Defects and symmetries in three-dimensional topological field theories

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Zusammenfassung

Trotz der umfassenden Erfolge von Quantenfeldtheorien (QFTs) in der Teilchen- und Festkörperphysik gibt es nach wie vor ungelöste konzeptionelle Fragen, insbesondere über die zugrundeliegende mathematische Struktur. Die jüngere Forschung hat sich auf Spezialfälle wie *topologische* QFTs (TFTs) konzentriert, in denen eine rigorose mathematische Beschreibung durch die Kategorientheorie möglich ist. Darüber hinaus kann eine auf *Bordismen* sowie eine auf *höheren Kategorien* basierende Klasse mathematischer Modelle für TFTs *Defekte* wie Schnittstellen zwischen verschiedenen TFTs, Ränder und Punktdefekte beschreiben.

Die Übersetzung einer durch die Sprache der Physik beschriebenen Defekt-TFT in ein rigoroses mathematisches Modell stellt jedoch eine große Herausforderung dar. Ein facettenreiches Beispiel hierfür ist das *affine Rozansky–Witten-Modell*, welches in der Physik eine *topologisch getwistete* 3D $\mathcal{N}=4$ supersymmetrische QFT ist. In der Mathematik ist es durch eine höhere Kategorie $\mathcal{RW}$ modelliert, die u. a. die Struktur der Defekte beschreibt. Bislang erfolgte grundlegende Analysen von $\mathcal{RW}$ haben bereits ergeben, dass seine zweidimensionalen Defekte nah verwandt sind mit dem topologischen *Landau–Ginzburg-Modell*, welches eine durch eine Bikategorie $\mathcal{LG}$ beschriebene, umfassend analysierte 2D Defekt-TFT bildet. Dennoch sind viele Aspekte der Trikategorie $\mathcal{RW}$ noch nicht genau erforscht worden.

Diese Dissertation ist in zwei Teile gegliedert: Der erste Teil beginnt mit einer Zusammenfassung der mathematischen Beschreibung von TFTs im Allgemeinen und $\mathcal{RW}$ im Speziellen. Ein zentraler Aspekt von $\mathcal{RW}$ ist die Theorie der *Matrixfaktorisierungen*, deren Grundlagen ausführlich eingeführt werden. Im nächsten Schritt werden neue Aussagen darüber bewiesen. Anschließend erfolgt eine Vorstellung der Beschreibung von $\mathcal{RW}$ als Trikategorie inklusive bislang nicht behandelter Details, die u. a. für einen zukünftigen Beweis der Trikategorie-Axiome erforderlich sind.

Mit dem Ziel, die *Orbifold-Prozedur* anzuwenden, werden danach *Adjunktionen* und *pivotale Strukturen* in $\mathcal{RW}$ diskutiert. Als erstes wichtiges Ergebnis dieser Dissertation wird an dieser Stelle eine Verallgemeinerung bekannter Resultate in $\mathcal{LG}$ (wie z. B. der Kapustin–Li-Formel) formuliert und bewiesen. Das zweite zentrale Ergebnis der Arbeit ist die Konstruktion einer *pivotalen Trikategorie mit Dualitäten* $\mathcal{T} \subset \mathcal{RW}$. Zum Abschluss wird ein *Orbifold-Datum* in $\mathcal{T}$ postuliert und ein Großteil seiner definierenden Eigenschaften bewiesen.

Der zweite Teil dieser Dissertation behandelt Modelle mit weniger Supersymmetrie wie 3D $\mathcal{N}=2$, in denen ein holomorpher *Halb-Twist* möglich ist. Der Halb-Twist macht QFTs nur partiell topologisch, weswegen die oben eingeführte mathematische Beschreibung nicht anwendbar ist. Dennoch ermöglicht er in durch *Lagrange-Funktionen* beschriebenen QFTs mehrere exakte (nicht-pertubative) Konstruktionen wie *supercurrent multiplets*. Ein weiteres Resultat dieser Forschungsarbeit ist eine Verallgemeinerung Letzterer auf 3D $\mathcal{N}=2$ QFTs mit Rändern und Freiheitsgraden auf dem Rand, mit einer beispielhaften Anwendung auf dreidimensionale Landau–Ginzburg-Modelle.

Relevante Veröffentlichungen

- Abschnitt 1.5 und Kapitel 2 und 3 basieren auf Forschungsarbeit mit Ilka Brunner, Nils Carqueville und Pantelis Fragkos, die in [7] erscheinen wird.
- Kapitel 4 ist eine gekürzte Version der mit Ilka Brunner und Alexander Tabler publizierten Forschungsarbeit [13].

Abstract

Despite the extensive success of quantum field theories (QFTs) in particle and solid state physics there are still unsolved conceptual problems, in particular regarding the underlying mathematical foundations. In recent years, research has focused on special cases like *topological* QFTs (TFTs) where mathematically rigorous descriptions in the language of category theory have been found. Two of these descriptions, namely those using *bordisms* and *higher categories*, are also capable of describing *defects* including boundaries, interfaces between different TFTs, and point insertions.

Translating examples of defect TFTs from a physics description to a rigorous mathematical model is, however, a challenging problem. A multifaceted example is given by the *affine Rozansky–Witten model*, which from a physics point of view is a *topologically twisted* supersymmetric 3D $\mathcal{N} = 4$ QFT. On the mathematics side, it features a description in terms of a higher category $\mathcal{RW}$ which covers many aspects of this model, in particular regarding its defects. For example, previous fundamental analysis of $\mathcal{RW}$ has shown that its two-dimensional defects are closely related to the topological *Landau–Ginzburg model* which forms a well-studied 2D defect TFT described by the bicategory $\mathcal{LG}$. However, many aspects of the tricategory $\mathcal{RW}$ have not yet been studied in detail.

This thesis consists of two parts: The first part begins with a summary of the mathematical description of TFTs in general and $\mathcal{RW}$ in particular. The latter prominently features *matrix factorisations* which are introduced in detail, followed by several new results. Afterwards, an introduction to the description of $\mathcal{RW}$ as a tricategory is presented, including novel details required for a future proof of the tricategory axioms.

With the goal of applying the *orbifold procedure*, *adjunctions* and *pivotal structures* in $\mathcal{RW}$ are discussed subsequently, yielding the first major result of this thesis: a generalisation of several established results in $\mathcal{LG}$ including the well-known Kapustin–Li formula. The second major result is the construction of a *pivotal tricategory with duals* $\mathcal{T} \subset \mathcal{RW}$. Finally, an *orbifold datum* in $\mathcal{T}$ is constructed and significant progress is made towards proving its defining relations.

The second part of this thesis discusses models with less supersymmetry, namely 3D $\mathcal{N} = 2$, which admit a holomorphic *half-twist*. While the latter is only capable of making QFTs partially topological, ruling out a mathematical description in the above sense, it nevertheless enables several exact (non-perturbative) constructions like *supercurrent multiplets* on the level of Lagrangians. The latter are generalised to 3D $\mathcal{N} = 2$ QFTs with boundaries and degrees of freedom on the boundary and then applied to three-dimensional Landau–Ginzburg models as an example.

Relevant publications

- Section 1.5 and Chapters 2 and 3 are based on joint work with Ilka Brunner, Nils Carqueville, and Pantelis Fragkos, to appear in [7].
- Chapter 4 is a shortened version of joint work with Ilka Brunner and Alexander Tabler published in [13].

Danksagung

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Contents

1 Introduction and background

1.1 Introduction

1.1.1 Topological quantum field theories with defects

Quantum field theories are ubiquitous in theoretical physics and have been among the most important tools in both particle and condensed matter physics for the past seven decades. Numerous real-world phenomena have been successfully described in the language of QFT, and methods like perturbation theory and lattice simulations enable highly accurate predictions in many examples. However, despite their great achievements, there are still unsolved conceptual problems regarding quantum field theories, most notably the lack of a mathematically rigorous description for most interacting QFTs. Especially the *path integral*, which from a physics point of view should describe a "sum" or "integral" over the space of field configurations, defies a rigorous description in most examples. Over the past few decades research has focused on special cases like *topological* QFTs (TFTs) [1, 93] and *conformal* QFTs [84] where rigorous mathematical descriptions in terms of category theory have been developed [81]. These descriptions are non-perturbative, i.e. they are valid even if the QFT is strongly coupled.

The defining property of a topological quantum field theory (TFT) on a given pseudo-Riemannian manifold $\mathcal{M}$ (i.e. a manifold with a metric) is that the correlation functions are invariant under continuous deformations of $\mathcal{M}$, i.e. they only depend on topological properties. TFTs appear in different areas of physics, for example in topological string theory [5], topological quantum computation [45], or the fractional quantum Hall effect in solid state physics [88]. Beyond practical applications, the study of toy model TFTs a fruitful research field in mathematical physics for improving the understanding of the underlying mathematical structure of QFTs.

The mathematical description of TFTs also encompasses *defects* [3, 30, 19] which are capable of describing boundaries, interfaces between different TFTs, point insertions, and general boundary TFTs localised on submanifolds of $\mathcal{M}$. An important consequence of topological invariance is that correlation functions in defect TFTs do not depend on the distances between defects. Consequently, reducing the distance between two defects A and B to an arbitrarily small amount does not change the physics, nor does it introduce singularities, allowing us to effectively *fuse* A and B. The ability to fuse defects introduces a rich algebraic structure into defect TFTs. Note that fusing defects is not possible in most non-topological QFTs due to the emergence of singularities.

The defect algebra in defect TFTs enables advanced constructions like the *orbifold procedure*, given in full generality in [27] with previous results in two dimensions in [46, 25, 8]. Starting from a TFT that is invariant under a symmetry, the orbifold procedure constructs a new TFT on which the symmetry acts trivially. The procedure resembles the well-known orbifold construction in string theory [34] and has a physical interpretation in terms of *gauging* a finite symmetry group [25, Remark 3.6].

The meaning of the term "topological field theory" is not unique, especially between mathematics and physics. To a physicist, "TFT" usually means the special case of a QFT (which might be described e.g. by a Lagrangian) whose observables are invariant under continuous deformations of the underlying manifold's metric [57, Chapter 16.2]. By contrast, a mathematician would define a (defect) TFT in the strict sense to be a functor from a (defect) bordism category into some target category (see Definition 1.2.1 for more details). The bordism approach has a physical interpretation in terms of an axiomatisation of the path integral [67, 24]. It should not be surprising that constructing a TFT in the strict mathematical sense is significantly harder than doing so in physics language.

Moreover, in two and three dimensions there exists another mathematical framework for rigorously describing aspects of defect TFTs using *higher categories* instead of functors on defect bordisms. These descriptions are not "complete" in the sense that there exists an algorithmic procedure to derive a higher category model from a defect bordism description, but the reverse is usually not possible (see Theorem 1.2.2). Nevertheless, the higher category models are capable of describing many aspects of a defect TFT, in particular regarding the structure of its defects. This dissertation uses one of two such descriptions that was first discussed in [30, 19], which should not be confused with the other approach called *extended TQFT* originating from [3, 44, 72, 16, 2]. The description used here is fundamentally based on the algebra of fusing defects in the TFT: Composition of 1-, 2-, and 3-morphisms corresponds to fusing defects of codimension 1, 2, or 3.

Given a defect TFT in physics language, constructing a higher category description has several uses like simplifying the evaluation of physical settings and helping understand the mathematical structure of the defects. Finding such a description also often amounts to an intermediate step in developing a bordism description.

1.1.2 Rozansky–Witten and Landau–Ginzburg models

A large class of TFTs (in the physics sense) can be generated from supersymmetric QFTs using the *topological twist* [93] (see Section 1.5.1), making supersymmetric theories an interesting field of study in the context of TFTs despite their phenomenological challenges. The main example discussed in this thesis is the *affine Rozansky–Witten model* which arises from the topological twist of a supersymmetric 3D $\mathcal{N} = 4$ quantum field theory [80]. It is a special case distinguished by its non-compact target space $T^*\mathbb{C}^n$, while other Rozansky–Witten models usually feature a compact holomorphic symplectic target manifold. There exists a description of the affine Rozansky–Witten model by a tricategory $\mathcal{RW}$ [64, 63], though the precise relation between the physics model and $\mathcal{RW}$ is not clear at this point.

The affine case is interesting in two regards: On the one hand, it is less complex than the compact case as one does not have to deal with curvature or non-trivial topology in the target space. Studying the details of this easier case may help formulating and proving statements about more difficult Rozansky–Witten models with curved target spaces. On the other hand, the non-compactness of the target space leads to some pathological properties of the affine model like an infinite-dimensional space of local operators (which is finite-dimensional in the most common type of bordism defect TFTs, see the discussion in Section 1.5.2). A priori it is not clear whether the constructions intended for "proper" TFTs also apply to the affine case. Our findings that constructions like *adjoints*, *pivotality*, and *orbifolds* do in fact work in the affine case are among the main results of this work.

The two-dimensional defects in $\mathcal{RW}$ are closely related to the topological *Landau–Ginzburg model* $\mathcal{LG}$ which is a well-studied two-dimensional defect TFT that was first discussed in [90] and is based on a related model for superconductivity first introduced in [50]. Like $\mathcal{RW}$

its physics description arises from a topological twist, though the underlying QFT has a 2D $\mathcal{N} = (2,2)$ supersymmetry. The structure and defects of Landau–Ginzburg models were analysed in numerous publications including [89, 62, 10, 70, 56, 11, 22, 20] that yielded a well-understood description in terms of a pivotal bicategory $\mathcal{LG}$.

Less research has been done on the (affine) Rozansky–Witten model: Its defect structure [64], the fundamentals of its tricategorical description [63], and aspects of its description in extended TQFT [9] have been worked out, yet a significant amount of details is still missing. In particular, *adjunctions* and *pivotality* in $\mathcal{RW}$ have not been discussed previously.

1.1.3 The structure of this dissertation

Section 1.2 provides a detailed summary of the description of defect TFTs by higher categories in two and three dimensions. Special attention is given to the diagrammatic calculus in higher categories which also forms a natural connection to physics. The relevance of *adjoints* and *pivotality* in higher categorical descriptions of defect TFTs is also discussed.

The Landau–Ginzburg model and the affine Rozansky–Witten model prominently feature *matrix factorisations* which are (roughly) given by pairs of matrices of polynomials (P, Q) such that for a given polynomial W, the following equations hold:

$$P \cdot Q = W \cdot \mathbb{1} \ , \quad Q \cdot P = W \cdot \mathbb{1} \ .$$

In other words, the pair (P, Q) factorises the identity matrix multiplied by some polynomial W. An extensive discussion of matrix factorisations can be found in Section 1.3, containing a summary of definitions and theorems needed by the subsequent chapters as well as several new results.

Section 1.5 then introduces the affine Rozansky–Witten model as a tricategory $\mathcal{RW}$ based on [63, 9]. In addition to a summary of previous work, we present new details regarding the structure of the tricategory and we provide a novel proof that the *truncation* (Definition 1.2.6) of $\mathcal{RW}$ is a bicategory, paving the way for proving that $\mathcal{RW}$ is a tricategory in the future. In order to simplify the tricategorical structure of $\mathcal{RW}$ we deviate from some conventions used by previous authors.

We subsequently attempt to prove the existence of a *pivotal structure* and *adjoints* in the tricategory $\mathcal{RW}$. Roughly speaking, these are conceptually required to "bend" straight 1- and 2-dimensional defects into a "Z"-shape and form the basis for deformation invariance in the tricategorical description (see Definition 1.2.15). In more practical terms, they are also necessary to describe "bubble"-shaped diagrams that are essential for the *orbifold construction* of Section 3.4.

One requirement for adjoint existence in $\mathcal{RW}$ is the existence of adjoints of 2-morphisms, i.e. that the homomorphism bicategories of $\mathcal{RW}$, which are generalisations of Landau–Ginzburg models, have adjoints. Adjoint existence in $\mathcal{LG}$ has been proven [22], but the proof is not general enough for $\mathcal{RW}$. The first major result of this thesis, presented in Chapter 2, is an adjoint existence proof for a sufficiently extensive generalisation of Landau–Ginzburg models. The discussion of pivotal structures in [22] is also slightly generalised to the case of an odd number of variables. These new results can be applied to find closed formulas for the *defect operators* and *quantum dimensions* (see Eqs. (2.5.9) and (2.5.10)), generalising the well-known Kapustin–Li disc correlator [62].

Having completed the discussion of adjoints of 2-morphisms in $\mathcal{RW}$, we begin Chapter 3 by showing that all 1-morphisms in $\mathcal{RW}$ have adjoints. We subsequently prove that the full tricategory $\mathcal{RW}$ cannot admit a pivotal structure, leading us to analyse the subcategories of $\mathcal{RW}$ that might be pivotal. The second major result of this thesis is the construction of a subcategory $\mathcal{T} \subset \mathcal{RW}$ that we conjecture and partially prove to be a *pivotal tricategory with duals*.

The structure of $\mathcal{T}$ is utilised in Section 3.4 where the prerequisites for applying the generalised orbifold procedure in $\mathcal{T}$ are discussed. A candidate for an *orbifold datum* $\mathcal{O}$ in $\mathcal{T}$ is constructed and most of its constraint equations are proven to hold (with the rest expected to hold as well), making extensive use of the formulas derived in Chapters 2 and 3.

A more general setting is discussed in Chapter 4 where the models feature less supersymmetry, namely 3D $\mathcal{N} = 2$. While this symmetry is insufficient to perform an ordinary (full) topological twist, there is a (holomorphic) *half-twist* making such theories partially topological. Due to the lack of full topological invariance the approaches of the previous chapters cannot be applied here. Nevertheless, exact (non-perturbative) constructions like *supercurrent multiplets* are possible even in this more general setting. We generalise known constructions for bulk supercurrent multiplets in 3D QFTs to admit not only boundary conditions, but also localised degrees of freedom on the boundary. As an example, the newly developed framework is then applied to three-dimensional Landau–Ginzburg models with boundaries.

1.1.4 Outlook

Rozansky–Witten models

An obvious starting point for future research is to complete some proofs that are left as conjectures in this thesis, i.e. completing the proof that $\mathcal{RW}$ is a tricategory (Conjecture 1.5.15), that the subcategory $\mathcal{T} \subset \mathcal{RW}$ is pivotal (Conjecture 3.2.15), and that the remaining orbifold datum identities of Conjecture 3.4.3 hold. The latter (and possibly Conjecture 3.2.15) will be discussed in [7], completing the construction of the orbifold datum in $\mathcal{RW}$.

Moving forward, it would be highly interesting to analyse the constructed orbifold of the affine Rozansky–Witten model and compare it to known structures like other tricategories or QFTs in physics language. Both finding and not finding a relation to known TFTs in tricategorical or physics language would be exciting new results.

Finally, it might be possible to generalise the presented results to larger subcategories of the affine Rozansky–Witten model $\mathcal{T} \subset \mathcal{T}' \subset \mathcal{RW}$ or even to Rozansky–Witten models with compact target spaces.

Supercurrent multiplets

The methods used to construct supercurrent multiplets are not restricted to 3D $\mathcal{N} = 2$ supersymmetry and could be generalised to other dimensions and symmetries in a straightforward manner. It would be particularly interesting to study the case of 3D $\mathcal{N} = 4$ and analyse the Rozansky–Witten model in this context.

Furthermore, one could generalise the boundary conditions to defect gluing conditions, making potentially contact with [33, 36]. In the Landau–Ginzburg example it is easy to see that defects between theories with different superpotentials involve factorisations of the difference of the two superpotentials on the two sides of the defect.

In the example of Landau–Ginzburg models, we have exhibited in some detail the symmetries of models involving matrix factorisations. In the case of *two-dimensional* Landau–Ginzburg models, matrix factorisations provided the key to fully solve the theories in situations with boundaries, in the sense that the full bulk and boundary spectrum and all correlation functions [62] were determined. It would be interesting to see to what extend these features have analogons in three dimensions. As the theory cannot be fully twisted, one would expect that a holomorphic dependence has to remain.

1.2 Categories and topological QFTs with defects

1.2.1 Introduction

From a physics point of view, a classical field theory consists of fields that map from some (pseudo-)Riemannian manifold $\mathcal{M}$ (sometimes called "spacetime", or "worldsheet" in the case of string theory) into some target space $\mathcal{S}$. In the simplest examples $\mathcal{M}$ is a flat Minkowski space and $\mathcal{S} = \mathbb{C}$. Roughly speaking, a *quantum field theory* (QFT) can be constructed from such a classical field theory, hence the former contains quantised versions of fields $\phi\colon \mathcal{M} \to \mathcal{S}$. *Topological* quantum field theories [93] in the physics sense (abbreviated as "topological field theories" or "TFTs") then are, roughly, the subset of all quantum field theories whose correlation functions are invariant under continuous deformations of $\mathcal{M}$ and thus only depend on topological structures on $\mathcal{M}$.

Let $n := \dim \mathcal{M}$. The simplest version of topological field theory is *closed TFT* which restricts the discussion to manifolds $\mathcal{M}$ without boundary.[1] A generalisation is given by *open-closed TFT* which allows $\mathcal{M}$ to have an optional boundary. Even more general is *defect TFT* where the closed n-dimensional manifold $\mathcal{M}$ is divided into n-dimensional submanifolds, and each submanifold may be home to a different bulk TFT. The interfaces (or domain walls) between the different bulk TFTs form TFTs on $(n-1)$-dimensional submanifolds of $\mathcal{M}$ that are coupled to the bulk on either side. These interfaces may be subdivided in the same manner, creating TFTs on $(n-2)$-dimensional submanifolds, and the process can be iterated all the way down to dimension zero. Various physical settings can be expressed in the language of defect TFTs:

- Closed TFT is a special case of defect TFT where the only submanifold of $\mathcal{M}$ is $\mathcal{M}$ itself and there are no defects of lower dimension.
- One may describe two (or more) different n-dimensional TFTs a, b that interact via an $(n-1)$-dimensional interface X. In the language of defect TFTs, one subdivides $\mathcal{M}$ into several parts labelled a or b, and the $(n-1)$-dimensional submanifolds between a and b are labelled X. In physical applications, this may e.g. describe interfaces between different vacuum phases or boundary layers between different materials.
- Open-closed TFT is another special case of defect TFT: We use the same setting as the previous case with $\{a, b\} := \{\emptyset, t\}$, with t representing an open-closed TFT and $\emptyset$ representing the empty (trivial) TFT. Defects labelled X are interfaces between an open-closed TFT and the empty TFT, which is the same as a boundary of an open-closed TFT.
- For every dimension $m < n$ there are special types of defects called *identity* defects whose $(m+1)$-dimensional domain and codomain agree. Their defining property is that they are "invisible": They can be added to and removed from $\mathcal{M}$ without changing correlation functions or observables.
- Defect TFTs also contain point defects, which can be used to describe bulk field insertions or point-like impurities in materials. They may be located on higher-dimensional

[1] More precisely, every slice of constant time $\mathcal{M}|_{t=t'}$ is a closed $(n-1)$-dimensional manifold for all times t'; see [17] for more details.

defects or in the bulk; in the latter case one formally puts the bulk point defect onto an identity defect.

The main reason why defects are especially fruitful in topological field theories is the ability to *fuse* defects: We may reduce the distance of two m-dimensional defects $X: a \to b$, $Y: b \to c$ to an arbitrarily small value; doing this does not change the physics due to topological invariance. This effectively removes the $(m+1)$-dimensional defect b between X and Y, creating a new m-dimensional defect $Y \otimes X: a \to c$. Fusing defects in TFTs thus induces a rich algebraic structure. The construction of similar defect algebras is usually not possible in non-topological QFTs due to the appearance of singularities when reducing the distance between defects to arbitrarily small values.

1.2.2 Mathematical description

One special property of (defect) TFTs is the availability of a rigorous mathematical description. As the details will not be of importance for the rest of this thesis, only the definition will be stated here; see [27] for the full details. First developments towards this description were made in [1] for TFTs and in [84] for CFTs.

Definition 1.2.1. An n-dimensional defect TFT (in the mathematical sense) is a symmetric monoidal functor $\mathcal{Z}^{\mathrm{def}}$ from a category of n-dimensional *decorated defect bordisms* $\mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D})$ to a target category which is often taken to be the category of vector spaces Vect_k [27].

In two and three dimensions, many aspects of defect TFT $\mathcal{Z}^{\mathrm{def}}$ can also be described by a *higher category* $\mathcal{B}_{\mathcal{Z}^{\mathrm{def}}}$. While this description does not incorporate all aspects of Definition 1.2.1 in the sense that $\mathcal{Z}^{\mathrm{def}}$ usually cannot be reconstructed from $\mathcal{B}_{\mathcal{Z}^{\mathrm{def}}}$, it is often easier to work with in practice. Furthermore, finding a higher category description of a defect TFT given in physics language may help towards finding a bordism description.

Theorem 1.2.2. *Let $n \in \{2, 3\}$. For every n-dimensional defect TFT in bordism language $\mathcal{Z}^{\mathrm{def}}$ there is an associated pivotal (weak)*[2] *n-category $\mathcal{B}_{\mathcal{Z}^{\mathrm{def}}}$ which can be constructed algorithmically from $\mathcal{Z}^{\mathrm{def}}$ [30, 19].*

Conjecture 1.2.3. *A relation analogous to Theorem 1.2.2 is expected to hold for all $n \geq 4$ as well [24].*

See [24] for a pedagogical introduction to this topic including a proof of Theorem 1.2.2 for $n = 2$.

1.2.3 A short summary on higher categories

In the following a short summary of definitions from category theory is provided; see e.g. [82, Appendix A] for the precise definitions. Basic familiarity with category theory will be assumed (see e.g. [73]).

- A *1-category* is a category, and a *1-morphism* is a morphism in the usual sense.

[2] $\mathcal{B}_{\mathcal{Z}^{\mathrm{def}}}$ is a 2-category for $n = 2$ [30] and a Gray category for $n = 3$ [19]. For $n > 3$ the details of this category have not been worked out.

- Let $n \geq 2$. An *n-category* $\mathcal{C}$ is a category such that $\mathrm{Hom}_{\mathcal{C}}(A, B)$ forms an $(n-1)$-category for all $A, B \in \mathsf{Obj}(\mathcal{C})$, subject to consistency conditions that hold exactly.
- Let $n \geq 2$. A *weak n-category* $\mathcal{C}$ (also called a *bicategory* for $n = 2$ and a *tricategory* for $n = 3$) is a category such that $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a weak $(n-1)$-category, subject to consistency conditions that hold up to coherent equivalences (or isomorphisms). The precise definition of a bicategory will be discussed in Section 1.5.
- An m-morphism in a (weak) n-category $\mathcal{C}$ is an $(m-1)$-morphism in $\mathrm{Hom}_{\mathcal{C}}(\dots)$. Because all m-morphisms are morphisms in some category, compatible pairs of m-morphisms can be composed to form a new m-morphism (which is used in formulating the coherence conditions of the previous point).
- A *Gray category* is a tricategory that is "almost as strict" as a 3-category: One specific consistency condition is allowed to hold up to 2-equivalence, the others must hold exactly. See [82, Def. 5.3.2] for the full details.

Notation 1.2.4.

(i) The term *vertical composition*, denoted by $-\circ-$, refers to the composition of 2-morphisms in a bicategory or the composition of 3-morphisms in a tricategory.

(ii) The term *horizontal composition*, denoted by $-\otimes-$, refers to the composition of 1-morphisms in bicategories or of 2-morphisms in tricategories. The term is also used for the induced composition of higher morphisms (see Definition 1.5.3).

(iii) The term *box product* refers to the composition of 1-morphisms in a tricategory as well as the induced composition of 2- and 3-morphisms.

Example 1.2.5. The horizontal composition in a 2-category $\mathcal{C}$ is required to be strictly associative, i.e.

$$(A \otimes B) \otimes C = A \otimes (B \otimes C) \quad \text{for all } A, B, C \in \mathsf{Obj}(\mathcal{C}) \ .$$

By contrast, in a bicategory $\mathcal{B}$ the horizontal composition is required to be associative only up to coherent 2-isomorphisms, i.e. for all composable 1-morphisms A, B, C there exist 2-morphisms

$$\begin{aligned}&\Phi_{A,B,C}\colon (A \otimes B) \otimes C \to A \otimes (B \otimes C) \ , \quad \Psi_{A,B,C}\colon A \otimes (B \otimes C) \to (A \otimes B) \otimes C\\ &\text{such that } \ \Phi_{A,B,C} \circ \Psi_{A,B,C} = 1_{A\otimes(B\otimes C)} \ , \quad \Psi_{A,B,C} \circ \Phi_{A,B,C} = 1_{(A\otimes B)\otimes C} \ ,\end{aligned}$$

with Φ and Ψ subject to coherence conditions.

Furthermore, it is possible to *truncate* a tricategory to a bicategory [82, p. 118] (or, more generally, truncate a weak n-category to a weak m-category for all $1 \leq m < n$):

Definition 1.2.6. Let $\mathcal{T}$ be a tricategory. The *truncation* of $\mathcal{T}$, denoted by $h\mathcal{T}$, is the bicategory defined by the following:

(i) The objects and 1-morphisms of $h\mathcal{T}$ are the objects and 1-morphisms of $\mathcal{T}$.

(ii) The 2-morphisms of $h\mathcal{T}$ are the isomorphism classes of 2-morphisms in $\mathcal{T}$.

(iii) The composition of morphisms in $h\mathcal{T}$ is induced by the respective composition in $\mathcal{T}$.

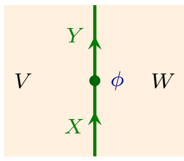
1.2.4 Diagrams in bi- and tricategories

Objects and morphisms in 2-categories and Gray categories can be described by a rich diagrammatic calculus [4, 59], even allowing some proofs to be stated entirely in pictures. The diagrams also form a natural connection between mathematical and physical objects: All constituent parts of a diagram correspond to physical objects in a defect TFT like bulk TFTs, defects, or local operator insertions. The colour scheme of the diagrams is borrowed from [27].

Notation 1.2.7 (Conventions for diagrams in bicategories).
The conventions below can be found e.g. in [22, p. 486].

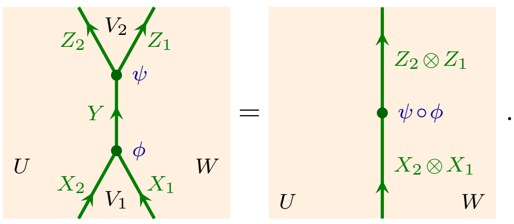
- Objects are drawn as two-dimensional volumes, identified by a label.
- 1-morphisms are drawn as labelled oriented lines which map from the right to the left, whose standard orientation is denoted by an arrow pointing up. There are no horizontal 1-morphisms.
- 2-morphisms are drawn as labelled points on lines which map from the bottom to the top line.
- Adjacent 1-morphisms Y, X may be replaced by their horizontal composition $Y \otimes X$, and adjacent 2-morphisms ϕ, ψ located on the same 1-morphism may be replaced by their vertical composition $\psi \circ \phi$.

For example, the diagram



(1.2.1)

describes a 2-morphism $\phi\colon X \to Y$ where $X, Y\colon W \to V$ are 1-morphisms and W, V are objects. Furthermore, by the composition rules the following diagrams are equal:



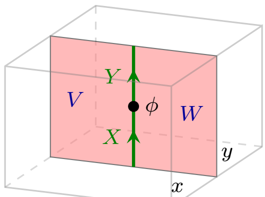
(1.2.2)

Notation 1.2.8 (Conventions for diagrams in Gray categories).
The conventions below follow [19, pp. 21–22]; see also [4]:

- Every diagram is bounded by a cubical frame.
- Objects are drawn as labels on three-dimensional volumes that are bounded either by the diagram's frame or by 1-morphisms.
- 1-morphisms are drawn as labelled oriented surfaces which map from the front to the back. There are no 1-morphisms orthogonal to the viewing plane. A surface is drawn with or without hatching depending on whether it is in reverse or standard orientation.

- 2-morphisms are drawn as labelled oriented lines on 1-morphisms that map from the right to the left and must not be horizontal, analogous to the 1-morphisms of bicategories.
- 3-morphisms are drawn as labelled points on 2-morphisms which map from the bottom to the top, analogous to the 2-morphisms of bicategories.
- Adjacent 1-morphisms W, V may be replaced by their box product $V \boxtimes W$, adjacent 2-morphisms X, Y located on the same 1-morphism may be replaced by their horizontal composition $Y \otimes X$, and adjacent 3-morphisms ϕ, ψ located on the same 2-morphism may be replaced by their vertical composition $\psi \circ \phi$.

For example, the diagram



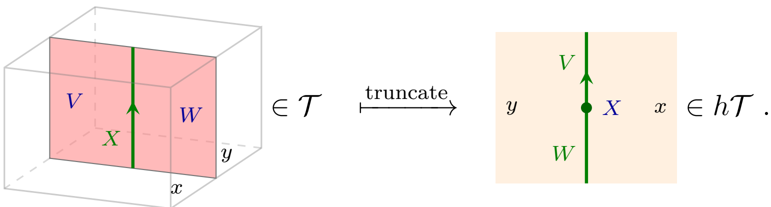
(1.2.3)

describes a 3-morphism $\phi\colon X \to Y$ where $X, Y : W \to V$ are 2-morphisms, $W, V : x \to y$ are 1-morphisms, and x, y are objects.

Notation 1.2.9 (The value of a diagram). Every diagram in a (weak) n-category can be collapsed similar to (1.2.2): First, all adjacent 1-morphisms are composed, then all adjacent 2-morphisms are composed, and so forth. The resulting diagram contains no more than two objects, two m-morphisms for all $1 \leq m < n$, and at most one n-morphism. If the diagram contains no n-morphism, we add identity m-morphisms to the diagram such that there are exactly two objects, two m-morphisms for all $1 \leq k < n$, and one n-morphism. This n-morphism is called the *value* of the diagram.

Horizontal slices and truncations

The truncation (Definition 1.2.6) maps diagrams without 3-morphisms in a tricategory $\mathcal{T}$ to diagrams in the bicategory $h\mathcal{T}$, which amounts to "projecting out the y-axis". For example, we identify



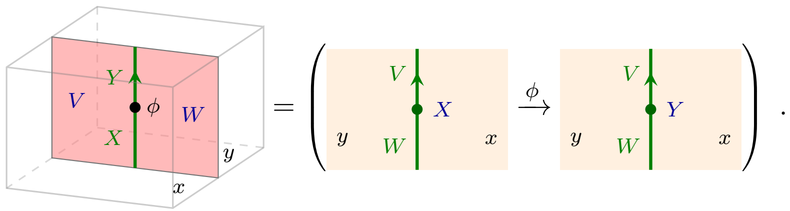
(1.2.4)

Note the slightly unintuitive order — the truncated diagram corresponds to "looking at the 3D diagram from below, with the formerly front side facing right", which is a direct consequence of the mapping orders defined in Notations 1.2.7 and 1.2.8.[3]

Furthermore, it is possible to visualise a 3-morphism in $\mathcal{T}$ as a map between two truncated diagrams, each of which can be thought of as a "horizontal slice" through the three-dimensional

[3]The mapping order in all diagrams is "front to back (if applicable), right to left, bottom to top". The truncation turns a "front to back" 1-morphism into a "right to left" 1-morphism, and a "right to left" 2-morphism into a "bottom to top" 2-morphism.

diagram that is viewed from below. For example, the following diagrams describe the same 3-morphism:



(1.2.5)

1.2.5 Higher categories as models for defect TFTs

The diagrams of the previous section already suggest a relation between mathematical and physical objects, which we now state precisely.

Remark 1.2.10. Let $\mathcal{C}$ be a 2-category. The data of $\mathcal{C}$ correspond to the following physical objects:

(i) The objects $W \in \mathsf{Obj}(\mathcal{C})$ correspond to two-dimensional TFTs.

(ii) For objects $W, V \in \mathsf{Obj}(\mathcal{C})$, the 1-morphisms $X \in \mathcal{C}(W, V)$ correspond to codimension 1 defects (interfaces / line defects) separating the TFTs W and V.

(iii) For 1-morphisms $X, Y : W \to V$, the 2-morphisms $\phi\colon X \to Y$ correspond to codimension 2 defects (point defects) separating the line defects X and Y.

(iv) The diagrams in $\mathcal{C}$ directly translate to physical settings in flat two-dimensional space (or in charts of two-dimensional manifolds).

Remark 1.2.11. Let $\mathcal{C}$ be a Gray category. The data of $\mathcal{C}$ correspond to the following physical objects:

(i) The objects $x \in \mathsf{Obj}(\mathcal{C})$ correspond to three-dimensional TFTs.

(ii) For objects $x, y \in \mathsf{Obj}(\mathcal{C})$, the 1-morphisms $W \in \mathcal{C}(x, y)$ correspond to codimension 1 defects (interfaces / surface defects) between the TFTs x and y.

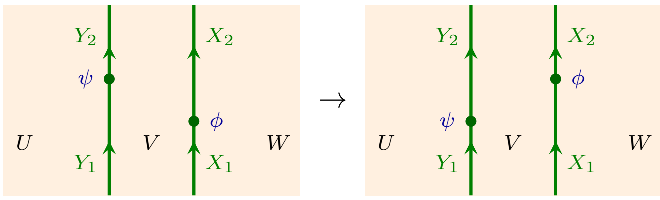
(iii) For 1-morphisms $W, V : x \to y$, the 2-morphisms $X : W \to V$ correspond to codimension 2 defects (line defects) separating the surface defects W and V.

(iv) For 2-morphisms $X, Y : W \to V$, the 3-morphisms $\phi\colon X \to Y$ correspond to codimension 3 defects (point defects) separating the line defects X and Y.

(v) The diagrams in $\mathcal{C}$ directly translate to physical settings in flat three-dimensional space (or in charts of three-dimensional manifolds).

Remark 1.2.12. The dual association is also possible, i.e. identifying objects with point defects, 1-morphisms with 1-dimensional defects, and so forth. This approach leads to a *different* construction called *extended TQFT*, which historically is the older of the two approaches and was introduced in [3, 44]. The approach of Remark 1.2.10 was first discussed rigorously in [30]; this source also contains a comparison of both approaches in Section 1. A discussion of the affine Rozansky–Witten model in the extended TQFT approach can be found in [9].

Let us assume for now that we have a defect TFT with both a physics description and a description in terms of higher categories such that a one-to-one correspondence between them is apparent. The description on the physics side is by assumption invariant under general diffeomorphisms, which we would expect from the mathematical description as well. Several aspects of diffeomorphism invariance are manifest in the language of higher categories: For example, there is no notion of "distance" between morphisms, corresponding to the ability to expand and shrink volumes and 1-morphisms using diffeomorphisms. Fusing defects (as explained in Section 1.2.1) is also naturally expressed in terms of horizontal and vertical composition, and it is clear from Notation 1.2.9 that fusing defects does not change the values of diagrams. Furthermore, the invariance of diagrams under deformations like



(1.2.6)

is incorporated into the axioms of a 2-category (discussed in more detail in Section 1.5). However, the properties of a 2-category do not imply the invariance of its diagrams under arbitrary deformations. Furthermore, while every 2-categorical diagram can be translated to a physical setting, more work is needed so arbitrary physical settings can be translated to 2-categorical diagrams. The structure necessary to do so will be introduced below in Section 1.2.7.

1.2.6 Weak and strict higher categories

Most examples of higher categories describing defect TFTs are *weak* n-categories. However, there are several reasons to prefer descriptions in terms of stricter categories:

- The graphical calculus is formally defined for strict n-categories only.
- The higher categories constructed from bordism defect TFTs (see Theorem 1.2.2) are quite[4] strict.
- Certain structures like *orbifold data* which are central in Section 3.4 are only defined on sufficiently strict categories.

These issues can be solved by the following *coherence* (or *strictification*) theorems:

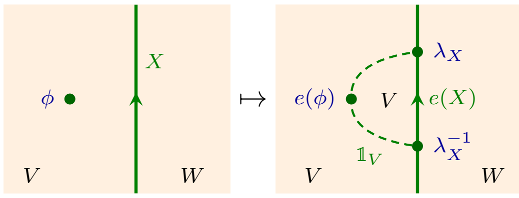
Theorem 1.2.13 (Strictification of bicategories)**.** *Every bicategory $\mathcal{C}$ is biequivalent to a 2-category $s\mathcal{C}$ which has the same objects as $\mathcal{C}$ [86].*

Theorem 1.2.14 (Strictification of tricategories)**.** *Every tricategory $\mathcal{T}$ is triequivalent to a Gray category $g\mathcal{T}$ which has the same objects as $\mathcal{T}$ [51].*

While the existence of the strictification $s\mathcal{C}$ of a weak n-category $\mathcal{C}$ is conceptually important, it is usually not feasible to work with the explicit form of $s\mathcal{C}$ due to its complexity and technicality. Instead, it is often possible to reduce statements in $s\mathcal{C}$ to statements in $\mathcal{C}$ which

[4] see Theorem 1.2.2

we will do in Section 3.4. Another important consequence of the coherence theorems is their ability to extend the diagrammatic formalism to bi- and tricategories. One subtlety of this procedure is the introduction of structure morphisms. For example, consider a bicategory $\mathcal{B}$ and its strictification $s\mathcal{B}$. Then the biequivalence $e\colon s\mathcal{B} \to \mathcal{B}$ maps

$e\colon$ 

,

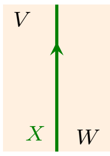
as the identity 1-morphism and the unitor 2-morphisms are trivial in $s\mathcal{B}$ but not in $\mathcal{B}$.

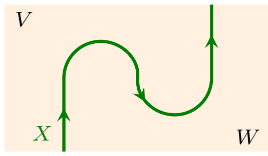
1.2.7 Adjunctions in bicategories

Intuitive motivation

As discussed above, a 2-category (or a bicategory) $\mathcal{B}$ needs additional structure in order to comprehensively describe a physical defect TFT, which can be seen in the following examples:

(i) Consider the following diagrams, which we could either regard as physical settings or as diagrams in the 2-category:

$D_1 :=$ 

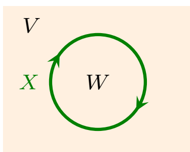
, $D_2 :=$ 

. (1.2.7)

From a physics point of view D_1 and D_2 are clearly diffeomorphic and therefore describe the same physical TFT setting, so a consistent 2-categorical description should also identify D_1 and D_2 as diagrams. However, a priori there exists no "reverse" version of X (1-morphisms always map from the global right to the global left, implying that the central line of D_2 is a 1-morphism mapping $V \to W$), so D_2 is not even a well-defined diagram in $\mathcal{B}$.

(ii) Straight horizontal line defects (1-morphisms) are not allowed in diagrams of 2-categories. This problem can be remedied easily by applying a small diffeomorphism that deforms each straight horizontal line into a slightly curved one. However, implementing this solution suffers from the same problem as the previous example.

(iii) *Loops* or *bubbles* of the form

$D_3 :=$ 

(1.2.8)

are well-defined settings in physics, but also require turnarounds of 1-morphisms to be well-defined.

The first two problems can be solved by defining *adjoints* in the bicategory $\mathcal{B}$, while solving the third problem additionally requires *pivotality.* In order for diagrams shaped like D_2 to be well-defined in a bicategory $\mathcal{B}$, for every 1-morphism $X: W \to V$ we must define

(i) a 1-morphism $X^\dagger: V \to W$ called the *right adjoint*, which can be interpreted as a downwards oriented version of X,

(ii) a pair of 2-morphisms $\widetilde{\mathrm{ev}}_X: X \otimes X^\dagger \to \mathbb{1}_V$ and $\widetilde{\mathrm{coev}}_X: \mathbb{1}_W \to X^\dagger \otimes X$ forming the turn-around points.

One of the consistency conditions of $\widetilde{\mathrm{ev}}_X$ and $\widetilde{\mathrm{coev}}_X$ is that the diagrams D_1 and D_2 must be equal in $\mathcal{B}$, establishing the desired diffeomorphism invariance. Similar considerations can be made for bending X in the opposite direction, leading to the notion of a *left adjoint* ${}^\dagger X: V \to W$ which does not necessarily agree with $X^\dagger$.

Formal definition

This section is based on [22, Section 2.1]. Adjunctions are formally defined as follows:

Definition 1.2.15 (Adjunctions in bicategories)**.** Let $\mathcal{B}$ be a bicategory with strictly associative[5] horizontal composition $-\otimes-$. Let $W, V \in \mathcal{B}$ be objects and let $\mathbb{1}_W$ denote the identity 1-morphism of W. An *adjunction* between 1-morphisms $X: W \to V$ and $Y: V \to W$ is a tuple $(X, Y, \mathrm{ev}, \mathrm{coev})$ with 2-morphisms

$$\mathrm{ev}: Y \otimes X \to \mathbb{1}_W \ , \qquad \mathrm{coev}: \mathbb{1}_V \to X \otimes Y \ , \tag{1.2.9}$$

such that the following maps evaluate to identity 2-morphisms:

$$\begin{aligned} &X \xrightarrow{\lambda_X^{-1}} \mathbb{1}_V \otimes X \xrightarrow{\mathrm{coev}\otimes 1_X} X \otimes Y \otimes X \xrightarrow{1_X \otimes \mathrm{ev}} X \otimes \mathbb{1}_W \xrightarrow{\rho_X} X \ , \\ &Y \xrightarrow{\rho_Y^{-1}} Y \otimes \mathbb{1}_V \xrightarrow{1_Y \otimes \mathrm{coev}} Y \otimes X \otimes Y \xrightarrow{\mathrm{ev}\otimes 1_Y} \mathbb{1}_W \otimes Y \xrightarrow{\lambda_Y} Y \ . \end{aligned} \tag{1.2.10}$$

We say that Y is *left adjoint* to X and X is *right adjoint* to Y. The 2-morphisms ev and coev are called *evaluation* and *coevaluation*. The identities (1.2.10) are called the *left Zorro moves* of X or *right Zorro moves* of Y.

Remark 1.2.16.

- The literature is not consistent in what is considered to be the “left” and “right” adjoint. The present work follows the convention of [22, Def. 2.2] in contrast to e.g. [82, p. 111].
- The terms “duals” and “adjoints” are used (mostly) interchangeably in the literature. We use “adjoints” most of the time but keep the term “duals” in fixed expressions like “pivotal tricategory with duals” (Definition 1.2.21).

Definition 1.2.17. Let $\mathcal{B}$ be a bicategory.

(i) $\mathcal{B}$ is a *bicategory with left adjoints* (or $\mathcal{B}$ *has left adjoints*) if every 1-morphism X in $\mathcal{B}$ is assigned a 1-morphism ${}^\dagger X$ and 2-morphisms ev_X, coev_X such that $(X, {}^\dagger X, \mathrm{ev}_X, \mathrm{coev}_X)$ is an adjunction.

[5] This definition is slightly more complicated if $-\otimes-$ is not strictly associative.

(ii) $\mathcal{B}$ is a *bicategory with right adjoints* (or $\mathcal{B}$ *has right adjoints*) if every 1-morphism X in $\mathcal{B}$ is assigned a 1-morphism $X^\dagger$ and 2-morphisms $\widetilde{\mathrm{ev}}_X$, $\widetilde{\mathrm{coev}}_X$ such that $(X^\dagger, X, \widetilde{\mathrm{ev}}_X, \widetilde{\mathrm{coev}}_X)$ is an adjunction.

(iii) $\mathcal{B}$ is a *bicategory with adjoints* (or $\mathcal{B}$ *has adjoints*) if $\mathcal{B}$ is a bicategory with left and right adjoints.

Remark 1.2.18. If the left (right) adjoint of a 1-morphism X exists, it is unique up to unique isomorphism, see e.g. [82, Lemma 5.1.2]. It is nevertheless convenient to define "with adjoints" to be a statement about the *structure* and not about *existence*, i.e. there is a standard choice of left (right) adjoint for every 1-morphism in $\mathcal{B}$.

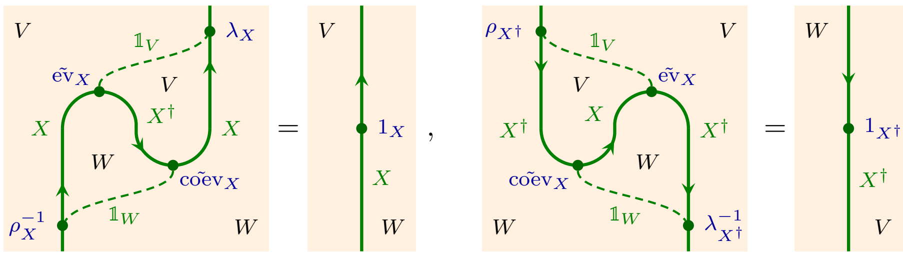
Notation 1.2.19.

- For a 1-morphism $X\colon W \to V$ the notation ${}^\dagger X\colon V \to W$ refers to the standard left adjoint of X and $X^\dagger\colon V \to W$ refers to one canonical right adjoint of X. The adjunction 2-morphism are written as follows:

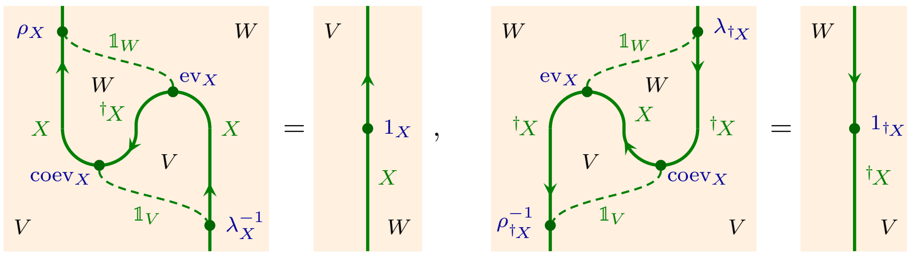
$$\mathrm{ev}_X\colon {}^\dagger X \otimes X \to \mathbb{1}_W \ , \qquad \mathrm{coev}_X\colon \mathbb{1}_V \to X \otimes {}^\dagger X \ , \tag{1.2.11}$$

$$\widetilde{\mathrm{ev}}_X\colon X \otimes X^\dagger \to \mathbb{1}_V \ , \qquad \widetilde{\mathrm{coev}}_X\colon \mathbb{1}_W \to X^\dagger \otimes X \ . \tag{1.2.12}$$

- The 1-morphisms ${}^\dagger X$ and $X^\dagger$ are drawn in diagrams with arrows pointing down. The Zorro moves (1.2.10) of $X\colon W \to V$ can be visualised as follows:



(1.2.13)



(1.2.14)

These diagrams formally encode the intuition of "$D_1 = D_2$" in Eq. (1.2.7).

- In diagrams of 2-categories the identity 1-morphisms ending on ev_X, coev_X etc. can be omitted because 2-categories are strictly unital. While it is not possible to formally remove the identity 1-morphisms in a bicategory, omitting them does not introduce any ambiguities due to coherence. In order to translate a diagram in a bicategory with omitted 1-morphisms to a formula, one must reintroduce an identity 1-morphism for every instance of ev_X, coev_X, $\widetilde{\mathrm{ev}}_X$, $\widetilde{\mathrm{coev}}_X$. The other end of every reintroduced identity 1-morphism can be placed onto any other 1-morphism (see also Remark 1.5.5).

EXAMPLE 1.2.20. The identity 1-morphism $\mathbb{1}_W : W \to W$ is left and right self-adjoint for all choices of adjunction 2-morphisms

$$\mathrm{ev}_{\mathbb{1}_W}, \tilde{\mathrm{ev}}_{\mathbb{1}_W} \in \{\rho_{\mathbb{1}_W}, \lambda_{\mathbb{1}_W}\} , \quad \mathrm{coev}_{\mathbb{1}_W}, \tilde{\mathrm{coev}}_{\mathbb{1}_W} \in \{\lambda^{-1}_{\mathbb{1}_W}, \rho^{-1}_{\mathbb{1}_W}\} . \tag{1.2.15}$$

This can either be shown explicitly from the bicategory axioms, or using coherence: The Zorro maps of $\mathbb{1}_W$ are networks $\mathbb{1}_W \to \mathbb{1}_W$ consisting only of the nodes λ, ρ, λ^{-1}, ρ^{-1}, and all edges are given by identity lines, so the network reduces to $1_{\mathbb{1}_W}$ according to coherence.

Application: Loops and bubbles

Now we turn our attention to the bubble diagram (1.2.8). Being able to efficiently evaluate diagrams of this shape is essential for the orbifold construction in Section 3.4. A significant part of Chapters 2 and 3 is therefore devoted to finding a closed formula for bubbles in the Rozansky–Witten model.

Evidently, adjoints are necessary to describe bubble diagrams like D_3 since the latter contains the adjunction 2-morphisms $\tilde{\mathrm{ev}}_X$ and coev_X. However, the codomain of coev_X is given by $X \otimes {}^\dagger X$ while the domain of $\tilde{\mathrm{ev}}_X$ is given by $X \otimes X^\dagger$, so "$\tilde{\mathrm{ev}}_X \circ \mathrm{coev}_X$" is not well-defined. The missing piece is an isomorphism $\delta_X : {}^\dagger X \to X^\dagger$ called a *pivotal structure*. For consistency reasons δ_X must be compatible with $- \otimes -$ which leads to tight constraints; see the discussion in Section 2.4. The higher categories constructed from bordism defect TFTs (see Theorem 1.2.2) also feature pivotal structures.

1.2.8 Adjunctions in tricategories

Adjunctions and pivotality in tricategories resemble those in bicategories in most regards, the most notable difference being that the concept of adjunctions applies to both 1- and 2-morphisms in tricategories.

Definition 1.2.21. A tricategory $\mathcal{T}$ is a *pivotal tricategory with duals* [83, Def. 4.5] (sometimes called a *tricategory with weak duals* [82, Def. 5.2.2]) if

(i) for every pair of objects $b, c \in \mathcal{T}$ the bicategory $\mathcal{T}(b, c)$ is a pivotal bicategory, with the pivotal structure given by a monoidal isomorphism $\delta : \mathrm{Id} \Rightarrow {}^{\dagger\dagger}(-)$,

(ii) for all 1-morphisms $W : c \to d$ the 2-functors

$$W \boxtimes - : \mathcal{T}(b, c) \to \mathcal{T}(b, d) \quad \text{and} \quad - \boxtimes W : \mathcal{T}(d, e) \to \mathcal{T}(c, e) \tag{1.2.16}$$

are pivotal 2-functors (see e.g. [82, Def. 5.1.9] for the definition), and

(iii) the truncation $h\mathcal{T}$ (which is a bicategory, see Definition 1.2.6) is a bicategory with left adjoints (i.e. for every 1-morphism W there exists a designated left adjoint $W^\#$).

REMARK 1.2.22. Left adjoints are called "right duals" in [82], see Remark 1.2.16. This thesis is consistent with the language of [22] and the formulas (but not the language) of [82].

NOTATION 1.2.23. Let $\mathcal{T}$ be a pivotal tricategory with duals. We write $W^\#$ for the left adjoint of a 1-morphism W with adjunction 2-morphisms $(\mathrm{ev}_W, \mathrm{coev}_W)$, and ${}^\dagger X$ for the left adjoint of a 2-morphism X with adjunction 3-morphisms $(\mathrm{ev}_X, \mathrm{coev}_X)$. The pivotal structure is given by a 3-isomorphism $\delta_X : X \to {}^{\dagger\dagger}X$ for all 2-morphisms X.

The reader might be surprised to find that Definition 1.2.21 does not require $h\mathcal{T}$ to have right adjoints or be pivotal. It turns out that this is not needed, as the following theorem shows (first stated in [72, Remark 3.4.22] and proven in [35, Lemma 1.4.4], see also [4]):

Theorem 1.2.24. *Let $\mathcal{T}$ be a pivotal tricategory with duals and let $W \in \mathcal{T}(b,c)$. By definition W has a left adjoint $W^\#$ and adjunction 2-morphisms*

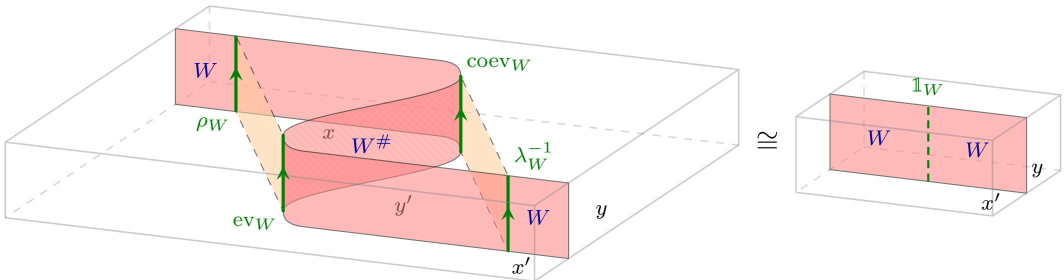
$$\mathrm{ev}_W : W^\# \boxtimes W \to \mathbb{1}_b \ , \quad \mathrm{coev}_W : \mathbb{1}_c \to W \boxtimes W^\# \ .$$

Then $W^\#$ is also a right adjoint of W with adjunction 2-morphisms

$$\tilde{\mathrm{ev}}_W := {}^\dagger(\mathrm{coev}_W) \ , \quad \tilde{\mathrm{coev}}_W := {}^\dagger(\mathrm{ev}_W) \ . \tag{1.2.17}$$

Additionally, there is a (usually non-trivial) pivotal structure on $h\mathcal{T}$ that is compatible with Theorem 1.2.24, which we elaborate on in Section 3.2.3.

Notation 1.2.25. Definition 1.2.21 (iii) can be expressed by the three-dimensional versions of Eq. (1.2.14) called the *Zorro movies*,[6] the first being



(1.2.18)

and the second corresponding to the second diagram of Eq. (1.2.14). The 3-isomorphisms of the Zorro movies, called the *triangulators* [82, p. 124], will be written as follows:

$$\begin{aligned} \Upsilon_W &: \rho_W \otimes (\mathbb{1}_W \boxtimes \mathrm{ev}_W) \otimes (\mathrm{coev}_W \boxtimes \mathbb{1}_W) \otimes \lambda_W^{-1} \to \mathbb{1}_W \ , \\ \tilde{\Upsilon}_W &: \mathbb{1}_{W^\#} \to \lambda_{W^\#} \otimes (\mathrm{ev}_W \boxtimes \mathbb{1}_{W^\#}) \otimes (\mathbb{1}_{W^\#} \boxtimes \mathrm{coev}_W) \otimes \rho_{W^\#}^{-1} \ . \end{aligned} \tag{1.2.19}$$

The corresponding 3-isomorphism mediating the right Zorro movie (see Theorem 1.2.24) is given by $({}^\dagger\Upsilon_W)^{-1}$ which is called τ_W in [19, 27].

1.2.9 Strictifications of pivotal bi- and tricategories

In analogy to the strictification theorems of Section 1.2.6 there are also strictification theorems for pivotal bi- and tricategories.

Theorem 1.2.26. *Every pivotal bicategory $\mathcal{B}$ is biequivalent to a pivotal 2-category $s\mathcal{B}$, and the biequivalence $e\colon s\mathcal{B} \to \mathcal{B}$ is a pivotal 2-functor [77, Thm. 2.2].*

Definition 1.2.27. Two pivotal tricategories with duals $\mathcal{S}$, $\mathcal{T}$ are *equivalent*[7] if there is a triequivalence $F\colon \mathcal{S} \to \mathcal{T}$ such that the 2-functors $F_{a,b}\colon \mathcal{S}(a,b) \to \mathcal{T}(F(a), F(b))$ are pivotal.

[6]The 3-isomorphism Υ_W of Notation 1.2.25 maps from the left to the right. If one draws Υ_W in a diagram and interprets the y-axis as "time", the 3-isomorphism Υ_W corresponds to evolving a Z-shaped line into a straight line, hence the name "Zorro movie".

[7]This notion of "equivalent" is used in Theorem 1.2.28, hence this work assumes that it is the most natural way of identifying pivotal tricategories with duals. It is possible that a more rigid definition exists.

A pivotal tricategory with duals can be strictified to a *Gray category with strict duals*, whose definition is rather technical and not needed for this thesis (see [82, Def. 5.3.7]).

Theorem 1.2.28. *Every pivotal tricategory with duals $\mathcal{T}$ is equivalent to a Gray category with strict duals $s\mathcal{T}$ in the sense of Definition 1.2.27, and $\mathcal{T}$ and $s\mathcal{T}$ have the same objects [82, Thm. 7.2.1]. Furthermore, the triequivalence $e\colon s\mathcal{T} \to \mathcal{T}$ is of the shape*

$$s\mathcal{T} \xrightarrow{e_1} g\mathcal{T} \xrightarrow{e_2} \mathcal{T} \ , \tag{1.2.20}$$

where $g\mathcal{T}$ and e_2 form the strictification of $\mathcal{T}$ as an ordinary Gray category (without duals).

Gray categories with strict duals are arguably the most "standard" way of representing a three-dimensional defect TFT in higher category language. For example, the generalised orbifold procedure of Section 3.4 is defined on Gray categories with strict duals, and the category constructed from a 3D defect TFT in bordism language by Theorem 1.2.2 is also a Gray category with strict duals.

1.3 Matrix factorisations

1.3.1 Definitions

Definitions and results from [21] and [63, pp. 10–18] are summarised here. Throughout this thesis k is a commutative ring. The relevant examples for later chapters are $k = \mathbb{C}$ and the polynomial ring in d variables $k = \mathbb{C}[w_1, \ldots, w_d]$.

Notation 1.3.1. Letters in boldface denote finite lists of variables:

$$\boldsymbol{x} = \{x_1, \ldots, x_n\} . \tag{1.3.1}$$

Curly braces are used for both sets and lists to better distinguish them from ideals, which are written using parentheses. We denote by $\ell(\boldsymbol{x})$ the length of $\boldsymbol{x}$, i.e.

$$\ell(\{x_1, \ldots, x_n\}) := n . \tag{1.3.2}$$

For lists $\boldsymbol{a}, \boldsymbol{b}$ with $\ell(\boldsymbol{a}) = \ell(\boldsymbol{b})$ we define

$$\boldsymbol{a} \cdot \boldsymbol{b} := \textstyle\sum_{i=1}^{\ell(\boldsymbol{a})} a_i b_i . \tag{1.3.3}$$

Definition 1.3.2. Let k be a commutative ring and let $\boldsymbol{x} = \{x_1, \ldots, x_n\}$ be a list of variables. Set $R := k[x_1, \ldots, x_n] = k[\boldsymbol{x}]$, and let $W \in R$ be a polynomial.

(i) A *linear factorisation* of W over R is a $\mathbb{Z}_2$-graded R-module X together with an odd module endomorphism $d_X \in \mathrm{End}_R(X)$ called a *twisted differential*, such that $d_X^2 = W \cdot \mathbb{1}_X$.

(ii) If X is a free R-module, (X, d_X) is called a *matrix factorisation* [21].

(iii) Furthermore, if the rank of X is finite, (X, d_X) is called a *finite-rank matrix factorisation.*

Notation 1.3.3. Let (X, d_X) be a linear factorisation. By the definition of a graded module, X is split into an even and odd part:

$$X = X_0 \oplus X_1 , \quad |X_0| = 0 , \ |X_1| = 1 . \tag{1.3.4}$$

In this representation, the differential d_X has the form

$$d_X = \begin{pmatrix} 0 & p_1 \\ p_0 & 0 \end{pmatrix} , \qquad p_1 \circ p_0 = W \cdot 1_{X_0} , \qquad p_0 \circ p_1 = W \cdot 1_{X_1} , \tag{1.3.5}$$

which will also sometimes be written in the notation

$$(X, d_X) = X_1 \underset{p_0}{\overset{p_1}{\rightleftarrows}} X_0 \tag{1.3.6}$$

found e.g. in [11]. If $X_0 = X_1 = R$, we also use the notation (used e.g. in [9])

$$(X, d_X) = [p_1, p_0] , \quad p_i \in R . \tag{1.3.7}$$

Notation 1.3.4. In the following text, linear factorisations (X, d_X) are often denoted by just X, as it is usually clear from context whether the module X or the pair (X, d_X) is meant.

Definition 1.3.5. Let (X, d_X) and (Y, d_Y) be linear factorisations of W over R, and let $\phi \in \operatorname{Hom}_R(X, Y)$ be a homogeneous module homomorphism, i.e.

$$|\phi(x)| = |\phi| + |x| \quad \text{for all homogeneous } x \in X \ . \tag{1.3.8}$$

The *differential* on ϕ is defined as follows:

$$\mathrm{d}\phi := d_Y \circ \phi - (-1)^{|\phi|} \phi \circ d_X \in \operatorname{Hom}_R(X, Y) \ . \tag{1.3.9}$$

It is evident that $|\mathrm{d}\phi| = |\phi| + 1$. For endomorphisms $\phi \in \operatorname{End}_R(X)$, we can also write $\mathrm{d}\phi$ using the *graded commutator*

$$\mathrm{d}\phi = \{d_X, \phi\} := d_X \circ \phi - (-1)^{|\phi|} \phi \circ d_X \ . \tag{1.3.10}$$

Lemma 1.3.6. *The differential on the morphisms of linear factorisations squares to zero.*

Proof. Let $x \in X$.

$$\begin{aligned} \mathrm{d}^2\phi(x) &= \mathrm{d}\big(d_Y \circ \phi - (-1)^{|\phi|} \phi \circ d_X\big)(x) \\ &= \Big(d_Y^2 \circ \phi - (-1)^{|\phi|} d_Y \circ \phi \circ d_X - (-1)^{|\phi| + |d_X|} d_Y \circ \phi \circ d_X + (-1)^{2|\phi| + |d_X|} \phi \circ d_X^2\Big)(x) \\ &= W \cdot \phi(x) - \phi(W \cdot x) = 0 \end{aligned}$$

using R-linearity of ϕ. □

Definition 1.3.7. The *morphisms between linear factorisations* (X, d_X) and (Y, d_Y) are the even [22, p. 490] elements of the cohomology

$$\mathrm{H}^0_{\mathrm{d}}\left(\operatorname{Hom}_R(X, Y)\right) = \frac{\{\phi \in \operatorname{Hom}_R(X, Y) \colon |\phi| = 0,\ \mathrm{d}\phi = 0\}}{\{\mathrm{d}\chi \colon \chi \in \operatorname{Hom}_R(X, Y),\ |\chi| = 1\}} \tag{1.3.11}$$

(the even d-closed module homomorphisms modulo the d-exact module homomorphisms).

Theorem 1.3.8. *The matrix factorisations (X, d_X) of a polynomial $W \in k[\boldsymbol{x}]$ form a category*

$$\mathsf{MF}_k(\boldsymbol{x};\, W) \tag{1.3.12}$$

whose objects are matrix factorisations of W over $k[\boldsymbol{x}]$, and whose morphisms are as defined in Definition 1.3.7.

The ring k is sometimes omitted when the statement makes no assumptions about k or it is clear which ring is meant.

Remark 1.3.9. The above structure is natural in the following way: Consider linear factorisations (X, d_X) and (Y, d_Y). Those can be written as a 2-periodic twisted complexes:

$$\ldots \xrightarrow{p_0} X_1 \xrightarrow{p_1} X_0 \xrightarrow{p_0} X_1 \xrightarrow{p_1} \ldots$$

(the word "twisted" meaning that $p_{i+1} \circ p_i = W \cdot \mathbb{1}_{X_i} \neq 0$). Now we study the homotopy category of chain complexes:

$$\begin{tikzcd} \ldots \arrow[r, "p_0"] & X_1 \arrow[r, "p_1"] \arrow[d, "\phi_1"] \arrow[dl, "\chi_1"'] & X_0 \arrow[r, "p_0"] \arrow[d, "\phi_0"] \arrow[dl, "\chi_0"'] & X_1 \arrow[r, "p_1"] \arrow[d, "\phi_1"] \arrow[dl, "\chi_1"'] & \ldots \arrow[dl, "\chi_0"'] \\ \ldots \arrow[r, "q_0"'] & Y_1 \arrow[r, "q_1"'] & Y_0 \arrow[r, "q_0"'] & Y_1 \arrow[r, "q_1"'] & \ldots \end{tikzcd} \tag{1.3.13}$$

The chain map $\phi = (\phi_0, \phi_1)$ corresponds to a morphism between linear factorisations, as the condition $\mathrm{d}\phi = 0$ is precisely the commutativity of Eq. (1.3.13) without the χ_i. The chain map ϕ is, by definition, exact if there is a chain homotopy $\phi \sim 0$, i.e.

$$\phi_i = q_{i+1} \circ \chi_i + \chi_{i+1} \circ p_i \ ,$$

which corresponds to $\phi = \mathrm{d}\chi$ for morphisms of matrix factorisations.

Lemma 1.3.10. *Let (X, d_X) be a linear factorisation. Then the linear factorisations (X, d_X) and $(X, -d_X)$ are isomorphic.*

Proof. Split $X = X_0 \oplus X_1$ into its even and odd part. Define

$$\phi\colon X_0 \oplus X_1 \to X_0 \oplus X_1 \ , \quad (x_0, x_1) \mapsto (x_0, -x_1) \tag{1.3.14}$$

which will also be written as

$$\phi(x) = (-1)^{|x|} x \ . \tag{1.3.15}$$

Clearly $|\phi| = 0$ and $\phi^2 = \mathbb{1}_X$, so ϕ is an even self-inverse automorphism of the module X. The image of d_X under ϕ is given by

$$(\phi \circ d_X \circ \phi^{-1})(x) = (-1)^{|x|+|d_X(x)|} d_X(x) = -d_X(x) \ ,$$

so ϕ is an isomorphism of linear factorisations between (X, d_X) and $(X, -d_X)$. □

1.3.2 The tensor product

There are various operations on linear factorisations. Their definitions and properties will be discussed, starting with the tensor product of matrix factorisations. We start with some prerequisites for the modules:

Lemma 1.3.11. *Let X be a non-zero, free, finite-rank module over $k[\boldsymbol{x}, \boldsymbol{y}]$ and let $\ell(\boldsymbol{y}) > 0$. Then X is a free, infinite-rank module over $k[\boldsymbol{x}]$.*

Proof. $k[\boldsymbol{x}]$ is a subring of $k[\boldsymbol{x}, \boldsymbol{y}]$, so X is a module over $k[\boldsymbol{x}]$. X is also clearly free over $k[\boldsymbol{x}]$. Let $\{e_1, \ldots, e_n\}$ be a basis of X over $k[\boldsymbol{x}, \boldsymbol{y}]$. Then

$$\left\{ \boldsymbol{y}^I \cdot e_i \colon 1 \le i \le n, \ I \in \mathbb{N}_0^{\ell(\boldsymbol{y})} \right\} \tag{1.3.16}$$

is a basis of X as a $k[\boldsymbol{x}]$-module. □

Example 1.3.12. Consider the special case $\boldsymbol{y} = \{y\}$. X is free and finite-rank, so $X \cong k[\boldsymbol{x}, y]^{\oplus n}$ for some n. Then we can write X as the following $k[\boldsymbol{x}]$-module:

$$X \cong k[\boldsymbol{x}]^{\oplus n} \oplus y \cdot k[\boldsymbol{x}]^{\oplus n} \oplus y^2 \cdot k[\boldsymbol{x}]^{\oplus n} \oplus \ldots \quad . \tag{1.3.17}$$

Different expansions are possible: For every $p \in k[\boldsymbol{x}]$ we find

$$X \cong k[\boldsymbol{x}]^{\oplus n} \oplus (y-p) \cdot k[\boldsymbol{x}]^{\oplus n} \oplus (y-p)^2 \cdot k[\boldsymbol{x}]^{\oplus n} \oplus \ldots \quad . \tag{1.3.18}$$

Notation 1.3.13. Let X and Y be modules over R. Then we define

$$X \otimes_R Y \tag{1.3.19}$$

to be the tensor product of X and Y as R-modules.

Example 1.3.14. Consider rings

$$k := \mathbb{C}[\boldsymbol{w}] \ , \quad R := k[\boldsymbol{x}, \boldsymbol{y}] \cong \mathbb{C}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}] \ , \quad S := k[\boldsymbol{y}, \boldsymbol{z}] \cong \mathbb{C}[\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w}] \ ,$$

and free, finite-rank modules X over R and Y over S with $\ell(\boldsymbol{y}) > 0$. Then

(i) $R \cong k[\boldsymbol{x}] \otimes_k k[\boldsymbol{y}]$ where each term is regarded as an infinite-rank k-module,

(ii) $X \otimes_{k[\boldsymbol{y}]} Y$ is a free, finite-rank module over $R \otimes_{k[\boldsymbol{y}]} S \cong k[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$ and a free, infinite-rank module over $k[\boldsymbol{x}, \boldsymbol{z}]$,

(iii) $X \otimes_k Y$ is a free, finite-rank module over $R \otimes_k S \cong k[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}', \boldsymbol{z}]$. Note that multiplication by $\boldsymbol{y}$ only acts on X and multiplication by $\boldsymbol{y}'$ only acts on Y.

Definition 1.3.15 (Tensor product of matrix factorisations)**.** Let

$$(X, d_X) \in \mathsf{MF}_k(\boldsymbol{x}, \boldsymbol{y};\, W_1) \ , \quad (Y, d_Y) \in \mathsf{MF}_k(\boldsymbol{y}, \boldsymbol{z};\, W_2) \ .$$

Then the *tensor product of matrix factorisations* $X \otimes_{k[\boldsymbol{y}]} Y$ is defined by

$$X \otimes_{k[\boldsymbol{y}]} Y := (X \otimes_{k[\boldsymbol{y}]} Y,\ d_X \otimes 1_Y + 1_X \otimes d_Y) \in \mathsf{MF}_k(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z};\, W_1 + W_2) \tag{1.3.20}$$

with the Koszul sign convention [21]

$$(\phi \otimes \psi)(\alpha \otimes \beta) := (-1)^{|\psi||\alpha|} \phi(\alpha) \otimes \psi(\beta) \tag{1.3.21}$$

for homogeneous ϕ, ψ, α, β.

Remark 1.3.16. Like in Example 1.3.14, one needs to pay attention over which ring R the tensor product $X \otimes_R Y$ is taken, as it specifies which variables are "shared" between the matrix factorisations. For example, let $W \in k[x]$ and $X, Y \in \mathsf{MF}_k(x, W)$. Then

$$X \otimes_{k[x]} Y \in \mathsf{MF}_k(x;\, 2 \cdot W) \ , \quad X \otimes_k Y \in \mathsf{MF}_k(x, x';\, W(x) + W(x')) \ .$$

Example 1.3.17. For matrix factorisations

$$X \in \mathsf{MF}_k(\boldsymbol{x};\, W_1(\boldsymbol{x})) \ , \quad X_{12} \in \mathsf{MF}_k(\boldsymbol{x}, \boldsymbol{y};\, W_2(\boldsymbol{y}) - W_1(\boldsymbol{x}))$$

we find

$$X_{12} \otimes_{k[\boldsymbol{x}]} X \in \mathsf{MF}_k(\boldsymbol{x}, \boldsymbol{y};\, W_2(\boldsymbol{y})) \ . \tag{1.3.22}$$

Because $W_2(\boldsymbol{y})$ does not depend on $\boldsymbol{x}$, it is possible to remove the dependency on $\boldsymbol{x}$ in Eq. (1.3.22). This turns the finite-rank $k[\boldsymbol{x}, \boldsymbol{y}]$-module X into an infinite-rank $k[\boldsymbol{y}]$-module according to Lemma 1.3.11. The tensor product with X_{12} is a functor acting on $\mathsf{MF}_k(\boldsymbol{x};\, W_1)$:

$$\mathsf{MF}_k(\boldsymbol{x};\, W_1) \to \mathsf{MF}_k(\boldsymbol{y};\, W_2) \ , \quad X \mapsto X \otimes_{k[\boldsymbol{x}]} X_{12} \in \mathsf{MF}_k(\boldsymbol{y};\, W_2) \ . \tag{1.3.23}$$

In almost all cases $X \otimes_{k[\boldsymbol{x}]} X_{12}$ turns out to be isomorphic to a finite-rank matrix factorisation (see the discussion in Section 2.2.3).

Notation 1.3.18. Unless explicitly specified otherwise, tensor products of matrix factorisations are taken over all variables that appear on both sides of the tensor product, and the symbol $\cong$ means "isomorphic as linear factorisations of some polynomial $W(\boldsymbol{a})$ over the ring $k[\boldsymbol{a}]$". For example, let $X(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w})$ factorise $V(\boldsymbol{y}, \boldsymbol{w}) - U(\boldsymbol{x}, \boldsymbol{w})$ and let $Y(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w})$ factorise $W(\boldsymbol{z}, \boldsymbol{w}) - V(\boldsymbol{y}, \boldsymbol{w})$. By convention, in the expression

$$Y(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w}) \otimes X(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}) \cong Z(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{w})$$

the symbol $\otimes$ means $\otimes_{k[\boldsymbol{y},\boldsymbol{w}]}$ and $\cong$ means "isomorphic as linear factorisations of $W(\boldsymbol{z}, \boldsymbol{w}) - U(\boldsymbol{x}, \boldsymbol{w})$ over $k[\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{w}]$".

Lemma 1.3.19. *The following identities hold in general for $p_i, q_i, \alpha \in k[\boldsymbol{x}]$:*

$$[p_1, p_0] \otimes [q_1, q_0] \cong [p_1, p_0 + \alpha q_0] \otimes [q_1 - \alpha p_1, q_0] \;, \tag{1.3.24}$$
$$[p_1, p_0] \otimes [q_1, q_0] \cong [p_1 + \alpha q_1, p_0] \otimes [q_1, q_0 - \alpha p_0] \;, \tag{1.3.25}$$
$$[p_1, p_0] \otimes [q_1, q_0] \cong [p_1 + \alpha q_0, p_0] \otimes [q_1 - \alpha p_0, q_0] \;, \tag{1.3.26}$$
$$[p_1, p_0] \otimes [q_1, q_0] \cong [p_1, p_0 + \alpha q_1] \otimes [q_1, q_0 - \alpha p_1] \;, \tag{1.3.27}$$
$$[p_1, p_0] \cong [\alpha^2 p_1, \alpha^{-2} p_0] \quad \textit{(if } \alpha \textit{ is invertible).} \tag{1.3.28}$$

Proof. The isomorphisms can be constructed as follows: For the first four identities, use Eq. (1.3.39) to write the differential on both sides as a 4×4 matrix. Both are related by a change-of-basis matrix that differs from the identity matrix by an off-diagonal $\pm\alpha$. For Eq. (1.3.28), use the change-of-basis matrix $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right)$. Alternatively, all statements can be shown using Theorem 1.3.47 below. $\square$

1.3.3 Duals and grade shift

Definition 1.3.20 (Dual factorisation)**.** Let (X, d_X) be a linear factorisation of W over R. The dual module of X is given by

$$X^\vee := \mathrm{Hom}_R(X, R) \;. \tag{1.3.29}$$

The *dual factorisation* of X is defined by [22, p. 490]

$$(X, d_X)^\vee := (X^\vee, d_{X^\vee}) \;, \quad d_{X^\vee}(\nu) := -(-1)^{|\nu|} \nu \circ d_X \tag{1.3.30}$$

with the usual linear extension to non-homogeneous ν. This is a linear factorisation of $-W$.

Proof. Let $x \in X$.

$$\begin{aligned} d^2_{X^\vee}(\nu)(x) &= d_{X^\vee}(-(-1)^{|\nu|} \nu \circ d_X)(x) = ((-1)^{2|\nu| + |d_X|} \nu \circ d_X^2)(x) \\ &= -\nu(W \cdot x) = -W \cdot \nu(x) = -W \cdot 1_{X^\vee}(\nu)(x) \;. \end{aligned}$$ $\square$

Definition 1.3.21 (Grade shift)**.**

(i) Let (X, d_X) be a linear factorisation of W over R and let $n \in \mathbb{Z}_2$. The linear factorisation $(X[n], d_{X[n]})$ of W, called the *grade shift* of X, is defined by [65, p. 22]

$$X[n] := X, \qquad |\cdot|_{X[n]} := |\cdot|_X + n \mod 2 \;, \qquad d_{X[n]} := (-1)^n d_X \;. \tag{1.3.31}$$

An even element $x \in X$ is thus odd in $X[1]$ and vice versa.

(ii) Let $R[n] := (R[n], d_{R[n]} = 0)$ be the matrix factorisation whose module consists of R in degree n and $\{0\}$ in degree $n+1$. We denote the single basis element by

$$1_n \in R[n] \ , \qquad |1_n| = n \ . \tag{1.3.32}$$

An equivalent way of writing $X[n]$ is

$$X[n] \cong R[n] \otimes_R X \ , \tag{1.3.33}$$

as

$$d_{R[n]\otimes X}(1_n \otimes x) = (1_{R[n]} \otimes d_X)(1_n \otimes x) = (-1)^n 1_n \otimes d_X(x).$$

Lemma 1.3.22. *The identity map* $\mathrm{id}\colon X \to X[n]$ *is a closed invertible map of degree* n.

Proof. This is clear for n even as $X[n] = X$, $d_{X[n]} = d_X$. Let us therefore only consider the case $n = 1$. For all x, $|\mathrm{id}(x)|_{X[1]} = |x|_{X[1]} = |x|_X + 1$, hence the identity map has degree 1. For closedness, we find

$$\mathrm{d}(\mathrm{id}) = (d_{X[1]} \circ \mathrm{id} + \mathrm{id} \circ d_X)(x) = -d_X(x) + d_X(x) = 0 \ . \qquad \square$$

Remark 1.3.23. We will not call odd closed invertible maps "isomorphisms of odd degree" because morphisms of matrix factorisations are always even by Definition 1.3.7. This implies that, in general, $X \ncong X[1]$ (see [23, Remark 7.3] for an example).

1.3.4 Matrix representations

The differentials of finite-rank matrix factorisations can be represented as matrices. We give explicit formulas for $d_{X\otimes Y}$, $d_{X^\vee}$, and $d_{X[1]}$ in terms of these explicit matrices.

Definition 1.3.24. A basis $\{b_1, \dots, b_{l+m}\} = \{e_1, \dots, e_l, f_1, \dots, f_m\}$ of a $\mathbb{Z}_2$-graded module X is *canonically ordered* if all b_i are homogeneous and $|e_i| = 0$, $|f_j| = 1$ for all i, j.

Definition 1.3.25. Let

$$\begin{aligned} \{b_1, \dots, b_{l+m}\} &= \{e_1, \dots, e_l, f_1, \dots, f_m\} \quad \subset X \ , \\ \{c_1, \dots, c_{l'+m'}\} &= \{g_1, \dots, g_{l'}, h_1, \dots, h_{m'}\} \subset Y \end{aligned}$$

be canonically ordered bases of finite-rank matrix factorisations X and Y. Then the following are canonically ordered bases:

(i) The *dual basis* $\{b_i^*\}$ is given by

$$\{b_1^*, \dots, b_{l+m}^*\} = \{e_1^*, \dots, e_l^*, f_1^*, \dots, f_m^*\} \subset X^\vee \ , \qquad b_i^*(b_j) = \delta_{i,j} \ . \tag{1.3.34}$$

(ii) The *grade-shifted basis* is given by

$$\{f_1, \dots, f_l, e_1, \dots, e_k\} = \{1_1 \otimes f_1, \dots, 1_1 \otimes e_k\} \subset X[1] \tag{1.3.35}$$

where the former notation will be used with $|f_i|_{X[1]} = 0$, $|e_i|_{X[1]} = 1$ if it is clear from context that the grade-shifted matrix factorisation is meant.

(iii) The *tensor product basis* is given by

$$\begin{aligned} &\{e_1 \otimes g_1, e_1 \otimes g_2, \ldots, e_l \otimes g_{l'}, f_1 \otimes h_1, f_1 \otimes h_2, \ldots, f_m \otimes h_{m'}, \\ &\ e_1 \otimes h_1, e_1 \otimes h_2, \ldots, e_l \otimes h_{m'}, f_1 \otimes g_1, f_1 \otimes g_2, \ldots, f_m \otimes g_{l'}\} \subset X \otimes Y \,. \end{aligned} \tag{1.3.36}$$

Lemma 1.3.26. *Let X and Y be finite-rank matrix factorisations with*

$$d_X = \begin{pmatrix} 0 & p_1 \\ p_0 & 0 \end{pmatrix} , \quad d_Y = \begin{pmatrix} 0 & q_1 \\ q_0 & 0 \end{pmatrix} .$$

With respect to the bases of Definition 1.3.25 we find the following matrix representations:

$$d_{X^\vee} = \begin{pmatrix} 0 & p_0^\top \\ -p_1^\top & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} d_X^\top \,, \tag{1.3.37}$$

$$d_{X[1]} = \begin{pmatrix} 0 & -p_0 \\ -p_1 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} d_X \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,, \tag{1.3.38}$$

$$d_{X\otimes Y} = \left(\begin{array}{cc|cc} 0 & 0 & p_1 \otimes 1_{Y_0} & 1_{X_0} \otimes q_1 \\ 0 & 0 & -1_{X_1} \otimes q_0 & p_0 \otimes 1_{Y_1} \\ \hline p_0 \otimes 1_{Y_0} & -1_{X_1} \otimes q_1 & 0 & 0 \\ 1_{X_0} \otimes q_0 & p_1 \otimes 1_{Y_1} & 0 & 0 \end{array}\right) . \tag{1.3.39}$$

Let (X, d_X) be a finite-rank matrix factorisation. In the following chapters, matrix factorisations of the form $X^\dagger = R[n] \otimes_R X^\vee$ and ${}^\dagger X = X^\vee \otimes_R R[m]$ will be relevant.

Notation 1.3.27. Let $\{b_i\}$ be a canonically ordered basis of X with its dual $\{b_i^*\} \subset X^\vee$. Then the notation $\{b_i^*\}$ will also be used for the following bases:

$$\{b_i^*\} \equiv \{1_n \otimes b_i^*\} \subset R[n] \otimes_R X^\vee = X^\dagger \,, \tag{1.3.40}$$

$$\{b_i^*\} \equiv \{b_i^* \otimes 1_m\} \subset X^\vee \otimes_R R[m] = {}^\dagger X \,. \tag{1.3.41}$$

Note that the basis $\{b_i^*\}$ on $X^\dagger$ (${}^\dagger X$) is *not* canonically ordered for odd n (m).

Lemma 1.3.28. *Let X be a finite-rank matrix factorisation with canonically ordered basis $\{b_1, \ldots, b_{r+s}\} = \{e_1, \ldots, e_r, f_1, \ldots, f_s\}$ and differential $d_X = \left(\begin{smallmatrix} 0 & p_1 \\ p_0 & 0 \end{smallmatrix}\right)$. The differentials $d_{X^\dagger}$ and $d_{{}^\dagger X}$ take the following form with respect to the (not necessarily canonically ordered) basis $\{b_i^*\}$ of Notation 1.3.27:*

$$d_{X^\dagger} = (-1)^n \begin{pmatrix} 0 & p_0^\top \\ -p_1^\top & 0 \end{pmatrix} , \qquad d_{{}^\dagger X} = \begin{pmatrix} 0 & p_0^\top \\ -p_1^\top & 0 \end{pmatrix} . \tag{1.3.42}$$

With respect to the respective canonically ordered bases, the matrices take the form

$$d_{X^\dagger} = \begin{cases} \begin{pmatrix} 0 & p_0^\top \\ -p_1^\top & 0 \end{pmatrix} & n \text{ even} \\ \begin{pmatrix} 0 & p_1^\top \\ -p_0^\top & 0 \end{pmatrix} & n \text{ odd} \end{cases} , \quad d_{{}^\dagger X} = \begin{cases} \begin{pmatrix} 0 & p_0^\top \\ -p_1^\top & 0 \end{pmatrix} & m \text{ even} \\ \begin{pmatrix} 0 & -p_1^\top \\ p_0^\top & 0 \end{pmatrix} & m \text{ odd} \end{cases} . \tag{1.3.43}$$

Proof. Applying the Koszul sign rule (1.3.21) yields

$$d_{X^\dagger}(1_n \otimes b_i) = (1 \otimes d_{X^\vee})(1_n \otimes b_i) = (-1)^n 1_n \otimes d_{X^\vee}(b_i) \; ,$$
$$d_{{}^\dagger X}(b_i \otimes 1_m) = (d_{X^\vee} \otimes 1)(b_i \otimes 1_m) = d_{X^\vee}(b_i) \otimes 1_m \; .$$

The first identity then follows from Eq. (1.3.37). For n, m even, $X^\dagger = {}^\dagger X = X^\vee$, and the bases are equal. For n, m odd, the $\{e_i^*\}$ are exchanged with the $\{f_j^*\}$, so the blocks of the matrices are swapped as well. □

1.3.5 On the associativity of the tensor product

The tensor product of modules is strictly associative. Therefore, if one disregards the graded structure of matrix factorisations, one finds $(X \otimes' Y) \otimes' Z = X \otimes' (Y \otimes' Z)$ and $d_{(X\otimes' Y)\otimes' Z} = d_{X\otimes'(Y\otimes' Z)}$, corresponding to the tensor product basis

$$\{e_i \otimes g_j, e_i \otimes h_j, f_i \otimes g_j, f_i \otimes h_j\} \; . \tag{1.3.44}$$

However, the tensor product of *graded* modules is defined differently in order to respect the graded structure (see Eq. (1.3.36)), and this tensor product is only associative up to a permutation of basis elements. Both tensor products are consistent with each other in the following sense: Let $\{e_i\} \subset X$, $\{f_j\} \subset Y$, $\{g_l\} \subset Z$ be bases. Then

$$d_{X\otimes' Y\otimes' Z}(e_i \otimes' f_j \otimes' g_l) = \sum_{i',j',l'} \alpha_{i,j,l}^{i',j',l'} e_{i'} \otimes' f_{j'} \otimes' g_{l'} \; ,$$
$$d_{(X\otimes Y)\otimes Z}((e_i \otimes f_j) \otimes g_l) = \sum_{i',j',l'} \alpha_{i,j,l}^{i',j',l'} (e_{i'} \otimes f_{j'}) \otimes g_{l'} \; ,$$
$$d_{X\otimes (Y\otimes Z)}(e_i \otimes (f_j \otimes g_l)) = \sum_{i',j',l'} \alpha_{i,j,l}^{i',j',l'} e_{i'} \otimes (f_{j'} \otimes g_{l'})$$

for coefficients $\alpha_{i,j,l}^{i',j',l'}$. Hence, the different tensor products are "equal on bases" and their only difference is how the tensor product basis $\{e_i \otimes f_j \otimes g_l\}_{i,j,l}$ is ordered. While this distinction is conceptually important, it is only of practical relevance whenever the tensor product basis order matters (e.g. when writing $d_{(X\otimes Y)\otimes Z}$ in matrix form). As this turns out not to matter for the remainder of this work, we treat the tensor product of matrix factorisations as if it was strictly associative, keeping in mind that one has to be careful when working with explicit bases.

1.3.6 Compatibility

We may now study how the different operations on matrix factorisations commute and how they act on bases.

Lemma 1.3.29. *Let X and Y be matrix factorisations over $k[\boldsymbol{x}, \boldsymbol{y}]$ resp. $k[\boldsymbol{y}, \boldsymbol{z}]$. Define $R := k[\boldsymbol{y}]$. Then the following identities hold:*

$$X \otimes_R Y \cong Y \otimes_R X \; , \qquad x \otimes y \mapsto (-1)^{|x||y|} y \otimes x \; , \tag{1.3.45}$$
$$X[n] \otimes_R Y[m] \cong (X \otimes_R Y)[m+n] \; , \quad 1_n \otimes x \otimes 1_m \otimes y \mapsto (-1)^{m|x|} 1_{m+n} \otimes x \otimes y \; . \tag{1.3.46}$$

If X is finite-rank with homogeneous basis $\{e_i\}$, the following holds:

$$X^\vee[n] \cong (X[n])^\vee \,, \qquad 1_n \otimes e_i^* \mapsto (-1)^{n|e_i|}(1_n \otimes e_i)^* \,. \tag{1.3.47}$$

If Y is also finite-rank with basis $\{f_j\}$ and both $X \otimes_R Y$ and $Y^\vee \otimes_R X^\vee$ are regarded as finite-rank matrix factorisations over $k[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$, the following holds:

$$(X \otimes_R Y)^\vee \cong Y^\vee \otimes_R X^\vee \,, \qquad (f_j \otimes e_i)^* \mapsto e_i^* \otimes f_j^* \,. \tag{1.3.48}$$

*Note that (1.3.48) does **not** hold if one regards $X \otimes_R Y$ as an infinite-rank matrix factorisation over $k[\boldsymbol{x}, \boldsymbol{z}]$, as shown in Appendix A.4.1.*

Proof. All the maps are clearly bijective. Their closedness can be seen directly from the definitions of dual, grade-shifted, and tensor product matrix factorisations. □

Definition 1.3.30. The permutation isomorphism of Eq. (1.3.45) will be denoted by

$$\sigma^{(i_1, \dots, i_n)} \colon X_1 \otimes_R \cdots \otimes_R X_n \to X_{\sigma(1)} \otimes_R \cdots \otimes_R X_{\sigma(n)} \,. \tag{1.3.49}$$

In line with Notation 1.3.18 we also write $\sigma^{(i_1, \dots, i_n)}$ if the tensor products are taken over different rings, like $X_1 \otimes_{R_1} \dots \otimes_{R_{n-1}} X_n$.

1.3.7 Resolutions and cokernels

Constructing explicit morphisms between matrix factorisations can be difficult. Even in simple cases, finding the isomorphism between the infinite-rank matrix factorisation arising from a tensor product and its finite-rank presentation is quite involved; an explicit example is shown in [11, Appendix B]. For practical purposes, the method introduced in this section is a significantly easier. Starting with a matrix factorisation

$$X = X_0 \oplus X_1, \quad d_X = \left(\begin{smallmatrix} 0 & p_1 \\ p_0 & 0 \end{smallmatrix}\right), \quad d_X^2 = W \cdot 1 \,,$$

we define $\tilde{X}_i := X_i/(W)$. Then the following is a 2-periodic exact sequence [40, Prop. 5.1]:

$$\dots \xrightarrow{p_0} \tilde{X}_1 \xrightarrow{p_1} \tilde{X}_0 \xrightarrow{p_0} \tilde{X}_1 \xrightarrow{p_1} \dots \tag{1.3.50}$$

Proof.

$$a \in \mathrm{Ker}(p_1 \colon \tilde{X}_1 \to \tilde{X}_0) \implies p_1(a) = Wb \quad \text{for some } b \in X_0 \,,$$
$$W p_0(b) = p_0(Wb) = p_0(p_1(a)) = Wa \implies p_0(b) = a \implies a \in \mathrm{Im}(p_0)$$

using that W is a non-zero-divisor. $a \in \mathrm{Ker}(p_0) \Rightarrow a \in \mathrm{Im}(p_1)$ can be shown analogously. □

It immediately follows that the following sequence is exact [42]:

$$\dots \xrightarrow{p_0} \tilde{X}_1 \xrightarrow{p_1} \tilde{X}_0 \xrightarrow{p_0} \tilde{X}_1 \xrightarrow{p_1} \tilde{X}_0 \twoheadrightarrow \mathrm{coker}(p_1 \colon \tilde{X}_1 \to \tilde{X}_0) \longrightarrow 0 \,. \tag{1.3.51}$$

The $\tilde{X}_i$ are free modules over $\tilde{R} := k[\boldsymbol{x}]/(W)$ as they are only annihilated by the ideal (W) which is equal to 0 in $\tilde{R}$. Therefore, Eq. (1.3.51) is an *$\tilde{R}$-free* and thus *$\tilde{R}$-projective resolution*[8] of $\mathrm{coker}(p_1 \colon \tilde{X}_1 \to \tilde{X}_0)$.

[8] Free modules are always projective. By the Quillen–Suslin theorem, the reverse also holds for modules over $k[\boldsymbol{x}]$ when k is a principal ideal domain.

Definition 1.3.31. Let $X \in \mathsf{MF}_k(\boldsymbol{x}; W)$. A $k[\boldsymbol{x}]/(W)$-module M is *associated* to X if there is a $k[\boldsymbol{x}]/(W)$-projective resolution of M of the form

$$\dots \xrightarrow{p_1} \tilde{X}_0 \xrightarrow{p_0} \tilde{X}_1 \xrightarrow{p_1} \tilde{X}_0 \xrightarrow{\delta_{2m+1}} M_{2m} \xrightarrow{\delta_{2m}} \dots \xrightarrow{\delta_2} M_1 \xrightarrow{\delta_1} M \longrightarrow 0 \tag{1.3.52}$$

for some integer $m \in \mathbb{N}_0$. Spelled out, there is an exact sequence of projective modules ending on $\ldots \to M \to 0$ that turns into the 2-periodic exact sequence (1.3.50) after a finite number of steps, with an even number of steps from M to any $\tilde{X}_1$.

Corollary 1.3.32. *For every matrix factorisation X of W over k with $d_X = \begin{pmatrix} 0 & p_1 \\ p_0 & 0 \end{pmatrix}$, the module*

$$\operatorname{coker}(p_1 \colon X_1/(W) \to X_0/(W)) = X_0/\{p_1(x) \mid x \in X_1\} \subset \tilde{X}_0 \tag{1.3.53}$$

is associated to X.

Proof. As discussed above, Eq. (1.3.51) is a $k[\boldsymbol{x}]/(W)$-free resolution of the cokernel. This meets the requirements of Definition 1.3.31 for $m = 0$ and δ_1 being the projection from $\tilde{X}_0$ to $\operatorname{coker}(p_1)$. □

Remark 1.3.33. A module M is a *maximal Cohen–Macaulay module* if and only if it is associated to some matrix factorisation with $m = 0$, i.e. $M = \operatorname{coker}(p_1)$ [14, 42, p. 8].

The following is a classic result:

Theorem 1.3.34. *Let X, Y be matrix factorisations of $W(\boldsymbol{x})$. Let M and N be isomorphic $k[\boldsymbol{x}]/(W)$-modules associated to X resp. Y, i.e. there are projective resolutions of the form*

$$\begin{tikzcd} \dots \arrow[r, "p_1"] & \tilde{X}_0 \arrow[r, "p_0"] & \tilde{X}_1 \arrow[r, "p_1"] & \tilde{X}_0 \arrow[r, "\delta_{2m+1}"] & M_{2m} \arrow[r, "\delta_{2m}"] & \dots \arrow[r, "\delta_2"] & M_1 \arrow[r, two heads, "\delta_1"] & M \arrow[r] \arrow[d, shift left, "\cong"] & 0 \\ \dots \arrow[r, "q_1"] & \tilde{Y}_0 \arrow[r, "q_0"] & \tilde{Y}_1 \arrow[r, "q_1"] & \tilde{Y}_0 \arrow[r, "\delta'_{2n+1}"] & N_{2n} \arrow[r, "\delta'_{2n}"] & \dots \arrow[r, "\delta'_2"] & N_1 \arrow[r, two heads, "\delta'_1"] & N \arrow[r] \arrow[u, shift left] & 0 \end{tikzcd} \tag{1.3.54}$$

Then $X \cong Y$ as matrix factorisations.

Proof. For $n = m = 0$, this was first shown in [40, Section 6]. A mathematical discussion of the generalisation to arbitrary m and n can be found in [14]. Applications and a discussion in physics language can be found in [11, 42]. □

Remark 1.3.35. The isomorphism between X and Y in Theorem 1.3.34 can be constructed in the following way [12, p. 14]:

$$\begin{tikzcd} \dots \arrow[r, "p_1"] & \tilde{X}_0 \arrow[r, "p_0"] \arrow[d, shift left, "r_{2k+1}"] & \tilde{X}_1 \arrow[r, "p_1"] \arrow[d, shift left, "r_{2k}"] & \dots \arrow[r, "\delta_3"] & M_2 \arrow[r, "\delta_2"] \arrow[d, shift left, "r_2"] & M_1 \arrow[r, two heads, "\delta_1"] \arrow[d, shift left, "r_1"] & M \arrow[r] \arrow[d, shift left, "\pi"] & 0 \\ \dots \arrow[r, "p_0"] & \tilde{Y}_0 \arrow[r, "q_0"'] \arrow[u, shift left, "r^*_{2k+1}"] & \tilde{Y}_1 \arrow[r, "q_1"'] \arrow[u, shift left, "r^*_{2k}"] & \dots \arrow[r, "\delta'_3"'] & N_2 \arrow[r, "\delta'_2"'] \arrow[u, shift left, "r^*_2"] & N_1 \arrow[r, two heads, "\delta'_1"'] \arrow[u, shift left, "r^*_1"] & N \arrow[r] \arrow[u, shift left, "\psi"] & 0 \end{tikzcd} \tag{1.3.55}$$

By assumption there exist isomorphisms $\psi \circ \pi = 1$, $\pi \circ \psi = 1$. It is possible to construct the r_i, r_i^* in such a way that the diagram commutes.[9] This yields a pair of isomorphisms (r_i, r_i^*) on

[9] Both the subset of arrows pointing down or right and the subset of arrows point up or right must commute. The down-up and up-down cycles are homotopic, but not equal to the identity.

every level, i.e. $r_i^* \circ r_i = 1_{M_i}$, $r_i \circ r_i^* = 1_{N_i}$ up to homotopy. Once we reach the periodic part in both resolutions, we find the explicit isomorphism (r_{2k+i}, r_{2k+i}^*) between $\tilde{X}_i$ and $\tilde{Y}_i$. The assumption of an even number of steps in Definition 1.3.31 is important here — without it, the r_i might be odd maps, which are not considered to be isomorphisms. This procedure is applied to a concrete example in Appendix A.2.

An important application of the resolution method is computing explicit finite-rank presentations of $X \otimes Y$ for finite-rank matrix factorisations X and Y. As discussed in Example 1.3.17, the rank of $X \otimes Y$ is, a priori, infinite in most cases.

Lemma 1.3.36. *Let X and Y be finite-rank matrix factorisations with $d_X = \left(\begin{smallmatrix} 0 & p_1 \\ p_0 & 0 \end{smallmatrix}\right)$, $d_Y = \left(\begin{smallmatrix} 0 & q_1 \\ q_0 & 0 \end{smallmatrix}\right)$ for finite-rank matrices p_i, q_i. Then*

$$V := \operatorname{coker}(p_1 \otimes 1_{Y_0}, -1_{X_0} \otimes q_1) \tag{1.3.56}$$

is associated to $X \otimes Y$ [11, p. 21].

In many examples, V is already of finite rank, and it is often easy to find a finite-rank matrix factorisation associated to V. A generalisation to n-fold tensor products is discussed in the next section, which involves additional machinery and a class of matrix factorisations called *Koszul* matrix factorisations.

1.3.8 Koszul matrix factorisations and resolutions

We start by defining the Koszul complex, which is the main ingredient of Koszul matrix factorisations.

The Koszul complex

Definition 1.3.37 (Koszul complex)**.** Let $\{p_1, \ldots, p_n\} \subset k[\boldsymbol{x}]$. We define

$$V := \bigoplus_{i=1}^{n} k[\boldsymbol{x}] \cdot \theta_i \ , \qquad K_i := \bigwedge^i V \ , \qquad \delta := \sum_{i=1}^{n} p_i(\boldsymbol{x}) \cdot \theta_i^* \colon K_j \to K_{j-1} \tag{1.3.57}$$

with formal anti-commuting variables $\{\theta_1, \ldots, \theta_n\}$. The inclusion map θ_i^* is defined by

$$\theta_i^*(\theta_{j_1} \wedge \cdots \wedge \theta_{j_l}) = \begin{cases} (-1)^{m-1}\theta_{j_1} \wedge \ldots \theta_{j_{m-1}} \wedge \theta_{j_{m+1}} \wedge \cdots \wedge \theta_{j_l} & i = j_m \\ 0 & i \notin \{j_m\} \end{cases} . \tag{1.3.58}$$

The *Koszul complex* of $\boldsymbol{p} = \{p_1, \ldots, p_n\}$ is defined by

$$K_\bullet(\boldsymbol{p}) : \ 0 \longrightarrow K_n \xrightarrow{\delta} K_{n-1} \xrightarrow{\delta} \ldots \xrightarrow{\delta} K_1 \xrightarrow{\delta} K_0 \longrightarrow 0 \ . \tag{1.3.59}$$

The cohomology of the Koszul complex is well studied. We start with the following definition:

Definition 1.3.38. A sequence $\{p_1, \ldots, p_n\}$ is called *Koszul-regular* if the Koszul complex $K_\bullet(p_1, \ldots, p_n)$ is exact except in degree zero, i.e. at K_0 [87, Remark 061T].

