5) shows that for any $1 \leq i \leq n$

$$\begin{aligned} |\langle x_i, \phi\rangle| &\leq \frac{1}{\frac{1}{\varepsilon}\|h^{\frac{1}{2}}v\|}\frac{\sqrt{\lambda_{\max}}}{\varepsilon - \lambda_{\max}^2}|\langle x_i, v\rangle| = \frac{\varepsilon}{\varepsilon - \lambda_{\max}^2}\frac{\sqrt{\lambda_{\max}}}{\sqrt{\langle v, hv\rangle}}|\langle x_i, v\rangle| \qquad (11.2.7)\\ &\leq \frac{2\langle v, hv\rangle}{2\langle v, hv\rangle - \lambda_{\max}^2}\frac{\sqrt{\lambda_{\max}}}{\sqrt{\langle v, hv\rangle}}|\langle x_i, v\rangle| = \frac{2\sqrt{\langle v, hv\rangle \lambda_{\max}}}{2\langle v, hv\rangle - \lambda_{\max}^2}|\langle x_i, v\rangle| . \end{aligned}$$

□

Under the same assumption we can also control ε well:

Proposition 11.2.2. *Provided* $2\langle v, hv\rangle > \lambda_{\max}^2$ *it holds that*

$$2\langle v, hv\rangle + \frac{\langle v, h^3 v\rangle}{\langle v, hv\rangle} \leq \varepsilon \leq 2\langle v, hv\rangle + \frac{\langle v, h^3 v\rangle}{\langle v, hv\rangle} + \frac{4\langle v, h^3 v\rangle \lambda_{\max}^2}{(2\langle v, hv\rangle - \lambda_{\max}^2)^2}.$$

Proof: We noted the lower bound above. For the upper bound we estimate

$$\begin{aligned} \left\langle \phi, h^2\phi\right\rangle &= \frac{\left\langle v, h^3(\varepsilon - h^2)^{-2}v\right\rangle}{\left\langle v, h(\varepsilon - h^2)^{-2}v\right\rangle} \leq \left(\frac{\varepsilon}{\varepsilon - \lambda_{\max}^2}\right)^2\frac{\langle v, h^3 v\rangle}{\langle v, hv\rangle} \leq \left(\frac{2\langle v, hv\rangle}{2\langle v, hv\rangle - \lambda_{\max}^2}\right)^2\frac{\langle v, h^3 v\rangle}{\langle v, hv\rangle}\\ &= \frac{4\langle v, hv\rangle^2}{(2\langle v, hv\rangle - \lambda_{\max}^2)^2}\frac{\langle v, h^3 v\rangle}{\langle v, hv\rangle} = \frac{\langle v, h^3 v\rangle}{\langle v, hv\rangle} + \frac{4\langle v, hv\rangle^2 - (2\langle v, hv\rangle - \lambda_{\max}^2)^2}{(2\langle v, hv\rangle - \lambda_{\max}^2)^2}\frac{\langle v, h^3 v\rangle}{\langle v, hv\rangle}\\ &\leq \frac{\langle v, h^3 v\rangle}{\langle v, hv\rangle} + \frac{4\langle v, hv\rangle \lambda_{\max}^2}{(2\langle v, hv\rangle - \lambda_{\max}^2)^2}\frac{\langle v, h^3 v\rangle}{\langle v, hv\rangle} = \frac{\langle v, h^3 v\rangle}{\langle v, hv\rangle} + \frac{4\langle v, h^3 v\rangle \lambda_{\max}^2}{(2\langle v, hv\rangle - \lambda_{\max}^2)^2} \end{aligned}$$

and see that by the eigevalue equation for ε and the Cauchy-Schwarz inequality in the form $\left|\left\langle h^{\frac{1}{2}}v, \phi\right\rangle\right|^2 \leq \langle v, hv\rangle$,

$$\varepsilon = \left\langle \phi, h^2\phi\right\rangle + 2\left|\left\langle h^{\frac{1}{2}}v, \phi\right\rangle\right|^2 \leq 2\langle v, hv\rangle + \frac{\langle v, h^3 v\rangle}{\langle v, hv\rangle} + \frac{4\langle v, h^3 v\rangle \lambda_{\max}^2}{(2\langle v, hv\rangle - \lambda_{\max}^2)^2}. \tag{11.2.8}$$

□

Lastly we bound $\|\tilde{E} - h\|_{\mathrm{HS}}^2$:

Proposition 11.2.3. *It holds that*

$$\|\tilde{E} - h\|_{\mathrm{HS}}^2 \leq \min\left\{2\langle v, hv\rangle, \|v\|^4\right\}.$$

Proof: The first bound is easily obtained as

$$\begin{aligned}\|\tilde{E} - h\|_{\mathrm{HS}}^2 &= \mathrm{tr}\Big(\big(\tilde{E} - h\big)^2\Big) = \mathrm{tr}\Big(\tilde{E}^2 - 2\{\tilde{E}, h\} + h^2\Big) = 2\,\mathrm{tr}\Big(h^2 + P_{h^{\frac{1}{2}}v} - h^{\frac{1}{2}}\tilde{E}h^{\frac{1}{2}}\Big) \\ &\leq 2\,\mathrm{tr}\Big(P_{h^{\frac{1}{2}}v}\Big) = 2\langle v, hv\rangle\end{aligned} \tag{11.2.9}$$

since $h \leq \tilde{E}$ implies that $h^2 - h^{\frac{1}{2}}\tilde{E}h^{\frac{1}{2}} \leq 0$. For the second, note that $\tilde{E} = h^{\frac{1}{2}}e^{-2K}h^{\frac{1}{2}}$ whence Proposition 6.1.4 affords us the elementwise estimate

$$\left|\left\langle x_i, \big(\tilde{E} - h\big)x_j\right\rangle\right| = \sqrt{\lambda_i\lambda_j}\left|\left\langle x_i, \big(e^{-2K} - 1\big)x_j\right\rangle\right| \leq \frac{2\sqrt{\lambda_i\lambda_j}}{\lambda_i + \lambda_j}\langle x_i, v\rangle\langle v, x_j\rangle \leq \langle x_i, v\rangle\langle v, x_j\rangle \tag{11.2.10}$$

for $1 \leq i, j \leq n$, so

$$\|\tilde{E} - h\|_{\mathrm{HS}}^2 = \sum_{i,j=1}^n \left|\left\langle x_i, \big(\tilde{E} - h\big)x_j\right\rangle\right|^2 \leq \sum_{i,j=1}^n |\langle x_i, v\rangle\langle v, x_j\rangle|^2 = \|v\|^4. \tag{11.2.11}$$

□

11.3 Final Details

We now insert the particular operators h_k and P_k. For the quantity $2\langle v_k, h_k v_k\rangle - \lambda_{k,\max}^2$, we note that the inequalities defining L_k imply that

$$\lambda_{k,p} = k \cdot p - \frac{1}{2}|k|^2 = k \cdot (p - k) + \frac{1}{2}|k|^2 \leq |k|\left(k_F + \frac{1}{2}|k|\right), \quad p \in L_k, \tag{11.3.1}$$

so

$$\lambda_{k,\max}^2 \leq C k_F^2 |k|^2 \tag{11.3.2}$$

as we assumed that $k \in B_F$. The quantity $2\langle v_k, h_k v_k\rangle$ is

$$2\langle v_k, h_k v_k\rangle = \frac{s\hat{V}_k}{(2\pi)^3}\sum_{p \in L_k}\lambda_{k,p} \tag{11.3.3}$$

and for a lower bound we prove the following in appendix section B.4:

Proposition 11.3.1. *For all $k \in B_F$ and $\beta \in \{0\} \cup [1, \infty)$ it holds that*

$$\sum_{p \in L_k}\lambda_{k,p}^\beta \geq c k_F^{2+\beta}|k|^{1+\beta}, \quad k_F \to \infty,$$

for a $c > 0$ depending only on β.

It follows that

$$2\langle v_k, h_k v_k\rangle - \lambda_{k,\max}^2 \geq c_1 s k_F^3 |k|^2 \left(\hat{V}_k - c_1^{-1} k_F^{-1}\right), \tag{11.3.4}$$

so if $\hat{V}_k > c k_F$ for $c = 2c_1^{-1}$, say, it holds that

$$2\langle v_k, h_k v_k\rangle - \lambda_{k,\max}^2 \geq c' s k_F^3 |k|^2 \hat{V}_k \tag{11.3.5}$$

for some $c' > 0$ independent of all quantities. For $\beta \in [0, \infty)$ we also have that

$$\left\langle v_k, h_k^\beta v_k\right\rangle = \frac{s\hat{V}_k}{(2\pi)^3} \sum_{p\in L_k} \lambda_{k,p}^\beta \leq \frac{s\hat{V}_k}{(2\pi)^3} |L_k| \lambda_{k,\max}^\beta \leq C' s k_F^{2+\beta} |k|^{1+\beta} \hat{V}_k \tag{11.3.6}$$

so Corollary 11.2.1 allows us to bound $\|\phi\|_\infty$ as

$$\|\phi\|_\infty \leq \frac{2\sqrt{\langle v_k, h_k v_k\rangle \lambda_{k,\max}}}{2\langle v_k, h_k v_k\rangle - \lambda_{k,\max}^2} \|v_k\|_\infty \leq C' \frac{\sqrt{\left(s k_F^3 |k|^2 \hat{V}_k\right)(k_F |k|)}}{s k_F^3 |k|^2 \hat{V}_k} \left(s\hat{V}_k\right)^{\frac{1}{2}} = \frac{C'}{k_F |k|^{\frac{1}{2}}}. \tag{11.3.7}$$

Note that since $|L_k| \sim O(k_F^2 |k|)$, ϕ is indeed almost completely delocalized, and we can estimate that

$$\|\phi\|_\infty^{-2} - M \geq C' k_F^2 |k| \tag{11.3.8}$$

for all $M \in \mathbb{N}$ such that $M \leq C k_F^2 |k|$ for some C also independent of all quantities.

Finally, by Proposition 11.2.3,

$$\begin{aligned} \|\tilde{E}_l - h_l\|_{\mathrm{HS}}^2 &\leq \min\left\{2\langle v_l, h_l v_l\rangle, \|v_l\|^4\right\} \leq C \min\left\{s k_F^3 \hat{V}_l |l|^2, s^2 \hat{V}_l^2 |L_l|^2\right\} \\ &\leq C s^2 \min\left\{k_F^3 \hat{V}_l |l|^2, k_F^4 \hat{V}_l^2 |l|^2, k_F^6 \hat{V}_l^2\right\} \\ &= C s^2 \min\left\{1, k_F \hat{V}_l, k_F^3 \hat{V}_l |l|^{-2}\right\} k_F^3 \hat{V}_l |l|^2 \end{aligned} \tag{11.3.9}$$

for any $l \in \mathbb{Z}_*^3$. Inserting these bounds into Proposition 11.1.6 yields the first claim of Theorem 11.0.1:

Proposition 11.3.2. *There exists constants $c, C > 0$ such that if $\hat{V}_k > c k_F^{-1}$ it holds for all $M \leq C k_F^2 |k|$ that $\hat{\Psi}_M = \|\Psi_M\|^{-1} \Psi_M$ obeys*

$$\|(H_{\mathrm{eff}} - M\epsilon_k)\hat{\Psi}_M\| \leq C' \sqrt{\sum_{l\in\mathbb{Z}_*^3} \min\left\{1, k_F \hat{V}_l, k_F^3 \hat{V}_l |l|^{-2}\right\} \hat{V}_l |l|^2} \frac{M^{\frac{5}{2}}}{\sqrt{k_F} |k|}$$

for a constant $C' > 0$. c, C, C' are independent of all quantities.

The Eigenvalue ϵ_k

For ϵ_k we have by Proposition 11.2.2 that (recalling the relation $\epsilon = 2\sqrt{\varepsilon}$)

$$2\sqrt{2\langle v_k, h_k v_k\rangle + \frac{\langle v_k, h_k^3 v_k\rangle}{\langle v_k, h_k v_k\rangle}} \leq \epsilon_k \leq 2\sqrt{2\langle v_k, h_k v_k\rangle + \frac{\langle v_k, h_k^3 v_k\rangle}{\langle v_k, h_k v_k\rangle} + \frac{4\langle v_k, h_k^3 v_k\rangle \lambda_{k,\max}^2}{\left(2\langle v_k, h_k v_k\rangle - \lambda_{k,\max}^2\right)^2}}. \tag{11.3.10}$$

The lower bound given in Theorem 11.0.1 is then immediate since

$$\epsilon_k \geq 2\sqrt{2\langle v_k, h_k v_k\rangle} \geq c' s^{\frac{1}{2}} k_F^{\frac{3}{2}} |k| \hat{V}_k^{\frac{1}{2}} \tag{11.3.11}$$

as above, while the inequality $\sqrt{a+b} - \sqrt{a} \leq \frac{b}{2\sqrt{a}}$ yields the upper bound

$$\begin{aligned} \epsilon_k - 2\sqrt{2\langle v_k, h_k v_k\rangle + \frac{\langle v_k, h_k^3 v_k\rangle}{\langle v_k, h v_k\rangle}} &\leq \frac{1}{\sqrt{2\langle v_k, h_k v_k\rangle + \frac{\langle v_k, h_k^3 v_k\rangle}{\langle v_k, h v_k\rangle}}} \frac{4\langle v_k, h_k^3 v_k\rangle \lambda_{k,\max}^2}{\left(2\langle v_k, h_k v_k\rangle - \lambda_{k,\max}^2\right)^2} \\ &\leq C \frac{1}{\sqrt{s k_F^3 |k|^2 \hat{V}_k}} \frac{\left(s k_F^5 |k|^4 \hat{V}_k\right)(k_F |k|)^2}{\left(s k_F^3 |k|^2 \hat{V}_k\right)^2} \\ &= C k_F^{-\frac{1}{2}} |k| \hat{V}_k^{-\frac{3}{2}}. \end{aligned} \tag{11.3.12}$$

For the form given in Theorem 1.1.4 for $\hat{V}_k = g|k|^{-2}$, note that expanding the inner products gives

$$\begin{aligned} \epsilon \sim 2\sqrt{2\langle v_k, h_k v_k\rangle + \frac{\langle v_k, h_k^3 v_k\rangle}{\langle v_k, h v_k\rangle}} &= 2\sqrt{2\frac{s\hat{V}_k}{2(2\pi)^3}\sum_{p\in L_k}\lambda_{k,p} + \frac{\frac{s\hat{V}_k}{2(2\pi)^3}\sum_{p\in L_k}\lambda_{k,p}^3}{\frac{s\hat{V}_k}{2(2\pi)^3}\sum_{p\in L_k}\lambda_{k,p}}} \\ = 2\sqrt{\frac{s}{(2\pi)^3}\frac{g}{|k|^2}\sum_{p\in L_k}\lambda_{k,p} + \frac{\sum_{p\in L_k}\lambda_{k,p}^3}{\sum_{p\in L_k}\lambda_{k,p}}} \end{aligned} \tag{11.3.13}$$

and formally replacing the Riemann sums by integrals according to equation (B.3.28) shows that

$$\sum_{p\in L_k}\lambda_{k,p} \sim \frac{2\pi}{3}k_F^3|k|^2, \quad \sum_{p\in L_k}\lambda_{k,p}^3 \sim \frac{2\pi}{5}k_F^5|k|^4 + \frac{\pi}{6}k_F^3|k|^6 \approx \frac{2\pi}{5}k_F^5|k|^4, \tag{11.3.14}$$

whence

$$\begin{aligned} \epsilon_k \sim 2\sqrt{\frac{s}{(2\pi)^3}\frac{g}{|k|^2}\frac{2\pi}{3}k_F^3|k|^2 + \frac{\frac{2\pi}{5}k_F^5|k|^4}{\frac{2\pi}{3}k_F^3|k|^2}} &= \sqrt{2g\left(\frac{1}{(2\pi)^3}\frac{4\pi s}{3}k_F^3\right) + \frac{12}{5}k_F^2|k|^2} \\ \sim \sqrt{2gn + \frac{12}{5}k_F^2|k|^2} \end{aligned} \tag{11.3.15}$$

for $n = \frac{N}{(2\pi)^3} = \frac{s|B_F|}{(2\pi)^3} \sim \frac{1}{(2\pi)^3}\frac{4\pi s}{3}k_F^3$.

Appendix A

Some Functional Analysis Results

A.1 The Square Root of a Rank One Perturbation

Let $\langle V, \langle\cdot,\cdot\rangle\rangle$ be an n-dimensional Hilbert space. With the notation

$$P_w(\cdot) = \langle w, \cdot\rangle\, w, \quad w \in V, \tag{A.1.1}$$

we recall the Sherman-Morrison formula:

Lemma A.1.1. *Let $A : V \to V$ be an invertible operator. Then for any $w \in V$ and $g \in \mathbb{C}$, the operator $A + gP_w$ is invertible if and only if $\langle w, A^{-1}w\rangle \neq g^{-1}$, in which case the inverse is given by*

$$(A + gP_w)^{-1} = A^{-1} - \frac{g}{1 + g\langle w, A^{-1}w\rangle} P_{A^{-1}w}.$$

By applying this we conclude the following representation (first presented in [8]):

Proposition (6.1.2)**.** *Let $A : V \to V$ be a positive self-adjoint operator. Then for any $w \in V$ and $g \in \mathbb{R}$ such that $A + gP_w > 0$ it holds that*

$$(A + gP_w)^{\frac{1}{2}} = A^{\frac{1}{2}} + \frac{2g}{\pi}\int_0^\infty \frac{t^2}{1 + g\left\langle w, (A + t^2)^{-1}w\right\rangle} P_{(A+t^2)^{-1}w}dt$$

and

$$\operatorname{tr}\Big((A + gP_w)^{\frac{1}{2}}\Big) = \operatorname{tr}\Big(A^{\frac{1}{2}}\Big) + \frac{1}{\pi}\int_0^\infty \log\Big(1 + g\left\langle w, \left(A + t^2\right)^{-1} w\right\rangle\Big)dt.$$

Proof: For any $a > 0$ there holds the integral identity

$$\sqrt{a} = \frac{2}{\pi}\int_0^\infty \frac{a}{a + t^2}dt = \frac{2}{\pi}\int_0^\infty \left(1 - \frac{t^2}{a + t^2}\right)dt \tag{A.1.2}$$

so by the spectral theorem the same is true for a positive operator A, provided the fraction is understood as a resolvent. As the Sherman-Morrison formula lets us write

$$\left(A + gP_w + t^2\right)^{-1} = \left(A + t^2\right)^{-1} - \frac{g}{1 + g\left\langle w, (A + t^2)^{-1}w\right\rangle} P_{(A+t^2)^{-1}w} \tag{A.1.3}$$

for any $t \geq 0$, we thus conclude that

$$
\begin{aligned}
(A+gP_w)^{\frac{1}{2}} &= \frac{2}{\pi}\int_0^\infty \Big(1 - t^2\big(A+gP_w+t^2\big)^{-1}\Big)dt \qquad \text{(A.1.4)}\\
&= \frac{2}{\pi}\int_0^\infty \left(1 - t^2\left(\big(A+t^2\big)^{-1} - \frac{g}{1+g\left\langle w,(A+t^2)^{-1}w\right\rangle}P_{(A+t^2)^{-1}w}\right)\right)dt\\
&= \frac{2}{\pi}\int_0^\infty \Big(1 - t^2\big(A+t^2\big)^{-1}\Big)dt + \frac{2g}{\pi}\int_0^\infty \frac{t^2}{1+g\left\langle w,(A+t^2)^{-1}w\right\rangle}P_{(A+t^2)^{-1}w}dt\\
&= A^{\frac{1}{2}} + \frac{2g}{\pi}\int_0^\infty \frac{t^2}{1+g\left\langle w,(A+t^2)^{-1}w\right\rangle}P_{(A+t^2)^{-1}w}dt.
\end{aligned}
$$

The trace formula now follows by partial integration as

$$
\begin{aligned}
&\operatorname{tr}\Big((A+gP_w)^{\frac{1}{2}} - A^{\frac{1}{2}}\Big) = \frac{2g}{\pi}\int_0^\infty \frac{t^2}{1+g\left\langle w,(A+t^2)^{-1}w\right\rangle}\|\big(A+t^2\big)^{-1}w\|^2dt\\
&= \frac{1}{\pi}\int_0^\infty t\frac{2gt\left\langle w,(A+t^2)^{-2}w\right\rangle}{1+g\left\langle w,(A+t^2)^{-1}w\right\rangle}dt \qquad \text{(A.1.5)}\\
&= -\frac{1}{\pi}\left[t\log\Big(1+g\left\langle w,\big(A+t^2\big)^{-1}w\right\rangle\Big)\right]_0^\infty + \frac{1}{\pi}\int_0^\infty \log\Big(1+g\left\langle w,\big(A+t^2\big)^{-1}w\right\rangle\Big)dt\\
&= \frac{1}{\pi}\int_0^\infty \log\Big(1+g\left\langle w,\big(A+t^2\big)^{-1}w\right\rangle\Big)dt
\end{aligned}
$$

since $\left|\log\Big(1+g\left\langle w,(A+t^2)^{-1}w\right\rangle\Big)\right| \leq \left|g\left\langle w,(A+t^2)^{-1}w\right\rangle\right| \leq Ct^{-2}$ for $t \to \infty$.

□

A.2 A Square Root Estimation Result

Lemma (8.2.1)**.** *Let A, B, Z be given with $A > 0$, $Z \geq 0$ and $[A, Z] = 0$. Then if $\pm[A,[A,B]] \leq Z$ it holds that*

$$\pm[A^{\frac{1}{2}},[A^{\frac{1}{2}},B]] \leq \frac{1}{4}A^{-1}Z.$$

Proof: Applying the identity $A^{\frac{1}{2}} = \frac{2}{\pi}\int_0^\infty\Big(1 - t^2(A+t^2)^{-1}\Big)\,dt$ as above, we find that

$$
\begin{aligned}
[A^{\frac{1}{2}},B] &= \frac{2}{\pi}\int_0^\infty \left[1 - t^2\big(A+t^2\big)^{-1}, B\right]dt = -\frac{2}{\pi}\int_0^\infty \left[\big(A+t^2\big)^{-1}, B\right]t^2dt\\
&= \frac{2}{\pi}\int_0^\infty \big(A+t^2\big)^{-1}\left[A+t^2,B\right]\big(A+t^2\big)^{-1}t^2dt \qquad \text{(A.2.1)}\\
&= \frac{2}{\pi}\int_0^\infty \big(A+t^2\big)^{-1}[A,B]\big(A+t^2\big)^{-1}t^2dt
\end{aligned}
$$

where we also used the general identity $[A^{-1}, B] = -A^{-1}[A,B]A^{-1}$. Iterating this formula we conclude that

$$\begin{aligned}[A^{\frac{1}{2}}, [A^{\frac{1}{2}}, B]] &= \frac{2}{\pi}\int_0^\infty \left(A+t^2\right)^{-1}[A,[A^{\frac{1}{2}},B]]\left(A+t^2\right)^{-1}t^2dt \\ &= \left(\frac{2}{\pi}\right)^2\int_0^\infty \left(A+t^2\right)^{-1}\left(A+s^2\right)^{-1}[A,[A,B]]\left(A+s^2\right)^{-1}\left(A+t^2\right)^{-1}s^2t^2dt\end{aligned} \tag{A.2.2}$$

whence the asumptions imply that

$$\begin{aligned}\pm[A^{\frac{1}{2}}, [A^{\frac{1}{2}}, B]] &\leq \left(\frac{2}{\pi}\right)^2\int_0^\infty \left(A+t^2\right)^{-1}\left(A+s^2\right)^{-1}Z\left(A+s^2\right)^{-1}\left(A+t^2\right)^{-1}s^2t^2dt \\ &= \left(\frac{2}{\pi}\int_0^\infty \left(A+t^2\right)^{-2}t^2dt\right)^2 Z = \left(\frac{1}{2}A^{-\frac{1}{2}}\right)^2 Z = \frac{1}{4}A^{-1}Z\end{aligned} \tag{A.2.3}$$

as the identity $\int_0^\infty \frac{t^2}{(a+t^2)^2}dt = \frac{\pi}{4}a^{-\frac{1}{2}}$, $a>0$, similarly yields that $\int_0^\infty (A+t^2)^{-2}t^2dt = \frac{\pi}{4}A^{-\frac{1}{2}}$.

