Higher gauge theories: Extended Chern–Simons forms and L_∞ formulation

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Zusammenfassung

In dieser Dissertation konzentrieren wir uns auf zwei Hauptthemen: Das erste ist die Konstruktion höherdimensionaler Differentialpolynome, die unter den Transformationen freier Differentialalgebren invariant sind. Ein solches Problem wird im Rahmen einer bestimmten freien Differentialalgebra, bekannt als FDA1, behandelt, die man erhält, wenn man die einfachste Erweiterung einer Lie-Algebra durch Einführung eines nicht-trivialen Vertreters einer Chevalley–Eilenberg-Kohomologieklasse in ihre Maurer-Cartan-Gleichungen betrachtet. Die Studie umfasst die explizite Formulierung einer invarianten FDA1-Form, analog zur invarianten Chern–Pontryagin-Dichte einer Lie-Gruppe. Das Hauptmerkmal freier Differentialalgebren, und damit einer FDA1, ist das Vorhandensein von Differentialformen höheren Grades als Basiselemente, in direkter Analogie zu den linksinvarianten Maurer-Cartan-Eins-Formen, deren Differentialgleichungen eine Lie-Algebra definieren. Infolgedessen enthält die von uns vorgeschlagene verallgemeinerte invariante Form dieselben Differentialformen höheren Grades als Bausteine. Die Untersuchung ihrer erweiterten Invarianzeigenschaften wird uns zu verallgemeinerten Definitionen der kovarianten Ableitung und der invarianten Tensoren im Kontext einer FDA1 führen, was die Berechnungen in den folgenden Schritten erleichtern wird und es uns ermöglicht, explizite Ausdrücke für Chern–Simons und Transgressionsformen zu erhalten, die mit der verallgemeinerten invarianten Dichte assoziiert und unter einer FDA1 invariant sind. Die Eichinvarianz dieser erweiterten Formen macht sie zu Kandidaten für Lagrange-Dichten bei der Konstruktion verallgemeinerter Wirkungen für Eichtheorien, deren Feldinhalte Eins-Formen und Differentialformen höheren Grades enthalten, die auf nichttriviale Weise koppeln. Als Beispiel betrachten wir die Konstruktion einer fünfdimensionalen Chern–Simons-Wirkung, die unter einem besonderen Fall von FDA1 invariant ist. Zu diesem Zweck untersuchen wir die Existenz nichttrivialer Kozyklen einiger bestimmter bosonischer Lie-Algebren. Diese Studie führt uns zur Formulierung von drei nicht-äquivalenten Fällen von FDA1 in beliebigen Dimensionen. Einer von

ihnen ist eine Erweiterung der bosonischen Poincaré-Algebra, und die beiden anderen sind Erweiterungen der bosonischen Maxwell-Algebra. Für einen dieser letzten Fälle stellen wir ein Beispiel für eine FDA1-Chern–Simons-Theorie vor, die Eins- mit Drei-Formen koppelt, ohne die Eichsymmetrie unter der Transformation der Maxwell-Algebra zu brechen, sondern sie, wie oben erwähnt, auf den Fall einer FDA1 erweitert. Die Standard-Chern–Simons-Formen sind nicht nur hilfreich bei der Konstruktion von Wirkungen, sondern auch bei der Untersuchung von Eichtheorie-Anomalien in der Quantenfeldtheorie relevant. Wir beenden den ersten Teil der Forschung dieser Dissertation, indem wir eine solche Analyse auf den Fall einer FDA1 ausweiten und dabei zwei unabhängige Verallgemeinerungen der nicht-abelschen Standardanomalie erhalten.

Das zweite Hauptthema dieser Dissertation befasst sich mit der Formulierung klassischer Eichtheorien in Form von L_∞-Algebren. Konkret schlagen wir drei explizite Beispiele für eine solche abstrakte Formulierung vor, indem wir die L_∞-Algebren, die solche Theorien beschreiben, explizit aufschreiben. Der erste zu analysierende Fall ist die ungerade-dimensionale Chern–Simons-Theorie, deren Feldinhalt ein Eins-Formen-Eichfeld ist und deren Eichsymmetrie durch eine Lie-Algebra beschrieben wird. Im zweiten Fall betrachten wir das einfachste Beispiel einer eichinvarianten Theorie, deren Symmetrie durch eine FDA1 beschrieben wird, die wir als flache FDA1-Theorie bezeichnen. Die Dynamik einer solchen Theorie wird durch die Maurer-Cartan-Gleichungen einer FDA1 bestimmt, d.h. durch die Null-Krümmungs-Bedingungen. Der letzte untersuchte Fall ist die verallgemeinerte, unter einer FDA1 invariante Chern–Simons-Theorie, die im ersten Teil der Dissertation eingeführt wurde. Dieser letzte Fall erweist sich aus technischer Sicht als anspruchsvoller, da sich die Dynamik der Theorie in Abhängigkeit von der Dimensionalität stark verändert. Außerdem analysieren wir die Bedingungen, unter denen die entsprechende Eichalgebra geschlossen ist. Folglich untersuchen wir die Anforderungen, die die Wirkung erfüllen muss, um zu einer wohldefinierten Eichtheorie zu führen und somit durch eine wohldefinierte L_∞-Algebra beschrieben zu werden. Wir stellen fest, dass diese Anforderungen nur in bestimmten Dimensionalitäten oder in Theorien erfüllt sind, deren Dynamik bestimmte Kopplungsbedingungen zwischen den Standard- und erweiterten Feldern erfüllt.

Summary

In this thesis, we focus on two main topics: the first one is the construction of higher-dimensional differential polynomials invariant under the transformations of free differential algebras. Such a problem will be treated in the framework of a particular free differential algebra, known as FDA1, which is the simplest extension of a Lie algebra that can be obtained by introducing a representative of a non-trivial Chevalley–Eilenberg cohomology class into its Maurer–Cartan equations. The study of this thesis covers the explicit formulation of a FDA1 invariant form, analogue to the Chern–Pontryagin invariant density of a Lie group. The main feature of a free differential algebra, and therefore of a FDA1, is the presence of higher-degree differential forms as basis elements in direct analogy to the Maurer–Cartan left-invariant one-forms that describe a Lie algebra through its differential equations. Consequently, the generalized invariant form that we propose includes the same higher-degree differential forms as building blocks. The study of its extended invariance properties will lead us to generalized definitions of covariant derivative and invariant tensors in the context of a FDA1. These definitions will be helpful in facilitating calculations in the following steps and allowing us to obtain explicit expressions for Chern–Simons and transgression forms that are associated to the generalized invariant density, and that are invariant under the mentioned FDA1. The gauge invariance of these extended forms makes them candidates to be considered Lagrangian densities in the construction of generalized action principles for gauge theories whose field contents include one-forms and higher-degree differential forms that couple in a non-trivial way. As an example, we consider the construction of a five-dimensional Chern–Simons action, invariant under a particular case of FDA1. With this purpose, we study the existence of non-trivial cocycles of some particular bosonic Lie algebras. This study leads us to the formulation of three non-equivalent cases of FDA1 in arbitrary dimensions. One of them is an extension of the bosonic Poincaré algebra, and the remaining two are extensions of the bosonic Maxwell algebra. For one of these last cases, we present an example of FDA1-Chern–Simons theory that couples one-forms with three-forms without breaking the gauge symmetry under the transformation of Maxwell algebra but extending it to the case of the abovementioned FDA1. In addition to being helpful in constructing action principles, standard Chern–Simons forms are relevant in the study of gauge anomalies in quantum field theory. We finish the first part of the research of this thesis by extending such analysis to the case of a FDA1, obtaining two independent generalizations to the standard non-abelian gauge anomaly.

The second main topic of this thesis is the formulation of classical gauge theories in terms of L_∞ algebras. Specifically, we propose three examples of such abstract formulation by explicitly writing down the L_∞ algebras that describe such theories. The first case that will be analyzed is the standard odd-dimensional Chern–Simons theory, whose field content is given by one-form gauge field and whose gauge symmetry is described by a Lie algebra. In the second case, we consider the simplest example of a gauge-invariant theory whose symmetry is described by a FDA1, which we refer to as flat FDA1 theory. The dynamics of such theory is governed by the Maurer–Cartan equations of a FDA1, i.e., the zero-curvature conditions. The last case of study is the generalized Chern–Simons theory invariant under a FDA1, which was introduced in the first part of the research of the thesis. Such a last case turns out to be more challenging, from a technical point of view, due to the strong changes shown by the dynamics of the theory depending on the dimensionality. Moreover, we analyze the conditions under which the corresponding gauge algebra is closed. Consequently, we study the requirements that the action principle must satisfy in order to lead to a well-defined gauge theory and, therefore, to be described by a well-defined L_∞ algebra, finding that such requirements are only fulfilled in certain dimensionalities or in theories whose dynamics satisfy specific coupling conditions between the standard and extended fields.

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Contents

Chapter 1

Introduction

In this chapter, we introduce the main concepts that motivate the research of this thesis. We begin with a description of the role of symmetry algebras in standard gauge theories. We then introduce in simple words the idea of free differential algebra, which constitutes the central concept related to the first part of the research of this thesis, and explain its importance in the construction of generalized physical theories. We continue by presenting the motivation for the construction of gauge theories that involve higher degree tensors as gauge fields and their relation with the aforementioned algebras. The following section of the chapter is devoted to describing the importance of two particular gauge theories, namely, Chern–Simons and Yang–Mills theories, in the context of standard and extended symmetry algebras. Later, we introduce L_∞ algebras, the central concept associated to the second part of the research of this thesis, and explain the motivation for such study. We finish the chapter with a description of the structure of the thesis, in which we shortly summarize the content of each chapter.

1.1 Symmetries and gauge theories

Many physical systems of interest have some degree of symmetry. Physical theories themselves may have redundancy in their formulation such that a set of transformations can be performed without modifying the measurable results. In most cases, this symmetry can be mathematically described by a Lie algebra. On the other hand, we constantly look for generalizations of theories such that they comply the Bohr's correspondence principle, i.e., such that they are able to reproduce the results of the already known theories as particular cases corresponding to some specific regime of the generalized theory. Therefore, once we know a mechanism of obtaining a theory from a particular symmetry, it becomes relevant to know how to generalize Lie algebras in order to describe enlarged symmetries and, through them, to study generalized theories. Two well-known ways to formulate a theory for a particular Lie algebra are Yang–Mills and Chern–Simons theories. These are theories that admit a gauge formulation, i.e., theories on which some transformations on the fundamental dynamical variable leave invariant the corresponding action principle and, consequently, the corresponding dynamics. In both cases, the action of such transformations on the fundamental variable is ruled by a Lie algebra.

In the context of standard gauge theories, the fundamental gauge potential μ is locally represented by a one-form taking values on a Lie algebra. Its corresponding field strength is consequently described by a two-form. Thus, the one-form gauge potential allows describing the parallel transport of a point particle, i.e., a zero-dimensional object, whose trajectory is represented by a one-dimensional curve in spacetime. In the same way, antisymmetric tensor fields of higher degrees allow carrying out the parallel transport of extended objects, such as strings or branes, along a higher-dimensional trajectory. A detailed analysis in the context of the so-called p-form electrodynamics can be found in refs. [1, 2]. Another example of a higher gauge field is the so-called Kalb–Ramond field in string theory, which is described by a two-form [3]. The presence of gauge fields in physical theories that are described by higher-degree tensors motivates the generalization of the gauge principle by means of the use of enlarged algebraic structures whose gauging contains such field content in a natural way.

1.2 Free differential algebras

A Lie algebra is a vector space endowed with a bilinear antisymmetric product that satisfies the Jacobi identity. Such antisymmetric product is usually realized as a commutator between the algebraic vectors, implying that the Jacobi identity of the antisymmetric product is equivalent to the associativity condition of the secondary product with which such commutator is defined. On the other hand, any Lie algebra admits an equivalent dual formulation in terms of differential one-forms (or Maurer–Cartan potentials). The information regarding the Lie product is then codified into a set of differential equations known as

Maurer–Cartan equations for the corresponding potentials. Therefore, one straightforward way to generalize Lie algebras is to introduce new Maurer–Cartan potentials, given not only by one-forms but by higher-degree differential forms satisfying new differential equations that must be consistent with the structure equations of the original one-forms. This procedure leads to a new mathematical structure known as free differential algebra (FDA). In the following, we present our ideas for applying generalized symmetries in the context of gauge theories.

FDAs were first introduced in physics in 1980 by R. D'Auria, P. Fré and T. Regge [4] as mathematical structures that allow formulating supergravity in the superspace in a geometric way. In 1982, R. D'Auria and P. Fré made use of such structure to unveil a hidden symmetry algebra in the eleven-dimensional supergravity previously constructed by Cremmer, Julia and Scherk [5,6]. On the other hand, the field content of supergravity theories in six or more dimensions includes higher-degree differential forms in the bosonic sector when it is formulated in terms of differential forms in the first-order formalism. The presence of such higher-degree forms is a consequence of the consistency requirement of an equal number of bosonic and fermionic degrees of freedom in supersymmetry [7]. Therefore, the field content of these theories cannot be consistently introduced in terms of one-forms, making impossible a formulation in terms of a one-form gauge field valued in a Lie algebra (or superalgebra) as it happens in standard Yang–Mills and Chern–Simons theories. This motivates to introduce FDAs in substitution of Lie algebras in the first-order formulation of gauge theories for gravity and supergravity that inherently involve higher-degree forms as fundamental fields.

1.3 Higher gauge theories

As it was mentioned, the field content of a standard gauge theory is given by a one-form gauge field that is valued in the Lie algebra that describes the gauge symmetry. On the other hand, higher gauge theories are those in which the fundamental field is not only given by a one-form gauge field but a composite gauge field whose components are tensors of different degrees [8–10]. Since FDAs are, basically, generalizations of Lie algebras that involve differential forms of different degrees as Maurer–Cartan potentials, their gauging provide a natural mechanism in the construction of higher gauge theories. The most simple example of a higher gauge theory is the so-called p-form electrodynamics [1,2], a theory that describes the dynamics of extended objects, such as p-branes. The gauge symmetry algebra of p-form electrodynamics is also a Lie algebra, specifically the U(1) algebra, and not a non-trivial generalization of a Lie algebra. The field content is described only by a p-form that is valued on U(1), such as it happens in standard electrodynamics, and therefore, it does not carry an algebraic index. To introduce higher-degree differential forms as gauge fields in a theory is therefore possible without extending the underlying algebraic structure; it is enough to extend the gauge principle and consider higher degree tensors valued in a Lie algebra. Examples of this type were introduced and studied in refs. [11–19]. In these cases, the field contents include several differential forms of different degree, being each one valued in the same Lie algebra. Such forms are used in the construction of generalized Chern–Simons forms, leading to theories that couple differential forms of different degrees and whose gauge invariance is also described by Lie algebras.

In the aforementioned examples, the gauge symmetry is described by a Lie algebra, exactly as it happens in standard Yang–Mills and Chern–Simons theories. Although their field content include differential forms of different degree, in every case, they are valued in a Lie algebra without the need to introduce generalized gauge algebras. However, the presence of higher-degree forms as gauge fields allows us to understand the gauge algebras of these theories as FDAs by interpreting the one-forms and the higher-degree differential forms as different Maurer–Cartan potentials of a FDA instead of gauge potentials valued in the same Lie algebra. In these cases, the difference is merely a matter of interpretation. Consequently, FDAs that describe the gauge symmetry these type of theories are equivalent to Lie algebras and can be obtained from them without introducing an additional mathematical structure. These are FDAs whose main feature is to be trivial extensions of Lie algebras. In this context, the triviality lies in the fact that the Maurer–Cartan potentials that are higher-degree differential forms can be decomposed as linear combinations of tensorial products of the Maurer–Cartan potentials of a Lie algebra, i.e., in terms of products of their one-forms. A FDA is therefore non-trivial when we introduce another element into its Maurer–Cartan equations, namely, a higher-degree differential form in a way that does not contradict the structure equations of the original Lie algebra (i.e., without contradicting its Jacobi identity) and whose role is to make it impossible to decompose the higher-degree forms in terms of the one-forms without contradicting the structure equations. Algebraic elements that satisfy these conditions are known as non-trivial cocycles, representatives of the Chevalley–Eilenberg cohomology classes of the Lie algebra [20]. When a Lie algebra is extended to a FDA through the introduction of a non-trivial cocycle into the Maurer–Cartan equations, the new structure can be considered a non-trivial FDA extension of the original Lie algebra, and it can be considered as a candidate to describe richer symmetry in a physical system. It follows that there are so many non-equivalent and non-trivial FDA extensions for a Lie algebra as Chevalley–Eilenberg cohomology classes such Lie algebra has.

1.4 Yang–Mills & Chern–Simons

The field content of standard Yang–Mills and Chern–Simons theories is codified into a one-form that takes values on certain gauge algebra. In particular, Yang–Mills theory, the basis of the standard model, has SU(N) as gauge group and describes three of the four fundamental interactions, namely, the electromagnetic, weak nuclear, and strong nuclear interactions. Supersymmetric SU(N) Yang–Mills theories also appear to be a very interesting case for $\mathcal{N} = 4$ because of its conformal symmetry and therefore its relation with string theory. Moreover, according with the well-celebrated AdS/CFT correspondence [21], this theory is dynamically equivalent to type IIB superstring theory. On the other hand, the Chern–Simons action principle makes use of a Chern–Simons form as Lagrangian density and, although it has been extensively studied in the three-dimensional case, it can be formulated in arbitrary odd dimensions for any Lie gauge group or supergroup [22–24]. The $(2n+1)$-dimensional Chern–Simons form, denoted by Q_{2n+1}, has a topological origin, and it can be mathematically derived from the $2n+2$ dimensional Chern–Pontryagin invariant density, denoted by χ_{2n+2}, through the well-known relation [25, 26]

$$\chi_{2n+2} = \mathrm{d}Q_{2n+1}. \tag{1.1}$$

The relation between the Chern–Pontryagin invariant density and the Chern–Simons form constitutes the so-called Chern–Weil theorem, which provides the mathematical tools for the explicit derivation of Chern–Simons (and transgression) forms necessary in the construction of higher-dimensional action principles in odd dimensions. Due to their invariance properties, Chern–Simons theories have been extensively studied in the construction of gauge theories in arbitrary odd dimensions for different Lie algebras. For details on their use as Lagrangian densities for gravity theories, see refs. [27–29]. Moreover, their corresponding supersymmetric formulations can be found in refs. [30,31]. The consequent study of the dynamics of these gravity theories has led to different applications in black hole theory. See for example refs. [32–34].

Generalized Chern–Simons forms (and transgression forms) whose invariance is not described by Lie algebras but by enlarged structures is an active area of research. See, for example, refs. [35, 36]. In addition to being helpful in the construction of action principles for odd-dimensional topological gauge theories, standard Chern–Simons forms are relevant in the study of non-abelian gauge anomalies in quantum field theory. Gauge anomalies appear in quantum field theory when classical symmetries associated to gauge fields are broken in the quantization process. As a consequence, the currents that are associated with such symmetries are not conserved anymore because of the appearance of anomalous terms in the calculation of their divergences. An important result due to B. Zumino in refs. [37–39] shows that, when considering a quantum field theory in even-dimensional spacetime, the non-abelian gauge anomaly can be derived from the gauge-variation of the Chern–Simons form corresponding to such gauge fields. This motivates us to find explicit expressions for Chern–Simons and transgression forms, not only for the construction of action principles but also for the research of the generalized anomalies that emerge from the Chern–Simons forms whose gauge field has components of different differential degree, and whose invariance is described by FDAs that are non-trivially obtained from Lie algebras. It is therefore expected that generalized versions of Chern–Simons forms and their corresponding gauge anomalies reproduce their standard versions as the first terms of their expansions and include new terms that depend on the extended components of the gauge field, i.e., on higher-degree differential forms, due to the natural origin of FDAs as extensions of the Lie algebras from which such standard quantities were originaly derived. Moreover, to consider higher-degree forms as building blocks in the construction of generalized Chern–Simons forms makes possible their formulation in both odd and even dimensionalities, making also possible the study of the gauge anomalies that emerge from them in the context of odd and even-dimensional spacetimes. As mentioned before, the presence of the cocycle in the construction of the FDA is essential to obtain results that cannot be trivially reduced to the standard expressions originated in the study of gauge theories and Lie algebras. In this thesis, we calculate the explicit expressions for Chern–Simons and transgression forms, as well as the anomalies emerging from the Chern–Simons theory (see chapters 5 and 6, respectively).

1.5 L_∞ algebras

We can understand FDAs as the generalizations of Lie algebras in their dual formulation, i.e., as differential algebras whose Maurer–Cartan equations non-trivially extend the ones of Lie algebras by using algebraic elements, representatives of their Chevalley–Eilenberg cohomology classes. This is also possible in the basis of contravariant tangent vectors by defining not only a bilinear antisymmetric product but multilinear and graded-symmetric products acting on a generalized vector space. As it happens with FDAs, some conditions must be satisfied by consistency with the generalized version of the Jacobi identity that FDAs satisfy. This 'Lie algebra analogue' is called L_∞ algebra (see for example [40]). L_∞ algebras have been extensively studied in the formulation of physical theories. In ref. [41], it was shown that it is possible to write the relevant information of an arbitrary classical gauge theory in terms of an L_∞ algebra. Previous results regarding the relation between classical gauge theories and L_∞ algebras can be found in ref. [42]. Moreover, for extensive reviews on the role of L_∞ algebras in the context of the Batalin–Vilkovisky formalism, see refs. [43, 44].

In most cases, when formulating a classical gauge theory, the gauge symmetry is described by an algebraic structure (usually a Lie algebra) whose relevant information is encoded into its corresponding structure constants. The information concerning the dynamics of the theory is encoded in the equations of motion. In contrast, the L_∞ formulation of gauge theories allows to write down all the information of a classical gauge theory in terms of the structure constants of an algebra denoted by L_∞^{full}, consisting on a graded vector space endowed with a set of multilinear products. Thus, the information regarding the covariance, the gauge algebra and the dynamics of the interacting theory is encoded into the products between elements of different subspaces. The gauge symmetry remains described by a certain subalgebra $L_\infty^{\text{gauge}} \subset L_\infty^{\text{full}}$. If the symmetry algebra of the theory is a Lie algebra, then L_∞^{gauge} will also be, even if L_∞^{full} is not. However, by considering a gauge theory whose gauge invariance is governed by a FDA, the L_∞ formulation of gauge theories becomes more appropiate, due to the dual relation between FDAs and L_∞ algebras. This further motivates us to study the formulation of arbitrary odd-dimensional Chern–Simons forms in terms of L_∞ algebras and to extend the results to generalized Chern–Simons theories that involve higher-degree differential forms invariant under FDAs (see chapters 7 and 8 respectively).

1.6 Structure of the thesis

In this thesis we consider two main research topics. The first one is the construction of gauge theories for FDAs. We will therefore start by focusing on the reproduction of the Chern–Weil theorem for such algebras, and the formulation of gauge-invariant action principles that make use of extended Chern–Simons and transgression forms as Lagrangian densities. Moreover we will study the existence of gauge anomalies based on such generalization. The second main topic is the formulation of gauge invariant theories that involve higher-degree differential forms as fundamental fields in terms of L_∞ algebras. To cover such topics, this thesis is presented in the following structure:

- Chapter 2 is entirely devoted to mathematical preliminaries. We begin with a review of differential geometry on Lie group manifolds and introduce the concept of Chevalley–Eilenberg cohomology, which will be of particular importance in chapters 3 and 5. In addition, we introduce mathematical tools, necessary for the construction of Lie gauge theories such as the invariant tensors and the Chern–Weil theorem.

- In chapter 3, we review the concept of free differential algebra. We include its mathematical definition, its gauging and the detailed study of the properties that emerge for a particular case, known as FDA1, which is of special importance in the research developed in chapter 5.

- In chapter 4, we carry out a review of L_∞ algebras. We introduce their definition in two equivalent pictures and show their dual relation with the already mentioned free differential algebras. Moreover, we study the formulation of classical gauge theories in terms of L_∞ algebras, which is of significant relevance for the second part of the research of this thesis, developed in chapters 7 and 8.

- Chapter 5 contains the main results concerning the first part of the research of this thesis (see ref. [45]). We generalize the Chern–Weil theorem to the case of the above mentioned algebra FDA1. This leads to definitions of generalized Chern–Simons and transgression forms. Furthermore, we study the equations of motion of the FDA1 invariant gauge theories that make use of such forms as Lagrangian densities. The chapter ends with the study of the existence of non-trivial FDAs that can be obtained as extensions of the bosonic Poincaré and Maxwell algebras, and the construction of an action principle for one of these cases.

- In chapter 6 we will study the existence of gauge anomalies in higher gauge theories. For this purpose, we consider the generalized Chern–Simons forms, introduced in chapter 5, and from them, we derive explicit expressions for generalized gauge anomalies, in theories that couple the standard one-form from Yang–Mills theories with higher degree tensors (see ref. [45]).

- In chapter 7 the formulation of Chern–Simons theories is considered in the language of L_∞ algebras. For this purpose, we begin by finding the L_∞ subspace structure that is shared by every Lie gauge theory, i.e., the sectors that encode the gauge transformations and the gauge algebra. We then study the subspace that encodes the dynamics of standard Chern–Simons theory in arbitrary odd dimensions. We finish the chapter with a summary that shows the L_∞ algebras for such theories, including a remaining structure that was not explicitly shown in the first sections of the chapter and that, for consistency, must be non-trivial (see ref. [46]) .

- Finally, in chapter 8 we study the L_∞ formulation of gauge theories whose symmetry is described by a FDA1. We begin by finding the subspace structure that is shared by every FDA1 gauge theory, which includes the information regarding the definition of FDA1 gauge transformations and the corresponding gauge algebra. We then study the particular cases of a flat FDA1 theory (in which the dynamics is governed by the zero-curvature conditions) and the FDA1-Chern–Simons theories introduced in chapter 5. We explicitly write down the L_∞ algebras for both types of theories (see ref. [46]).

- Chapter 9 contains the conclusions of the research presented in this thesis.

Chapter 2

Mathematical preliminaries

An important part of this thesis is based on the study of an algebraic structure known as free differential algebra (FDA). These are generalizations of Lie algebras that allow the construction of gauge theories involving higher-degree differential forms as gauge fields. In this chapter we introduce concepts of Lie groups and standard gauge theories, such as left- and right-invariant vector fields, Lie derivatives, the dual formulation of Lie algebras, and the so-called Chern–Weil theorem, which relates the Chern–Pontryagin density and Chern–Simons form. These concepts will be necessary in the later introduction of FDAs and their corresponding higher gauge theories.

2.1 Symmetry algebras

Symmetry algebras are mathematical structures that have an important role in the formulation of physical theories. To introduce them, let us consider a physical theory on an arbitrary n-dimensional manifold M with a set of coordinates $x^0, \ldots, x^{n-1}$, and a set of fundamental fields $\mu^A(x)$ with $A = 1, \ldots, N$. The dynamics of the theory is determined by the general field variation of an action principle which is given by a functional of the fundamental fields

$$S[\mu] = \int_M \mathrm{d}x^n \mathcal{L}(\mu). \tag{2.1}$$

This action principle is defined in terms of the integration over M of the Lagrangian density $\mathcal{L}(\mu)$ of the theory. Let us now introduce a symmetry variation $\delta_\varepsilon \mu$ with respect to a scalar parameter ε. The variation of the action principle is given by

$$\delta_{\varepsilon} S\left[\mu\right] = \int_M \mathrm{d}x^n \frac{\delta \mathcal{L}\left(\mu\right)}{\delta \mu^A} \frac{\delta \mu^A}{\delta \varepsilon} \delta \varepsilon. \tag{2.2}$$

By considering two successive variations with independent parameters ε_1 and ε_2, it is direct to see that

$$\delta_{\varepsilon_2} \delta_{\varepsilon_1} S\left[\mu\right] = \int_M \mathrm{d}x^n \delta_{\varepsilon_2} \left(\frac{\delta \mathcal{L}\left(\mu\right)}{\delta \mu^B \delta \mu^A} \delta_{\varepsilon_2} \mu^B \delta_{\varepsilon_1} \mu^A + \frac{\delta \mathcal{L}\left(\mu\right)}{\delta \mu^A} \delta_{\varepsilon_2} \delta_{\varepsilon_1} \mu^A \right), \tag{2.3}$$

where we have used the eq. (2.2). By assuming that the parameters ε_1 and ε_2 commute, we obtain the following expression for the commutator of two independent symmetry variations acting on the action principle:

$$\left[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}\right] S\left[\mu\right] = \int_M \mathrm{d}x^n \frac{\delta \mathcal{L}\left(\mu\right)}{\delta \mu^A} \left[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}\right] \mu^A. \tag{2.4}$$

As we have introduced both δ_{ε_1} and δ_{ε_2} as symmetries, the variations on eqs. (2.2) and (2.4) must vanish. This shows that $\left[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}\right]$ is also a symmetry of the action principle and allows us to define a new gauge parameter $\left[\varepsilon_1, \varepsilon_2\right]$ as follows

$$\delta_{\left[\varepsilon_1, \varepsilon_2\right]} = \left[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}\right]. \tag{2.5}$$

Since $\left[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}\right]$ is a commutator defined by using an associative product, it satisfies the Jacobi identity. On the other hand, the composite parameter $\left[\varepsilon_1, \varepsilon_2\right]$ defines a product between parameters that is not necessarily a commutator. However, due to eq. (2.5), it is also antisymmetric and satisfies the Jacobi identity. Notice that the symmetry parameters have not been introduced as vectors of a previously defined space, but they will obey the axioms of Lie algebras if we demand that the symmetry transformations leave the action principle invariant and continuously depend on the parameters. This shows that Lie algebras are the mathematical structure that describes the symmetries of physical systems.

2.2 Group manifolds

In this section, we shortly review some basic concepts on Lie groups. For extensive reviews on this subject, see refs. [7, 47–49]. Let us consider a Lie group $\mathcal{G}$. This is a soft manifold on which the map defined by $\mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$ with $\left(g', g\right) \longrightarrow g'g$ and the inverse map defined by $\mathcal{G} \longrightarrow \mathcal{G}$ with $g \longrightarrow g^{-1}$, are both smooth. For the sake of introducing the corresponding notation, we mention some definitions in the theory of tensors on Lie groups.

Let g be a fixed point of $\mathcal{G}$. The left-translation L_g and the right-translation R_g are both diffeomorphisms $\mathcal{G} \longrightarrow \mathcal{G}$ defined by the following action on a generic point $h \in \mathcal{G}$:

$$R_g h = hg, \tag{2.6}$$

$$L_g h = gh. \tag{2.7}$$

The definition of left- (and right-) translation induces a definition of left- (and right) invariant vectors and one-forms. Indeed, by considering the tangent space of contravariant vectors $T_e(\mathcal{G})$ at the identity point $e \in \mathcal{G}$, the left-translation $L_g e = g$ induces a mapping over every vector $v_e \in T_e(\mathcal{G})$ given by the pull-back of the left-translation $(L_g)_*$

$$\begin{aligned} (L_g)_* : \quad T_e(\mathcal{G}) &\longrightarrow T_g(\mathcal{G}), \\ v_e &\longmapsto v_g \equiv (L_g)_* v_e. \end{aligned}$$

where $T_g(\mathcal{G})$ is the tangent space of contravariant vectors in g. Since the transformed vector $v_g \in T_g(\mathcal{G})$ transforms in the same functional way under a secondary left-translation $L_{g'} v_g = v_{g'g}$, it is said to be left-invariant. In this way, the left-translation induces a notion of left-invariant vector field over the manifold $\mathcal{G}$. Analogously, the right-translation $R_g e = g$ induces a mapping $(R_g)_* : T_e(\mathcal{G}) \longrightarrow T_g(\mathcal{G})$ by means of its corresponding pull-back on the contravariant tangent space. The resulting vector field over $\mathcal{G}$ is said to be right-invariant. These definitions are immediately generalized to the dual tangent spaces of covariant vectors (or one-forms) providing a notion of left- (and right-) invariant one-forms. Moreover, the antisymmetric wedge product of differential forms provides a definition of left- (and right-) invariant differential forms over the group manifold $\mathcal{G}$. From now on we will denote the space of left-invariant q-forms by $\Lambda^q(\mathcal{G})$ unless we specify another notation.

The exterior derivative operator, denoted by d is defined as a mapping

$$\mathrm{d} : \Lambda^q(\mathcal{G}) \longrightarrow \Lambda^{q+1}(\mathcal{G}), \tag{2.8}$$

such that its action on an arbitrary differential form

$$\omega^{(q)} = \frac{1}{q!}\omega^{(q)}_{\mu_1\cdots\mu_q}\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_q}, \tag{2.9}$$

is given by

$$\mathrm{d}\omega^{(q)} = \frac{1}{q!}\left(\frac{\partial}{\partial x^{\nu}}\omega^{(q)}_{\mu_1\cdots\mu_q}\right)\mathrm{d}x^{\nu}\wedge\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_q}. \tag{2.10}$$

From this definition, it follows that the exterior derivative satisfies the Leibniz rule

$$\mathrm{d}\left(\omega^{(r)}\wedge\omega^{(s)}\right) = \mathrm{d}\omega^{(r)}\wedge\omega^{(s)} + (-1)^{r}\,\omega^{(r)}\wedge\mathrm{d}\omega^{(s)}, \tag{2.11}$$

and nilpotence $\mathrm{d}^2 = 0$.

2.3 Lie algebras and Chevalley–Eilenberg cohomology

A Lie algebra is a vector space G endowed with a bilinear antisymmetric product

$$G \times G \longrightarrow G, \tag{2.12}$$

$$(A, B) \longmapsto [A, B], \tag{2.13}$$

called Lie product, which verifies the Jacobi identity

$$[[A, B], C] + [[C, A], B] + [[B, C], A] = 0. \tag{2.14}$$

One can prove that the vector space of left- (and right-) invariant vector fields on a Lie group $\mathcal{G}$ becomes a Lie algebra by defining the commutator as Lie product $[A, B] = AB - BA$. As a consequence, every Lie group has an associated Lie algebra. From now on, we will denote G to the Lie algebra associated with a Lie group $\mathcal{G}$. Thus, since every left-invariant vector field on $\mathcal{G}$ is univocally determined by its value on the identity e, the Lie algebra can be identified by the space of tangent vectors $G \equiv T_e(\mathcal{G})$.

Let $\{t_A\}_{A=1}^{n}$ now be a basis for an arbitrary n-dimensional Lie algebra G. By writing the Lie product in terms of this basis one finds

$$[t_A, t_B] = C_{AB}^{C} t_C, \tag{2.15}$$

where the coefficients C_{AB}^{C} are called structure constants. They are constants due to the left-invariance of the basis vectors, and antisymmetric in the lower indices due to the antisymmetry of the Lie product. Writing the Lie algebra in terms of a certain basis, also allows to write the Jacobi identity in terms of the structure constants in such basis, which gives

$$C^E_{D[A}C^D_{BC]} = 0. \tag{2.16}$$

The commutation relations of eq. (2.15) define a Lie algebra in terms of the basis elements of the contravariant basis. It is also possible to define the Lie algebra in terms of a set of differential equations relating the dual basis of one-forms. Let $\{x^\mu\}$ be a local coordinate system on $\mathcal{G}$. Every basis vector t_A can be written as a linear combination of the coordinate basis associated to $\{x^\mu\}$

$$t_A(x) = e^\mu_A(x)\frac{\partial}{\partial x^\mu}, \tag{2.17}$$

where the matrix coefficients $e^\mu_A(x)$ involve indices corresponding to both bases. Moreover, the corresponding dual bases $\left\{\omega^A\right\}_{A=1}^{\dim G}$ and $\{\mathrm{d}x^\mu\}_{\mu=1}^{\dim G}$ are related by the inverse matrix, whose coefficients e^A_μ satisfy

$$\omega^A = e^A_\mu \mathrm{d}x^\mu. \tag{2.18}$$

The matrices e^{-1} with entries e^A_μ constitute the so-called left-invariant vielbein group. By directly replacing eqs. (2.17) and (2.18) in eq. (2.15) it is possible to write down the information regarding the Lie algebra into a set of differential equations for e^μ_A and e^A_μ, namely

$$2e^\mu_{[A}e^\nu_{B]}\partial_\mu e^C_\nu = -C^C_{AB}. \tag{2.19}$$

By directly contracting eq. (2.19) with $\omega^A \wedge \omega^B = e^A_\mu e^B_\nu \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ one gets

$$(\mathrm{d}x^\nu\partial_\nu)\wedge\left(e^C_\mu\mathrm{d}x^\mu\right) = -\frac{1}{2}C^C_{AB}\left(e^A_\mu\mathrm{d}x^\mu\right)\wedge\left(e^B_\nu\mathrm{d}x^\nu\right). \tag{2.20}$$

It is now possible to identify the exterior derivative operator and the elements of the basis of one-forms on eq. (2.20), and write the information coming from the commutation relations of the contravariant vectors t_A without using the coordinate basis but only the dual basis of one-forms, as follows

$$\mathrm{d}\omega^A + \frac{1}{2}C^A_{BC}\omega^B\wedge\omega^C = 0. \tag{2.21}$$

The differential equations in eq. (2.21) are known as Maurer–Cartan equations. They provide a dual formulation of the Lie algebra in terms of the covariant basis. The Jacobi identity for the structure constants becomes equivalent to the integrability condition $\mathrm{d}^2 = 0$ in the Maurer–Cartan equations. Both ways of describing a Lie algebra are completely equivalent. However, considering the dual formulation is especially useful when one generalizes the algebraic structure to a new one that involves higher-degree differential forms.

At this point, we have considered the basis of covariant one-forms on the group manifold. The use of such differential forms in the construction of a physical theory requires the definition of a more sophisticated mathematical structure, that allows the transformations of the Lie algebra to be applied to differential forms defined on a space of different dimensionality. We now discuss the mathematical idea of a principal bundle, which is a pair $(M, \mathcal{G})$, where M is a manifold describing the base space (in our case, spacetime), and $\mathcal{G}$ is the symmetry group of another manifold, the so-called fiber. This mathematical framework allows us to define a gauge theory in terms of its base manifold and symmetry Lie group by introducing a gauge connection instead of a vielbein one-form, as follows

$$e_\mu^A \mathrm{d}x^\mu \longrightarrow \mu_\mu^A \mathrm{d}x^\mu.$$

Unlike e_μ^A , the quantity μ_μ^A is not a vielbein, but a connection defined on the bundle. The indices A and μ no longer take the same values. The algebraic index A takes values between 1 and $\dim G$, while μ, being a coordinate spacetime index, takes values between 0 and $\dim M - 1$. A formal definition of a bundle involves a definition of the connection in terms of local charts due to the possibility of non-trivial topology of $\mathcal{G}$. For the purposes of this chapter, this heuristic definition is sufficient; however, a rigorous definition can be found in ref. [49].

We aim now to gauge the Lie group manifold. As usual, one considers a smooth deformation $\tilde{\mathcal{G}}$ of $\mathcal{G}$. An arbitrary basis of one-forms $\left\{\mu^A\right\}$ on $\tilde{\mathcal{G}}$ is defined by

$$\mu^A(x) = \mu_\mu^A(x)\mathrm{d}x^\mu, \tag{2.22}$$

which, in general, do not satisfy the Maurer–Cartan equations of the Lie algebra G associated to $\mathcal{G}$. The failure of the one-forms μ^A in satisfying such set of Maurer–Cartan equations defines a two-form called curvature form, which is given by

$$R^A(\mu) = \mathrm{d}\mu^A + \frac{1}{2} C_{BC}^A \mu^B \wedge \mu^C. \tag{2.23}$$

Eq. (2.23) is usually called Cartan structure equation.

Let us now introduce a matrix representation for the Lie algebra; this is, a mapping from G in a set of matrix operators $D(t_A)$ acting on a representation space

$$D : t_A \longrightarrow D(t_A)^i{}_j\,, \tag{2.24}$$

satisfying the commutation relations of the Lie algebra. Moreover, let us introduce polynomials of the type

$$\Omega^i = \Omega^i_{A_1\cdots A_p}\omega^{A_1}\wedge\cdots\wedge\omega^{A_p}, \tag{2.25}$$

where the index i takes values in the irreducible representations of G. Ω^i is therefore a p-form in $\Lambda^p(\mathcal{G})$ that takes values in the representation space. These objects are called Chevalley–Eilenberg cochains. The matrix representation D induces a G-covariant derivative given by

$$(\nabla)^i{}_j = \delta^i_j \mathrm{d} + \omega^A \wedge D(t_A)^i{}_j\,, \tag{2.26}$$

acting on elements with index i in the representation space. It is direct to verify that the Jacobi identity implies the nilpotence of the operator ∇. The action of the covariant derivative on Chevalley–Eilenberg cochains define certain equivalence classes. A cochain that is covariantly closed is called a cocycle. It is important to point out the existence of two types of cocycles. A coboundary (or trivial cocycle) is a cocycle that is covariantly closed due to the structure of the Lie algebra, i.e., it can be written as the covariant derivative of a secondary $(p-1)$-form Φ^i in the same representation, and therefore, it is a cocycle because of the nilpotence of the covariant derivative in the absence of curvature, namely

$$\Omega^i = \nabla\Phi^i, \tag{2.27}$$

$$\nabla\Omega^i = \nabla^2\Phi^i \equiv 0. \tag{2.28}$$

On the other hand, the cocycles that cannot be written as the covariant derivative of a secondary form are said to be non-trivial. They are representatives of the Chevalley–Eilenberg cohomology classes of the Lie algebra. It is important to notice that, with this notion of cohomology, two cocycles that differ by a coboundary are equivalent; that is, they constitute an equivalence class. Moreover, if G is a graded semi-simple Lie algebra, there are no non-trivial cohomology classes. For details on the Chevalley–Eilenberg cohomology of Lie algebras, see ref. [20].