We will come back to the characterisation of Koszul-regular sequences in a moment. For now, assuming we do have a sequence that is Koszul-regular, we may use the Koszul complex to construct a $k[\boldsymbol{x}]$-free resolution of $\operatorname{coker}(\delta\colon K_1 \to K_0)$:

Definition 1.3.39 (Koszul resolution)**.** Let $\{p_1, \ldots, p_n\}$ be a Koszul-regular sequence in $k[\boldsymbol{x}]$. Then the *Koszul resolution* of

$$N := k[\boldsymbol{x}]/(p_1, \ldots, p_n) \tag{1.3.60}$$

is given by the projective resolution

$$0 \longrightarrow K_n \xrightarrow{\delta} K_{n-1} \xrightarrow{\delta} \ldots \xrightarrow{\delta} K_1 \xrightarrow{\delta} K_0 \longrightarrow N \longrightarrow 0 \, . \tag{1.3.61}$$

This resolution will also denoted by $K_\bullet(\boldsymbol{p})$ if it does not cause any confusion.

Regular, Koszul-regular, and quasi-regular sequences

There are different kinds of regular sequences, Definition 1.3.38 being one of them. "Classical" regular sequences are defined as follows [75, Chapter 16]:

Definition 1.3.40. Let k be a ring and M be a module over R. An element $r \in R$ is called *M-regular* if $rm \neq 0$ for all $0 \neq m \in M$. A sequence $\{f_1, \ldots, f_n\} \subset R$ is called an *M-regular sequence* if

(i) f_i is $M/(f_1, \ldots, f_{i-1})$-regular for all i,

(ii) $M/(f_1, \ldots, f_n) \neq 0$.

An R-regular sequence is simply called a *regular sequence*.

As we will not need the definition of a quasi-regular sequence, we refer to [75, Chapter 16]. There is a simple hierarchy between the three notions of regularity:

Theorem 1.3.41. *Regular sequences are always Koszul-regular [75, Theorem 16.5 (i)], and Koszul-regular sequences are always quasi-regular [87, Lemma 09CC].*

The reverse is not true in general. However, the following holds:

Theorem 1.3.42. *Let k be a Noetherian ring (e.g. $k = \mathbb{C}[\boldsymbol{w}]$) and let $\{p_1, \ldots, p_n\} \subset k[\boldsymbol{x}]$ be a quasi-regular sequence. Then $\boldsymbol{p}$ is Koszul-regular [22, p. 489].*

A natural question to ask is whether the different notions of regularity depend on the order of a given sequence. For regular sequences, the answer is yes — their permutations are, in general, not regular sequences. On the other hand, Koszul- and quasi-regular sequences are order-independent by construction. In fact, an even stronger statement holds.

Lemma 1.3.43. *Let $\{f_1, \ldots, f_n\} \subset k[\boldsymbol{x}]$ and let $M \in k[\boldsymbol{x}]^{n\times n}$ be an invertible matrix. Define $\boldsymbol{g} := M \cdot \boldsymbol{f}$. Then both sequences generate the same ideal, i.e. $(g_1, \ldots, g_n) = (f_1, \ldots, f_n)$.*

Proof.

$$a \in (\boldsymbol{f}) \implies a = \sum_i a_i f_i = \sum_{i,j} a_i M^{-1}_{ij} g_j \in (\boldsymbol{g}) \, ,$$
$$a \in (\boldsymbol{g}) \implies a = \sum_i a_i g_i = \sum_{i,j} a_i M_{ij} f_j \in (\boldsymbol{f}) \, . \qquad \square$$

Lemma 1.3.44. *Let* $\{f_1, \ldots, f_n\} \subset k[\boldsymbol{x}]$ *be a Koszul-regular sequence and let* $M \in k[\boldsymbol{x}]^{n\times n}$ *be an invertible matrix. Then* $\boldsymbol{g} := M \cdot \boldsymbol{f}$ *is also Koszul-regular.*

Proof. By [87, Lemma 066A] it is sufficient to show $(\boldsymbol{f}) = (\boldsymbol{g})$ which holds by Lemma 1.3.43. □

Corollary 1.3.45. *Let* $\{f_1, \ldots, f_n\} \subset k[\boldsymbol{x}]$ *be a regular sequence. Then every permutation* $\{f_{\sigma(1)}, \ldots, f_{\sigma(n)}\}$ *is Koszul-regular.*

Koszul matrix factorisations

There is a class of matrix factorisations called *Koszul* matrix factorisations which are closely related to the Koszul complex. This introduction follows [63, p. 13].

Definition 1.3.46. Let $\{p_1, \ldots, p_n\}, \{q_1, \ldots, q_n\} \subset k[\boldsymbol{x}]$. The *Koszul matrix factorisation* $K(\boldsymbol{p}; \boldsymbol{q})$ is the following matrix factorisation of $W := \boldsymbol{p}\cdot\boldsymbol{q} = \sum_{i=1}^n p_i q_i$:

$$K(\boldsymbol{p};\, \boldsymbol{q}) := \bigotimes_{i=1}^{n}{}_{k[\boldsymbol{x}]} \left(k[\boldsymbol{x}] \underset{q_i}{\overset{p_i}{\rightleftarrows}} k[\boldsymbol{x}] \right) . \tag{1.3.62}$$

An equivalent way of writing this matrix factorisation is as follows [21], with the K_i from Eq. (1.3.57):

$$K(\boldsymbol{p};\, \boldsymbol{q}) = \bigoplus_{i=0}^{n} K_i = \bigwedge \left(\bigoplus_{i=1}^{n} k[\boldsymbol{x}] \cdot \theta_i \right) , \quad d_{K(\boldsymbol{p};\, \boldsymbol{q})} := \sum_{i=1}^{n} [p_i \cdot \theta_i^* + q_i \cdot \theta_i] \tag{1.3.63}$$

with the grading $|K_i| = i$, and $\theta_i := \theta_i \wedge -$ acts as a wedge product from the left.

In the latter notation, the relation to the Koszul complex $K_\bullet(\boldsymbol{p})$ in Eq. (1.3.59) is apparent: The module of $K(\boldsymbol{p};\, \boldsymbol{q})$ consists of all modules in $K_\bullet(\boldsymbol{p})$, and $d_{K(\boldsymbol{p};\boldsymbol{q})} = \delta_{K_\bullet(\boldsymbol{p})} + \sigma$ with $|\delta_{K_\bullet(\boldsymbol{p})}| = -1$ and a co-differential σ with $|\sigma| = 1$.

Theorem 1.3.47. *For fixed* $\boldsymbol{p}$ *and* $W = \boldsymbol{p}\cdot\boldsymbol{q}$*, the isomorphism class of* $K(\boldsymbol{p};\, \boldsymbol{q})$ *is independent of* $\boldsymbol{q}$*. We write* $K_W(\boldsymbol{p})$ *for this equivalence class (or for a representative).*

This has the following important implication:

Lemma 1.3.48. *Let* $U \in k[\boldsymbol{x}]$*,* $V \in k[\boldsymbol{y}]$*,* $W \in k[\boldsymbol{z}]$*,* $\boldsymbol{p} \subset k[\boldsymbol{x}, \boldsymbol{y}]$*, and* $\boldsymbol{q} \subset k[\boldsymbol{y}, \boldsymbol{z}]$*, such that* $V - U \in (\boldsymbol{p})$ *and* $W - V \in (\boldsymbol{q})$*. Then*

$$K_{W-V}(\boldsymbol{q}) \otimes_{k[\boldsymbol{y}]} K_{V-U}(\boldsymbol{p}) = K_{W-U}(\boldsymbol{p}, \boldsymbol{q}) . \tag{1.3.64}$$

Proof. By assumption, there are $\boldsymbol{p}' \subset k[\boldsymbol{x}, \boldsymbol{y}]$ and $\boldsymbol{q}' \subset k[\boldsymbol{y}, \boldsymbol{z}]$ such that $\boldsymbol{p}\cdot\boldsymbol{p}' = V - U$ and $\boldsymbol{q}\cdot\boldsymbol{q}' = W - V$. From the definition of the Koszul matrix factorisation it is clear that the tensor product is equal to

$$K(\boldsymbol{p}, \boldsymbol{q};\, \boldsymbol{p}', \boldsymbol{q}') \in K_{\boldsymbol{p}\cdot\boldsymbol{p}' + \boldsymbol{q}\cdot\boldsymbol{q}'}(\boldsymbol{p}, \boldsymbol{q}) = K_{W-U}(\boldsymbol{p}, \boldsymbol{q}) .$$ □

The rank of a Koszul matrix factorisation grows exponentially with the length of $\boldsymbol{p}$, so working with the definition directly is usually not feasible. However, in many cases there is a simple module associated to $K_W(\boldsymbol{p})$.

Theorem 1.3.49. *Let $\{p_1, \dots, p_n\}$ be a Koszul-regular sequence in $k[\boldsymbol{x}]$, and let $\{q_1, \dots, q_n\}$ be polynomials in $k[\boldsymbol{x}]$. Then the module*

$$N := k[\boldsymbol{x}]/(p_1, \dots, p_n) \tag{1.3.65}$$

is associated to $K(\boldsymbol{p};\boldsymbol{q}) \in K_W(\boldsymbol{p})$ for $W := \boldsymbol{p} \cdot \boldsymbol{q}$.

Proof. This proof is in close analogy to [42, Section 4.3], which is based on a more general argument in [40]. Because very few modifications to the argument in [42] are necessary, only the basic structure and the main differences will be spelled out here.

We start with the Koszul resolution $K_\bullet(\boldsymbol{p})$ defined in Eq. (1.3.61). By assumption, $\boldsymbol{p}$ is Koszul-regular, therefore $K_\bullet(\boldsymbol{p})$ is a $k[\boldsymbol{x}]$-free resolution of N. We define the co-differential σ on $K_\bullet(\boldsymbol{p})$ by

$$\sigma\colon K_j \to K_{j+1} \ , \qquad \omega \mapsto \Big(\sum_{i=1}^{n} q_i(\boldsymbol{x}) \cdot \theta_i\Big) \wedge \omega$$

in agreement with Definition 1.3.46. Furthermore, we define $\tilde{\delta} := \delta + \sigma = d_{K(\boldsymbol{p};\boldsymbol{q})}$ and find $\tilde{\delta}^2 = \delta\sigma + \sigma\delta = W \cdot 1_{K_\bullet(\boldsymbol{p})}$. These are all the ingredients required to construct a $k[\boldsymbol{x}]/(W)$-free resolution of N, which is of the form

$$\dots \longrightarrow \tilde{F}_i \xrightarrow{\tilde{\delta}} \tilde{F}_{i-1} \xrightarrow{\tilde{\delta}} \dots \xrightarrow{\tilde{\delta}} \tilde{F}_1 \xrightarrow{\tilde{\delta}} \tilde{F}_0 \longrightarrow N \longrightarrow 0 \ .$$

The precise definition of the $\tilde{F}_i$ can be found in [42]. Using the notation $\tilde{K}_i := K_i/(W)$, the construction yields $\tilde{F}_i \cong \bigoplus_j \tilde{K}_{i-2j}$ for $i \geq n$. This implies

$$\tilde{F}_{2i} \cong \bigoplus_{j \text{ even}} \tilde{K}_j = K(\boldsymbol{p};\, \boldsymbol{q})_0/(W) \ , \qquad \tilde{F}_{2i+1} \cong \bigoplus_{j \text{ odd}} \tilde{K}_j = K(\boldsymbol{p};\, \boldsymbol{q})_1/(W)$$

for $2i \geq n$, so the 2-periodic part is given by $K(\boldsymbol{p};\boldsymbol{q})$. Note that there are $2i+2$ steps from N to $\tilde{F}_{2i+1} \cong K(\boldsymbol{p};\boldsymbol{q})_1/(W)$ because the $\tilde{F}_i$ start at $i = 0$. Thus N is associated to $K(\boldsymbol{p};\boldsymbol{q})$. □

The following statement can now be shown easily:

Corollary 1.3.50. *Let $\boldsymbol{f} = \{f_1, \dots, f_n\} \subset k[\boldsymbol{x}]$ be Koszul-regular, let $M \in k[\boldsymbol{x}]^{n\times n}$ be an invertible matrix, and let $W \in k[\boldsymbol{x}]$. Then*

$$K_W(\boldsymbol{f}) = K_W(M \cdot \boldsymbol{f}) \ . \tag{1.3.66}$$

Proof. $\boldsymbol{f}$ and $\boldsymbol{g} := M \cdot \boldsymbol{f}$ generate the same ideal by Lemma 1.3.43. By assumption, $\boldsymbol{f}$ is Koszul-regular, so by Lemma 1.3.44, $\boldsymbol{g}$ is also Koszul-regular. This implies that $K_W(\boldsymbol{f})$ is associated to $k[\boldsymbol{x}]/(\boldsymbol{f})$ and $K_W(M\cdot\boldsymbol{f})$ is associated to $k[\boldsymbol{x}]/(M \cdot \boldsymbol{f}) = k[\boldsymbol{x}]/(\boldsymbol{f})$. The equality of both equivalence classes then follows from Theorem 1.3.34. □

Finally, we will prove the regularity of a class of sequences defined below.

Lemma 1.3.51. *Let $\boldsymbol{f} = \{f_1, \dots, f_n\}$ be a sequence in $\mathbb{C}[\boldsymbol{x}]$ with the following property: There exist non-overlapping indices $\alpha_1, \dots, \alpha_n$ such that for all i*

$$f_i = a_i \cdot x_{\alpha_i}^{d_i} + g_i, \quad a_i \in \mathbb{C}\setminus\{0\}, \quad d_i \geq 1, \quad \frac{\partial g_j}{\partial x_{\alpha_l}} = 0 \text{ for all } j \leq l \ ,$$

i.e. x_{α_i} first appears in f_i, and $f_i = a \cdot (x_{\alpha_i})^d + (x_{\alpha_i}\text{-independent})$ for $a \in \mathbb{C}\setminus\{0\}$. Then $\boldsymbol{f}$ is a regular sequence on $\mathbb{C}[\boldsymbol{x}]$.

Proof. By contraposition, we need to show that on $\mathbb{C}[\boldsymbol{x}]/(f_1, \ldots, f_{i-1})$, $f_i y \equiv 0$ implies $y \equiv 0$, i.e.

$$f_i y = \sum_{j=1}^{i-1} h_j f_j \implies y \in (f_1, \ldots, f_{i-1}) \ .$$

We expand each term in powers of x_{α_i}:

$$y = \sum_{m=0}^{\beta} y^{(m)} x_{\alpha_i}^m, \quad h_j = \sum_{m=0}^{\gamma_j} h_j^{(m)} x_{\alpha_i}^m \ .$$

By assumption, f_1 to f_{i-1} as well as g_i are independent of x_{α_i}. Thus we find (where we set $h_j^{(m)} := 0$ for $m > \gamma_j$):

$$\begin{aligned} (a_i x_{\alpha_i}^{d_i} + g_i) \sum_{m=0}^{\beta} y^{(m)} x_{\alpha_i}^m &= \sum_{j=1}^{i-1} f_j \sum_{m=0}^{\gamma_j} h_j^{(m)} x_{\alpha_i}^m \\ \implies \sum_{m=0}^{\beta} (a_i x_{\alpha_i}^{m+d_i} + g_i x_{\alpha_i}^m) y^{(m)} &= \sum_{m=0}^{\max\{\gamma_j\}} \sum_{j=1}^{i-1} f_j h_j^{(m)} x_{\alpha_i}^m \ . \end{aligned}$$

Expanding the equation in coefficients $x_{\alpha_i}^{\beta+d_i}, \ldots, x_{\alpha_i}^{d_i}$ yields

$$\begin{aligned} a_i y^{(\beta)} &= \textstyle\sum_{j=1}^{i-1} f_j h_j^{(\beta+d_i)} \ , & a_i y^{(\beta-d_i)} + g_i y^{(\beta)} &= \textstyle\sum_{j=1}^{i-1} f_j h_j^{(\beta)} \ , \\ a_i y^{(\beta-1)} &= \textstyle\sum_{j=1}^{i-1} f_j h_j^{(\beta+d_i-1)} \ , & a_i y^{(\beta-d_i-1)} + g_i y^{(\beta-1)} &= \textstyle\sum_{j=1}^{i-1} f_j h_j^{(\beta-1)} \ , \\ &\vdots & &\vdots \\ a_i y^{(\beta-d_i+1)} &= \textstyle\sum_{j=1}^{i-1} f_j h_j^{(\beta+1)} \ , & a_i y^{(0)} + g_i y^{(d_i)} &= \textstyle\sum_{j=1}^{i-1} f_j h_j^{(d_i)} \end{aligned}$$

(if $\beta < d_i$, the left column stops at $a_i y^{(0)}$ and the right column does not exist). We argue via complete induction that each $y^{(m)}$ is part of the ideal $(f_1, \ldots, f_{i-1})$: This is trivial for $y^{(\beta)}$ to $y^{(\beta-d_i+1)}$ as a_i is a unit. For the remaining $y^{(m)}$ we use that we have already shown $\{y^{(j)}\} \subset (f_1, \ldots, f_{i-1})$ for $j > m$. This proves $y \equiv 0$ on $\mathbb{C}[\boldsymbol{x}]/(f_1, \ldots, f_{i-1})$.

Finally, we need to show that $\mathbb{C}[\boldsymbol{x}]/(f_1, \ldots, f_n) \neq \{0\} \Leftrightarrow 1 \notin (f_1, \ldots, f_n)$. Let us try to solve

$$1 = \sum_{i=1}^{n} h_i f_i \ .$$

If we compare coefficients in x_{α_n}, we find 0 on the left and $h_n a_n x_{\alpha_n}^{d_n}$ on the right, implying $h_n = 0$. Now we can argue similarly for $h_{n-1} = 0$ all the way down to $h_1 = 0$, and find the contradiction $1 = 0$. □

Lemma 1.3.51 and Corollary 1.3.45 can be used in conjunction to prove the Koszul-regularity of a given sequence, which we will be done extensively in subsequent chapters.

1.3.9 The identity matrix factorisation

The defining property of the identity matrix factorisation I is that it is unital with respect to the tensor product, i.e. $I \otimes X \cong X$ for all matrix factorisations X. To construct the identity and various related matrix factorisations, we first need to define the following operator.

Definition 1.3.52 (Divided difference operator)**.** Let $A \in M$ where M is a $k[\boldsymbol{x}]$-module.[10] Then we define [21, p. 2]

$$\partial^{\boldsymbol{x},\boldsymbol{x}'}_{[i]} A := \frac{A(x'_1, \ldots, x'_{i-1}, x_i, \ldots, x_n) - A(x'_1, \ldots, x'_i, x_{i+1}, \ldots, x_n)}{x_i - x'_i} \tag{1.3.67}$$
$$\in k[\boldsymbol{x}, \boldsymbol{x}'] \otimes_{k[\boldsymbol{x}]} M \ ,$$

$$\partial^{\boldsymbol{x},\boldsymbol{x}'} A := \left\{ \partial^{\boldsymbol{x},\boldsymbol{x}'}_{[1]} A, \ldots, \partial^{\boldsymbol{x},\boldsymbol{x}'}_{[\ell(\boldsymbol{x})]} A \right\} \in k[\boldsymbol{x}, \boldsymbol{x}'] \otimes_{k[\boldsymbol{x}]} M^{\oplus \ell(\boldsymbol{x})} \ , \tag{1.3.68}$$

which is well-defined because the numerator is anti-symmetric under $x_i \leftrightarrow x'_i$. Furthermore, let $f(\boldsymbol{x}) = \{f_1(\boldsymbol{x}), \ldots, f_{\ell(\boldsymbol{x})}(\boldsymbol{x})\}$ and $g(\boldsymbol{x}) = \{g_1(\boldsymbol{x}), \ldots, g_{\ell(\boldsymbol{x})}(\boldsymbol{x})\}$ be lists of polynomials in $\boldsymbol{x}$ (which can also be interpreted as ring endomorphisms of $k[\boldsymbol{x}]$, see Lemma 1.4.11 below). Then we define

$$\partial^{f(\boldsymbol{x}),g(\boldsymbol{x}')}_{[i]} A := \partial^{\boldsymbol{y},\boldsymbol{y}'}_{[i]} A\big|_{\boldsymbol{y}=f(\boldsymbol{x}),\boldsymbol{y}'=g(\boldsymbol{x}')} \in k[\boldsymbol{x}, \boldsymbol{x}'] \otimes_{k[\boldsymbol{x}]} M \ . \tag{1.3.69}$$

It is easy to see that

$$(f(\boldsymbol{x}) - g(\boldsymbol{x}')) \cdot \partial^{f(\boldsymbol{x}),g(\boldsymbol{x}')} A = A(f(\boldsymbol{x})) - A(g(\boldsymbol{x}')) \ . \tag{1.3.70}$$

Definition 1.3.53 (Identity matrix factorisation)**.** Let $W \in k[\boldsymbol{x}]$ be a polynomial. Then the *identity matrix factorisation* $I^{\boldsymbol{x}\leftarrow\boldsymbol{x}'}_W$ is defined to be the following Koszul matrix factorisation [63, p. 13]:

$$I^{\boldsymbol{x}\leftarrow\boldsymbol{x}'}_W := K(\boldsymbol{x} - \boldsymbol{x}'; \partial^{\boldsymbol{x},\boldsymbol{x}'} W) \in \mathsf{MF}_k(\boldsymbol{x}, \boldsymbol{x}'; W(\boldsymbol{x}) - W(\boldsymbol{x}')) \ . \tag{1.3.71}$$

Notation 1.3.54. In cases where a divided difference or the identity matrix factorisation is not taken over all variables that appear in W (e.g. $k = k_0[\boldsymbol{w}]$, $W(\boldsymbol{x}, \boldsymbol{w}) \in k[\boldsymbol{x}]$), we write

$$\partial^{\boldsymbol{x},\boldsymbol{x}'}_{[i]} W(\bullet, \boldsymbol{w}) := \partial^{\boldsymbol{x},\boldsymbol{x}'}_{[i]} \tilde{W} \quad \text{with } \tilde{W}(\boldsymbol{x}) := W(\boldsymbol{x}, \boldsymbol{w}) \ , \tag{1.3.72}$$

$$I^{\boldsymbol{x}\leftarrow\boldsymbol{x}'}_{W(\bullet,\boldsymbol{w})} := I^{\boldsymbol{x}\leftarrow\boldsymbol{x}'}_{\tilde{W}} \in \mathsf{MF}_{k_0[\boldsymbol{w}]}(\boldsymbol{x}, \boldsymbol{x}'; W(\boldsymbol{x}, \boldsymbol{w}) - W(\boldsymbol{x}', \boldsymbol{w})) \ . \tag{1.3.73}$$

Lemma 1.3.55 (Basic properties of the identity matrix factorisation)**.** *The following identities hold for all $W \in k[\boldsymbol{x}, \boldsymbol{y}]$, $V \in k[\boldsymbol{x}]$, $U \in k[\boldsymbol{y}]$:*

$$I^{\boldsymbol{x}\leftarrow\boldsymbol{x}'}_V \cong I^{\boldsymbol{x}'\leftarrow\boldsymbol{x}}_{-V} \ , \tag{1.3.74}$$

$$I^{\boldsymbol{x}\leftarrow\boldsymbol{x}'}_V = \bigotimes_{i=1}^{\ell(\boldsymbol{x})} I^{x_i\leftarrow x'_i}_{V(x'_1,\ldots,x'_{i-1},\bullet,x_{i+1},\ldots,x_{\ell(\boldsymbol{x})})} \ , \tag{1.3.75}$$

$$I^{\boldsymbol{x}\leftarrow\boldsymbol{x}'}_{W(\bullet,\boldsymbol{y})} = I^{\boldsymbol{x}\leftarrow\boldsymbol{x}'}_{W(\bullet,\boldsymbol{y})+U(\boldsymbol{y})} \ , \tag{1.3.76}$$

$$I^{\{\boldsymbol{x},\boldsymbol{y}\}\leftarrow\{\boldsymbol{x}',\boldsymbol{y}'\}}_W = I^{\boldsymbol{x}\leftarrow\boldsymbol{x}'}_{W(\bullet,\boldsymbol{y})} \otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{W(\boldsymbol{x}',\bullet)} \cong I^{\boldsymbol{x}\leftarrow\boldsymbol{x}'}_{W(\bullet,\boldsymbol{y}')} \otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{W(\boldsymbol{x},\bullet)} \ . \tag{1.3.77}$$

Proof. Eq. (1.3.76) follows from

$$\partial^{\boldsymbol{x},\boldsymbol{x}'}_{[i]} V(\boldsymbol{y}) = 0 \ , \quad W(\boldsymbol{x}, \boldsymbol{y}) + V(\boldsymbol{y}) - (W(\boldsymbol{x}', \boldsymbol{y}) + V(\boldsymbol{y})) = W(\boldsymbol{x}, \boldsymbol{y}) - W(\boldsymbol{x}', \boldsymbol{y}) \ .$$

To prove Eq. (1.3.75) we write $K(\boldsymbol{p}; \boldsymbol{q})$ according to Eq. (1.3.62) and realise

$$\partial^{x_i,x'_i} W(x'_1, \ldots, x'_{i-1}, \bullet, x_{i+1}, \ldots, x_n) = \partial^{\boldsymbol{x},\boldsymbol{x}'}_{[i]} W(\bullet) \ .$$

[10] usually $M = k[\boldsymbol{x}]$ or $M = \mathrm{End}_{k[\boldsymbol{x}]}\left(k[\boldsymbol{x}]^{\oplus d}\right)$

The first identity of Eq. (1.3.77) follows from Eq. (1.3.75), the latter isomorphy holds according to Theorem 1.3.47 as all three of them are elements of the equivalence class

$$K_{W(\boldsymbol{x},\boldsymbol{y})-W(\boldsymbol{x}',\boldsymbol{y}')}(\boldsymbol{x}-\boldsymbol{x}',\boldsymbol{y}-\boldsymbol{y}')\ .$$

Alternatively, one can argue using Lemma 1.3.57 below. Finally, Eq. (1.3.74) can be shown using Lemmas 1.3.10 and 1.3.57, or one can argue that both are elements of $K_{W(\boldsymbol{x})-W(\boldsymbol{x}')}(\boldsymbol{x}-\boldsymbol{x}')$. □

Theorem 1.3.56. *Let $X(\boldsymbol{x},\boldsymbol{z})\in \mathsf{MF}_k(\boldsymbol{x},\boldsymbol{z};V(\boldsymbol{z})-W(\boldsymbol{x}))$ be a matrix factorisation. Then the morphisms*

$$\lambda_X\colon I_V^{\boldsymbol{z}\leftarrow\boldsymbol{z}'}\otimes_{k[\boldsymbol{z}']}X(\boldsymbol{x},\boldsymbol{z}')\to X(\boldsymbol{x},\boldsymbol{z})\ ,\qquad a\cdot\theta_{\alpha_1}\dots\theta_{\alpha_l}\otimes e_i\mapsto \delta_{l,0}(a|_{\boldsymbol{z}'\mapsto\boldsymbol{z}})\,e_i\ , \tag{1.3.78}$$
$$\rho_X\colon X(\boldsymbol{x},\boldsymbol{z})\otimes_{k[\boldsymbol{x}]}I_W^{\boldsymbol{x}\leftarrow\boldsymbol{x}'}\to X(\boldsymbol{x}',\boldsymbol{z})\ ,\qquad b\cdot e_i\otimes\theta_{\alpha_1}\dots\theta_{\alpha_l}\mapsto \delta_{l,0}(b|_{\boldsymbol{x}\mapsto\boldsymbol{x}'})\,e_i \tag{1.3.79}$$

are isomorphisms, and their inverses are given by

$$\lambda_X^{-1}\colon e_i\mapsto\sum_{l\geqslant0}\sum_{\alpha_1<\dots<\alpha_l}\sum_j\theta_{\alpha_1}\dots\theta_{\alpha_l}\left\{\partial^{\boldsymbol{z},\boldsymbol{z}'}_{[\alpha_l]}d_X\dots\partial^{\boldsymbol{z},\boldsymbol{z}'}_{[\alpha_1]}d_X\right\}_{ji}\otimes e_j\ , \tag{1.3.80}$$
$$\rho_X^{-1}\colon e_i\mapsto\sum_{l\geqslant0}\sum_{\alpha_1<\dots<\alpha_l}\sum_j(-1)^{\binom{l}{2}+l|e_i|}e_j\otimes\left\{\partial^{\boldsymbol{x},\boldsymbol{x}'}_{[\alpha_1]}d_X\dots\partial^{\boldsymbol{x},\boldsymbol{x}'}_{[\alpha_l]}d_X\right\}_{ji}\theta_{\alpha_1}\dots\theta_{\alpha_l}\ . \tag{1.3.81}$$

Proof. This is summarised in [21, pp. 3–4] and proven in [22, Section 4], where the unitors are constructed in terms of Atiyah classes. While the entire reference [22] makes some assumptions about W, V, and X, these assumptions are not used in the cited section. In fact, the statement also holds if X has infinite rank (by adapting the above formulas in the obvious way, or using the coordinate-free formulas in terms of Atiyah classes in loc. cit.). □

Note that Theorem 1.3.56 does not apply if W depends on $\boldsymbol{z}$ or V depends on $\boldsymbol{x}$; in those cases, the following isomorphy holds.

Lemma 1.3.57. *Let $V,W\in k[\boldsymbol{x}]$ and let $X(\boldsymbol{x})\in\mathsf{MF}_k(\boldsymbol{x};V(\boldsymbol{x})-W(\boldsymbol{x}))$. Then*

$$\lambda_{X(\boldsymbol{x})\otimes I_W^{\boldsymbol{x}\leftarrow\boldsymbol{x}''}}\circ\rho^{-1}_{I_V^{\boldsymbol{x}\leftarrow\boldsymbol{x}''}\otimes X(\boldsymbol{x}'')}\colon I_V^{\boldsymbol{x}\leftarrow\boldsymbol{x}''}\otimes_{k[\boldsymbol{x}'']}X(\boldsymbol{x}'')\to X(\boldsymbol{x})\otimes_{k[\boldsymbol{x}]}I_W^{\boldsymbol{x}\leftarrow\boldsymbol{x}''} \tag{1.3.82}$$

is an isomorphism on $\mathsf{MF}(\boldsymbol{x},\boldsymbol{x}'';V(\boldsymbol{x})-W(\boldsymbol{x}''))$.

Proof. We start from

$$I_V^{\boldsymbol{x}\leftarrow\boldsymbol{x}'}\otimes_{k[\boldsymbol{x}']}X(\boldsymbol{x}')\otimes_{k[\boldsymbol{x}']}I_W^{\boldsymbol{x}'\leftarrow\boldsymbol{x}''}\in\mathsf{MF}_k(\boldsymbol{x},\boldsymbol{x}'';\,V(\boldsymbol{x})-W(\boldsymbol{x}''))\ .$$

Applying $\lambda_{X(\boldsymbol{x})\otimes I_W^{\boldsymbol{x}\leftarrow\boldsymbol{x}''}}$ yields $X(\boldsymbol{x})\otimes I_W^{\boldsymbol{x}\leftarrow\boldsymbol{x}''}$, and applying $\rho_{I_V^{\boldsymbol{x}\leftarrow\boldsymbol{x}''}\otimes X(\boldsymbol{x}'')}$ yields $I_V^{\boldsymbol{x}\leftarrow\boldsymbol{x}''}\otimes X(\boldsymbol{x}'')$. Both morphisms are linear in $\boldsymbol{x}$ and $\boldsymbol{x}''$, and are isomorphisms by Theorem 1.3.56. □

Notation 1.3.58. Later we will use matrix factorisations that are of a similar shape as identity matrix factorisations. For $W\in k[\boldsymbol{x}]$ and $\boldsymbol{a},\boldsymbol{b}$ with $a_i,b_i\colon k[\boldsymbol{x}]\to k[\boldsymbol{x}]$, we use the analogous notation

$$I_W^{\boldsymbol{b}(\boldsymbol{x})\leftarrow\boldsymbol{a}(\boldsymbol{x})}:=K\big(\boldsymbol{b}(\boldsymbol{x})-\boldsymbol{a}(\boldsymbol{x});\,\partial^{\boldsymbol{b}(\boldsymbol{x}),\boldsymbol{a}(\boldsymbol{x})}W\big)\in K_{W(\boldsymbol{b}(\boldsymbol{x}))-W(\boldsymbol{a}(\boldsymbol{x}))}\big(\boldsymbol{b}(\boldsymbol{x})-\boldsymbol{a}(\boldsymbol{x})\big)\ . \tag{1.3.83}$$

Note that matrix factorisations of this type are *not* necessarily equivalences, so they do not share all properties of $I_W^{\boldsymbol{x}\leftarrow\boldsymbol{x}'}$.

1.4 Bicategories of matrix factorisation categories

1.4.1 Definition

All categories of matrix factorisations $\mathsf{MF}_k(\boldsymbol{x};W)$ over a ring k can be unified into a bicategory $\ddot{\mathsf{MF}}_k$: The objects are pairs $(\boldsymbol{x};W)$, and the morphisms between two objects $(\boldsymbol{x};W)$ and $(\boldsymbol{z};V)$ are given by $\mathsf{MF}_k(\boldsymbol{x},\boldsymbol{z};V(\boldsymbol{z})-W(\boldsymbol{x}))$. This naturally introduces a notion of 1- and 2-morphisms. The composition of 1-morphisms is given by the tensor product (Definition 1.3.15). Furthermore, we introduce a set of variables $\boldsymbol{w}$ that are shared by all objects and morphisms in the bicategory.

Definition 1.4.1. The bicategory $\ddot{\mathsf{MF}}_k(\boldsymbol{w})$ is defined as follows [63, p. 12]:

(i) Objects are pairs $(\boldsymbol{a};W)=(\boldsymbol{a};W(\boldsymbol{a},\boldsymbol{w}))$ with a list of variables $\boldsymbol{a}$ and a polynomial $W\in k[\boldsymbol{a},\boldsymbol{w}]$.

(ii) 1-morphisms between objects $(\boldsymbol{a};W(\boldsymbol{a},\boldsymbol{w}))$ and $(\boldsymbol{b};V(\boldsymbol{b},\boldsymbol{w}))$ are matrix factorisations of $V(\boldsymbol{b},\boldsymbol{w})-W(\boldsymbol{a},\boldsymbol{w})$:

$$\mathrm{Hom}_{\ddot{\mathsf{MF}}_k(\boldsymbol{w})}((\boldsymbol{a};W),(\boldsymbol{b};V)):=\mathsf{MF}_k(\boldsymbol{a},\boldsymbol{b},\boldsymbol{w};V(\boldsymbol{b},\boldsymbol{w})-W(\boldsymbol{a},\boldsymbol{w}))\ . \tag{1.4.1}$$

(iii) The 2-morphisms of $\ddot{\mathsf{MF}}_k(\boldsymbol{w})$ are given by the 1-morphisms of MF_k:

$$X,X'\colon(\boldsymbol{a};W)\to(\boldsymbol{b};V)\ ,\qquad(\phi\colon X\to X')\in\mathrm{Hom}_{\mathsf{MF}_k(\boldsymbol{a},\boldsymbol{b},\boldsymbol{w};V-W)}(X,X') \tag{1.4.2}$$

with the morphisms between matrix factorisations of Definition 1.3.7.

The morphisms can be composed in the following ways:

(i) *Vertical composition* of 2-morphisms is denoted by $-\circ-$ and is given by the composition of module homomorphisms, consistent with composition of morphisms in $\mathsf{MF}_k(\boldsymbol{x};W)$.

(ii) *Horizontal composition* of 1-morphisms is denoted by $-\otimes-$ and is given by the tensor product of matrix factorisations of Definition 1.3.15: For 1-morphisms

$$X\colon(\boldsymbol{a};W)\to(\boldsymbol{b};V),\qquad Y\colon(\boldsymbol{b};V)\to(\boldsymbol{c};U)$$

we define

$$Y\otimes X:=Y\otimes_{k[\boldsymbol{b},\boldsymbol{w}]}X\colon(\boldsymbol{a};W)\to(\boldsymbol{c};U)\ . \tag{1.4.3}$$

There must not be overlaps between the names of domain and codomain variables. If there are any, the variables in the domain must be relabelled (e.g. by adding a prime).[11]

(iii) For 1- and 2-morphisms

$$X,X'\colon(\boldsymbol{a};W)\to(\boldsymbol{b};V),\quad Y,Y'\colon(\boldsymbol{b};V)\to(\boldsymbol{c};U),\quad\phi\colon X\to X',\quad\psi\colon Y\to Y'$$

we define

$$\phi\otimes\psi\colon Y\otimes_{k[\boldsymbol{b},\boldsymbol{w}]}X\to Y'\otimes_{k[\boldsymbol{b},\boldsymbol{w}]}X' \tag{1.4.4}$$

to be the tensor product of module homomorphisms.

[11] Mathematically, we find structures of the form $k[\boldsymbol{x}]\otimes_k k[\boldsymbol{x}]\cong k[\boldsymbol{x},\boldsymbol{x}']$, $x_i\otimes 1\mapsto x_i$, $1\otimes x_i\mapsto x_i'$ [22, p. 492].

There are the following special morphisms:

(i) The identity 1-morphism $\mathbb{1}_W$ (or Δ_W) of an object $(\boldsymbol{a};\, W)$ is given by the identity matrix factorisation

$$\mathbb{1}_W = \Delta_W = I_W^{\boldsymbol{a}\leftarrow\boldsymbol{a}'} \in \mathrm{Hom}_{\ddot{\mathsf{MF}}_k(\boldsymbol{x})}((\boldsymbol{a}';\, W), (\boldsymbol{a};\, W)) \ . \tag{1.4.5}$$

Note the relabelling of $\boldsymbol{a}$ in the domain, consistent with the convention above.

(ii) The unitor 2-morphisms

$$\lambda_X \colon \mathbb{1}_V \otimes X \to X \ , \quad \rho_X \colon X \otimes \mathbb{1}_W \to X \tag{1.4.6}$$

and their inverses are defined in Eqs. (1.3.78) to (1.3.81). See Definition 1.5.3 for the general properties of the unitor 2-morphisms in a bicategory.

Remark 1.4.2. The bicategory $\ddot{\mathsf{MF}}_k := \ddot{\mathsf{MF}}_k(\emptyset)$ is sometimes called the category of *Landau–Ginzburg models* [63]. We will follow [22] instead and define the Landau–Ginzburg models as a subcategory of $\ddot{\mathsf{MF}}_k$ in Definition 2.1.2.

Remark 1.4.3. Shared variables can be interpreted in different ways in both MF_k and $\ddot{\mathsf{MF}}_k$. We first realise that we can identify matrix factorisations

$$X \in \mathsf{MF}_k(\boldsymbol{a};\, W(\boldsymbol{a})) \quad \text{and} \quad \hat{X} \in \mathsf{MF}_{k[\boldsymbol{a}]}(\emptyset;\, W(\boldsymbol{a})) \ , \tag{1.4.7}$$

as both X and $\hat{X}$ are $k[\boldsymbol{a}]$-modules with the same $k[\boldsymbol{a}]$-linear differential $d_X^2 = W(\boldsymbol{a})$. Furthermore, the bicategories $\ddot{\mathsf{MF}}_{k_0}(\boldsymbol{w})$ and $\ddot{\mathsf{MF}}_{k_0[\boldsymbol{w}]}$ can be identified. Let

$$(\boldsymbol{a};\, W), (\boldsymbol{b};\, V) \in \ddot{\mathsf{MF}}_{k_0}(\boldsymbol{w}) \ , \quad X, Y \colon W \to V \ , \quad \phi \colon X \to Y \ .$$

Then we identify objects and morphisms as follows:

$\ddot{\mathsf{MF}}_{k_0}(\boldsymbol{w})$	$\ddot{\mathsf{MF}}_{k_0[\boldsymbol{w}]}$
$k = k_0$	$k = k_0[\boldsymbol{w}]$
$W \in k_0[\boldsymbol{a}, \boldsymbol{w}]$	$W \in k[\boldsymbol{a}] = k_0[\boldsymbol{a}, \boldsymbol{w}]$
$V \in k_0[\boldsymbol{b}, \boldsymbol{w}]$	$V \in k[\boldsymbol{b}] = k_0[\boldsymbol{b}, \boldsymbol{w}]$
X, Y modules over $k_0[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{w}]$	X, Y modules over $k[\boldsymbol{a}, \boldsymbol{b}] = k_0[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{w}]$
$\phi \colon X \to Y$ $k_0[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{w}]$-linear	$\phi \colon X \to Y$ $k[\boldsymbol{a}, \boldsymbol{b}]$-linear

Therefore, all statements that hold in $\ddot{\mathsf{MF}}_k$ for all k automatically hold in $\ddot{\mathsf{MF}}_k(\boldsymbol{w})$ as well. Consequently, all proofs may set $\boldsymbol{w} \mapsto \emptyset$ without loss of generality as long as they do not make assumptions about k that do not apply to $k[\boldsymbol{w}]$.

1.4.2 Properties of $\ddot{\mathsf{MF}}_k$

We have already seen that the identity matrix factorisation $\mathbb{1}_W$ behaves like a "multiplicative identity" in $\ddot{\mathsf{MF}}_k$. There is also a matrix factorisation which behaves like an "additive identity":

Definition 1.4.4. Let $V, W \in \ddot{\mathsf{MF}}_k$. The *trivial* matrix factorisation is given by

$$0_{V\leftarrow W} := K(1;\ V - W) \in K_{V-W}(1) \subset \mathrm{Hom}_{\ddot{\mathsf{MF}}_k}(W, V)\ . \tag{1.4.8}$$

Lemma 1.4.5. *For all $U, V, W \in \ddot{\mathsf{MF}}_k$ and all $X \in \mathrm{Hom}_{\ddot{\mathsf{MF}}_k}(V, W)$ the trivial matrix factorisation fulfils*

$$X \oplus 0_{V\leftarrow W} \cong X\ , \qquad X \otimes 0_{W\leftarrow U} \cong 0_{V\leftarrow U}, \qquad 0_{U\leftarrow V} \otimes X \cong 0_{U\leftarrow W}\ . \tag{1.4.9}$$

Furthermore, it is invariant under grade shifts, i.e.

$$0_{V\leftarrow W} \cong 0_{V\leftarrow W}[1]\ , \tag{1.4.10}$$

and all its homomorphisms are null-homotopic:

$$\mathrm{Hom}_{\ddot{\mathsf{MF}}_k(W,V)}(X, 0_{V\leftarrow W}) = \mathrm{Hom}_{\ddot{\mathsf{MF}}_k(W,V)}(0_{V\leftarrow W}, X) = \{0\}\ . \tag{1.4.11}$$

Proof sketch. Eq. (1.4.11) implies the other statements: Choose zero maps for all isomorphisms where a trivial matrix factorisation is involved. These zero maps compose to the identity up to homotopy because the identity is also null-homotopic. The proof of Eq. (1.4.11) is straightforward: Using that the top right component of $d_{0_{V\leftarrow W}}$ is equal to 1, it is not hard to construct null-homotopies for every morphism mapping into or out of $0_{V\leftarrow W}$. □

Remark 1.4.6. In the language of category theory $0_{V\leftarrow W}$ is a *zero morphism*. The existence of zero morphisms is a requirement to construct direct sums in $\ddot{\mathsf{MF}}_k$, as we do in Section 1.5.8.

Lemma 1.4.7. *There are multiple ways to interpret a given matrix factorisation X as a 1-morphism in $\ddot{\mathsf{MF}}$.*

(i) $X \in \mathsf{MF}_k(\boldsymbol{a}, \boldsymbol{b};\ V(\boldsymbol{b}) - W(\boldsymbol{a}))$ *can be interpreted as a 1-morphism*

$$X\colon (\boldsymbol{a};\ W(\boldsymbol{a}) + U) \to (\boldsymbol{b};\ V(\boldsymbol{b}) + U) \quad \textit{for all } U \in k \textit{ (including } U = 0\textit{)}. \tag{1.4.12}$$

(ii) $X \in \mathsf{MF}_k(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c};\ U(\boldsymbol{c}) - V(\boldsymbol{b}) - W(\boldsymbol{a}))$ *can be interpreted as a 1-morphism*

$$\begin{aligned} &X\colon (\boldsymbol{a}, \boldsymbol{b};\ W(\boldsymbol{a}) + V(\boldsymbol{b})) \to (\boldsymbol{c};\ U(\boldsymbol{c})) \\ \textit{or}\quad &X\colon (\boldsymbol{a};\ W(\boldsymbol{a})) \to (\boldsymbol{b}, \boldsymbol{c};\ U(\boldsymbol{c}) - V(\boldsymbol{b}))\ . \end{aligned} \tag{1.4.13}$$

(iii) $X \in \mathsf{MF}_k(\boldsymbol{a}, \boldsymbol{b};\ V(\boldsymbol{b}) - W(\boldsymbol{a})) = \mathsf{MF}_{k[\boldsymbol{a},\boldsymbol{b}]}(\emptyset;\ V(\boldsymbol{b}) - W(\boldsymbol{a}))$ *can be interpreted as*

$$\begin{aligned} &X \in \ddot{\mathsf{MF}}_k\Big((\boldsymbol{a};\ W(\boldsymbol{a})),\ (\boldsymbol{b};\ V(\boldsymbol{b}))\Big) \\ \textit{or}\quad &X \in \ddot{\mathsf{MF}}_{k[\boldsymbol{a},\boldsymbol{b}]}\Big((\emptyset;\ W(\boldsymbol{a})),\ (\emptyset;\ V(\boldsymbol{b}))\Big)\ . \end{aligned} \tag{1.4.14}$$

Proof. In Eq. (1.4.12) we find

$$\ddot{\mathsf{MF}}_k\Big((\boldsymbol{a};\ W(\boldsymbol{a}) + U),\ (\boldsymbol{b};\ V(\boldsymbol{b}) + U)\Big) = \mathsf{MF}_k(\boldsymbol{a}, \boldsymbol{b};\ V(\boldsymbol{b}) - W(\boldsymbol{a}))$$

for all $U \in k$. In Eq. (1.4.13), both homomorphism classes are given by

$$\mathsf{MF}_k(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c};\ U(\boldsymbol{c}) - V(\boldsymbol{b}) - W(\boldsymbol{a}))\ .$$

Eq. (1.4.14) follows directly from Eq. (1.4.7). □

1.4.3 Equivalences between objects in $\ddot{\mathsf{MF}}_k$

The classification of the equivalence classes of the objects in $\ddot{\mathsf{MF}}_k$ is, in general, a hard problem (see [60]). However, some objects can be shown to be equivalent with relative ease. We start with a statement about the non-uniqueness of the equivalence 1-morphisms, and then construct some equivalences explicitly.

On the non-uniqueness of equivalence 1-morphisms

Let V, W be equivalent[12] objects in some (bi)category $\mathcal{C}$, i.e. there is at least one equivalence 1-morphism $X\colon W \to V$. It is a natural question to ask if there are others, i.e. we wish to classify all equivalences $W \to V$, which can be rephrased into classifying the automorphisms $\mathrm{Aut}(W)$. If $\mathcal{C}$ is a bicategory, we may also study the quotient $\mathrm{Aut}(W)/\sim$ with respect to isomorphy of 1-morphisms, i.e. we consider two equivalences $X, Y\colon W \to W$ to be equal if $X \cong Y$ as 1-morphisms. An interesting question to ask is whether $\mathrm{Aut}(W)/\sim$ is non-trivial, i.e. whether there are equivalences $X, Y\colon W \to W$ with $X \ncong Y$.

Many matrix factorisations $X \in \mathsf{MF}_k(\boldsymbol{a};\, W)$ like the identity matrix factorisation $\mathbb{1}_W$ only have one automorphism (up to scalar multiples). The following lemma shows that the situation is different for objects of $\ddot{\mathsf{MF}}_k$:

Lemma 1.4.8 (Non-uniqueness of equivalence 1-morphisms between objects of $\ddot{\mathsf{MF}}_k$)**.** *For every equivalence $X \in \mathrm{Hom}_{\ddot{\mathsf{MF}}_k}(W, V)$ with inverse Y, the grade shift $X[1]$ is also an equivalence from W to V with inverse $Y[1]$.*

Proof. By assumption there exist 1-morphisms

$$X\colon (\boldsymbol{a};\, W(\boldsymbol{a})) \to (\boldsymbol{b};\, V(\boldsymbol{b}))\ , \quad Y\colon (\boldsymbol{b};\, V(\boldsymbol{b})) \to (\boldsymbol{a};\, W(\boldsymbol{a}))\ ,$$

such that $Y \otimes X \cong I_W^{\boldsymbol{a}\leftarrow\boldsymbol{a}'}$ and $X \otimes Y \cong I_V^{\boldsymbol{b}\leftarrow\boldsymbol{b}'}$. Applying a grade shift of 1 (Definition 1.3.21) to both X and Y preserves this property:

$$\begin{aligned} Y[1] \otimes X[1] &\cong (Y \otimes X)[2] = Y \otimes X \cong I_W^{\boldsymbol{a}\leftarrow\boldsymbol{a}'}\ ,\\ X[1] \otimes Y[1] &\cong (X \otimes Y)[2] = X \otimes Y \cong I_V^{\boldsymbol{b}\leftarrow\boldsymbol{b}'}\ . \end{aligned}$$ □

In general, $X \ncong X[1]$, implying that most pairs of equivalent objects (W, V) have at least two non-isomorphic equivalence 1-morphisms.

Knörrer periodicity

Theorem 1.4.9 (Knörrer periodicity)**.** *Let $W \in k[\boldsymbol{a}]$ and let $\boldsymbol{u}$, $\boldsymbol{v}$ be lists of variables of equal length. Then the following objects are equivalent in $\ddot{\mathsf{MF}}_k$:*

$$(\boldsymbol{a};\, W(\boldsymbol{a})) \cong (\boldsymbol{a}', \boldsymbol{u}, \boldsymbol{v};\, W(\boldsymbol{a}') + \boldsymbol{u}\cdot\boldsymbol{v})\ . \tag{1.4.15}$$

The equivalence is given by

$$I_W^{\boldsymbol{a}'\leftarrow\boldsymbol{a}} \otimes_k I_{\boldsymbol{v}\cdot\bullet}^{\boldsymbol{u}\leftarrow 0}\colon (\boldsymbol{a};\, W(\boldsymbol{a})) \to (\boldsymbol{a}', \boldsymbol{u}, \boldsymbol{v};\, W(\boldsymbol{a}') + \boldsymbol{u}\cdot\boldsymbol{v})\ , \tag{1.4.16}$$

$$I_W^{\boldsymbol{a}\leftarrow\boldsymbol{a}'} \otimes_k I_{\boldsymbol{u}\cdot\bullet}^{0\leftarrow\boldsymbol{v}}\colon (\boldsymbol{a}', \boldsymbol{u}, \boldsymbol{v};\, W(\boldsymbol{a}') + \boldsymbol{u}\cdot\boldsymbol{v}) \to (\boldsymbol{a};\, W(\boldsymbol{a}))\ , \tag{1.4.17}$$

with the matrix factorisations as defined in Eq. (1.3.83).

[12] If $\mathcal{C}$ is not a bicategory, the term “equivalent” may be replaced by “isomorphic”.

Remark 1.4.10. By Lemma 1.4.8 we may replace $I^{\boldsymbol{u}\leftarrow 0}_{\boldsymbol{v}\cdot\bullet}$ by $I^{\boldsymbol{v}\leftarrow 0}_{\boldsymbol{u}\cdot\bullet}$ and $I^{0\leftarrow\boldsymbol{v}}_{\boldsymbol{u}\cdot\bullet}$ by $I^{0\leftarrow\boldsymbol{u}}_{\boldsymbol{v}\cdot\bullet}$, showing the symmetry between $\boldsymbol{u}$ and $\boldsymbol{v}$.

Proof. This theorem was first stated and proven in [66]. We will follow the presentation in [25], where the following 1-morphism is shown to be an equivalence:

$$I^{\boldsymbol{a}'\leftarrow\boldsymbol{a}}_W \otimes \begin{pmatrix} 0 & x_1-x_2 \\ x_1+x_2 & 0 \end{pmatrix} \colon (\boldsymbol{a};\, W(\boldsymbol{a})) \to (\boldsymbol{a}', x_1, x_2;\, W(\boldsymbol{a}') + x_1^2 - x_2^2)$$

The argument can adapted easily to show that

$$I^{\boldsymbol{a}'\leftarrow\boldsymbol{a}}_W \otimes \begin{pmatrix} 0 & y_1 \\ y_2 & 0 \end{pmatrix} \colon (\boldsymbol{a};\, W(\boldsymbol{a})) \to (\boldsymbol{a}', y_1, y_2;\, W(\boldsymbol{a}') + y_1 y_2)$$

is an equivalence. Furthermore, [25] proves that the inverse is given by the (coinciding) left and right adjoint (see Definition 1.2.15):

$$I^{\boldsymbol{a}\leftarrow\boldsymbol{a}'}_W \otimes \begin{pmatrix} 0 & x_2 \\ -x_1 & 0 \end{pmatrix} \colon (\boldsymbol{a}', x_1, x_2;\, W(\boldsymbol{a}') + x_1^2 - x_2^2) \to (\boldsymbol{a};\, W(\boldsymbol{a}))\ .$$

Knörrer periodicity as defined above now follows by applying the previous case $\ell(\boldsymbol{u})$ times. We find the equivalences

$$I^{\boldsymbol{a}'\leftarrow\boldsymbol{a}}_W \otimes_k \bigotimes_{i=1}^{\ell(\boldsymbol{u})} \begin{pmatrix} 0 & u_i \\ v_i & 0 \end{pmatrix} = I^{\boldsymbol{a}'\leftarrow\boldsymbol{a}}_W \otimes_k I^{\boldsymbol{u}\leftarrow 0}_{\boldsymbol{v}\cdot\bullet}\ ,$$

$$I^{\boldsymbol{a}\leftarrow\boldsymbol{a}'}_W \otimes_k \bigotimes_{i=1}^{\ell(\boldsymbol{u})} \begin{pmatrix} 0 & v_i \\ -u_i & 0 \end{pmatrix} \cong I^{\boldsymbol{a}\leftarrow\boldsymbol{a}'}_W \otimes_k I^{0\leftarrow\boldsymbol{v}}_{\boldsymbol{u}\cdot\bullet}\ . \qquad \square$$

Ring automorphisms

The following theorems will be about variable transformations in $\ddot{\mathsf{MF}}_k$, i.e.

$$(\boldsymbol{a};\, W(\boldsymbol{a})) \to (\boldsymbol{a}';\, W(F(\boldsymbol{a}')))\ . \tag{1.4.18}$$

The natural setting for such transformations is to require F to be a ring automorphism of $k[\boldsymbol{a}]$. Rather than studying the full group of such automorphisms (which is an unsolved problem), we consider some well-understood subgroups that are sufficient for our purposes. For further reading, see [69] and the references therein.

Lemma 1.4.11. *Let $\boldsymbol{a} = \{a_1, \dots, a_n\}$ be a list of variables. The ring endomorphisms of $k[\boldsymbol{a}]$ are represented by lists $\{a'_1, \dots, a'_n\} \subset k[\boldsymbol{a}]$ and act as $F\colon k[\boldsymbol{a}] \to k[\boldsymbol{a}]$, $a_i \mapsto a'_i(\boldsymbol{a})$.*

Proof. Clearly F is a ring endomorphism for every choice of $\boldsymbol{a}'$: Additivity, multiplicativity, unit preservation, and linearity in k hold by definition. Now let G be an arbitrary endomorphism of $k[\boldsymbol{a}]$. Set $a'_i := G(a_i)$. By the properties of ring endomorphisms, it is easy to see that $G = F$. $\square$

Definition 1.4.12 (Invertible triangular transformations)**.** Let $\boldsymbol{a} = \{a_1, \dots, a_n\}$ be a list of variables, and let $\boldsymbol{b}' = \{b'_1, \dots, b'_n\} \subset k[\boldsymbol{a}]$.

(i) The ring endomorphism of $k[\boldsymbol{a}]$ induced by $\boldsymbol{a} \mapsto \boldsymbol{b}'(\boldsymbol{a})$ is an *invertible lower triangular (ILT) transformation* if

$$b'_i(\boldsymbol{a}) = d_i \cdot a_i + t_i(a_1, \dots, a_{i-1}) \tag{1.4.19}$$

for units $d_i \in k$ and polynomials $t_i \in k[a_1, \dots, a_{i-1}]$.

(ii) The ring endomorphism of $k[\boldsymbol{a}]$ induced by $\boldsymbol{a} \mapsto \boldsymbol{b}'(\boldsymbol{a})$ is an *invertible triangular (IT) transformation* if there is a permutation $\sigma\colon (a_1, \ldots, a_n) \mapsto (a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ such that

$$\sigma^{-1} \circ \boldsymbol{b}' \circ \sigma \colon k[\boldsymbol{a}] \to k[\boldsymbol{a}] \tag{1.4.20}$$

is an invertible lower triangular transformation.

(iii) The polynomials $V \in k[\boldsymbol{a}]$ and $W \in k[\boldsymbol{a}]$ are *related by an I(L)T transformation* if $W(\boldsymbol{b}'(\boldsymbol{a})) = V(\boldsymbol{a})$ for an I(L)T transformation $\boldsymbol{a} \mapsto \boldsymbol{b}'(\boldsymbol{a})$.

Remark 1.4.13. In the mathematics literature on ring automorphisms there is a notion of *triangular* ring automorphisms [69] closely related to Definition 1.4.12. The "invertible triangular transformations" defined above form a subset of the *tame* ring automorphisms of $k[\boldsymbol{a}]$.

Remark 1.4.14. The set of all transformations

$$\{\boldsymbol{a} \mapsto M \cdot \boldsymbol{a} \mid M \in k^{n\times n} \text{ invertible, lower triangular}\}$$

is a subset of the ILT transformations. However, its generalisation to $M \in k[\boldsymbol{a}]^{n\times n}$ is neither a sub- nor a superset of the IT transformations. For example, $a_1 \mapsto a_1 + r$ for $r \in k$ is an ILT transformation, but cannot be written as $\boldsymbol{a} \mapsto M \cdot \boldsymbol{a}$. On the other hand, the transformation

$$\boldsymbol{a} \mapsto \left(\begin{smallmatrix} 1 & 0 & 0 \\ a_3 & 1 & 0 \\ 1 & 1 & 1 \end{smallmatrix}\right) \cdot \boldsymbol{a} = \left(\begin{smallmatrix} a_1 \\ a_1 a_3 + a_2 \\ a_1 + a_2 + a_3 \end{smallmatrix}\right) =: \boldsymbol{b}$$

is not an IT transformation because both b_2 and b_3 depend on all a_i's.

Lemma 1.4.15. *The inverse of an I(L)T transformation $\boldsymbol{a} \mapsto \boldsymbol{b}'(\boldsymbol{a})$ is again an I(L)T transformation.*

Proof. Let us study the ILT case first. Let $\boldsymbol{a} \mapsto \boldsymbol{b}'(\boldsymbol{a})$ be an ILT transformation in the notation of Eq. (1.4.19). We need to show that there is an ILT transformation $\boldsymbol{b} \mapsto \boldsymbol{a}'(\boldsymbol{b})$ such that $\boldsymbol{a}'(\boldsymbol{b}'(\boldsymbol{a})) = \boldsymbol{a}$ and $\boldsymbol{b}'(\boldsymbol{a}'(\boldsymbol{b})) = \boldsymbol{b}$. We define the ILT transformation $\boldsymbol{b} \mapsto \boldsymbol{a}'(\boldsymbol{b})$ recursively:

$$\begin{aligned} a'_1(\boldsymbol{b}) &:= d_1^{-1} \cdot b_1 - d_1^{-1} \cdot t_1 \ , \\ a'_m(\boldsymbol{b}) &:= d_m^{-1} \cdot b_m - d_m^{-1} \cdot t_m(a'_1(\boldsymbol{b}), \ldots, a'_{m-1}(\boldsymbol{b})) \ . \end{aligned} \tag{1.4.21}$$

$\boldsymbol{b}'(\boldsymbol{a}'(\boldsymbol{b})) = \boldsymbol{b}$ is straightforward to show:

$$\begin{aligned} b'_m(\boldsymbol{a}'(\boldsymbol{b})) &= d_m \cdot a'_m(\boldsymbol{b}) + t_m(a'_1(\boldsymbol{b}), \ldots, a'_{m-1}(\boldsymbol{b})) \\ &= b_m - t_m(a'_1(\boldsymbol{b}), \ldots, a'_{m-1}(\boldsymbol{b})) + t_m(a'_1(\boldsymbol{b}), \ldots, a'_{m-1}(\boldsymbol{b})) = b_m \ . \end{aligned}$$

We prove $a'_m(\boldsymbol{b}'(\boldsymbol{a})) = a_m$ by complete induction on m:

$$m = 1\colon \qquad a'_1(\boldsymbol{b}'(\boldsymbol{a})) = d_1^{-1} \cdot (b'_1(\boldsymbol{a}) - t_1) = d_1^{-1} \cdot (d_1 \cdot a_1 + t_1 - t_1) = a_1 \ .$$

$$\begin{aligned} (m-1) \to m\colon \qquad a'_m(\boldsymbol{b}'(\boldsymbol{a})) &= d_m^{-1} \cdot \Big(b'_m(\boldsymbol{a}) - t_m(a'_1(\boldsymbol{b}'(\boldsymbol{a})), \ldots, a'_{m-1}(\boldsymbol{b}'(\boldsymbol{a})))\Big) \\ &= d_m^{-1} \cdot (d_m \cdot a_m + t_m(a_1, \ldots, a_{m-1}) - t_m(a_1, \ldots, a_{m-1})) \\ &= a_m \ . \end{aligned}$$

For an IT transformation $\boldsymbol{a} \mapsto \boldsymbol{b}'(\boldsymbol{a})$ there exists a permutation σ and an ILT transformation $\boldsymbol{b}''$ such that

$$\boldsymbol{b}' = \sigma \circ \boldsymbol{b}'' \circ \sigma^{-1} \implies \boldsymbol{b}'^{-1} = \sigma \circ \boldsymbol{b}''^{-1} \circ \sigma^{-1} \ .$$

By the above argument, $\boldsymbol{b}''^{-1}$ is an ILT transformation, so $\boldsymbol{b}'^{-1}$ is an IT transformation. $\square$

Lemma 1.4.16. *Let $\boldsymbol{a} \mapsto \boldsymbol{b}'(\boldsymbol{a})$ be an IT transformation with inverse $\boldsymbol{b} \mapsto \boldsymbol{a}'(\boldsymbol{b})$. Then the following ideals in $k[\boldsymbol{a}, \boldsymbol{b}]$ are equal:*

$$(\boldsymbol{b} - \boldsymbol{b}'(\boldsymbol{a})) = (\boldsymbol{a}'(\boldsymbol{b}) - \boldsymbol{a}) \ . \tag{1.4.22}$$

Proof. Let $\boldsymbol{b}''$ be an ILT transformation with inverse $\boldsymbol{a}''$ such that

$$\boldsymbol{b}' = \sigma^{-1} \circ \boldsymbol{b}'' \circ \sigma \ , \quad \boldsymbol{a}' = \sigma^{-1} \circ \boldsymbol{a}'' \circ \sigma \ .$$

Ideals are invariant under permutations of its generators:

$$\begin{aligned}
(\boldsymbol{b} - \boldsymbol{b}'(\boldsymbol{a})) &= (\sigma(\boldsymbol{b} - \boldsymbol{b}'(\boldsymbol{a}))) = (\sigma(\boldsymbol{b}) - (\boldsymbol{b}'' \circ \sigma)(\boldsymbol{a})) =: I \ , \\
(\boldsymbol{a}'(\boldsymbol{b}) - \boldsymbol{a}) &= (\sigma(\boldsymbol{a}'(\boldsymbol{b}) - \boldsymbol{a})) = ((\boldsymbol{a}'' \circ \sigma)(\boldsymbol{b}) - \sigma(\boldsymbol{a})) =: J \ .
\end{aligned}$$

It is sufficient to show $J \subset I$, as $I \subset J$ then follows from exchanging $\boldsymbol{b}' \leftrightarrow \boldsymbol{a}'$. The following argument proves $(a''_m \circ \sigma)(\boldsymbol{b}) - a_{\sigma(m)} \in I$ via complete induction on m. Using the notation of Eq. (1.4.19) for $\boldsymbol{b}''$ and the explicit formula for $\boldsymbol{a}'' = \boldsymbol{b}''^{-1}$ in Eq. (1.4.21), we find the following for $m = 1$:

$$\begin{aligned}
& b_{\sigma(1)} - b''_1(\sigma(\boldsymbol{a})) = b_{\sigma(1)} - (d_1 a_{\sigma(1)} + t_1) \in I \\
\implies \ & d_1^{-1} \cdot (b_{\sigma(1)} - b''_1(\sigma(\boldsymbol{a}))) = d_1^{-1}(b_{\sigma(1)} - t_1) - a_{\sigma(1)} = a''_1(\sigma(\boldsymbol{b})) - a_{\sigma(1)} \in I \ .
\end{aligned}$$

For the induction step $m-1 \to m$, we first show

$$\begin{aligned}
& t_m((\boldsymbol{a}'' \circ \sigma)(\boldsymbol{b})) - t_m(\sigma(\boldsymbol{a})) \\
&= t_m((a''_1 \circ \sigma)(\boldsymbol{b}), \ \ldots, (a''_{m-1} \circ \sigma)(\boldsymbol{b})) - t_m(a_{\sigma(1)}, \ \ldots, a_{\sigma(m-1)}) \\
&= \sum_{i=1}^{m-1} \underbrace{\left((a''_i \circ \sigma)(\boldsymbol{b}) - a_{\sigma(i)}\right)}_{\in I \text{ by assumption}} \cdot \partial_{[i]}^{\sigma(\boldsymbol{a}), (\boldsymbol{a}'' \circ \sigma)(\boldsymbol{b})} t_m \\
&\in I \ .
\end{aligned} \tag{1.4.23}$$

Then

$$\begin{aligned}
& b_{\sigma(m)} - b''_m(\sigma(\boldsymbol{a})) \\
&= b_{\sigma(m)} - (d_m a_{\sigma(m)} + t_m(a_{\sigma(1)}, \ \ldots, a_{\sigma(m-1)})) \in I && \Big| - (1.4.23) \\
\implies \ & b_{\sigma(m)} - t_m(a''_1(\sigma(\boldsymbol{b})), \ \ldots, a''_{m-1}(\sigma(\boldsymbol{b}))) - d_m a_{\sigma(m)} \in I && \Big| \cdot d_m^{-1} \\
\implies \ & \underbrace{d_m^{-1}\Big(b_{\sigma(m)} - t_m(a''_1(\sigma(\boldsymbol{b})), \ \ldots, a''_{m-1}(\sigma(\boldsymbol{b})))\Big) - a_{\sigma(m)}}_{= a''_m(\sigma(\boldsymbol{b})) - a_{\sigma(m)}} \in I \ .
\end{aligned}$$

$\square$

Corollary 1.4.17. *For an IT transformation $\boldsymbol{a} \mapsto \boldsymbol{b}'(\boldsymbol{a})$ with inverse $\boldsymbol{b} \mapsto \boldsymbol{a}'(\boldsymbol{b})$, the following matrix factorisations are isomorphic:*

$$I_{W(\bullet)}^{\boldsymbol{b} \leftarrow \boldsymbol{b}'(\boldsymbol{a})} \cong I_{W(\boldsymbol{b}'(\bullet))}^{\boldsymbol{a}'(\boldsymbol{b}) \leftarrow \boldsymbol{a}} \ . \tag{1.4.24}$$

Proof. Both sides factorise $W(\boldsymbol{b}) - W(\boldsymbol{b}'(\boldsymbol{a}))$. By Lemma 1.3.51, both $\boldsymbol{b} - \boldsymbol{b}'(\boldsymbol{a})$ and $\boldsymbol{a}'(\boldsymbol{b}) - \boldsymbol{a}$ are regular sequences in $k[\boldsymbol{a}, \boldsymbol{b}]$. Therefore, by Theorem 1.3.49,

$$I_{W(\bullet)}^{\boldsymbol{b} \leftarrow \boldsymbol{b}'(\boldsymbol{a})} \in K_{W(\boldsymbol{b}) - W(\boldsymbol{b}'(\boldsymbol{a}))}(\boldsymbol{b} - \boldsymbol{b}'(\boldsymbol{a}))$$

is associated to $k[\boldsymbol{a}, \boldsymbol{b}]/(\boldsymbol{b} - \boldsymbol{b}'(\boldsymbol{a}))$, which is, by Lemma 1.4.16, equal to $k[\boldsymbol{a}, \boldsymbol{b}]/(\boldsymbol{a}'(\boldsymbol{b}) - \boldsymbol{a})$, which is associated to

$$I_{W(\boldsymbol{b}'(\bullet))}^{\boldsymbol{a}'(\boldsymbol{b}) \leftarrow \boldsymbol{a}} \in K_{W(\boldsymbol{b}) - W(\boldsymbol{b}'(\boldsymbol{a}))}(\boldsymbol{a}'(\boldsymbol{b}) - \boldsymbol{a}) \ .$$

□

Theorem 1.4.18. *Let V and $W \in k[\boldsymbol{a}]$ be related by an IT transformation $\boldsymbol{a} \mapsto \boldsymbol{b}'(\boldsymbol{a})$, i.e.*

$$V, W \in k[\boldsymbol{a}] \ , \quad W(\boldsymbol{b}'(\boldsymbol{a})) = V(\boldsymbol{a}) \ , \quad \boldsymbol{a}' := \boldsymbol{b}'^{-1} \ . \tag{1.4.25}$$

Then the following objects of $\ddot{\mathsf{MF}}_k$ are equivalent:

$$(\boldsymbol{b};\ V(\boldsymbol{a}'(\boldsymbol{b}))) = (\boldsymbol{b};\ W(\boldsymbol{b})) \cong (\boldsymbol{a};\ V(\boldsymbol{a})) = (\boldsymbol{a};\ W(\boldsymbol{b}'(\boldsymbol{a}))) \tag{1.4.26}$$

The equivalences are given by

$$I_W^{\boldsymbol{b} \leftarrow \boldsymbol{b}'(\boldsymbol{a})} \cong I_V^{\boldsymbol{a}'(\boldsymbol{b}) \leftarrow \boldsymbol{a}} \colon (\boldsymbol{a};\ V(\boldsymbol{a})) \to (\boldsymbol{b};\ W(\boldsymbol{b})) \ , \tag{1.4.27}$$

$$I_W^{\boldsymbol{b}'(\boldsymbol{a}) \leftarrow \boldsymbol{b}} \cong I_V^{\boldsymbol{a} \leftarrow \boldsymbol{a}'(\boldsymbol{b})} \colon (\boldsymbol{b};\ W(\boldsymbol{b})) \to (\boldsymbol{a};\ V(\boldsymbol{a})) \ . \tag{1.4.28}$$

Proof. The two different versions of the equivalences are isomorphic by Corollary 1.4.17. We compute

$$\begin{aligned} &I_W^{\boldsymbol{d} \leftarrow \boldsymbol{b}'(\boldsymbol{a})} \otimes_{k[\boldsymbol{a}]} I_V^{\boldsymbol{a} \leftarrow \boldsymbol{a}'(\boldsymbol{b})} \colon (\boldsymbol{b};\ W(\boldsymbol{b})) \to (\boldsymbol{d};\ W(\boldsymbol{d})) \\ \in\ &K_{W(\boldsymbol{d}) - W(\boldsymbol{b})}(\boldsymbol{a} - \boldsymbol{a}'(\boldsymbol{b}),\ \boldsymbol{d} - \boldsymbol{b}'(\boldsymbol{a})) \ . \end{aligned}$$

The sequence $\{\boldsymbol{a} - \boldsymbol{a}'(\boldsymbol{b}),\ \boldsymbol{d} - \boldsymbol{b}'(\boldsymbol{a})\} \subset k[\boldsymbol{b}, \boldsymbol{a}, \boldsymbol{d}]$ is regular by Lemma 1.3.51, so by Theorem 1.3.49, $I_W^{\boldsymbol{d} \leftarrow \boldsymbol{b}'(\boldsymbol{a})} \otimes I_V^{\boldsymbol{a} \leftarrow \boldsymbol{a}'(\boldsymbol{b})}$ is associated to the following $k[\boldsymbol{b}, \boldsymbol{d}]$-module:

$$\frac{k[\boldsymbol{b}, \boldsymbol{a}, \boldsymbol{d}]}{(\boldsymbol{a} - \boldsymbol{a}'(\boldsymbol{b}),\ \boldsymbol{d} - \boldsymbol{b}'(\boldsymbol{a}))} \cong \frac{k[\boldsymbol{b}, \boldsymbol{d}]}{(\boldsymbol{d} - \boldsymbol{b}'(\boldsymbol{a}'(\boldsymbol{b})))} = \frac{k[\boldsymbol{b}, \boldsymbol{d}]}{(\boldsymbol{d} - \boldsymbol{b})}$$

which is associated to the identity matrix factorisation $I_W^{\boldsymbol{d} \leftarrow \boldsymbol{b}}$. Similarly,

$$\begin{aligned} &I_V^{\boldsymbol{c} \leftarrow \boldsymbol{a}'(\boldsymbol{b})} \otimes_{k[\boldsymbol{b}]} I_W^{\boldsymbol{b} \leftarrow \boldsymbol{b}'(\boldsymbol{a})} \colon (\boldsymbol{a};\ V(\boldsymbol{a})) \to (\boldsymbol{c};\ V(\boldsymbol{c})) \\ \in\ &K_{V(\boldsymbol{c}) - V(\boldsymbol{a})}(\boldsymbol{b} - \boldsymbol{b}'(\boldsymbol{a}),\ \boldsymbol{c} - \boldsymbol{a}'(\boldsymbol{b})) \ . \end{aligned}$$

Analogously, $\{\boldsymbol{b} - \boldsymbol{b}'(\boldsymbol{a}),\ \boldsymbol{c} - \boldsymbol{a}'(\boldsymbol{b})\} \subset k[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$ is a regular sequence, thus $I_V^{\boldsymbol{c} \leftarrow \boldsymbol{a}'(\boldsymbol{b})} \otimes I_W^{\boldsymbol{b} \leftarrow \boldsymbol{b}'(\boldsymbol{a})}$ is associated to the $k[\boldsymbol{a}, \boldsymbol{c}]$-module

$$\frac{k[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}{(\boldsymbol{b} - \boldsymbol{b}'(\boldsymbol{a}),\ \boldsymbol{c} - \boldsymbol{a}'(\boldsymbol{b}))} \cong \frac{k[\boldsymbol{a}, \boldsymbol{c}]}{(\boldsymbol{c} - \boldsymbol{a}'(\boldsymbol{b}'(\boldsymbol{a})))} = \frac{k[\boldsymbol{a}, \boldsymbol{c}]}{(\boldsymbol{c} - \boldsymbol{a})}$$

which is associated to the identity matrix factorisation $I_V^{\boldsymbol{c} \leftarrow \boldsymbol{a}}$. □

1.5 Affine Rozansky–Witten models and tricategories

The topological Rozansky–Witten model was first introduced in [80] and is constructed as a *topological twist* of a 3D $\mathcal{N} = 4$ supersymmetric sigma model with holomorphic symplectic target manifolds [9, p. 5]. The defects in this model are discussed in [64] and [63], the former using physics language, the latter using higher categories. This section contains a brief summary of the topological twist followed by a detailed introduction to the higher category description of the *affine* Rozansky–Witten model.

1.5.1 The topological twist

Starting from a physics description of a QFT T with a sufficient amount of supersymmetry (specified e.g. by a Lagrangian), the topological twist yields a TFT in physics language that is closely related to T. This summary follows [57, Chapter 16].

(i) We start with a supersymmetric QFT T in d-dimensional Euclidean flat space, i.e. the spacetime symmetry group of T is given by $\mathrm{SO}(d)_e$.

(ii) We furthermore require T to have an R-symmetry whose group is isomorphic to $\mathrm{SO}(d)_e$, i.e. T is invariant under some R-symmetry group $\mathrm{SO}(d)_R$.[13]

(iii) Even without supersymmetry it is possible to construct a QFT in curved space T' by gauging the $\mathrm{SO}(d)_e$ spacetime symmetry of T with the gauge field taking the role of the spin connection of the curved manifold. If the manifold is chosen to be flat, T' agrees with T.

(iv) Now we construct a QFT in curved space T'' by a similar gauge procedure, but instead of gauging the spacetime symmetry $\mathrm{SO}(d)_e$ of T, we gauge the diagonal group $\mathrm{SO}(d)'_e \subset \mathrm{SO}(d)_e \times \mathrm{SO}(d)_R$. This has two major consequences for T'':

- Because the energy-momentum tensor is defined by the variation of the action with respect to the metric, T'' differs from T even in flat space.
- In T and T' the conserved quantities of the supersymmetry algebra (called the *supercharges*) transform as spinors under $\mathrm{SO}(d)_e$. However, because the supercharges also transform as spinors under $\mathrm{SO}(d)_R$, in the twisted theory T'' there is at least one supercharge component Q that has spin 0 under $\mathrm{SO}(d)'_e$, i.e. there is a conserved *fermionic scalar* Q. If there is more than one such supercharge, different twists (like the *A-twist* and *B-twist* in 2D $\mathcal{N} = (2, 2)$ supersymmetry) are possible.

(v) The twist of T is now given by the following subset of T'': The physical operators are defined to be the operators in T'' that commute with Q, and the physical states are defined to be the Q-cohomology.

(vi) In many examples of twisted theories the energy-momentum tensor is Q-exact. If that is the case, the correlation functions can be shown to be invariant under deformations of the metric and hence may only depend on topological properties of the manifold, making the twist of T a TFT.

[13] This imposes restrictions on the supersymmetry algebra: Depending on the dimension, a 2D $\mathcal{N} = (2, 2)$, a 3D $\mathcal{N} = 4$, or a 4D $\mathcal{N} = 4$ SUSY algebra is required [41].

It is noteworthy that the topological twist as explained above yields a TFT in physics description and a priori without defects. Further analysis is required to understand the defect structure of the twisted theory. Once this has been done, one may try to find a description of this defect TFT in terms of higher categories. Formal proofs that both describe the same physics are possible in principle, but quite hard in practice; for example, the proof for Landau–Ginzburg models can be found in [20].

1.5.2 Affine topological Rozansky–Witten models

Similar to [9] we discuss *affine* Rozansky–Witten models whose target manifold is $T^*\mathbb{C}^n$, which is not compact. While this choice of target simplifies some aspects of the model due to the absence of a target manifold metric, it also introduces additional complications: For example, the Hilbert space of bulk operators is finite-dimensional for compact target spaces, but is infinite-dimensional in the affine case. It follows that there can be no description of the affine Rozansky–Witten model in terms of a functor $\mathcal{Z}\colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Vect}_k$ because every such TFT has a finite-dimensional space of bulk operators[14] [67, Section 1.2.25] [79]. This thesis demonstrates that several constructions like a tricategorical description, adjunctions, pivotality, and even orbifold constructions are nevertheless possible in the affine case.

The three-dimensional bulk theories in the affine Rozansky–Witten model are represented by lists of variables $\boldsymbol{x} = \{x_1, \dots, x_n\}$ corresponding to n free hypermultiplets. When introducing boundaries or more general two-dimensional defects into topologically twisted 3D $\mathcal{N} = 4$ theories, a 2D $\mathcal{N} = (2, 2)$ subalgebra of the full SUSY algebra can be preserved in a way that is consistent with the topological twist; a similar procedure will be discussed in more detail in Chapter 4. Two-dimensional defects with this symmetry may thus be introduced into $\mathcal{RW}$. These defects are allowed to have their own localised degrees of freedom which happen to be closely related to Landau–Ginzburg models (the special case of a two-dimensional defect between two trivial Rozansky–Witten bulk TFTs precisely describes a Landau–Ginzburg model). Numerous results from the analysis of topological Landau–Ginzburg models can be applied here as well; for example, one-dimensional defects between different two-dimensional Landau–Ginzburg models are given by fermionic superfields with E- and J-potentials (in the Lagrangian description) or matrix factorisations of the difference in superpotentials (in the categorical description) [10].

The following discussion of the affine Rozansky–Witten model in categorical language is based on [63, 9] with some conventions changed and additional details filled in.

1.5.3 Definition

Below we define the structure $\mathcal{RW}$, which one can intuitively think of as "the tricategory of all bicategories $\ddot{\mathsf{MF}}_{\mathbb{C}}(\boldsymbol{w})$" [63]. It is conjectured, but not proven, that $\mathcal{RW}$ can indeed be endowed with the structure of a tricategory [9]. Furthermore, the objects and morphisms in $\mathcal{RW}$ can be matched to the constituent parts of affine Rozansky-Witten models described in physics language, hence it is believed that $\mathcal{RW}$ is related to a tricategorical description of affine Rozansky-Witten models.

[14] It is likely possible to describe the affine Rozansky–Witten model by a functor $\mathcal{Z}\colon \mathrm{Bord}_n^{\mathrm{def}}(\mathbb{D}) \to \mathrm{Mod}_R$ for an appropriate ring R [18].

Definition 1.5.1. The objects and morphisms of the structure $\mathcal{RW}$ are given by the following [63, pp. 12–18]:

(i) Objects are lists of variables called *bulk variables* of arbitrary length:

$$(\boldsymbol{x}) = (x_1, \dots, x_n) \in \mathsf{Obj}(\mathcal{RW}) \ .$$

These correspond to n hypermultiplets of the bulk affine Rozansky–Witten model. To keep the notation tidy, the parentheses around an object will be omitted if it is surrounded by other parentheses, e.g. we write

$$(\boldsymbol{x}) \in \mathcal{RW} \ , \quad W \in \mathrm{Hom}(\boldsymbol{x}, \boldsymbol{y}) = \mathrm{Hom}((\boldsymbol{x}), (\boldsymbol{y})) \ .$$

(ii) For objects $(\boldsymbol{x})$, $(\boldsymbol{y})$ the bicategory of 1-morphisms is defined by

$$\mathcal{RW}(\boldsymbol{x}, \boldsymbol{y}) := \mathrm{Hom}_{\mathcal{RW}}(\boldsymbol{x}, \boldsymbol{y}) := \ddot{\mathsf{MF}}_{\mathbb{C}}(\boldsymbol{x}, \boldsymbol{y}) \ , \tag{1.5.1}$$

implying that 1-morphisms are pairs consisting of

(a) a list of variables $\boldsymbol{a}$ called *surface variables*,

(b) a polynomial W in the surface and bulk variables called *superpotential*:

$$(\boldsymbol{a};\ W(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y})) \in \mathcal{RW}(\boldsymbol{x}, \boldsymbol{y}) \ . \tag{1.5.2}$$

Physically, we may interpret the a_i as chiral superfields living on a codimension 1 surface separating two affine Rozansky–Witten models, with the a_i coupled to each other and to the bulk hypermultiplets by the superpotential $W(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y})$.

If there are overlaps between domain and codomain bulk variable names, the domain bulk variables must be renamed, reminiscent of a similar rule in $\ddot{\mathsf{MF}}_k$ (see Definition 1.4.1). The *identity 1-morphism* of $(\boldsymbol{x}) = (x_1, \dots, x_n)$ is given by

$$\mathbb{1}_{\boldsymbol{x}} = (a_1, \dots, a_n;\ \boldsymbol{a} \cdot (\boldsymbol{x} - \boldsymbol{x}')) \in \mathrm{Hom}_{\mathcal{RW}}(\boldsymbol{x}', \boldsymbol{x}) \ . \tag{1.5.3}$$

(iii) The 2-morphisms of $\mathcal{RW}$ are given by matrix factorisations, consistent with the 1-morphisms of $\ddot{\mathsf{MF}}_{\mathbb{C}}(\boldsymbol{x}, \boldsymbol{y})$:

$$\begin{gathered}(\boldsymbol{a};\ W(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y})), (\boldsymbol{b};\ V(\boldsymbol{b}, \boldsymbol{x}, \boldsymbol{y})) \in \mathcal{RW}(\boldsymbol{x}, \boldsymbol{y}) \ , \\ \mathrm{Hom}_{\mathcal{RW}(\boldsymbol{x},\boldsymbol{y})}\big((\boldsymbol{a};\ W), (\boldsymbol{b};\ V)\big) = \mathsf{MF}_{\mathbb{C}}\big(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{x}, \boldsymbol{y};\ V(\boldsymbol{b}, \boldsymbol{x}, \boldsymbol{y}) - W(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y})\big) \ .\end{gathered} \tag{1.5.4}$$

Explicitly, a 2-morphism

$$(X, d_X) : (\boldsymbol{a};\ W(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y})) \to (\boldsymbol{b};\ V(\boldsymbol{b}, \boldsymbol{x}, \boldsymbol{y})) \tag{1.5.5}$$

is a matrix factorisation of $V(\boldsymbol{b}, \boldsymbol{x}, \boldsymbol{y}) - W(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y})$ over $\mathbb{C}[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{x}, \boldsymbol{y}]$. This construction can be translated to the following physical setting: $(\boldsymbol{a}; W)$ and $(\boldsymbol{b}; V)$ describe two Landau–Ginzburg models coupled to Rozansky–Witten bulk theories. Now consider a one-dimensional defect X between them:

$$X \colon (\boldsymbol{a};\ W) \to (\boldsymbol{b};\ V) \ , \quad X = X_0 \oplus X_1 \ , \quad d_X = \left(\begin{smallmatrix} 0 & p_1 \\ p_0 & 0 \end{smallmatrix}\right) \ , \quad m = \operatorname{rank} X_0 \cdot \operatorname{rank} X_1 \ ,$$

which corresponds to m fermionic multiplets localised on X whose E- and J-potentials are given by the components of p_1 and p_0 (see [61, Sect. 7.2]).

Note that the bulk variable names are shared between W and V, while the surface variables are not. For example, the identity 2-morphism is given by

$$(\boldsymbol{a};\, W) \in \mathcal{RW}(\boldsymbol{x}, \boldsymbol{y}) \ , \qquad \mathbb{1}_W \in \mathsf{MF}_{\mathbb{C}}(\boldsymbol{a}, \boldsymbol{a}', \boldsymbol{x}, \boldsymbol{y};\, W(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y}) - W(\boldsymbol{a}', \boldsymbol{x}, \boldsymbol{y})) \ .$$

(iv) The 3-morphisms of $\mathcal{RW}$ are morphisms of matrix factorisations, consistent with the 2-morphisms of $\ddot{\mathsf{MF}}_{\mathbb{C}}(\boldsymbol{x}, \boldsymbol{y})$:

$$\phi\colon X \to Y, \quad X, Y\colon (\boldsymbol{a};\, W(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y})) \to (\boldsymbol{b};\, V(\boldsymbol{b}, \boldsymbol{x}, \boldsymbol{y})) \ .$$

There are the following compositions:

- *Vertical composition* of 3-morphisms is denoted by $-\circ-$ and is given by the composition of module homomorphisms, consistent with vertical composition of 2-morphisms in $\ddot{\mathsf{MF}}_{\mathbb{C}}$.
- *Horizontal composition* of 2-morphisms is denoted by $-\otimes-$ and is given by the tensor product of matrix factorisations, consistent with horizontal composition of 1-morphisms in $\ddot{\mathsf{MF}}_{\mathbb{C}}$.
- Composition of 1-morphisms, called the *box product*, is denoted by $-\boxtimes-$ and is defined below.

The action of the box product on 1-, 2-, and 3-morphisms is defined as follows:

(i) The box product of two 1-morphisms

$$(\boldsymbol{a};\, W(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y}))\colon (\boldsymbol{x}) \to (\boldsymbol{y}) \ , \quad (\boldsymbol{b};\, V(\boldsymbol{b}, \boldsymbol{y}, \boldsymbol{z}))\colon (\boldsymbol{y}) \to (\boldsymbol{z}) \ ,$$

is defined to be the following 1-morphism in $\mathcal{RW}(\boldsymbol{x}, \boldsymbol{z})$:

$$(\boldsymbol{b};\, V(\boldsymbol{b}, \boldsymbol{y}, \boldsymbol{z})) \boxtimes (\boldsymbol{a};\, W(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y})) := (\boldsymbol{b}, \boldsymbol{y}, \boldsymbol{a};\, V(\boldsymbol{b}, \boldsymbol{y}, \boldsymbol{z}) + W(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y})) \ . \tag{1.5.6}$$

The rule can be summarised as "bulk variables in between the surfaces are turned into surface variables, and the order of surface variables in the product is 'left to right' (in truncated pictures) or 'back to front' (in three-dimensional pictures)". Note that we deviate slightly from the conventions in [63] in order to make $-\boxtimes-$ strictly associative.

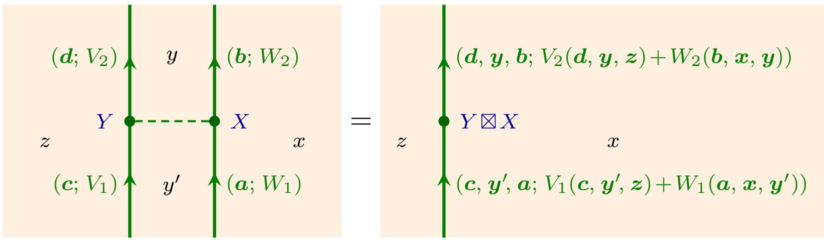
(ii) To define the action of the box product on 2-morphisms, consider 1- and 2-morphisms

$$\begin{aligned} X &: (\boldsymbol{a};\, W_1(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y})) \to (\boldsymbol{b};\, W_2(\boldsymbol{b}, \boldsymbol{x}, \boldsymbol{y})) \ , \\ Y &: \quad (\boldsymbol{c};\, V_1(\boldsymbol{c}, \boldsymbol{y}, \boldsymbol{z})) \to (\boldsymbol{d};\, V_2(\boldsymbol{d}, \boldsymbol{y}, \boldsymbol{z})) \ . \end{aligned}$$

The domain and codomain of $Y \boxtimes X\colon V_1 \boxtimes W_1 \to V_2 \boxtimes W_2$ are given by

$$\begin{aligned} &(\boldsymbol{c};\, V_1(\boldsymbol{c}, \boldsymbol{y}, \boldsymbol{z})) \boxtimes (\boldsymbol{a};\, W_1(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y})) = (\boldsymbol{c}, \boldsymbol{y}, \boldsymbol{a};\, V_1(\boldsymbol{c}, \boldsymbol{y}, \boldsymbol{z}) + W_1(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y})) \\ \to\ &(\boldsymbol{d};\, V_2(\boldsymbol{d}, \boldsymbol{y}, \boldsymbol{z})) \boxtimes (\boldsymbol{b};\, W_2(\boldsymbol{b}, \boldsymbol{x}, \boldsymbol{y})) = (\boldsymbol{d}, \boldsymbol{y}, \boldsymbol{b};\, V_2(\boldsymbol{d}, \boldsymbol{y}, \boldsymbol{z}) + W_2(\boldsymbol{b}, \boldsymbol{x}, \boldsymbol{y})) \ . \end{aligned}$$

According to the definition of $-\boxtimes-$ on 1-morphisms, $\boldsymbol{y}$ becomes a *surface* variable in both domain and codomain of $Y \boxtimes X$, causing an illegal overlap of surfaces variable names. By the rules of $\ddot{\mathsf{MF}}_{\mathbb{C}[\boldsymbol{x},\boldsymbol{z}]}$ we must rename $\boldsymbol{y} \mapsto \boldsymbol{y}'$ in the domain, and we find the corresponding truncated diagram



. (1.5.7)

The dashed horizontal line between Y and X is drawn to visualise the split between $\boldsymbol{y}$ and $\boldsymbol{y}'$ and does not correspond to a 1-morphism. In these conventions we define $Y \boxtimes X$ by

$$Y \boxtimes X := Y(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{y}, \boldsymbol{z}) \otimes I^{\boldsymbol{y} \leftarrow \boldsymbol{y}'}_{V_1(\boldsymbol{c}, \bullet, \boldsymbol{z}) + W_2(\boldsymbol{b}, \boldsymbol{x}, \bullet)} \otimes X(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{x}, \boldsymbol{y}') \ , \tag{1.5.8}$$

Note that according to Lemma 1.3.57, this matrix factorisation is isomorphic to

$$Y(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{y}', \boldsymbol{z}) \otimes I^{\boldsymbol{y} \leftarrow \boldsymbol{y}'}_{V_2(\boldsymbol{d}, \bullet, \boldsymbol{z}) + W_1(\boldsymbol{a}, \boldsymbol{x}, \bullet)} \otimes X(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{x}, \boldsymbol{y}) \tag{1.5.9}$$

$$\cong Y(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{y}', \boldsymbol{z}) \otimes I^{\boldsymbol{y} \leftarrow \boldsymbol{y}'}_{V_2(\boldsymbol{d}, \bullet, \boldsymbol{z}) + W_2(\boldsymbol{b}, \boldsymbol{x}, \bullet)} \otimes X(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{x}, \boldsymbol{y}') \tag{1.5.10}$$

$$\cong Y(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{y}, \boldsymbol{z}) \otimes I^{\boldsymbol{y} \leftarrow \boldsymbol{y}'}_{V_1(\boldsymbol{c}, \bullet, \boldsymbol{z}) + W_1(\boldsymbol{a}, \boldsymbol{x}, \bullet)} \otimes X(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{x}, \boldsymbol{y}) \ . \tag{1.5.11}$$

The rule can be summarised as follows:

- In truncated pictures we write the *left* 2-morphism in the *top* variable and the *right* 2-morphism in the *bottom* variable.
- In 3D pictures we write the *back* 2-morphism in the *left* variable and the *front* 2-morphism in the *right* variable.

As we will see later, it is advantageous to choose (1.5.8) over (1.5.9), (1.5.10), or (1.5.11) for the definition of $Y \boxtimes X$.

(iii) To define the box product on 3-morphisms, consider 2-morphisms (for $i = 1, 2$)

$$X_i \colon (\boldsymbol{a};\, W_1(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y})) \to (\boldsymbol{b};\, W_2(\boldsymbol{b}, \boldsymbol{x}, \boldsymbol{y})) \ ,$$
$$Y_i \colon \ (\boldsymbol{c};\, V_1(\boldsymbol{c}, \boldsymbol{y}, \boldsymbol{z})) \to (\boldsymbol{d};\, V_2(\boldsymbol{d}, \boldsymbol{y}, \boldsymbol{z})) \ ,$$

and 3-morphisms $\phi \colon X_1 \to X_2$, $\psi \colon Y_1 \to Y_2$. Then

$$Y_i \boxtimes X_i = Y_i(\boldsymbol{y}) \otimes I^{\boldsymbol{y} \leftarrow \boldsymbol{y}'}_{V_1(\boldsymbol{c}, \bullet, \boldsymbol{z}) + W_2(\boldsymbol{b}, \boldsymbol{x}, \bullet)} \otimes X_i(\boldsymbol{y}') \ ,$$

and we define the 3-morphism

$$\psi \boxtimes \phi \colon Y_1 \boxtimes X_1 \to Y_2 \boxtimes X_2 \ , \quad \psi \boxtimes \phi := \psi \otimes 1_{I^{\boldsymbol{y} \leftarrow \boldsymbol{y}'}_{V_1(\boldsymbol{c}, \bullet, \boldsymbol{z}) + W_2(\boldsymbol{b}, \boldsymbol{x}, \bullet)}} \otimes \phi \ . \tag{1.5.12}$$

Remark 1.5.2. A different definition of $\mathbb{1}_V \boxtimes X$ for

$$X\colon (\boldsymbol{a};\, W_1) \to (\boldsymbol{b};\, W_2)\ , \quad \mathbb{1}_V\colon (\boldsymbol{c}';\, V) \to (\boldsymbol{c};\, V)$$

is given by [63, p. 16]

$$\begin{aligned}\mathbb{1}_V \boxtimes X &= X(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{x}, \boldsymbol{y}) \otimes_{\mathbb{C}[\boldsymbol{a},\boldsymbol{b},\boldsymbol{x},\boldsymbol{y}]} K_{V^{\mathrm{tot}}}(\boldsymbol{y}-\boldsymbol{y}'', \boldsymbol{b}'-\boldsymbol{b}, \boldsymbol{y}-\boldsymbol{y}', \boldsymbol{a}'-\boldsymbol{a}, \boldsymbol{c}-\boldsymbol{c}')\ ,\\ V^{\mathrm{tot}} &= W_2(\boldsymbol{b}', \boldsymbol{x}, \boldsymbol{y}') + V(\boldsymbol{c}, \boldsymbol{y}', \boldsymbol{z}) - (W_1(\boldsymbol{a}', \boldsymbol{x}, \boldsymbol{y}'') + V(\boldsymbol{c}', \boldsymbol{y}'', \boldsymbol{z}))\\ &\quad - (W_2(\boldsymbol{b}, \boldsymbol{x}, \boldsymbol{y}) - W_1(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y}))\end{aligned}$$

which is a 2-morphism

$$\mathbb{1}_V \boxtimes X\colon (\boldsymbol{c}', \boldsymbol{y}'', \boldsymbol{a}';\, V(\boldsymbol{c}', \boldsymbol{y}'', \boldsymbol{z}) + W_1(\boldsymbol{a}', \boldsymbol{x}, \boldsymbol{y}'')) \to (\boldsymbol{c}, \boldsymbol{b}', \boldsymbol{y}';\, V(\boldsymbol{c}, \boldsymbol{y}', \boldsymbol{z}) + W_2(\boldsymbol{b}', \boldsymbol{x}, \boldsymbol{y}'))\ .$$

Integrating out $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{y}\}$ yields a matrix factorisation isomorphic to Eq. (1.5.8).

1.5.4 The truncation and the bicategory axioms

A necessary condition for $\mathcal{RW}$ to be a tricategory is that the truncation $h\mathcal{RW}$ (see Definition 1.2.6) is a bicategory, which will be proven in this section (see [9] for a different proof).

Truncating $\mathcal{RW}$ means identifying all matrix factorisations that are in the same isomorphism class, and "forgetting" about the existence of morphisms between matrix factorisations. As explained in Section 1.2.4, this visually corresponds to projecting out the vertical dimension of 3D diagrams without 3-morphisms in $\mathcal{RW}$, turning them into 2D diagrams in $h\mathcal{RW}$.

We will apply the bicategory axioms presented in [58, pp. 25–26] to $h\mathcal{RW}$. In order to avoid confusion, we will write $Y \cong X$ for isomorphic but unequal matrix factorisations Y and X, keeping in mind that they are formally (by definition) equal in $h\mathcal{RW}$.

Definition 1.5.3. $h\mathcal{RW}$ is a *bicategory* if the following axioms are fulfilled:

(i) For each object $(\boldsymbol{x}) \in h\mathcal{RW}$ there is a 1-morphism

$$\mathbb{1}_{\boldsymbol{x}} \in h\mathcal{RW}(\boldsymbol{x}', \boldsymbol{x}) \tag{1.5.13}$$

called the *identity 1-morphism of* $(\boldsymbol{x})$.

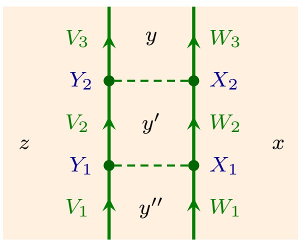
(ii) The box product is *functorial*, that is, it preserves identity 2-morphisms

$$\mathbb{1}_V \boxtimes \mathbb{1}_W \cong \mathbb{1}_{V\boxtimes W} \tag{1.5.14}$$

and composition

$$(Y_2 \otimes Y_1) \boxtimes (X_2 \otimes X_1) \cong (Y_2 \boxtimes X_2) \otimes (Y_1 \boxtimes X_1) \tag{1.5.15}$$

for 1-morphisms $W_i\colon (\boldsymbol{x}) \to (\boldsymbol{y})$, $V_i\colon (\boldsymbol{y}) \to (\boldsymbol{z})$ and 2-morphisms $X_i\colon W_i \to W_{i+1}$, $Y_i\colon V_i \to V_{i+1}$. Diagrammatically, horizontal and vertical composition must commute in the following diagram:



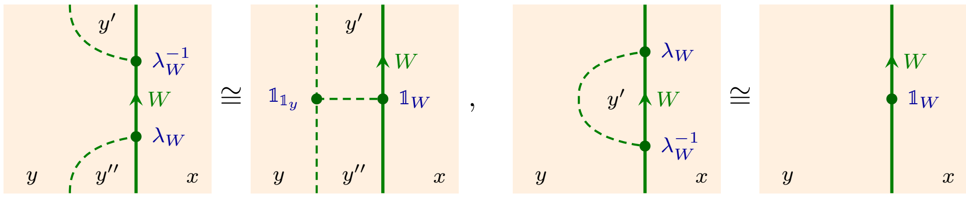
(1.5.16)

(iii) The box product is strictly associative,[15] i.e. $U \boxtimes (V \boxtimes W) = (U \boxtimes V) \boxtimes W$.

(iv) For each 1-morphism $W \in h\mathcal{RW}(\boldsymbol{x}, \boldsymbol{y})$ there are two invertible 2-morphisms

$$\lambda_W \colon \mathbb{1}_{\boldsymbol{y}} \boxtimes W \to W \ , \qquad \rho_W \colon W \boxtimes \mathbb{1}_{\boldsymbol{x}} \to W \ , \tag{1.5.17}$$

whose invertibility is equivalent to the diagram identities



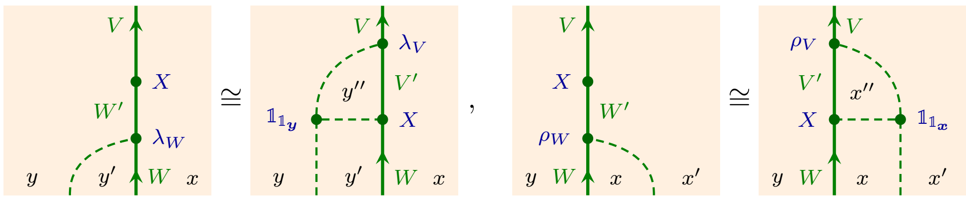
(1.5.18)

plus the mirror images for ρ_W.