□

A.3 Operators of the Form $e^{zK}Ae^{-zK}$ for Unbounded A

We prove the following:

Proposition A.3.1. *Let X be a Banach space, $A : D(A) \to X$ be a closed operator and let $K : X \to X$ be a bounded operator which preserves $D(A)$. Suppose that $AK : D(A) \to X$ is A-bounded.*

Then for every $z \in \mathbb{C}$ the operator $e^{zK} : X \to X$ likewise preserves $D(A)$ and $e^{zK}Ae^{-zK} : D(A) \to X$ is closed. If additionally X is a Hilbert space, A is self-adjoint and K is skew-symmetric then $e^{tK}Ae^{-tK}$ is self-adjoint for all $t \in \mathbb{R}$.

Furthermore, for every $x \in D(A)$ the mapping $z \mapsto e^{zK}Ae^{-zK}x$ is complex differentiable and C^1 with

$$\frac{d}{dz}e^{zK}Ae^{-zK}x = e^{zK}[K,A]e^{-zK}x.$$

For the remainder of this section we impose the following assumptions: $A : D(A) \to X$ is a closed operator on a Banach space X and $K : X \to X$ is a bounded operator on X, which preserves $D(A)$ such that $AK : D(A) \to X$ is A-bounded according to

$$\|AKx\| \leq a\|Ax\| + b\|x\|, \quad x \in D(A), \tag{A.3.1}$$

for some $a, b \geq 0$.

Well-Definedness of $e^{zK}Ae^{-zK}$

We begin with a lemma:

Lemma A.3.2. *Under the assumptions on A and K, the operator $AK^m : D(A) \to X$ is A-bounded for any $m \in \mathbb{N}_0$ with*

$$\|AK^m x\| \leq a^m \|Ax\| + mc^{m-1}b\|x\|, \quad x \in D(A),$$

for $c = \max\left\{a, \|K\|_{\mathrm{Op}}\right\}$.

Proof: The claim is clearly true for $m = 0, 1$ (by assumption). We prove the general claim by induction: Suppose that case $m-1$ holds. Then we obtain case m by estimating

$$\begin{aligned}\|AK^m x\| &= \left\|AK\left(K^{m-1}x\right)\right\| \leq a\left\|AK^{m-1}x\right\| + b\left\|K^{m-1}x\right\| \\ &\leq a\left(a^{m-1}\|Ax\| + (m-1)c^{m-2}b\|x\|\right) + b\left\|K^{m-1}\right\|_{\mathrm{Op}}\|x\| \\ &\leq a^m\|Ax\| + \left((m-1)ac^{m-2} + \|K\|_{\mathrm{Op}}^{m-1}\right)b\|x\| \\ &\leq a^m\|Ax\| + mc^{m-1}b\|x\|.\end{aligned} \tag{A.3.2}$$

□

We can now conclude the first part of Proposition A.3.1, namely that e^{zK} preserves $D(A)$ for any $z \in \mathbb{C}$, so that $e^{zK}Ae^{-zK} : D(A) \to X$ is well-defined. For use below we prove the following more general statement:

Proposition A.3.3. *Under the assumptions on A and K, it holds for any entire function $f(z) = \sum_{m=0}^{\infty} d_m z^m$ with $d_m \geq 0$, $m \in \mathbb{N}_0$, that $f(zK) : X \to X$ also preserves $D(A)$ for any $z \in \mathbb{C}$, and that $Af(zK) : D(A) \to X$ is A-bounded as*

$$\|Af(zK)x\| \leq f(a|z|)\|Ax\| + b|z|f'(c|z|)\|x\|, \quad x \in D(A),$$

for $c = \max\left\{a, \|K\|_{\mathrm{Op}}\right\}$.

Proof: By definition of $f(zK) = \sum_{m=0}^{\infty} d_m (zK)^m$ we can for any $x \in D(A)$ express $f(zK)x$ as the limit

$$f(zK)x = \sum_{m=0}^{\infty} d_m (zK)^m x = \lim_{k\to\infty} \sum_{m=0}^{k} d_m z^m K^m x = \lim_{k\to\infty} y_k \tag{A.3.3}$$

where $y_k = \sum_{m=0}^{k} d_m z^m K^m x$, $k \in \mathbb{N}$.

Since K preserves $D(A)$, so too does K^m for any $m \in \mathbb{N}_0$, whence $y_k \in D(A)$ for every $k \in \mathbb{N}$. In order to prove that $f(zK)x$ is an element of $D(A)$ it thus suffices to prove that the sequence

$$Ay_k = \sum_{m=0}^{k} d_m z^m AK^m x, \quad k \in \mathbb{N}, \tag{A.3.4}$$

converges. As X is a Banach space this is ensured if $\sum_{m=0}^{\infty} \|d_m z^m A K^m x\| < \infty$. By the lemma this is indeed the case, as we may estimate

$$\begin{aligned} \sum_{m=0}^{\infty} \|d_m z^m A K^m x\| &= \sum_{m=0}^{\infty} d_m |z|^m \|A K^m x\| \leq \sum_{m=0}^{\infty} d_m |z|^m a^m \|Ax\| + \sum_{m=0}^{\infty} m d_m |z|^m c^{m-1} b \|x\| \\ &= \sum_{m=0}^{\infty} d_m (a|z|)^m \|Ax\| + b|z| \sum_{m=0}^{\infty} m d_m (c|z|)^{m-1} \|x\| \\ &= f(a|z|) \|Ax\| + b|z| f'(c|z|) \|x\| . \end{aligned} \tag{A.3.5}$$

We can then similarly conclude the A-boundedness as

$$\|Af(zK)x\| = \lim_{n\to\infty} \|Ay_n\| \leq \sum_{m=0}^{\infty} \|d_m z^m A K^m x\| \leq f(a|z|)\|Ax\| + b|z| f'(c|z|) \|x\| . \tag{A.3.6}$$

□

Qualitative Properties of $e^{zK}Ae^{-zK}$

Having ensured that $e^{zK}Ae^{-zK}$ is well-defined, we now show the second part of Proposition A.3.1, i.e. that $e^{zK}Ae^{-zK}$ also inherits the properties of A:

Proposition A.3.4. *Under the assumptions on A and K, the operator $e^{zK}Ae^{-zK} : D(A) \to X$ is closed for any $z \in \mathbb{C}$.*

Proof: Let $(x_k)_{k=1}^{\infty} \subset D(A)$ be a sequence such that $x_k \to x$ and $e^{zK}Ae^{-zK}x_k \to y$ for some $x, y \in X$. We must show that $x \in D(A)$ and $y = e^{zK}Ae^{-zK}x$.

By boundedness of K, hence of e^{-zK}, it holds that also $e^{-zK}x_k \to e^{-zK}x$, and similarly

$$Ae^{-zK}x_k = e^{-zK}\left(e^{zK}Ae^{-zK}x\right) \to e^{-zK}y, \tag{A.3.7}$$

so by closedness of A, $e^{-zK}x \in D(A)$ and $Ae^{-zK}x = e^{-zK}y$. Since e^{zK} preserves $D(A)$, it follows that also $x = e^{zK}\left(e^{-zK}x\right) \in D(A)$, and furthermore

$$e^{zK}Ae^{-zK}x = e^{zK}\left(Ae^{-zK}x\right) = e^{zK}\left(e^{-zK}y\right) = y \tag{A.3.8}$$

as was to be shown.

□

If A is a self-adjoint operator on a Hilbert space, self-adjointness is also inherited (for appropriate tK):

Proposition A.3.5. *Suppose that X is a Hilbert space, that A is self-adjoint and that K is skew-symmetric. Then under the assumptions on A and K, the operator $e^{tK}Ae^{-tK} : D(A) \to X$ is self-adjoint for any $t \in \mathbb{R}$.*

Proof: The assumptions clearly imply that $e^{tK}Ae^{-tK}$ is at least symmetric. Letting $x \in D\Big(\big(e^{tK}Ae^{-tK}\big)^*\Big)$ be arbitrary, we must thus show that $x \in D\big(e^{tK}Ae^{-tK}\big) = D(A)$.

The assumption is that there exists a $z \in X$ such that

$$\left\langle x, e^{tK}Ae^{-tK}y\right\rangle = \langle z, y\rangle, \quad y \in D(A). \tag{A.3.9}$$

Rearranging this, we have

$$\left\langle e^{-tK}x, A\Big(e^{-tK}y\Big)\right\rangle = \left\langle x, e^{tK}Ae^{-tK}y\right\rangle = \langle z, y\rangle = \left\langle e^{-tK}z, \Big(e^{-tK}y\Big)\right\rangle, \quad y \in D(A), \tag{A.3.10}$$

which implies that $e^{-tK}x \in D(A^*) = D(A)$ by self-adjointness of A, hence $x \in D(A)$ as in the previous proposition.

□

Differentiability of $z \mapsto e^{zK}Ae^{-zK}x$

Finally we come to the last part of Proposition A.3.1, which is the statement regarding the mapping $z \mapsto e^{zK}Ae^{-zK}x$ for $x \in D(A)$. We begin by observing that this is indeed differentiable:

Proposition A.3.6. *Under the assumptions on A and K, it holds for every $x \in D(A)$ that the mapping $z \mapsto e^{zK}Ae^{-zK}$, $z \in \mathbb{C}$, is complex differentiable with derivative*

$$\frac{d}{dz}e^{zK}Ae^{-zK}x = e^{zK}\,[K, A]\,e^{-zK}x.$$

Proof: The claim is that for any $z_0 \in \mathbb{C}$

$$\left\|\frac{e^{zK}Ae^{-zK}x - e^{z_0K}Ae^{-z_0K}x}{z - z_0} - e^{z_0K}\,[K, A]\,e^{-z_0K}x\right\| \to 0, \quad z \to z_0. \tag{A.3.11}$$

By the identity

$$\begin{aligned} e^{zK}Ae^{-zK}x - e^{z_0K}Ae^{-z_0K}x &= \Big(e^{zK} - e^{z_0K}\Big)Ae^{-z_0K}x + e^{z_0K}A\Big(e^{-zK} - e^{-z_0K}\Big)x \\ &\quad + \Big(e^{zK} - e^{z_0K}\Big)A\Big(e^{-zK} - e^{-z_0K}\Big)x \\ &= e^{z_0K}\Big(e^{(z-z_0)K} - 1\Big)Ae^{-z_0K}x + e^{z_0K}A\Big(e^{-(z-z_0)K} - 1\Big)e^{-z_0K}x \\ &\quad + \Big(e^{zK} - e^{z_0K}\Big)A\Big(e^{-(z-z_0)K} - 1\Big)e^{-z_0K}x \end{aligned} \tag{A.3.12}$$

we see that we can write the argument of $\|\cdot\|$ of the previous equation as a sum of three terms:

$$\begin{aligned} &\frac{e^{zK}Ae^{-zK}x - e^{z_0K}Ae^{-z_0K}x}{z - z_0} - e^{z_0K}\,[K, A]\,e^{-z_0K}x \\ &= e^{z_0K}\left(\frac{e^{(z-z_0)K} - 1}{z - z_0} - K\right)Ae^{-z_0K}x + e^{z_0K}A\left(\frac{e^{-(z-z_0)K} - 1}{z - z_0} + K\right)e^{-z_0K}x \end{aligned} \tag{A.3.13}$$

$$+\left(e^{zK}-e^{z_0K}\right)A\left(\frac{e^{-(z-z_0)K}-1}{z-z_0}\right)e^{-z_0K}x.$$

We show that each term converges to 0 separately as $z \to z_0$. First we have

$$\left\|e^{z_0K}\left(\frac{e^{(z-z_0)K}-1}{z-z_0}-K\right)Ae^{-z_0K}x\right\| \le \left\|e^{z_0K}\right\|_{\mathrm{Op}}\left\|\frac{e^{(z-z_0)K}-1}{z-z_0}-K\right\|_{\mathrm{Op}}\left\|Ae^{-z_0K}x\right\| \tag{A.3.14}$$

which vanishes as $\frac{d}{dz}\Big|_{z=0} e^{zK} = K$ in operator norm by boundedness of K. For the second we estimate using Proposition A.3.3 with $f(z') = e^{z'} - 1 - z'$ and $z' = -(z-z_0)$ that

$$\left\|e^{z_0K}A\left(\frac{e^{-(z-z_0)K}-1}{z-z_0}+K\right)e^{-z_0K}x\right\| \le \frac{\left\|e^{z_0K}\right\|_{\mathrm{Op}}}{|z-z_0|}\left\|A\left(e^{-(z-z_0)K}-1+K(z-z_0)\right)e^{-z_0K}x\right\|$$
$$\le \frac{\left\|e^{z_0K}\right\|_{\mathrm{Op}}}{|z-z_0|}\left(\left(e^{a|z-z_0|}-1-a\,|z-z_0|\right)\left\|Ae^{-z_0K}x\right\|+b\,|z-z_0|\left(e^{c|z-z_0|}-1\right)\left\|e^{-z_0K}x\right\|\right)$$
$$= \left\|e^{z_0K}\right\|_{\mathrm{Op}}\left(\left(\frac{e^{a|z-z_0|}-1}{|z-z_0|}-1\right)\left\|Ae^{-z_0K}x\right\|+b\left(e^{c|z-z_0|}-1\right)\left\|e^{-z_0K}x\right\|\right) \tag{A.3.15}$$

which likewise vanishes since $z \mapsto e^z$ is continuous and $\frac{d}{dz}\Big|_{z=0} e^z = 1$. Similarly, for the last term we can apply Proposition A.3.3 with $f(z') = e^{z'} - 1$ to bound

$$\left\|\left(e^{zK}-e^{z_0K}\right)A\left(\frac{e^{-(z-z_0)K}-1}{z-z_0}\right)e^{-z_0K}x\right\| \le \frac{\left\|e^{zK}-e^{z_0K}\right\|_{\mathrm{Op}}}{|z-z_0|}\left\|A\left(e^{-(z-z_0)K}-1\right)e^{-z_0K}x\right\|$$
$$\le \frac{\left\|e^{zK}-e^{z_0K}\right\|_{\mathrm{Op}}}{|z-z_0|}\left(\left(e^{a|z-z_0|}-1\right)\left\|Ae^{-z_0K}x\right\|+b\,|z-z_0|\,e^{c|z-z_0|}\left\|e^{-z_0K}x\right\|\right) \tag{A.3.16}$$
$$= \left\|e^{zK}-e^{z_0K}\right\|_{\mathrm{Op}}\left(\frac{e^{a|z-z_0|}-1}{|z-z_0|}\left\|Ae^{-z_0K}x\right\|+be^{c|z-z_0|}\left\|e^{-z_0K}x\right\|\right)$$

which vanishes since the term in parenthesis is uniformly bounded for z near z_0 by differentiability of $z \mapsto e^z$ while $e^{zK} \to e^{z_0K}$ as $z \to z_0$ by boundedness of K.

□

A similar argument now shows that the derivative is even continuous:

Proposition A.3.7. *Under the assumptions on A and K, it holds for every $x \in D(A)$ that the mapping $z \mapsto e^{zK}Ae^{-zK}$, $z \in \mathbb{C}$, is C^1.*

Proof: We must show that for any $z_0 \in \mathbb{C}$

$$\left\|e^{zK}\,[K,A]\,e^{-zK}x - e^{z_0K}\,[K,A]\,e^{-z_0K}x\right\| \to 0, \quad z \to z_0. \tag{A.3.17}$$

As in the previous proposition we can write the argument of $\|\cdot\|$ as a sum of three terms:

$$e^{zK}\,[K,A]\,e^{-zK}x - e^{z_0K}\,[K,A]\,e^{-z_0K}x = e^{z_0K}\left(e^{(z-z_0)K}-1\right)[K,A]\,e^{-z_0K}x$$

$$+ e^{z_0 K}\,[K,A]\left(e^{-(z-z_0)K} - 1\right)e^{-z_0 K}x \qquad \text{(A.3.18)}$$
$$+ \left(e^{zK} - e^{z_0 K}\right)[K,A]\left(e^{-(z-z_0)K} - 1\right)e^{-z_0 K}x.$$

The first term vanishes as

$$\left\|e^{z_0 K}\left(e^{(z-z_0)K} - 1\right)[K,A]\,e^{-z_0 K}x\right\| \le \left\|e^{z_0 K}\right\|_{\mathrm{Op}}\left\|\left(e^{(z-z_0)K} - 1\right)[K,A]\,e^{-z_0 K}x\right\| \qquad \text{(A.3.19)}$$

and $e^{(z-z_0)K} \to 1$ as $z \to z_0$ while $[K,A]\,e^{-z_0 K}x$ is a fixed vector. For the other two terms we note that since

$$\left\|e^{z_0 K}\,[K,A]\left(e^{-(z-z_0)K} - 1\right)e^{-z_0 K}x\right\| \le \left\|e^{z_0 K}\right\|_{\mathrm{Op}}\left\|[K,A]\left(e^{-(z-z_0)K} - 1\right)e^{-z_0 K}x\right\|$$
$$\left\|\left(e^{zK} - e^{z_0 K}\right)[K,A]\left(e^{-(z-z_0)K} - 1\right)e^{-z_0 K}x\right\| \le \left\|\left(e^{zK} - e^{z_0 K}\right)\right\|_{\mathrm{Op}}\left\|[K,A]\left(e^{-(z-z_0)K} - 1\right)e^{-z_0 K}x\right\|$$

it suffices to prove that $\left\|[K,A]\left(e^{-(z-z_0)K} - 1\right)e^{-z_0 K}x\right\| \to 0$. By boundedness of K, the assumed A-boundedness of AK implies that $[K,A] = KA - AK$ is also A-bounded, since

$$\|[K,A]\,x\| \le \|KAx\| + \|AKx\| \le \left(\|K\|_{\mathrm{Op}} + a\right)\|Ax\| + b\,\|x\|\,, \quad x \in D(A), \qquad \text{(A.3.20)}$$

so

$$\left\|[K,A]\left(e^{-(z-z_0)K} - 1\right)e^{-z_0 K}x\right\| \le \left(\|K\|_{\mathrm{Op}} + a\right)\left\|A\left(e^{-(z-z_0)K} - 1\right)e^{-z_0 K}x\right\| \qquad \text{(A.3.21)}$$
$$+ b\left\|\left(e^{-(z-z_0)K} - 1\right)e^{-z_0 K}x\right\|$$

and again $\left\|\left(e^{-(z-z_0)K} - 1\right)e^{-z_0 K}x\right\| \to 0$ while $\left\|A\left(e^{-(z-z_0)K} - 1\right)e^{-z_0 K}x\right\|$ is seen to vanish when $z \to z_0$ as in equation (A.3.16).

□

Appendix B

Riemann Sum Estimates

In this section we establish three results. The first is the following general bound on sums of the form $\sum_{p\in L_k}\lambda_{k,p}^{\beta}$:

Proposition B.0.1 (6.2.6)**.** *For any $k\in\mathbb{Z}_*^3$ and $\beta\in[-1,0]$ it holds that*

$$\sum_{p\in L_k}\lambda_{k,p}^{\beta}\leq C\begin{cases}k_F^{2+\beta}\,|k|^{1+\beta} & |k|<2k_F\\ k_F^3\,|k|^{2\beta} & |k|\geq 2k_F\end{cases},\quad k_F\to\infty,$$

for a constant $C>0$ depending only on β.

The second result is the precise asymptotic behaviour of $\sum_{p\in L_k}\lambda_{k,p}^{-1}$ for small k:

Proposition B.0.2 (9.0.1)**.** *For any $\gamma\in\left(0,\frac{1}{11}\right)$ and $k\in\overline{B}(0,k_F^{\gamma})$ it holds that*

$$\sum_{p\in L_k}\lambda_{k,p}^{-1}=2\pi k_F+O\Big(\log(k_F)^{\frac{5}{3}}k_F^{\frac{1}{3}(2+11\gamma)}\Big),\quad k_F\to\infty.$$

Finally we prove the following lower bounds for the sums $\sum_{p\in L_k}\lambda_{k,p}^{\beta}$:

Proposition B.0.3 (11.3.1)**.** *For all $k\in B_F$ and $\beta\in\{0\}\cup[1,\infty)$ it holds that*

$$\sum_{p\in L_k}\lambda_{k,p}^{\beta}\geq ck_F^{2+\beta}\,|k|^{1+\beta},\quad k_F\to\infty,$$

for a $c>0$ depending only on β.

Some General Riemann Sum Estimation Results

To prove these propositions we first note some general Riemann sum estimation results.

Let $S\subset\mathbb{R}^n$, $n\in\mathbb{N}$, be given, define for $k\in\mathbb{Z}^n$ the translated unit cube $\mathcal{C}_k$ by

$$\mathcal{C}_k=\left[-2^{-1},2^{-1}\right]+k \tag{B.0.1}$$

and let $\mathcal{C}_S=\bigcup_{k\in S\cap\mathbb{Z}^n}\mathcal{C}_k$ denote the union of the cubes centered at the lattice points contained in S. We then note the following:

Lemma B.0.4. *Let $f \in C(\mathcal{C}_S)$ be a function which is convex on $\mathcal{C}_k$ for all $k \in S \cap \mathbb{Z}^n$. Then*

$$\sum_{k \in S \cap \mathbb{Z}^n} f(k) \leq \int_{\mathcal{C}_S} f(p)\, dp.$$

Proof: As a convex function admits a supporting hyperplane at every interior point of its domain, there exists for every $k \in S \cap \mathbb{Z}^n$ a $c \in \mathbb{R}^n$ such that

$$f(p) \geq f(k) + c \cdot (p - k), \quad p \in \mathcal{C}_k, \tag{B.0.2}$$

and so integration yields

$$\int_{\mathcal{C}_k} f(p)\, dp \geq \int_{\mathcal{C}_k} f(k)\, dp + \int_{\mathcal{C}_k} c \cdot (p - k)\, dp = f(k) \tag{B.0.3}$$

as $\int_{\mathcal{C}_k} c \cdot (p - k)\, dp = 0$ by antisymmetry. Consequently

$$\sum_{k \in S \cap \mathbb{Z}^n} f(k) \leq \sum_{k \in S \cap \mathbb{Z}^n} \int_{\mathcal{C}_k} f(p)\, dp = \int_{\mathcal{C}_S} f(k)\, dp. \tag{B.0.4}$$

□

This lemma lets us replace a sum by an integral, but over an integration domain $\mathcal{C}_S$ which will generally be complicated. An exception is the $n = 1$ case which we record in the following (generalizing also the statement to any lattice spacing l):

Corollary B.0.5. *Let for $a, b \in \mathbb{Z}$ and $l > 0$ a convex function $f \in C\Big(\Big[la - \frac{1}{2}l, lb + \frac{1}{2}l\Big]\Big)$ be given. Then*

$$\sum_{m=a}^{b} f(lm)l \leq \int_{la - \frac{1}{2}l}^{lb + \frac{1}{2}l} f(x)\, dx.$$

For $n \neq 1$ we instead require an additional step that lets us replace $\mathcal{C}_S$ by a simpler integration domain. Define $S_+ \subset \mathbb{R}^n$ by

$$S_+ = \left\{ p \in \mathbb{R}^n \mid \inf_{q \in S} |p - q| \leq \frac{\sqrt{n}}{2} \right\}. \tag{B.0.5}$$

Observe that $\mathcal{C}_S \subset S_+$: Indeed, for any $p \in \mathcal{C}_S$ there exists by assumption a $k \in S \cap \mathbb{Z}^n$ such that $p \in \mathcal{C}_S$; consequently

$$\inf_{q \in S} |p - q| \leq |p - k| \leq \frac{\sqrt{n}}{2} \tag{B.0.6}$$

since every point of a unit cube is a distance at most $\frac{\sqrt{n}}{2}$ from its center. The containment $\mathcal{C}_S \subset S_+$ and the lemma now easily imply the following:

Corollary B.0.6. *Let $f \in C(S_+)$ be a positive function which is convex on $\mathcal{C}_k$ for all $k \in S \cap \mathbb{Z}^n$. Then*

$$\sum_{k \in S \cap \mathbb{Z}^n} f(k) \leq \int_{S_+} f(p)\, dp.$$

Note that in the particular case that f is identically 1 this yields a bound on the lattice points contained in S:

$$|S \cap \mathbb{Z}^n| \leq \mathrm{Vol}(S_+). \tag{B.0.7}$$

B.1 Simple Upper Bounds for $\beta \in [-1, 0]$

We now consider the sums $\sum_{p\in L_k} \lambda_{k,p}^{\beta}$. In this subsection we prove the $\beta \in (-1, 0]$ statement of Proposition B.0.1, i.e. that

$$\sum_{p\in L_k} \lambda_{k,p}^{\beta} \leq C \begin{cases} k_F^{2+\beta} \left|k\right|^{1+\beta} & |k| < 2k_F \\ k_F^3 \left|k\right|^{2\beta} & |k| \geq 2k_F \end{cases}, \quad \beta \in (-1, 0], \tag{B.1.1}$$

as well as the partial statement for $\beta = -1$ that

$$\sum_{p\in L_k} \lambda_{k,p}^{-1} \leq C \begin{cases} \left(1 + \left|k\right|^{-1} \log(k_F)\right) k_F & |k| < 2k_F \\ k_F^3 \left|k\right|^{-2} & |k| \geq 2k_F \end{cases}. \tag{B.1.2}$$

The improvement of the latter estimate to $\sum_{p\in L_k} \lambda_{k,p}^{-1} \leq C k_F$ for $|k| < 2k_F$ will be handled by more precise estimates later in the section.