The covariant derivative of a generic form ω^i can be defined in the deformed manifold $\tilde{\mathcal{G}}$ as follows

$$\nabla\omega^i = \mathrm{d}\omega^i + \mu^A \wedge D(t_A)^i{}_j\,\omega^j. \tag{2.29}$$

By applying the same procedure, it is direct to verify that, in this case, the covariant derivative is not nilpotent. This is a natural consequence of considering a deformation of the original manifold. Since the one-forms μ^A do not satisfy the Maurer–Cartan equations because of the presence of the curvature form R^A, they do not describe a Lie algebra and the second covariant derivative turns out to be proportional to the curvature two-form

$$\nabla^2 \omega^i = R^A \wedge D(t_A)^i{}_j \omega^j. \tag{2.30}$$

In particular, by taking the exterior derivative of the curvature R^A it is also direct to verify that R^A is a covariantly closed form

$$\nabla R^A = \mathrm{d}R^A + C^A_{BC}\mu^B \wedge R^C = 0, \tag{2.31}$$

where we have used the adjoint matrix representation $D(t_B)^A{}_C = C^A_{BC}$. Eq. (2.23) is usually known as the Bianchi identity.

Let us now consider an arbitrary contravariant vector on G, denoted by $X = X^A t_A$, and let $\varepsilon^{(p)} = \varepsilon_{A_1 \cdots A_p} \omega^{A_1} \wedge \cdots \wedge \omega^{A_p} \in \Lambda^p(\mathcal{G})$ be an arbitrary p-form (with $p > 0$). The contraction mapping i_X associated with X is defined as

$$i_X \quad : \quad \Lambda^p(\mathcal{G}) \longrightarrow \Lambda^{p-1}(\mathcal{G}), \tag{2.32}$$

$$i_X \varepsilon^{(p)} \quad = \quad p X^A \varepsilon_{AA_1 \cdots A_{p-1}} \omega^{A_1} \wedge \cdots \wedge \omega^{A_p}, \tag{2.33}$$

where t_A and ω^A belong to dual bases. Moreover, if X is a contravariant vector and $\omega^{(p)}$ is an arbitrary p-form, the Lie derivative of $\omega^{(p)}$ along X is a p-form defined as the anticommutator between the exterior derivative operator and the contraction i_X

$$\mathcal{L}_X \omega^{(p)} = i_X \mathrm{d}\omega^{(p)} + \mathrm{d}\left(i_X \omega^{(p)}\right). \tag{2.34}$$

Notice that, by writing the vector in a certain basis $X = X^A t_A$, the Lie covariant derivative from eq. (2.34) can be written as

$$\mathcal{L}_t \omega^{(p)} = \mathrm{d}X^A i_{t_A} \omega^{(p)} + X^A \mathcal{L}_{t_A} \omega^{(p)}. \tag{2.35}$$

Let us consider the behavior of μ^A under a general transformation of coordinates $x \to x + \delta x$. The variation of $\mu^A(x)$ is given in terms of the zero-form parameter $\delta x = \varepsilon$ by

$$\delta_\varepsilon \mu^A(x) = \mathrm{d}x^\nu \left\{\partial_\mu \mu^A_\nu(x)\delta x^\mu + \mu^A_\mu(x)\partial_\nu \delta x^\mu\right\}. \tag{2.36}$$

Integrating the second term at the r.h.s of eq. (2.36) by parts, we have

$$\delta_\varepsilon \mu^A(x) = \mathrm{d}x^\nu \partial_\nu \left(\mu^A_\mu(x)\varepsilon^\mu\right) + \mathrm{d}x^\nu \varepsilon^\mu \left[\partial_\mu \mu^A_\nu(x) - \partial_\nu \mu^A_\mu(x)\right]. \tag{2.37}$$

On the other hand, the components of the two-form $\mathrm{d}\mu^A$ can be explicitly written as

$$\left(\mathrm{d}\mu^A\right)_{\mu\nu} = \frac{1}{2}\left(\partial_\mu \mu_\nu^A \ - \partial_\nu \mu_\mu^A\right). \tag{2.38}$$

By plugging in eq. (2.38) into eq. (2.37), it is possible to write the variation of μ^A in terms of the anholonomic components of the parameter $\varepsilon^A = \mu_\mu^A \varepsilon^\nu$ as

$$\delta_\varepsilon \mu^A(x) = \mathrm{d}\varepsilon^A - 2\left(\mu^B\right)\left(\varepsilon^C\right)\left(\mathrm{d}\mu^A\right)_{BC}. \tag{2.39}$$

We can now identify the contraction operation $i_\varepsilon\left(\mathrm{d}\mu^A\right) = 2\mu^B\varepsilon^C\left(\mathrm{d}\mu^A\right)_{CB}$, so that the variation of μ^A turns out to be proportional to its Lie derivative along the direction of the parameter, namely

$$\delta_\varepsilon \mu^A(x) = \mathrm{d}\left(\varepsilon^A\right) + i_\varepsilon\left(\mathrm{d}\mu^A\right) = \mathcal{L}_\varepsilon \mu^A. \tag{2.40}$$

This means that the Lie derivative of μ^A is given by its infinitesimal change under a general transformation of anholonomic coordinates. In general if $\omega^{(p)}$ is an arbitrary p-form then, under a general coordinate transformation, it experiences a local infinitesimal change, given by its Lie derivative:

$$\delta_\varepsilon \omega^{(p)} = \mathcal{L}_\varepsilon \omega^{(p)}. \tag{2.41}$$

Eq. (2.41) shows that the Lie derivative is the generator of the general transformations of coordinates, and leads to an interesting result: by performing two successive Lie derivatives along the basis vectors t_A and t_B, it is possible to prove that they generate the Lie algebra G

$$\left[\mathcal{L}_{t_A}, \mathcal{L}_{t_B}\right] = C_{AB}^C \mathcal{L}_{t_C}. \tag{2.42}$$

2.4 Invariant tensors

Let us now consider the linear representation of the Lie group $\mathcal{G}$. To each element $g \in \mathcal{G}$ there is an associated mapping $[g] : G \longrightarrow G$ such that for any vector $v \in G$ we have $g \longmapsto [g]\, v = gvg^{-1}$. It follows that for any group elements $h, g \in \mathcal{G}$ and vector $v \in V$ we have

$$[h]\,[g]\, v = [hg]\, v. \tag{2.43}$$

By using this action of the group elements on the algebraic vectors, we can define an invariant tensor of the Lie algebra (or G-invariant polynomial) as an n-linear mapping of the form $\langle \cdots \rangle : G \times \cdots \times G \longrightarrow \mathbb{R}$ satisfying the following invariance condition

$$\langle v_1, \ldots, v_n \rangle = \left\langle \left(g v_1 g^{-1}\right), \ldots, \left(g v_n g^{-1}\right) \right\rangle . \tag{2.44}$$

The invariance condition of a polynomial can be expressed in terms of the structure constants of the corresponding Lie algebra, as follows

$$\sum_{i=1}^{n} C_{A_0 A_i}^{C} \left\langle t_{A_1}, \ldots, \hat{t}_{A_i}, t_C, \ldots, t_{A_n} \right\rangle = 0, \tag{2.45}$$

where $\{t_A\}_{A=1}^{\dim G}$ is the basis of vectors satisfying the Lie algebra from eq. (2.37), and where a basis vector with hat, $\hat{t}_A$, denotes the absence of such vector in the sequence. For details on invariant tensors of Lie algebras and proof of this theorem, see refs. [49–51].

2.5 Chern–Weil theorem

Chern–Simons theories are topological theories that make use of a Chern–Simons form as Lagrangian density, leading to gauge theories defined on odd-dimensional spacetimes. The gauge symmetry of such theories is ruled by the Lie algebra, on which the fundamental field of the theory, a one-form gauge connection, is evaluated. Let us therefore consider a one-form gauge connection $\mu^A = \mu_\mu^A \mathrm{d}x^\mu$. The corresponding field-strength (or curvature) two-form can be found by considering a perturbated group manifold, i.e., by means of the gauging of the Maurer–Cartan equation and the introduction of a non-vanishing curvature on it:

$$R^A = \mathrm{d}\mu^A + \frac{1}{2} C_{BC}^{A} \mu^B \wedge \mu^C \tag{2.46}$$

The Chern–Simons $(2n+1)$-form, denoted by Q_{2n+1} can be derived from the Chern–Pontryagin topological invariant density, which is a $(2n+2)$-form given by the symmetrized trace of its Lie valued curvature forms [52, 53], as follows

$$\chi_{2n+2}(\mu) = \mathrm{Str}\left(R^{n+1}\right), \tag{2.47}$$

where Str denotes the symmetrized trace that acts over the matrix representation of a basis of vectors of the corresponding Lie algebra, $R = R^A t_A$. In terms of the components R^A, the Chern–Pontryagin form can be written as

$$\chi_{2n+2}(\mu) = g_{A_1 \cdots A_{n+1}} R^{A_1} \wedge \cdots \wedge R^{A_{n+1}}, \tag{2.48}$$

where the coefficients $g_{A_1 \cdots A_{n+1}}$ are the components of the degree-$(n+1)$ invariant tensor in the chosen basis. By taking the exterior derivative of χ_{2n+2} and using the Bianchi identities, it is direct to prove that it is indeed a closed form. Moreover, by replacing the gauge transformation rule on eq. (2.48) and using the invariant tensor condition for Lie algebras, it is also direct to prove its gauge invariance.

Let us consider two independent gauge fields μ and $\bar{\mu}$ valued in the same Lie algebra G. We also introduce a third gauge field depending on them, given by $\mu_t = \bar{\mu} + tu$, with $u = \mu - \bar{\mu}$. $\bar{\mu}$ is known as homotopic gauge field, and it is interpolated from $\bar{\mu}$ to μ by the parameter $t \in [0, 1]$. In the same way, the homotopic two-form field-strength associated to μ_t, denoted and given by

$$R_t = \bar{R} + t\bar{\nabla}u + \frac{t^2}{2}\left[u, u\right], \tag{2.49}$$

is interpolated between $\bar{R}$ and R as the parameter changes between 0 and 1. A useful property of the homotopic field-strength is that its derivative along the homotopic parameter is given by a total covariant homotopic derivative, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}R_t = \nabla_t u. \tag{2.50}$$

On the other hand, the difference between the Chern–Pontryagin invariant densities corresponding to both gauge fields can be written in a convenient way by using Stoke's theorem and the homotopic gauge field

$$\chi_{2n+2}\left(\mu\right) - \chi_{2n+2}\left(\bar{\mu}\right) = \int_0^1 \mathrm{d}t\ \frac{\mathrm{d}}{\mathrm{d}t}\chi_{2n+2}\left(\mu_t\right). \tag{2.51}$$

The invariant density $\chi_{2n+2}\left(\mu_t\right)$ is constructed with the homotopic gauge curvatures. By directly applying the derivative along the homotopic parameter and using eq. (2.50), it is possible to prove that the difference in the l.h.s. of eq. (2.51) is given by the following total derivative

$$\chi_{2n+2}\left(\mu\right) - \chi_{2n+2}\left(\bar{\mu}\right) = \mathrm{d}Q_{2n+1}\left(\bar{\mu}, \mu\right). \tag{2.52}$$

The $(2n+1)$-form inside the exterior derivative at the r.h.s. of eq. (2.52) depends on both gauge fields μ and $\bar{\mu}$. It is globally gauge invariant, known as transgression form, and explicitly given by the following expression:

$$Q_{2n+1}\left(\mu, \bar{\mu}\right) = (n+1)\int_0^1 \mathrm{d}t\ g_{A_1 \cdots A_{n+1}} u^{A_1} R_t^{A_2} \wedge \cdots \wedge R_t^{A_{n+1}}. \tag{2.53}$$

The Chern–Simons form can be obtained from eq. (2.53) by locally setting $\bar{\mu} = 0$, namely

$$\begin{aligned}Q_{2n+1}(\mu) &\equiv Q_{2n+1}(\mu,0)\\ &=(n+1)\int_0^1 \mathrm{d}t\; g_{A_1\cdots A_{n+1}}\mu^{A_1}R_t^{A_2}\wedge\cdots\wedge R_t^{A_{n+1}}.\end{aligned}\tag{2.54}$$

This condition cannot be globally fixed, and therefore, the Chern–Simons form is only locally defined. The homotopy rule is also reduced to a particular case in which $\mu_t = t\mu$ and $R_t = tR + \left(t^2 - t\right)\mu^2$. Notice that R_t and μ_t take values between 0 and μ in the first case and 0 and R in the latter. An important feature of transgression forms is that they are invariant under gauge transformations. The Chern–Simons form inherits such gauge-invariance only partially because of the non-covariance of the setting $\bar{\mu} = 0$. For this reason, action principles that use transgression forms as Lagrangian densities are fully gauge invariant, but the Chern–Simons action principles are invariant only up to boundary terms [25, 26]. For recent examples of the use of Chern–Simons and transgression forms in the construction of gauge theories for gravity, see refs. [54, 55]. The relation between the Chern–Pontryagin invariant density and transgression (and Chern–Simons) forms is a celebrated result known as the Chern–Weil theorem.

When using a transgression form as Lagrangian density, the transgression form in eq. (2.53) has, in principle, all the necessary information about the theory. However, in practice, we work with well-determined gauge groups containing different subgroups, each one having a clear physical meaning. For this reason, it is helpful to split the transgression form $Q_{2n+1}(\mu,\bar{\mu})$ into parts that explicitly reflect such subgroup structure. This split describes the relation between the Chern–Simons and transgression forms: one way to see this relation is to consider the Chern–Weil theorem. From eq. (2.52) it is straightforward to check the following identity

$$\mathrm{d}Q_{2n+1}(\mu,\bar{\mu}) + \mathrm{d}Q_{2n+1}(\tilde{\mu},\mu) + \mathrm{d}Q_{2n+1}(\bar{\mu},\tilde{\mu}) = 0,\tag{2.55}$$

where μ, $\bar{\mu}$ and $\tilde{\mu}$ are three independent one-form connections valued in the same Lie algebra. From Poincaré's lemma we find that the sum of the three transgression forms in eq. (2.55) can be locally written as a total derivative:

$$Q_{2n+1}(\mu,\bar{\mu}) + Q_{2n+1}(\tilde{\mu},\mu) + Q_{2n+1}(\bar{\mu},\tilde{\mu}) = -\mathrm{d}Q_{2n}(\tilde{\mu},\mu,\bar{\mu}),\tag{2.56}$$

where $Q_{2n}(\tilde{\mu},\mu,\bar{\mu})$ is a $2n$-form that depends on the three connections. However, it is not possible to determine the explicit form of $Q_{2n}(\tilde{\mu},\mu,\bar{\mu})$ using only the Chern–Weil theorem. Eq. (2.56) is known as the triangle equation and can be more conveniently written in the following way

$$Q_{2n+1}\left(\tilde{\mu},\bar{\mu}\right)=Q_{2n+1}\left(\mu,\bar{\mu}\right)+Q_{2n+1}\left(\tilde{\mu},\mu\right)+\mathrm{d}Q_{2n}\left(\tilde{\mu},\mu,\bar{\mu}\right), \tag{2.57}$$

which allows us to understand $Q_{2n+1}\left(\tilde{\mu},\bar{\mu}\right)$ as a transgression form that interpolates between $\tilde{\mu}$ and $\bar{\mu}$ and, in consequence, this differential form can be written as the sum of two transgression forms by introducing an auxiliary one-form μ and a total derivative. It is interesting to note that μ is completely arbitrary and can be conveniently chosen. The mathematical foundation on which the previous result rests is given in the extended Cartan homotopy formula. Such formula shows that the triangle equation and the Chern–Weil theorem have a common origin and allows to obtain an explicit expression for the $2n$-form $Q_{2n}\left(\tilde{\mu},\mu,\bar{\mu}\right)$.

2.6 Triangle equation

In 1985 J. Mañes, R. Stora and B. Zumino [39] showed that the Chern–Weil theorem corresponds to a special case of the extended Cartan homotopy formula (ECHF) [39]. To analyze this, let us consider the following elements. Let $\left\{\mu_i^A\right\}_{i=0}^{r+1}$ be a set of one-form gauge connections on a d-dimensional bundle based on a manifold M. Let also T_{r+1} be an oriented $(r+1)$-dimensional simplex, parameterized by the set $\left\{t^i\right\}_{i=0}^{r+1}$, where the parameters t^i satisfy

$$t^i \in [0,1], \tag{2.58}$$

$$\sum_{i=0}^{r+1} t^i = 1. \tag{2.59}$$

Eq. (2.59) implies that the linear combination

$$\mu_t=\sum_{i=0}^{r+1} t^i \mu_i, \tag{2.60}$$

transforms as a gauge connection in the same way as each μ_i does. It is possible to consider each μ_i as an element associated with the i-th vertex of the simplex T_{r+1}, so that it can be denoted as $T_{r+1}=\left(\mu_0,\ldots,\mu_{r+1}\right)$. The exterior derivatives on M and T_{r+1} are denoted by d and d_t. We will also consider an antiderivative operator l_t, which increases the degree in $\mathrm{d}t$ and decreases the degree in $\mathrm{d}x$; that is

$$l_t:\Lambda^p\left(M\right)\times\Lambda^q\left(T_{r+1}\right)\longrightarrow\Lambda^{p-1}\left(M\right)\times\Lambda^{q+1}\left(T_{r+1}\right), \tag{2.61}$$

and satisfies Leibniz rule just as d and d_t do. This operator is defined so that it constitutes a graded algebra together with d and d_t

$$\mathrm{d}^2 = \mathrm{d}_t^2 = \{\mathrm{d}, \mathrm{d}_t\} = 0, \tag{2.62}$$

$$[l_t, \mathrm{d}] = \mathrm{d}_t, \tag{2.63}$$

$$[\mathrm{d}_t, l_t] = 0. \tag{2.64}$$

The action of l_t on the algebra of polynomials generated by $\{\mu_t, R_t\}$ is defined in such a way that the algebra from eqs. (2.62)-(2.64) is satisfied and the algebra of polynomials is stable under the application of the operators d, d_t and l_t. In summary, the operators d, d_t and l_t change the degree of an (r,s)-form in $\left(\mathrm{d}x^\mu, \mathrm{d}t^i\right)$ as follows

$$(r,s) \xrightarrow{\mathrm{d}} (r+1, s)\,, \tag{2.65}$$

$$(r,s) \xrightarrow{\mathrm{d}_t} (r, s+1)\,, \tag{2.66}$$

$$(r,s) \xrightarrow{l_t} (r-1, s+1)\,. \tag{2.67}$$

Moreover, the action of the exterior derivatives d and d_t on the elements of the algebra of polynomials is defined in the usual way, according to eq. (2.10), while the only choice for l_t that satisfies the algebra (2.62)-(2.64) and keeps the algebra of polynomials closed is given by

$$l_t \mu_t = 0, \tag{2.68}$$

$$l_t R_t = \mathrm{d}_t \mu_t. \tag{2.69}$$

Let now π be an arbitrary $(m+q)$-form given by an arbitrary polynomial in the forms $\{\mu_t, R_t\}$, such that m and q are its differential degree in $\mathrm{d}x^\mu$ and $\mathrm{d}t$ respectively. Using the algebra from eqs. (2.62)-(2.64) we have [39]

$$(p+1)\,\mathrm{d}_t l_t^p \pi = \left[l_t^{p+1}, \mathrm{d}\right]\pi, \tag{2.70}$$

where, since the operator l_t decreases the degree of the differential form in M, for consistency one gets $m \geq p$. Integrating eq. (2.70) over T_{r+1} we have

$$\frac{1}{p!}\int_{T_{r+1}} \mathrm{d}_t l_t^p \pi = \frac{1}{(p+1)!}\int_{T_{r+1}} \left[l_t^{p+1}, \mathrm{d}\right]\pi. \tag{2.71}$$

By applying Stokes' theorem on the simplex, we can directly integrate the left side of the eq. (2.71), obtaining

$$\frac{1}{p!}\int_{\partial T_{r+1}} l_t^p \pi = \frac{1}{(p+1)!}\int_{T_{r+1}} \left(l_t^{p+1}\mathrm{d}\pi - \mathrm{d}l_t^{p+1}\pi\right). \tag{2.72}$$

Note that the quantity $l_t^{p+1}\pi$ must be of differential degree $(m-p)$ in $\mathrm{d}x$ and $(q+p+1)$ in $\mathrm{d}t$. Denoting $r=q+p$, we have that $l_t^{p+1}\pi$ is a $(r+1)$-form over T_{r+1}, therefore eq. (2.72) takes the form

$$\frac{1}{p!}\int_{\partial T_{r+1}} l_t^p \pi = \frac{1}{(p+1)!}\int_{T_{r+1}} l_t^{p+1}\mathrm{d}\pi - \frac{(-1)^{r+1}}{(p+1)!}\mathrm{d}\int_{T_{r+1}} l_t^{p+1}\pi. \tag{2.73}$$

Eq. (2.73) is known as ECHF. If we consider that π is given by the $2n$-form $\langle R_t^n\rangle$ with degree zero in $\mathrm{d}t$, we have $q=0$ and $p=r$. Thus, this formula is reduced to

$$\frac{1}{p!}\int_{\partial T_{p+1}} l_t^p \langle R_t^n\rangle = \frac{(-1)^p}{(p+1)!}\mathrm{d}\int_{T_{p+1}} l_t^{p+1}\langle R_t^n\rangle . \tag{2.74}$$

The study of eq. (2.74) for particular cases leads to interesting results:

- For $p=0$, the simplex is parametrized by one parameter t. In this case, the formula reproduces the Chern–Weil theorem from eqs. (2.51) and (2.52).
- For $p=1$, the simplex is parametrized by two parameters t^0 and t^1. The homotopic gauge field μ_t depends on them according $\mu_t=\bar{\mu}+t^0(\mu-\bar{\mu})+t^1(\tilde{\mu}-\mu)$. The resulting formula reproduces the triangle equation from eq. (2.56) and provides an explicit expression for the $2n$-form $Q_{2n}(\tilde{\mu},\mu,\bar{\mu})$ at the r.h.s of eq. (2.57), given by

$$Q_{2n}(\tilde{\mu},\mu,\bar{\mu}) = n(n+1)\int_0^1 \mathrm{d}t^0\int_0^{t^0}\mathrm{d}t^1\left\langle R_t^{n-1}(\tilde{\mu}-\mu)(\mu-\bar{\mu})\right\rangle , \tag{2.75}$$

  where the homotopic curvature two-form R_t is, as usual, defined from μ_t and, hence, depends on both homotopic parameters. For details on these particular cases, see refs. [49, 56–58].

Chapter 3

Free differential algebras

FDAs are helpful when describing symmetries in theories that intrinsically contain higher degree tensors in the field content. They were introduced in theoretical physics in the context of the formulation of supergravity theories in six or more dimensions, where the supersymmetry requirement of an equal number of bosonic and fermionic degrees of freedom leads to the introduction of new gauge fields given by higher degree tensors. An example can be found in the formulation of eleven-dimensional supergravity, where the field content includes the vielbein field, the spin connection, and the spin 3/2 gravitino field. The algebraic structure that allows interpreting such field content as a multiplet is the eleven-dimensional Poincaré superalgebra, being each field associated to a symmetry generator; specifically the generators of translations, rotations, and supersymmetry transformations, respectively. However, the counting of the corresponding degrees of freedom shows a deficit of 84 bosonic degrees of freedom, leading to the introduction of a new bosonic three-form, whose number of degrees of freedom is exactly 84. Therefore, the standard eleven-dimensional supergravity multiplet (or CJS supergravity) is described by the aforementioned field content, in addition to a bosonic three-form that cannot be interpreted as the gauge field associated to a symmetry generator of a Lie algebra. Thus, the resulting algebraic structure turns out to be a FDA for a composite one-form and a bosonic three-form that is non-trivially included as an extension of Poincaré Lie superalgebra by means of a non-trivial cocycle [7, 59–63].

In chapter 5, we will formulate a higher degree gauge theory. For this, we need to introduce the mathematical tools regarding FDAs: in this chapter, we will review the mathematical formulation of FDAs with particular emphasis on the simplest non-trivial case; this is, a differential algebra that includes only a one-form and a higher-degree form.

3.1 Generalized Maurer–Cartan equations

As it was mentioned in chapter 2, a Lie algebra can be described in two equivalent ways. The standard way consists of a vector space of contravariant vectors endowed with a bilinear antisymmetric product, while the so-called dual formulation consists of a set of differential equations for the covariant dual vectors (or left-invariant one-forms). The dual formulation has the advantage that it can be naturally extended to the case of p-forms.

Let M be an arbitrary N-dimensional manifold and let $\left\{\theta^{A(p)}\right\}_{p=1}^{N}$ be a basis of exterior forms defined on M. The basis elements are labeled by a set of indices $A\left(p\right)$ where p labels the degree of the differential form $\theta^{A(p)}$. Notice that if $p \neq q$, the indices $A\left(p\right)$ and $A\left(q\right)$ run on different domains and they cannot be contracted. Since $\left\{\theta^{A(p)}\right\}_{p=1}^{N}$ is a basis of differential forms, the exterior derivative $\mathrm{d}\theta^{A(p)}$ can be expressed in terms of their elements. This immediately allows to write down a generalized Maurer–Cartan equation [4, 7, 64]:

$$\mathrm{d}\theta^{A(p)} + \sum_{n=1}^{N+1} \frac{1}{n} C^{A(p)}_{B_1(p_1)\cdots B_n(p_n)} \theta^{B_1(p_1)} \wedge \cdots \wedge \theta^{B_n(p_n)} = 0. \tag{3.1}$$

At this point, there are no restrictions on the coefficients $C^{A(p)}_{B_1(p_1)\cdots B_n(p_n)}$, however, they must be reduced to the structure constants of a Lie algebra in the simplest case. A factor $1/n$ is introduced for later convenience. These coefficients are called generalized structure constants, and their symmetry in the lower indices is induced by the permutation of the forms in the wedge product from eq. (3.1). Notice that, there is a $(p+1)$-form at the l.h.s. of eq. (3.1) and therefore, the generalized structure constants are non-zero only if $p_1 + \cdots + p_n = p+1$. In the dual formulation of Lie algebras, demanding the structure constants to verify the Jacobi identity is equivalent to demanding the exterior derivative operator to be nilpotent, and therefore, that the differential calculus to be well defined. In the same way, eq. (3.1) is self-consistent only if the generalized structure constants are such that the second derivative $\mathrm{d}^2\theta^{A(p)}$ vanishes identically for every basis element. By directly applying the exterior derivative in eq (3.1), one finds the following condition

$$\sum_{n,m=1}^{N+1} \frac{1}{m} C^{A(p)}_{B_1(p_1)\cdots B_n(p_n)} C^{B_1(p_1)}_{D_1(q_1)\cdots D_m(q_m)} \theta^{D_1(q_1)} \wedge \cdots \wedge \theta^{D_m(q_m)} \wedge \theta^{B_2(p_2)} \wedge \cdots \wedge \theta^{B_n(p_n)} = 0. \tag{3.2}$$

Eq. (3.2) constitutes the generalized Jacobi identity for FDAs and, together with eq. (3.1), defines the algebraic structure known as free differential algebra. As it happens with Lie algebras, the generalized Jacobi identity can be written in terms of the structure constants, removing the basis of p-forms and adding the corresponding antisymmetrization. However, it is convenient to express it as in eq. (3.2) and remove the differential forms when studying particular examples case by case.

3.2 FDAs and Lie algebras

FDAs were first introduced in physics as extensions of Lie algebras by a well-defined procedure. Such procedure, extensively studied in refs. [4, 7, 64], make use of the non-trivial cocycles of a Lie algebra, representatives of its Chevalley–Eilenberg cohomology classes to provide new structure to its Maurer–Cartan equations.

Let us consider a Lie algebra G with a basis of vectors $\{t_A\}_{A=1}^{\dim G}$ satisfying eq. (2.15), and its corresponding dual basis of one-forms $\left\{\omega^A\right\}_{A=1}^{\dim G}$ satisfying the Maurer–Cartan equations from eq. (2.21). Let us also consider the spectrum of irreducible n-dimensional matrix representations $D^{(n)}\left(t_A\right)^i{}_j$ of G and the following non-trivial $p+1$-form cocycle

$$\Omega^i_{n,p+1} = \Omega^i_{A_1\cdots A_{p+1}} \omega^{A_1} \wedge \cdots \wedge \omega^{A_{p+1}}. \tag{3.3}$$

Since the $p+1$-form $\Omega^i_{n,p+1}$ is a cocycle, it is covariantly closed. Moreover, the non-triviality implies the non-existence of a p-cochain Φ^i such that $\Omega^i_{n,p+1} = \nabla^{(n)} \Phi^i$. For each cocycle in this representation, representative of a Chevalley–Eilenberg cohomology class, there is a possible extension of the Lie algebra of eq. (2.21) to a non-trivial FDA. Indeed, given the $(p+1)$-cocycle from eq. (3.3), we can introduce a new p-form $A^i_{n,p}$ in the same representation and write the following generalized Maurer–Cartan equation

$$\nabla^{(n)} A^i_{n,p} + \Omega^i_{n,p+1} = 0, \tag{3.4}$$

where $\nabla^{(n)}$ is the covariant derivative, as it was defined in eq. (2.29) in the mentioned representation. Eqs. (2.21) and (3.4) constitute a new FDA for the differential forms

$\theta^{A(1)} = \omega^A$ and $\theta^{A(p)} = A^i_{n,p}$. Eq. (3.4) guarantees that the operator $\nabla^{(n)}$ is still nilpotent in the extended space, and therefore, we can still talk about cocycles and cohomology classes for the new structure.

At this point, we can repeat the process by using eq. (3.4) and start over. By recalling the definition of covariant derivative in this representation, we can consider the construction of new non-trivial cocycles using the basis elements ω^A and $A^i_{n,p}$

$$\Omega^{i'}_{n',p'+1}(\omega, A) = \Omega^i_{A_1\cdots A_r i_1 \cdots i_s}\omega^{A_1}\wedge\cdots\wedge\omega^{A_r}\wedge A^{i_1}_{n,p}\wedge\cdots\wedge A^{i_s}_{n,p}. \tag{3.5}$$

Notice that, at the r.h.s. of eq. (3.5) is a $(r+sp)$-form. The new cocycle $\Omega^{i'}_{n',p'+1}(\omega, A)$ is therefore a $(p'+1)$-form with $p' = r+sp-1$ in a n'-dimensional representation with index i'. It satisfies closure and it is non-exact under the covariant derivative of the corresponding representation, denoted by $\nabla^{(n')}$. In this way, each cohomology class corresponds to a new extension of the FDA that can be found by introducing new differential forms. This FDA is then determined by the following Maurer–Cartan equations

$$\begin{aligned}
\mathrm{d}\omega^A + \frac{1}{2}C^A{}_{BC}\omega^B\wedge\omega^C &= 0, &(3.6)\\
\nabla^{(n)}A^i_{n,p} + \Omega^i_{n,p+1}(\omega) &= 0, &(3.7)\\
\nabla^{(n')}A^{i'}_{n',p'} + \Omega^{i'}_{n',p'+1}(\omega, A) &= 0. &(3.8)
\end{aligned}$$

The repetition of this procedure allows obtaining the most general FDA for a given Lie algebra G.

3.3 Gauging free differential algebras

As it happens with Lie algebras, physical applications of FDAs require a gauging of the Maurer–Cartan equations [4,7]. This means to consider a deformation of the FDA manifold, on which the basis of differential forms $\left\{\theta^{A(p)}\right\}$ satisfies a modified version of the generalized Maurer–Cartan equations:

$$R^{A(p)} = \mathrm{d}\theta^{A(p)} + \sum_{n=1}^{N}\frac{1}{n}C^{A(p)}_{B_1(p_1)\cdots B_n(p_n)}\theta^{B_1(p_1)}\wedge\cdots\wedge\theta^{B_n(p_n)}. \tag{3.9}$$

The forms $R^{A(p)}$ are called $(p+1)$-curvatures. Notice that, for $p=1$, eq. (3.10) reproduces the definition of curvature from eq. (2.23). By directly applying the exterior derivative operator to eq. (3.9) it is straightforward to find the following generalized Bianchi identity [7]

$$\mathrm{d}R^{A(p)} + \sum_{n=1}^{N} C^{A(p)}_{B_1(p_1)\cdots B_n(p_n)} R^{B_1(p_1)} \wedge \Theta^{B_2(p_2)} \wedge \cdots \wedge \Theta^{B_n(p_n)} = 0. \tag{3.10}$$

Eq. (3.10) turns out to be particularly useful by inducing a notion of covariant derivative for FDAs. In this way, the covariant derivative can be defined such that eq. (3.10) is equivalent to $\nabla R = 0$.

3.4 The FDA1 algebra

Let us consider a particular case of FDA, known in the literature as a FDA1 [65,66]. This FDA has only two collections of differential forms as basis, namely the one-forms μ^A and the p-forms B^i. This means that, in eq. (3.9), we fix $\theta^{A(1)} = \mu^A$ and $\theta^{A(p)} = B^i$. We will study the 'flat'-case i.e., the non-gauged algebra, and continue by introducing a non-vanishing curvature later. For convenience, we will explicitly mention which of these two cases we are dealing with and indistinctly denote the basis of differential forms as μ^A and B^i in both cases.

By considering the given basis of differential forms, eq. (3.1) is reduced to the following set of Maurer–Cartan equations

$$\mathrm{d}\mu^A + \frac{1}{2} C^A_{BC} \mu^B \mu^C = R^A = 0, \tag{3.11}$$

$$\mathrm{d}B^i + C^i_{Aj} \mu^A B^j + \frac{1}{(p+1)!} C^i_{A_1\cdots A_{p+1}} \mu^{A_1} \cdots \mu^{A_{p+1}} = R^i = 0. \tag{3.12}$$

Eqs. (3.11) and (3.12) define a FDA1. Notice that we have considered that the structure constants of the type $C^{A(p)}_{B(p+1)}$ vanish. FDAs that share this property are called minimal algebras. The choice of FDA1 as a minimal algebra allows eq. (3.11) to reproduce the Maurer–Cartan equations of a Lie algebra. In consequence, the generalized Jacobi identity of eq. (3.2) is reduced to a set of three algebraic equations for the structure constants of a FDA1, as follows

$$C^A_{B[C} C^B_{DE]} = 0, \tag{3.13}$$

$$C^i_{Aj} C^j_{Bk} - C^i_{Bj} C^j_{Ak} - C^i_{Ck} C^C_{AB} = 0, \tag{3.14}$$

$$2C^i_{[A_1|j} C^j_{|A_2\cdots A_{p+2}]} - (p+1)\, C^i_{A_0[A_1\cdots A_p} C^{A_0}_{A_{p+1}A_{p+2}]} = 0. \tag{3.15}$$

Notice that, eq. (3.13) is the standard Jacobi identity of Lie algebras, showing that the one-forms μ^A describe a Lie subalgebra of the FDA1. Moreover, eq. (3.14) states the structure constants C^i_{Aj} as a representation of the Lie subalgebra. In fact, by defining the matrix operators $(t_A)^i{}_j \equiv C^i_{Aj}$, eq. (3.14) immediately becomes equivalent to the commutation relations of a Lie algebra. Lastly, the third Jacobi identity in eq. (3.15) is equivalent to demand that the standard covariant derivative of the structure constants $C^i_{A_1\cdots A_{p+1}}$ vanishes, showing that a FDA1 is only consistent if $C^i_{A_1\cdots A_{p+1}}$ defines a cocycle of the original Lie subalgebra.

As we have seen in chapter 2, in the study of Lie groups, the diffeomorphism transformations are given by Lie derivatives along all the possible directions on the group manifold. To study the diffeomorphism transformations in the case of a FDA1, it is therefore necessary to define a regular Lie derivative along the standard directions of the Lie subalgebra described by eq. (3.11), and an extended Lie derivative that determines the transformations along the extended directions. For this purpose, we define the generalized contraction operators i_{t_A} and i_{t_j} whose action on the one-form μ^A and the p-form B^i is given by [65, 66]

$$i_{t_A}\mu^B = \delta^B_A, \tag{3.16}$$
$$i_{t_A}B^j = 0, \tag{3.17}$$
$$i_{t_j}\mu^A = 0, \tag{3.18}$$
$$i_{t_j}B^i = \delta^i_j. \tag{3.19}$$

t_A denote the basis vectors, generators of the Lie algebra described by eq. (3.11) (Lie subalgebra of a FDA1). Moreover, we also have to introduce the corresponding basis vectors along the extended directions of the FDA manifold, t_j. The new vectors are dual to B^j in the same way in which t_A is dual to μ^A.

Let now ε^A be a zero-form and ε^i a $(p-1)$-form. As it happens with Lie groups, the Lie derivative operators along $\varepsilon^A t_A$ and $\varepsilon^i t_i$ are defined as the anticommutator of the corresponding contraction operators and the exterior derivatives:

$$\mathcal{L}_{\varepsilon^A t_A} = \mathrm{d}i_{\varepsilon^A t_A} + i_{\varepsilon^A t_A}\mathrm{d}, \tag{3.20}$$
$$\mathcal{L}_{\varepsilon^j t_j} = \mathrm{d}i_{\varepsilon^j t_j} + i_{\varepsilon^j t_j}\mathrm{d}, \tag{3.21}$$

with $i_{\varepsilon^A t_A} = \varepsilon^A i_{t_A}$ and $i_{\varepsilon^i t_i} = \varepsilon^i i_{t_i}$.

Let us now consider the gauging of a FDA1. This means to introduce non-vanishing curvatures R^A and R^i, explicitly given by eqs. (3.11) and (3.12). We split the components of the curvature in the chosen basis of one-forms and p-forms, and denote them as follows

$$R^A = R^A_{BC}\mu^B\mu^C + R^A_j B^j, \tag{3.22}$$
$$R^i = R^i_{A_1\cdots A_{p+1}}\mu^{A_1}\cdots\mu^{p+1} + R^i_{Aj}\mu^A B^j, \tag{3.23}$$

where the second term at the r.h.s. of eq. (3.22) is non-zero only for $p = 2$.

Let us now proceed to the application of the Lie derivatives on the gauge potentials μ^A and B^i along the standard zero-form parameter $\varepsilon^A t_A$. Such derivatives are explicitly given by:

$$\mathcal{L}_{\varepsilon^A t_A}\mu^A = \mathrm{d}\varepsilon^A + C^A_{BC}\mu^B\varepsilon^C + 2R^A_{BC}\varepsilon^B\mu^C, \tag{3.24}$$
$$\mathcal{L}_{\varepsilon^A t_A}B^i = \left(R^i_{Aj} - C^i_{Aj}\right)\varepsilon^A B^j + \left((p+1)\,R^i_{AA_1\cdots A_p} - \frac{1}{p!}C^i_{AA_1\cdots A_p}\right)\varepsilon^A\mu^{A_1}\cdots\mu^{A_p}. \tag{3.25}$$

On the other hand, the Lie derivative of the gauge fields along the extended $(p-1)$-form $\varepsilon^i t_i$ parameter is given by

$$\mathcal{L}_{\varepsilon^j t_j}\mu^A = \varepsilon^j R^A_j, \tag{3.26}$$
$$\mathcal{L}_{\varepsilon^j t_j}B^i = \mathrm{d}\varepsilon^j + C^i_{Aj}\mu^A\varepsilon^j - R^i_{Aj}\mu^A\varepsilon^j. \tag{3.27}$$

Eqs. (3.24)-(3.27) contain the complete set of diffeomorphism transformations along all the independent directions of the FDA manifold, which can be summarized as follows [65, 66]

$$\delta\mu^A = \mathrm{d}\varepsilon^A + C^A_{BC}\mu^B\varepsilon^C + 2R^A_{BC}\varepsilon^B\mu^C + \varepsilon^j R^A_j, \tag{3.28}$$
$$\begin{aligned}\delta B^i &= \mathrm{d}\varepsilon^j + C^i_{Aj}\mu^A\varepsilon^j - R^i_{Aj}\mu^A\varepsilon^j + \left(R^i_{Aj} - C^i_{Aj}\right)\varepsilon^A B^j \\ &+ \left((p+1)\,R^i_{AA_1\cdots A_p} - \frac{1}{p!}C^i_{AA_1\cdots A_p}\right)\varepsilon^A\mu^{A_1}\cdots\mu^{A_p}.\end{aligned} \tag{3.29}$$

Notice that the diffeomorphism transformations depend on the standard and extended components of the curvature. The imposition of certain horizontality conditions of the curvatures in some directions of the FDA-manifold leads to the transformations of eqs. (3.28) and (3.29) becoming gauge transformations in an analogous way to what happens in the case of Lie algebras (for details see refs. [65, 66]). Indeed, by imposing the so-called 'FDA1 horizontality conditions', given by

$$i_{\varepsilon^B t_B}\left(R^A_{CD}\mu^C\mu^D\right) = 0, \tag{3.30}$$
$$i_{\varepsilon^i t_i}\left(R^A_j B^j\right) = 0, \tag{3.31}$$

$$i_{\varepsilon^B t_B}\left(R^j_{Ak}\mu^A B^k\right) = 0, \tag{3.32}$$
$$i_{\varepsilon^i t_i}\left(R^j_{Ak}\mu^A B^k\right) = 0, \tag{3.33}$$
$$i_{\varepsilon^B t_B}\left(R^i_{A_1\cdots A_{p+1}}\mu^{A_1}\cdots\mu^{A_{p+1}}\right) = 0, \tag{3.34}$$

for arbitrary parameters ε^A and ε^i, the diffeomorphism transformations from eqs. (3.28) and (3.29) become gauge transformations, namely

$$\delta\mu^A = \mathrm{d}\varepsilon^A + C^A_{BC}\mu^B\varepsilon^C, \tag{3.35}$$
$$\delta B^i = \mathrm{d}\varepsilon^i + C^i_{Aj}\mu^A\varepsilon^j - C^i_{Aj}\varepsilon^A B^j - \frac{1}{p!}C^i_{A_1\cdots A_{p+1}}\varepsilon^{A_1}\mu^{A_2}\cdots\mu^{A_{p+1}}. \tag{3.36}$$

Eq. (3.35) corresponds to the usual Lie covariant derivative of the zero-form parameter. The second equation is the natural extension to the case of a p-form. Notice that the transformation of μ^A is the same that in ordinary groups and depends only on ε^A. However, the transformation of B^i depends on both parameters, ε^A and ε^i [67,68].