(v) For every pair of 1-morphisms $V, W \in h\mathcal{RW}(\boldsymbol{x}, \boldsymbol{y})$ and every 2-morphism $X \colon W \to V$, the 1-morphisms λ and ρ fulfil the *naturality axioms*

$$X \otimes \lambda_W \cong \lambda_V \otimes (\mathbb{1}_{\mathbb{1}_{\boldsymbol{y}}} \boxtimes X) \ , \quad X \otimes \rho_W \cong \rho_V \otimes (X \boxtimes \mathbb{1}_{\mathbb{1}_{\boldsymbol{x}}}) \ , \tag{1.5.19}$$

corresponding to the diagram identities

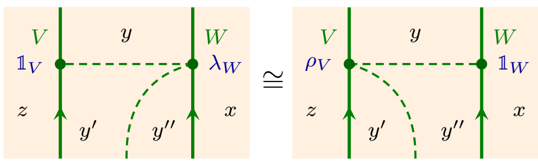


(1.5.20)

(vi) For all 1-morphisms $W \in h\mathcal{RW}(\boldsymbol{x}, \boldsymbol{y})$, $V \in h\mathcal{RW}(\boldsymbol{y}, \boldsymbol{z})$, the following *unity axiom* holds:[16]

$$\mathbb{1}_V \boxtimes \lambda_W \cong \rho_V \boxtimes \mathbb{1}_W \colon V \boxtimes \mathbb{1}_{\boldsymbol{y}} \boxtimes W \to V \boxtimes W \ , \tag{1.5.21}$$

corresponding to the diagram identity



(1.5.22)

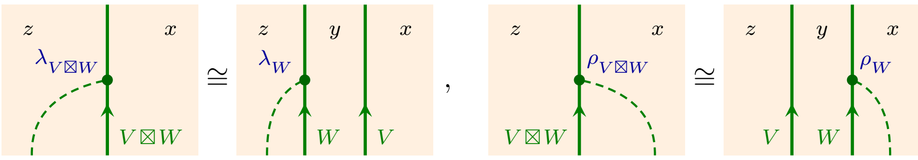
Theorem 1.5.4. *The unitors λ and ρ are consistent with horizontal composition, i.e. for $W \colon (\boldsymbol{x}) \to (\boldsymbol{y})$, $V \colon (\boldsymbol{y}) \to (\boldsymbol{z})$, we find*

$$\lambda_V \boxtimes \mathbb{1}_W \cong \lambda_{V \boxtimes W} \ , \quad \mathbb{1}_V \boxtimes \rho_W \cong \rho_{V \boxtimes W} \ , \tag{1.5.23}$$

[15] The most general definition of a bi- and tricategory requires associativity to hold only up to a natural (resp. pseudonatural) transformation a, which we require to be the identity. This significantly reduces the complexity of the remaining bicategory axioms; for example, the pentagon axiom [82, Eq. (A.30)] holds trivially.

[16] This expression is more complex if $-\boxtimes-$ is not strictly associative.

which corresponds to the diagram identities



. (1.5.24)

Proof. This is shown in [58, Prop. 2.2.4] and only uses the bicategory axioms. □

Remark 1.5.5. The combination of Theorem 1.5.4 and all axioms involving λ and ρ imply the "invisibility" of identity 1-morphisms: Adding, removing, and relocating 1-morphisms which start and end on λ, ρ, or their inverses does not change the value of diagrams. This makes it possible to entirely omit identity 1-morphisms from diagrams of bicategories as there are no ambiguities in their reintroduction. While many authors choose to do so, this work will only omit identity 1-morphisms in rare cases.

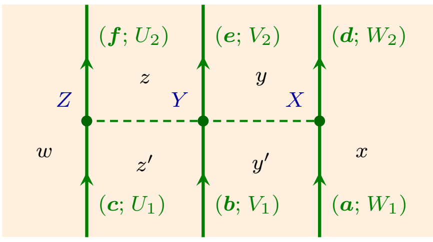
We now verify the axioms of Definition 1.5.3 step by step. The identity 1-morphism of an object $(\boldsymbol{x})$ was already defined in Eq. (1.5.3), so the first axiom is taken care of.

Lemma 1.5.6. *As required by Definition 1.5.3 (iii), the box product is strictly associative.*[17]

Proof. For the box product of 1-morphisms we find

$$\begin{aligned} W &:= (\boldsymbol{a};\, W(\boldsymbol{a},\boldsymbol{x},\boldsymbol{y}))\ , \quad V := (\boldsymbol{b};\, V(\boldsymbol{b},\boldsymbol{y},\boldsymbol{z}))\ , \quad U := (\boldsymbol{c};\, U(\boldsymbol{c},\boldsymbol{z},\boldsymbol{w}))\ , \\ U\boxtimes(V\boxtimes W) &= U\boxtimes(\boldsymbol{b},\boldsymbol{y},\boldsymbol{a};\, V+W) = (\boldsymbol{c},\boldsymbol{z},\boldsymbol{b},\boldsymbol{y},\boldsymbol{a};\, U+V+W)\ , \\ (U\boxtimes V)\boxtimes W &= (\boldsymbol{c},\boldsymbol{z},\boldsymbol{b};\, U+V)\boxtimes W = (\boldsymbol{c},\boldsymbol{z},\boldsymbol{b},\boldsymbol{y},\boldsymbol{a};\, U+V+W)\ . \end{aligned}$$

For 2-morphisms, consider $X\colon W_1\to W_2$, $Y\colon V_1\to V_2$, $Z\colon U_1\to U_2$ with 1-morphisms W_i, V_i, U_i as above:



(1.5.25)

To keep the notation tidy, we only spell out the dependencies on $\boldsymbol{y}$, $\boldsymbol{y}'$, $\boldsymbol{z}$, and $\boldsymbol{z}'$. According to Eq. (1.5.8), we find

$$\begin{aligned} Z\boxtimes(Y\boxtimes X) &= Z(\boldsymbol{z})\otimes I^{\boldsymbol{z}\leftarrow\boldsymbol{z}'}_{U_1(\bullet)+V_2(\boldsymbol{y},\bullet)+W_2(\boldsymbol{y})}\otimes(Y\boxtimes X)(\boldsymbol{z}') \\ &= Z(\boldsymbol{z})\otimes I^{\boldsymbol{z}\leftarrow\boldsymbol{z}'}_{U_1(\bullet)+V_2(\boldsymbol{y},\bullet)+W_2(\boldsymbol{y})}\otimes\left(Y(\boldsymbol{y},\boldsymbol{z}')\otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{V_1(\bullet,\boldsymbol{z}')+W_2(\bullet)}\otimes X(\boldsymbol{y}')\right)\ , \\ (Z\boxtimes Y)\boxtimes X &= (Z\boxtimes Y)(\boldsymbol{y})\otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{U_1(\boldsymbol{z}')+V_1(\bullet,\boldsymbol{z}')+W_2(\bullet)}\otimes X(\boldsymbol{y}') \\ &= \left(Z(\boldsymbol{z})\otimes I^{\boldsymbol{z}\leftarrow\boldsymbol{z}'}_{U_1(\bullet)+V_2(\boldsymbol{y},\bullet)}\otimes Y(\boldsymbol{y},\boldsymbol{z}')\right)\otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{U_1(\boldsymbol{z}')+V_1(\bullet,\boldsymbol{z}')+W_2(\bullet)}\otimes X(\boldsymbol{y}') \\ &= Z(\boldsymbol{z})\otimes I^{\boldsymbol{z}\leftarrow\boldsymbol{z}'}_{U_1(\bullet)+V_2(\boldsymbol{y},\bullet)+W_2(\boldsymbol{y})}\otimes Y(\boldsymbol{y},\boldsymbol{z}')\otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{V_1(\bullet,\boldsymbol{z}')+W_2(\bullet)}\otimes X(\boldsymbol{y}') \\ &= Z\boxtimes(Y\boxtimes X)\ , \end{aligned}$$

[17] up to the subtlety explained in Section 1.3.5

where the penultimate step is an equality by Lemma 1.3.55, and not merely an isomorphism. □

Remark 1.5.7. For the tricategorical structure on $\mathcal{RW}$ it matters that each step in this proof is an equality, not just an isomorphism. In Appendix A.4.2 we show that if we had defined $Y \boxtimes X$ by (1.5.10) or (1.5.11) instead of (1.5.8) or (1.5.9), the use of an isomorphism would have been unavoidable and hence the box product on $\mathcal{RW}$ would not have been be strictly associative. This is permitted in a tricategory, but makes its structure significantly more complicated. Furthermore, below we show that $\mathbb{1}_{V\boxtimes W} = \mathbb{1}_V \boxtimes \mathbb{1}_W$ holds exactly for (1.5.8) but only up to isomorphism for (1.5.9), making (1.5.8) the best choice. None of this matters in the truncation $h\mathcal{RW}$ since there is no notion of unequal but isomorphic matrix factorisations.

Lemma 1.5.8. *The box product on $h\mathcal{RW}$ is functorial according to Definition 1.5.3 (ii).*

Proof. We start with the identity 2-morphisms of

$$(\boldsymbol{a};\, W)\colon (\boldsymbol{x}) \to (\boldsymbol{y})\ , \quad (\boldsymbol{b};\, V)\colon (\boldsymbol{y}) \to (\boldsymbol{z})\ , \quad V \boxtimes W = (\boldsymbol{b}, \boldsymbol{y}, \boldsymbol{a};\, W + V)\ .$$

The variables $\boldsymbol{x}$ and $\boldsymbol{z}$ will be omitted for readability.

$$\begin{aligned}
\mathbb{1}_{V\boxtimes W} &= I^{\{\boldsymbol{b},\boldsymbol{y},\boldsymbol{a}\}\leftarrow\{\boldsymbol{b}',\boldsymbol{y}',\boldsymbol{a}'\}}_{V+W} \\
&= I^{\boldsymbol{b}\leftarrow\boldsymbol{b}'}_{V(\bullet,\boldsymbol{y})+W(\boldsymbol{y},\boldsymbol{a})} \otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{V(\boldsymbol{b}',\bullet)+W(\bullet,\boldsymbol{a})} \otimes I^{\boldsymbol{a}\leftarrow\boldsymbol{a}'}_{V(\boldsymbol{b}',\boldsymbol{y}')+W(\boldsymbol{y}',\bullet)} \\
&= I^{\boldsymbol{b}\leftarrow\boldsymbol{b}'}_{V(\bullet,\boldsymbol{y})} \otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{V(\boldsymbol{b}',\bullet)+W(\bullet,\boldsymbol{a})} \otimes I^{\boldsymbol{a}\leftarrow\boldsymbol{a}'}_{W(\boldsymbol{y}',\bullet)} \\
&= I^{\boldsymbol{b}\leftarrow\boldsymbol{b}'}_{V(\bullet,\boldsymbol{y})} \boxtimes I^{\boldsymbol{a}\leftarrow\boldsymbol{a}'}_{W(\boldsymbol{y},\bullet)} \\
&= \mathbb{1}_V \boxtimes \mathbb{1}_W
\end{aligned}$$

using Eqs. (1.3.76) and (1.3.77). The fact that this is an equality, not just an isomorphism, depends on the two conventions (1.5.6) and (1.5.8).

For the composition rule we write the dependencies on $\boldsymbol{y}$, $\boldsymbol{y}'$, and $\boldsymbol{y}''$ explicitly. The task at hand is to construct an isomorphism between the following matrix factorisations:

$$\begin{aligned}
\Phi^{\boxtimes}\colon (Y_2 \boxtimes X_2) \otimes (Y_1 \boxtimes X_1) &= \big(Y_2(\boldsymbol{y}) \otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{V_2+W_3} \otimes X_2(\boldsymbol{y}')\big) \otimes \big(Y_1(\boldsymbol{y}') \otimes I^{\boldsymbol{y}'\leftarrow\boldsymbol{y}''}_{V_1+W_2} \otimes X_1(\boldsymbol{y}'')\big) \\
\to (Y_2 \otimes Y_1) \boxtimes (X_2 \otimes X_1) &= Y_2(\boldsymbol{y}) \otimes Y_1(\boldsymbol{y}) \otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}''}_{V_1+W_3} \otimes X_2(\boldsymbol{y}'') \otimes X_1(\boldsymbol{y}'')\ .
\end{aligned}$$

See also Eq. (A.5.23) for the meaning of this morphism in the context of a tricategory. We construct $\Phi^{\boxtimes}$ step by step using the permutation isomorphism (1.3.49) and the unitors of Theorem 1.3.56:

$$\Phi^{\boxtimes}\colon (Y_2 \boxtimes X_2) \otimes (Y_1 \boxtimes X_1) = Y_2(\boldsymbol{y}) \otimes \underbrace{I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{V_2+W_3} \otimes X_2(\boldsymbol{y}') \otimes Y_1(\boldsymbol{y}')}_{=:\, Z_1\colon (\boldsymbol{y}';(V_1+W_2)(\boldsymbol{y}'))\to(\boldsymbol{y};(V_2+W_3)(\boldsymbol{y}))} \otimes I^{\boldsymbol{y}'\leftarrow\boldsymbol{y}''}_{V_1+W_2} \otimes X_1(\boldsymbol{y}'')$$

$Z_1 \otimes I^{\boldsymbol{y}'\leftarrow\boldsymbol{y}''}_{V_1+W_2}$ is the domain of the unitor ρ_{Z_1}:

$$\begin{aligned}
&\xrightarrow{1_{Y_2}\otimes\rho_{Z_1}\otimes 1_{X_1}} Y_2(\boldsymbol{y}) \otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}''}_{V_2+W_3} \otimes X_2(\boldsymbol{y}'') \otimes Y_1(\boldsymbol{y}'') \otimes X_1(\boldsymbol{y}'') \\
&\xrightarrow{\sigma^{(1,2,4,3,5)}} Y_2(\boldsymbol{y}) \otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}''}_{V_2+W_3} \otimes Y_1(\boldsymbol{y}'') \otimes X_2(\boldsymbol{y}'') \otimes X_1(\boldsymbol{y}'')
\end{aligned}$$

Using Lemma 1.4.7 we interpret $Z_2(\boldsymbol{y}'') := Y_1(\boldsymbol{y}'')$ as a matrix factorisation

$$Z_2(\boldsymbol{y}'') \in \mathsf{MF}(\boldsymbol{y}'';\ (V_2(\boldsymbol{y}'') + W_3(\boldsymbol{y}'')) - (V_1(\boldsymbol{y}'') + W_3(\boldsymbol{y}''))) \ ,$$

allowing the application of Lemma 1.3.57 to the second and third term:

$$\xrightarrow{1_{Y_2} \otimes (\lambda_{Z_2 \otimes I} \circ \rho^{-1}_{I \otimes Z_2}) \otimes 1_{X_2 \otimes X_1}} Y_2(\boldsymbol{y}) \otimes Y_1(\boldsymbol{y}) \otimes I^{\boldsymbol{y} \leftarrow \boldsymbol{y}''}_{V_1 + W_3} \otimes X_2(\boldsymbol{y}'') \otimes X_1(\boldsymbol{y}'')$$
$$= (Y_2 \otimes Y_1) \boxtimes (X_2 \otimes X_1) \ . \qquad \square$$

1.5.5 The unitor 2-morphisms λ_W and ρ_W

In order to prove the remaining properties of $h\mathcal{RW}$, we first need to define the 2-morphisms λ_W, ρ_W, and their inverses for all 1-morphisms W.

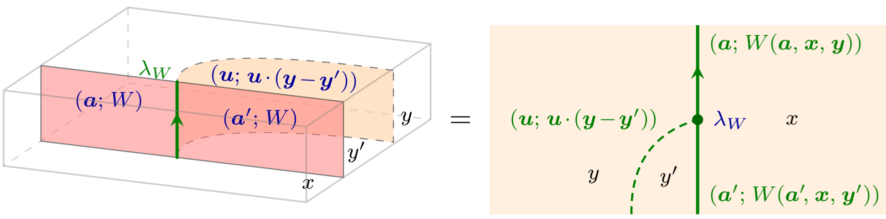
Definition

The essential property of the identity 1-morphism is that its action on other 1-morphisms is unital up to equivalence, i.e. $\mathbb{1}_{\boldsymbol{y}} \boxtimes W \cong W \cong W \boxtimes \mathbb{1}_{\boldsymbol{x}}$ for all $W\colon (\boldsymbol{x}) \to (\boldsymbol{y})$. To show this, we define the *unitor 2-morphisms* below.

Definition 1.5.9 (Unitor 2-morphisms in $\mathcal{RW}$)**.** We define the following 2-morphisms in $\mathcal{RW}$:

- $$\begin{aligned} \lambda_W &:= K\big(\boldsymbol{a} - \boldsymbol{a}';\ \partial^{\boldsymbol{a},\boldsymbol{a}'} W(\bullet, \boldsymbol{x}, \boldsymbol{y})\big) \otimes K\big(\boldsymbol{y} - \boldsymbol{y}';\ -\boldsymbol{u} + \partial^{\boldsymbol{y},\boldsymbol{y}'} W(\boldsymbol{a}', \boldsymbol{x}, \bullet)\big) \\ &= I^{\boldsymbol{a} \leftarrow \boldsymbol{a}'}_{W(\bullet, \boldsymbol{x}, \boldsymbol{y})} \otimes I^{\boldsymbol{y} \leftarrow \boldsymbol{y}'}_{W(\boldsymbol{a}', \boldsymbol{x}, \bullet) - \boldsymbol{u} \cdot \bullet} \\ &\in K_{W(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y}) - W(\boldsymbol{a}', \boldsymbol{x}, \boldsymbol{y}') - \boldsymbol{u} \cdot (\boldsymbol{y} - \boldsymbol{y}')}(\boldsymbol{a} - \boldsymbol{a}', \boldsymbol{y} - \boldsymbol{y}') \ , \end{aligned} \tag{1.5.26}$$

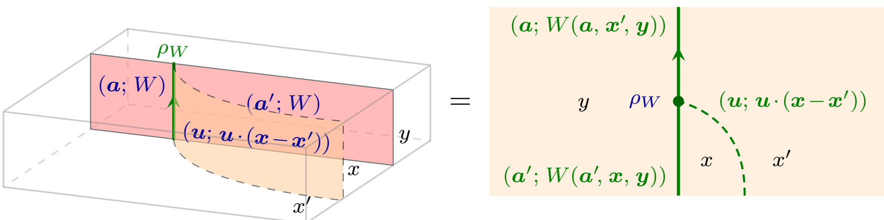
  displayed as

  ,

- $$\begin{aligned} \rho_W &:= K\big(\boldsymbol{a} - \boldsymbol{a}';\ \partial^{\boldsymbol{a},\boldsymbol{a}'} W(\bullet, \boldsymbol{x}', \boldsymbol{y})\big) \otimes K\big(\boldsymbol{x}' - \boldsymbol{x};\ \boldsymbol{u} + \partial^{\boldsymbol{x}',\boldsymbol{x}} W(\boldsymbol{a}', \bullet, \boldsymbol{y})\big) \\ &= I^{\boldsymbol{a} \leftarrow \boldsymbol{a}'}_{W(\bullet, \boldsymbol{x}', \boldsymbol{y})} \otimes I^{\boldsymbol{x}' \leftarrow \boldsymbol{x}}_{W(\boldsymbol{a}', \bullet, \boldsymbol{y}) + \boldsymbol{u} \cdot \bullet} \\ &\in K_{W(\boldsymbol{a}, \boldsymbol{x}', \boldsymbol{y}) - W(\boldsymbol{a}', \boldsymbol{x}, \boldsymbol{y}) - \boldsymbol{u} \cdot (\boldsymbol{x} - \boldsymbol{x}')}(\boldsymbol{a} - \boldsymbol{a}', \boldsymbol{x} - \boldsymbol{x}') \ , \end{aligned} \tag{1.5.27}$$

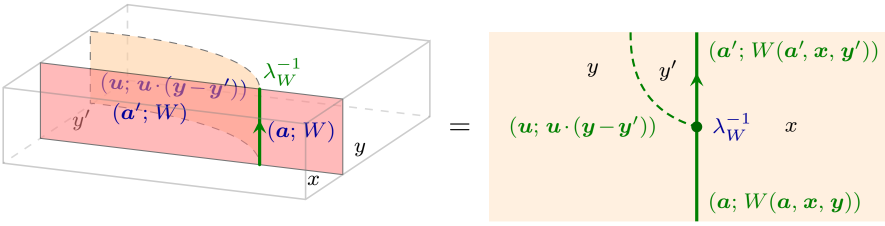
  displayed as

  ,

- $$\lambda_W^{-1} := K\big(\boldsymbol{a}' - \boldsymbol{a};\ \partial^{\boldsymbol{a},\boldsymbol{a}'} W(\bullet, \boldsymbol{x}, \boldsymbol{y})\big) \otimes K\big(\boldsymbol{u} + \partial^{\boldsymbol{y},\boldsymbol{y}'} W(\boldsymbol{a}', \boldsymbol{x}, \bullet);\ \boldsymbol{y} - \boldsymbol{y}'\big) \tag{1.5.28}$$
  $$\in K_{W(\boldsymbol{a}',\boldsymbol{x},\boldsymbol{y}') + \boldsymbol{u}\cdot(\boldsymbol{y}-\boldsymbol{y}') - W(\boldsymbol{a},\boldsymbol{x},\boldsymbol{y})}\big(\boldsymbol{a}' - \boldsymbol{a},\, \boldsymbol{u} + \partial^{\boldsymbol{y},\boldsymbol{y}'} W(\boldsymbol{a}', \boldsymbol{x}, \bullet)\big)\ ,$$

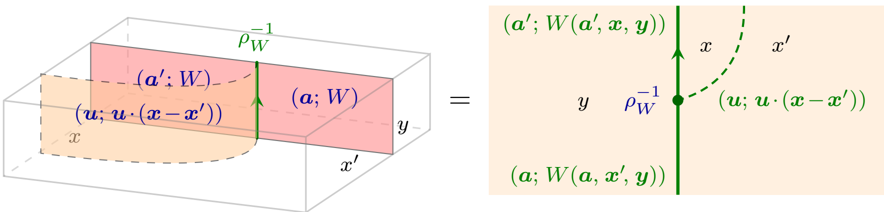
  displayed as

  ,

- $$\rho_W^{-1} := K\big(\boldsymbol{a}' - \boldsymbol{a};\ \partial^{\boldsymbol{a},\boldsymbol{a}'} W(\bullet, \boldsymbol{x}', \boldsymbol{y})\big) \otimes K\big(\boldsymbol{u} + \partial^{\boldsymbol{x},\boldsymbol{x}'} W(\boldsymbol{a}', \bullet, \boldsymbol{y});\ \boldsymbol{x} - \boldsymbol{x}'\big) \tag{1.5.29}$$
  $$\in K_{W(\boldsymbol{a}',\boldsymbol{x},\boldsymbol{y}) + \boldsymbol{u}\cdot(\boldsymbol{x}-\boldsymbol{x}') - W(\boldsymbol{a},\boldsymbol{x}',\boldsymbol{y})}\big(\boldsymbol{a}' - \boldsymbol{a},\, \boldsymbol{u} + \partial^{\boldsymbol{x},\boldsymbol{x}'} W(\boldsymbol{a}', \bullet, \boldsymbol{y})\big)\ ,$$

  displayed as

  .

Remark 1.5.10. Writing down λ_W and ρ_W involves some arbitrary choices regarding grade shifts — some grade-shifted versions of λ_W and ρ_W are just as valid and result in an equivalent tricategory. A detailed discussion can be found in Appendix A.5.

Invertibility

Lemma 1.5.11. *Definition 1.5.3 (iv) is fulfilled: The 2-morphisms λ_W and ρ_W as defined in (1.5.26) and (1.5.27) are invertible, and their inverses are given by (1.5.28) and (1.5.29).*

Proof. The argument will be shown for λ_W, which has domain and codomain

$$\lambda_W\colon\ (\boldsymbol{u}, \boldsymbol{y}', \boldsymbol{a}';\ \tilde{W}(\boldsymbol{a}', \boldsymbol{y}', \boldsymbol{u}, \boldsymbol{x}, \boldsymbol{y})) \to (\boldsymbol{a};\ W(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y}))\ ,$$
$$\tilde{W}(\boldsymbol{a}', \boldsymbol{y}', \boldsymbol{u}, \boldsymbol{x}, \boldsymbol{y}) := W(\boldsymbol{a}', \boldsymbol{x}, \boldsymbol{y}') + \boldsymbol{u}\cdot(\boldsymbol{y} - \boldsymbol{y}')\ .$$

We define the following invertible triangular transformation (see Definition 1.4.12):

$$\begin{pmatrix} \boldsymbol{u}'' \\ \boldsymbol{y}'' \\ \boldsymbol{a}'' \end{pmatrix} \mapsto \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{y}' \\ \boldsymbol{a}' \end{pmatrix} = \begin{pmatrix} \boldsymbol{u}'' - \partial^{\boldsymbol{y},\boldsymbol{y}-\boldsymbol{y}''} W(\boldsymbol{a}'', \boldsymbol{x}, \bullet) \\ \boldsymbol{y} - \boldsymbol{y}'' \\ \boldsymbol{a}'' \end{pmatrix}$$

(note that $\boldsymbol{a}'$ does not depend on $\boldsymbol{y}''$ or $\boldsymbol{u}''$, and $\boldsymbol{y}'$ does not depend on $\boldsymbol{u}''$). This transformation maps $\tilde{W}$ to

$$\begin{aligned}
&W(\boldsymbol{a}'(\boldsymbol{a}''), \boldsymbol{x}, \boldsymbol{y}'(\boldsymbol{y}'')) + \boldsymbol{u}(\boldsymbol{a}'', \boldsymbol{y}'', \boldsymbol{u}'')\cdot(\boldsymbol{y} - \boldsymbol{y}'(\boldsymbol{y}'')) \\
&= W(\boldsymbol{a}'', \boldsymbol{x}, \boldsymbol{y} - \boldsymbol{y}'') + (\boldsymbol{u}'' - \partial^{\boldsymbol{y},\boldsymbol{y}-\boldsymbol{y}''} W(\boldsymbol{a}'', \boldsymbol{x}, \bullet))\cdot(\boldsymbol{y} - (\boldsymbol{y} - \boldsymbol{y}'')) \\
&= W(\boldsymbol{a}'', \boldsymbol{x}, \boldsymbol{y} - \boldsymbol{y}'') + \partial^{\boldsymbol{y},\boldsymbol{y}-\boldsymbol{y}''} W(\boldsymbol{a}'', \boldsymbol{x}, \bullet)\cdot(\boldsymbol{y} - (\boldsymbol{y} - \boldsymbol{y}'')) + \boldsymbol{u}''\cdot\boldsymbol{y}'' \\
&= W(\boldsymbol{a}'', \boldsymbol{x}, \boldsymbol{y} - \boldsymbol{y}'') - W(\boldsymbol{a}'', \boldsymbol{x}, \boldsymbol{y} - \boldsymbol{y}'') + W(\boldsymbol{a}'', \boldsymbol{x}, \boldsymbol{y}) + \boldsymbol{u}''\cdot\boldsymbol{y}'' \\
&= W(\boldsymbol{a}'', \boldsymbol{x}, \boldsymbol{y}) + \boldsymbol{u}''\cdot\boldsymbol{y}''\ .
\end{aligned}$$

Theorem 1.4.18 therefore implies

$$(\boldsymbol{u}, \boldsymbol{y}', \boldsymbol{a}'; \tilde{W}(\boldsymbol{a}', \boldsymbol{y}', \boldsymbol{u}, \boldsymbol{x}, \boldsymbol{y})) \cong (\boldsymbol{u}'', \boldsymbol{y}'', \boldsymbol{a}''; W(\boldsymbol{a}'', \boldsymbol{x}, \boldsymbol{y}) + \boldsymbol{u}'' \cdot \boldsymbol{y}'')$$

which, in turn, is equivalent to $(\boldsymbol{a}; W(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y}))$ by Knörrer periodicity (Theorem 1.4.9). The overall equivalence is given by

$$\begin{aligned} X &= I_W^{\boldsymbol{a}\leftarrow \boldsymbol{a}''} \otimes I_{\boldsymbol{u}''\cdot\bullet}^{0\leftarrow \boldsymbol{y}''} \otimes I_{\tilde{W}}^{\{\boldsymbol{a}'', \boldsymbol{y}-\boldsymbol{y}'', \boldsymbol{u}''-\partial^{\boldsymbol{y},\boldsymbol{y}-\boldsymbol{y}''}W(\boldsymbol{a}'',\boldsymbol{x},\bullet)\}\leftarrow\{\boldsymbol{a}',\boldsymbol{y}',\boldsymbol{u}\}} \\ &\in K_{W(\boldsymbol{a},\boldsymbol{x},\boldsymbol{y})-\tilde{W}}(\boldsymbol{a}''-\boldsymbol{a}', \boldsymbol{y}-\boldsymbol{y}''-\boldsymbol{y}', \boldsymbol{u}''-\partial^{\boldsymbol{y},\boldsymbol{y}-\boldsymbol{y}''}W(\boldsymbol{a}'', \boldsymbol{x}, \bullet) - \boldsymbol{u}, \boldsymbol{a}-\boldsymbol{a}'', \boldsymbol{y}'') \ . \end{aligned}$$

The sequence in parentheses can be permuted to

$$\{\boldsymbol{y}'', \boldsymbol{a}''-\boldsymbol{a}', \boldsymbol{y}-\boldsymbol{y}''-\boldsymbol{y}', \boldsymbol{u}''-\partial^{\boldsymbol{y},\boldsymbol{y}-\boldsymbol{y}''}W(\boldsymbol{a}'', \boldsymbol{x}, \bullet) - \boldsymbol{u}, \boldsymbol{a}-\boldsymbol{a}''\}$$

which is a regular sequence by Lemma 1.3.51. Therefore, by Corollary 1.3.45 and Theorem 1.3.49, X is associated to

$$\frac{\mathbb{C}[\boldsymbol{a}', \boldsymbol{y}', \boldsymbol{u}', \boldsymbol{a}'', \boldsymbol{y}'', \boldsymbol{u}'', \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y}]}{(\boldsymbol{y}'', \boldsymbol{a}''-\boldsymbol{a}', \boldsymbol{y}-\boldsymbol{y}''-\boldsymbol{y}', \boldsymbol{u}''-\partial^{\boldsymbol{y},\boldsymbol{y}-\boldsymbol{y}''}W(\boldsymbol{a}'', \boldsymbol{x}, \bullet) - \boldsymbol{u}, \boldsymbol{a}-\boldsymbol{a}'')} \cong \frac{\mathbb{C}[\boldsymbol{a}', \boldsymbol{y}', \boldsymbol{u}', \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y}]}{(\boldsymbol{a}-\boldsymbol{a}', \boldsymbol{y}-\boldsymbol{y}')}$$

which is associated to λ_W as defined in Eq. (1.5.26). Because X is an equivalence and $\lambda_W \cong X$, it follows that λ_W is an equivalence, too.

The construction of λ_W^{-1} is analogous. Using the explicit form of the inverse 2-morphisms in Theorems 1.4.9 and 1.4.18, we find

$$X^{-1} = I_{\tilde{W}}^{\{\boldsymbol{a}',\boldsymbol{y}',\boldsymbol{u}\}\leftarrow\{\boldsymbol{a}'', \boldsymbol{y}-\boldsymbol{y}'', \boldsymbol{u}''-\partial^{\boldsymbol{y},\boldsymbol{y}-\boldsymbol{y}''}W(\boldsymbol{a}'',\boldsymbol{x},\bullet)\}} \otimes I_W^{\boldsymbol{a}''\leftarrow \boldsymbol{a}} \otimes I_{\boldsymbol{y}''\cdot\bullet}^{\boldsymbol{u}''\leftarrow 0}$$

which is associated to

$$\begin{aligned} &\frac{\mathbb{C}[\boldsymbol{a}', \boldsymbol{y}', \boldsymbol{u}', \boldsymbol{a}'', \boldsymbol{y}'', \boldsymbol{u}'', \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y}]}{(\boldsymbol{a}''-\boldsymbol{a}, \boldsymbol{u}'', \boldsymbol{a}'-\boldsymbol{a}'', \boldsymbol{y}'-(\boldsymbol{y}-\boldsymbol{y}''), \boldsymbol{u}-(\boldsymbol{u}''-\partial^{\boldsymbol{y},\boldsymbol{y}-\boldsymbol{y}''}W(\boldsymbol{a}'', \boldsymbol{x}, \bullet)))} \\ \cong\ &\frac{\mathbb{C}[\boldsymbol{a}', \boldsymbol{y}', \boldsymbol{u}', \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y}]}{(\boldsymbol{a}'-\boldsymbol{a}, \boldsymbol{u}+\partial^{\boldsymbol{y},\boldsymbol{y}'}W(\boldsymbol{a}'', \boldsymbol{x}, \bullet))} \end{aligned}$$

which is associated to λ_W^{-1} of Eq. (1.5.28). The proof for ρ_W and ρ_W^{-1} is analogous. □

In Section 3.1.2 we discuss how to find explicit formulas for the isomorphisms

$$\alpha_l(W)\colon \lambda_W \otimes \lambda_W^{-1} \to \mathbb{1}_W \ , \qquad \beta_l(W)\colon \mathbb{1}_{\mathbb{1}_{\boldsymbol{y}}\boxtimes W} \to \lambda_W^{-1} \otimes \lambda_W \ , \tag{1.5.30}$$

$$\alpha_r(W)\colon \rho_W \otimes \rho_W^{-1} \to \mathbb{1}_W \ , \qquad \beta_r(W)\colon \mathbb{1}_{W\boxtimes\mathbb{1}_{\boldsymbol{x}}} \to \rho_W^{-1} \otimes \rho_W \ . \tag{1.5.31}$$

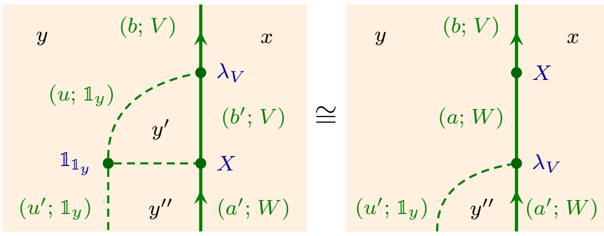
1.5.6 The remaining axioms

Lemma 1.5.12. *The naturality axiom (Definition 1.5.3 (v)) is fulfilled.*

Proof. We show the first identity of Eq. (1.5.20) for an arbitrary 2-morphism

$$X(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{y})\colon (\boldsymbol{a}; W(\boldsymbol{a}, \boldsymbol{y})) \to (\boldsymbol{b}; V(\boldsymbol{b}, \boldsymbol{y}))$$

(the dependencies on $\boldsymbol{x}$ will not be spelled out in this proof). We first assign names to all variables:



. (1.5.32)

See Appendix A.5 for the meaning of this isomorphism in the tricategorical structure, where it is called $l_{W,V}(X)$. We define it to be the composite of the following isomorphisms:

$$
\begin{aligned}
l_{W,V}(X)\colon \quad & \lambda_V \otimes (\mathbb{1}_{\mathbb{1}_y} \boxtimes X) \\
&= (I^{\boldsymbol{b}\leftarrow\boldsymbol{b}'}_{V(\bullet,\boldsymbol{y})} \otimes \underbrace{I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{V(\boldsymbol{b}',\bullet)-\boldsymbol{u}\cdot\bullet}) \otimes (I^{\boldsymbol{u}\leftarrow\boldsymbol{u}'}_{(\boldsymbol{y}-\boldsymbol{y}')\cdot\bullet} \otimes I^{\boldsymbol{y}'\leftarrow\boldsymbol{y}''}_{\boldsymbol{u}'\cdot(\boldsymbol{y}-\bullet)+V(\boldsymbol{b}',\bullet)} \otimes X(\boldsymbol{a}', \boldsymbol{b}', \boldsymbol{y}''))}_{=:\, Z_1\colon\, (\emptyset;\, W(\boldsymbol{a}',\boldsymbol{y}'')+\boldsymbol{u}'\cdot(\boldsymbol{y}-\boldsymbol{y}''))\to(\boldsymbol{b}';\, V(\boldsymbol{b}',\boldsymbol{y}))}
\end{aligned}
$$

using a reinterpretation of surface and global variables in Z_1 following Lemma 1.4.7. The codomain of Z_1 matches the domain of $I^{\boldsymbol{b}\leftarrow\boldsymbol{b}'}_{V(\bullet,\boldsymbol{y})}$, allowing us to apply the 3-morphism λ_{Z_1}:

$$
\xrightarrow{\lambda_{Z_1}} \underbrace{I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{V(\boldsymbol{b},\bullet)-\boldsymbol{u}\cdot\bullet}}_{=:\, Z_2\colon\, (\boldsymbol{u};\boldsymbol{u}\cdot(\boldsymbol{y}-\boldsymbol{y}'))\to(\emptyset;V(\boldsymbol{b},\boldsymbol{y})-V(\boldsymbol{b},\boldsymbol{y}'))} \otimes I^{\boldsymbol{u}\leftarrow\boldsymbol{u}'}_{(\boldsymbol{y}-\boldsymbol{y}')\cdot\bullet} \otimes I^{\boldsymbol{y}'\leftarrow\boldsymbol{y}''}_{\boldsymbol{u}'\cdot(\boldsymbol{y}-\bullet)+V(\boldsymbol{b},\bullet)} \otimes X(\boldsymbol{a}', \boldsymbol{b}, \boldsymbol{y}'')
$$

The domain of Z_2 matches the codomain of $I^{\boldsymbol{u}\leftarrow\boldsymbol{u}'}_{(\boldsymbol{y}-\boldsymbol{y}')\cdot\bullet}$ and both the third and fourth term are independent of $\boldsymbol{u}$, thus ρ_{Z_2} may be applied to the first two terms. We may furthermore use Eq. (1.3.76) to remove $\boldsymbol{u}'\cdot\boldsymbol{y}$ from the third term:

$$
\begin{aligned}
&\xrightarrow{\rho_{Z_2}\otimes 1_{I\otimes X}} I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{V(\boldsymbol{b},\bullet)-\boldsymbol{u}'\cdot\bullet} \otimes I^{\boldsymbol{y}'\leftarrow\boldsymbol{y}''}_{V(\boldsymbol{b},\bullet)-\boldsymbol{u}'\cdot\bullet} \otimes X(\boldsymbol{a}', \boldsymbol{b}, \boldsymbol{y}'') \\
&\xrightarrow{\rho_I\otimes 1_X} I^{\boldsymbol{y}\leftarrow\boldsymbol{y}''}_{V(\boldsymbol{b},\bullet)-\boldsymbol{u}'\cdot\bullet} \otimes X(\boldsymbol{a}', \boldsymbol{b}, \boldsymbol{y}'') \\
&\xrightarrow{1_I\otimes\rho_X^{-1}} I^{\boldsymbol{y}\leftarrow\boldsymbol{y}''}_{V(\boldsymbol{b},\bullet)-\boldsymbol{u}'\cdot\bullet} \otimes \underbrace{X(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{y}'') \otimes I^{\boldsymbol{a}\leftarrow\boldsymbol{a}'}_{W(\bullet,\boldsymbol{y}'')}}_{=:\, Z_3\colon\, (\emptyset;W(\boldsymbol{a}',\boldsymbol{y}'')-\boldsymbol{u}'\cdot\boldsymbol{y}'')\to(\emptyset;V(\boldsymbol{b},\boldsymbol{y}'')-\boldsymbol{u}'\cdot\boldsymbol{y}'')}
\end{aligned}
$$

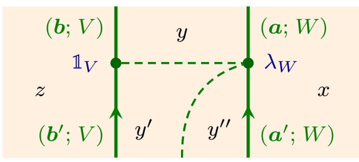
again using multiple statements of Lemma 1.4.7. We finally apply Lemma 1.3.57:

$$
\begin{aligned}
&\xrightarrow{\lambda_{Z_3\otimes I}\circ\rho^{-1}_{I\otimes Z_3}} X(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{y}) \otimes I^{\boldsymbol{a}\leftarrow\boldsymbol{a}'}_{W(\bullet,\boldsymbol{y})} \otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}''}_{W(\boldsymbol{a}',\bullet)-\boldsymbol{u}'\cdot\bullet} \\
&= X(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{y}) \otimes \lambda_W \ .
\end{aligned}
$$

The construction of $r_{W,V}(X)$ for ρ_W is similar. $\square$

Lemma 1.5.13. *The unity axiom (Definition 1.5.3 (vi)) is fulfilled.*

Proof. We assign the following variable names:



(1.5.33)

The dependencies on $\boldsymbol{x}$ and $\boldsymbol{z}$ will be left implicit in this proof. In these conventions, both sides of Eq. (1.5.21) are matrix factorisations of

$$V(\boldsymbol{b},\boldsymbol{y},\boldsymbol{z})+W(\boldsymbol{a},\boldsymbol{x},\boldsymbol{y})-\big(V(\boldsymbol{b}',\boldsymbol{y}',\boldsymbol{z})+\boldsymbol{u}\cdot(\boldsymbol{y}'-\boldsymbol{y}'')+W(\boldsymbol{a}',\boldsymbol{x},\boldsymbol{y}'')\big)\ ,$$

given by

$$\begin{aligned}
\mathbb{1}_V\boxtimes\lambda_W &= I^{\boldsymbol{b}\leftarrow\boldsymbol{b}'}_{V(\bullet,\boldsymbol{y})}\otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{V(\boldsymbol{b}',\bullet)+W(\boldsymbol{a},\bullet)}\otimes\lambda_W(\boldsymbol{y}')\\
&= I^{\boldsymbol{b}\leftarrow\boldsymbol{b}'}_{V(\bullet,\boldsymbol{y})}\otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{V(\boldsymbol{b}',\bullet)+W(\boldsymbol{a},\bullet)}\otimes\big(I^{\boldsymbol{a}\leftarrow\boldsymbol{a}'}_{W(\bullet,\boldsymbol{y}')}\otimes I^{\boldsymbol{y}'\leftarrow\boldsymbol{y}''}_{W(\boldsymbol{a}',\bullet)-\boldsymbol{u}\cdot\bullet}\big)\ ,\\
\rho_V\boxtimes\mathbb{1}_W &= \rho_V(\boldsymbol{y})\otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}''}_{(V(\boldsymbol{b}',\boldsymbol{y}')+\boldsymbol{u}\cdot(\boldsymbol{y}'-\bullet))+W(\boldsymbol{a},\bullet)}\otimes I^{\boldsymbol{a}\leftarrow\boldsymbol{a}'}_{W(\bullet,\boldsymbol{y}'')}\\
&= \big(I^{\boldsymbol{b}\leftarrow\boldsymbol{b}'}_{V(\bullet,\boldsymbol{y})}\otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{V(\boldsymbol{b}',\bullet)+\boldsymbol{u}\cdot\bullet}\big)\otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}''}_{-\boldsymbol{u}\cdot\bullet+W(\boldsymbol{a},\bullet)}\otimes I^{\boldsymbol{a}\leftarrow\boldsymbol{a}'}_{W(\bullet,\boldsymbol{y}'')}\ .
\end{aligned}$$

We now construct an isomorphism that is linear in all variables:[18]

$$\begin{aligned}
\hat{\mu}_{V,W}\colon\qquad & \mathbb{1}_V\boxtimes\lambda_W\\
&= I^{\boldsymbol{b}\leftarrow\boldsymbol{b}'}_{V(\bullet,\boldsymbol{y})}\otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{V(\boldsymbol{b}',\bullet)+W(\boldsymbol{a},\bullet)}\otimes I^{\boldsymbol{a}\leftarrow\boldsymbol{a}'}_{W(\bullet,\boldsymbol{y}')-\boldsymbol{u}\cdot\boldsymbol{y}'}\otimes I^{\boldsymbol{y}'\leftarrow\boldsymbol{y}''}_{W(\boldsymbol{a}',\bullet)-\boldsymbol{u}\cdot\bullet}\\
&\xrightarrow{1_{I\otimes I}\otimes(\lambda_{I\otimes I}\circ\rho^{-1}_{I\otimes I})} I^{\boldsymbol{b}\leftarrow\boldsymbol{b}'}_{V(\bullet,\boldsymbol{y})}\otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{V(\boldsymbol{b}',\bullet)+W(\boldsymbol{a},\bullet)}\otimes I^{\boldsymbol{y}'\leftarrow\boldsymbol{y}''}_{W(\boldsymbol{a},\bullet)-\boldsymbol{u}\cdot\bullet}\otimes I^{\boldsymbol{a}\leftarrow\boldsymbol{a}'}_{W(\bullet,\boldsymbol{y}'')-\boldsymbol{u}\cdot\boldsymbol{y}''}\\
&= I^{\boldsymbol{b}\leftarrow\boldsymbol{b}'}_{V(\bullet,\boldsymbol{y})}\otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{V(\boldsymbol{b}',\bullet)+W(\boldsymbol{a},\bullet)+\boldsymbol{u}\cdot\boldsymbol{y}''}\otimes I^{\boldsymbol{y}'\leftarrow\boldsymbol{y}''}_{W(\boldsymbol{a},\bullet)-\boldsymbol{u}\cdot\bullet}\otimes I^{\boldsymbol{a}\leftarrow\boldsymbol{a}'}_{W(\bullet,\boldsymbol{y}'')}
\end{aligned}$$

Using Lemma 1.4.7 we interpret $Z(\boldsymbol{y}') := I^{\boldsymbol{y}'\leftarrow\boldsymbol{y}''}_{W(\boldsymbol{a},\bullet)-\boldsymbol{u}\cdot\bullet}$ as a matrix factorisation

$$Z(\boldsymbol{y}')\in\mathsf{MF}\big(\boldsymbol{y}';\ (W(\boldsymbol{a},\boldsymbol{y}')+V(\boldsymbol{b}',\boldsymbol{y}')+\boldsymbol{u}\cdot\boldsymbol{y}'')-(W(\boldsymbol{a},\boldsymbol{y}'')+V(\boldsymbol{b}',\boldsymbol{y}')+\boldsymbol{u}\cdot\boldsymbol{y}')\big)\ ,$$

allowing the application of Lemma 1.3.57 to the second and third term:

$$\begin{aligned}
&\xrightarrow{1_I\otimes(\lambda_{Z\otimes I}\circ\rho^{-1}_{I\otimes Z})\otimes 1_I} I^{\boldsymbol{b}\leftarrow\boldsymbol{b}'}_{V(\bullet,\boldsymbol{y})}\otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}''}_{W(\boldsymbol{a},\bullet)-\boldsymbol{u}\cdot\bullet}\otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{W(\boldsymbol{a},\boldsymbol{y}'')+V(\boldsymbol{b}',\bullet)+\boldsymbol{u}\cdot\bullet}\otimes I^{\boldsymbol{a}\leftarrow\boldsymbol{a}'}_{W(\bullet,\boldsymbol{y}'')}\\
&\xrightarrow{\sigma^{(1,3,2,4)}} I^{\boldsymbol{b}\leftarrow\boldsymbol{b}'}_{V(\bullet,\boldsymbol{y})}\otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{V(\boldsymbol{b}',\bullet)+\boldsymbol{u}\cdot\bullet}\otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}''}_{-\boldsymbol{u}\cdot\bullet+W(\boldsymbol{a},\bullet)}\otimes I^{\boldsymbol{a}\leftarrow\boldsymbol{a}'}_{W(\bullet,\boldsymbol{y}'')}\\
&= \rho_V\boxtimes\mathbb{1}_W\ .
\end{aligned}$$

□

Remark 1.5.14. The isomorphism $\hat{\mu}_{V,W}$ is closely related to the invertible modification μ of [82, Def. A.4.1.vii)] evaluated on the pair $(V,W)\in\mathcal{RW}(\boldsymbol{y},\boldsymbol{z})\times\mathcal{RW}(\boldsymbol{x},\boldsymbol{y})$. The domain and codomain of μ (which are pseudonatural transformations) both contain some additional identity matrix factorisations, therefore $\mu(V,W)$ is equal to $\hat{\mu}$ pre- and post-composed with some unitor 3-morphisms.

1.5.7 The full tricategory

All ingredients for the proof that $\mathcal{RW}$ is a tricategory have been presented in this section. The missing statements are coherence conditions between the isomorphisms constructed in this section like $\Phi^{\boxtimes}$, $\alpha_{l,r}$, $\beta_{l,r}$, $l_{W,V}(X)$, $\hat{\mu}_{V,W}$, and the isomorphisms of Theorem 1.5.4. Our main reference for the axioms of a tricategory is [82, Def. A.4.1].

[18] For example, applying $\lambda_X\colon I^{\boldsymbol{b}\leftarrow\boldsymbol{b}'}_{\dots}\otimes Y(\boldsymbol{b}')\to Y(\boldsymbol{b})$ in the first step is not allowed since λ_X is not linear in the variables $\boldsymbol{b}'$.

Conjecture 1.5.15. *The category $\mathcal{RW}$ is a tricategory.*

We will show the following lemma which simplifies the axioms of the tricategory $\mathcal{RW}$ significantly, thus reducing the amount of work necessary to prove Conjecture 1.5.15.

Lemma 1.5.16. *The box product $-\boxtimes-$ in $\mathcal{RW}$ is strictly associative.*

Proof. For objects, 1-morphisms, and 2-morphisms this was already proven in Lemma 1.5.6. The argument for 3-morphisms goes as follows: Consider

$$\begin{aligned}
\text{objects}\colon & \ \boldsymbol{x},\, \boldsymbol{y},\, \boldsymbol{z},\, \boldsymbol{w}\ , \\
\text{1-morphisms}\colon & \ W_i\colon \boldsymbol{x}\to\boldsymbol{y},\ V_i\colon \boldsymbol{y}\to\boldsymbol{z},\ U_i\colon \boldsymbol{z}\to\boldsymbol{w}\ , \\
\text{2-morphisms}\colon & \ X_i\colon W_1\to W_2,\ Y_i\colon V_1\to V_2,\ Z_i\colon U_1\to U_2\ , \\
\text{3-morphisms}\colon & \ \phi\colon X_1\to X_2,\ \psi\colon Y_1\to Y_2,\ \chi\colon Z_1\to Z_2\ .
\end{aligned}$$

Now we find (again only writing out the dependencies on $\boldsymbol{y}, \boldsymbol{y}', \boldsymbol{z}, \boldsymbol{z}'$)

$$\begin{aligned}
\chi\boxtimes(\psi\boxtimes\phi) &= \chi\boxtimes\Big(\psi\otimes 1_{I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{V_1(\bullet,\boldsymbol{z}')+W_2(\bullet)}}\otimes\phi\Big) \\
&= \chi\otimes 1_{I^{\boldsymbol{z}\leftarrow\boldsymbol{z}'}_{U_1(\bullet)+V_2(\boldsymbol{y},\bullet)+W_2(\boldsymbol{y})}}\otimes\Big(\psi\otimes 1_{I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{V_1(\bullet,\boldsymbol{z}')+W_2(\bullet)}}\otimes\phi\Big) \\
&= \chi\otimes 1_{I^{\boldsymbol{z}\leftarrow\boldsymbol{z}'}_{U_1(\bullet)+V_2(\boldsymbol{y},\bullet)}}\otimes\psi\otimes 1_{I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{V_1(\bullet,\boldsymbol{z}')+W_2(\bullet)}}\otimes\phi\ , \\
(\chi\boxtimes\psi)\boxtimes\phi &= \Big(\chi\otimes 1_{I^{\boldsymbol{z}\leftarrow\boldsymbol{z}'}_{U_1(\bullet)+V_2(\boldsymbol{y},\bullet)}}\otimes\psi\Big)\boxtimes\phi \\
&= \Big(\chi\otimes 1_{I^{\boldsymbol{z}\leftarrow\boldsymbol{z}'}_{U_1(\bullet)+V_2(\boldsymbol{y},\bullet)}}\otimes\psi\Big)\otimes 1_{I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{U_1(\boldsymbol{z}')+V_1(\bullet,\boldsymbol{z}')+W_2(\bullet)}}\otimes\phi \\
&= \chi\otimes 1_{I^{\boldsymbol{z}\leftarrow\boldsymbol{z}'}_{U_1(\bullet)+V_2(\boldsymbol{y},\bullet)}}\otimes\psi\otimes 1_{I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{V_1(\bullet,\boldsymbol{z}')+W_2(\bullet)}}\otimes\phi \\
&= \chi\boxtimes(\psi\boxtimes\phi)\ .
\end{aligned}$$

□

1.5.8 The direct sum completion of $\mathcal{RW}$

Definition

We formally introduce a notion of direct sums in $\mathcal{RW}$ which we will need in Chapter 3. The relation to the general construction of direct sums in categories is discussed in [7]; only the result and a few proofs will be shown here. A physical interpretation of a direct sum of surface defects is a superposition of different defect states.

Definition 1.5.17 (The direct sum completion of $\mathcal{RW}$)**.** We define the structure $\overline{\mathcal{RW}}^{\oplus}$ as follows:

(i) The objects of $\overline{\mathcal{RW}}^{\oplus}$ are the objects of $\mathcal{RW}$.

(ii) A 1-morphism $W\in\overline{\mathcal{RW}}^{\oplus}(\boldsymbol{x},\boldsymbol{y})$ is a finite, ordered list of 1-morphisms $W_i\in\mathcal{RW}(\boldsymbol{x},\boldsymbol{y})$, which will be written as

$$\bigoplus_{i=1}^{n} W_i = W = \{W_i\}_i\ . \tag{1.5.34}$$

(iii) For a pair of 1-morphisms $W, V \in \overline{\mathcal{RW}}^{\oplus}(\boldsymbol{x}, \boldsymbol{y})$, a 2-morphism $X\colon W \to V$ is a matrix of matrix factorisations $\{X_{j,i}\}_{j,i}$ where $X_{j,i}\colon W_i \to V_j$ is a 2-morphism in $\mathcal{RW}$.

(iv) For a pair of 2-morphisms

$$\{X_{j,i}\}, \{Y_{j,i}\}\colon W \to V \ , \tag{1.5.35}$$

a 3-morphism $\phi\colon \{X_{j,i}\} \to \{Y_{j,i}\}$ is a table $\phi = \{\phi_{j,i}\}$ where each $\phi_{j,i}\colon X_{j,i} \to Y_{j,i}$ is a 3-morphism in $\mathcal{RW}$. Note that "off-diagonal" 3-morphisms $X_{j,i} \to Y_{l,k}$ do not make sense for $(j, i) \neq (l, k)$, as the domains and/or codomains of the 2-morphisms disagree.

Compositions are defined as follows:

(i) Vertical composition of two 3-morphisms is defined by the component-wise composition

$$\{\phi_{j,i}\} \circ \{\psi_{j,i}\} = \{\phi_{j,i} \circ \psi_{j,i}\} \ . \tag{1.5.36}$$

Note that we do *not* have the structure of a matrix multiplication here, but a component-wise composition of tables.

(ii) For 2-morphisms $X\colon W \to V$, $Y\colon V \to W$, the horizontal composition is given by

$$Y \otimes X = \{(Y \otimes X)_{k,i}\} \ , \quad (Y \otimes X)_{k,i} = \bigoplus_{j=1}^{\ell(V)} Y_{k,j} \otimes X_{j,i} \tag{1.5.37}$$

where the sum is the direct sum of matrix factorisations. The action of horizontal composition on 3-morphisms is defined in the obvious way.

(iii) For 1-morphisms $W\colon (\boldsymbol{x}) \to (\boldsymbol{y})$, $V\colon (\boldsymbol{y}) \to (\boldsymbol{z})$, the box product is given by

$$\bigoplus_{j=1}^{\ell(V)} V_j \boxtimes \bigoplus_{i=1}^{\ell(W)} W_i = \bigoplus_{(j,i)\in I} V_j \boxtimes W_i \ , \quad I = \{1, \ldots, \ell(V)\} \times \{1, \ldots, \ell(W)\} \ . \tag{1.5.38}$$

The action of the box product on 2-morphisms is defined as follows: let

$$X\colon \bigoplus_{i=1}^{n} W_i \to \bigoplus_{i'=1}^{n'} W'_i \ , \quad Y\colon \bigoplus_{j=1}^{m} V_j \to \bigoplus_{j=1}^{m'} V'_j \ . \tag{1.5.39}$$

Then

$$Y \boxtimes X = \big\{(Y \boxtimes X)_{(j',i'),(j,i)}\big\} \ , \qquad (Y \boxtimes X)_{(j',i'),(j,i)} = Y_{j',j} \boxtimes X_{i',i} \ . \tag{1.5.40}$$

The action of the box product on 3-morphisms is again defined in the obvious way.

Lemma 1.5.18. *The bicategory $\ddot{\mathsf{MF}}_k$ admits (category theoretical) direct sums that are consistent with the direct sum in $\overline{\mathcal{RW}}^{\oplus}$ [7].*

Remark 1.5.19. The direct sum on $\overline{\mathcal{RW}}^{\oplus}$ is strictly associative and is commutative up to equivalence, i.e. for $U, V, W \in \overline{\mathcal{RW}}^{\oplus}(\boldsymbol{x}, \boldsymbol{y})$, we find

$$(U \oplus V) \oplus W = U \oplus (V \oplus W) \ , \quad V \oplus W \cong W \oplus V \ . \tag{1.5.41}$$

Notation 1.5.20 (Conventions in $\overline{\mathcal{RW}}^{\oplus}$).

(i) Let $A, B \in \overline{\mathcal{RW}}^{\oplus}(\boldsymbol{x}, \boldsymbol{y})$ with $\ell(A) = \ell(B)$ and let $X_i\colon A_i \to B_i$ be a list of 2-morphisms in $\mathcal{RW}$. Then we define the 2-morphism

$$\{\delta_{j,i} \cdot X_i\}_{j,i}\colon A \to B \ , \qquad \delta_{j,i} \cdot X_i := \begin{cases} X_i & i = j \\ 0_{A_i \to B_j} & i \neq j \end{cases} \tag{1.5.42}$$

with the trivial matrix factorisation $0_{A_i \to B_j}$ of Definition 1.4.4. The 2-morphisms

$$\delta_{j,i} \cdot Y_i\colon C_j \boxtimes A_i \to D \ , \qquad \delta_{j,i} \cdot Z_i\colon D \to C_j \boxtimes A_i \tag{1.5.43}$$

are defined analogously for appropriate C_j, D, Y_i, Z_i.

(ii) For a 2-morphism $X = \{X_{j,i}\}_{j,i}\colon \bigoplus_{i=1}^{n} A_i \to \bigoplus_{j=1}^{m} B_j$, the notation

$$X|_{A_i \to B_j} := X_{j,i} \tag{1.5.44}$$

refers to one matrix element of X.

(iii) For a 3-morphism $\phi\colon \{X_{j,i}\} \to \{Y_{j,i}\}$, the notation

$$\phi|_{X_{j,i}}\colon X_{j,i} \to Y_{j,i} \tag{1.5.45}$$

refers to one element of ϕ.

(iv) 3-morphisms are implicitly defined to be 0 on trivial matrix factorisations. For example, let $X = Y = \{\delta_{j,i} \cdot Z_i\}_{j,i}$ and $\psi_i\colon Z_i \to Z_i$. Then the assignment

$$\phi\colon X \to Y \ , \quad \phi|_{Z_i} := \psi_i$$

implicitly defines $\phi|_{\delta_{j,i} \cdot Z_i} := 0$ for $j \neq i$ (by Lemma 1.4.5 it does not matter how ϕ is defined on trivial matrix factorisations since all morphisms are exact).

Identity morphisms and unitors

Lemma 1.5.21. *Let $A = \bigoplus_{i=1}^{n} A_i$, $B = \bigoplus_{k=1}^{m} B_k \in \overline{\mathcal{RW}}^{\oplus}(\boldsymbol{x}, \boldsymbol{y})$ and let $\{X_{k,i}\}\colon A \to B$.*

(i) The identity 2-morphism is given by

$$\mathbb{1}_A|_{A_i \to A_j} = \delta_{j,i} \cdot \mathbb{1}_{A_j} \ . \tag{1.5.46}$$

(ii) We find

$$(X \otimes \mathbb{1}_A)|_{A_i \to B_k} = \bigoplus_{j=1}^{n} X_{k,j} \otimes \delta_{j,i} \cdot \mathbb{1}_{A_i} \cong X_{k,i} \boxtimes \mathbb{1}_{A_i} \ ,$$

and the unitor 3-morphisms are given by

$$\begin{aligned} \rho_X|_{X_{k,i} \boxtimes \mathbb{1}_{A_i}} &= \rho_{X_{k,i}} \ , & \rho_X^{-1}|_{X_{k,i}} &= \rho_{X_{k,i}}^{-1} \ , \\ \lambda_X|_{\mathbb{1}_{B_k} \boxtimes X_{k,i}} &= \lambda_{X_{k,i}} \ , & \lambda_X^{-1}|_{X_{k,i}} &= \lambda_{X_{k,i}}^{-1} \ . \end{aligned} \tag{1.5.47}$$

Proof. The direct sum collapses to just one element by Lemma 1.4.5. The properties of λ_X and ρ_X follow immediately from the properties of the unitor 3-morphisms of $\mathcal{RW}$. $\square$

Lemma 1.5.22. *The identity 1-morphism $\mathbb{1}_{\boldsymbol{x}} \in \overline{\mathcal{RW}}^{\oplus}(\boldsymbol{x}', \boldsymbol{x})$ is the list containing only the identity 1-morphism $\mathbb{1}_{\boldsymbol{x}} \in \mathcal{RW}(\boldsymbol{x}', \boldsymbol{x})$. For a 1-morphism $A = \bigoplus_{i=1}^{n} A_i \in \overline{\mathcal{RW}}^{\oplus}(\boldsymbol{x}, \boldsymbol{y})$, we find*

$$\mathbb{1}_{\boldsymbol{y}} \boxtimes A = \bigoplus_{i=1}^{n} \mathbb{1}_{\boldsymbol{y}} \boxtimes A_i \ , \qquad A \boxtimes \mathbb{1}_{\boldsymbol{x}} = \bigoplus_{i=1}^{n} A_i \boxtimes \mathbb{1}_{\boldsymbol{x}} \ . \tag{1.5.48}$$

The unitor 2-morphisms of $\overline{\mathcal{RW}}^{\oplus}$ are given by

$$\begin{aligned} \lambda_A\big|_{\mathbb{1}_{\boldsymbol{y}}\boxtimes A_i \to A_j} &:= \delta_{i,j} \cdot \lambda_{A_i} \ , & \rho_A\big|_{A_i\boxtimes \mathbb{1}_{\boldsymbol{x}} \to A_j} &:= \delta_{i,j} \cdot \rho_{A_i} \ , \\ \lambda_A^{-1}\big|_{A_i \to \mathbb{1}_{\boldsymbol{y}}\boxtimes A_j} &:= \delta_{i,j} \cdot \lambda_{A_i}^{-1} \ , & \rho_A^{-1}\big|_{A_i \to A_j\boxtimes \mathbb{1}_{\boldsymbol{x}}} &:= \delta_{i,j} \cdot \rho_{A_i}^{-1} \ . \end{aligned} \tag{1.5.49}$$

Proof. All proofs from Section 1.5.4 can be generalised easily. As an example, we will show $\lambda_A \otimes \lambda_A^{-1} \cong \mathbb{1}_A$.

$$\begin{aligned} \lambda_A \otimes \lambda_A^{-1}\big|_{A_i \to A_k} &= \bigoplus_{j=1}^{n} (\delta_{j,k}\lambda_{A_j}) \otimes (\delta_{i,j}\lambda_{A_i}^{-1}) \cong \delta_{i,k}(\lambda_{A_i} \otimes \lambda_{A_i}^{-1}) \oplus \bigoplus_{j \neq i} (\delta_{j,k}\lambda_{A_j}) \otimes 0_{A_i \to \mathbb{1}_{\boldsymbol{y}}\boxtimes A_j} \\ &\cong \delta_{i,k}\mathbb{1}_{A_i} \oplus \bigoplus_{j\neq i} 0_{A_i \to A_k} \cong \delta_{i,k}\mathbb{1}_{A_i} = \mathbb{1}_A\big|_{A_i \to A_k} \end{aligned}$$

using the properties of λ_{A_i} and $0_{A\to B}$. $\square$

Corollary 1.5.23. *If Conjecture 1.5.15 holds, then $\overline{\mathcal{RW}}^{\oplus}$ is a tricategory with the unitor 2- and 3-morphisms of Eqs. (1.5.47) and (1.5.49).*

Proof sketch. While somewhat arduous to show, this ultimately follows from the lemmas above. $\square$

2 Adjunctions in the bicategory $\ddot{\mathsf{M}}\mathsf{F}_k$

For the reasons explained in Section 1.2.7 we wish to construct adjoints and a pivotal structure on $\mathcal{RW}$, turning it into a pivotal tricategory with duals (see Definition 1.2.21) assuming Conjecture 1.5.15 holds. In particular we have to show that the Hom-bicategories $\mathcal{RW}(\boldsymbol{x}, \boldsymbol{y})$ are pivotal bicategories (and hence have adjoints). Understanding adjunctions and pivotality in $\ddot{\mathsf{M}}\mathsf{F}_k$ is therefore essential for the constructions in Chapter 3.

2.1 Adjunctions in Landau–Ginzburg models

2.1.1 The admissible superpotentials in Landau–Ginzburg models

This section summarises several results from [22]. We start with a definition:

Definition 2.1.1. A polynomial $W \in k[x_1, \ldots, x_n]$ is called a *potential* if for $f_i := \partial_{x_i} W$ the following holds:

(i) $\{f_1, \ldots, f_n\}$ is a Koszul-regular sequence,

(ii) The *Jacobi ring* $k[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ is a finitely generated free k-module.

The bicategory of Landau–Ginzburg models as discussed in [22] is restricted to such potentials:

Definition 2.1.2. Let k be a commutative ring. The bicategory of Landau–Ginzburg models $\mathcal{LG}_k$ is the subcategory of $\ddot{\mathsf{M}}\mathsf{F}_k$ with the following data:

(i) The objects $(\boldsymbol{a}; W(\boldsymbol{a})) \in \mathcal{LG}_k$ are restricted to those where W is a potential according to Definition 2.1.1.

(ii) The set of 1-morphisms $\mathcal{LG}_k((\boldsymbol{x}; W), (\boldsymbol{z}; V))$ is the idempotent closure[1] of all finite-rank matrix factorisations of $V(\boldsymbol{z}) - W(\boldsymbol{x})$ over $k[\boldsymbol{x}, \boldsymbol{z}]$.

(iii) The set of 2-morphisms is unchanged.

Essentially, the objects are restricted to potentials in the above sense and the 1-morphisms are restricted to those that are isomorphic to direct sums of finite-rank matrix factorisations up to homotopy.

Notation 2.1.3. In Section 1.5 and Chapter 3 we use the variable names $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ for objects of $\mathcal{RW}$ (i.e. for bulk variables) and the variable names $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \ldots$ in objects $(\boldsymbol{a}; W(\boldsymbol{a})) \in \ddot{\mathsf{M}}\mathsf{F}_k$ (i.e. for surface variables). In this chapter we deviate from this convention in order to ensure notational consistency with [22] and use the variable names $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ in objects $(\boldsymbol{x}; W(\boldsymbol{x})) \in \ddot{\mathsf{M}}\mathsf{F}_k$. Later in this chapter we will introduce the variables $\boldsymbol{w}$ which corresponds to the union of all bulk variables involved. Schematically, we translate as follows:

$$(\boldsymbol{a}'; W(\boldsymbol{a}', \boldsymbol{x}', \boldsymbol{y}')) \in \mathcal{RW}(\boldsymbol{x}', \boldsymbol{y}') \quad \longmapsto \quad (\boldsymbol{x}; W(\boldsymbol{x}, \boldsymbol{w})) \in \ddot{\mathsf{M}}\mathsf{F}_k(\boldsymbol{w}) \ , \quad \boldsymbol{w} = \{\boldsymbol{x}', \boldsymbol{y}'\} \ . \tag{2.1.1}$$

[1] See [22, Section 2.2] and the discussion in Section 3.2.5.

2.1.2 Residue operators

Residue operators are central in proving that $\mathcal{LG}_k$ has adjoints as they show up ev_X and $\tilde{\mathrm{ev}}_X$ for 1-morphisms X in $\mathcal{LG}_k$. Only a few key properties will be summarised here. General references are [28, pp. 15 ff.], [39], [71, pp. 239–241].

Lemma 2.1.4. *Let k be a commutative ring and let $\boldsymbol{f} = \{f_1, \dots, f_n\} \subset k[x_1, \dots, x_n]$ be a Koszul-regular sequence[2] such that $k[\boldsymbol{x}]/(\boldsymbol{f})$ is a finitely generated projective k-module. Then the k-linear **residue operator***

$$\operatorname{Res}_{k[\boldsymbol{x}]/k}\left[\frac{\bullet\, \mathrm{d}\boldsymbol{x}}{f_1, \dots, f_n}\right] : k[\boldsymbol{x}] \to k \ , \tag{2.1.2}$$

is well-defined [22, Section 2.4] [71, pp. 16, 19] (see the cited references for a formal definition). The subscript $k[\boldsymbol{x}]/k$ denotes the domain and codomain of the residue operator and is often omitted when they can be inferred from context.

Remark 2.1.5. The constraints of Lemma 2.1.4 are fundamental — it is not possible to define the residue operator in a consistent way if the sequence is not at least quasi-regular or the quotient is not finitely generated and projective.

Lemma 2.1.6. *Let $n = \ell(\boldsymbol{x})$, $c, d \in k$, $\boldsymbol{y} \in k^{\oplus n}$, $g = g(\boldsymbol{x}) \in k[\boldsymbol{x}]$, $C \in k[\boldsymbol{x}]^{\oplus n \times n}$. The following identities hold for the residue operator:*

$$\operatorname{Res}_{k[\boldsymbol{x}]/k}\left[\frac{g\, \mathrm{d}\boldsymbol{x}}{f_1, \dots, f_n}\right] = 0 \quad \text{for } g \in (f_1, \dots, f_n) \ , \tag{2.1.3}$$

$$\operatorname{Res}_{k[\boldsymbol{x}]/k}\left[\frac{1\, \mathrm{d}\boldsymbol{x}}{x_1^{i_1}, \dots, x_n^{i_n}}\right] = \delta_{i_1,1} \dots \delta_{i_n,1} \ , \tag{2.1.4}$$

$$\operatorname{Res}_{k[\boldsymbol{x}]/k}\left[\frac{g\, \mathrm{d}\boldsymbol{x}}{f_1, \dots, f_n}\right] = \operatorname{Res}_{k[\boldsymbol{x}]/k}\left[\frac{\det(C) g\, \mathrm{d}\boldsymbol{x}}{f_1', \dots, f_n'}\right] \quad \text{with} \quad \boldsymbol{f}' := C \cdot \boldsymbol{f} \ , \tag{2.1.5}$$

$$\operatorname{Res}_{k[\boldsymbol{x}]/k}\left[\frac{g\, \mathrm{d}x_1\, \mathrm{d}x_2}{x_1 - c \cdot x_2, x_2 - d}\right] = \operatorname{Res}_{k[\boldsymbol{x}]/k}\left[\frac{g\, \mathrm{d}x_1\, \mathrm{d}x_2}{x_1 - c \cdot d, x_2 - d}\right] \ , \tag{2.1.6}$$

$$\operatorname{Res}_{k[\boldsymbol{x}]/k}\left[\frac{g(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}}{x_1 - y_1, \dots, x_n - y_n}\right] = g(\boldsymbol{y}) \ . \tag{2.1.7}$$

Further properties of the residue operator can be found in [28, Appendix A].

Proof. Eq. (2.1.3) is stated in [22, Section 2.4]. Eqs. (2.1.4) and (2.1.5) were originally stated in [54, pp. 197 ff.] with minor corrections made in [28, Appendix A]. Eq. (2.1.6) follows from Eq. (2.1.5) with $C = \left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right)$. Finally, for Eq. (2.1.7) we use $g(\boldsymbol{x}) = (\boldsymbol{x} - \boldsymbol{y}) \cdot \partial^{\boldsymbol{x},\boldsymbol{y}} g + g(\boldsymbol{y})$ and Eq. (2.1.3) to find

$$g(\boldsymbol{y}) \cdot \operatorname{Res}_{k[\boldsymbol{x}]/k}\left[\frac{1\, \mathrm{d}\boldsymbol{x}}{x_1 - y_1, \dots, x_n - y_n}\right] \ .$$

It remains to be shown that the residue evaluates to 1. Here we may use that under certain conditions, the residue operator agrees with the ordinary residue [54, §9] as defined e.g. in

[2] There are weaker conditions that suffice. In the present definition, "Koszul-regular" can be relaxed to "H_1-regular", meaning that $K_\bullet(\boldsymbol{f})$ is exact in degree one. A slightly different set of constraints is used in [39].

[52, Chapter 5]. Specifically, each function $(x_i - y_i)$ has an isolated zero at $x_i = y_i$, $1 \cdot \mathrm{d}\boldsymbol{x}$ is a regular differential form on $X = k[\boldsymbol{x}]$, and X is regular over k. Then we find

$$\begin{aligned}\mathrm{Res}_{k[\boldsymbol{x}]/k}\left[\frac{1\,\mathrm{d}\boldsymbol{x}}{x_1 - y_1,\ \ldots,\ x_n - y_n}\right] &= \mathrm{Res}_{\{\boldsymbol{y}\}}\frac{\mathrm{d}x_1 \ldots \mathrm{d}x_n}{(x_1-y_1)\ldots(x_n-y_n)} \\ &= \frac{1}{(2\pi i)^n}\int_\Gamma \frac{\mathrm{d}x_1 \ldots \mathrm{d}x_n}{x_1 \ldots x_n} = 1\ ,\end{aligned}$$

where the latter two terms follow the definitions and notation of [52]. □

2.1.3 Known result: $\mathcal{LG}_k$ has adjoints

In the bicategory $\mathcal{LG}_k$ as defined above, the following central result of [22] holds:

Theorem 2.1.7. *The bicategory $\mathcal{LG}_k$ has left and right adjoints. For a 1-morphism*

$$X\colon (x_1,\ \ldots,\ x_n;\ W) \to (z_1,\ \ldots,\ z_m;\ V)$$

we define $R := k[\boldsymbol{x}]$, $S := k[\boldsymbol{z}]$. Then the adjoints of X are given by

$$X^\dagger = R[n] \otimes_R X^\vee = X^\vee[n]\ , \qquad {}^\dagger X = X^\vee \otimes_S S[m] \cong X^\vee[m]\ . \tag{2.1.8}$$

Let $\{e_i\}$ be a homogeneous basis of X and let $\{e_i^*\}$ be the respective dual basis on $X^\dagger$ resp. ${}^\dagger X$ according to Notation 1.3.27. Then there are the following closed formulas for the evaluation and coevaluation maps [21]:

$$\begin{aligned}\widetilde{\mathrm{ev}}_X(a \cdot e_j \otimes e_i^*) = \sum_{l \geq 0}\sum_{\alpha_1 < \ldots < \alpha_l} (-1)^{l + (n+1)|e_j|}\,\theta_{\alpha_1}\ldots\theta_{\alpha_l} \\ \cdot\ \mathrm{Res}_{k[\boldsymbol{x},\boldsymbol{z},\boldsymbol{z}']/k[\boldsymbol{z},\boldsymbol{z}']}\left[\frac{a \cdot \{\partial^{\boldsymbol{z},\boldsymbol{z}'}_{[\alpha_l]} d_X \ldots \partial^{\boldsymbol{z},\boldsymbol{z}'}_{[\alpha_1]} d_X \Lambda^{(x)}\}_{ij}\,\mathrm{d}\boldsymbol{x}}{\partial_{x_1}W,\ \ldots,\ \partial_{x_n}W}\right]\ ,\end{aligned} \tag{2.1.9}$$

$$\begin{aligned}\mathrm{ev}_X(b \cdot e_i^* \otimes e_j) = \sum_{l \geq 0}\sum_{\alpha_1 < \ldots < \alpha_l} (-1)^{\binom{l}{2} + l|e_j|}\,\theta_{\alpha_1}\ldots\theta_{\alpha_l} \\ \cdot\ \mathrm{Res}_{k[\boldsymbol{x},\boldsymbol{x}',\boldsymbol{z}]/k[\boldsymbol{x},\boldsymbol{x}']}\left[\frac{b \cdot \{\Lambda^{(z)}\partial^{\boldsymbol{x},\boldsymbol{x}'}_{[\alpha_1]} d_X \ldots \partial^{\boldsymbol{x},\boldsymbol{x}'}_{[\alpha_l]} d_X\}_{ij}\,\mathrm{d}\boldsymbol{z}}{\partial_{z_1}V,\ \ldots,\ \partial_{z_m}V}\right]\ ,\end{aligned} \tag{2.1.10}$$

$$\widetilde{\mathrm{coev}}_X(\bar{\gamma}) = \sum_{i,j}(-1)^{(\bar{r}+1)|e_j| + s_n}\left\{\partial^{\boldsymbol{x},\boldsymbol{x}'}_{[\bar{\beta}_{\bar{r}}]} d_X \ldots \partial^{\boldsymbol{x},\boldsymbol{x}'}_{[\bar{\beta}_1]} d_X\right\}_{ji} e_i^* \otimes e_j\ , \tag{2.1.11}$$

$$\mathrm{coev}_X(\gamma) = \sum_{i,j}(-1)^{\binom{r+1}{2} + mr + s_m}\left\{\partial^{\boldsymbol{z},\boldsymbol{z}'}_{[\beta_1]} d_X \ldots \partial^{\boldsymbol{z},\boldsymbol{z}'}_{[\beta_r]} d_X\right\}_{ij} e_i \otimes e_j^* \tag{2.1.12}$$

with

$$a \in k[\boldsymbol{x},\,\boldsymbol{z},\,\boldsymbol{z}']\ , \quad b \in k[\boldsymbol{x},\,\boldsymbol{x}',\,\boldsymbol{z}]\ , \tag{2.1.13}$$

$$\Lambda^{(x)} = (-1)^n \partial_{x_1} d_X(\boldsymbol{x},\,\boldsymbol{z}) \ldots \partial_{x_n} d_X(\boldsymbol{x},\,\boldsymbol{z})\ , \tag{2.1.14}$$

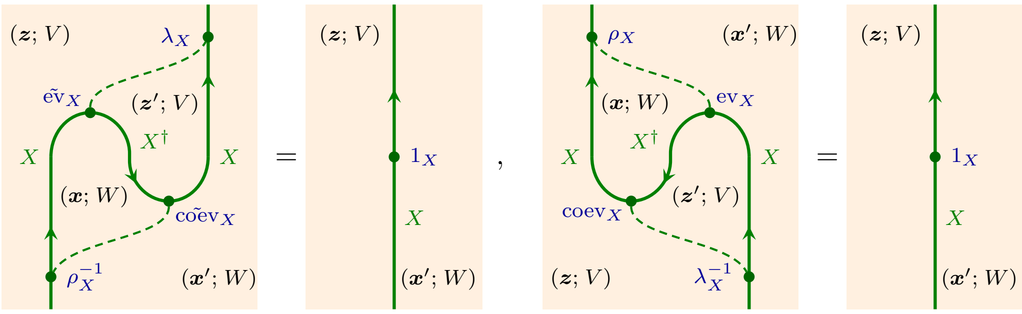
$$\Lambda^{(z)} = \partial_{z_1} d_X(\boldsymbol{x},\,\boldsymbol{z}) \ldots \partial_{z_m} d_X(\boldsymbol{x},\,\boldsymbol{z})\ , \tag{2.1.15}$$

and β_i, $\bar{\beta}_{\bar{j}}$, s_n, $s_m \in \mathbb{Z}_2$ uniquely determined by $b_1 < \ldots < b_r$, $\bar{b}_1 < \ldots < \bar{b}_{\bar{r}}$,

$$\bar{\gamma} \wedge \theta_{\bar{b}_1} \ldots \theta_{\bar{b}_{\bar{r}}} = (-1)^{s_n}\theta_1 \ldots \theta_n\ , \quad \gamma \wedge \theta_{b_1} \ldots \theta_{b_r} = (-1)^{s_m}\theta_1 \ldots \theta_m\ . \tag{2.1.16}$$

Remark 2.1.8. We follow the sign conventions of [21] and [23] instead of [22] which differ in $(\mathrm{ev}_X, \mathrm{coev}_X)$ by a factor of $(-1)^m$, as noted in [22, Footnote 4]. Therefore, some formulas cited from [22] will have different prefactors in this work. While the Zorro moves are invariant under this change, the pivotal structure discussed in Section 2.4 is subject to slight changes.

Diagrammatically, we find



(2.1.17)

plus the other two Zorro diagrams. The explicit formulas in Eqs. (2.1.9) to (2.1.12) depend on the locations of the primes, which follow the conventions explained in Definition 1.4.1.

2.2 Problems for non-potentials

2.2.1 Non-potentials in Rozansky–Witten models

If we wish to find adjunctions for all 2-morphisms in $\mathcal{RW}$, the assumption that W is a potential in the sense of Definition 2.1.1 must be dropped. The following examples illustrate how non-potentials appear even in the basic building blocks of $\mathcal{RW}$.

Example 2.2.1. We consider the following setup in $\mathcal{RW}$:

- two objects (x) and (x') with one bulk variable each,
- two copies of the identity 1-morphism $\mathbb{1}_x$: $(a;\, a(x-x')),\ (b;\, b(x-x'))\colon (x') \to (x)$,
- the identity 2-morphism $X := \mathbb{1}_{\mathbb{1}_x} = I^{b\leftarrow a}_{(x-x')\cdot\bullet}\colon \mathbb{1}_x \to \mathbb{1}_x$.

By Example 1.2.20 we know that X has adjoints. However, Theorem 2.1.7 cannot be applied to X: According to the rules of $\mathcal{RW}$, the left adjunction is given by 2-morphisms

$$\mathrm{ev}_X\colon\ {}^\dagger X(a, b', x, x') \otimes_{\mathbb{C}[b', x, x']} X(a', b', x, x') \to I^{a\leftarrow a'}_{(x-x')\cdot\bullet}\ ,$$
$$\mathrm{coev}_X\colon\ I^{b\leftarrow b'}_{(x-x')\cdot\bullet} \to X(a, b, x, x') \otimes_{\mathbb{C}[a, x, x']} {}^\dagger X(a, b', x, x')\ .$$

In the language of Landau–Ginzburg models, we regard X as a matrix factorisation

$$X \in \ddot{\mathsf{MF}}_{\mathbb{C}[x, x']}\big((a;\ a(x-x')),\ (b;\ b(x-x'))\big) \tag{2.2.1}$$

in line with Remark 1.4.3. Now it is easy to see that $a(x-x')$ is not a potential: We find $f_1 = \partial_a(a(x-x')) = x - x'$ and the quotient

$$\mathbb{C}[a, x, x']/(f_1) = \mathbb{C}[a, x, x']/(x-x') \cong \bigoplus_{i=0}^{\infty} a^i\, \mathbb{C}[x, x']/(x-x') \tag{2.2.2}$$

where "$\cong$" means "isomorphic as $\mathbb{C}[x, x']$-modules". Clearly, this module is neither free nor finitely generated as a $\mathbb{C}[x, x']$-module. Consequently, the evaluation maps (2.1.9) and (2.1.10) are not well-defined for this 2-morphism.

Example 2.2.2. We consider the following setting in $\mathcal{RW}$:

- three objects (x), (x'), (x'') with one bulk variable each,
- three identity 1-morphisms

$$\begin{aligned}&(a;\, a(x-x'))\colon (x') \to (x)\ , \qquad (b;\, b(x'-x''))\colon (x'') \to (x')\ ,\\ &(c;\, c(x-x''))\colon (x'') \to (x)\ ,\end{aligned}$$

  with the box product

$$(a;\, a(x-x')) \boxtimes (b;\, b(x'-x'')) = (a, x', b;\, a(x-x') + b(x'-x''))\colon (x'') \to (x)\ ,$$

- the 2-morphism

$$\lambda_{\mathbb{1}_x}\colon (a, x', b;\, a(x-x') + b(x'-x'')) \to (c;\, c(x-x'')) \tag{2.2.3}$$

  as defined in Eq. (1.5.26).

In the previous example we have shown that the codomain is not a potential, and neither is the domain:

$$\begin{aligned}\mathbb{C}[a, x', b, x, x'']/(\partial_a W, \partial_{x'} W, \partial_b W) &= \mathbb{C}[b, x', a, x, x'']/(x-x', b-a, x'-x'')\\ &\cong \mathbb{C}[a, x, x'']/(x-x'')\end{aligned}$$

which is neither free nor finitely generated over $\mathbb{C}[x, x'']$. However, it turns out that $\lambda_{\mathbb{1}_x}$ does have a left and right adjoint. This can be shown by constructing the adjunction 2-morphisms manually using one of several techniques, two of which are presented in Appendices A.2 and A.3. This works well for simple matrix factorisations, but quickly becomes infeasible for larger numbers of bulk and surface variables.

These examples illustrate that there are 1-morphisms in $\ddot{\mathsf{MF}}_k$ that have adjoints, but their adjunction 2-morphisms are not given by Eqs. (2.1.9) and (2.1.10).

2.2.2 Not all matrix factorisations have adjoints

Given that there are matrix factorisations which map between non-potentials but still have adjoints, a natural question to ask is whether *all* finite-rank matrix factorisations have adjoints. The answer is no. Let us first look at a simple example:

Example 2.2.3. The matrix factorisation

$$X := (\mathbb{C}[b],\, d=0)\colon (\emptyset;\, 0) \to (b;\, 0) \tag{2.2.4}$$

does not have a left adjoint. Intuitively, one can argue as follows: For every defect $Y\colon A \to B$ that has a left adjoint there exist isomorphisms

$${}^{\dagger}Y \otimes Y \cong \mathbb{1}_B \oplus \dots\ , \quad Y \otimes {}^{\dagger}Y \cong \mathbb{1}_A \oplus \dots\ . \tag{2.2.5}$$

The adjunction 2-morphisms map between the tensor product on the left and one identity matrix factorisations on the right. However, the variable b does not show up in the differential d_X of Eq. (2.2.4), so $Z \otimes X$ cannot have a summand that is proportional to $I_0^{b \leftarrow b'}$ (whose differential has a term $(b-b')$) no matter what the matrix factorisation Z looks like.

More generally, we find:

Theorem 2.2.4. *If the matrix factorisation X is of the form*

$$(X, d_X) \colon (\boldsymbol{x};\, W) \to (\boldsymbol{z}, b;\, V) \ \text{ with } d_X = d_X(\boldsymbol{x}, \boldsymbol{z}) \ , \tag{2.2.6}$$

i.e. the variable b does not appear in d_X (implying that b does not appear in V either),[3] *then X does not have a left adjoint. Analogously, if X is of the form*

$$(X, d_X) \colon (\boldsymbol{x}, b;\, W) \to (\boldsymbol{z};\, V) \ \text{ with } d_X = d_X(\boldsymbol{x}, \boldsymbol{z}) \tag{2.2.7}$$

then it does not have a right adjoint.

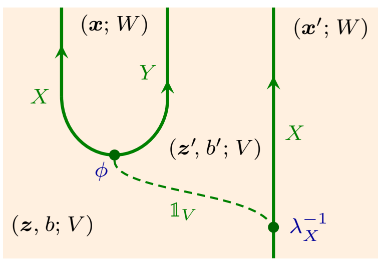
Remark 2.2.5.

- The statement of Theorem 2.2.4 is that *no* matrix factorisation can be left (right) adjoint to X; it would be insufficient to show that $X^\vee[s]$ is not left (right) adjoint to X for $s \in \mathbb{Z}_2$.
- If the domain and codomain of X are potentials in the sense of Definition 2.1.1, X has adjoints according to Theorem 2.1.7. This is consistent with the fact that Eq. (2.2.6) implies that V is not a potential: $k[\boldsymbol{z}, b]/(V(\boldsymbol{z})) \cong k[b] \otimes_k k[\boldsymbol{z}]/(V(\boldsymbol{z}))$ has infinite rank over k.

The following lemma is sufficient to prove Theorem 2.2.4:

Lemma 2.2.6. *Let $W = W(\boldsymbol{x})$ and $V = V(\boldsymbol{z})$. Then for all $Y \in \mathsf{MF}(\boldsymbol{x}, \boldsymbol{z}', b';\, W - V)$ and all $\phi \colon I_V^{\{\boldsymbol{z}, b\} \leftarrow \{\boldsymbol{z}', b'\}} \to X \otimes Y$, we find $(\phi \otimes 1) \circ \lambda_X^{-1} \equiv 0$.*

Lemma 2.2.6 implies that the Zorro map (2.1.17) cannot evaluate to the identity because it already evaluates to zero after the first two maps:



Lemma 2.2.6 thus implies Theorem 2.2.4. We introduce the notation

$$\tilde{\boldsymbol{z}} := (\boldsymbol{z}, b) \ , \qquad \mathbb{1}_V = I_V^{\tilde{\boldsymbol{z}} \leftarrow \tilde{\boldsymbol{z}}'} = I_V^{\boldsymbol{z} \leftarrow \boldsymbol{z}'} \otimes I_0^{b \leftarrow b'} =: \mathbb{1}_{\boldsymbol{z}} \otimes \mathbb{1}_b \tag{2.2.8}$$

and use the basis $(1, \theta_b)$ for $\mathbb{1}_b$. In order to show Lemma 2.2.6 we first prove the following lemma:

[3]Taking the derivative ∂_b on both sides of $d_X(\boldsymbol{x}, \boldsymbol{z})^2 = (V - W) \cdot 1$ yields $\partial_b V = 0$.

Lemma 2.2.7. *For X as defined in Eq. (2.2.6), terms of the form*

$$\alpha \otimes \theta_b \otimes e_i \in I_V^{\boldsymbol{x}' \leftarrow \boldsymbol{x}} \otimes I_0^{b \leftarrow b'} \otimes X \tag{2.2.9}$$

are not in the image of λ_X^{-1}.

Proof. We apply Eq. (1.3.80) to X:

$$\lambda_X^{-1} \colon e_i \mapsto \sum_{l \geqslant 0} \sum_{\alpha_1 < \cdots < \alpha_l} \sum_j \theta_{\alpha_1} \ldots \theta_{\alpha_l} \left\{ \partial_{[\alpha_l]}^{\tilde{\boldsymbol{z}}, \tilde{\boldsymbol{z}}'} d_X \ldots \partial_{[\alpha_1]}^{\tilde{\boldsymbol{z}}, \tilde{\boldsymbol{z}}'} d_X \right\}_{ji} \otimes e_j \ .$$

Now we find that $\theta_{\ell(\boldsymbol{z})+1} = \theta_b$ is not in the image of λ_X^{-1}, since any term with a θ_b is proportional to

$$(d_X(\boldsymbol{z}', b) - d_X(\boldsymbol{z}', b'))/(b - b') = 0 \ . \qquad \square$$

Proof of Lemma 2.2.6. Assume that $\phi \colon I_V^{\{\boldsymbol{z}, b\} \leftarrow \{\boldsymbol{z}', b'\}} \to X \otimes Y$ is a morphism of matrix factorisations. We spell out the closedness condition $\mathrm{d}\phi = 0$ acting on $\alpha \otimes \theta_b \in \mathbb{1}_{\boldsymbol{z}} \otimes \mathbb{1}_b$ (the condition on $\alpha \otimes 1$ does not matter for this argument):

$$\begin{aligned} 0 &= \left(d_{X \otimes Y} \circ \phi - \phi \circ d_{\mathbb{1}_V}\right)(\alpha \otimes \theta_b) \\ &= d_{X \otimes Y}\left(\phi(\alpha \otimes \theta_b)\right) - \phi(d_{\mathbb{1}_{\boldsymbol{z}}}(\alpha) \otimes \theta_b) - (-1)^{|\alpha|} \phi(\alpha \otimes (b - b')1) \\ &= (-1)^{|\alpha|+1}(b - b')\phi(\alpha \otimes 1) + d_{X \otimes Y}\left(\phi(\alpha \otimes \theta_b)\right) - \phi(d_{\mathbb{1}_{\boldsymbol{z}}}(\alpha) \otimes \theta_b) \ . \end{aligned} \tag{2.2.10}$$

Now we define ϕ_0 and ϕ_1 such that

$$\phi = (b - b')\phi_1 + \phi_0 \tag{2.2.11}$$

and that the matrix representation of ϕ_0 does not contain b. This can be done by writing ϕ as a (potentially infinite-dimensional) matrix and expanding each coefficient

$$\phi_{ij} = \underbrace{\gamma_0}_{(\phi_0)_{ij}} + (b - b') \underbrace{\textstyle\sum_{i=0}^{\infty} \gamma_{i+1}(b - b')^i}_{(\phi_1)_{ij}} \quad \text{for } \gamma_i \in k[\boldsymbol{z}, \boldsymbol{z}', b'] \ .$$

In this notation Eq. (2.2.10) reads

$$\begin{aligned} (-1)^{|\alpha|}(b - b')\phi(\alpha \otimes 1) = d_{X \otimes Y}&\left(\phi_0(\alpha \otimes \theta_b) + (b - b')\phi_1(\alpha \otimes \theta_b)\right) \\ &- \phi_0(d_{\mathbb{1}_{\boldsymbol{z}}}(\alpha) \otimes \theta_b) - (b - b')\phi_1(d_{\mathbb{1}_{\boldsymbol{z}}}(\alpha) \otimes \theta_b) \ . \end{aligned} \tag{2.2.12}$$

A detailed study of this condition reveals that on the right hand side, only $(b - b')\phi_1$ is able to introduce further terms proportional to b:

- ϕ_0 does not contain any b-terms by construction.
- The variable b does not appear in $d_{\mathbb{1}_{\boldsymbol{z}}} = d_{I_V^{\boldsymbol{z} \leftarrow \boldsymbol{z}'}}$ because b does not appear in V by assumption.
- $d_{X \otimes Y} = d_X \otimes 1 + 1 \otimes d_Y$, so any b-terms introduced by $d_{X \otimes Y}$ must come from d_X or d_Y.
- d_X does not introduce any b-terms by assumption.

- d_Y cannot introduce b-terms because Y is a matrix factorisation over $k[\boldsymbol{x}, \boldsymbol{z}', b']$ and so does not "know" about b. Visually, the line Y is not adjacent to the surface where b lives.

To solve Eq. (2.2.12), there are thus two options: Either $\phi(\alpha \otimes 1) = 0$, or the b-terms on the left hand side must be cancelled by $(b - b')\phi_1$ on the right. Comparing coefficients in b thus yields the two identities

$$(-1)^{|\alpha|}\phi(\alpha \otimes 1) = d_{X\otimes Y}(\phi_1(\alpha \otimes \theta_b)) - \phi_1(d_{\mathbb{1}_{\boldsymbol{z}}}(\alpha) \otimes \theta_b) \ , \tag{2.2.13}$$

$$0 = d_{X\otimes Y}(\phi_0(\alpha \otimes \theta_b)) - \phi_0(d_{\mathbb{1}_{\boldsymbol{z}}}(\alpha) \otimes \theta_b) \ . \tag{2.2.14}$$

These conditions must be fulfilled for all α by any closed map $\phi\colon \mathbb{1}_{\boldsymbol{z}} \to X \otimes Y$. We now apply a homotopy $\phi \mapsto \phi' = \phi + \mathrm{d}\psi$, the latter being defined by

$$\begin{aligned} &\psi(\delta \otimes 1) := (-1)^{|\delta|+1}\phi_1(\delta \otimes \theta_b) \quad \text{for all } \delta \in \mathbb{1}_{\boldsymbol{z}} \ , \\ &\psi(\delta \otimes \theta_b) := 0 \ . \end{aligned}$$

For ϕ' we thus find (using Eq. (2.2.13), the definition of ψ, and $|d_{\mathbb{1}_{\boldsymbol{z}}}(\alpha)| = |\alpha| + 1$)

$$\begin{aligned} \phi'(\alpha \otimes 1) &= \phi(\alpha \otimes 1) + d_{X\otimes Y}(\psi(\alpha \otimes 1)) + \psi(d_{\mathbb{1}_{\boldsymbol{z}}}(\alpha) \otimes 1) \\ &= (-1)^{|\alpha|}\big(d_{X\otimes Y}(\phi_1(\alpha \otimes \theta_b)) - \phi_1(d_{\mathbb{1}_{\boldsymbol{z}}}(\alpha) \otimes \theta_b)\big) \\ &\quad + d_{X\otimes Y}\big((-1)^{|\alpha|+1}\phi_1(\alpha \otimes \theta_b)\big) + (-1)^{|\alpha|+2}\phi_1(d_{\mathbb{1}_{\boldsymbol{z}}}(\alpha) \otimes \theta_b) \\ &= 0 \ . \end{aligned}$$

We have thus proven that every morphism $\phi\colon \mathbb{1}_{\boldsymbol{z}} \to X \otimes Y$ can be represented by a map that is zero on all $\alpha \otimes 1$ and non-zero only on $\alpha \otimes \theta_b$. As we have seen in Lemma 2.2.7, the image of λ_X^{-1} does not contain any $\alpha \otimes \theta_b$, so indeed $(\phi \otimes \mathbb{1}) \circ \lambda_X^{-1} \equiv 0$ up to homotopy. □

2.2.3 Tensor products and infinite rank

There is a well-established theorem about the tensor product of matrix factorisations in $\mathcal{LG}_k$, which makes the horizontal composition $- \otimes -$ in $\mathcal{LG}_k$ well-defined:

Theorem 2.2.8. *Let $W \in k[\boldsymbol{x}]$, $V \in k[\boldsymbol{z}]$, $U \in k[\boldsymbol{y}]$ be potentials (Definition 2.1.1). Then for all matrix factorisations*

$$X \in \mathsf{MF}(\boldsymbol{x}, \boldsymbol{z};\ V - W) \ , \quad Y \in \mathsf{MF}(\boldsymbol{z}, \boldsymbol{y};\ U - V) \ ,$$

the tensor product $X \otimes_{k[\boldsymbol{z}]} Y$ is isomorphic to a direct sum of finite-rank matrix factorisations in $\mathsf{MF}(\boldsymbol{x}, \boldsymbol{y}; U - W)$ [22, p. 491] [39, Section 12] [11, p. 19].

This statement does not generalise to arbitrary superpotentials. Curiously, the counterexample to the above statement for non-potentials is also a matrix factorisation without adjoints, see Theorem 2.2.4.

Example 2.2.9. Consider objects

$$A = (a;\ 0),\ B = (b;\ 0),\ C = (c;\ 0) \in \ddot{\mathsf{MF}}_{\mathbb{C}}$$

and 1-morphisms

$$X = (\mathbb{C}[a,\, b],\, d_X = 0) \colon A \to B \ , \quad Y = (\mathbb{C}[b,\, c],\, d_Y = 0) \colon B \to C \ .$$

Then

$$(X \otimes_{\mathbb{C}[b]} Y,\, d_{X\otimes Y}) \in \mathrm{Hom}_{\ddot{\mathsf{MF}}_{\mathbb{C}}}(A,\, C) = \mathsf{MF}(a,\, c;\, W = 0) \ , \tag{2.2.15}$$

$$(X \otimes_{\mathbb{C}[b]} Y,\, d_{X\otimes Y}) = (\mathbb{C}[a,\, b,\, c],\, d = 0) = \bigoplus_{n\in\mathbb{N}} (\mathbb{C}[a,\, c],\, d = 0) \ , \tag{2.2.16}$$

i.e. we find that the tensor product $X \otimes_{\mathbb{C}[b]} Y$ is equal to an infinite sum of rank 1 matrix factorisations.

It is apparent that the bicategory $\ddot{\mathsf{MF}}_k$ contains "unphysical" examples. Therefore, rather than studying the entire bicategory $\ddot{\mathsf{MF}}_k$, we will focus on a well-behaved subclass that contains the potentials of $\mathcal{LG}_k$ as special cases.

2.3 Adjoint existence in $\ddot{\mathsf{MF}}_k(\boldsymbol{w})$

As discussed in the previous section, not all 1-morphisms in $\ddot{\mathsf{MF}}_k(\boldsymbol{x})$ have adjoints. Nevertheless, it is possible to prove adjoint existence on a subset of all matrix factorisations. This subset contains Examples 2.2.1 and 2.2.2 as well as all 1-morphisms in $\mathcal{LG}_k$, thus generalising the main result of [22].

2.3.1 Potentials and admissible variables

First we need to loosen the definition of a "potential" from Definition 2.1.1. In this entire section, W and V are given by

$$W := (x_1,\, \dots\, ,\, x_n;\, W(\boldsymbol{x},\, \boldsymbol{w})) \ , \quad V := (z_1,\, \dots\, ,\, z_m;\, V(\boldsymbol{z},\, \boldsymbol{w})) \in \ddot{\mathsf{MF}}_k(\boldsymbol{w}) \ , \tag{2.3.1}$$

implying

$$\mathrm{Hom}_{\ddot{\mathsf{MF}}_k(\boldsymbol{w})}(W,\, V) = \mathsf{MF}_k(\boldsymbol{x},\, \boldsymbol{z},\, \boldsymbol{w};\, V(\boldsymbol{z},\, \boldsymbol{w}) - W(\boldsymbol{x},\, \boldsymbol{w})) \ .$$

Definition 2.3.1 (Admissible variables)**.**

(i) The bicategory $\mathrm{Hom}_{\ddot{\mathsf{MF}}_k(\boldsymbol{w})}(W,\, V)$ has *right admissible variables*

$$\boldsymbol{u} = \{u_1,\, \dots\, ,\, u_n\} \subset \{x_1,\, \dots\, ,\, x_n,\, \boldsymbol{w}\} \tag{2.3.2}$$

if the following holds:

(a) $\boldsymbol{f} := \{\partial_{u_1}(W - V),\, \dots\, ,\, \partial_{u_n}(W - V)\}$ is a Koszul-regular sequence,

(b) $k[\boldsymbol{x},\, \boldsymbol{z},\, \boldsymbol{w}]/(\boldsymbol{f})$ is a free, finite-rank $k[\boldsymbol{z},\, \boldsymbol{w}]$-module.

(ii) The bicategory $\mathrm{Hom}_{\ddot{\mathsf{MF}}_k(\boldsymbol{w})}(W,\, V)$ has *left admissible variables*

$$\boldsymbol{v} = \{v_1,\, \dots\, ,\, v_m\} \subset \{z_1,\, \dots\, ,\, z_m,\, \boldsymbol{w}\} \tag{2.3.3}$$

if the following holds:

(a) $\boldsymbol{g} := \{\partial_{v_1}(V - W),\, \dots\, ,\, \partial_{v_m}(V - W)\}$ is a Koszul-regular sequence,

(b) $k[\boldsymbol{x},\, \boldsymbol{z},\, \boldsymbol{w}]/(\boldsymbol{g})$ is a free, finite-rank $k[\boldsymbol{x},\, \boldsymbol{w}]$-module.

Remark 2.3.2. Admissible variables are properties of the Hom-categories. We say that a matrix factorisation $X \in \mathcal{C}$ has right (left) admissible variables if $\mathcal{C}$ does.

Example 2.3.3. For $\boldsymbol{w} = \emptyset$, Definition 2.3.1 reduces to W resp. V being a potential: The only possible choice for $\{u_1, \ldots, u_n\}$ is $\{x_1, \ldots, x_n\}$ (or a permutation thereof), and

$$\partial_{u_i}(W(\boldsymbol{x}) - V(\boldsymbol{z})) = \partial_{x_i}(W(\boldsymbol{x}) - V(\boldsymbol{z})) = \partial_{x_i} W \ .$$

Example 2.3.4. The matrix factorisations from Examples 2.2.1 and 2.2.2 have left and right admissible variables: Consider again

$$\lambda_{\mathbb{1}_x} \colon (a, x', b;\ a(x-x') + b(x'-x'')) \to (c;\ c(x-x'')) \ .$$

Now we choose the following admissible variables (several other choices are possible):

$$\begin{aligned} \{u_1, u_2, u_3\} &:= \{x, a, x''\} &&\Longrightarrow \boldsymbol{f} = \{a-c,\ x-x',\ c-b\} \ , \\ \{v_1\} &:= \{x\} &&\Longrightarrow \boldsymbol{g} = \{c-a\} \ . \end{aligned}$$

These sequences are regular and thus Koszul-regular by Lemma 1.3.51, and we find the quotients

$$\begin{aligned} k[b, x', a, c, x, x'']/(\boldsymbol{f}) &= k[b, x', a, c, x, x'']/(c-b,\ x-x',\ a-c) \cong k[c, x, x''] \ , \\ k[b, x', a, c, x, x'']/(\boldsymbol{g}) &= k[b, x', a, c, x, x'']/(c-a) = k[b, x', a, x, x''] \ , \end{aligned}$$

which are clearly free and finite-rank over $k[c, x, x'']$ resp. $k[b, x', a, x, x'']$.