Recall that the lunes L_k are given by

$$L_k = \left\{p \in \mathbb{Z}^3 \mid |p - k| \leq k_F < |p|\right\} = S \cap \mathbb{Z}^3 \tag{B.1.3}$$

where $S = \overline{B}(k, k_F) \backslash \overline{B}(0, k_F)$. The relevant integrand for our Riemann sums,

$$p \mapsto \lambda_{k,p}^{\beta} = \left(\frac{1}{2}\left(|p|^2 - |p - k|^2\right)\right)^{\beta} = |k|^{\beta} \left(\hat{k}\cdot p - \frac{1}{2}\left|k\right|\right)^{\beta} \tag{B.1.4}$$

is convex on $\left\{p \in \mathbb{R}^3 \mid \hat{k} \cdot p > \frac{1}{2}\left|k\right|\right\}$ but singular when $\hat{k} \cdot p = \frac{1}{2}\left|k\right|$. We must therefore introduce a cut-off to the Riemann sum $\sum_{p\in L_k} \lambda_{k,p}^{\beta}$: We write $L_k = L_k^1 \cup L_k^2$ for

$$\begin{aligned} L_k^1 &= \left\{p \in L_k \mid \hat{k} \cdot p \leq \frac{1}{2}\left|k\right| + 1 + \frac{\sqrt{3}}{2}\right\} \\ L_k^2 &= \left\{p \in L_k \mid \hat{k} \cdot p > \frac{1}{2}\left|k\right| + 1 + \frac{\sqrt{3}}{2}\right\}. \end{aligned} \tag{B.1.5}$$

Then also $L_k^i = S^i \cap \mathbb{Z}^3$, $i = 1, 2$, for

$$\begin{aligned} S^1 &= \left\{p \in S \mid \hat{k} \cdot p \leq \frac{1}{2}\left|k\right| + 1 + \frac{\sqrt{3}}{2}\right\} \\ S^2 &= \left\{p \in S \mid \hat{k} \cdot p > \frac{1}{2}\left|k\right| + 1 + \frac{\sqrt{3}}{2}\right\} \end{aligned} \tag{B.1.6}$$

so we can by Corollary B.0.6 estimate that

$$\sum_{p\in L_k} \lambda_{k,p}^{\beta} = \sum_{p\in L_k^1} \lambda_{k,p}^{\beta} + \sum_{p\in L_k^2} \lambda_{k,p}^{\beta} \leq \left(\sup_{p\in L_k^1} \lambda_{k,p}^{\beta}\right) \left|L_k^1\right| + \int_{S_+^2} |k|^{\beta} \left(\hat{k}\cdot p - \frac{1}{2}\left|k\right|\right)^{\beta} dp \tag{B.1.7}$$

$$\leq 2^{-\beta}\,\mathrm{Vol}\Big(S_+^1\Big) + |k|^\beta \int_{S_+^2} \left(\hat{k}\cdot p - \frac{1}{2}\,|k|\right)^\beta dp.$$

Here we also used the observation that

$$\lambda_{k,p} = \frac{1}{2}\Big(|p|^2 - |p-k|^2\Big) \geq \frac{1}{2} \tag{B.1.8}$$

for all $k \in \mathbb{Z}^3_*$ and $p \in L_k$, as $|p|^2$ and $|p-k|^2$ are then non-equal integers.

To estimate $\mathrm{Vol}\Big(S_+^1\Big)$ and the integral over S_+^2 we will replace these by simpler sets once more: Let $\mathcal{S} \subset \mathbb{R}^3$ be given by

$$\mathcal{S} = \overline{B}\left(k, k_F + \frac{\sqrt{3}}{2}\right) \backslash B\left(0, k_F - \frac{\sqrt{3}}{2}\right) \tag{B.1.9}$$

and define the subsets $\mathcal{S}^1, \mathcal{S}^2 \subset \mathcal{S}$ by

$$\mathcal{S}^1 = \left\{p \in \mathcal{S} \mid -\frac{\sqrt{3}}{2} \leq \hat{k}\cdot p - \frac{1}{2}\,|k| \leq 1 + \sqrt{3}\right\} \tag{B.1.10}$$
$$\mathcal{S}^2 = \left\{p \in \mathcal{S} \mid 1 \leq \hat{k}\cdot p - \frac{1}{2}\,|k|\right\}.$$

Then we have the following:

Proposition B.1.1. *It holds that*

$$S_+^1 \subset \mathcal{S}^1 \quad \textit{and} \quad S_+^2 \subset \mathcal{S}^2.$$

Proof: We first show that $S_+ \subset \mathcal{S}$: Let $p \in S_+ = \left\{p' \in \mathbb{R}^3 \mid \inf_{q\in S} |p'-q| \leq \frac{\sqrt{3}}{2}\right\}$ be arbitrary. Then we can for any $q \in S$ estimate that

$$|p| \geq |q| - |p-q| > k_F - |p-q| \tag{B.1.11}$$
$$|p-k| \leq |q-k| + |p-q| \leq k_F + |p-q|$$

whence taking the supremum and infimum over $q \in S$ yields

$$|p| \geq k_F - \frac{\sqrt{3}}{2}, \quad |p-k| \leq k_F + \frac{\sqrt{3}}{2}, \tag{B.1.12}$$

which is to say that $p \in \mathcal{S}$ as claimed. Supposing then that $p \in S_+^1$ we furthermore note that for any $q \in S^1$, Cauchy-Schwarz implies that

$$\hat{k}\cdot p - \frac{1}{2}\,|k| = \hat{k}\cdot q - \frac{1}{2}\,|k| + \hat{k}\cdot(p-q) \leq 1 + \frac{\sqrt{3}}{2} + |p-q| \tag{B.1.13}$$

and similarly

$$\hat{k}\cdot p - \frac{1}{2}\,|k| = \hat{k}\cdot q - \frac{1}{2}\,|k| + \hat{k}\cdot(p-q) \geq -|p-q| \tag{B.1.14}$$

so taking the supremum and infimum over $q \in S^1$ again yields

$$-\frac{\sqrt{3}}{2} \leq \hat{k} \cdot p - \frac{1}{2}|k| \leq 1 + \sqrt{3} \tag{B.1.15}$$

i.e. $p \in \mathcal{S}^1$. That $S_+^2 \subset \mathcal{S}^2$ follows similarly.

□

By this proposition it now follows from equation (B.1.7) that

$$\sum_{p \in L_k} \lambda_{k,p}^{\beta} \leq 2^{-\beta}\,\mathrm{Vol}\Big(\mathcal{S}^1\Big) + |k|^{\beta} \int_{\mathcal{S}^2} \left(\hat{k} \cdot p - \frac{1}{2}|k|\right)^{\beta} dp. \tag{B.1.16}$$

To compute $\mathrm{Vol}(\mathcal{S}^1)$ and the integral over $\mathcal{S}^2$ we will integrate along the $\hat{k}$-axis, so we must now consider the behaviour of the "slices"

$$\mathcal{S}_t = \mathcal{S} \cap \left\{p \in \mathbb{R}^3 \mid \hat{k} \cdot p = t\right\}. \tag{B.1.17}$$

The Case $|k| < 2k_F$

Suppose first that $|k| < 2k_F$. Then when moving along the $\hat{k}$-axis, it holds that

$$\inf(\{t \mid \mathcal{S}_t \neq \emptyset\}) = \begin{cases} -\Big(k_F + \frac{\sqrt{3}}{2}\Big) + |k| & |k| \leq \sqrt{3} \\ \frac{1}{2}|k| - |k|^{-1}\sqrt{3}\,k_F & |k| > \sqrt{3} \end{cases} \tag{B.1.18}$$

where the first case corresponds to the case that $B\Big(0, k_F - \frac{\sqrt{3}}{2}\Big)$ is entirely contained in $\overline{B}\Big(0, k_F + \frac{\sqrt{3}}{2}\Big)$.

As the lower end of $\mathcal{S}^1$ is at $t = \frac{1}{2}|k| - \frac{\sqrt{3}}{2}$, we need not consider this case, since $\Big\{\hat{k} \cdot p = \frac{1}{2}|k| - \frac{\sqrt{3}}{2}\Big\}$ will intersect both $\overline{B}\Big(k, k_F + \frac{\sqrt{3}}{2}\Big)$ and $B\Big(0, k_F - \frac{\sqrt{3}}{2}\Big)$ anyway. In this case the slice $\mathcal{S}_t$ forms an annulus, and elementary trigonometry shows that

$$\begin{aligned} \mathrm{Area}(\mathcal{S}_t) &= \pi\left(\left(k_F + \frac{\sqrt{3}}{2}\right)^2 - (t - |k|)^2\right) - \pi\left(\left(k_F - \frac{\sqrt{3}}{2}\right)^2 - t^2\right) \\ &= \pi\Big(2\sqrt{3}\,k_F - \Big(|k|^2 - 2\,|k|\,t\Big)\Big) = 2\pi\left(|k|\left(t - \frac{1}{2}|k|\right) + \sqrt{3}\,k_F\right) \end{aligned} \tag{B.1.19}$$

for $\frac{1}{2}|k| - \frac{\sqrt{3}}{2} \leq t \leq k_F - \frac{\sqrt{3}}{2}$, with $t = k_F - \frac{\sqrt{3}}{2}$ corresponding to the "upper end" of $B\Big(0, k_F - \frac{\sqrt{3}}{2}\Big)$. Thereafter the planes intersect only $\overline{B}\Big(k, k_F + \frac{\sqrt{3}}{2}\Big)$, whence

$$\begin{aligned} \mathrm{Area}(\mathcal{S}_t) &= \pi\left(\left(k_F + \frac{\sqrt{3}}{2}\right)^2 - (t - |k|)^2\right) = \pi\left(\left(k_F + \frac{\sqrt{3}}{2}\right)^2 - t^2 + 2\,|k|\left(t - \frac{1}{2}|k|\right)\right) \\ &= 2\pi\left(|k|\left(t - \frac{1}{2}|k|\right) + \sqrt{3}\,k_F\right) + \pi\left(\left(k_F - \frac{\sqrt{3}}{2}\right)^2 - t^2\right) \end{aligned} \tag{B.1.20}$$

$$\leq 2\pi\Big(|k|\left(t-\frac{1}{2}|k|\right)+\sqrt{3}\,k_F\Big)$$

for $k_F-\frac{\sqrt{3}}{2}\leq t\leq k_F+\frac{\sqrt{3}}{2}+|k|$.

With this we can now prove the $|k|<2k_F$ bounds:

Proposition B.1.2. *For all $k\in B(0,2k_F)\cap\mathbb{Z}^3_*$ and $\beta\in[-1,0]$ it holds that*

$$\sum_{p\in L_k}\lambda_{k,p}^{\beta}\leq C\begin{cases}k_F^{2+\beta}\,|k|^{1+\beta} & \beta\in(-1,0]\\ \big(1+|k|^{-1}\log(k_F)\big)k_F & \beta=-1\end{cases},\quad k_F\to\infty,$$

for a constant $C>0$ depending only on β.

Proof: Recall that

$$\sum_{p\in L_k}\lambda_{k,p}^{\beta}\leq 2^{-\beta}\,\mathrm{Vol}\big(\mathcal{S}^1\big)+|k|^{\beta}\int_{\mathcal{S}^2}\left(\hat{k}\cdot p-\frac{1}{2}|k|\right)^{\beta}dp. \tag{B.1.21}$$

The volume of $\mathcal{S}^1$ obeys

$$\begin{aligned}\mathrm{Vol}\big(\mathcal{S}^1\big)&=\int_{\frac{1}{2}|k|-\frac{\sqrt{3}}{2}}^{\frac{1}{2}|k|+1+\sqrt{3}}\mathrm{Area}(\mathcal{S}_t)\,dt\leq 2\pi\int_{\frac{1}{2}|k|-\frac{\sqrt{3}}{2}}^{\frac{1}{2}|k|+1+\sqrt{3}}\Big(|k|\left(t-\frac{1}{2}|k|\right)+\sqrt{3}\,k_F\Big)\,dt \qquad \text{(B.1.22)}\\&=2\pi\int_{-\frac{\sqrt{3}}{2}}^{1+\sqrt{3}}\big(|k|\,t+\sqrt{3}\,k_F\big)\,dt\leq C(|k|+k_F)\leq Ck_F,\quad k_F\to\infty,\end{aligned}$$

which is $O\Big(k_F^{2+\beta}\,|k|^{1+\beta}\Big)$ for all $\beta\in[-1,0]$. For $\beta\in(-1,0]$ the integral is

$$\begin{aligned}\int_{\mathcal{S}^2}\left(\hat{k}\cdot p-\frac{1}{2}|k|\right)^{\beta}dp&=\int_{\frac{1}{2}|k|+1}^{k_F+\frac{\sqrt{3}}{2}+|k|}\left(t-\frac{1}{2}|k|\right)^{\beta}\mathrm{Area}(\mathcal{S}_t)\,dt\\&\leq 2\pi\int_{\frac{1}{2}|k|+1}^{k_F+\frac{\sqrt{3}}{2}+|k|}\left(t-\frac{1}{2}|k|\right)^{\beta}\Big(|k|\left(t-\frac{1}{2}|k|\right)+\sqrt{3}\,k_F\Big)\,dt\\&=2\pi\left(|k|\int_1^{k_F+\frac{\sqrt{3}}{2}+\frac{1}{2}|k|}t^{1+\beta}dt+\sqrt{3}\,k_F\int_1^{k_F+\frac{\sqrt{3}}{2}+\frac{1}{2}|k|}t^{\beta}dt\right) \qquad \text{(B.1.23)}\\&\leq 2\pi\left(\frac{|k|}{2+\beta}\left(k_F+\frac{\sqrt{3}}{2}+\frac{1}{2}|k|\right)^{2+\beta}+\frac{\sqrt{3}\,k_F}{1+\beta}\left(k_F+\frac{\sqrt{3}}{2}+\frac{1}{2}|k|\right)^{1+\beta}\right)\\&\leq 2\pi\left(\frac{1}{2+\beta}k_F^{2+\beta}\,|k|+\frac{\sqrt{3}}{1+\beta}k_F^{2+\beta}\right)\leq Ck_F^{2+\beta}\,|k|,\quad k_F\to\infty,\end{aligned}$$

while the $\beta=-1$ case is

$$\int_{\mathcal{S}^2}\left(\hat{k}\cdot p-\frac{1}{2}|k|\right)^{-1}dp\leq 2\pi\left(|k|\int_1^{k_F+\frac{\sqrt{3}}{2}+\frac{1}{2}|k|}1\,dt+\sqrt{3}\,k_F\int_1^{k_F+\frac{\sqrt{3}}{2}+\frac{1}{2}|k|}t^{-1}dt\right)$$

$$\leq 2\pi\left(|k|\left(k_F+\frac{\sqrt{3}}{2}+\frac{1}{2}|k|\right)+\sqrt{3}\,k_F\log\left(k_F+\frac{\sqrt{3}}{2}+\frac{1}{2}|k|\right)\right)$$
$$\leq C\,|k|\left(1+|k|^{-1}\log(k_F)\right)k_F. \tag{B.1.24}$$

Combining the estimates yields the claim.

□

The Case $|k| \geq 2k_F$

Now suppose instead that $|k| \geq 2k_F$. In this case the lune $S = \overline{B}(k,k_F)\backslash\overline{B}(0,k_F)$ degenerates into a ball, and so we simply have that

$$S_+ = \mathcal{S} = \overline{B}\left(k, k_F + \frac{\sqrt{3}}{2}\right). \tag{B.1.25}$$

Now, if $\frac{1}{2}|k| \geq k_F + 1 + \frac{\sqrt{3}}{2}$ then every $p \in \mathcal{S}$ satisfies $\hat{k}\cdot p - \frac{1}{2}|k| \geq 1$ and the cut-off set $\mathcal{S}^1$ is unnecessary. If this is not the case then it still holds that

$$\sum_{p\in L_k}\lambda_{k,p}^{\beta} \leq 2^{-\beta}\,\mathrm{Vol}\left(\mathcal{S}^1\right) + |k|^{\beta}\int_{\mathcal{S}^2}\left(\hat{k}\cdot p - \frac{1}{2}|k|\right)^{\beta}dp \tag{B.1.26}$$

for

$$\mathcal{S}^1 = \left\{p\in\mathcal{S} \mid \hat{k}\cdot p - \frac{1}{2}|k| \leq 1+\sqrt{3}\right\} \tag{B.1.27}$$
$$\mathcal{S}^2 = \left\{p\in\mathcal{S} \mid 1 \leq \hat{k}\cdot p - \frac{1}{2}|k|\right\},$$

and we may easily estimate $\mathrm{Vol}(\mathcal{S}^1)$ as $\mathcal{S}^1$ is now seen to be a spherical cap of radius $k_F + \frac{\sqrt{3}}{2}$ and height

$$\left(\frac{1}{2}|k|+1+\sqrt{3}\right)-\left(|k|-k_F-\frac{\sqrt{3}}{2}\right) = k_F - \frac{1}{2}|k| + 1 + \frac{3\sqrt{3}}{2} \leq 1+\frac{3\sqrt{3}}{2} \tag{B.1.28}$$

whence

$$\mathrm{Vol}\left(\mathcal{S}^1\right) \leq \frac{\pi}{3}\left(1+\frac{3\sqrt{3}}{2}\right)\left(3\left(k_F+\frac{\sqrt{3}}{2}\right)-1-\frac{3\sqrt{3}}{2}\right) \leq Ck_F \tag{B.1.29}$$

which is again $O\left(k_F^{2+\beta}|k|^{1+\beta}\right)$. We thus only need to estimate the integral for the $|k| \geq 2k_F$ bounds:

Proposition B.1.3. *For all $k \in \mathbb{Z}^3_*\backslash B(0,2k_F)$ and $\beta \in [-1,0]$ it holds that*

$$\sum_{p\in L_k}\lambda_{k,p}^{\beta} \leq Ck_F^3\,|k|^{2\beta}, \quad k_F\to\infty,$$

for a constant $C > 0$ depending only on β.

Proof: We again note that

$$\text{Area}(\mathcal{S}_t) = \pi\left(\left(k_F + \frac{\sqrt{3}}{2}\right)^2 - (t - |k|)^2\right), \tag{B.1.30}$$

now for $|k| - k_F - \frac{\sqrt{3}}{2} \leq t \leq |k| + k_F + \frac{\sqrt{3}}{2}$. If $\frac{1}{2}|k| \leq k_F + 1 + \frac{\sqrt{3}}{2}$ we just saw that the contribution coming from the cut-off set $\mathcal{S}^1$ is negligible, while the integral term is

$$\begin{aligned} |k|^\beta \int_{\mathcal{S}^2}\left(\hat{k}\cdot p - \frac{1}{2}|k|\right)^\beta dp &= |k|^\beta \int_{\frac{1}{2}|k|+1}^{k_F+\frac{\sqrt{3}}{2}+|k|}\left(t - \frac{1}{2}|k|\right)^\beta \text{Area}(\mathcal{S}_t)\,dt \\ &\leq Ck_F^{2+\beta}|k|^{1+\beta}, \quad k_F \to \infty, \end{aligned} \tag{B.1.31}$$

as calculated in the previous proposition, which is $O\left(k_F^3|k|^{2\beta}\right)$ since $\frac{1}{2}|k| \leq k_F + 1 + \frac{\sqrt{3}}{2}$. (Here we also used that for $\beta = -1$, the term $|k|^{-1}\log(k_F)$ can be disregarded when $|k| \geq 2k_F$.)

If $\frac{1}{2}|k| > k_F + 1 + \frac{\sqrt{3}}{2}$ then we simply have

$$\sum_{p\in L_k}\lambda_{k,p}^\beta \leq |k|^\beta\int_{\mathcal{S}}\left(\hat{k}\cdot p - \frac{1}{2}|k|\right)^\beta dp = |k|^\beta\int_{|k|-k_F-\frac{\sqrt{3}}{2}}^{|k|+k_F+\frac{\sqrt{3}}{2}}\left(t - \frac{1}{2}|k|\right)^\beta\text{Area}(\mathcal{S}_t)\,dt, \tag{B.1.32}$$

and noting that

$$(t - |k|)^2 = \left(t - \frac{1}{2}|k|\right)^2 - |k|\left(t - \frac{1}{2}|k|\right) + \frac{1}{4}|k|^2 \tag{B.1.33}$$

we can now estimate $\text{Area}(\mathcal{S}_t)$ as

$$\begin{aligned} \text{Area}(\mathcal{S}_t) &= \pi\left(\left(k_F + \frac{\sqrt{3}}{2}\right)^2 - (t-|k|)^2\right) \\ &= \pi\left(\left(k_F + \frac{\sqrt{3}}{2}\right)^2 - \left(t - \frac{1}{2}|k|\right)^2 + |k|\left(t - \frac{1}{2}|k|\right) - \frac{1}{4}|k|^2\right) \\ &= \pi\left(|k|\left(t - \frac{1}{2}|k|^2\right) - \left(\frac{1}{4}|k|^2 - \left(k_F + \frac{\sqrt{3}}{2}\right)^2\right) - \left(t - \frac{1}{2}|k|\right)^2\right) \\ &\leq \pi|k|\left(t - \frac{1}{2}|k|\right). \end{aligned} \tag{B.1.34}$$

Consequently

$$\begin{aligned} \sum_{p\in L_k}\lambda_{k,p}^\beta &\leq \pi|k|^{1+\beta}\int_{|k|-k_F-\frac{\sqrt{3}}{2}}^{|k|+k_F+\frac{\sqrt{3}}{2}}\left(t - \frac{1}{2}|k|\right)^{1+\beta}dt = \pi|k|^{1+\beta}\int_{\frac{1}{2}|k|-k_F-\frac{\sqrt{3}}{2}}^{\frac{1}{2}|k|+k_F+\frac{\sqrt{3}}{2}} t^{1+\beta}dt \\ &= \frac{\pi}{2+\beta}|k|^{1+\beta}\left(\left(\frac{1}{2}|k| + k_F + \frac{\sqrt{3}}{2}\right)^{2+\beta} - \left(\frac{1}{2}|k| - k_F - \frac{\sqrt{3}}{2}\right)^{2+\beta}\right) \end{aligned} \tag{B.1.35}$$

$$\leq C\,|k|^{3+2\beta}.$$

If additionally $|k| \leq 3\,k_F$ (say) then this is $O\Big(k_F^3\,|k|^{2\beta}\Big)$, and if not then we can nonetheless trivially estimate

$$\begin{aligned}\sum_{p\in L_k}\lambda_{k,p}^{\beta} &\leq |k|^{\beta}\int_{\mathcal{S}}\left(\hat{k}\cdot p-\frac{1}{2}\,|k|\right)^{\beta}dp \leq |k|^{\beta}\left(\inf_{p\in\mathcal{S}}\left(\hat{k}\cdot p-\frac{1}{2}\,|k|\right)\right)^{\beta}\int_{\mathcal{S}}1\,dp \\ &\leq |k|^{\beta}\left(|k|-k_F-\frac{\sqrt{3}}{2}-\frac{1}{2}\,|k|\right)^{\beta}\mathrm{Vol}\left(\overline{B}\left(0,k_F+\frac{\sqrt{3}}{2}\right)\right) \qquad \text{(B.1.36)} \\ &\leq Ck_F^3\,|k|^{\beta}\left(\frac{1}{2}\,|k|-\frac{1}{3}\,|k|-\frac{\sqrt{3}}{2}\right)^{\beta} \leq Ck_F^3\,|k|^{2\beta}, \quad k_F\to\infty,\end{aligned}$$

for the claim.