As we have seen in eqs. (3.22) and (3.23), the curvature forms admit the splitting $R^A = R^A_1 + R^A_2$, $R^i = R^i_1 + R^i_2$, with

$$R^A_1 = R^A_{BC}\mu^B\mu^C, \tag{3.37}$$
$$R^A_2 = R^A_j B^j, \tag{3.38}$$
$$R^i_1 = R^i_{A_1\cdots A_{p+1}}\mu^{A_1}\cdots\mu^{A_{p+1}}, \tag{3.39}$$
$$R^i_2 = R^i_{Aj}\mu^A B^j. \tag{3.40}$$

Using this notation, the FDA1 horizontality conditions take the form

$$i_{\varepsilon^B t_B}R^A_1 = i_{\varepsilon^B t_B}R^i_1 = i_{\varepsilon^B t_B}R^i_2 = 0, \tag{3.41}$$
$$i_{\varepsilon^j t_j}R^A_2 = i_{\varepsilon^j t_j}R^i_1 = 0. \tag{3.42}$$

Moreover, by defining the contraction operator along the composite parameter ε as $i_\varepsilon = \varepsilon^A i_{t_A} + \varepsilon^i i_{t_i}$, eqs. (3.41) and (3.42) can be written in a more convenient way:

$$i_\varepsilon R^A = 0, \tag{3.43}$$
$$i_{\varepsilon^A t_A}R^i = 0. \tag{3.44}$$

As it happens with the generalized Jacobi identity, eqs. (3.35) and (3.36) induce a definition of covariant derivative on differential forms of degree zero and $p-1$. In this way, it is possible to write the variation of the gauge fields as the covariant derivative of the corresponding gauge parameters.

3.4.1 Covariant derivative

In order to construct an invariant gauge theory under a FDA, it is necessary to define a covariant derivative that involves all the components of the gauge field. For this purpose, we will use the transformation law for the curvatures and the Bianchi identities.

From eqs. (3.35) and (3.36), it is straightforward to find the following gauge transformation law for the components of the curvature [67, 68]

$$\delta R^A = C^A_{BC} R^B \varepsilon^C , \tag{3.45}$$

$$\delta R^i = C^i_{Aj} R^A \varepsilon^j - C^i_{Aj} \varepsilon^A R^j - \frac{1}{(p-1)!} C^i_{AA_1 \cdots A_p} \varepsilon^A R^{A_1} \mu^{A_2} \cdots \mu^{A_p} . \tag{3.46}$$

Notice that eq. (3.45) corresponds to the usual Lie bracket between the two-form curvature and the zero-form parameter, written in components for the chosen basis and involving only the structure constants of the Lie subalgebra of the FDA1. On the other hand, eq. (3.46) corresponds to the natural generalization of the Lie product between the curvature and the parameter and involves both components, standard and extended, of each one. Moreover, the extended component R^i transforms homogeneously, i.e. not depending on the derivatives of the parameters, such as R^A.

On the other hand, let us now consider the second source of information regarding the covariant derivative, namely, the Bianchi identity. By imposing the conditions of FDA1 in eq. (3.10), we obtain the Bianchi identities involving the curvatures R^A and R^i

$$\mathrm{d}R^A - C^A_{BC} R^B \mu^C = 0, \tag{3.47}$$

$$\mathrm{d}R^i + C^i_{Aj} \mu^A R^j - C^i_{Aj} R^A B^j - \frac{1}{p!} C^i_{A_1 \cdots A_{p+1}} R^{A_1} \mu^{A_2} \cdots \mu^{A_{p+1}} = 0. \tag{3.48}$$

As it happens with the gauge transformation laws, eq. (3.45) reproduces the Bianchi identity regarding R^A, which is the same that appears for Lie algebras. It turns out to be the statement that the Lie covariant derivative of the two-form curvature vanishes. This is a natural consequence of the fact that eq. (3.11) is the Maurer–Cartan equation for a Lie algebra. Furthermore, eq. (3.48) generalizes the definition of covariant derivative for a $(p+1)$-form.

In summary, eqs. (3.35) and (3.36) provide a definition of the covariant derivative of a vector of the FDA1 ε whose components are $\left(\varepsilon^A, \varepsilon^i\right)$, being ε^A a zero-form and ε^i a $(p-1)$-form. The covariant derivative of ε is therefore also a vector given by $\nabla\varepsilon = \left((\nabla\varepsilon)^A, (\nabla\varepsilon)^i\right)$, whose components are explicitly given by

$$(\nabla\varepsilon)^A = \mathrm{d}\varepsilon^A + C^A_{BC}\mu^B\varepsilon^C, \tag{3.49}$$

$$(\nabla\varepsilon)^i = \mathrm{d}\varepsilon^j + C^i_{Aj}\mu^A\varepsilon^j - C^i_{Aj}\varepsilon^A B^j - \frac{1}{p!}C^i_{AA_1\cdots A_p}\varepsilon^A\mu^{A_1}\cdots\mu^{A_p}. \tag{3.50}$$

Moreover, eqs. (3.47) and (3.48) provide a definition of covariant derivative of a vector of the type $R = \left(R^A, R^i\right)$, being R^A a two-form and R^i a $(p+1)$-form. The components of the covariant derivative of R, denoted by $\nabla R = \left((\nabla R)^A, (\nabla R)^i\right)$ are therefore given by

$$(\nabla R)^A = \mathrm{d}R^A + C^A_{BC}\mu^B R^C, \tag{3.51}$$

$$(\nabla R)^i = \mathrm{d}R^i + C^i_{Aj}\mu^A R^j - C^i_{Aj}R^A B^j - \frac{1}{p!}C^i_{A_1\cdots A_{p+1}}R^{A_1}\mu^{A_2}\cdots\mu^{A_{p+1}}. \tag{3.52}$$

These results can be understood as the equations that define the covariant derivative of every set of differential forms $x = \left(x^A, x^i\right)$ in the (A, i)-representation of the FDA1, where the index A corresponds to the adjoint representation of the Lie algebra with which the FDA1 was constructed, while the index i corresponds to the irreducible representation of the corresponding cocycle with which the algebra was constructed. However, directly generalizing eqs. (3.50) and (3.52) to any vector of the FDA1 leads to the following problems:

- The covariant derivative defined in this way does not satisfy a homogeneity condition, i.e., the second covariant derivative $\nabla^2 x$ depends on the degree-one derivatives of x. Such homogeneity condition holds only for differential forms of even degree.
- The gauge variation of the $(p+1)$-form curvature cannot be written as $\delta R^i = \nabla\delta B^i$, not even reproducing the original results for Lie algebras.

To solve these problems, we must define a covariant derivative depending on the degree of the differential forms on which the operator is applied. Let us consider a vector $x = \left(x^A, x^i\right)$ in the (A, i)-representation of the FDA1. The components of x, denoted as x^A and x^i are q-forms and $(p+q-1)$-forms, respectively. We define the FDA1-covariant derivative of x as a vector $\nabla x = \left((\nabla x)^A, (\nabla x)^i\right)$, whose components are given by

$$(\nabla x)^A = \mathrm{d}x^A + C^A_{BC}\mu^B x^C, \tag{3.53}$$

$$\begin{aligned}(\nabla x)^i =& \mathrm{d}x^i + C^i_{Aj}\left(\mu^A x^j - (-1)^{f(q)}\, x^A B^j\right) \\ &- \frac{(-1)^{g(q)}}{p!} C^i_{A_1\cdots A_{p+1}} x^{A_1}\mu^{A_2}\cdots\mu^{A_{p+1}}.\end{aligned} \tag{3.54}$$

The arbitrary scalar functions $f(q)$ and $g(q)$ are introduced for later consistency. From the particular cases given by the covariant derivatives of the curvatures and parameters in eqs. (3.50) and (3.52), it is direct to see that

$$(-1)^{f(0)} = (-1)^{f(2)} = (-1)^{g(0)} = (-1)^{g(2)} = 1. \tag{3.55}$$

Moreover, in order to verify the homogeneity condition, it is necessary to remove the dependency on the derivatives of x^A and x^i in the second covariant derivatives of x. Such requirement leads to the following conditions for $f(q)$ and $g(q)$

$$f(q+1) = f(q) + 1, \tag{3.56}$$

$$g(q+1) = g(q) + 1. \tag{3.57}$$

A valid solution to this equation system is given by $f(q) = g(q) = q$, which leads us to the following definition of covariant derivative

$$(\nabla x)^A = \mathrm{d}x^A + C^A_{BC}\mu^B x^C, \tag{3.58}$$

$$\begin{aligned}(\nabla x)^i =& \mathrm{d}x^i + C^i_{Aj}\left(\mu^A x^j - (-1)^q\, x^A B^j\right) \\ &- \frac{(-1)^q}{p!} C^i_{A_1\cdots A_{p+1}} x^{A_1}\mu^{A_2}\cdots\mu^{A_{p+1}}.\end{aligned} \tag{3.59}$$

Eqs. (3.58) and (3.59) reproduce eqs. (3.49)-(3.52), fulfill the requirements of homogeneity in the second covariant derivative and allow to write the gauge variations of the components of R as $\delta R^A = \nabla\delta\mu^A$ and $\delta R^i = \nabla\delta B^i$.

Chapter 4

L_∞ algebras and field theory

In this chapter, we review the mathematical structure known as L_∞ algebra. We consider an introduction in two pictures known as b-picture and ℓ-picture. Such pictures are equivalent and will be useful in this and following chapters.

L_∞ algebras are generalizations of Lie algebras defined as graded vector spaces endowed with a set of multilinear products whose symmetry rules depend on the degree of the subspaces on which they act [41, 69]. This set of multilinear products directly generalizes the bilinear products of Lie algebras. In this sense, Lie algebras are particular cases of L_∞ algebras in which the only non-vanishing product is the bilinear one, and the subspace structure of the vector space carries only one subspace, making the bilinear product always antisymmetric. The products of an L_∞ algebra satisfy an enlarged version of the Jacobi identity. It is interesting to notice that the presence of multilinear products, in general, allows the bilinear product not to satisfy the standard Jacobi identity. Consequently, when the bilinear product is defined as a commutator, the product with which such commutator is defined can be non-associative [70, 71].

4.1 L_∞ algebras in the b-picture

An L_∞ algebra is defined as a vector space $\bar{X}$ endowed with a set of products $\{b_k\}_{k\in\mathbb{N}}$ where:

- $\bar{X}$ is a graded vector space
$$\bar{X} = \bigoplus_{n\in\mathbb{Z}} \bar{X}_n. \tag{4.1}$$
Eq. (4.1) defines the gradation of $\bar{X}$, i.e., given an element $\bar{x} \in \bar{X}_n$ we denote $\deg \bar{x} = n$.

- $\{b_k\}_{k\in\mathbb{N}}$ is a set of k-linear products of degree -1 defined on $\bar{X}$, i.e., given an arbitrary set of elements $\bar{x}_1, \ldots, \bar{x}_k \in \bar{X}$, we have
$$\deg b_k\left(\bar{x}_1, \ldots, \bar{x}_k\right) = \deg \bar{x}_1 + \cdots + \deg \bar{x}_k - 1. \tag{4.2}$$

- The products are graded symmetric
$$b_k\left(\bar{x}_1, \ldots, \bar{x}_k\right) = \epsilon\left(\sigma, \bar{x}\right) b_k\left(\bar{x}_{\sigma(1)}, \ldots, \bar{x}_{\sigma(k)}\right), \tag{4.3}$$
where $\epsilon\left(\sigma, \bar{x}\right)$ is the Koszul sign defined by means of a graded symmetric product $\bar{x} \wedge \bar{y} = (-1)^{\deg \bar{x} \deg \bar{y}}\, y \wedge x$. It depends on the degree of the elements on $\bar{X}$ and the order of the permutation σ through the following relation
$$\bar{x}_1 \wedge \cdots \wedge \bar{x}_k = \epsilon\left(\sigma, \bar{x}\right) \bar{x}_{\sigma(1)} \wedge \cdots \wedge \bar{x}_{\sigma(k)}. \tag{4.4}$$

- The products b_k satisfy the so-called L_∞ identities in the b-picture
$$\sum_{\sigma\in U_n(i,j)} \epsilon\left(\sigma, \bar{x}\right) b_j\left(b_i\left(\bar{x}_{\sigma(1)}, \ldots, \bar{x}_{\sigma(i)}\right), \bar{x}_{\sigma(i+1)}, \ldots, \bar{x}_{\sigma(n)}\right) = 0, \tag{4.5}$$
with $n \geq 1$, $i+j = n+1$, and $U_n\left(i,j\right)$ being the set of unshuffled permutations of n elements that satisfy the following ordering relations
$$\sigma\left(1\right) < \cdots < \sigma\left(i\right), \tag{4.6}$$
$$\sigma\left(i+1\right) < \cdots < \sigma\left(n\right). \tag{4.7}$$

4.2 L_∞ algebras in the ℓ-picture

L_∞ algebras can be equivalently formulated in the so-called ℓ-picture. In this picture, an L_∞ algebra is defined as a pair $\left(X, \{\ell_k\}_{k\in\mathbb{N}}\right)$ where:

- X is a graded vector space
$$X = \bigoplus_{n\in\mathbb{Z}} X_n. \tag{4.8}$$

Eq. (4.8) defines the gradation of X, i.e., a vector $x \in X_n$ has degree n, $\deg x = n$.

- $\{\ell_k\}_{k\in\mathbb{N}}$ is a set of k-linear products of degree $k-2$ defined on X, i.e., for k vectors $x_1, \ldots, x_k \in X$, we have

$$\deg \ell_k\left(x_1, \ldots, x_k\right) = k - 2 + \deg x_1 + \cdots + \deg x_k. \tag{4.9}$$

- The products verify the following graded symmetry depending on the order of the permutation and the Koszul sign depending on the degrees of the vectors in the ℓ-picture

$$\ell_k\left(x_1, \ldots, x_k\right) = (-1)^\sigma \epsilon\left(\sigma, x\right) \ell_k\left(x_{\sigma(1)}, \ldots, x_{\sigma(k)}\right). \tag{4.10}$$

- The products ℓ_k satisfy the L_∞ identities in the ℓ-picture $(n \geq 1 :)$

$$\sum_{i+j=n+1} (-1)^{i(j-1)} \sum_{\sigma\in U_n(i,j)} (-1)^\sigma \epsilon\left(\sigma, x\right) \ell_j\left(\ell_i\left(x_{\sigma(1)}, \ldots, x_{\sigma(i)}\right), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}\right) = 0. \tag{4.11}$$

It is possible to define a mapping between both formulations. Let us consider an L_∞ algebra in the ℓ-picture. We define the suspension map s as follows

$$s : X_n \longrightarrow \bar{X}_{n+1}, \tag{4.12}$$

$$x \mapsto sx \equiv \bar{x}. \tag{4.13}$$

The operation s maps vectors from X into vectors on $\bar{X}$, such that, the image of a vector x of degree n in X is a vector $\bar{x}$ of degree $n+1$ in $\bar{X}$

$$\deg \bar{x} = \deg x + 1\,. \tag{4.14}$$

The product of an arbitrary number of vectors on X in the ℓ-picture is also mapped to the product of their corresponding images on $\bar{X}$ in the b-picture up to a -1 factor depending on the degree of the vectors, as follows [41]:

$$b_n(\bar{x}_1, \ldots \bar{x}_n) = (-1)^{(n-1)\deg x_1 + (n-2)\deg x_2 + \cdots + \deg x_{n-1}} s\ell_n(x_1, ..., x_n). \tag{4.15}$$

Notice that the relation from eq. (4.15) is written using the algebraic degrees of vectors in X. The mapping s is invertible. By introducing s^{-1} as

$$s^{-1} : \bar{X}_{n+1} \longrightarrow X_n, \tag{4.16}$$

$$\bar{x} \mapsto s^{-1}\bar{x} \equiv x, \tag{4.17}$$

we can immediately obtain the inverse relation for the mapping of products from $\bar{X}$ into X in terms of the degrees of $\bar{X}$

$$\ell_n(x_1, ..., x_n) = (-1)^{\frac{n(n-1)}{2}+(n-1)\deg \bar{x}_1+(n-2)\deg \bar{x}_2+\cdots+\deg x_{n-1}} s^{-1} b_n(\bar{x}_1, \ldots \bar{x}_n). \tag{4.18}$$

In the b-picture, the products always have algebraic degree -1. In contrast, in the ℓ-picture, the degrees of the products depend case by case. Such degrees correspond to different pictures, and therefore, the Koszul signs defined in terms of them $\epsilon\,(\sigma, x)$ and $\epsilon\,(\sigma, \bar{x})$ are different, even if they correspond to the same permutation σ.

The L_∞ identities from eq. (4.11) relate n vectors of X using products of different numbers of them in a single equation. Let us analyze the first cases:

Identity of one vector:

$$\ell_1\left(\ell_1\left(x\right)\right) = 0. \tag{4.19}$$

The first identity defines ℓ_1 as a nilpotent operator on the entire vector space X.

Identity of two vectors:

$$\ell_1\left(\ell_2\left(x_1, x_2\right)\right) = \ell_2\left(\ell_1\left(x_1\right), x_2\right) + (-1)^{\deg x_1} \ell_2\left(x_1, \ell_1\left(x_2\right)\right). \tag{4.20}$$

The second identity defines ℓ_1 as a derivation of ℓ_2.

Identity of three vectors:

$$\begin{aligned}&\ell_2\left(\ell_2\left(x_1, x_2\right), x_3\right) + (-1)^{x_1(x_2+x_3)} \ell_2\left(\ell_2\left(x_2, x_3\right), x_1\right) + (-1)^{x_1x_3+x_2x_3} \ell_2\left(\ell_2\left(x_3, x_1\right), x_2\right)\\ &= -\ell_3\left(\ell_1\left(x_1\right), x_2, x_3\right) - (-1)^{x_1} \ell_3\left(x_1, \ell_1\left(x_2\right), x_3\right) - (-1)^{x_1+x_2} \ell_3\left(x_1, x_2, \ell_1\left(x_3\right)\right)\\ &- \ell_1\left(\ell_3\left(x_1, x_2, x_3\right)\right).\end{aligned} \tag{4.21}$$

The l.h.s. of eq. (4.21) correspond to the Jacobiator of the bilinear product ℓ_2. Moreover, on the r.h.s, we identify the action of ℓ_1 on ℓ_3. If ℓ_1 is a derivation of ℓ_3, the r.h.s. of eq. (4.21) vanishes. Therefore, this identity states that the bilinear bracket is, in general, not a Lie bracket. The failure of its Jacobi identity is measured by the failure of ℓ_1 as a derivation on ℓ_3. Notice that in the absence of a three-linear product, the Jacobi identity is immediately satisfied by the bilinear product.

4.3 L_∞ algebras and FDAs

The dual relation between Lie algebras and Lie differential algebras can also be extended to FDAs. Through the introduction of a set of products acting on a dual basis of vectors, it is possible to find a dual relation between FDAs and L_∞ algebras. Let us introduce a graded vector space $\bar{X}$, and a basis for $\bar{X}$ denoted by $\left\{t_{A(p)}\right\}_{p=1}^{N}$. The gradation of the vectors in $\bar{X}$ is labeled according $\deg_{\bar{X}} t_{A(p)} = p$, and each index $A(p)$ runs over a different domain depending on the value of p, allowing each subspace to have different dimensionality. Let us also introduce a set of n-linear products ($n \geq 1$) acting on $\bar{X}$, denoted by

$$\left[t_{A_1(p_1)}, \ldots, t_{A_n(p_n)}\right]_n \in \bar{X}. \tag{4.22}$$

Since the product of n basis elements $\left[t_{A_1(p_1)}, \ldots, t_{A_n(p_n)}\right]_n$ is also an element of $\bar{X}$, it can be written in terms of the basis. We denote its components by $\left[t_{A_1(p_1)}, \ldots, t_{A_n(p_n)}\right]_n^{A(p)}$. Such components are chosen to be proportional to the structure constants of a FDA, as follows

$$\left[t_{A_1(p_1)}, \ldots, t_{A_n(p_n)}\right]_n^{A(p)} = (n-1)! C^{A(p)}_{A_1(p_1)\cdots A_n(p_n)}. \tag{4.23}$$

As a consequence, the products on $\bar{X}$ satisfy the same graded-symmetry relation than the structure constants of the FDA, which can be conveniently written as follows

$$\left[t_{A_{\sigma(1)}(p_{\sigma(1)})}, \ldots, t_{A_{\sigma(n)}(p_{\sigma(n)})}\right]_n = \epsilon(\sigma, T) \left[t_{A_1(p_1)}, \ldots, t_{A_n(p_n)}\right]_n, \tag{4.24}$$

where T denotes the gradation of the vectors in the argument of the products. Notice that, for $n = 2$, eq. (4.24) leads to the following rule for the bilinear product

$$\left[t_{A(r)}, t_{B(s)}\right]_2 = (-1)^{rs} \left[t_{B(s)}, t_{A(r)}\right]_2. \tag{4.25}$$

Since the new products are proportional to the generalized structure constants of a FDA, they inherit some of their properties. As we have seen, the generalized structure constants in eq. (4.24) are non-vanishing only if their indices satisfy $p_1 + \cdots + p_n = p + 1$. Consequently, a product $\left[t_{A_1(p_1)}, \ldots, t_{A_n(p_n)}\right]_n$ is an vector of the subspace $\bar{X}_p$. Moreover, by directly plugging in eq. (4.23) into eq. (3.2) it is possible to write down the generalized Jacobi identity in terms of the products in

$\bar{X}$, as follows

$$\sum_{m,n=1}^{N+1} \frac{1}{m!\,(n-1)!} \left[\left[t_{C_1(q_1)}, \ldots, t_{C_m(q_m)}\right]_m, t_{B_2(p_2)} \ldots, t_{B_n(p_n)}\right]_n^{A(p)}$$
$$\theta^{C_1(q_1)} \ldots \theta^{C_m(q_m)} \theta^{B_2(p_2)} \ldots \theta^{B_n(p_n)} = 0 \tag{4.26}$$

Here, the sum runs over the integer values of p_i and q_i satisfying $q_1 + \cdots + q_m = p_1 + 1$ and $p_1 + \cdots + p_n = p + 1$. Let us now remove the dependence of eq. (4.26) on the differential forms. To this end, we separate the terms that are wedge products of the same number of basis elements. Each term on the l.h.s. of eq. (4.26) is a product of $m+n-1$ differential forms $\theta^{A(q)}$. This allows separating eq. (4.26) in equations that contain the product of the same number of basis forms and perform the sum over unsuffles. Thus, we can isolate the Jacobi equation corresponding to $l = m + n + 1$ basis elements, as follows

$$\sum_{m+n=l-1} \sum_{\sigma \in U(l)} \epsilon\,(\sigma, T) \left[\left[t_{B_{\sigma(1)}\left(q_{\sigma(1)}\right)}, \ldots, t_{B_{\sigma(m)}\left(q_{\sigma(m)}\right)}\right]_m\right.$$
$$\left., t_{B_{\sigma(m+1)}\left(q_{\sigma(m+1)}\right)}, \ldots, t_{B_{\sigma(l)}\left(p_{\sigma(l)}\right)}\right]_n = 0. \tag{4.27}$$

Notice that, since there is no repetition of forms $\theta^{A(q)}$ in eq. (4.26) (otherwise, the wedge product vanishes), we have performed the sum on $m!\,(n-1)!$ equivalent terms, removing the factorial factor. Moreover, by considering the reordering of differential forms when performing that sum, we have introduced a Koszul sign depending on the order of the permutation and the differential degrees of the elements.

Let us now consider a new $\mathbb{Z}$-graded vector space, denoted by $X = \oplus_n X_n$. Let $\left\{t_{A(p)}\right\}_{p=1}^{N}$ be a basis of X, with the gradation $\deg_X t_{A(p)} = p - 1$. By last, in terms of the products in eq. (4.23), we define the following set of products on $\bar{X}$:

$$\ell_n\left(t_{A_1(p_1)}, \ldots, t_{A_n(p_n)}\right) = (-1)^{(p_1-1)(n-1)+\cdots+(p_{n-1}-1)} \left[t_{A_1(p_1)}, \ldots, t_{A_n(p_n)}\right]_n. \tag{4.28}$$

By plugging in eq. (4.28) into eqs. (4.27) and (4.24), one immediately obtains that the products ℓ_n satisfy the L_∞ identities and the graded symmetry relations of L_∞ algebras in the ℓ-picture. Therefore, the vector space X, endowed with the products of eq. (4.28) defines an L_∞ algebra in the ℓ-picture. On the other hand, eq. (4.27) turns on to be equivalent to the L_∞ identities in the b-picture, being eq. (4.28) the mapping of the products between both pictures. This shows the dual relation between FDAs and L_∞ algebras. For later convenience, we explicitly show the relation between FDAs and L_∞ algebras in the ℓ-picture. Extensive analyses of the relation between L_∞ algebras and graded differential algebras can be found in refs. [70–72]. Moreover, for recent reviews on the role of non-associative algebras in physics, with a strong emphasis on their historical development and dual formulation, see refs. [73, 74] .

4.3.1 Example: five-dimensional case

Let us consider an example of a FDA, namely, a FDA1 carrying a one form $\theta^{A(1)}$ and a two-form $\theta^{A(2)}$. This setup reduces the Maurer–Cartan equations from eq. (3.1) to a set of two differential equations, as follows[1]

$$\mathrm{d}\theta^{A(1)} + \frac{1}{2} C^{A(1)}_{B(1)C(1)} \theta^{B(1)} \wedge \theta^{C(1)} = 0, \tag{4.29}$$

$$\mathrm{d}\theta^{A(2)} + \frac{1}{2}\left(C^{A(2)}_{B(2)C(1)} \theta^{B(2)} \wedge \theta^{C(1)} + C^{A(2)}_{B(1)C(2)} \theta^{B(1)} \wedge \theta^{C(2)} \right) + \frac{1}{3} C^{A(2)}_{B(1)C(1)D(1)} \theta^{B(1)} \wedge \theta^{C(1)} \wedge \theta^{D(1)} = 0. \tag{4.30}$$

The graded symmetry in eq. (4.24) shows that the first generalized structure constants $C^{A(2)}_{B(2)C(1)}$ are symmetric in the lower indices, i.e.,

$$C^{A(2)}_{B(2)C(1)} = C^{A(2)}_{C(1)B(2)}, \tag{4.31}$$

The second generalized structure constants $C^{A(2)}_{B(1)C(1)D(1)}$ are the components of the three-cocycle with which the FDA1 is constructed. The wedge product between one-forms in the r.h.s of eq. (4.24) induces antisymmetry in their lower indices. By imposing that the three-cocycle is in the adjoint representation of the Lie algebra, the FDA1 is reduced to a particular case in which every algebraic index takes values in the same domain. In this way, the indices $A(1)$ and $A(2)$ can be simply denoted by the same letter without their labels i.e., $A(1) \to A$ and $A(2) \to A$. This also allows renaming the FDA1 potentials as $\theta^{A(1)} = \theta^A_1$ and $\theta^{A(2)} = \theta^A_2$. Notice that we have included a new label to distinguish them. By last, the generalized structure constants of the FDA can also be renamed. Notice that the symmetry rule in eq. (4.31) include both the indices B and C and the labels 1 and 2, being the permutation of both of them necessary to hold the symmetry. This allows to define the components of $C^{A(1)}_{B(1)C(1)}$ in terms of the antisymmetric structure constants of a

[1]For this example, we explicitly write the wedge product between differential forms.

Lie algebra, as follows

$$C^{A(1)}_{B(1)C(1)} = \left[t_{B(1)}, t_{C(1)}\right]^{A(1)}_2 = C^A_{BC}, \tag{4.32}$$
$$C^{A(2)}_{B(2)C(1)} = \left[t_{B(2)}, t_{C(1)}\right]^{A(2)}_2 = C^A_{BC}, \tag{4.33}$$
$$C^{A(2)}_{C(1)B(2)} = \left[t_{C(1)}, t_{B(2)}\right]^{A(2)}_2 = -C^A_{CB}, \tag{4.34}$$
$$C^{A(2)}_{B(1)C(1)D(1)} = \frac{1}{2}\left[t_{B(1)}, t_{C(1)}, t_{D(1)}\right]^{A(2)}_3 = C^A_{BCD}. \tag{4.35}$$

Eqs. (4.33) and (4.34) show that, although the structure generalized structure constants C^A_{BC} are antisymmetric in the lower indices, the product between FDA1 forms, given by

$$\left[\theta^{(2)}, \theta^{(1)}\right]^{A(2)} = C^{A(2)}_{B(2)C(1)}\theta^{B(2)} \wedge \theta^{C(1)}, \tag{4.36}$$

is indeed symmetric due to the symmetry of the generalized structure constants and the permutation of an even-degree form. In this case, we have chosen the structure constants of the same Lie subalgebra with which the FDA1 is defined. In this notation and with these choices, the Maurer–Cartan equations of the FDA1 take the following form

$$0 = \mathrm{d}\theta^A_1 + \frac{1}{2}C^A_{BC}\theta^B_1 \wedge \theta^C_1, \tag{4.37}$$
$$0 = \mathrm{d}\theta^A_2 + C^A_{BC}\theta^B_2 \wedge \theta^C_1 + \frac{1}{3}C^A_{BCD}\theta^B_1 \wedge \theta^C_1 \wedge \theta^D_1. \tag{4.38}$$

Let us now write down this FDA as an L_∞ algebra in the ℓ-picture. To this end, we introduce a graded vector space

$$X = X_0 \oplus X_1, \tag{4.39}$$

and denote its basis elements as $\left\{t_A, \tilde{t}_A\right\}$, with $t_A \in X_0$ and $\tilde{t}_A \in X_1$. By writing down the information of the FDA1 structure constants of eqs. (4.32)-(4.35) into the L_∞ products according to eq. (4.28), we find the following L_∞ products:

$$\ell_2\left(t_B, t_C\right) = C^A_{BC}t_A, \tag{4.40}$$
$$\ell_2\left(t_B, \tilde{t}_C\right) = -C^A_{BC}\tilde{t}_A, \tag{4.41}$$
$$\ell_3\left(t_B, t_C, t_D\right) = 2C^A_{BCD}\tilde{t}_A, \tag{4.42}$$
$$\text{Others} = 0. \tag{4.43}$$

Here, we have identified the vectors on X as $t_{A(1)} = t_A$ and $t_{A(2)} = \tilde{t}_A$. Thus, eqs. (4.40) and (4.41) contain the information of the Lie subalgebra of eq. (4.29), while the product in eq. (4.42) carry the information about the cocycle with which the FDA1 is defined. They satisfy the L_∞ identities due the Jacobi identities in eq.

(4.27).

A similar approach for the derivation of a dual of FDAs was performed in ref. [68]. Such dual formulation leads to a non-associative algebra by defining a set of products in terms of the Lie derivatives acting on the FDA manifold. The commutation relations of that algebra arise by directly calculating the commutator between the Lie derivatives along the standard and extended directions on the FDA manifold. The motivation for investigating these dual non-associative structures lies in the fact that, in closed string theory, non-associative structures in double phase space describe flux backgrounds [75]. In the case of these dual non-associative algebras, as well as in L_∞ algebras, the non-associativity appears when the bilinear algebraic product does not satisfy the standard Jacobi identity. The failure of such Jacobiator is proportional to a cocycle, being the non-associativity controlled by the Chevalley–Eilenberg cohomology (see refs. [76–78]).

4.4 L_∞ formulation of gauge theories

L_∞ algebras provide a mathematical structure that allows to write down the complete information about an arbitrary classical gauge theory [41]. This means to include not only the gauge algebra but also the information regarding the dynamics into the definition of algebraic products. The information regarding the gauge symmetry is still encoded into a certain subalgebra L_∞^{gauge} of the entire algebraic structure L_∞^{full}, being the dynamics codified into the remaining subspace. Standard gauge theories, such as Yang–Mills and Chern–Simons theories, have classical symmetries described by Lie algebras. As a consequence, when writing them in terms of L_∞ algebras, the gauge subalgebra L_∞^{gauge} is a Lie algebra. However, higher-gauge theories present higher-degree tensors as gauge fields [79] and describe the dynamics of extended objects such as string and branes. Their corresponding gauge symmetry is therefore described by enlarged algebraic structures such as FDAs. Thus, when describing higher-gauge theories in L_∞ formulation, the corresponding gauge subalgebra is an L_∞ algebra that could not be trivially reduced to a Lie algebra. For additional information about the use of L_∞ algebras in string theory and supergravity, see refs. [80–85].

Let us consider a classical gauge theory with fundamental field μ, whose dynamics is governed by the equation of motion $\mathcal{F} = 0$, and the gauge symmetry is induced by a set of transformations $\delta_\varepsilon \mu$ for a gauge parameter ε. Moreover, we consider that the field μ takes values in a vector space X_{-1}, while the parameter ε takes values in another vector space X_0. By last, we consider a third space, denoted by X_{-2}, in which the off-shell non-vanishing function $\mathcal{F}$ takes values. It is then possible to encode the information of the theory into an L_∞ algebra by introducing a vector space given by

$$X = X_0 \oplus X_{-1} \oplus X_{-2}, \tag{4.44}$$

and a set of products ℓ_n (with $n \geq 1$) defined on X, such that they include the information related to the definition of gauge transformations, the closed gauge algebra, and the equations of motion. In the following, we will see how the information of the gauge theory is encoded in these products.

4.4.1 Gauge transformations

Let us begin by writing the gauge transformations of an arbitrary theory in terms of algebraic products. At this point, we have introduced a gauge field $\mu \in X_{-1}$ and a set of parameters $\varepsilon \in X_0$. The gauge variation $\delta_\varepsilon \mu$ is defined in terms of the L_∞ products as follows

$$\delta_\varepsilon \mu = \sum_{n=0}^{\infty} \frac{(-1)^{\frac{n(n-1)}{2}}}{n!} \ell_{n+1}\left(\varepsilon, \mu^n\right). \tag{4.45}$$

Moreover, it is also possible to consider trivial gauge transformations, namely, equations of motion symmetries. A particular case of a trivial transformation that depends on two independent parameters $\varepsilon_1, \varepsilon_2 \in X_0$, and that will be important in chapter 7, is given by the following transformation

$$\delta^T_{\varepsilon_1,\varepsilon_2} \mu = \sum_{n=0}^{\infty} \frac{(-1)^{\frac{n(n-1)}{2}}}{n!} \ell_{n+3}\left(\varepsilon_1, \varepsilon_2, \mathcal{F}, \mu^n\right). \tag{4.46}$$

Since the variation of the gauge field is proportional to $\mathcal{F}$, it vanishes on-shell. Notice that the transformation depends on products of three or more elements, and therefore, it does not appear in the case of gauge transformations described by Lie algebras, due to the absence of higher products.

4.4.2 Equations of motion

Once introduced the gauge transformations (4.45), it is possible to define a gauge invariant action by introducing an inner product $\langle\ ,\ \rangle_{L_\infty}$ on X, such that, for a given collection of $n+1$ vectors $x_0, \ldots, x_n \in X$, the inner product satisfies the following properties

$$\langle x_0, \ell_n(x_1, x_2, \ldots, x_n)\rangle_{L_\infty} = (-1)^{1+\deg x_0 \deg x_1} \langle x_1, \ell_n(x_0, x_2, \ldots, x_n)\rangle_{L_\infty}, \quad (4.47)$$

$$\langle x_0, x_1\rangle_{L_\infty} = (-1)^{\deg x_0 \deg x_1} \langle x_1, x_0\rangle_{L_\infty}. \quad (4.48)$$

Notice that we introduce a label in the inner product to explicitly show that it is defined for L_∞ algebras, and distinguish it from the inner products of Lie algebras. By using this inner product, it is possible to define the following action principle:

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{\frac{n(n-1)}{2}}}{(n+1)!} \langle \mu, \ell_n(\mu^n)\rangle_{L_\infty}, \quad (4.49)$$

which is invariant under the gauge transformations defined on eq. (4.45) due to the properties of eqs. (4.47) and (4.48). By taking the field-variation of this action principle and plugging in the L_∞ identities, one obtains

$$\delta S = \langle \delta\mu, \mathcal{F}\rangle_{L_\infty}, \quad (4.50)$$

where

$$\mathcal{F} = \sum_{n=1}^{\infty} \frac{(-1)^{\frac{n(n-1)}{2}}}{n!} \ell_n(\mu^n). \quad (4.51)$$

This provides a definition for the equation of motion term $\mathcal{F}$. Notice that it is necessary to assume non-degeneracy of the inner product [41, 69], otherwise, the variation of the action principle does not lead to $\mathcal{F} = 0$ as the equation of motion. From now on, this non-degeneracy will be considered as a requirement in the studied theories.

4.4.3 Gauge algebra

As the third step, let us now consider two independent gauge parameters ε_1 and ε_2 belonging to X_0. By applying two successive and independent gauge transformations with these parameters and plugging in the definition from eq. (4.45), it is possible to prove that the commutator of two gauge transformations is given by

$$[\delta_{\varepsilon_2}, \delta_{\varepsilon_1}]\mu = \delta_{\varepsilon_3}\mu + \delta^T_{\varepsilon_1,\varepsilon_2}\mu. \quad (4.52)$$

The first term at the r.h.s. of eq. (4.52) is a gauge transformation of μ as introduced in eq. (4.45), where the parameter $\varepsilon_3 \in X_0$ is a function of ε_1 and ε_2, explicitly given by

$$\varepsilon_3 = \sum_{n=0}^{\infty} \frac{(-1)^{\frac{n(n-1)}{2}}}{n!} \ell_{n+2}\left(\varepsilon_1, \varepsilon_2, \mu^n\right). \tag{4.53}$$

On the other hand, the second term at the r.h.s. of eq. (4.52) is a trivial gauge transformation. It becomes important to notice that this trivial transformation involves the equation of motion term $\mathcal{F}$, providing information about the dynamics, even before having specified an action principle for the theory. More precisely, when writing a classical gauge theory involving products of three or more vectors in terms of an L_∞ algebra, the definition of covariant derivative induces a definition of gauge transformation. By directly calculating the commutator between two gauge transformations, it turns out to be possible to find a trivial contribution to such commutator (given by the second term at the r.h.s. of eq. (4.52)), becoming possible to find information about the equation of motion by inspection of the algebraic vector $\mathcal{F}$. It is therefore relevant to study the presence of these trivial transformations when introducing the action principle. In order to write down a consistent gauge theory, the allowed action principles must be only those that are not inconsistent with the trivial gauge transformation that appears in the commutator. In other words, the action principle must be such that their equations of motions effectively imply that the trivial transformation vanishes on-shell. As it was mentioned, since Lie algebras do not carry products of more than two vectors, action principles can be introduced for Lie gauge theories without taking this issue in account.

In summary, to write down the L_∞ algebra of a classical gauge theory, it is necessary to consider the relevant information coming from the definition of gauge transformations, the equation of motion, and the gauge algebra. From eq. (4.45) we can see that, given a gauge theory, the definition of gauge variations determines the products of the corresponding L_∞ algebra that involve vectors in X_0 and X_{-1}. On the other hand, eqs. (4.46) and (4.53) show that the gauge algebra provides the information about the L_∞ products between two or more vectors in X_0. By last, the dynamics of the theory is determined by the equation of motion term in eq. (4.50), which exclusively involves products between vectors in the subspace X_{-1}. These products can be obtained by direct inspection of these equations. However, they do not necessarily satisfy the L_∞ identities by themselves. In order to obtain the L_∞ algebra that describes a particular theory, it is also necessary to plug the obtained products into the L_∞ identities and demand them to be verified. This consequently leads to new information regarding algebraic products that must be non-vanishing for consistency. From now on, we will refer to the products that are not obtained by inspection of the three main sources of information mentioned before, but by their consistency with the L_∞ identities as consistency products.

This procedure was used in ref. [41] to write down the L_∞ algebras that describe three-dimensional Chern–Simons theory and Yang–Mills theory.