2.3.2 The main result

Let k_0 be a commutative ring and let

$$\begin{gathered} k = k_0[\boldsymbol{w}] \ , \quad R = k[\boldsymbol{x}] \cong k_0[\boldsymbol{x}, \boldsymbol{w}] \ , \quad R^e = R \otimes_k R \cong k_0[\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{w}] \ , \\ S = k[\boldsymbol{z}] \cong k_0[\boldsymbol{z}, \boldsymbol{w}] \ , \quad S^e = S \otimes_k S \cong k_0[\boldsymbol{z}, \boldsymbol{z}', \boldsymbol{w}] \ . \end{gathered}$$

We first establish the existence of the residue operators used in the generalisations of ev_X and $\widetilde{\mathrm{ev}}_X$.

Lemma 2.3.5.

(i) Let $\mathrm{Hom}_{\ddot{\mathsf{MF}}_k(\boldsymbol{w})}(W, V)$ have right admissible variables $\{u_1, \ldots, u_n\}$. Then the residue operator

$$\mathrm{Res}_{S^e[\boldsymbol{x}]/S^e} \left[\frac{\bullet \, \mathrm{d}\boldsymbol{x}}{\partial_{u_1}(W-V), \ \ldots, \ \partial_{u_n}(W-V)} \right] \tag{2.3.4}$$

is well-defined, where $V(\boldsymbol{z}, \boldsymbol{z}', \boldsymbol{w}) := V(\boldsymbol{z}, \boldsymbol{w})$.

(ii) Let $\mathrm{Hom}_{\ddot{\mathsf{MF}}_k(\boldsymbol{w})}(W, V)$ have left admissible variables $\{v_1, \ldots, v_m\}$. Then the residue operator

$$\mathrm{Res}_{R^e[\boldsymbol{z}]/R^e} \left[\frac{\bullet \, \mathrm{d}\boldsymbol{z}}{\partial_{v_1}(V-W), \ \ldots, \ \partial_{v_m}(V-W)} \right] \tag{2.3.5}$$

is well-defined, where $W(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{w}) := W(\boldsymbol{x}, \boldsymbol{w})$.

Proof. Set $V' := (\boldsymbol{z}, \boldsymbol{z}'; V(\boldsymbol{z}, \boldsymbol{w})) \in \ddot{\mathsf{MF}}_k(\boldsymbol{w})$. Then $\mathrm{Hom}_{\ddot{\mathsf{MF}}_k(\boldsymbol{w})}(W, V')$ also has right admissible variables $\{u_1, \dots, u_n\}$: The sequence $\boldsymbol{f}$ is unchanged, so it is still Koszul-regular, and

$$k[\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{z}', \boldsymbol{w}]/(\boldsymbol{f}) \cong k[\boldsymbol{z}'] \otimes_k (k[\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{w}]/(\boldsymbol{f}))$$

is a free, finite-rank $k[\boldsymbol{z}, \boldsymbol{z}', \boldsymbol{w}]$-module because $k[\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{w}]/(\boldsymbol{f})$ is a free, finite-rank $k[\boldsymbol{z}, \boldsymbol{w}]$-module by assumption. The existence of right admissible variables is sufficient for the residue operator (2.3.4) to be well-defined (see Lemma 2.1.4). The other case is analogous. □

We now define a generalisation of the adjunction 2-morphisms (2.1.9) and (2.1.10):

Definition 2.3.6. Let $X \in \mathrm{Hom}_{\ddot{\mathsf{MF}}_k(\boldsymbol{w})}(W, V)$ be a finite-rank matrix factorisation.

(i) If $\mathrm{Hom}_{\ddot{\mathsf{MF}}_k(\boldsymbol{w})}(W, V)$ has right admissible variables $\{u_1, \dots, u_n\}$, we define the morphism

$$\widetilde{\mathrm{ev}}_X \colon X(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{w}) \otimes_R X^\dagger(\boldsymbol{x}, \boldsymbol{z}', \boldsymbol{w}) \to I_V^{\boldsymbol{z} \leftarrow \boldsymbol{z}'} ,$$

$$\widetilde{\mathrm{ev}}_X(a \cdot e_j \otimes e_i^*) = \sum_{l \geq 0} \sum_{\substack{\alpha_1 < \dots < \alpha_l \\ \{\alpha_i\} \subset \boldsymbol{z}}} (-1)^{l + (n+1)|e_j|} \theta_{\alpha_1} \dots \theta_{\alpha_l}$$
$$\cdot \operatorname{Res}_{S^e[\boldsymbol{x}]/S^e} \left[\frac{a \cdot \{\partial_{[\alpha_l]}^{\boldsymbol{z}, \boldsymbol{z}'} d_X \dots \partial_{[\alpha_1]}^{\boldsymbol{z}, \boldsymbol{z}'} d_X \Lambda^{(x)}\}_{ij} \, \mathrm{d}\boldsymbol{x}}{\partial_{u_1}(W - V), \ \dots, \ \partial_{u_n}(W - V)} \right] \tag{2.3.6}$$

with $\Lambda^{(x)} := (-1)^n \partial_{u_1} d_X(\boldsymbol{x}, \boldsymbol{z}) \dots \partial_{u_n} d_X(\boldsymbol{x}, \boldsymbol{z})$ and $V = V(\boldsymbol{z})$ (so d_X and V are written in the variables of X in $X \otimes X^\dagger$).

(ii) If $\mathrm{Hom}_{\ddot{\mathsf{MF}}_k(\boldsymbol{w})}(W, V)$ has left admissible variables $\{v_1, \dots, v_m\}$, we define the morphism

$$\mathrm{ev}_X \colon {}^\dagger X(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{w}) \otimes_S X(\boldsymbol{x}', \boldsymbol{z}, \boldsymbol{w}) \to I_W^{\boldsymbol{x} \leftarrow \boldsymbol{x}'} ,$$

$$\mathrm{ev}_X(b \cdot e_i^* \otimes e_j) = \sum_{l \geq 0} \sum_{\substack{\alpha_1 < \dots < \alpha_l \\ \{\alpha_i\} \subset \boldsymbol{x}}} (-1)^{\binom{l}{2} + l|e_j|} \theta_{\alpha_1} \dots \theta_{\alpha_l}$$
$$\cdot \operatorname{Res}_{R^e[\boldsymbol{z}]/R^e} \left[\frac{b \cdot \{\Lambda^{(z)} \partial_{[\alpha_1]}^{\boldsymbol{x}, \boldsymbol{x}'} d_X \dots \partial_{[\alpha_l]}^{\boldsymbol{x}, \boldsymbol{x}'} d_X\}_{ij} \, \mathrm{d}\boldsymbol{z}}{\partial_{v_1}(V - W), \ \dots, \ \partial_{v_m}(V - W)} \right] \tag{2.3.7}$$

with $\Lambda^{(z)} := \partial_{v_1} d_X(\boldsymbol{x}, \boldsymbol{z}) \dots \partial_{v_m} d_X(\boldsymbol{x}, \boldsymbol{z})$ and $W = W(\boldsymbol{x})$ (so d_X and W are written in the variables of ${}^\dagger X$ in ${}^\dagger X \otimes X$).

Now we have all ingredients to state the main result.

Theorem 2.3.7 (Adjoint existence in $\ddot{\mathsf{MF}}_k(\boldsymbol{w})$)**.**
If $\mathrm{Hom}_{\ddot{\mathsf{MF}}_k(\boldsymbol{w})}(W, V)$ has right (left) admissible variables, then all of its objects each have a right (left) adjoint in $\ddot{\mathsf{MF}}_k(\boldsymbol{w})$. The formulas for $X^\dagger$, ${}^\dagger X$, $\widetilde{\mathrm{coev}}_X$, and coev_X are identical to those in $\mathcal{LG}_k$ (see Eqs. (2.1.8), (2.1.11), and (2.1.12)), and the formulas for $\widetilde{\mathrm{ev}}_X$ and ev_X are given by Eqs. (2.3.6) and (2.3.7).

Remark 2.3.8. There are still matrix factorisations in $\ddot{\mathsf{MF}}_k(\boldsymbol{w})$ which have adjoints, but Theorem 2.3.7 does not apply to them. One example is $I_0^{b \leftarrow b'} \colon (b'; 0) \to (b; 0)$, which has adjoints according to Example 1.2.20, but there are no admissible variables on $\mathrm{Hom}((b'; 0), (b; 0))$.

The following sections will prove Theorem 2.3.7.

2.3.3 Structure of the proof

Reference [22] proves Theorem 2.1.7, i.e. that $\mathcal{LG}_k$ has adjoints. The full argument is quite extensive, and its vast majority needs no modification for the proof of Theorem 2.3.7. Therefore, the present work will explicitly spell out the statements and proofs that differ significantly, and explain how the remaining statements and proofs can be generalised easily.

In contrast to the present work, [22] assumes that for all objects $(\boldsymbol{a}; W) \in \mathcal{LG}_k$, the superpotential W is a *potential* in the sense of Definition 2.1.1. Thus, it is important to understand which statements of [22] depend on this assumption. A careful study reveals that this assumption on W is used explicitly only in [22, Section 5.2] (and implicitly in the subsequent sections that utilise the results of the cited section). Specifically, two statements are inferred from W being a potential:

(i) The residue operators of Eqs. (2.1.9) and (2.1.10) exist.

(ii) [39, Theorem 7.4] applies.

No other special properties of W are used anywhere else in the proof, implicitly or explicitly (including [22, Prop. 2.19] whose proof cites a statement from [29]). In particular, the whole machinery of bar complexes, Atiyah classes, and homological perturbation works in the present setting without any modifications. The proof of the unitor properties also makes no such assumptions, as pointed out in the proof of Theorem 1.3.56.

2.3.4 The general setup

Let k_0 be a commutative ring (which we relabel to avoid confusion). Following Remark 1.4.3 we will consider the bicategory $\ddot{\mathsf{MF}}_{k_0[\boldsymbol{w}]}$ instead of $\ddot{\mathsf{MF}}_{k_0}(\boldsymbol{w})$. In a Rozansky–Witten context one can think of the $\boldsymbol{w}$ as bulk variables, which may be located on either side of the 1-morphisms described by $\ddot{\mathsf{MF}}_{k_0}(\boldsymbol{w})$. In the language of [22], we have

$$k = k_0[\boldsymbol{w}] \ , \quad R = k[x_1, \ldots, x_n] \ , \quad S = k[z_1, \ldots, z_m] \ , \quad W \in R \ , \quad V \in S \ , \qquad (2.3.8)$$

and a finite-rank matrix factorisation

$$(X, d_X) \in \mathsf{MF}_k(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{w};\, V(\boldsymbol{z}, \boldsymbol{w}) - W(\boldsymbol{x}, \boldsymbol{w})) \qquad (2.3.9)$$

with X being a free, finite-rank $S \otimes_k R$-module. We will also use the following definitions with $R_i := R$, $S_i := S$:

$$R^e := R_1 \otimes_k R_2 = k[\boldsymbol{x}, \boldsymbol{x}'] \ , \qquad \tilde{W} := W(\boldsymbol{x}) - W(\boldsymbol{x}') \in R^e, \qquad (2.3.10)$$

$$S^e := S_1 \otimes_k S_2 = k[\boldsymbol{z}, \boldsymbol{z}'] \ , \qquad \tilde{V} := V(\boldsymbol{z}) - V(\boldsymbol{z}') \quad \in S^e \ . \qquad (2.3.11)$$

The general assumption is that $\mathrm{Hom}_{\ddot{\mathsf{MF}}_k(\boldsymbol{w})}(W, V)$ has right admissible variables $\{u_1, \ldots, u_n\}$ in all arguments involving the right Zorro move, and left admissible variables $\{v_1, \ldots, v_m\}$ for the left Zorro move. We write

$$\boldsymbol{f} := \{\partial_{u_1}(W - V), \ldots, \partial_{u_n}(W - V)\} \qquad (2.3.12)$$

and we define

$$\lambda_i := -\partial_{u_i} d_X(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{w}) \ , \qquad \Lambda^{(x)} := \lambda_1 \ldots \lambda_n \ . \qquad (2.3.13)$$

λ_i is a null-homotopy for $f_i \cdot 1_X$:

$$\begin{aligned} d_X^2 &= (V(\boldsymbol{z}, \boldsymbol{w}) - W(\boldsymbol{x}, \boldsymbol{w})) \cdot 1_X \qquad \mid \partial/\partial_{u_i} \\ \partial_{u_i} d_X \circ d_X + d_X \circ \partial_{u_i} d_X &= \partial_{u_i}(V - W) \cdot 1_X \\ \{d_X, -\partial_{u_i} d_X\} &= f_i \cdot 1_X \ . \end{aligned} \tag{2.3.14}$$

2.3.5 The construction of $\tilde{\mathrm{ev}}_0$

We start with a few definitions and lemmas:

Definition 2.3.9. We define the ring

$$\bar{M} := (R \otimes_k S)/(f_i) = k[\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{w}]/(f_1, \ \ldots \ , f_n) \ . \tag{2.3.15}$$

By the assumptions of admissible variables, $\bar{M}$ is a free, finite-rank S-module. We further define

$$\bar{X} := X \otimes_{R \otimes_k S} \bar{M} \tag{2.3.16}$$

which is a free, finite-rank module over both S and $\bar{M}$.

Proof. Let $\{e_i\}$ be a finite basis of X over $S \otimes_k R$. For $r \in R$, $s \in S$, $m \in \bar{M}$, an arbitrary element of $\bar{X}$ as an $\bar{M}$-module is given by

$$(r \cdot s \cdot e_i) \otimes m = e_i \otimes (r \cdot s \cdot m) = (r \cdot s \cdot m)\, e_i \otimes 1 \in \bar{X} \ ,$$

so $\{e_i\}$ is a finite $\bar{M}$-basis of $\bar{X}$. Now let $\{m_\alpha\}$ be a finite S-basis of $\bar{M}$. Then

$$(r \cdot s \cdot e_i) \otimes m = s \cdot (e_i \otimes (r \cdot m)) \in \bar{X}$$

so $\{e_i \otimes m_\alpha\}$ is a finite S-basis of $\bar{X}$. Because X and $\bar{M}$ are free as S-modules, $\bar{X}$ is free as well. □

Lemma 2.3.10. *S is a torsion module over S^e via $(s_1, s_2) \cdot s := s_1 \cdot s_2 \cdot s$, implying $S \cong k[\boldsymbol{z}, \boldsymbol{z}']/(\boldsymbol{z} - \boldsymbol{z}')$. It is also a linear factorisation of $\tilde{V}$ in the following sense:*

$$S = (k[\boldsymbol{z}, \boldsymbol{z}']/(\boldsymbol{z} - \boldsymbol{z}'), d_S = 0) \ . \tag{2.3.17}$$

Proof. $d_S^2 = 0 \equiv (V(\boldsymbol{z}) - V(\boldsymbol{z}')) \cdot 1_S \mod (\boldsymbol{z} - \boldsymbol{z}')$. □

Definition 2.3.11. The morphism

$$\pi_\Delta \colon \Delta_V = I_V^{\boldsymbol{z} \leftarrow \boldsymbol{z}'} \to k[\boldsymbol{z}, \boldsymbol{z}']/(\boldsymbol{z} - \boldsymbol{z}') = S \ , \quad \theta_{i_1} \ldots \theta_{i_k} \mapsto \delta_{k,0} \tag{2.3.18}$$

is a closed even morphism of linear factorisations [22, Eq. (2.16)].

Lemma 2.3.12.

$$\mathrm{d}\Lambda^{(x)} = \{d_X, \Lambda^{(x)}\} = \sum_{i=1}^{n} (-1)^{n+i-1} f_i \cdot \lambda_1 \ldots \widehat{\lambda_i} \ldots \lambda_n \ . \tag{2.3.19}$$

In particular, $\Lambda^{(x)}$ is closed as a map $\Lambda^{(x)} \colon \bar{X} \to \bar{X}$.

Proof.

$$\begin{aligned}\{d_X, \Lambda^{(x)}\} &= (-1)^n d_X \lambda_1 \dots \lambda_n - \lambda_1 \dots \lambda_n d_X \\ &= \sum_{i=1}^{n} (-1)^{n+i-1} \lambda_1 \dots \lambda_{i-1} (d_X \lambda_i - \lambda_i d_X) \lambda_{i+1} \dots \lambda_n \\ &= \sum_{i=1}^{n} (-1)^{n+i-1} f_i \cdot \lambda_1 \dots \widehat{\lambda_i} \dots \lambda_n \ . \end{aligned}$$ □

In analogy to [22, Section 5.2], the next step is to construct $\tilde{\mathrm{ev}}_0 \colon X \otimes_R X^\dagger \to S$ as a composite of simpler maps, with S being the linear factorisation of Lemma 2.3.10. Each constituent map is a morphism of linear factorisations: They are all closed, some of them may be odd, $\tilde{\mathrm{ev}}_0$ as a whole is even.

Lemma 2.3.13. *The map*

$$\tilde{\mathrm{ev}}_0(\eta \otimes \nu) = (-1)^{n+n|\eta|} \operatorname{Res}_{R\otimes S/S} \left[\frac{\operatorname{str}(\Lambda^{(x)} \circ \eta \circ \nu)\big|_{z' \mapsto z} \, \mathrm{d}\boldsymbol{x}}{f_1, \ \dots \ , f_n} \right] \tag{2.3.20}$$

is a morphism of linear factorisations of $\tilde{V}$ over S^e.

Proof. In this setting, X is an $S_1 \otimes_k R$-module, $X^\vee$ is an $R \otimes_k S_2$-module, and $X \otimes_R X^\vee$ is an infinite-rank $S^e = S_1 \otimes_k S_2$-module. The first part of $\tilde{\mathrm{ev}}_0$ is given by a projection

$$X \otimes_R X^\vee[n] \cong (X \otimes_R X^\vee)[n] \longrightarrow \bar{M} \otimes_{(S_1 \otimes_k R)} (X \otimes_R X^\vee)[n] =: N[n] \ . \tag{2.3.21}$$

This step is analogous to [22, Eq. (5.15)], although N cannot be interpreted as $\bar{X} \otimes_R \bar{X}^\vee$ here. The subsequent map, in analogy to [22, Eq. (5.16)],[4] is defined by

$$N[n] = \bar{M} \otimes_{(S_1 \otimes_k R)} (X \otimes_R X^\vee)[n] \xrightarrow{1 \otimes \Lambda^{(x)} \otimes 1} \bar{M} \otimes_{(S_1 \otimes_k R)} (X \otimes_R X^\vee) = N \ . \tag{2.3.22}$$

Its closedness can be seen as follows: $\{d_X, \Lambda^{(x)}\} = \sum_{i=1}^n f_i \cdot \dots$ by Lemma 2.3.12, $f_i \in S_1 \otimes_k R$ so by bilinearity of $\otimes$ we can move f_i into $\bar{M}$, and $f_i \equiv 0$ in $\bar{M}$.

We proceed with a projection and three isomorphisms:

$$\begin{aligned} N \xrightarrow{\text{can}} S \otimes_{S^e} N &= S \otimes_{S^e} (\bar{M} \otimes_{S_1 \otimes_k R} (X \otimes_R X^\vee)) \\ &\cong \bar{M} \otimes_{S \otimes_k R} (X \otimes_{S \otimes_k R} X^\vee) \\ &\cong (X \otimes_{S \otimes_k R} \bar{M}) \otimes_{\bar{M}} (X^\vee \otimes_{S \otimes_k R} \bar{M}) \\ &\cong \operatorname{Hom}_{\bar{M}}(\bar{X}, \bar{X}) \end{aligned} \tag{2.3.23}$$

with $d_{\operatorname{Hom}(\bar{X},\bar{X})}(\phi) = \{d_X, \phi\}$. A detailed explanation of the isomorphisms is presented in Appendix A.1.1. The final two maps are given by

$$\operatorname{Hom}_{\bar{M}}(\bar{X}, \bar{X}) \xrightarrow{\text{str}} \bar{M} \xrightarrow{\text{Res}} S \ . \tag{2.3.24}$$

The supertrace is closed because $d_{\bar{M}} = 0$ and

$$(\mathrm{d}\,\mathrm{str})(\phi) \sim \operatorname{str}\big(d_X \circ \phi - (-1)^{|\phi|} \phi \circ d_X\big) = \operatorname{str}\big(d_X \circ \phi - d_X \circ \phi\big) = 0 \ .$$

By Eq. (2.1.3), the residue operator acts trivially on f_i and is thus well-defined on $\bar{M}$. Its closedness is trivial because the differential is zero on both sides. □

[4] In [22, Eqs. (5.16), (5.17)], a few instances of R should read $\bar{R}$, so $\bar{X}$ is a free $S \otimes_k \bar{R}$-module.

2.3.6 The lift to $\tilde{\mathrm{ev}}_X$

The argument in [22] uses homological perturbation to lift the morphism $\tilde{\mathrm{ev}}_0 \colon X \otimes X^\dagger \to S$ to a morphism $\tilde{\mathrm{ev}}_X \colon X \otimes_R X^\dagger \to I_V^{\boldsymbol{z}\leftarrow\boldsymbol{z}'}$. The main ingredient is a lifting theorem in [22, Section 4] which can be applied to finite-rank matrix factorisations. However, as the rank of $X \otimes_R X^\dagger$ over S^e is infinite, some work is needed to be able to apply said theorem.

The idempotent pushforward

A central part of reducing the infinite-rank matrix factorisations to finite rank is the idempotent pushforward [39, Theorem 4.2]:

Theorem 2.3.14. *Let (Y, d_Y) be a finite-rank matrix factorisation over $k[\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{z}']$, and let $\{f_1, \dots, f_n\} \subset k[\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{z}']$ be a Koszul-regular sequence. Furthermore, assume that*

(i) there are $\lambda_i \in \mathrm{End}_R(Y)$ such that $\{d_Y, \lambda_i\} = f_i \cdot 1_Y$,

(ii) $k[\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{z}']/(\boldsymbol{f})$ is projective as a $k[\boldsymbol{z}, \boldsymbol{z}']$-module.

Then there is a diagram of morphisms between linear factorisations with $\nu \circ \vartheta = 1$ and $\vartheta = \Lambda^{(x)} \otimes 1$:

$$Y[n] \underset{\vartheta}{\overset{\nu}{\leftrightarrows}} Y \otimes_R R/(\boldsymbol{f}) \ . \tag{2.3.25}$$

Details on how to derive Theorem 2.3.14 from the formulation in [39] are given in Appendix A.1.2. We may now apply Theorem 2.3.14 to $Y = X \otimes_R X^\vee$: Both X and $X^\vee$ are finite-rank, so over $k[\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{z}']$, their tensor product is finite-rank. We choose $\{f_1, \dots, f_n\}$ according to Eq. (2.3.12) and $\lambda_i := \lambda_i \otimes 1_{X^\vee}$ with the latter λ_i defined by Eq. (2.3.13). By the assumptions of admissible variables (Definition 2.3.1), $\boldsymbol{f}$ is Koszul-regular and $k[\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{z}']/(\boldsymbol{f})$ is free and thus projective as a $k[\boldsymbol{z}, \boldsymbol{z}']$-module. We thus get the following diagram:

$$(X \otimes_R X^\vee)[n] \underset{\vartheta}{\overset{\nu}{\leftrightarrows}} (X \otimes_R X^\vee) \otimes_{S_1 \otimes_k R} \bar{M} \quad (= N) \tag{2.3.26}$$

where we have used that $\boldsymbol{f} \subset k[\boldsymbol{x}, \boldsymbol{z}] \subset k[\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{z}']$. This allows us to work with finite-rank modules instead of $X \otimes_R X^\dagger$ which is infinite-rank over S^e.

The lift construction

We start with the following lemma:

Lemma 2.3.15. *The module N of Eq. (2.3.21) is finite-rank over S^e.*

Proof. By assumption, X is finite-rank over $S \otimes_k R$ and thus has a finite $S \otimes_k R$-basis $\{e_i\}$, and $\bar{M}$ is finite-rank over S so it has a finite S-basis $\{m_\alpha\}$. It follows that $\{e_i^*\}$ is a finite $R \otimes_k S$-basis of $X^\vee$. Now for $r \in R$, $s_1 \in S_1$, $s_2 \in S_2$, $m \in \bar{M}$, $x \in X$, $y \in X^\vee$, an arbitrary term

$$s_1 r s_2 (x \otimes y \otimes m) = s_1 s_2 (x \otimes y \otimes (r \cdot m)) \in (X \otimes_R X^\vee) \otimes_{S_1 \otimes_k R} \bar{M} = N$$

can be generated by the finite S^e-basis $\{e_i \otimes e_j^* \otimes m_\alpha\}_{i,j,\alpha}$. □

In analogy to [22] we define ψ as the map in Eq. (2.3.21) and $\tilde{\mathrm{ev}}_0'$ as the composite of Eqs. (2.3.23) and (2.3.24). We then study the diagram

$$\begin{tikzcd} \mathrm{Hom}_{S^e}((X \otimes_R X^\vee)[n], \Delta) \arrow[d, "\pi_\Delta^\bullet"] & \mathrm{Hom}_{S^e}(N, \Delta) \arrow[l, "(\vartheta\circ\psi)_\bullet"'] \arrow[d, "\pi_\Delta^\bullet"] \\ \mathrm{Hom}_{S^e}((X \otimes_R X^\vee)[n], S) & \mathrm{Hom}_{S^e}(N, S) \arrow[l, "(\vartheta\circ\psi)_\bullet"'] \end{tikzcd} . \tag{2.3.27}$$

By using the finite rank of N and [22, Prop. 4.11], the right $\pi_\Delta^\bullet$ has a homotopy inverse $(\pi_\Delta^\bullet)^{-1}$. The construction of $\tilde{\mathrm{ev}}_X$ is then fully analogous to [22]: $\tilde{\mathrm{ev}}_0'$ lives in the bottom right of Eq. (2.3.27), and we define $\tilde{\mathrm{ev}}'$ to be its image under $(\pi_\Delta^\bullet)^{-1}$. Mapping to the left in both rows, we find $\tilde{\mathrm{ev}}_X$ in the top row and $\tilde{\mathrm{ev}}_0$ in the bottom row. The closed formula for $\tilde{\mathrm{ev}}_X$ can be derived in analogy to [22], and we find the generalised formula

$$\tilde{\mathrm{ev}}_X(\eta \otimes \nu) = \sum_{l \geq 0} (-1)^{n+n|\eta|} \operatorname{Res}_{R \otimes S^e/S^e} \left[\frac{\Psi \langle 1\mathrm{At}_{S_1}(X \otimes_R X^\vee)^l (\Lambda^{(x)} \eta \otimes \nu) \rangle \, \mathrm{d}\boldsymbol{x}}{\partial_{u_1}(W - V), \ \ldots, \ \partial_{u_n}(W - V)} \right] , \tag{2.3.28}$$

whose basis representation is given by Eq. (2.3.6). A fully analogous construction yields

$$\mathrm{ev}_X(\nu \otimes \eta) = \sum_{l \geq 0} \operatorname{Res}_{S \otimes R^e/R^e} \left[\frac{\Psi \langle 1\mathrm{At}_{R_1}(X^\vee \otimes_S X)^l (\nu \Lambda^{(z)} \otimes \eta) \rangle \, \mathrm{d}\boldsymbol{z}}{\partial_{v_1}(V - W), \ \ldots, \ \partial_{v_m}(V - W)} \right] , \tag{2.3.29}$$

whose basis representation is given by Eq. (2.3.7). Note that in accordance with Remark 2.1.8 there is a factor of $(-1)^m$ in ev_X relative to [21, Eq. (5.7)].

2.3.7 The Zorro move

This section will prove the first identity of Eq. (2.1.17) for all finite-rank matrix factorisations X with right admissible variables $\{u_1, \ldots, u_n\}$. Only a few modifications of the proof in [22, Section 6] are required, which are explained below. The notation is consistent between the present work and [22], the only differences being that f_i and λ_i are defined according to Eqs. (2.3.12) and (2.3.13), and that the domain and definition of $\langle\langle - \rangle\rangle$ needs to be changed:

Definition 2.3.16. In analogy to [22, Definition 6.1], we define

$$\langle\langle - \rangle\rangle \colon S \otimes_k \mathbb{B} \to S \otimes_k R[n], \quad \langle\langle s \otimes \alpha \rangle\rangle = s \cdot \operatorname{Res}_{S \otimes_k R^e / S \otimes_k R_2} \left[\frac{\epsilon \Psi(\alpha) \, \mathrm{d}\boldsymbol{x}}{f_1, \ \ldots, \ f_n} \right] . \tag{2.3.30}$$

This change is needed because in contrast to [22], the f_i are elements of $k[\boldsymbol{x}, \boldsymbol{z}]$ instead of $k[\boldsymbol{x}]$. Note that this change is quite minor, as [22] defines $\langle\langle - \rangle\rangle$ implicitly[5] on $S \otimes_k \mathbb{B}$ as $1 \otimes \langle\langle - \rangle\rangle$.

The formulations and proofs of Lemmas 6.2 and 6.3 as well as Remark 6.4 of loc. cit. can be copied verbatim, i.e. they are "covariant" under the changes. The subsequent lemmas involving the Atiyah operator require some formal modifications.

[5] This can be seen in [22, Eqs. (6.7), (6.8)]: The codomain of str is $S \otimes_k \mathbb{B}$, and the Zorro map $\mathcal{Z}$ is of the form $\langle\langle - \rangle\rangle \circ \mathrm{str} \circ \ldots$.

The domain of the Atiyah operator

Two structures are central in the proof presented in [21, Section 6]: the bar complex, which is the k-module

$$\mathbb{B} = \Omega R_1 \otimes_k R_2 = R_1 \otimes_k \bar{R}^{\otimes n} \otimes_k R_2 \ , \qquad \bar{R} = R/k \ , \ R_1 = R_2 = R \ , \tag{2.3.31}$$

and the Atiyah operator

$$\mathrm{At}\colon \operatorname{End}(X) \otimes_{R_1} \mathbb{B} \to \operatorname{End}(X) \otimes_{R_1} \mathbb{B} \ . \tag{2.3.32}$$

The original formulation does not permit $f_i \in k[\boldsymbol{x}, \boldsymbol{z}]$, but only $f_i \in k[\boldsymbol{z}]$: For example, [21, Corollary 6.9] contains expressions $df_i \in \mathbb{B}$, and $\mathbb{B}$ is not S-linear. Therefore, changes are required to ensure df_i is well-defined in our generalisation. It turns out that this is only a formal, not a fundamental problem: The Atiyah operator does not need to be modified, only its domain and codomain need to be rewritten.

Lemma 2.3.17. *The Atiyah class as defined in [22, Eq. (6.6)] can be understood as an operator*

$$\mathrm{At}\colon \operatorname{End}(X) \otimes_{R'_1} \mathbb{B}' \to \operatorname{End}(X) \otimes_{R'_1} \mathbb{B}' \ , \tag{2.3.33}$$

$$k' := k[\boldsymbol{z}] = S \ , \quad R' := k'[\boldsymbol{x}] \ , \quad R'_1 := R'_2 := R' \ , \quad \mathbb{B}' = R' \otimes_{k'} \bar{R}'^{\otimes n} \otimes_{k'} R' \ . \tag{2.3.34}$$

In particular, $\mathbb{B}'$ is the bar complex of $R' = k[\boldsymbol{x}, \boldsymbol{z}]$ over the ring $k' = k[\boldsymbol{z}]$.

The subsequent supertrace operator also needs a slight formal change:

$$\mathrm{str} = \mathrm{str} \otimes 1\colon \operatorname{End}(X) \otimes_{R'_1} \mathbb{B}' \to (S \otimes_k R) \otimes_{R'_1} \mathbb{B}' \cong \mathbb{B}' \ . \tag{2.3.35}$$

Proof of Lemma 2.3.17. Let $(Y, D) = (\operatorname{End}(X), \{d_X, -\})$. The Atiyah operator is defined to be the following graded commutator [22, Eq. (6.6)]:

$$\mathrm{At} = \mathrm{At}_R(Y) = \{d, D \otimes 1_{\Omega R}\}\colon Y \otimes_R \Omega R \to Y \otimes_R \Omega R \ . \tag{2.3.36}$$

According to the detailed explanation in [22, Example 3.8], the map d and thus At is linear with respect to S-multiplication on Y. Furthermore, the following isomorphy holds:

$$Y \otimes_R \Omega R \cong Y \otimes_{S \otimes_k R} \Omega_S(S \otimes_k R) = Y \otimes_{k[\boldsymbol{x},\boldsymbol{z}]} \Omega_{k[\boldsymbol{z}]}(k[\boldsymbol{x}, \boldsymbol{z}])$$

Using that k is commutative, $\Omega_{k[\boldsymbol{x}]}(k[\boldsymbol{x}, \boldsymbol{z}])$ corresponds to $\Omega(R')$ over the ring $k' = k[\boldsymbol{z}]$ [22, below Lemma 2.9]. Therefore, we can naturally regard $\mathrm{At}_R(Y)$ as an operator on

$$\operatorname{End}(X) \otimes_{S \otimes_k R} \Omega_{k'}(R') = \operatorname{End}(X) \otimes_{R'} \Omega_{k'}(R') \ . \tag{2.3.37}$$

As in [22], we relabel $R' \mapsto R'_1$ and take the tensor product with $\otimes_k R_2$ to the right:

$$(\operatorname{End}(X) \otimes_{R'_1} \Omega_{k'}(R'_1)) \otimes_k R_2 \tag{2.3.38}$$

This module is isomorphic (as a k-module) to

$$\operatorname{End}(X) \otimes_{R'_1} (\Omega_{k'}(R'_1) \otimes_{k'} R'_2) = \operatorname{End}(X) \otimes_{R'_1} \mathbb{B}' \ . \qquad \square$$

The Zorro move

The remaining arguments in [22, Section 6] need very little modification. Lemmas 6.5 to 6.7 as well as Proposition 6.8 can be copied in both formulation and proof — they only use algebraic properties and the fact that λ_i is a null-homotopy for f_i. With the redefined Atiyah operator, Proposition 6.9 can also be copied in formulation and proof. Most of Proposition 6.10 needs no adaptation, only one detail requires attention: The expression

$$\sum_{\sigma \in S_n} (-1)^{|\sigma|} \langle\langle df_{\sigma(1)} \dots df_{\sigma(n)\,\mathrm{str}(-)} \rangle\rangle \tag{2.3.39}$$

evaluates to

$$\begin{aligned} &\mathrm{Res}_{k[\boldsymbol{x},\boldsymbol{x}',\boldsymbol{z}]/k[\boldsymbol{x}',\boldsymbol{z}]} \left[\frac{\sum_{\sigma \in S_n} (-1)^{|\sigma|} \partial^{\boldsymbol{x},\boldsymbol{x}'}_{[1]} f_{\sigma(1)} \dots \partial^{\boldsymbol{x},\boldsymbol{x}'}_{[n]} f_{\sigma(n)} \,\mathrm{str}(-)\, \mathrm{d}\boldsymbol{x}}{f_1, \ \dots\ , f_n} \right] \\ = {} &\mathrm{Res}_{k'[\boldsymbol{x},\boldsymbol{x}']/k'[\boldsymbol{x}']} \left[\frac{\det\big((\partial^{\boldsymbol{x},\boldsymbol{x}'}_{[i]} f_j)_{ij}\big) \,\mathrm{str}(-)\, \mathrm{d}\boldsymbol{x}}{f_1, \ \dots\ , f_n} \right] \end{aligned}$$

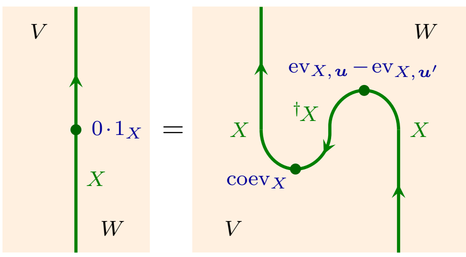
which is of a form where [22, Prop. 2.17] can be directly applied, completing the proof.

Uniqueness of adjunction 2-morphisms

Defining the evaluation and coevaluation operators involves making several (arbitrary) choices under which the operators should be invariant. In [22], it is shown that $\widetilde{\mathrm{ev}}_X$ and ev_X are independent of the choice of λ_i up to homotopy and independent of the order of $\{x_1, \ \dots\ , x_n\}$ up to a sign. The present work introduces another ambiguity: As discussed in Example 2.3.4 there can be different sets of admissible variables for a given Hom-category. It turns out that $\widetilde{\mathrm{ev}}_X$ and ev_X are independent of this choice as well.

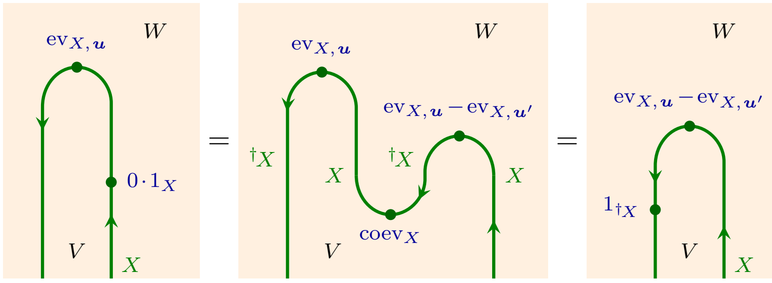
Theorem 2.3.18. *The evaluation operator is independent of the choice of admissible variables up to homotopy.*

Proof. Let $X \in \mathrm{Hom}(W, V)$ with two sets of left admissible variables $\boldsymbol{u}$, $\boldsymbol{u}'$. By Theorem 2.3.7, both $(\mathrm{ev}_{X,\boldsymbol{u}}, \mathrm{coev}_X)$ and $(\mathrm{ev}_{X,\boldsymbol{u}'}, \mathrm{coev}_X)$ are an adjunction between X and ${}^\dagger X$, implying that the respective Zorro move evaluates to the identity 2-morphism. Using linearity in k, we find



where, as usual, $=$ between diagrams means "equal as morphisms of matrix factorisations", so the module homomorphisms are homotopic. Now we compose both sides horizontally with

${}^{\dagger}X$ and then vertically with $\mathrm{ev}_{X,\boldsymbol{u}}$:



,

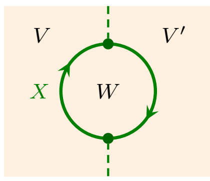
where we used the second Zorro move of X in the last step. Using linearity in k and the unit properties of $1_{{}^{\dagger}X}$ and 1_X, we find (up to homotopy)

$$\mathrm{ev}_{X,\boldsymbol{u}} - \mathrm{ev}_{X,\boldsymbol{u}'} = 0 \cdot \mathrm{ev}_{X,\boldsymbol{u}} \implies \mathrm{ev}_{X,\boldsymbol{u}} = \mathrm{ev}_{X,\boldsymbol{u}'} \ .$$

The argument for the right Zorro move is fully analogous. □

2.4 Pivotality

We have seen that a large class of 1-morphisms in $\ddot{\mathsf{MF}}_k(\boldsymbol{w})$ has adjoints. One standard application of adjoints is to compute "bubbles" of 1-morphisms, i.e. diagrams of the form



.

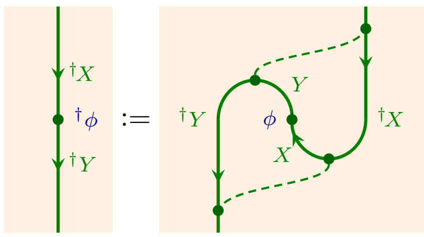
However, adjoints alone are not sufficient for a well-defined bubble. By definition,

$$\mathrm{coev}_X \colon \mathbb{1}_W \to X \otimes {}^{\dagger}X \ , \quad \widetilde{\mathrm{ev}}_X \colon X \otimes X^{\dagger} \to \mathbb{1}_W \ , \tag{2.4.1}$$

so a canonical isomorphism from ${}^{\dagger}X$ to $X^{\dagger}$ must be inserted between $\widetilde{\mathrm{coev}}_X$ and ev_X.

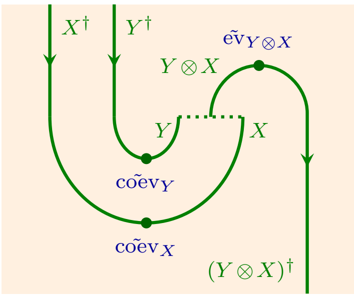
2.4.1 Definition

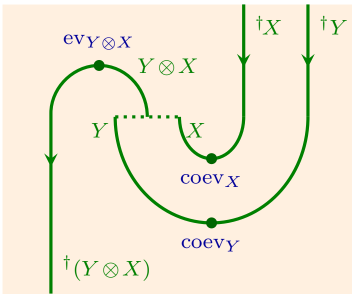
Let $\mathcal{B}$ be a bicategory with adjoints. A *pivotal structure* on $\mathcal{B}$ consists of a 2-isomorphism $\delta_X \colon {}^{\dagger}X \to X^{\dagger}$ for every 1-morphism X, subject to naturality and monoidality conditions. To formulate these conditions, we first realise that right (or left) adjunction can be understood as a contravariant functor ${}^{\dagger}(-)$ [82, Lemma 5.1.2] mapping 1-morphisms $X \mapsto {}^{\dagger}X$ and 2-morphisms $\phi \mapsto {}^{\dagger}\phi$, the latter being defined by [22, Section 6]



. (2.4.2)

The left and right adjunction functors are also *monoidal*, i.e. they are compatible with horizontal composition by means of the following (natural) isomorphisms:

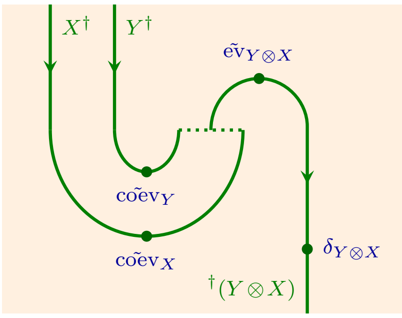
$\mathscr{R} :=$ 

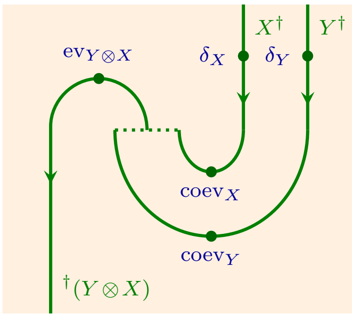
, $\mathscr{L} :=$ 

. (2.4.3)

The 1-morphism $Y \otimes X$ can either be interpreted as two adjacent 1-morphisms Y and X or as a single 1-morphism $Z = Y \otimes X$. The dotted horizontal line visualises a switch from the former to the latter interpretation. We may now define pivotality:

Definition 2.4.1. A bicategory with adjoints $\mathcal{B}$ is *pivotal* if there is a monoidal transformation $\delta\colon {}^\dagger(-) \Rightarrow (-)^\dagger$ (or, equivalently, a monoidal isomorphism $\mathrm{Id} \Rightarrow (-)^{\dagger\dagger}$ or $\mathrm{Id} \Rightarrow {}^{\dagger\dagger}(-)$) [22, Section 7] called an *(ordinary) pivotal structure*. Spelled out, a pivotal structure is given by a natural isomorphism $\delta_X\colon {}^\dagger X \to X^\dagger$ for every 1-morphism X that is compatible with $\mathscr{L}$ and $\mathscr{R}$, i.e. $\mathscr{R} \circ \delta_{Y\otimes X} = (\delta_X \otimes \delta_Y) \circ \mathscr{L}$ (see also [26, Sect. 2.3]):



$=$ 

. (2.4.4)

Remark 2.4.2.

- In a bicategory with adjoints, the functors Id, ${}^\dagger((-)^\dagger)$, and $({}^\dagger(-))^\dagger$ are monoidally isomorphic (with the isomorphism given by [22, Eq. (7.14)]), rendering all three formulations of Definition 2.4.1 equivalent.
- Defining pivotality in terms of a monoidal isomorphism $\delta'\colon \mathrm{Id} \Rightarrow (-)^{\dagger\dagger}$ does not require left adjoints to be defined at all. From the right adjunction functor $(-)^\dagger$ and the monoidal isomorphism δ' one can construct left adjoints via

$$ {}^\dagger X := X^\dagger \;, \quad \mathrm{ev}_X := \widetilde{\mathrm{ev}}_{X^\dagger} \circ (1_{X^\dagger} \otimes \delta'_X) \;, \quad \mathrm{coev}_X := (1 \otimes \delta'^{-1}_X) \circ \widetilde{\mathrm{coev}}_{X^\dagger} \;. \tag{2.4.5}$$

  Analogously, one can construct right adjoints from the left adjunction functor ${}^\dagger(-)$ and a monoidal isomorphism $\delta''\colon \mathrm{Id} \Rightarrow {}^{\dagger\dagger}(-)$.
- In the first formulation, one may also redefine left adjoints to be equal to right adjoints using the pivotal structure $\delta_X\colon {}^\dagger X \to X^\dagger$:

$$ {}^\dagger X := X^\dagger \;, \quad \mathrm{ev}_X := \mathrm{ev}_{X,\mathrm{old}} \circ (\delta_X \otimes 1_X) \;, \quad \mathrm{coev}_X := (1 \otimes \delta_X^{-1}) \circ \mathrm{coev}_{X,\mathrm{old}} \;. \tag{2.4.6}$$

  This way, ${}^\dagger X = X^\dagger$ for all 1-morphisms X.

It turns out that $\mathcal{LG}_k$ (and, by extension, $\ddot{\mathsf{MF}}_k$) is *not* pivotal: For odd $\ell(\boldsymbol{x})+\ell(\boldsymbol{z})$ the left and right adjoint $X^\dagger = X^\vee[\ell(\boldsymbol{x})]$, ${}^\dagger X \cong X^\vee[\ell(\boldsymbol{z})]$ are, in general, not isomorphic (see Remark 1.3.23). There is, however, a weaker notion of *graded pivotality* which is fulfilled in subcategories of $\ddot{\mathsf{MF}}_k$ that have adjoints ($\mathcal{LG}_k$ being one example).

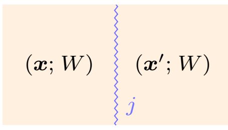
2.4.2 Graded bicategories and shifted identity lines

We will only discuss the special case of $\ddot{\mathsf{MF}}_k$ here and refer to [22, Section 7] for the general definitions of graded bicategories and graded pivotality. This section generalises results from [22] and [23, Section 7.2], the latter being an earlier version of [22]. Throughout this section we use the following setup:

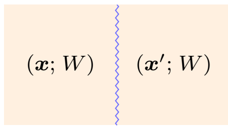
$$X\colon (\boldsymbol{x};\, W) \to (\boldsymbol{z};\, V)\ , \quad Y\colon (\boldsymbol{z};\, V) \to (\boldsymbol{y};\, U)\ , \quad n := \ell(\boldsymbol{x})\ ,\ m := \ell(\boldsymbol{z})\ ,\ p := \ell(\boldsymbol{y})\ .$$

We first introduce diagrammatic rules for grade-shifted identity lines.

Notation 2.4.3. The identity 1-morphism grade-shifted by $j \in \mathbb{Z}$ is displayed as follows:

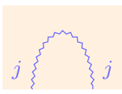


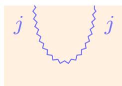
$:= \Delta_W[j]\ ,$



$:= \Delta_W[\ell(\boldsymbol{x})]\ , \qquad (2.4.7)$

i.e. the grade shift of a wiggly line without label is given by $\ell(\boldsymbol{x})$. For $\Delta_W[j]$ there are 2-isomorphisms

$\mu =$


$: \Delta_W[j] \otimes \Delta_W[j] \to \Delta_W\ ,\qquad \mu^{-1} =$


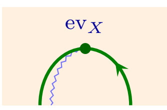
$: \Delta_W \to \Delta_W[j] \otimes \Delta_W[j]$

defined in [22, p. 538], which make $\Delta_W[j]$ left and right adjoint to itself. Therefore, $\Delta_W[j]$ may be drawn without an orientation.

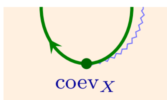
Notation 2.4.4. We may rewrite the adjoints of X as follows:

$$\begin{aligned}{}^\dagger X \overset{\rho^{-1}}{\cong} {}^\dagger X \otimes \Delta_V \quad &= \quad X^\vee \otimes S[m] \otimes \Delta_V = X^\vee \otimes \Delta_V[m]\ ,\\ X^\dagger = R[n] \otimes X^\vee \overset{1\otimes\lambda^{-1}}{\cong} &\quad R[n] \otimes \Delta_W \otimes X^\vee = \Delta_W[n] \otimes X^\vee\ .\end{aligned}$$

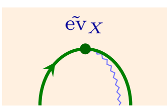
For the remainder of this chapter we define a downwards oriented line in $\ddot{\mathsf{MF}}_k$ labelled X to be $X^\vee$ (instead of ${}^\dagger X$ or $X^\dagger$). This way, downwards oriented lines are unique and no longer depend on whether they originate from left or right adjunction. In this modified convention, the adjunction 2-morphisms are visualised as follows:



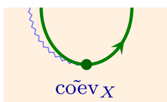
$: {}^\dagger X \otimes X \to \Delta_W\ ,$



$: \Delta_V \to X \otimes {}^\dagger X\ ,$



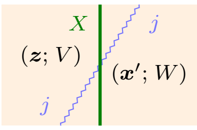
$: X \otimes X^\dagger \to \Delta_V\ ,$



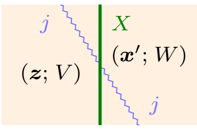
$: \Delta_W \to X^\dagger \otimes X\ .$

The next step is to introduce rules that allow grade-shifted identity lines to cross other 1-morphisms, first discussed in [23] and restated in the language of [22] here.

Definition 2.4.5. Crossings between shifted identity lines and other 1-morphisms are defined as follows:

$\omega_X :=$ 

$$:= \rho^{-1}_{X\otimes R[j]} \circ \sigma^{(2,1)} \circ (1_{S[j]} \otimes \lambda_X) : \\ (S[j] \otimes \Delta_V) \otimes X \to X \otimes (R[j] \otimes \Delta_W) \ , \tag{2.4.8}$$

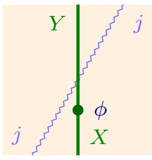
$\omega_X^{-1} :=$ 

$$:= (1_{S[j]} \otimes \lambda_X^{-1}) \circ \sigma^{(2,1)} \circ \rho_{X\otimes R[j]} : \\ X \otimes (R[j] \otimes \Delta_W) \to (S[j] \otimes \Delta_V) \otimes X \ , \tag{2.4.9}$$

with $\sigma^{(2,1)}$ of Eq. (1.3.49) being the canonical isomorphism between $S[j] \otimes_S X$ and $X \otimes_R R[j]$.

Eqs. (2.4.8) and (2.4.9) are clearly inverse to each other. Furthermore, ω_X is natural with respect to the action of other 2-morphisms:

Lemma 2.4.6. *For all $\phi\colon X \to Y$, the following holds:*



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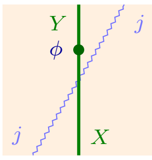
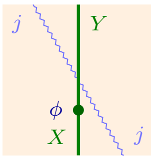
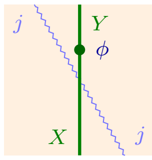


$,$


$=$


$.$ (2.4.10)

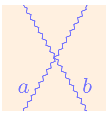
Proof. σ and ϕ commute in the expected way:

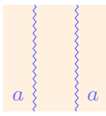
$$\begin{aligned}(\sigma \circ (1_{S[j]} \otimes \phi))(1_j \otimes x) &= \sigma\big((-1)^{j|\phi|} 1_j \otimes \phi(x)\big) &&= (-1)^{j|x|} \phi(x) \otimes 1_j \ , \\ ((\phi \otimes 1_{R[j]}) \circ \sigma)(1_j \otimes x) &= (\phi \otimes 1_{S[j]})\big((-1)^{j|x|} x \otimes 1_j\big) &&= (-1)^{j|x|} \phi(x) \otimes 1_j \ ,\end{aligned}$$

implying $\sigma \circ (1_{S[j]} \otimes \phi) = (\phi \otimes 1_{R[j]}) \circ \sigma$. The rest of the argument follows from the naturality properties of λ and ρ presented in Eq. (1.5.20), which apply to all bicategories. □

Finally, under certain conditions it is possible to resolve crossings of shifted identity lines [23, Eq. (7.2)]:

Lemma 2.4.7. *If $a \equiv b \bmod 2$, the following identity holds:*



$= (-1)^a$


$.$ (2.4.11)

Proof. Note that $a \equiv b \bmod 2$ is required for Eq. (2.4.11) to be well-defined, as otherwise the domains and codomains do not match. We define the isomorphism

$$\tau := \sigma^{(2,1)} \colon R[b] \otimes \Delta_W \to \Delta_W \otimes R[b] \ , \quad 1_b \otimes \alpha \mapsto (-1)^{b|\alpha|} \alpha \otimes 1_b$$

and precompose both sides of Eq. (2.4.11) with $1_{R[a]} \otimes \tau \otimes 1_{\Delta_W}$. Let $\alpha \in \Delta_W$. We find

$$\begin{aligned}(\lambda_{\Delta_W[b]} \circ (\tau \otimes 1_{\Delta_W}))(1_b \otimes \theta_{i_1} \dots \theta_{i_k} \otimes \alpha) &= (-1)^{b\cdot k} \lambda_{\Delta_W[b]}(\theta_{i_1} \dots \theta_{i_k} \otimes (1_b \otimes \alpha)) \\ &= \delta_{k,0}\, 1_b \otimes \alpha = (1_{R[b]} \otimes \lambda_{\Delta_W})(1_b \otimes \theta_{i_1} \dots \theta_{i_k} \otimes \alpha) \ ,\end{aligned}$$

implying $(\lambda_{\Delta_W[b]} \circ (\tau \otimes 1_{\Delta_W})) = 1_{R[b]} \otimes \lambda_{\Delta_W}$. The modified left hand side is now given by

$$\begin{aligned} &\rho^{-1}_{R[b]\otimes\Delta_W\otimes R[a]} \circ \sigma \circ (1_{R[a]} \otimes \lambda_{R[b]\otimes\Delta_W}) \circ (1_{R[a]} \otimes \tau \otimes 1_{\Delta_W}) \\ &= \rho^{-1}_{R[b]\otimes\Delta_W\otimes R[a]} \circ \sigma \circ (1_{R[a]} \otimes 1_{R[b]} \otimes \lambda_{\Delta_W}) \end{aligned}$$

with $\sigma\colon R[a] \otimes (R[b] \otimes \Delta_W) \to (R[b] \otimes \Delta_W) \otimes R[a]$. We use the naturality of ρ^{-1}:

$$= (\sigma \otimes 1_{\Delta_W}) \circ \rho^{-1}_{R[a]\otimes R[b]\otimes\Delta_W} \circ (1_{R[a]} \otimes 1_{R[b]} \otimes \lambda_{\Delta_W})$$

We now use $R[a] \otimes R[b] = R[a+b] = R$:

$$= (\sigma \otimes 1_{\Delta_W}) \circ (1_{R[a]} \otimes 1_{R[b]} \otimes \rho^{-1}_{\Delta_W}) \circ (1_{R[a]} \otimes 1_{R[b]} \otimes \lambda_{\Delta_W})$$

λ_{Δ_W} and $\rho^{-1}_{\Delta_W}$ compose to $1_{\Delta_W\otimes\Delta_W}$, as discussed in Example 1.2.20:

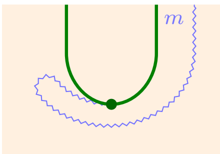
$$= \sigma \otimes 1_{\Delta_W} \ .$$

We compare the modified left and right hand side by acting on $\alpha, \beta \in \Delta_W$:

$$\begin{aligned} (\sigma \otimes 1_{\Delta_W})(1_a \otimes 1_b \otimes \alpha \otimes \beta) &= (-1)^{a(b+|\alpha|)}(1_b \otimes \alpha \otimes 1_a \otimes \beta) \ , \\ (1_{R[a]} \otimes \tau \otimes 1_{\Delta_W})(1_a \otimes 1_b \otimes \alpha \otimes \beta) &= (-1)^{b|\alpha|}(1_a \otimes \alpha \otimes 1_b \otimes \beta) \ , \\ \implies \text{LHS} &= (-1)^a \, \text{RHS} \end{aligned}$$

using $a \equiv b \bmod 2$ and $(-1)^{a\cdot b} = (-1)^a$ in the last step. □

Remark 2.4.8. Definition 2.4.5 and Lemma 2.4.6 plus self-adjointness allow us to move shifted identity lines across 1- and 2-morphisms. One does have to preserve the direction of their turnarounds — two left turns differ from the identity by a sign:



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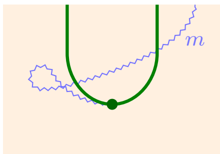
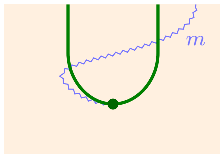


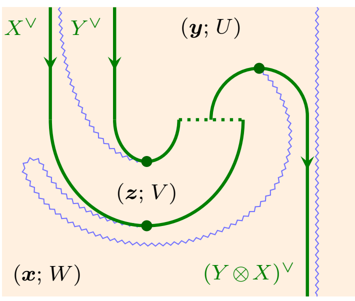
$= (-1)^m$


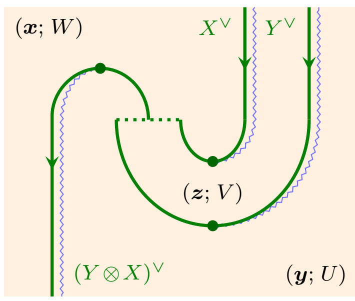
. (2.4.12)

2.4.3 Graded pivotality

Now we can formulate the statement of graded pivotality:

Theorem 2.4.9. *Let $(\boldsymbol{x}; W)$, $(\boldsymbol{z}; V)$, $(\boldsymbol{y}; U) \in \ddot{\mathsf{MF}}_k$ such that* $\mathrm{Hom}(W, V)$, $\mathrm{Hom}(V, U)$, *and* $\mathrm{Hom}(W, U)$ *have left and right admissible variables. Set $n := \ell(\boldsymbol{x})$, $m := \ell(\boldsymbol{z})$, $p := \ell(\boldsymbol{y})$. Then for all 1-morphisms $X\colon (\boldsymbol{x}; W) \to (\boldsymbol{z}; V)$ and $Y\colon (\boldsymbol{z}; V) \to (\boldsymbol{y}; U)$, the following holds:*



$= (-1)^m$


. (2.4.13)

Remark 2.4.10. The factor of $(-1)^m$ in Eq. (2.4.13) is also present in [23, Eq. (7.13)],[6] but not in [22, Eq. (7.5)] due to the different sign convention explained in Remark 2.1.8.

Proof. The special case where X, Y are 1-morphisms in $\mathcal{LG}_k$ is proven in [22, Section 7]. The generalisation to $\ddot{\mathsf{MF}}_k$ with admissible variables requires only slight modifications, which will be explained here. We define $\bar{Y}\otimes\bar{X}$ by

$$\bar{N} := k[\boldsymbol{x},\boldsymbol{z}]/(\partial_{\boldsymbol{v}}(V-W)) \ , \quad \bar{Y}\otimes\bar{X} := (Y\otimes_{k[\boldsymbol{z}]} X)\otimes_{k[\boldsymbol{x},\boldsymbol{z}]}\bar{N} \tag{2.4.14}$$

for left admissible variables $\{v_1,\,\dots\,,v_m\}$ of X. The split monomorphism $\kappa\colon Y\otimes X\to\bar{Y}\otimes\bar{X}$ can be constructed in analogy to [22, Appendix A] and Theorem 2.3.14, utilising the assumptions of admissible variables.

Let $\{g_\alpha\}_\alpha$ be a finite $k[\boldsymbol{x}]$-basis of $\bar{N}$. We employ an argument similar to the one in Lemma 2.3.17: With

$$k' := k[\boldsymbol{x}] \ , \quad S' := k'[\boldsymbol{z}] \ , \quad R' := k' \ , \quad \mathbb{B}' := \Omega_{k'}S_1\otimes_{k'}S_2' \tag{2.4.15}$$

we regard $\mathrm{At}_S(X^\dagger\otimes Y^\dagger)$ of [22, Eq. (7.8)] as a map

$$\mathrm{At}_S(X^\dagger\otimes Y^\dagger)\colon (X^\dagger\otimes_S Y^\dagger)\otimes_{S'}\mathbb{B}'\to(X^\dagger\otimes_S Y^\dagger)\otimes_{S'}\mathbb{B}' \ . \tag{2.4.16}$$

Now the image of $\epsilon\Psi$ can be written as

$$(X^\dagger\otimes_S Y^\dagger)\otimes_{S'}S'^e = (X^\dagger\otimes_S Y^\dagger)\otimes_{k[\boldsymbol{x},\boldsymbol{z}]}\big(k[\boldsymbol{x},\boldsymbol{z}]\otimes_{k[\boldsymbol{x}]}k[\boldsymbol{x},\boldsymbol{z}']\big) \ .$$

Then $1\otimes g_\alpha^*$ is well-defined on S'^e, and we find that [22, Eqs. (7.9), (7.10)] hold in this generalised setting (with an extra factor of $(-1)^m$ in $\bar{\mathscr{R}}$).

The rest of the proof is analogous: By the same argument as above, there is a split monomorphism

$$^\dagger X\otimes_S{}^\dagger Y \underset{\hat{\rho}}{\overset{\hat{\kappa}}{\rightleftarrows}} ({}^\dagger X\otimes_S{}^\dagger Y)\otimes_{k[\boldsymbol{x},\boldsymbol{z}]}\bar{N} =: {}^\dagger\bar{X}\otimes{}^\dagger\bar{Y}$$

with $\hat{\rho}\circ\hat{\kappa}=1$. Therefore, to show that both sides of Eq. (2.4.13) are homotopic, it is sufficient to show that they are equal after post-composing with $\hat{\kappa}$. As in [22], $\bar{X}\otimes\bar{Y}$ has a basis $\{e_i^*\otimes f_j^*\otimes g_\alpha\}_{i,j,\alpha}$. Furthermore, $g_\beta^*\otimes g_\alpha^*$ is well-defined on S'^e, and the rest of the proof can be copied verbatim. □

Remark 2.4.11. There are several ways one could define $\bar{Y}\otimes\bar{X}$. For example, one could use a different set of admissible variables $v_1',\,\dots\,,v_m'$ of X, or one could define $\bar{Y}\otimes\bar{X}$ to be

$$(Y\otimes_{k[\boldsymbol{z}]}X)\otimes_{k[\boldsymbol{x},\boldsymbol{y}]}k[\boldsymbol{x},\boldsymbol{y}]/(\partial_{w_1}(V-U),\,\dots\,,\partial_{w_m}(V-U))$$

for right admissible variables $w_1,\,\dots\,,w_m$ of Y. In general, these different definitions of $\bar{Y}\otimes\bar{X}$ are not isomorphic as a matrix factorisations. However, the adjoints of $Y\otimes X$ constructed from different definitions of $\bar{Y}\otimes\bar{X}$ are isomorphic according to the uniqueness theorem of adjoints.

As discussed above, we usually find ${}^\dagger X\not\cong X^\dagger$ if $\ell(\boldsymbol{x})+\ell(\boldsymbol{z})$ is odd, disproving the existence of an ordinary pivotal structure. However, in the even case we do find ordinary pivotality:

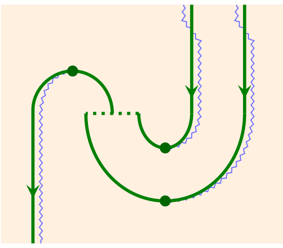
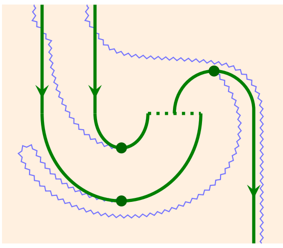
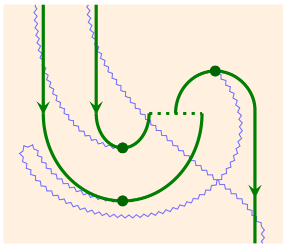
[6]There is a minor inconsistency in [23, Eq. (7.13)] which is related to Eq. (2.4.12) and was corrected here.

Corollary 2.4.12. *In the setting of Theorem 2.4.9, assume $n \equiv m \equiv p \bmod 2$. Define δ_X to be the crossing isomorphism ω_X^{-1} of Eq. (2.4.9), i.e.*

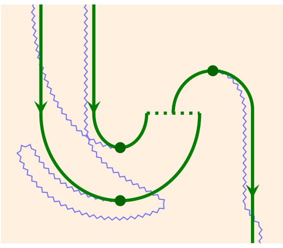
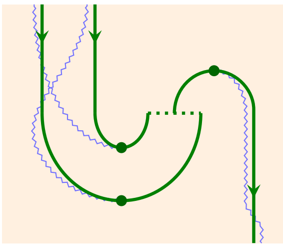
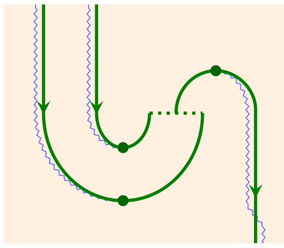
$$\delta_X \colon X^\vee \otimes S[m] \to R[m] \otimes X^\vee \ , \quad x \otimes 1_m \mapsto (-1)^{m|x|} 1_m \otimes x \ . \tag{2.4.17}$$

Then δ_X is an ordinary pivotal structure (Definition 2.4.1).

Proof. We need to show Eq. (2.4.4) with $\delta_X := \omega_X^{-1}$. All labels on shifted identity lines can be omitted here since n, m, and p are either all odd or all even. We post-compose both sides of Eq. (2.4.13) with $(-1)^m \delta_X \otimes \delta_Y$ and apply the rules summarised in Remark 2.4.8:

 $= (-1)^m$  $= (-1)^m$ 

Now we use Eq. (2.4.12) and Lemma 2.4.7:

$=$  $= (-1)^m$  $=$  ,

proving $(\delta_X \otimes \delta_Y) \circ \mathscr{L} = \mathscr{R} \circ \delta_{Y \otimes X}$. □

Remark 2.4.13. In the conventions of [22], the pivotal structure for $n+m$ even is given by

$$(-1)^{m(|-|+1)} \colon X^\vee \otimes S[m] \to R[m] \otimes X^\vee \ ,$$

which has a sign of $(-1)^m$ relative to δ_X as defined in Eq. (2.4.17).

Corollary 2.4.14. *Define the subcategories*

$$\ddot{\mathsf{MF}}_k^{\text{even}}, \ddot{\mathsf{MF}}_k^{\text{odd}} \subset \ddot{\mathsf{MF}}_{\mathbb{C}}(\boldsymbol{x}, \boldsymbol{y}) \tag{2.4.18}$$

to be the subcategories whose objects $(\boldsymbol{a}; W)$ all have an even (resp. odd) number of surface variables $\ell(\boldsymbol{a})$. Then for every subcategory $\mathcal{B} \subset \ddot{\mathsf{MF}}_k$ that is pivotal with the pivotal structure of Corollary 2.4.12, either $\mathcal{B} \subset \ddot{\mathsf{MF}}_k^{\text{even}}$ or $\mathcal{B} \subset \ddot{\mathsf{MF}}_k^{\text{odd}}$ holds.

2.5 Defect operators and quantum dimensions

2.5.1 Definition

Now that we have a pivotal structure we can turn our attention back to bubble-shaped diagrams. We start with the following general definitions:

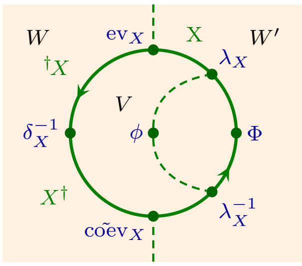
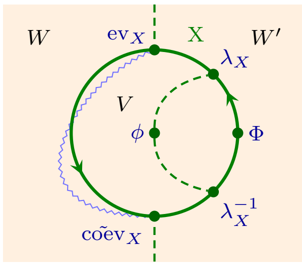
Definition 2.5.1. Let $\mathcal{B} \subset \ddot{\mathsf{MF}}_k$ be pivotal with the pivotal structure of Corollary 2.4.14, and let

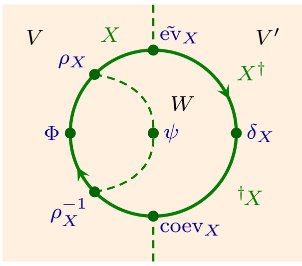
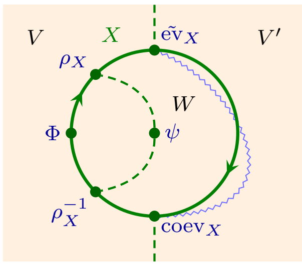
$$(\boldsymbol{x};\, W), (\boldsymbol{z};\, V) \in \mathcal{B}\ , \qquad X \in \mathrm{Hom}_{\mathcal{B}}(W, V)\ ,$$
$$\Phi \in \mathrm{End}(X)\ , \quad \phi \in \mathrm{End}(\Delta_V)\ , \quad \psi \in \mathrm{End}(\Delta_W)\ .$$

Then we define the *defect operators* [22, Eq. (8.1), 23, Eq. (8.2)]

$$\mathcal{D}_l^\Phi(X)\colon\ \mathrm{End}_{\ddot{\mathsf{MF}}_k}(\Delta_V) \to \mathrm{End}_{\ddot{\mathsf{MF}}_k}(\Delta_W)\ , \tag{2.5.1}$$
$$\mathcal{D}_r^\Phi(X)\colon\ \mathrm{End}_{\ddot{\mathsf{MF}}_k}(\Delta_W) \to \mathrm{End}_{\ddot{\mathsf{MF}}_k}(\Delta_V) \tag{2.5.2}$$

by

$\mathcal{D}_l^\Phi(X)(\phi) :=$  $=$  $,$ (2.5.3)

$\mathcal{D}_r^\Phi(X)(\psi) :=$  $=$  $.$ (2.5.4)

If the map Φ is omitted, it will be set to the identity on X, i.e.

$$\mathcal{D}_l(X)(\phi) := \mathcal{D}_l^{1_X}(X)(\phi)\ , \qquad \mathcal{D}_r(X)(\psi) := \mathcal{D}_r^{1_X}(X)(\psi)\ . \tag{2.5.5}$$

We further define the *quantum dimensions*

$$\dim_l X := \mathcal{D}_l(X)(1_{\mathbb{1}_V})\ , \qquad \dim_r X := \mathcal{D}_r(X)(1_{\mathbb{1}_W}) \tag{2.5.6}$$

to be the defect operators with Φ, ϕ, and ψ being identity 2-morphisms.

Remark 2.5.2. A pivotal structure δ_X fixes the value of the defect operators uniquely. Consider $\mathcal{D}_r^\Phi(X)$, which consists of the 2-morphisms $\widetilde{\mathrm{ev}}_X$, δ_X, and coev_X. Rescaling the pivotal structure $\delta_X \mapsto \alpha \cdot \delta_X$ is not possible: The left hand side of Eq. (2.4.4) is proportional to α while the right hand side is proportional to α^2, so $\alpha \cdot \delta_X$ is not a pivotal structure unless $\alpha = 1$. Furthermore, rescaling $(\mathrm{ev}_X, \mathrm{coev}_X) \mapsto (\alpha\, \mathrm{ev}_X, \alpha^{-1}\, \mathrm{coev}_X)$ necessitates a rescaling of $\delta_X \mapsto \alpha \cdot \delta_X$ to preserve Eq. (2.4.4), and the defect operators are invariant under this modification. The analogous statement holds for rescaling $(\widetilde{\mathrm{ev}}_X, \widetilde{\mathrm{coev}}_X)$. Therefore, the values of the defect operators are unique up to the existence of a different, inequivalent pivotal structure.

2.5.2 Closed formulas

The following known result on the spectrum $\Delta_W \to \Delta_W$ applies to $\ddot{\mathsf{MF}}_k(\boldsymbol{w})$ as well:

Lemma 2.5.3. *The morphisms on the identity line* $\Delta_W \colon (\boldsymbol{x}'; W(\boldsymbol{x}')) \to (\boldsymbol{x}; W(\boldsymbol{x}))$ *can be identified with [22, p. 545]*

$$\mathrm{End}_{\ddot{\mathsf{MF}}_k(\boldsymbol{w})}(\Delta_W) \cong k[\boldsymbol{x}, \boldsymbol{w}]/(\partial_{x_1} W, \, \dots \, , \partial_{x_n} W) \tag{2.5.7}$$

via

$$\theta_{i_1} \dots \theta_{i_k} \mapsto \alpha \cdot \theta_{i_1} \dots \theta_{i_k} \quad \textit{for } \alpha \in k[\boldsymbol{x}, \boldsymbol{w}]/(\partial_{x_1} W, \, \dots \, , \partial_{x_n} W) \ . \tag{2.5.8}$$

In particular, every 2-morphism on $\mathbb{1}_W$ *is equal to* $\alpha \cdot 1_{\mathbb{1}_W}$ *up to homotopy for some* α*, and* $\mathrm{End}_{\ddot{\mathsf{MF}}_k(\boldsymbol{w})}(\Delta_W)$ *is commutative.*

Remark 2.5.4. As discussed before, the spectrum of 2-morphisms on the identity 1-morphism corresponds to the Hilbert space of local bulk operators (field insertions), which is given by the Jacobi ring $k[\boldsymbol{x}, \boldsymbol{w}]/(\partial_{x_1} W, \, \dots \, , \partial_{x_n} W)$ in $\ddot{\mathsf{MF}}_k$ and $\mathcal{LG}$.

By Lemma 2.5.3 we may identify $\mathcal{D}_l^\Phi(X)(\phi)$ and $\mathcal{D}_r^\Phi(X)(\psi)$ with values in $k[\boldsymbol{x}, \boldsymbol{w}]/(\partial W)$ or $k[\boldsymbol{z}, \boldsymbol{w}]/(\partial V)$, respectively. The following central theorem establishes closed formulas for both.

Theorem 2.5.5. *Let*

$$R = k[x_1, \, \dots \, , x_n, \boldsymbol{w}] \ , \quad S = k[z_1, \, \dots \, , z_m, \boldsymbol{w}] \ , \quad n \equiv m \mod 2 \ ,$$

and let $X \colon (\boldsymbol{x}; W(\boldsymbol{x}, \boldsymbol{w})) \to (\boldsymbol{z}; V(\boldsymbol{z}, \boldsymbol{w}))$ *be a finite-rank matrix factorisation with right admissible variables* $\{u_1, \, \dots \, , u_n\}$ *and left admissible variables* $\{v_1, \, \dots \, , v_m\}$*. Then the defect operators of Definition 2.5.1 take the values*

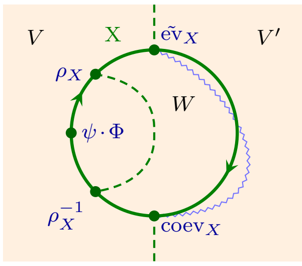
$$\mathcal{D}_l^\Phi(X)(\phi) = (-1)^{\binom{n+1}{2}} \operatorname{Res}_{R[\boldsymbol{z}]/R} \left[\frac{\phi \operatorname{str}\{\Phi(\prod_{i=1}^n \partial_{x_i} d_X)(\prod_{j=1}^m \partial_{v_j} d_X)\} \, \mathrm{d}\boldsymbol{z}}{\partial_{v_1}(V-W), \, \dots \, , \partial_{v_m}(V-W)} \right] \ , \tag{2.5.9}$$

$$\mathcal{D}_r^\Phi(X)(\psi) = (-1)^{\binom{m+1}{2}} \operatorname{Res}_{S[\boldsymbol{x}]/S} \left[\frac{\psi \operatorname{str}\{\Phi(\prod_{i=1}^n \partial_{u_i} d_X)(\prod_{j=1}^m \partial_{z_j} d_X)\} \, \mathrm{d}\boldsymbol{x}}{\partial_{u_1}(W-V), \, \dots \, , \partial_{u_n}(W-V)} \right] \ . \tag{2.5.10}$$

As in Example 2.3.3, these formulas agree with [21, Eq. (3.1)] for $\boldsymbol{w} = \emptyset$ *and thus generalise the result on* $\mathcal{LG}_k$*.*

Remark 2.5.6. While the right admissible variables $\{u_1, \, \dots \, , u_n\}$ do not appear in the formula of $\mathcal{D}_l^\Phi(X)$, there is no meaningful interpretation of $\mathcal{D}_l^\Phi(X)$ if X does not have a right adjoint. In that case $\widetilde{\mathrm{coev}}_X$ is still a well-defined morphism, but has no interpretation in terms of a right Zorro move. The analogous statement holds for $\{v_1, \, \dots \, , v_m\}$ and $\mathcal{D}_r^\Phi(X)$.

Proof of Theorem 2.5.5. Lemma 2.5.3 has several applications here: First, the action of ψ can be written as a multiplication by some $\psi \in k[\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{w}]$, which may, equivalently, be relocated to any point touching the surface W. In particular, we may relocate it to Φ:

$$\mathcal{D}_r^{\Phi}(X)(\psi) =$$


.

By the properties of Δ_W, ρ_X, and ρ_X^{-1}, the identity line in the bubble can then be removed. Invoking Lemma 2.5.3 a second time, we find that it is sufficient to evaluate $\mathcal{D}_r^{\Phi}(X)(\psi)$ on $1 \in \Delta_V$ (θ-order 0) and drop all terms $\theta_{i_1} \dots \theta_{i_k}$ for $k > 0$ from the result.

We now choose a homogeneous basis $\{e_i\} \subset X$, write Φ as a matrix $\Phi\colon e_i \mapsto \sum_j \Phi_{ji} e_j$, and evaluate $(\psi \cdot \Phi \otimes \delta_X) \circ \text{coev}_X$ on 1.

$$\begin{aligned}
&\big((\psi \cdot \Phi \otimes \delta_X) \circ \text{coev}_X\big)(1) \\
&= \psi \cdot (\Phi \otimes \delta_X)\Big(\sum_{i,j} (-1)^{\binom{m+1}{2}+m^2} \{\partial_{[1]}^{\boldsymbol{z},\boldsymbol{z}'} d_X \dots \partial_{[m]}^{\boldsymbol{z},\boldsymbol{z}'} d_X\}_{ij} e_i \otimes e_j^*\Big) \\
&= \psi \sum_{i,j,k} (-1)^{\binom{m+1}{2}+m} \{\partial_{[1]}^{\boldsymbol{z},\boldsymbol{z}'} d_X \dots \partial_{[m]}^{\boldsymbol{z},\boldsymbol{z}'} d_X\}_{ij} (\Phi_{ki} e_k) \otimes (-1)^{n|e_j|} e_j^* \\
&= \psi \sum_{j,k} (-1)^{\binom{m+1}{2}+m} \{\Phi(\textstyle\prod_i \partial_{[i]}^{\boldsymbol{z},\boldsymbol{z}'} d_X)(-1)^{n|-|}\}_{kj} e_k \otimes e_j^* \ .
\end{aligned}$$

Now we find

$$\begin{aligned}
\mathcal{D}_r^{\Phi}(X)(\psi) &= \big(\tilde{\text{ev}}_X \circ (\psi \cdot \Phi \otimes \delta_X) \circ \text{coev}_X\big)(1)\big|_{\{\theta_i\}_i = 0} \\
&= \sum_{i,k} (-1)^{\binom{m+1}{2}+m} \operatorname{Res}\left[\frac{\psi\, (-1)^{(n+1)|e_k|} \{\Phi(\prod_i \partial_{[i]}^{\boldsymbol{z},\boldsymbol{z}'} d_X)(-1)^{n|-|}\}_{kj} \Lambda_{jk}^{(x)}\, \mathrm{d}\boldsymbol{x}}{\partial_{u_1}(W-V),\ \dots\ ,\ \partial_{u_n}(W-V)}\right] .
\end{aligned}$$

Let us simplify the numerator separately. According to Lemma 2.5.3, $z_i - z_i'$ is exact in $\text{End}(\Delta_V)$. Because of $\partial_{[i]}^{\boldsymbol{z},\boldsymbol{z}'} d_X = \partial_{z_i} d_X(\boldsymbol{x}, \boldsymbol{z}) + (\boldsymbol{z} - \boldsymbol{z}') \cdot \dots$ and the linearity of the residue operator in $k[\boldsymbol{z}, \boldsymbol{z}']$, we may replace $\partial_{[i]}^{\boldsymbol{z},\boldsymbol{z}'} d_X \mapsto \partial_{z_i} d_X$ here [22, p. 546]. We find

$$\begin{aligned}
&\textstyle\sum_{j,k} (-1)^{(n+1)|e_k|} \{\Phi(\prod_i \partial_{z_i} d_X)(-1)^{n|-|}\}_{kj} \Lambda_{jk}^{(x)} \\
&= \operatorname{tr}\{(-1)^{(n+1)|-|} \Phi(\textstyle\prod_{j=1}^m \partial_{z_j} d_X)(-1)^{n|-|} \Lambda^{(x)}\} \\
&= \operatorname{str}\{(-1)^{n|-|} \Phi(\textstyle\prod_{j=1}^m \partial_{z_j} d_X)(-1)^{n|-|} \Lambda^{(x)}\}
\end{aligned}$$

As $|\prod_{j=1}^m \partial_{z_j} d_X| = m$ and $|\Phi| = 0$, the conjugation with $(-1)^{n|-|}$ yields a factor of $(-1)^{m \cdot n}$:

$$\begin{aligned}
&= (-1)^{m \cdot n} \operatorname{str}\{\Phi(\textstyle\prod_{j=1}^m \partial_{z_j} d_X)(-1)^n (\prod_{k=1}^n \partial_{u_k} d_X)\} \\
&= (-1)^n \operatorname{str}\{(\textstyle\prod_{k=1}^n \partial_{u_k} d_X)\Phi(\prod_{j=1}^m \partial_{z_j} d_X)\} \ ,
\end{aligned}$$

where we get another factor of $(-1)^{m \cdot n}$ from $\operatorname{str} AB = (-1)^{|A||B|} \operatorname{str} BA$. Putting all back together and using $(-1)^{m+n} = 1$, we find (with $S^e = k[\boldsymbol{z}, \boldsymbol{z}', \boldsymbol{w}]$)

$$\mathcal{D}_r^{\Phi}(X)(\psi) = (-1)^{\binom{m+1}{2}} \operatorname{Res}_{S^e[\boldsymbol{x}]/S^e}\left[\frac{\psi \operatorname{str}\{(\prod_{k=1}^n \partial_{u_k} d_X)\Phi(\prod_{j=1}^m \partial_{z_j} d_X)\}\, \mathrm{d}\boldsymbol{x}}{\partial_{u_1}(W-V),\ \dots\ ,\ \partial_{u_n}(W-V)}\right] . \tag{2.5.11}$$

A short argument shows that we may move Φ to an arbitrary location inside the supertrace:

$$\begin{aligned} & 0 = [d_X, \Phi] = d_X\Phi - \Phi d_X && \mid \partial/\partial_s \\ \implies \quad & 0 = \partial_s d_X \Phi + d_X \partial_s \Phi - \partial_s \Phi d_X - \Phi \partial_s d_X \\ \implies \quad & [\partial_s d_X, \Phi] = [d_X, -\partial_s \Phi] = d(-\partial_s \Phi) \end{aligned}$$

for all $s \in \{\boldsymbol{z}, \boldsymbol{u}\}$. Thus, commuting Φ with $\partial_s d_X$ yields an exact term which does not contribute to the defect action,[7] so we may commute Φ to the front of Eq. (2.5.11). Finally, $\boldsymbol{z} - \boldsymbol{z}'$ is exact on $\mathrm{Hom}(\Delta_V, \Delta_V)$ and $\boldsymbol{z}'$ no longer appears in Eq. (2.5.11), so we may remove $\boldsymbol{z}'$ entirely and get Eq. (2.5.10).

The argument for $\mathcal{D}_l^{\Phi}(X)$ is analogous; the only extra step needed is to replace $\Phi(\boldsymbol{x}', \boldsymbol{z})$ by $\Phi(\boldsymbol{x}, \boldsymbol{z}) + (\boldsymbol{x}' - \boldsymbol{x}) \cdot \partial^{\boldsymbol{x}', \boldsymbol{x}} \Phi$, and the latter part can be dropped because $\boldsymbol{x}' - \boldsymbol{x}$ is exact on $\mathrm{Hom}(\Delta_W, \Delta_W)$. □

2.5.3 Properties of the defect operators

Grade shifts

The following result on the quantum dimension will be needed in Chapter 3.

Lemma 2.5.7. *Under the assumptions of Theorem 2.5.5 we find*

$$\dim_r(X[j]) = (-1)^j \dim_r X \ , \qquad \dim_l(X[j]) = (-1)^j \dim_l X \ . \tag{2.5.12}$$

Proof. The case of even j is trivial, so we consider $j = 1$. We study the odd identity map (see Lemma 1.3.22)

$$\begin{gathered} \phi\colon (X, d_X) \to (X[1], -d_X) \ , \quad e_i \mapsto e_i \ , \quad |\phi| = 1 \ , \\ \mathrm{d}\phi = 0 \implies d_{X[1]} = -\phi^{-1} \circ d_X \circ \phi \ . \end{gathered}$$

Let $\{u_1, \ldots, u_n\}$ be right admissible variables of $\ddot{\mathsf{MF}}_k(W, V)$. We compute the quantum dimension

$$\begin{aligned} \dim_r X[1] &= (-1)^{\binom{m+1}{2}} \operatorname{Res} \left[\frac{\mathrm{str}\{ (\prod_k \partial_{u_k} d_{X[1]}) (\prod_j \partial_{z_j} d_{X[1]}) \} \, \mathrm{d}\boldsymbol{x}}{\partial_{u_1}(W - V), \ \ldots, \ \partial_{u_n}(W - V)} \right] \\ &= (-1)^{\binom{m+1}{2}} \operatorname{Res} \left[\frac{\mathrm{str}\{ (\prod_k \partial_{u_k} (-\phi^{-1} d_X \phi)) (\prod_j \partial_{z_j} (-\phi^{-1} d_X \phi)) \} \, \mathrm{d}\boldsymbol{x}}{\partial_{u_1}(W - V), \ \ldots, \ \partial_{u_n}(W - V)} \right] \\ &= (-1)^{\binom{m+1}{2} + n + m} \operatorname{Res} \left[\frac{\mathrm{str}\{ \phi^{-1} (\prod_k \partial_{u_k} d_X) (\prod_j \partial_{z_j} d_X) \phi \} \, \mathrm{d}\boldsymbol{x}}{\partial_{u_1}(W - V), \ \ldots, \ \partial_{u_n}(W - V)} \right] \end{aligned}$$

Now $|\phi| = 1$, $|\phi^{-1} (\prod_k \partial_{u_k} d_X) (\prod_j \partial_{z_j} d_X)| = n + m + 1$

$$\begin{aligned} &= (-1)^{\binom{m+1}{2} + n + m + 1 \cdot (m + n + 1)} \operatorname{Res} \left[\frac{\mathrm{str}\{ \phi \phi^{-1} (\prod_k \partial_{u_k} d_X) (\prod_j \partial_{z_j} d_X) \} \, \mathrm{d}\boldsymbol{x}}{\partial_{u_1}(W - V), \ \ldots, \ \partial_{u_n}(W - V)} \right] \\ &= (-1)^{\binom{m+1}{2} + 1} \operatorname{Res} \left[\frac{\mathrm{str}\{ (\prod_k \partial_{u_k} d_X) (\prod_j \partial_{z_j} d_X) \} \, \mathrm{d}\boldsymbol{x}}{\partial_{u_1}(W - V), \ \ldots, \ \partial_{u_n}(W - V)} \right] \\ &= -\dim_r X \ . \end{aligned}$$

[7] $\mathrm{str}(A\, d(B)) = (-1)^{\cdots} \mathrm{str}(d(A) B)$, and $\{d_X, \partial_s d_X\}$ for $s \in \{\boldsymbol{z}, \boldsymbol{u}\}$ generates either $\partial_{z_i} V$, which is exact in $\mathrm{End}(\Delta_V)$, or $\partial_{u_i}(W - V)$, which is set to zero by the residue operator.

The argument for $\dim_l X$ is analogous. $\square$

Remark 2.5.8. It is easy to prove

$$X \cong Y \implies \dim_{l,r} X = \dim_{l,r} Y \tag{2.5.13}$$

in close analogy to the proof of Lemma 2.5.7.

Anti-bubbles

Furthermore, the following "reverse" versions of the quantum dimensions will also be needed:

Definition 2.5.9. The *anti-bubbles* of a 1-morphism $X\colon (\boldsymbol{x}; W) \to (\boldsymbol{z}; V)$ in $\ddot{\mathsf{MF}}_k$ are defined by

$$\mathrm{ab}_l(X) := \widetilde{\mathrm{coev}}_X \circ \mathrm{ev}_X \circ (\delta_X^{-1} \otimes 1_X)\colon X^\dagger \otimes X \to X^\dagger \otimes X \ , \tag{2.5.14}$$

$$\mathrm{ab}_r(X) := (1_X \otimes \delta_X) \circ \mathrm{coev}_X \circ \widetilde{\mathrm{ev}}_X\colon X \otimes X^\dagger \to X \otimes X^\dagger \ . \tag{2.5.15}$$

Under certain conditions, the anti-bubbles can be evaluated easily:

Lemma 2.5.10. *Let* $X\colon (\boldsymbol{x}; W) \to (\boldsymbol{z}; V)$ *such that*

$$X \otimes X^\dagger \cong \mathbb{1}_V \quad \textit{and} \quad X^\dagger \otimes X \cong \mathbb{1}_W \ . \tag{2.5.16}$$

Then $\dim_l X$ *and* $\dim_r X$ *are invertible, and*

$$\mathrm{ab}_l(X) = \dim_l X \cdot 1_{X^\dagger \otimes X} \ , \qquad \mathrm{ab}_r(X) = \dim_r X \cdot 1_{X \otimes X^\dagger} \tag{2.5.17}$$

up to homotopy.

Proof. In general, $X \otimes X^\dagger \cong \mathbb{1}_V \oplus \ldots$, and $\widetilde{\mathrm{ev}}_X$ is a projector from $X \otimes X^\dagger$ to one of its $\mathbb{1}_V$-components (there must be at least one, otherwise X has no right adjoint and $\widetilde{\mathrm{ev}}_X$ does not exist). Analogous statements hold for $\widetilde{\mathrm{coev}}_X$, ev_X, and coev_X. Therefore, under the assumptions of Lemma 2.5.10, all adjunction 2-morphisms of X are isomorphisms. All constituent maps of the right quantum dimension

$$\dim_r X = \widetilde{\mathrm{ev}}_X \circ (\delta_X \otimes 1) \circ \mathrm{coev}_X \tag{2.5.18}$$

are thus isomorphisms, so $\dim_r X$ is also an isomorphism and can be represented by a non-zero complex number in $k[\boldsymbol{z}]/(\partial_{\boldsymbol{z}} V)$, proving the invertibility statement. For the second statement we transform Eq. (2.5.18):

$$\begin{aligned}
\widetilde{\mathrm{ev}}_X \circ (\delta_X \otimes 1) \circ \mathrm{coev}_X &= \dim_r X \cdot 1_V && \big|\ \widetilde{\mathrm{ev}}_X^{-1} \circ \\
(\delta_X \otimes 1) \circ \mathrm{coev}_X &= \dim_r X \cdot \widetilde{\mathrm{ev}}_X^{-1} && \big|\ \circ\, \widetilde{\mathrm{ev}}_X \\
(\delta_X \otimes 1) \circ \mathrm{coev}_X \circ \widetilde{\mathrm{ev}}_X &= \dim_r X \cdot \widetilde{\mathrm{ev}}_X^{-1} \circ \widetilde{\mathrm{ev}}_X && \\
\mathrm{ab}_r(X) &= \dim_r X \cdot 1_{X \otimes X^\dagger} \ . &&
\end{aligned}$$

By assumption, X is a 1-morphism in a pivotal bicategory (otherwise the quantum dimension would not be well-defined), so ${}^\dagger X \cong X^\dagger$, hence we may replace $X^\dagger$ by ${}^\dagger X$ in Eq. (2.5.16). The formula for $\mathrm{ab}_l(X)$ can then be shown analogously. $\square$

Covariance under pivotal 2-functors

The following statement holds in every pivotal bicategory.

Lemma 2.5.11. *Let $\mathcal{B}$ and $\mathcal{C}$ be pivotal bicategories, and let $F\colon \mathcal{B} \to \mathcal{C}$ be a pivotal 2-functor (see [82, Def. 5.1.9] for the definition). Consider the following data in $\mathcal{B}$:*

(i) objects W, V,

(ii) a 1-morphism $X\colon W \to V$,

(iii) 2-morphisms $\Phi \in \mathrm{End}(X)$, $\phi \in \mathrm{End}(\mathbb{1}_V)$, $\psi \in \mathrm{End}(\mathbb{1}_W)$.

Then we find the following identities of defect operators:

$$F(\mathcal{D}_l^{\Phi}(X)(\phi)) = \mathcal{D}_l^{F(\Phi)}(F(X))(F(\phi)) \ , \quad F(\mathcal{D}_r^{\Phi}(X)(\psi)) = \mathcal{D}_r^{F(\Phi)}(F(X))(F(\psi)) \ . \tag{2.5.19}$$

Proof sketch. While slightly cumbersome to show, this ultimately follows directly from the properties of pivotal 2-functors and Eq. (2.4.2). □

3 Adjunctions and orbifolds in $\mathcal{RW}$

Having studied adjunctions and pivotality in the bicategory $\ddot{\mathsf{MF}}_k$, we may resume the discussion of $\mathcal{RW}$. As mentioned in Section 1.2.9, our goal is to construct a Gray category with strict duals $\underline{\mathcal{G}}$ related to $\mathcal{RW}$ to be able to apply the orbifold procedure.

This chapter assumes Conjecture 1.5.15, i.e. that $\mathcal{RW}$ as defined in Definition 1.5.1 is a tricategory. To construct a Gray category with strict duals $\underline{\mathcal{G}}$, we first restrict our discussion to a subcategory $\mathcal{T} \subset \mathcal{RW}$ that is conjectured to be a pivotal tricategory with duals. To do so, we construct adjoints of the 1-morphisms of $\mathcal{RW}$ in Section 3.2.1 and subsequently discuss general properties of pivotal subcategories $\mathcal{T} \subset \mathcal{RW}$ in Sections 3.2.2 to 3.2.4, including a proof that $\mathcal{RW}$ itself (unlike subcategories of $\mathcal{RW}$) cannot be pivotal. In Section 3.2.5 we explicitly construct a candidate $\mathcal{T} \subset \mathcal{RW}$ and present most of the proof that $\mathcal{T}$ is a pivotal tricategory with duals (assuming Conjecture 1.5.15). We may then apply Theorem 1.2.28 to $\mathcal{T}$ which yields the desired Gray category with strict duals.

Next we turn our attention to an application: Starting from a known defect TFT that has some finite symmetry group (subject to constraints discussed later), the *orbifold procedure* for n-dimensional defect TFTs [27] yields a new defect TFT on which the symmetry group acts trivially (one can interpret this procedure as *gauging* a finite symmetry [25, Remark 3.6]). We discuss the general aspects and requirements of the orbifold procedure in Section 3.3. These are then applied to $\mathcal{T} \subset \mathcal{RW}$ in Section 3.4 where we conjecture the existence of an *orbifold datum* and prove most of its constraint equations.

3.1 Pivotal tricategories revisited

We begin with a discussion of some subtleties regarding pivotal tricategories and their strictifications which will be important in Section 3.4.

3.1.1 Properties of the strictification triequivalence

Let $\mathcal{T}$ be a pivotal tricategory with duals. Then the Gray category with strict duals $s\mathcal{T}$ constructed from Theorem 1.2.28 can, in principle, be written down explicitly. However, for most applications the explicit form is far too complex to feasibly work with. The full details can be found in [82, Thm. 7.2.1], [82, Prop. 7.1.2], [53, Thm. 10.3.3]. Roughly speaking, the structure of $s\mathcal{T}$ is as follows:

- The objects of $s\mathcal{T}$ are the objects of $\mathcal{T}$.
- The 1- and 2-morphisms of $s\mathcal{T}$ are nested lists of composable 1- resp. 2-morphisms of $\mathcal{T}$ plus additional data.
- The identity 1- and 2-morphisms are given by empty lists, denoted by $\emptyset_{\boldsymbol{x}}$ resp. $\emptyset_W$.
- A 3-morphism $\phi \colon X \to Y$ in $s\mathcal{T}$ is represented by a 3-morphism $e(\phi) \colon e(X) \to e(Y)$ in $\mathcal{T}$.

The triequivalence e is defined to be the identity on objects and 3-morphisms. On 1- and 2-morphisms, it approximately maps lists of composable morphisms to their respective product, i.e.

$$\text{“}e(\{\ldots\{X_1\}\ldots\},\ \ldots,\{\ldots\{X_m\}\ldots\}) = X_1 \otimes \ldots \otimes X_m\text{”}\ , \qquad e(\emptyset_V) = \mathbb{1}_V\ ,$$
$$\text{“}e(\{\ldots\{W_1\}\ldots\},\ \ldots,\{\ldots\{W_n\}\ldots\}) = W_1 \boxtimes \ldots \boxtimes W_n\text{”}\ , \qquad e(\emptyset_{\boldsymbol{x}}) = \mathbb{1}_{\boldsymbol{x}}$$

for an object $(\boldsymbol{x})$, 1-morphisms $\{W_i\}$, V, and 2-morphisms $\{X_i\}$. A pseudoinverse $f\colon \mathcal{T} \to s\mathcal{T}$ of e is given by (again omitting a lot of details)

$$f(W) = \text{“}\{\ldots\{W\}\ldots\}\text{”}\ , \qquad f(X) = \text{“}\{\ldots\{X\}\ldots\}\text{”}\ , \tag{3.1.1}$$

i.e. f maps 1- and 2-morphisms to lists containing one element. Evidently, $e\circ f\colon \mathcal{T} \to \mathcal{T}$ is the identity 3-functor; however, $f\circ e\colon s\mathcal{T} \to s\mathcal{T}$ is the identity map on objects and 3-morphisms but only preserves equivalence classes of 1- and 2-morphisms (with the 2- and 3-morphisms possibly being pre- and post-composed with structure morphisms). In particular, f does *not* necessarily map the structure 1- and 2-morphisms of $\mathcal{T}$ to the respective 1- and 2-morphisms of $s\mathcal{T}$:

$$\begin{aligned} f(\mathbb{1}_{\boldsymbol{x}}) &= \text{“}\{\ldots\{\mathbb{1}_{\boldsymbol{x}}\}\ldots\}\text{”} \neq \emptyset_{\boldsymbol{x}} \in s\mathcal{T}(\boldsymbol{x},\boldsymbol{x})\ , \\ f(\lambda_W) &= \text{“}\{\ldots\{\lambda_W\}\ldots\}\text{”} \in \mathrm{Hom}_{s\mathcal{T}(\boldsymbol{x},\boldsymbol{y})}(\text{“}\{\mathbb{1}_{\boldsymbol{y}} \boxtimes W\}\text{”}, \text{“}\{W\}\text{”}) \\ &\neq \emptyset_W \in \mathrm{Hom}_{s\mathcal{T}(\boldsymbol{x},\boldsymbol{y})}(\text{“}\{W\}\text{”}, \text{“}\{W\}\text{”}) \quad (\text{with } \emptyset_{\boldsymbol{y}} \boxtimes \{W\} = \{W\}). \end{aligned}$$

3.1.2 Adjunctions of unitor 2-morphisms in tricategories

Even in a non-pivotal tricategory, the unitor 2-morphisms λ_W, ρ_W always have adjoints: According to the definition of a tricategory [82, Def. A.4.1] there exists an *adjoint equivalence*[1] $r\colon \boxtimes(1\times I_a) \Rightarrow 1$ whose components are the unitor 2-morphisms, i.e. $r(W) = \rho_W$. We spell out the definition:

(i) $r\colon \boxtimes(1\times I_a) \Rightarrow 1$ is a pseudonatural transformation,

(ii) there is another pseudonatural transformation $r^-\colon 1 \Rightarrow \boxtimes(1\times I_a)$ that corresponds to ρ^{-1}, i.e. $r^-(W) = \rho_W^{-1}$,

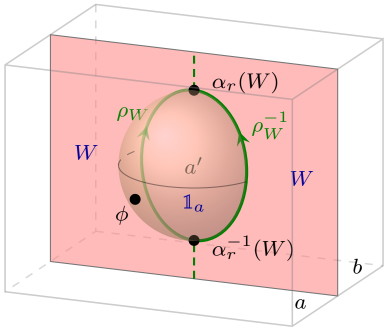
(iii) there are invertible modifications $\alpha_r\colon rr^- \Rrightarrow 1$, $\beta_r\colon 1 \Rrightarrow r^-r$ which fulfil the two right Zorro moves (1.2.13).

In components, the 3-morphisms $\alpha_r(W)$ and $\beta_r(W)$ exhibit ρ_W^{-1} as the right adjoint of ρ_W, and $\beta_r^{-1}(W)$ and $\alpha_r^{-1}(W)$ exhibit ρ_W^{-1} as the left adjoint of ρ_W. The analogous statement holds for ρ_W.

These morphisms can e.g. be used to insert 3-morphisms $\phi \in \mathrm{End}(I_{I_a})$ close to some non-trivial 1-morphism W in a tricategory *without* additional structure (see Lemma 3.4.2 for a

[1]We use the notation of [82]: 1 is the identity 2-functor, $I_a\colon I \to \mathcal{T}(a,a)$ is the 2-functor selecting the unit 1-morphism of a, $\boxtimes\colon \mathcal{T}(b,c)\times\mathcal{T}(a,b) \to \mathcal{T}(a,c)$ is the box product 2-functor, the composite of 2-functors is denoted by juxtaposition, and r is a pseudonatural transformation between two functors $\mathcal{T}(a,b) \to \mathcal{T}(a,b)$. See also Appendix A.5.

concrete example):



(3.1.2)

Consequently, in a pivotal tricategory with duals there are *two* right (and left) adjunctions for ρ_W, which do not necessarily coincide:

$$(\rho_W,\, \rho_W^{-1},\, \alpha_r(W),\, \beta_r(W)) \ , \qquad (\rho_W,\, \rho_W^{\dagger},\, \tilde{\mathrm{ev}}_{\rho_W},\, \tilde{\mathrm{coev}}_{\rho_W}) \ . \tag{3.1.3}$$

However, by uniqueness of adjoints in bicategories $\rho_W^{\dagger} \cong \rho_W^{-1}$ is guaranteed.

The adjunction $(\lambda_W, \lambda_W^{-1}, \alpha_l(W), \beta_l(W))$ is the one appearing "naturally" in the following sense: Consider a pivotal tricategory with duals $\mathcal{T}$ which is not strictly unital. When the strictification triequivalence $e\colon s\mathcal{T} \to \mathcal{T}$ (see Theorem 1.2.28) maps diagrams of $s\mathcal{T}$ to diagrams of $\mathcal{T}$ it sometimes has to introduce identity half-spheres. We claim that these half-spheres are bounded by α_r and β_r: By definition, e is equal to $s\mathcal{T} \xrightarrow{e_1} g\mathcal{T} \xrightarrow{e_2} \mathcal{T}$, where $g\mathcal{T}$ is the strictification of $\mathcal{T}$ as a Gray category *without duals* constructed by Theorem 1.2.14. The triequivalence e_1 maps between Gray categories (which are strictly unital and thus have trivial unitor 2-morphisms), so e_1 does not need to introduce any identity half-spheres. The triequivalence e_2, on the other hand, does not know about the dual structure, hence so it cannot introduce 3-morphisms like $\tilde{\mathrm{ev}}_{\rho_W}$.