□

B.2 Some Lattice Concepts

To improve upon our bound on $\sum_{p\in L_k}\lambda_{k,p}^{-1}$ (and in particular to establish its asymptotic behaviour) we will need some results regarding lattices, which we now review.

A lattice Λ in a real n-dimensional vector space V is defined to be a subset of V with the following property: There exists a basis $(v_i)_{i=1}^n$ of V such that Λ equals the integral span of $(v_i)_{i=1}^n$, i.e.

$$\Lambda = \left\{\sum_{i=1}^{n} m_i v_i \mid m_1,\ldots,m_n\in\mathbb{Z}\right\}. \qquad \text{(B.2.1)}$$

Given a basis $(v_i)_{i=1}^n$, the right-hand side of this equation always defines a lattice, called the lattice generated by $(v_i)_{i=1}^n$, and denoted by $\langle v_1,\ldots,v_n\rangle$. Two different bases $(v_i)_{i=1}^n$ and $(w_i)_{i=1}^n$ may generate the same lattice, in which case the following is well-known:

Proposition B.2.1. *Let $(v_i)_{i=1}^n$ and $(w_i)_{i=1}^n$ be bases of V. Then $\langle v_1,\ldots,v_n\rangle = \langle w_1,\ldots,w_n\rangle$ if and only if the transition matrix $(T_{i,j})_{i,j=1}^n$, defined by the relation*

$$w_i = \sum_{j=1}^{n} T_{i,j}v_j, \quad 1\leq i\leq n,$$

has integer entries and determinant ± 1.

This result has an important consequence when V is endowed with an inner product: Then one can define the hypervolume of the parallelepiped spanned by $(v_i)_{i=1}^n$ by

$$\left|\det\begin{pmatrix}\langle e_1,v_1\rangle & \cdots & \langle e_n,v_1\rangle \\ \vdots & \ddots & \vdots \\ \langle e_1,v_n\rangle & \cdots & \langle e_n,v_n\rangle\end{pmatrix}\right| = \sqrt{\det\begin{pmatrix}\langle v_1,v_1\rangle & \cdots & \langle v_n,v_1\rangle \\ \vdots & \ddots & \vdots \\ \langle v_1,v_n\rangle & \cdots & \langle v_n,v_n\rangle\end{pmatrix}} \qquad \text{(B.2.2)}$$

for any orthonormal basis $(e_i)_{i=1}^n$ (the expression on the right-hand side follows by orthonormal expansion). It is however a general fact that if two bases $(v_i)_{i=1}^n$ and $(w_i)_{i=1}^n$ are related by a transition matrix T, then

$$\det\begin{pmatrix} \langle e_1, w_1\rangle & \cdots & \langle e_n, w_1\rangle \\ \vdots & \ddots & \vdots \\ \langle e_1, w_n\rangle & \cdots & \langle e_n, w_n\rangle \end{pmatrix} = \det(T)\det\begin{pmatrix} \langle e_1, v_1\rangle & \cdots & \langle e_n, v_1\rangle \\ \vdots & \ddots & \vdots \\ \langle e_1, v_n\rangle & \cdots & \langle e_n, v_n\rangle \end{pmatrix} \tag{B.2.3}$$

whence one concludes the following:

Proposition B.2.2. *Let Λ be a lattice in $(V, \langle\cdot,\cdot\rangle)$ and let $(v_i)_{i=1}^n$ generate Λ. Then the quantity*

$$d(\Lambda) = \left|\det\begin{pmatrix} \langle e_1, v_1\rangle & \cdots & \langle e_n, v_1\rangle \\ \vdots & \ddots & \vdots \\ \langle e_1, v_n\rangle & \cdots & \langle e_n, v_n\rangle \end{pmatrix}\right| = \sqrt{\det\begin{pmatrix} \langle v_1, v_1\rangle & \cdots & \langle v_n, v_1\rangle \\ \vdots & \ddots & \vdots \\ \langle v_1, v_n\rangle & \cdots & \langle v_n, v_n\rangle \end{pmatrix}}$$

is an invariant of Λ, independent of the choice of generators $(v_i)_{i=1}^n$.

The quantity $d(\Lambda)$ is referred to as the covolume of Λ.

For a lattice Λ in an inner product space V, one defines the succesive minima (relative to $\overline{B}(0,1)$), $(\lambda_i)_{i=1}^n$, by

$$\lambda_i = \inf\Big(\Big\{\lambda \mid \overline{B}(0,\lambda)\cap\Lambda \text{ contains } i \text{ linearly independent vectors}\Big\}\Big), \quad 1 \le i \le n. \tag{B.2.4}$$

A well-known theorem due to Minkowski relates succesive minima and covolumes:

Theorem B.2.3 (Minkowski's Second Theorem)**.** *Let Λ be a lattice in an n-dimensional inner product space V. Then it holds that*

$$\frac{2^n d(\Lambda)}{n!\mathrm{Vol}\big(\overline{B}(0,1)\big)} \le \lambda_1\cdots\lambda_n \le \frac{2^n d(\Lambda)}{\mathrm{Vol}\big(\overline{B}(0,1)\big)}.$$

Note that although the quantity λ_n is such that $\overline{B}(0,\lambda_n)\cap\Lambda$ contains n linearly independent vectors, it is not ensured that these can be chosen to generate Λ. For $n = 2$ this is nonetheless the case:

Proposition B.2.4. *Let Λ be a lattice in a 2-dimensional inner product space V. Then there exists vectors $v_1, v_2 \in \Lambda$ which generate Λ such that*

$$\|v_1\|\,\|v_2\| \le \frac{4}{\pi} d(\lambda).$$

Proof: By definition there exists linearly independent vectors $v_1, v_2 \in \Lambda$ such that $\|v_1\| \le \lambda_1$, $\|v_2\| \le \lambda_2$, and by Minkowski's second theorem $\|v_1\|\,\|v_2\| \le \frac{4}{\pi}d(\lambda)$. We argue that v_1 and v_2 must generate Λ.

Suppose otherwise, i.e. let $v \in \Lambda$ be such that $v \neq m_1 v_1 + m_2 v_2$ for $m_1, m_2 \in \mathbb{Z}$. As v_1 and v_2 are linearly independent and $\dim(V) = 2$ they span V, so we can nonetheless write $v = c_1 v_1 + c_2 v_2$ for some $c_1, c_2 \in \mathbb{R}$. By subtracting integer multiples of v_1 and v_2 we may further assume that $|c_1|, |c_2| \leq \frac{1}{2}$.

As $\langle v_1, v_2 \rangle < \|v_1\| \|v_2\|$ by Cauchy-Schwarz (the inequality being strict due to linear independence) we can then estimate that

$$\begin{aligned} \|v\|^2 &= |c_1|^2 \|v_1\|^2 + |c_2|^2 \|v_2\|^2 + 2c_1c_2 \langle v_1, v_2 \rangle < |c_1|^2 \|v_1\|^2 + |c_2|^2 \|v_2\|^2 + 2|c_1||c_2| \|v_1\| \|v_2\| \\ &= (|c_1| \|v_1\| + |c_2| \|v_2\|)^2 \leq \left(\frac{1}{2}\lambda_2 + \frac{1}{2}\lambda_2\right)^2 = \lambda_2^2, \end{aligned} \tag{B.2.5}$$

i.e. $\|v\| < \lambda_2$. But this contradicts the minimality of λ_2, so such a v can not exist.

□

The Sublattice Orthogonal to a Vector $k \in \mathbb{Z}^3$

Consider $\mathbb{Z}^3$ as a lattice in $\mathbb{R}^3$, endowed with the usual dot product. Let $k = (k_1, k_2, k_3) \in \mathbb{Z}^3_*$ be arbitrary, and write $\hat{k} = |k|^{-1} k$. We now characterize sets of the form

$$\{p \in \mathbb{Z}^3 \mid \hat{k} \cdot p = t\}, \quad t \in \mathbb{R}. \tag{B.2.6}$$

For this we note the following well-known result on linear Diophantine equations:

Theorem B.2.5. *Let $(k_1, k_2, k_3) \in \mathbb{Z}^3_*$ and $c \in \mathbb{Z}$ be given. Then the linear Diophantine equation*

$$k_1 m_1 + k_2 m_2 + k_3 m_3 = c$$

is solvable with $(m_1, m_2, m_3) \in \mathbb{Z}^3$ if and only if c is a multiple of $\gcd(k_1, k_2, k_3)$.

If this is the case then there exists linearly independent vectors $v_1, v_2 \in \mathbb{Z}^3$, which are independent of c, such that if (m_1^*, m_2^*, m_3^*) is any particular solution of the equation then all solutions are given by

$$\left\{(m_1, m_2, m_3) \in \mathbb{Z}^3 \mid k_1 m_1 + k_2 m_2 + k_3 m_3 = c\right\} = (m_1^*, m_2^*, m_3^*) + \{a_1 v_1 + a_2 v_2 \mid a_1, a_2 \in \mathbb{Z}\}.$$

This theorem implies the following:

Proposition B.2.6. *Let $k = (k_1, k_2, k_3) \in \mathbb{Z}^3_*$ and define $l = |k|^{-1} \gcd(k_1, k_2, k_3)$. Then there holds the disjoint union of non-empty sets*

$$\mathbb{Z}^3 = \bigcup_{m \in \mathbb{Z}} \{p \in \mathbb{Z}^3 \mid \hat{k} \cdot p = lm\}$$

and $\{p \in \mathbb{Z}^3 \mid \hat{k} \cdot p = 0\}$ is a lattice in $k^\perp = \{p \in \mathbb{R}^3 \mid \hat{k} \cdot p = 0\}$.

Proof: Clearly $\mathbb{Z}^3 = \bigcup_{t\in\mathbb{R}}\{p \in \mathbb{Z}^3 \mid \hat{k}\cdot p = t\}$ so we must determine for which values of t the set $\{p \in \mathbb{Z}^3 \mid \hat{k}\cdot p = t\}$ is non-empty. For an arbitrary $p = (p_1, p_2, p_3) \in \mathbb{Z}^3$ the equation $\hat{k} \cdot p = t$ is equivalent with

$$k_1p_1 + k_2p_2 + k_3p_3 = |k|\, t \tag{B.2.7}$$

and as the left-hand side is the sum of products of integers, the right-hand side must likewise be an integer, i.e. $t = |k|^{-1} c$ for some $c \in \mathbb{Z}$. By the theorem it must then hold that $c = \gcd(k_1, k_2, k_3) \cdot m$ for some $m \in \mathbb{Z}$, i.e.

$$t = |k|^{-1} \gcd(k_1, k_2, k_3) \cdot m = lm. \tag{B.2.8}$$

As p was arbitrary we see that $\mathbb{Z}^3 = \bigcup_{m\in\mathbb{Z}}\{p \in \mathbb{Z}^3 \mid \hat{k}\cdot p = lm\}$ as claimed. That all sets $\{p \in \mathbb{Z}^3 \mid \hat{k}\cdot p = lm\}$ are non-empty likewise follows from the theorem, as does the existence of linearly independent $v_1, v_2 \in \mathbb{Z}^3$ such that

$$\{p \in \mathbb{Z}^3 \mid \hat{k}\cdot p = lm\} = q + \{a_1v_1 + a_2v_2 \mid a_1, a_2 \in \mathbb{Z}\} \tag{B.2.9}$$

for any particular $q \in \{p \in \mathbb{Z}^3 \mid \hat{k}\cdot p = lm\}$. Taking $q = 0$ as a particular solution, we see that

$$\{p \in \mathbb{Z}^3 \mid \hat{k}\cdot p = 0\} = \{a_1v_1 + a_2v_2 \mid a_1, a_2 \in \mathbb{Z}\} \tag{B.2.10}$$

which is precisely the statement that $\{p \in \mathbb{Z}^3 \mid \hat{k}\cdot p = 0\}$ is a lattice (in $k^\perp$).

□

The covolume $d\Big(\{p \in \mathbb{Z}^3 \mid \hat{k}\cdot p = 0\}\Big) = \sqrt{\|v_1\|^2 \|v_2\|^2 - (v_1 \cdot v_2)^2}$ is given by the following:

Proposition B.2.7. *For any generators $v_1, v_2 \in \mathbb{Z}^3$ of $\{p \in \mathbb{Z}^3 \mid \hat{k}\cdot p = 0\}$ it holds that*

$$d\Big(\{p \in \mathbb{Z}^3 \mid \hat{k}\cdot p = 0\}\Big) = \sqrt{\|v_1\|^2 \|v_2\|^2 - (v_1 \cdot v_2)^2} = l^{-1}.$$

Proof: Let $w \in \{p \in \mathbb{Z}^3 \mid \hat{k}\cdot p = l\}$ be arbitrary. Then by linearity

$$\{p \in \mathbb{Z}^3 \mid \hat{k}\cdot p = lm\} = mw + \{p \in \mathbb{Z}^3 \mid \hat{k}\cdot p = 0\} \tag{B.2.11}$$

for any $m \in \mathbb{Z}$, so by the previous proposition

$$\mathbb{Z}^3 = \bigcup_{m\in\mathbb{Z}} \Big(mw + \{p \in \mathbb{Z}^3 \mid \hat{k}\cdot p = 0\}\Big) = \{m_1v_1 + m_2v_2 + m_3w \mid m_1, m_2, m_3 \in \mathbb{Z}\}\,, \tag{B.2.12}$$

i.e. (v_1, v_2, w) is a set of generators for $\mathbb{Z}^3$. Let (e_1, e_2) be an orthonormal basis for $k^\perp$ so that $(e_1, e_2, \hat{k})$ forms an orthonormal basis for $\mathbb{R}^3$. Then

$$d\Big(\mathbb{Z}^3\Big) = \left|\det\begin{pmatrix} (e_1\cdot v_1) & (e_2\cdot v_1) & (\hat{k}\cdot v_1) \\ (e_1\cdot v_2) & (e_2\cdot v_2) & (\hat{k}\cdot v_2) \\ (e_1\cdot w) & (e_w\cdot w) & (\hat{k}\cdot w) \end{pmatrix}\right| = \left|\det\begin{pmatrix} (e_1\cdot v_1) & (e_2\cdot v_1) & 0 \\ (e_1\cdot v_2) & (e_2\cdot v_2) & 0 \\ (e_1\cdot w) & (e_w\cdot w) & l \end{pmatrix}\right| \tag{B.2.13}$$

$$= l\left|\det\begin{pmatrix} (e_1 \cdot v_1) & (e_2 \cdot v_1) \\ (e_1 \cdot v_2) & (e_2 \cdot v_2) \end{pmatrix}\right| = l \cdot d\Big(\{p \in \mathbb{Z}^3 \mid \hat{k}\cdot p = 0\}\Big)$$

and as it is clear that $d(\mathbb{Z}^3) = 1$ the result follows.

□

Finally we note that Proposition B.2.4 implies a bound on the norms of a generating set of $\{p \in \mathbb{Z}^3 \mid \hat{k}\cdot p = 0\}$:

Corollary B.2.8. *There exists a constant $C > 0$ independent of k such that $\{p \in \mathbb{Z}^3 \mid \hat{k}\cdot p = 0\}$ admits generators v_1 and v_2 obeying*

$$\|v_1\|^2 + \|v_2\|^2 \leq Cl^{-2}.$$

Proof: By the proposition there exists generators v_1, v_2 such that

$$\|v_1\| \, \|v_2\| \leq \frac{4}{\pi} d(\lambda) = \frac{4}{\pi} l^{-1}, \tag{B.2.14}$$

and as every $v \in \mathbb{Z}^3_*$ obeys $\|v\| \geq 1$ this implies that $\|v_1\|, \|v_2\| \leq \frac{4}{\pi} l^{-1}$. Consequently

$$\|v_1\|^2 + \|v_2\|^2 \leq \frac{32}{\pi^2} l^{-2} = Cl^{-2}. \tag{B.2.15}$$

□

B.3 Precise Estimates

Throughout this section we let $k = (k_1, k_2, k_3) \in B(0, 2k_F) \cap \mathbb{Z}^3_*$ be fixed and write $\hat{k} = |k|^{-1} k$ for brevity.

We now decompose the lune

$$L_k = \Big\{p \in \mathbb{Z}^3 \mid |p - k| \leq k_F < |p|\Big\} \tag{B.3.1}$$

along the $\{\hat{k}\cdot p = t\}$ planes. Note that for any $p \in L_k$ it holds that

$$k\cdot p = \frac{1}{2}\Big(|p|^2 - |p-k|^2 + |k|^2\Big) > \frac{1}{2}|k|^2 \tag{B.3.2}$$

and that

$$k\cdot p = k\cdot (p - k) + |k|^2 \leq |k|\,(k_F + |k|) \tag{B.3.3}$$

so

$$\frac{1}{2}|k| < \hat{k}\cdot p \leq k_F + |k|\,. \tag{B.3.4}$$

Let $l = |k|^{-1}\gcd(k_1, k_2, k_3)$ as in Proposition B.2.6, and let m^* be the least integer and M^* the greatest integer such that

$$\frac{1}{2}|k| < lm^*, \quad lM^* \leq k_F + |k|\,. \tag{B.3.5}$$

It then follows by the decomposition of Proposition B.2.6 that L_k can be expressed as the disjoint union

$$L_k = \bigcup_{m=m^*}^{M^*} L_k^m \tag{B.3.6}$$

where the subsets L_k^m are given by

$$L_k^m = \{p \in L_k \mid \hat{k} \cdot p = lm\}, \quad m^* \leq m \leq M^*. \tag{B.3.7}$$

Consequently, a Riemann sum of the form $\sum_{p\in L_k} f(\lambda_{k,p})$ can be written as

$$\sum_{p\in L_k} f(\lambda_{k,p}) = \sum_{m=m^*}^{M^*} \sum_{p\in L_k^m} f\Big(|k| \left(\hat{k} \cdot p - \frac{1}{2}|k|\right)\Big) = \sum_{m=m^*}^{M^*} f\Big(|k| \left(lm - \frac{1}{2}|k|\right)\Big) |L_k^m| . \tag{B.3.8}$$

To proceed we must analyze $|L_k^m|$, the number of points contained in L_k^m. For this, note that by expanding and rearranging the inequalities defining L_k, we may equivalently express it as

$$L_k = \left\{p \in \mathbb{Z}^3 \mid k_F^2 < |p|^2 \leq k_F^2 - |k|^2 + 2k \cdot p\right\}. \tag{B.3.9}$$

Letting $P_\perp : \mathbb{R}^3 \to k^\perp$ denote the orthogonal projection onto $k^\perp$, it holds that $|p|^2 = |P_\perp p|^2 + (\hat{k} \cdot p)^2$, whence

$$\begin{aligned} L_k &= \{p \in \mathbb{Z}^3 \mid k_F^2 - (\hat{k} \cdot p)^2 < |P_\perp p|^2 \leq k_F^2 - |k|^2 + 2k \cdot p - (\hat{k} \cdot p)^2\} \\ &= \{p \in \mathbb{Z}^3 \mid k_F^2 - (\hat{k} \cdot p)^2 < |P_\perp p|^2 \leq k_F^2 - (\hat{k}\cdot p - |k|)^2\} \end{aligned} \tag{B.3.10}$$

so the sets $L_k^m = L_k \cap \{p \in \mathbb{Z}^3 \mid \hat{k} \cdot p = lm\}$ can be written as

$$\begin{aligned} L_k^m &= \left\{p \in \mathbb{Z}^3 \mid \hat{k}\cdot p = lm,\, k_F^2 - (lm)^2 < |P_\perp p|^2 \leq k_F^2 - (lm - |k|)^2\right\} \\ &= \left\{p \in \mathbb{Z}^3 \mid \hat{k}\cdot p = lm,\, (R_1^m)^2 < |P_\perp p|^2 \leq (R_2^m)^2\right\} \end{aligned} \tag{B.3.11}$$

where the real numbers R_1^m and R_2^m are given by

$$R_1^m = \sqrt{k_F^2 - (lm)^2}, \quad R_2^m = \sqrt{k_F^2 - (lm - |k|)^2}, \quad m^* \leq m \leq M^*. \tag{B.3.12}$$

Let $v_1, v_2 \in \mathbb{Z}^3$ generate $\{p \in \mathbb{Z}^3 \mid \hat{k} \cdot p = 0\}$. For a fixed m, let $q \in \{p \in \mathbb{Z}^3 \mid \hat{k} \cdot p = lm\}$ be arbitrary. Then Proposition B.2.6 asserts that $p \in \mathbb{Z}^3$ is an element of $\{p \in \mathbb{Z}^3 \mid \hat{k}\cdot p = lm\}$ if and only if it can be written as

$$p = a_1 v_1 + a_2 v_2 + q, \quad a_1, a_2 \in \mathbb{Z}. \tag{B.3.13}$$

As v_1 and v_2 span $k^\perp$ it must hold that $P_\perp q = b_1 v_1 + b_2 v_2$ for some $b_1, b_2 \in \mathbb{R}$, whence we see that $P_\perp p$ is of the form

$$P_\perp p = a_1 v_1 + a_2 v_2 + P_\perp q = (a_1 + b_1) v_1 + (a_2 + b_2) v_2, \tag{B.3.14}$$

and so we can express $|L_k^m|$ as

$$\begin{aligned}|L_k^m| &= \Big|\Big\{(a_1,a_2)\in\mathbb{Z}^2 \mid (R_1^m)^2 < (a_1+b_1)^2\|v_1\|^2 + (a_2+b_2)^2\|v_2\|^2 \\ &\qquad +2(a_1+b_1)(a_2+b_2)(v_1\cdot v_2) \le (R_2^m)^2\Big\}\Big| \\ &= \Big|\Big\{(x,y)\in\Big(\mathbb{R}^2+(b_1,b_2)\Big) \mid (R_1^m)^2 < \|v_1\|^2 x^2 + \|v_2\|^2 y^2 + 2(v_1\cdot v_2)xy \le (R_2^m)^2\Big\}\cap\mathbb{Z}^2\Big| \\ &= \Big|(E_2^m\backslash E_1^m + (b_1,b_2))\cap\mathbb{Z}^2\Big|\end{aligned} \tag{B.3.15}$$

where the sets E_1^m and E_2^m are given by

$$E_i^m = \Big\{(x,y)\in\mathbb{R}^2 \mid \|v_1\|^2 x^2 + \|v_2\|^2 y^2 + 2(v_1\cdot v_2)xy \le (R_i^m)^2\Big\}, \quad i=1,2. \tag{B.3.16}$$

Lattice Point Estimation

The sets E_i^m are seen to be (closed interiors of) ellipses, and analyzing $|L_k^m|$ amounts to estimating the lattice points enclosed by these. To do this we will apply the following general result:

Theorem B.3.1 ([18])**.** *Let $K\subset\mathbb{R}^2$ be a compact, strictly convex set with C^2 boundary and let ∂K have minimal and maximal radii of curvature $0<r_1\le r_2$. If $r_2\ge 1$ then*

$$\Big|\big|K\cap\mathbb{Z}^2\big| - \mathrm{Area}(K)\Big| \le C\frac{r_2}{r_1}r_2^{\frac{2}{3}}\log\Big(1+2\sqrt{2}r_2^{\frac{1}{2}}\Big)^{\frac{2}{3}}$$

for a constant $C>0$ independent of all quantities.

This result follows from the techniques of Chapter 8 of [18], but is not explicitly stated in this fashion. Giving a proof of this result is out of the scope of this thesis, but a detailed derivation is available upon request.