Chapter 5

Chern–Weil theorem and FDA1-Chern–Simons forms

In this chapter, we will use the formulation of FDAs introduced in chapter 4 to extend the results from chapter 3. This means to formulate an extension of the Chern–Weil theorem for a FDA1. In order to accomplish this goal, we begin by considering the definition of gauge curvatures that emerges from the gauging of the Maurer–Cartan equations, and then we postulate a generalized invariant density. Let us then consider a composite gauge field $\mu = \left(\mu^A, \mu^i\right)$ having as components a one form μ^A and a p-form μ^i. In the same way, its corresponding field strength is given by a pair $R = \left(R^A, R^i\right)$, being R^A the standard two-form curvature and R^i the extended $(p+1)$-form curvature defined by means of the inclusion of a cocycle $(p+1)$-form. They are both non-vanishing off-shell and explicitly given by the gauging of the Maurer–Cartan equations in eqs. (3.11) and (3.12).

We propose a generalized invariant density (analogous to the Chern–Pontryagin invariant density of a Lie group from eq. (2.48)), as a multilinear product of the components of the field strength. We set the form-degree of the new invariant density as q and consider the most general q-form that can be constructed using R^A and R^i as building blocks [45]

$$\chi_q = \sum_{m,n} g_{A_1 \cdots A_m i_1 \cdots i_n} R^{A_1} \cdots R^{A_m} R^{i_1} \cdots R^{i_n}. \tag{5.1}$$

To write the product of curvatures on the r.h.s of eq. (5.1), we introduce a set of constant coefficients $g_{A_1 \cdots A_m i_1 \cdots i_n}$ in analogy to the higher-rank invariant tensors of Lie algebras. The new coefficients contain both types of algebraic indices in the appropriate number for each term. Notice that, since each term on the r.h.s. of eq. (5.1) can have a different power on the curvatures, we are also including coefficients $g_{A_1\cdots A_m i_1\cdots i_n}$ with different numbers of indices. The sum at the r.h.s. of eq. (5.1) runs over all the possible combinations such that the result is a q-form. Since the two-form R^i appears m times and the $(p+1)$-form appears n times, the result is always a differential form of degree $2m+(p+1)\,n$. Therefore, the sum runs over all the non-negative integer solutions (m,n) to the algebraic equation

$$2m+(p+1)\,n=q. \tag{5.2}$$

At this point coefficients $g_{A_1\cdots A_m i_1\cdots i_n}$ are introduced such that χ_q is completely gauge invariant. Therefore, the transformations of the FDA1 induce generalized invariant conditions on these coefficients.

For example, in order to find a 12-form using only a two-form R^A and a four-form R^i, we have four possible terms, given by the integer solutions to the equation $m+2n=6$ namely

$$\begin{aligned}\chi_{12} &= g_{A_1\cdots A_6}R^{A_1}\cdots R^{A_6}+g_{A_1\cdots A_4 i}R^{A_1}\cdots R^{A_4}R^i+g_{ABij}R^AR^BR^iR^j \\ &\quad +g_{ijk}R^iR^jR^k. \end{aligned}\tag{5.3}$$

5.1 FDA1 invariant tensor

The coefficients $g_{A_1\cdots A_m i_1\cdots i_n}$ were introduced as the generalization of the invariant tensors of Lie algebras to the case of a FDA1. They are defined by the requirement of gauge invariance of χ_q. By considering the gauge variation of eq. (5.1) and plugging in the transformation laws from eqs. (3.45) and (3.46) we find

$$\begin{aligned}\delta\chi_q(\mu,B) &= \sum_{m,n} g_{A_1\cdots A_m i_1\cdots i_n}\Big[mC^{A_1}_{BC}R^B\varepsilon^C R^{A_2}\cdots R^{A_m}R^{i_1}\cdots R^{i_n} \\ &\quad +nR^{A_1}\cdots R^{A_m}\Big(C^{i_1}_{Bj}R^B\varepsilon^j-C^{i_1}_{Bj}\varepsilon^BR^j \\ &\quad -\frac{1}{(p-1)!}C^{i_1}_{BB_1\cdots B_p}\varepsilon^BR^{B_1}\mu^{B_2}\cdots\mu^{B_p}\Big)R^{i_2}\cdots R^{i_n}\Big]. \end{aligned}\tag{5.4}$$

The variation of χ_q includes both types of gauge transformations on eq. (5.4), standard and extended. Thus, the variation depends on the gauge parameters ε^A and ε^j. Since they are independent, and the total variation vanishes, we can decompose eq. (5.4) in parts that are proportional to each parameter and set both as equal to zero, as follows

$$\sum_{m,n} g_{A_1\cdots A_m i_1\cdots i_n}\Big[mC^{A_1}_{BC}R^B\varepsilon^C R^{A_2}\cdots R^{A_m}R^{i_1}\cdots R^{i_n} + \frac{n}{(p-1)!}C^{i_1}_{BB_1\cdots B_p}R^{A_1}\cdots R^{A_m}\varepsilon^B R^{B_1}\mu^{B_2}\cdots\mu^{B_p}R^{i_2}\cdots R^{i_n} - nC^{i_1}_{Bj}R^{A_1}\cdots R^{A_m}\varepsilon^B R^j R^{i_2}\cdots R^{i_n} = 0, \tag{5.5}$$

$$\sum_{m,n} g_{A_1\cdots A_m i_1\cdots i_n} nC^{i_1}_{Bj}R^{A_1}\cdots R^{A_m}R^B\varepsilon^j R^{i_2}\cdots R^{i_n} = 0, \tag{5.6}$$

Eqs. (5.5) and (5.6) are related to the variations with respect to ε^A and ε^i respectively. Since $p > 1$ (otherwise, the FDA is a Lie algebra), there is always a dependence on μ^A factor in the second term of (5.5). However, the first and third terms contain only curvatures, meaning that both contributions are always functionally different and therefore, in order to not impose any condition on the fields and curvature, both contributions should vanish independently. Therefore, eq. (5.5) can be split into two independent equations, as follows

$$\sum_{m,n} g_{A_1\cdots A_m i_1\cdots i_n}\Big[mC^{A_1}_{BC}R^B\varepsilon^C R^{A_2}\cdots R^{A_m}R^{i_1}\cdots R^{i_n} - nC^{i_1}_{Bj}R^{A_1}\cdots R^{A_m}\varepsilon^B R^j R^{i_2}\cdots R^{i_n} = 0, \tag{5.7}$$

$$\sum_{m,n} n g_{A_1\cdots A_m i_1\cdots i_n} C^{i_1}_{BB_1\cdots B_p}R^{A_1}\cdots R^{A_m}\varepsilon^B R^{B_1}\mu^{B_2}\cdots\mu^{B_p}R^{i_2}\cdots R^{i_n} = 0, \tag{5.8}$$

The invariance condition of χ_q is then summarized into eqs. (5.6), (5.7) and (5.8). Having the independent equations already separated, we can remove the dependence on the gauge fields and curvatures, resulting in three conditions that the coefficients $g_{A_1\cdots A_m i_1\cdots i_n}$ must verify in relation with the structure constants of the FDA1 to ensure the invariance of χ_q. Therefore, for each pair (m,n) in the solutions of eq. (5.6), we have the following conditions

$$\sum_{r=1}^{m} C^C_{A_0A_r} g_{A_1\cdots\hat{A}_r C\cdots A_m i_1\cdots i_n} + \sum_{s=1}^{n} C^k_{A_0 i_s} g_{A_1\cdots A_m i_1\cdots\hat{\imath}_s k\cdots i_n} = 0, \tag{5.9}$$

$$\sum_{r=1}^{m+1} C^{i_1}_{A_r B_1\cdots B_p} g_{A_1\cdots\hat{A}_r\cdots A_{m+1} i_1\cdots i_n} = 0, \tag{5.10}$$

$$\sum_{r=1}^{m+1} C^{i_1}_{A_r j} g_{A_1\cdots\hat{A}_r\cdots A_{m+1} i_1\cdots i_n} = 0, \tag{5.11}$$

where $\hat{A}$ and $\hat{\imath}$ denote the absence of such indices in the sequence. Eqs. (5.9), (5.10) and (5.11) provide the definition of extended invariant tensor for a FDA1. Notice that, in the absence of p-form (this sets $n = 0$), eqs. (5.10) and (5.11) are not present while eq. (5.9) becomes equivalent to the standard definition of the invariant tensor of a Lie group.

In general, from its original definition in eq. (5.1), the generalized invariant tensor is symmetric in the standard indices $A_1 \cdots A_m$. Moreover, it is symmetric in the extended indices $i_1 \cdots i_n$ for p odd and antisymmetric for p even. Since there are no symmetry rules for the interchange of mixed indices, we indistinctly denoted both sets of indices in a different order, i.e.,

$$g_{A_1 \cdots A_m i_1 \cdots i_n} \equiv g_{i_1 \cdots i_n A_1 \cdots A_m}. \tag{5.12}$$

An important feature of the standard invariant density in (2.48) is to be a closed form, allowing the existence of locally defined Chern–Simons forms, due to the Poincaré lemma. By directy applying the exterior derivative operator on χ_q and plugging in the Bianchi identities for a FDA1, given by eqs. (3.47) and (3.48), we find the following relation

$$\begin{aligned} \mathrm{d}\chi_q (\mu, B) = & \sum_{m,n} g_{A_1 \cdots A_m i_1 \cdots i_n} \Big[m C^{A_1}_{BC} R^B \mu^C R^{A_2} \cdots R^{A_m} R^{i_1} \cdots R^{i_n} \\ & + n R^{A_1} \cdots R^{A_m} \left(C^{i_1}_{Aj} R^A B^j + \frac{1}{p!} C^{i_1}_{A_1 \cdots A_{p+1}} R^{A_1} \mu^{A_2} \cdots \mu^{A_{p+1}} \right. \\ & \left. - C^{i_1}_{Aj} \mu^A R^j \right) R^{i_2} \cdots R^{i_n} \Big]. \end{aligned} \tag{5.13}$$

We can see that the exterior derivative of χ_q vanishes as a consequence of the invariant tensor conditions from eqs. (5.9)-(5.11). This proves that the invariant tensor conditions for a FDA1 lead to the construction of an analogue of the Chern–Pontryagin form for a FDA1 that inherits the properties of being closed and fully gauge invariant. These properties also allow us to find an analogue to the Chern–Simons and transgression forms for a FDA1.

5.2 Chern–Weil theorem

At this point, we have introduced a closed gauge invariant q-form. Due to the Poincaré lemma, it should be possible to write down a Chern–Simons form that generalizes eq. (2.54) to the case of a FDA1. Moreover, as we will see in this section, it is also possible to define fully gauge invariant transgression forms. To show this, in direct analogy to section 2.5, let us introduce two independent FDA1-valued gauge fields, each one with components given by a one-form and a p-form, and denoted by $\left(\mu_0^A, B_0^i\right)$ and $\left(\mu_1^A, B_1^i\right)$. Moreover, in terms of them, we also introduce a third homotopic gauge field, denoted by $\left(\mu_t^A, B_t^i\right)$, whose components are defined as follows

$$\mu_t^A = (1-t)\,\mu_0^A + t\mu_1^A, \tag{5.14}$$

$$B_t^i = (1-t)\,B_0^i + tB_1^i, \tag{5.15}$$

where the parameter t takes values in $[0,1]$, such that $\left(\mu_t^A, B_t^i\right)$ is interpolated between $\left(\mu_0^A, B_0^i\right)$ and $\left(\mu_1^A, B_1^i\right)$. The Stoke's theorem allows us to write the difference between the invariant densities corresponding to $\left(\mu_0^A, B_0^i\right)$ and $\left(\mu_1^A, B_1^i\right)$ in terms of an integral and a total derivative with respect to the parameter

$$\chi_q\left(\mu_1, B_1\right) - \chi_q\left(\mu_0, B_0\right) = \int_0^1 \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t}\chi_q\left(\mu_t, B_t\right). \tag{5.16}$$

As it was introduced in eq. (5.1), the invariant form $\chi_q\left(\mu_t, B_t\right)$ makes use of the components of $R_t = \left(R_t^A, R_t^i\right)$ as building blocks. By directly applying the derivative operator, it is possible to show that the components of the homotopic curvature satisfy the following useful relations

$$\frac{\mathrm{d}R_t^A}{\mathrm{d}t} = \left(\nabla_t u\right)^A, \tag{5.17}$$

$$\frac{\mathrm{d}R_t^i}{\mathrm{d}t} = \left(\nabla_t b\right)^i, \tag{5.18}$$

where we define $u^A = \mu_1^A - \mu_0^A$ and $b^i = B_1^i - B_0^i$ for convenience, and where ∇_t is the FDA1 covariant derivative, as it was defined in eqs. (3.58) and (3.59) with respect to the homotopic gauge field. We now apply the derivative along the parameter t into the expression for $\chi_q\left(\mu_t, B_t\right)$ and plug in the relations from eq. (5.17) to isolate a total derivative on the r.h.s. of eq. (5.16). This allows us to write eq. (5.16) as follows

$$\chi_q\left(\mu_1, B_1\right) - \chi_q\left(\mu_0, B_0\right) = \mathrm{d}Q_{q-1}\left(\mu_1, B_1; \mu_0, B_0\right). \tag{5.19}$$

where

$$\begin{aligned} Q_{q-1}\left(\mu_1, B_1; \mu_0, B_0\right) \quad = \quad & \sum_{m,n} g_{A_1\cdots A_m i_1\cdots i_n} \int_0^1 \mathrm{d}t \left(m u^{A_1} R_t^{A_2}\cdots R_t^{A_m} R_t^{i_1}\cdots R_t^{i_n}\right. \\ & \left. + n R_t^{A_1}\cdots R_t^{A_m} b^{i_1} R_t^{i_2}\cdots R_t^{i_n}\right). \end{aligned} \tag{5.20}$$

Eq. (5.19) generalizes the Chern–Weil theorem for a FDA1, specifically, by means of the relation between the gauge invariant densities and the transgression forms in eq. (2.52). Moreover, eq. (5.20) is the definition of transgresion form for a FDA1. As it happens with Lie groups, if we locally fix one set of gauge fields as zero, i.e., $\left(\mu_0^A, B_0^i\right) = (0,0)$, and rename the remaining gauge field as $\left(\mu_1^A, B_1^i\right) = \left(\mu^A, B^i\right)$, we obtain a definition for Chern–Simons forms for a FDA1 (or FDA1-Chern–Simons forms)

$$\begin{aligned} Q_{q-1}(\mu, B) \;=\; & \sum_{m,n} g_{A_1 \cdots A_m i_1 \cdots i_n} \int_0^1 \mathrm{d}t \left(m\mu^{A_1} R_t^{A_2} \cdots R_t^{A_m} R_t^{i_1} \cdots R_t^{i_n} \right. \\ & \left. + n R_t^{A_1} \cdots R_t^{A_m} B^{i_1} R_t^{i_2} \cdots R_t^{i_n} \right). \end{aligned} \tag{5.21}$$

Transgression forms as defined in eq. (5.20) are fully gauge invariant under the FDA1. In the same way, action principles that use FDA1-Chern–Simons forms as Lagrangian densities are invariant up to boundary terms.

5.3 Dynamics

Let us now consider the construction of transgression and Chern–Simons action principles. In order to derive their corresponding field equations, we introduce a change of notation that simplify the calculations. Let us therefore consider a $q-1$ dimensional manifold M_{q-1} on which we define two independent FDA1 valued gauge fields μ and $\bar{\mu}$, to whose components we denote as follows

$$\mu = \left(\mu^A, \mu^i\right), \tag{5.22}$$

$$\bar{\mu} = \left(\bar{\mu}^A, \bar{\mu}^i\right). \tag{5.23}$$

Notice that, for simplicity, we use the same letter to denote both components of the gauge fields. We distinguish them by their corresponding algebraic indices. Moreover, when we write a gauge field (or any other FDA1 vector) without algebraic index, it denotes the complete set of fields, as given in eqs. (5.22) and (5.23). We also consider a compact notation for FDA1 algebraic vectors and its contraction with the components of the invariant tensor in order to write the action principles in terms of index-free expressions. Let $x_1, \ldots, x_{m+n}$ be FDA1 vectors. Each one can be split in components as $x = \left(x^A, x^i\right)$, being x^A and x^i differential forms of different degrees. We denote their contraction with the FDA1 invariant tensor, as follows

$$\langle B_1, \ldots, B_m; B_{m+1}, \ldots, B_{m+n} \rangle = g_{A_1 \cdots A_m i_1 \cdots i_n} B_1^{A_1} \cdots B_m^{A_m} B_{m+1}^{i_1} \cdots B_{m+n}^{i_n}. \quad (5.24)$$

Notice that the semicolon separates both algebraic sectors. The details of this change in the notation can be found in appendix A. The new notation allows writing the invariant density from eq. (5.1) as

$$\chi_q \left(\mu^A, \mu^i \right) = \sum_{m,n} \langle R^m; R^n \rangle , \quad (5.25)$$

while the transgression forms from eq. (5.20) can be used to define a transgression action principle by integrating it over the base q-dimensional spacetime M_{q-1}, as follows:

$$S_\mathrm{T} = \int_{M_{q-1}} \sum_{m,n} \int_0^1 \mathrm{d}t \left(m \left\langle u, R_t^{m-1}; R_t^n \right\rangle + n \left\langle R_t^m; u, R_t^{n-1} \right\rangle \right), \quad (5.26)$$

where we define $u = \mu - \bar{\mu}$. For the purpose of finding the equations of motion for this action principle, we consider the general field variation of S_T with respect to μ and $\bar{\mu}$ simultaneously. By performing integration by parts in the variation of S_T, we obtain:

$$\begin{aligned}\delta S_\mathrm{T} = \int_{M_{q-1}} \sum_{m,n} \int_0^1 \mathrm{d}t \, & \Big(m \left\langle \delta u, R_t^{m-1}; R_t^n \right\rangle + m (m-1) \left\langle \nabla_t u, \delta\mu_t, R_t^{m-2}; R_t^n \right\rangle \\ & + mn \left\langle \nabla_t u, R_t^{m-1}; \delta\mu_t, R_t^{n-1} \right\rangle + mn \left\langle \delta\mu_t, R_t^{m-1}; \nabla_t u, R_t^{n-1} \right\rangle + n \left\langle R_t^m; \delta u, R_t^{n-1} \right\rangle \\ & - (-1)^p n (n-1) \left\langle R_t^m; \nabla_t u, \delta\mu_t, R_t^{n-2} \right\rangle \Big) + \text{Boundary terms.} \end{aligned} \quad (5.27)$$

The homotopic covariant derivatives and field variations in the r.h.s. of eq. (5.27) can be more conveniently written by using the following identities: the total derivative of the homotopic gauge fields and curvatures are given by

$$\frac{\mathrm{d}R_t}{\mathrm{d}t} = \nabla_t u, \quad (5.28)$$

$$\frac{\mathrm{d}\mu_t}{\mathrm{d}t} = \delta u. \quad (5.29)$$

Eqs. (5.28) and (5.29) are valid for both standard and extended components and allow to isolate the components of $\delta\mu$ and $\delta\bar{\mu}$ in the variation of the action principle. Indeed, by plugging them in eq. (5.27) and performing integration by parts, eq. (5.27) takes the following form

$$\delta S_\mathrm{T} = \int_{M_{q-1}} \sum_{m,n} \left(m \left\langle \delta\mu, R^{m-1}; R^n \right\rangle + n \left\langle R^m; \delta\mu, R^{n-1} \right\rangle \right.$$

$$-m\left\langle\delta\bar{\mu},\bar{R}^{m-1};\bar{R}^{n}\right\rangle-n\left\langle\bar{R}_t^{m};\delta\bar{\mu},\bar{R}^{n-1}\right\rangle\big). \tag{5.30}$$

For convenience, we have neglected the boundary terms. Finally, eq. (5.30) leads to the following two equations of motion

$$\delta\mu: \qquad \sum_{m,n}\left(m\left\langle\delta\mu,R^{m-1};R^{n}\right\rangle+n\left\langle R^{m};\delta\mu,R^{n-1}\right\rangle\right)=0, \tag{5.31}$$

$$\delta\bar{\mu}: \qquad \sum_{m,n}\left(m\left\langle\delta\bar{\mu},\bar{R}^{m-1};\bar{R}^{n}\right\rangle+n\left\langle\bar{R}_t^{m};\delta\bar{\mu},\bar{R}^{n-1}\right\rangle\right)=0. \tag{5.32}$$

Notice that we have split the resulting equation of motion in two equations by recalling that the variations with respect to μ and $\bar{\mu}$ are independent. Eqs. (5.31) and (5.32) are the equations of motion of the general FDA1 transgression theory, and therefore, it is interesting to analyze them in some particular cases. By setting $n=0$, eqs. (5.31) and (5.32) are reduced to the equations of motion of standard transgression theory (see ref. [49]). Moreover, since the functional in eq. (5.26) is reduced to the FDA1-Chern–Simons action principle if we locally set $\bar{\mu}=0$, eq. (5.31) becomes the equation of motion of FDA1-Chern–Simons theory with that setup (in such case, eq. (5.32) becomes trivial). In this last case, the first and second terms at the l.h.s. of eq. (5.31) depend on $\delta\mu^A$ and $\delta\mu^i$ respectively, due to the positions of $\delta\mu$ with respect to the semicolon. Therefore, the resulting equation of motion can be split again into two independent equations, related to the variations $\delta\mu^A$ and $\delta\mu^i$ respectively. In terms of the index-dependent notation, the equations of motions of the FDA1-Chern–Simons theory are therefore given by

$$\delta\mu^A: \qquad \sum_{m,n} m g_{A_1A_2\cdots A_m i_1\cdots i_n}R^{A_2}\cdots R^{A_m}R^{i_1}\cdots R^{i_n}=0, \tag{5.33}$$

$$\delta\mu^i: \qquad \sum_{m,n} n g_{A_1\cdots A_m i_1 i_2\cdots i_n}R^{A_1}\cdots R^{A_m}R^{i_2}\cdots R^{i_n}=0. \tag{5.34}$$

By last, notice that in the absence of FDA1 extension (i.e., by setting $n=0$) eq. (5.33) becomes the equation of motion of standard Chern–Simons theory and eq. (5.34) becomes trivial. For details on the dynamics of standard Chern–Simons theories, see refs. [86–88].

5.4 Triangle equation

In this section, we use the ECHF (see refs. [39, 56, 57]) to write down a triangle relation for Chern–Simons and transgression forms for a FDA1. This turns out to be a completely analogous procedure to the one introduced in section 2.6. In order to simplify the calculation, we denote again the gauge fields of the FDA1 as $\mu = \left(\mu^{A}, \mu^{i}\right)$. We can therefore denote the gauge invariant form simply as $\chi_q\left(\mu\right)$, understanding μ as a multiplet of one-forms and p-form in the (A, i)-representation of the FDA1. With this notation, let us consider a collection of $r+2$ gauge fields, labeled by the index $J = 0, \ldots, r+1$

$$\mu_J = \left(\mu_J^A, \mu_J^i\right), \tag{5.35}$$

defined on a fiber bundle over the base manifold M and a $(r+1)$-dimensional simplex T_{r+1} parametrized by $r+2$ parameters t_J, each one taking values in the one-dimensional segment $[0,1]$. Moreover, let us introduce a homotopic gauge field defined as

$$\mu_t = \sum_{J=0}^{r+1} t_J \mu_J. \tag{5.36}$$

Since each gauge field μ_J has a one-form and a p-form as components that transform according to eqs. (3.35) and (3.36), it is necessary to impose the following constraint on the homotopic parameters

$$\sum_{J=0}^{r+1} t_J = 1. \tag{5.37}$$

Eq. (5.37) defines trajectories over the simplex, on which the components of μ_t also transform according to eqs. (3.35) and (3.36), and therefore it is a well-defined gauge field for a FDA1 theory. Notice that this constraint reduces the number of independent parameters from $r+1$ to r. It is now possible to introduce the gauge invariant form $\chi_q\left(\mu_t\right)$, which is interpolated between the values $\chi_q\left(\mu_0\right), \ldots, \chi_q\left(\mu_{r+1}\right)$ as well as μ_t is interpolated between $\mu_0, \ldots, \mu_{r+1}$ along the mentioned trajectories.

By applying the ECHF to $\chi_q\left(\mu_t\right)$, one finds the following relation, analogue to eq. (2.74)

$$\int_{\partial T_{r+1}} \frac{l_t^r}{r!} \chi_q\left(\mu_t\right) = (-1)^r \, \mathrm{d} \int_{T_{r+1}} \frac{l_t^{r+1}}{(r+1)!} \chi_q\left(\mu_t\right), \tag{5.38}$$

where the allowed values for r are $r = 0, \ldots, q$. As it happens with Lie algebras, the operators d, d_t and l_t define a graded algebra given by eqs. (2.62)-(2.64), while the action of l_t on $\{\mu_t^A, R_t^A\}$ is determined by eqs. (2.68) and (2.69). However, the action of l_t on the extended components μ_t^i and R_t^i remains to be determined.

As it was mentioned in section 2.6, particular cases of eq. (2.74) reproduce the Chern–Weil theorem and the triangle equation for Lie algebras, relating transgression forms depending on different sets of gauge fields in the same representation of the Lie algebra. We demand that the same happens for the FDA1, i.e., we define the action of the homotopy operator l_t on the extended components of μ_t and R_t, such that the particular cases of eq. (5.38) reproduce already found results of the Chern–Weil theorem. Let us now consider eq. (5.38) case by case.

First equation:

By setting $r = 0$, the simplex T_1 becomes a one-dimensional segment parametrized by $t_0 \in [0, 1]$. In this case, eq. (5.38) takes the form

$$\int_{\partial T_1} \chi_q (\mu_t) = \mathrm{d} \int_{T_1} l_t \chi_q (\mu_t) \,, \tag{5.39}$$

where the components of the homotopic gauge fields are given by

$$\mu_t^A = \mu_0^A + t \left(\mu_1^A - \mu_0^A \right) , \tag{5.40}$$
$$\mu_t^i = \mu_0^i + t \left(\mu_1^i - \mu_0^i \right) . \tag{5.41}$$

The l.h.s. of eq. (5.39) is immediately given by the values of $\chi_q (\mu_t)$ in the extreme points $t = 0, 1$:

$$\int_{\partial T_1} \chi_q (\mu_t) = \chi_q (\mu_1) - \chi_q (\mu_0) \,, \tag{5.42}$$

while for r.h.s of eq. (5.39) we get

$$\begin{aligned} \mathrm{d} \int_{T_1} l_t \chi_q (\mu_t) \quad &= \quad \mathrm{d} \sum_{m,n \in q(p)} g_{A_1 \cdots A_m i_1 \cdots i_n} \int_{T_1} \left[m \left(l_t R_t^{A_1} \right) R_t^{A_1} \cdots R_t^{A_m} R_t^{i_1} \cdots R_t^{i_n} \right. \\ & \left. + n R_t^{A_1} \cdots R_t^{A_m} \left(l_t R_t^{i_1} \right) R_t^{i_2} \cdots R_t^{i_n} \right] . \end{aligned} \tag{5.43}$$

This equation must be consistent with the Chern–Weil theorem for the FDA1. By comparing eq. (5.43) with eq. (2.52) we can see that the action of the homotopy operator on μ_t^i and R_t^i is given by the following relations

$$l_t R_t^i = \mathrm{d}_t \mu_t^i, \tag{5.44}$$
$$l_t \mu_t^i = 0. \tag{5.45}$$

Second equation:

By setting $r = 1$, eq. (5.38) takes the form

$$\int_{\partial T_2} l_t \chi_q (\mu_t) = -\mathrm{d} \int_{T_2} \frac{l_t^2}{2} \chi_q (\mu_t) \,, \tag{5.46}$$

where the homotopic gauge fields depend on two parameters t_0 and t_2 taking values between 0 and 1, and three independent gauge fields μ_0, μ_1 and μ_2, as follows

$$\mu_t = t_0 \left(\mu_0 - \mu_1\right) + t_2 \left(\mu_2 - \mu_1\right) + \mu_1 . \tag{5.47}$$

The l.h.s of eq. (5.46) corresponds to an integral along the boundary of the simplex T_2 that can be immediately integrated as follows

$$\int_{\partial T_2} \chi_q \left(\mu_t\right) = Q^{(q-1)} \left(\mu_2, \mu_1\right) - Q^{(q-1)} \left(\mu_2, \mu_0\right) + Q^{(q-1)} \left(\mu_1, \mu_0\right) . \tag{5.48}$$

By plugging in eq. (5.48) into eq. (5.46) we obtain a triangle relation for the FDA1 that is analogue to eq. (2.56) for the standard case

$$Q^{(q-1)} \left(\mu_0, \mu_1\right) + Q^{(q-1)} \left(\mu_1, \mu_2\right) + Q^{(q-1)} \left(\mu_2, \mu_0\right) = \mathrm{d}Q^{(q-2)} \left(\mu_2, \mu_1, \mu_0\right) , \tag{5.49}$$

where the $(q-2)$-form inside the total derivative is defined as follows

$$Q^{(q-2)} \left(\mu_2, \mu_1, \mu_0\right) = \int_{T_2} \frac{l_t^2}{2} \chi_q \left(\mu_t\right) . \tag{5.50}$$

The direct application of the homotopy operator into the invariant density, leads to the following explicit expression for the term inside the total derivative at the r.h.s. of eq. (5.49)

$$\begin{aligned} Q^{(q-2)} \left(\mu_2, \mu_1, \mu_0\right) = & \sum_{m,n} g_{A_1 \cdots A_m i_1 \cdots i_n} \int_0^1 \mathrm{d}t_0 \int_0^1 \mathrm{d}t_2 \\ & \Big[m \left(m-1\right) \left(\mu_2^{A_2} - \mu_1^{A_2}\right) \left(\mu_0^{A_1} - \mu_1^{A_1}\right) R_t^{A_3} \cdots R_t^{A_m} R_t^{i_1} \cdots R_t^{i_n} \\ & + mn \left(\mu_0^{A_1} - \mu_1^{A_1}\right) R_t^{A_2} \cdots R_t^{A_m} \left(\mu_2^{i_1} - \mu_1^{i_1}\right) R_t^{i_2} \cdots R_t^{i_n} \\ & - mn \left(\mu_2^{A_1} - \mu_1^{A_1}\right) R_t^{A_2} \cdots R_t^{A_m} \left(\mu_0^{i_1} - \mu_1^{i_1}\right) R_t^{i_2} \cdots R_t^{i_n} \\ & + n \left(n-1\right) R_t^{A_1} \cdots R_t^{A_m} \left(\mu_0^{i_1} - \mu_1^{i_1}\right) \left(\mu_2^{i_2} - \mu_1^{i_2}\right) R_t^{i_3} \cdots R_t^{i_n} \Big] , \end{aligned} \tag{5.51}$$

where R_t is the homotopic curvature associated to the homotopic gauge field from eq. (5.47), and therefore, it depends on both parameters t_0 and t_2. Eq. (5.51) generalized the standard boundary term from eq. (2.75) to the case of a FDA1. Notice that such boundary term can be obtained as the first term of the expansion at the r.h.s. of eq. (5.51), in which the p-form is not present.

5.5 Adjoint representation

One of the main difficulties in constructing Chern–Simons and transgression gauge theories is obtaining the invariant tensors of the corresponding Lie algebra. This issue is also present in the study of FDA invariant gauge theories. In the case of a FDA1, eqs. (5.9)-(5.11) show the conditions that a tensor must satisfy to be considered in constructing an invariant action. Let us consider the components of an invariant tensor of a FDA1, given by $g_{A_1\cdots A_m i_1\cdots i_n}$. There are m standard indices and n extended ones. By inspection of the first term in the l.h.s of eq. (5.9) we can see that, if $n = 0$, $g_{A_1\cdots A_m i_1\cdots i_n}$ satisfies the invariant tensor conditions of the Lie subalgebra of the FDA1. Therefore, an invariant tensor the Lie subalgebra is also an invariant tensor of the FDA1. However, these tensors are not useful in the construction of actions that involve higher-degree differential forms. Moreover, if $n \neq 0$, the invariant tensor of the Lie subalgebra is not necessarily an invariant tensor of the FDA1 due to the non-vanishing second term in the l.h.s. of eq. (5.9). A case of FDA1, in which this situation becomes particularly convenient, can be found when the p-form gauge field B^i is also in the adjoint representation of the Lie subalgebra. In that case, the structure constants C^i_{Ak} become equivalent to the structure constants of the Lie subalgebra, i,e, $C^i_{Ak} \to C^C_{AB}$ and $C^i_{A_1\cdots A_{p+1}} \to C^C_{A_1\cdots A_{p+1}}$. In order to avoid ambiguity, we use a comma to separate the indices corresponding to different sectors of the algebra in the components of the invariant tensor, i.e., $g_{A_1\cdots A_m i_1\cdots i_n} \to g_{A_1\cdots A_m, B_1\cdots B_n}$. Thus, the invariant tensor remains completely symmetric on the first set of indices, while the symmetry rule on the second set of indices is not determined and depends on the differential degree p of the extended component of the gauge field B^i. With this notation, the invariant tensor conditions in eqs. (5.9)-(5.11) take the following form

$$\sum_{r=1}^{m} C^C_{A_0 A_r} g_{A_1\cdots\hat{A}_r C\cdots A_m, B_1\cdots B_n} + \sum_{s=1}^{n} C^C_{A_0 B_s} g_{A_1\cdots A_m, B_1\cdots\hat{B}_s C\cdots B_n} = 0, \tag{5.52}$$

$$\sum_{r=1}^{m+1} C^{C_1}_{A_r B_1\cdots B_p} g_{A_1\cdots\hat{A}_r\cdots A_{m+1} C_1\cdots C_n} = 0, \tag{5.53}$$

$$\sum_{r=1}^{m+1} C^{C_1}_{A_0 A_r} g_{A_1\cdots\hat{A}_r\cdots A_{m+1} C_1\cdots C_n} = 0. \tag{5.54}$$

The first invariant tensor condition, given by eq. (5.52), is equivalent to the invariant tensor condition of the Lie subalgebra even for $n \neq 0$. The main difference of the case mentioned before (for $n = 0$) is that the invariance of the tensor takes into account both sets of indices. It is not an invariant tensor of the Lie subalgebra in the first set of indices but it is with respect to all of them, and therefore, an action constructed with this invariant tensor will also couple the higher-degree differential forms. This shows that the invariant tensors of the Lie subalgebra are candidates to be invariant tensors of the FDA1 because they immediately verify the first condition. However. eqs. (5.53) and (5.54) still must be verified in order to find an action principle invariant under both transformations, standard and extended ones.

5.6 Poincaré FDA

We now consider the goal of finding a FDA1 invariant gauge invariant theory for a particular case. The first step is to construct a non-trivial FDA1 extension for a Lie algebra. For this purpose, we consider candidates for FDA1 extensions in the adjoint representation of the original Lie algebra, in order to facilitate the identification of invariant tensors.

As the first candidate, let us consider the bosonic arbitrary-dimensional Poincaré Lie algebra, whose Maurer–Cartan one-form can be decomposed in components as $\mu^A = \left(e^a, \omega^{ab}\right)$. Notice that, the standard index is composite as $A = (a, ab)$. The Maurer–Cartan differential equations that define the Poincaré algebra are given by

$$\mathrm{d}e^a + \omega^a{}_b e^b = 0, \tag{5.55}$$

$$\mathrm{d}\omega^{ab} + \omega^a{}_c \omega^{cb} = 0. \tag{5.56}$$

In order to extend this algebra to a FDA1, let us consider a new Maurer–Cartan equation of the type $\nabla B^A + \Omega^A = 0$, where $B^A = \left(b^a, b^{ab}\right)$ is a three-form gauge field, and $\Omega^A = \left(\Omega^a, \Omega^{ab}\right)$ is a four-form cocycle representative of a non-trivial Chevalley–Eilenberg cohomology class of the Poincaré algebra, being both in the adjoint representation of the Poincaré algebra. Since Ω^A is a cocycle, it can be written in terms of the one-form μ^A. As a candidate to be a cocycle, we postulate the most general four-form that we can write using only combinations of wedge products of the one-forms ω^{ab} and e^a without using the Levi–Civita pseudotensor. Thus, we write down its components as follows

$$\Omega^a = \left(a_1 \omega^a{}_b \omega^b{}_c \omega^c{}_d e^d + a_2 e^a \omega_{cd} e^c e^d + a_3 e^a \omega^{bc} \omega_{bd} \omega^d{}_c\right), \tag{5.57}$$

$$\Omega^{ab} = \left(b_1 \omega^{ab} \omega_{cd} e^c e^d + 2 b_2 \omega^{[a|}{}_c \omega^c{}_d e^d e^{|b]}\right), \tag{5.58}$$

where we introduce the arbitrary constants a_1, a_2, a_3, and b_1, b_2. In order to be a cocycle, its covariant derivative must vanish [4], leading to the following equations

$$(\nabla\Omega)^a = \mathrm{d}\Omega^a + \omega^a{}_c\Omega^c - \Omega^a{}_c e^c = 0 \tag{5.59}$$

$$(\nabla\Omega)^{ab} = \mathrm{d}\Omega^{ab} + \omega^a{}_c\Omega^{cb} - \omega^b{}_c\Omega^{ca} = 0. \tag{5.60}$$

Eqs. (5.59) and (5.60) impose conditions on the expansion from eqs. (5.57) and (5.58) in terms of one-forms. This fix $a_1 = b_1 = 0$ and $a_2 = b_2$ leaving only two independent forms on Ω that are proportional to a_2 and a_3 respectively. These forms are cocycles of the Poincaré algebra that must still be proved as non-trivial. We denote them as $\Omega_1^A = \left(\Omega_1^a, \Omega_1^{ab}\right)$ and $\Omega_2^A = \left(\Omega_1^a, \Omega_1^{ab}\right)$ respectively. They are explicitly given in components by the following expressions

$$\Omega_1^a = e^a\omega_{cd}e^c e^d, \tag{5.61}$$

$$\Omega_1^{ab} = 2\omega^{[a|}{}_c\omega^c{}_d e^d e^{|b]}, \tag{5.62}$$

$$\Omega_2^a = e^a\omega^{bc}\omega_{bd}\omega^d{}_c, \tag{5.63}$$

$$\Omega_2^{ab} = 0. \tag{5.64}$$

In order to find out whether the cocycles are trivial, we introduce the most general three-form in the adjoint representation of the Poincaré algebra $\phi = \left(\phi^a, \phi^{ab}\right)$ that can be built with wedge products of the one forms e^a and ω^{ab} in the adjoint representation, as we did for Ω^A

$$\phi^a = \alpha_1\omega^a{}_b\omega^b{}_c e^c, \tag{5.65}$$

$$\phi^{ab} = 2\beta_1\omega^{[a|}{}_c e^c e^{|b]}. \tag{5.66}$$

The components of the covariant derivative of ϕ give us a notion of the most general trivial-cocycle that can be found with this procedure. The resulting four-form is covariantly closed and covariantly exact

$$(\nabla\phi)^a = e^a\omega^b{}_c e_b e^c \tag{5.67}$$

$$(\nabla\phi)^{ab} = 2\omega^{[a|}{}_c\omega^c{}_e e^e e^{|b]} \tag{5.68}$$

By comparing eqs. (5.67) and (5.68) with eqs. (5.59) and (5.60), it follows that Ω_1 is a trivial cocycle but Ω_2 is not. We can therefore formulate a new algebraic structure that we call Poincaré-FDA by adding a Maurer–Cartan equation for the three-form B^A

$$\mathrm{d}e^a + \omega^a{}_b e^b = 0, \tag{5.69}$$

$$\mathrm{d}\omega^{ab} + \omega^a{}_c\omega^{cb} = 0, \tag{5.70}$$

$$
\mathrm{d}b^a + \omega^a{}_c b^c + b^a{}_c e^c + e^a \omega^{bc} \omega_{bd} \omega^d{}_c = 0, \tag{5.71}
$$

$$
\mathrm{d}b^{ab} + \omega^a{}_c b^{cb} - \omega^b{}_c b^{ca} = 0. \tag{5.72}
$$

5.7 Maxwell FDA

We can repeat the procedure from the last section for a larger algebra, namely the arbitrary-dimensional bosonic Maxwell algebra. In this case, the Maurer–Cartan one-form is decomposed in components as $\mu^A = \left(e^a, \omega^{ab}, k^{ab}\right)$, and the Maurer–Cartan differential equations that define the algebra are given by

$$
\mathrm{d}e^a + \omega^a{}_c e^c = 0, \tag{5.73}
$$

$$
\mathrm{d}\omega^{ab} + \omega^a{}_c \omega^{cb} = 0, \tag{5.74}
$$

$$
\mathrm{d}k^{ab} + \omega^a{}_c k^{cb} + k^a{}_c \omega^{cb} + \frac{1}{l^2} e^a e^b = 0. \tag{5.75}
$$

For details on the derivation of this algebra and its physical meaning, see refs. [89–91]. Notice that in this case, the index A is decomposed in three sectors, leading to three independent components of the gauge field, namely e^a, ω^{ab} and k^{ab}. As before, we postulate the most general four-form that can be built with wedge products of the components of μ^A, without using the Levi–Civita pseudotensor, as a candidate for being a cocycle. We denote its components in the different algebraic sectors as follows

$$
\Omega = \left(\Omega^a, \Omega^{ab}, \Theta^{ab}\right) \tag{5.76}
$$

In order to be a cocycle, the covariant derivative $\nabla\Omega$ must vanish. After imposing such condition on the expansion of Ω^A in terms of one forms, the resulting four-form Ω^A is a linear combination of cocycles. In order to isolate the components of Ω^A that are not covariantly exact we compare with the most general trivial cocycle that can be found by applying the covariant derivative on a three form built using the components of μ^A as building blocks. This procedure is analogue to the one shown in the previous section and shows the existence of two non-trivial 4-cocycles for the Maxwell Lie algebra. We denote their components of the first one as $\Omega_1 = \left(\Omega_1^a, \Omega_1^{ab}, \Theta_1^{ab}\right)$. They are explicitly given by the following equations

$$
\Omega_1^a = 0, \tag{5.77}
$$

$$
\Omega_1^{ab} = 0, \tag{5.78}
$$

$$
\Theta_1^{ab} = k^a{}_c k^c{}_d e^d e^b - k^b{}_c k^c{}_d e^d e^a - 2k^{ab} k_{cd} e^c e^d. \tag{5.79}
$$

By using Ω_1, we introduce the following differential algebra, which we will call Maxwell-FDA1. We denote the components of the three-form B^A corresponding to different algebraic sectors as $B^A = \left(b^a, b^{ab}, B^{ab}\right)$ and extend eqs. (5.73)-(5.75) by including the following Maurer–Cartan equations

$$\mathrm{d}e^a + \omega^a{}_c e^c = 0, \tag{5.80}$$
$$\mathrm{d}\omega^{ab} + \omega^a{}_c \omega^{cb} = 0, \tag{5.81}$$
$$\mathrm{d}k^{ab} + \omega^a{}_c k^{cb} - \omega^b{}_c k^{ca} + \frac{1}{l^2} e^a e^b = 0, \tag{5.82}$$
$$\mathrm{d}b^a + \omega^a{}_c b^c + b^a{}_c e^c = 0, \tag{5.83}$$
$$\mathrm{d}b^{ab} + \omega^a{}_c b^{bc} - \omega^b{}_c b^{ba} = 0, \tag{5.84}$$

$$\begin{aligned}\mathrm{d}B^{ab} + \omega^a{}_c B^{cb} - \omega^b{}_c B^{ca} + b^a{}_c k^{cb} - b^b{}_c k^{ca} + \frac{1}{l^2}\left(e^a b^b - e^b b^a\right)& \\ + k^a{}_c k^c{}_d e^d e^b - k^b{}_c k^c{}_d e^d e^a - 2k^{ab} k_{cd} e^c e^d = 0.&\end{aligned} \tag{5.85}$$

Any rescaling of the cocycle in the Maurer–Cartan equations leads to an equivalent algebra through a redefinition of the extended gauge fields. By applying the exterior derivative to eqs. (5.80)-(5.85) we can check that it satisfies the integrability condition.