If one removes ϕ from diagram (3.1.2), the 3-morphisms α_r and α_r^{-1} cancel and the diagram evaluates to the identity 3-morphism $1_{\mathbb{1}_W}$. This is consistent with the "invisibility" of the surface $\mathbb{1}_{\boldsymbol{x}}$ (see also the discussion in Remark 1.5.5). Note this cancellation does not necessarily happen for $\dim_r \rho_W = \tilde{\mathrm{ev}}_{\rho_W} \circ \mathrm{coev}_{\rho_W}$, which is another indicator why the second adjunction in Eq. (3.1.3) is "less natural".

The existence of two adjunctions can also be utilised in building a tricategory: Suppose we have a candidate for a tricategory $\mathcal{T}$ with candidates $\rho_W\colon W \boxtimes \mathbb{1}_{\boldsymbol{x}} \to W$ for all 1-morphisms W, and we have already shown that $\mathrm{Hom}_{\mathcal{T}}(\boldsymbol{x}, \boldsymbol{y})$ is a pivotal bicategory for all objects $(\boldsymbol{x})$ and $(\boldsymbol{y})$. Then we can construct candidates for $\alpha_r(W)$ and $\beta_r(W)$ in the following way:

$$\rho_W^{-1} := \rho_W^{\dagger} \ , \quad \alpha_r(W) := \tilde{\mathrm{ev}}_{\rho_W} \ , \quad \beta_r(W) := \tilde{\mathrm{coev}}_{\rho_W} \ , \tag{3.1.4}$$

$$\begin{aligned} \alpha_r^{-1}(W) &= (1 \otimes \delta_{\rho_W}) \circ \mathrm{coev}_{\rho_W} \circ (\dim_r \rho_W)^{-1} \ , \\ \beta_r^{-1}(W) &= (\dim_l \rho_W)^{-1} \circ \mathrm{ev}_{\rho_W} \circ (\delta_{\rho_W}^{-1} \otimes 1) \ . \end{aligned} \tag{3.1.5}$$

3.2 Adjunctions in the affine Rozansky–Witten model

We first define adjunction 2-morphisms $(\mathrm{ev}_W, \mathrm{coev}_W)$ for all $W \in \mathcal{RW}(\boldsymbol{x}, \boldsymbol{y})$ and show that $h\mathcal{RW}$ has left adjoints, hence $\mathcal{RW}$ fulfils the third axiom of Definition 1.2.21. Next we will show that $\mathcal{RW}$ cannot fulfil the first axiom and discuss necessary conditions for a subcategory $\mathcal{T} \subset \mathcal{RW}$ to be pivotal.

3.2.1 The adjoints of 1-morphisms in $\mathcal{RW}$

Definition 3.2.1. For a 1-morphism $(\boldsymbol{a};\, W(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y})) \in \mathcal{RW}(\boldsymbol{x}, \boldsymbol{y})$ we define the (left) adjoint

$$W^{\#} := (\boldsymbol{a}';\, W^{\#}(\boldsymbol{a}', \boldsymbol{y}, \boldsymbol{x}))\colon (\boldsymbol{y}) \to (\boldsymbol{x}) \ , \quad W^{\#}(\boldsymbol{a}', \boldsymbol{y}, \boldsymbol{x}) := -W(\boldsymbol{a}', \boldsymbol{x}, \boldsymbol{y}) \ , \tag{3.2.1}$$

and the left adjunction 2-morphisms

- $$\begin{aligned} \mathrm{ev}_W &= K(\partial^{\boldsymbol{a}',\boldsymbol{a}} W(\bullet, \boldsymbol{x}, \boldsymbol{y});\, \boldsymbol{a}' - \boldsymbol{a}) \otimes K(\boldsymbol{u} + \partial^{\boldsymbol{x},\boldsymbol{x}'} W(\boldsymbol{a}, \bullet, \boldsymbol{y});\, \boldsymbol{x} - \boldsymbol{x}') \\ &\cong I^{\boldsymbol{a}'\leftarrow \boldsymbol{a}}_{W(\bullet,\boldsymbol{x},\boldsymbol{y})} \otimes I^{\boldsymbol{x}\leftarrow\boldsymbol{x}'}_{W(\boldsymbol{a},\bullet,\boldsymbol{y})+\boldsymbol{u}\cdot\bullet}[\ell(\boldsymbol{a}) + \ell(\boldsymbol{x})] \\ &\in K_{\boldsymbol{u}\cdot(\boldsymbol{x}-\boldsymbol{x}') - W(\boldsymbol{a},\boldsymbol{x}',\boldsymbol{y}) + W(\boldsymbol{a}',\boldsymbol{x},\boldsymbol{y})}(\partial^{\boldsymbol{a}',\boldsymbol{a}} W(\bullet, \boldsymbol{x}, \boldsymbol{y}), \boldsymbol{u} + \partial^{\boldsymbol{x},\boldsymbol{x}'} W(\boldsymbol{a}, \bullet, \boldsymbol{y})) \ , \end{aligned}$$

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  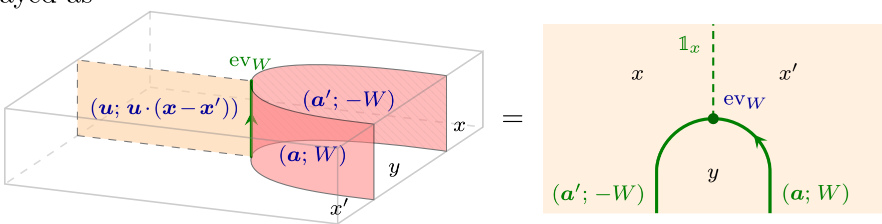

  , (3.2.2)

- $$\begin{aligned} \mathrm{coev}_W &= (\partial^{\boldsymbol{a},\boldsymbol{a}'} W(\bullet, \boldsymbol{x}, \boldsymbol{y});\, \boldsymbol{a} - \boldsymbol{a}') \otimes K(\boldsymbol{y} - \boldsymbol{y}';\, -\boldsymbol{u} + \partial^{\boldsymbol{y},\boldsymbol{y}'} W(\boldsymbol{a}', \boldsymbol{x}, \bullet)) \\ &\cong I^{\boldsymbol{a}\leftarrow\boldsymbol{a}'}_{W(\bullet,\boldsymbol{x},\boldsymbol{y})} \otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{W(\boldsymbol{a}',\boldsymbol{x},\bullet)-\boldsymbol{u}\cdot\bullet}[\ell(\boldsymbol{a})] \\ &\in K_{W(\boldsymbol{a},\boldsymbol{x},\boldsymbol{y}) - W(\boldsymbol{a}',\boldsymbol{x}',\boldsymbol{y}) - \boldsymbol{u}\cdot(\boldsymbol{y}-\boldsymbol{y}')}(\partial^{\boldsymbol{a},\boldsymbol{a}'} W(\bullet, \boldsymbol{x}, \boldsymbol{y}), \boldsymbol{y} - \boldsymbol{y}') \ , \end{aligned}$$

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  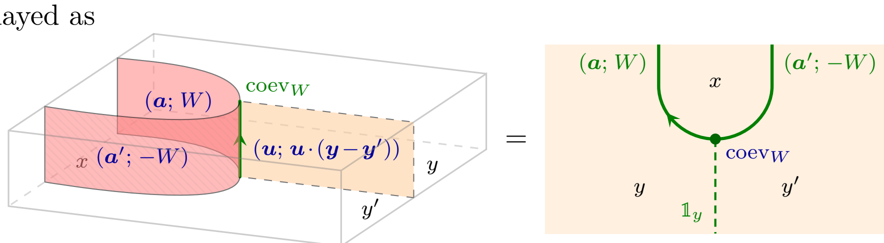

  . (3.2.3)

We also define the following right adjunction 2-morphisms motivated by Theorem 1.2.24:

- $$\begin{aligned} \tilde{\mathrm{ev}}_W &:= \mathrm{coev}_W^\vee[\ell(\boldsymbol{x})] \\ &\cong K(\boldsymbol{a}' - \boldsymbol{a};\, \partial^{\boldsymbol{a},\boldsymbol{a}'} W(\bullet, \boldsymbol{x}, \boldsymbol{y}')) \otimes K(-\boldsymbol{u} + \partial^{\boldsymbol{y},\boldsymbol{y}'} W(\boldsymbol{a}, \boldsymbol{x}, \bullet);\, \boldsymbol{y}' - \boldsymbol{y})[l(\boldsymbol{x})] \\ &\cong I^{\boldsymbol{a}'\leftarrow\boldsymbol{a}}_{W(\bullet,\boldsymbol{x},\boldsymbol{y}')} \otimes I^{\boldsymbol{y}'\leftarrow\boldsymbol{y}}_{W(\boldsymbol{a},\boldsymbol{x},\bullet)-\boldsymbol{u}\cdot\bullet}[\ell(\boldsymbol{x}) + \ell(\boldsymbol{y})] \\ &\in K_{\boldsymbol{u}\cdot(\boldsymbol{y}-\boldsymbol{y}') - W(\boldsymbol{a},\boldsymbol{x},\boldsymbol{y}) + W(\boldsymbol{a}',\boldsymbol{x},\boldsymbol{y}')}(\boldsymbol{a}' - \boldsymbol{a},\, -\boldsymbol{u} + \partial^{\boldsymbol{y},\boldsymbol{y}'} W(\boldsymbol{a}, \boldsymbol{x}, \bullet))[\ell(\boldsymbol{x})] \ , \end{aligned}$$

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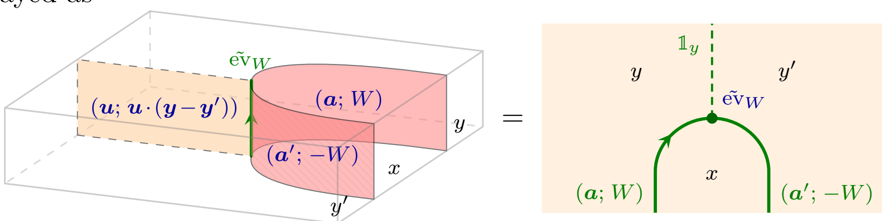
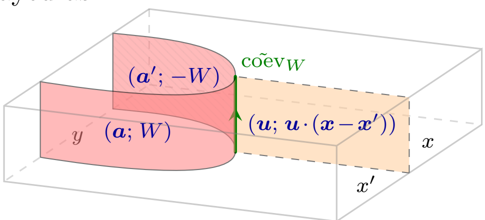
  

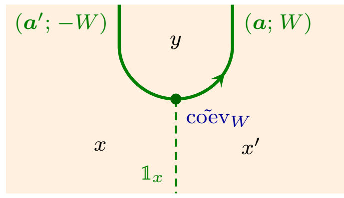
  , (3.2.4)

- $$\begin{aligned} \widetilde{\mathrm{coev}}_W := \mathrm{ev}_W^\vee[\ell(\boldsymbol{y})] &\cong K\big(\boldsymbol{a}-\boldsymbol{a}';\, \partial^{\boldsymbol{a}',\boldsymbol{a}} W(\bullet,\boldsymbol{x}',\boldsymbol{y})\big) \otimes K\big(\boldsymbol{u}+\partial^{\boldsymbol{x},\boldsymbol{x}'} W(\boldsymbol{a}',\bullet,\boldsymbol{y});\, \boldsymbol{x}'-\boldsymbol{x}\big) \\ &\cong I^{\boldsymbol{a}\leftarrow\boldsymbol{a}'}_{W(\bullet,\boldsymbol{x}',\boldsymbol{y})} \otimes I^{\boldsymbol{x}'\leftarrow\boldsymbol{x}}_{W(\boldsymbol{a}',\bullet,\boldsymbol{y})+\boldsymbol{u}\cdot\bullet}[\ell(\boldsymbol{x})] \\ &\in K_{W(\boldsymbol{a},\boldsymbol{x}',\boldsymbol{y})-W(\boldsymbol{a}',\boldsymbol{x},\boldsymbol{y})-\boldsymbol{u}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\big(\boldsymbol{a}-\boldsymbol{a}',\, \boldsymbol{u}+\partial^{\boldsymbol{x}',\boldsymbol{x}} W(\boldsymbol{a}',\bullet,\boldsymbol{y})\big)\ , \end{aligned}$$

  displayed as



$=$



$.$ (3.2.5)

Remark 3.2.2. It turns out that $\widetilde{\mathrm{ev}}_W$ and $\widetilde{\mathrm{coev}}_W$ are well-defined for all 1-morphisms in $\mathcal{RW}$ and always exhibit $W^\#$ as a right adjoint of W. However, some further properties like $\widetilde{\mathrm{ev}}_W \cong {}^\dagger\mathrm{coev}_W \cong \mathrm{coev}_W^\dagger$ only hold if the assumptions of Theorem 1.2.24 are met, i.e. we are considering a pivotal tricategory with duals $\mathcal{T} \subset \mathcal{RW}$.

Theorem 3.2.3. *The bicategory $h\mathcal{RW}$ has left adjoints with the data of Definition 3.2.1.*

We will present the basic idea here and refer [7] for the full details.

Proof sketch. We need to show the two Zorro movies (Notation 1.2.25). While a bit cumbersome, the proof is ultimately straightforward with the methods introduced in earlier chapters: We combine the left hand side into a single 2-morphism which turns out to be a large Koszul matrix factorisation. Its sequence can be shown to be Koszul-regular using Lemma 1.3.51 and Corollary 1.3.45. Theorem 1.3.49 then yields an associated module which can be shown to be isomorphic to the module associated to the identity matrix factorisation. □

3.2.2 $\mathcal{RW}$ is not a pivotal tricategory with duals

Even under the assumption that $\mathcal{RW}$ is a tricategory, it fails to a be pivotal tricategory with duals in several ways:

- Some 2-morphisms do not have adjoints at all, as we showed in Theorem 2.2.4.
- In Section 2.4 we have seen that there exist subcategories of $\ddot{\mathsf{MF}}_{\mathbb{C}}(\boldsymbol{x},\boldsymbol{y})$ that have adjoints and are graded pivotal, but do not admit an ordinary pivotal structure; see Corollary 2.4.14 for a discussion of the (ordinarily) pivotal subcategories of $\ddot{\mathsf{MF}}_k$. Consequently, even if all 2-morphisms in a subcategory $\mathcal{T} \subset \mathcal{RW}$ have adjoints, its Hom-bicategories $\mathcal{T}(\boldsymbol{x},\boldsymbol{y}) \subset \ddot{\mathsf{MF}}_{\mathbb{C}}(\boldsymbol{x},\boldsymbol{y})$ might not admit ordinary pivotal structures.
- Let $\mathcal{T} \subset \mathcal{RW}$ be a subcategory with objects $(\boldsymbol{x}), (\boldsymbol{y}) \in \mathcal{T}$ such that

  $$\ell(\boldsymbol{x}) + \ell(\boldsymbol{y}) \equiv 1 \bmod 2\ ,$$

  and consider an arbitrary 1-morphism $(\boldsymbol{a}; W) \in \mathcal{T}(\boldsymbol{x},\boldsymbol{y})$. The domain of ev_W is given by $(\boldsymbol{a}'; -W) \boxtimes (\boldsymbol{a}; W) = (\boldsymbol{a}', \boldsymbol{y}, \boldsymbol{a}; \dots)$ and the codomain is given by $\mathbb{1}_{\boldsymbol{x}} = (\boldsymbol{u}; \boldsymbol{u}\cdot(\boldsymbol{x}-\boldsymbol{x}'))$. We find

  $$\ell(\{\boldsymbol{a}', \boldsymbol{y}, \boldsymbol{a}\}) + \ell(\boldsymbol{u}) \equiv \ell(\boldsymbol{y}) + \ell(\boldsymbol{x}) \equiv 1 \mod 2\ . \tag{3.2.6}$$

Now we can explicitly see that ${}^\dagger\mathrm{ev}_W \cong \mathrm{ev}_W^\dagger[1] \ncong \mathrm{ev}_W^\dagger$. It follows that either $\mathcal{T}(\boldsymbol{x},\boldsymbol{x})$ is not pivotal or $\mathrm{Hom}_{\mathcal{T}(\boldsymbol{x},\boldsymbol{x})}(W^\# \boxtimes W, \mathbb{1}_{\boldsymbol{x}})$ is empty, showing that $\mathcal{T}$ cannot simultaneously fulfil Definition 1.2.21 (i) and (iii). It follows that if $\mathcal{T} \subset \mathcal{RW}$ is a pivotal tricategory with duals, then $\ell(\boldsymbol{x})$ must be equal modulo 2 for all objects $(\boldsymbol{x}) \in \mathcal{T}$.

This leads us to the following definition:

Definition 3.2.4. We define the non-full subcategories $\mathcal{RW}^{\mathrm{even}}, \mathcal{RW}^{\mathrm{odd}} \subset \mathcal{RW}$ by

$$\mathsf{Obj}(\mathcal{RW}^{\mathrm{even}}) := \{\boldsymbol{x} \in \mathcal{RW} \mid \ell(\boldsymbol{x}) \text{ even}\} \,, \qquad \mathcal{RW}^{\mathrm{even}}(\boldsymbol{x},\boldsymbol{y}) := \ddot{\mathsf{MF}}{}^{\mathrm{even}}_{\mathbb{C}}(\boldsymbol{x},\boldsymbol{y}) \,, \tag{3.2.7}$$

$$\mathsf{Obj}(\mathcal{RW}^{\mathrm{odd}}) := \{\boldsymbol{x} \in \mathcal{RW} \mid \ell(\boldsymbol{x}) \text{ odd}\} \,, \qquad \mathcal{RW}^{\mathrm{odd}}(\boldsymbol{x},\boldsymbol{y}) := \ddot{\mathsf{MF}}{}^{\mathrm{odd}}_{\mathbb{C}}(\boldsymbol{x},\boldsymbol{y}) \,, \tag{3.2.8}$$

with the $\ddot{\mathsf{MF}}_{\mathbb{C}}$-subcategories defined in Eq. (2.4.18).

The above argument can thus be summarised as follows:

Corollary 3.2.5. *If $\mathcal{T} \subset \mathcal{RW}$ is a pivotal tricategory with duals, then either $\mathcal{T} \subset \mathcal{RW}^{\mathrm{even}}$ or $\mathcal{T} \subset \mathcal{RW}^{\mathrm{odd}}$.*

Remark 3.2.6. $\mathcal{RW}^{\mathrm{even}}$ and $\mathcal{RW}^{\mathrm{odd}}$ are closed under $-\boxtimes-$: Let

$$(\boldsymbol{a};\, W)\colon (\boldsymbol{x}) \to (\boldsymbol{y}) \,, \quad (\boldsymbol{b};\, V)\colon (\boldsymbol{y}) \to (\boldsymbol{z}) \,.$$

Then we find by assumption

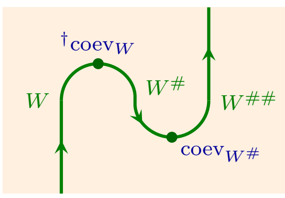
$$\ell(\boldsymbol{a}) \equiv \ell(\boldsymbol{b}) \equiv \ell(\boldsymbol{x}) \equiv \ell(\boldsymbol{y}) \equiv \ell(\boldsymbol{z}) \mod 2 \,,$$
$$(\boldsymbol{b};\, V) \boxtimes (\boldsymbol{a};\, W) = (\boldsymbol{b}, \boldsymbol{y}, \boldsymbol{a};\, W + V) \,, \quad \ell(\boldsymbol{b}) + \ell(\boldsymbol{y}) + \ell(\boldsymbol{a}) \equiv 3\,\ell(\boldsymbol{a}) \equiv \ell(\boldsymbol{a}) \mod 2 \,.$$

Furthermore, the identity surface (being the only structure 1-morphism of $\mathcal{RW}$) has $\ell(\boldsymbol{a}) = \ell(\boldsymbol{x}) = \ell(\boldsymbol{x}')$, so all objects have an identity surface in $\mathcal{RW}^{\mathrm{even}}(\boldsymbol{x},\boldsymbol{y})$ and $\mathcal{RW}^{\mathrm{odd}}(\boldsymbol{x},\boldsymbol{y})$.

3.2.3 The induced pivotal structure on $h\mathcal{T}$

Let $\mathcal{T} \subset \mathcal{RW}$ be a pivotal tricategory with duals. Then Corollary 3.2.5 implies $\mathcal{T} \subset \mathcal{RW}^{\mathrm{even}}$ or $\mathcal{T} \subset \mathcal{RW}^{\mathrm{odd}}$, and Theorem 1.2.24 yields 2-morphisms which exhibit $W^\#$ as the *right* adjoint of W. Furthermore, $h\mathcal{T}$ has an induced pivotal structure:

Lemma 3.2.7. *Let $\mathcal{T} \subset \mathcal{RW}$ be a pivotal tricategory with duals. Then a pivotal structure on $h\mathcal{T}$ is given by*

$\delta_W :=$ 

$\cong \mathbb{1}_W \colon W \to W = W^{\#\#}$. (3.2.9)

The right adjunction 2-morphisms induced by this pivotal structure are given by

$$\widetilde{\mathrm{ev}}_W \cong \mathrm{ev}_{W^\#} \cong {}^\dagger\mathrm{coev}_W \,, \qquad \widetilde{\mathrm{coev}}_W \cong \mathrm{coev}_{W^\#} \cong {}^\dagger\mathrm{ev}_W \,, \tag{3.2.10}$$

consistent with Theorem 1.2.24.

Proof. The proof of δ_W being a pivotal structure is similar to the proof of Theorem 1.2.24 presented in [35, Lemma 1.4.4] and will not be shown here. To prove the isomorphy in Eq. (3.2.9) we compare ${}^\dagger\mathrm{coev}_W$ and $\mathrm{ev}_{W^\#}$ as 2-morphisms $(\boldsymbol{a}; W) \boxtimes (\boldsymbol{a}'; -W) \to (\boldsymbol{u}; \mathbb{1}_{\boldsymbol{y}})$, which is possible since $W^{\#\#} = W$. We also use Corollary 3.2.5 which implies $\ell(\boldsymbol{a}) \equiv \ell(\boldsymbol{x}) \equiv \ell(\boldsymbol{y}) \bmod 2$:

$$\begin{aligned}
\mathrm{ev}_{W^\#} &\cong I^{\boldsymbol{a}\leftarrow\boldsymbol{a}'}_{W^\#(\bullet,\boldsymbol{y},\boldsymbol{x})} \otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{W^\#(\boldsymbol{a}',\bullet,\boldsymbol{x})+\boldsymbol{u}\cdot\bullet}[\ell(\boldsymbol{a}) + \ell(\boldsymbol{x})] \cong I^{\boldsymbol{a}'\leftarrow\boldsymbol{a}}_{W(\bullet,\boldsymbol{x},\boldsymbol{y}')} \otimes I^{\boldsymbol{y}'\leftarrow\boldsymbol{y}}_{W(\boldsymbol{a},\boldsymbol{x},\bullet)-\boldsymbol{u}\cdot\bullet} \ , \\
{}^\dagger\mathrm{coev}_W &\cong I^{\boldsymbol{a}'\leftarrow\boldsymbol{a}}_{W(\bullet,\boldsymbol{x},\boldsymbol{y}')} \otimes I^{\boldsymbol{y}'\leftarrow\boldsymbol{y}}_{W(\boldsymbol{a},\boldsymbol{x},\bullet)-\boldsymbol{u}\cdot\bullet}[\ell(\boldsymbol{x}) + \ell(\boldsymbol{y})] \cong \mathrm{ev}_{W^\#} \ .
\end{aligned}$$

We may therefore replace ${}^\dagger\mathrm{coev}_W$ by $\mathrm{ev}_{W^\#}$ in Eq. (3.2.9) and then apply the second Zorro movie of $W^\#$, yielding $\delta_W \cong \mathbb{1}_{W^{\#\#}} = \mathbb{1}_W$. Furthermore, by Remark 2.4.2 we find the induced right adjunction 2-morphisms

$$\tilde{\mathrm{ev}}_W = \mathrm{ev}_{W^\#} \otimes (\delta_W \boxtimes \mathbb{1}_{W^\#}) \cong {}^\dagger\mathrm{coev}_W \ , \quad \tilde{\mathrm{coev}}_W = (1_{W^\#} \boxtimes \delta_W^{-1}) \otimes \mathrm{coev}_{W^\#} \cong {}^\dagger\mathrm{ev}_W \ . \quad \square$$

Remark 3.2.8.

(i) The full bicategory $h\mathcal{RW}$ has left and right adjoints by Remark 3.2.2. Furthermore, one can manually verify that $\delta_W := \mathbb{1}_W$ defines a pivotal structure on $h\mathcal{RW}$. This computation will not be shown here as it does not matter for the rest of this thesis.

(ii) For $(\boldsymbol{a}; W)\colon (\boldsymbol{x}) \to (\boldsymbol{y})$, the collection of 2-morphisms

$$\delta'_W := \mathbb{1}_W[\ell(\boldsymbol{x}) + \ell(\boldsymbol{y})] \tag{3.2.11}$$

defines a different, inequivalent pivotal structure on $h\mathcal{RW}$: The naturality condition is easy to see. For the monoidality condition, consider $(\boldsymbol{b}; V)\colon (\boldsymbol{y}) \to (\boldsymbol{z})$ and compare

$$\begin{aligned}
\delta'_{V\boxtimes W} &= \delta_{V\boxtimes W}\left[\ell(\boldsymbol{x}) + \ell(\boldsymbol{z})\right] \ , \\
\delta'_V \boxtimes \delta'_W &\cong \delta_V \boxtimes \delta_W\left[\ell(\boldsymbol{x}) + \ell(\boldsymbol{y}) + \ell(\boldsymbol{y}) + \ell(\boldsymbol{z})\right] \ ,
\end{aligned}$$

so both sides of the monoidality condition are shifted by the same amount. While this is conceptually interesting for $h\mathcal{RW}$, it does not matter for pivotal subcategories $\mathcal{T} \subset \mathcal{RW}$ because the induced pivotal structure (3.2.9) on $h\mathcal{T}$ is canonical, and also because $\ell(\boldsymbol{x}) \equiv \ell(\boldsymbol{y}) \bmod 2$ in pivotal subcategories.

3.2.4 Duals and grade shifts

The structure of a pivotal tricategory with duals $\mathcal{T} \subset \mathcal{RW}$ is not very "rigid" in the sense that there is still some freedom to replace structure 2-morphisms by grade-shifted versions of themselves.

Lemma 3.2.9. *Let $\mathcal{T} \subset \mathcal{RW}$ be a pivotal tricategory with duals which is closed under grade shifts of 2-morphisms. Let $\mathcal{T}'$ be a copy of the tricategory $\mathcal{T}$. We endow $\mathcal{T}'$ with the same adjunctions and pivotal structure on 2-morphisms. Furthermore, we define the left adjoints of 1-morphisms in $\mathcal{T}'$ to be the same as in $\mathcal{T}$, but we apply a grade shift to the adjunction 2-morphisms*

$$(\mathrm{ev}_W, \mathrm{coev}_W) \to (\mathrm{ev}_W[s_W], \mathrm{coev}_W[s_W]) \tag{3.2.12}$$

with an arbitrary $s_W \in \mathbb{Z}_2$ for every 1-morphism W. Then $\mathcal{T}'$ is also a pivotal tricategory with duals. Furthermore, $\mathcal{T}$ and $\mathcal{T}'$ are equivalent in the sense of Definition 1.2.27.

Proof. The truncation $h\mathcal{T}'$ has left adjoints because the total grade shift of the Zorro movie (1.2.18) is even. The other axioms of a pivotal tricategory with duals are independent of the 2-morphisms $(\mathrm{ev}_W, \mathrm{coev}_W)$, showing that $\mathcal{T}'$ is a pivotal tricategory with duals. The identity 3-functor $\mathcal{T} \to \mathcal{T}'$ is well-defined and is pivotal on the Hom-bicategories, so $\mathcal{T}$ and $\mathcal{T}'$ are equivalent as pivotal tricategories with duals. □

Remark 3.2.10.

- By definition, the grade shift (3.2.12) induces a grade shift on $\widetilde{\mathrm{ev}}_W$ and $\widetilde{\mathrm{coev}}_W$:

$$(\widetilde{\mathrm{ev}}_W, \widetilde{\mathrm{coev}}_W) = ({}^\dagger\mathrm{coev}_W, {}^\dagger\mathrm{ev}_W) \mapsto ({}^\dagger(\mathrm{coev}_W[s_W]), {}^\dagger(\mathrm{ev}_W[s_W])) \cong (\widetilde{\mathrm{ev}}_W[s_W], \widetilde{\mathrm{coev}}_W[s_W]) \ .$$

- The grade shifts of $(\mathrm{ev}_W, \mathrm{coev}_W)$ may influence other quantities: Consider for example $\dim_r \mathrm{ev}_W$, which can be interpreted as a W-sphere without line or point defects. By Lemma 2.5.7, shifting the grade of ev_W changes the sign of $\dim_r \mathrm{ev}_W$. Since different grade choices lead to equivalent pivotal tricategories with duals by Lemma 3.2.9, $\dim_r \mathrm{ev}_W$ is not preserved under such equivalences.

- The equivalence of a pair of pivotal tricategories with duals $\mathcal{T} \cong \mathcal{T}'$ clearly implies the equivalence of their strictifications $s\mathcal{T}$ and $s\mathcal{T}'$ as pivotal tricategories with duals. However, there exists a more rigid definition of equivalence for Gray categories with strict duals (see e.g. [82, Thm. 7.3.2]), so $s\mathcal{T}$ and $s\mathcal{T}'$ may be *inequivalent* as Gray categories with strict duals. Consequently, there are different ways to strictify a given pivotal tricategory with duals since the relation between $\mathcal{T}$ and $s\mathcal{T}$ is not fundamentally different from the relation between $\mathcal{T}$ and $s\mathcal{T}'$ for any $\mathcal{T}' \cong \mathcal{T}$.

- Our interpretation of Lemma 3.2.9 is that it gives us some freedom of choice in the way we strictify $\mathcal{T}$. For example, we will later discuss some diagram identities in Gray categories with duals that do not hold in $s\mathcal{T}$ but do hold in $s\mathcal{T}'$ if s_W is chosen correctly. Since $s\mathcal{T}' \cong \mathcal{T} \cong s\mathcal{T}$ as pivotal tricategories with duals, we may interpret $s\mathcal{T}'$ as a "different strictification" of $\mathcal{T}$.

- For the identity surface $W = \mathbb{1}_{\boldsymbol{x}}$ it is slightly unexpected that we are allowed to freely grade-shift $\mathrm{ev}_{\mathbb{1}_{\boldsymbol{x}}}$ and hence change the sign of $\dim_r \mathrm{ev}_{\mathbb{1}_{\boldsymbol{x}}}$; in contrast to Gray categories with strict duals, there are no axioms constraining $\mathrm{ev}_{\mathbb{1}_{\boldsymbol{x}}}$. One might be tempted to argue that $\dim_r \mathrm{ev}_{\mathbb{1}_{\boldsymbol{x}}}$ has a visual interpretation as the 2-sphere of the identity surface which should be invisible and thus evaluate to 1. However, such diagrammatic arguments can only be made in the strictification $s\mathcal{T}$, where we always find $\mathrm{ev}'_{\mathbb{1}'_{\boldsymbol{x}}} = \mathbb{1}'_{\mathbb{1}'_{\boldsymbol{x}}}$ and $\dim'_r \mathrm{ev}'_{\mathbb{1}'_{\boldsymbol{x}}} = 1$. The triequivalence $e\colon s\mathcal{T} \to \mathcal{T}$ maps the identity bubble $\dim'_r \mathrm{ev}'_{\mathbb{1}'_{\boldsymbol{x}}} \in s\mathcal{T}$ to $1_{\mathbb{1}_{\mathbb{1}_{\boldsymbol{x}}}} \in \mathcal{T}$ which is independent of $\mathrm{ev}_{\mathbb{1}_{\boldsymbol{x}}} \in \mathcal{T}$. Furthermore, as discussed in Section 3.1.1, the pseudoinverse $f\colon \mathcal{T} \to s\mathcal{T}$ does not map the identity surfaces and lines of $\mathcal{T}$ to the identities of $s\mathcal{T}$, so there are no special constraints for $f(\dim_r \mathrm{ev}_{\mathbb{1}_{\boldsymbol{x}}}) \in s\mathcal{T}$ and everything is consistent.

3.2.5 The pivotal subcategory $\mathcal{T}$

In the remainder of this chapter we will assume that Conjecture 1.5.15 holds, i.e. that $\mathcal{RW}$ is a tricategory. Motivated by the application in Section 3.4 we now construct a subcategory

$\mathcal{T} \subset \mathcal{RW}$ which we conjecture to be a pivotal tricategory with duals in Conjecture 3.2.15, and we provide a partial proof for the latter assuming Conjecture 1.5.15. The aforementioned application requires $\mathcal{T}$ to contain all 1-morphisms of the form

$$\left(\boldsymbol{a};\ \textstyle\sum_{i=1}^{n} \zeta_g a_i (x_i - g_{ij} x'_j)\right) \colon (\boldsymbol{x}') \to (\boldsymbol{x}) \tag{3.2.13}$$

with an invertible matrix $g \in \mathbb{C}^{n \times n}$ and $\zeta_g \in \mathbb{C} \setminus \{0\}$.

Definition 3.2.11. Fix $n \in \mathbb{N}$. We define the structure $\mathcal{T} \subset \mathcal{RW}$ as follows:

- There is only one object $* := (\boldsymbol{x}) = (x_1, \ldots, x_n)$,
- the 1-morphisms $V \in \mathcal{T}(\boldsymbol{x}', \boldsymbol{x})$ are of the form

$$V = V_m \boxtimes \cdots \boxtimes V_1 \ , \quad V_i \colon (\boldsymbol{y}^{i-1}) \to (\boldsymbol{y}^i) \ , \quad \boldsymbol{y}^0 := \boldsymbol{x}', \quad \boldsymbol{y}^m := \boldsymbol{x} \ ,$$
$$V_i = \left(\boldsymbol{a}^i;\ \boldsymbol{a}^i \cdot (C_i \cdot \boldsymbol{y}^i - D_i \cdot \boldsymbol{y}^{i-1})\right) \tag{3.2.14}$$

  for invertible matrices $C_i, D_i \in \mathrm{Gl}_n(\mathbb{C})$ (note that each $\boldsymbol{a}^j$ is a list with $\ell(\boldsymbol{a}^j) = n$),
- the 2-morphisms are restricted to the idempotent closure of finite-rank matrix factorisations (see [22, Section 2.2], [9, Sect. 2.1]), implying that each matrix factorisation is isomorphic to a direct sum of finite-rank matrix factorisations (see Remark 3.2.14),
- the 3-morphisms are those of $\mathcal{RW}$,
- the structure morphisms of $\mathcal{T}$ are those of $\mathcal{RW}$, with the exception of $(\mathrm{ev}_W, \mathrm{coev}_W)$ where we apply a global grade shift by $s_{\mathrm{ev}} \in \mathbb{Z}_2$ according to Lemma 3.2.9:

$$(\mathrm{ev}_W, \mathrm{coev}_W) \mapsto (\mathrm{ev}_W[s_{\mathrm{ev}}], \mathrm{coev}_W[s_{\mathrm{ev}}]) \ .$$

  At this point we regard s_{ev} as a free parameter of $\mathcal{T}$; in Section 3.4.5 we will find $s_{\mathrm{ev}} := n + \binom{n}{2}$ to be a good choice.

Lemma 3.2.12. *The 1-morphisms of $\mathcal{T}$ are closed under $- \boxtimes -$ and $(-)^{\#}$. Furthermore, the 1-morphisms of Eq. (3.2.13) are contained in $\mathcal{T}$.*

Proof. The closedness under $- \boxtimes -$ is clear by construction. The closedness under $(-)^{\#}$ is easy to see since $-C_i, -D_i \in \mathrm{Gl}_n(\mathbb{C})$. For the second statement we only need to check that Eq. (3.2.13) is of the form (3.2.14), which is obvious. $\square$

Lemma 3.2.13. *The Hom-category $\mathrm{Hom}_{\mathcal{T}(*,*)}(V, U)$ has left and right admissible variables for all 1-morphisms $V, U \in \mathcal{T}(*, *)$.*

Proof of Lemma 3.2.13. We spell out two arbitrary 1-morphisms $V, U \in \mathcal{T}(*, *) = \mathcal{T}(\boldsymbol{x}', \boldsymbol{x})$:

$$V_i \colon (\boldsymbol{y}_{i-1}) \to (\boldsymbol{y}_i) \quad \text{for } 1 \le i \le m \ , \quad \boldsymbol{y}^0 := \boldsymbol{x}',\ \boldsymbol{y}^m := \boldsymbol{x} \ ,$$
$$U_j \colon (\boldsymbol{z}_{j-1}) \to (\boldsymbol{z}_j) \quad \text{for } 1 \le j \le l \ , \quad \boldsymbol{z}^0 := \boldsymbol{x}',\ \boldsymbol{z}^l := \boldsymbol{x} \ ,$$
$$V = V_m \boxtimes \ldots \boxtimes V_1 = \left(\boldsymbol{a}^m, \boldsymbol{y}^{m-1}, \boldsymbol{a}^{m-1}, \ldots, \boldsymbol{y}^1, \boldsymbol{a}^1;\ \textstyle\sum_{i=1}^{m} V_i(\boldsymbol{y}^i, \boldsymbol{a}^i, \boldsymbol{y}^{i-1})\right) \ ,$$
$$U = U_l \boxtimes \ldots \boxtimes U_1 = \left(\boldsymbol{b}^l, \boldsymbol{z}^{l-1}, \boldsymbol{b}^{l-1}, \ldots, \boldsymbol{z}^1, \boldsymbol{b}^1;\ \textstyle\sum_{j=1}^{l} U_j(\boldsymbol{z}^j, \boldsymbol{b}^j, \boldsymbol{z}^{j-1})\right) \ .$$

We define candidates for left ($\boldsymbol{v}$) and right ($\boldsymbol{u}$) admissible variables:

$$\boldsymbol{u} := \{\boldsymbol{y}^0, \dots, \boldsymbol{y}^{m-1}, \boldsymbol{a}^1, \dots, \boldsymbol{a}^{m-1}\}, \qquad \boldsymbol{v} := \{\boldsymbol{z}^0, \dots, \boldsymbol{z}^{l-1}, \boldsymbol{b}^1, \dots, \boldsymbol{b}^{l-1}\}.$$

From the explicit formula (3.2.14) we find the derivatives

$$\begin{aligned}\partial_{\boldsymbol{y}^i} V &= \partial_{\boldsymbol{y}^i}(V_i(\boldsymbol{y}^i, \boldsymbol{a}^i, \boldsymbol{y}^{i-1}) + V_{i+1}(\boldsymbol{y}^{i+1}, \boldsymbol{a}^{i+1}, \boldsymbol{y}^i)) = C_i^\top \cdot \boldsymbol{a}^i - D_{i+1}^\top \cdot \boldsymbol{a}^{i+1}, \\ \partial_{\boldsymbol{a}^i} V &= \partial_{\boldsymbol{a}^i} V_i(\boldsymbol{y}^i, \boldsymbol{a}^i, \boldsymbol{y}^{i-1}) = C_i \cdot \boldsymbol{y}^i - D_i \cdot \boldsymbol{y}^{i-1},\end{aligned}$$

with $\boldsymbol{a}^0$ and $\boldsymbol{a}^{m+1}$ implicitly set to zero. The sequence $\boldsymbol{f}$ of Definition 2.3.1 is thus given by

$$\begin{aligned}\boldsymbol{f} = \partial_{\boldsymbol{u}}(V-U) = \{&-\partial U/\partial \boldsymbol{x}' - D_1^\top \cdot \boldsymbol{a}^1, C_1^\top \cdot \boldsymbol{a}^1 - D_2^\top \cdot \boldsymbol{a}^2, \dots, C_{m-1}^\top \cdot \boldsymbol{a}^{m-1} - D_m^\top \cdot \boldsymbol{a}^m, \\ &C_1 \cdot \boldsymbol{y}^1 - D_1 \cdot \boldsymbol{x}', C_2 \cdot \boldsymbol{y}^2 - D_2 \cdot \boldsymbol{y}^1, \dots, C_{m-1} \cdot \boldsymbol{y}^{m-1} - D_{m-1} \cdot \boldsymbol{y}^{m-2}\}.\end{aligned}$$

The Koszul-regularity (Definition 2.3.1 (a)) can be seen as follows: We define the invertible matrix

$$M := \mathrm{diag}((-D_1^\top)^{-1}, \dots, (-D_m^\top)^{-1}, C_1^{-1}, \dots, C_{m-1}^{-1})$$

which maps $\boldsymbol{f}$ to the sequence

$$\begin{aligned}\boldsymbol{f}' := M \cdot \boldsymbol{f} = \{&\boldsymbol{a}^1 + (D_1^\top)^{-1} \cdot \partial U/\partial \boldsymbol{x}', \boldsymbol{a}^2 - (D_2^\top)^{-1} \cdot C_2^\top \cdot \boldsymbol{a}^1, \dots, \boldsymbol{a}^m - (D_m^\top)^{-1} \cdot C_m^\top \cdot \boldsymbol{a}^{m-1}, \\ &\boldsymbol{y}^1 - C_1^{-1} \cdot D_1 \cdot \boldsymbol{x}', \boldsymbol{y}^2 - C_2^{-1} \cdot D_2 \cdot \boldsymbol{y}^1, \dots, \boldsymbol{y}^{m-1} - C_{m-1}^{-1} \cdot D_{m-1} \cdot \boldsymbol{y}^{m-2}\}\end{aligned}$$

which is regular by Lemma 1.3.51 as each element introduces a new variable. Lemma 1.3.44 then implies that $\boldsymbol{f}$ is Koszul-regular.

For the finite free quotient property (Definition 2.3.1 (b)), we define

$$S := \mathbb{C}[\boldsymbol{b}^1, \dots, \boldsymbol{b}^l, \boldsymbol{z}^1, \dots, \boldsymbol{z}^{l-1}, \boldsymbol{x}', \boldsymbol{x}]$$

and show

$$S[\boldsymbol{a}^1, \dots, \boldsymbol{a}^m, \boldsymbol{y}^1, \dots, \boldsymbol{y}^{m-1}]/(\boldsymbol{f}) \cong S \quad (\text{as } S\text{-modules})$$

by a similar argument: Lemma 1.3.43 tells us $(\boldsymbol{f}) = (\boldsymbol{f}')$. Dividing out the first n generators of $(\boldsymbol{f}')$ amounts to replacing $\boldsymbol{a}^1 \mapsto -(D_1^\top)^{-1} \cdot \partial U/\partial \boldsymbol{x}' \in S$. By induction, all $\boldsymbol{a}^i$ can be divided out successively since $\boldsymbol{a}^{i-1}$ will have been replaced by some element in S. The argument is analogous for the $\boldsymbol{y}^i$: The first n remaining generators replace $\boldsymbol{y}^1 \mapsto C_1^{-1} \cdot D_1 \cdot \boldsymbol{x}' \in S$, and we may successively replace all the $\boldsymbol{y}^i$ by elements in S. At the end, all $\boldsymbol{a}^i$ and $\boldsymbol{y}^i$ will have been removed from the numerator and the denominator will be empty. Only the module S remains, which is finite-rank and free over itself.

The proof of $\boldsymbol{v}$ being a set of left admissible variables is analogous. □

Remark 3.2.14. The idempotent closure of finite-rank matrix factorisations in the full bicategory $\ddot{\mathsf{MF}}_k$ contains infinite-rank matrix factorisations as seen in Example 2.2.9. In the case of $\mathcal{T}$, however, the existence of left and right admissible variables keeps the ranks of 2-morphisms finite: The construction below Theorem 2.3.14 can be adapted easily to show that horizontal compositions $X \otimes Y$ are isomorphic to direct sums of finite-rank matrix factorisations (by replacing $X^\vee \mapsto Y$).

Conjecture 3.2.15. *The structure $\mathcal{T}$ as defined in Definition 3.2.11 is a pivotal tricategory with duals.*

Partial proof. We see that the identity 1-morphism (which is the only structure 1-morphism) of the only object $* \in \mathcal{T}$ is an element of $\mathcal{T}(*,*)$ by inserting $m = 1, C_1 = D_1 = \mathbb{1}$ into Eq. (3.2.14). Because all structure 2-morphisms of $\mathcal{RW}$ are finite-rank matrix factorisations, $\mathcal{T}$ is closed under both $-\boxtimes-$ and $-\otimes-$, and $\mathcal{T}$ has the same 3-morphisms as $\mathcal{RW}$, $\mathcal{T}$ is a tricategory assuming Conjecture 1.5.15 holds. We proceed by proving the axioms of a pivotal tricategory with duals (Definition 1.2.21):

- The existence of left adjoints in the truncation $h\mathcal{T}$ follows from Theorem 3.2.3, proving the third axiom.
- For the first axiom we first use Lemma 3.2.13 and Theorem 2.3.7 to show that all 2-morphisms in $\mathcal{T}$ have left and right adjoints. Furthermore, we note that if n is even (odd), then $\mathcal{T}$ is a subcategory of $\mathcal{RW}^{\text{even}}$ ($\mathcal{RW}^{\text{odd}}$): the object $*$ has n bulk variables, each I_g has n surface variables, the $\{I_g\}$ generate the 1-morphisms of $\mathcal{T}$, and both $\mathcal{RW}^{\text{even}}$ and $\mathcal{RW}^{\text{odd}}$ are closed under $-\boxtimes-$. Therefore, all $X \in \mathrm{Hom}_{\mathcal{T}(*,*)}(V, W)$ have an even (odd) number of variables on both sides, hence $\mathcal{T}(*,*)$ has a pivotal structure by Corollary 2.4.12.
- It remains to be shown that the 2-functors $W \boxtimes -$ and $- \boxtimes W$ are pivotal for all 1-morphisms $W \in \mathcal{T}(*,*)$, which we conjecture to be true. □

Corollary 3.2.16. *If Conjecture 3.2.15 holds, then the direct sum completion of $\mathcal{T}$ (denoted by $\overline{\mathcal{T}}^{\oplus}$) is a pivotal tricategory with duals.*

Proof. This is a direct consequence of Lemma 3.2.19. □

3.2.6 Adjunctions in the direct sum completion $\overline{\mathcal{RW}}^{\oplus}$

Having studied adjunctions in $\mathcal{T} \subset \mathcal{RW}$ we now turn our attention to $\overline{\mathcal{RW}}^{\oplus}$ (see Definition 1.5.17).

Lemma 3.2.17. *The truncated bicategory $h\overline{\mathcal{RW}}^{\oplus}$ has left adjoints. For $A \in \overline{\mathcal{RW}}^{\oplus}(\boldsymbol{x}, \boldsymbol{y})$ we define*

$$A^{\#} := \bigoplus_{i=1}^{\ell(A)} A_i^{\#} \in \overline{\mathcal{RW}}^{\oplus}(\boldsymbol{x}, \boldsymbol{y}) \tag{3.2.15}$$

and the adjunction 2-morphisms

$$\mathrm{coev}_A\big|_{\mathbb{1}_{\boldsymbol{y}} \to A_i \boxtimes A_j^{\#}} := \delta_{i,j} \cdot \mathrm{coev}_{A_i} \ , \qquad \mathrm{ev}_A\big|_{A_i^{\#} \boxtimes A_j \to \mathbb{1}_{\boldsymbol{x}}} := \delta_{i,j} \cdot \mathrm{ev}_{A_i} \ , \tag{3.2.16}$$

$$\widetilde{\mathrm{coev}}_A\big|_{\mathbb{1}_{\boldsymbol{x}} \to A_i^{\#} \boxtimes A_j} := \delta_{i,j} \cdot \widetilde{\mathrm{coev}}_{A_i} \ , \qquad \widetilde{\mathrm{ev}}_A\big|_{A_i \boxtimes A_j^{\#} \to \mathbb{1}_{\boldsymbol{y}}} := \delta_{i,j} \cdot \widetilde{\mathrm{ev}}_{A_i} \ . \tag{3.2.17}$$

Proof. We will show the first Zorro movie:

$$\begin{aligned}
&(\rho_A \otimes (\mathbb{1}_A \boxtimes \mathrm{ev}_A) \otimes (\mathrm{coev}_A \boxtimes \mathbb{1}_A) \otimes \lambda_A^{-1})|_{A_i \to A_n} \\
&= \bigoplus_{j,l,m} (\rho_A \otimes (\mathbb{1}_A \boxtimes \mathrm{ev}_A))|_{A_j \boxtimes A_l^{\#} \boxtimes A_m \to A_n} \otimes (\mathrm{coev}_A \boxtimes \mathbb{1}_A) \otimes \lambda_A^{-1}|_{A_i \to A_j \boxtimes A_l^{\#} \boxtimes A_m} \\
&\cong \bigoplus_{j,l,m} \delta_{j,n} \cdot \rho_{A_j} \otimes (\mathbb{1}_{A_j} \boxtimes \delta_{l,m} \mathrm{ev}_{A_l}) \otimes \delta_{i,m} \delta_{j,l} (\mathrm{coev}_{A_j} \boxtimes \mathbb{1}_{A_i}) \otimes \lambda_{A_i}^{-1} \\
&\cong \delta_{n,i} \cdot \rho_{A_i} \otimes (\mathbb{1}_{A_i} \boxtimes \mathrm{ev}_{A_i}) \otimes (\mathrm{coev}_{A_i} \boxtimes \mathbb{1}_{A_i}) \otimes \lambda_{A_i}^{-1} \\
&\cong \delta_{n,i} \cdot \mathbb{1}_{A_i}
\end{aligned}$$

using Lemma 1.4.5 to contract $\delta_{i,j}$-terms and using the Zorro movie of A_i in the last step. The second Zorro movie is analogous. $\square$

Lemma 3.2.18. *Let $A, B \in \overline{\mathcal{RW}}^{\oplus}(\boldsymbol{x}, \boldsymbol{y})$ and $X = \{X_{j,i}\}\colon A \to B$. If all $X_{j,i}$ have left (right) adjoints, then X has a left (right) adjoint, given by*

$$({}^{\dagger}X)_{i,j} = {}^{\dagger}(X_{j,i}) \ , \qquad (X^{\dagger})_{i,j} = (X_{j,i})^{\dagger} \ , \tag{3.2.18}$$

together with the adjunction 3-morphisms

$$\mathrm{ev}_X\big|_{{}^{\dagger}X_{j,l}\otimes X_{j,i}} := \delta_{l,i}\cdot \mathrm{ev}_{X_{j,i}} \ , \qquad \mathrm{coev}_X\big|_{\mathbb{1}_{B_j}} := \sum_l \mathrm{coev}_{X_{j,l}} \ , \tag{3.2.19}$$

$$\widetilde{\mathrm{ev}}_X\big|_{X_{j,i}\otimes X^{\dagger}_{m,i}} := \delta_{j,m}\cdot \widetilde{\mathrm{ev}}_{X_{j,i}}, \qquad \widetilde{\mathrm{coev}}_X\big|_{\mathbb{1}_{A_i}} := \sum_m \widetilde{\mathrm{coev}}_{X_{m,i}} \ . \tag{3.2.20}$$

Proof. The domain of ev_X and the codomain of coev_X are given by

$$({}^{\dagger}X\otimes X)_{l,i} = \bigoplus_{j=1}^{\ell(B)} {}^{\dagger}(X_{j,l})\otimes X_{j,i} \ , \qquad (X\otimes {}^{\dagger}X)_{j,m} = \bigoplus_{i=1}^{\ell(A)} X_{j,i}\otimes {}^{\dagger}X_{m,i} \ .$$

We evaluate the left Zorro map of X on one component $X_{j,i}$:

$$\begin{tikzcd} X_{j,i} \arrow[r, "\lambda^{-1}_{X_{j,i}}"] & \mathbb{1}_{B_j}\otimes X_{j,i} \arrow[r, "(\sum_l \mathrm{coev}_{X_{j,l}})\otimes 1_{X_{j,i}}"] & (\bigoplus_j X_{j,l}\otimes {}^{\dagger}X_{j,l})\otimes X_{j,i} \arrow[dll, "1_{X_{j,l}}\otimes \delta_{l,i}\cdot \mathrm{ev}_{X_{j,i}}"] \\ X_{j,i}\otimes \mathbb{1}_{A_i} \arrow[r, "\rho_{X_{j,i}}"'] & X_{j,i} \ , & \end{tikzcd}$$

yielding the map

$$\rho_{X_{j,i}}\circ(1_{X_{j,l}}\otimes \mathrm{ev}_{X_{j,i}})\circ(\mathrm{coev}_{X_{j,i}}\otimes 1_{X_{j,i}})\circ\lambda^{-1}_{X_{j,i}}\colon X_{j,i}\to X_{j,i}$$

which is precisely the left Zorro map of $X_{j,i}$ and evaluates to $1_{X_{j,i}}$ by assumption. The argument for the right Zorro move is analogous. $\square$

Lemma 3.2.19. *Let $\mathcal{T}\subset\mathcal{RW}$ be a pivotal tricategory with duals and pivotal structure $\delta_X\colon X\to {}^{\dagger\dagger}X$ for all 2-morphisms X. Then $\overline{\mathcal{T}}^{\oplus}\subset\overline{\mathcal{RW}}^{\oplus}$ is a pivotal tricategory with duals, with the pivotal structure given by*

$$\delta_X\colon \{X_{j,i}\}\to {}^{\dagger\dagger}\{X_{j,i}\} = \{{}^{\dagger\dagger}X_{j,i}\} \ , \quad \delta_X\big|_{X_{j,i}} = \delta_{X_{j,i}} \tag{3.2.21}$$

for all 2-morphisms $X = \{X_{j,i}\}\colon\{W_i\}\to\{V_j\}$ in $\overline{\mathcal{T}}^{\oplus}$.

Proof. Consider $\{W_i\}, \{V_j\}\in\overline{\mathcal{T}}^{\oplus}(\boldsymbol{x},\boldsymbol{y})$ and $\{X_{j,i}\}\colon\{W_i\}\to\{V_j\}$. All $X_{j,i}$ are 2-morphisms in $\mathcal{T}$, so they have adjoints by assumption, so $\{X_{j,i}\}$ has adjoints as well by Lemma 3.2.18. The properties of the pivotal structure δ_X follow from the properties of the $\delta_{X_{j,i}}$ in analogy to the proof of Lemma 3.2.18. The pivotality of the 2-functors $-\boxtimes W$ and $W\boxtimes -$ on $\overline{\mathcal{T}}^{\oplus}$ also follows directly from the properties of the analogous functors on $\mathcal{T}$. $\square$

The following lemma is essential for evaluating bubble diagrams in Section 3.4.

Lemma 3.2.20. *Let $\overline{\mathcal{T}}^{\oplus} \subset \overline{\mathcal{RW}}^{\oplus}$ be a pivotal tricategory with duals. Consider 1-morphisms $W = \bigoplus_{i=1}^{n} W_i$ and $V = \bigoplus_{j=1}^{m} V_i$, a 2-morphism $X \colon W \to V$, and 3-morphisms*

$$\Phi = \{\Phi_{j,i}\} \in \mathrm{End}(X) \ , \quad \phi = \{\phi_j\} \in \mathrm{End}(\mathbb{1}_V) \ , \quad \psi = \{\psi_i\} \in \mathrm{End}(\mathbb{1}_W) \ .$$

Then the defect operators on $\mathrm{Hom}_{\overline{\mathcal{T}}^{\oplus}}$*, defined by the diagrams (2.5.3) and (2.5.4), take the values*

$$\mathcal{D}_l^{\Phi}(X)(\phi)\big|_{\mathbb{1}_{W_i}} = \sum_{j=1}^{m} \mathcal{D}_l^{\Phi_{j,i}}(X_{j,i})(\phi_j) \ , \qquad \mathcal{D}_r^{\Phi}(X)(\psi)\big|_{\mathbb{1}_{V_j}} = \sum_{i=1}^{n} \mathcal{D}_r^{\Phi_{j,i}}(X_{j,i})(\psi_i) \ . \tag{3.2.22}$$

Proof. We show the formula for $\mathcal{D}_r^{\Phi}(X)(\psi)$. As in the proof of Theorem 2.5.5, we relocate the action of ψ to the 2-morphism X. Then we find

$$\begin{aligned}
\mathcal{D}_r^{\Phi}(X)(\psi)\big|_{\mathbb{1}_{V_j}} &= \big(\tilde{\mathrm{ev}}_X \circ (\psi \cdot \Phi \otimes \delta_X) \circ \mathrm{coev}_X\big)\big|_{\mathbb{1}_{V_j}} \\
&= \sum_{i=1}^{n} \big(\tilde{\mathrm{ev}}_X \circ (\psi \cdot \Phi \otimes \delta_X)\big)\big|_{X_{j,i} \otimes {}^{\dagger}X_{j,i}} \circ \mathrm{coev}_{X_{j,i}} \\
&= \sum_{i=1}^{n} \tilde{\mathrm{ev}}_{X_{j,i}} \circ (\psi_i \cdot \Phi_{j,i} \otimes \delta_{X_{j,i}}) \circ \mathrm{coev}_{X_{j,i}} \\
&= \sum_{i=1}^{n} \mathcal{D}_r^{\Phi_{j,i}}(X_{j,i})(\psi_i) \ .
\end{aligned}$$

The argument for $\mathcal{D}_l^{\Psi}(X)(\psi)$ is analogous. □

3.3 Orbifolds and group action defects

3.3.1 Introduction

Orbifold constructions in high energy physics were first discussed in the context of string theory [34]. The basic idea is as follows [96, pp. 296 ff.]:

- Choose a finite symmetry group G of the original string theory T.
- Define a new theory T' which consists of the G-invariant states of T.
- Add the *twisted sectors* to T', which consist of states in T that violate the boundary conditions (or other constraints) of T but fulfil them modulo the action of G.

If the group G is a symmetry of the target S of the string theory T, the theory T' is defined on the orbifold S/G. The basic idea is the same in every kind of orbifold construction in physics: Starting from some theory T that is invariant under some group action G, one constructs a new theory T' on which the group G acts trivially.

There exists an orbifold construction for n-dimensional defect TFTs in bordism language [27] as well as compatible orbifold constructions in pivotal 2-categories [25, 46] and Gray categories with strict duals [27]; the present work will only discuss the latter. To apply the generalised orbifold procedure to a Gray category with strict duals $\mathcal{G}$, a *special orbifold* *datum* is required, which is a set consisting of one object, one 1- and 2-morphism, and four 3-morphisms

$$\mathcal{O} = \{*, \mathcal{A}, T, \alpha, \bar{\alpha}, \psi, \phi\}$$

subject to a list of constraint equations[2] discussed below in Definition 3.3.1. The relation between special orbifold data in Gray categories and the aforementioned construction of orbifolds from symmetry groups is not obvious. It turns out that not all special orbifold data can be interpreted as coming from a symmetry group, hence the term "*generalised* orbifold procedure" is used; we will elaborate on this point in Section 3.3.3.

3.3.2 Orbifold data in three-dimensional defect TFTs

We refer to [27] for the general theory of orbifolds in n-dimensional defect TQFTs and repeat the definition of an orbifold datum in Gray categories with strict duals.

Definition 3.3.1. Let $\mathcal{G}$ be a Gray category with strict duals. A *special orbifold datum* in $\mathcal{G}$ is a set consisting of [27, Def. 4.2]

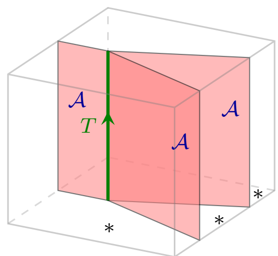
(i) an object $* \in \mathcal{G}$,

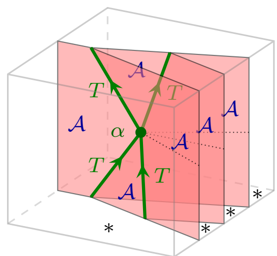
(ii) a 1-morphism $\mathcal{A}\colon * \to *$,

(iii) a 2-morphism $T\colon \mathcal{A} \boxtimes \mathcal{A} \to \mathcal{A}$,

(iv) two 3-isomorphisms $\alpha\colon T \otimes (\mathbb{1}_{\mathcal{A}} \boxtimes T) \leftrightarrows T \otimes (T \boxtimes \mathbb{1}_{\mathcal{A}}) : \bar{\alpha}$,

displayed as



,


, (3.3.1)

together with 3-isomorphisms $\phi \in \mathrm{Aut}(1_{1_*})$, $\psi \in \mathrm{Aut}(1_{\mathcal{A}})$, such that the following constraint equations hold:

(i) The 2-3 move identity [27, Def. 4.2 (i)],

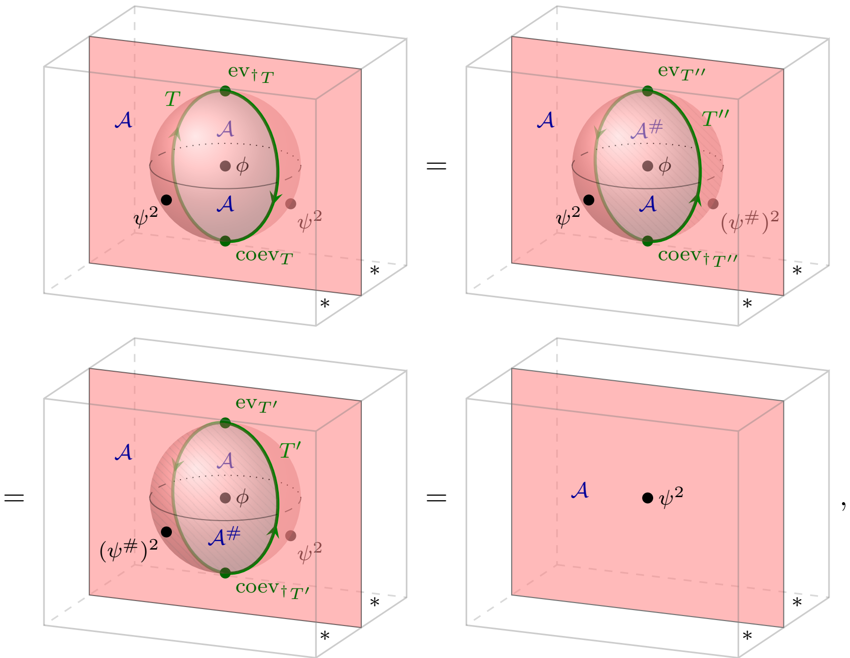
(ii) the normal associator identity and its opposite [27, Def. 4.2 (ii) & Figure 5],

(iii) the partially reversed associator identity and its opposite [27, Def. 4.2 (iii) & Figure 6],

(iv) the Frobenius type associator identity and its opposite [27, Def. 4.2 (iv) & Figure 7],

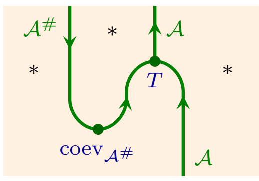
[2] The constraints assure that sufficiently fine-grained networks consisting only of the object and morphisms of the orbifold datum have equal values; see [27, 25] for the role of these networks in the orbifold procedure.

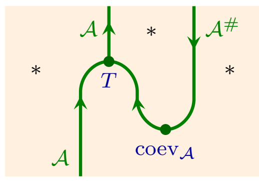
(v) the bubble identities [27, Eq. (3.51) & Figure 8], given by



(3.3.2)

with the primed versions of T defined as follows:

$T'' =$ 

, $T' =$ 

. (3.3.3)

No identity morphisms are used in these diagrams because $\mathcal{G}$ is strictly unital.

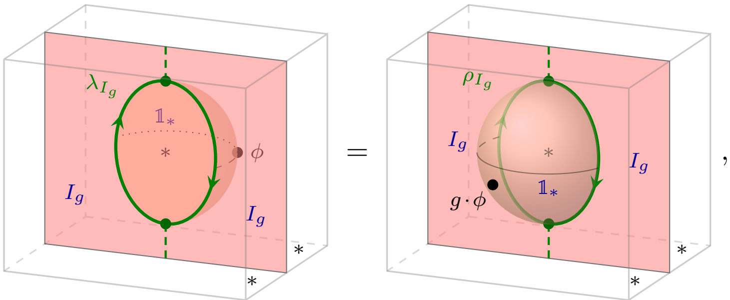
3.3.3 Group action type special orbifold data

As discussed above, some (but not all) special orbifold data can be constructed from a symmetry group. To see this relation explicitly, we first define the structure of group actions on defect TFTs in higher category description:

Definition 3.3.2. Let $\mathcal{T}$ be a pivotal tricategory with duals and let $* \in \mathcal{T}$ be an object. Let G be a group with a group action on bulk point insertions of $*$, i.e.

$$(g, \phi) \mapsto g \cdot \phi \quad \text{for} \quad \phi \in \mathrm{End}(\mathbb{1}_{\mathbb{1}_*}) \ . \tag{3.3.4}$$

Then a *group action defect* is a 1-morphism $I_g \in \mathcal{T}(*, *)$ such that



, (3.3.5)

i.e. "moving ϕ across I_g implements the group action $\phi \mapsto g \cdot \phi$ for all $\phi \in \mathrm{End}(\mathbb{1}_{\mathbb{1}_*})$".

We may now define the following subclass of special orbifold data:

Definition 3.3.3 (Group action type special orbifold data in Gray categories)**.** Let $\mathcal{G}$ be a Gray category with strict duals that has a notion of direct sums of 1-morphisms. Let $* \in \mathcal{G}$ and let G be a finite group according to Definition 3.3.2, i.e. there is a group action on bulk point insertions on $*$ and a group action 1-morphisms $I_g \in \mathcal{G}(*, *)$ for all $g \in G$. Then a *group action type special orbifold datum* in $\mathcal{G}$ is a special orbifold datum $\mathcal{O}$ whose 1-morphism is given by $\mathcal{A} = \bigoplus_{g \in G} I_g$ [27, Section 1].

The morphisms T, α, $\bar{\alpha}$ of $\mathcal{O}$ describe the intersections in networks of $\mathcal{A}$: For example, T has components $\mu_{g,h} \colon I_g \boxtimes I_h \to I_{gh}$ that describe the "multiplication" of two group action defects. It should be noted that it is not necessarily possible to construct an orbifold datum from every symmetry group or even from a given set of group action 1-morphisms.

3.4 Orbifolds in $\mathcal{RW}$

In this section we will construct a candidate for a group action orbifold datum in the affine Rozansky–Witten model according to Definition 3.3.3. To do so, we have to work with the tricategory $\overline{\mathcal{T}}^{\oplus} \subset \overline{\mathcal{RW}}^{\oplus}$ (see Section 1.5.8) since $\mathcal{RW}$ is not pivotal, and direct sums of 1-morphisms are not defined in $\mathcal{T}$. Furthermore, we have to use the strictification $s\overline{\mathcal{T}}^{\oplus}$ because orbifold data are only defined on Gray categories with strict duals. The following steps have to be taken:

(i) Choose a finite group G with a group action $(g, \phi) \mapsto g \cdot \phi$ for $\phi \in \mathbb{1}_{\mathbb{1}_*}$ (with $*$ being the only object of $\mathcal{T}$).

(ii) Construct group action 1-morphisms I_g for all $g \in G$ implementing the group action.

(iii) Construct multiplication 2-morphisms $\mu_{g,h} \colon I_g \boxtimes I_h \to I_{gh}$.

(iv) Find 3-morphisms α, $\bar{\alpha}$, ψ, ϕ in $s\overline{\mathcal{T}}^{\oplus}$ such that the constraints of Definition 3.3.1 are fulfilled.

A full proof of the bubble identities (3.3.2) and the main idea for the associator diagrams will be shown here, the rest will be discussed in [7].

3.4.1 Group action defects

Definition 3.4.1. Let $n \in \mathbb{N}$, let $G \subset \mathrm{Gl}_n(\mathbb{C})$ be a finite matrix group, $\ell(\boldsymbol{a}) = \ell(\boldsymbol{x}) = \ell(\boldsymbol{x}') = n$, and $g, h \in G$. We define the following building blocks of an orbifold datum in a subcategory of $\overline{\mathcal{RW}}^{\oplus}$:

(i) The group action 1-morphisms are defined as follows:

$$\begin{aligned} W_g(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{x}') &:= \zeta_g \textstyle\sum_{i=1}^n a_i (x_i - \sum_j g_{ij} x'_j) = \zeta_g \boldsymbol{a} \cdot (\boldsymbol{x} - g \cdot \boldsymbol{x}') \ , \\ I_g &:= (\boldsymbol{a};\, W_g) = (\boldsymbol{a};\, \zeta_g \boldsymbol{a} \cdot (\boldsymbol{x} - g \cdot \boldsymbol{x}')) \colon (\boldsymbol{x}') \to (\boldsymbol{x}) \ , \\ \zeta_g &:= (\det g)^{-1/2n} \in \mathbb{C} \setminus \{0\} \implies \zeta_e = 1 \ , \quad \zeta_{gh} = \zeta_g \zeta_h \ . \end{aligned} \tag{3.4.1}$$

The reason for this value of $\{\zeta_g\}$ is not obvious a priori and only becomes clear after evaluating the constraint equations; one could also regard the $\{\zeta_g\}$ as free parameters and make a choice at a later point.

(ii) To define the multiplication 2-morphism $\mu_{g,h} \colon I_g \boxtimes I_h \to I_{gh}$ we first spell out its domain and codomain, i.e.

$$\begin{aligned} I_g \boxtimes I_h &= \big(\boldsymbol{a}, \boldsymbol{x}', \boldsymbol{b};\, \zeta_g \boldsymbol{a} \cdot (\boldsymbol{x} - g \cdot \boldsymbol{x}') + \zeta_h \boldsymbol{b} \cdot (\boldsymbol{x}' - h \cdot \boldsymbol{x}'')\big) \ , \\ I_{gh} &= \big(\boldsymbol{c};\, \zeta_{gh} \boldsymbol{c} \cdot (\boldsymbol{x} - g \cdot h \cdot \boldsymbol{x}'')\big) \ , \end{aligned}$$

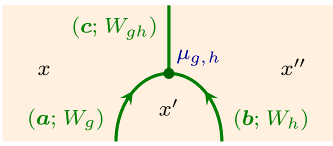
and we define

$$\mu_{g,h} := K\big(\zeta_{gh}\boldsymbol{c} - \zeta_g \boldsymbol{a};\, \boldsymbol{x} - g \cdot h \cdot \boldsymbol{x}''\big) \otimes K\big(\boldsymbol{x}' - h \cdot \boldsymbol{x}'';\, \zeta_g \boldsymbol{a} \cdot g - \zeta_h \boldsymbol{b}\big)[s_\mu] \tag{3.4.2}$$

$$\in K_{W_{gh} - W_g - W_h}\big(\zeta_{gh}\boldsymbol{c} - \zeta_g \boldsymbol{a},\, \boldsymbol{x}' - h \cdot \boldsymbol{x}''\big)[s_\mu] \ , \tag{3.4.3}$$

$$s_\mu := \tbinom{n+1}{2} + n \ ,$$

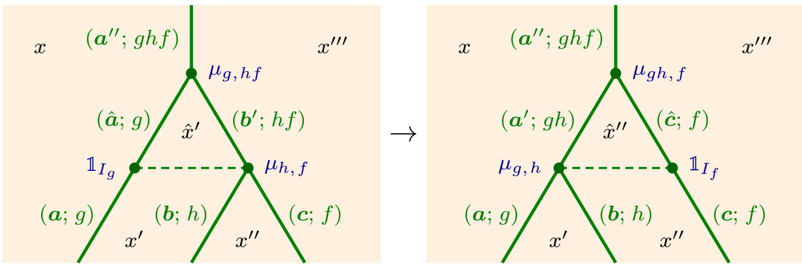
with a global grade shift s_μ (whose value, like the values of the $\{\zeta_g\}$, is not obvious a priori). The conventions are summarised in the following (truncated) picture:



(3.4.4)

(iii) The associator 3-morphism $\alpha_{g,h,f}$ has the following domain and codomain:

$\alpha_{g,h,f}$:



(3.4.5)

using the shorthand notation $(\boldsymbol{a};\, g) := (\boldsymbol{a};\, W_g(\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{x}'))$. To construct $\alpha_{g,h,f}$, we first apply multiple isomorphisms to $\mu_{g,h}$:

$$\begin{aligned}
\eta_{\mu_{g,h}} \colon \mu_{g,h} &= K\big(\zeta_{gh}\boldsymbol{c} - \zeta_g \boldsymbol{a};\; \boldsymbol{x} - g \cdot h \cdot \boldsymbol{x}''\big) \otimes K\big(\boldsymbol{x}' - h \cdot \boldsymbol{x}'';\; \zeta_g \boldsymbol{a} \cdot g - \zeta_h \boldsymbol{b}\big)[s_\mu] \\
&\xrightarrow{(1.3.46)} K\big(\zeta_{gh}\boldsymbol{c} - \zeta_g \boldsymbol{a};\; \boldsymbol{x} - g \cdot h \cdot \boldsymbol{x}''\big) \otimes K\big(\zeta_g \boldsymbol{a} \cdot g - \zeta_h \boldsymbol{b};\; \boldsymbol{x}' - h \cdot \boldsymbol{x}''\big)[s_\mu + n] \\
&\xrightarrow{(1.3.24)} K\big(\zeta_{gh}\boldsymbol{c} - \zeta_g \boldsymbol{a};\; \boldsymbol{x} - g \cdot \boldsymbol{x}'\big) \otimes K\big(\zeta_{gh} \boldsymbol{c} \cdot g - \zeta_h \boldsymbol{b};\; \boldsymbol{x}' - h \cdot \boldsymbol{x}''\big)[s_\mu + n] \\
&\xrightarrow{(1.3.28)} K\big(\zeta_h\boldsymbol{c} - \boldsymbol{a};\; \zeta_g(\boldsymbol{x} - g \cdot \boldsymbol{x}')\big) \otimes K\big(\zeta_g \boldsymbol{c} \cdot g - \boldsymbol{b};\; \zeta_h(\boldsymbol{x}' - h \cdot \boldsymbol{x}'')\big)[s_\mu + n] \\
&= I^{\zeta_h \boldsymbol{c} \leftarrow \boldsymbol{a}}_{W_g(\boldsymbol{x}, \bullet, \boldsymbol{x}')} \otimes I^{\zeta_g \boldsymbol{c} \cdot g \leftarrow \boldsymbol{b}}_{W_h(\boldsymbol{x}', \bullet, \boldsymbol{x}'')}[s_\mu + n] \;.
\end{aligned}$$

Only the dependencies on variables that appear multiple times will be spelled out, i.e. $\{\boldsymbol{a}, \hat{\boldsymbol{a}}, \boldsymbol{x}', \hat{\boldsymbol{x}}'\}$ for the domain of $\alpha_{g,h,f}$ and $\{\boldsymbol{c}, \hat{\boldsymbol{c}}, \boldsymbol{x}'', \hat{\boldsymbol{x}}''\}$ for the codomain. We define the following constituent parts of $\alpha_{g,h,f}$:

$$\begin{aligned}
&\beta_{g,h,f} \colon \mu_{g,hf} \otimes (\mathbb{1}_{I_g} \boxtimes \mu_{h,f}) \\
&\quad = \mu_{g,hf}(\hat{\boldsymbol{x}}', \hat{\boldsymbol{a}}) \otimes \big(I^{\hat{\boldsymbol{a}} \leftarrow \boldsymbol{a}}_{W_g(\bullet, \hat{\boldsymbol{x}}')} \otimes I^{\hat{\boldsymbol{x}}' \leftarrow \boldsymbol{x}'}_{W_g(\boldsymbol{a}, \bullet) + W_{hf}(\bullet)} \otimes \mu_{h,f}(\boldsymbol{x}')\big) \\
&\quad = \mu_{g,hf}(\hat{\boldsymbol{x}}', \hat{\boldsymbol{a}}) \otimes I^{\hat{\boldsymbol{a}} \leftarrow \boldsymbol{a}}_{W_g(\bullet, \hat{\boldsymbol{x}}') + W_{hf}} \otimes I^{\hat{\boldsymbol{x}}' \leftarrow \boldsymbol{x}'}_{W_g(\boldsymbol{a}, \bullet) + W_{hf}(\bullet)} \otimes \mu_{h,f}(\boldsymbol{x}') \\
&\quad = \mu_{g,hf}(\hat{\boldsymbol{x}}', \hat{\boldsymbol{a}}) \otimes I^{\{\hat{\boldsymbol{a}}, \hat{\boldsymbol{x}}'\} \leftarrow \{\boldsymbol{a}, \boldsymbol{x}'\}}_{W_g + W_{hf}} \otimes \mu_{h,f}(\boldsymbol{x}') \\
&\xrightarrow{\rho_{\mu_{g,hf}} \otimes 1_{\mu_{h,f}}} \mu_{g,hf}(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{x}''', \boldsymbol{a}, \boldsymbol{b}', \boldsymbol{a}'') \otimes \mu_{h,f}(\boldsymbol{x}', \boldsymbol{x}'', \boldsymbol{x}''', \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{b}') \\
&\xrightarrow{\eta_{\mu_{g,hf}} \otimes \eta_{\mu_{h,f}}} \Big(I^{\zeta_{hf} \boldsymbol{a}'' \leftarrow \boldsymbol{a}}_{W_g(\boldsymbol{x}, \bullet, \boldsymbol{x}')} \otimes I^{\zeta_g \boldsymbol{a}'' \cdot g \leftarrow \boldsymbol{b}'}_{W_{hf}(\boldsymbol{x}', \bullet, \boldsymbol{x}''')}\Big)[s_\mu + n] \otimes \Big(I^{\zeta_f \boldsymbol{b}' \leftarrow \boldsymbol{b}}_{W_h(\boldsymbol{x}', \bullet, \boldsymbol{x}'')} \otimes I^{\zeta_h \boldsymbol{b}' \cdot h \leftarrow \boldsymbol{c}}_{W_f(\boldsymbol{x}'', \bullet, \boldsymbol{x}''')}\Big)[s_\mu + n] \\
&\xrightarrow{(1.3.46)} I^{\zeta_{hf} \boldsymbol{a}'' \leftarrow \boldsymbol{a}}_{W_g(\boldsymbol{x}, \bullet, \boldsymbol{x}')} \otimes I^{\zeta_g \boldsymbol{a}'' \cdot g \leftarrow \boldsymbol{b}'}_{W_{hf}(\boldsymbol{x}', \bullet, \boldsymbol{x}''')} \otimes I^{\zeta_f \boldsymbol{b}' \leftarrow \boldsymbol{b}}_{W_h(\boldsymbol{x}', \bullet, \boldsymbol{x}'')} \otimes I^{\zeta_h \boldsymbol{b}' \cdot h \leftarrow \boldsymbol{c}}_{W_f(\boldsymbol{x}'', \bullet, \boldsymbol{x}''')} \;.
\end{aligned}$$

Starting from the codomain, we may apply similar isomorphisms:

$$\begin{aligned}
&\beta'_{g,h,f} \colon \mu_{gh,f} \otimes (\mu_{g,h} \boxtimes \mathbb{1}_{I_f}) \\
&\quad = \mu_{gh,f}(\hat{\boldsymbol{x}}'', \hat{\boldsymbol{c}}) \otimes \big(\mu_{g,h}(\hat{\boldsymbol{x}}'') \otimes I^{\hat{\boldsymbol{x}}'' \leftarrow \boldsymbol{x}''}_{W_g + W_h(\bullet) + W_f(\bullet, \hat{\boldsymbol{c}})} \otimes I^{\hat{\boldsymbol{c}} \leftarrow \boldsymbol{c}}_{W_f(\boldsymbol{x}'', \bullet)}\big) \\
&\quad = \mu_{gh,f}(\hat{\boldsymbol{x}}'', \hat{\boldsymbol{c}}) \otimes \mu_{g,h}(\hat{\boldsymbol{x}}'') \otimes I^{\hat{\boldsymbol{x}}'' \leftarrow \boldsymbol{x}''}_{W_g + W_h(\bullet) + W_f(\bullet, \hat{\boldsymbol{c}})} \otimes I^{\hat{\boldsymbol{c}} \leftarrow \boldsymbol{c}}_{W_g + W_h + W_f(\boldsymbol{x}'', \bullet)} \\
&\quad = \mu_{gh,f}(\hat{\boldsymbol{x}}'', \hat{\boldsymbol{c}}) \otimes \mu_{g,h}(\hat{\boldsymbol{x}}'') \otimes I^{\{\hat{\boldsymbol{x}}'', \hat{\boldsymbol{c}}\} \leftarrow \{\boldsymbol{x}'', \boldsymbol{c}\}}_{W_g + W_h + W_f} \\
&\xrightarrow{\rho_{\mu_{gh,f}} \otimes \mu_{g,h}} \mu_{gh,f}(\boldsymbol{x}, \boldsymbol{x}'', \boldsymbol{x}''', \boldsymbol{a}', \boldsymbol{c}, \boldsymbol{a}''') \otimes \mu_{g,h}(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{x}'', \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a}') \\
&\xrightarrow{\eta_{\mu_{gh,f}} \otimes \eta_{\mu_{g,h}}} \Big(I^{\zeta_f \boldsymbol{a}'' \leftarrow \boldsymbol{a}'}_{W_{gh}(\boldsymbol{x}, \bullet, \boldsymbol{x}'')} \otimes I^{\zeta_{gh} \boldsymbol{a}'' \cdot g \cdot h \leftarrow \boldsymbol{c}}_{W_f(\boldsymbol{x}'', \bullet, \boldsymbol{x}''')}\Big)[s_\mu + n] \otimes \Big(I^{\zeta_h \boldsymbol{a}' \leftarrow \boldsymbol{a}}_{W_g(\boldsymbol{x}, \bullet, \boldsymbol{x}')} \otimes I^{\zeta_g \boldsymbol{a}' \cdot g \leftarrow \boldsymbol{b}}_{W_h(\boldsymbol{x}', \bullet, \boldsymbol{x}'')}\Big)[s_\mu + n] \\
&\xrightarrow{(1.3.46)} I^{\zeta_f \boldsymbol{a}'' \leftarrow \boldsymbol{a}'}_{W_{gh}(\boldsymbol{x}, \bullet, \boldsymbol{x}'')} \otimes I^{\zeta_{gh} \boldsymbol{a}'' \cdot g \cdot h \leftarrow \boldsymbol{c}}_{W_f(\boldsymbol{x}'', \bullet, \boldsymbol{x}''')} \otimes I^{\zeta_h \boldsymbol{a}' \leftarrow \boldsymbol{a}}_{W_g(\boldsymbol{x}, \bullet, \boldsymbol{x}')} \otimes I^{\zeta_g \boldsymbol{a}' \cdot g \leftarrow \boldsymbol{b}}_{W_h(\boldsymbol{x}', \bullet, \boldsymbol{x}'')} \;.
\end{aligned}$$

Define $R := \mathbb{C}[\boldsymbol{a}, \boldsymbol{a}'', \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{x}, \boldsymbol{x}', \boldsymbol{x}'', \boldsymbol{x}''']$. The codomain of $\beta_{g,h,f}$ is associated to

$$\begin{aligned}
&R[\boldsymbol{b}']/(\zeta_{hf}\boldsymbol{a}'' - \boldsymbol{a},\, \zeta_g \boldsymbol{a}'' \cdot g - \boldsymbol{b}',\, \zeta_f \boldsymbol{b}' - \boldsymbol{b},\, \zeta_h \boldsymbol{b}' \cdot h - \boldsymbol{c}) \\
\cong\; &R/(\zeta_{hf}\boldsymbol{a}'' - \boldsymbol{a},\, \zeta_f \zeta_g \boldsymbol{a}'' \cdot g - \boldsymbol{b},\, \zeta_h \zeta_g \boldsymbol{a}'' \cdot g \cdot h - \boldsymbol{c}) \\
\cong\; &R[\boldsymbol{a}']/(\zeta_h \boldsymbol{a}' - \boldsymbol{a},\, \zeta_f \boldsymbol{a}'' - \boldsymbol{a}',\, \zeta_g \boldsymbol{a}' \cdot g - \boldsymbol{b},\, \zeta_{gh} \boldsymbol{a}'' \cdot g \cdot h - \boldsymbol{c})
\end{aligned}$$

which is associated to the codomain of $\beta'_{g,h,f}$, so there exists an isomorphism

$$
\begin{aligned}
\gamma_{g,h,f}\colon I^{\zeta_{hf}\boldsymbol{a}''\leftarrow\boldsymbol{a}}_{W_g(\boldsymbol{x},\bullet,\boldsymbol{x}')} \otimes I^{\zeta_g\boldsymbol{a}''\cdot g\leftarrow\boldsymbol{b}'}_{W_{hf}(\boldsymbol{x}',\bullet,\boldsymbol{x}''')} \otimes I^{\zeta_f\boldsymbol{b}'\leftarrow\boldsymbol{b}}_{W_h(\boldsymbol{x}',\bullet,\boldsymbol{x}'')} \otimes I^{\zeta_h\boldsymbol{b}'\cdot h\leftarrow\boldsymbol{c}}_{W_f(\boldsymbol{x}'',\bullet,\boldsymbol{x}''')} \\
\to I^{\zeta_f\boldsymbol{a}''\leftarrow\boldsymbol{a}'}_{W_{gh}(\boldsymbol{x},\bullet,\boldsymbol{x}'')} \otimes I^{\zeta_{gh}\boldsymbol{a}''\cdot g\cdot h\leftarrow\boldsymbol{c}}_{W_f(\boldsymbol{x}'',\bullet,\boldsymbol{x}''')} \otimes I^{\zeta_h\boldsymbol{a}'\leftarrow\boldsymbol{a}}_{W_g(\boldsymbol{x},\bullet,\boldsymbol{x}')} \otimes I^{\zeta_g\boldsymbol{a}'\cdot g\leftarrow\boldsymbol{b}}_{W_h(\boldsymbol{x}',\bullet,\boldsymbol{x}'')}
\end{aligned}
$$

which can be constructed explicitly using Remark 1.3.35. We then define

$$
\alpha_{g,h,f} := C_\alpha \cdot (\beta'_{g,h,f})^{-1} \circ \gamma_{g,h,f} \circ \beta_{g,h,f} \ , \qquad \bar{\alpha}_{g,h,f} := \alpha^{-1}_{g,h,f} \tag{3.4.6}
$$

with a free[3] parameter $C_\alpha \in \mathbb{C} \setminus \{0\}$.

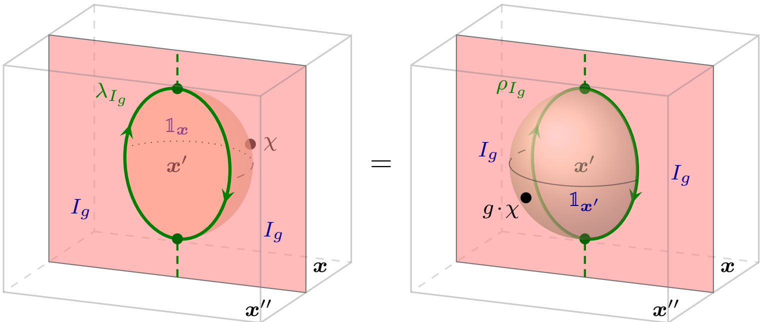
Lemma 3.4.2. *The 1-morphisms defined in Eq. (3.4.1) are group action defects in the sense of Definition 3.3.2, and the group action on a 3-morphism*

$$
\chi(\boldsymbol{a}, \boldsymbol{x}) \in \mathbb{C}[\boldsymbol{a}, \boldsymbol{x}] \cong \mathbb{C}[\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{x}']/(\boldsymbol{x} - \boldsymbol{x}') = \operatorname{End}(\mathbb{1}_{\mathbb{1}_{\boldsymbol{x}}})
$$

is given by

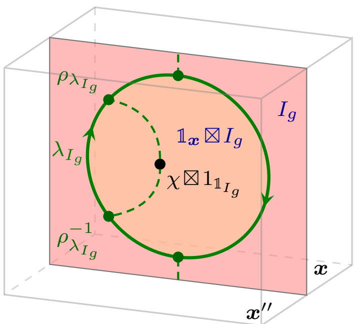
$$
\chi(\boldsymbol{a}, \boldsymbol{x}) \mapsto g \cdot \chi := \chi(\boldsymbol{a} \cdot g^{-1}, g \cdot \boldsymbol{x}'') \in \operatorname{End}(\mathbb{1}_{\mathbb{1}_{\boldsymbol{x}''}}) \ . \tag{3.4.7}
$$

Proof. We repeat Eq. (3.3.5) with the tricategory $\mathcal{T}$ inserted:



. (3.4.8)

Flattening the left hand side yields



. (3.4.9)

The 1-morphism inside the circle is given by

$$
\mathbb{1}_{\boldsymbol{x}} \boxtimes I_g = \big(\boldsymbol{a}, \boldsymbol{x}', \boldsymbol{b};\ \boldsymbol{a} \cdot (\boldsymbol{x} - \boldsymbol{x}') + \zeta_g \boldsymbol{b} \cdot (\boldsymbol{x}' - g \cdot \boldsymbol{x}'')\big) \ ,
$$

[3]The only constraint equation that depends on C_α is the 2-3 move (Definition 3.3.1 (i)) which is not discussed in this work, hence the value of C_α cannot be determined here. As far as the computations of this work are concerned one may simply set $C_\alpha := 1$.

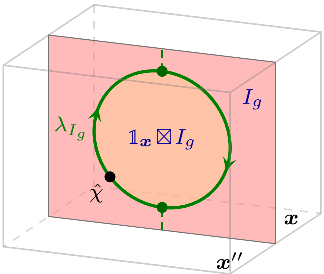
and outside of the circle we find

$$I_g = (\boldsymbol{c};\ \zeta_g \boldsymbol{c} \cdot (\boldsymbol{x} - g \cdot \boldsymbol{x}''))\ . \tag{3.4.10}$$

By Lemma 2.5.3 we can write the 3-morphisms χ and $\chi \boxtimes 1_{\mathbb{1}_{I_g}}$ in the form of a multiplication, i.e.

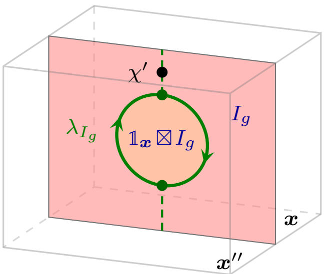
$$\chi \boxtimes 1_{\mathbb{1}_{I_g}} \colon \alpha \mapsto \chi(\boldsymbol{a}, \boldsymbol{x}) \cdot \alpha \quad \text{for } \alpha \in \mathbb{1}_{\mathbb{1}_{\boldsymbol{x}}} \boxtimes \mathbb{1}_{I_g}\ ,\ \ \chi(\boldsymbol{a}, \boldsymbol{x}) \in \mathbb{C}[\boldsymbol{a}, \boldsymbol{x}] \cong \operatorname{End}(\mathbb{1}_{I_g})\ .$$

This multiplication may also be performed on $\operatorname{End}(\lambda_{I_g})$, hence Eq. (3.4.9) is equal to



$$\text{with}\quad \hat{\chi} \colon \beta \mapsto \chi(\boldsymbol{a}, \boldsymbol{x}) \cdot \beta \quad \text{for } \beta \in \lambda_{I_g}\ . \tag{3.4.11}$$

On $\operatorname{End}(\lambda_{I_g})$, both $(\zeta_g \boldsymbol{b} - \boldsymbol{a})$ and $(\boldsymbol{c} - \boldsymbol{b})$ are exact,[4] so $\hat{\chi}$ is homotopic to $\alpha \mapsto \chi(\zeta_g \boldsymbol{c}, \boldsymbol{x}) \cdot \alpha$. This multiplication can also be performed on $\operatorname{End}(\mathbb{1}_{I_g})$, thus Eq. (3.4.9) is equal to



$$\text{with}\quad \chi' \colon \gamma \mapsto \chi(\zeta_g \boldsymbol{c}, \boldsymbol{x}) \cdot \gamma \quad \text{for } \gamma \in \mathbb{1}_{I_g}\ . \tag{3.4.12}$$

According to the discussion in Section 3.1.2, the half-sphere on the left hand side of Eq. (3.4.8) is bounded at the top and bottom by $(\alpha_l(I_g), \alpha_l^{-1}(I_g))$, and the half-sphere on the right hand side is bounded by $(\alpha_r(I_g), \alpha_r^{-1}(I_g))$. Therefore, the (now empty) λ_{I_g}-bubble simply evaluates to 1, so Eq. (3.4.9) reduces to just χ' of Eq. (3.4.12).

Applying the analogous procedure on the right hand side, we find

$$I_g \boxtimes \mathbb{1}_{\boldsymbol{x}''} = (\boldsymbol{b}, \boldsymbol{x}', \boldsymbol{a};\ \zeta_g \boldsymbol{b} \cdot (\boldsymbol{x} - g \cdot \boldsymbol{x}') + \boldsymbol{a} \cdot (\boldsymbol{x}' - \boldsymbol{x}''))\ ,$$
$$1_{\mathbb{1}_{I_g}} \boxtimes (g \cdot \chi) \colon \alpha \mapsto \chi(\boldsymbol{a} \cdot g^{-1}, g \cdot \boldsymbol{x}'') \cdot \alpha\ .$$

After relocating the multiplication onto ρ_{I_g} where $(\boldsymbol{a} - \zeta_g \boldsymbol{b} \cdot g)$, $(\boldsymbol{c} - \boldsymbol{b})$, and $(\boldsymbol{x} - g \cdot \boldsymbol{x}'')$ are exact, we apply a homotopy turning χ into $\beta \mapsto \chi(\zeta_g \boldsymbol{c}, \boldsymbol{x}) \cdot \beta$. This multiplication can be relocated to $\operatorname{End}(\mathbb{1}_{I_g})$, and the bubble evaluates to 1. Both sides of Eq. (3.4.8) are therefore equal to χ' of Eq. (3.4.12). $\square$

[4]Let $X := K(\boldsymbol{p}; \boldsymbol{q})$ be a Koszul matrix factorisation. Then on $\operatorname{End}(X)$, both $\alpha \mapsto p_i \alpha$ and $\alpha \mapsto q_i \alpha$ are exact for all i as they are the d-images of $\alpha \mapsto \theta_i \wedge \alpha$ and $\alpha \mapsto \theta_i^* \alpha$, respectively. Inserting the explicit formula of λ_{I_g} in Eq. (1.5.26) proves the claim.

3.4.2 The orbifold datum in $\overline{\mathcal{RW}}^{\oplus}$

Conjecture 3.4.3. *Let $n \in \mathbb{N}$ and let $G \subset \mathrm{Gl}_n(\mathbb{C})$ be a finite matrix group. Let $\overline{\mathcal{T}}^{\oplus}$ denote the pivotal tricategory with duals of Corollary 3.2.16 and let $f\colon \overline{\mathcal{T}}^{\oplus} \to s\overline{\mathcal{T}}^{\oplus}$ denote the strictification triequivalence of Eq. (3.1.1). Then the set*

$$\mathcal{O} = \{f(*), f(\mathcal{A}), f(T), f(\alpha), f(\bar{\alpha}), f(\psi), f(\phi)\} \subset s\overline{\mathcal{T}}^{\oplus} \tag{3.4.13}$$

is a special orbifold datum in the Gray category with strict duals $s\overline{\mathcal{T}}^{\oplus}$. Its components consist of the morphisms of Definition 3.4.1 as follows:

(i) $* := (\boldsymbol{x}) = (x_1, \ldots, x_n) \in \overline{\mathcal{RW}}^{\oplus}$,

(ii) $\mathcal{A} := \bigoplus_{g\in G} I_g$,

(iii) $T|_{I_g \boxtimes I_h \to I_f} := \delta_{f,gh} \cdot \mu_{g,h}$,

(iv) $\alpha = \bar{\alpha}^{-1}$ *defined by*

$$\begin{aligned} T \otimes (\mathbb{1}_{\mathcal{A}} \boxtimes T)|_{I_g \boxtimes I_h \boxtimes I_f \to I_{ghf}} &= \mu_{g,hf} \otimes (\mathbb{1}_{I_h} \boxtimes \mu_{h,f}) \ , \\ T \otimes (T \boxtimes \mathbb{1}_{\mathcal{A}})|_{I_g \boxtimes I_h \boxtimes I_f \to I_{ghf}} &= \mu_{gh,f} \otimes (\mu_{g,h} \boxtimes \mathbb{1}_{I_f}) \ , \\ \alpha\colon T \otimes (\mathbb{1}_{\mathcal{A}} \boxtimes T) \to T \otimes (T \boxtimes \mathbb{1}_{\mathcal{A}}) \ , \quad & \alpha|_{\mu_{g,hf} \otimes (\mathbb{1}_{I_h} \boxtimes \mu_{h,f})} := \alpha_{g,h,f} \ , \end{aligned}$$

(v) $\psi = 1_{\mathbb{1}_{\mathcal{A}}}$, $\quad \phi = |G|^{-1} \cdot 1_{\mathbb{1}_{\mathbb{1}_*}}$.

In the following sections we will prove most of Conjecture 3.4.3.

3.4.3 Evaluating bubble diagrams

We first assign the names

$$\chi^T, \chi^{T''}, \chi^{T'} : \mathbb{1}_{\mathcal{A}} \to \mathbb{1}_{\mathcal{A}} \tag{3.4.14}$$

to the three bubble diagrams (3.3.2) with the special orbifold datum of Conjecture 3.4.3 inserted. As a first step we will evaluate χ^T, which is a diagram in $s\overline{\mathcal{T}}^{\oplus}$. Since 3-morphisms in $s\overline{\mathcal{T}}^{\oplus}$ are defined by their image under the triequivalence $e\colon s\overline{\mathcal{T}}^{\oplus} \to \overline{\mathcal{T}}^{\oplus}$ of Theorem 1.2.28, we must first understand the image of χ^T under e. We find that $e(\chi^T)$ looks very similar to χ^T: all e does is to insert the pivotal structure δ_T into the T-loop, and insert identity 1- and 2-morphisms bounded by unitor 2- and 3-morphisms through ϕ and ψ^2. We will not distinguish χ^T and $e(\chi^T)$ from now on as the former is represented by the latter anyway.