In our present case we note that this implies that for any ellipse $E\subset\mathbb{R}^2$, it holds that

$$\Big|\big|E\cap\mathbb{Z}^2\big| - \mathrm{Area}(E)\Big| \le C\bigg(1+\frac{r_2}{r_1}r_2^{\frac{2}{3}}\log\Big(1+2\sqrt{2}r_2^{\frac{1}{2}}\Big)^{\frac{2}{3}}\bigg), \tag{B.3.17}$$

the $r_2\le 1$ case being accounted for by the constant term. It follows that $|L_k^m|$ obeys

$$|L_k^m| = \mathrm{Area}(E_2^m\backslash E_1^m) + O\bigg(1+\frac{r_2}{r_1}r_2^{\frac{2}{3}}\log\Big(1+2\sqrt{2}r_2^{\frac{1}{2}}\Big)^{\frac{2}{3}} + \frac{r_2'}{r_1'}(r_2')^{\frac{2}{3}}\log\Big(1+2\sqrt{2}(r_2')^{\frac{1}{2}}\Big)^{\frac{2}{3}}\bigg) \tag{B.3.18}$$

where r_1, r_1' and r_2, r_2' are the minimal and maximal radii of curvature of E_1^m and E_2^m, respectively.

We thus need to obtain some information on the geometry of the ellipses E_i^m. Consulting a reference on conic sections, one finds that the semi axes $a_i\ge b_i>0$ of E_i^m, as defined by equation (B.3.16), are given by

$$a_i = \sqrt{2}R_i^m\bigg(\|v_1\|^2+\|v_2\|^2-\sqrt{\Big(\|v_1\|^2-\|v_2\|^2\Big)^2+4(v_1\cdot v_2)^2}\bigg)^{-\frac{1}{2}} \tag{B.3.19}$$

$$b_i = \sqrt{2}R_i^m\left(\|v_1\|^2 + \|v_2\|^2 + \sqrt{\left(\|v_1\|^2 - \|v_2\|^2\right)^2 + 4(v_1\cdot v_2)^2}\right)^{-\frac{1}{2}}.$$

We can then describe the geometry of the ellipses in terms of k and m:

Proposition B.3.2. *It holds that*

$$\mathrm{Area}(E_2^m\backslash E_1^m) = \begin{cases} 2\pi\,|k|\left(lm - \frac{1}{2}\,|k|\right)l & lm^* \leq lm \leq k_F \\ \pi\left(k_F^2 - (lm-|k|)^2\right)l & k_F < lm \leq lM^* \end{cases}$$

and the minimal and maximal radii of curvature $0 < r_1 \leq r_2$ of either of E_1^m, E_2^m can be assumed to obey the estimates

$$\frac{r_2}{r_1} \leq Cl^{-3}, \quad r_2 \leq Cl^{-1}k_F,$$

for a constant $C > 0$ independent of all quantities.

Proof: The area enclosed by an ellipse with semi-axes a and b is πab, so for $lm^* \leq lm \leq k_F$, when $\emptyset \neq E_1^m \subset E_2^m$,

$$\begin{aligned}\mathrm{Area}(E_2^m\backslash E_1^m) &= \frac{2\pi\left((R_2^m)^2 - (R_1^m)^2\right)}{\sqrt{\left(\|v_1\|^2 + \|v_2\|^2\right)^2 - \left(\left(\|v_1\|^2 - \|v_2\|^2\right)^2 + 4(v_1\cdot v_2)^2\right)}} \\ &= \frac{2\pi\left(k_F^2 - (lm-|k|)^2 - \left(k_F^2 - (lm)^2\right)\right)}{\sqrt{4\,\|v_1\|^2\,\|v_2\|^2 - 4(v_1\cdot v_2)^2}} \\ &= \frac{2\pi(|k|\,(2lm - |k|))}{2l^{-1}} = 2\pi\,|k|\left(lm - \frac{1}{2}\,|k|\right)l,\end{aligned} \tag{B.3.20}$$

where we used that $\sqrt{\|v_1\|^2\,\|v_2\|^2 - (v_1\cdot v_2)^2} = l^{-1}$ by Proposition B.2.7, while for $k_F < lm \leq lM^*$, when $E_1^m = \emptyset$,

$$\mathrm{Area}(E_2^m\backslash E_1^m) = \mathrm{Area}(E_2^m) = \frac{2\pi(R_2^m)^2}{2l^{-1}} = \pi\left(k_F^2 - (lm-|k|)^2\right)l. \tag{B.3.21}$$

For the radii of curvature we note that for an ellipse with semi axes $a \geq b > 0$ these are given by $r_1 = a^{-1}b^2$ and $r_2 = b^{-1}a^2$, respectively, so for either of E_i^m we can estimate that

$$\begin{aligned}\frac{r_2}{r_1} = \left(\frac{a_i}{b_i}\right)^3 &= \left(\frac{\|v_1\|^2 + \|v_2\|^2 + \sqrt{\left(\|v_1\|^2 - \|v_2\|^2\right)^2 + 4(v_1\cdot v_2)^2}}{\|v_1\|^2 + \|v_2\|^2 - \sqrt{\left(\|v_1\|^2 - \|v_2\|^2\right)^2 + 4(v_1\cdot v_2)^2}}\right)^{\frac{3}{2}} \\ &= \left(\frac{\left(\|v_1\|^2 + \|v_2\|^2 + \sqrt{\left(\|v_1\|^2 - \|v_2\|^2\right)^2 + 4(v_1\cdot v_2)^2}\right)^2}{\|v_1\|^2 + \|v_2\|^2 - \left(\left(\|v_1\|^2 - \|v_2\|^2\right)^2 + 4(v_1\cdot v_2)^2\right)}\right)^{\frac{3}{2}}\end{aligned} \tag{B.3.22}$$

$$= \left(\frac{1}{4} \frac{\left(\|v_1\|^2 + \|v_2\|^2 + \sqrt{\left(\|v_1\|^2 + \|v_2\|^2\right)^2 - 4\left(\|v_1\|^2 \|v_2\|^2 - (v_1 \cdot v_2)^2\right)} \right)^2}{\|v_1\|^2 \|v_2\|^2 - (v_1 \cdot v_2)^2} \right)^{\frac{3}{2}}$$

$$\leq \left(\frac{l^2}{4} \left(2\left(\|v_1\|^2 + \|v_2\|^2\right) \right)^2 \right)^{\frac{3}{2}} = \left(\|v_1\|^2 + \|v_2\|^2 \right)^3 l^3$$

and that

$$r_2 = \frac{a_i^2}{b_i} = \sqrt{2} R_i^m \frac{\sqrt{\|v_1\|^2 + \|v_2\|^2 + \sqrt{\left(\|v_1\|^2 - \|v_2\|^2\right)^2 + 4(v_1 \cdot v_2)^2}}}{\|v_1\|^2 + \|v_2\|^2 - \sqrt{\left(\|v_1\|^2 - \|v_2\|^2\right)^2 + 4(v_1 \cdot v_2)^2}}$$

$$= \sqrt{2} R_i^m \frac{\left(\|v_1\|^2 + \|v_2\|^2 + \sqrt{\left(\|v_1\|^2 - \|v_2\|^2\right)^2 + 4(v_1 \cdot v_2)^2} \right)^{\frac{3}{2}}}{\|v_1\|^2 + \|v_2\|^2 - \left(\left(\|v_1\|^2 - \|v_2\|^2\right)^2 + 4(v_1 \cdot v_2)^2 \right)} \tag{B.3.23}$$

$$\leq \sqrt{2} R_i^m \frac{\left(2\left(\|v_1\|^2 + \|v_2\|^2\right) \right)^{\frac{3}{2}}}{4l^{-2}} = \left(\|v_1\|^2 + \|v_2\|^2 \right)^{\frac{3}{2}} l^2 R_i^m.$$

Corollary B.2.8 asserts that v_1 and v_2 can be chosen to obey $\|v_1\|^2 + \|v_2\|^2 \leq Cl^{-2}$, in which case these estimates become

$$\frac{r_2}{r_1} \leq \left(Cl^{-2}\right)^3 l^3 \leq Cl^{-3}, \quad r_2 \leq \left(Cl^{-2}\right)^{\frac{3}{2}} l^2 R_i^m \leq Cl^{-1} k_F, \tag{B.3.24}$$

as claimed (using also that $R_i^m \leq k_F$ for all $m^* \leq m \leq M^*$).

□

Noting that l obeys

$$l^{-1} = \frac{|k|}{\gcd(k_1, k_2, k_3)} \leq |k| \tag{B.3.25}$$

we can by equation (B.3.18) and the proposition estimate that

$$\begin{aligned} \left| |L_k^m| - \mathrm{Area}(E_2^m \backslash E_1^m) \right| &\leq C \left(1 + l^{-3} \left(l^{-1} k_F\right)^{\frac{2}{3}} \log\left(1 + \left(l^{-1} k_F\right)^{\frac{1}{2}} \right)^{\frac{2}{3}} \right) \\ &\leq C \left(1 + |k|^{3+\frac{2}{3}} k_F^{\frac{2}{3}} \log\left(1 + \sqrt{|k|\, k_F} \right)^{\frac{2}{3}} \right) \\ &\leq C\, |k|^{3+\frac{2}{3}} \log(k_F)^{\frac{2}{3}} k_F^{\frac{2}{3}}, \quad k_F \to \infty, \end{aligned} \tag{B.3.26}$$

for a constant $C > 0$ independent of all quantities, which is to say

$$|L_k^m| = \begin{cases} 2\pi\, |k| \left(lm - \frac{1}{2}|k| \right) l & lm^* \leq lm \leq k_F \\ \pi \left(k_F^2 - (lm - |k|)^2 \right) l & k_F < lm \leq lM^* \end{cases} + O\left(|k|^{3+\frac{2}{3}} \log(k_F)^{\frac{2}{3}} k_F^{\frac{2}{3}} \right). \tag{B.3.27}$$

The Summation Formula

From equation (B.3.8) we can now conclude a general summation formula:

Proposition B.3.3. *For all $k = (k_1, k_2, k_3) \in \mathbb{Z}^3_*$ with $|k| < 2k_F$ and $f : (0, \infty) \to \mathbb{R}$ it holds that*

$$\begin{aligned}\sum_{p\in L_k} f(\lambda_{k,p}) &= 2\pi\,|k| \sum_{m=m^*}^{M} f\Big(|k|\Big(lm - \frac{1}{2}|k|\Big)\Big)\Big(lm - \frac{1}{2}|k|\Big)l \\ &+ \pi \sum_{m=M+1}^{M^*} f\Big(|k|\Big(lm - \frac{1}{2}|k|\Big)\Big)\Big(k_F^2 - (lm - |k|)^2\Big)l \\ &+ O\left(|k|^{3+\frac{2}{3}}\log(k_F)^{\frac{2}{3}} k_F^{\frac{2}{3}} \sum_{m=m^*}^{M^*} \left|f\Big(|k|\Big(lm - \frac{1}{2}|k|\Big)\Big)\right|\right)\end{aligned}$$

as $k_F \to \infty$, where $l = |k|^{-1}\gcd(k_1, k_2, k_3)$ and m^* is the least integer and M, M^* the greatest integers for which

$$\frac{1}{2}|k| < lm^*, \quad lM \le k_F, \quad lM^* \le k_F + |k|.$$

Note that the two first terms are exactly what one would expect from the continuum case, since

$$\begin{aligned}\int_{\overline{B}(k,k_F)\backslash\overline{B}(0,k_F)} f\Big(k\cdot p - \frac{1}{2}|k|^2\Big)dp &= 2\pi\,|k| \int_{\frac{1}{2}|k|}^{k_F} f\Big(|k|\Big(t - \frac{1}{2}|k|\Big)\Big)\Big(t - \frac{1}{2}|k|\Big)dt \\ &+ \pi \int_{k_F}^{k_F+|k|} f\Big(|k|\Big(t - \frac{1}{2}|k|\Big)\Big)\Big(k_F^2 - (t - |k|)^2\Big)dt.\end{aligned} \tag{B.3.28}$$

The summation formula thus allows us to convert the 3-dimensional Riemann sum $\sum_{p\in L_k} f(\lambda_{k,p})$ into the 1-dimensional Riemann sums corresponding to the integrals above, up to an additional error term.

We can then finally conclude the precise estimate of Proposition B.0.2:

Proposition B.3.4. *For all $k \in B(0, 2k_F)$ it holds that*

$$\left|\sum_{p\in L_k} \lambda_{k,p}^{-1} - 2\pi k_F\right| \le C\,|k|^{3+\frac{2}{3}}\log(k_F)^{\frac{5}{3}} k_F^{\frac{2}{3}}, \quad k_F \to \infty,$$

for a constant $C > 0$ independent of all quantities.

Proof: By the summation formula we have that

$$\sum_{p\in L_k} \lambda_{k,p}^{-1} = 2\pi\,|k| \sum_{m=m^*}^{M} \frac{lm - \frac{1}{2}|k|}{|k|\Big(lm - \frac{1}{2}|k|\Big)} l + \pi \sum_{m=M+1}^{M^*} \frac{k_F^2 - (lm - |k|)^2}{|k|\Big(lm - \frac{1}{2}|k|\Big)} l \tag{B.3.29}$$

$$+\,O\left(|k|^{2+\frac{2}{3}}\log(k_F)^{\frac{2}{3}}k_F^{\frac{2}{3}}\sum_{m=m^*}^{M^*}\frac{1}{lm-\frac{1}{2}|k|}\right).$$

The first sum is what contributes the term $2\pi k_F$, as we can estimate

$$\begin{aligned}\left|2\pi|k|\sum_{m=m^*}^{M}\frac{lm-\frac{1}{2}|k|}{|k|\left(lm-\frac{1}{2}|k|\right)}l-2\pi k_F\right| &= 2\pi\left|\sum_{m=m^*}^{M}l-k_F\right| = 2\pi\left|(lM-lm^*+l)-k_F\right| \\ &\leq 2\pi(l(m^*-1)+|lM-k_F|+2l) \\ &\leq 2\pi\left(\frac{1}{2}|k|+3\right)\leq C|k|\end{aligned}\tag{B.3.30}$$

which is $O\left(|k|^{3+\frac{2}{3}}\log(k_F)^{\frac{5}{3}}k_F^{\frac{2}{3}}\right)$ as $k_F\to\infty$ (above we also used that $l\leq 1$). Noting that

$$k_F^2-(lm-|k|)^2=k_F^2-(lm)^2+2|k|\left(lm-\frac{1}{2}|k|\right)\leq 2|k|\left(lm-\frac{1}{2}|k|\right)\tag{B.3.31}$$

for $m\geq M+1$, we can similarly estimate the second sum as

$$\begin{aligned}0\leq\pi\sum_{m=M+1}^{M^*}\frac{k_F^2-(lm-|k|)^2}{|k|\left(lm-\frac{1}{2}|k|\right)}l &= 2\pi\sum_{m=M+1}^{M^*}l=2\pi(lM^*-lM+l) \\ = 2\pi(lM^*-l(M+1)+2l)&\leq 2\pi(k_F+|k|-k_F+2)\leq C|k|.\end{aligned}\tag{B.3.32}$$

For the main error term we first note that $lm^*-\frac{1}{2}|k|\geq\frac{1}{2}|k|^{-1}$, as the definition of m^* implies that

$$2\gcd(k_1,k_2,k_3)m^*>|k|^2\tag{B.3.33}$$

so as both sides are integers

$$2\gcd(k_1,k_2,k_3)m^*\geq|k|^2+1\Leftrightarrow lm^*\geq\frac{1}{2}|k|+\frac{1}{2}|k|^{-1}.\tag{B.3.34}$$

We can thus apply Corollary B.0.5 to estimate

$$\begin{aligned}\sum_{m=m^*}^{M^*}\frac{1}{lm-\frac{1}{2}|k|} &= \frac{1}{lm^*-\frac{1}{2}|k|}+l^{-1}\sum_{m=m^*+1}^{M^*}\frac{1}{lm-\frac{1}{2}|k|}l\leq 2|k|+l^{-1}\int_{lm^*+\frac{1}{2}l}^{lM^*+\frac{1}{2}l}\frac{1}{x-\frac{1}{2}|k|}dx \\ &\leq C|k|\left(1+\log\left(\frac{lM^*+\frac{1}{2}l-\frac{1}{2}|k|}{lm^*+\frac{1}{2}l-\frac{1}{2}|k|}\right)\right) \\ &\leq C|k|\left(1+\log\left(\frac{k_F+|k|+\frac{1}{2}l-\frac{1}{2}|k|}{\frac{1}{2}l}\right)\right) \\ &\leq C|k|(1+\log(|k|k_F))\leq C|k|\log(k_F),\quad k_F\to\infty,\end{aligned}\tag{B.3.35}$$

where we also used that $l^{-1}\leq|k|$. In all the last error term thus obeys

$$|k|^{2+\frac{2}{3}}\log(k_F)^{\frac{2}{3}}k_F^{\frac{2}{3}}\sum_{m=m^*}^{M^*}\frac{1}{lm-\frac{1}{2}|k|}\leq C|k|^{3+\frac{2}{3}}\log(k_F)^{\frac{5}{3}}k_F^{\frac{2}{3}}\tag{B.3.36}$$

and the claim follows by combining the estimates.

□

Note that the condition $|k| \leq k_F^\gamma$, $\gamma \in \left(0, \frac{1}{11}\right)$, of the statement of Proposition B.0.2 arises to ensure that the error term is always $o(k_F)$. Although we must require this condition to control the precise asymptotics, we can however still conclude the bound

$$\sum_{p \in L_k} \lambda_{k,p}^{-1} \leq C k_F, \quad |k| < 2k_F, \tag{B.3.37}$$

of Proposition B.0.1, since it at least shows that $\sum_{p \in L_k} \lambda_{k,p}^{-1}$ is $O(k_F)$ for $|k| < k_F^{\frac{1}{20}}$ (say), and we previously established the bound

$$\sum_{p \in L_k} \lambda_{k,p}^{-1} \leq C\Big(1 + |k|^{-1} \log(k_F)\Big) k_F, \quad |k| < 2k_F, \tag{B.3.38}$$

of which the right-hand side is also $O(k_F)$ if $|k| \geq k_F^{\frac{1}{20}}$, so either way the claimed estimate holds.

B.4 Lower Bounds for $\beta \in \{0\} \cup [1, \infty)$

For the lower bound of Proposition B.0.3 we must similarly divide our analysis into a "small k" and a "large k" part. The result of Proposition B.3.3 is sufficiently precise that we can obtain the small k estimate almost immediately by the following lower bound for 1-dimensional Riemann sums of convex functions:

Lemma B.4.1. *Let for $a, b \in \mathbb{Z}$ and $l > 0$ a convex function $f \in C([la, lb])$ be given. Then*

$$\sum_{m=a}^{b} f(lm) l \geq \int_{la}^{lb} f(x)\, dx + \frac{l}{2}(f(la) + f(lb)).$$

Proof: Convexity implies that for every $m \in \{a, a+1, \ldots, b-1\}$,

$$f(x) \leq \Big(1 - \big(l^{-1}x - m\big)\Big) f(lm) + \big(l^{-1}x - m\big) f(l(m+1)), \quad x \in [lm, l(m+1)], \tag{B.4.1}$$

so

$$\begin{aligned} \int_{lm}^{l(m+1)} f(x)\, dx &\leq \left(\int_{lm}^{l(m+1)} \Big(1 - \big(l^{-1}x - m\big)\Big)\, dx\right) f(lm) + \left(\int_{lm}^{l(m+1)} \big(l^{-1}x - m\big)\, dx\right) f(l(m+1)) \\ &= f(lm) l \int_0^1 (1 - x)\, dx + f(l(m+1)) l \int_0^1 x\, dx \qquad \text{(B.4.2)} \\ &= \frac{1}{2}(f(lm) l + f(l(m+1)) l) \end{aligned}$$

whence

$$\sum_{m=a}^{b} f(lm) l = \frac{l}{2}(f(la) + f(lb)) + \sum_{m=a}^{b-1} \frac{1}{2}(f(lm) l + f(l(m+1)) l) \tag{B.4.3}$$

$$\geq \frac{l}{2}(f(la) + f(lb)) + \int_{la}^{lb} f(x)\,dx.$$

□

By applying this we obtain the following:

Proposition B.4.2. *For all $k \in B(0, 2k_F)$ and $\beta \in \{0\} \cup [1, \infty)$ it holds that*

$$\sum_{p \in L_k} \lambda_{k,p}^{\beta} \geq c\left(\left(1 - \frac{1}{2} k_F^{-1} |k|\right)^{2+\beta} - C\, |k|^{3+\frac{2}{3}} \log(k_F)^{\frac{2}{3}} k_F^{-\frac{1}{3}}\right) k_F^{2+\beta} |k|^{1+\beta}, \quad k_F \to \infty,$$

for constants $c, C > 0$ depending only on β.

Proof: By Proposition B.3.3 it holds that

$$\sum_{p \in L_k} \lambda_{k,p}^{\beta} \geq 2\pi\, |k|^{1+\beta} \sum_{m=m^*}^{M} \left(lm - \frac{1}{2}|k|\right)^{1+\beta} l - C\, |k|^{\beta+3+\frac{2}{3}} \log(k_F)^{\frac{2}{3}} k_F^{\frac{2}{3}} \sum_{m=m^*}^{M^*} \left(lm - \frac{1}{2}|k|\right)^{\beta} \tag{B.4.4}$$

where we discarded the second sum as every term of this is non-negative. By the previous lemma we can bound

$$\begin{aligned} \sum_{m=m^*}^{M} \left(lm - \frac{1}{2}|k|\right)^{1+\beta} l &\geq \int_{lm^*}^{lM} \left(x - \frac{1}{2}|k|\right)^{1+\beta} dx + \frac{l}{2}\left(\left(lM - \frac{1}{2}|k|\right)^{1+\beta} + \left(lm^* - \frac{1}{2}|k|\right)^{1+\beta}\right) \\ &\geq \frac{1}{2+\beta}\left(\left(lM - \frac{1}{2}|k|\right)^{2+\beta} - \left(lm^* - \frac{1}{2}|k|\right)^{2+\beta}\right) \qquad \text{(B.4.5)} \\ &\geq \frac{1}{2+\beta}\left(\left(k_F - \frac{1}{2}|k| - l\right)^{2+\beta} - l^{2+\beta}\right) \geq c\left(1 - \frac{1}{2}k_F^{-1}|k|\right)^{2+\beta} k_F^{2+\beta} \end{aligned}$$

as $k_F \to \infty$, where we used that $l \leq 1$ and that by the definition of m^* and M,

$$l(m^* - 1) \leq \frac{1}{2}|k|, \quad k_F < l(M+1). \tag{B.4.6}$$

Meanwhile, Corollary B.0.5 lets us bound the sum of the error term as

$$\begin{aligned} \sum_{m=m^*}^{M^*} \left(lm - \frac{1}{2}|k|\right)^{\beta} &\leq l^{-1} \int_{lm^* - \frac{1}{2}l}^{lM^* + \frac{1}{2}l} \left(x - \frac{1}{2}|k|\right)^{\beta} dx \\ &= \frac{l^{-1}}{1+\beta}\left(\left(lM^* - \frac{1}{2}|k|\right)^{1+\beta} - \left(lm^* - \frac{1}{2}|k|\right)^{1+\beta}\right) \qquad \text{(B.4.7)} \\ &\leq \frac{|k|}{1+\beta}\left(k_F + \frac{1}{2}|k|\right)^{1+\beta} \leq C\,|k|\,k_F^{1+\beta}, \quad k_F \to \infty, \end{aligned}$$

and combining the estimates yields the claim.

□

As was the case for our precise bound on $\sum_{p \in L_k} \lambda_{k,p}^{-1}$, this implies that

$$\sum_{p \in L_k} \lambda_{k,p}^{\beta} \geq c k_F^{2+\beta} |k|^{1+\beta}, \quad k_F \to \infty, \tag{B.4.8}$$

uniformly for $|k| \leq k_F^{\gamma}$, $\gamma \in \left(0, \frac{1}{11}\right)$, but to extend this to all $k \in B_F$ we must also establish some simpler bounds for larger k.