On the other hand, we denote the components of the second non-trivial Maxwell cocycle as $\Omega_2 = \left(\Omega_2^a, \Omega_2^{ab}, \Theta_2^{ab}\right)$. Their components are explicitly given by

$$\Omega_2^a = e^a k_{cd} e^c e^d, \tag{5.86}$$
$$\Omega_2^{ab} = 0, \tag{5.87}$$
$$\Theta_2^{ab} = 2k^{ab} k_{cd} e^c e^d. \tag{5.88}$$

In the same way, using Ω_2 we define a secondary algebra that we call Maxwell-FDA2 through the following differential equations

$$\mathrm{d}e^a + \omega^a{}_c e^c = 0, \tag{5.89}$$
$$\mathrm{d}\omega^{ab} + \omega^a{}_c \omega^{cb} = 0, \tag{5.90}$$
$$\mathrm{d}k^{ab} + \omega^a{}_c k^{cb} - \omega^b{}_c k^{ca} + \frac{1}{l^2} e^a e^b = 0, \tag{5.91}$$
$$\mathrm{d}b^a + \omega^a{}_c b^c + b^a{}_c e^c + e^a k_{cd} e^c e^d P_a = 0, \tag{5.92}$$
$$\mathrm{d}b^{ab} + \omega^a{}_c b^{bc} - \omega^b{}_c b^{ba} = 0, \tag{5.93}$$

$$\begin{aligned} \mathrm{d}B^{ab} + \omega^{a}{}_{c}B^{cb} - \omega^{b}{}_{c}B^{ca} + b^{a}{}_{c}k^{cb} - b^{b}{}_{c}k^{ca} \\ + \frac{1}{l^2}\left(e^{a}b^{b} - e^{b}b^{a}\right) + k^{ab}k_{cd}e^{c}e^{d} = 0. \end{aligned} \tag{5.94}$$

Both algebras, Maxwell-FDA1 and Maxwell-FDA2, are differential algebras of the type FDA1 introduced in section 3.4. Since they are extended from the Maxwell algebra by the inclusion of different four-forms, representatives of the Chevalley–Eilenberg cohomology classes of the Maxwell algebra, they are non-equivalent structures, i.e., it is not possible to obtain one of them from the other one by redefinitions of the gauge fields.

At this point, we have found three non-trivial FDA1 extensions of the Poincaré and Maxwell Lie algebras. From now on, we will focus on the construction of a gauge invariant theory whose symmetry is described by the Maxwell-FDA1. Similar results, in which Poincaré and Maxwell algebras are extended (for both the bosonic and supersymmetric cases) and in which generalized action principles are constructed, can be found in refs. [92–94].

5.7.1 Gauge transformations

Let us consider a one-form gauge field $\mu^A = \left(e^a, \omega^{ab}, k^{ab}\right)$ and a three-form gauge field $B^A = \left(b^a, b^{ab}, B^{ab}\right)$. We proceed to gauge the Maxwell-FDA1 by considering that the components of μ^A and B^A do not satisfy the Maurer–Cartan equations, i.e., by introducing non-vanishing curvatures. In this case, $p = 3$ and the algebraic indices take the same values in both algebraic sectors $A = (a, ab, ab)$. In order to introduce gauge variations, we consider a zero-form parameter ε^A and a two-form parameter ξ^A, and denote their components corresponding to different algebraic sectors as follows:

$$\varepsilon^A = \left\{\varepsilon^a, \varepsilon^{ab}, \rho^{ab}\right\}, \tag{5.95}$$

$$\xi^i = \left\{\xi^a, \xi^{ab}, \lambda^{ab}\right\}. \tag{5.96}$$

By replacing the structure constants from the Maurer–Cartan equations (5.80)-(5.85) into the general definition of gauge variation in eq. (4.45), we find the following gauge variations for the components of μ^A

$$e^a = \mathrm{d}\varepsilon^a + \omega^{a}{}_{c}\varepsilon^{c} - \varepsilon^{a}{}_{c}e^{c}, \tag{5.97}$$

$$\omega^{ab} = \mathrm{d}\varepsilon^{ab} + \omega^{a}{}_{c}\varepsilon^{cb} - \omega^{b}{}_{c}\varepsilon^{ca}, \tag{5.98}$$

$$\delta k^{ab} = \mathrm{d}\rho^{ab} Z_{ab} + \omega^{a}{}_{c}\rho^{cb} - \omega^{b}{}_{c}\rho^{ca} - \varepsilon^{a}{}_{c}k^{cb} + \varepsilon^{b}{}_{c}k^{ca} + \frac{1}{l^2}\left(e^{a}\varepsilon^{b} - e^{b}\varepsilon^{a}\right). \tag{5.99}$$

These variations are ruled by the Lie subalgebra (in this case, Maxwell algebra). Moreover, the variations of the components of B^A are given by

$$\delta b^{a} = \mathrm{d}\xi^{a} + \omega^{a}{}_{c}\xi^{c} - \xi^{a}{}_{c}e^{c} \tag{5.100}$$

$$\delta b^{ab} = \mathrm{d}\xi^{ab} + \omega^{a}{}_{c}\xi^{cb} - \omega^{b}{}_{c}\xi^{ca}, \tag{5.101}$$

$$\begin{aligned}\delta B^{ab} &= \mathrm{d}\lambda^{ab} + \omega^{a}{}_{c}\lambda^{cb} - \omega^{b}{}_{c}\lambda^{a} - \varepsilon^{a}{}_{c}B^{cb} + \varepsilon^{b}{}_{c}B^{ca} \\ &\quad + k^{a}{}_{c}\varepsilon^{c}{}_{d}e^{d}e^{b} - k^{b}{}_{c}\varepsilon^{c}{}_{d}e^{d}e^{a} - \varepsilon^{a}{}_{c}k^{c}{}_{d}e^{d}e^{b} + \varepsilon^{b}{}_{c}k^{c}{}_{d}e^{d}e^{a} \\ &\quad + 2\varepsilon^{ab}k_{cd}e^{c}e^{d} - 2k^{ab}\varepsilon_{cd}e^{c}e^{d}.\end{aligned} \tag{5.102}$$

Notice that the information concerning the cocycle Ω_1 is present on the transformation of B^{ab}, while the first two components of B^A transform according to the Lie covariant derivative of the extended parameter.

5.8 Maxwell-FDA1 Chern–Simons action

The gauging of the Maxwell-FDA1 implies considering non-zero field strengths. We therefore introduce a two-form field strength, to whose components we denote $R^A = \left(R^a, R^{ab}, F^{ab}\right)$ and a four-form field strength, denoted by $H^A = \left(h^a, h^{ab}, H^{ab}\right)$. Explicit expressions of the components of R^A and H^A in terms of the components of μ^A and B^A and their derivatives, can be found on the l.h.s. of eqs. (5.80)-(5.85).

Let us now consider the construction of the five-dimensional Chern–Simons action given by a functional of the standard and extended gauge fields μ^A and B^A defined in a five-dimensional manifold M_5

$$S_{\mathrm{CS}}\left[\mu, B\right] = \int_{M_5} Q_5\left(\mu, B\right). \tag{5.103}$$

From eq. (5.20), we can write down a general expression for a five-dimensional Chern–Simons form invariant under an arbitrary FDA1 with a three-form gauge field, as follows

$$Q_5\left(\mu, B\right) = 3\int_0^1 \mathrm{d}t\ g_{ABC}\mu^{A}R_t^{B}R_t^{C} + g_{AB}\int_0^1 \mathrm{d}t\ \left(B^{A}H_t^{B} + R_t^{A}B^{B}\right). \tag{5.104}$$

However, a more convenient expression for $Q_5\left(\mu, B\right)$ can be found using the triangle equation (2.56). Let us then consider a new set of independent gauge fields taking values in certain subspaces of the Maxwell-FDA1. We define $\bar{\mu} = \left(\bar{\mu}^A, \bar{B}^A\right)$ as a set of gauge fields whose components take non-vanishing values in the Maxwell-FDA1 as follows

$$\bar{\mu}^A = \left(0, \omega^{ab}, k^{ab}\right), \tag{5.105}$$

$$\bar{B}^A = \left(0, b^{ab}, B^{ab}\right), \tag{5.106}$$

i.e., $\bar{\mu}$ take values in the rotational sectors of the Maxwell-FDA1, but not in the translation subspace associated with the vielbein field e^a. By considering this new set of gauge fields, it is possible to write down the Chern–Simons form from eq. (5.104) in terms of a transgression form, another Chern–Simons form and total derivatives, as follows

$$Q_5\left(\mu, B\right) = Q_5\left(\mu, B; \bar{\mu}, \bar{B}\right) + Q_5\left(\bar{\mu}\right) + \text{total derivative}. \tag{5.107}$$

The first term at the r.h.s. of eq. (5.107) is a transgression from that, according with eq. (5.20) can be explicitly written as

$$\begin{aligned} Q_5\left(\mu, B; \bar{\mu}, \bar{B}\right) &= \int_0^1 \mathrm{d}t \left\{3 g_{ABC}\left(\mu^A - \bar{\mu}^A\right) R_t^B R_t^C \right. \\ &\quad \left. + g_{AB}\left[\left(\mu^A - \bar{\mu}^A\right) H_t^B + R_t^{A_1}\left(B^B - \bar{B}^B\right)\right]\right\}. \end{aligned} \tag{5.108}$$

where we introduce the homotopic gauge field μ_t, whose standard and extended components are given by

$$\mu_t^A = \left(te^a, \omega^{ab}, k^{ab}\right), \tag{5.109}$$

$$B_t^A = \left(tb^a, b^{ab}, B^{ab}\right). \tag{5.110}$$

The second term at the r.h.s. of eq. (5.107) is a Chern–Simons form depending only on the gauge field $\bar{\mu}$, and given by

$$Q_5\left(\bar{\mu}, \bar{B}\right) = \int_0^1 \mathrm{d}t \; g_{AB}\left(\bar{\mu}^A \bar{H}_t^B + \bar{R}_t^A \bar{B}^i\right), \tag{5.111}$$

where we have introduced another set of homotopic gauge fields and homotopic curvatures, explicitly given by

$$\bar{\mu}_t = t\bar{\mu} = \left(t\bar{\mu}^A, t\bar{B}^B\right), \tag{5.112}$$

$$\bar{R}_t = \left(\bar{R}_t^A, \bar{H}_t^B\right). \tag{5.113}$$

Finally, the last step in constructing the action principle is finding the components of the invariant tensor of the Maxwell-FDA1. Therefore, we consider a degree-2 tensor, carrying one standard index and one extended index, denoted by $g_{Ai} \equiv g_{AB}$. Since both indices are in the adjoint representation of the Maxwell algebra, it is easy to confuse the mentioned tensor with a tensor carrying two standard indices. However, since a different number of standard and extended indices leads to different invariance conditions in eqs. (5.9)-(5.11), it is important to remark the difference between both cases. In this case, such invariant tensor conditions are reduced to

$$g_{AD}C^{D}_{BC} + g_{BD}C^{D}_{AC} = 0, \tag{5.114}$$

$$g_{AB}C^{B}_{CDEF} + g_{DB}C^{B}_{CAEF} = 0, \tag{5.115}$$

Notice that two (5.9) and (5.10) have become equivalent, reducing the invariant tensor conditions to a set of only two equations. As we have mentioned before, the first condition (5.114) implies that g_{AB} is a degree-2 invariant tensor of the Lie subalgebra (the Maxwell algebra, in this case) which also satisfies eq. (5.115). Therefore, we propose the usual degree-2 invariant tensor. To avoid confusion, we denote with square brackets the indices in the third sector of the algebra (the one associated to k^{ab})

$$g_{ab,cd} = \alpha_0 \left(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}\right), \tag{5.116}$$

$$g_{ab,[cd]} = \alpha_1 \left(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}\right), \tag{5.117}$$

$$g_{a,b} = \alpha_1 \eta_{ab}. \tag{5.118}$$

The coefficients α_0 and α_1 are arbitrary constants, each constant corresponding to an independent invariant tensor. We now impose g_{AB} to verify the invariance requirement from eq. (5.115). Such condition sets the constants as $\alpha_1 = 0$. Thus, the resulting tensor proportional to α_0 (whose only non-vanishing component is given by eq. (5.116)) is an invariant tensor of the complete algebra. Moreover, we also need the degree-3 invariant tensor of the Maxwell-FDA1 that carries three standard indices. This is equivalent to the invariant tensor of the Maxwell Lie algebra, which is well-known (see ref. [95]), and given by the Levi–Civita symbol $g_{ab,cd,e} = \epsilon_{abcde}$. By plugging in eqs. (5.116), (5.108) and (5.111) into the general expression in eq. (5.104), we find the following Chern–Simons action principle

$$S_{\mathrm{CS}}\left[\mu, B\right] = \int_{M_5} \frac{3}{4}\epsilon_{abcde}R^{ab}R^{cd}e^{e} + \frac{\alpha_0}{2}\left(\omega^{ab}h_{ab} + R^{ab}b_{ab}\right). \tag{5.119}$$

The first term on the integral at the r.h.s. of eq. (5.119) corresponds to the usual Chern–Simons form invariant under the transformations of the Maxwell Lie algebra.

Moreover, the second term extends the action, including the three-form gauge field without breaking the invariance under transformations of the Maxwell algebra but extending it and modifying the resulting dynamics in the corresponding theory. It is important to notice that the cocycle is not present in the Lagrangian. Since the only non-vanishing component of the invariant tensor in the extended sector of the algebra is given by (5.116), the B^{ab} field is not present in the Chern–Simons form. To find an example of a Chern–Simons Lagrangian action principle, whose invariance is described by a FDA, involving the information of a non-trivial cocycle into the symmetry transformations is still an open problem.

Chapter 6

Gauge anomalies and FDAs

There is a strong relation between the appearance of anomalies in gauge theories and the invariant densities of the corresponding gauge group. The breaking of symmetries in the quantization process of a classical theory triggers the so-called gauge anomalies. In refs. [96–99], the concept of the chiral anomaly was introduced: this anomaly emerges in gauge theories in which the gauge fields interact with Weyl fermions. The abelian anomaly, a consequence of the non-conservation of the classically conserved current J^μ, has a topological origin and can be calculated from the Chern–Pontryagin four-form, as follows

$$\partial_\mu J^\mu \propto \mathrm{Tr}\ \left[\varepsilon_{\mu\nu\rho\sigma} R^{\mu\nu} R^{\rho\sigma}\right] = \mathrm{Tr}\ \left[\partial^\mu \varepsilon_{\mu\nu\rho\sigma} \left(\mu^\nu \partial^\rho \mu^\sigma + \frac{2}{3} \mu^\nu \mu^\rho \mu^\sigma\right)\right]. \tag{6.1}$$

Here, the gauge connection valued on the Lie algebra is represented by $\mu_\nu = \mu_\nu^A t_A$, and t_A are the generators of the Lie algebra corresponding to an internal Lie group [37]. We can also express eq. (6.1) in a compact way by defining the one-form $\mu = \mu_\mu \mathrm{d}x^\mu$. With this, we get, in terms of the Hodge operator, a relation between the divergence of the current one-form $J = J_\mu \mathrm{d}x^\mu$ and the Chern–Simons three-form $Q_3(\mu)$, namely

$$\mathrm{d} * J \propto \mathrm{Tr}\ R^2 = \mathrm{d}Q_3(\mu)\,. \tag{6.2}$$

This can also be done for the non-abelian vector current $J_{\mu A}$. In this case, the covariant divergence of $J_{\mu A}$ can also be written in terms of a topological originated quantity:

$$\mathrm{D}^\mu J_{\mu A} \propto \mathrm{Tr}\ \partial^\mu \left[t_A \varepsilon_{\mu\nu\rho\sigma} \left(\mu^\nu \partial^\rho \mu^\sigma + \frac{1}{2} \mu^\nu \mu^\rho \mu^\sigma\right)\right]. \tag{6.3}$$

As before, in the language of differential forms, we get the following relation for the covariant divergence of the vector current $J_A = J_{\mu A}\mathrm{d}x^\mu$

$$\mathrm{D} * J_A \propto \mathrm{d}\left[\mathrm{Tr}\ \left(t_A\left(\mu\mathrm{d}\mu + \frac{1}{2}\mu^3\right)\right)\right]. \tag{6.4}$$

Notice that there is an issue when writing the r.h.s. of eq. (6.4) using only field-strengths: in contrast to the abelian anomaly, this is, in general, not possible. In refs. [37, 38] it is shown that the covariant divergence of J_A, can also be written in terms of a topological originated quantity, namely, the gauge-variation of the Chern–Simons three-form in eq. (6.2). It turns out that this variation can be written as the exterior derivative of a two-form $Q_2^1(\varepsilon, \mu)$, linear on the components of the zero-form parameter ε^A. In $2n$ dimensions, one gets a similar result for both abelian and non-abelian anomalies: in general, the divergence of the current $J = J_\mu \mathrm{d}x^\mu$ and the covariant divergence of the current $J_A = J_{\mu A}\mathrm{d}x^\mu$ can be written in terms of the Chern–Pontryagin invariant form and the variation of the Chern–Simons form respectively, i.e.,

$$\begin{aligned}
\mathrm{d} * J \quad &\propto \quad \mathrm{Tr}\ (R^n) = \mathrm{d}Q_{2n-1}(\mu)\,, \\
\mathrm{D} * J_A \quad &\propto \quad \delta Q_{2n-1}(\mu) = \mathrm{d}Q^1_{2n-2}(\varepsilon, \mu)\,.
\end{aligned}$$

The $(2n-2)$-form $Q^1_{2n-2}(\varepsilon, \mu)$ is linear in the components of the zero-form ε^A, which plays the role of constant of proportionality. Moreover, $Q^1_{2n-2}(\varepsilon, \mu)$ is explicitly given in terms of ε and μ as follows [37, 38]

$$Q^1_{2n-2}(\varepsilon, \mu) = n(n-1)\int_0^1 \mathrm{d}t(1-t)\mathrm{Str}\left(\varepsilon, \mathrm{d}\left(\mu, R_t^{n-2}\right)\right). \tag{6.5}$$

Here, Str denotes the symmetrized trace, acting on the generators of the Lie algebra.

6.1 Anomalies in higher gauge theory

The study of gauge invariant forms in higher gauge theories has also introduced the possibility of finding new gauge anomalies in quantum field theory [12–14, 16–19]. In standard gauge theory, the standard one-form gauge field is valued in a Lie algebra. Generalizations of Lie algebras allow the construction of physical theories involving gauge fields that replace or extend the one-form by higher-degree tensors. As a consequence, the gauge-invariant densities are also modified in order to extend the gauge invariance to the generalized algebraic structure and include the higher-degree tensors in the field content. An example of this is given in chapter 5 by the generalized invariant densities for FDA1 in arbitrary dimensions and their corresponding Chern–Simons forms. In this chapter, we study the existence of generalized anomaly terms that generalize the $(2n-2)$-form $Q^1_{2n-2}(\varepsilon, \mu)$ to arbitrary dimensions by replacing Lie algebras with FDA1. In the new case, the standard gauge field is modified in order to include the p-form μ^i, while the scalar parameters ε^A is modified to also include a higher parameter ε^i. The new anomaly terms will reproduce the standard ones as the first terms of their expansion (at least in even dimensions). Moreover, due to the existence of higher gauge fields and parameters, odd-dimensional anomalies are also allowed.

Let us begin by considering the following FDA1-valued gauge field

$$\mu = \left(\mu^A, \mu^i\right), \tag{6.6}$$

consisting on one-forms μ^A and p-forms μ^i. As we have seen, the corresponding curvature is given by a $(2, p+1)$-form

$$R = \left(R^A, R^i\right). \tag{6.7}$$

From now on, we use the compact notation for FDA1 algebraic vectors and invariant tensors. The details can be found in appendix A. In this notation, the Chern–Weil theorem for Chern–Simons forms can be written in a convenient compact manner

$$\chi_q(\mu) = \mathrm{d}Q_{q-1}(\mu), \tag{6.8}$$

where the Chern–Simons form is given by the following expression

$$Q_{q-1}(\mu) = \sum_{m,n} \int_0^1 \mathrm{d}t \left(m \left\langle \mu, R_t^{m-1}; R_t^n \right\rangle + n \left\langle R_t^m; \mu, R_t^{n-1} \right\rangle\right). \tag{6.9}$$

As before, the sum runs over all the positive integer solutions m and n to the equation $2m + (p+1)\,n = q$. The next goal is to generalize eq. (6.5). Furthermore, we propose that, as it happens with Lie algebras, the total gauge variation of the FDA1-Chern–Simons form $Q^{(q-1)}(\mu)$ can be written in terms of the exterior derivative of a $(q-2)$-form depending on the gauge fields and parameters of the transformation, i.e.,

$$\delta Q_{q-1}(\mu) = \mathrm{d}\omega^1_{q-2}(\varepsilon, \mu). \tag{6.10}$$

In this case, we deal with two sets of parameters valued in the corresponding Lie subalgebra and in the extended subspace of the FDA1 respectively. Thus, we separate the variations concerning the parameters ε^A and ε^i to find two independent generalizations of the gauge anomaly from eq. (6.5). We begin with the study of the extended variation, i.e., the one proportional to ε^i.

6.2 Extended variations

Let us consider a gauge transformation with parameter ε^i. This means to consider no transformation along the standard parameter ε^A. Therefore, the components of the gauge field and curvature transform as follows

$$\delta\mu^A = 0, \tag{6.11}$$

$$\delta\mu^i = \mathrm{d}\varepsilon^i + [\mu, \varepsilon]^i, \tag{6.12}$$

$$\delta R^A = 0, \tag{6.13}$$

$$\delta R^i = [R, \varepsilon]^i. \tag{6.14}$$

Using the generalized Jacobi identity, it is straightforward to show that the variation of the components of the homotopic curvature R_t change according to the following rules

$$\delta R_t^A = 0, \tag{6.15}$$

$$\delta R_t^i = [R_t, \varepsilon]^i + t\,(t-1)\,[\mu, \mathrm{d}\varepsilon]^i. \tag{6.16}$$

By taking the variation of the Chern–Simons form in eq. (6.9) and directly plugging in the transformation laws from eqs. (6.11)-(6.16) we obtain

$$\begin{aligned}\delta Q_{q-1}(\mu) = \sum_{m,n}\int_0^1 \mathrm{d}t\,\Big[&nm\left\langle \mu, R_t^{m-1}; [R_t,\varepsilon], R_t^{n-1}\right\rangle + n\left\langle R_t^m; [\mu,\varepsilon], R_t^{n-1}\right\rangle \\ &+ n\left\langle R_t^m; \mathrm{d}\varepsilon, R_t^{n-1}\right\rangle + t\,(t-1)\,mn\left\langle \mu, R_t^{m-1}; [\mu,\mathrm{d}\varepsilon], R_t^{n-1}\right\rangle \\ &+ n\,(n-1)\left\langle R_t^m; \mu, [R_t,\varepsilon], R_t^{n-2}\right\rangle \\ &+ t\,(t-1)\,n\,(n-1)\left\langle R_t^m; \mu, [\mu,\mathrm{d}\varepsilon], R_t^{n-2}\right\rangle\Big]. \end{aligned} \tag{6.17}$$

Eqs. (B.4)-(B.6) from appendix B can be used to write $\delta Q_{q-1}(\mu)$ in a more convenient way. By plugging in R_t and ε into these equations, it is possible to show that they satisfy the following identities, when contracting with the invariant tensor of the FDA1

$$0 = \left\langle R_t^m; [R_t,\varepsilon], \mu, R_t^{n-2}\right\rangle, \tag{6.18}$$

$$0 = \left\langle R_t^m; [\mu,\varepsilon], R_t^{n-1}\right\rangle + m\left\langle \mu, R_t^{m-1}; [R_t,\varepsilon], R_t^{n-1}\right\rangle, \tag{6.19}$$

$$0 = m\left\langle [\mu, R_t], R_t^{m-1}; \mathrm{d}\varepsilon, \mu, R_t^{n-2}\right\rangle + \left\langle R_t^m; [\mu,\mathrm{d}\varepsilon], \mu, R_t^{n-2}\right\rangle$$

$$+ (-1)^p \left\langle R_t^m; \mathrm{d}\varepsilon, [\mu, \mu], R_t^{n-2} \right\rangle + (n-2) \left\langle R_t^m; \mathrm{d}\varepsilon, \mu, [\mu, R_t], R_t^{n-3} \right\rangle, \qquad (6.20)$$

$$\begin{aligned} 0 = &\left\langle [\mu, \mu], R_t^{m-1}; \mathrm{d}\varepsilon, R_t^{n-1} \right\rangle - (m-1) \left\langle \mu, [\mu, R_t], R_t^{m-2}; \mathrm{d}\varepsilon, R_t^{n-1} \right\rangle \\ &- \left\langle \mu, R_t^{m-1}; [\mu, \mathrm{d}\varepsilon], R_t^{n-1} \right\rangle + (-1)^{p+1} (n-1) \left\langle \mu, R_t^{m-1}; \mathrm{d}\varepsilon, [\mu, R_t], R_t^{n-2} \right\rangle. \end{aligned} \qquad (6.21)$$

Eqs. (6.18)-(6.21) allow to remove the terms that include FDA1 products between the gauge field and the derivatives of the parameters. Therefore, the extended gauge variation of the Chern–Simons form takes the following form

$$\begin{aligned} \delta Q_{q-1}(\mu) = \sum_{m,n} \int_0^1 \mathrm{d}t\, t(t-1) \Big[& mn \left\langle [\mu, \mu], R_t^{m-1}; \mathrm{d}\varepsilon, R_t^{n-1} \right\rangle \\ & - mn (m-1) \left\langle \mu, [\mu, R_t], R_t^{m-2}; \mathrm{d}\varepsilon, R_t^{n-1} \right\rangle \\ & + mn (n-1) (-1)^{p+1} \left\langle \mu, R_t^{m-1}; \mathrm{d}\varepsilon, [\mu, R_t], R_t^{n-2} \right\rangle \\ & + n \left\langle R_t^m; \mathrm{d}\varepsilon, R_t^{n-1} \right\rangle + n(n-1)\, m (-1)^p \left\langle [\mu, R_t], R_t^{m-1}; \mathrm{d}\varepsilon, \mu, R_t^{n-2} \right\rangle \\ & + n(n-1) \left\langle R_t^m; \mathrm{d}\varepsilon, [\mu, \mu], R_t^{n-2} \right\rangle \\ & + n(n-1)(n-2)(-1)^p \left\langle R_t^m; \mathrm{d}\varepsilon, \mu, [\mu, R_t], R_t^{n-3} \right\rangle \Big]. \end{aligned} \qquad (6.22)$$

In order to write the r.h.s. of eq. (6.22) as a total exterior derivative, it is necessary to use the following properties:

1. Let us recall the generalized Bianchi identities for a FDA1. From eqs. (3.47) and (3.48) it follows that the components of the FDA1 bracket between the homotopic gauge fields and curvatures can be written as follows

$$[\mu_t, R_t]^A = -\mathrm{d}R_t^A, \qquad (6.23)$$

$$[\mu_t, R_t]^i = -\mathrm{d}R_t^i + [R_t, \mu_t]^i + \frac{1}{p!} [R_t, \mu_t^p]^i. \qquad (6.24)$$

2. Let us now consider the definition of gauge curvatures from eqs. (3.11) and (3.12) corresponding to the homotopic gauge fields $\mu_t = t\mu$. By directly taking the derivative with respect to the parameter t one finds

$$t[\mu, \mu]^A = \frac{\partial R_t^A}{\partial t} - \mathrm{d}\mu^A, \qquad (6.25)$$

$$t[\mu, \mu]^i = \frac{1}{2} \frac{\partial R_t^i}{\partial t} - \frac{1}{2} \mathrm{d}\mu^i - \frac{t^p}{2p!} \left[\mu^{p+1}\right]^i. \qquad (6.26)$$

3. The generalized invariant tensor conditions for the FDA1 (B.4)-(B.6) lead to

the following identities

$$\left\langle R_t^m; [\mu,\mu_t], \mathrm{d}\varepsilon, R_t^{n-2}\right\rangle - m\left\langle \mu, R_t^{m-1}; \mathrm{d}\varepsilon, [R_t,\mu_t], R_t^{n-2}\right\rangle = 0, \tag{6.27}$$

$$\left\langle R_t^m; [\mu,\mu_t], \mathrm{d}\varepsilon, R_t^{n-2}\right\rangle + m\left\langle \mu, R_t^{m-1}; [R_t,\mu_t], \mathrm{d}\varepsilon, R_t^{n-2}\right\rangle = 0. \tag{6.28}$$

By plugging in the relations from eqs. (6.23)-(6.28) into eq. (6.22), it is possible to write the extended variation of the Chern–Simons form in terms of an exterior derivative and a total derivative with respect to the homotopic parameter $\partial/\partial t$

$$\begin{aligned}\delta Q_{q-1}(\mu) = \sum_{m,n}\int_0^1 \mathrm{d}t \Big[& n\frac{\partial}{\partial t}\left\{(t-1)\left\langle R_t^m; \mathrm{d}\varepsilon, R_t^{n-1}\right\rangle\right\} \\ & -(t-1)\, mn\,(-1)^{(p+1)(n-1)}\left\langle \mathrm{d}\left(\mu, R_t^{m-1}; R_t^{n-1}\right), \mathrm{d}\varepsilon\right\rangle \\ & -(t-1)\,(-1)^{(p+1)(n-1)}\, n\,(n-1)\left\langle \mathrm{d}\left(R_t^m;\mu, R_t^{n-2}\right), \mathrm{d}\varepsilon\right\rangle\Big].\end{aligned} \tag{6.29}$$

The first term on the r.h.s. of eq. (6.29) can be immediately integrated due to Stoke's theorem. The second and third terms are exact forms. This removes the dependence of $\delta Q_{q-1}(\mu)$ on non-exact forms and allows to write it in terms of a secondary $(q-2)$-form. This secondary form is analogue to the anomaly term from eq. (6.10). Since it turns out to be proportional to the extended gauge parameter ε^i, we denote it as $\omega^1_{q-2}\left(\varepsilon^i,\mu\right)$. This is summarized in the following relation

$$\delta_{\text{Extended}} Q_{q-1}(\mu) = \mathrm{d}\omega^1_{q-2}\left(\varepsilon^i,\mu\right). \tag{6.30}$$

Here, the extended anomalous term for the FDA1 is explicitly given by

$$\begin{aligned}\omega^1_{q-2}\left(\varepsilon^i,\mu\right) &= \sum_{m,n}\int_0^1 \mathrm{d}t\,(1-t)\,n\left(m\left\langle \mathrm{d}\left(\mu, R_t^{m-1}; R_t^{n-1}\right), \varepsilon\right\rangle\right. \\ &\quad \left. +(n-1)\left\langle \mathrm{d}\left(R_t^m;\mu,R_t^{n-2}\right),\varepsilon\right\rangle\right).\end{aligned} \tag{6.31}$$

As it happens with Lie algebras, the $(q-2)$-form $\omega^1_{q-2}\left(\varepsilon^i,\mu\right)$ is linear in the gauge parameter, in this case the extended one ε^i. Since it does not involve the standard parameter ε^A, eq. (6.31) does not reproduce the standard equation from eq. (6.5) as a particular case. However, it shares its functional form and topological origin. In order to recover the standard anomaly term, it is therefore necessary to study the variation of the Chern–Simons form with respect to the standard parameter.

6.3 Standard variations

In the case of the standard gauge variations, i.e., those with parameters $\varepsilon^i = 0$ and $\varepsilon^A \neq 0$, there is an important difference with respect to the previous case. In order to isolate the anomaly term that comes from the variation of the FDA1-Chern–Simons form, it is convenient to write the FDA1-Chern–Simons form in terms of new homotopic gauge fields and curvatures. Let us then consider a FDA1-valued gauge field $\mu = \left(\mu^A, \mu^i\right)$ and its corresponding gauge field strength $R = \left(R^A, R^i\right)$. We now redefine the exterior derivative operator by means of its action on the components of R and μ as follows [37]

$$\mathrm{d}\mu^A = R^A - \frac{1}{2}\left[\mu, \mu\right]^A, \tag{6.32}$$

$$\mathrm{d}\mu^i = R^i - \left[\mu, \mu\right]^i - \frac{1}{(p+1)!}\left[\mu^{p+1}\right]^i. \tag{6.33}$$

Notice that the nilpotence condition of d is automatically verified because of the Jacobi identity of the FDA1 products at the r.h.s of eqs. (6.32) and (6.33).