To simplify the diagram χ^T, we first study the automorphisms ϕ and ψ. According to Lemma 2.5.3 we may represent ϕ by a complex number and $\psi = \{\psi_g\}_{g\in G}$ by $|G|$ complex numbers:

$$\phi \in \mathrm{Aut}(\mathbb{1}_{\mathbb{1}_*}) \cong \mathbb{C} \ , \quad \psi \in \mathrm{Aut}(\mathbb{1}_{\mathcal{A}}) \ , \quad \mathbb{1}_{\mathcal{A}}|_{I_g \to I_g} = \mathbb{1}_{I_g} \ , \quad \psi_g := \psi|_{\mathbb{1}_{I_g}} \in \mathrm{Aut}(\mathbb{1}_{I_g}) \cong \mathbb{C} \ .$$

Formally, the two copies of ψ are located on identity lines $\mathbb{1}_{\mathcal{A}}$ bounded by the 3-morphisms ρ_T and ρ_T^{-1}, and ϕ is located on an identity half-sphere extending inwards from one of the two $\mathcal{A}$-surfaces. However, because the actions of ϕ and ψ are merely simple multiplications, we can use a trick similar to the one used in the proof of Theorem 2.5.5 to first relocate ϕ to one $\mathcal{A}$-surface and then fuse ϕ and both copies of ψ^2 after flattening the diagram. The result is the 3-morphism

$$\psi^4\phi \in \mathrm{Aut}(\mathbb{1}_{\mathcal{A}\boxtimes\mathcal{A}}) \ , \quad \psi^4\phi|_{\mathbb{1}_{I_g\boxtimes I_h}} = \psi_g^2\psi_h^2\phi \cdot 1_{\mathbb{1}_{I_g\boxtimes I_h}} \ .$$

The identity half-sphere ϕ lived on can be removed subsequently. The modified diagram now has the shape of a defect operator:

$$\chi^T = \mathcal{D}_r(T)(\psi^4\phi) \ . \tag{3.4.15}$$

Now we apply Lemma 3.2.20 to find

$$\begin{aligned} \chi_g^T := \chi^T|_{\mathbb{1}_{I_g}} &= \sum_{(h,f)\in G\times G} \mathcal{D}_r\big(\delta_{g,hf}\cdot\mu_{h,f}\big)\big(\psi_h^2\psi_f^2\phi\big) \\ &= \sum_{h\in G} \psi_h^2\psi_{h^{-1}g}^2\phi\cdot\mathcal{D}_r\big(\mu_{h,h^{-1}g}\big)(1) \\ &= \sum_{h\in G} \psi_h^2\psi_{h^{-1}g}^2\phi\cdot\dim_r\big(\mu_{h,h^{-1}g}\big) \ . \end{aligned} \tag{3.4.16}$$

The orbifold constraint equation (3.3.2) demands $\chi^T \stackrel{!}{=} \psi^2$, yielding the system of equations

$$\sum_{h\in G} \psi_h^2\psi_{h^{-1}g}^2\phi \ \dim_r(\mu_{h,h^{-1}g}) \stackrel{!}{=} \psi_g^2 \quad \text{for all } g\in G. \tag{3.4.17}$$

3.4.4 The quantum dimension of $\mu_{g,h}$

We now evaluate the right quantum dimension of

$$\mu_{g,h}\colon \big(\boldsymbol{a},\boldsymbol{x}',\boldsymbol{b};\ W_g(\boldsymbol{x},\boldsymbol{a},\boldsymbol{x}') + W_h(\boldsymbol{x}',\boldsymbol{b},\boldsymbol{x}'')\big) \to \big(\boldsymbol{c};\ W_{gh}(\boldsymbol{x},\boldsymbol{c},\boldsymbol{x}'')\big)$$

as defined in Eq. (3.4.2). The first step is to choose admissible variables:

$$\{u_1,\,\dots\,,u_{3n}\} := \{\boldsymbol{x},\boldsymbol{b},\boldsymbol{x}'\} \ , \qquad \{v_1,\,\dots\,,v_n\} := \{\boldsymbol{x}\} \ ,$$

yielding the sequences

$$\begin{aligned} \boldsymbol{f} &= \partial_{\boldsymbol{u}}(W_g + W_h - W_{gh}) = \{\zeta_g\boldsymbol{a} - \zeta_{gh}\boldsymbol{c},\ \zeta_h(\boldsymbol{x}' - h\cdot\boldsymbol{x}''),\ \zeta_h\boldsymbol{b} - \zeta_g\boldsymbol{a}\cdot g\} \ , \\ \boldsymbol{g} &= \partial_{\boldsymbol{v}}(W_{gh} - W_g - W_h) = \zeta_{gh}\boldsymbol{c} - \zeta_g\boldsymbol{a} \ . \end{aligned}$$

The proof that $\boldsymbol{u}$ and $\boldsymbol{v}$ are admissible variables is analogous to the proof of Lemma 3.2.13. Now we apply Lemma 2.5.7 and find

$$\dim_r \mu_{g,h} = (-1)^{s_\mu}\dim_r(\mu_{g,h}[s_\mu]) \ ,$$

so we may compute the right quantum dimension of $\mu_{g,h}$ as defined in Eq. (3.4.2) *without* the grade shift s_μ and multiply the result by $(-1)^{s_\mu}$. We evaluate the defect operator formula (2.5.10) on $\mu_{g,h}[s_\mu]$, starting with the numerator

$$\mathrm{str}\big\{\big(\textstyle\prod_{i=1}^{3n}\partial_{u_i}d_X\big)\big(\prod_{j=1}^{n}\partial_{c_j}d_X\big)\big\} \ .$$

Let $X := \mu_{g,h}[s_\mu]$, $R := k[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{x}, \boldsymbol{x}', \boldsymbol{x}'']$, $M := R \oplus R[1]$. Then

$$\begin{aligned} X &= M^{\otimes n} \otimes M^{\otimes n} \ , \\ d_X &= \sum_{i=1}^{n} \Big(1_M^{\otimes(i-1)} \otimes \alpha_i \otimes 1_M^{\otimes(n-i)} \otimes 1_{M^{\otimes n}} + 1_{M^{\otimes n}} \otimes 1_M^{\otimes(i-1)} \otimes \beta_i \otimes 1_M^{\otimes(n-i)} \Big) \ , \\ \alpha_i &= \left(\begin{smallmatrix} & \zeta_{gh} c_i - \zeta_g a_i \\ x_i - (g \cdot h \cdot \boldsymbol{x}'')_i & \end{smallmatrix} \right) \ , \quad \beta_i = \left(\begin{smallmatrix} & x_i' - (h \cdot \boldsymbol{x}'')_i \\ \zeta_g (\boldsymbol{a} \cdot g)_i - \zeta_h b_i & \end{smallmatrix} \right) \end{aligned}$$

with the usual Koszul sign convention. In this notation, we find

$$\begin{aligned} \partial_{c_i} d_X &= \zeta_{gh} \Big(1_M^{\otimes(i-1)} \otimes \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right) \otimes 1_M^{\otimes(n-i)} \Big) \otimes 1_{M^{\otimes n}} \ , \\ \partial_{x_i} d_X &= \quad \Big(1_M^{\otimes(i-1)} \otimes \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} \right) \otimes 1_M^{\otimes(n-i)} \Big) \otimes 1_{M^{\otimes n}} \ , \\ \partial_{b_i} d_X &= \zeta_h 1_{M^{\otimes n}} \otimes \Big(1_M^{\otimes(i-1)} \otimes \left(\begin{smallmatrix} 0 & 0 \\ -1 & 0 \end{smallmatrix} \right) \otimes 1_M^{\otimes(n-i)} \Big) \ , \\ \partial_{x_i'} d_X &= \quad 1_{M^{\otimes n}} \otimes \Big(1_M^{\otimes(i-1)} \otimes \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right) \otimes 1_M^{\otimes(n-i)} \Big) \ . \end{aligned} \tag{3.4.18}$$

We now compute the following products (which do not acquire Koszul signs because no odd maps are permuted):

$$\begin{aligned} &\textstyle\prod_{i=1}^n \partial_{c_i} d_X = \zeta_{gh}^n \cdot \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right)^{\otimes n} \otimes 1_{M^{\otimes n}} \ , \qquad &&\textstyle\prod_{i=1}^n \partial_{b_i} d_X = \zeta_h^n \cdot 1_{M^{\otimes n}} \otimes \left(\begin{smallmatrix} 0 & 0 \\ -1 & 0 \end{smallmatrix} \right)^{\otimes n} \ , \\ &\textstyle\prod_{i=1}^n \partial_{x_i} d_X = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} \right)^{\otimes n} \otimes 1_{M^{\otimes n}} \ , &&\textstyle\prod_{i=1}^n \partial_{x_i'} d_X = 1_{M^{\otimes n}} \otimes \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right)^{\otimes n} \ . \end{aligned}$$

However, the composition of two maps of the form

$$|d_i| = |d_i'| = 1, \quad (d_1 \otimes \cdots \otimes d_n) \circ (d_1' \otimes \cdots \otimes d_n') = (-1)^{\binom{n}{2}} (d_1 \circ d_1') \otimes \cdots \otimes (d_n \circ d_n')$$

does introduce a Koszul sign. Thus

$$\begin{aligned} \big(\textstyle\prod_{i=1}^{3n} \partial_{u_i} d_X\big)\big(\prod_{j=1}^{n} \partial_{c_j} d_X\big) &= (-1)^{2\binom{n}{2}} \zeta_h^n \zeta_{gh}^n \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \right)^{\otimes n} \otimes \left(\begin{smallmatrix} 0 & 0 \\ 0 & -1 \end{smallmatrix} \right)^{\otimes n} \\ &= (-1)^n (\zeta_h \zeta_{gh})^n \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \right)^{\otimes n} \otimes \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \right)^{\otimes n} \ . \end{aligned} \tag{3.4.19}$$

Let $\{e_1, e_2\}$ be the homogeneous standard basis of M. Then (3.4.19) is diagonal and acts non-trivially only on $e_2^{\otimes n} \otimes e_2^{\otimes n}$, which has degree $2n$. Therefore, the supertrace on the matrices evaluates to $(-1)^{2n} = 1$, and we find

$$\mathrm{str}\big\{ \big(\textstyle\prod_{i=1}^{3n} \partial_{u_i} d_X\big)\big(\prod_{j=1}^{n} \partial_{c_j} d_X\big) \big\} = (-1)^n (\zeta_h \zeta_{gh})^n \ .$$

The right quantum dimension then turns out to be

$$\begin{aligned} \dim_r \mu_{g,h} &= (-1)^{\binom{n+1}{2} + s_\mu} \operatorname{Res} \left[\frac{(-1)^n (\zeta_h^2 \zeta_g)^n \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}\boldsymbol{x}' \, \mathrm{d}\boldsymbol{b}}{\zeta_g \boldsymbol{a} - \zeta_{gh} \boldsymbol{c}, \, \zeta_h (\boldsymbol{x}' - h \cdot \boldsymbol{x}''), \, \zeta_h \boldsymbol{b} - \zeta_g \boldsymbol{a} \cdot g} \right] \\ &= (-1)^{\binom{n+1}{2} + n + s_\mu} \operatorname{Res} \left[\frac{1 \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}\boldsymbol{x}' \, \mathrm{d}\boldsymbol{b}}{\boldsymbol{a} - \zeta_h \boldsymbol{c}, \, \boldsymbol{x}' - h \cdot \boldsymbol{x}'', \, \boldsymbol{b} - \zeta_h^{-1} \zeta_g \boldsymbol{a} \cdot g} \right] \\ &= (-1)^{\binom{n+1}{2} + n + s_\mu} \end{aligned}$$

where multiple identities of Lemma 2.1.6 were used.

We find that $\dim_r \mu_{g,h}$ is independent of the $\{\zeta_g\}$ and only depends on n and s_μ. An obvious choice for s_μ is

$$s_\mu := \binom{n+1}{2} + n \implies \dim_r \mu_{g,h} = 1\ . \tag{3.4.20}$$

This choice can also be justified by considering the special case $g = h = e$: in that case, $\dim_r \mu_{e,e}$ describes a bubble of the identity surface in $\mathcal{T}$, which should, intuitively, be invisible (however, $\dim_r \mu_{e,e}$ is *not* the identity bubble in the *strictification* $s\mathcal{T}$, which is always invisible by construction; see the discussion in Remark 3.2.10).

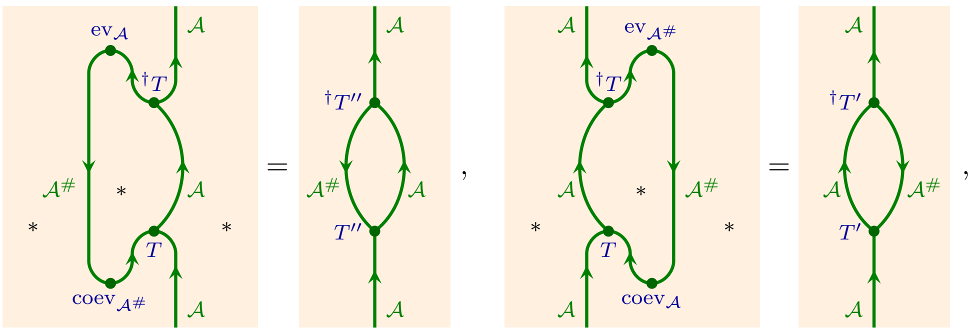
Inserting this result into Eq. (3.4.17), we find the system of equations

$$\textstyle\sum_{h\in G} \psi_h^2 \psi_{h^{-1}g}^2 \phi \overset{!}{=} \psi_g^2 \quad \text{for all } g \in G\ , \tag{3.4.21}$$

which is solved by our choice $\psi_g = 1$, $\phi = |G|^{-1}$ in Conjecture 3.4.3.

3.4.5 The "beer belly" bubbles

Let us proceed with $\chi^{T''}$ and $\chi^{T'}$, i.e. the second and third bubble diagram of Eq. (3.3.2). A horizontal slice through each diagram's centre looks like

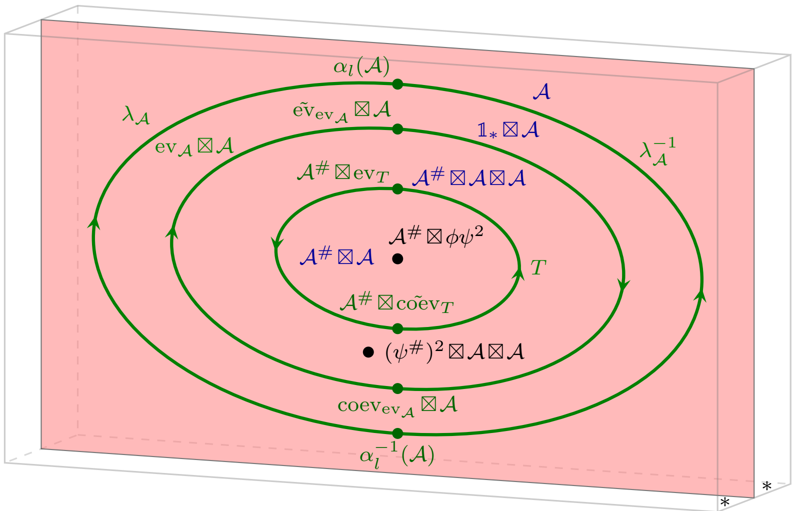


hence the name. Note that $\mathcal{A}^\#$ in this diagram refers to the adjoint of $\mathcal{A}$ in the tricategory $s\overline{\mathcal{T}}^\oplus$. We again evaluate $\chi^{T''}$ and $\chi^{T'}$ by applying $e\colon s\overline{\mathcal{T}}^\oplus \to \overline{\mathcal{T}}^\oplus$ with $e = e_2 \circ e_1$ factoring over $g\overline{\mathcal{T}}^\oplus$ (see Theorem 1.2.28).

The 2-morphisms T and $^\dagger T$ are bounded from above and below by the 3-morphisms $(\mathrm{ev}_T, \mathrm{coev}_{^\dagger T})$, and the pair $(\mathrm{ev}_\mathcal{A}, \mathrm{coev}_{\mathcal{A}^\#})$ is bounded by $(\mathrm{ev}_{^\dagger\mathrm{ev}_\mathcal{A}}, \mathrm{coev}_{\mathrm{ev}_\mathcal{A}})$. From the definition of e_1 we find $e_1(\mathrm{ev}_{\mathcal{A}^\#}) = {^\dagger\mathrm{coev}_\mathcal{A}}$ and $e_1(\mathrm{coev}_{\mathcal{A}^\#}) = {^\dagger\mathrm{ev}_\mathcal{A}}$. Overall, $e_1(\chi^{T''})$ is given by the following 3-morphism in $g\overline{\mathcal{T}}^\oplus$:

$$\begin{aligned}
\emptyset_\mathcal{A} &\xrightarrow{\mathrm{coev}_{\mathrm{ev}_\mathcal{A}} \boxtimes \mathcal{A}} (\mathrm{ev}_\mathcal{A} \otimes {^\dagger\mathrm{ev}_\mathcal{A}}) \boxtimes \mathcal{A} = (\mathrm{ev}_\mathcal{A} \boxtimes \mathcal{A}) \otimes ({^\dagger\mathrm{ev}_\mathcal{A}} \boxtimes \mathcal{A}) \\
&\xrightarrow{1 \otimes (\mathcal{A}^\# \boxtimes \mathrm{coev}_{^\dagger T}) \otimes 1} (\mathrm{ev}_\mathcal{A} \boxtimes \mathcal{A}) \otimes (\mathcal{A}^\# \boxtimes ({^\dagger T} \otimes T)) \otimes ({^\dagger\mathrm{ev}_\mathcal{A}} \boxtimes \mathcal{A}) \\
&\xrightarrow{1 \otimes \left((\psi^\#)^2 \boxtimes \phi \boxtimes (1 \otimes \psi^2 \otimes 1)\right) \otimes 1} (\mathrm{ev}_\mathcal{A} \boxtimes \mathcal{A}) \otimes (\mathcal{A}^\# \boxtimes ({^\dagger T} \otimes T)) \otimes ({^\dagger\mathrm{ev}_\mathcal{A}} \boxtimes \mathcal{A}) \\
&\xrightarrow{1 \otimes (\mathcal{A}^\# \boxtimes \mathrm{ev}_T) \otimes 1} (\mathrm{ev}_\mathcal{A} \otimes {^\dagger\mathrm{ev}_\mathcal{A}}) \boxtimes \mathcal{A} \\
&\xrightarrow{\mathrm{ev}_{^\dagger\mathrm{ev}_\mathcal{A}} \boxtimes \mathcal{A}} \emptyset_\mathcal{A}\ ,
\end{aligned}$$

using the notation $\mathcal{A} \boxtimes X := \mathbb{1}_\mathcal{A} \boxtimes X$ and $\mathcal{A} \boxtimes \psi := 1_{\mathbb{1}_\mathcal{A}} \boxtimes \psi$ for a 1-morphism $\mathcal{A}$, 2-morphism X, and 3-morphism ψ, found e.g. in [82]. The next step is to apply e_2, yielding a 3-morphism in $\overline{\mathcal{T}}^{\oplus}$ and introducing additional complexity. As the codomain of $e_2(\mathrm{ev}_{\mathcal{A}})$ and the domain of $e_2({}^{\dagger}\mathrm{ev}_{\mathcal{A}})$ are given by the identity 1-morphism $e_2(\emptyset_*) = \mathbb{1}_* \in \overline{\mathcal{T}}^{\oplus}(*,*)$, e_2 introduces an identity 1-morphism that is bounded by the 2-morphisms $(\lambda_{\mathcal{A}}, \lambda_{\mathcal{A}}^{-1})$ on the left and right and by the 3-morphisms $(\alpha_l(\mathcal{A}), \alpha_l^{-1}(\mathcal{A}))$ at the top and bottom. Leaving ϕ in the centre bulk would introduce another identity half-sphere within the central bubble for ϕ to live on. Similar to the argument in Section 3.4.3 we relocate ϕ to the 1-morphism $\mathcal{A}$, removing this complexity. Furthermore, we relocate $(\psi^{\#})^2$ slightly so it ends up outside the innermost bubble when the diagram is flattened. Overall, the flattened diagram $e(\chi^{T''})$ in $\overline{\mathcal{T}}^{\oplus}$ (with some identity lines and other details omitted) looks like



. (3.4.22)

We assign the name s_1 to the innermost circle, where we can split off a factor of $A^{\#} \boxtimes -$:

$$s_1 := \mathcal{D}_l(T)(\phi\psi^2) \in \mathrm{End}(\mathbb{1}_{\mathcal{A}} \boxtimes \mathbb{1}_{\mathcal{A}}) .$$

For the middle circle we use Lemma 2.5.11 and the fact that $- \boxtimes \mathcal{A}$ is a pivotal 2-functor by the assumption that $\overline{\mathcal{T}}^{\oplus}$ is a pivotal tricategory. This allows us to replace

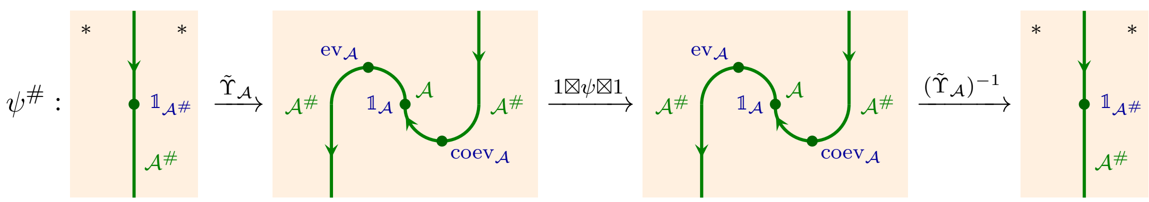
$$(\tilde{\mathrm{ev}}_{\mathrm{ev}_{\mathcal{A}}} \boxtimes 1_{\mathbb{1}_{\mathcal{A}}}, \mathrm{coev}_{\mathrm{ev}_{\mathcal{A}}} \boxtimes 1_{\mathbb{1}_{\mathcal{A}}}) \to (\tilde{\mathrm{ev}}_{\mathrm{ev}_{\mathcal{A}} \boxtimes \mathbb{1}_{\mathcal{A}}}, \mathrm{coev}_{\mathrm{ev}_{\mathcal{A}} \boxtimes \mathbb{1}_{\mathcal{A}}}) , \tag{3.4.23}$$

which induces slight changes to the identity 2-morphisms and unitor 3-morphisms that were suppressed in diagram (3.4.22). Now we can spell out the formula for the middle circle, which we name s_2, in a compact form:

$$s_2 := \mathcal{D}_r(\mathrm{ev}_{\mathcal{A}} \boxtimes \mathbb{1}_{\mathcal{A}})\Big((1_{\mathbb{1}_{\mathcal{A}^{\#}}} \boxtimes s_1) \circ ((\psi^{\#})^2 \boxtimes 1_{\mathbb{1}_{\mathcal{A}}} \boxtimes 1_{\mathbb{1}_{\mathcal{A}}})\Big) . \tag{3.4.24}$$

The action of $\psi^{\#}$

The first step is to understand how ψ is defined on $\mathcal{A}^{\#}$, i.e. how the adjunction of 1-morphisms acts on 3-morphisms. The definition of $\psi^{\#}$ is reminiscent of the definition of ${}^{\dagger}\psi$ in Eq. (2.4.2):



using the triangulator $\Upsilon'_{\mathcal{A}}$ of Eq. (1.2.19) and a suppressed pair of unitor 3-morphisms bounding the 2-morphism $\mathbb{1}_{\mathcal{A}}$. Evidently, $\psi^{\#}$ acts on $\mathbb{1}_{I_g^{\#}}$ with a factor of $\psi_g \in \mathbb{C}\setminus\{0\}$. We turn this action into a prefactor, thus removing $\psi^{\#}|_{I_g}$ from the diagram. Then we use the unitor properties and the Zorro movie to simplify the rest of the diagram to the identity 3-morphism $1_{\mathbb{1}_{\mathcal{A}^{\#}}}$. Overall we find

$$\psi^{\#}|_{\mathbb{1}_{I_g^{\#}}} : \alpha \mapsto \psi_g \cdot \alpha \ , \tag{3.4.25}$$

so the action of $\psi^{\#}$ on $I_g^{\#}$ is "the same" as the action of ψ on I_g.

The innermost bubble s_1 and the left quantum dimension of T

We evaluate s_1 using Lemma 3.2.20:

$$s_1|_{\mathbb{1}_{I_g \boxtimes I_h}} = \sum_{f\in G}^{m} \mathcal{D}_l(\delta_{f,gh}\cdot \mu_{g,h})(\phi\psi_f^2) = \phi\psi_{gh}^2 \cdot \mathcal{D}_l(\mu_{g,h})(1) = \phi\psi_{gh}^2 \dim_l(\mu_{g,h}) \ .$$

In the second step we use the explicit formula (2.5.9) and the fact that we may pull complex numbers out of residue operators. To evaluate the left quantum dimension of $\mu_{g,h}$ we reuse the notation and some results of Section 3.4.4. First we give the name $\boldsymbol{w} := \{\boldsymbol{a}, \boldsymbol{x}', \boldsymbol{b}\}$ to the list of the variables to the right of $\mu_{g,h}$. Then we choose left admissible variables $\boldsymbol{v} := \boldsymbol{x}$ and compute the numerator of the left quantum dimension (2.5.9):

$$\operatorname{str}\left\{ \left(\prod_{i=1}^{3n} \partial_{w_i} d_X\right) \left(\prod_{j=1}^{n} \partial_{v_j} d_X\right) \right\} \ .$$

The only derivative we have not yet computed in Eq. (3.4.18) is given by

$$\begin{aligned}\partial_{a_i} d_X &= \zeta_g \left(1_M^{\otimes(i-1)} \otimes \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \otimes 1_M^{\otimes(n-i)} \right) \otimes 1_{M^{\otimes n}} \\ &\quad + \zeta_g \sum_{j=1}^{n} 1_{M^{\otimes n}} \otimes \left(1_M^{\otimes(j-1)} \otimes \left(\begin{smallmatrix} 0 & 0 \\ g_{ij} & 0 \end{smallmatrix}\right) \otimes 1_M^{\otimes(n-j)} \right) \ .\end{aligned}$$

We permute the terms in the supertrace:

$$\begin{aligned}&\operatorname{str}\left\{ \left(\prod_{i=1}^{3n} \partial_{w_i} d_X\right) \left(\prod_{j=1}^{n} \partial_{v_j} d_X\right) \right\} \\ &= (-1)^{3n\cdot n} \operatorname{str}\left\{ \left(\prod_j \partial_{x'_j} d_X\right) \left(\prod_k \partial_{b_k} d_X\right) \left(\prod_l \partial_{x_l} d_X\right) \left(\prod_i \partial_{a_i} d_X\right) \right\} \ .\end{aligned}$$

The product of the first three terms is given by

$$\left(\prod_j \partial_{x'_j} d_X\right) \left(\prod_k \partial_{b_k} d_X\right) \left(\prod_l \partial_{x_l} d_X\right) = (-1)^{\binom{n}{2}} \zeta_h^n \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)^{\otimes n} \otimes \left(\begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix}\right)^{\otimes n} \ .$$

Now we multiply by $\partial_{a_1} d_X$ from the right. The second term of $\partial_{a_1} d_X$ does not contribute since $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ g_{ij} & 0 \end{smallmatrix}\right) = 0$. By the same argument, the second term of $\partial_{a_2} d_X$ does not contribute either, and so forth. We therefore find

$$\left(\prod_j \partial_{x'_j} d_X\right) \left(\prod_k \partial_{b_k} d_X\right) \left(\prod_l \partial_{x_l} d_X\right) \left(\prod_i \partial_{a_i} d_X\right) = (-1)^{2\binom{n}{2}} \zeta_g^n \zeta_h^n \left(\begin{smallmatrix} 0 & 0 \\ 0 & -1 \end{smallmatrix}\right)^{\otimes n} \otimes \left(\begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix}\right)^{\otimes n} \ ,$$

hence the supertrace evaluates to

$$\operatorname{str}\left\{ \left(\prod_{i=1}^{3n} \partial_{w_i} d_X\right) \left(\prod_{j=1}^{n} \partial_{v_j} d_X\right) \right\} = (-1)^n \zeta_g^n \zeta_h^n \operatorname{str}\left\{ \left(\begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix}\right)^{\otimes n} \otimes \left(\begin{smallmatrix} 0 & 0 \\ 0 & -1 \end{smallmatrix}\right)^{\otimes n} \right\} = \zeta_g^n \zeta_h^n \ .$$

Note that the prefactor of the left quantum dimension depends on the number of variables to the *right* of $\mu_{g,h}$, which is given by $3n$. We compute

$$\tbinom{3n+1}{2} = \tfrac{(3n+1)3n}{2} = 4n^2 + n + \tfrac{(n+1)n}{2} \equiv \tbinom{n+1}{2} + n \mod 2 \ . \tag{3.4.26}$$

The left quantum dimension then turns out to be

$$\begin{aligned} \dim_l \mu_{g,h} &= (-1)^{\binom{3n+1}{2}+s_\mu} \operatorname{Res}\left[\frac{\zeta_g^n \zeta_h^n \, \mathrm{d}\boldsymbol{c}}{\zeta_{gh}\boldsymbol{c} - \zeta_g \boldsymbol{a}}\right] \\ &= (-1)^{\binom{n+1}{2}+n+s_\mu} \operatorname{Res}\left[\frac{1 \, \mathrm{d}\boldsymbol{c}}{\boldsymbol{c} - \zeta_h^{-1}\boldsymbol{a}}\right] \\ &= (-1)^{\binom{n+1}{2}+n+s_\mu} \\ &= 1 \end{aligned} \tag{3.4.27}$$

using Eq. (3.4.20) for s_μ in the last step. Overall we find that the central 3-morphisms evaluate to

$$s_1|_{\mathbb{1}_{I_g \boxtimes I_h}} = \phi \psi_{gh}^2 \ . \tag{3.4.28}$$

The right quantum dimension of $\mathrm{ev}_{\mathcal{A}}$

Now we evaluate the middle circle (3.4.24) using Lemma 3.2.20 and the result for s_1 in Eq. (3.4.28):

$$\begin{aligned} s_2|_{\mathbb{1}_{\mathbb{1}_* \boxtimes I_h}} &= \sum_{f,g,j \in G^{\times 3}} \mathcal{D}_r\Big(\mathrm{ev}_{\mathcal{A}} \boxtimes \mathbb{1}_{\mathcal{A}}|_{\mathbb{1}_* \boxtimes I_h \to I_f^\# \boxtimes I_g \boxtimes I_j}\Big)\Big(\big(1_{\mathbb{1}_{I_f^\#}} \boxtimes s_1|_{\mathbb{1}_{I_g \boxtimes I_j}}\big) \circ \big(\psi_f^2 \boxtimes 1_{\mathbb{1}_{I_g}} \boxtimes 1_{\mathbb{1}_{I_j}}\big)\Big) \\ &= \sum_{f,g,j \in G^{\times 3}} \mathcal{D}_r\Big((\delta_{f,g} \cdot \mathrm{ev}_{I_g}) \boxtimes (\delta_{h,j} \cdot \mathbb{1}_{I_h})\Big)\Big(\big(1_{\mathbb{1}_{I_f^\#}} \boxtimes \phi \psi_{gj}^2\big) \circ \big(\psi_f^2 \boxtimes 1_{\mathbb{1}_{I_g}} \boxtimes 1_{\mathbb{1}_{I_j}}\big)\Big) \\ &= \sum_{g \in G} \phi \psi_g^2 \psi_{gh}^2 \cdot \mathcal{D}_r\Big(\mathrm{ev}_{I_g} \boxtimes \mathbb{1}_{I_h}\Big)\Big(1_{\mathbb{1}_{I_f^\#}} \boxtimes 1_{\mathbb{1}_{I_g}} \boxtimes 1_{\mathbb{1}_{I_h}}\Big) \\ &= \sum_{g \in G} \phi \psi_g^2 \psi_{gh}^2 \cdot \mathcal{D}_r\big(\mathrm{ev}_{I_g}\big)\Big(1_{\mathbb{1}_{I_f^\#}} \boxtimes 1_{\mathbb{1}_{I_g}}\Big) \boxtimes 1_{\mathbb{1}_{I_h}} \\ &= \sum_{g \in G} \phi \psi_g^2 \psi_{gh}^2 \cdot \big(\dim_r \mathrm{ev}_{I_g}\big) \boxtimes 1_{\mathbb{1}_{I_h}} \end{aligned}$$

using Lemma 2.5.11 in the penultimate step. To compute $\dim_r \mathrm{ev}_{I_g}$ we again use a setup similar to the one in Section 3.4.4. Note the global grade shift of ev_{I_g} from Definition 3.2.11.

$$\begin{aligned} I_g^\# \boxtimes I_g &= \big(\boldsymbol{a}', \boldsymbol{x}', \boldsymbol{a};\ -\zeta_g \boldsymbol{a}' \cdot (\boldsymbol{x}' - g \cdot \boldsymbol{x}) + \zeta_g \boldsymbol{a} \cdot (\boldsymbol{x}' - g \cdot \boldsymbol{x}'')\big) \ , \\ \mathbb{1}_* &= \big(\boldsymbol{c};\ \boldsymbol{c} \cdot (\boldsymbol{x} - \boldsymbol{x}'')\big) \ , \\ \mathrm{ev}_{I_g} &= K\big(\zeta_g(\boldsymbol{x}' - g \cdot \boldsymbol{x});\ \boldsymbol{a}' - \boldsymbol{a}\big) \otimes K\big(\boldsymbol{c} - \zeta_g \boldsymbol{a} \cdot g;\ \boldsymbol{x} - \boldsymbol{x}''\big)[s_{\mathrm{ev}}] \ . \end{aligned}$$

We choose admissible variables

$$\{u_1, \dots, u_{3n}\} = \{\boldsymbol{x}', \boldsymbol{a}', \boldsymbol{x}''\} \ , \qquad \{v_1, \dots, v_n\} := \{\boldsymbol{x}\} \ ,$$

resulting in the sequence

$$\boldsymbol{f} = \{\zeta_g(\boldsymbol{a} - \boldsymbol{a}'),\, \zeta_g(g \cdot \boldsymbol{x} - \boldsymbol{x}'),\, \boldsymbol{c} - \zeta_g \boldsymbol{a} \cdot g\} \ .$$

The properties of admissible variables can be shown as in the proof of Lemma 3.2.13. We proceed with the supertrace:

$$\begin{gathered}
\alpha_i = \begin{pmatrix} & \zeta_g(\boldsymbol{x}'-g\cdot\boldsymbol{x})_i \\ a_i'-a_i & \end{pmatrix} , \qquad \beta_i = \begin{pmatrix} & c_i-\zeta_g(\boldsymbol{a}\cdot g)_i \\ x_i-x_i'' & \end{pmatrix} , \\
\textstyle\prod_{i=1}^n \partial_{x_i'} d_X = \zeta_g^n \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)^{\otimes n} \otimes 1_{M^{\otimes n}} \ , \quad \prod_{i=1}^n \partial_{x_i''} d_X = 1_{M^{\otimes n}} \otimes \left(\begin{smallmatrix} 0 & 0 \\ -1 & 0 \end{smallmatrix}\right)^{\otimes n} , \\
\textstyle\prod_{i=1}^n \partial_{a_i'} d_X = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)^{\otimes n} \otimes 1_{M^{\otimes n}} \ , \qquad \prod_{i=1}^n \partial_{c_i} d_X = 1_{M^{\otimes n}} \otimes \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)^{\otimes n} , \\
\operatorname{str}\{(\textstyle\prod_{i=1}^{3n} \partial_{u_i} d_X)(\prod_{j=1}^n \partial_{c_j} d_X)\} = \zeta_g^n \operatorname{str}\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)^{\otimes n} \otimes \left(\begin{smallmatrix} 0 & 0 \\ 0 & -1 \end{smallmatrix}\right)^{\otimes n}\} = \zeta_g^n \ .
\end{gathered}$$

Now we find the quantum dimension

$$\begin{aligned}
\dim_r \mathrm{ev}_{I_g} &= (-1)^{\binom{n+1}{2} + s_{\mathrm{ev}}} \operatorname{Res} \left[\frac{\zeta_g^n \, \mathrm{d}\boldsymbol{a}' \, \mathrm{d}\boldsymbol{x}' \, \mathrm{d}\boldsymbol{a}}{\boldsymbol{c} - \zeta_g \boldsymbol{a} \cdot g,\, \zeta_g(g \cdot \boldsymbol{x} - \boldsymbol{x}'),\, \zeta_g(\boldsymbol{a} - \boldsymbol{a}')} \right] \\
&= (-1)^{\binom{n+1}{2} + s_{\mathrm{ev}}} \operatorname{Res} \left[\frac{(-1)^n \zeta_g^{-2n} \, \mathrm{d}\boldsymbol{a}' \, \mathrm{d}\boldsymbol{x}' \, \mathrm{d}\boldsymbol{a}}{\boldsymbol{a} \cdot g - \zeta_g^{-1} \boldsymbol{c},\, \boldsymbol{x}' - g \cdot \boldsymbol{x},\, \boldsymbol{a}' - \boldsymbol{a}} \right] \\
&= (-1)^{\binom{n+1}{2} + n + s_{\mathrm{ev}}} \operatorname{Res} \left[\frac{\det(g^{-1}) \zeta_g^{-2n} \, \mathrm{d}\boldsymbol{a}' \, \mathrm{d}\boldsymbol{x}' \, \mathrm{d}\boldsymbol{a}}{\boldsymbol{a} - \zeta_g^{-1} \boldsymbol{c} \cdot g^{-1},\, \boldsymbol{x}' - g \cdot \boldsymbol{x},\, \boldsymbol{a}' - \boldsymbol{a}} \right] \\
&= (-1)^{\binom{n+1}{2} + n + s_{\mathrm{ev}}} (\det g)^{-1} \zeta_g^{-2n} \ .
\end{aligned}$$

We find a dependency on the grade shift s_{ev} and on the parameters $\{\zeta_g\}$. A natural choice for these parameters is

$$s_{\mathrm{ev}} := \binom{n+1}{2} + n \ , \quad \zeta_g := (\det g)^{-\frac{1}{2n}} \quad \Longrightarrow \dim_r \mathrm{ev}_{I_g} = 1 \ , \tag{3.4.29}$$

confirming the value of ζ_g in Eq. (3.4.1). The choice for s_{ev} can be justified in a similar way as the choice for s_μ, since $\dim_r \mathrm{ev}_{I_e}$ also describes an identity bubble in $\overline{\mathcal{T}}^{\oplus}$. Furthermore, there is an intuitive explanation for the choice of $\{\zeta_g\}$. We compare the group action defects $I_g^{\#}$ and $I_{g^{-1}}$ with matching domain and codomain:

$$\begin{aligned}
W_{g^{-1}} &= \boldsymbol{a} \cdot \big(\underbrace{(\zeta_{g^{-1}} \cdot 1_{\mathbb{C}^n})}_{\det = (\det g)^{1/2}} \cdot \boldsymbol{x} - \underbrace{(\zeta_{g^{-1}} \cdot g^{-1})}_{\det = (\det g)^{-1/2}} \cdot \boldsymbol{x}'\big) \ , \\
W_g^{\#} &= \boldsymbol{a} \cdot \big(\overbrace{(\zeta_g \cdot g)} \cdot \boldsymbol{x} \qquad - \overbrace{(\zeta_g \cdot 1_{\mathbb{C}^n})} \cdot \boldsymbol{x}'\big) \ .
\end{aligned}$$

We find that the choice (3.4.29) makes the actions of $I_{g^{-1}}$ and $I_g^{\#}$ on $\boldsymbol{x}$ and $\boldsymbol{x}'$ "as closely related as possible" in that the determinants of the matrices agree. In simple examples like $n = 1$, $G = \mathbb{Z}_d$, both defects are even equal:

$$W_{g^{-1}} = a(e^{\pi i g/d} x - e^{-\pi i g/d} x') = W_g^{\#} \ .$$

Yet another perspective is to interpret ev_{I_g} as a composite of $\mu_{g^{-1},g}$ and an equivalence $\eta_g \colon I_g^{\#} \to I_{g^{-1}}$. We have seen before that $\dim_r \mu_{g,h} = 1$, implying $\dim_r \eta_g = (\det g)^{-1} \zeta_g^{-2n}$. Our choice of $\{\zeta_g\}$ sets $\dim_r \eta_g$ to 1.

Inserting everything back into s_2, we find

$$s_2|_{\mathbb{1}_* \boxtimes I_h} = \sum_{g \in G} \psi_g^2 \phi \psi_{gh}^2 \cdot \dim_r(\mathrm{ev}_{I_g}) \boxtimes 1_{\mathbb{1}_{I_h}} = \sum_{g \in G} \psi_g^2 \phi \psi_{gh}^2 \cdot 1_{\mathbb{1}_* \boxtimes I_h} \ .$$

The rest of the beer belly diagram

The process has to be repeated one more time to get the final result for the first beer belly diagram. The 3-morphism s_2 is surrounded by the 2-morphisms $(\lambda_{\mathcal{A}}, \lambda_{\mathcal{A}}^{-1})$, which are bounded from above and below by $(\alpha_l(\mathcal{A}), \alpha_l^{-1}(\mathcal{A}))$. As s_2 is just an identity operator times some prefactor, so we may first move it onto $\lambda_{\mathcal{A}}$ and then pull out the prefactor.

$$\chi_h^{T''} = \alpha_l(\mathcal{A}) \circ s_2 \circ \alpha_l^{-1}(\mathcal{A})\big|_{I_h} = \sum_{g\in G} \psi_g^2 \phi \psi_{gh}^2 \, \alpha_l(I_h) \circ \alpha_l^{-1}(I_h) = \sum_{g\in G} \psi_g^2 \phi \psi_{gh}^2 \overset{!}{=} \psi_h^2 \ . \tag{3.4.30}$$

Inserting the values of ϕ and ψ defined in Conjecture 3.4.3 also solves this constraint equation. Note that $\det(g)$ and $(-1)^{\binom{n+1}{2}+n}$ would show up in this equation (and would render it unsolvable in some examples) if we had not introduced the parameters s_{ev} and $\{\zeta_g\}$.

The other beer belly diagram

The diagram $\chi^{T'}$ can be evaluated in the same manner, the main difference being the appearance of $\dim_l \mathrm{coev}_{\mathcal{A}}$ (whose components are given by $\dim_l \mathrm{coev}_{I_g}$). We use the setup

$$\begin{aligned}
\mathbb{1}_* &= (\boldsymbol{c};\ \boldsymbol{c}\cdot(\boldsymbol{x}-\boldsymbol{x}''))\ , \\
I_g \boxtimes I_g^{\#} &= (\boldsymbol{a}, \boldsymbol{x}', \boldsymbol{a}';\ \zeta_g \boldsymbol{a}\cdot(\boldsymbol{x}-g\cdot\boldsymbol{x}') - \zeta_g \boldsymbol{a}'\cdot(\boldsymbol{x}''-g\cdot\boldsymbol{x}'))\ , \\
\mathrm{coev}_{I_g} &= K(\zeta_g(\boldsymbol{x}-g\cdot\boldsymbol{x}');\ \boldsymbol{a}-\boldsymbol{a}') \otimes K(\boldsymbol{x}-\boldsymbol{x}'';\ -\boldsymbol{c}+\zeta_g\boldsymbol{a}')[s_{\mathrm{ev}}]
\end{aligned}$$

and we choose admissible variables

$$\{v_i\} = \{\boldsymbol{x}, \boldsymbol{a}, \boldsymbol{x}''\}\ , \qquad \{u_1, \dots, u_n\} := \{\boldsymbol{x}\}\ ,$$

resulting in the sequence

$$\boldsymbol{g} = \{\zeta_g\boldsymbol{a}-\boldsymbol{c},\ \zeta_g(\boldsymbol{x}-g\cdot\boldsymbol{x}'),\ \boldsymbol{c}-\zeta_g\boldsymbol{a}'\}\ .$$

That $\boldsymbol{v}$ and $\boldsymbol{u}$ are admissible variables can be shown as in the proof of Lemma 3.2.13. We proceed with the supertrace:

$$\begin{aligned}
\alpha_i &= \begin{pmatrix} & \zeta_g(\boldsymbol{x}-g\cdot\boldsymbol{x}')_i \\ a_i-a_i' & \end{pmatrix}\ , \qquad \beta_i = \begin{pmatrix} & x_i-x_i'' \\ -c_i+\zeta_g a_i' & \end{pmatrix}\ , \\
\textstyle\prod_{i=1}^n \partial_{x_i''} d_X &= 1_{M^{\otimes n}} \otimes \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}^{\otimes n}\ , \quad \textstyle\prod_{i=1}^n \partial_{a_i} d_X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^{\otimes n} \otimes 1_{M^{\otimes n}}\ , \\
\textstyle\prod_{i=1}^n \partial_{c_i} d_X &= 1_{M^{\otimes n}} \otimes \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}^{\otimes n}\ , \\
\partial_{x_i} d_X &= \left(1_M^{\otimes(i-1)} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes 1_M^{\otimes(n-i)}\right) \otimes 1_{M^{\otimes n}} + 1_{M^{\otimes n}} \otimes \left(1_M^{\otimes(i-1)} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes 1_M^{\otimes(n-i)}\right)\ ,
\end{aligned}$$

thus we find

$$\begin{aligned}
&\mathrm{str}\{(\textstyle\prod_{j=1}^n \partial_{c_j} d_X)(\prod_{i=1}^{3n} \partial_{v_i} d_X)\} \\
&= (-1)^{3n\cdot n}\, \mathrm{str}\{(\textstyle\prod_{i=1}^n \partial_{x_i} d_X)(\prod_{i=1}^n \partial_{a_i} d_X)(\prod_{i=1}^n \partial_{x_i''} d_X)(\prod_{j=1}^n \partial_{c_j} d_X)\} \\
&= (-1)^{\binom{n}{2}+n}\, \mathrm{str}\{(\textstyle\prod_{i=1}^n \partial_{x_i} d_X)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^{\otimes n} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^{\otimes n}\}
\end{aligned}$$

Each $\partial_{x_i} d_X$ consists of two summands, but the second one cannot contribute to the supertrace because the resulting matrix will be off-diagonal:

$$\begin{aligned}&= (-1)^n \zeta_g^n \operatorname{str}\{\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)^{\otimes n} \otimes \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)^{\otimes n} + \text{off-diagonal}\} \\ &= (-1)^n \zeta_g^n \ .\end{aligned}$$

We note that in contrast to $\mu_{g,h}$ and ev_{I_g}, coev_{I_g} has a surface with n variables to the right and $3n$ variables to the left. Plugging our results into Eq. (2.5.3) yields

$$\begin{aligned}\dim_l \mathrm{coev}_{I_g} &= (-1)^{\binom{n+1}{2}+s_{\mathrm{ev}}} \operatorname{Res}\left[\frac{(-1)^n \zeta_g^n \,\mathrm{d}\boldsymbol{a}\,\mathrm{d}\boldsymbol{x}'\,\mathrm{d}\boldsymbol{a}'}{\zeta_g \boldsymbol{a} - \boldsymbol{c},\ \zeta_g(\boldsymbol{x} - g\cdot \boldsymbol{x}'),\ \boldsymbol{c} - \zeta_g \boldsymbol{a}'}\right] \\ &= (-1)^{\binom{n+1}{2}+n+s_{\mathrm{ev}}} \operatorname{Res}\left[\frac{\zeta_g^{-2n} \,\mathrm{d}\boldsymbol{a}\,\mathrm{d}\boldsymbol{x}'\,\mathrm{d}\boldsymbol{a}'}{\boldsymbol{a} - \zeta_g^{-1}\boldsymbol{c},\ g\cdot \boldsymbol{x}' - \boldsymbol{x},\ \boldsymbol{a}' - \zeta_g^{-1} \boldsymbol{c}}\right] \\ &= (-1)^{\binom{n+1}{2}+n+s_{\mathrm{ev}}} \operatorname{Res}\left[\frac{\det(g^{-1})\zeta_g^{-2n} \,\mathrm{d}\boldsymbol{a}\,\mathrm{d}\boldsymbol{x}'\,\mathrm{d}\boldsymbol{a}'}{\boldsymbol{a} - \zeta_g^{-1}\boldsymbol{c},\ \boldsymbol{x}' - g\cdot \boldsymbol{x},\ \boldsymbol{a}' - \zeta_g^{-1} \boldsymbol{c}}\right] \\ &= (-1)^{\binom{n+1}{2}+n+s_{\mathrm{ev}}} (\det g)^{-1} \zeta_g^{-2n} \\ &= 1\end{aligned}$$

using Eq. (3.4.29) in the last step. Note that there are no additional constraints or free parameters in this quantum dimension — the value of $\dim_l \mathrm{coev}_{I_g}$ is uniquely determined by $\dim_r \mathrm{ev}_{I_g}$ here.

In analogy to Eq. (3.4.30) we find the constraint equation

$$\chi_g^{T'} = \sum_{h\in G} \psi_{gh}^2 \phi \psi_h^2 \overset{!}{=} \psi_g^2 \ , \tag{3.4.31}$$

which has the same form and hence the same solutions as Eq. (3.4.30).

3.4.6 Preview: The associator-type diagrams

We will present the basic idea for the associator type diagrams (Definition 3.3.1 (ii) to (iv)) with the full details following in [7].

The normal associator identity

The normal associator identity (Definition 3.3.1 (ii)) is of the form

$$\psi \circ \alpha \circ \psi^2 \circ \bar{\alpha} \circ \psi \overset{!}{=} 1 \tag{3.4.32}$$

with some details concerning the ψ omitted. We evaluate the left hand side in components using the setup of Eq. (3.4.5):

$$\psi \circ \alpha \circ \psi^2 \circ \bar{\alpha} \circ \psi|_{\mu_{gh,f}\otimes(\mu_{g,h}\boxtimes 1_{I_f})} = \psi_{gh}\psi_{hf}^2\psi_{gh} \cdot \alpha \circ \bar{\alpha} = \psi_{gh}^2 \psi_{hf}^2 \overset{!}{=} 1$$

using that $\bar{\alpha}$ was constructed as the inverse of α. Inserting $(h, f) := (e, g)$ yields $\psi_g^4 = 1$, implying $\psi_g \in \{1, -1, i, -i\}$ for all $g \in G$. Therefore, the associator identity holds for

$$\psi_g := 1 \quad \text{for all } g \in G \tag{3.4.33}$$

which was our choice in Conjecture 3.4.3. Choosing the value 1 appears to be the most natural as it means we can remove ψ from all diagrams. The opposite version of [27, Def. 4.2 (ii)] (which can also be found in [27, Figure 5]) then simply demands

$$\bar{\alpha} \circ \alpha \stackrel{!}{=} 1 \tag{3.4.34}$$

which again holds because $\bar{\alpha}$ is the inverse of α by definition.

The partially reversed associator identity

The main difference in Definition 3.3.1 (iii) compared to (ii) is the presence of a reversed 1-morphism, i.e. the domain 1-morphism of [27, Def. 4.2 (iii)] is given by $\mathcal{A} \boxtimes \mathcal{A}^{\#} \boxtimes \mathcal{A}$. Consequently, the 2-morphisms contain additional factors of $\mathrm{ev}_{\mathcal{A}}$ or $\tilde{\mathrm{ev}}_{\mathcal{A}}$: for example, the bottom (domain) 2-morphism of the diagram is given by

$$\begin{aligned} X &:= T \otimes (\mathbb{1}_{\mathcal{A}} \boxtimes \mathrm{ev}_{\mathcal{A}} \boxtimes \mathbb{1}_{\mathcal{A}}) \otimes (\mathbb{1}_{\mathcal{A}} \boxtimes \mathbb{1}_{\mathcal{A}^{\#}} \boxtimes {}^{\dagger}T) \ , \\ X_{g,h^{\#},hf} &:= X|_{I_g \boxtimes I_h^{\#} \boxtimes I_{hf} \to I_j} = \delta_{gf,j} \cdot \mu_{g,f} \otimes (\mathbb{1}_{I_g} \boxtimes \mathrm{ev}_{I_h} \boxtimes \mathbb{1}_{I_f}) \otimes (\mathbb{1}_{I_g} \boxtimes \mathbb{1}_{I_h^{\#}} \boxtimes {}^{\dagger}\mu_{h,f}) \ . \end{aligned}$$

The modified 3-morphism α' is given by a composite of six 3-morphisms [27, Figure 9], schematically given by

$$\alpha' = \tilde{\mathrm{ev}}_T \circ \bar{\alpha} \circ \sigma \circ \tilde{\mathrm{ev}}_{\tilde{\mathrm{coev}}_{\mathcal{A}}} \circ \tau_{\mathcal{A}}^{-1} \circ \mathrm{coev}_T \ . \tag{3.4.35}$$

Inserting the definitions of the adjunction 1- and 2-morphisms in $\mathcal{T}^{\oplus}$ yields the expected component 3-morphism

$$\alpha'|_{X_{g,h^{\#},hf}} = \tilde{\mathrm{ev}}_{\mu_{h,f}} \circ \alpha^{-1}_{gh^{-1},h,f} \circ \sigma_{\dots} \circ \tilde{\mathrm{ev}}_{\tilde{\mathrm{coev}}_{I_h}} \circ \tau_{I_h}^{-1} \circ \mathrm{coev}_{\mu_{gh^{-1},h}} \ . \tag{3.4.36}$$

where some sums are introduced and then cancelled against Kronecker deltas in intermediate steps. Let $Y_{g,h^{\#},hf}$ be the codomain of α'. Then the corresponding 3-morphism $\bar{\alpha}'$ is given by

$$\bar{\alpha}'|_{Y_{g,h^{\#},hf}} = \tilde{\mathrm{ev}}_{\mu_{gh^{-1},h}} \circ \tau_{I_h} \circ \mathrm{coev}_{\tilde{\mathrm{coev}}_{I_h}} \circ \sigma_{\dots}^{-1} \circ \alpha_{gh^{-1},h,f} \circ \mathrm{coev}_{\mu_{h,f}} \ . \tag{3.4.37}$$

The partially reversed associator identities then demand

$$\alpha' \circ \bar{\alpha}' \stackrel{!}{=} 1 \ , \qquad \bar{\alpha}' \circ \alpha' \stackrel{!}{=} 1 \ . \tag{3.4.38}$$

We evaluate both equations in components using Eqs. (3.4.36) and (3.4.37) and find the right *anti-bubble* $\mathrm{ab}_r(\mu_{g,h})$ (Definition 2.5.9) in the middle of both expressions. Under the assumption that $\mu_{g,h}$ fulfils the requirements of Lemma 2.5.10 we can replace the anti-bubble $\mathrm{ab}_r(\mu_{g,h})$ by $\dim_r(\mu_{g,h})$. Using that both the left and right quantum dimensions of $\mu_{g,h}$ are equal to 1 in our choice of s_μ, both $\alpha' \circ \bar{\alpha}'$ and $\bar{\alpha}' \circ \alpha'$ can easily be simplified to the identity, proving (3.4.38). It remains to be shown that the assumptions of Lemma 2.5.10 are met.

The anti-bubbles of $\mu_{g,h}$

Lemma 3.4.4. $X := \mu_{g,h}$ *fulfils the assumptions of Lemma 2.5.10.*

Proof. We use Eqs. (1.3.45) and (1.3.46) to rewrite

$$\begin{aligned} X^\dagger \otimes X &\cong K(\zeta_g \boldsymbol{a} - \zeta_{gh}\boldsymbol{c};\ \boldsymbol{x} - g\cdot h\cdot \boldsymbol{x}'') \otimes K(\zeta_h \boldsymbol{b} - \zeta_g \boldsymbol{a}\cdot g;\ \boldsymbol{x}' - h\cdot \boldsymbol{x}'') \\ &\qquad \otimes K(\zeta_{gh}\boldsymbol{c} - \zeta_g \hat{\boldsymbol{a}};\ \boldsymbol{x} - g\cdot h\cdot \boldsymbol{x}'') \otimes K(\hat{\boldsymbol{x}}' - h\cdot \boldsymbol{x}'';\ \zeta_g \hat{\boldsymbol{a}}\cdot g - \zeta_h \hat{\boldsymbol{b}}) \\ &\cong \bigotimes_{i=1}^{n} K\big(\zeta_g a_i - \zeta_{gh} c_i,\ \zeta_h b_i - \zeta_g(\boldsymbol{a}\cdot g)_i,\ \zeta_{gh}c_i - \zeta_g \hat{a}_i,\ \hat{x}'_i - (h\cdot \boldsymbol{x}'')_i\,; \\ &\qquad x_i - (g\cdot h\cdot \boldsymbol{x}'')_i,\ x'_i - (h\cdot\boldsymbol{x}'')_i,\ x_i - (g\cdot h\cdot \boldsymbol{x}'')_i,\ \zeta_g(\hat{\boldsymbol{a}}\cdot g)_i - \zeta_h \hat{b}_i\big)\ . \end{aligned}$$

Note that the two grade shifts of s_μ cancel. In a procedure analogous to the one presented in Appendix A.2.2, the matrix factorisations in the tensor product can be rewritten to

$$\begin{aligned} &\cong \bigotimes_{i=1}^{n} K\big(b_i - \hat{b}_i,\ x'_i - \hat{x}'_i,\ a_i - \hat{a}'_i,\ \zeta_g a_i - \zeta_{gh} c_i\,; \\ &\qquad \zeta_h(\boldsymbol{x}' - h\cdot \boldsymbol{x}'')_i,\ \zeta_h \hat{b}_i - \zeta_g(\boldsymbol{a}\cdot g)_i,\ \zeta_g(\boldsymbol{x} - g\cdot \boldsymbol{x}')_i,\ 0\big) \end{aligned}$$

which is associated to

$$\begin{aligned} &\mathbb{C}[\boldsymbol{b}, \boldsymbol{x}', \boldsymbol{a}, \boldsymbol{c}, \hat{\boldsymbol{b}}, \hat{\boldsymbol{x}}', \hat{\boldsymbol{a}}, \boldsymbol{x}, \boldsymbol{x}'']/(\boldsymbol{b} - \hat{\boldsymbol{b}},\ \boldsymbol{x}' - \hat{\boldsymbol{x}}',\ \boldsymbol{a} - \hat{\boldsymbol{a}}',\ \zeta_g \boldsymbol{a} - \zeta_{gh}\boldsymbol{c}) \\ \cong\ &\mathbb{C}[\boldsymbol{b}, \boldsymbol{x}', \boldsymbol{a}, \hat{\boldsymbol{b}}, \hat{\boldsymbol{x}}', \hat{\boldsymbol{a}}, \boldsymbol{x}, \boldsymbol{x}'']/(\boldsymbol{b} - \hat{\boldsymbol{b}},\ \boldsymbol{x}' - \hat{\boldsymbol{x}}',\ \boldsymbol{a} - \hat{\boldsymbol{a}}') \end{aligned}$$

which is associated to the identity matrix factorisation $\mathbb{1}_{I_g \boxtimes I_h}$, so

$$\mu_{g,h}^\dagger \otimes \mu_{g,h} \cong {}^\dagger\mu_{g,h} \otimes \mu_{g,h} \cong \mathbb{1}_{I_g\boxtimes I_h}\ .$$

By a similar argument, we find

$$\begin{aligned} X \otimes X^\dagger &\cong K(\zeta_{gh}\boldsymbol{c} - \zeta_g \boldsymbol{a};\ \boldsymbol{x} - g\cdot h\cdot \boldsymbol{x}'') \otimes K(\boldsymbol{x}' - h\cdot \boldsymbol{x}'';\ \zeta_g \boldsymbol{a}\cdot g - \zeta_h \boldsymbol{b}) \\ &\qquad \otimes K(\zeta_g \boldsymbol{a} - \zeta_{gh}\hat{\boldsymbol{c}};\ \boldsymbol{x} - g\cdot h\cdot \boldsymbol{x}'') \otimes K(\zeta_h \boldsymbol{b} - \zeta_g \boldsymbol{a}\cdot g;\ \boldsymbol{x}' - h\cdot \boldsymbol{x}'') \\ &\cong \bigotimes_{i=1}^{n} K\big(c_i - \hat{c}_i,\ (\zeta_g \boldsymbol{a}\cdot g)_i - \zeta_h b_i,\ x'_i - (h\cdot \boldsymbol{x}'')_i,\ \zeta_{gh}c_i - \zeta_g a_i\,; \\ &\qquad x_i - (g\cdot h\cdot \boldsymbol{x}'')_i,\ 0,\ 0,\ 0\big) \end{aligned}$$

which is associated to

$$\begin{aligned} &\mathbb{C}[\boldsymbol{c}, \boldsymbol{b}, \boldsymbol{x}', \boldsymbol{a}, \hat{\boldsymbol{c}}, \boldsymbol{x}, \boldsymbol{x}'']/(\boldsymbol{c} - \hat{\boldsymbol{c}},\ \zeta_g \boldsymbol{a}\cdot g - \zeta_h \boldsymbol{b},\ \boldsymbol{x}' - h\cdot \boldsymbol{x}'',\ \zeta_{gh}\boldsymbol{c} - \zeta_g \boldsymbol{a}) \\ \cong\ &\mathbb{C}[\boldsymbol{c}, \hat{\boldsymbol{c}}, \boldsymbol{x}, \boldsymbol{x}'']/(\boldsymbol{c} - \hat{\boldsymbol{c}}) \end{aligned}$$

which is associated to the identity matrix factorisation $\mathbb{1}_{I_{gh}}$, so

$$\mu_{g,h} \otimes \mu_{g,h}^\dagger \cong \mu_{g,h} \otimes {}^\dagger\mu_{g,h} \cong \mathbb{1}_{I_{gh}}\ . \qquad \square$$

Corollary 3.4.5. *The anti-bubbles of $\mu_{g,h}$ take the values*

$$\mathrm{ab}_r(\mu_{g,h}) = 1_{\mu_{g,h}\otimes \mu_{g,h}^\dagger}\ , \qquad \mathrm{ab}_l(\mu_{g,h}) = 1_{\mu_{g,h}^\dagger \otimes \mu_{g,h}}\ . \tag{3.4.39}$$

Proof. By Eqs. (3.4.20) and (3.4.27), both quantum dimensions of $\mu_{g,h}$ are equal to 1. The above formulas then follow from Lemmas 2.5.10 and 3.4.4. $\square$

3.4.7 Summary of the orbifold datum constraints

We combine the results (3.4.20) and (3.4.33) to find the solution

$$s_\mu := s_{\mathrm{ev}} := \binom{n+1}{2} + n \ , \qquad \zeta_g := (\det g)^{-1/2n} \ , \qquad \psi_g := 1 \ , \qquad \phi := |G|^{-1} \ . \tag{3.4.40}$$

We have shown that in these choices the first two associator identities and the bubble identities (3.3.2) are fulfilled for the orbifold datum of Conjecture 3.4.3. The third associator identity and the 2-3 move identity are expected to hold as well, and the latter is expected to fix C_α of Eq. (3.4.6).

4 Boundaries in 3D $\mathcal{N}=2$ SUSY QFTs

This chapter is a shortened version of joint work with Ilka Brunner and Alexander Tabler published in [13].

4.1 Introduction

In this chapter we study $\mathcal{N}=2$ supersymmetric theories in flat $(2+1)$-dimensional spacetime with (spacelike) boundaries. The boundary necessarily breaks translational invariance and hence can only preserve part of the bulk supersymmetry. Explicitly, the $\mathcal{N}=2$ algebra in 3 dimensions is

$$\{Q_\pm, \bar{Q}_\pm\} = -4P_\pm, \quad \{Q_+, \bar{Q}_-\} = 2P_\perp, \tag{4.1.1}$$

where spacetime has coordinates $x^\pm$ and $x_\perp$, see Appendix A.6.1 for a summary of our conventions. We want to consider the case with a boundary in $x_\perp$-direction, breaking supersymmetry to a subalgebra of (4.1.1) that does not contain $P_\perp$, the generator of translations in that direction. As has been analysed before [33, 78, 47], in the case of $\mathcal{N}=2$ supersymmetry in 3 dimensions, there are two types of supersymmetric boundary conditions, referred to as A-type and B-type. Each of them is associated to a subalgebra of the initial bulk supersymmetry algebra, containing two momentum operators and two supersymmetry charges. A-type boundary conditions preserve $(1,1)$ supersymmetry, whereas B-type boundary conditions preserve a chiral $\mathcal{N}=(0,2)$ subalgebra, generated by Q_+ and $\bar{Q}_+$. We will focus on $(0,2)$ boundary conditions and analyse them from two points of view: On the one hand, for theories defined by a Lagrangian, we employ a Noether procedure. On the other hand, we discuss the structure of the supercurrent multiplets [68, 38] and formulate boundary multiplets. The two points of view are interrelated, as the (improved) Noether currents form components of the current multiplets.

From a Lagrangian point of view, the supersymmetric bulk Lagrangian transforms under SUSY-variations into a total derivative. In the presence of a boundary, this generically yields a boundary term which must be cancelled for the symmetry to be preserved. This can be achieved by choosing boundary conditions on the fields, such that the boundary variation vanishes. Alternatively, and this is the main focus of the present chapter, one can cancel the boundary variation by adding a suitable boundary part to the action, such that the action is invariant under symmetry variations without reference to the boundary conditions on the fields. This term can contain extra boundary degrees of freedom that are not inherited from the bulk. The full invariant action thus contains a bulk and boundary term.

$$S = \int_M \mathcal{L}^B + \int_{\partial M} \mathcal{L}^\partial. \tag{4.1.2}$$

Given an action which is invariant under a symmetry, Noether's procedure yields a set of conserved charges and currents. In the case of supersymmetry, this includes the (canonical) energy momentum tensor and the supersymmetry currents. After imposing canonical commutation relations between the fields, the Noether charges provide a representation of the symmetry algebra in terms of the fields.

In the case of pure bulk theories, it is very useful to arrange the supercurrents together with other conserved currents into multiplets. This can be done independently of a Lagrangian definition of a theory. The supercurrents form a representation of the supersymmetry algebra, and their most general form has been discussed recently in [68, 38]. Supercurrent multiplets are indecomposable SUSY multiplets that contain the stress energy tensor $T_{\mu\nu}$ and the supercurrents $S_{\alpha\mu}$ as part of their components. In addition, there are brane currents, whose integrals yield brane charges. The components of the supercurrent multiplets appear in a local version of the supersymmetry algebra, which very schematically takes the form

$$\{\bar{Q}_\alpha, S_{\beta\mu}\} = 2\gamma^\nu_{\alpha\beta} T_{\mu\nu} + \ldots, \tag{4.1.3}$$

$$\{Q_\alpha, S_{\beta\mu}\} = \ldots, \tag{4.1.4}$$

where the dots indicate various currents that will be explained later in this chapter. As is well-known, the stress tensor for a theory is not unique, but can be modified by improvement transformations. Indeed, the same holds for the supercurrents, and the notion of improvement transformations can be lifted to the full multiplet. In any three-dimensional $\mathcal{N} = 2$ supersymmetric theory (and also in other dimensions with the same amount of supersymmetry), there exists a so-called $\mathcal{S}$-multiplet. Under special conditions, the $\mathcal{S}$-multiplet can be decomposable, such that there exist smaller multiplets. Of special interest in the context of the current work is the $\mathcal{R}$-multiplet, which exists in theories which exhibit an R-symmetry.

The notion of supercurrent multiplets has been extended to theories with defects in [37], where a new so-called defect multiplet was constructed. As a consequence of the violation of translation symmetry perpendicular to the defect, the stress tensor is no longer conserved. This violation is encoded in the displacement operator. The defect multiplet contains the displacement operator as one of its components [48, 37].

In the current chapter, we consider supermultiplets in situations with boundary, focusing on the *preserved* symmetries. As mentioned above, to formulate boundary conditions means to specify a subalgebra of the SUSY algebra such that the momentum operator in the direction perpendicular to the boundary is not contained. The supercurrent multiplet is in particular a representation of the larger (bulk) algebra and hence also of the smaller algebra. In the case of the $\mathcal{N} = (0, 2)$ subalgebra, we show how the bulk supercurrent multiplets decompose under the smaller algebra. Of course, due to the presence of the boundary, the currents contained in the multiplet are no longer conserved by themselves. To formulate a consistent multiplet for a theory with boundary, we discuss how to add boundary parts to the $(0, 2)$-components of the initial bulk multiplet. Our ansatz for a full $\mathcal{R}$-multiplet is

$$\mathcal{R}^{\text{full}}_\mu = \mathcal{R}^B_\mu + \mathcal{P}_\mu{}^{\hat{\mu}} \delta(x_\perp) \mathcal{R}^\partial_{\hat{\mu}}, \tag{4.1.5}$$

where $\mathcal{R}^B_\mu$ is the bulk part, $\mathcal{R}^\partial_{\hat{\mu}}$ is the boundary part and $\mathcal{P}_\mu{}^{\hat{\mu}}$ denotes an embedding. Both parts decompose into $(0, 2)$-components. The boundary part is added to the bulk multiplet in such a way that the initial divergence-freeness of the bulk currents is completed to bulk-boundary conservation laws. We do not discuss possible modifications of the bulk currents corresponding to the *broken* symmetries.

One important feature of supercurrent multiplets is that they fall into short representations of the supersymmetry algebra. Therefore, they are protected under RG flow and retain their form [38]. The supercurrents of the quantum theory can thus be used to constrain the IR behaviour of a theory using the UV information. In the case of *two-dimensional* $\mathcal{N} = (0, 2)$

models, the supercurrent multiplet for theories with an R-symmetry was used to study renormalisation group invariants in [31]. In particular, it was shown that an RG invariant chiral algebra exists, extending earlier works [91, 85, 94]. The chiral algebra arises as the cohomology of the supercharge $\bar{Q}_+$. Using only the form of the supercurrent multiplet, [31] shows that there is a half-twisted stress tensor (the original stress tensor modified by the R-current) in the cohomology. As a consequence, conformal symmetry is part of the chiral algebra.

Given these findings, we consider the consequences of the supercurrent multiplets for the case that the $\mathcal{N} = (0, 2)$ supersymmetry is the symmetry preserved at the *boundary* of a three-dimensional theory. We do not find a stress-tensor in the cohomology following the steps in [31], however, there is a weaker statement. For this, one makes the $(0, 2)$-structure manifest by regarding the three-dimensional $\mathcal{N} = 2$ theory as a two-dimensional theory living on the boundary $\mathbb{R}^{1,1}$. The bosonic fields of this theory are valued in maps from $\mathbb{R}_{\leq 0}$ to the original target. The currents are obtained from the original three-dimensional ones by integrating over the direction perpendicular to the boundary and are preserved in the boundary theory. In this theory, we then do have a stress energy tensor that is part of the cohomology. Formulated in the initial theory, this cohomology element is obtained by an integration in the perpendicular direction from infinity (or a second boundary, which we do not discuss here) to the boundary. Note that the action of any charge computed from the currents applied to an insertion at the boundary would involve an integration over this direction, as well as all other spatial directions. In this sense we can also interpret the partially integrated currents in the original theory.

The integration along the perpendicular direction also gives another perspective on the boundary multiplets. The theory effectively becomes a two-dimensional theory with $\mathcal{N} = (0, 2)$ supersymmetry, and the integrated multiplets are genuine $\mathcal{N} = (0, 2)$ multiplets.

In the second part of the chapter, we study a specific example, namely a theory of three-dimensional chiral multiplets with a superpotential. We restrict our explicit discussion to the case of a single chiral field with a monomial superpotential. However, a generalisation to more than one superfield and an arbitrary superpotential preserving an R-symmetry is straightforward and our discussion applies to this case as well. As has already been shown in [47, 95], the condition of preserving $\mathcal{N} = (0, 2)$ supersymmetry leads to a three-dimensional generalisation of matrix factorisations [61, 10]. In this case, boundary terms are cancelled by adding fermionic degrees of freedom and a potential at the boundary. Using Noether's procedure, we compute the conserved currents for different boundary conditions. These currents contain pure bulk as well as boundary pieces and only the combination of both is conserved. We also discuss the current multiplets, following [38] for the bulk case. Starting from the Noether currents, one needs to apply improvement transformations to symmetrise the stress energy tensor and subsequently organise the currents into supercurrent multiplets. As we consider the case where R-symmetry is preserved, the relevant multiplet is the $\mathcal{R}$-multiplet. We spell out all components of the bulk-boundary multiplet in the example following the strategy outlined above.

To complete our understanding of the symmetries in the model, we study the algebra of the supercharges and supercurrents (4.1.3) in the example. We start with the explicit expressions of the currents in terms of fields and impose canonical commutation relations on the fields. We then verify the relations (4.1.3) as well as the $\mathcal{N} = (0, 2)$ superalgebra in the specific representation of the example. In the computation, it is essential to make use of the correct factorisation properties of the superpotential to recover the correct form of the algebra.

This chapter is organised as follows: In Section 4.2 we review and elaborate on various aspects of theories formulated on Minkowski space $\mathbb{R}^{1,N-1}$ with a flat boundary $\mathbb{R}^{1,N-2}$, following and extending [32]. In Section 4.3 we discuss supercurrent multiplets. We review the constraints that have to be satisfied by such a multiplet from [38]. As explained there, the most general supercurrent multiplets consists of certain superfields satisfying defining constraint relations. Their solutions are unique up to improvement transformations. In special situations, such improvements can be used to formulate shorter multiplets, in particular the $\mathcal{R}$-multiplet. We decompose the multiplets and constraints according to the $(0, 2)$-substructure and formulate consistent bulk-boundary multiplets. Here, current conservation of the combined bulk-boundary system is imposed. We then discuss an integrated structure, where we integrate in the direction perpendicular to the boundary. This provides a two-dimensional version of the conservation equations, taking the familiar form of divergence-freeness of the currents.

We then turn to a discussion of the Landau–Ginzburg example in Section 4.4. First, in 4.4.1, we introduce the bulk model, then introduce a boundary in 4.4.2. We distinguish the cases with and without superpotential and show that without specifying boundary conditions, $\mathcal{N} = (0, 2)$ SUSY can be preserved by introducing boundary fermions and matrix factorisations. Boundary conditions do however have to be imposed to make the action stationary and we discuss Dirichlet and Neumann conditions. In particular, there is a possibility to make boundary conditions dynamical, imposing them as equations of motion. We then turn to the formulation of currents in 4.4.3. Here, Noether's procedure is employed to compute the conserved currents in the combined bulk boundary system. We discuss improvements to symmetrise the stress-energy tensor. Section 4.4.4 contains a discussion of the supercurrent multiplets in the example.

In Section 4.5 we study the realisation of the symmetry algebra in the Landau–Ginzburg model in terms of the fields. By imposing canonical commutation relations for the bulk and boundary fields, we verify that the supercharges implement the correct symmetry transformations on the fields and their derivatives, and we compute the brackets between supercharges and supercurrents. Special attention is paid to the contributions from the boundary. We do not impose any explicit boundary conditions on the fields in our computations. In 4.5.1 we use the action of the supercharge on the derivative of the scalars to verify how the bulk fermions decompose into boundary fermions — a decomposition that was seen in earlier sections from the point of view of the action. In the following subsections 4.5.2 and 4.5.3 we verify the brackets between supercharges and currents and finally integrate them to the global supersymmetry algebra. The factorisation condition of the superpotential arises as a consistency condition on the SUSY algebra.

4.2 Currents and charges in theories with boundaries

In this section we want to study (classical) theories on flat Minkowski half-space with spacelike boundary. In particular, we will consider

$$M = \left\{ x^\mu = (x^0, \ldots, x^n, \ldots, x^{N-1}) \in \mathbb{R}^{1,N-1} \,\middle|\, x^n \leq 0 \right\}, \quad \partial M = \{x^n = 0\}. \tag{4.2.1}$$

On the flat boundary we will denote the tangential coordinates by $x^{\hat\mu}$ while the normal coordinate is x^n, i.e. $\hat\mu$ takes all values except n. While our main focus will be $N = 3$ later on, we will keep the discussion more general in this section.

4.2.1 Bulk and boundary Lagrangians

We want to study theories that have both bulk and boundary fields with an action of the form[1]

$$S = S^B + S^\partial = \int_M \mathcal{L}^B + \int_{\partial M} \mathcal{L}^\partial, \quad \mathcal{L}^B = \mathcal{L}^B[\Phi, \partial_\mu \Phi], \quad \mathcal{L}^\partial = \mathcal{L}^\partial[X, \partial_{\hat{\mu}} X, \Phi|_\partial, \partial_{\hat{\mu}} \Phi|_\partial], \tag{4.2.2}$$

where Φ and X denotes bulk and boundary fields, respectively. Furthermore, one has to impose *boundary conditions*. We follow the similar discussion from [32], which we generalise here. The most general boundary conditions will be of the form

$$G(\text{fields}|_\partial, \text{derivatives of fields}|_\partial) = 0. \tag{4.2.3}$$

Under a (rigid) variation of the fields and an integration by parts, we get

$$\begin{aligned} \delta S &= \int_M \delta\mathcal{L}^B + \int_{\partial M} \delta\mathcal{L}^\partial \\ &= \int_M \Big[\underbrace{\tfrac{\partial \mathcal{L}^B}{\partial \Phi} - \partial_\mu \tfrac{\partial \mathcal{L}^B}{\partial(\partial_\mu \Phi)}}_{\text{Bulk EoM}^B[\Phi]}\Big]\delta\Phi \\ &\quad + \int_{\partial M} \Big[\underbrace{\big(\tfrac{\partial \mathcal{L}^\partial}{\partial X} - \partial_{\hat{\mu}} \tfrac{\partial \mathcal{L}^\partial}{\partial(\partial_{\hat{\mu}} X)}\big)}_{\text{Boundary EoM}^\partial[X]}\delta X + \underbrace{\big(\tfrac{\partial \mathcal{L}^\partial}{\partial \Phi|_\partial} - \partial_{\hat{\mu}} \tfrac{\partial \mathcal{L}^\partial}{\partial(\partial_{\hat{\mu}} \Phi|_\partial)} + \tfrac{\partial \mathcal{L}^B}{\partial(\partial_n \Phi|_\partial)}\big)}_{=:\mathcal{A}}\delta\Phi|_\partial + \partial_{\hat{\mu}}(\cdots)\Big]. \end{aligned} \tag{4.2.4}$$

Stationarity of the action requires that

$$\left[\mathcal{A} \cdot \delta\Phi|_\partial\right]_{G=0} = \partial_{\hat{\mu}} A^{\hat{\mu}}, \tag{4.2.5}$$

for some boundary vector field $A^{\hat{\mu}}$. Note that in any case, the variations $\delta\Phi|_\partial$, δX we consider must be *compatible* with the chosen boundary condition, i.e. we may only consider such variations that satisfy

$$\delta G|_{G=0} = 0. \tag{4.2.6}$$

A special kind of boundary condition is the *dynamical boundary condition*, which amounts to requiring

$$G := \mathcal{A} \stackrel{!}{=} 0. \tag{4.2.7}$$

This is equivalent to the paradigm (e.g. found in [33, 6]) not to impose static boundary conditions, but instead adopt the boundary conditions that are naturally imposed by the tendency of the system to make the action stationary.

4.2.2 Symmetries, currents and charges in boundary theories

Symmetries in boundary theories

Let us try to understand symmetries in theories with a boundary. If we want the full theory to be invariant under some symmetry transformation of the fields, the boundary condition must be compatible with the symmetry transformation, as mentioned in [32]:

$$\delta_{\text{sym}} G|_{G=0} = 0. \tag{4.2.8}$$

[1] We assume that the boundary Lagrangian $\mathcal{L}^\partial$ contains only tangential derivatives $\partial_{\hat{\mu}}$. We also assume for simplicity that there are only first order derivatives of any kind.

If this is satisfied, G is called a *symmetric* boundary condition with respect to δ_{sym}. Conveniently, this requirement is equivalent to demanding that the symmetry variation δ_{sym} is a *permitted variation* in the sense of Eq. (4.2.6).

The definition of a symmetry in the presence of a boundary is analogous to the case of a pure bulk theory: A symmetry is an off-shell transformation of both bulk and boundary fields that leaves the action invariant, *possibly* after using boundary conditions:

$$0 = \delta_{\text{sym}} S|_{G=0} = (\delta_{\text{sym}} S^B + \delta_{\text{sym}} S^\partial)|_{G=0} = 0. \tag{4.2.9}$$

It is natural to restrict to symmetries that arise from a symmetry of the bulk theory, i.e. $\delta_{\text{sym}}\mathcal{L}^B = \partial_\mu V^\mu$ holds for some bulk vector fields V^μ. Noether's theorem in the bulk then ensures that $\partial_\mu J^\mu_B = 0$, where $J^\mu_B = -\frac{\partial \mathcal{L}^B}{\partial(\partial_\mu \Phi)}\delta_{\text{sym}}\Phi + V^\mu$ still holds. In terms of Lagrangians, we can then write

$$0 = \delta_{\text{sym}} S|_{G=0} = \int_M \delta_{\text{sym}}\mathcal{L}^B + \int_{\partial M}\delta_{\text{sym}}\mathcal{L}^\partial = \int_{\partial M}(V^n + \delta_{\text{sym}}\mathcal{L}^\partial)|_{G=0}. \tag{4.2.10}$$

If the above condition holds without imposing any boundary condition $G = 0$, one says that the symmetry is preserved *without reference to specific boundary conditions*. As we start from a bulk theory $\mathcal{L}^B$ with a symmetry, it is interesting to investigate whether a *boundary compensating term* $\mathcal{L}^\partial$ exists, so that (4.2.9) holds without referring to a boundary condition [32, 6]. However, in general cases, one must impose specific symmetric boundary conditions *and* add boundary terms so that the full action is stationary (4.2.5) and symmetric (4.2.9).

Currents and charges

It is clear that the bulk theory charge $Q_B = \int_\Sigma J^0_B$ of the aforementioned symmetry is, in general, no longer conserved after introducing a boundary, since the constant-time slice Σ now has a boundary $\partial\Sigma$. As a physical interpretation, the bulk current "leaks" from the boundary, and this "leakage" must be compensated by a boundary term. More precisely, what we need in addition to the bulk current J^μ_B is a *boundary current* $J^{\hat\mu}_\partial$ which lives on the boundary of the full theory, such that the equations

$$\partial_\mu J^\mu_B = 0, \quad \partial_{\hat\mu} J^{\hat\mu}_\partial = J^n_B|_\partial \tag{4.2.11}$$

are satisfied. We also introduce the *total* conserved current[2]

$$J^\mu_{\text{full}} = J^\mu_B + \delta(x^n)\mathcal{P}^\mu{}_{\hat\mu} J^{\hat\mu}_\partial, \tag{4.2.12}$$

where $\mathcal{P}^\mu{}_{\hat\mu}$ is a projector/embedding with $\mathcal{P}^n{}_{\hat\mu} = 0$, $\mathcal{P}^{\hat\nu}{}_{\hat\mu} = \delta^{\hat\nu}{}_{\hat\mu}$. The conservation equations (4.2.11) can be expressed as:

$$\partial_\mu J^\mu_{\text{full}} = \delta(x^n) J^n_{\text{full}}. \tag{4.2.13}$$

Note that the (boundary) conservation equation might only hold modulo boundary conditions. The full conserved charge of the theory is then given by

$$Q = \int_\Sigma J^0_B + \int_{\partial\Sigma} J^0_\partial = \int_\Sigma J^0_{\text{full}}, \tag{4.2.14}$$

whose conservation is easy to see from (4.2.11).

[2]See Appendix A.6.2 for details on δ-distributions at the boundary.

Just as in pure bulk theories, there is more than one current leading to the same conserved charge associated to a given symmetry. Transformations of the currents that preserve the conservation equations and the charges are called the *improvements* of the currents.

For a pure bulk theory, an improvement locally takes the form

$$J^\mu_B \mapsto \tilde{J}^\mu_B = J^\mu_B + \partial_\nu M^{[\mu\nu]}, \tag{4.2.15}$$

which preserves the conservation equations and charges: $\partial_\mu \tilde{J}^\mu_B = 0$ and $\tilde{Q}_B = \int_\Sigma \tilde{J}^0_B \equiv Q_B$. For a theory with boundary, an improvement takes the form

$$\begin{Bmatrix} J^\mu_B \\ J^{\hat{\mu}}_\partial \end{Bmatrix} \mapsto \begin{Bmatrix} \tilde{J}^\mu_B = J^\mu_B + \partial_\nu M^{[\mu\nu]} \\ \tilde{J}^{\hat{\mu}}_\partial = J^{\hat{\mu}}_\partial + M^{n\hat{\mu}} + \partial_{\hat{\nu}} m^{[\hat{\mu}\hat{\nu}]} \end{Bmatrix}, \tag{4.2.16}$$

which preserves the conservation equations (4.2.11) and the charge (4.2.14). The improvement of the bulk current induces an improvement on the boundary current, and the boundary current may be further improved by a pure boundary improvement.

Note that it is sometimes possible to completely "improve away" the boundary part of the conserved current, in particular if there are no degrees of freedom on the boundary. In that case the bulk charge Q_B is conserved even in the presence of a boundary, but then it is, in general, sensitive to bulk improvements. This is the approach of the authors of [32].

4.2.3 Noether's theorem on manifolds with boundary

Now that we have discussed the properties of conserved currents and charges in boundary theories, let us investigate how we can compute them in a particular model. We present a modification of Noether's theorem that yields bulk and boundary currents in the sense we defined above. The special case of a theory with boundary (and boundary terms) but without boundary dynamics is discussed in detail in [32].

Currents and charges without boundary

For completeness, let us quickly repeat Noether's theorem in pure bulk theories. A symmetry is an off-shell transformation of the fields that leaves the action invariant:

$$\delta_{\text{sym}} S = 0. \tag{4.2.17}$$

If the transformation is rigid (i.e. leaves spacetime invariant) and assuming that fields "fall off" at infinity the above condition is equivalent to

$$\delta_{\text{sym}} \mathcal{L}^B = \partial_\mu V^\mu \tag{4.2.18}$$

for some bulk vector field V^μ. On the other hand, a generic variation of the Lagrangian is also given by

$$\delta_{\text{sym}} \mathcal{L}^B = \text{EoM}^B[\Phi]\delta_{\text{sym}}\Phi + \partial_\mu\big(\tfrac{\partial \mathcal{L}^B}{\partial(\partial_\mu \Phi|_\partial)}\delta_{\text{sym}}\Phi\big) \overset{\text{on-shell}}{=} \partial_\mu\big(\tfrac{\partial \mathcal{L}^B}{\partial(\partial_\mu \Phi|_\partial)}\delta_{\text{sym}}\Phi\big). \tag{4.2.19}$$

We find that on-shell $0 = \partial_\mu(V^\mu - \frac{\partial \mathcal{L}^B}{\partial(\partial_\mu \Phi)}\delta_{\text{sym}}\Phi)$, thus the *bulk Noether current*

$$J^\mu_B = -\tfrac{\partial \mathcal{L}^B}{\partial(\partial_\mu \Phi)}\delta_{\text{sym}}\Phi + V^\mu, \quad \partial_\mu J^\mu_B = 0, \tag{4.2.20}$$

is divergence-free (on-shell). Since there is no boundary present, divergence-freeness implies the conservation of the charge

$$Q_B = \int_\Sigma J_B^0, \tag{4.2.21}$$

where Σ is a constant-time slice of M, since $\partial_0 Q_B = \int_\Sigma \partial_0 J^0 = -\int_\Sigma \partial_i J^i = 0$.

Currents and charges with boundary

As we restricted to symmetries of boundary theories that come from a bulk theory, J_B^μ from (4.2.20) is still a valid divergence-free current by the same argument as above, so we can use it as the bulk part of the full current (4.2.12). The task at hand is to now find a boundary current which satisfies $\partial_{\hat\mu} J_\partial^{\hat\mu} = J_B^n|_\partial$. We want to apply a similar strategy as in the pure bulk theory: compare the (off-shell) symmetry variation of the action with an on-shell variation. On the off-shell side we get

$$0 = \delta_{\text{sym}} S|_{G=0} = \int_M \partial_\mu V^\mu + \int_{\partial M} \delta_{\text{sym}} \mathcal{L}^\partial|_{G=0} = \int_{\partial M} [V^n + \delta_{\text{sym}} \mathcal{L}^\partial]_{G=0}, \tag{4.2.22}$$

which implies that

$$[V^n + \delta_{\text{sym}} \mathcal{L}^\partial]_{G=0} = \partial_{\hat\mu} K^{\hat\mu} \tag{4.2.23}$$

for some boundary vector field $K^{\hat\mu}$.

On the on-shell side, we now use equations of motion *and* the stationarity condition (4.2.5). By varying the boundary Lagrangian directly and assuming $G[\ldots]|_\partial = 0$ we get

$$\begin{aligned} \delta_{\text{sym}} \mathcal{L}^\partial = {} & \left[\tfrac{\partial \mathcal{L}^\partial}{\partial \Phi|_\partial} - \partial_{\hat\mu} \tfrac{\partial \mathcal{L}^\partial}{\partial(\partial_{\hat\mu}\Phi|_\partial)}\right] \delta_{\text{sym}} \Phi|_\partial + \text{EoM}^\partial[X] \delta_{\text{sym}} X \\ & + \partial_{\hat\mu}\big(\tfrac{\partial \mathcal{L}^\partial}{\partial(\partial_{\hat\mu} X)} \delta_{\text{sym}} X + \tfrac{\partial \mathcal{L}^\partial}{\partial(\partial_{\hat\mu}\Phi|_\partial)} \delta_{\text{sym}} \Phi|_\partial\big). \end{aligned} \tag{4.2.24}$$

To rewrite the first term, let us plug in $\delta = \delta_{\text{sym}}$ into (4.2.5) (still assuming $G[\ldots] = 0$) and use the definition of the bulk Noether current (4.2.20) to rewrite it:

$$\partial_{\hat\mu} A^{\hat\mu} \overset{\text{on-shell}}{=} \left[\tfrac{\partial \mathcal{L}^\partial}{\partial \Phi|_\partial} - \partial_{\hat\mu} \tfrac{\partial \mathcal{L}^\partial}{\partial(\partial_{\hat\mu}\Phi|_\partial)}\right] \delta_{\text{sym}} \Phi|_\partial + [V^n - J_B^n]_\partial. \tag{4.2.25}$$

Plugging this into the previous equation and going on-shell, we get

$$\delta_{\text{sym}} \mathcal{L}^\partial \overset{\text{on-shell}}{=} [J_B^n - V^n]_\partial + \partial_{\hat\mu}\big(A^{\hat\mu} + \tfrac{\partial \mathcal{L}^\partial}{\partial(\partial_{\hat\mu} X)} \delta_{\text{sym}} X + \tfrac{\partial \mathcal{L}^\partial}{\partial(\partial_{\hat\mu}\Phi|_\partial)} \delta_{\text{sym}} \Phi|_\partial\big). \tag{4.2.26}$$

We can now compare this on-shell variation to the off-shell variation in (4.2.23) and see

$$J_B^n|_\partial = \partial_{\hat\mu}\big(K^{\hat\mu} - A^{\hat\mu} - \tfrac{\partial \mathcal{L}^\partial}{\partial(\partial_{\hat\mu} X)} \delta_{\text{sym}} X - \tfrac{\partial \mathcal{L}^\partial}{\partial(\partial_{\hat\mu}\Phi|_\partial)} \delta_{\text{sym}} \Phi|_\partial\big). \tag{4.2.27}$$

Thus, we can read off the boundary Noether current

$$J_\partial^{\hat\mu} = K^{\hat\mu} - A^{\hat\mu} - \tfrac{\partial \mathcal{L}^\partial}{\partial(\partial_{\hat\mu} X)} \delta_{\text{sym}} X - \tfrac{\partial \mathcal{L}^\partial}{\partial(\partial_{\hat\mu}\Phi|_\partial)} \delta_{\text{sym}} \Phi|_\partial, \quad \partial_{\hat\mu} J_\partial^{\hat\mu} = J_B^n|_\partial. \tag{4.2.28}$$

Together with the bulk current (4.2.20), this forms a conserved boundary theory current in the sense of (4.2.11). We recall that $K^{\hat\mu}$ is defined by the symmetry condition (4.2.23) and $A^{\hat\mu}$ is defined by the stationarity condition (4.2.5). Notice that through the dependency on $A^{\hat\mu}$, the boundary Noether current may explicitly depend on the boundary condition, even if the bulk variation is compensated at the boundary in a boundary-condition-independent way (cf. Section 4.2.2).

Special case: Energy-momentum tensor

In the presence of a boundary, Noether's theorem applies to spacetime translations as well: The total energy-momentum tensor is[3]

$$T_\nu{}^\mu = (T^B)_\nu{}^\mu + \delta(x^n)\mathcal{P}^\mu{}_{\hat\mu}\mathcal{P}_\nu{}^{\hat\nu}(T^\partial)_{\hat\nu}{}^{\hat\mu}. \tag{4.2.29}$$

Here, the bulk contribution is

$$(T^B)_\nu{}^\mu = -\frac{\partial\mathcal{L}^B}{\partial(\partial_\mu\Phi)}\partial_\nu\Phi + \delta_\nu{}^\mu\mathcal{L}^B, \tag{4.2.30}$$

while the boundary contribution is

$$(T^\partial)_{\hat\nu}{}^{\hat\mu} = -\frac{\partial\mathcal{L}^\partial}{\partial(\partial_{\hat\mu}X)}\partial_{\hat\nu}X - \frac{\partial\mathcal{L}^\partial}{\partial(\partial_{\hat\mu}\Phi)}\partial_{\hat\nu}\Phi + \delta^{\hat\mu}{}_{\hat\nu}\mathcal{L}^\partial, \tag{4.2.31}$$

where summation over fields is implied. The conservation equations are given by

$$\begin{aligned} &\partial_\mu(T^B)_\nu{}^\mu = 0, \\ &\partial_{\hat\mu}(T^\partial)_{\hat\nu}{}^{\hat\mu} = T_{\hat\nu}{}^n|_\partial, \end{aligned} \tag{4.2.32}$$

and the total tensor satisfies $\partial_\mu T_\nu{}^\mu = \delta(x^n)\mathcal{P}_\nu{}^{\hat\nu}T_{\hat\nu}{}^n$. The momenta along the tangential $\hat\nu$-directions are conserved

$$P_{\hat\nu} = \int_\Sigma (T^B)_{\hat\nu}{}^0 + \int_{\partial\Sigma}(T^\partial)_{\hat\nu}{}^0, \qquad \partial_0 P_{\hat\nu} = 0, \tag{4.2.33}$$

while $P_n = \int_\Sigma (T^B)_n{}^0$ is clearly not conserved in general: $\partial_0 P_n = -\int_\Sigma \partial_i (T^B)_n{}^0 = -T_n{}^n|_\partial$. This is consistent with a flat boundary: The theory is only invariant under spacetime translations tangential to the boundary.

As far as improvements are concerned, the most general improvement takes the form

$$\begin{Bmatrix} (T^B)_{\mu\nu} \\ (T^\partial)_{\hat\mu\hat\nu} \end{Bmatrix} \mapsto \begin{Bmatrix} (T^B)_{\mu\nu} + \partial^\rho M_{\nu[\mu\rho]} \\ (T^\partial)_{\hat\mu\hat\nu} + M_{\hat\nu n\hat\mu} + \partial^{\hat\rho} m_{\hat\nu[\hat\mu\hat\rho]} \end{Bmatrix}, \tag{4.2.34}$$

which, as before, leads to the same charges. However, if we restrict to improvements of symmetric tensors containing up to spin 1 components [38], the allowed improvements take the form:

$$\begin{Bmatrix} (T^B)_{\mu\nu} \\ (T^\partial)_{\hat\mu\hat\nu} \end{Bmatrix} \mapsto \begin{Bmatrix} (T^B)_{\mu\nu} + \partial_\nu U_\mu - \eta_{\mu\nu}\partial^\rho U_\rho \\ (T^\partial)_{\hat\mu\hat\nu} + \eta_{\hat\mu\hat\nu}U_n - \eta_{n\hat\nu}U_{\hat\mu} + \partial_{\hat\nu}u_{\hat\mu} - \eta_{\hat\mu\hat\nu}\partial^{\hat\rho}u_{\hat\rho} \end{Bmatrix}, \tag{4.2.35}$$

where U_μ is the bulk improvement and $u_{\hat\mu}$ the boundary improvement.

[3] Strictly speaking, the index ν does not take the value n: Translations in x^n-direction are no longer symmetries. However, we may still consider this part of the tensor even though it does not lead to a conserved charge.

4.3 Boundary supercurrent multiplets in 3D

We want to study supercurrent multiplets of theories on manifolds with boundary. In particular, we consider the special case of bulk theories with 3D $\mathcal{N} = 2$ supersymmetry, broken to 2D $\mathcal{N} = (0, 2)$ due to the boundary. While our discussion is limited to this particular case, the strategy is expected to work in greater generality. We start by recalling the definitions and some facts about supercurrent multiplets, following [38] (see also [74] for a connection to a superspace Noether procedure).

The defining properties of a supercurrent multiplet are:

(i) The energy-momentum tensor $(T^B)^{\mu\nu}$ should be a component of the multiplet. It is also the only component with spin 2.

(ii) The supercurrents, i.e. conserved currents associated to supersymmetry, are components of the multiplet. They are the only components with spin 3/2. No component other than the supercurrents and the energy-momentum tensor are allowed to have spin larger than 1.

(iii) The supercurrent multiplet is not unique: It allows for (supersymmetrically complete) improvements of its components.

(iv) The multiplet is *indecomposable*, so it may have non-trivial submultiplets, but it may not be decomposed into two independent decoupled multiplets.

The components of a supercurrent multiplet (in particular, the conserved currents) are only unique up to improvements. However, improving one component and not the others breaks the structure of the multiplet. Hence, to consistently improve the supercurrent multiplet, we must restrict to improvements of all components which are related in a certain way (specifically, the improvement terms have to form a supersymmetry multiplet themselves; details are in [38]). In other words, if one is given two components (e.g. a supercurrent and an energy-momentum tensor), one may have to improve one of them such that they can be part of the same supercurrent multiplet. We say two conserved currents which have been improved such that they are part of a consistent supercurrent multiplet are in the same *improvement frame*.

For some theories the supercurrent multiplet may be improved into a smaller multiplet (e.g. to obtain an $\mathcal{R}$-multiplet or a Ferrara–Zumino multiplet). There are still improvements that preserve this smaller multiplet [38, 68, 31]. We will recall the case of 3D $\mathcal{N} = 2$ theories in more detail.

4.3.1 In bulk theories

We will focus on three-dimensional theories with two-dimensional boundaries. In this section we recall the defining relations and properties of supercurrent multiplets in three-dimensional bulk theories with $\mathcal{N} = 2$ supersymmetry from [38].

The most general supercurrent multiplet satisfying the conditions (i)–(iv) (called the $\mathcal{S}$-*multiplet*) consists of three superfields, $\mathcal{S}_{\alpha\beta}$, χ_α, $\mathcal{Y}_\alpha$ with $\mathcal{S}_{\alpha\beta}$ real, χ_α, $\mathcal{Y}_\alpha$ fermionic, and a complex constant C. They must satisfy the defining relations:

$$\begin{aligned}
\bar{D}^{\beta}\mathcal{S}_{\alpha\beta} &= \chi_\alpha + \mathcal{Y}_\alpha,\\
\bar{D}_\alpha\chi_\beta &= \tfrac{1}{2}C\epsilon_{\alpha\beta},\\
D^\alpha\chi_\alpha + \bar{D}^\alpha\bar{\chi}_\alpha &= 0,\\
D_\alpha\mathcal{Y}_\beta + D_\beta\mathcal{Y}_\alpha &= 0,\\
\bar{D}^\alpha\mathcal{Y}_\alpha + C &= 0.
\end{aligned} \tag{4.3.1}$$

These defining relations are solved by the following expansions (using bispinor relations (A.6.13)):

$$\begin{aligned}
\mathcal{S}_\mu = {} & j_\mu - i\theta(S_\mu + \tfrac{i}{\sqrt{2}}\gamma_\mu\bar{\omega}) - i\overline{\theta}(\overline{S}_\mu - \tfrac{i}{\sqrt{2}}\gamma_\mu\omega) + \tfrac{i}{2}\theta^2\bar{Y}_\mu + \tfrac{i}{2}\overline{\theta}^2 Y_\mu\\
& - (\theta\gamma^\nu\overline{\theta})(2T_{\nu\mu} - \eta_{\mu\nu}A - \tfrac{1}{4}\epsilon_{\mu\nu\rho}H^\rho) - i\theta\overline{\theta}(\tfrac{1}{4}\epsilon_{\mu\nu\rho}F^{\nu\rho} + \epsilon_{\mu\nu\rho}\partial^\nu j^\rho)\\
& + \tfrac{1}{2}\theta^2\overline{\theta}(\gamma^\nu\partial_\nu S_\mu - \tfrac{i}{\sqrt{2}}\gamma_\mu\gamma_\nu\partial^\nu\bar{\omega}) + \tfrac{1}{2}\overline{\theta}^2\theta(\gamma^\nu\partial_\nu\overline{S}_\mu + \tfrac{i}{\sqrt{2}}\gamma_\mu\gamma_\nu\partial^\nu\omega)\\
& - \tfrac{1}{2}\theta^2\overline{\theta}^2(\partial_\mu\partial^\nu j_\nu - \tfrac{1}{2}\partial^2 j_\mu),
\end{aligned} \tag{4.3.2a}$$

$$\begin{aligned}
\chi_\alpha = {} & -i\lambda_\alpha(y) + \theta_\beta[\delta_\alpha{}^\beta D(y) - (\gamma^\mu)_\alpha{}^\beta(H_\mu(y) - \tfrac{i}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho}(y))]\\
& + \tfrac{1}{2}\overline{\theta}_\alpha C - \theta^2(\gamma^\mu\partial_\mu\bar{\lambda})_\alpha(y),
\end{aligned} \tag{4.3.2b}$$

$$\begin{aligned}
\mathcal{Y}_\alpha = {} & \sqrt{2}\omega_\alpha + 2\theta_\alpha B + 2i\gamma^\mu_{\alpha\beta}\overline{\theta}^\beta Y_\mu + \sqrt{2}i(\theta\gamma^\mu\overline{\theta})\epsilon_{\mu\nu\rho}(\gamma^\nu\partial^\rho\omega)_\alpha\\
& + \sqrt{2}i\theta\overline{\theta}(\gamma^\mu\partial_\mu\omega)_\alpha + i\theta^2\gamma^\mu_{\alpha\beta}\overline{\theta}^\beta\partial_\mu B - \overline{\theta}^2\theta_\alpha\partial_\mu Y^\mu + \tfrac{1}{2\sqrt{2}}\theta^2\overline{\theta}^2\partial^2\omega_\alpha,
\end{aligned} \tag{4.3.2c}$$

where $(S^\mu)_\alpha$, $(\overline{S}^\mu)_\alpha$ are conserved supercurrents, $T_{\mu\nu}$ is a symmetric energy-momentum tensor, and

$$\begin{aligned}
\lambda_\alpha &= -2(\gamma^\mu\overline{S}_\mu)_\alpha + 2\sqrt{2}i\omega_\alpha,\\
D &= -4T^\mu{}_\mu + 4A,\\
B &= A + i\partial_\mu j^\mu,\\
\mathrm{d}H = 0, &\quad \mathrm{d}Y = 0, \quad \mathrm{d}F = 0,
\end{aligned} \tag{4.3.3}$$

where H, F, Y are forms with components H_μ, $F_{\mu\nu}$, Y_μ. Additionally, y is the "chiral" coordinate $y^\mu = x^\mu - i\theta\gamma^\mu\overline{\theta}$. If the forms Y or H are exact, the superfields $\mathcal{Y}_\alpha$ or χ_α may be written as covariant derivatives: If $Y_\mu = \partial_\mu x$, then $\mathcal{Y}_\alpha = D_\alpha X$ where $X|_{\theta^0} = x$, and if $H_\mu = \partial_\mu g$, then $\chi_\alpha = i\bar{D}_\alpha G$ where $G|_{\theta^0} = g$.

Improvements

The expansions (4.3.2) together with the relations (4.3.3) and the conservation of currents $\partial_\mu(S^\mu)_\alpha = 0$, $\partial^\mu T_{\mu\nu} = 0$ do not form the only solution of the constraints (4.3.1). We may improve without violating the constraints

$$\begin{aligned}
\mathcal{S}_\mu &\mapsto \mathcal{S}_\mu + \tfrac{1}{4}\gamma_\mu^{\alpha\beta}[D_\alpha, \bar{D}_\beta]U,\\
\chi_\alpha &\mapsto \chi_\alpha - \bar{D}^2 D_\alpha U,\\
\mathcal{Y}_\alpha &\mapsto \mathcal{Y}_\alpha - \tfrac{1}{2}D_\alpha\bar{D}^2 U,
\end{aligned} \tag{4.3.4}$$

where $U = u+\theta\eta-\bar{\theta}\bar{\eta}+\theta^2 N-\bar{\theta}^2\bar{N}+(\theta\gamma^\mu\bar{\theta})V_\mu-i\theta\bar{\theta}K+\ldots$ is a real superfield. The improvement transforms

$$\begin{aligned}(S_\mu)_\alpha &\mapsto (S_\mu)_\alpha + \epsilon_{\mu\nu\rho}(\gamma^\nu\partial^\rho\eta)_\alpha,\\ T_{\mu\nu} &\mapsto T_{\mu\nu} + \tfrac{1}{2}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)u,\\ H_\mu &\mapsto H_\mu - 4\partial_\mu K,\\ F_{\mu\nu} &\mapsto F_{\mu\nu} - 4(\partial_\mu V_\nu - \partial_\nu V_\mu),\\ Y_\mu &\mapsto Y_\mu - 2\partial_\mu \bar{N}.\end{aligned} \tag{4.3.5}$$

The multiplet $\mathcal{S}_\mu$ may be improved into smaller multiplets. In particular:

(i) If $C=0$, $\chi_\alpha = i\bar{D}_\alpha G$ (i.e. H is exact) and there exists a well-defined improvement U such that $G = 2i\bar{D}^\alpha D_\alpha U$, then it sends χ_α to zero and we obtain a *Ferrara–Zumino multiplet* [43]:

$$\begin{aligned}\bar{D}^\beta \mathcal{J}_{\alpha\beta} &= \mathcal{Y}_\alpha,\\ D_\alpha\mathcal{Y}_\beta + D_\beta\mathcal{Y}_\alpha = 0, &\quad \bar{D}^\alpha\mathcal{Y}_\alpha = 0.\end{aligned} \tag{4.3.6}$$

(ii) If $C=0$, $\mathcal{Y}_\alpha = D_\alpha X$ (i.e. Y is exact) and there exists a well-defined improvement U such that $X = \frac{1}{2}\bar{D}^2 U$, then it sends $\mathcal{Y}_\alpha$ to zero and we obtain an $\mathcal{R}$*-multiplet* [49]:

$$\begin{aligned}\bar{D}^\beta \mathcal{R}_{\alpha\beta} &= \chi_\alpha,\\ \bar{D}_\alpha\chi_\beta = 0, &\quad D^\alpha\chi_\alpha + \bar{D}^\alpha\bar{\chi}_\beta = 0.\end{aligned} \tag{4.3.7}$$

In this case, the lowest component j_μ of the multiplet $\mathcal{R}_\mu$ (we relabel $\mathcal{S}_\mu$ to $\mathcal{R}_\mu$) is a conserved R-current (in the general $\mathcal{S}_\mu$-multiplet, j^μ is not conserved; however, we still call it a "non-conserved R-current"). The $\mathcal{R}$-multiplet will be the primary focus of our example in Section 4.4.