Large k Estimates

We begin by observing that

$$\sum_{p\in L_k}\lambda_{k,p}^{\beta}\geq|k|^{\beta}\int_{\bigcup_{q\in L_k}\mathcal{C}_q}\max\left\{\left(\hat{k}\cdot p-\frac{1}{2}\left|k\right|-\frac{\sqrt{3}}{2}\right)^{\beta},0\right\}dp \tag{B.4.9}$$

where we recall that $\mathcal{C}_q=[-2^{-1},2^{-1}]+q$. Indeed, for any $p\in\mathcal{C}_q$ it holds that

$$\begin{aligned}\lambda_{k,q}&=\frac{1}{2}\Big(|q|^2-|q-k|^2\Big)=|k|\left(\hat{k}\cdot q-\frac{1}{2}\left|k\right|\right)\\&=|k|\left(\hat{k}\cdot p-\frac{1}{2}\left|k\right|-\hat{k}\cdot(p-q)\right)\geq|k|\left(\hat{k}\cdot p-\frac{1}{2}\left|k\right|-\frac{\sqrt{3}}{2}\right)\end{aligned} \tag{B.4.10}$$

by Cauchy-Schwarz, as $p\in\mathcal{C}_q$ implies that $|p-q|\leq\frac{\sqrt{3}}{2}$ as also used earlier. We then note the following inclusion:

Proposition B.4.3. *For any $\epsilon>0$ it holds that*

$$\mathcal{S}_\epsilon=\overline{B}\left(k,k_F-\frac{\sqrt{3}}{2}-\epsilon\right)\backslash\overline{B}\left(0,k_F+\frac{\sqrt{3}}{2}+\epsilon\right)\subset\bigcup_{q\in L_k}\mathcal{C}_q.$$

Proof: We first show that $S_-\subset\bigcup_{q\in L_k}\mathcal{C}_q$ where S_- is given by

$$S_-=\left\{p\in\mathbb{R}^3\mid\inf_{q\in\mathbb{R}^3\backslash\left(\overline{B}(k,k_F)\backslash\overline{B}(0,k_F)\right)}|p-q|>\frac{\sqrt{3}}{2}\right\}. \tag{B.4.11}$$

Indeed, for any $p\in\mathbb{R}^3$ we have that $\mathcal{C}_p\cap\mathbb{Z}^3\neq\emptyset$, so if additionally $p\in S_-$ then it holds for $q'\in\mathcal{C}_p\cap\mathbb{Z}^3$ that

$$\inf_{q\in\mathbb{R}^3\backslash\left(\overline{B}(k,k_F)\backslash\overline{B}(0,k_F)\right)}|q'-q|\geq\inf_{q\in\mathbb{R}^3\backslash\left(\overline{B}(k,k_F)\backslash\overline{B}(0,k_F)\right)}|p-q|-|q'-p|>0 \tag{B.4.12}$$

hence $q'\in\mathbb{Z}^3\cap\left(\overline{B}(k,k_F)\backslash\overline{B}(0,k_F)\right)=L_k$. As $q'\in\mathcal{C}_p\Leftrightarrow p\in\mathcal{C}_{q'}$ by symmetry of the cube, this shows that $p\in\bigcup_{q\in L_k}\mathcal{C}_q$.

Now it holds that $\mathcal{S}_\epsilon\subset S_-$, as $p\in\mathcal{S}_\epsilon$ implies that if $q\in\mathbb{R}^3\backslash\left(\overline{B}(k,k_F)\backslash\overline{B}(0,k_F)\right)=\left(\mathbb{R}^3\backslash\overline{B}(k,k_F)\right)\cup\overline{B}(0,k_F)$ then at least one of the inequalities

$$\begin{aligned}&|p-q|\geq||p-k|-|q-k||=|q-k|-|p-k|>k_F-k_F+\frac{\sqrt{3}}{2}+\epsilon=\frac{\sqrt{3}}{2}+\epsilon\\&|p-q|\geq||p|-|q||=|p|-|q|>k_F+\frac{\sqrt{3}}{2}+\epsilon-k_F=\frac{\sqrt{3}}{2}+\epsilon\end{aligned} \tag{B.4.13}$$

are valid, according to whether $q \in \mathbb{R}^3 \backslash \overline{B}(k, k_F)$ or $q \in \overline{B}(0, k_F)$, hence

$$\inf_{q \in \mathbb{R}^3 \backslash \left(\overline{B}(k,k_F) \backslash \overline{B}(0,k_F)\right)} |p - q| \geq \frac{\sqrt{3}}{2} + \epsilon > \frac{\sqrt{3}}{2} \tag{B.4.14}$$

i.e. $p \in S_- \subset \bigcup_{q \in L_k} \mathcal{C}_q$.

□

From equation (B.4.9) we can now obtain

$$\begin{aligned} \sum_{p \in L_k} \lambda_{k,p}^{\beta} &\geq \limsup_{\epsilon \to 0^+} |k|^{\beta} \int_{\mathcal{S}_\epsilon} \max\left\{ \left(\hat{k} \cdot p - \frac{1}{2} |k| - \frac{\sqrt{3}}{2} \right)^{\beta}, 0 \right\} dp \\ &= |k|^{\beta} \int_{\mathcal{S}} \left(\hat{k} \cdot p - \frac{1}{2} |k| - \frac{\sqrt{3}}{2} \right)^{\beta} dp \end{aligned} \tag{B.4.15}$$

for $\mathcal{S} = \overline{B}\left(k, k_F - \frac{\sqrt{3}}{2}\right) \backslash \overline{B}\left(0, k_F + \frac{\sqrt{3}}{2}\right)$, where we also used that $\hat{k} \cdot p \geq \frac{1}{2} |k| + \frac{\sqrt{3}}{2}$ for $p \in \mathcal{S}$. Note that $\mathcal{S} = \emptyset$ unless $|k| > \sqrt{3}$.

Similar to what we did for the simple upper bounds, we consider the slices $\mathcal{S}_t = \mathcal{S} \cap \left\{ \hat{k} \cdot p = t \right\}$: The area of $\mathcal{S}_t$ is

$$\begin{aligned} \text{Area}(\mathcal{S}_t) &= \pi\left(\left(k_F - \frac{\sqrt{3}}{2} \right)^2 - (t - |k|) \right) - \pi\left(\left(k_F + \frac{\sqrt{3}}{2} \right)^2 - t^2 \right) \\ &= 2\pi\left(|k| \left(t - \frac{1}{2} |k| \right) - \sqrt{3} k_F \right) \end{aligned} \tag{B.4.16}$$

for $\frac{1}{2} |k| + \sqrt{3} |k|^{-1} k_F \leq t \leq k_F + \frac{\sqrt{3}}{2}$; the area for $t \geq k_F + \frac{\sqrt{3}}{2}$ is unnecessary since the integrand under consideration is non-negative and we are looking for a lower bound. We can then estimate as follows:

Proposition B.4.4. *For all $k \in B(0, 2k_F) \backslash \overline{B}\left(0, \sqrt{3}\right)$ and $\beta \in \{0\} \cup [1, \infty)$ it holds that*

$$\sum_{p \in L_k} \lambda_{k,p}^{\beta} \geq c\left(\left(1 - \frac{1}{2} k_F^{-1} |k| \right)^{2+\beta} - C\left(|k|^{-1} \left(1 - \frac{1}{2} k_F^{-1} |k| \right)^{1+\beta} + |k|^{-(2+\beta)} \right) \right) k_F^{2+\beta} |k|^{1+\beta}$$

as $k_F \to \infty$ for constants $c, C > 0$ depending only on β.

Proof: By the considerations above

$$\begin{aligned} \sum_{p \in L_k} \lambda_{k,p}^{\beta} &\geq 2\pi |k|^{\beta} \int_{\frac{1}{2}|k| + \sqrt{3}|k|^{-1} k_F}^{k_F + \frac{\sqrt{3}}{2}} \left(t - \frac{1}{2} |k| - \frac{\sqrt{3}}{2} \right)^{\beta} \left(|k| \left(t - \frac{1}{2} |k| \right) - \sqrt{3} k_F \right) dt \\ &\geq 2\pi |k|^{\beta} \left(|k| \int_{\sqrt{3}|k|^{-1} k_F - \frac{\sqrt{3}}{2}}^{k_F - \frac{1}{2}|k|} t^{1+\beta} dt - \sqrt{3} k_F \int_{\sqrt{3}|k|^{-1} k_F - \frac{\sqrt{3}}{2}}^{k_F - \frac{1}{2}|k|} t^{\beta} dt \right) \end{aligned} \tag{B.4.17}$$

$$\geq 2\pi\left|k\right|^{\beta}\left(\frac{\left|k\right|}{2+\beta}\left(\left(k_F-\frac{1}{2}\left|k\right|\right)^{2+\beta}-\left(\sqrt{3}\left|k\right|^{-1}k_F\right)^{2+\beta}\right)-\frac{\sqrt{3}k_F}{1+\beta}\left(k_F-\frac{1}{2}\left|k\right|\right)^{1+\beta}\right)$$
$$\geq c\left(\left(1-\frac{1}{2}k_F^{-1}\left|k\right|\right)^{2+\beta}-C\left(\left|k\right|^{-1}\left(1-\frac{1}{2}k_F^{-1}\left|k\right|\right)^{1+\beta}+\left|k\right|^{-(2+\beta)}\right)\right)k_F^{2+\beta}\left|k\right|^{1+\beta}.$$

□

This implies that
$$\sum_{p\in L_k}\lambda_{k,p}^{\beta}\geq ck_F^{2+\beta}\left|k\right|^{1+\beta},\quad k_F\to\infty, \tag{B.4.18}$$
uniformly for $k_F^{\gamma}<\left|k\right|<k_F$, $\gamma>0$, which combined with the small k result yields Proposition B.0.3.

Appendix C

Careful Justification of the Transformation Formulas

In this section we give a more detailed justification of the transformation identities which we derived in the sections 4 and 8 for the operator $\mathcal{K}$. Although we proved in Section 5 that

$$\mathcal{K} = \frac{1}{2}\sum_{l\in\mathbb{Z}^3_*}\sum_{p,q\in L_l}\langle e_p, K_l e_q\rangle\left(b_{l,p}b_{-l,-q} - b^*_{-l,-q}b^*_{l,p}\right) \tag{C.0.1}$$

defines a bounded operator whenever $\sum_{l\in\mathbb{Z}^3_*}\|K_l\|^2_{\mathrm{HS}} < \infty$, and so most of the subleties involving unbounded operators can be avoided, the fact that the operators we apply the transformation to are themselves unbounded still raises some technical questions.

The first transformation rules we consider are those for the bosonizable terms

$$H_{\mathrm{B}} = H'_{\mathrm{kin}} + \sum_{k\in\mathbb{Z}^3_*}\left(2Q^k_1(P_k) + Q^k_2(P_k)\right). \tag{C.0.2}$$

In this section we prove the following precise statement for these:

Proposition C.0.1. *The transformation $e^{-\mathcal{K}}$ preserves $D(H'_{\mathrm{kin}})$, $e^{\mathcal{K}}H_{\mathrm{B}}e^{-\mathcal{K}} : D(H'_{\mathrm{kin}}) \to \mathcal{H}_N$ is self-adjoint and both $H_{\mathrm{B}} - H'_{\mathrm{kin}}$ and $e^{\mathcal{K}}H_{\mathrm{B}}e^{-\mathcal{K}} - H'_{\mathrm{kin}}$ extend to bounded operators on all of $\mathcal{H}_N$.*

In words, the transformation of the bosonizable terms does indeed make rigorous sense, and the tranformation does not generate any "new" unboundedness, in so far as H'_{kin} is the only unbounded part of H_{B} both before and after the transformation.

The second transformation formula we consider is the one concerning $\mathcal{Q}_{\mathrm{SR}}$. Here we will prove the following:

Proposition C.0.2. *$\mathcal{Q}_{\mathrm{SR}}$ and $e^{\mathcal{K}}\mathcal{Q}_{\mathrm{SR}}e^{-\mathcal{K}}$ are well-defined in quadratic form sense on $D(H'_{\mathrm{kin}})$ and $e^{\mathcal{K}}\mathcal{Q}_{\mathrm{SR}}e^{-\mathcal{K}} - Q_{\mathrm{SR}}$ extends to a bounded operator on all of $\mathcal{H}_N$.*

Due to a technical point we will not verify whether the transformation identity is valid on an operator level, but it is valid in the quadratic form sense (which is all we apply in the main text) and again the transformation does not generate any new unboundedness.

As we are chiefly concerned with qualitative properties of operators in this section, we will generally estimate rather roughly and not keep track of k_F and s dependencies. In this case the bound of Proposition 5.0.1 can simply be summarized as

$$\|\mathcal{K}\|_{\mathrm{Op}} \leq C\sqrt{\sum_{l\in\mathbb{Z}^3_*}\|K_l\|^2_{\mathrm{HS}}} \tag{C.0.3}$$

for any operator of the form of equation (C.0.1), since (as also remarked in Section 5) $\mathcal{N}_E \leq N \leq Csk_F^3 \leq C'$.

Elaboration on the Well-Definedness of $\mathcal{K}$

On the same note, let us also elaborate on *how* this bound implies that $\mathcal{K}$ is well-defined - since this is a sum of infinitely many terms, this is not immediately clear, and so the bound of equation (C.0.3) might only constitute a formal calculation.

The reason this is not so is that Proposition 5.0.1 applies to *any* operator of the form of equation (C.0.1), and so if we for $R \in \mathbb{N}$ define $\mathcal{K}_R$ by

$$\mathcal{K}_R = \frac{1}{2}\sum_{l\in\overline{B}(0,R)\cap\mathbb{Z}^3_*}\sum_{p,q\in L_l}\langle e_p, K_l e_q\rangle\left(b_{l,p}b_{-l,-q} - b^*_{-l,-q}b^*_{l,p}\right), \tag{C.0.4}$$

i.e. let $\mathcal{K}_R$ be a cut-off version of $\mathcal{K}$, then this is *a priori* well-defined, as the summation is now only over finitely many terms. The bound then certainly applies in this case to show that

$$\|\mathcal{K}_R\|_{\mathrm{Op}} \leq C\sqrt{\sum_{l\in\overline{B}(0,R)\cap\mathbb{Z}^3_*}\|K_l\|^2_{\mathrm{HS}}}. \tag{C.0.5}$$

This implies that if the limit $\mathcal{K} = \lim_{R\to\infty}\mathcal{K}_R$ exists then it obeys the claimed bound. Existence is however automatically guaranteed by the same argument, as $(\mathcal{K}_R)_{R=1}^\infty$ is in fact Cauchy: For any $r, R \in \mathbb{N}$, the difference $\mathcal{K}_R - \mathcal{K}_r$ is also of the form of equation (C.0.1), whence (assuming that $r \leq R$ for definiteness)

$$\|\mathcal{K}_R - \mathcal{K}_r\|_{\mathrm{Op}} \leq C\sqrt{\sum_{l\in\overline{B}(0,R)\setminus\overline{B}(0,r)\cap\mathbb{Z}^3_*}\|K_l\|^2_{\mathrm{HS}}} \leq C\sqrt{\sum_{l\in\mathbb{Z}^3_*\setminus\overline{B}(0,r)}\|K_l\|^2_{\mathrm{HS}}} \tag{C.0.6}$$

which implies the Cauchy property.

For our argument we considered the particular cut-off sets $\overline{B}(0,R)\cap\mathbb{Z}^3_*$, but an argument similar to this last one shows that the limit exists for, and is independent of, any particular exhaustion of $\mathbb{Z}^3_*$, so $\mathcal{K}$ is indeed unambigously defined.

C.1 Transformation of Quadratic Operators

We begin by considering the transformation law for quadratic operators. This is greatly simplified by the fact that these are in fact bounded - not only are

$$Q^1_k(A) = \sum_{p,q\in L_k}\langle e_p, Ae_q\rangle\, b^*_{k,p}b_{k,q} \tag{C.1.1}$$

$$Q_k^2(B) = \sum_{p,q\in L_k} \langle e_p, Be_q\rangle \left(b_{k,p}b_{-k,-q} + b_{-k,-q}^* b_{k,p}^*\right)$$

bounded for any $k \in \mathbb{Z}_*^3$ and $A, B : \ell^2(L_k) \to \ell^2(L_k)$ simply by virtue of being sums of finitely many terms of bounded operators, the infinite sums $\sum_{k\in\mathbb{Z}_*^3} Q_1^k(A_k)$ and $\sum_{k\in\mathbb{Z}_*^3} Q_2^k(B_k)$ also define bounded operators, as we claim the following holds:

Proposition C.1.1. *For any collections of symmetric operators (A_k), (B_k) and $\Psi \in \mathcal{H}_N$ it holds that*

$$\left|\sum_{k\in\mathbb{Z}_*^3} \left\langle \Psi, Q_1^k(A_k)\Psi\right\rangle\right| \leq \sqrt{3}\sqrt{\sum_{k\in\mathbb{Z}_*^3} \|A_k\|_{\mathrm{HS}}^2}\,\langle \Psi, \mathcal{N}_E\Psi\rangle$$

$$\left|\sum_{k\in\mathbb{Z}_*^3} \left\langle \Psi, Q_2^k(B_k)\Psi\right\rangle\right| \leq 2\sqrt{5}\sqrt{\sum_{k\in\mathbb{Z}_*^3} \|B_k\|_{\mathrm{HS}}^2}\,\langle \Psi, (\mathcal{N}_E+1)\Psi\rangle .$$

Qualitatively this implies that

$$\left\|\sum_{k\in\mathbb{Z}_*^3} Q_1^k(A_k)\right\|_{\mathrm{Op}} \leq C\sqrt{\sum_{k\in\mathbb{Z}_*^3} \|A_k\|_{\mathrm{HS}}^2}, \quad \left\|\sum_{k\in\mathbb{Z}_*^3} Q_2^k(B_k)\right\|_{\mathrm{Op}} \leq C\sqrt{\sum_{k\in\mathbb{Z}_*^3} \|B_k\|_{\mathrm{HS}}^2}. \tag{C.1.2}$$

(Here we also use the assumed symmetry of (A_k) and (B_k), though this isn't necessary.)

The same argument we just illustrated with $\mathcal{K}$ thus implies that these sums are well-defined bounded operators provided the right-hand sides are finite.

Before we turn to the transformation law, let us prove this proposition. First we note that we have effectively already proven the Q_2^k bound, since we can write

$$Q_2^k(B) = \sum_{p,q\in L_k} \langle e_p, Be_q\rangle \left(b_{k,p}b_{-k,-q} + b_{-k,-q}^* b_{k,p}^*\right) = 2\,\mathrm{Re}\Big(\tilde{Q}_2^k(B)\Big) \tag{C.1.3}$$

for

$$\tilde{Q}_2^k(B) = \sum_{p,q\in L_k} \langle e_p, Be_q\rangle\, b_{k,p}b_{-k,-q}, \tag{C.1.4}$$

and $\sum_{k\in\mathbb{Z}_*^3} \tilde{Q}_2^k(B_k)$ is (up to a factor of 2) of the same form as $\tilde{\mathcal{K}}$ in Proposition 5.1.3, whence

$$\left|\sum_{k\in\mathbb{Z}_*^3} \left\langle \Psi, Q_2^k(B_k)\Psi\right\rangle\right| \leq 2\left|\sum_{k\in\mathbb{Z}_*^3} \left\langle \Psi, \tilde{Q}_2^k(B_k)\Psi\right\rangle\right| \leq 2\sqrt{5}\sqrt{\sum_{k\in\mathbb{Z}_*^3} \|B_k\|_{\mathrm{HS}}^2}\,\langle \Psi, (\mathcal{N}_E+1)\Psi\rangle . \tag{C.1.5}$$

The Q_1^k bound follows similarly to how we obtained Proposition 5.1.3 (although simpler, as there is less computation necessary): Writing

$$\sum_{k\in\mathbb{Z}_*^3} Q_1^k(A_k) = \sum_{k\in\mathbb{Z}_*^3}\sum_{p,q\in L_k} \langle e_p, A_ke_q\rangle\, b_{k,p}^* b_{k,q} = \frac{1}{\sqrt{s}}\sum_{k\in\mathbb{Z}_*^3}\sum_{p,q\in L_k}^{\sigma} \langle e_p, A_ke_q\rangle\, b_{k,p}^* c_{q-k,\sigma}^* c_{q,\sigma} \tag{C.1.6}$$

$$= \frac{1}{\sqrt{s}} \sum_{q \in B_F^c}^{\sigma} \left(\sum_{k \in \mathbb{Z}_*^3} \sum_{p \in L_k} 1_{L_k}(q) \langle e_p, A_k e_q \rangle b_{k,p}^* c_{q-k,\sigma}^* \right) c_{q,\sigma}$$

we can bound

$$\sum_{k \in \mathbb{Z}_*^3} \left\langle \Psi, Q_1^k(A_k)\Psi \right\rangle = \frac{1}{\sqrt{s}} \sum_{q \in B_F^c}^{\sigma} \left\langle \left(\sum_{k \in \mathbb{Z}_*^3} \sum_{p \in L_k} 1_{L_k}(q) \langle A_k e_q, e_p \rangle c_{q-k,\sigma} b_{k,p} \right) \Psi, c_{q,\sigma}\Psi \right\rangle \tag{C.1.7}$$
$$\leq \frac{1}{\sqrt{s}} \sqrt{\sum_{q \in B_F^c}^{\sigma} \left\| \sum_{k \in \mathbb{Z}_*^3} \sum_{p \in L_k} 1_{L_k}(q) \langle A_k e_q, e_p \rangle c_{q-k,\sigma} b_{k,p} \Psi \right\|^2} \sqrt{\sum_{q \in B_F^c}^{\sigma} \|c_{q,\sigma}\Psi\|^2}$$
$$= \frac{1}{\sqrt{s}} \sqrt{\sum_{q \in B_F^c}^{\sigma} \left\| \sum_{k \in \mathbb{Z}_*^3} \sum_{p \in L_k} 1_{L_k}(q) \langle A_k e_q, e_p \rangle c_{q-k,\sigma} b_{k,p} \Psi \right\|^2} \sqrt{\langle \Psi, \mathcal{N}_E \Psi \rangle}$$

and note that

$$\sum_{k \in \mathbb{Z}_*^3} \sum_{p \in L_k} 1_{L_k}(q) \langle A_k e_q, e_p \rangle c_{q-k,\sigma} b_{k,p} = \frac{1}{\sqrt{s}} \sum_{k \in \mathbb{Z}_*^3} \sum_{p \in L_k}^{\tau} 1_{L_k}(q) \langle A_k e_q, e_p \rangle c_{q-k,\sigma} c_{p-k,\tau}^* c_{p,\tau}$$
$$= \frac{1}{\sqrt{s}} \sum_{p' \in B_F^c}^{\tau} \sum_{q', r' \in B_F} \left(\sum_{k \in \mathbb{Z}_*^3} \sum_{p \in L_k} \delta_{p,p'} \delta_{p-k,q'} \delta_{q-k,r'} 1_{L_k}(q) \langle A_k e_q, e_p \rangle \right) c_{r',\sigma} c_{q',\tau}^* c_{p',\tau} \tag{C.1.8}$$

so that it suffices to consider expressions of the form

$$\frac{1}{\sqrt{s}} \sum_{p \in B_F^c}^{\tau} \sum_{q, r \in B_F} A_{p,q,r} c_{r,\sigma} c_{q,\tau}^* c_{p,\tau}. \tag{C.1.9}$$

We calculate the following commutator:

Lemma C.1.2. *For any $p, p' \in B_F^c$, $q, q', r, r' \in B_F$ and $1 \leq \sigma, \tau, \tau' \leq s$ it holds that*

$$\left\{ \left(c_{r,\sigma} c_{q,\tau}^* c_{p,\tau} \right)^*, c_{r',\sigma} c_{q',\tau'}^* c_{p',\tau'} \right\} = \delta_{p,p'}^{\tau,\tau'} c_{r',\sigma} c_{q',\tau'}^* c_{q,\tau} c_{r,\sigma}^* + \delta_{q,q'}^{\tau,\tau'} c_{r',\sigma} c_{p',\tau'} c_{p,\tau}^* c_{r,\sigma}^*$$
$$+ \delta_{r,r'} c_{p,\tau}^* c_{q,\tau} c_{q',\tau'}^* c_{p',\tau'} - \delta_{p,p'}^{\tau,\tau} \delta_{q,q'}^{\tau,\tau'} c_{r',\sigma} c_{r,\sigma}^*.$$