Let us now introduce an arbitrary variation of μ (and therefore of R) denoted by $\delta\mu$. With respect to that variation, we introduce the homotopy operator ℓ, such that its action on μ and R is given by

$$\ell\mu = 0, \tag{6.34}$$

$$\ell R = \delta\mu. \tag{6.35}$$

By directly applying the homotopy operator and the exterior derivative operator into eqs. (6.32) and (6.33), it is possible to prove that it satisfies the following anticommutation relations

$$\left(\ell \mathrm{d} + \mathrm{d}\ell\right)\mu = \delta\mu, \tag{6.36}$$

$$\left(\ell \mathrm{d} + \mathrm{d}\ell\right)R = \delta R. \tag{6.37}$$

We now introduce homotopic gauge fields μ_t^A and μ_t^i depending on a real parameter $t \in [0,1]$. The one-form μ^A is parametrized as usual, i.e.,

$$\mu_t^A = t\mu^A. \tag{6.38}$$

Moreover, the extended gauge field carries a different parametrization. It is defined as proportional to a power of the parameter, such that μ_t is interpolated along a convenient trajectory in the parametric space

$$\mu_t^i = t^p \mu^i. \tag{6.39}$$

As before, the homotopic gauge field is interpolated between 0 and μ_1 as the parameter t varies between 0 and 1. We now consider the variation δ as the parametric variation i.e., $\delta = \mathrm{d}_t$. In consequence, the homotopic operator from eqs. (6.34) and (6.35) defined with respect to such variation (from now on denoted by l_t) satisfies the anticommutation relation $l_t \mathrm{d} + \mathrm{d} l_t = \mathrm{d}_t$. Therefore, the action of l_t on gauge field and field strength in eqs. (6.34) and (6.35) take the following form

$$l_t \mu_t = 0, \tag{6.40}$$

$$l_t R_t = \mathrm{d}_t \mu_t = \mathrm{d}t \frac{\partial \mu_t}{\partial t}. \tag{6.41}$$

From these definitions, it follows that the homotopic operator satisfies the following property when it is integrated along the parametric space $[0,1]$

$$\int_0^1 (l_t \mathrm{d} + \mathrm{d} l_t) = \int_0^1 \mathrm{d}_t. \tag{6.42}$$

By applying the l.h.s. of (6.42) into the gauge invariant $\chi_q(\mu_t)$ constructed with the homotopic gauge field, and by using the Stokes' theorem, we recover the generalized Chern–Weil theorem for the FDA1, which relates the gauge invariant density with the total exterior derivative of a $(q-1)$-form

$$\chi_q(\mu) = \mathrm{d} \int_0^1 l_t \chi_q(\mu_t) = \mathrm{d} Q_{q-1}(\mu). \tag{6.43}$$

Notice that in this case, the integration is performed along a different parametric trajectory. Eq. (6.43) allows finding alternative expressions for the FDA1-Chern–Simons form by directly applying the homotopy operator l_t in $\chi_q(\mu_t)$ and neglecting the exterior derivatives on both sides of the equation. This particular choice of the homotopic trajectory leads to an expression for the FDA1-Chern–Simons form that functionally depends on the parameter according to eqs. (6.38) and (6.39), as follows

$$Q_{q-1}(\mu) = \sum_{m,n} \int_0^1 \mathrm{d}t \left(m \left\langle \mu, R_t^{m-1}; R_t^n \right\rangle + npt^{p-1} \left\langle R_t^m; \mu, R_t^{n-1} \right\rangle \right). \tag{6.44}$$

The new expression for the FDA1-Chern–Simons form is equivalent to the one previously obtained in eq. (6.9). Notice that the standard component of the homotopic curvature R_t^A remains the same but the extended one changes. However, since the final expression is independent of the integration, it is natural to have different choices. The new homotopic path, and therefore, the new expression in eq. (6.44) is chosen such that the anomaly is more easily obtained. In this case, the variation of the FDA1-Chern–Simons form along the standard parameter ε^A takes the following form

$$\begin{aligned}\delta Q_{q-1}(\mu) = \sum_{m,n}\int_0^1 \mathrm{d}t \big[& m \left\langle \delta\mu, R_t^{m-1}; R_t^n \right\rangle + m(m-1)\left\langle \mu, \delta R_t, R_t^{m-2}; R_t^n \right\rangle \\ & + mn\left\langle \mu, R_t^{m-1}; \delta R_t, R_t^{n-1} \right\rangle + mnpt^{p-1}\left\langle \delta R_t, R_t^{m-1}; \mu, R_t^{n-1} \right\rangle \\ & + npt^{p-1}\left\langle R_t^m; \delta\mu, R_t^{n-1} \right\rangle + n(n-1)pt^{p-1}\left\langle R_t^m; \mu, \delta R_t, R_t^{n-2} \right\rangle \big] . \end{aligned} \tag{6.45}$$

Eq. (6.45) depends only on the variations of the homotopic field strengths and the non-homotopic gauge fields. The homotopic gauge fields are not explicitly included in the variation. Therefore, let us consider the gauge variation of the gauge fields along the standard parameter ε^A. By setting $\varepsilon^i = 0$ in eqs (3.35) and (3.36) one finds

$$\delta\mu^A = \mathrm{d}\varepsilon^A + [\mu, \varepsilon]^A , \tag{6.46}$$

$$\delta\mu^i = -[\varepsilon, \mu_t]^i - \frac{1}{p!}[\varepsilon, \mu^p]^i . \tag{6.47}$$

On the other hand, from eqs. (3.45) and (3.46) one finds that the standard variations of the components of the homotopic field strength are given by

$$\delta R_t^A = [R_t, \varepsilon]^A + \left(t^2 - t\right)[\mathrm{d}\varepsilon, \mu]^A , \tag{6.48}$$

$$\begin{aligned}\delta R_t^i = & -[\varepsilon, R_t]^i - \frac{t^{p-1}}{(p-1)!}\left[\varepsilon, R_t, \mu^{p-1}\right]^i + t^p(t-1)[\mathrm{d}\varepsilon, \mu]^i \\ & + \frac{t^p(t-1)}{p!}[\mathrm{d}\varepsilon, \mu^p]^i . \end{aligned} \tag{6.49}$$

Different choices of the homotopy rule in eqs. (6.38) and (6.39) lead to a different expressions for δR_t^i. The chosen homotopic trajectory is particularly useful because it allows writing eq. (6.49) in terms of the components of R_t, without explicit dependence on the derivatives of the gauge fields. Notice that this situation does not happen when calculating the extended anomaly term; therefore, we only introduce the new homotopy rule for the anomaly resulting from the standard variations. As before, the total derivative in eq. (6.45) has to be isolated by using the invariant tensor conditions in eqs. (B.4)-(B.6). Such equations allow us to prove the following relations for the components of ε, μ and R:

$$\begin{aligned}0 = & \left\langle [\mu, \mu], \mathrm{d}\varepsilon, R_t^{m-2}; R_t^n \right\rangle - \left\langle \mu, [\mathrm{d}\varepsilon, \mu], R_t^{m-2}; R_t^n \right\rangle \\ & + (m-2)\left\langle \mu, \mathrm{d}\varepsilon, [\mu, R_t], R_t^{m-3}; R_t^n \right\rangle + n\left\langle \mu, \mathrm{d}\varepsilon, R_t^{m-2}; [\mu, R_t], R_t^{n-1} \right\rangle , \end{aligned} \tag{6.50}$$

$$\begin{aligned} 0 &= \left\langle [\mathrm{d}\varepsilon, \mu], R_t^{m-1}; \mu, R_t^{n-1} \right\rangle - (m-1) \left\langle \mathrm{d}\varepsilon, [\mu, R_t], R_t^{m-2}; \mu, R_t^{n-1} \right\rangle \\ &- \left\langle \mathrm{d}\varepsilon, R_t^{m-1}; [\mu, \mu], R_t^{n-1} \right\rangle - (-1)^p (n-1) \left\langle \mathrm{d}\varepsilon, R_t^{m-1}; \mu, [\mu, R_t], R_t^{n-2} \right\rangle, \end{aligned} \tag{6.51}$$

$$\begin{aligned} 0 &= \left\langle \mu, R_t^{m-1}; [\mathrm{d}\varepsilon, \mu^p], R_t^{n-1} \right\rangle - \left\langle \mathrm{d}\varepsilon, R_t^{m-1}; \left[\mu^{p+1}\right], R_t^{n-1} \right\rangle \\ &- (m-1) \left\langle \mathrm{d}\varepsilon, \mu, R_t^{m-2}; [R_t, \mu^p], R_t^{n-1} \right\rangle, \end{aligned} \tag{6.52}$$

$$\begin{aligned} 0 &= \left\langle \mu, R_t^{m-1}; [\mathrm{d}\varepsilon, \mu], R_t^{n-1} \right\rangle - \left\langle \mathrm{d}\varepsilon, R_t^{m-1}; [\mu, \mu], R_t^{n-1} \right\rangle \\ &- (m-1) \left\langle \mathrm{d}\varepsilon, \mu, R_t^{m-2}; [R_t, \mu], R_t^{n-1} \right\rangle, \end{aligned} \tag{6.53}$$

$$0 = \left\langle R_t^m; \mu, [\mathrm{d}\varepsilon, \mu], R_t^{n-2} \right\rangle - (-1)^p m \left\langle \mathrm{d}\varepsilon, R_t^{m-1}; [R_t, \mu], \mu, R_t^{n-2} \right\rangle, \tag{6.54}$$

$$0 = \left\langle R_t^m; [\mathrm{d}\varepsilon, \mu^p], \mu, R_t^{n-2} \right\rangle + m \left\langle \mathrm{d}\varepsilon, R_t^{m-1}; [R_t, \mu^p], \mu, R_t^{n-2} \right\rangle. \tag{6.55}$$

Eqs. (6.50)-(6.55) are analogue to eqs. (6.27)-(6.28) for the standard case and allow to remove the dependence of $\delta Q_{q-1}(\mu)$ on terms carrying components of the FDA1 product between $\mathrm{d}\varepsilon$ and μ. By plugging them in into the standard variation of the FDA1-Chern–Simons form, it takes the form

$$\begin{aligned} \delta Q_{q-1}(\mu) = \sum_{m,n} \int_0^1 & \mathrm{d}t \; m \left[\left\langle \mathrm{d}\varepsilon, R_t^{m-1}; R_t^n \right\rangle \right. \\ &+ (t-1) \Big((m-1)\, t\, (m-2) \left\langle \mu, \mathrm{d}\varepsilon, [\mu, R_t], R_t^{m-3}; R_t^n \right\rangle \\ &+ (m-1)\, tn \left\langle \mu, \mathrm{d}\varepsilon, R_t^{m-2}; [\mu, R_t], R_t^{n-1} \right\rangle \\ &+ t^p n (m-1) \left\langle \mathrm{d}\varepsilon, \mu, R_t^{m-2}; [R_t, \mu], R_t^{n-1} \right\rangle \\ &+ \frac{t^p}{p!} n (m-1) \left\langle \mathrm{d}\varepsilon, \mu, R_t^{m-2}; [R_t, \mu^p], R_t^{n-1} \right\rangle \\ &+ n (n-1)\, p t^{2p-1} (-1)^p \left\langle \mathrm{d}\varepsilon, R_t^{m-1}; [R_t, \mu], \mu, R_t^{n-2} \right\rangle \\ &+ \frac{t^p}{p!} n (n-1)\, p t^{p-1} (-1)^p \left\langle \mathrm{d}\varepsilon, R_t^{m-1}; [R_t, \mu^p], \mu, R_t^{n-2} \right\rangle \\ &+ npt^{p-1} t (m-1) \left\langle \mathrm{d}\varepsilon, [\mu, R_t], R_t^{m-2}; \mu, R_t^{n-1} \right\rangle \\ &+ (m-1)\, t \left\langle [\mu, \mu], \mathrm{d}\varepsilon, R_t^{m-2}; R_t^n \right\rangle \\ &+ t^p (1+p)\, n \left\langle \mathrm{d}\varepsilon, R_t^{m-1}; \{[\mu, \mu]\}, R_t^{n-1} \right\rangle \\ &+ \frac{t^p}{p!} n \left\langle \mathrm{d}\varepsilon, R_t^{m-1}; \left[\mu^{p+1}\right], R_t^{n-1} \right\rangle \\ &+ \left. (-1)^p npt^{p-1} t (n-1) \left\langle \mathrm{d}\varepsilon, R_t^{m-1}; \mu, [\mu, R_t], R_t^{n-2} \right\rangle \Big) \right]. \end{aligned} \tag{6.56}$$

The last step consists of isolating the exterior derivative by using the analogue of eqs. (6.23)-(6.26). In this case, using the definition of homotopic gauge curvatures and the generalized Bianchi identities, we get the following identities for the new homotopy rule

$$t\left[\mu,\mu\right]^{A} = \frac{\partial R_t^A}{\partial t} - \mathrm{d}\mu^A, \tag{6.57}$$

$$(p+1)\,t^p\left[\mu,\mu\right]^{i} + \frac{t^p}{p!}\left[\mu,\ldots,\mu\right]^{i} = \frac{\partial R_t^i}{\partial t} - pt^{p-1}\mathrm{d}\mu^i, \tag{6.58}$$

$$\left[\mu_t,R_t\right]^{A} = -\mathrm{d}R_t^A, \tag{6.59}$$

$$\left[\mu_t,R_t\right]^{i} - \left[R_t,\mu_t\right]^{i} - \frac{1}{p!}\left[R_t,\mu_t^p\right]^{i} = -\mathrm{d}R_t^i. \tag{6.60}$$

We now perform an integration by parts with respect to the exterior derivatives d and d_t. This allows us to write the variation of $Q_{q-1}(\mu)$ as an exact form. The anomalous term emerges in the following way

$$\delta_{\text{Standard}}Q_{q-1}(\mu) = \mathrm{d}\omega_{q-2}^1\left(\varepsilon^A,\mu\right). \tag{6.61}$$

The secondary form $\omega_{q-2}^1\left(\varepsilon^A,\mu\right)$ is explicitly given in terms of the new homotopy rule as follows

$$\begin{aligned}\omega_{q-2}^1\left(\varepsilon^A,\mu\right) \;=\; & \sum_{m,n}\int_0^1 \mathrm{d}t\,(1-t)\,m\left\{(m-1)\left\langle\varepsilon,\mathrm{d}\left(\mu,R_t^{m-2};R_t^n\right)\right\rangle\right.\\ & \left.+npt^{p-1}\left\langle\varepsilon,\mathrm{d}\left(R_t^{m-1};\mu,R_t^{n-1}\right)\right\rangle\right\}.\end{aligned} \tag{6.62}$$

In summary, eqs. (6.31) and (6.62) explicitly show the total gauge variation of the Chern–Simons form for the FDA1. The total variation takes in account both independent parameters ε^i and ε^A. Although both expressions are similar in their integral forms, it is important to recall that they use different definitions of the homotopic curvature R_t. Each case performs the integration along a different homotopic trajectory: these two choices are equivalent and, in both cases, they allow convenient isolation of the exterior derivatives, and consequently, of the anomalous term. The generalized anomaly term from eq. (6.62) originated in the standard variations, reproduces the anomaly term corresponding to Lie algebras from eq. (6.5). This particular case is reobtained in the first term on the sum for $n=0$ which does not depend on the p-form gauge field.

Chapter 7

L_∞ formulation of Chern–Simons theories

In this chapter, we consider the formulation of arbitrary-dimensional Chern–Simons theories in terms of L_∞ algebras. For this purpose, we consider the L_∞ formulation of classical gauge theories introduced in section 4.4. Let us therefore consider a $2m-1$ dimensional Chern–Simons theory invariant under a Lie algebra G. We define a basis of vectors for G, denoted by $\{t_A\}_{A=1}^{\dim G}$ and introduce the fundamental field of the theory as a one-form gauge connection, denoted by $\mu = \mu_\mu^A \mathrm{d}x^\mu t_A$ defined on a principal bundle (G, M), where M is a $2m-1$ dimensional spacetime manifold. The corresponding action principle is defined as a functional integral over M that makes use of the Chern–Simons $(2m-1)$-form from eq. (2.54) as Lagrangian density

$$S_{\mathrm{CS}} = m \int_M \int_0^1 \mathrm{d}t \, \langle \mu, R_t^{m-1} \rangle_{\mathrm{Lie}} \, . \tag{7.1}$$

Here, the bracket $\langle , \rangle_{\mathrm{Lie}}$ denotes the symmetrized trace acting on the vectors of the Lie algebra G, being the differential form inside the integrals of eqs. (7.1) and (2.54) equivalent. The details of the change of notation from the invariant tensor to the bracket $\langle , \rangle_{\mathrm{Lie}}$ can be found in appendix A. Moreover, in this case, the homotopic gauge field is defined as $\mu_t = t\mu$, being $R_t = \left(t^2 - t\right) R$ its corresponding gauge curvature. This theory is odd-dimensional and does not depend on a background metric. In the following sections, we extract the information concerning the definition of gauge transformations, the gauge algebra, and the equations of motion that come from the variation of eq. (7.1), and write it in terms of the products of an L_∞ algebra.

7.1 Gauge transformations

We begin by considering the definition of gauge transformations. In this case, the gauge symmetry is described by a Lie algebra, and the gauge variations of the fundamental field are given by the Lie-covariant derivative of a zero-form gauge parameter ε valued in the Lie algebra G. In components, the transformations of the gauge field μ_μ^A is given by eq. (3.35). Therefore, we consider an L_∞ algebra with a graded vector space $X = X_0 \oplus X_{-1} \oplus X_{-2}$ and identify the parameters of the transformation and gauge fields as vectors lying on the first and second subspaces, respectively, i.e., $\varepsilon^A \in X_0$ and $\mu_\mu^A \in X_{-1}$. In the L_∞ formulation of gauge theories, the gauge transformation of μ_μ^A is given in eq. (4.45) as an expansion in terms of the L_∞ products. By inspection of eq. (3.35) it is direct to see that every contribution depending on μ^2 or higher powers of the gauge field must vanish in the expansion of eq. (4.45), leading to the following expression

$$\delta\mu_\mu^A = \left[\ell_1\left(\varepsilon\right)\right]_\mu^A + \left[\ell_2\left(\varepsilon,\mu\right)\right]_\mu^A . \tag{7.2}$$

Then, by comparing eqs. (4.45) and (7.2) we obtain the following information concerning the L_∞ products of elements belonging to the subspaces X_0 and X_{-1}

$$\left[\ell_1\left(\varepsilon\right)\right]_\mu^A = \partial_\mu\varepsilon^A, \tag{7.3}$$

$$\left[\ell_2\left(\varepsilon,\mu\right)\right]_\mu^A = \left[\mu_\mu,\varepsilon\right]^A . \tag{7.4}$$

This shows that any other product originated in the expansion of eq. (4.45), and therefore involving one vector from X_0 and any number of vectors from X_{-1}, vanishes.

7.2 Gauge algebra

The second source of information that must be written in terms of the L_∞ products is the gauge algebra. Thus, it is necessary to consider the commutator between two independent gauge transformations. If the gauge theory is well-defined, the gauge transformations close an algebra whose relations can be written in terms of the products of the subalgebra L_∞^{gauge}, which in this case can be immediately identified as the Lie algebra G. From eq. (3.35) it is possible to prove that the commutator between two independent gauge transformations δ_1 and δ_2 defined in terms of independent gauge parameters ε_1^A and ε_2^A, is given by

$$(\delta_2\delta_1 - \delta_1\delta_2)\,\mu_\mu^A = \partial_\mu\varepsilon_3^A + C^A_{DB}\mu_\mu^D\varepsilon_3^B. \tag{7.5}$$

Thus, the commutator of two gauge transformations turns out to be equivalent to a third gauge transformation depending on a composite gauge parameter ε_3 given by the Lie product between the original parameters, whose components are

$$\varepsilon_3^A = C^A_{BC}\varepsilon_2^B\varepsilon_1^C. \tag{7.6}$$

Let us now consider the writing of the same commutator in terms of its the expansion in L_∞ products. Eqs. (4.52) and (4.53) show that the expansion can be truncated by considering only those terms that are linear or do not depend on the components of the gauge field, as follows

$$(\delta_2\delta_1 - \delta_1\delta_2)\,\mu_\mu^A = \left[\ell_1\left(\ell_2\left(\varepsilon_1,\varepsilon_2\right)\right)\right]_\mu^A + \left[\ell_1\left(\ell_3\left(\varepsilon_1,\varepsilon_2,\mu\right)\right)\right]_\mu^A + \left[\ell_2\left(\ell_2\left(\varepsilon_1,\varepsilon_2\right),\mu\right)\right]_\mu^A. \tag{7.7}$$

Notice that $\ell_2(\varepsilon_1,\varepsilon_2)$ and $\ell_3(\varepsilon_1,\varepsilon_2,\mu)$ are vectors belonging to X_0, and therefore, eq. (7.3) implies that ℓ_1 acts in the following way on them

$$\left[\ell_1\left(\ell_2\left(\varepsilon_1,\varepsilon_2\right)\right)\right]_\mu^A = \partial_\mu\left[\ell_2\left(\varepsilon_1,\varepsilon_2\right)\right]^A, \tag{7.8}$$

$$\left[\ell_1\left(\ell_3\left(\varepsilon_1,\varepsilon_2,\mu\right)\right)\right]_\mu^A = \partial_\mu\left[\ell_3\left(\varepsilon_1,\varepsilon_2,\mu\right)\right]^A. \tag{7.9}$$

By replacing eqs. (7.8) and (7.9) into eq. (7.7) and comparing with eq. (7.5), we obtain the following information concerning L_∞ products between vectors in the subspace X_0

$$\left[\ell_2\left(\varepsilon_1,\varepsilon_2\right)\right]^A = C^A_{BC}\varepsilon_2^B\varepsilon_1^C, \tag{7.10}$$

Any other product of vectors lying exclusively in X_0, or in both subspaces X_0 and X_{-1}, vanishes. As it was anticipated, this shows that the gauge subalgebra L_∞^{gauge} is indeed the Lie algebra G. Eqs. (7.3), (7.4) and (7.10) contain the information about the gauge transformations and gauge algebra, and therefore, this subspace structure is shared by every Lie gauge theory. The next step is to include the information that is intrinsic to the theory, namely, the dynamics, and then to demand the resulting products to satisfy the L_∞ identities.

7.3 Equations of motion

Let us now consider the equations of motion coming from the field variation of the $2m-1$ dimensional Chern–Simons action. With this purpose, we start by identifying the inner product of the L_∞ algebra in the variation of the action principle. By taking the variation in eq. (6.9) with respect to the gauge field and performing the integration with respect to the parameter, one finds

$$\begin{aligned}\delta S_{\rm CS} = \int \mathrm{d}x^{2m-1} \sum_{k=0}^{m-1} \frac{1}{2^k} \binom{m-1}{k} \varepsilon^{\mu_1 \cdots \mu_{2m-1}} \\ g_{AB_1\cdots B_{m-1}} \delta\mu^A_{\mu_1} \partial_{\mu_2}\mu^{B_1}_{\mu_3} \cdots \partial_{\mu_{2m-2k-3}} \mu^{B_{m-k-1}}_{\mu_{m-k+1}} \\ \left[\mu_{\mu_{2m-2k-1}}, \mu_{\mu_{2m-2k}}\right]^{B_{m-k}} \cdots \left[\mu_{\mu_{2m-2}}, \mu_{\mu_{2m-1}}\right]^{B_{m-1}} .\end{aligned} \tag{7.11}$$

Here, $\varepsilon^{\mu_1 \cdots \mu_{2m-1}}$ denotes the Levi–Civita pseudotensor and $[,]^A$ denotes A-component of the Lie bracket. This means that, given zero forms valued in the Lie algebra $x, y \in G$ with components x^A and y^A in the basis of vectors $\{t_A\}_{A=1}^{\dim G}$, such component is given by

$$[x, y]^A = C^A_{BC} x^B y^C. \tag{7.12}$$

The components of the Lie algebra invariant tensor, denoted by $g_{A_1 \cdots A_m}$, are given by the trace over the basis of the Lie algebra in the mentioned basis, and it can be understood as a multilinear product of Lie valued vectors. In contrast, the inner product of the L_∞ algebra is bilinear. In order to write one of them in terms of the other, let us consider two vectors valued on the Lie algebra, belonging to the following subspaces in the L_∞ formalism

$$x_\mu \in X_{-1}, \tag{7.13}$$

$$y_\mu \in X_{-2}. \tag{7.14}$$

By comparing both relations expressing the variation of the action principle in eqs. (4.50) and (7.11), we define the inner product of the L_∞ algebra in terms of the invariant tensor of the Lie algebra, as follows [41]

$$\langle x, y \rangle_{L_\infty} = \int \mathrm{d}x^{2m-1} \eta^{\mu\nu} \langle x_\mu, y_\nu \rangle_{\rm Lie} . \tag{7.15}$$

This identification is consistent with the general definition of inner product for L_∞ algebras and reproduces eq. (4.50) for the case of the Chern–Simons theory, i.e., it allows writing the variation of the action principle in terms of an L_∞ inner product, as follows

$$\delta S_{\mathrm{CS}} = \int \mathrm{d}x^{2m-1} \eta^{\mu\nu} g_{AB} \delta\mu^A_\mu \mathcal{F}^B_\nu, \tag{7.16}$$

Hence, eq. (7.11) isolates the equation of motion term in eq. (7.11), obtaining an explicit expression for $\mathcal{F}^A_\nu$, as follows

$$\begin{aligned}\mathcal{F}^A_\nu =& \sum_{k=0}^{m-1} \frac{1}{2^k} \binom{m-1}{k} \varepsilon_\nu{}^{\mu_1\cdots\mu_{2m-2}} g^A{}_{B_1\cdots B_{m-1}} \partial_{\mu_1}\mu^{B_1}_{\mu_2} \cdots \partial_{\mu_{2m-2k-3}} \mu^{B_{m-k-1}}_{\mu_{2m-2k-2}} \\ &\times \left[\mu_{\mu_{2m-2k-1}}, \mu_{\mu_{2m-2k}}\right]^{B_{m-k}} \cdots \left[\mu_{\mu_{2m-3}}, \mu_{\mu_{2m-2}}\right]^{B_{m-1}}.\end{aligned} \tag{7.17}$$

Once obtained the equation of motion term $\mathcal{F}$ in terms of the gauge fields, the next step is to identify the contributions corresponding to different L_∞ products in eq. (4.51). Thus, we write down the expansion of $\mathcal{F}^A_\nu$ in terms of l-linear products and compare both expressions term by term, as follows

$$\begin{aligned}&\sum_{l=1}^{\infty} \frac{(-1)^{\frac{l(l-1)}{2}}}{l!} \left[\ell_l\left(\mu^l\right)\right]^A_\nu \\ &= \sum_{k=0}^{m-1} \frac{1}{2^k} \binom{m-1}{k} \varepsilon_\nu{}^{\mu_1\cdots\mu_{2m-2}} g^A{}_{B_1\cdots B_{m-1}} \partial_{\mu_1}\mu^{B_1}_{\mu_2}, \cdots \partial_{\mu_{2m-2k-3}} \mu^{B_{m-k-1}}_{\mu_{2m-2k-2}} \\ &\quad \times \left[\mu_{\mu_{2m-2k-1}}, \mu_{\mu_{2m-2k}}\right]^{B_{m-k}} \cdots \left[\mu_{\mu_{2m-3}}, \mu_{\mu_{2m-2}}\right]^{B_{m-1}}.\end{aligned} \tag{7.18}$$

By inspection of both sides of eq. (7.18) one gets that the series is truncated, giving place to a certain finite number of non-vanishing products in the dynamical sector of the L_∞ algebra. The number of these products depends on the dimension of the Chern–Simons theory. In order to obtain an expression for the L_∞ product of k gauge fields (i.e., k vectors in X_{-1}), we compare terms of equal powers of μ. The k-th element in the sum of the r.h.s. of eq. (7.18) depends on powers of degree $m+k-1$ in the components of the gauge field. Since there is only one term of that power on each side of the equation, we match them one by one. Thus, given a fixed value of m, k take values in a different domain ($k = 0, \ldots, m-1$). Then, for a given value k, the value of l is fixed and given by

$$l = m + k - 1. \tag{7.19}$$

Therefore, the non-vanishing contributions to the equation of motion terms in eq. (4.51) come from ℓ_l products relating vectors of the subspace X_{-1} with $l \in [m-1, \ldots, 2m-2]$, which are explicitly given by

$$
\begin{aligned}
\left[\ell_l\left(\mu^l\right)\right]^A_\nu = {} & (-1)^{\frac{l(l-1)}{2}} \frac{1}{2^{l-m+1}} \frac{l!\,(m-1)!}{(2m-l-2)!\,(l-m+1)!} \varepsilon_\nu{}^{\mu_1\cdots\mu_{2m-2}} \\
& \times g^A{}_{B_1\cdots B_{m-1}} \partial_{\mu_1}\mu^{B_1}_{\mu_2}\cdots\partial_{\mu_{4m-2l-5}}\mu^{B_{2m-l-2}}_{\mu_{4m-2l-4}} \\
& \times \left[\mu_{\mu_{4m-2l-3}}, \mu_{\mu_{4m-2l-2}}\right]^{B_{2m-l-1}} \cdots \left[\mu_{\mu_{2m-3}}, \mu_{\mu_{2m-2}}\right]^{B_{m-1}} .
\end{aligned} \tag{7.20}
$$

7.4 Summary

In summary, the L_∞ algebra of the standard $2m+1$ dimensional Chern–Simons theory is constructed as a vector space $X = X_0 \oplus X_{-1} \oplus X_{-2}$ endowed with the following products encoding the gauge transformations and gauge symmetry

$$
\boxed{
\begin{aligned}
\left[\ell_1(\varepsilon)\right]^A_\mu &= \partial_\mu \varepsilon^A, \\
\left[\ell_2(\varepsilon,\mu)\right]^A_\mu &= \left[\mu_\mu, \varepsilon\right]^A, \\
\left[\ell_2(\varepsilon_1,\varepsilon_2)\right]^A &= \left[\varepsilon_2, \varepsilon_1\right]^A, \\
\left[\ell_2(\varepsilon, E)\right]^A_\nu &= \left[E_\nu, \varepsilon\right]^A .
\end{aligned}
} \tag{7.21}
$$

Here, $\varepsilon, \varepsilon_1, \varepsilon_2 \in X_0$, $\mu \in X_{-1}$ and $E \in X_{-2}$ are arbitrary vectors. Notice that, in order to obtain more compact expressions, we have changed $m \longrightarrow m+1$. The last product in eqs. (7.21) relates one vector of the subspace X_{-2}. This product is obtained by consistency, by demanding the already found products to satisfy the L_∞ identities (4.11). The calculation of this consistency product can be found in appendix C. Moreover the L_∞ algebra of this theory shows a dynamical sector described by a set of non-vanishing products ℓ_l with l taking integer values between m and $2m$. These products exclusively relate vectors in the subspace X_{-1} and are given by

$$
\boxed{
\begin{aligned}
&\left[\ell_l\left(\mu_1, ..., \mu_l\right)\right]^A_\nu = \tfrac{1}{2^{l-m}} \tfrac{(-1)^{\frac{l(l-1)}{2}}\, l!m!}{(2m-l)!(l-m)!} \varepsilon_\nu{}^{\mu_1\cdots\mu_{2m}} \\
&\quad \times g^A{}_{B_1\cdots B_m} \partial_{\mu_1}\left(\mu_{\{1}\right)^{B_1}_{\mu_2} \cdots \partial_{\mu_{4m-2l-1}}\left(\mu_{2m-l}\right)^{B_{2m-l}}_{\mu_{4m-2l}} \\
&\quad \times \left[\left(\mu_{2m-l+1}\right)_{\mu_{4m-2l+1}}, \left(\mu_{2m-l+2}\right)_{\mu_{4m-2l+2}}\right]^{B_{2m-l+1}} \cdots \left[\left(\mu_{l-1}\right)_{\mu_{2m-1}}, \left(\mu_{l\}}\right)_{\mu_{2m}}\right]^{B_m} ,
\end{aligned}
} \tag{7.22}
$$

with $\mu_1, \ldots, \mu_l \in X_{-1}$, Eq. (7.22) is directly derived from eq. (7.20) by considering arbitrary vectors in X_{-1} and including the corresponding normalized symmetrization (denoted with braces) in order to preserve the symmetry properties of the L_∞ products.

Let us finish the chapter by considering a particular case, namely, a three-dimensional Chern–Simons theory. Since eqs. (7.21) do not depend on the dimensionality, it is only necessary to study the dynamical sector of the L_∞ algebra. The corresponding products can be obtained by setting $m = 1$ in eq. (7.22). This leads to two products, corresponding to the values $l = 1, 2$, given by

$$\boxed{\begin{array}{rcl} \left[\ell_1\left(\mu_1\right)\right]^A_\nu & = & \varepsilon_\nu{}^{\mu_1\mu_2}\partial_{\mu_1}\left(\mu_1^A\right)_{\mu_2}, \\ \left[\ell_2\left(\mu_1,\mu_2\right)\right]^A_\nu & = & -\varepsilon_\nu{}^{\mu_1\mu_2}\left[\left(\mu_1\right)_{\mu_1},\left(\mu_2\right)_{\mu_2}\right]^A. \end{array}} \tag{7.23}$$

Therefore, the entire L_∞ algebra for this theory is given by eqs. (7.21), describing the gauge transformations and gauge algebra, and by eq. (7.23) describing the dynamics. This particular case reproduces the formulation of three-dimensional Chern–Simons theory from ref. [41].

Chapter 8

L_∞ formulation of FDA1 gauge theories

At this point, we have studied the formulation of higher-dimensional Chern–Simons theories in terms of L_∞ algebras. As we have seen, this means to write down the relevant information of an entire theory into a single algebra L_∞^{full} that encodes the gauge symmetry in a certain subalgebra $L_\infty^{\text{gauge}} \subset L_\infty^{\text{full}}$. The gauge invariance of standard Chern–Simons theories is described by Lie algebras, and therefore, the gauge subalgebras in these cases are indeed Lie algebras. In this chapter, we focus on writing down the L_∞ algebras that describe FDA1 gauge theories, i.e., gauge theories whose symmetry is not described by a Lie algebra but by a FDA1. Due to the dual relation between FDAs and L_∞ algebras, the new gauge subalgebras are not Lie algebras anymore but also L_∞ algebras whose defining products cannot be decomposed in Lie brackets. The mathematical obstacle that prevents this from happening is the non-trivial cocycle, representative of a Chevalley–Eilenberg cohomology class, with which the FDA1 is constructed. Moreover, as we will see, the presence of higher-degree differential forms as gauge fields in the FDA1 gauge theories is naturally described by the multilinear products of an L_∞ algebra. We study two separated cases, namely, the arbitrary-dimensional FDA1-Chern–Simons theory introduced in chapter 5, and a gauge theory whose dynamics is determined by the zero-curvature conditions. We will refer to the later as ‘flat FDA1 theory’.

8.1 FDA1 gauge theories

Let us consider the gauging of a FDA1. This means considering a composite gauge field whose components are given by a one-form and a p-form respectively, namely

$$\mu = \left(\mu_\mu^A, \mu^i_{\mu_1 \dots \mu_p}\right). \tag{8.1}$$

Let us also introduce a set of gauge parameters composed by a zero-form and a $(p-1)$-form and denoted as follows

$$\varepsilon = \left(\varepsilon^A, \varepsilon^i_{\mu_1 \cdots \mu_{p-1}}\right). \tag{8.2}$$

As before, we consider an L_∞ algebra with a vector space endowed with the following subspace structure

$$X = X_0 \oplus X_{-1} \oplus X_{-2}. \tag{8.3}$$

An important difference with respect to the previous case must be pointed out. The FDA1 has two separated algebraic sectors that we call the standard sector (or A-sector) and the extended sector (or i-sector). As a consequence, the vector space X and each one of its subspaces X_0, X_{-1} and X_{-2} can be split into two subspaces, as follows

$$X_0 = X_0^{\text{standard}} \oplus X_0^{\text{extended}}, \tag{8.4}$$

$$X_{-1} = X_{-1}^{\text{standard}} \oplus X_{-1}^{\text{extended}}, \tag{8.5}$$

$$X_{-2} = X_{-2}^{\text{standard}} \oplus X_{-2}^{\text{extended}}. \tag{8.6}$$

Therefore, in the L_∞ formulation of FDA1 gauge theories, each vector in X carry two components, being each one a differential form of different degree. For X_0 and X_{-1} this is summarized as follows:

- Every vector $x \in X_0$ carries two components corresponding to two subspaces of X_0 (a zero-form and a $(p-1)$-form)

$$x = \left(x^A, x^i_{\mu_1 \dots \mu_{p-1}}\right). \tag{8.7}$$

- Every vector $y \in X_{-1}$ carries two components corresponding to two subspaces of X_{-1} (a one-form and a p-form)

$$y = \left(y_\mu^A, y^i_{\mu_1 \dots \mu_p}\right). \tag{8.8}$$

- Vectors in X_{-2} also carry two components. Since this subspace carry the vectors that describe the dynamics of the theory, the differential degrees of their components must be analyzed case by case depending on the dynamics of the theory.

We identify $\mu = \left(\mu^A_\mu, \mu^i_{\mu_1 \ldots \mu_p}\right)$ and $\varepsilon = \left(\varepsilon^A, \varepsilon^i_{\mu_1 \ldots \mu_{p-1}}\right)$ as vectors in X_{-1} and X_0 respectively. Moreover, we identify the equation of motion term $\mathcal{F} = \left(\mathcal{F}^A, \mathcal{F}^i\right)$ as a vector of X_{-2}.

As we have seen, in order to find the L_∞ algebras that describe FDA1-Chern–Simons theory and flat FDA1 theory, we need to study three aspects of them: their definition of gauge transformations, their gauge algebra, and their equations of motions. Both theories have the same gauge symmetry; therefore, they share the first two sources of information. For this reason, we will extract the information regarding the definition of gauge transformations and gauge algebra first and then study case by case the algebraic sectors that encode the dynamics.

In general, when writing a FDA1 gauge theory, the algebraic vectors carry two components. As we have seen in chapter 5, the gauge parameters carry a zero-form ε^A and a $(p-1)$-form ε^i as components. In the same way, for gauge fields, these components are a one-form μ^A and a p-form μ^i. In general, every FDA1 vector is given by a q-form and a $(p+q-1)$-form carrying algebraic indices A and i respectively. Since the L_∞ products between vectors in X are also vectors in X, their components carry both types of algebraic indices A and i depending on which algebraic sector they lie, and a different number of antisymmetric spacetime indices, depending on which subspace of X they belong. In general, an L_∞ product between r vectors in X is a vector in X whose components are a q-form with algebraic index A, and a $(p+q-1)$-form with algebraic index i, i.e.,

$$\ell_r\left(x_1, \ldots, x_r\right) = \left(\left[\ell_r\left(x_1, \ldots, x_r\right)\right]^A_{\mu_1 \cdots \mu_q}, \left[\ell_r\left(x_1, \ldots, x_r\right)\right]^i_{\mu_1 \cdots \mu_{p+q-1}}\right), \tag{8.9}$$

From now on, we will write down products of the type $\ell_r\left(x_1, \ldots, x_r\right)$ in components, i.e., writing them in terms of the FDA1 and spacetime indices. For convenience, it is useful to introduce differential form products. We write the components of the product (8.9) in terms of differential forms as follows

$$\left[\ell_r\left(x_1, \ldots, x_r\right)\right]^A = \frac{1}{q!}\left[\ell_r\left(x_1, \ldots, x_r\right)\right]^A_{\mu_1 \ldots \mu_q} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_q}, \tag{8.10}$$

$$[\ell_r(x_1,\dots,x_r)]^i = \frac{1}{(p+q-1)!}[\ell_r(x_1,\dots,x_r)]^i_{\mu_1\cdots\mu_{(p+q-1)}}\,\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_{(p+q-1)}}, \tag{8.11}$$

It is important to notice that the product in eq. (8.9) (and therefore also its corresponding components) satisfy the original graded symmetry rule of L_∞ algebras in the ℓ-picture of eq. (4.10). However, its components, when they are written in terms of differential forms, satisfy a modified symmetry rule due to the inclusion of new algebraic structure in the wedge products. Although this notation does not explicitly hold the symmetry of the products, it allows to write the product between a large number of vectors without overloading of indices, and it turns out to be especially useful in the formulation of FDA1 gauge theories, due to the natural presence of higher-degree differential forms in them. From now on, we will write down all the components of the L_∞ products in terms of differential forms.

8.1.1 Gauge transformations

Let us consider the definition of gauge transformations for a FDA1. The variation of the fundamental gauge field $\delta\mu$ is a vector in X_{-1} that carry a one-form and a p-form as components. Since the transformations of the one-form are given by the Lie covariant derivative of the zero-form ε^A, the information in terms of L_∞ products originated in that definition is the same that was obtained in the study of standard Chern–Simons theories. The same happens with the corresponding gauge algebra, which is a Lie algebra in the standard sector. Therefore, we will focus on obtaining the information regarding the extended sector. The transformation rule of the extended gauge field $\mu^i_{\mu_1\cdots\mu_p}$ is given in terms of the gauge parameters by eq. (3.36). On the other hand, this variation can also be written in terms a sum of L_∞ products, as in eq. (4.45). From both equations, we can see that such sum is truncated, resulting only in those terms that are powers of degree zero, one, and p in the gauge fields, as follows

$$\delta\mu^i_{\mu_1\cdots\mu_p} = [\ell_1(\varepsilon)]^i_{\mu_1\cdots\mu_p} + [\ell_2(\varepsilon,\mu)]^i_{\mu_1\cdots\mu_p} + \frac{(-1)^{\frac{p(p-1)}{2}}}{p!}[\ell_{p+1}(\varepsilon,\mu,\dots,\mu)]^i_{\mu_1\cdots\mu_p}\,. \tag{8.12}$$

By writing eq. (8.12) in terms of differential forms, as in eq. (8.11), we obtain a spacetime index free expression

$$\delta\mu^i = [\ell_1(\varepsilon)]^i + [\ell_2(\varepsilon,\mu)]^i + \frac{(-1)^{\frac{p(p-1)}{2}}}{p!}[\ell_{p+1}(\varepsilon,\mu,\dots,\mu)]^i\,. \tag{8.13}$$

Thus, by directly comparing eqs. (3.36) and (8.13), we obtain the following information concerning the components of products of gauge fields and parameters in the extended sector, written in terms of differential forms

$$\left[\ell_1\left(\varepsilon\right)\right]^i = \mathrm{d}\varepsilon^i, \tag{8.14}$$

$$\left[\ell_2\left(\varepsilon,\mu\right)\right]^i = \left[\mu,\varepsilon\right]^i - \left[\varepsilon,\mu\right]^i, \tag{8.15}$$

$$\left[\ell_{p+1}\left(\varepsilon,\mu^p\right)\right]^i = (-1)^{1+\frac{p(p-1)}{2}}\left[\varepsilon,\mu^p\right]^i. \tag{8.16}$$

It is important to recall that those are not the L_∞ products of the theory by themselves, but their components in the extended sector (or i-sector) written in terms of differential forms. Every other product originated in the gauge transformations and, therefore, involving one vector of X_0 and any number of vectors of X_{-1} has zero components in this sector. The next step is to obtain the information concerning the gauge algebra.

8.1.2 Gauge algebra

The commutator of two consecutive gauge transformations must lead to a third gauge transformation depending on a composite parameter. However, when describing a gauge theory in terms of an L_∞ algebra, the presence of equation of motion symmetries is allowed in the commutator (see eq. (4.52)). Therefore, in order to have a well-defined L_∞ algebra, it is necessary to ensure the closure of the commutator of two gauge transformations and study the presence of trivial gauge transformations. Since such transformations must be on-shell vanishing, their functional form can impose constraints on the action principle. By applying two consecutive transformations δ_1 and δ_2 on the gauge field, with parameters $\varepsilon_1 = \left(\varepsilon_1^A, \varepsilon_1^i\right)$ and $\varepsilon_2 = \left(\varepsilon_2^A, \varepsilon_2^i\right)$ respectively, we calculate the commutator $\left(\delta_2\delta_1 - \delta_1\delta_2\right)\mu$. This is also a vector in X_{-1}, whose standard components close a Lie algebra, such as in the Chern–Simons case. Moreover, by taking the extended component of the commutator, we find

$$\begin{aligned}
&\left(\delta_2\delta_1 - \delta_1\delta_2\right)\mu^i \\
&= \left[\mathrm{d}\varepsilon_2,\varepsilon_1\right]^i - \left[\varepsilon_1,\mathrm{d}\varepsilon_2\right]^i - \left[\mathrm{d}\varepsilon_1,\varepsilon_2\right]^i + \left[\varepsilon_2,\mathrm{d}\varepsilon_1\right]^i + \left[\varepsilon_1,\left[\varepsilon_2,\mu\right]\right]^i \\
&- \left[\varepsilon_2,\left[\varepsilon_1,\mu\right]\right]^i + \left[\left[\mu,\varepsilon_2\right],\varepsilon_1\right]^i - \left[\left[\mu,\varepsilon_1\right],\varepsilon_2\right]^i + \left[\varepsilon_2,\left[\mu,\varepsilon_1\right]\right]^i - \left[\varepsilon_1,\left[\mu,\varepsilon_2\right]\right]^i \\
&+ \frac{1}{p!}\left[\varepsilon_1,\left[\varepsilon_2,\mu^p\right]\right]^i - \frac{1}{(p-1)!}\left[\varepsilon_1,\mathrm{d}\varepsilon_2,\mu^{p-1}\right]^i - \frac{1}{(p-1)!}\left[\varepsilon_1,\left[\mu,\varepsilon_2\right],\mu^{p-1}\right]^i \\
&- \frac{1}{p!}\left[\varepsilon_2,\left[\varepsilon_1,\mu^p\right]\right]^i + \frac{1}{(p-1)!}\left[\varepsilon_2,\mathrm{d}\varepsilon_1,\mu^{p-1}\right]^i + \frac{1}{(p-1)!}\left[\varepsilon_2,\left[\mu,\varepsilon_1\right],\mu^{p-1}\right]^i.
\end{aligned} \tag{8.17}$$

This expression must be treated using the generalized Jacobi identity for the FDA1. Indeed, from eqs. (3.13)-(3.15), it is possible to prove the following relations:

$$[\varepsilon_1, [\varepsilon_2, \mu]]^i - [\varepsilon_2, [\varepsilon_1, \mu]]^i - [[\varepsilon_1, \varepsilon_2], \mu]^i = 0, \tag{8.18}$$

$$[\varepsilon_1, [\mu, \varepsilon_2]]^i + [[\mu, \varepsilon_1], \varepsilon_2]^i - [\mu, [\varepsilon_1, \varepsilon_2]]^i = 0. \tag{8.19}$$

Eqs. (8.18) and (8.19) are directly obtained from the FDA1 Jacobi identities in the compact notation of brackets. For details on the writing of the Jacobi identities in such notation, see appendix B. By plugging in eqs. (8.18) and (8.19), and the relation $\mathrm{d}\mu^A = R^A - \frac{1}{2}[\mu, \mu]^A$ into eq. (8.17), the extended component of the commutator takes a compact form, namely

$$(\delta_2\delta_1 - \delta_1\delta_2)\,\mu^i = \delta_3\mu^i - \frac{1}{(p-2)!}\left[\varepsilon_2, \varepsilon_1, R, \mu^{p-2}\right]^i, \tag{8.20}$$

As it happens with Lie algebras, the commutator gives place to a new gauge transformation δ_3 for which we introduce a third composite parameter $\varepsilon_3 = \left(\varepsilon_3^A, \varepsilon_3^i\right)$ whose components depend on the original parameters and gauge fields as follows

$$\varepsilon_3^A = [\varepsilon_2, \varepsilon_1]^A, \tag{8.21}$$

$$\varepsilon_3^i = [\varepsilon_2, \varepsilon_1]^i - [\varepsilon_1, \varepsilon_2]^i + \frac{1}{(p-1)!}\left[\varepsilon_2, \varepsilon_1, \mu^{p-1}\right]^i. \tag{8.22}$$

In contrast, the second term on the r.h.s. of eq. (8.20) is not present in the study of Lie algebras. Its presence is due to the cocycle extension in the FDA1, and it involves the standard two-form curvature R^A but not its extended components R^i. This presence of this term is important in the definition of the action principle of a FDA1 gauge theory because it must vanish on-shell in order to close the gauge algebra without contradictions with the equations of motion.