(iii) If $C=0$ and the improvements from (1) and (2) coincide, we can improve both superfields χ_α, $\mathcal{Y}_\alpha$ to zero *simultaneously*. In that case we obtain a *superconformal multiplet*

$$\bar{D}^\beta \mathcal{S}_{\alpha\beta} = 0. \tag{4.3.8}$$

Note that even if smaller multiplets exist, they are still not unique: We may further improve the smaller multiplets without violating the respective additional constraints. For example, in the case of the $\mathcal{R}$-multiplet, the improvements that preserve the defining constraints are transformations

$$\begin{aligned}\mathcal{R}_\mu &\mapsto \mathcal{R}_\mu + \tfrac{1}{4}\gamma_\mu^{\alpha\beta}[D_\alpha, \bar{D}_\beta]U,\\ \chi_\alpha &\mapsto \chi_\alpha - \bar{D}^2 D_\alpha U,\\ D_\alpha\bar{D}^2 U &= 0.\end{aligned} \tag{4.3.9}$$

Brane currents

We may associate to the closed forms F, H, Y, C the *brane currents* defined by taking their Hodge dual:

$$C_\mu \sim \epsilon_{\mu\nu\rho}F^{\nu\rho}, \quad C_{\mu\nu} \sim \epsilon_{\mu\nu\rho}H^\rho, \quad C'_{\mu\nu} \sim \epsilon_{\mu\nu\rho}\bar{Y}^\rho, \quad C_{\mu\nu\rho} \sim \epsilon_{\mu\nu\rho}\overline{C}. \tag{4.3.10}$$

Note that these are conserved by construction, $\partial_\mu C^\mu{}_{\mu_1\ldots\mu_k} = 0$, since the forms are closed.[4] Then, the *brane charges* defined by $Z_{\mu_1\ldots\mu_k} = \int_\Sigma C^0{}_{\mu_1\ldots\mu_k}$ are conserved as well. In addition, they are also invariant under the improvements (4.3.5). The brane charges, if they are non-trivial, are central charges of the supersymmetry algebra (but not of the Poincaré algebra). This is motivated by studying the explicit commutators [38] that follow from the multiplet structure of $\mathcal{S}_\mu$:[5]

$$\begin{aligned}\{\bar{\mathcal{Q}}_\alpha, (S_\mu)_\beta\} &= \gamma^\nu_{\alpha\beta}(2T_{\nu\mu} - \tfrac{1}{4}\epsilon_{\mu\nu\rho}H^\rho) + i\epsilon_{\alpha\beta}\tfrac{1}{4}\epsilon_{\mu\nu\rho}F^{\nu\rho} + \text{total derivatives},\\ \{\mathcal{Q}_\alpha, (S_\mu)_\beta\} &= \tfrac{1}{4}(\gamma_\mu)_{\alpha\beta}\overline{C} + i\epsilon_{\mu\nu\rho}\gamma^\nu_{\alpha\beta}\bar{Y}^\rho,\end{aligned} \tag{4.3.11}$$

where we may find non-trivial central charges in the supersymmetry algebra upon integration. Each current $C_{\mu\mu_1\ldots\mu_k}$ and the corresponding charge $Z_{\mu_1\ldots\mu_m}$ is associated to an m-brane. Hence, the brane charges form a physical obstruction to improvements into smaller multiplets. In particular, a non-zero charge associated to F or H obstructs the existence of a Ferrara–Zumino multiplet, and a non-zero charge associated to Y obstructs the existence of an $\mathcal{R}$-multiplet.

4.3.2 In theories with boundary

The introduction of a boundary affects supercurrent multiplets in two obvious ways.

First, supersymmetry is broken to a subalgebra like 2D $\mathcal{N} = (0, 2)$, which will be our main focus. The bulk supercurrent multiplets, previously 3D $\mathcal{N} = 2$ superfields, now decompose under the subalgebra to $(0, 2)$-superfields. We spell out this decomposition in Appendix A.7.2. Similarly, the constraints (4.3.1) now decompose into constraints of the $(0, 2)$-superfields. We will spell out this decomposition in the next subsection.

Second, a boundary changes the conserved currents of *remaining* symmetries by supplementing the bulk currents (4.2.20) with boundary currents (4.2.28) satisfying appropriate conservation equations. The conservation equations must follow from the constraints that define the supercurrent multiplets, as in bulk theories, and the *full* supercurrent multiplets will now consist of bulk and boundary pieces. The schematic form of full supercurrent multiplets reads

$$\begin{aligned}\mathcal{S}^{\text{full}}_\mu &= \mathcal{S}^B_\mu + \delta(x^n)\mathcal{P}^{\hat\mu}{}_\mu\mathcal{S}^\partial_{\hat\mu},\\ \chi^{\text{full}}_\alpha &= \chi^B_\alpha + \delta(x^n)\chi^\partial_\alpha,\\ \mathcal{Y}^{\text{full}}_\alpha &= \mathcal{Y}^B_\alpha + \delta(x^n)\mathcal{Y}^\partial_\alpha,\end{aligned} \tag{4.3.12}$$

where once again $\mathcal{P}^\mu{}_{\hat\mu}$ is an embedding.

Let us briefly discuss how the conditions (i)–(iv) are modified. It is clear that the new superfields should contain the *full* conserved currents of unbroken symmetries, in the sense of Section 4.2.3 (conditions (i), (ii)). Furthermore, improvements of the full conserved currents in the sense of (4.2.16) that form consistent multiplets under the smaller subalgebra are improvements of the $\mathcal{N} = (0, 2)$ supercurrent multiplets (condition (iii)). However, under the smaller symmetry algebra, the previously indecomposable (bulk) multiplet decomposes into

[4] In coordinate-free notation this is written as $\mathrm{d}{*C} = 0$ which follows trivially if $C = {*A}$, with $\mathrm{d}A = 0$.

[5] Recall, the action of physical supercharges via commutators is related to the action via (super)-differential operators by

$$[\xi^\alpha\mathcal{Q}_\alpha - \bar\xi^\alpha\bar{\mathcal{Q}}_\alpha, X] = i(\xi^\alpha Q_\alpha - \bar\xi^\alpha\bar{Q}_\alpha)X =: i\delta^{\xi,\bar\xi}_{\text{sym}}X.$$

possibly several indecomposable multiplets of the remaining symmetry subalgebra. Therefore condition (iv) is not preserved in general.

4.3.3 Bulk and boundary constraints by decomposition

Let us recall the structure of subalgebras of the 3D $\mathcal{N}=2$ algebra which may be preserved after the introduction of a boundary. The (unbroken) symmetry algebra is generated by tangential translations $P_{\hat{\mu}}$, Lorentz transformations $M_{\hat{\mu}\hat{\nu}}$ in the unbroken directions, and one of the following:

(i) supercharges Q_+, $\bar{Q}_+$ corresponding to a 2D $(0,2)$-subalgebra satisfying

$$(Q_+)^2 = 0, \quad \{Q_+, \bar{Q}_+\} = -4i\partial_+, \tag{4.3.13}$$

(ii) their left-moving $(2,0)$ counterparts Q_-, $\bar{Q}_-$,

(iii) (real) supercharges $\mathbb{Q}_-$, $\mathbb{Q}_+$ corresponding to a 2D $(1,1)$-subalgebra satisfying

$$(\mathbb{Q}_\pm)^2 = -i\partial_\pm, \quad \{\mathbb{Q}_-, \mathbb{Q}_+\} = 0. \tag{4.3.14}$$

In this work we consider the only the first case. We want to determine constraint equations that define supercurrent multiplets in a 3D theory with boundary and 2D $\mathcal{N}=(0,2)$ supersymmetry. To do so, we first decompose the 3D $\mathcal{N}=2$ bulk constraints into $\mathcal{N}=(0,2)$ bulk constraints, and then investigate possible $\mathcal{N}=(0,2)$ boundary constraints.

We supplement the superspace operators Q_+, $\bar{Q}_+$ with covariant derivatives $D_+^{(0,2)}$, $\bar{D}_+^{(0,2)}$ defined in Appendix A.7.1. Let us emphasise that these are *not* the operators D_α, $\bar{D}_\alpha$ acting on 3D $\mathcal{N}=2$ superspace, but are related to them by (A.7.4). We will omit the label $(0,2)$ from now on; it should be clear from context whether a 3D $\mathcal{N}=2$ or a $\mathcal{N}=(0,2)$ covariant derivative is meant.

Bulk constraints

Since the bulk conservation equations remain unchanged, the constraints on the bulk pieces in our ansatz (4.3.12) will remain the same component-wise. We merely have to decompose the multiplets and their constraints into constraints of $(0,2)$-submultiplets. For simplicity we choose to do so for the case where the supermultiplet is an $\mathcal{R}$-multiplet:[6] We will decompose the superfields $(\mathcal{R}_\mu, \chi_\alpha)$ and the constraints (4.3.7). We can achieve this using the *branching coordinate* ξ^μ. It has the defining property that in the coordinates $(\xi^\mu, \theta^+, \theta^-)$, the preserved supercharges Q_+ and $\bar{Q}_+$ commute with θ^- and $\overline{\theta}^-$; see Appendix A.7.1 for details. Another property is that Q_+, $\bar{Q}_+$ do not involve a derivative in $\perp$-direction. In terms of ξ we can decompose

$$\begin{aligned} \mathcal{R}_\mu^B(x,\theta,\overline{\theta}) &= \mathcal{R}_\mu^{B(0)} + \theta^- \mathcal{R}_\mu^{B(1)} - \overline{\theta}^- \mathcal{R}_\mu^{\bar{B}(1)} + \theta^- \overline{\theta}^- \mathcal{R}_\mu^{B(2)}, \\ \chi_\alpha^B(x,\theta,\overline{\theta}) &= \chi_\alpha^{B(0)} + \theta^- \chi_\alpha^{B(1a)} + \overline{\theta}^- \chi_\alpha^{B(1b)} + \theta^- \overline{\theta}^- \chi_\alpha^{B(2)}, \end{aligned} \tag{4.3.15}$$

where we now denote bulk fields by a superscript B, and boundary fields (to appear later) with a superscript ∂. The number superscripts in parentheses refer to the order in θ^-, $\overline{\theta}^-$ we

[6] The more general case of the $\mathcal{S}$-multiplet is quite similar and can be found in [13, Appendix C.3].

have expanded in. Here, each field on each right-hand side is a (super)function of $(\xi, \theta^+, \overline{\theta}^+)$. Furthermore, because Q_+, $\bar{Q}_+$ commute with θ^-, $\overline{\theta}^-$, the coefficient at each order in θ^-, $\overline{\theta}^-$ is a $(0, 2)$-submultiplet — the remaining supersymmetry group acts independently on each of them. This is a constructive way to decompose 3D $\mathcal{N} = 2$ superfields with respect to the 2D $\mathcal{N} = (0, 2)$ subalgebra. In the appendix, we write the above decomposition explicitly for the $\mathcal{S}$-multiplet (A.7.12)–(A.7.14), from which the $\mathcal{R}$-multiplet follows by setting appropriate terms to zero.

In terms of the $(0, 2)$-submultiplets, the constraints (4.3.7) are then written as the following collection of equations, where we use coordinates $\xi^+ = \xi^0 + \xi^1$, $\xi^- = \xi^0 - \xi^1$ and $\xi^\perp = x^\perp + i(\theta^+\overline{\theta}^- - \theta^-\overline{\theta}^+)$:[7]
From $\bar{D}_-\chi_\alpha = 0$:

$$\chi_\alpha^{B(1b)} = 0, \tag{4.3.16a}$$
$$\chi_\alpha^{B(2)} + 2i\partial_-\chi_\alpha^{B(0)} = 0. \tag{4.3.16b}$$

From $\bar{D}_+\chi_\alpha = 0$:

$$\bar{D}_+\chi_\alpha^{B(0)} = 0, \tag{4.3.17a}$$
$$\bar{D}_+\chi_\alpha^{B(1a)} + 2i\partial_\perp\chi_\alpha^{B(0)} = 0, \tag{4.3.17b}$$
$$\bar{D}_+\chi_\alpha^{B(2)} = 0. \tag{4.3.17c}$$

From $\operatorname{Im} D^\alpha\chi_\alpha = 0$:

$$\operatorname{Im}\left(D_+\chi_-^{B(0)} - \chi_+^{B(1a)}\right) = 0, \tag{4.3.18a}$$
$$\bar{D}_+\chi_-^{\overline{B(1a)}} + \chi_+^{B(2)} - 2i\partial_-\chi_+^{B(0)} - 2i\partial_\perp\chi_-^{B(0)} = 0, \tag{4.3.18b}$$
$$\operatorname{Im}\left(D_+\chi_-^{B(2)} - 2i\partial_-\chi_+^{B(1a)} - 2i\partial_\perp\chi_-^{B(1a)}\right) = 0. \tag{4.3.18c}$$

Finally, the relation $\bar{D}^\beta\mathcal{R}_{\alpha\beta} = \chi_\alpha$ yields:[8]

$$\chi_\alpha^{B(0)} = \bar{D}_+\mathcal{R}_{\alpha-}^{B(0)} - \mathcal{R}_{\alpha+}^{\bar{B}(1)}, \tag{4.3.19a}$$
$$-\chi_\alpha^{B(1a)} = \bar{D}_+\mathcal{R}_{\alpha-}^{B(1)} + \mathcal{R}_{\alpha+}^{B(2)} + 2i\partial_\perp\mathcal{R}_{\alpha-}^{B(0)} + 2i\partial_-\mathcal{R}_{\alpha+}^{B(0)}, \tag{4.3.19b}$$
$$0 = \bar{D}_+\bar{\mathcal{R}}_{\alpha\beta}^{B(1)}, \tag{4.3.19c}$$
$$\chi_\alpha^{B(2)} = \bar{D}_+\mathcal{R}_{\alpha-}^{B(2)} + 2i\partial_\perp\mathcal{R}_{\alpha-}^{\bar{B}(1)} + 2i\partial_-\mathcal{R}_{\alpha+}^{\bar{B}(1)}. \tag{4.3.19d}$$

Note that we have not introduced *any* new structure here: Component-wise, equations (4.3.7) have identical content as (4.3.16a)–(4.3.19d). In particular, the bulk conservation equations follow from these constraints. Let us explicitly verify this in the example of the conservation of the R-current j_μ^B. We start with Eq. (4.3.19b) setting $\alpha = +$ and taking the imaginary part. Using the reality of $\mathcal{R}_{\alpha\beta}$ (which implies the reality of $\mathcal{R}_{\alpha\beta}^{B(0)}$ and $\mathcal{R}_{\alpha\beta}^{B(2)}$), we arrive at

$$-\operatorname{Im}\left(\chi_+^{B(1a)}\right) = \operatorname{Im}\left(\bar{D}_+\mathcal{R}_{+-}^{B(1)}\right) + 2\partial_\perp\mathcal{R}_{+-}^{B(0)} + 2\partial_-\mathcal{R}_{++}^{B(0)}. \tag{4.3.20}$$

[7] Note that covariant derivatives acting on $(0, 2)$-superfields are $(0, 2)$-covariant derivatives.
[8] These have already been simplified by some relations we found, e.g. (4.3.16a).

Now consider Eq. (4.3.19a); setting $\alpha=-$, conjugating, applying $\bar{D}_+$ on both sides and finally taking the imaginary part we obtain

$$\mathrm{Im}\Big(\bar{D}_+\chi_-^{\bar{B}(0)}\Big)=\mathrm{Im}\Big(\bar{D}_+D_+\mathcal{R}_{--}^{B(0)}\Big)-\mathrm{Im}\Big(\bar{D}_+\mathcal{R}_{-+}^{B(1)}\Big). \tag{4.3.21}$$

From the reality of $\mathcal{R}_{\alpha\beta}^{B(0)}$ we get $\mathrm{Im}\Big(\bar{D}_+D_+\mathcal{R}_{--}^{B(0)}\Big)=2\partial_+\mathcal{R}_{--}^{B(0)}$. Finally, we use (4.3.18a) to combine (4.3.20) and (4.3.21) into the bulk conservation equation for the R-current:

$$2\partial_+\mathcal{R}_{--}^{B(0)}+2\partial_\perp\mathcal{R}_{+-}^{B(0)}+2\partial_-\mathcal{R}_{++}^{B(0)}=0. \tag{4.3.22}$$

This equation also implies the bulk conservation of $(S_\mu^B)_+$, $(\overline{S}_\mu^B)_+$ and $T_{\mu+}^B$, as can be verified by the expansions (A.7.12)–(A.7.14).

In an analogous fashion, we may derive the bulk conservation for $\mathcal{R}_{\alpha\beta}^{B(1)}$ and $\mathcal{R}_{\alpha\beta}^{B(2)}$. The conservation of $\mathcal{R}_{\alpha\beta}^{B(1)}$ follows from (4.3.19a) to (4.3.19d) together with (4.3.18b), and implies the conservation of bulk supercurrents $(S_\mu^B)_-$, $(\overline{S}_\mu^B)_-$ and the tensor $T_{\mu\perp}^B$. The conservation of $\mathcal{R}_{\alpha\beta}^{B(2)}$ follows from (4.3.18c), (4.3.19b), and (4.3.19d), and implies the conservation of the bulk tensor $T_{\mu-}^B$.

Boundary constraints

We now want to discuss constraints that we need to impose on the boundary parts $\mathcal{R}_\mu^\partial$ and χ_α^∂ (and $\mathcal{Y}_\alpha^\partial$ in the case of the $\mathcal{S}_\mu$-multiplet). Our guiding principle is of course the fact that the constraints need to impose boundary conservation (4.2.28) on the components of the boundary multiplets. The constraints can, in part, be deduced from the bulk constraints (4.3.16)–(4.3.19). Let us elaborate on this point: Bulk and boundary superfields are combined to our *total* supercurrent multiplet

$$\mathcal{R}_\mu^{\text{full}}=\mathcal{R}_\mu^B+\delta(\xi^\perp)\mathcal{P}_\mu{}^{\hat{\mu}}\mathcal{R}_{\hat{\mu}}^\partial, \tag{4.3.23}$$

where both bulk and boundary pieces can be decomposed into $(0,2)$-multiplets:[9]

$$\begin{aligned}\mathcal{R}_\mu^B(x,\theta,\bar{\theta})&=\mathcal{R}_\mu^{B(0)}+\theta^-\mathcal{R}_\mu^{B(1)}-\bar{\theta}^-\mathcal{R}_\mu^{\bar{B}(1)}+\theta^-\bar{\theta}^-\mathcal{R}_\mu^{B(2)},\\ \mathcal{R}_{\hat{\mu}}^\partial(x,\theta,\bar{\theta})&=\mathcal{R}_{\hat{\mu}}^{\partial(0)}+\theta^-\bar{\theta}^-\mathcal{R}_{\hat{\mu}}^{\partial(2)},\end{aligned} \tag{4.3.24}$$

and similarly for auxiliary fields

$$\begin{aligned}\chi_\alpha^B(x,\theta,\bar{\theta})&=\chi_\alpha^{B(0)}+\theta^-\chi_\alpha^{B(1a)}+\bar{\theta}^-\chi_\alpha^{B(1b)}+\theta^-\bar{\theta}^-\chi_\alpha^{B(2)},\\ \chi_\alpha^\partial(x,\theta,\bar{\theta})&=\chi_\alpha^{\partial(0)}+\theta^-\chi_\alpha^{\partial(1a)}+\bar{\theta}^-\chi_\alpha^{\partial(1b)}+\theta^-\bar{\theta}^-\chi_\alpha^{\partial(2)}.\end{aligned} \tag{4.3.25}$$

As mentioned before, the bulk part is unchanged compared to the full 3D $\mathcal{N}=2$ current multiplet; it is simply decomposed into its $(0,2)$-submultiplets. The ansatz for the boundary part is directly motivated by the expansions (A.7.12)–(A.7.14) of the bulk multiplet, which show that

$$\begin{aligned}\mathcal{R}_\mu^{B(0)}&=j_\mu^B+\dots,\\ \mathcal{R}_\mu^{B(1)}&=-i(S_\mu^B)_-+\dots,\\ \mathcal{R}_\mu^{B(2)}&=-2K_{\mu-}+\dots,\end{aligned} \tag{4.3.26}$$

[9] This is essentially an embedding into 3D $\mathcal{N}=2$ superspace, see [37, 36].

where for the $\mathcal{R}$-multiplet, $K_{\mu\nu} = 2T_{\nu\mu} - \frac{1}{4}\epsilon_{\mu\nu\rho}H^{\rho}$. We can conclude

$$\begin{aligned} \mathcal{R}^{\partial(0)}_{\hat{\mu}} &= j^{\partial}_{\hat{\mu}} + \dots, \\ \mathcal{R}^{\partial(2)}_{\hat{\mu}} &= -2K^{\partial}_{\hat{\mu}-} + \dots, \end{aligned} \tag{4.3.27}$$

as the bulk conserved currents have to be paired with their respective boundary currents. Note that we do not consider a boundary contribution to the "broken" $(S^B_\mu)_-$ currents, as we have no guiding principle in this framework. Then, due to the form of the full supercurrent multiplet (4.3.23), we postulate that the constraints applied to the boundary piece must be of similar form as (4.3.16a)–(4.3.19d), but instead of imposing divergence-freeness (4.2.20), they should impose (4.2.28) on the *remaining, conserved* boundary currents.

We postulate the following adjustments on the constraints obtained from the bulk (4.3.16)–(4.3.19), now applied to boundary multiplets $\mathcal{R}^{\partial(*)}_{\alpha\alpha}$, in order to obtain correct conservation equations:

(i) Firstly, since the boundary is two-dimensional with directions x_{++}, x_{--} (in bispinor notation, cf. (A.6.13)) we only have superfields $\mathcal{R}^{\partial(*)}_{++}$, $\mathcal{R}^{\partial(*)}_{--}$, and no superfield $\mathcal{R}^{\partial(*)}_{+-}$.

(ii) Secondly, we do not consider boundary contributions to the "broken" $(S^B_\mu)_-$ currents, and hence no $\mathcal{R}^{\partial(1)}_{\alpha\alpha}$ should appear.

(iii) Lastly, to impose the correct conservation equation on the boundary, we must replace terms of the form $\partial_\perp A^\partial$ with $-A^B|_\partial$ whenever such terms appear. This transformation precisely maps divergence-free equations (4.2.20) into boundary conservation equations (4.2.28). In addition, this replacement parses well with the fact that derivatives in the perpendicular direction make little sense when they act on boundary currents, in particular when the boundary currents are functions of purely boundary fields.

The preliminary constraints on the boundary pieces then read:
Analogons to (4.3.16):

$$\chi^{\partial(1b)}_\alpha = 0, \tag{4.3.28a}$$

$$\chi^{\partial(2)}_- + 2i\partial_-\chi^{\partial(0)}_- = 0. \tag{4.3.28b}$$

Analogons to (4.3.17):

$$\bar{D}_+\chi^{\partial(0)}_- = 0, \tag{4.3.29a}$$

$$\bar{D}_+\chi^{\partial(1a)}_+ - 2i\chi^{B(0)}_+|_\partial = 0, \tag{4.3.29b}$$

$$\bar{D}_+\chi^{\partial(2)}_\alpha = 0. \tag{4.3.29c}$$

Analogons to (4.3.18):

$$\mathrm{Im}\Big(D_+\chi^{\partial(0)}_- - \chi^{\partial(1a)}_+\Big) = 0, \tag{4.3.30a}$$

$$\mathrm{Im}\Big(D_+\chi^{\partial(2)}_- - 2i\partial_-\chi^{\partial(1a)}_+ + 2i\chi^{B(1a)}_-|_\partial\Big) = 0. \tag{4.3.30b}$$

Lastly, the analogons to (4.3.19):

$$\chi_{+}^{\partial(0)} = 0, \tag{4.3.31a}$$
$$\chi_{-}^{\partial(0)} = \bar{D}_{+}\mathcal{R}_{--}^{\partial(0)}, \tag{4.3.31b}$$
$$\chi_{+}^{\partial(1a)} = -\mathcal{R}_{++}^{\partial(2)} + 2i\mathcal{R}_{+-}^{B(0)}|_{\partial} - 2i\partial_{-}\mathcal{R}_{++}^{\partial(0)}, \tag{4.3.31c}$$
$$\chi_{-}^{\partial(1a)} = 2i\mathcal{R}_{--}^{B(0)}|_{\partial}, \tag{4.3.31d}$$
$$\chi_{+}^{\partial(2)} = -2i\mathcal{R}_{+-}^{\bar{B}(1)}|_{\partial}, \tag{4.3.31e}$$
$$\chi_{-}^{\partial(2)} = \bar{D}_{+}\mathcal{R}_{--}^{\partial(2)} - 2i\mathcal{R}_{--}^{\bar{B}(1)}|_{\partial}. \tag{4.3.31f}$$

Note that the naive application of adjustments (i)–(iii) leads to three further relations, which we have intentionally omitted above. These are: the analogon of (4.3.16b) for $\alpha=+$, which reads

$$\chi_{+}^{\partial(2)} + 2i\partial_{-}\chi_{+}^{\partial(0)} = 0, \tag{4.3.32}$$

the analogon of (4.3.17b) for $\alpha=-$, which reads

$$\bar{D}_{+}\chi_{-}^{\partial(1a)} - 2i\chi_{-}^{B(0)}|_{\partial}, \tag{4.3.33}$$

and lastly, the analogon of (4.3.18b) reads

$$\bar{D}_{+}\chi_{-}^{\partial(\bar{1}a)} + \chi_{+}^{\partial(2)} - 2i\partial_{-}\chi_{+}^{\partial(0)} + 2i\chi_{-}^{B(0)}|_{\partial} = 0. \tag{4.3.34}$$

We argue that these relations must be discarded from the set of constraints of boundary multiplets. To see this, note that the first relation (4.3.32) is compatible with Eqs. (4.3.31a) and (4.3.31e) *only if* $\mathcal{R}_{+-}^{B(1)}|_{\partial}=0$. Similarly, the second relation (4.3.33) in agreement with Eqs. (4.3.19a) and (4.3.31d) again *only if* $\mathcal{R}_{+-}^{B(1)}|_{\partial}=0$. Lastly, the third relation (4.3.34) is consistent with Eqs. (4.3.19a), (4.3.31a), (4.3.31d), and (4.3.31e), once more *only if* $\mathcal{R}_{+-}^{B(1)}|_{\partial}=0$. Hence, including any of the three relations in the constraints of boundary $(0,2)$-supermultiplets would impose

$$\mathcal{R}_{+-}^{B(1)}|_{\partial} = -i(S_{\perp}^{B})_{-}|_{\partial} + \ldots = 0. \tag{4.3.35}$$

However, imposing this condition would imply the conservation of the "broken" charge Q_{-}, as the current $(S_{\mu}^{B})_{-}$ would fulfil (4.2.11) with a trivial boundary part. Therefore, if we did not omit relations (4.3.32)–(4.3.34), we would impose the conservation of the "broken" charges $Q_{-}, \bar{Q}_{-}$, which is inconsistent with the "breaking" of $P_{\perp}$ and the explicit conservation of the charges $Q_{+}, \bar{Q}_{+}$. As a last argument, we note that the three omitted relations are *not* required to obtain the boundary conservation equations for the remaining symmetries ($\mathcal{N}=(0,2)$ supersymmetry, R-symmetry, 2D Poincaré symmetry). We will verify this in the remainder of this subsection by explicitly checking that the boundary conservation equations indeed follow from constraints (4.3.28)–(4.3.31).

We spell out the derivation of the boundary conservation equation in the example of $\mathcal{R}_{\alpha\alpha}^{\partial(0)}$: Taking the imaginary part of (4.3.31c) and using the reality of the multiplet, we obtain

$$\mathrm{Im}\Big(\chi_{+}^{\partial(1a)}\Big) = 2\mathcal{R}_{+-}^{B(0)}|_{\partial} - 2\partial_{-}\mathcal{R}_{++}^{\partial(0)}. \tag{4.3.36}$$

Now we take Eq. (4.3.31b), conjugate it, apply $\bar{D}_+$ on both sides and finally take imaginary part again to obtain:

$$\mathrm{Im}\Big(\bar{D}_+\chi_-^{\bar{\partial}(0)}\Big) = \mathrm{Im}\Big(\bar{D}_+D_+\mathcal{R}_{--}^{\partial(0)}\Big). \tag{4.3.37}$$

Again, due to the reality of $\mathcal{R}_{--}^{\partial(0)}$, we have that $\mathrm{Im}\Big(\bar{D}_+D_+\mathcal{R}_{--}^{\partial(0)}\Big) = 2\partial_+\mathcal{R}_{--}^{\partial(0)}$. Finally, we can combine equations (4.3.36) and (4.3.37) using (4.3.30a) into the conservation equation for the boundary R-current:

$$2\mathcal{R}_{+-}^{B(0)}|_\partial - 2\partial_-\mathcal{R}_{++}^{\partial(0)} - 2\partial_+\mathcal{R}_{--}^{\partial(0)} = 0. \tag{4.3.38}$$

Like its bulk counterpart, this superfield equation also implies the boundary conservation of $(S^\partial_{\hat{\mu}})_+$, $(\overline{S}^\partial_{\hat{\mu}})_+$ and $T^\partial_{\hat{\mu}+}$.

In a similar fashion, we may derive boundary conservation of $\mathcal{R}^{\partial(2)}_{\alpha\alpha}$ using Eqs. (4.3.30b), (4.3.31c), (4.3.31d), and (4.3.31f). Component-wise it implies the conservation of the boundary tensor $T^\partial_{\hat{\mu}-}$. No boundary analogue to bulk conservation of $(S^B_\mu)_-$ follows from the boundary constraints, which is what we expect.

4.3.4 Integrated supercurrent multiplets

Comparison to pure 2D theories

The supercurrent multiplets that satisfy the bulk (4.3.16)–(4.3.19) and boundary (4.3.28)–(4.3.31) constraints are $(0,2)$-multiplets of a three-dimensional theory with boundary, with supersymmetry algebra isomorphic to that of a 2D $(0,2)$-theory. It is interesting to compare and contrast the structure of this theory to a pure bulk 2D theory with $(0,2)$-supersymmetry. We review some generic aspects of such bulk theories following [31] (see also [76] for a comprehensive review on 2D $\mathcal{N}=(0,2)$ models).

In the case of a 2D theory with $\mathcal{N}=(0,2)$ supersymmetry, the most general $\mathcal{S}$-multiplet is given by superfields $(\mathcal{S}^{(0,2)}_{++}, \mathcal{W}^{(0,2)}_{-}, \mathcal{T}^{(0,2)}_{----}, C)$ and corresponding defining constraints in 2D $(0,2)$-theories, such that conditions (i)–(iv) are satisfied. For details on their structure see [38, 31]. If the $(0,2)$-model we consider has an R-symmetry, there exists a corresponding smaller $\mathcal{R}$-multiplet $(\mathcal{R}^{(0,2)}_{\hat{\mu}}, \mathcal{T}^{(0,2)}_{----})$ containing an improved energy momentum tensor $T_{\mu\nu}$. Furthermore, the structure of the multiplet guarantees that we can define the *half-twisted* energy-momentum tensor $\widetilde{T}_{\mu\nu}$

$$\begin{aligned}\widetilde{T}_{++} &= T_{++} + \tfrac{i}{2}\partial_+ j_+,\\ \widetilde{T}_{+-} &= T_{+-} - \tfrac{i}{2}\partial_- j_+,\\ \widetilde{T}_{--} &= T_{--} - \tfrac{i}{2}\partial_- j_-,\end{aligned} \tag{4.3.39}$$

which satisfies

$$\begin{aligned}\{\bar{Q}_+, \cdots\} &= \widetilde{T}_{+\hat{\mu}},\\ \{\bar{Q}_+, \widetilde{T}_{--}\} &= 0, \quad \text{but } \{\bar{Q}_+,\cdots\} \neq \widetilde{T}_{--}.\end{aligned} \tag{4.3.40}$$

In other words, the components of the twisted energy-momentum tensor are $\bar{Q}_+$-cohomology elements, and $\widetilde{T}_{--}$ is a non-trivial element. Starting from these identities, one can show that the $\bar{Q}_+$-cohomology of observables is invariant under renormalisation group flow and thus carries information about possible IR fixed points of the model under consideration [31]. In particular, there is an emergent conformal symmetry on the level of cohomology.

Secondly, it is well known for (0, 2)-theories that the cohomology of $\bar{Q}_+$ as an operator on fields is isomorphic to the cohomology of $\bar{D}_+$ as an operator on superfields. In this language, the non-trivial components of the twisted energy-momentum tensor are given by an appropriate $\bar{D}_+$-closed linear combination of superfields from the supercurrent multiplet:

$$\bar{D}_+\Big(\mathcal{T}^{(0,2)}_{----} - \tfrac{i}{2}\partial_{--}\mathcal{R}^{(0,2)}_{--}\Big) = 0. \tag{4.3.41}$$

An energy-momentum tensor (not) in the cohomology

Since our three-dimensional theory with boundary has the same supersymmetry algebra, we might expect a similar structure as far as $\bar{Q}_+$-cohomology is concerned. Indeed, we can identify the analogue of (4.3.41) in 3D: We combine bulk equations (4.3.16b), (4.3.19a), and (4.3.19d) for $\alpha = -$ to:[10]

$$\bar{D}_+\Big(\mathcal{R}^{B(2)}_{--} + 2i\partial_-\mathcal{R}^{B(0)}_{--}\Big) = -2i\partial_\perp\mathcal{R}^{\bar{B}(1)}_{--}, \tag{4.3.42}$$

as well as boundary equations (4.3.28b), (4.3.31b), and (4.3.31f) to

$$\bar{D}_+\Big(\mathcal{R}^{\partial(2)}_{--} + 2i\partial_-\mathcal{R}^{\partial(0)}_{--}\Big) = 2i\mathcal{R}^{\bar{B}(1)}_{--}|_\partial. \tag{4.3.43}$$

Furthermore, we can also combine (4.3.17b), (4.3.19a), and (4.3.19b) for $\alpha = +$ into

$$\bar{D}_+\Big(\mathcal{R}^{B(2)}_{++} + 2i\mathcal{R}^{B(0)}_{++}\Big) = -2i\partial_\perp\mathcal{R}^{\bar{B}(1)}_{++}. \tag{4.3.44}$$

Similarly, we combine boundary equations (4.3.19a), (4.3.29b), and (4.3.30b) into

$$\bar{D}_+\Big(\mathcal{R}^{\partial(2)}_{++} + 2i\partial_-\mathcal{R}^{\partial(0)}_{++}\Big) = 2i\mathcal{R}^{\bar{B}(1)}_{++}|_\partial. \tag{4.3.45}$$

We can rewrite these relations using the full $\mathcal{R}$-multiplet $\mathcal{R}^{\text{full}(*)}_{\alpha\beta} = \mathcal{R}^{B(*)}_{\alpha\beta} + \delta(\xi^\perp)\mathcal{R}^{\partial(*)}_{\alpha\beta}$:

$$\begin{aligned}\bar{D}_+\Big(\mathcal{R}^{\text{full}(2)}_{--} + 2i\partial_-\mathcal{R}^{\text{full}(0)}_{--}\Big) &= -2i\partial_\perp\mathcal{R}^{\bar{B}(1)}_{--} + 2i\delta(\xi^\perp)\mathcal{R}^{\bar{B}(1)}_{--}|_\partial,\\ \bar{D}_+\Big(\mathcal{R}^{\text{full}(2)}_{++} + 2i\partial_-\mathcal{R}^{\text{full}(0)}_{++}\Big) &= -2i\partial_\perp\mathcal{R}^{\bar{B}(1)}_{++} + 2i\delta(\xi^\perp)\mathcal{R}^{\bar{B}(1)}_{++}|_\partial.\end{aligned} \tag{4.3.46}$$

The first equation is the analogue of (4.3.41) in 3D, as we already stated. The second equation is the analogue of the $\bar{Q}$-closedness (equivalent to $\bar{D}_+$-closedness) of the half-twisted tensor $T_{+-} - \frac{i}{2}\partial_- j_+$, which in 2D follows from $\bar{Q}$-exactness.

Integrated currents and multiplets

Eq. (4.3.46) shows that one cannot, in general, repeat the pure 2D argument to produce a local energy-momentum tensor twisted by the R-symmetry such that it is a $\bar{Q}_+$-cohomology element. However, there is a different point of view which is helpful here: A three-dimensional quantum field theory with a finite number of fields can instead be regarded as a two-dimensional quantum field theory with an infinite number of fields. More precisely, instead of viewing bulk fields,

[10] The factor discrepancy on the left-hand side of (4.3.41) and the above equations is merely due to switching between spacetime and bispinor notation (cf. (A.6.14)).

loosely speaking, as maps $\partial M \times \mathbb{R}_{\leq 0} \to T$, we view them as maps $\partial M \to \{\text{maps: } \mathbb{R}_{\leq 0} \to T\}$ [33, 15]. Now, instead of considering separate bulk and boundary actions, we can write a single Lagrangian for the full theory:

$$\mathcal{L}^{\text{int.}} := \mathcal{L}^{\partial} + \int_{\mathbb{R}_{\leq 0}} \mathrm{d}x^n \mathcal{L}^B; \quad S = \int_{\partial M} \mathrm{d}x^{N-1} \mathcal{L}^{\text{int.}}. \tag{4.3.47}$$

We see that the action is the same as before, but integration along x^n is now a conceptually different operation: Before, it used to be an integral on the spacetime on which the field theory is defined; now it is an operation on the new target space (i.e. a functional). The integration along x^n also translates to conserved currents: As the theory is now formally two-dimensional, applying Noether's theorem to the above Lagrangian yields a two-dimensional current of the form

$$J^{\text{int.}}_{\hat{\mu}} = J^{\partial}_{\hat{\mu}} + \int_{\mathbb{R}_{\leq 0}} \mathrm{d}x^n J^B_{\hat{\mu}}. \tag{4.3.48}$$

We see that its conserved charge

$$Q = \int_{\partial\Sigma} J^0_{\text{int.}} \tag{4.3.49}$$

is identical to the one belonging to the local current (4.2.14). This also provides an argument why the integrated currents are "natural" from the point of view of the three-dimensional QFT: To find the conserved charge of a current, one has to integrate all spatial directions, and the integrated current is an "intermediate step" of this integration. The conservation equations (4.2.28) now take the familiar form

$$\partial^{\hat{\mu}} J^{\text{int.}}_{\hat{\mu}} = 0, \tag{4.3.50}$$

where boundary conditions are possibly used. Extending supersymmetrically, we introduce the *integrated supercurrent multiplets*, in our conventions (recall, mixed indices correspond to the $\perp$-direction, which does not appear here, see (A.6.13))

$$\mathcal{R}^{\text{int.}}_{\alpha\alpha} = \mathcal{R}^{\partial}_{\alpha\alpha} + \int_{\mathbb{R}_{\leq 0}} \mathrm{d}x^{\perp} \mathcal{R}^B_{\alpha\alpha}. \tag{4.3.51}$$

In terms of integrated currents, the right-hand side of (4.3.46) cancels exactly due to the integral:

$$\begin{aligned} \bar{D}_+\left(\mathcal{R}^{\text{int.}(2)}_{--} + 2i\partial_- \mathcal{R}^{\text{int.}(0)}_{--}\right) &= 0, \\ \bar{D}_+\left(\mathcal{R}^{\text{int.}(2)}_{++} + 2i\partial_- \mathcal{R}^{\text{int.}(0)}_{++}\right) &= 0. \end{aligned} \tag{4.3.52}$$

The general arguments from [31] presented in Section 4.3.4 then imply that the lowest component $-16(T^{\text{int.}}_{--} - \frac{i}{2}\partial_- j^{\text{int.}}_-)$ of the first equation is a non-trivial $\bar{Q}_+$-cohomology element and the lowest component $-16(T^{\text{int.}}_{-+} - \frac{i}{2}\partial_- j^{\text{int.}}_+)$ of the second equation is a trivial $\bar{Q}_+$-cohomology element in the integrated 2D theory. In fact, a stronger statement holds: The integrated multiplets are genuine 2D $\mathcal{N} = (0, 2)$ supersymmetry multiplets. Setting

$$\begin{aligned} \mathcal{R}^{(0,2)}_{\alpha\alpha} &:= \mathcal{R}^{\text{int.}(0)}_{\alpha\alpha}, \\ \mathcal{T}^{(0,2)}_{----} &:= -\mathcal{R}^{\text{int.}(2)}_{--}, \end{aligned} \tag{4.3.53}$$

the constraints (4.3.16)–(4.3.19) and (4.3.28)–(4.3.31) imply that these are indeed 2D (0, 2)-supercurrent multiplets in the sense of [38, 31].

4.3.5 Summarising the results on boundary multiplets

We look at three-dimensional theories with $\mathcal{N}=2$ supersymmetry, broken to a 2D (0, 2)-subalgebra due to a boundary. Currents associated to (remaining) symmetries now consist of bulk and boundary pieces.

First we study 3D bulk supercurrent multiplets, in particular the $\mathcal{R}$-multiplet. Its structure remains unchanged, as the bulk parts of conserved currents are still divergence-free; we merely decompose the bulk multiplets into their (0, 2)-submultiplets

$$\begin{aligned}\mathcal{R}^B_\mu(x,\theta,\bar\theta) &= \mathcal{R}^{B(0)}_\mu + \theta^-\mathcal{R}^{B(1)}_\mu - \bar\theta^-\mathcal{R}^{\bar B(1)}_\mu + \theta^-\bar\theta^-\mathcal{R}^{B(2)}_\mu,\\ \chi^B_\alpha(x,\theta,\bar\theta) &= \chi^{B(0)}_\alpha + \theta^-\chi^{B(1a)}_\alpha + \bar\theta^-\chi^{B(1b)}_\alpha + \theta^-\bar\theta^-\chi^{B(2)}_\alpha.\end{aligned} \tag{4.3.54}$$

The defining constraints (4.3.7) now decompose under the $\mathcal{N}=(0,2)$-subalgebra into equations (4.3.16)–(4.3.19).

We investigate possible defining constraints for the boundary parts, using as guiding principles that

- bulk and boundary pieces combine into a full multiplet $\mathcal{R}^{\text{full}}_\mu = \mathcal{R}^B_\mu + \delta(\xi^\perp)\mathcal{P}_\mu{}^{\hat\mu}\mathcal{R}^\partial_{\hat\mu}$, where the boundary pieces are also decomposed as in (4.3.24) and (4.3.25), hence the boundary constraints must be of the same form as the bulk constraints (4.3.16)–(4.3.19),
- boundary constraints must impose boundary conservation (4.2.28) on the *remaining, conserved* boundary currents.

We obtain the following list of constraints:

$$0 = \chi^{\partial(2)}_- + 2i\partial_-\chi^{\partial(0)}_-, \tag{4.3.55a}$$
$$0 = \bar D_+\chi^{\partial(1a)}_+ - 2i\chi^{B(0)}_+|_\partial, \tag{4.3.55b}$$
$$0 = \operatorname{Im}\Big(D_+\chi^{\partial(0)}_- - \chi^{\partial(1a)}_+\Big), \tag{4.3.55c}$$
$$0 = \operatorname{Im}(D_+\chi^{\partial(2)}_- - 2i\partial_-\chi^{\partial(1a)}_+ + 2i\chi^{B(1a)}_-|_\partial), \tag{4.3.55d}$$
$$\chi^{\partial(0)}_- = \bar D_+\mathcal{R}^{\partial(0)}_{--}, \tag{4.3.55e}$$
$$\chi^{\partial(1a)}_+ = -\mathcal{R}^{\partial(2)}_{++} + 2i\mathcal{R}^{B(0)}_{+-}|_\partial - 2i\partial_-\mathcal{R}^{\partial(0)}_{++}, \tag{4.3.55f}$$
$$\chi^{\partial(1a)}_- = 2i\mathcal{R}^{B(0)}_{--}|_\partial, \tag{4.3.55g}$$
$$\chi^{\partial(2)}_- = \bar D_+\mathcal{R}^{\partial(2)}_{--} - 2i\mathcal{R}^{\bar B(1)}_{--}|_\partial, \tag{4.3.55h}$$
$$\chi^{\partial(2)}_+ = -2i\mathcal{R}^{\bar B(1)}_{+-}|_\partial, \tag{4.3.55i}$$
$$\chi^{\partial(0)}_+ = 0, \tag{4.3.55j}$$
$$\chi^{\partial(1b)}_\alpha = 0. \tag{4.3.55k}$$

The constraints (4.3.55a)–(4.3.55h) are necessary to derive boundary conservation equations and equations (4.3.43). The last three relations (4.3.55i)–(4.3.55k) are not used in any conservation equation and are independent of the rest of the constraints.

These constraints imply, in particular, equations (4.3.46) on full currents:

$$\bar D_+\Big(\mathcal{R}^{\text{full}(2)}_{\pm\pm} + 2i\partial_-\mathcal{R}^{\text{full}(0)}_{\pm\pm}\Big) = -2i\partial_\perp\mathcal{R}^{\bar B(1)}_{\pm\pm} + 2i\delta(\xi^\perp)\mathcal{R}^{\bar B(1)}_{\pm\pm}|_\partial, \tag{4.3.56}$$

which motivates the introduction of integrated current multiplets

$$\mathcal{R}^{\text{int.}}_{\alpha\alpha} = \mathcal{R}^{\partial}_{\alpha\alpha} + \int \mathrm{d}x^{\perp} \mathcal{R}^{B}_{\alpha\alpha}. \tag{4.3.57}$$

These multiplets are genuine 2D $\mathcal{N} = (0, 2)$ supercurrent multiplets in the usual sense. In particular, the integration sets the right-hand side of relation (4.3.56) to zero. This implies that $T^{\text{int.}}_{--} - \frac{i}{2}\partial_- j^{\text{int.}}_{-}$ is a non-trivial $\bar{Q}_+$-cohomology element and $T^{\text{int.}}_{-+} - \frac{i}{2}\partial_- j^{\text{int.}}_{+}$ is a trivial $\bar{Q}_+$-cohomology element in the effective (integrated) 2D theory.

4.4 Three-dimensional Landau–Ginzburg models

4.4.1 Bulk theory

We now study a particular model where the framework we developed above can be applied. Our bulk theory should be a 3D $\mathcal{N} = 2$ Landau–Ginzburg model which lives on three-dimensional Minkowski space. At a later point, we will introduce a boundary and restrict the theory to the half-space

$$M = \{x \in \mathbb{R}^{1,2} \mid x^{\perp} := x^1 \leq 0\}. \tag{4.4.1}$$

We will formulate the bulk theory in 3D $\mathcal{N} = 2$ superspace. A generic chiral field is given by

$$\Phi_{3D}(x, \theta, \overline{\theta}) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y), \qquad y^{\mu} = x^{\mu} - i\theta\gamma^{\mu}\overline{\theta}, \tag{4.4.2}$$

where, as usual, ϕ is a complex scalar field, ψ_α is a complex fermion, and F is a complex auxiliary field. Under the bulk supersymmetry, the components transform as follows:

$$\begin{aligned} \delta_{\text{sym}}\phi &= \sqrt{2}\epsilon\psi, \\ \delta_{\text{sym}}\psi_\alpha &= \sqrt{2}\epsilon_\alpha F - \sqrt{2}i(\gamma^{\mu}\bar{\epsilon})_\alpha \partial_\mu \phi, \\ \delta_{\text{sym}}F &= -\sqrt{2}i\bar{\epsilon}\gamma^{\mu}\partial_\mu\psi. \end{aligned} \tag{4.4.3}$$

Let us now consider the simplest non-trivial theory: a Landau–Ginzburg model of a single chiral superfield. Its kinetic (Kähler) term is given by

$$\mathcal{L}_{\text{kin.}} = \int \mathrm{d}^4\theta\, \Phi_{3D}\bar{\Phi}_{3D} = -\partial_\mu\bar{\phi}\partial^\mu\phi + \tfrac{i}{2}(\bar{\psi}\gamma^\mu\partial_\mu\psi) - \tfrac{i}{2}(\partial_\mu\bar{\psi}\gamma^\mu\psi) + \bar{F}F + \tfrac{1}{4}\partial^2(\bar{\phi}\phi). \tag{4.4.4}$$

At a later point, the term $\frac{1}{4}\partial^2(\bar{\phi}\phi)$ will be removed; as it is a total derivative, it does not influence the bulk theory, but will be relevant once a boundary is introduced.
The superpotential is of the well-known form

$$\mathcal{L}_W = \int \mathrm{d}^2\theta\, W(\Phi_{3D}) + \text{cc.} = W'(\phi)F - \tfrac{1}{2}W''(\phi)\psi\psi + \text{cc.} \tag{4.4.5}$$

The bulk equations of motion are given by

$$\bar{D}^2\bar{\Phi} = -4W'(\Phi) \Leftrightarrow \begin{Bmatrix} 0 = \bar{F} + W'(\phi) \\ 0 = \partial_\mu\partial^\mu\bar{\phi} + W''(\phi)F - \frac{1}{2}W'''(\phi)\psi\psi \\ 0 = i(\gamma^\mu\partial_\mu\bar{\psi})_\alpha - W''(\phi)\psi_\alpha \end{Bmatrix}. \tag{4.4.6}$$

4.4.2 Introducing a boundary and breaking to (0, 2)

As we already saw in Section 4.3.3, the supersymmetry algebra breaks at least to a 2D $\mathcal{N}=(0,2)$ (or 2D $\mathcal{N}=(1,1)$, which we do not consider here) subalgebra when we introduce a boundary. Let us study sufficient conditions to preserve exactly $\mathcal{N}=(0,2)$.

Decomposition of the bulk fields

Under the $\mathcal{N}=(0,2)$ subalgebra, the chiral field Φ_{3D} decomposes into a (0, 2) chiral multiplet and a Fermi multiplet. More details of this decomposition are written in Appendix A.7.1. The resulting (0, 2)-superfields are

$$\begin{aligned}\Phi &= \phi + \sqrt{2}\theta^+\psi_+ - 2i\theta^+\bar{\theta}^+\partial_+\phi,\\ \Psi &= \psi_- - \sqrt{2}\theta^+ F - 2i\theta^+\bar{\theta}^+\partial_+\psi_- + \sqrt{2}i\bar{\theta}^+\partial_\perp\phi - 2i\theta^+\bar{\theta}^+\partial_\perp\psi_+.\end{aligned} \tag{4.4.7}$$

The Fermi superfield satisfies

$$\bar{D}_+\Psi = \sqrt{2}E_\Psi, \quad E_\Psi = -i\partial_\perp\Phi, \tag{4.4.8}$$

and the chirality condition reads

$$\bar{D}_+\Phi = 0. \tag{4.4.9}$$

The supersymmetry variation in the smaller algebra is given by $\delta_{\text{sym}} := \epsilon Q_+ - \bar{\epsilon}\bar{Q}_+$ (see Eq. (A.7.2) for the definition of the superspace operators). The component fields now transform as

$$\begin{aligned}\delta_{\text{sym}}\phi &= \sqrt{2}\epsilon\psi_+, & \delta_{\text{sym}}\psi_+ &= -2\sqrt{2}i\bar{\epsilon}\partial_+\phi,\\ \delta_{\text{sym}}F &= 2\sqrt{2}i\bar{\epsilon}\partial_+\psi_- + \sqrt{2}i\bar{\epsilon}\partial_\perp\psi_+, & \delta_{\text{sym}}\psi_- &= -\sqrt{2}\epsilon F + \sqrt{2}i\bar{\epsilon}\partial_\perp\phi,\end{aligned} \tag{4.4.10}$$

which is precisely the restriction of (4.4.3) to the (0, 2)-subalgebra, given by choosing $\epsilon^\alpha = \left(\begin{smallmatrix}0\\ \epsilon\end{smallmatrix}\right)$. We may rewrite the Lagrangian in terms of (0, 2)-superspace:

$$\begin{aligned}\mathcal{L}_{\text{kin.}} &= \frac{1}{2}\int \mathrm{d}^2\theta^+\Big[i\bar{\Phi}\partial_-\Phi - i\partial_-\bar{\Phi}\Phi + \bar{\Psi}\Psi\\ &\qquad + \partial_\perp\left(\tfrac{1}{2}\theta^+\bar{\theta}^+\partial_\perp(\bar{\Phi}\Phi) + \tfrac{i}{\sqrt{2}}\theta^+\bar{\Phi}\Psi + \tfrac{i}{\sqrt{2}}\bar{\theta}^+\bar{\Psi}\Phi\right)\Big],\\ \mathcal{L}_W &= -\tfrac{1}{\sqrt{2}}\int \mathrm{d}\theta^+\Psi W'(\Phi) + \text{cc.}\end{aligned} \tag{4.4.11}$$

Note that $\mathcal{L}_{\text{kin.}}$ consists of two parts. The first part is invariant under (0, 2)-supersymmetry even in the presence of a boundary, as its (0, 2)-variation is just a total x^+-derivative. The second term (trivially) transforms into an $x^\perp$-derivative, so it breaks (0, 2)-supersymmetry in the presence of a boundary, and hence dictates part of the "boundary compensating term" (cf. discussion at the end of Section 4.2.2).

The equations of motion may again be written as superfield equations, now in (0, 2)-superspace:

$$\begin{aligned}0 &= 2i\partial_-\bar{D}_+\bar{\Phi} + \sqrt{2}i\partial_\perp\bar{\Psi} - \sqrt{2}W''(\Phi)\Psi,\\ 0 &= \bar{D}_+\bar{\Psi} + \sqrt{2}W'(\Phi).\end{aligned} \tag{4.4.12}$$

Recovering partial supersymmetry

As it stands, the pure bulk action (4.4.11) is not even (0, 2)-supersymmetric in the presence of a boundary:

$$\delta_{\text{sym}} S = \int_M \delta_{\text{sym}}(\mathcal{L}_{\text{kin.}} + \mathcal{L}_W) = \int_M \partial_\mu (V^\mu_{\text{kin.}} + V^\mu_W) = \int_{\partial M} (V^\perp_{\text{kin.}} + V^\perp_W), \tag{4.4.13}$$

which, in general, does not vanish. To recover at least $\mathcal{N} = (0, 2)$ supersymmetry, we must compensate these bulk variations.

For the kinetic term, we can add a boundary compensating term $\widetilde{\Delta}_{\text{kin.}}$ to the boundary Lagrangian (in a boundary-condition-independent way) in the spirit of [10, 32]. The boundary term is precisely minus the total $\perp$-derivative from the bulk Lagrangian in (0, 2)-superspace (4.4.11):

$$\widetilde{\Delta}_{\text{kin.}} := -\tfrac{1}{4}\partial_\perp(\bar{\phi}\phi) - \tfrac{i}{2}\bar{\psi}_+\psi_- + \tfrac{i}{2}\bar{\psi}_-\psi_+. \tag{4.4.14}$$

We see that the $-\frac{1}{4}\partial_\perp(\bar{\phi}\phi)$ cancels the bulk total derivative in $x^\perp$ direction when pulled into the bulk. This means that we can just drop $\frac{1}{4}\partial^2(\bar{\phi}\phi)$ from the bulk and $-\frac{1}{4}\partial_\perp(\bar{\phi}\phi)$ from the boundary simultaneously, leaving us with bulk and boundary Lagrangians:

$$\begin{aligned} \mathcal{L}^B &= -\partial_\mu\bar{\phi}\partial^\mu\phi + \tfrac{i}{2}(\psi\gamma^\mu\partial_\mu\bar{\psi}) - \tfrac{i}{2}(\partial_\mu\psi\gamma^\mu\bar{\psi}) \\ &\quad + \bar{F}F + W'(\phi)F + \bar{W}'(\bar{\phi})\overline{F} - \tfrac{1}{2}W''(\phi)\psi\psi + \tfrac{1}{2}\bar{W}''(\bar{\phi})\bar{\psi}\bar{\psi}, \\ \Delta_{\text{kin.}} &= -\tfrac{i}{2}\bar{\psi}_+\psi_- + \tfrac{i}{2}\bar{\psi}_-\psi_+ = -\tfrac{i}{2}\bar{\psi}\psi. \end{aligned} \tag{4.4.15}$$

For the bulk superpotential term, the supersymmetry variation yields

$$\delta_{\text{sym}}\mathcal{L}_W = \partial_\perp\Big(-i\int \mathrm{d}\theta^+\bar{\epsilon}W(\Phi) + \text{cc.}\Big) + \partial_+(\dots) = \partial_\perp(-i\bar{\epsilon}\psi_+W'(\phi) + \text{cc.}) + \partial_+(\dots), \tag{4.4.16}$$

where the right-hand side needs to be compensated. To do this in a boundary-condition-independent way, one can use a bulk R-symmetry (see [32]), or one can add boundary degrees of freedom, which we will discuss in detail.

Boundary Fermi multiplet and factorisation

To compensate the superpotential term variation (4.4.16), we introduce a 2D boundary Fermi multiplet with E- and J-potential terms, analogously to [62, 10, 55, 70], where a 1D Fermi multiplet was used to compensate bulk 2D superpotential terms (see also [95, 47] for equivalent, three-dimensional examples). The general superspace expansion of a 2D Fermi multiplet is given by

$$H = \eta - \sqrt{2}\theta^+ G - 2i\theta^+\overline{\theta}^+\partial_+\eta - \sqrt{2}\overline{\theta}^+ E(\phi) + 2\theta^+\overline{\theta}^+ E'(\phi)\psi_+, \tag{4.4.17}$$

so it has an E-potential of

$$\bar{D}_+ H = \sqrt{2}E(\Phi). \tag{4.4.18}$$

The (0, 2)-supersymmetry variation of the components is given by

$$\begin{aligned} \delta_{\text{sym}}\eta &= -\sqrt{2}(\epsilon G + \bar{\epsilon}E), \\ \delta_{\text{sym}}G &= \sqrt{2}\bar{\epsilon}(2i\partial_+\eta - E'\psi_+). \end{aligned} \tag{4.4.19}$$

Its (boundary) Lagrangian is

$$\begin{aligned}\mathcal{L}_H &= \int \mathrm{d}^2\theta^+ \tfrac{1}{2}\bar{H}H - \int \mathrm{d}\theta^+ \tfrac{i}{\sqrt{2}} J(\Phi)H + \int \mathrm{d}\bar{\theta}^+ \tfrac{i}{\sqrt{2}} \bar{J}(\bar{\Phi})\bar{H} \\ &= i\bar{\eta}\partial_+\eta - i\partial_+\bar{\eta}\eta - E'\bar{\eta}\psi_+ - \bar{E}'\bar{\psi}_+\eta + iJ'\eta\psi_+ - i\bar{J}'\bar{\psi}_+\bar{\eta} - |E|^2 - |J|^2.\end{aligned} \tag{4.4.20}$$

It consists of a kinetic term, two boundary potentials E and J of the *bulk* chiral field ϕ, and interactions between the boundary and bulk fermions. The boundary equations of motion are

$$\bar{D}_+\bar{H} + \sqrt{2}iJ(\Phi) = 0 \Leftrightarrow \begin{Bmatrix} G = i\bar{J} \\ 2i\partial_+\eta = E'(\phi)\psi_+ - i\bar{J}'(\bar{\phi})\bar{\psi}_+ \end{Bmatrix}. \tag{4.4.21}$$

The supersymmetry variation is

$$\delta_{\mathrm{sym}}\mathcal{L}_H = i\int \mathrm{d}\theta^+ \bar{\epsilon} J(\Phi)E(\Phi) + \mathrm{cc.} + \partial_+(\dots). \tag{4.4.22}$$

We thus find that in case of a *matrix factorisation*

$$W(\Phi)|_\partial = E(\Phi)J(\Phi)|_\partial, \tag{4.4.23}$$

the bulk term from (4.4.16) will be compensated precisely, and (0, 2)-supersymmetry is preserved. As stated before [92] and can be seen from (4.4.22), a pure 2D $\mathcal{N}=(0,2)$ theory must fulfil $E \cdot J = 0$ in order to preserve supersymmetry. However, in our case, the "failure" of the boundary Fermi multiplet to meet this condition cancels the failure of the bulk theory to preserve $\mathcal{N}=(0,2)$-supersymmetry at the boundary.

The total action of the factorised Landau–Ginzburg model then reads

$$\begin{aligned} S &= \int_M \mathcal{L}^B + \int_{\partial M} \mathcal{L}^\partial \\ &= \frac{1}{2}\int_M \left\{ \int \mathrm{d}^2\theta^+ \left[i\bar{\Phi}\partial_-\Phi - i\partial_-\bar{\Phi}\Phi + \bar{\Psi}\Psi + \partial_\perp \Delta \right] - \sqrt{2}\int \mathrm{d}\theta^+ \Psi W(\Phi) + \mathrm{cc.} \right\} \\ &\quad + \frac{1}{2}\int_{\partial M} \left\{ \int \mathrm{d}^2\theta^+ \left[-\Delta|_\partial + \bar{H}H \right] - \sqrt{2}i \int \mathrm{d}\theta^+ J(\Phi)H + \mathrm{cc.} \right\}, \end{aligned} \tag{4.4.24}$$

where $\frac{1}{2}\int \mathrm{d}^2\theta^+ \Delta = \frac{i}{2\sqrt{2}}\int \mathrm{d}^2\theta^+ (\theta^+\bar{\Phi}\Psi + \bar{\theta}^+\bar{\Psi}\Phi) = \frac{i}{2}(\bar{\psi}_+\psi_- - \bar{\psi}_-\psi_+)$ (cf. (4.4.15)). After using the algebraic equations of motion, we get the following component expansions:

$$\begin{aligned} \mathcal{L}^B &= -\partial_\mu\bar{\phi}\partial^\mu\phi + \tfrac{i}{2}(\psi\gamma^\mu\partial_\mu\bar{\psi}) - \tfrac{i}{2}(\partial_\mu\psi\gamma^\mu\bar{\psi}) \\ &\quad - |W(\phi)|^2 - \tfrac{1}{2}W''(\phi)\psi\psi + \tfrac{1}{2}\bar{W}''(\bar{\phi})\bar{\psi}\bar{\psi}, \\ \mathcal{L}^\partial &= i\bar{\eta}\partial_+\eta - i\partial_+\bar{\eta}\eta - |J|^2 - |E|^2 - \bar{E}'\bar{\psi}_+\eta - E'\bar{\eta}\psi_+ \\ &\quad - iJ'\psi_+\eta - i\bar{J}'\bar{\psi}_+\bar{\eta} - \tfrac{i}{2}(\bar{\psi}_+\psi_- - \bar{\psi}_-\psi_+)|_\partial. \end{aligned} \tag{4.4.25}$$

The (0, 2)-variation of the total action is zero, hence $\mathcal{N}=(0,2)$ supersymmetry is preserved in a boundary-condition-independent way.

Symmetric boundary conditions

In every theory with a boundary it is necessary to introduce boundary conditions such that the action can be made stationary. Requiring the boundary condition to be compatible with the $\mathcal{N} = (0, 2)$ subalgebra in the sense of (4.2.8) further restricts the number of options. We now discuss some explicit boundary conditions for our LG model. We consider boundary conditions without superpotential (previously discussed in [33]) and with superpotential separately.

Without superpotential

- *(generalised) Dirichlet*: $\Phi = 0$ or more generally $\Phi = c$ (in components $\phi = c$ and $\psi_+ = 0$) is a symmetric boundary condition (the action may require the addition of some boundary terms to be symmetric).
- *Neumann*: $\Psi = 0$ (in components $\partial_\perp \phi = 0$ and $\psi_- = 0$) is also symmetric. It is also the dynamical boundary condition in the sense of (4.2.7) for the action (4.4.15) without superpotential. Note that one can also obtain the (generalised) Dirichlet as a dynamical boundary condition by adding appropriate boundary terms [33].
- *Mixed conditions*: In models with more than one 3D chiral superfield, we may assign Dirichlet conditions to some and Neumann conditions to others [33].

With superpotential

- *(generalised) Dirichlet*: Setting $\Phi = c$ is symmetric and also statically cancels the supervariation of the potential (4.4.16) (albeit in a boundary-condition-*dependent* way). However, if $W'(c)|_\partial \neq 0$, supersymmetry is broken spontaneously, as the vacuum expectation value of ψ_- then transforms non-trivially under supersymmetry.[11]
- *Mixed conditions*: Setting $\Psi = 0$ (Neumann) is only symmetric if $W'(\phi)|_\partial = 0$. For one bosonic field, this holds only if $W = 0$, as ϕ is unconstrained on the boundary. If $W \neq 0$ and the theory has more than one chiral superfield, one can assign Dirichlet conditions to some and Neumann conditions to others while maintaining supersymmetry (a requirement the authors in [33] call "sufficiently Dirichlet").
- *Factorised Neumann*: If we introduce additional degrees of freedom on the boundary as in Section 4.4.2, we may again choose dynamical boundary conditions. In the case without superpotential this lead to the Neumann boundary condition. For the action (4.4.25) the dynamical boundary condition is the analogue of the Neumann boundary condition, now with superpotential:

$$\bar{\Psi} = -i\bar{H}E'(\Phi) - HJ'(\Phi) \Leftrightarrow \begin{Bmatrix} \bar{\psi}_- = -i\bar{\eta}E' - \eta J', \\ \partial_\perp \bar{\phi} = -\bar{E}E' - \bar{J}J' - (\bar{\eta}E'' - i\eta J'')\psi_+ \end{Bmatrix}. \tag{4.4.26}$$

  One can check that it is indeed symmetric if the factorisation condition (4.4.23) is met. We use this boundary condition in our computations for currents and current multiplets.

[11]We note that if there is a bulk R-symmetry, one can also compensate the superpotential variation boundary-condition-independently, see [32]. However, in the case of one chiral field, one can then only impose Dirichlet boundary conditions, as the Neumann condition with $W \neq 0$ is not symmetric.

This choice of boundary condition in fact encodes a collection of boundary conditions labelled by the choices of *matrix factorisations* of W (since the boundary condition depends explicitly on E and J).

4.4.3 Currents

Here we present conserved currents associated to the symmetries of the Landau–Ginzburg theory with one chiral field in the bulk, a Fermi multiplet on the boundary, and factorised Neumann boundary conditions. We will compute the currents in various improvement frames in order to place them into consistent multiplets in the following subsection.

R-current

If the superpotentials W, E, J are (quasi-)homogeneous functions of Φ — in the case of one chiral field, monomials —, then the action is invariant under the R-symmetry transformation[12]

$$\begin{aligned} \theta^+ &\mapsto e^{-i\tau}\theta^+, & \Phi &\mapsto e^{-2i\tau\alpha}\Phi, \\ \Psi &\mapsto e^{-i\tau(2\alpha-1)}\Psi, & H &\mapsto e^{-i\tau(\ell_E-\ell_J)\alpha}H, \end{aligned} \tag{4.4.27}$$

where τ is the symmetry variation parameter and we have defined

$$\alpha \coloneqq (\deg W)^{-1}, \quad \ell_E \coloneqq \deg E, \quad \ell_J \coloneqq \deg J. \tag{4.4.28}$$

Note that factorisation implies

$$\alpha(\ell_E+\ell_J)=1. \tag{4.4.29}$$

The bulk contribution to the R-current is given by

$$j^B_\mu = 2i\alpha(\bar\phi\partial_\mu\phi - \partial_\mu\bar\phi\phi) + (1-2\alpha)\bar\psi\gamma_\mu\psi, \tag{4.4.30}$$

while the boundary contribution is given by

$$j^\partial_\mu = \begin{pmatrix} j^\partial_+ \\ j^\partial_- \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha(\ell_E-\ell_J)\bar\eta\eta \end{pmatrix}. \tag{4.4.31}$$

Supercurrents

After introducing the boundary (with the aforementioned choices), only $(0,2)$-supersymmetry is preserved. We may however still discuss the full 3D $\mathcal{N}=2$ supersymmetry in the bulk, as the $(0,2)$-restrictions of the bulk currents remain identical (and covariant notation can be conveniently used).

Noether frame ($\mathcal{S}$-frame) The bulk supercurrent induced by $\delta_{\text{sym}}=\epsilon Q$ (full supersymmetry is $\delta_{\text{sym}}=\epsilon Q-\bar\epsilon\bar Q$) in the Noether frame is given by:[13]

$$(S^B_\mu)_\alpha = \sqrt{2}(\gamma^\nu\gamma_\mu\psi)_\alpha\partial_\nu\bar\phi - \sqrt{2}i(\gamma_\mu\bar\psi)_\alpha\bar W'. \tag{4.4.32}$$

[12] For multiple chiral fields Φ_i, the condition for quasi-homogeneity reads $W(\Phi_1,\dots,\Phi_k)=\sum_i\alpha_i\Phi_i\partial_{\Phi_i}W$ for some choice of R-charges α_i.

[13] One finds this supercurrent by applying Noether's theorem to (4.4.15) and improving the boundary part to zero.

Its (0, 2)-restriction $\delta_{\text{sym}} = \epsilon^+ Q_+$ is given by setting $\alpha = +$:

$$(S^B_\mu)_+ = \begin{pmatrix} (S^B_+)_+ \\ (S^B_-)_+ \\ (S^B_\perp)_+ \end{pmatrix} = \begin{pmatrix} 2\sqrt{2}\psi_+\partial_+\bar{\phi} \\ -\sqrt{2}(\psi_-\partial_\perp\bar{\phi} + i\bar{\psi}_-\bar{W}') \\ \sqrt{2}(\psi_+\partial_\perp\bar{\phi} - 2\psi_-\partial_+\bar{\phi} + i\bar{\psi}_+\bar{W}') \end{pmatrix}. \tag{4.4.33}$$

The boundary contribution is induced by $\delta_{\text{sym}} = \epsilon^+ Q_+$ and reads in the Noether frame:

$$(S^\partial_{\hat{\mu}})_+ = \begin{pmatrix} (S^\partial_+)_+ \\ (S^\partial_-)_+ \end{pmatrix} = \begin{pmatrix} 0 \\ -\sqrt{2}(\bar{J}\bar{\eta} - i\bar{E}\eta) \end{pmatrix}. \tag{4.4.34}$$

$\mathcal{R}$-frame If the Lagrangian has an R-symmetry (4.4.27), we may improve the above supercurrent to a supercurrent which is part of the $\mathcal{R}$-multiplet. We call this improvement frame the $\mathcal{R}$-frame. The bulk components are:

$$\begin{aligned} (S^B_\mu)^{\mathcal{R}}_\alpha &= (S^B_\mu)^{\mathcal{S}}_\alpha - 2\sqrt{2}\alpha\epsilon_{\mu\nu\rho}(\gamma^\nu\partial^\rho(\bar{\phi}\psi))_\alpha \\ &= \sqrt{2}(1-2\alpha)\big((\gamma^\nu\gamma_\mu\psi)_\alpha\partial_\nu\bar{\phi} + i(\gamma_\mu\bar{\psi})_\alpha\bar{W}'\big) + 2\sqrt{2}\alpha(\partial_\mu\bar{\phi}\psi_\alpha - \bar{\phi}\partial_\mu\psi_\alpha), \end{aligned} \tag{4.4.35}$$

where $(S^B_\mu)^{\mathcal{S}}_\alpha$ denotes the supercurrent in the Noether frame, $\alpha = (\deg W)^{-1}$ and the last equality uses equations of motion (4.4.6) and homogeneity of W.

The boundary components are

$$(S^\partial_{\hat{\mu}})^{\mathcal{R}}_+ = (S^\partial_{\hat{\mu}})^{\mathcal{S}}_+ + 2\sqrt{2}\alpha\epsilon_{\hat{\mu}\nu n}(\gamma^\nu\psi)_+\bar{\phi} = \begin{pmatrix} 0 \\ \sqrt{2}\alpha(\ell_J - \ell_E)(\bar{J}\bar{\eta} + i\bar{E}\eta) \end{pmatrix}, \tag{4.4.36}$$

where the last equality uses boundary conditions (4.4.26).

Energy-momentum tensor

Similarly to the case of supercurrents, we stick to covariant notation for the bulk pieces, even though certain directions are no longer symmetries. Let us start by simplifying the Lagrangians (4.4.25) on-shell:[14]

$$\mathcal{L}^B \overset{\text{on-shell}}{=} -\partial_\rho\bar{\phi}\partial^\rho\phi - |W'|^2, \tag{4.4.37}$$

$$\mathcal{L}^\partial \overset{\text{on-shell}}{=} -|E|^2 - |J|^2. \tag{4.4.38}$$

Noether frame ($\mathcal{S}$-frame) Using the Noether procedure, in the bulk we find the non-symmetric energy-momentum tensor

$$\widehat{T}^B_{\mu\nu} = \partial_\mu\bar{\phi}\partial_\nu\phi + \partial_\nu\bar{\phi}\partial_\mu\phi + \tfrac{i}{2}\partial_\mu\bar{\psi}\gamma_\nu\psi - \tfrac{i}{2}\bar{\psi}\gamma_\nu\partial_\mu\psi - \eta_{\mu\nu}(\partial_\rho\bar{\phi}\partial^\rho\phi + |W'|^2), \tag{4.4.39}$$

[14] Note that the second equation also uses boundary conditions (4.4.26). Without using them, we get

$$\mathcal{L}^\partial \overset{\text{on-shell}}{=} -|E|^2 - |J|^2 - \tfrac{1}{2}(\bar{E}'\bar{\psi}_+\eta + E'\bar{\eta}\psi_+ + iJ'\psi_+\eta + i\bar{J}'\bar{\psi}_+\bar{\eta}) - \tfrac{i}{2}(\bar{\psi}_+\psi_- - \bar{\psi}_-\psi_+)|_\partial.$$

and in the boundary we find (using equations of motion but *not* boundary conditions)

$$\begin{aligned}
\widehat{T}^{\partial}_{++} &= 0,\\
\widehat{T}^{\partial}_{--} &= \tfrac{i}{2}\bar{\eta}\partial_-\eta - \tfrac{i}{2}\partial_-\bar{\eta}\eta,\\
\widehat{T}^{\partial}_{+-} &= \tfrac{i}{4}\bar{\psi}_+\psi_- - \tfrac{i}{4}\bar{\psi}_-\psi_+ + \tfrac{1}{2}|E|^2 + \tfrac{1}{2}|J|^2\\
&\quad + \tfrac{1}{2}(E'\bar{\eta}\psi_+ + \bar{E}'\bar{\psi}_+\eta - iJ'\eta\psi_+ + i\bar{J}'\bar{\psi}_+\bar{\eta}),\\
\widehat{T}^{\partial}_{-+} &= \tfrac{i}{4}\bar{\psi}_+\psi_- - \tfrac{i}{4}\bar{\psi}_-\psi_+ + \tfrac{1}{2}|E|^2 + \tfrac{1}{2}|J|^2\\
&\quad + \tfrac{1}{4}(E'\bar{\eta}\psi_+ + \bar{E}'\bar{\psi}_+\eta - iJ'\eta\psi_+ + i\bar{J}'\bar{\psi}_+\bar{\eta}).
\end{aligned} \tag{4.4.40}$$

If we utilise the boundary conditions (4.4.26), the expressions simplify to

$$\begin{aligned}
\widehat{T}^{\partial}_{++} &= 0,\\
\widehat{T}^{\partial}_{--} &= \tfrac{i}{2}\bar{\eta}\partial_-\eta - \tfrac{i}{2}\partial_-\bar{\eta}\eta,\\
\widehat{T}^{\partial}_{+-} &= \tfrac{i}{2}\bar{\eta}\partial_+\eta - \tfrac{i}{2}\partial_+\bar{\eta}\eta + \tfrac{1}{2}(|E|^2 + |J|^2),\\
\widehat{T}^{\partial}_{-+} &= \tfrac{1}{2}(|E|^2 + |J|^2).
\end{aligned} \tag{4.4.41}$$

Symmetrisation These can by made symmetric using an improvement. In the bulk we find

$$\begin{aligned}
T^B_{\mu\nu} &= \widehat{T}^B_{\mu\nu} - \tfrac{1}{8}\epsilon_{\mu\nu\rho}H^\rho\\
&= (\partial_\mu\bar{\phi}\partial_\nu\phi + \partial_\nu\bar{\phi}\partial_\mu\phi) - \eta_{\mu\nu}(|\partial\phi|^2 + |W'|^2) + \tfrac{i}{2}(\partial_{(\mu}\bar{\psi}\gamma_{\nu)}\psi) - \tfrac{i}{2}(\bar{\psi}\gamma_{(\nu}\partial_{\mu)}\psi),
\end{aligned} \tag{4.4.42}$$

where $H^\rho = -2i\partial^\rho(\bar{\psi}\psi)$.[15] The induced boundary improvement is $T^{\partial}_{\hat{\mu}\hat{\nu}} = \widehat{T}^{\partial}_{\hat{\mu}\hat{\nu}} - \frac{i}{4}\epsilon_{\hat{\mu}\hat{\nu}n}\bar{\psi}\psi|_\partial$, so

$$\begin{aligned}
T^{\partial}_{++} &= 0,\\
T^{\partial}_{--} &= \tfrac{i}{2}\bar{\eta}\partial_-\eta - \tfrac{i}{2}\partial_-\bar{\eta}\eta,\\
T^{\partial}_{+-} &= \tfrac{i}{2}\bar{\eta}\partial_+\eta - \tfrac{i}{2}\partial_+\bar{\eta}\eta + \tfrac{1}{2}(|E|^2 + |J|^2) - \tfrac{i}{8}(\bar{\psi}_-\psi_+ - \bar{\psi}_+\psi_-)|_\partial,\\
T^{\partial}_{-+} &= \tfrac{1}{2}(|E|^2 + |J|^2) + \tfrac{i}{8}(\bar{\psi}_-\psi_+ - \bar{\psi}_+\psi_-)|_\partial.
\end{aligned} \tag{4.4.43}$$

Note that using boundary conditions (4.4.26) and equations of motion for η (4.4.21) we find that

$$\tfrac{i}{2}(\bar{\psi}_-\psi_+ - \bar{\psi}_+\psi_-)|_\partial = i\bar{\eta}\partial_+\eta - i\partial_+\bar{\eta}\eta, \tag{4.4.44}$$

which shows that the boundary components are symmetric modulo boundary conditions in this frame as well.

***$\mathcal{R}$*-frame** Again, as in the case of the supercurrent, there is an improved energy-momentum tensor in the $\mathcal{R}$-frame. We find

$$\begin{aligned}
(T^B_{\mu\nu})^{\mathcal{R}} &= (T^B_{\mu\nu})^{\mathcal{S}} + \tfrac{1}{2}[\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2](-2\alpha\bar{\phi}\phi)\\
&= (1-\alpha)(\partial_\nu\phi\partial_\mu\bar{\phi} + \partial_\mu\phi\partial_\nu\bar{\phi}) - \alpha(\partial_\mu\partial_\nu\bar{\phi}\phi + \bar{\phi}\partial_\mu\partial_\nu\phi) + \tfrac{i}{2}(\partial_{(\nu}\bar{\psi}\gamma_{\mu)}\psi)\\
&\quad - \tfrac{i}{2}(\bar{\psi}\gamma_{(\mu}\partial_{\nu)}\psi) - (1-2\alpha)\eta_{\mu\nu}(|\partial\phi|^2 - |W'|^2) + \alpha\eta_{\mu\nu}(i\psi\gamma^\rho\partial_\rho\bar{\psi} - i\partial_\rho\psi\gamma^\rho\bar{\psi}),
\end{aligned} \tag{4.4.45}$$

[15] This is precisely the brane current from the supercurrent multiplet, see Appendix A.7.2. To obtain the desired form for $T^B_{\mu\nu}$ we use equations of motion and the Clifford algebra.

where for the last equality we have used equations of motion. The boundary contributions are given by $(T^\partial)^R_{\hat\mu\hat\nu} = (T^\partial)^S_{\hat\mu\hat\nu} + \frac{1}{2}\eta_{\hat\mu\hat\nu}\partial_\perp(-2\alpha\bar\phi\phi)$, hence

$$\begin{aligned}
(T^\partial_{++})^{\mathcal{R}} &= 0,\\
(T^\partial_{--})^{\mathcal{R}} &= \tfrac{i}{2}\bar\eta\partial_-\eta - \tfrac{i}{2}\partial_-\bar\eta\eta,\\
(T^\partial_{+-})^{\mathcal{R}} &= \tfrac{i}{2}\bar\eta\partial_+\eta - \tfrac{i}{2}\partial_+\bar\eta\eta + \tfrac{1}{2}(|E|^2+|J|^2) + \tfrac{\alpha}{2}\partial_\perp(\bar\phi\phi)|_\partial - \tfrac{i}{8}(\bar\psi_-\psi_+ - \bar\psi_+\psi_-)|_\partial,\\
(T^\partial_{-+})^{\mathcal{R}} &= \tfrac{1}{2}(|E|^2+|J|^2) + \tfrac{\alpha}{2}\partial_\perp(\bar\phi\phi)|_\partial + \tfrac{i}{8}(\bar\psi_-\psi_+ - \bar\psi_+\psi_-)|_\partial.
\end{aligned} \tag{4.4.46}$$

Note that the symmetry of the boundary stress tensor (modulo boundary conditions) was preserved by the improvement.

4.4.4 Supercurrent multiplets of the LG model

Let us now assemble the conserved currents of the Landau–Ginzburg model from the previous subsection into supercurrent multiplets. We first recall the supercurrent multiplets of a pure bulk theory, as well as its possible smaller multiplets. After that we will present a valid supercurrent multiplet in the Landau–Ginzburg model with boundary, and also discuss integrated supercurrent multiplets.

Bulk theory

Here we study a pure bulk theory with Lagrangian $\mathcal{L} = \mathcal{L}_{\text{kin.}} + \mathcal{L}_W$ as in (4.4.4) and (4.4.5). In such a theory a valid $\mathcal{S}$-multiplet is given by

$$\mathcal{S}_{\alpha\beta} = D_\alpha\Phi_{3D}\bar D_\beta\bar\Phi_{3D} + D_\beta\Phi_{3D}\bar D_\alpha\bar\Phi_{3D}. \tag{4.4.47}$$

It contains the supercurrent and energy-momentum tensor (in the $\mathcal{S}$-frame) in its components. We explicitly compute the components to verify this in the appendix (cf. (A.7.17)). The multiplet satisfies

$$\bar D^\alpha\mathcal{S}_{\alpha\beta} = \underbrace{-D_\beta\Phi_{3D}\bar D^2\bar\Phi_{3D}}_{=\mathcal{Y}_\beta} + \underbrace{(-\tfrac{1}{2})\bar D^2 D_\beta(\Phi_{3D}\bar\Phi_{3D})}_{=\chi_\beta}. \tag{4.4.48}$$

Using the equations of motion (4.4.6), one may rewrite $\mathcal{Y}_\beta = 4D_\beta W(\Phi_{3D})$. The defining equations in (4.3.1) can be verified easily, proving that this is indeed an $\mathcal{S}$-multiplet. The central charge C is zero. This $\mathcal{S}$-multiplet can be improved to a Ferrara–Zumino multiplet using the improvement $U_{\text{FZ}} = -\frac{1}{2}\bar\Phi_{3D}\Phi_{3D}$ (4.3.4), as this implies

$$\chi'_\alpha = -\tfrac{1}{2}\bar D^2 D_\alpha(\bar\Phi_{3D}\Phi_{3D}) - \bar D^2 D_\alpha U = 0. \tag{4.4.49}$$

The multiplet is then given by

$$\mathcal{J}_{\alpha\beta} = \tfrac{1}{2}(D_\alpha\Phi_{3D}\bar D_\beta\bar\Phi_{3D} + D_\beta\Phi_{3D}\bar D_\alpha\bar\Phi_{3D}) + \tfrac{1}{2}(i\bar\Phi_{3D}\partial_{\alpha\beta}\Phi_{3D} - i\partial_{\alpha\beta}\bar\Phi_{3D}\Phi_{3D}). \tag{4.4.50}$$

If the R-symmetry (cf. Section 4.4.3) is present, one can instead apply the improvement $U_{\mathcal{R}} = -2\alpha\bar\Phi_{3D}\Phi_{3D}$ ($\alpha = (\deg W)^{-1}$) to the $\mathcal{S}$-multiplet, which sets $\mathcal{Y}_\alpha$ to zero modulo equations of motion:

$$\mathcal{Y}'_\alpha = 4D_\alpha W(\Phi_{3D}) - \tfrac{1}{2}D_\alpha\bar D^2 U = 4D_\alpha W(\Phi_{3D}) - 4\alpha D_\alpha(\Phi_{3D}W'(\Phi_{3D})) = 0. \tag{4.4.51}$$

Now $\mathcal{S}_{\alpha\beta}$ transforms to

$$\mathcal{R}_{\alpha\beta} = (1-2\alpha)(D_\alpha\Phi_{3D}\bar{D}_\beta\bar{\Phi}_{3D} + D_\beta\Phi_{3D}\bar{D}_\alpha\bar{\Phi}_{3D}) + 2\alpha(i\bar{\Phi}_{3D}\partial_{\alpha\beta}\Phi_{3D} - i\partial_{\alpha\beta}\bar{\Phi}_{3D}\Phi_{3D}). \quad (4.4.52)$$

We see that the lowest component of this multiplet is exactly the R-current (4.4.30), and one can check that the remaining currents in the $\mathcal{R}$-multiplet are in the $\mathcal{R}$-frame.

Adding a boundary

Now that we have studied the bulk, let us go back to our Landau–Ginzburg theory with a boundary and a boundary Fermi multiplets whose potentials factorises the superpotential (4.4.23). We want to extend the above bulk supercurrent multiplet to a full (bulk and boundary) supercurrent multiplet as described in Section 4.3.3.

We already computed the bulk and boundary conserved currents in the sense of (4.2.11) in various improvements frames in the previous subsection, and now have to organise the components into admissible (0, 2)-multiplets. We choose to do so in the case of the $\mathcal{R}$-multiplet.

We consider the embedding [37] into 3D $\mathcal{N}=2$ superspace:

$$\begin{aligned}
\mathcal{R}^B_{\alpha\beta} &= \mathcal{R}^{B(0)}_{\alpha\beta} + \theta^-\mathcal{R}^{B(1)}_{\alpha\beta} - \bar{\theta}^-\mathcal{R}^{\bar{B}(1)}_{\alpha\beta} + \theta^-\bar{\theta}^-\mathcal{R}^{B(2)}_{\alpha\beta},\\
\mathcal{R}^\partial_{\alpha\alpha} &= \mathcal{R}^{\partial(0)}_{\alpha\alpha} + \theta^-\underbrace{\mathcal{R}^{\partial(1)}_{\alpha\alpha}}_{=0} - \bar{\theta}^-\underbrace{\mathcal{R}^{\bar{\partial}(1)}_{\alpha\alpha}}_{=0} + \theta^-\bar{\theta}^-\mathcal{R}^{\partial(2)}_{\alpha\alpha}.
\end{aligned} \quad (4.4.53)$$

First, we decompose the bulk contribution to the $\mathcal{R}$-multiplet into its (0, 2)-submultiplets. The zeroth-order bulk (0, 2)-superfields are

$$\begin{aligned}
\mathcal{R}^{B(0)}_{++} &= 8\alpha(i\bar{\Phi}\partial_+\Phi - i\partial_+\bar{\Phi}\Phi) - 2(1-2\alpha)\bar{D}_+\bar{\Phi}D_+\Phi\\
&= 4j^B_+ + \dots,\\
\mathcal{R}^{B(0)}_{--} &= 8\alpha(i\bar{\Phi}\partial_-\Phi - i\partial_-\bar{\Phi}\Phi) - 4(1-2\alpha)\bar{\Psi}\Psi\\
&= 4j^B_- + \dots,\\
\mathcal{R}^{B(0)}_{+-} &= -4\alpha(i\bar{\Phi}\partial_\perp\Phi - i\partial_\perp\bar{\Phi}\Phi) - \sqrt{2}(1-2\alpha)(\bar{D}_+\bar{\Phi}\Psi + \bar{\Psi}D_+\Phi)\\
&= -2j^B_\perp + \dots.
\end{aligned} \quad (4.4.54)$$

The first-order bulk (0, 2)-superfields are

$$\begin{aligned}
\mathcal{R}^{B(1)}_{++} &= 4(1-2\alpha)(i\partial_\perp\bar{\Phi}D_+\Phi - \bar{D}_+\bar{\Phi}\bar{W}'(\bar{\Phi})) - 8i\sqrt{2}\alpha(\partial_+\bar{\Phi}\Psi - \bar{\Phi}\partial_+\Psi)\\
&= -4i(S^B_+)^{\mathcal{R}}_- + \dots,\\
\mathcal{R}^{B(1)}_{--} &= -8i\sqrt{2}(1-\alpha)\Psi\partial_-\bar{\Phi} + 8i\sqrt{2}\alpha\bar{\Phi}\partial_-\Psi\\
&= -4i(S^B_-)^{\mathcal{R}}_- + \dots,\\
\mathcal{R}^{B(1)}_{+-} &= 2\sqrt{2}i(\partial_\perp\bar{\Phi}\Psi - \bar{\Phi}\partial_\perp\Psi) + 2(1-2\alpha)(iD_+\Phi\partial_-\bar{\Phi} - \sqrt{2}\bar{\Psi}\bar{W}'(\bar{\Phi}) + \sqrt{2}i\bar{\Phi}\partial_\perp\Psi)\\
&= 2i(S^B_\perp)^{\mathcal{R}}_- + \dots.
\end{aligned} \quad (4.4.55)$$

The second-order bulk (0, 2)-superfields are lengthy, but are a straightforward (0, 2)-completion of their lowest components:[16]

$$\begin{aligned}
\mathcal{R}^{B(2)}_{++} &= -16\big(\partial_+\bar\Phi\partial_-\Phi + \partial_-\bar\Phi\partial_+\Phi + \alpha\partial_+\partial_-(\bar\Phi\Phi) - \tfrac{\alpha}{2}\partial_\perp^2(\bar\Phi\Phi) \\
&\qquad - \tfrac{i}{4}\partial_-\bar D_+\bar\Phi D_+\Phi + \tfrac{i}{4}\bar D_+\bar\Phi\partial_- D_+\Phi - \tfrac{1}{2}\widetilde{\mathcal{L}^B}\big) \\
&= -16\big(\alpha\partial_+\partial_-(\bar\Phi\Phi) + \tfrac{1}{2}\partial_\perp\bar\Phi\partial_\perp\Phi - \tfrac{\alpha}{2}\partial_\perp^2(\bar\Phi\Phi) + \tfrac{1}{2}|W'(\Phi)|^2 \\
&\qquad - \tfrac{i}{4}\partial_-\bar D_+\bar\Phi D_+\Phi + \tfrac{i}{4}\bar D_+\bar\Phi\partial_- D_+\Phi\big) \\
&= -16(T^B_{-+})^{\mathcal{R}} + 2(C^B_{+-})^{\mathcal{R}} + \dots, \\
\mathcal{R}^{B(2)}_{--} &= -16\big(2\partial_-\bar\Phi\partial_-\Phi - \alpha\partial_-^2(\bar\Phi\Phi) - \tfrac{i}{2}\partial_-\bar\Psi\Psi + \tfrac{i}{2}\bar\Psi\partial_-\Psi\big) \\
&= -16(T^B_{--})^{\mathcal{R}} + \dots, \\
\mathcal{R}^{B(2)}_{+-} &= 8\big(\partial_-\bar\Phi\partial_\perp\Phi + \partial_\perp\bar\Phi\partial_-\Phi - \alpha\partial_-\partial_\perp(\bar\Phi\Phi) \\
&\qquad + \tfrac{i}{2\sqrt{2}}(\partial_-\bar D_+\bar\Phi\Psi + \partial_-\Psi D_+\Phi - \bar D_+\bar\Phi\partial_\Psi - \bar\Psi\partial_- D_+\Phi)\big) \\
&= 8(T^B_{-\perp})^{\mathcal{R}} - (C^B_{\perp-})^{\mathcal{R}}\dots,
\end{aligned} \tag{4.4.56}$$

where the lowest components are given by the energy-momentum tensor (4.4.45). The brane current $(C^B_{\mu\nu})^{\mathcal{R}} = \epsilon_{\mu\nu\rho}(H^\rho)^{\mathcal{R}}$ in the $\mathcal{R}$-frame is given by $H^{\mathcal{R}}_\mu = -2i(1-4\alpha)\partial_\mu(\bar\psi\psi)$ where we have used (4.3.5) and the explicit improvement $U_{\mathcal{R}}$. We have also (0, 2)-completed the bulk Lagrangian on-shell

$$\widetilde{\mathcal{L}^B} = 2\partial_+\bar\Phi\partial_-\Phi + 2\partial_-\bar\Phi\partial_+\Phi - \partial_\perp\bar\Phi\partial_\perp\Phi - |W'(\Phi)|^2. \tag{4.4.57}$$

Note that one may also interpret the sum of the tensor and the brane current as a non-symmetric energy-momentum tensor $\widehat{T}^B_{\mu\nu}$ (cf. (4.4.42)).

For the zeroth component $\mathcal{R}^{\partial(0)}_{\hat\mu}$, we simply (0, 2)-supersymmetrically complete the boundary R-current (4.4.31), where again $\alpha = (\deg W)^{-1}$, $\ell_E = \deg E$ and $\ell_J = \deg J$:

$$\mathcal{R}^{\partial(0)}_{\hat\mu} = \alpha(\ell_E - \ell_J)\delta^-_{\hat\mu}\bar H H, \tag{4.4.58}$$

or, in bispinor notation,

$$\begin{aligned}
\mathcal{R}^{\partial(0)}_{--} &= 4\alpha(\ell_E - \ell_J)\bar H H, \\
\mathcal{R}^{\partial(0)}_{++} &= 0.
\end{aligned} \tag{4.4.59}$$

Note that the (0, 2)-completion $(\mathcal{R}^{\partial(0)})_{\hat\mu}$ of $(j^\partial)_{\hat\mu}$ does not contain *all* the boundary contributions necessary: We need the boundary corrections T^∂_{--} to the energy-momentum tensor, which are not contained in our boundary multiplet $(\mathcal{R}^{\partial(0)})_{++}$, as can be checked.[17] Hence, we must also compute the correction for the second-order terms $(\mathcal{R}^{\partial(2)})_{\alpha\alpha}$:

$$\begin{aligned}
\mathcal{R}^{\partial(2)}_{++} &= 8\widetilde{\mathcal{L}^\partial} - 8\alpha\partial_\perp(\bar\Phi\Phi) + 4\sqrt{2}i\alpha(\bar D_+\bar\Phi\Psi - \bar\Psi D_+\Phi)|_\partial \\
&= -8|J(\Phi)|^2 - 8|E(\Phi)|^2 - 8\alpha\partial_\perp(\bar\Phi\Phi) + 4\sqrt{2}i\alpha(\bar D_+\bar\Phi\Psi - \bar\Psi D_+\Phi)|_\partial \\
&= -16(T^\partial_{-+})^{\mathcal{R}} + 2C^\partial_{+-} + \dots, \\
\mathcal{R}^{\partial(2)}_{--} &= 8i\partial_-\bar H H - 8i\bar H\partial_- H \\
&= -16(T^\partial_{--})^{\mathcal{R}} + \dots,
\end{aligned} \tag{4.4.60}$$

[16]Recall the general expansions (A.7.12c), (A.7.13c), and (A.7.14c), in particular the definition of $K_{\mu\nu}$ (A.7.15).

[17]This can be directly verified by the explicit expansions (A.7.12)–(A.7.14): T_{+-} and T_{++} are contained in the (0)-pieces, while T_{--} is contained in the (2)-piece.

where the boundary contribution $(C^{\partial}_{\hat{\mu}\hat{\nu}})^{\mathcal{R}}$ to the brane current $(C^{B}_{\mu\nu})^{\mathcal{R}}$ in the $\mathcal{R}$-frame can be found to be $(C^{\partial}_{+-})^{\mathcal{R}} = -i(1-4\alpha)\bar{\psi}\psi$. It is essentially the induced boundary improvement corresponding to symmetrisation of the energy-momentum tensor (cf. page 158), now in the $\mathcal{R}$-frame. We have also (0, 2)-completed the on-shell boundary Lagrangian:

$$\widetilde{\mathcal{L}^{\partial}} = -|J(\Phi)|^2 - |E(\Phi)|^2. \tag{4.4.61}$$

Integrated supercurrent multiplets

We now discuss integrated supercurrent multiplets as in Section 4.3.4. The integration along $x^{\perp}$ will make our Landau–Ginzburg model effectively two-dimensional and we will recover genuine 2D $\mathcal{N}=(0,2)$ (integrated) supercurrent multiplets.

We thus find, according to (4.3.51):

$$\begin{aligned}
\mathcal{R}^{\text{int.}(0)}_{++} &= \int \mathrm{d}x^{\perp}\ \big[8\alpha(i\bar{\Phi}\partial_+\Phi - i\partial_+\bar{\Phi}\Phi) - 2(1-2\alpha)\bar{D}_+\bar{\Phi}D_+\Phi\big],\\
\mathcal{R}^{\text{int.}(0)}_{--} &= 4\alpha(\ell_E - \ell_J)\bar{H}H + \int \mathrm{d}x^{\perp}\ \big[8\alpha(i\bar{\Phi}\partial_-\Phi - i\partial_-\bar{\Phi}\Phi) - 4(1-2\alpha)\bar{\Psi}\,\big],
\end{aligned} \tag{4.4.62}$$

as well as

$$\begin{aligned}
\mathcal{R}^{\text{int.}(2)}_{++} &= -8|J(\Phi)|^2 - 8|E(\Phi)|^2 + 4\sqrt{2}i\alpha(\bar{D}_+\bar{\Phi}\Psi - \bar{\Psi}D_+\Phi)|_{\partial}\\
&\quad - 16\int \mathrm{d}x^{\perp}\ \big[\alpha\partial_+\partial_-(\bar{\Phi}\Phi) + \tfrac{1}{2}\partial_{\perp}\bar{\Phi}\partial_{\perp}\Phi + \tfrac{1}{2}|W'(\Phi)|^2\\
&\qquad\qquad - \tfrac{i}{4}\partial_-\bar{D}_+\bar{\Phi}D_+\Phi + \tfrac{i}{4}\bar{D}_+\bar{\Phi}\partial_-D_+\Phi\big],\\
\mathcal{R}^{\text{int.}(2)}_{--} &= 8i\partial_-\bar{H}H - 8i\bar{H}\partial_-H\\
&\quad - 16\int \mathrm{d}x^{\perp}\ \big[2\partial_-\bar{\Phi}\partial_-\Phi - \alpha\partial_-^2(\bar{\Phi}\Phi) - \tfrac{i}{2}\partial_-\bar{\Psi}\Psi + \tfrac{i}{2}\bar{\Psi}\partial_-\Psi\big].
\end{aligned} \tag{4.4.63}$$

Note that from a 2D perspective, the superfields $\mathcal{R}^{\text{int.}(0)}_{++}$, $\mathcal{R}^{\text{int.}(0)}_{--}$ and $\mathcal{R}^{\text{int.}(2)}_{--}$ are enough to form a 2D supercurrent multiplet.

After using equations of motion (4.4.6), (4.4.21), boundary conditions (4.4.26), factorisation condition (4.4.23) and homogeneity of superpotential terms, we find that these integrated current multiplets indeed satisfy the relations

$$\begin{aligned}
\bar{D}_+(\mathcal{R}^{\text{int.}(2)}_{--} + 2i\partial_-\mathcal{R}^{\text{int.}(0)}_{--}) &= 0,\\
\bar{D}_+\Big(\mathcal{R}^{\text{int.}(2)}_{++} + 2i\partial_-\mathcal{R}^{\text{int.}(0)}_{++}\Big) &= 0.
\end{aligned} \tag{4.4.64}$$

which shows that the respective lowest components are $\bar{Q}_+$-cohomology elements (cf. Section 4.3.4).

4.5 Quantisation

Similarly to [32], we can follow a canonical quantisation approach in this model and verify the supersymmetry conservation, the boundary conditions, and the factorisation condition in an independent way. We impose the following canonical quantisation conditions:

$$[\partial_0\phi(x),\, \bar{\phi}(y)] = -i\delta^{(2)}(x-y), \tag{4.5.1a}$$
$$\{\bar{\psi}_\alpha(x),\, \psi_\beta(y)\} = -\gamma^0_{\alpha\beta}\delta^{(2)}(x-y), \tag{4.5.1b}$$
$$\{\bar{\eta}(x),\, \eta(y)\} = \delta(x^2-y^2). \tag{4.5.1c}$$

In general, the commutation relations are modified by the introduction of a boundary. Here, however, we follow the point of view of the authors in [32] and use the "naive" commutators even after introducing the boundary. This can be justified by considering a full bulk theory first, quantising it, then introducing a boundary and studying the effect of the boundary on the old bulk fields. Using this method, some properties of the model can be verified independently of the approach in Section 4.4. If we used static boundary conditions and the respective modified commutators instead, these properties would hold trivially. Notice that there are no singularities when moving component fields of chiral multiplets to the boundary as a consequence of supersymmetry.

The following relations hold in the bulk [38]:

$$\{\bar{\mathcal{Q}}_\alpha,\, S_{\beta\mu}\} = \gamma^\nu_{\alpha\beta}(2T_{\nu\mu} + \tfrac{1}{4}\epsilon_{\nu\mu\rho}H^\rho + i\partial_\nu j_\mu - i\eta_{\mu\nu}\partial_\rho j^\rho) + i\epsilon_{\alpha\beta}\epsilon_{\mu\nu\rho}(\tfrac{1}{4}F^{\nu\rho} + \partial^\nu j^\rho), \tag{4.5.2}$$
$$\{\mathcal{Q}_\alpha,\, S_{\beta\mu}\} = \tfrac{1}{4}\bar{C}(\gamma_\mu)_{\alpha\beta} + i\epsilon_{\mu\nu\rho}\gamma^\nu_{\alpha\beta}\bar{Y}^\rho. \tag{4.5.3}$$

In the absence of a boundary, integration of these relations yields the expected supersymmetry algebra. However, this changes under the introduction of a boundary for two reasons: First, there are additional degrees of freedom at the boundary which appear in $\mathcal{Q}$ and S, and second, there are boundary contributions from pure *bulk* terms as well.

Notice that the half-integrated commutators like $\{\mathcal{Q}_\alpha,\, S_{\alpha\mu}\}$ or $\{\bar{\mathcal{Q}}_\alpha,\, S_{\beta\mu}\}$ are affected by improvements, but the fully integrated commutators like $\{\mathcal{Q}_+,\, \mathcal{Q}_+\}$ and $\{\bar{\mathcal{Q}}_+,\, \mathcal{Q}_+\}$ must be invariant under them. This is easy to see in pure bulk theories, but with a boundary, it holds as well. A generic commutator of a charge $\mathcal{Q}$ and a current J^μ improves as follows:

$$\{\mathcal{Q},\, J'^\mu(x)\} = \{\mathcal{Q},\, J^\mu(x)\} + \{\mathcal{Q},\, \partial_\nu M^{[\mu\nu]}(x)\} + \delta(x^n)\{\mathcal{Q},\, M^{n\hat{\mu}}(x)\}. \tag{4.5.4}$$

Notice that $\mathcal{Q}$ is a charge and thus invariant under improvements. If $\mathcal{Q}$ commutes with ∂_ν, we find that the integrated algebra $\int \mathrm{d}x\{\mathcal{Q},\, J'^\mu(x)\}$ is unchanged.

4.5.1 General properties of the supercharge

Let us now restrict to the case $\alpha = +$, $\beta = +$ as the other supercharge and -current will be broken by the introduction of the boundary. The supercurrent from (4.4.32), (4.4.34) (which we repeat for convenience)

$$S_{+\mu}(x) = -\sqrt{2}i(\gamma_\mu\bar{\psi})_+\bar{W}'(\bar{\phi}) + \sqrt{2}(\psi\gamma_\mu\gamma^\nu)_+\partial_\nu\bar{\phi} - \sqrt{2}\,\delta(x^\perp)\delta^-_\mu(\bar{J}\bar{\eta} - i\bar{E}\eta) \tag{4.5.5}$$

integrates to the full supercharge

$$\mathcal{Q}_+ = \int_\Sigma \sqrt{2}(\psi_-\partial_\perp\bar{\phi} + i\bar{\psi}_-\bar{W}'(\bar{\phi}) - 2\psi_+\partial_+\bar{\phi}) + \int_{\partial\Sigma}\sqrt{2}(\bar{J}\bar{\eta} - i\bar{E}\eta), \tag{4.5.6}$$

where $\int_\Sigma = \int_{\mathbb{R}\times(-\infty,0]}\mathrm{d}x^2\mathrm{d}x^\perp$ and $\int_{\partial\Sigma} = \int_{\mathbb{R}}\mathrm{d}x^2$. Let us start by studying the action of the bulk part of $\mathcal{Q}_+$ on component fields. Analogous to [32], the action of $[\mathcal{Q}_{+,\mathrm{bulk}},\, \cdot]$ on bulk component fields (ϕ, $\bar\phi$, $\psi_\pm$, $\bar\psi_\pm$) is the same as in the pure bulk theory with one exception: The commutator with $\partial_0\phi$ receives an extra boundary term

$$[\mathcal{Q}_{+,\mathrm{bulk}}, \partial_0\phi(x)] = \sqrt{2}i\partial_0\psi_+(x) + \sqrt{2}i\delta(x^\perp)\psi_-(x). \tag{4.5.7}$$

This result can be derived using the quantisation conditions (4.5.1) and the delta distribution rule in Appendix A.6.2. Notice that for $\nu \neq 0$, the identity $\frac{\partial}{\partial x^\nu}[\mathcal{Q}_+, \phi(x)] = [\mathcal{Q}_+, \partial_\nu\phi(x)]$ is only true if the correct delta distribution rules derived in Appendix A.6.2 are applied.