Proof: Repeatedly applying the CAR we find

$$\left(c_{r,\sigma} c_{q,\tau}^* c_{p,\tau} \right)^* c_{r',\sigma} c_{q',\tau'}^* c_{p',\tau'} = c_{p,\tau}^* c_{q,\tau} c_{r,\sigma}^* c_{r',\sigma} c_{q',\tau'}^* c_{p',\tau'}$$
$$= -c_{p,\tau}^* c_{q,\tau} c_{r',\sigma} c_{r,\sigma}^* c_{q',\tau'}^* c_{p',\tau'} + \delta_{r,r'} c_{p,\tau}^* c_{q,\tau} c_{q',\tau'}^* c_{p',\tau'}$$
$$= -c_{r',\sigma} c_{p,\tau}^* c_{q,\tau} c_{q',\tau'}^* c_{p',\tau'} c_{r,\sigma}^* + \delta_{r,r'} c_{p,\tau}^* c_{q,\tau} c_{q',\tau'}^* c_{p',\tau'} \tag{C.1.10}$$
$$= c_{r',\sigma} c_{p,\tau}^* c_{q',\tau'}^* c_{q,\tau} c_{p',\tau'} c_{r,\sigma}^* - \delta_{q,q'}^{\tau,\tau'} c_{r',\sigma} c_{p,\tau}^* c_{p',\tau'} c_{r,\sigma}^* + \delta_{r,r'} c_{p,\tau}^* c_{q,\tau} c_{q',\tau'}^* c_{p',\tau'}$$
$$= c_{r',\sigma} c_{q',\tau'}^* c_{p,\tau}^* c_{p',\tau'} c_{q,\tau} c_{r,\sigma}^* + \delta_{q,q'}^{\tau,\tau'} c_{r',\sigma} c_{p',\tau'} c_{p,\tau}^* c_{r,\sigma}^* + \delta_{r,r'} c_{p,\tau}^* c_{q,\tau} c_{q',\tau'}^* c_{p',\tau'} - \delta_{p,p'}^{\tau,\tau'} \delta_{q,q'}^{\tau,\tau'} c_{r',\sigma} c_{r,\sigma}^*$$
$$= -c_{r',\sigma} c_{q',\tau'}^* c_{p',\tau'} c_{p,\tau}^* c_{q,\tau} c_{r,\sigma}^* + \delta_{p,p'}^{\tau,\tau'} c_{r',\sigma} c_{q',\tau'}^* c_{q,\tau} c_{r,\sigma}^* + \delta_{q,q'}^{\tau,\tau'} c_{r',\sigma} c_{p',\tau'} c_{p,\tau}^* c_{r,\sigma}^* + \delta_{r,r'} c_{p,\tau}^* c_{q,\tau} c_{q',\tau'}^* c_{p',\tau'}$$

$$- \delta_{p,p'}^{\tau,\tau} \delta_{q,q'}^{\tau,\tau'} c_{r',\sigma} c_{r,\sigma}^*.$$

□

The bound on $\sum_{p \in B_F^c}^{\tau} \sum_{q,r \in B_F} A_{p,q,r} c_{r,\sigma} c_{q,\tau}^* c_{p,\tau}$ now follows:

Proposition C.1.3. *Let $A_{p,q,r} \in \mathbb{C}$ for $p \in B_F^c$ and $q, r \in B_F$ with $\sum_{p \in B_F} \sum_{q,r \in B_F} |A_{p,q,r}|^2 < \infty$ be given. Then for any $\Psi \in \mathcal{H}_N$*

$$\frac{1}{s} \sum_{\sigma=1}^{s} \left\| \sum_{p \in B_F^c}^{\tau} \sum_{q,r \in B_F} A_{p,q,r} c_{r,\sigma} c_{q,\tau}^* c_{p,\tau} \Psi \right\|^2 \leq 3s \sum_{p \in B_F^c} \sum_{q,r \in B_F} |A_{p,q,r}|^2 \langle \Psi, \mathcal{N}_E \Psi \rangle .$$

Proof: Arguing as in Proposition 5.1.2 and applying the lemma, we estimate

$$\begin{aligned}
&\frac{1}{s} \sum_{\sigma=1}^{s} \left\| \sum_{p \in B_F^c}^{\tau} \sum_{q,r \in B_F} A_{p,q,r} c_{r,\sigma} c_{q,\tau}^* c_{p,\tau} \Psi \right\|^2 \\
&\leq \frac{1}{s} \sum_{p,p' \in B_F^c}^{\sigma,\tau,\tau'} \sum_{q,q',r,r' \in B_F} \overline{A_{p,q,r}} A_{p',q',r'} \left\langle \Psi, \left\{ \left(c_{r,\sigma} c_{q,\tau}^* c_{p,\tau} \right)^*, c_{r',\sigma} c_{q',\tau'}^* c_{p',\tau'} \right\} \Psi \right\rangle \\
&= \frac{1}{s} \sum_{p \in B_F^c}^{\sigma,\tau} \left\| \sum_{q,r \in B_F} \overline{A_{p,q,r}} c_{q,\tau} c_{r,\sigma}^* \Psi \right\|^2 + \frac{1}{s} \sum_{q \in B_F}^{\sigma,\tau} \left\| \sum_{p \in B_F^c} \sum_{r \in B_F} \overline{A_{p,q,r}} c_{p,\tau}^* c_{r,\sigma}^* \Psi \right\|^2 \\
&+ \frac{1}{s} \sum_{r \in B_F}^{\sigma} \left\| \sum_{p' \in B_F^c}^{\tau'} \sum_{q' \in B_F} A_{p',q',r} c_{q',\tau'}^* c_{p',\tau'} \Psi \right\|^2 - \frac{1}{s} \sum_{p \in B_F^c}^{\sigma,\tau} \sum_{q \in B_F} \left\| \sum_{r \in B_F} \overline{A_{p,q,r}} c_{r,\sigma}^* \Psi \right\|^2 \qquad \text{(C.1.11)} \\
&\leq \frac{1}{s} \sum_{p \in B_F^c}^{\sigma,\tau} \left(\sum_{r \in B_F} \sqrt{\sum_{q \in B_F} |A_{p,q,r}|^2} \left\| c_{r,\sigma}^* \Psi \right\| \right)^2 + \frac{1}{s} \sum_{q \in B_F}^{\sigma,\tau} \left(\sum_{r \in B_F} \sqrt{\sum_{p \in B_F^c} |A_{p,q,r}|^2} \left\| c_{r,\sigma}^* \Psi \right\| \right)^2 \\
&+ \frac{1}{s} \sum_{r \in B_F}^{\sigma} \left(\sum_{p' \in B_F^c}^{\tau'} \sqrt{\sum_{q' \in B_F} |A_{p',q',r}|^2} \left\| c_{p',\tau'} \Psi \right\| \right)^2 \\
&\leq (2+s) \langle \Psi, \mathcal{N}_E \Psi \rangle \leq 3s \langle \Psi, \mathcal{N}_E \Psi \rangle .
\end{aligned}$$

□

Applying this to equation (C.1.8) we conclude the desired bound:

$$\begin{aligned}
&\sum_{k \in \mathbb{Z}_*^3} \left\langle \Psi, Q_1^k(A_k) \Psi \right\rangle \\
&\leq \frac{1}{\sqrt{s}} \sqrt{3s \sum_{q \in B_F^c} \sum_{p' \in B_F^c} \sum_{q',r' \in B_F} \left| \sum_{k \in \mathbb{Z}_*^3} \sum_{p \in L_k} \delta_{p,p'} \delta_{p-k,q'} \delta_{q-k,r'} 1_{L_k}(q) \langle A_k e_q, e_p \rangle \right|^2} \langle \Psi, \mathcal{N}_E \Psi \rangle \\
&= \sqrt{3 \sum_{q \in B_F^c} \sum_{p' \in B_F^c} \sum_{q' \in B_F} \sum_{k \in \mathbb{Z}_*^3} \left| \sum_{p \in L_k} \delta_{p,p'} \delta_{p-k,q'} 1_{L_k}(q) \langle A_k e_q, e_p \rangle \right|^2} \langle \Psi, \mathcal{N}_E \Psi \rangle \qquad \text{(C.1.12)}
\end{aligned}$$

$$= \sqrt{3\sum_{q\in B_F^c}\sum_{p'\in B_F^c}\sum_{k\in\mathbb{Z}_*^3}\sum_{p\in L_k}\left|\delta_{p,p'}1_{L_k}(q)\left\langle A_k e_q, e_p\right\rangle\right|^2}\left\langle\Psi,\mathcal{N}_E\Psi\right\rangle$$
$$= \sqrt{3\sum_{k\in\mathbb{Z}_*^3}\sum_{p,q\in L_k}\left|\left\langle e_p, A_k e_q\right\rangle\right|^2}\left\langle\Psi,\mathcal{N}_E\Psi\right\rangle = \sqrt{3}\sqrt{\sum_{k\in\mathbb{Z}_*^3}\|A_k\|_{\mathrm{HS}}^2}\left\langle\Psi,\mathcal{N}_E\Psi\right\rangle.$$

Justification of the Transformation

We can now justify the transformation. First note that the expression we consider,

$$\sum_{k\in\mathbb{Z}_*^3}\Big(2\,Q_1^k(P_k)+Q_2^k(P_k)\Big), \tag{C.1.13}$$

defines a bounded operator as

$$\sqrt{\sum_{k\in\mathbb{Z}_*^3}\|P_k\|_{\mathrm{HS}}^2} = \sqrt{\sum_{k\in\mathbb{Z}_*^3}\|v_k\|^4} = \sqrt{\sum_{k\in\mathbb{Z}_*^3}\left(\frac{s\hat{V}_k k_F^{-1}}{2\,(2\pi)^3}\,|L_k|\right)^2} \leq C\sqrt{\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2} < \infty, \tag{C.1.14}$$

where we simply estimate that $|L_k| \leq Ck_F^3$.

Now we note that the transformation rules of Proposition 4.3.1, i.e.

$$e^{\mathcal{K}}\Big(2\,Q_1^k(T_k)+2\,Q_1^{-k}(T_{-k})\Big)e^{-\mathcal{K}} = \mathrm{tr}\Big(T_k^1(1)-T_k\Big)+2\,Q_1^k\Big(T_k^1(1)\Big)+Q_2^k\Big(T_k^2(1)\Big) \tag{C.1.15}$$
$$+\int_0^1 e^{(1-t)\mathcal{K}}\Big(\varepsilon_k\Big(\Big\{K_k,T_k^2(t)\Big\}\Big)+2\,\mathrm{Re}\Big(\mathcal{E}_k^1\Big(T_k^1(t)\Big)\Big)+2\,\mathrm{Re}\Big(\mathcal{E}_k^2\Big(T_k^2(t)\Big)\Big)\Big)e^{-(1-t)\mathcal{K}}dt+(k\to -k)$$

and

$$e^{\mathcal{K}}\Big(Q_2^k(T_k)+Q_2^{-k}(T_{-k})\Big)e^{-\mathcal{K}} = \mathrm{tr}\Big(T_k^2(1)\Big)+2\,Q_1^k\Big(T_k^2(1)\Big)+Q_2^k\Big(T_k^1(1)\Big) \tag{C.1.16}$$
$$+\int_0^1 e^{(1-t)\mathcal{K}}\Big(\varepsilon_k\Big(\Big\{K_k,T_k^1(t)\Big\}\Big)+2\,\mathrm{Re}\Big(\mathcal{E}_k^1\Big(T_k^2(t)\Big)\Big)+2\,\mathrm{Re}\Big(\mathcal{E}_k^2\Big(T_k^1(t)\Big)\Big)\Big)e^{-(1-t)\mathcal{K}}dt+(k\to -k)$$

for

$$T_k^1(t) = \frac{1}{2}\Big(e^{tK_k}T_k e^{tK_k}+e^{-tK_k}T_k e^{-tK_k}\Big) \tag{C.1.17}$$
$$T_k^2(t) = \frac{1}{2}\Big(e^{tK_k}T_k e^{tK_k}-e^{-tK_k}T_k e^{-tK_k}\Big)$$

do actually hold without further justification by boundedness[1], so it is the summation over $k\in\mathbb{Z}_*^3$ that must be justified. Again we consider a cut-off: The above implies that for any $R\in\mathbb{N}$ (taking the Q_1^k case for definiteness)

$$e^{\mathcal{K}}\left(2\sum_{k\in\overline{B}(0,R)\cap\mathbb{Z}_*^3}Q_1^k(T_k)\right)e^{-\mathcal{K}}$$

[1]Strictly speaking, as the $\mathcal{E}_k^1(A)$ and $\mathcal{E}_k^2(B)$ operators are also defined as infinite sums (due to the sum over l in their definition), one should also justify that these are bounded operators. This can be done by considering limits of cut-offs in l and the kind of estimation we perform in Section 7 - we omit the details.

$$= \sum_{k\in\overline{B}(0,R)\cap\mathbb{Z}^3_*} \operatorname{tr}\Big(T^1_k(1) - T_k\Big) + 2\sum_{k\in\overline{B}(0,R)\cap\mathbb{Z}^3_*} Q^k_1\Big(T^1_k(1)\Big) + \sum_{k\in\overline{B}(0,R)\cap\mathbb{Z}^3_*} Q^k_2\Big(T^2_k(1)\Big) \quad (C.1.18)$$

$$+ \sum_{k\in\overline{B}(0,R)\cap\mathbb{Z}^3_*} \int_0^1 e^{(1-t)\mathcal{K}}\Big(\varepsilon_k\Big(\Big\{K_k, T^2_k(t)\Big\}\Big) + 2\operatorname{Re}\Big(\mathcal{E}^1_k\Big(T^1_k(t)\Big)\Big) + 2\operatorname{Re}\Big(\mathcal{E}^2_k\Big(T^2_k(t)\Big)\Big)\Big)e^{-(1-t)\mathcal{K}}dt$$

and we must argue that the limit $R \to \infty$ is well-defined. By Proposition C.1.1 and the estimates of Section 7 this is assured if (for $j = 1, 2$)

$$\sum_{k\in\mathbb{Z}^3_*} \Big|\operatorname{tr}\Big(T^1_k(1) - T_k\Big)\Big|, \sum_{k\in\mathbb{Z}^3_*} \|T^j_k(1)\|^2_{\mathrm{HS}} < \infty \quad (C.1.19)$$

and

$$\sum_{k\in\mathbb{Z}^3_*} \sup_{p\in L_k} \Big|\Big\langle e_p, \Big\{K_k, T^j_k(t)\Big\} e_p\Big\rangle\Big|, \sum_{k\in\mathbb{Z}^3_*}\sum_{p\in L_k}\max_{q\in L_k}\Big|\Big\langle e_p, T^j_k(t)e_q\Big\rangle\Big|^2, \sum_{k\in\mathbb{Z}^3_*} \|T^j_k(t)h_k^{-\frac{1}{2}}\|^2_{\mathrm{HS}} < \infty. \quad (C.1.20)$$

In our particular case $T_k = P_k$ and $P^1_k(t)$, $P^2_k(t)$ can be written as

$$P^1_k(t) = P_k + \frac{1}{2}P^+_k(t) + \frac{1}{2}P^-_k(t), \quad P^2_k(t) = \frac{1}{2}P^+_k(t) - \frac{1}{2}P^-_k(t), \quad (C.1.21)$$

for

$$P^{\pm}_k = e^{\pm tK_k}P_k e^{\pm tK_k} - P_k. \quad (C.1.22)$$

Arguing as in Proposition 6.2.1 (which really concerns $P^+_k(t)$) one can see that

$$\Big|\Big\langle e_p, P^{\pm}_k(t)e_q\Big\rangle\Big| \le C\hat{V}^2_k \quad (C.1.23)$$

independently of t, and naturally $|\langle e_p, P_k e_q\rangle| \le C\hat{V}_k$ which implies finiteness of the sums above.

In conclusion:

Proposition C.1.4. *The expression* $\sum_{k\in\mathbb{Z}^3_*}\Big(2\,Q^k_1(P_k) + Q^k_2(P_k)\Big)$ *defines a bounded operator on* $\mathcal{H}_N$ *and it holds that*

$$e^{\mathcal{K}}\left(\sum_{k\in\mathbb{Z}^3_*}\Big(2\,Q^k_1(P_k) + Q^k_2(P_k)\Big)\right)e^{-\mathcal{K}}$$

$$= \sum_{k\in\mathbb{Z}^3_*} \operatorname{tr}\Big(P^1_k(1) + P^2_k(1) - P_k\Big) + \sum_{k\in\mathbb{Z}^3_*}\Big(2\,Q^k_1\Big(P^1_k(1) + P^2_k(1)\Big) + Q^k_2\Big(P^1_k(1) + P^2_k(1)\Big)\Big)$$

$$+ \sum_{k\in\mathbb{Z}^3_*}\int_0^1 e^{(1-t)\mathcal{K}}\Big(\varepsilon_k\Big(\Big\{K_k, P^1_k(t) + P^2_k(t)\Big\}\Big) + \mathcal{E}^1_k\Big(P^1_k(t) + P^2_k(t)\Big) + \mathcal{E}^2_k\Big(P^1_k(t) + P^2_k(t)\Big)\Big)e^{-(1-t)\mathcal{K}}dt$$

with the right-hand side likewise defining a bounded operator.

C.2 Transformation of H'_{kin}

We now come to H'_{kin}. As this is a proper unbounded operator we must exercise more care in working with this than we did with the quadratic operators.

To work with H'_{kin} we will apply the following general result, which we prove in appendix section A.3:

Proposition (A.3.1)**.** *Let X be a Banach space, $A : D(A) \to X$ be a closed operator and let $K : X \to X$ be a bounded operator which preserves $D(A)$. Suppose that $AK : D(A) \to X$ is A-bounded.*

Then for every $z \in \mathbb{C}$ the operator $e^{zK} : X \to X$ likewise preserves $D(A)$ and $e^{zK}Ae^{-zK} : D(A) \to X$ is closed. If additionally X is a Hilbert space, A is self-adjoint and K is skew-symmetric then $e^{tK}Ae^{-tK}$ is self-adjoint for all $t \in \mathbb{R}$.

Furthermore, for every $x \in D(A)$ the mapping $z \mapsto e^{zK}Ae^{-zK}x$ is complex differentiable and C^1 with

$$\frac{d}{dz}e^{zK}Ae^{-zK}x = e^{zK}\left[K, A\right]e^{-zK}x.$$

To apply the result we must show that $\mathcal{K}$ preserves $D(H'_{\text{kin}})$ and that $H'_{\text{kin}}\mathcal{K}$ is H'_{kin}-bounded. To do this we will work with the cut-off operators $\mathcal{K}_R$, and obtain the corresponding results for $\mathcal{K}$ by the following lemma:

Lemma C.2.1. *Let X be a Banach space, $A : D(A) \to X$ be a closed operator and $(B_k)_{k=1}^{\infty} \subset \mathcal{B}(X)$ a collection of bounded operators such that $B_k \to B \in \mathcal{B}(X)$ (in norm).*

Suppose that all B_k preserve $D(A)$ and that the commutators $[B_k, A] : D(A) \to X$ converge pointwise to some $C : D(A) \to X$.

Then B also preserves $D(A)$ and $[B, A] = C$.

Proof: Let $x \in D(A)$ be arbitrary. Then $B_k x \to Bx$ by assumption, and likewise

$$AB_k x = B_k Ax - \left[B_k, A\right]x \to BAx - Cx. \tag{C.2.1}$$

It follows by closedness of A that $Bx \in D(A)$, i.e. that B preserves $D(A)$, and that

$$ABx = BAx - Cx \tag{C.2.2}$$

i.e. $[B, A] = C$.

□

We consider the operators $\mathcal{K}_R$. For this we require another general result:

Lemma C.2.2. *Let $A : D(A) \to X$ be a closed operator with core $\mathcal{C}$ and let $K : X \to X$ be a bounded operator which maps $\mathcal{C}$ into $D(A)$. Suppose that $AK|_{\mathcal{C}} : \mathcal{C} \to X$ is $A|_{\mathcal{C}}$-bounded. Then K preserves $D(A)$ and $AK : D(A) \to X$ is A-bounded (with the same relative bounds).*

Proof: Let $x \in D(A)$ be arbitrary. As $\mathcal{C}$ is a core for A, there exists a sequence $(x_k)_{k=1}^{\infty} \subset \mathcal{C}$ such that

$$x_k \to x \quad \text{and} \quad Ax_k \to Ax, \quad k \to \infty. \tag{C.2.3}$$

Since K is bounded, $Kx_k \to Kx$, and as $AK|_{\mathcal{C}}$ is $A|_{\mathcal{C}}$-bounded, the fact that $(Ax_k)_{k=1}^{\infty}$ converges implies that $(AKx_k)_{k=1}^{\infty}$ also converges. By closedness of A it then follows that $Kx \in D(A)$ and $AKx_k \to AKx$. The first statement shows that K indeed preserves $D(A)$, while the second implies that $AK : D(A) \to X$ is A-bounded, since if $\|AKx'\| \leq a\|Ax'\| + b\|x'\|$ for $x' \in \mathcal{C}$ then also

$$\|AKx\| = \lim_{k\to\infty} \|AKx_k\| \leq \limsup_{k\to\infty}(a\|Ax_k\| + b\|x_k\|) = a\|Ax\| + b\|x\| \tag{C.2.4}$$

for $x \in D(A)$.

□

We can now prove that $[\mathcal{K}_R, H'_{\text{kin}}]$ behaves as expected:

Proposition C.2.3. *For any $R \in \mathbb{N}$ it holds that $\mathcal{K}_R$ preserves $D(H'_{\text{kin}})$ and*

$$[\mathcal{K}_R, H'_{\text{kin}}] = \sum_{k\in\overline{B}(0,R)\cap\mathbb{Z}^3_*} Q_2^k(\{K_k, h_k\})\bigg|_{D\left(H'_{\text{kin}}\right)}.$$

Proof: First we note that $\mathcal{K}_R$ maps $\bigwedge^N_{\text{alg}} H^2(\mathbb{T}^3;\mathbb{C}^s)$, which is a core for H'_{kin}, into $D(H'_{\text{kin}})$: The operator $b_{k,p}$ can be written as

$$\begin{aligned} b_{k,p} &= \frac{1}{\sqrt{s}}\sum_{\sigma=1}^{s} c^*_{p-k,\sigma}c_{p,\sigma} = \frac{1}{\sqrt{s}}\sum_{\sigma=1}^{s}\sum_{p',q'\in\mathbb{Z}^3}\delta_{p',p-k}\delta_{q',p}c^*_{p',\sigma}c_{q',\sigma} \\ &= \frac{1}{\sqrt{s}}\sum_{\sigma=1}^{s}\sum_{p',q'\in\mathbb{Z}^3}^{\tau,\tau'}\langle u_{p',\tau}, u_{p-k,\sigma}\rangle\langle u_{p,\sigma}, u_{q',\tau'}\rangle c^*_{p',\tau}c_{q',\tau'} \\ &= \frac{1}{\sqrt{s}}\sum_{\sigma=1}^{s}\sum_{p',q'\in\mathbb{Z}^3}^{\tau,\tau'}\left\langle u_{p,\tau}, P^{(\sigma)}_{p-k,p}u_{q',\tau'}\right\rangle c^*_{p',\tau}c_{q',\tau'} = \frac{1}{\sqrt{s}}\sum_{\sigma=1}^{s}\mathrm{d}\Gamma\left(P^{(\sigma)}_{p-k,p}\right) \end{aligned} \tag{C.2.5}$$

where $P^{(\sigma)}_{p-k,p} = |u_{p-k,\sigma}\rangle\langle u_{p,\sigma}|$. Now, $\mathrm{d}\Gamma\left(P^{(\sigma)}_{p-k,p}\right)$ preserves $\bigwedge^N_{\text{alg}} H^2(\mathbb{T}^3;\mathbb{C}^s)$ for any $k,p \in \mathbb{Z}^3$ and $1 \leq \sigma \leq s$, as $P^{(\sigma)}_{p-k,p}$ simply takes an inner product and projects onto $u_{p,\sigma} \in H^2(\mathbb{T}^3;\mathbb{C}^s)$, so $b_{k,p}$ likewise preserves $\bigwedge^N_{\text{alg}} H^2(\mathbb{T}^3;\mathbb{C}^s)$. The same argument applies to $b^*_{k,p}$, so as a finite sum of products of operators which preserve $\bigwedge^N_{\text{alg}} H^2(\mathbb{T}^3;\mathbb{C}^s)$, $\mathcal{K}_R$ also preserves this, hence certainly maps it into $D(H'_{\text{kin}}) = D(H_{\text{kin}}) = \bigwedge^N H^2(\mathbb{T}^3;\mathbb{C}^s)$.