As we have seen, when writing the commutator of two gauge transformations in terms of their expansions in L_∞ products, the result is given by eq. (4.52) as the sum of a new gauge transformation and a trivial one that involves higher products (of three or more vectors) and relates the equation of motion term of the theory. We identify the second term in eq. (8.20) as the trivial transformation, while the first one can be more conveniently written by separating $\delta_{\varepsilon_3}\mu$ in terms that depend on powers of the same degree in the gauge field. By truncating general expansion in terms of L_∞ products from eq. (4.45), the variation $\delta_{\varepsilon_3}\mu$ can be then written as follows

$$\begin{aligned}\delta_{\varepsilon_3}\mu &= \ell_1\left(\varepsilon_3\right)+\ell_2\left(\varepsilon_3,\mu\right)+\frac{1}{p!}\left(-1\right)^{\frac{p(p-1)}{2}}\ell_{p+1}\left(\varepsilon_3,\mu^p\right)\\ &=\left[\delta_{\varepsilon_3}\mu\right]_0+\left[\delta_{\varepsilon_3}\mu\right]_1+\left[\delta_{\varepsilon_3}\mu\right]_{p-1}+\left[\delta_{\varepsilon_3}\mu\right]_p\,, \end{aligned}\tag{8.23}$$

where we denote as $\left[\delta_{\varepsilon_3}\mu\right]_k$ to the contribution to $\delta_{\varepsilon_3}\mu$ that depends on powers of degree k in μ. Therefore, by inspection of eqs. (8.21)-(8.23, we match the terms in $\delta_{\varepsilon_3}\mu$ depending on the same powers of μ, and write down an expression for each contribution $\left[\delta_{\varepsilon_3}\mu\right]_k$ to the variation of the gauge field:

$$\left[\delta_{\varepsilon_3}\mu\right]_0=\ell_1\left(\ell_2\left(\varepsilon_1,\varepsilon_2\right)\right),\tag{8.24}$$

$$\left[\delta_{\varepsilon_3}\mu\right]_1=\ell_1\left(\ell_3\left(\varepsilon_1,\varepsilon_2,\mu\right)\right)+\ell_2\left(\ell_2\left(\varepsilon_1,\varepsilon_2\right),\mu\right),\tag{8.25}$$

$$\begin{aligned}\left[\delta_{\varepsilon_3}\mu\right]_{p-1}=&\frac{\left(-1\right)^{\frac{(p-1)(p-2)}{2}}}{(p-1)!}\ell_1\left(\ell_{p+1}\left(\varepsilon_1,\varepsilon_2,\mu^{p-1}\right)\right)\\ &+\frac{\left(-1\right)^{\frac{(p-2)(n-3)}{2}}}{(p-2)!}\ell_2\left(\ell_p\left(\varepsilon_1,\varepsilon_2,\mu^{p-2}\right),\mu\right),\end{aligned}\tag{8.26}$$

$$\begin{aligned}\left[\delta_{\varepsilon_3}\mu\right]_p=&\frac{\left(-1\right)^{\frac{p(p-1)}{2}}}{p!}\ell_1\left(\ell_{p+2}\left(\varepsilon_1,\varepsilon_2,\mu^p\right)\right)+\frac{\left(-1\right)^{\frac{(p-1)(p-2)}{2}}}{(p-1)!}\ell_2\left(\ell_{p+1}\left(\varepsilon_1,\varepsilon_2,\mu^{p-1}\right),\mu\right)\\ &+\frac{\left(-1\right)^{\frac{p(p-1)}{2}}}{p!}\ell_{p+1}\left(\ell_2\left(\varepsilon_1,\varepsilon_2\right),\mu^p\right).\end{aligned}\tag{8.27}$$

On the other hand, by directly replacing the components of ε_3, given by eqs. (8.21) and (8.22) into the definition of the FDA1 gauge transformation, it is possible to explicitly write down $\delta_3\mu^i$ without using the L_∞ products:

$$\begin{aligned}\delta_3\mu^i=&\mathrm{d}\left\{\left[\varepsilon_2,\varepsilon_1\right]^i-\left[\varepsilon_1,\varepsilon_2\right]^i+\frac{1}{(p-1)!}\left[\varepsilon_2,\varepsilon_1,\mu^{p-1}\right]^i\right\}\\ &+\left[\mu,\left\{\left[\varepsilon_2,\varepsilon_1\right]-\left[\varepsilon_1,\varepsilon_2\right]+\frac{1}{(p-1)!}\left[\varepsilon_2,\varepsilon_1,\mu^{p-1}\right]\right\}\right]^i\\ &-\left[\left[\varepsilon_2,\varepsilon_1\right],\mu\right]^i-\frac{1}{p!}\left[\left[\varepsilon_2,\varepsilon_1\right],\mu^p\right]^i.\end{aligned}\tag{8.28}$$

By comparing term by term eq. (8.28) with eqs. (8.24)-(8.27) we obtain four relations, each one proportional to a different power of the gauge field. In order to obtain the information regarding different L_∞ products on them, we proceed case by case:

First relation (power 0 in μ):

$$\left[\ell_1\left(\ell_2\left(\varepsilon_1,\varepsilon_2\right)\right)\right]^i=\mathrm{d}\left\{\left[\varepsilon_2,\varepsilon_1\right]^i-\left[\varepsilon_1,\varepsilon_2\right]^i\right\}.\tag{8.29}$$

Second relation (power 1 in μ):

$$\left[\ell_1\left(\ell_3\left(\varepsilon_1,\varepsilon_2,\mu\right)\right)+\ell_2\left(\ell_2\left(\varepsilon_1,\varepsilon_2\right),\mu\right)\right]^i=\left[\mu,\left(\left[\varepsilon_2,\varepsilon_1\right]-\left[\varepsilon_1,\varepsilon_2\right]\right)\right]^i-\left[\left[\varepsilon_2,\varepsilon_1\right],\mu\right]^i. \tag{8.30}$$

Third relation (power $p-1$ in μ):

$$\begin{aligned}&(-1)^{\frac{(p-1)(p-2)}{2}}\left[\ell_1\left(\ell_{p+1}\left(\varepsilon_1,\varepsilon_2,\mu^{p-1}\right)\right)\right]^i\\&+(-1)^{\frac{(p-2)(n-3)}{2}}(p-1)\left[\ell_2\left(\ell_p\left(\varepsilon_1,\varepsilon_2,\mu^{p-2}\right),\mu\right)\right]^i=\mathrm{d}\left[\varepsilon_2,\varepsilon_1,\mu^{p-1}\right]^i.\end{aligned} \tag{8.31}$$

Fourth relation (power p in μ):

$$\begin{aligned}&\left[\ell_1\left(\ell_{p+2}\left(\varepsilon_1,\varepsilon_2,\mu^p\right)\right)\right]^i+p\left[\ell_2\left(\ell_{p+1}\left(\varepsilon_1,\varepsilon_2,\mu^{p-1}\right),\mu\right)\right]^i+\left[\ell_{p+1}\left(\ell_2\left(\varepsilon_1,\varepsilon_2\right),\mu^p\right)\right]^i\\&=(-1)^{\frac{(p-1)(p-2)}{2}}p\left[\mu,\left[\varepsilon_2,\varepsilon_1,\mu^{p-1}\right]\right]^i-(-1)^{\frac{(p-1)(p-2)}{2}}\left[\left[\varepsilon_2,\varepsilon_1\right],\mu^p\right]^i.\end{aligned} \tag{8.32}$$

We now use the previously obtained information about the products. Since $\ell_2\left(\varepsilon_1,\varepsilon_2\right)$, $\ell_3\left(\varepsilon_1,\varepsilon_2,\mu\right)$, $\ell_p\left(\varepsilon_1,\varepsilon_2,\mu^{p-2}\right)$, $\ell_{p+1}\left(\varepsilon_1,\varepsilon_2,\mu^{p-1}\right)$ and $\ell_{p+2}\left(\varepsilon_1,\varepsilon_2,\mu^p\right)$ are vectors of X_0, ℓ_1 acts on them according eqs. (8.29)-(8.32). This provides an explicit expression for these products. On the other hand, we identify the trivial transformation of eq. (4.52) with the last term on the r.h.s. of eq. (8.20), resulting in a $(p+1)$-linear product involving one vector in X_{-2}, as follows

$$\ell_{p+1}\left(\varepsilon_1,\varepsilon_2,R,\mu^{p-2}\right)=(-1)^{1+\frac{(p-2)(p-3)}{2}}\left[\varepsilon_2,\varepsilon_1,R,\mu^{p-2}\right]^i. \tag{8.33}$$

8.1.3 *Summary:*

The L_∞ products previously obtained contain the information of a FDA1 gauge theory regarding its definition of gauge transformations and gauge algebra. Since the dynamics has not been yet specified, the structure of these sectors is shared for any FDA1 gauge theory and can be summarized as follows:

Gauge transformations

$$\boxed{\begin{array}{rcl}
\left[\ell_1\left(\varepsilon_1\right)\right]^A & = & \mathrm{d}\varepsilon_1^A, \\
\left[\ell_1\left(\varepsilon_1\right)\right]^i & = & \mathrm{d}\varepsilon_1^i, \\
\left[\ell_2\left(\varepsilon_1,\mu_1\right)\right]^A & = & \left[\mu_1,\varepsilon_1\right]^A, \\
\left[\ell_2\left(\varepsilon_1,\mu_1\right)\right]^i & = & \left[\mu_1,\varepsilon_1\right]^i - \left[\varepsilon_1,\mu_1\right]^i, \\
\left[\ell_{p+1}\left(\varepsilon_1,\mu_1,\ldots,\mu_p\right)\right]^i & = & (-1)^{1+\frac{p(p-1)}{2}}\left[\varepsilon,\mu_1,\ldots,\mu_p\right]^i.
\end{array}} \tag{8.34}$$

Gauge algebra

$$\boxed{\begin{array}{rcl}
\left[\ell_2\left(\varepsilon_1,\varepsilon_2\right)\right]^A & = & \left[\varepsilon_2,\varepsilon_1\right]^A, \\
\left[\ell_2\left(\varepsilon_1,\varepsilon_2\right)\right]^i & = & \left[\varepsilon_2,\varepsilon_1\right]^i - \left[\varepsilon_1,\varepsilon_2\right]^i, \\
\left[\ell_{p+1}\left(\varepsilon_1,\varepsilon_2,\mu_1,\ldots,\mu_{p-1}\right)\right]^i & = & (-1)^{\frac{(p-1)(p-2)}{2}}\left[\varepsilon_2,\varepsilon_1,\mu_1,\ldots,\mu_{p-1}\right]^i, \\
\left[\ell_{p+1}\left(\varepsilon_1,\varepsilon_2,E,\mu_1,\ldots,\mu_{p-2}\right)\right]^i & = & (-1)^{1+\frac{(p-2)(p-3)}{2}}\left[\varepsilon_1,\varepsilon_2,E,\mu_1,\ldots,\mu_{p-2}\right]^i.
\end{array}} \tag{8.35}$$

Here, $\varepsilon_1,\varepsilon_2 \in X_0$, $\mu_1,\ldots,\mu_p \in X_{-1}$ and $E \in X_{-2}$ are arbitrary vectors. Any other product originated in these two sources of information, and therefore relating at least one vector of the subspace X_0, vanishes. These products do not close an L_∞ algebra by themselves. In order to specify a theory, it is necessary to introduce the products coming from the third source of information, namely, the equation of motion. They exclusively involve vectors of the subspace X_{-1} and determine the dynamics of the different theories that share the FDA1 symmetry. Consequently, they must be introduced case by case for the chosen theory, in a way that satisfies the L_∞ identities in combination with the products defined by eqs. (8.34) and (8.35).

8.2 Flat FDA1 gauge theory

The first case to analyze is the flat FDA1 theory. This is a FDA1 gauge theory in which the equations of motion are the Maurer–Cartan equations of the fundamental field $\mu = \left(\mu^A,\mu^i\right)$, i.e., on-shell vanishing curvatures. In contrast to the first two sources of information, we need to study both algebraic sectors. The corresponding field equation term $\mathcal{F}$ is a vector in the subspace X_{-2}. In this case, we define the components of the vectors in X_{-2} as a two-form and a $(p+1)$-form. This choice allows to write the components of $\mathcal{F}$ easily, in terms of differential forms for both standard and extended subspaces, as follows

$$\mathcal{F}^A = R^A = \mathrm{d}\mu^A + \frac{1}{2}C^A_{BC}\mu^B\mu^C, \tag{8.36}$$

$$\mathcal{F}^i = R^i = \mathrm{d}\mu^i + C^i_{Aj}\mu^A\mu^j + \frac{1}{(p+1)!}C^i_{A_1\cdots A_{p+1}}\mu^{A_1}\cdots\mu^{A_{p+1}}. \tag{8.37}$$

Notice that this choice for the dynamics is consistent and does not contradict the constraint imposed by the trivial gauge transformation found in the gauge algebra. The second term on the r.h.s. of eq. (8.20) depends on the standard component of the curvature R^A which vanishes on-shell, and therefore it satisfies the condition of being an equation of motion symmetry, guaranteeing the closure of the gauge subalgebra.

Since the equation of motion is immediately obtained from the FDA1 curvature, there is no need to introduce an action principle or an L_∞ inner product. Eqs. (8.36) and (8.37) show that the general expansion of the equation of motion term $\mathcal{F}$ in terms of L_∞ products is truncated, removing every element not linear, bilinear or $(p+1)$-linear in the gauge field, as follows

$$\mathcal{F} = \ell_1(\mu) - \frac{1}{2}\ell_2(\mu^2) + \frac{(-1)^{\frac{p(p+1)}{2}}}{(p+1)!}\ell_{p+1}(\mu^{p+1}).$$

Notice that the $(p+1)$-linear product has non-vanishing components only in the extended sector, i.e., the expansion is truncated in different ways depending on the algebraic sector:

$$\mathcal{F}^A = [\ell_1(\mu)]^A - \frac{1}{2}\left[\ell_2(\mu^2)\right]^A, \tag{8.38}$$

$$\mathcal{F}^i = [\ell_1(\mu)]^i - \frac{1}{2}\left[\ell_2(\mu^2)\right]^i + \frac{(-1)^{\frac{p(p+1)}{2}}}{(p+1)!}\left[\ell_{p+1}(\mu^{p+1})\right]^i. \tag{8.39}$$

Therefore, we separately analyze the standard and extended sectors. By matching eqs. (8.36) and (8.38) term by term, we obtain the following information about the components of the products in the standard sector written in terms of differential forms:

$$[\ell_1(\mu)]^A = \mathrm{d}\mu^A, \tag{8.40}$$

$$\left[\ell_2(\mu^2)\right]^A = -C^A_{BC}\mu^B\mu^C. \tag{8.41}$$

On the other hand, from eqs. (8.37) and (8.39) we obtain the components of the corresponding products in the extended sector also in terms of differential forms. Notice that the $(p+1)$-linear product that carries the information about the cocycle has non-vanishing components only in this sector:

$$[\ell_1(\mu)]^i = \mathrm{d}\mu^i, \tag{8.42}$$

$$\left[\ell_2\left(\mu^2\right)\right]^i = -2C^i_{Aj}\mu^A\mu^j, \tag{8.43}$$

$$\left[\ell_{p+1}\left(\mu^{p+1}\right)\right]^i = (-1)^{\frac{p(p+1)}{2}} C^i_{A_1\cdots A_{p+1}}\mu^{A_1}\cdots\mu^{A_{p+1}}. \tag{8.44}$$

Thus, the information regarding the third source of information can be summarized as follows; let $\varepsilon \in X_0$, $\mu, \mu_1, \ldots, \mu_{p+1} \in X_{-1}$ and $E \in X_{-2}$ be arbitrary vectors. The dynamics of a flat FDA1 gauge theory is encoded into the following components of L_∞ products, written in terms of differential forms:

Flat FDA1 dynamical sector

$$\boxed{\begin{array}{rcl} \left[\ell_1\left(\mu\right)\right]^A &=& \mathrm{d}\mu^A, \\ \left[\ell_1\left(\mu\right)\right]^i &=& \mathrm{d}\mu^i, \\ \left[\ell_2\left(\mu_1,\mu_2\right)\right]^A &=& -C^A_{BC}\mu_1^B\mu_2^C, \\ \left[\ell_2\left(\mu_1,\mu_2\right)\right]^i &=& -C^i_{Aj}\left(\mu_1^A\mu_2^j+\mu_2^A\mu_1^j\right), \\ \left[\ell_{p+1}\left(\mu_1,\ldots,\mu_{p+1}\right)\right]^i &=& (-1)^{\frac{p(p+1)}{2}} C^i_{A_1\cdots A_{p+1}}\mu_1^{A_1}\cdots\mu_{p+1}^{A_{p+1}}. \end{array}} \tag{8.45}$$

Consistency products

$$\boxed{\begin{array}{rcl} \left[\ell_2\left(\varepsilon,E\right)\right]^A &=& \left[E,\varepsilon\right]^A, \\ \left[\ell_2\left(\varepsilon,E\right)\right]^i &=& \left[E,\varepsilon\right]^i-\left[\varepsilon,E\right]^i, \\ \left[\ell_{p+1}\left(\varepsilon,E,\mu_1,\ldots,\mu_{p-1}\right)\right]^i &=& (-1)^{1+\frac{(p-1)(p-2)}{2}}\left[\varepsilon,E,\mu_1,\ldots,\mu_{p-1}\right]^i. \end{array}} \tag{8.46}$$

As it was mentioned in the previous chapter, we include a set of consistency products involving at least one vector in the subspace X_{-2}. These products are not directly obtained from the three mentioned sources of information but by demanding the products obtained from them to satisfy the L_∞ identities. For an explicit calculation, see appendix C. The inclusion of the consistency products allows eqs. (8.34), (8.35), (8.46) and (8.45) to define an L_∞ algebra and to consistently formulate the flat FDA1 gauge theory.

8.3 L_∞ formulation of FDA1-Chern–Simons theory

The second case to analyze is the extended Chern–Simons theory invariant under FDA1 introduced in chapter 5. The writing of the corresponding L_∞ algebra is slightly more complicated for this case because, although the gauge symmetry remains the same that in the flat FDA1 theory, the equations of motion take a different functional form depending on the dimensionality. As a consequence, the subspace X_{-2}, encoding the dynamics, has a different structure in different dimensions. Let us begin by introducing the algebraic vectors belonging to the subspaces X_0, X_{-1} and X_{-2}. Since the gauge transformations of the theory are defined by eqs. (3.35) and (3.36), an arbitrary vector in X_0 is given by $\varepsilon = \left(\varepsilon^A, \varepsilon^i_{\mu_1\cdots\mu_{p-1}}\right)$, i.e., its the A-, and i-components are a zero-form and a $(p-1)$-form respectively. Moreover, vectors in the subspaces X_{-1} and X_{-2} have components in the standard and extended sectors that also can be split in the following way:

$$u = \left(u^A_\mu, u^i_{\mu_1\cdots\mu_p}\right) \in X_{-1}, \tag{8.47}$$

$$v = \left(v^A_{\mu_1\cdots\mu_{q-2}}, v^i_{\mu_1\cdots\mu_{q-p-1}}\right) \in X_{-2}. \tag{8.48}$$

Notice that since the first two sources of information of the theory (the gauge transformations and gauge algebra) are the same that in the flat FDA1 theory, vectors in X_0 and X_{-1} are defined in the same way that in the previous section. Unlike the previous case, vectors in X_{-2} are not given by a two-form and a $(p+1)$-form. We define them as a $(q-2)$-form and a $(q-p-1)$-form, where $q-1$ is de dimensionality of the theory. This new definition does not contradict the definition of FDA1 algebraic vectors from chapter 3. The reason for this choice is that we need to define algebraic vectors $\mathcal{F} \in X_{-2}$ such that, the variation of the action can be written as eq. (4.50). Different choices of the components of $\mathcal{F}$ would lead to different but equivalent definitions of the L_∞ inner product that is used in the definition of the action principle. Once we defined the components of the vectors, we define the L_∞ inner product between two arbitrary vectors $u \in X_{-1}$ and $v \in X_{-2}$ as follows

$$\langle u, v\rangle_{L_\infty} = \int \mathrm{d}x^{2m-1} \varepsilon^{\mu_1\cdots\mu_{q-1}} \left(g_{AB} u^A_{\mu_1} v^B_{\mu_2\cdots\mu_{q-2}} + g_{ij} u^i_{\mu_1\cdots\mu_p} v^j_{\mu_{p+1}\cdots\mu_{q-1}}\right). \tag{8.49}$$

Notice that the coefficients g_{AB} and g_{ij} were defined in eqs. (5.9)-(5.11) as the components of the rank-2 FDA1 invariant tensor. With this definition of inner product, the general variation of the action principle in eq. (4.49), reproduces the variation of the FDA1-Chern–Simons action principle obtained by setting $\bar{\mu} = 0$ in eq. (5.27). Thus, according to eqs. (4.50) and (8.49), this variation can be written terms of the inner product and the components of the equation of motion term, as follows

$$\langle \delta\mu, \mathcal{F}\rangle_{L_\infty} = \int \mathrm{d}x^{2m-1} \varepsilon^{\mu_1\cdots\mu_{q-1}} \left(\frac{1}{(q-2)!} g_{AB} \delta\mu^A_{\mu_1} \mathcal{F}^B_{\mu_2\cdots\mu_{q-2}}\right.$$

$$+\frac{1}{p!\,(q-p-1)!}g_{ij}\delta\mu^{i}_{\mu_1\cdots\mu_p}\mathcal{F}^{j}_{\mu_{p+1}\cdots\mu_{q-1}}\Bigg). \tag{8.50}$$

Eq. (5.33) allows to identify the components of the vector $\mathcal{F}\in X_{-2}$ by direct inspection of the field variation of the action principle. In terms of differential forms, these components are explicitly given by

$$\mathcal{F}^{A}=\sum_{m,n}mg^{AA_1}g_{A_1A_2\cdots A_m i_1\cdots i_n}R^{A_2}\cdots R^{A_m}R^{i_1}\cdots R^{i_n}, \tag{8.51}$$

$$\mathcal{F}^{i}=\sum_{m,n}ng^{ii_1}g_{A_1\cdots A_m i_1 i_2\cdots i_n}R^{A_1}\cdots R^{A_m}R^{i_2}\cdots R^{i_n}. \tag{8.52}$$

Notice that $\mathcal{F}^A=0$ and $\mathcal{F}^i=0$ are indeed the equations of motion obtained in eqs. (5.33) and (5.34). Moreover, we have raised the algebraic indices A and i on these expressions by using the components of the rank-2 invariant tensor. We used g^{AB} to rise indices in the standard sector and g^{ij} for the extended one. For a specific explanation of how we raise FDA1 algebraic indices, see appendix A.

8.3.1 Standard dynamical sector

Since our purpose is to obtain the information regarding the dynamics, let us begin by splitting the equation of motion term $\mathcal{F}$ in components and first consider the standard ones $\mathcal{F}^A$. Eqs. (8.51) and (4.51) imply that $\mathcal{F}^A$ can be written as an expansion of the standard components of L_∞ products, as follows

$$\mathcal{F}^{A}=\sum_{l=1}^{\infty}\frac{(-1)^{\frac{l(l-1)}{2}}}{l!}\left[\ell_l\left(\mu^l\right)\right]^A. \tag{8.53}$$

Eq. (8.53) expands $\mathcal{F}^A$ in terms that are linear products of the gauge fields. This expansion is truncated in different ways depending on the dimensionality, i.e., in terms of the allowed values of the coefficients m and n in the equation of motion (8.51). In order to separate the contributions to $\mathcal{F}^A$ corresponding to different L_∞ products, we plug in the definition of gauge curvatures from eqs. (3.11) and (3.12) into the generic expression from eq. (8.51), and write down $\mathcal{F}^A$ in terms of the gauge fields and their derivatives

$$\begin{aligned}\mathcal{F}^{A}=&\sum_{m,n}\sum_{k=0}^{m-1}\sum_{r+s+t=n}\frac{1}{2^k\,(p+1)!^t}\frac{m!}{k!\,(m-k-1)!}\frac{n!}{r!s!t!}\\&\times g^{A}{}_{A_1\cdots A_{m-1}i_1\cdots i_n}\mathrm{d}\mu^{A_1}\cdots\mathrm{d}\mu^{m-k-1}\left[\mu,\mu\right]^{A_{m-k}}\cdots\left[\mu,\mu\right]^{A_{m-1}}\\&\times\mathrm{d}\mu^{i_1}\cdots\mathrm{d}\mu^{i_r}\left[\mu,\mu\right]^{i_{r+1}}\cdots\left[\mu,\mu\right]^{i_{r+s}}\left[\mu^{p+1}\right]^{i_{s+r+1}}\cdots\left[\mu^{p+1}\right]^{i_n}.\end{aligned} \tag{8.54}$$

As we proceeded with the equation of motion term in standard Chern–Simons theory, we split the terms of the sum at the r.h.s. of eq. (8.54) that depend on powers of the same degree in the gauge fields, i.e.,

$$\mathcal{F}^A = \sum_{l=1}^{\infty} \left[\mathcal{F}^A\right]_l . \tag{8.55}$$

where $\left[\mathcal{F}^A\right]_l$ denote the contributions to $\mathcal{F}^A$ depending on powers of degree l. We can therefore identify $\left[\mathcal{F}^A\right]_l$ as proportional to the l-linear L_∞ product, as follows

$$\left[\mathcal{F}^A\right]_l = \frac{(-1)^{\frac{l(l-1)}{2}}}{l!} \left[\ell_l\left(\mu^l\right)\right]^A . \tag{8.56}$$

With the purpose of isolating the term of the expansion depending on powers of degree l, we notice that each term on the r.h.s of eq. (8.54) is a power of degree $m+n+k+s+pt-1$ in μ. We therefore fix $l = m+n+k+s+pt-1$ and identify the l-linear term $\left[\mathcal{F}^A\right]_l$ as sum of those terms that satisfy that fixing. This allows us to write the l-linear contribution to $\mathcal{F}^A$:

$$\begin{aligned}\left[\mathcal{F}^A\right]_l = &\sum_{m,n}\sum_{r+s+t=n} \frac{1}{2^{k_{st}}\,(p+1)!^t}\,\frac{m!}{k_{st}!\,(m-k_{st}-1)!}\,\frac{n!}{r!s!t!} \\ &\times g^A{}_{A_1\cdots A_{m-1} i_1\cdots i_n} \mathrm{d}\mu^{A_1}\cdots \mathrm{d}\mu^{m-k_{st}-1}\,[\mu,\mu]^{A_{m-k_{st}}}\cdots[\mu,\mu]^{A_{m-1}} \\ &\times \mathrm{d}\mu^{i_1}\cdots\mathrm{d}\mu^{i_r}\,[\mu,\mu]^{i_{r+1}}\cdots[\mu,\mu]^{i_{r+s}}\left[\mu^{p+1}\right]^{i_{s+r+1}}\cdots\left[\mu^{p+1}\right]^{i_n} .\end{aligned} \tag{8.57}$$

For convenience, we have introduced the coefficient $k_{st} = l+1-m-n-s-pt$. Finally, by inspection of eqs. (8.56) and (8.57) we identify the non vanishing L_∞ product that contributes to the dynamics in the standard sector:

$$\boxed{\begin{aligned}\left[\ell_l\left(\mu^l\right)\right]^A = \;& (-1)^{\frac{l(l-1)}{2}} \textstyle\sum_{m,n}\sum_{r+s+t=n} \frac{l!}{2^{k_{st}}(p+1)!^t}\,\frac{m!}{k_{st}!(m-k_{st}-1)!}\,\frac{n!}{r!s!t!} \\ \times\;& g^A{}_{A_1\cdots A_{m-1} i_1\cdots i_n} \mathrm{d}\mu^{A_1}\cdots \mathrm{d}\mu^{m-k_{st}-1}\,[\mu,\mu]^{A_{m-k_{st}}}\cdots[\mu,\mu]^{A_{m-1}} \\ \times\;& \mathrm{d}\mu^{i_1}\cdots\mathrm{d}\mu^{i_r}\,[\mu,\mu]^{i_{r+1}}\cdots[\mu,\mu]^{i_{r+s}}\left[\mu^{p+1}\right]^{i_{s+r+1}}\cdots\left[\mu^{p+1}\right]^{i_n} .\end{aligned}} \tag{8.58}$$

8.3.2 Extended dynamical sector

The procedure to obtain the L_∞ products in the extended algebraic sector is analogue. By explicitly writing eq. (8.52) in terms of the gauge fields and their derivatives, one gets the following expression for the extended component of $\mathcal{F}$ in terms of differential forms

$$\begin{aligned}\mathcal{F}^{i} = & \sum_{m,n}\sum_{k=0}^{m}\sum_{r+s+t=n-1}\frac{1}{2^{k}\left(p+1\right)!^{t}}\frac{m!}{k!\left(m-k\right)!}\frac{n!}{r!s!t!}g^{i}{}_{i_1\cdots i_{n-1}A_1\cdots A_m}\\ & \times \mathrm{d}\mu^{A_1}\cdots\mathrm{d}\mu^{A_{m-k}}\left[\mu,\mu\right]^{A_{m-k+1}}\cdots\left[\mu,\mu\right]^{A_m}\\ & \times \mathrm{d}\mu^{i_1}\cdots\mathrm{d}\mu^{i_r}\left[\mu,\mu\right]^{i_{r+1}}\cdots\left[\mu,\mu\right]^{i_{r+s}}\left[\mu^{p+1}\right]^{i_{r+s+1}}\cdots\left[\mu^{p+1}\right]^{i_{n-1}}. \end{aligned} \tag{8.59}$$

Notice that we write the algebraic indices of the invariant tensor with inverse ordering. This is simply notation; since the indices corresponding to the standard and extended sector take values in different domains, we can switch their positions in the invariant tensor without introducing ambiguity. As in the standard sector, we expand $\mathcal{F}^i$ in terms of the contributions depending on powers of degree l in μ, denoted by $\left[\mathcal{F}^i\right]_l$. In order to isolate the terms on the r.h.s. of eq. (8.59) depending on powers of the same degree on the gauge field, we notice that each term in the expansion is a power of degree $m+n+k+s+pt-1$ in μ. Therefore, by matching $l=m+n+k+s+pt-1$, we obtain

$$\begin{aligned}\left[\mathcal{F}^{i}\right]_l = & \sum_{m,n}\sum_{r+s+t=n-1}\frac{1}{2^{\bar{k}_{st}}\left(p+1\right)!^{t}}\frac{m!}{\bar{k}_{st}!\left(m-\bar{k}_{st}\right)!}\frac{n!}{r!s!t!}g^{i}{}_{i_1\cdots i_{n-1}A_1\cdots A_m}\\ & \times \mathrm{d}\mu^{A_1}\cdots\mathrm{d}\mu^{A_{m-\bar{k}_{st}}}\left[\mu,\mu\right]^{A_{m-\bar{k}_{st}+1}}\cdots\left[\mu,\mu\right]^{A_m}\\ & \times \mathrm{d}\mu^{i_1}\cdots\mathrm{d}\mu^{i_r}\left[\mu,\mu\right]^{i_{r+1}}\cdots\left[\mu,\mu\right]^{i_{r+s}}\left[\mu^{p+1}\right]^{i_{r+s+1}}\cdots\left[\mu^{p+1}\right]^{i_{n-1}}. \end{aligned} \tag{8.60}$$

For simplicity, we have defined the secondary coefficients $\bar{k}_{st}=l+1-m-n-s-pt$. From eqs. (8.60) and (4.51), we get that the extended component of a product of l gauge fields is given by

$$\boxed{\begin{aligned}\left[\ell_l\left(\mu^l\right)\right]^i = & \quad (-1)^{\frac{l(l-1)}{2}}\textstyle\sum_{m,n}\sum_{r+s+t=n-1}\frac{l!}{2^{\bar{k}_{st}}(p+1)!^t}\frac{m!}{\bar{k}_{st}!\left(m-\bar{k}_{st}\right)!}\frac{n!}{r!s!t!}\\ \times & \quad g^{i}{}_{i_1\cdots i_{n-1}A_1\cdots A_m}\mathrm{d}\mu^{A_1}\cdots\mathrm{d}\mu^{A_{m-\bar{k}_{st}}}\left[\mu,\mu\right]^{A_{m-\bar{k}_{st}+1}}\cdots\left[\mu,\mu\right]^{A_m}\\ \times & \quad \mathrm{d}\mu^{i_1}\cdots\mathrm{d}\mu^{i_r}\left[\mu,\mu\right]^{i_{r+1}}\cdots\left[\mu,\mu\right]^{i_{r+s}}\left[\mu^{p+1}\right]^{i_{r+s+1}}\cdots\left[\mu^{p+1}\right]^{i_{n-1}}.\end{aligned}} \tag{8.61}$$

Thus, eqs. (8.58) and (8.61) provide the information regarding the L_∞ products for the complete dynamical sector of the theory, and therefore, regarding the L_∞ products of vectors in X_{-1}. As it happens in standard Chern–Simons theory, the equations of motion in eqs. (5.33) and (5.34) show strong functional changes depending on the dimensionality of the theory. Consequently, the number of non-vanishing products in the dynamical sector depends on the value of q. Let us recall that the original Chern–Simons action is $q-1$ dimensional, and that the values of m and n are the integer non-negative solutions of the equation

$$2m + (p+1)\, n = q. \tag{8.62}$$

For each solution of the type (m, n) to eq. (8.61), there is a different set of allowed values for l in eqs. (8.58) and (8.61). In the standard sector, i.e., for eq. (8.58), they take integer values between $l_{\min} = n$ and $l_{\max} = 2m - 2$. In the extended sector, i.e., for eq. (8.61), l take values between $l_{\min} = n - 1$ and $l_{\max} = 2m$.

The L_∞ products previously obtained from the three original sources of information do not necessarily combine to make an L_∞ algebra by themselves. Therefore, we impose them to satisfy the L_∞ identities. For an explicit calculation, see appendix C. Thus, this requirement leads to two consistency products, whose components (in terms of differential forms) are given by:

$$\boxed{\begin{aligned}
\left[\ell_2\left(\varepsilon, E\right)\right]^A &= C^A_{BC} E^B \varepsilon^C - g^{AB} g_{ij} C^i_{Bk} E^j \varepsilon^k, \\
\left[\ell_2\left(\varepsilon, E\right)\right]^i &= g^{ij} g_{kl} C^k_{Bj} \varepsilon^B E^l, \\
\left[\ell_{p+1}\left(\varepsilon, E, \mu^{p-1}\right)\right]^A &= (-1)^{1+\frac{(p-1)(p-2)}{2}} g^{AB_1} g_{ij} C^i_{B_1 \cdots B_{p+1}} \varepsilon^{B_2} \mu^{B_3} \cdots \mu^{B_{p+1}} E^j, \\
\left[\ell_{p+1}\left(\varepsilon, E, \mu^{p-1}\right)\right]^i &= 0.
\end{aligned}} \tag{8.63}$$

Here, $\varepsilon \in X_0$, $\mu \in X_{-1}$ and $E \in X_{-2}$ are arbitrary vectors.

This completes the writing of the algebraic products corresponding to the L_∞ formulation of the FDA1-Chern–Simons theory. The L_∞ algebra that describes this theory in arbitrary dimensions is given by eqs. (8.34) and (8.35) encoding the information of the gauge transformations and gauge algebra respectively, in addition with eqs. (8.58), (8.61) and (8.63) encoding the dynamics. Notice that, in contrast with the writing of the L_∞ algebra of standard Chern–Simons theory in chapter 7, we have written the higher-degree products that involve many vectors from the subspace X_{-1} by only using one vector μ instead of a set of arbitrary independent vectors. This choice allows for writing the products more compactly. The general expressions for the products describing the dynamical sector can be obtained from eqs. (8.58) and (8.61) by considering independent vectors instead of the repetition of the same gauge field and including the corresponding normalized symmetrization (as we did in the case of standard Chern–Simons theory). Moreover, in order to avoid overloading of indices, we have written every product in terms of differential forms. The resulting expressions are, in this way, compact due to the natural presence of higher-degree differential forms in FDA1 gauge theories. It is important to point out that the equations of motion do not guarantee the on-shell closure of the gauge subalgebra. This issue in the formulation is a consequence of the non-covariance in the gauge transformation law of the extended curvature form R^i. In order to have a well-defined L_∞ algebra for the theory, it is necessary that the second term on the r.h.s of eq. (8.20) vanishes on-shell. This condition is fulfilled only in special cases. For example, in three-dimensional theories or those whose gauge algebra does not carry a non-trivial cocycle, this happens by default. Moreover, if the invariant tensors of the FDA1 do not carry mixed indices of the type g_{Ai} the issue is also avoided. Chern–Simons theories that do not satisfy these conditions cannot be properly described by L_∞ algebras since they are not completely well-defined gauge theories. However, they still have action principles that are invariant under the transformations of the FDA1.

8.3.3 Five-dimensional example

At this point, we have written the general L_∞ algebra that allows formulating a FDA1-Chern–Simons theory in arbitrary dimensions. As an example, let us consider a particular case given by a five-dimensional theory that couples the standard one-form with a three-form. For simplicity, we will consider trivial cohomology in the symmetry algebra, i.e., that the corresponding FDA1 carries no cocycle in the Maurer–Cartan equations. This choice simplifies the long expressions in the final L_∞ product, allowing us to write their components without using differential forms. Moreover, in contrast with the general expressions for the products describing the dynamical sector of the theory in eqs. (8.58), (8.61), in this case we will write the products in a completely general way. This means that, when writing the products of a large number of vectors in X_{-1}, we will use arbitrary independent vectors instead of one vector multiple times.

Let us therefore consider a FDA1 with $p = 3$ with trivial cohomology, i.e., in absence of four-cocycle . By imposing these conditions in eqs. (3.11) and (3.12), we obtain the following Maurer–Cartan equations

$$\mathrm{d}\mu^A + \frac{1}{2} C^A_{BC} \mu^B \mu^C = R^A = 0, \tag{8.64}$$

$$\mathrm{d}\mu^i + C^i_{Aj} \mu^A \mu^j = R^i = 0. \tag{8.65}$$

Notice that eq. (8.64) describes the Lie subalgebra, while eq. (8.65) just states that the covariant derivative of the three-form μ^i, defined through the arbitrary representation $(t_A)^i{}_j = C^i_{Aj}$, vanishes. The gauging of this FDA1 leads to a non-vanishing curvature with components R^A and R^i that allows the construction of a six-dimensional invariant density χ_6 and, consequently, a Chern–Simons action principle that makes use of its corresponding Chern–Simons form as Lagrangian density

$$S_{\mathrm{CS}_5}[\mu] = \int_{M_5} \int_0^1 \mathrm{d}t \left(3g_{A_1A_2A_3}\mu^{A_1}R_t^{A_2}R_t^{A_3} + g_{A_1i_1}\mu^{A_1}R_t^{i_1} + g_{A_1i_1}R_t^{A_1}\mu^{i_1}\right). \quad (8.66)$$

In this case, the homotopic gauge field is defined as $\mu_t = t\mu = \left(t\mu^A, t\mu^i\right)$, being R_t its corresponding field strength. An example of this action for a particular bosonic FDA can be found in ref. [45]. This allows to formulate a gauge theory with non-trivial coupling between a one-form and a three-form.

The L_∞ algebra that describes this theory is given by a vector space $X = X_0 \oplus X_{-1} \oplus X_{-2}$ endowed with a finite set of products. Every subspace can be split into two subspaces, standard and extended. Thus, arbitrary vectors in X_0 can be split as

$$\varepsilon = \left(\varepsilon^A, \varepsilon^i_{\mu\nu}\right), \quad (8.67)$$

i.e., their components are given by a zero-form in the adjoint representation of the Lie subalgebra and a two-form in the arbitrary representation space. In the same way, vectors in X_{-1} are decomposed in terms of a one-form and a three-form in the same respective representations, i.e.

$$\mu = \left(\mu^A_\mu, \mu^i_{\mu\nu\rho}\right). \quad (8.68)$$

Finally, an arbitrary vector in X_{-2} also carries two components, being the standard one a four-form in the adjoint representation of the Lie algebra and the extended one a two-form in the arbitrary representation

$$E = \left(E^A_{\mu\nu\rho\sigma}, E^i_{\mu\nu}\right). \quad (8.69)$$

Notice that there is an important difference in the definition of vectors in X_{-2} with respect to the other subspaces. Every other vector is, in general, given by an r-form in the standard sector and a $(p+r-1)$-form in the extended one. However, since the subspace X_{-2} is defined in order to introduce the equation of motion term $\mathcal{F}$, there is an ambiguity in the definition of these vectors that is compensated by the same ambiguity in the definition of the inner product of the L_∞ algebra. In this case, we have chosen to define vectors in X_{-2}, such that it becomes easy to identify the components of $\mathcal{F}$ from eq. (5.27).