Having established that $H'_{\text{kin}}\mathcal{K}_R$ is well-defined on $\bigwedge^N_{\text{alg}} H^2(\mathbb{T}^3;\mathbb{C}^s)$, we note that the calculation we performed in Proposition 4.2.4 shows that

$$[\mathcal{K}_R, H'_{\text{kin}}] = \sum_{k\in\overline{B}(0,R)\cap\mathbb{Z}^3_*} Q_2^k(\{K_k, h_k\}), \tag{C.2.6}$$

at least on this domain. It follows that $H'_{\mathrm{kin}}\mathcal{K}_R$ is H'_{kin}-bounded here, since for any $\Psi \in \bigwedge_{\mathrm{alg}}^N H^2(\mathbb{T}^3;\mathbb{C}^s)$

$$\left\|H'_{\mathrm{kin}}\mathcal{K}_R\Psi\right\| \leq \left\|\mathcal{K}_R H'_{\mathrm{kin}}\Psi\right\| + \left\|\left[\mathcal{K}_R, H'_{\mathrm{kin}}\right]\Psi\right\| \leq \left\|\mathcal{K}_R\right\|_{\mathrm{Op}}\left\|H'_{\mathrm{kin}}\Psi\right\| + \left\|\left[\mathcal{K}_R, H'_{\mathrm{kin}}\right]\right\|_{\mathrm{Op}}\left\|\Psi\right\|. \tag{C.2.7}$$

Lemma C.2.2 now implies that $\mathcal{K}_R$ in fact preserves all of $D(H'_{\mathrm{kin}})$ and the commutator identity continues to hold.

□

We can now extend this to $\mathcal{K}$ proper:

Proposition C.2.4. *$\mathcal{K}$ preserves $D(H'_{\mathrm{kin}})$, the commutator*

$$[\mathcal{K}, H'_{\mathrm{kin}}] = \sum_{k\in\mathbb{Z}^3_*} Q_2^k(\{K_k, h_k\})\bigg|_{D\left(H'_{\mathrm{kin}}\right)}$$

extends to a bounded operator on all of $\mathcal{H}_N$, and $H'_{\mathrm{kin}}\mathcal{K}$ is H'_{kin}-bounded.

Proof: By Lemma C.2.1 it only remains to be shown that $\lim_{R\to\infty}\sum_{k\in\overline{B}(0,R)\cap\mathbb{Z}^3_*} Q_2^k(\{K_k, h_k\})$ exists on $D(H'_{\mathrm{kin}})$. In fact this exists everywhere, since Proposition C.1.1 says that this is ensured if $\sum_{k\in\mathbb{Z}^3_*}\left\|\{K_k, h_k\}\right\|_{\mathrm{HS}}^2 < \infty$, and by the one-body operator estimates of Section 6,

$$\begin{aligned}\left\|\{K_k, h_k\}\right\|_{\mathrm{HS}}^2 &= \sum_{p,q\in L_k}\left|\langle e_p, \{K_k, h_k\}\, e_q\rangle\right|^2 = \sum_{p,q\in L_k}\left|(\lambda_{k,p}+\lambda_{k,q})\,\langle e_p, K_k e_q\rangle\right|^2 \\ &\leq \sum_{p,q\in L_k}\left|(\lambda_{k,p}+\lambda_{k,q})\frac{\langle e_p, v_k\rangle\,\langle v_k, e_q\rangle}{\lambda_{k,p}+\lambda_{k,q}}\right|^2 = \left(\sum_{p\in L_k}\left|\langle e_p, v_k\rangle\right|^2\right)^2 \\ &= \left(\sum_{p\in L_k}\frac{s\hat{V}_k k_F^{-1}}{2\,(2\pi)^3}\right)^2 \leq C\Big(\hat{V}_k\,|L_k|\Big)^2 \leq C'\hat{V}_k^2, \quad k\in\mathbb{Z}^3_*.\end{aligned} \tag{C.2.8}$$

□

Proposition A.3.1 now gives us the following:

Corollary C.2.5. *The operator $e^{t\mathcal{K}}H'_{\mathrm{kin}}e^{-t\mathcal{K}} : D(H'_{\mathrm{kin}}) \to \mathcal{H}_N$ is a well-defined, self-adjoint operator for all $t\in\mathbb{R}$, and for any $\Psi\in D(H'_{\mathrm{kin}})$ it holds that*

$$\frac{d}{dt}e^{t\mathcal{K}}H'_{\mathrm{kin}}e^{-t\mathcal{K}}\Psi = e^{t\mathcal{K}}\left[\mathcal{K}, H'_{\mathrm{kin}}\right]e^{-t\mathcal{K}}\Psi = \sum_{k\in\mathbb{Z}^3_*} e^{t\mathcal{K}}Q_2^k(\{K_k, h_k\})e^{-t\mathcal{K}}\Psi$$

and this is continuous in t.

We now have all the necessary prerequisites to carefully implement Proposition 4.3.2:

Proposition C.2.6. *The statement of Proposition 4.3.2 holds pointwise on $D(H'_{\mathrm{kin}})$ and $e^{\mathcal{K}}H'_{\mathrm{kin}}e^{-\mathcal{K}} - H'_{\mathrm{kin}}$ extends continuously to all of $\mathcal{H}_N$.*

Proof: For any $R \in \mathbb{N}$, $\sum_{k\in\overline{B}(0,R)\cap\mathbb{Z}^3_*} Q^k_1(h_k)$ defines a bounded operator. Given $\Psi \in D(H'_{\text{kin}})$ we can then conclude by the corollary that

$$\begin{aligned}
&\frac{d}{dt}e^{t\mathcal{K}}\left(H'_{\text{kin}} - 2\sum_{k\in\overline{B}(0,R)\cap\mathbb{Z}^3_*} Q^k_1(h_k)\right)e^{-t\mathcal{K}}\Psi \\
&= e^{t\mathcal{K}}\left(\sum_{k\in\mathbb{Z}^3_*} Q^k_2(\{K_k,h_k\}) - \sum_{k\in\overline{B}(0,R)\cap\mathbb{Z}^3_*}\left[\mathcal{K}, Q^k_1(h_k)\right]\right)e^{-t\mathcal{K}}\Psi \qquad \text{(C.2.9)}\\
&= e^{t\mathcal{K}}\left(\sum_{k\in\mathbb{Z}^3_*\backslash\overline{B}(0,R)} Q^k_2(\{K_k,h_k\}) - \sum_{k\in\overline{B}(0,R)\cap\mathbb{Z}^3_*} 2\,\mathrm{Re}\left(\mathcal{E}^1_k(h_k)\right)\right)e^{-t\mathcal{K}}\Psi,
\end{aligned}$$

which upon rearrangement reads

$$\begin{aligned}
\frac{d}{dt}e^{t\mathcal{K}}H'_{\text{kin}}e^{-t\mathcal{K}}\Psi &= \frac{d}{dt}e^{t\mathcal{K}}\left(2\sum_{k\in\overline{B}(0,R)\cap\mathbb{Z}^3_*} Q^k_1(h_k)\right)e^{-t\mathcal{K}}\Psi \qquad \text{(C.2.10)}\\
&+ e^{t\mathcal{K}}\left(\sum_{k\in\mathbb{Z}^3_*\backslash\overline{B}(0,R)} Q^k_2(\{K_k,h_k\}) - \sum_{k\in\overline{B}(0,R)\cap\mathbb{Z}^3_*} 2\,\mathrm{Re}\left(\mathcal{E}^1_k(h_k)\right)\right)e^{-t\mathcal{K}}\Psi.
\end{aligned}$$

As the corollary also ensures that this is continuous in t, hence Riemann integrable, the fundamental theorem of calculus together with equation (C.1.18) shows that

$$\begin{aligned}
&e^{\mathcal{K}}H'_{\text{kin}}e^{-\mathcal{K}}\Psi \\
&= H'_{\text{kin}}\Psi + e^{\mathcal{K}}\left(2\sum_{k\in\overline{B}(0,R)\cap\mathbb{Z}^3_*} Q^k_1(h_k)\right)e^{-\mathcal{K}}\Psi - 2\sum_{k\in\overline{B}(0,R)\cap\mathbb{Z}^3_*} Q^k_1(h_k)\Psi \\
&+ \int_0^1 e^{t\mathcal{K}}\left(\sum_{k\in\mathbb{Z}^3_*\backslash\overline{B}(0,R)} Q^k_2(\{K_k,h_k\}) - \sum_{k\in\overline{B}(0,R)\cap\mathbb{Z}^3_*} 2\,\mathrm{Re}\left(\mathcal{E}^1_k(h_k)\right)\right)e^{-t\mathcal{K}}\Psi dt \qquad \text{(C.2.11)}\\
&= \sum_{k\in\overline{B}(0,R)\backslash\mathbb{Z}^3_*} \mathrm{tr}\left(h^1_k(1) - h_k\right)\Psi + H'_{\text{kin}}\Psi + \sum_{k\in\overline{B}(0,R)\backslash\mathbb{Z}^3_*}\left(2\,Q^k_1\left(h^1_k(1)-h_k\right) + Q^k_2\left(h^2_k(1)\right)\right)\Psi \\
&+ \sum_{k\in\overline{B}(0,R)\backslash\mathbb{Z}^3_*}\int_0^1 e^{(1-t)\mathcal{K}}\left(\varepsilon_k\left(\left\{K_k, h^2_k(t)\right\}\right) + 2\,\mathrm{Re}\left(\mathcal{E}^1_k\left(h^1_k(t)-h_k\right)\right) + 2\,\mathrm{Re}\left(\mathcal{E}^2_k\left(h^2_k(t)\right)\right)\right)e^{-(1-t)\mathcal{K}}\Psi dt \\
&+ \sum_{k\in\mathbb{Z}^3_*\backslash\overline{B}(0,R)}\int_0^1 e^{t\mathcal{K}}Q^k_2(\{K_k,h_k\})e^{-t\mathcal{K}}\Psi dt.
\end{aligned}$$

The formula of Proposition 4.3.2 now follows provided we can take $R \to \infty$. As in the previous subsection, this is possible if various sums involving the one-body operators $h^1_k(t)$ and $h^2_k(t)$ are finite - but with respect to the notation in Section 6,

$$h^1_k(t) - h_k = A_{h_k}(t), \quad h^2_k(t) = B_{h_k}(t) - tP_{v_k}, \qquad \text{(C.2.12)}$$

and the bounds derived in that section for these operators yield the desired estimates. The same bounds also imply the boundedness of $e^{\mathcal{K}} H'_{\mathrm{kin}} e^{-\mathcal{K}} - H'_{\mathrm{kin}}$ by the same argument.

□

C.3 Transformation of $\mathcal{Q}_{\mathrm{SR}}$

For the short-range quartic terms

$$\mathcal{Q}_{\mathrm{SR}} = \frac{k_F^{-1}}{2\,(2\pi)^3} \sum_{k\in\mathbb{Z}^3_*} \hat{V}_k \sum_{p,q\in B_F^c\cap\left(B_F^c-k\right)}^{\sigma,\tau} c^*_{p+k,\sigma} c^*_{q,\tau} c_{q+k,\tau} c_{p,\sigma} \tag{C.3.1}$$

we will switch our argument around and rather than cutting-off $\mathcal{K}$, cut-off $\mathcal{Q}_{\mathrm{SR}}$ instead, and so consider for $R\in\mathbb{N}$ the bounded operators

$$\mathcal{Q}^{(R)}_{\mathrm{SR}} = \frac{k_F^{-1}}{2\,(2\pi)^3} \sum_{k\in\overline{B}(0,R)\cap\mathbb{Z}^3_*} \hat{V}_k \sum_{p,q\in B_F^c\cap\left(B_F^c-k\right)}^{\sigma,\tau} c_{p+k,\sigma} c_{q,\tau} c_{q+k,\tau} c_{p,\sigma}. \tag{C.3.2}$$

Now, we would like to say that $\mathcal{Q}_{\mathrm{SR}}\Psi = \lim_{R\to\infty} \mathcal{Q}^{(R)}_{\mathrm{SR}}\Psi$ for any $\Psi\in D(H'_{\mathrm{kin}})$, but here arises a technical point: How is $\mathcal{Q}_{\mathrm{SR}}\Psi$ defined? We obtained $\mathcal{Q}_{\mathrm{SR}}$ by manipulating the second-quantized form of H_N, but *a priori* the action of this representation need only be defined for elements of $\bigwedge^N_{\mathrm{alg}} H^2(\mathbb{T}^3;\mathbb{C}^s)$, with the general action captured by extension arguments. Manipulating such forms can therefore be a delicate issue (had we not included the additional quadratic terms in our definition of $\mathcal{Q}$, for instance, this would not be a well-defined operator, as an unavoidable infinity then appears for unbounded V).

We must therefore clarify what we mean by $\mathcal{Q}_{\mathrm{SR}}$. We note the following:

Proposition C.3.1. *Let $\sum_{k\in\mathbb{Z}^3_*} \hat{V}_k^2 < \infty$. Then for any $\Psi\in D(H'_{\mathrm{kin}}) = D(H_{\mathrm{kin}})$ it holds that*

$$\left|\left\langle \Psi, \mathcal{Q}^{(R)}_{\mathrm{SR}}\Psi\right\rangle\right| \le C\left(\|\Psi\|^2 + \|H_{\mathrm{kin}}\Psi\|^2\right)$$

for a $C>0$ independent of R.

Proof: By Cauchy-Schwarz and the triangle inequality in the form $|k| = |p+k-p| \le |p+k| + |p|$ we can estimate

$$\begin{aligned}
\left|\left\langle \Psi, \mathcal{Q}^{(R)}_{\mathrm{SR}}\Psi\right\rangle\right| &\le \frac{k_F^{-1}}{2\,(2\pi)^3} \sum_{k\in\overline{B}(0,R)\cap\mathbb{Z}^3_*} \hat{V}_k \sum_{p,q\in B_F^c\cap\left(B_F^c-k\right)}^{\sigma,\tau} \|c_{q,\tau}c_{p+k,\sigma}\Psi\|\,\|c_{q+k,\tau}c_{p,\sigma}\Psi\| \qquad \text{(C.3.3)}\\
&\le C \sum_{k\in\overline{B}(0,R)\cap\mathbb{Z}^3_*} \hat{V}_k \sum_{p,q\in B_F^c\cap\left(B_F^c-k\right)}^{\sigma,\tau} \frac{|p|+|p+k|}{|k|}\frac{|q|+|q+k|}{|k|} \|c_{q,\tau}c_{p+k,\sigma}\Psi\|\,\|c_{q+k,\tau}c_{p,\sigma}\Psi\|\\
&\le C \sum_{k\in\overline{B}(0,R)\cap\mathbb{Z}^3_*} \frac{\hat{V}_k}{|k|^2} \sum_{p,q\in B_F^c\cap\left(B_F^c-k\right)}^{\sigma,\tau} \left(|p|\,|q| + |p|\,|q+k|\right) \|c_{q,\tau}c_{p+k,\sigma}\Psi\|\,\|c_{q+k,\tau}c_{p,\sigma}\Psi\|
\end{aligned}$$

where we apply the symmetry of the summations to reduce the consideration of

$$(|p|+|p+k|)(|q|+|q+k|) \tag{C.3.4}$$

to the two terms $|p|\,|q|$ and $|p|\,|q+k|$. For the first kind of terms we bound as

$$\begin{aligned}
&\sum_{p,q\in B_F^c\cap\left(B_F^c-k\right)}^{\sigma,\tau} |p|\,|q|\,\|c_{q,\tau}c_{p+k,\sigma}\Psi\|\,\|c_{q+k,\tau}c_{p,\sigma}\Psi\| \\
&\leq \sqrt{\sum_{p,q\in B_F^c\cap\left(B_F^c-k\right)}^{\sigma,\tau} |p|^2\,\|c_{q+k,\tau}c_{p,\sigma}\Psi\|^2}\sqrt{\sum_{p,q\in B_F^c\cap\left(B_F^c-k\right)}^{\sigma,\tau} |q|^2\,\|c_{q,\tau}c_{p+k,\sigma}\Psi\|^2} \\
&\leq \sum_{p\in B_F^c\cap\left(B_F^c-k\right)}^{\sigma} |p|^2\,\|\mathcal{N}_E^{\frac{1}{2}}c_{p,\sigma}\Psi\|^2 \leq C\sum_{p\in B_F^c\cap\left(B_F^c-k\right)}^{\sigma} |p|^2\,\|c_{p,\sigma}\Psi\|^2 \\
&\leq C\|H_{\mathrm{kin}}^{\frac{1}{2}}\Psi\|^2 \leq C\left(\|\Psi\|^2+\|H_{\mathrm{kin}}\Psi\|^2\right).
\end{aligned} \tag{C.3.5}$$

For the second, observe that in the same manner one can show that $\sum_{\sigma=1}^{s}\|\mathcal{N}_E^{\frac{1}{2}}c_{p,\sigma}\Psi\|^2 \leq \sum_{\sigma=1}^{s}\|c_{p,\sigma}\mathcal{N}_E^{\frac{1}{2}}\Psi\|^2$, as noted in equation (7.1.18), it follows that

$$\sum_{\sigma=1}^{s}\|H_{\mathrm{kin}}^{\frac{1}{2}}c_{p,\sigma}\Psi\|^2 \leq \sum_{\sigma=1}^{s}\|c_{p,\sigma}H_{\mathrm{kin}}^{\frac{1}{2}}\Psi\|^2. \tag{C.3.6}$$

We may then estimate

$$\begin{aligned}
&\sum_{p,q\in B_F^c\cap\left(B_F^c-k\right)}^{\sigma,\tau} |p|\,|q+k|\,\|c_{q,\tau}c_{p+k,\sigma}\Psi\|\,\|c_{q+k,\tau}c_{p,\sigma}\Psi\| \\
&\leq \sqrt{\sum_{p,q\in B_F^c\cap\left(B_F^c-k\right)}^{\sigma,\tau} |p|^2\,|q+k|^2\,\|c_{q+k,\tau}c_{p,\sigma}\Psi\|^2}\sqrt{\sum_{p,q\in B_F^c\cap\left(B_F^c-k\right)}^{\sigma,\tau} \|c_{q,\tau}c_{p+k,\sigma}\Psi\|^2} \\
&\leq \sqrt{\sum_{p\in B_F^c\cap\left(B_F^c-k\right)}^{\sigma} |p|^2\,\|H_{\mathrm{kin}}^{\frac{1}{2}}c_{p,\sigma}\Psi\|}\,\|\mathcal{N}_E\Psi\| \leq C\,\|\Psi\|\,\|H_{\mathrm{kin}}\Psi\| \leq C\left(\|\Psi\|^2+\|H_{\mathrm{kin}}\Psi\|^2\right),
\end{aligned} \tag{C.3.7}$$

so in all

$$\begin{aligned}
|\langle\Psi,\mathcal{Q}_{\mathrm{SR}}\Psi\rangle| &\leq C\left(\sum_{k\in\overline{B}(0,R)\cap\mathbb{Z}_*^3}\frac{\hat{V}_k}{|k|^2}\right)\left(\|\Psi\|^2+\|H_{\mathrm{kin}}\Psi\|^2\right) \\
&\leq C\sqrt{\sum_{k\in\mathbb{Z}_*^3}\hat{V}_k^2}\sqrt{\sum_{k\in\mathbb{Z}_*^3}|k|^{-4}}\left(\|\Psi\|^2+\|H_{\mathrm{kin}}\Psi\|^2\right) \leq C\left(\|\Psi\|^2+\|H_{\mathrm{kin}}\Psi\|^2\right).
\end{aligned} \tag{C.3.8}$$

□

By the proposition (or rather, its argument) it follows as we have used repeatedly throughout this section that for any $\Psi \in D(H'_{\mathrm{kin}})$, the sequence $\left(\left\langle \Psi, \mathcal{Q}_{\mathrm{SR}}^{(R)} \Psi \right\rangle\right)_{R=1}^{\infty}$ is Cauchy, hence converges, and so we can define $\mathcal{Q}_{\mathrm{SR}}$ in quadratic form sense on all of $D(H'_{\mathrm{kin}})$ by this limiting procedure[2].

Having clarified $\mathcal{Q}_{\mathrm{SR}}$, the transformation formula now follows by the calculations of the main text: For any $R \in \mathbb{N}$ we have

$$e^{\mathcal{K}} \mathcal{Q}_{\mathrm{SR}}^{(R)} e^{-\mathcal{K}} = \mathcal{Q}_{\mathrm{SR}}^{(R)} + \int_0^1 e^{t\mathcal{K}} \Big(2 \operatorname{Re}\Big(\mathcal{G}^{(R)}\Big)\Big) e^{t\mathcal{K}} dt \tag{C.3.9}$$

where $\mathcal{G}^{(R)}$ is given by

$$\begin{aligned}
\mathcal{G}^{(R)} = {} & \frac{s^{-\frac{1}{2}} k_F^{-1}}{(2\pi)^3} \sum_{k \in \overline{B}(0,R) \cap \mathbb{Z}_*^3} \sum_{l \in \mathbb{Z}_*^3} \hat{V}_k \sum_{p,q \in B_F^c \cap \left(B_F^c + k\right)}^{\sigma,\tau} 1_{L_l}(q) c_{p,\sigma}^* b_l(K_l e_q) c_{-q+l,\tau}^* c_{-q+k,\tau} c_{p-k,\sigma} \\
& + \frac{s^{-1} k_F^{-1}}{2\,(2\pi)^3} \sum_{k \in \overline{B}(0,R) \cap \mathbb{Z}_*^3} \sum_{l \in \mathbb{Z}_*^3} \hat{V}_k \sum_{p,q \in B_F^c \cap \left(B_F^c + k\right)}^{\sigma,\tau} 1_{L_l}(p) 1_{L_l}(q) \left\langle K_l e_q, e_p \right\rangle c_{p-l,\sigma}^* c_{-q+l,\tau}^* c_{-q+k,\tau} c_{p-k,\sigma}.
\end{aligned}$$

The same estimates used in Proposition 8.1.4 now apply to show $\mathcal{G}^{(R)} \to \mathcal{G}$ in norm as $R \to \infty$, so for any $\Psi \in D(H'_{\mathrm{kin}})$

$$\begin{aligned}
\left\langle \Psi, e^{\mathcal{K}} \mathcal{Q}_{\mathrm{SR}} e^{-\mathcal{K}} \Psi \right\rangle &= \lim_{R \to \infty} \left\langle \Psi, e^{\mathcal{K}} \mathcal{Q}_{\mathrm{SR}}^{(R)} e^{-\mathcal{K}} \Psi \right\rangle \\
&= \lim_{R \to \infty} \left(\left\langle \Psi, \mathcal{Q}_{\mathrm{SR}}^{(R)} \Psi \right\rangle + \int_0^1 \left\langle \Psi, e^{t\mathcal{K}} \Big(2 \operatorname{Re}\Big(\mathcal{G}^{(R)}\Big)\Big) e^{t\mathcal{K}} \Psi \right\rangle dt \right) \\
&= \left\langle \Psi, \mathcal{Q}_{\mathrm{SR}} \Psi \right\rangle + \int_0^1 \left\langle \Psi, e^{t\mathcal{K}} (2 \operatorname{Re}(\mathcal{G})) e^{t\mathcal{K}} \Psi \right\rangle dt
\end{aligned} \tag{C.3.10}$$

which is the claim.

[2]The cubic terms $\mathcal{C}$ arguably warrant a similar justification, but this can be handled by the same kind of arguments we have just used, so we omit this.

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