The definition of gauge transformation leads to the following L_∞ products, regarding vectors in the subspaces X_0 and X_{-1} :

$$\boxed{\begin{array}{rcl} [\ell_1(\varepsilon)]^A_\mu &=& \partial_\mu \varepsilon^A, \\ [\ell_1(\varepsilon)]^i_\mu &=& \partial_\mu \varepsilon^i, \\ [\ell_2(\varepsilon,\mu)]^A_\mu &=& C^A_{BC}\mu^B_\mu \varepsilon^C, \\ [\ell_2(\varepsilon,\mu)]^i_{\mu\nu\rho} &=& C^i_{Aj}\left(3\mu^A_{[\mu}\varepsilon^j_{\nu\rho]} - \varepsilon^A \mu^j_{\mu\nu\rho}\right). \end{array}} \tag{8.70}$$

On the other hand, the gauge algebra acts as a second source of information, leading to the following products between vectors in X_0

$$\boxed{\begin{array}{rcl} [\ell_2(\varepsilon_1,\varepsilon_2)]^A &=& C^A_{BC}\varepsilon^B_2\varepsilon^C_1, \\ [\ell_2(\varepsilon_1,\varepsilon_2)]^i_{\mu\nu} &=& C^i_{Aj}\left(\varepsilon^A_2(\varepsilon_1)^j_{\mu\nu} - \varepsilon^A_1(\varepsilon_2)^j_{\mu\nu}\right). \end{array}} \tag{8.71}$$

Let us now consider the dynamics. In this case, the allowed values for the indices (m,n) in eq. (8.62) are $(3,0)$ and $(1,1)$. Therefore, from eqs. (8.58) and (8.61), we obtain three non-vanishing products for the standard dynamical sector, and one product for the extended dynamical sector of the theory, whose components are given by

$$\boxed{\begin{array}{l} [\ell_4(\mu_1,\mu_2,\mu_3,\mu_4)]^A_{\mu\nu\rho\sigma} = 3\times 3!\times 4! g^{AD} g_{DBC} C^B_{B_1B_2} C^C_{C_1C_2} \\ \qquad\qquad \times \left(\mu_{\{1}\right)^{B_1}_{[\mu} (\mu_2)^{B_2}_\nu (\mu_3)^{C_1}_\rho \left(\mu_{4\}}\right)^{C_2}_{\sigma]}, \\ [\ell_3(\mu_1,\mu_2,\mu_3)]^A_{\mu\nu\rho\sigma} = -3\times 3!\times 4! g^{AF} g_{FBC} C^B_{DE} \left(\mu_{\{1}\right)^D_{[\mu} (\mu_2)^E_\nu \partial_\rho \left(\mu_{3\}}\right)^C_{\sigma]}, \\ [\ell_2(\mu_1,\mu_2)]^A_{\mu\nu\rho\sigma} = -3!\times 4! g^{AD} g_{DBC} \partial_{[\mu}\left(\mu_{\{1}\right)^B_\nu \partial_\rho (\mu)^C_{\sigma]} \\ \qquad\qquad -8 g^A{}_i C^i_{Bj} \left(\mu_{\{1}\right)^B_{[\mu} \left(\mu_{2\}}\right)^j_{\nu\rho\sigma]}, \\ [\ell_2(\mu_1,\mu_2)]^i_{\mu\nu} = -2 g^{ij} g_{Aj} C^A_{BC} (\mu_1)^B_{\{\mu} (\mu_2)^C_{\nu\}}. \end{array}} \tag{8.72}$$

Finally, we consider the consistency products that come from plugging in the previously obtained products into the L_∞ identities in the ℓ-picture. These products involve vectors in X_0 and X_{-2} and are given by

$$\boxed{\begin{array}{rcl} [\ell_2(\varepsilon,E)]^A_{\mu\nu\rho\sigma} &=& C^A_{BC}E^B_{\mu\nu\rho\sigma}\varepsilon^C - 3! g^{AB} g_{ij} C^i_{Bk} E^j_{[\mu\nu}\varepsilon^k_{\rho\sigma]}, \\ [\ell_2(\varepsilon,E)]^i_{\mu\nu} &=& 2 g^{ij} g_{kl} C^k_{Bj} \varepsilon^B E^l_{\mu\nu}. \end{array}} \tag{8.73}$$

This completes the formulation of the mentioned five-dimensional Chern–Simons theory in terms of an L_∞ algebra. Notice that, by inspection of the action principle in eq. (8.66), we can see that the so-called BF five-dimensional theory can be obtained as a particular case (see ref. [100]). This case is obtained by considering a FDA1 for which the only non-vanishing component of the rank-2 invariant tensor is g_{Ai}, thus removing the first and second terms in the integral at the right side of eq. (8.66). The corresponding L_∞ algebra that describes this theory can also be obtained by imposing the same conditions in the L_∞ products from eqs. (8.70)-(8.73).

Chapter 9

Conclusions

In this dissertation, we have studied the dual relation between free differential algebras and L_∞ algebras, and their role in standard and higher gauge theories. The first goal accomplished in the thesis has been the generalization of the so-called Chern–Weil theorem to the case of a particular free differential algebra, known as FDA1, which is the most simple extension that can be found of a Lie algebra by means of the use of one of its own Chevalley–Eilenberg cohomology classes. The inclusion of such cohomology class extends the symmetry algebra, making it possible to couple higher-degree differential forms to a gauge theory in a non-trivial way. This has led to the following results:

- The study of the generalized Chern–Weil theorem has been achieved, in first place, by generalizing the Chern–Pontryagin invariant density to the case of a FDA1. The new density inherits the properties of gauge-invariance and differential closeness of its standard analogue, making the local existence of Chern–Simons forms possible. Moreover, its invariance properties have provided us with a definition of generalized invariant tensors, essential in the construction of action principles for FDA1 gauge theories.
- The gauging of a FDA1 and the consequent study of the invariance properties of the generalized Chern–Pontryagin invariant density have led to a definition of generalized covariant derivative for a FDA1 that has simplified further calculations.
- The new covariant derivative allows performing calculations in a completely analogous way to those presented in chapter 2 in the context of standard

gauge theories and Lie algebras. Thus, we have derived explicit expressions for transgression and Chern–Simons forms for a FDA1 that allow the construction of gauge invariant action principles that couple differential forms of different degrees as gauge fields. The corresponding equations of motion have also been derived.

Secondly, the results mentioned above allow studying the properties of FDA1-Chern–Simons forms from a different point of view. Standard Chern–Simons forms play an important role in the study of gauge anomalies, consequences of the breaking of classical symmetries in the quantization process of Yang–Mills theories. In particular, the chiral abelian anomaly is proportional to the Chern–Pontryagin invariant density. Moreover, the non-abelian gauge anomaly shares that topological origin; it is proportional to the gauge variation of the corresponding Chern–Simons form that is related to the Chern–Pontryagin invariant density by means of the standard version of the Chern–Weil theorem. On the other hand, the study of the generalized Chern–Pontryagin densities and Chern–Simons forms for a FDA1 shows that the generalized expressions contain the standard ones as the first terms of an expansion that can be understood as the sum over all the possible combinations of the standard and extended gauge fields, holding the FDA1 symmetry. Moreover, by studying the properties of invariance and differential closeness of these topological quantities and their gauge variations, we have found generalized versions of the non-abelian gauge anomaly. As mentioned, a FDA1 is an algebra that allows considering not only a one-form gauge field but also a higher-degree differential form. As a consequence, there are also two types of gauge variations: standard and extended. The standard one is performed with respect to a zero-form gauge parameter that takes values along all the possible directions of the Lie group manifold, corresponding to the Lie subalgebra of the FDA1. The extended variation is performed with respect to an extended parameter given by a higher-degree differential form that takes values in the remaining subspace of the FDA1. The existence of two gauge transformations has been proven to allow two types of generalizations of the non-abelian gauge anomaly that we called primary and secondary anomaly terms. The first of them is related to the standard parameter, and therefore, only that one reproduces the standard non-abelian anomaly as a particular case in the first term of its expansion. It must be pointed out that, since the extended gauge field (and also the extended parameter) is a higher-degree tensor, both anomaly terms exist in even and odd dimensionality, being possible to recover the standard case only for even-dimensional spacetimes. In contrast, the secondary anomaly term does not reproduce the standard non-abelian anomaly for either odd or even dimensionality. This is a natural consequence of its origin. Since the secondary anomaly term is obtained from the variations of the Chern–Simons form along the extended directions of the FDA1 manifold, it has an entirely different functional form which, however, naturally generalizes the primary anomaly term and shares its topological properties.

The second main goal of this thesis is the study of the aforementioned classical gauge theories in terms of L_∞ algebras. L_∞ algebras have been studied as deformations of Lie algebras that naturally appear when one moves from describing a theory in terms of the Poisson brackets of its functionals, such as the action principle or the Hamiltonian functional, to describe it in terms of the algebra of its local functions [42]. On the other hand, free differential algebras turn out to be dual to L_∞ algebras, which have been found in a completely different context by R. D'Auria, P. Fré and T. Regge [4] when studying the cohomology classes of the algebra of higher-dimensional supergravities. The results regarding the second part of this dissertation make use of a recent work of O. Hohm and B. Zwiebach [41], in which a formulation of classical gauge theories in terms of L_∞ algebras is introduced. Such formalism allows writing down the information of a theory, regarding its gauge transformations, gauge algebra, and the dynamics of the interacting theory into a single L_∞ algebra that contains the original gauge algebra as a subalgebra. In this thesis, we hace proposed some explicit examples of the abstract result of O. Hohm and B. Zwiebach, with particular emphasis on theories whose gauge symmetry is described by a FDA1, and therefore its gauge subalgebra is not necessarily a Lie algebra but a genuine L_∞ algebra. The natural presence of higher degree differential forms as gauge fields in FDA1 gauge theories makes it convenient to write the relations of the L_∞ algebras in terms of differential forms. We have therefore written the L_∞ algebras of FDA1 gauge theories in these terms, which can be understood as a compact notation that allows to easily obtain the true L_∞ relations by removing the dependence on differential forms. This has led to the following results:

- The L_∞ algebra that describes standard Chern–Simons theory in arbitrary dimensions has been found. In this case, the corresponding gauge subalgebra is a Lie algebra, as expected, and does not depend on the dimensionality of the theory. Moreover, the closure of the gauge subalgebra is ensured for every case without imposing any restriction on the dynamics. In contrast, the dynamics of the theory has been encoded into a subspace in which the number of non-vanishing products has been enumerated in a way that explicitly depends on the dimensionality of the theory.

- We have formulated the L_∞ algebra that describes the most simple case of a

FDA1 gauge theory, namely the flat FDA1 theory. This is the case in which the dynamics is governed by the Maurer–Cartan equations of the FDA1, and therefore, there is no on-shell gauge curvature. In this case, the gauge subalgebra is not a Lie algebra but a FDA1, and its closure is not ensured for every action principle. We have proved that the commutator of two gauge transformations leads to a third composite gauge transformation that includes a term that must be interpreted as an equation-of-motion symmetry. Therefore, in order to introduce a well-defined gauge theory, it is necessary to consider an action principle for which such extra term vanishes on-shell. This condition is always satisfied by a flat FDA1 theory.

- The L_∞ algebra that describes FDA1-Chern–Simons theory in arbitrary dimensions has been found. In contrast to the standard case, this formulation is valid in both odd and even dimensionality. Although the gauge symmetry is, as in flat FDA1 theory, described by a FDA1, its closure is not verified for every case. In order to write down a completely well-defined gauge theory, it is necessary to verify that the equation-of-motion symmetry that appears in the gauge algebra is indeed vanishing on-shell. The L_∞ algebra that describes FDA1-Chern–Simons theory is therefore well-defined only in the cases in which this condition is verified. There are some criteria that allow ensuring the closure of the gauge subalgebra. For instance, this requirement is immediately satisfied by every three-dimensional theory and also by every theory described by an L_∞ algebra without non-vanishing products of three or more elements. Lie gauge theories are examples of these cases. Moreover, if the gauge symmetry is described by a FDA1 carrying a trivial cocycle, i.e., in the absence of cohomology, the theory is immediately consistent. There is also possible to find examples of well-defined higher-dimensional FDA1-Chern–Simons theories with non-trivial cohomology. However, the consistency between the closure of the gauge algebra and the dynamics must be verified case by case.

Appendix A

Notation

In this thesis, we use two notations for vectors and invariant tensors for a FDA1, both useful in different contexts. We refer to them as index-dependent notation, and index-free notation, respectively. In this appendix, we review the translation between both notations.

Let us consider an arbitrary FDA and an algebraic vector x, given by a collection of differential forms, and denoted by $x = \left(x^{A(1)}, \ldots, x^{A(N)}\right)$, where each index $A\,(p)$ runs on a different domain. We introduce a FDA-degree that will be useful to identify the differential degree of each component. The FDA-degree of x is the differential degree of its first component, i.e., if $x^{A(1)}$ is a r-form, we say that $\deg_{\mathrm{FDA}} x = r$. For instance, the FDA1 field strength carries two components, a two-form R^A and a $(p+1)$-form R^i. The FDA-degree of R is therefore 2. This gradation of the vector space is useful in the following definitions that allow introducing the index-free notation, and it has no relation with the gradations of the dual L_∞ algebra from chapter 4. In general, a FDA1-valued vector with FDA-degree r can be split into its standard and extended components, as follows

$$x = \left(x^{A(1)}, x^{A(p)}\right). \tag{A.1}$$

The first component $x^{A(1)}$ is a r-form, while the extended component $x^{A(p)}$ is a $(r+p-1)$-form. For later convenience, we refer in general to the extended components as $\bar{r}$-forms, with $\bar{r} = r + p - 1$.

We now introduce a product between algebraic vectors in terms of the brackets. These brackets encode the information regarding the generalized structure constants of a FDA1 and provide an index-free notation for their contraction. Let $B_1, \ldots B_{p+1}$ be FDA1 vectors, each one of FDA-degree $b_1, \ldots, b_{p+1}$ respectively. In terms of the structure constants of the FDA1, we define:

1. A bilinear product $[B_1, B_2]$, whose standard and extended components are given by

$$[B_1, B_2]^A = C^A_{BC} B^B_1 B^C_2, \tag{A.2}$$

$$[B_1, B_2]^i = C^i_{Bj} B^B_1 B^j_2. \tag{A.3}$$

2. A $(p+1)$-linear product $[B_1, \ldots, B_{p+1}]$, whose standard and extended components are given by

$$[B_1, \ldots, B_{p+1}]^A = 0, \tag{A.4}$$

$$[B_1, \ldots, B_{p+1}]^i = C^i_{A_1 \cdots A_{p+1} 1} B^{A_1}_1 \cdots B^{A_{p+1}}_{p+1}. \tag{A.5}$$

The resulting products are both FDA1 valued vectors, with

$$\deg_{\mathrm{FDA}} [B_1, B_2] = b_1 + b_2, \tag{A.6}$$

$$\deg_{\mathrm{FDA}} [B_1, \ldots, B_{p+1}] = b_1 + \cdots + b_{p+1} - p + 1. \tag{A.7}$$

The index-free notation allows us to write the Maurer–Cartan equations of a FDA1 from eqs. (3.11) and (3.12) as follows

$$\mathrm{d}\mu^A + \frac{1}{2} [\mu, \mu]^A = 0, \tag{A.8}$$

$$\mathrm{d}B^i + [\mu, \mu]^i + \frac{1}{(p+1)!} \left[\mu^{p+1}\right]^i = 0, \tag{A.9}$$

with $\left[\mu^{p+1}\right] \equiv \left[\mu, \overset{p+1}{\ldots}, \mu\right]$.

Let us now consider two sets of algebraic vectors $B_1, \ldots B_m$ and $E_1, ..., E_n$, with FDA-degrees $b_1, \ldots, b_m$ and $e_1, \ldots e_n$ respectively. We introduce the following compact notation for the contraction of their components with the FDA1 invariant tensor

$$\langle B_1, \ldots, B_m; E_1, \ldots, E_n \rangle = g_{A_1 \cdots A_m i_1 \cdots i_n} B^{A_1}_1 \cdots B^{A_m}_m E^{i_1}_1 \cdots E^{i_n}_n. \tag{A.10}$$

Notice that in the absence of extended components (fixing $n = 0$), the FDA1 becomes a Lie algebra, and eq. (A.10) becomes the usual notation of the symmetrized trace.

Moreover, the bracket in the l.h.s of eq. (A.10) separates the algebraic sectors before and after the semicolon, being the first ones valued in the standard sector and the latter in the extended sector. The invariant tensor of the FDA1 inherits the symmetry in the first set of indices from the invariant tensor of Lie algebras. As a consequence, the bracket in the l.h.s. of eq. (A.10) has the following graded symmetry due to the permutation of differential forms in the standard sector:

$$\langle \ldots, B_r, B_{r+1}, \ldots; E_1, \ldots, E_n \rangle = (-1)^{b_r b_{r+1}} \langle \ldots, B_{r+1}, B_r, \ldots; E_1, \ldots, E_n \rangle \, . \tag{A.11}$$

On the other hand, the FDA1 invariant tensor presents graded symmetry in the extended indices, depending on the degree of the p-form with which the FDA1 is defined, as follows

$$g_{A_1 \cdots A_m i_1 \cdots ij \cdots i_n} = (-1)^{p+1} \, g_{A_1 \cdots A_m i_1 \cdots ji \cdots i_n} \, . \tag{A.12}$$

Such symmetry rule, in addition to the permutation of differential forms in the extended sector, leads to the following graded symmetry for the bracket in index-free notation:

$$\langle B_1, \ldots, B_m; \ldots, E_s, E_{s+1}, \ldots \rangle = (-1)^{\bar{e}_s \bar{e}_{s+1} + p + 1} \langle B_1, \ldots, B_m; \ldots, E_{s+1}, E_s, \ldots \rangle \, . \tag{A.13}$$

The invariant properties of $g_{A_1 \cdots A_m i_1 \cdots i_n}$ provide us with a notion of covariance and contravariance for the algebraic indices of a FDA1. To clarify this, let us consider an arbitrary vector $B = \left(B^A, B^i\right)$ with FDA-degree b. In analogy with the case of Lie algebras, we call B^A and B^i the covariant components of the vector. We define the contravariant duals of B as $B_A = g_{AB} B^B$ and $B_i = g_{ij} B^i$. Although the components with mixed indices g_{Ai} are in general non-vanishing, we do not include them into the definition in order not to change the differential form degree of the components of B. In this way, B_A and B_i are also a b-form and $\bar{b}$-form respectively, which are univocally determined due to the assumption of non-degeneracy in g_{AB} and g_{ij}. In the same way, we define the inverse components g^{AB} and g^{ij} through the following relations

$$g_{AB} = g_{AC} g_{BD} g^{CD} , \tag{A.14}$$

$$g_{ij} = g_{ik} g_{jl} g^{kl} . \tag{A.15}$$

By setting $n = 0$, g_{AB} is reduced to the Cartan–Killing metric of the Lie algebra. However, it must be pointed out that this is not a rigorous definition of a generalized Cartan–Killing metric for a FDA1, but only a notation that turns out to be helpful when writing the components of FDA1 vectors with lower indices without introducing ambiguity.

Appendix B

Invariance identities

In this appendix, we consider the derivation of useful properties of FDA1 invariant tensors. In order to clarify the role of such properties in the study of extended gauge theories, we begin with their standard equivalents in the context of Lie algebras. Let G be a Lie algebra expanded by a basis of generators $\{t_A\}_{A=1}^{\dim G}$ with structure constants C_{AB}^{C}. Let us also consider a set of arbitrary differential forms $X_1, \ldots, X_n$ and Θ of degrees $x_1, \ldots, x_n$ and θ respectively, valued in G, and let us recall the degree-n invariant tensor condition from eq. (2.45). By contracting eq. (2.45) with the differential form $\Theta^{A_0} X_1^{A_1} \cdots X_n^{A_m}$ (which is given by the wedge product of components of $X_1, \ldots, X_n$ and Θ on the above mentioned basis), one finds

$$\sum_{k=1}^{n} C_{A_0 A_k}^{C} \left\langle t_{A_1}, \ldots, \hat{t}_{A_k}, t_C, \ldots, t_{A_n} \right\rangle \Theta^{A_0} X_1^{A_1} \cdots X_n^{A_m} = 0. \tag{B.1}$$

Eq. (B.1) allows to identify the algebraic vectors $X_1, \ldots, X_n$ and Θ, removing the dependence on the chosen basis of vectors $\{t_A\}_{A=1}^{\dim G}$. This allows writing the following identity

$$\sum_{k=1}^{n} (-1)^{\theta(x_1+\cdots+x_k)} \langle X_1, \ldots, [\Theta, X_k], \ldots, X_n \rangle = 0. \tag{B.2}$$

Eq. (B.2) is equivalent to the invariant tensor condition of eq. (2.45) in index-free notation and involves arbitrary differential forms, relating the Lie bracket with the symmetrized trace. We refer to eq. (B.2) as invariance identity.

Let us now repeat the above procedure for the case of a FDA1. Let $X_1, \ldots, X_m$, $Y_1, \ldots, Y_n$ and Θ be arbitrary FDA1 vectors whose FDA-degrees we denote by

$x_1, \ldots, x_m$, $y, \ldots, y_n$ and θ. We denote their components by the same letter with its corresponding algebraic index (e.g. $\Theta = \left(\Theta^A, \Theta^i\right)$). We now recall the invariant tensor conditions from eqs. (5.9)-(5.11) and proceed case by case.

By multiplying the first invariant tensor condition of eq. (5.9) by the wedge product of standard components $\Theta^{A_0} X^{A_1} \cdots X^{A_m} Y^{i_1} \cdots Y^{i_n}$, we get

$$\sum_{r=1}^{m} \Theta^{A_0} X_1^{A_1} \cdots X_m^{A_m} Y_1^{i_1} \cdots Y_n^{i_n} C_{A_0 A_r}^{C} g_{A_1 \cdots \hat{A}_r C \cdots A_m i_1 \cdots i_n} \\ + \sum_{s=1}^{n} \Theta^{A_0} X_1^{A_1} \cdots X_m^{A_m} Y_1^{i_1} \cdots Y_n^{i_n} C_{A_0 i_s}^{k} g_{A_1 \cdots A_m i_1 \cdots \hat{\imath}_s k \cdots i_n} = 0. \tag{B.3}$$

Thus, by plugging in eq. (A.10) into eq. (B.3) and identifying the argument of both sums as the components of the FDA1 products defined by eqs. (A.2)-(A.5), we obtain the following identity

$$\sum_{r=1}^{m} (-1)^{\theta(x_1+\cdots+x_{r-1})} \left\langle X_1, \ldots, X_{r-1}, \left[\Theta, X_r\right], X_{r+1}, \ldots, X_m; Y_1, \ldots, Y_n\right\rangle \\ + \sum_{s=1}^{n} (-1)^{\theta(x_1+\cdots+x_m+\bar{y}_1+\cdots \bar{y}_{s-1})} \left\langle X_1, \ldots, X_m; Y_1, \ldots, Y_{s-1}, \left[\Theta, Y_s\right], Y_{s+1}, \ldots, Y_n\right\rangle = 0. \tag{B.4}$$

Notice that eq. (B.4) relates both the standard and extended components of the bilinear product to the invariant tensor, and that it reproduces eq. (B.3) for $n = 0$.

The second invariant tensor condition from eq. (5.10) leads to a different identity, for which we need to introduce a new set of FDA1 vectors $\Theta_1, \ldots, \Theta_p$, and consider the wedge product of components $\Theta^{B_1} \cdots \Theta^{B_p} X_1^{A_1} \cdots X_{m+1}^{A_{m+1}} Y_1^{i_2} \cdots Y_n^{i_n}$. Notice that in this case we use the extended components of $Y_1, \ldots, Y_n$. By contracting such term with eq. (5.10) and identifying the resulting terms as the components of a FDA1 $(p+1)$ linear product, we obtain the following identity

$$\sum_{r=1}^{m+1} (-1)^{x_r(x_{r+1}+\cdots+x_{m+1})} \left\langle X_1, \ldots, X_{r-1}, X_{r+1}, \ldots, X_{m+1}; \right. \\ \left. \left[X_r, \Theta_1, \ldots, \Theta_p\right], Y_2, \ldots, Y_n\right\rangle = 0. \tag{B.5}$$

Finally, we consider the contraction of eq. (5.11) with the differential form $\Theta^j X^{A_1} \cdots X^{A_{m+1}} Y^{i_2} \cdots Y^{i_n}$. This relation implies the definition of one extra vector X_{m+1}. The repetition of the previous procedure leads to a third identity relating the extended component of the FDA1 bilinear product with the invariant tensor:

$$\sum_{r=1}^{m+1} (-1)^{x_r(x_{r+1}+\cdots+x_{m+1})} \langle X_1, \ldots, X_{r-1}, X_{r+1}, \ldots, X_{m+1}; [X_r, \Theta], Y_2, \ldots, Y_n \rangle = 0. \tag{B.6}$$

Eqs. (B.4)-(B.6) are the generalization to the case of a FDA1 of the invariant tensor property from eq. (5.11) studied in ref. [37] in the context of Lie algebras. They relate the FDA1 products with the invariant tensor in free-index notation, are equivalent to the requirement that $g_{A_1 \cdots A_m i_1 \cdots i_n}$ be an invariant tensor of the FDA1, and are valid for arbitrary values of m and n.

Appendix C

Consistency products

In this appendix, we consider the calculation of products not explicitly obtained in the L_∞ formulation of gauge theories from chapters 7 and 8. As it was mentioned in these chapters, the L_∞ formulation of gauge theories extracts the information of the theory from three sources of information and encodes it into the products of an L_∞ algebra. However, these products do not satisfy the L_∞ identities of eq. (4.11) by themselves, at least for the studied cases. By demanding them to satisfy such identities, we find new products acting on X_{-2} that must be non-vanishing for consistency. A simple procedure that allows to find those products consists of taking the gauge variation of the definition of the equation of motion term in eq. (4.51), namely

$$\delta\mathcal{F} = \sum_{k=1}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^{\frac{k(k-1)+r(r-1)}{2}}}{k!r!}\ell_k\left(\ell_{r+1}\left(\varepsilon,\mu^r\right),\mu^{n-1}\right). \tag{C.1}$$

The imposition of the L_∞ identities in the ℓ-picture of eq. (4.11) into eq. (C.1) allows to write $\delta\mathcal{F}$ in terms of a single sum, in which the nested products are absorbed into $\mathcal{F}$ as follows

$$\delta\mathcal{F} = \sum_{k=0}^{\infty}\frac{(-1)^{\frac{k(k-1)}{2}}}{k!}\ell_{k+2}\left(\varepsilon,\mathcal{F},\mu^k\right). \tag{C.2}$$

Thus, demanding eq. (C.2) to be satisfied becomes equivalent to demanding the already found products to satisfy the L_∞ identities. We now can directly compare this expression with the variation of $\mathcal{F}$ that is explicitly obtained case by case, and extract the information about the missing products that must be non-vanishing in order to close a well-defined L_∞ algebra. Since the gauge parameter ε lies in X_0 and the equation of motion term $\mathcal{F}$ lies in X_{-2}, every consistency product found using this procedure will involve at least one element in X_0 and one element in X_{-2}.

C.1 Products in Chern–Simons theory

We begin by studying the consistency products in the formulation of standard Chern–Simons theory. In this case, the equation of motion term is given by

$$\mathcal{F}^{A}_{\nu} = \varepsilon_{\nu}{}^{\mu_1\cdots\mu_{2m-2}} g^{A}{}_{B_1\cdots B_{m-1}} R^{B_1}_{\mu_1\mu_2} \cdots R^{B_{m-1}}_{\mu_{2m-3}\mu_{2m-2}}. \tag{C.3}$$

The gauge variation of the gauge curvature is given by the Lie bracket of the curvature and the parameter, i.e., through the well-known relation $\delta R^{A}_{\mu\nu} = C^{A}_{BC} R^{B}_{\mu\nu} \varepsilon^{C}$. This allows to write

$$\delta\mathcal{F}^{A}_{\nu} = (m-1)\,\varepsilon_{\nu}{}^{\mu_1\cdots\mu_{2m-2}} g^{A}{}_{B_1\cdots B_{m-1}} C^{B_1}_{BC} R^{B}_{\mu_1\mu_2} \varepsilon^{C} R^{B_2}_{\mu_2\mu_4} \cdots R^{B_{m-1}}_{\mu_{2m-3}\mu_{2m-2}}. \tag{C.4}$$

Finally, by plugging in the definition of invariant tensor for Lie algebras from eq. (2.45) into eq. (C.4), we find

$$\begin{aligned}\delta\mathcal{F}^{A}_{\nu} &= \varepsilon_{\nu}{}^{\mu_1\cdots\mu_{2m-2}} g^{B}{}_{B_1\cdots B_{m-1}} C^{A}_{BC} \varepsilon^{C} R^{B_1}_{\mu_1\mu_2} \cdots R^{B_{m-1}}_{\mu_{2m-3}\mu_{2m-2}} \\ &= C^{A}_{BC} \mathcal{F}^{B}_{\nu} \varepsilon^{C}.\end{aligned} \tag{C.5}$$

By inspection of eqs. (C.2) and (C.5), it follows that there is one non-vanishing product that involves vectors in X_0 and X_{-2}, necessary for the closure of the entire L_∞ algebra:

$$\left[\ell_2\left(\varepsilon, \mathcal{F}\right)\right]^{A} = C^{A}_{BC} \mathcal{F}^{B} \varepsilon^{C}. \tag{C.6}$$

Moreover, eq. (C.2) shows that the equation of motion term $\mathcal{F}^{A}_{\nu}$ inherits the transformation law of the gauge curvature. Since this feature does not come from the symmetry but the dynamics, it is only valid in standard Chern–Simons theory. In general, a theory with different symmetry algebra, or whose equation of motion term shows a different functional dependence on the fundamental field, may not share such property.

C.2 Products in flat FDA1 theory

The second case under study is the flat FDA1 theory. In this case, the equations of motions are equivalent to the Maurer–Cartan equations for the FDA1, and therefore, it is possible to immediately identify the components of $\mathcal{F}$ as $\left(\mathcal{F}^{A}, \mathcal{F}^{i}\right) = \left(R^{A}, R^{i}\right)$. Notice that, as it was mentioned in chapter 7, we write the equation of motion term and the corresponding L_∞ products in terms of differential forms. Thus, the gauge variation of $\mathcal{F}$ is given by eqs. (3.45) and (3.46) in terms of the components of the parameter ε^A and ε^i, as follows

$$\delta\mathcal{F}^A = C^A_{BC}\mathcal{F}^B\varepsilon^C, \tag{C.7}$$
$$\delta\mathcal{F}^i = C^i_{Aj}\mathcal{F}^A\varepsilon^j - C^i_{Aj}\varepsilon^A\mathcal{F}^j - \frac{1}{(p-1)!}C^i_{A_1\cdots A_{p+1}}\varepsilon^{A_1}\mathcal{F}^{A_2}\mu^{A_3}\cdots\mu^{A_{p+1}}. \tag{C.8}$$

By inspection of eqs. (C.7), (C.8) and (C.2) we immediately find two non-vanishing products $\ell_2(\varepsilon,\mathcal{F})$ and $\ell_{p+1}\left(\varepsilon,\mathcal{F},A^{p-1}\right)$, whose components (in terms of differential forms) are given by

$$[\ell_2(\varepsilon,\mathcal{F})]^A = C^A_{BC}\mathcal{F}^B\varepsilon^C, \tag{C.9}$$
$$[\ell_2(\varepsilon,\mathcal{F})]^i = C^i_{Aj}\left(\mathcal{F}^A\varepsilon^j - \varepsilon^A\mathcal{F}^j\right), \tag{C.10}$$
$$\left[\ell_{p+1}\left(\varepsilon,\mathcal{F},A^{p-1}\right)\right]^i = (-1)^{1+\frac{(p-1)(p-2)}{2}}\,C^i_{A_1\cdots A_{p+1}}\varepsilon^{A_1}\mathcal{F}^{A_2}\mu^{A_3}\cdots\mu^{A_{p+1}}. \tag{C.11}$$

Eq. (C.9) corresponds to the standard component of the first consistency product. Notice that it reproduces the consistency product of the previous case from eq. (C.6). This is a natural consequence of the presence of the Lie subalgebra in the FDA1.

C.3 Products in FDA1-Chern–Simons theory

In the case of FDA1-Chern–Simons theory, the equation of motion term does not transform as the field strength, leading to more complicated expressions for the consistency products. Hence, we separate $\mathcal{F}$ into its standard and extended components and analyze their gauge variations separately.

C.3.1 Standard sector

Let us consider the gauge variation of the standard component $\mathcal{F}^A$ in eq. (8.51). By plugging in the variation of the gauge curvatures of eqs. (C.7) and (C.8), we find

$$\begin{aligned}\delta\mathcal{F}^A = \sum_{m,n} m g^{AA_1}g_{A_1\cdots A_m i_1\cdots i_n}\Big(&(m-1)\,C^{A_2}_{BC}R^B\varepsilon^C R^{A_3}\cdots R^{A_m}R^{i_1}\cdots R^{i_n}\\ &+ nR^{A_2}\cdots R^{A_m}C^{i_1}_{Aj}R^A\varepsilon^j R^{i_2}\cdots R^{i_n} - nR^{A_2}\cdots R^{A_m}C^{i_1}_{Aj}\varepsilon^A R^j R^{i_2}\cdots R^{i_n}\\ &-\frac{n}{(p-1)!}R^{A_2}\cdots R^{A_m}C^{i_1}_{A_1\cdots A_{p+1}}\varepsilon^{A_1}R^{A_2}\mu^{A_3}\cdots\mu^{A_{p+1}}R^{i_2}\cdots R^{i_n}\Big).\end{aligned} \tag{C.12}$$

By using the invariant tensor conditions of eqs. (5.9)-(5.11), it is possible to prove the following relations:

$$\begin{aligned} 0 = g_{A_1 \cdots A_m i_1 \cdots i_n} & \Big((m-1)\, C^{A_2}_{BC} R^{B} \varepsilon^{C} R^{A_3} \cdots R^{A_m} ; R^{i_1} \cdots R^{i_n} \\ & - n R^{A_2} \cdots R^{A_m} C^{i_1}_{Bj} \varepsilon^{B} R^{j} R^{i_2} \cdots R^{i_n} \Big) \\ & - g_{A A_2 \cdots A_m i_1 \cdots i_n} C^{A}_{BA_1} \varepsilon^{B} R^{A_2} \cdots R^{A_m} R^{i_1} \cdots R^{i_n}, \end{aligned} \tag{C.13}$$

$$\begin{aligned} 0 = {} & m g_{A_1 \cdots A_m i_1 \cdots i_n} f^{A_1} R^{A_2} \cdots R^{A_m} C^{i_1}_{B_1 \cdots B_{p+1}} R^{B_1} \varepsilon^{B_2} \mu^{B_3} \cdots \mu^{B_{p+1}} R^{i_2} \cdots R^{i_n} \\ & + g_{A A_2 \cdots A_m i_1 \cdots i_n} R^{A} R^{A_2} \cdots R^{A_m} C^{i_1}_{A_1 B_2 \cdots B_{p+1}} \varepsilon^{B_2} \mu^{B_3} \cdots \mu^{B_{p+1}} R^{i_1} \cdots R^{i_n}, \end{aligned} \tag{C.14}$$

$$\begin{aligned} 0 = {} & m g_{A_1 \cdots A_m i_1 \cdots i_n} R^{A_2} \cdots R^{A_m} C^{i_1}_{Bj} R^{B} \varepsilon^{j} R^{i_2} \cdots R^{i_n} \\ & + g_{A A_2 \cdots A_m i_1 \cdots i_n} R^{A} R^{A_2} \cdots R^{A_m} C^{i_1}_{A_1 j} \varepsilon^{j} R^{i_2} \cdots R^{i_n}. \end{aligned} \tag{C.15}$$

Thus, by plugging in eqs. (C.13)-(C.15) into eq. (C.12) and by identifying the differential forms $\mathcal{F}^A$ and $\mathcal{F}^i$ in the resulting expression, we can write

$$\begin{aligned} \delta \mathcal{F}^{A} = {} & C^{A}_{BC} \mathcal{F}^{B} \varepsilon^{C} - g^{AB} g_{ik} C^{i}_{Bj} \mathcal{F}^{k} \varepsilon^{j} \\ & - \frac{1}{(p-1)!} g^{AB_1} g_{ik} C^{i}_{B_1 B_2 \cdots B_{p+1}} \varepsilon^{B_2} \mu^{B_3} \cdots \mu^{B_{p+1}} \mathcal{F}^{k}. \end{aligned} \tag{C.16}$$

Finally, by inspection of eq. (C.16) and the general variation in eq. (C.2), we obtain the following non-vanishing products

$$\left[\ell_2 \left(\varepsilon, \mathcal{F} \right) \right]^{A} = C^{A}_{BC} \mathcal{F}^{B} \varepsilon^{C} - g^{AB} C^{i}_{Bj} \mathcal{F}_{i} \varepsilon^{j}, \tag{C.17}$$

$$\left[\ell_{p+1} \left(\varepsilon, \mathcal{F}, A^{p-1} \right) \right]^{A} = (-1)^{1+\frac{(p-1)(p-2)}{2}} C^{i_1}_{AB_2 \cdots B_{p+1}} \varepsilon^{B_2} \mu^{B_3} \cdots \mu^{B_{p+1}} \mathcal{F}_{i_1}. \tag{C.18}$$

C.3.2 Extended sector

Let us now consider the gauge variation of the extended component $\mathcal{F}^i$ in eq. (8.52), written in terms of the variations of the gauge curvatures from eqs. (C.7) and (C.8)

$$\begin{aligned} \delta \mathcal{F}^{i} = \sum_{m,n} n g^{i i_1} g_{A_1 \cdots A_m i_1 \cdots i_n} & \Big(m C^{A_1}_{BC} R^{B} \varepsilon^{C} R^{A_2} \cdots R^{A_m} R^{i_1} \cdots R^{i_{n-1}} \\ & + (n-1)\, R^{A_1} \cdots R^{A_m} C^{i_1}_{Bj} R^{B} \varepsilon^{j} R^{i_2} \cdots R^{i_{n-1}} \\ & - (n-1)\, R^{A_1} \cdots R^{A_m} C^{i_1}_{Bj} \varepsilon^{B} R^{j} R^{i_2} \cdots R^{i_{n-1}} \\ & - \frac{(n-1)}{(p-1)!} R^{A_1} \cdots R^{A_m} C^{i_1}_{B_1 \cdots B_{p+1}} \varepsilon^{B_1} R^{B_2} \mu^{B_3} \cdots \mu^{B_{p+1}} R^{i_2} \cdots R^{i_{n-1}} \Big). \end{aligned} \tag{C.19}$$

By using again the invariant tensor conditions of eqs. (5.9)-(5.11), it is possible to prove the following relations

$$
\begin{aligned}
& g_{A_1\cdots A_m i_1\cdots i_n}\Big(mC^{A_1}_{BC}R^B\varepsilon^C R^{A_2}\cdots R^{A_m}f^{i_1}R^{i_2}\cdots R^{i_n} \\
& -(n-1)\,R^{A_1}\cdots R^{A_m}f^{i_1}C^{i_2}_{Bj}\varepsilon^B R^j R^{i_3}\cdots R^{i_n}\Big) \\
& -g_{A_1\cdots A_m i i_2\cdots i_n}R^{A_1}\cdots R^{A_m};C^{i}_{Bi_1}\varepsilon^B R^{i_2}\cdots R^{i_n}=0,
\end{aligned}
\tag{C.20}
$$

$$
g_{A_1\cdots A_m i_1\cdots i_n}R^{A_1}\cdots R^{A_m}C^{i_2}_{B_1\cdots B_{p+1}}\varepsilon^{B_1}R^{B_2}\mu^{B_3}\cdots\mu^{B_{p+1}}R^{i_3}\cdots R^{i_n}=0, \tag{C.21}
$$

$$
g_{A_1\cdots A_m i_1\cdots i_n}R^{A_1}\cdots R^{A_m}C^{i_2}_{Bj}R^B\varepsilon^j R^{i_3}\cdots R^{i_n}=0. \tag{C.22}
$$

By plugging in eqs. (C.20)-(C.22) into eq. (C.19), the variation of the extended component $\mathcal{F}^i$ takes a simple form, namely

$$
\delta\mathcal{F}^i=g^{ij}g_{kl}C^k_{Bj}\varepsilon^B\mathcal{F}^l. \tag{C.23}
$$

Finally, by inspection of eq. (C.23) and the general expression in eq. (C.2), we obtain one consistency product for the extended sector:

$$
\left[\ell_2\left(\varepsilon,\mathcal{F}\right)\right]^i=g^{ij}g_{kl}C^k_{Bj}\varepsilon^B\mathcal{F}^l. \tag{C.24}
$$

This completes the calculation of consistency products for the studied theories.

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