The boundary part $\mathcal{Q}_{+,\mathrm{bdy}}$ has a trivial action on the bulk component fields and on $\partial_\nu\phi$, $\nu \neq 0$. The action on the boundary fermions η, $\bar\eta$ is as expected. Again, we get extra terms for $\partial_0\phi$. Overall, the full charge acts as follows on $\partial_0\phi$:

$$[\mathcal{Q}_+, \partial_0\phi(x)] = \sqrt{2}i\partial_0\psi + \sqrt{2}(i\bar\eta\bar{J}' + \eta\bar{E}' + i\psi_-)\delta(x^\perp). \tag{4.5.8}$$

We see that the boundary contribution vanishes under the symmetric boundary condition (4.4.26), which is an independent way of verifying this boundary condition. However, for reasons outlined above, we will not impose this condition statically and thus treat this extra term like a genuine new contribution.

4.5.2 The $\{\bar{\mathcal{Q}}, S\}$ commutator

We now would like to compute $\{\bar{\mathcal{Q}}_+, S_{+\mu}(x)\}$ in the presence of a boundary and verify that it integrates to the expected preserved algebra $\{\bar{\mathcal{Q}}_+, \mathcal{Q}_+\} = -4P_+$ (see Section 4.3.3). We expect the known terms (4.5.2) in the bulk, and extra terms at the boundary. As both $\bar{\mathcal{Q}}_+$ and $S_{+\mu}(x)$ have bulk and boundary parts, there are four combinations from which new terms may arise: bulk-bulk, bulk-boundary, boundary-bulk, and boundary-boundary.

Boundary contributions from bulk-bulk terms

Let us now check how the changes introduced by the boundary affect the half-integrated algebra: The bulk-bulk term

$$\{\bar{\mathcal{Q}}_{\alpha,\mathrm{bulk}}, S_{\beta\mu,\mathrm{bulk}}\} = \{\bar{\mathcal{Q}}_\alpha, -\sqrt{2}i(\gamma_\mu\bar\psi)_\beta\bar{W}'(\bar\phi) + \sqrt{2}(\psi\gamma_\mu\gamma^\nu)_\beta\partial_\nu\bar\phi\} \tag{4.5.9}$$

is affected by the changed relation (4.5.7) in the term $\partial_0\bar\phi$, where we get an extra boundary term

$$\{\bar{\mathcal{Q}}_{\alpha,\mathrm{bulk}}, S_{\beta\mu,\mathrm{bulk}}(x)\} = \text{known bulk terms (4.5.2)} + \delta(x^\perp)\underbrace{2i(\psi\gamma_\mu\gamma^0)_\beta(\gamma^n\gamma^0\bar\psi(x))_\alpha}_{=:B_{\alpha\beta,\mu}}, \tag{4.5.10}$$

which is also consistent with a similar result in [32]. The component most relevant to us is ${B_{++}}^0$, as it appears in the integration of the part of the algebra that is unbroken. It can be rewritten to

$${B_{++}}^0 = 2i\psi_+\bar\psi_- = i\bar\psi\psi - i\bar\psi\gamma^\perp\psi = i\bar\psi\psi - ij^\perp, \tag{4.5.11}$$

where we have inserted a bulk component of the supercurrent multiplet in the last equality, see Appendix A.7.2. It is noteworthy that this boundary term is neither real nor imaginary.

Contributions from the boundary degrees of freedom

Again using the commutation relations but not boundary conditions, we find

$$\{\bar{\mathcal{Q}}_{+,\mathrm{bdy}}, S_{+,\mu,\mathrm{bdy}}(x)\} = -2\delta_\mu^- \delta(x^\perp)(|J|^2 + |E|^2), \tag{4.5.12a}$$
$$\{\bar{\mathcal{Q}}_{+,\mathrm{bulk}}, S_{+,\hat{\mu},\mathrm{bdy}}(x)\} = -2i\delta(x^\perp)\delta_{\hat{\mu}}^- \bar{\psi}_+(x)(\bar{J}'(x)\bar{\eta}(x) - i\bar{E}'(x)\eta(x)), \tag{4.5.12b}$$
$$\{\bar{\mathcal{Q}}_{+,\mathrm{bdy}}, S_{+,\mu,\mathrm{bulk}}(x)\} = 2i\delta(x^\perp)(J'(x)\eta(x) + iE'(x)\bar{\eta}(x))(\psi(x)\gamma_\mu\gamma^0)_+. \tag{4.5.12c}$$

It is noteworthy that modulo boundary conditions, the third term cancels the boundary contribution of $\{\bar{\mathcal{Q}}_{+,\mathrm{bulk}}, S_{+,\mu,\mathrm{bulk}}(x)\}$. This is expected since both extra terms have their origins in the changed relation (4.5.8) which is identical to the original bulk relation modulo boundary conditions.

Integrating the algebra

As the integral of $S_+{}^0$ over a constant time slice yields the supercharge $\mathcal{Q}_+$, an integration of the commutator $\int\{\bar{\mathcal{Q}}_+, S_+{}^0\} = \{\bar{\mathcal{Q}}_+, \mathcal{Q}_+\}$ is a commutator which appears in the preserved supersymmetry algebra (4.3.13). We can thus check (4.5.10) and (4.5.12) by integrating $\{\bar{\mathcal{Q}}_+, S_+{}^0\}$ and comparing the result to the known algebra. We will plug in the component expansions from Appendix A.7.2.

Let us first check the imaginary part, which is zero on the expected right-hand side of the equation. Interesting contributions come only from $\{\bar{\mathcal{Q}}_{+,\mathrm{bulk}}, S_{+,\mathrm{bulk}}^{\;0}\}$, as all other contributions together are trivially real.

$$\mathrm{Im}\int_\Sigma \{\bar{\mathcal{Q}}_+, S_+{}^0\} = \int_\Sigma 2(i\partial_+ j_0 - i\eta_{+0}\partial^\rho j_\rho) + \int_{\partial\Sigma} \mathrm{Im}\Big(B_{++}{}^0\Big) = \int_\Sigma i\partial_\perp j_\perp - \int_{\partial\Sigma} ij_\perp = 0. \tag{4.5.13}$$

A similar computation in four dimensions was done in [32].

For the real part we get (as expected from (4.5.2))

$$\{\bar{\mathcal{Q}}_+, S_+{}^0\}|_{\mathrm{bulk}} = -\gamma_{22}^\nu(2T_{\nu 0} + \tfrac{1}{4}\epsilon_{\nu 0\rho}H^\rho) = -4(\widehat{T}^B)_+{}^0, \tag{4.5.14}$$

where $\widehat{T}^B$ is the bulk Noether (non-symmetric) energy momentum tensor (4.4.39). All boundary terms together (Eqs. (4.5.10) and (4.5.12)) yield

$$\{\bar{\mathcal{Q}}_+, S_+{}^0\}|_\partial = i\bar{\psi}\psi + 2|J|^2 + 2|E|^2 + 2i\bar{\psi}_+(\bar{J}'\bar{\eta} - i\bar{E}'\eta) - 2i(J'\eta + iE'\bar{\eta})\psi_+ = -4(\widehat{T}^\partial)_+{}^0$$

using an explicit comparison to the non-symmetric boundary energy momentum tensor $\widehat{T}^\partial$ (4.4.40) which belongs to the bulk Noether energy momentum tensor $\widehat{T}^B$. Overall, we find that

$$\{\bar{\mathcal{Q}}_+, \mathcal{Q}_+\} = \int_\Sigma \{\bar{\mathcal{Q}}_+, S_+{}^0\} = -4\int_\Sigma ((\widehat{T}^B)_+{}^0 + \delta(x^\perp)(\widehat{T}^\partial)_+{}^0) = -4P_+, \tag{4.5.15}$$

which verifies the algebra. Notice that P_+ is independent of improvements, thus we may use improved versions of the energy-momentum tensor to compute the right-hand side of the equality. However, on the level of the half-integrated algebra, we see that the Noether ($\mathcal{S}$-frame) supercurrent generates the non-symmetric Noether energy momentum tensor, and both are sensitive to improvements.

Let us emphasise again that this argument works *without* explicitly assuming boundary conditions and modifying the bulk fields in the presence of a boundary. Rather, we study the bulk theory without a boundary, then introduce it, and verify that the supersymmetry algebra is preserved by the equal-time commutators. Notice that without assuming any boundary conditions, the charges $\mathcal{Q}_+$, $\bar{\mathcal{Q}}_+$ are not conserved. However, no reference to the specific choice of boundary conditions was made in this argument (although in this simple model with one bulk chiral field and one boundary Fermi, (4.4.26) is the only symmetric boundary condition compatible with stationarity).

4.5.3 The $\{\mathcal{Q}, S\}$ commutator

In a similar way, we can also verify that $\{\mathcal{Q}_+, S_{+\mu}(x)\}$ integrates to the expected algebra $\{\mathcal{Q}_+, \mathcal{Q}_+\} = 0$. In the bulk, we get from (4.5.3)

$$\{\mathcal{Q}_{+,\text{bulk}}, S_{+\mu,\text{bulk}}(x)\} = i\epsilon_{\mu+\rho}(\gamma^+)_{++}\bar{Y}^\rho = -8i\epsilon_{\mu+\rho}\partial^\rho\bar{W}. \tag{4.5.16}$$

From the commutation relations (4.5.1) we find that $\{\mathcal{Q}_{\alpha,\text{bulk}}, S_{\beta\mu,\text{bdy}}\}$ and $\{\mathcal{Q}_{\alpha,\text{bdy}}, S_{\beta\mu,\text{bulk}}\}$ are zero, but we do get a contribution from

$$\{\mathcal{Q}_{\alpha,\text{bdy}}, S_{\beta\mu,\text{bdy}}(x)\} = 4i\delta_\mu^- \delta(x^\perp)\bar{J}(x)\bar{E}(x). \tag{4.5.17}$$

Integrating the relation, we find

$$\{\mathcal{Q}_+, \mathcal{Q}_+\} = \int_\Sigma \{\mathcal{Q}_+, S_+{}^0(x)\} = 4i\int_{\partial\Sigma} (\bar{W} - \bar{J}(x)\bar{E}(x)), \tag{4.5.18}$$

so if the factorisation condition (4.4.23) is met, the algebra is preserved under the introduction of the boundary.

Appendix

A.1 Details on the proof of Theorem 2.3.7

A.1.1 The isomorphisms of $\tilde{\mathrm{ev}}_0$

Here we explain the isomorphisms in Eq. (2.3.23). The first isomorphism is easy to see: For an arbitrary element $1 \otimes m \otimes x \otimes x' \in S \otimes_{S^e} N$ we find

$$1 \otimes m \otimes x \otimes (sx') = s \otimes m \otimes x \otimes x' = 1 \otimes m \otimes (sx) \otimes x' \text{ for } s \in S \ .$$

The second isomorphism is obvious, so let us proceed with the third. We start by showing $X^\vee \otimes \bar{M} \cong (X \otimes \bar{M})^\vee = \bar{X}^\vee$, more precisely

$$X^\vee \otimes_{S \otimes_k R} \bar{M} = \mathrm{Hom}_{S \otimes R}(X, S \otimes R) \otimes_{S \otimes R} \bar{M} \cong \mathrm{Hom}_{\bar{M}}(X \otimes_{S \otimes R} \bar{M}, \bar{M}) = \mathrm{Hom}_{\bar{M}}(\bar{X}, \bar{M}) \ .$$

The isomorphism maps

$$\phi \otimes m_1 \in X^\vee \otimes_{S \otimes R} \bar{M} \mapsto \chi \in \mathrm{Hom}_{\bar{M}}(\bar{X}, \bar{M}), \quad \chi(x \otimes m_2) := m_1 m_2 \phi(x) \ .$$

χ is well-defined and linear in $\bar{M}$ by definition. The inverse is given by

$$\begin{aligned} \chi \in \mathrm{Hom}_{\bar{M}}(\bar{X}, \bar{M}) \mapsto \phi \otimes 1 \in X^\vee \otimes_{S \otimes R} \bar{M}, & \\ \phi(x) := \chi(x \otimes 1) \quad & (\text{choose a representative in } S \otimes R) \ . \end{aligned}$$

It remains to be checked that the image of ϕ is independent of the chosen representative:

$$\phi(x) \otimes 1 = (\chi(x \otimes 1) + \Sigma \alpha_i f_i) \otimes 1 = \chi(x \otimes 1) + 1 \otimes \Sigma \alpha_i f_i \equiv \chi(x \otimes 1) \ .$$

$\bar{X}$ is also free and finite-rank as an $\bar{M}$-module, thus we may apply [22, Section 2.6] to find that

$$\bar{X} \otimes_{\bar{M}} \bar{X}^\vee \cong \mathrm{Hom}_{\bar{M}}(\bar{X}, \bar{X})$$

which is what we wanted to show.

A.1.2 On our formulation of the idempotent pushforward

This section discusses how to derive Theorem 2.3.14 from its original formulation in [39, Theorem 7.4]. A related argument can be found in [22, Appendix A]. First, note that [39] makes no global assumptions on the superpotentials[18] in Sections 1 to 6. Therefore, the list of assumptions made at the beginning of Section 7 is all that remains to be checked for Theorem 7.4. Let us first translate our setup into their notation:

$$S = k[\boldsymbol{z}, \boldsymbol{z}'], \quad R = k[\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{z}'], \quad \varphi \colon S \to R, \quad 1 \mapsto 1 \ .$$

[39] then requires the existence of a quasi-regular sequence $\boldsymbol{f} \subset R$, which is implied by the assumption of a Koszul-regular sequence in Theorem 2.3.14. The next assumption of [39] is the existence of a *deformation retract* between $R/(\boldsymbol{f})$ and $K_\bullet(\boldsymbol{f})$. The following set of sufficient conditions is stated:

[18] The use of the word "potential" in [39] is not related to Definition 2.1.1, but is used for what this work calls "superpotential".

(i) Both R and $R/(\boldsymbol{f})$ are projective as modules over S. The former is true in our case since $R = k[\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{z}']$ is free over $S = k[\boldsymbol{z}, \boldsymbol{z}']$, and the projectivity of $R/(\boldsymbol{f})$ is explicitly assumed in Theorem 2.3.14.

(ii) Consider the following chain map, called the *augmentation* in [39]:

$$\begin{tikzcd} K_\bullet(\boldsymbol{f})\colon & 0 \arrow[r] & K_n \arrow[r, "\delta"] \arrow[d, "0"] & K_{n-1} \arrow[r, "\delta"] \arrow[d, "0"] & \dots \arrow[r, "\delta"] & K_1 \arrow[r, "\delta"] \arrow[d, "0"] & K_0 \arrow[r] \arrow[d, "\pi"] & 0 \\ R/(\boldsymbol{f})\colon & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots \arrow[r] & 0 \arrow[r] & R/(\boldsymbol{f}) \arrow[r] & 0 \end{tikzcd} \tag{A.1.1}$$

The action of π in the last component is the R-linear projection $R \to R/(\boldsymbol{f})$. π is a chain map for all R and $\boldsymbol{f}$:

$$\mathrm{Im}(\delta\colon K_1 \to K_0) = \boldsymbol{f}R \implies \pi \circ \delta = 0 \ .$$

Furthermore, π induces an isomorphism on cohomology in degree 0 for all $\boldsymbol{f}$, since

$$H^0(K_\bullet(\boldsymbol{p})) = \mathrm{Ker}(0\colon K_0 \to 0)/\,\mathrm{Im}(\delta\colon K_1 \to K_0) = R/(\boldsymbol{f}) = H^0(R/(\boldsymbol{f})) \ ,$$

and the map $H^0(K_\bullet(\boldsymbol{p})) \to H^0(R/(\boldsymbol{f}))$ induced by π is the identity on $R/(\boldsymbol{f})$.

The condition to be met for the deformation retract to exist is that π is a quasi-isomorphism, i.e. the chain map on cohomology induced by π is an isomorphism in *all* degrees. We now use that Theorem 2.3.14 assumes $\boldsymbol{f}$ to be Koszul-regular, meaning that the cohomology of $K_\bullet(\boldsymbol{f})$ is zero in all degrees except zero, implying that π is a quasi-isomorphism.

We thus find that the assumptions of Theorem 2.3.14 imply the assumptions of [39, Theorem 7.4].

A.2 Constructing ev_X using resolutions

This appendix and the following one each explain a method to construct ev_X and $\tilde{\mathrm{ev}}_X$ for a given 2-morphism X in $\mathcal{RW}$ without knowledge of Theorem 2.3.7. Both methods were used in the process of conjecturing, proving, and verifying Theorem 2.3.7. The example considered in both methods is the following:

$$\begin{aligned} W &:= a(x'-x) \ , \qquad V := u(y-x) - u'(y-x') \ , \\ X &:= K(u-u', x-x';\, y-x, a-u')\colon (a;\, W) \to (u, y, u';\, V) \end{aligned} \tag{A.2.1}$$

which is related to $\lambda^{-1}_{\mathbb{1}_x}$ of Eq. (1.5.28), $\mathrm{coev}_{\mathbb{1}_x}$ of Definition 3.2.1, and $\mu_{e,e}$ of Eq. (3.4.2) by grade shifts and simple isomorphisms.

A.2.1 The general idea

In order to construct the maps ev_X and $\tilde{\mathrm{ev}}_X$ using resolutions, the following general approach is taken:

(i) Start with the educated guess $X^\dagger = X^\vee[n]$, ${}^\dagger X = X^\vee \otimes_S S[m] \cong X^\vee[m]$.

(ii) Find a module associated to $X \otimes X^\dagger$ resp. ${}^\dagger X \otimes X$ which is isomorphic to a module associated to $\mathbb{1}_V \oplus \dots$ resp. $\mathbb{1}_W \oplus \dots$, e.g. using Theorem 1.3.49.

(iii) Lift the isomorphism on the associated modules to an isomorphism of matrix factorisations γ as explained in Remark 1.3.35.

(iv) A candidate for ev_X is the component of γ which maps ${}^\dagger X \otimes X \to \mathbb{1}_W$.

For the kind of matrix factorisations relevant to this thesis, there are two simplifications:

- We have $X \otimes X^\dagger \cong \mathbb{1}_V$ and ${}^\dagger X \otimes X \cong \mathbb{1}_W$ without extra terms.
- Before step (ii), it is possibly to simplify

$$ {}^\dagger X \otimes X \cong \mathbb{1}_W \otimes K(\boldsymbol{p};\, 0,\, \dots\, , 0) $$

  where the latter is a Koszul matrix factorisation of zero. This extra step simplifies step (iii) by a lot.

A.2.2 Constructing ev_X

We write X of Eq. (A.2.1) in the notation of Eq. (1.3.7):

$$ X := [u - u', y - x] \otimes [x - x', a - u'] \colon (a;\, a(x' - x)) \to (u, y, u';\, u(y - x) - u'(y - x')) \ . $$

The rings in the tensor products will be omitted in this section. As explained above, our ansatz is

$$ {}^\dagger X \cong X^\vee[3] \cong [u' - u, y - x] \otimes [a - u', x' - x] $$

where each $\cong$ is a change of basis. Let us write out and simplify ${}^\dagger X \otimes X$ while respecting the relabelling rules of Definition 1.4.1:

$$ \begin{aligned} {}^\dagger X \otimes X &\cong [u' - u, y - x] \otimes [a - u', x' - x] \otimes [u - u', y - x] \otimes [x - x', \hat{a} - u'] && \mid (1.3.27) \\ &\cong [u' - u, y - x] \otimes [a - u', 0] \otimes [u - u', y - x] \otimes [x - x', \hat{a} - a] && \mid (1.3.25) \\ &\cong [0, y - x] \otimes [a - u', 0] \otimes [u - u', 0] \otimes [x - x', \hat{a} - a] && \mid (1.3.46) \\ &\cong [x - y, 0] \otimes [a - u', 0] \otimes [u - u', 0] \otimes [a - \hat{a}, x' - x] && \mid (1.3.45) \\ &\cong [a - \hat{a}, x' - x] \otimes [u - u', 0] \otimes [x - y, 0] \otimes [a - u', 0] && \mid (1.3.25) \\ &\cong [a - \hat{a}, x' - x] \otimes [u - a, 0] \otimes [x - y, 0] \otimes [a - u', 0] && \mid \text{L. } 1.3.10 \\ &\cong [a - \hat{a}, x' - x] \otimes [a - u, 0] \otimes [y - x, 0] \otimes [u' - a, 0] \\ &= K(a - \hat{a}, a - u, y - x, u' - a;\, x' - x, 0, 0, 0) \ , \end{aligned} \tag{A.2.2} $$

motivating the definition

$$ \{p_1, p_2, p_3, p_4\} := \{a - \hat{a},\, -(u - a),\, y - x,\, u' - a\} \tag{A.2.3} $$

$$ \implies {}^\dagger X \otimes X \cong K(\boldsymbol{p};\, x' - x, 0, 0, 0) \cong \mathbb{1}_W \otimes K(p_2, p_3, p_4;\, 0) \ . \tag{A.2.4} $$

Remark A.2.1. We find $\{p_2, p_3, p_4\} = \{\partial_{\boldsymbol{v}}(V - W)\}$ for the set of left admissible variables $\boldsymbol{v} = \{x, u, x'\}$, showing a connection between this approach and the general formulas of Theorem 2.3.7. The freedom to choose different sets of admissible variables corresponds to the fact that there are several sets $\{p_2, p_3, p_4\}$ such that ${}^\dagger X \otimes X \cong \mathbb{1}_W \otimes K(p_2, p_3, p_4;\, 0, 0, 0)$ via “simple” isomorphisms.

Resolution setup

By Theorem 1.3.49, (A.2.2) is associated to the module

$$\mathbb{C}[a,\,\hat{a},\,x,\,x',\,u,\,u',\,y]/(a-\hat{a},\,-(u-a),\,y-x,\,u'-a)\ . \tag{A.2.5}$$

The variables u, u', and y are located in between the defect lines and can be integrated out. We find that over $\mathbb{C}[a,\,\hat{a},\,x,\,x']/((a-\hat{a})(x'-x))$, (A.2.5) is isomorphic to

$$\mathbb{C}[a,\,\hat{a},\,x,\,x']/(a-\hat{a}) \tag{A.2.6}$$

which is associated to the identity matrix factorisation $\mathbb{1}_W = [a-\hat{a},\,x'-x]$. For reasons that become clear soon, we will write the (trivial) isomorphisms between the modules (A.2.5) and (A.2.6) as the following morphisms over $\mathbb{C}[a,\,\hat{a},\,x,\,x']$:

$$\begin{aligned}
\pi\colon \mathbb{C}[a,\,\hat{a},\,x,\,x',\,u,\,u',\,y] &\to \mathbb{C}[a,\,\hat{a},\,x,\,x']\ , \\
\alpha &\mapsto -\operatorname{Res}\left[\frac{\alpha\,\mathrm{d}u\,\mathrm{d}y\,\mathrm{d}u'}{-(u-a),\,y-x,\,u'-a}\right] = \operatorname{Res}\left[\frac{\alpha\,\mathrm{d}u\,\mathrm{d}y\,\mathrm{d}u'}{u-a,\,y-x,\,u'-a}\right]\ , \\
\psi\colon \mathbb{C}[a,\,\hat{a},\,x,\,x'] &\to \mathbb{C}[a,\,\hat{a},\,x,\,x',\,u,\,u',\,y]\ , \quad \alpha \mapsto \alpha\ .
\end{aligned} \tag{A.2.7}$$

Dividing out the respective ideals turns π and ψ into identity maps.

Let us now lift this isomorphism from the modules to the matrix factorisations according to Remark 1.3.35. First we need to construct the resolutions explicitly. We define labels:

$$\begin{gathered}
R := \mathbb{C}[a,\,\hat{a},\,x,\,x',\,u,\,u',\,y]\ , \qquad S := \mathbb{C}[a,\,\hat{a},\,x,\,x']\ , \\
\tilde{V} := (a-\hat{a})(x'-x)\ , \qquad \tilde{R} := R/(\tilde{V})\ , \qquad \tilde{S} := S/(\tilde{V})\ .
\end{gathered}$$

We write the resolutions of the modules (A.2.5) and (A.2.6) explicitly using the Koszul complex $K_\bullet(\boldsymbol{p})$, consistent with the notation in Theorem 1.3.49:

$$\begin{aligned}
{}^\dagger X\otimes X &\cong \textstyle\bigoplus_{i=0}^4 K_i(\boldsymbol{p})\ , & \mathbb{1}_W &= \textstyle\bigoplus_{i=0}^1 K_i(\{a-\hat{a}\}) =: L_0\oplus L_1\ , \\
\delta &= p_1\theta_1^*+p_2\theta_2^*+p_3\theta_3^*+p_4\theta_4^*\ , & \delta' &= (a-\hat{a})\cdot\theta_a^*\ , \\
\sigma &= (x'-x)\cdot\theta_1+0\cdot(\theta_2+\theta_3+\theta_4)\ , & \sigma' &= (x'-x)\cdot\theta_a\ , \\
d_{{}^\dagger X\otimes X} &= \delta+\sigma\ , & d_{\mathbb{1}_W} &= \delta'+\sigma'\ .
\end{aligned}$$

Now Theorem 1.3.49 yields the following resolution (see [42, Section 4.3] for the full details):

$$\begin{tikzcd}
\dots \arrow[r, "d"] & \underbrace{\tilde{K}_1\oplus\tilde{K}_3}_{=({}^\dagger X\otimes X)_1} \arrow[r, "\left(\begin{smallmatrix}\delta & 0\\ \sigma & \delta\\ 0 & \sigma\end{smallmatrix}\right)"] & \underbrace{\tilde{K}_0\oplus\tilde{K}_2\oplus\tilde{K}_4}_{=({}^\dagger X\otimes X)_0} \arrow[r, "\left(\begin{smallmatrix}\sigma & \delta & 0\\ 0 & \sigma & \delta\end{smallmatrix}\right)"] & \tilde{K}_1\oplus\tilde{K}_3 \arrow[r, "\left(\begin{smallmatrix}\delta & 0\\ \sigma & \delta\end{smallmatrix}\right)"] & \tilde{K}_0\oplus\tilde{K}_2 \arrow[dlll, rounded corners, "(\sigma\ \delta)"'] \\
& \tilde{K}_1 \arrow[r, "\delta"] & \tilde{K}_0 \arrow[r, two heads] & \frac{R}{(p_1,p_2,p_3,p_4)} \arrow[r] & 0
\end{tikzcd}$$

The resolution of (A.2.6) is given by Eq. (1.3.51):

$$\dots \xrightarrow{\delta'} \tilde{L}_0 \xrightarrow{\sigma'} \tilde{L}_1 \xrightarrow{\delta'} \tilde{L}_0 \twoheadrightarrow \frac{S}{(a-\hat{a})} \longrightarrow 0$$

Lifting the isomorphism π

The main idea of Remark 1.3.35 is to construct commuting diagrams out of the two resolutions, starting with the isomorphisms between (A.2.5) and (A.2.6):

$$\begin{tikzcd}
\ldots \arrow[r] & \tilde{K}_0 \oplus \tilde{K}_2 \oplus \tilde{K}_4 \arrow[r, "\left(\begin{smallmatrix} \sigma & \delta & 0 \\ 0 & \sigma & \delta \end{smallmatrix}\right)"] \arrow[d, shift left, "r_4"] & \tilde{K}_1 \oplus \tilde{K}_3 \arrow[r, "\left(\begin{smallmatrix} \delta & 0 \\ \sigma & \delta \end{smallmatrix}\right)"] \arrow[d, shift left, "r_3"] & \tilde{K}_0 \oplus \tilde{K}_2 \arrow[r, "(\sigma\ \delta)"] \arrow[d, shift left, "r_2"] & \tilde{K}_1 \arrow[r, "\delta"] \arrow[d, shift left, "r_1"] & \tilde{K}_0 \arrow[r, two heads] \arrow[d, shift left, "r_0"] & \frac{R}{(p_1, p_2, p_3, p_4)} \arrow[r] \arrow[d, shift left, "\pi"] & 0 \\
\ldots \arrow[r] & \tilde{L}_0 \arrow[r, "\sigma'"'] \arrow[u, shift left, "r_4^*"] & \tilde{L}_1 \arrow[r, "\delta'"'] \arrow[u, shift left, "r_3^*"] & \tilde{L}_0 \arrow[r, "\sigma'"'] \arrow[u, shift left, "r_2^*"] & \tilde{L}_1 \arrow[r, "\delta'"'] \arrow[u, shift left, "r_1^*"] & \tilde{L}_0 \arrow[r, two heads] \arrow[u, shift left, "r_0^*"] & \frac{S}{(\hat{a}-a)} \arrow[r] \arrow[u, shift left, "\psi"] & 0
\end{tikzcd} \tag{A.2.8}$$

The isomorphism we are looking for consists of the maps r_3 and r_4. To construct them we first need to construct the other r_i such that the diagram commutes in the down-direction. We can use the similar structure of both differentials to make this step easier:

Lemma A.2.2. *The map*

$$\chi\colon \bigoplus_{i=0}^{4} \tilde{K}_i \to \bigoplus_{j=0}^{1} \tilde{L}_j \ , \quad (\theta_1, \theta_2, \theta_3, \theta_4) \mapsto (\theta_a, 0, 0, 0) \ , \quad \alpha\, \theta_{i_1} \ldots \theta_{i_k} \mapsto \pi(\alpha)\chi(\theta_{i_1} \ldots \theta_{i_k})$$

with π of Eq. (A.2.7) fulfils

$$\chi \circ \delta = \delta' \circ \chi \ , \quad \chi \circ \sigma = \sigma' \circ \chi \ .$$

Proof. This identity is trivial on most components. The non-trivial cases are

$$\begin{aligned} \chi \circ \delta(\theta_{2,3,4}) = \chi(p_{2,3,4}) = \pi(p_{2,3,4}) = 0 = \delta' \circ \chi(\theta_{2,3,4}) \ , \\ \chi \circ \delta(\theta_1) = \chi(a - \hat{a}) = \pi(a - \hat{a}) = a - \hat{a} = \delta'(\theta_a) = \delta' \circ \chi(\theta_1) \ . \end{aligned}$$

□

We choose $r_i := \chi$ for all i and find that in this choice, the diagram (A.2.8) commutes in the down direction. The isomorphism from (A.2.2) to $\mathbb{1}_W$ is therefore given by:

$$\begin{aligned} \alpha \cdot 1 &\mapsto \mathrm{Res}\left[\frac{\alpha\, \mathrm{d}u\, \mathrm{d}y\, \mathrm{d}u'}{u - a,\ y - x,\ u' - a}\right] \ , \\ \alpha \cdot \theta_1 &\mapsto \theta_a\ \mathrm{Res}\left[\frac{\alpha\, \mathrm{d}u\, \mathrm{d}y\, \mathrm{d}u'}{u - a,\ y - x,\ u' - a}\right] \ , \\ \text{other terms} &\mapsto 0 \ . \end{aligned} \tag{A.2.9}$$

Lifting the isomorphism ψ

We define similarly

$$\eta\colon \tilde{L}_0 \oplus \tilde{L}_1 \to \bigoplus_{i=0}^{4} \tilde{K}_i \ , \quad 1 \mapsto 1 \ , \ \theta_a \mapsto \theta_1 \ .$$

It is easy to see that

$$\eta \circ \delta' = \delta \circ \eta \ , \quad \eta \circ \sigma' = \sigma \circ \eta$$

since σ and σ' are equal and δ and δ' are equal on 1 and θ_1, which are the only terms that appear in this identity. The isomorphism making (A.2.8) commute in the up-direction is thus given by

$$r_i^* = \eta\colon 1 \mapsto 1,\ \theta_a \mapsto \theta_1 \ . \tag{A.2.10}$$

Going back to the natural basis of ${}^{\dagger}X$

We have applied numerous basis changes in the above procedure, some changing only the basis of ${}^{\dagger}X$, some changing the basis of ${}^{\dagger}X \otimes X$. To get a representation of ev_X in the natural basis of ${}^{\dagger}X \otimes X$, we need to apply the inverse of all these transformations to Eq. (A.2.9). In its canonically ordered basis $\{e_1, e_2, e_3, e_4\}$, the differential of X is given by

$$d_X = \begin{pmatrix} 0 & u-u' \\ y-x & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & x-x' \\ a-u' & 0 \end{pmatrix} = \left(\begin{array}{cc|cc} & & u-u' & x-x' \\ & & u'-\hat{a} & y-x \\ \hline y-x & x'-x & & \\ \hat{a}-u' & u-u' & & \end{array}\right) .$$

With respect to the canonically ordered basis $\{e_3^*, e_4^*, e_1^*, e_2^*\}$ of ${}^{\dagger}X$ and $X^{\dagger}$ (see Lemma 1.3.28) the differentials take the matrix form

$$-d_{X^{\dagger}} = d_{{}^{\dagger}X} = \left(\begin{array}{cc|cc} & & u'-u & a-u' \\ & & x'-x & x-y \\ \hline y-x & a-u' & & \\ x'-x & u-u' & & \end{array}\right) .$$

Next, we write all the transformations of Eq. (A.2.2) as one large matrix. The domain of this transformation matrix has the basis (note the use of the non-canonically ordered basis of Notation 1.3.27)

$$(e_1^* \otimes e_1, e_1^* \otimes e_2, \ldots, e_4^* \otimes e_4)$$

and the codomain $K(\boldsymbol{p}; x'-x, 0, 0, 0)$ has the basis

$$(1 \otimes 1 \otimes 1 \otimes 1,\ 1 \otimes 1 \otimes 1 \otimes \theta_4,\ \ldots,\ \theta_1 \otimes \theta_2 \otimes \theta_3 \otimes \theta_4) .$$

We find the transformation matrix

$$\alpha \cdot \left(\begin{array}{cccc|cccc|cccc|cccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}\right) \tag{A.2.11}$$

for an arbitrary invertible α. Using Eq. (A.2.11), we can now rewrite Eq. (A.2.9):

$$\mathrm{ev}_X(g\, e_i^* \otimes e_j) = \alpha \cdot \operatorname{Res}\left[\frac{g \begin{pmatrix} \theta_a & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} \mathrm{d}u\, \mathrm{d}y\, \mathrm{d}u'}{u-a,\ y-x,\ u'-a}\right] . \tag{A.2.12}$$

Evaluating the left Zorro move

To fix α and verify that we indeed have an adjunction between X and ${}^\dagger X$, we need to evaluate the Zorro map

$$\mathcal{Z} = \rho_X \circ (1 \otimes \mathrm{ev}_X) \circ (\mathrm{coev}_X \otimes 1) \circ \lambda_X^{-1} \ .$$

It is sufficient to evaluate $\mathcal{Z}$ on the basis $\{e_i\}$ of X, as the linearity of $\mathcal{Z}$ in a, $\hat{a}$, x, and x' is easy to see. Furthermore, the matrix representations of λ_X^{-1} and coev_X consist only out of -1, 0, and 1, and the residue operator of Eq. (A.2.12) fulfils $\mathrm{Res}(1) = 1$. Thus, the Zorro move reduces to a large, but conceptually easy matrix computation (best done on a computer due to the appearance of 16×16 matrices). We find

$$\mathcal{Z}(e_i) = \alpha \cdot e_i \ ,$$

verifying that we indeed have an adjunction between X and ${}^\dagger X$ for $\alpha = 1$.

Comparison to the general formula

We expect Eq. (A.2.12) to be consistent with the general evaluation map (2.3.7) constructed in Chapter 2. It turns out that they are not equal, but homotopic. There is no single "canonical" representative of ev_X in its homotopy class — the method presented here singles out one representative, and Eq. (2.3.7) singles out a different one. More details on the exact maps in this spectrum and the explicit homotopy are discussed in Appendix A.3.3.

Constructing $\widetilde{\mathrm{coev}}_X$

While we do not need it, we can also write $\mathrm{ev}_X^{-1} = \eta \colon \mathbb{1}_W \to {}^\dagger X \otimes X$ in both bases. We find

$$\mathrm{ev}_X^{-1} = -(\delta_X^{-1} \otimes 1) \circ \widetilde{\mathrm{coev}}_X \colon \mathbb{1}_W \to {}^\dagger X \otimes X$$

where $\delta_X \colon {}^\dagger X \to X^\dagger = (-1)^{|-|}$ is the pivotality isomorphism of Corollary 2.4.12. This procedure is therefore able to construct coev_X and $\widetilde{\mathrm{coev}}_X$ as well. This was not used in this thesis since the formulas (2.1.11) and (2.1.12) for coev_X and $\widetilde{\mathrm{coev}}_X$ in $\mathcal{LG}_k$ generalise to the relevant examples in $\ddot{\mathsf{MF}}_k$ without need for modifications.

A.2.3 Constructing $\widetilde{\mathrm{ev}}_X$

The right adjoint is given by

$$X^\dagger \cong [\hat{u} - \hat{u}', x - \hat{y}] \otimes [\hat{u}' - a, x - x']$$

where a change of basis was applied, and the variables of $X^\dagger$ in $X \otimes X^\dagger$ were relabelled. Now

$$\begin{aligned}
X \otimes X^\dagger &\cong [u - u', y - x] \otimes [x - x', a - u'] \otimes [\hat{u} - \hat{u}', x - \hat{y}] \otimes [\hat{u}' - a, x - x'] && \mid (1.3.46) \\
&\cong [u - u', y - x] \otimes [a - u', x - x'] \otimes [x - \hat{y}, \hat{u} - \hat{u}'] \otimes [\hat{u}' - a, x - x'] && \mid (1.3.24) \\
&\cong [u - u', y - x] \otimes [a - u', 0] \otimes [x - \hat{y}, \hat{u} - \hat{u}'] \otimes [\hat{u}' - u', x - x'] && \mid (1.3.27) \\
&\cong [u - u', y - x] \otimes [a - u', 0] \otimes [x - \hat{y}, \hat{u} - u'] \otimes [\hat{u}' - u', \hat{y} - x'] && \mid (1.3.26) \\
&\cong [u - \hat{u}, y - x] \otimes [a - u', 0] \otimes [y - \hat{y}, \hat{u} - u'] \otimes [\hat{u}' - u', \hat{y} - x'] && \mid \text{L. } 1.3.10 \\
&\cong [u - \hat{u}, y - x] \otimes [y - \hat{y}, \hat{u} - u'] \otimes [u' - \hat{u}', x' - \hat{y}] \otimes [u' - a, 0] \\
&= \mathbb{1}_W \otimes K(u' - a;\, 0) \ .
\end{aligned}$$

We find $\partial_{x'}(V-W) = u'-a$, again showing a connection to Theorem 2.3.7 with the right admissible variable x'. The Koszul resolutions associated to $X \otimes X^\dagger$ and $\mathbb{1}_W$ are constructed analogously, with the isomorphisms

$$\begin{aligned} \pi &:= \operatorname{Res}\left[\frac{\bullet\, \mathrm{d}a}{a-u'}\right] \colon \mathbb{C}[a, x, x', u, u', y, \hat{u}, \hat{u}', \hat{y}] \to \mathbb{C}[x, x', u, u', y, \hat{u}, \hat{u}', \hat{y}] \ , \\ \psi &:= 1 \colon \mathbb{C}[x, x', u, u', y, \hat{u}, \hat{u}', \hat{y}] \to \mathbb{C}[a, x, x', u, u', y, \hat{u}, \hat{u}', \hat{y}] \ , \end{aligned}$$

and the lift

$$\alpha \cdot \theta_{i_1} \wedge \ldots \wedge \theta_{i_k} \mapsto \begin{cases} \operatorname{Res}\left[\frac{\alpha\, \mathrm{d}a}{a-u'}\right] \theta_{i_1} \wedge \ldots \wedge \theta_{i_k} & \{i_j\} \subset \{1, 2, 3\} \\ 0 & \text{else} \end{cases} \ .$$

After constructing the explicit transformation matrix, we find the following presentation of $\tilde{\mathrm{ev}}_X$ in the basis $\{e_i \otimes e_j^*\}$, which agrees with Eq. (2.3.6):

$$\tilde{\mathrm{ev}}_X(g\, e_i \otimes e_j^*) = \alpha \cdot \operatorname{Res}\left[\frac{g \begin{pmatrix} 0 & 0 & 0 & 0 \\ \theta_u - \theta_{u'} - \theta_u\theta_y\theta_{u'} & \theta_u\theta_y\theta_{u'} + \theta_{u'} & -\theta_u\theta_y - 1 & \theta_u\theta_{u'} \\ 0 & 0 & 0 & 0 \\ 1 - \theta_y\theta_{u'} & \theta_y\theta_{u'} & -\theta_y & \theta_{u'} \end{pmatrix}_{ij} \mathrm{d}a}{a-u'}\right] \ . \tag{A.2.13}$$

The right Zorro move then fixes $\alpha = 1$.

A.3 Constructing ev_X using computer algebra systems

The presented method makes the following assumptions:

(i) $V \in \mathbb{C}[\boldsymbol{z}, \boldsymbol{w}]$ and $W \in \mathbb{C}[\boldsymbol{x}, \boldsymbol{w}]$ are polynomials of order 2,

(ii) X is a finite-rank matrix factorisation of $V(\boldsymbol{z}, \boldsymbol{w}) - W(\boldsymbol{x}, \boldsymbol{w})$ over $\mathbb{C}[\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{w}]$,

(iii) All matrix elements of d_X are polynomials of order 1 in $\{\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{w}\}$.

These assumptions hold for e.g. for X as defined in Eq. (A.2.1) and for μ_{g_1,g_2} of Eq. (3.4.2). The goal is to compute the spectrum of maps ${}^\dagger X \otimes X \to \mathbb{1}_W$, to find a suitable candidate for ev_X, and then to evaluate the Zorro move to verify the candidate.

A.3.1 Setup

We adapt the notation of the previous appendix with

$$\begin{gathered} {}^\dagger X \otimes X = \bigwedge\left(\textstyle\bigoplus_{i=1}^4 R \cdot \vartheta_i\right) \ , \qquad d_{{}^\dagger X \otimes X} = d_{{}^\dagger X} + d_X \ , \qquad \mathbb{1}_W = S \oplus S \cdot \theta_a \ , \\ \begin{aligned} d_{{}^\dagger X} &= (u'-u)\vartheta_1^* + (y-x)\vartheta_1 + (a-u')\vartheta_2^* + (x'-x)\vartheta_2 \ , \\ d_X &= (u-u')\vartheta_3^* + (y-x)\vartheta_3 + (x-x')\vartheta_4^* + (\hat{a}-u')\vartheta_4 \ , \\ d_{\mathbb{1}_W} &= (a-\hat{a})\theta_a^* + (x'-x)\theta_a \ . \end{aligned} \end{gathered}$$

The relation to the canonically ordered bases of X and ${}^\dagger X$ is as follows:

$$\{e_1, e_2, e_3, e_4\} = \{1, \vartheta_3\vartheta_4, \vartheta_3, \vartheta_4\} \ , \qquad \{e_3^*, e_4^*, e_1^*, e_2^*\} = \{1, \vartheta_2\vartheta_1, \vartheta_1, \vartheta_2\} \ .$$

Because $R = S[u, y, u']$ is an infinite-rank S-module, certain restrictions must be made to get a finite problem that can be solved on a computer. The first step is to choose projections $\boldsymbol{p} := \{p_u, p_y, p_{u'}\} \subset S$ and then identify R with the expansion

$$R = \bigoplus_{(i_1,i_2,i_3)\in\mathbb{N}_0^3} S_{i_1,i_2,i_3}(u-p_u)^{i_1}(y-p_y)^{i_2}(u'-p_{u'})^{i_3} \tag{A.3.1}$$

with $S_{i_1,i_2,i_3} = S$. Setting $\boldsymbol{p}$ to zero is possible, but turns out to be a bad choice for the given problem. We write

$$Y := {}^{\dagger}X \otimes X = \bigoplus_{(i_1,i_2,i_3)\in\mathbb{N}_0^3} Y_{i_1,i_2,i_3}(u-p_u)^{i_1}(y-p_y)^{i_2}(u'-p_{u'})^{i_3}$$

where we identify $Y \cong R^{\oplus 16}$ and expand R as in Eq. (A.3.1), inducing an isomorphism $Y_{i_1,i_2,i_3} \cong (S_{i_1,i_2,i_3})^{\oplus 16}$.

A.3.2 The evaluation-like closed maps

For the types of matrix factorisations X considered in this appendix, the matrix elements of the morphisms 1_X, ρ_X, λ_X^{-1}, and coev_X are order 0 polynomials: They either are already of order 0, or they are given by divided differences of order 1 polynomials. Hence, all of these morphisms map order zero polynomials in their domain to order zero polynomials in their codomain. Therefore, demanding the Zorro map to be equal to 1_X forces ev_X to map certain order zero polynomials to non-trivial order zero polynomials. We get no constraint on the action of ev_X on polynomials of order ≥ 1 from the Zorro move. Let us therefore consider the following ansatz:

$$f\colon \alpha \cdot \vartheta_{i_1} \ldots \vartheta_{i_k} \mapsto \sum_{\substack{t\subset\{a\}\\ \ell(t)\equiv k \bmod 2}} \mathrm{Res}\left[\frac{\alpha \cdot f_{\{i_1,\,\ldots\,,i_k\},t}\,\mathrm{d}u\,\mathrm{d}y\,\mathrm{d}u'}{u-p_u,\; y-p_y,\; u'-p_{u'}}\right]. \tag{A.3.2}$$

The map f has 16 free coefficients, acts non-trivially on $Y_{0,0,0}$, and acts trivially on Y_{i_1,i_2,i_3} for $i_1+i_2+i_3 \geq 1$. For a fixed set of projections $\boldsymbol{p}$, one can then evaluate

$$\mathrm{d}f = d_{\mathbb{1}_W} \circ f - f \circ d_Y \overset{!}{=} 0$$

which is a system of linear equations for the $\{f_{\{i_1,\,\ldots\,,i_k\},t}\}$:

$$\left\{F_i(\{f_{\{i_1,\,\ldots\,,i_k\},t}\}) = 0 \,\middle|\, i = 1,\,\ldots\,,16\right\} \tag{A.3.3}$$

This system of linear equations can be solved on a computer algebra system. To prevent the computer from dividing by polynomials which are not invertible in S, it is helpful to expand each equation F_i in coefficients of $\{a, \hat{a}, x, x'\}$, yielding a larger set of linear equations whose solution does not contain any “illegal” quotients. Each solution of this system of linear equations corresponds to a closed map of the form of Eq. (A.3.2).

This procedure does not provide a systematic way to derive the projections $\boldsymbol{p}$. One could use the projections from the method explained in Appendix A.2, or make an educated guess. The latter can be guided by formally evaluating $\mathrm{d}f$ with generic projections, looking at the resulting terms involving $\{p_u, p_y, p_{u'}\}$, and guessing which terms they might be able to cancel.

In case of a bad guess the non-trivial maps in the spectrum will have infinitely many non-zero matrix elements, hence f will be null-homotopic.

For the choice $\{p_u, p_y, p_{u'}\} = \{a, x, a\}$ we find four generators of closed maps:

$$\begin{aligned} &f\colon \alpha \cdot e_i^* \otimes e_j \mapsto \mathrm{Res}\left[\frac{\alpha\, \phi_{ij}\, \mathrm{d}u\, \mathrm{d}y\, \mathrm{d}u'}{u-a,\; y-x,\; u'-a}\right] , \\ &\phi \in \left\{ \begin{pmatrix} \theta_a & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & \theta_a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \begin{pmatrix} 0 & 0 & 0 & 0 \\ \theta_a & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & \theta_a & 0 \end{pmatrix} \right\} . \end{aligned} \tag{A.3.4}$$

A.3.3 Exact maps

The next step is to check which part of the spectrum found above is exact. Let us first analyse the action of d_Y on the Y_{i_1,i_2,i_3}. By assumption, each matrix element of d_Y is an order 1 polynomial in the variables of R. We may therefore split the differential into the following parts:

$$\begin{aligned} &d_Y^{0,0,0}\colon Y_{i_1,i_2,i_3} \to Y_{i_1,i_2,i_3} , && d_Y^{1,0,0}\colon Y_{i_1,i_2,i_3} \to Y_{i_1+1,i_2,i_3} , \\ &d_Y^{0,1,0}\colon Y_{i_1,i_2,i_3} \to Y_{i_1,i_2+1,i_3} , && d_Y^{0,0,1}\colon Y_{i_1,i_2,i_3} \to Y_{i_1,i_2,i_3+1} . \end{aligned}$$

We make the following ansatz for a preimage of f under d:

$$g := g^{0,0,0} + g^{1,0,0} + g^{0,1,0} + g^{0,0,1} , \quad g^{i_1,i_2,i_3}\colon Y_{i_1,i_2,i_3} \to Y_{0,0,0} . \tag{A.3.5}$$

Because f acts non-trivially only on $Y_{0,0,0}$ and d_Y can change the expansion order by at most 1, no higher order terms are required. For the differential of g we find

$$\begin{aligned} \mathrm{d}g|_{Y_{0,0,0}} &= (g^{1,0,0}d_Y^{1,0,0} + g^{0,1,0}d_Y^{0,1,0} + g^{0,0,1}d_Y^{0,0,1}) + d_{\mathbb{1}_W}g^{0,0,0} + g^{0,0,0}d_Y^{0,0,0} , \\ \mathrm{d}g|_{Y_{1,0,0}} &= d_{\mathbb{1}_W}g^{1,0,0} + g^{1,0,0}d_Y^{0,0,0} , \\ \mathrm{d}g|_{Y_{0,1,0}} &= d_{\mathbb{1}_W}g^{0,1,0} + g^{0,1,0}d_Y^{0,0,0} , \\ \mathrm{d}g|_{Y_{0,0,1}} &= d_{\mathbb{1}_W}g^{0,0,1} + g^{0,0,1}d_Y^{0,0,0} , \\ \mathrm{d}g|_{Y_{i_1,i_2,i_3}} &= 0 \quad \text{for } i_1+i_2+i_3 \geq 2 . \end{aligned}$$

By assumption, the matrix elements of $d_Y^{0,0,0}$ are order 1 polynomials, and the matrix elements of all other parts of d_Y are order 0 polynomials. We find the following constraints for g:

- We may set $g^{0,0,0}$ to zero for the following reason: The closed generators of Eq. (A.3.4) are matrices of order 0 polynomials, and $\mathrm{d}g^{0,0,0}$ only contains order 1 polynomials, so $g^{0,0,0}$ cannot generate null-homotopies for the closed generators.
- The map f acts trivially on Y_{i_1,i_2,i_3} for $i_1+i_2+i_3 > 0$ by construction, so in order to construct null homotopies for any f, we demand $\mathrm{d}g|_{Y_{i_1,i_2,i_3}} \overset{!}{=} 0$ for $i_1+i_2+i_3 = 1$.

The general ansatz for g is thus

$$\begin{aligned} g\colon \alpha \cdot \vartheta_{j_1} \ldots \vartheta_{j_k} \mapsto \sum_{\substack{i_1,i_2,i_3 \geq 0 \\ i_1+i_2+i_3=1}} \; \sum_{\substack{t \subset \{a\} \\ \ell(t)+k \equiv 1 \bmod 2}} \theta_{t_1} \ldots \theta_{t_{\ell(t)}} \\ \cdot \mathrm{Res}\left[\frac{\alpha \cdot g^{i_1,i_2,i_3}_{\{j_1,\ldots,j_k\},t}\, \mathrm{d}u\, \mathrm{d}y\, \mathrm{d}u'}{(u-p_u)^{i_1+1},\; (y-p_y)^{i_2+1},\; (u'-p_{u'})^{i_3+1}}\right] . \end{aligned} \tag{A.3.6}$$

After solving $\mathrm{d}g|_{Y_{i_1,i_2,i_3}} \overset{!}{=} 0$ for $i_1+i_2+i_3 = 1$ in analogy to solving $\mathrm{d}f \overset{!}{=} 0$, we find the following three linearly independent exact generators in the notation of Eq. (A.3.4):

$$\phi_{ij} \in \left\{ \begin{pmatrix} \theta_a & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -\theta_a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \begin{pmatrix} 0 & 0 & 0 & 0 \\ \theta_a & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & \theta_a & 0 \end{pmatrix} \right\} . \tag{A.3.7}$$

A.3.4 The spectrum and Zorro move

Comparing the closed maps in Eq. (A.3.4) and the exact maps in Eq. (A.3.7), we find that the first and second generator in Eq. (A.3.4) are homotopic and both non-trivial, and the third and fourth are both null-homotopic. Furthermore, we have constructed the explicit homotopy discussed at the end of Appendix A.2.2. Because we found a one-dimensional spectrum there is only one candidate for ev_X. Now we have all the ingredients to evaluate the Zorro map and determine the prefactor as shown in Appendix A.2.2.

A.3.5 The right Zorro move

The approach is identical to the left Zorro move, so only the results will be stated. We find a system of 64 linear equations for the ansatz

$$f\colon X \otimes X^\dagger \to \mathbb{1}_V \ , \quad \alpha \cdot \vartheta_{i_1} \dots \vartheta_{i_k} \mapsto \sum_{\substack{t \subset \{u,y,u'\} \\ \ell(t) \equiv k \bmod 2}} \operatorname{Res}\left[\frac{\alpha \, f_{\{i_1,\dots,i_k\},t}\, \mathrm{d}a}{a - p_a}\right] \tag{A.3.8}$$

with $p_a = u'$, whose solution yields one closed generator

$$\phi\colon \alpha \cdot e_i \otimes e_j^* \mapsto \operatorname{Res}\left[\frac{g \begin{pmatrix} 0 & 0 & 0 & 0 \\ \theta_u - \theta_{u'} - \theta_u\theta_y\theta_{u'} & \theta_u\theta_y\theta_{u'} + \theta_{u'} & -\theta_u\theta_y - 1 & \theta_u\theta_{u'} \\ 0 & 0 & 0 & 0 \\ 1 - \theta_y\theta_{u'} & \theta_y\theta_{u'} & -\theta_y & \theta_{u'} \end{pmatrix}_{ij} \mathrm{d}a}{a - u'}\right] \tag{A.3.9}$$

which agrees with Eq. (A.2.13). The exact maps are constructed in an analogous way, and we find no exact generators in the shape of f whose matrix elements are order 0 polynomials. The right Zorro move is computed as explained above, showing that $\phi = \tilde{\mathrm{ev}}_X$ for $\alpha = 1$.

A.4 Various examples

A.4.1 The dual of an infinite-rank matrix factorisation

The identity $(X \otimes_R Y)^\vee \cong Y^\vee \otimes_R X^\vee$ of Lemma 1.3.29 only holds in a setting where $X \otimes_R Y$ and $Y^\vee \otimes_R X^\vee$ are finite-rank matrix factorisations, e.g. over $k[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$ for $R = k[\boldsymbol{y}]$. Over $k[\boldsymbol{x}, \boldsymbol{z}]$, the set

$$\{f_j^* \otimes \boldsymbol{y}^I \otimes e_i^*\}_{i,j;I \in \mathbb{N}_0^{\ell(\boldsymbol{y})}} \subset Y^\vee \otimes_R X^\vee$$

is a (countably) infinite basis, hence the module is an infinite direct sum, while

$$(X \otimes_R Y)^\vee \cong \operatorname{Hom}_{k[\boldsymbol{x},\boldsymbol{z}]}\big(\textstyle\bigoplus_{i,j} k[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}], k[\boldsymbol{x}, \boldsymbol{z}]\big)$$

has the structure of an infinite direct product: A $k[\boldsymbol{x}, \boldsymbol{z}]$-linear form $\alpha\colon k[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}] \to k[\boldsymbol{x}, \boldsymbol{z}]$ may assume non-zero values on an infinite number of basis elements $y_1^{n_1} \dots y_k^{n_k}$. Thus, there is no obvious candidate for an isomorphism. An explicit counterexample can also be constructed: Let $W := (x; W)$ for some $W \in \mathbb{C}[x]$, $X := Y := \mathbb{1}_W$, where we find

$$(X \otimes Y)^\vee[1] = (\mathbb{1}_W \otimes \mathbb{1}_W)^\dagger \cong \mathbb{1}_W^\dagger \cong \mathbb{1}_W \ , \quad (Y^\vee \otimes X^\vee)[1] \cong \mathbb{1}_W^\dagger \otimes \mathbb{1}_W^\dagger[1] \cong \mathbb{1}_W[1] \ncong \mathbb{1}_W \ .$$

A common way to work with the dual of an infinite-rank matrix factorisations is to use the idempotent pushforward (Theorem 2.3.14), as can be seen in the proof of Theorem 2.4.9.

A.4.2 Associativity in other conventions

The box product on 2-morphisms in $\mathcal{RW}$ is defined by Eq. (1.5.8), and is strictly associative by Lemma 1.5.6. We now compare $Z \boxtimes' (Y \boxtimes' X)$ and $(Z \boxtimes' Y) \boxtimes' X$ with $- \boxtimes' -$ defined by Eq. (1.5.11):

$$\begin{aligned} Z \boxtimes' (Y \boxtimes' X) &= Z(\boldsymbol{z}) \otimes I^{\boldsymbol{z} \leftarrow \boldsymbol{z}'}_{U_1(\boldsymbol{y}') + V_1(\boldsymbol{y}', \bullet) + W_1(\bullet)} \otimes (Y \boxtimes' X)(\boldsymbol{z}) \\ &= Z(\boldsymbol{z}) \otimes I^{\boldsymbol{z} \leftarrow \boldsymbol{z}'}_{V_1(\boldsymbol{y}', \bullet) + W_1(\bullet)} \otimes Y(\boldsymbol{y}, \boldsymbol{z}) \otimes I^{\boldsymbol{y} \leftarrow \boldsymbol{y}'}_{U_1(\bullet) + V_1(\bullet, \boldsymbol{z})} \otimes X(\boldsymbol{y}) \ , \\ (Z \boxtimes' Y) \boxtimes' X &= (Z \boxtimes' Y)(\boldsymbol{y}) \otimes I^{\boldsymbol{y} \leftarrow \boldsymbol{y}'}_{U_1(\bullet) + V_1(\bullet, \boldsymbol{z}') + W_1(\boldsymbol{z}')} \otimes X(\boldsymbol{y}) \\ &= Z(\boldsymbol{y}) \otimes I^{\boldsymbol{z} \leftarrow \boldsymbol{z}'}_{V_1(\boldsymbol{y}, \bullet) + W_1(\bullet)} \otimes Y(\boldsymbol{y}, \boldsymbol{z}) \otimes I^{\boldsymbol{y} \leftarrow \boldsymbol{y}'}_{U_1(\bullet) + V_1(\bullet, \boldsymbol{z}')} \otimes X(\boldsymbol{y}) \ . \end{aligned}$$

We find that both are related by the (non-trivial) isomorphism of Eq. (1.3.77), thus $- \boxtimes' -$ is not strictly associative. An analogous computation shows the same problem for Eq. (1.5.10), while both (1.5.8) and (1.5.9) are strictly associative (up to the caveats discussed in Section 1.3.5).

A.5 Different unitor conventions

A.5.1 Grade ambiguities in $\mathcal{RW}$

In Section 1.5 we have constructed λ_W (ρ_W) are as an equivalence between the 1-morphisms W and $\mathbb{1} \boxtimes W$ ($W \boxtimes \mathbb{1}$). As discussed in Lemma 1.4.8, such equivalences are not unique — grade-shifting both $(\lambda_W, \lambda_W^{-1})$ yields another valid equivalence. In fact, as far as the equivalence property is concerned, we are free to grade-shift both $(\lambda_W, \lambda_W^{-1})$ and (ρ_W, ρ_W^{-1}) independently and by a different amount for every 1-morphism W:

$$(\lambda_W, \lambda_W^{-1}) \mapsto (\lambda_W[s_{\lambda,W}], \lambda_W^{-1}[s_{\lambda,W}]) \ , \quad (\rho_W, \rho_W^{-1}) \mapsto (\rho_W[s_{\rho,W}], \rho_W^{-1}[s_{\rho,W}])$$

for arbitrary $s_{\lambda,W}, s_{\rho,W} \in \mathbb{Z}_2$. The other axioms of Definition 1.5.3 constrain this freedom, which we will discuss in detail here.

Consider two 1-morphisms $W, V \colon (\boldsymbol{x}) \to (\boldsymbol{y})$ and a 2-morphism $X \colon W \to V$. Let us spell out the naturality axiom (Definition 1.5.3 (v)) for the shifted λ_W and ρ_W:

$$\begin{aligned} \lambda_V[s_{\lambda,V}] \otimes (\mathbb{1}_{\mathbb{1}_{\boldsymbol{y}}} \boxtimes X) &\cong (\lambda_V \otimes (\mathbb{1}_{\mathbb{1}_{\boldsymbol{y}}} \boxtimes X))[s_{\lambda,V}] \cong (X \otimes \lambda_W)[s_{\lambda,V}] \\ &\cong (X \otimes \lambda_W[s_{\lambda,W}])[s_{\lambda,V} - s_{\lambda,W}] \overset{!}{\cong} X \otimes \lambda_W[s_{\lambda,W}] \ , \\ \rho_W[s_{\rho,V}] \otimes (X \boxtimes \mathbb{1}_{\mathbb{1}_{\boldsymbol{x}}}) &\cong (X \otimes \rho_W[s_{\rho,W}])[s_{\rho,V} - s_{\rho,W}] \overset{!}{\cong} X \otimes \rho_W[s_{\rho,W}] \ . \end{aligned}$$

We thus find the constraints

$$s_{\lambda,V} - s_{\lambda,W} \overset{!}{\equiv} s_{\rho,V} - s_{\rho,W} \overset{!}{\equiv} 0 \mod 2 \quad \text{for all } W, V\colon (\boldsymbol{x}) \to (\boldsymbol{y}) \ . \tag{A.5.1}$$

This implies that $s_{\lambda,W}$ and $s_{\rho,W}$ are not allowed to depend on the specific 1-morphism W, but only on the Hom-category W is contained in. The only quantitative difference between different Hom-categories is the number of variables in their domain and codomain objects, so the only data $s_{\lambda,W}$ and $s_{\rho,W}$ may depend on are $\ell(\boldsymbol{x})$ and $\ell(\boldsymbol{y})$. On the other hand, the constraints are automatically fulfilled if we set both shifts to be the same for all elements of some Hom-category:

Lemma A.5.1. *The naturality axiom implies that $s_{\lambda,W}$ and $s_{\rho,W}$ may only depend on $\ell(\boldsymbol{x})$ and $\ell(\boldsymbol{y})$, i.e.*

$$s_{\lambda,W} = s_\lambda\big(\ell(\boldsymbol{x}), \ell(\boldsymbol{y})\big) \ , \quad s_{\rho,W} = s_\rho\big(\ell(\boldsymbol{x}), \ell(\boldsymbol{y})\big) \ . \tag{A.5.2}$$

for arbitrary functions s_λ, $s_\rho\colon \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}_2$. In particular, $s_{\lambda,W}$ and $s_{\rho,W}$ are independent of all properties of W, including the number of surface variables of W.

Let us now consider the unity axiom (Definition 1.5.3 (vi)) for the shifted λ and ρ applied to $W\colon (\boldsymbol{x}) \to (\boldsymbol{y})$ and $V\colon (\boldsymbol{y}) \to (\boldsymbol{z})$:

$$\mathbb{1}_V \boxtimes \lambda_W[s_{\lambda,W}] \cong (\rho_V[s_{\rho,V}] \boxtimes \mathbb{1}_W)[s_{\lambda,W} - s_{\rho,V}] \overset{!}{\cong} \rho_V[s_{\rho,V}] \boxtimes \mathbb{1}_W \ . \tag{A.5.3}$$

We thus find

$$s_{\lambda,W} - s_{\rho,V} = s_\lambda\big(\ell(\boldsymbol{x}), \ell(\boldsymbol{y})\big) - s_\rho\big(\ell(\boldsymbol{y}), \ell(\boldsymbol{z})\big) \overset{!}{\equiv} 0 \mod 2 \tag{A.5.4}$$

for all $W\colon (\boldsymbol{x}) \to (\boldsymbol{y})$, $V\colon (\boldsymbol{y}) \to (\boldsymbol{z})$. It is apparent that s_λ cannot depend on $\ell(\boldsymbol{x})$ and s_ρ cannot depend on $\ell(\boldsymbol{z})$. All these results can be summarised as follows:

Theorem A.5.2. *Let $s_{\lambda,\rho}\colon \mathbb{Z} \to \mathbb{Z}_2$ be an arbitrary function, and let $W \in h\mathcal{RW}(\boldsymbol{x}, \boldsymbol{y})$. Then the most general grade shift which preserves the bicategory axioms is given by*

$$\begin{aligned}(\lambda_W, \lambda_W^{-1}) &\mapsto \Big(\lambda_W[s_{\lambda,\rho}(\ell(\boldsymbol{y}))], \lambda_W^{-1}[s_{\lambda,\rho}(\ell(\boldsymbol{y}))]\Big) \ , \\ (\rho_W, \rho_W^{-1}) &\mapsto \Big(\rho_W[s_{\lambda,\rho}(\ell(\boldsymbol{x}))], \rho_W^{-1}[s_{\lambda,\rho}(\ell(\boldsymbol{x}))]\Big) \ .\end{aligned} \tag{A.5.5}$$

Definition A.5.3. Let $s\colon \mathbb{Z} \to \mathbb{Z}_2$. We define $\mathcal{RW}_s$ to be equal to $\mathcal{RW}$ with λ_W and ρ_W grade-shifted by $s_{\lambda,\rho}(n) := s(n)$ according to Eq. (A.5.5), and the rest of the tricategory data of $\mathcal{RW}_s$ adapted in the natural way from $\mathcal{RW}$ (see Definition A.5.10 for more details).

A.5.2 Adjunctions in $\mathcal{RW}_s$

The adjunction 2-morphisms of Definition 3.2.1 can also be adapted to $\mathcal{RW}_s$.

Definition A.5.4. Let $W \in \mathcal{RW}_s(\boldsymbol{x}, \boldsymbol{y})$. We define the adjunction 2-morphisms in $\mathcal{RW}_s$ by

$$\mathrm{ev}'_W := \mathrm{ev}_W[s(\ell(\boldsymbol{x}))] \ , \qquad \mathrm{coev}'_W := \mathrm{coev}_W[s(\ell(\boldsymbol{y}))] \ , \tag{A.5.6}$$

relative to $\mathcal{RW}$, implying

$$\widetilde{\mathrm{ev}}'_W \cong \widetilde{\mathrm{ev}}_W[s(\ell(\boldsymbol{y}))] \ , \qquad \widetilde{\mathrm{coev}}'_W \cong \widetilde{\mathrm{coev}}_W[s(\ell(\boldsymbol{x}))] \ . \tag{A.5.7}$$

Lemma A.5.5. *Let $\mathcal{T} \subset \mathcal{RW}$ be a pivotal tricategory with duals which is closed under grade shifts of 2-morphisms. Then we may also consider $\mathcal{T}' \subset \mathcal{RW}_s$ which has the same objects and morphisms as $\mathcal{T}$, but grade-shifted unitor 2-morphisms as in Definition A.5.3. Then $\mathcal{T}'$ is also a pivotal tricategory with duals with the adjunction 2-morphisms of Eq.* (A.5.6)*.*

Proof. Pivotality on $\mathcal{T}(\boldsymbol{x}, \boldsymbol{y})$ does not depend on λ_W and ρ_W, and neither do the 2-functors $W \boxtimes -$ and $- \boxtimes W$. All that remains to be checked is the Zorro movie (1.2.18), which acquires a total grade shift of

$$2 \cdot s(\ell(\boldsymbol{x})) + 2 \cdot s(\ell(\boldsymbol{y})) \equiv 0 \mod 2 \ . \qquad \square$$

A.5.3 All $\mathcal{RW}_s$ are equivalent

The obvious question to ask is whether the $\mathcal{RW}_s$ describe fundamentally different structures for different functions $s \colon \mathbb{Z} \to \mathbb{Z}_2$. It turns out that they are equivalent as tricategories for all functions s, suggesting that the grade shifts of the unitor 2-morphisms have no fundamental impact:

Theorem A.5.6. *Let $\mathcal{S} \subset \mathcal{RW}$ be a tricategory that is closed under grade shifts of matrix factorisations. Then $\mathcal{S}$ is triequivalent to $\mathcal{S}_s \subset \mathcal{RW}_s$ for all $s \colon \mathbb{Z} \to \mathbb{Z}_2$. Furthermore, if $\mathcal{S}$ is a pivotal tricategory with duals, then $\mathcal{S} \cong \mathcal{S}_s$ in the sense of Definition 1.2.27.*

This section will present the details of the proof.

A.5.4 Proof setup

Notation A.5.7. To stay consistent with the notation of [82] we write l, r for the unitor 2-morphisms (i.e. the pseudonatural transformations) and λ, ρ for the unitor 3-morphisms (i.e. the unitors of the bicategories $\mathcal{T}(\boldsymbol{x}, \boldsymbol{y})$). For a 1-morphism $W \colon (\boldsymbol{x}) \to (\boldsymbol{y})$, $l_W = l(W)$ corresponds to the 2-morphism λ_W of Definition 1.5.9.

Notation A.5.8. Let $\mathcal{S}$, $\mathcal{T}$ be bicategories.

- A 2-functor $F \colon \mathcal{S} \to \mathcal{T}$ will be written as F on every level, i.e. for objects W, V, 1-morphisms $X, Y \colon W \to V$, 2-morphisms $\phi \colon X \to Y$, we write

  $$F(W) \in \mathcal{T}, \quad F(X) \in \mathcal{T}(F(W), F(V)), \quad F(\phi) \colon F(X) \to F(Y) \ . \tag{A.5.8}$$

  The natural transformation which is part of the 2-functor data (see [82, Def. A.3.6.(iii)]) will be written as

  $$\Phi^F_{W_1,W_2,W_3} \colon \otimes \left(F_{W_2,W_3} \times F_{W_1,W_2}\right) \to F_{W_1,W_3} \otimes \tag{A.5.9}$$

  with 2-morphisms

  $$\Phi^F_{W_1,W_2,W_3}(X_2, X_1) \colon F(X_2) \otimes F(X_1) \to F(X_2 \otimes X_1) \ . \tag{A.5.10}$$

- The identity 2-functor of $\mathcal{S}$ will be written as $1_{\mathcal{S}} \colon \mathcal{S} \to \mathcal{S}$.

- Let $F, G \colon \mathcal{S} \to \mathcal{T}$ be 2-functors and let $f \colon F \Rightarrow G$ be a pseudonatural transformation. For its components, we write

  $$f_W := f(W) \colon F(W) \to G(W) \ , \quad f_{W,V}(X) \colon f(V) \otimes F(X) \to G(X) \otimes f(W) \ . \tag{A.5.11}$$

- The identity pseudonatural transformation will be written as

$$1^F\colon F \Rightarrow F\ , \quad 1^F(W) = \mathbb{1}_W\ , \quad 1^F_{W,V}(X) = \rho_X^{-1} \circ \lambda_X\ .$$

Notation A.5.9. All structure data of $\mathcal{T}$ are written with primes to distinguish them from those of $\mathcal{S}$. All non-primed structure data either belong to $\mathcal{S}$ or are identical in both $\mathcal{S}$ and $\mathcal{T}$.

Definition A.5.10. We define the tricategory $\mathcal{T}$ to be the same as $\mathcal{S}$ up to the following redefinitions:

- The pseudonatural transformations l and r are changed as follows:

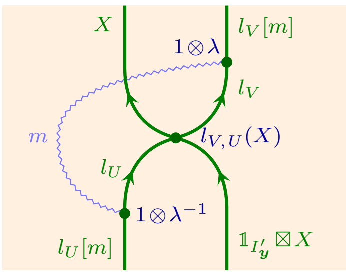
$$l'_W \in \mathrm{Hom}_{\mathcal{T}(\boldsymbol{x},\boldsymbol{y})}(I'_{\boldsymbol{y}} \boxtimes W \to W) := l_W\big[s(\ell(\boldsymbol{y}))\big]\ , \tag{A.5.12}$$
$$r'_W \in \mathrm{Hom}_{\mathcal{T}(\boldsymbol{x},\boldsymbol{y})}(W \boxtimes I'_{\boldsymbol{x}} \to W) := r_W\big[s(\ell(\boldsymbol{x}))\big]\ . \tag{A.5.13}$$

- The 3-morphism $l'_{V,U}(X)$ is defined by

$$l'_{V,U}(X)\colon l_U[m] \otimes (\mathbb{1}_{I'_{\boldsymbol{y}}} \boxtimes X) = S[m] \otimes l_U \otimes (\mathbb{1}_{I'_{\boldsymbol{y}}} \boxtimes X) \tag{A.5.14}$$
$$\xrightarrow{1\otimes\lambda^{-1}_{l_U}\otimes 1} \mathbb{1}_U[m] \otimes l_U \otimes (\mathbb{1}_{I'_{\boldsymbol{y}}} \boxtimes X) \tag{A.5.15}$$
$$\xrightarrow{1\otimes l_{V,U}(X)} \mathbb{1}_U[m] \otimes X \otimes l_V \tag{A.5.16}$$
$$\xrightarrow{\omega_X\otimes 1} X \otimes \mathbb{1}_V[m] \otimes l_V \tag{A.5.17}$$
$$\xrightarrow{1\otimes\lambda_{R[m]\otimes l_V}} X \otimes l_V[m]\ , \tag{A.5.18}$$

with ω of Definition 2.4.5 being the intersection between a shifted identity 2-morphism and another 2-morphism. Writing $m := s(\ell(\boldsymbol{y}))$ with $U, V\colon (\boldsymbol{x}) \to (\boldsymbol{y})$, $l'_{V,U}(X)$ corresponds to the diagram

$$l'_{V,U}(X) = \tag{A.5.19}$$



.

The definition of $r'_{V,U}(X)$ is analogous.

- The modifications μ', λ', and ρ' are adapted in the obvious way.
- If $\mathcal{S}$ is a pivotal tricategory with duals, all adjunction 2-morphisms are grade-shifted according to Definition A.5.4.

We now construct a triequivalence between $\mathcal{S}$ and $\mathcal{T}$.

Theorem A.5.11. *The following defines a triequivalence from $\mathcal{S}$ to $\mathcal{T}$ in the notation of [82, Def. A.4.3]:*

- *The function $F_0\colon \mathsf{Obj}(\mathcal{S}) \to \mathsf{Obj}(\mathcal{T})$ is the identity.*
- *The 2-functors $F_{\boldsymbol{x},\boldsymbol{y}}\colon \mathcal{S}(\boldsymbol{x}, \boldsymbol{y}) \to \mathcal{T}(\boldsymbol{x}, \boldsymbol{y})$ are identity 2-functors.*
- *The pseudonatural transformations $\chi_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}$ are identity 2-natural transformations.*
- *The invertible modifications $\omega_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{w}}$ are defined like in the identity 3-functor.*
- *The pseudonatural transformation $\iota_{\boldsymbol{x}}$ is defined as follows: We write $\{1, 1_1, 1_{1_1}\}$ for the unit 2-category. Then we define $W(\boldsymbol{a}) := \boldsymbol{a}\cdot(\boldsymbol{x}-\boldsymbol{x}')$, $m := \ell(\boldsymbol{x})$, and*

$$\begin{aligned} \iota_{\boldsymbol{x}}\colon (I'_{F_0(\boldsymbol{x})} =)\ I'_{\boldsymbol{x}} &\to I'_{\boldsymbol{x}}\ (= F\circ I_{\boldsymbol{x}})\ , \\ \iota_{\boldsymbol{x}}(1) := \mathbb{1}_W[m] = \mathbb{1}_{I'_{\boldsymbol{x}}}[m] &= I_W^{\boldsymbol{a}\leftarrow\boldsymbol{a}'}[m]\colon \\ &\qquad I'_{\boldsymbol{x}}(1) = (\boldsymbol{a}';\, W(\boldsymbol{a}')) \to (F\circ I_{\boldsymbol{x}})(1) = (\boldsymbol{a};\, W(\boldsymbol{a}))\ , \\ \iota_{\boldsymbol{x}}(1_1) := \lambda^{-1}_{\iota_{\boldsymbol{x}}(1)} \circ \rho_{\iota_{\boldsymbol{x}}(1)}\colon &\iota_{\boldsymbol{x}}(1) \otimes I'_{F_0(\boldsymbol{x})}(1_1) = \mathbb{1}_W[m] \otimes \mathbb{1}_W \\ &\qquad \to\ (F\circ I_{\boldsymbol{x}})(1_1) \otimes \iota_{\boldsymbol{x}}(1) = \mathbb{1}_W \otimes \mathbb{1}_W[m]\ . \end{aligned}$$

- *The modification γ is defined below in Eq. (A.5.24), and the modification δ is defined analogously.*

Theorem A.5.11 implies Theorem A.5.6: Because the function F_0 and the 2-functors $F_{\boldsymbol{x},\boldsymbol{y}}$ are identities, F is triessentially surjective. Furthermore, if $\mathcal{S}$ is pivotal, the $F_{\boldsymbol{x},\boldsymbol{y}}$ are pivotal because they are identities.

By [82, Def. A.4.3], the non-trivial statements of Theorem A.5.11 are the following:

- $\iota_{\boldsymbol{x}}$ is an adjoint equivalence,
- the two identities in [82, Def. A.4.3.(vi)] hold.

A.5.5 Verifying that ι is an adjoint equivalence

The following is required for $\iota_{\boldsymbol{x}}$ to be an adjoint equivalence:

- $\iota_{\boldsymbol{x}}(1_1)$ is a natural transformation,
- the assumptions a) and b) of [82, Def. A.3.10] hold,
- $\iota_{\boldsymbol{x}}$ has an adjoint inverse $\iota_{\boldsymbol{x}}^{-1}$.

The first of these four conditions is trivially fulfilled: The category $\{1_1, 1_{1_1}\}$ only has an identity morphism, on which the defining relation of natural transformations is trivial. We proceed with the second condition:

Lemma A.5.12. *[82, Def. A.3.10 a)] holds for $\iota_{\boldsymbol{x}}$.*

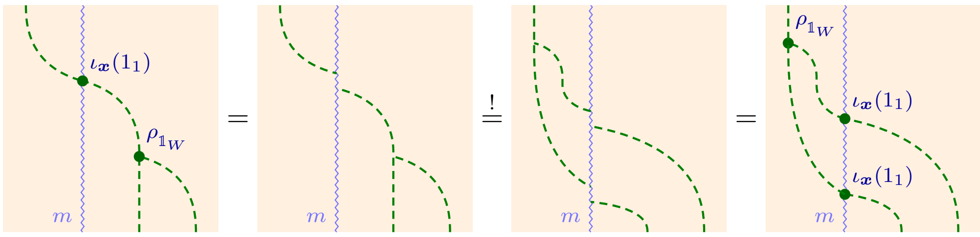
Proof. We first write out the functors $\Phi_{1,1}$ and $\Psi_{1,1}$ which are part of the 2-functors $G := I'_{F_0(\boldsymbol{x})} = I'_{\boldsymbol{x}}$ and $H := F \circ I_{\boldsymbol{x}} = I'_{\boldsymbol{x}}$ (so $\Phi_{1,1} = \Psi_{1,1}$):

$$\begin{aligned}\Phi_{1,1} = \rho_{\mathbb{1}_W} = \lambda_{\mathbb{1}_W} \colon I'_{\boldsymbol{x}}(1_1) \otimes I'_{\boldsymbol{x}}(1_1) = \mathbb{1}_W \otimes \mathbb{1}_W \\ \to I'_{\boldsymbol{x}}(1_1 \otimes 1_1) = I'_{\boldsymbol{x}}(1_1) = \mathbb{1}_W\end{aligned}$$

where the second equality is again up to homotopy. In the identity 2-category, there exists only one pair of morphisms $(1_1 \colon 1 \to 1, 1_1 \colon 1 \to 1)$. We insert this pair into the condition and find

$$\begin{aligned}\iota_{\boldsymbol{x}}(1_1) \circ (1 \otimes \rho_{\mathbb{1}_W}) \overset{!}{=} (\rho_{\mathbb{1}_W} \otimes 1) \circ (1 \otimes \iota_{\boldsymbol{x}}(1_1)) \circ (\iota_{\boldsymbol{x}}(1_1) \otimes 1) \colon \\ \mathbb{1}_W[m] \otimes \mathbb{1}_W \otimes \mathbb{1}_W \to \mathbb{1}_W \otimes \mathbb{1}_W[m] \ .\end{aligned}$$

This identity corresponds to the following diagram in $\mathcal{T}$, where we use the shifted identity lines defined in Notation 2.4.3:



(A.5.20)

Any junction without a label is one of λ, λ^{-1}, ρ, ρ^{-1}. As all points in these diagram are unitors, we can easily argue that both diagrams are identical using either the bicategory axioms of $\ddot{\mathsf{MF}}_k$, Remark 1.5.5, or a coherence argument. □

Lemma A.5.13. *[82, Def. A.3.10 b)] holds for* $\iota_{\boldsymbol{x}}$*.*

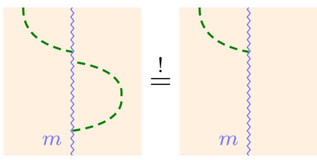
Proof. The morphisms $\Phi_1 = \Psi_1$ which are part of the 2-functors $G = H = I'_{\boldsymbol{x}}$ are defined by

$$\Phi_1 = 1_{I_W^{\boldsymbol{a} \leftarrow \boldsymbol{a}'}} \colon (I_W^{\boldsymbol{a} \leftarrow \boldsymbol{a}'} = \mathbb{1}_{I'_{\boldsymbol{x}}(1)}) \to (I'_{\boldsymbol{x}}(1_1) = I_W^{\boldsymbol{a} \leftarrow \boldsymbol{a}'}) \ .$$

The 3-morphisms λ and ρ are not spelled out in [82]. With these reintroduced, the condition reads

$$\iota_{\boldsymbol{x}}(1_1) \circ \rho^{-1}_{\iota_{\boldsymbol{x}}(1)} \overset{!}{=} \lambda^{-1}_{\iota_{\boldsymbol{x}}(1)} \colon \mathbb{1}_W[m] \to \mathbb{1}_W \otimes \mathbb{1}_W[m] \ .$$

The corresponding diagram identity is



which follows directly from the bicategory axioms (specifically, the property that ρ_W is an isomorphism for all W). □

Lemma A.5.14. *The pseudonatural transformation* $\iota_{\boldsymbol{x}} \colon I'_{\boldsymbol{x}} \to I'_{\boldsymbol{x}}$ *is self-adjoint (implying that it is self-inverse), meaning that there are invertible modifications*

$$\alpha \colon \iota_{\boldsymbol{x}} \circ \iota_{\boldsymbol{x}} \Rrightarrow 1_{I'_{\boldsymbol{x}}} \ , \qquad \beta \colon 1_{I'_{\boldsymbol{x}}} \Rrightarrow \iota_{\boldsymbol{x}} \circ \iota_{\boldsymbol{x}} \tag{A.5.21}$$

which fulfil the two Zorro moves (see Definition 1.2.15).

Proof. The identity pseudonatural transformation of the 2-functor $I'_{\boldsymbol{x}}$ is given by

$$\begin{aligned} 1^{I'_{\boldsymbol{x}}}(1) &= \mathbb{1}_{I'_{\boldsymbol{x}}(1)} = \mathbb{1}_W \colon I'_{\boldsymbol{x}}(1) \to I'_{\boldsymbol{x}}(1) \ , \\ 1^{I'_{\boldsymbol{x}}}_{1,1}(1_1) &= \rho^{-1}_{I'_{\boldsymbol{x}}(1_1)} \circ \lambda_{I'_{\boldsymbol{x}}(1_1)} \colon 1^{I'_{\boldsymbol{x}}}(1) \otimes I'_{\boldsymbol{x}}(1_1) = \mathbb{1}_W \otimes \mathbb{1}_W \\ &\qquad \to I'_{\boldsymbol{x}}(1_1) \otimes 1^{I'_{\boldsymbol{x}}}(1) = \mathbb{1}_W \otimes \mathbb{1}_W \ . \end{aligned}$$

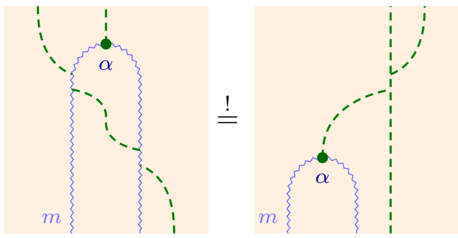
The modifications α and β map objects of the unit bicategory to 2-morphisms in $\mathcal{T}(\boldsymbol{x}, \boldsymbol{x})$ (which are 3-morphisms in $\mathcal{T}$):

$$\begin{aligned} 1_{I'_{\boldsymbol{x}}}(1) &= \mathbb{1}_{I'_{\boldsymbol{x}}} = \mathbb{1}_W \ , & \alpha(1) &\colon \mathbb{1}_W[m] \otimes \mathbb{1}_W[m] \to \mathbb{1}_W \ , \\ \iota_{\boldsymbol{x}}(1) \otimes \iota_{\boldsymbol{x}}(1) &= \mathbb{1}_W[m] \otimes \mathbb{1}_W[m] \ , & \beta(1) &\colon \mathbb{1}_W \to \mathbb{1}_W[m] \otimes \mathbb{1}_W[m] \ , \end{aligned}$$

which we define to be μ and μ^{-1} of Notation 2.4.3. The invertibility of $\alpha(1)$ and $\beta(1)$ as well as their Zorro moves are discussed in [22]. It only remains to be shown that α and β are modifications. For α we find the condition

$$\begin{aligned} (1 \otimes \alpha(1)) \circ (\iota_{\boldsymbol{x}}(1_1) \otimes 1) \circ (1 \otimes \iota_{\boldsymbol{x}}(1_1)) &\stackrel{!}{=} 1^{I'_{\boldsymbol{x}}}_{1,1}(1_1) \circ (\alpha(1) \otimes 1) \colon \\ \iota_{\boldsymbol{x}}(1) \otimes \iota_{\boldsymbol{x}}(1) \otimes I'_{\boldsymbol{x}}(1_1) &\to I'_{\boldsymbol{x}}(1_1) \otimes 1_{I'_{\boldsymbol{x}}}(1) \end{aligned}$$

which can be displayed as follows:



This again follows directly from Remark 1.5.5. The proof for β is analogous. □

Remark A.5.15. It is noteworthy that there is no idempocy condition for $\iota_{\boldsymbol{x}}(1)$, because in general $\iota_{\boldsymbol{x}}(1) \colon I'_{F(\boldsymbol{x})} \to F(I_{\boldsymbol{x}})$ cannot be composed with itself.

A.5.6 The remaining 3-functor axioms

Let us spell out the first axiom of [82, Def. A.4.3.(vi)] (the second axiom is analogous). Because $\mathcal{S}$ and $\mathcal{T}$ are strictly associative and χ is the identity pseudonatural transformation, the axiom simplifies to

$$\begin{tikzcd} & \mathcal{T}^2 \arrow[dr, "\boxtimes"] \arrow[d, phantom, "\Downarrow l"] & \\ \mathcal{T} \arrow[ur, bend left=50, "I'_{\boldsymbol{y}} \times 1"] \arrow[ur, "I'_{\boldsymbol{y}} \times 1"'] \arrow[ur, phantom, bend left=25, "\Downarrow \iota_{\boldsymbol{y}} \times 1"] \arrow[rr, "1"'] & {} & \mathcal{T} \end{tikzcd} \overset{\gamma}{\Rrightarrow} \begin{tikzcd} & \mathcal{T}^2 \arrow[dr, "\boxtimes"] \arrow[d, phantom, "\Downarrow l'"] & \\ \mathcal{T} \arrow[ur, "I'_{\boldsymbol{y}} \times 1"] \arrow[rr, "1"'] & {} & \mathcal{T} \end{tikzcd} \ . \tag{A.5.22}$$

We denote the composition of $\iota_{\boldsymbol{y}} \times 1$ and l by f. Spelling out f involves the natural isomorphism

$$\Phi^{\boxtimes \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}}_{(V_1, W_1), (V_2, W_2), (V_3, W_3)} \colon \otimes \left(\boxtimes^{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}}_{(V_2, W_2), (V_3, W_3)} \times \boxtimes^{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}}_{(V_1, W_1), (V_2, W_2)} \right) \to \boxtimes^{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}}_{(V_1, W_1), (V_3, W_3)} \otimes$$

which is part of the data of the 2-functor $\boxtimes^{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}\colon \mathcal{T}(\boldsymbol{y},\boldsymbol{z})\times\mathcal{T}(\boldsymbol{x},\boldsymbol{y})\to\mathcal{T}(\boldsymbol{x},\boldsymbol{z})$. In particular,

$$\Phi = \Phi^{\boxtimes^{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}}_{(V_1,W_1),(V_2,W_2),(V_3,W_3)}\big((Y_2,X_2),(Y_1,X_1)\big)\colon \\ (Y_2\boxtimes X_2)\otimes(Y_1\boxtimes X_1)\to(Y_2\otimes Y_1)\boxtimes(X_2\otimes X_1) \tag{A.5.23}$$

is the isomorphism of Lemma 1.5.8.

Now f has the following action on 1-morphisms:

$$f(V)=l(V)\otimes\boxtimes(\iota_{\boldsymbol{y}},1_{1_{\mathcal{T}(\boldsymbol{x},\boldsymbol{y})}})(1,V)=l_V\otimes(\mathbb{1}_{I'_{\boldsymbol{y}}}[m]\boxtimes\mathbb{1}_V)\colon I'_{\boldsymbol{y}}\boxtimes W\to W\ .$$

On a 2-morphism $X\colon V\to U$, we find the action

$$\begin{aligned} f_{V,U}(X)\colon\quad f(U)\otimes(\boxtimes(I'_{\boldsymbol{y}}\times 1))(1_1,X) &= l_U\otimes(\mathbb{1}_{I'_{\boldsymbol{y}}}[m]\boxtimes\mathbb{1}_U)\otimes(\mathbb{1}_{I'_{\boldsymbol{y}}}\boxtimes X)\\ &\xrightarrow{1_{l(U)}\otimes\Phi} l_U\otimes\boxtimes\big((\mathbb{1}_{I'_{\boldsymbol{y}}}[m]\otimes\mathbb{1}_{I'_{\boldsymbol{y}}}),(\mathbb{1}_U\otimes X)\big)\\ &\xrightarrow{1_{l(U)}\otimes\boxtimes(\iota_{\boldsymbol{y}}(1_1),\,\rho_X^{-1}\circ\lambda_X)} l_U\otimes\boxtimes\big((\mathbb{1}_{I'_{\boldsymbol{y}}}\otimes\mathbb{1}_{I'_{\boldsymbol{y}}}[m]),(X\otimes\mathbb{1}_V)\big)\\ &\xrightarrow{1_{l(U)}\otimes\Phi^{-1}} l_U\otimes(\mathbb{1}_{I'_{\boldsymbol{y}}}\boxtimes X)\otimes(\mathbb{1}_{I'_{\boldsymbol{y}}}[m]\boxtimes\mathbb{1}_V)\\ &\xrightarrow{l_{V,U}(X)\otimes 1_{\boxtimes(\iota_{\boldsymbol{y}}\times 1)(V)}} X\otimes l_V\otimes(\mathbb{1}_{I'_{\boldsymbol{y}}}[m]\boxtimes\mathbb{1}_V)\\ &= 1_{\mathcal{T}(\boldsymbol{x},\boldsymbol{y})}(X)\otimes f(V)\ . \end{aligned}$$

We first realise that $\mathbb{1}_{I'_{\boldsymbol{y}}}[m]\boxtimes\mathbb{1}_V\cong\mathbb{1}_{I'_{\boldsymbol{y}}\boxtimes V}[m]$ is the grade-shifted identity matrix factorisation, and the isomorphism looks like the identity map (the only difference between both expressions is the grade-shifted ring to the left). Now we construct an invertible modification $\gamma\colon f\Rrightarrow l'$:

$$\gamma(V)\colon f(V)=l_V\otimes(\mathbb{1}_{I'_{\boldsymbol{y}}}[m]\boxtimes\mathbb{1}_V)\to l'(V)=l_V[m]\ ,$$

$$l_V\otimes(\mathbb{1}_{I'_{\boldsymbol{y}}}[m]\boxtimes\mathbb{1}_V)\cong l_V\otimes\mathbb{1}_{I'_{\boldsymbol{y}}\boxtimes V}[m]\xrightarrow{\omega^{-1}_{l_V}}\mathbb{1}_V[m]\otimes l_V\xrightarrow{1\otimes\lambda_{l_V}} l_V[m]\ . \tag{A.5.24}$$

In particular, $\gamma(V)$ is an isomorphism. It remains to be checked that γ is a modification, meaning that the following must hold for all $X\colon V\to U$:

$$(1_X\otimes\gamma(V))\circ f_{V,U}(X)\overset{!}{=}l'_{V,U}(X)\circ(\gamma(U)\otimes 1_{\mathbb{1}_{I'_{\boldsymbol{y}}}\boxtimes X})\colon \\ l_U\otimes(\mathbb{1}_{I'_{\boldsymbol{y}}}[m]\boxtimes\mathbb{1}_U)\otimes(\mathbb{1}_{I'_{\boldsymbol{y}}}\boxtimes X)\to X\otimes l_V[m]\ . \tag{A.5.25}$$

We first show the following lemma in order to simplify $f_{V,U}(X)$:

Lemma A.5.16. *The identity*

$$\Phi^{-1}\circ(\iota_{\boldsymbol{y}}(1_1)\boxtimes(\rho_X^{-1}\circ\lambda_X))\circ\Phi=\omega_{\mathbb{1}_{I'_{\boldsymbol{y}}}\boxtimes X} \tag{A.5.26}$$

holds, where $\mathbb{1}_{I'_{\boldsymbol{y}}}[m]\boxtimes\mathbb{1}_U\cong\mathbb{1}_{I'_{\boldsymbol{y}}\boxtimes U}[m]$ *is used implicitly.*

Proof sketch. While a bit arduous, the proof is ultimately straightforward. We first post-compose both sides of Eq. (A.5.26) with the isomorphism $(\lambda_{\mathbb{1}[m]}\boxtimes\rho_X)\circ\Phi$, yielding

$$(\rho_{\mathbb{1}[m]}\boxtimes\lambda_X)\circ\Phi=(\lambda_{\mathbb{1}[m]}\boxtimes\rho_X)\circ\Phi\circ\omega_{\mathbb{1}_{I'_{\boldsymbol{y}}}\boxtimes X}\ . \tag{A.5.27}$$

Next we write out both sides of the equation explicitly. After performing some non-trivial cancellations and permutations, we find that both sides agree up to the following term:

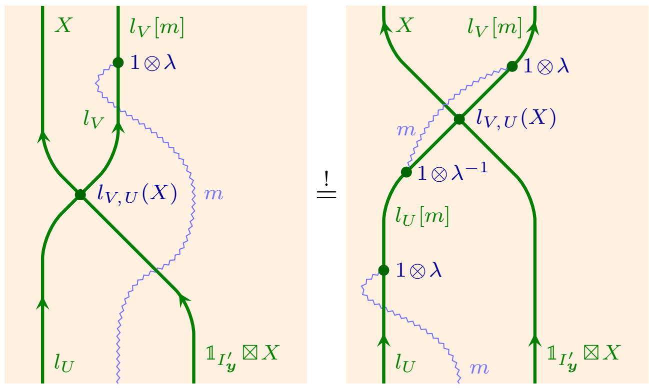
$$\begin{aligned}&\rho_{I\otimes I}^{-1}\circ(\lambda_I\otimes 1_I)\circ(1_I\otimes\lambda_{I\otimes I}) \overset{?}{=} \rho_{I^{\otimes 3}}:\\ &\qquad I^{\boldsymbol{u}\leftarrow\boldsymbol{u}'}_{\dots(\boldsymbol{y})}\otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{\dots(\boldsymbol{u}')}\otimes I^{\boldsymbol{u}'\leftarrow\boldsymbol{u}''}_{\dots(\boldsymbol{y}')}\otimes I^{\boldsymbol{y}'\leftarrow\boldsymbol{y}''}_{\dots(\boldsymbol{u}'')}\to I^{\boldsymbol{u}\leftarrow\boldsymbol{u}'}_{\dots(\boldsymbol{y})}\otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}''}_{\dots(\boldsymbol{u}')}\otimes I^{\boldsymbol{u}'\leftarrow\boldsymbol{u}''}_{\dots(\boldsymbol{y}'')}\ .\end{aligned} \tag{A.5.28}$$

We post-compose both sides with the isomorphism $\rho_{I\otimes I}$ and with π_Δ (see Definition 2.3.11 for the latter), yielding

$$\begin{aligned}&\pi_\Delta\circ(\lambda_I\otimes 1)\circ(1\otimes\lambda_{I\otimes I}) \overset{?}{=} \pi_\Delta\circ\rho_{I\otimes I}\circ\rho_{I^{\otimes 3}}:\\ &\qquad I^{\boldsymbol{u}\leftarrow\boldsymbol{u}'}_{\dots(\boldsymbol{y})}\otimes I^{\boldsymbol{y}\leftarrow\boldsymbol{y}'}_{\dots(\boldsymbol{u}')}\otimes I^{\boldsymbol{u}'\leftarrow\boldsymbol{u}''}_{\dots(\boldsymbol{y}')}\otimes I^{\boldsymbol{y}'\leftarrow\boldsymbol{y}''}_{\dots(\boldsymbol{u}'')}\to k[\boldsymbol{u},\boldsymbol{u}'',\boldsymbol{y},\boldsymbol{y}'']/(\boldsymbol{u}-\boldsymbol{u}'',\boldsymbol{y}-\boldsymbol{y}'')\ .\end{aligned} \tag{A.5.29}$$

It is easy to see that both sides project to θ-order zero in all four identity matrix factorisations and identify $\boldsymbol{u}=\boldsymbol{u}'=\boldsymbol{u}''$, $\boldsymbol{y}=\boldsymbol{y}'=\boldsymbol{y}''$, hence Eq. (A.5.29) holds. Furthermore, by the unique lifting theorem of [22, Section 4], post-composing with π_Δ is an isomorphism on the homomorphisms, thus Eq. (A.5.29) implies Eq. (A.5.28). □

Using the formula of γ_V given in Eq. (A.5.24), the formula of $l'_{V,U}(X)$ given in Eq. (A.5.14), and Lemma A.5.16, we can transform Eq. (A.5.25) into the following diagram identity:



(A.5.30)

This identity can be proven using the invertibility of λ and the wiggly line calculus of Section 2.4, concluding the proof of Theorem A.5.6.

A.6 Conventions in 3D $\mathcal{N}=2$

A.6.1 Notation

We mostly follow the notation and conventions of [38] which we recall for convenience.

Spacetime Our (half-)spacetime is given by

$$M=\{(x^0,x^1,x^2)\mid x^1\le 0\}, \tag{A.6.1}$$

with mostly-plus metric $\eta_{\mu\nu}=\mathrm{diag}(-1,1,1)$. We most frequently use light-cone coordinates

$$x^{\pm}=x^0\pm x^2,\quad x_{\pm}=\tfrac{1}{2}(x_0\pm x_2),\quad x^1=x_1=x^{\perp}, \tag{A.6.2}$$

where the metric reads

$$\eta_{\mu\nu} = \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad \eta^{\mu\nu} = \begin{pmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{A.6.3}$$

The Levi–Civita symbol is defined by $\epsilon_{012} = -1$, $\epsilon^{012} = 1$. In light-cone coordinates it is $\epsilon_{+-\perp} = -\frac{1}{2}$, $\epsilon^{+-\perp} = 2$. It satisfies

$$\begin{aligned} \epsilon_{\mu\nu\lambda}\epsilon^{\sigma\rho\lambda} &= \delta_\mu{}^\rho \delta_\nu{}^\sigma - \delta_\mu{}^\sigma \delta_\nu{}^\rho, \\ \epsilon_{\mu\rho\lambda}\epsilon^{\nu\rho\lambda} &= -2\delta_\mu{}^\nu. \end{aligned} \tag{A.6.4}$$

Spinors Spinors in 3D are $SL(2,\mathbb{R})$ fundamental representations, i.e. two component spinors ψ_α, $\alpha \in \{1,2\} = \{-,+\}$. Indices are raised and lowered by $\epsilon_{\alpha\beta}$, $\epsilon^{\alpha\beta}$, where $\epsilon_{12} = -1$, $\epsilon^{12} = 1$ according to the rule

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta \ , \quad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta. \tag{A.6.5}$$

Explicitly, we have

$$\psi^\alpha = \begin{pmatrix} \psi^- \\ \psi^+ \end{pmatrix} = \begin{pmatrix} \psi_+ \\ -\psi_- \end{pmatrix}, \quad \psi_\alpha = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} = \begin{pmatrix} -\psi^+ \\ \psi^- \end{pmatrix}. \tag{A.6.6}$$

Indices that are contracted "from top to bottom" are omitted:

$$\psi\chi \coloneqq \psi^\alpha\chi_\alpha = \psi^-\chi_- + \psi^+\chi_+. \tag{A.6.7}$$

Note that $\psi\chi = \chi\psi$. Since Hermitian conjugation flips the order of spinors without flipping index position, we have that $\overline{\psi\chi} = -\bar{\psi}\bar{\chi}$.

Some useful identities are given by

$$\begin{aligned} \psi\psi &= 2\psi^+\psi^-, \\ \psi^\alpha\psi^\beta &= -\tfrac{1}{2}(\psi\psi)\epsilon^{\alpha\beta}, \\ \psi_\alpha\psi_\beta &= \tfrac{1}{2}(\psi\psi)\epsilon_{\alpha\beta}. \end{aligned} \tag{A.6.8}$$

Clifford algebra We use the real gamma matrices

$$\gamma^\mu_{\alpha\beta} = (\gamma^0_{\alpha\beta}, \gamma^1_{\alpha\beta}, \gamma^2_{\alpha\beta}) = (-\mathbb{1}, \sigma^1, \sigma^3). \tag{A.6.9}$$

In light-cone coordinates these read explicitly

$$\gamma^\mu_{\alpha\beta} = (\gamma^+_{\alpha\beta}, \gamma^-_{\alpha\beta}, \gamma^\perp_{\alpha\beta}) = \left(\left(\begin{smallmatrix} 0 & 0 \\ 0 & -2 \end{smallmatrix}\right), \left(\begin{smallmatrix} -2 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \right). \tag{A.6.10}$$

They are symmetric $\gamma^\mu_{\alpha\beta} = \gamma^\mu_{\beta\alpha}$, real, and satisfy the Clifford algebra

$$(\gamma^\mu\gamma^\nu)_\alpha{}^\beta = \eta^{\mu\nu}\delta_\alpha{}^\beta + \epsilon^{\mu\nu\rho}(\gamma_\rho)_\alpha{}^\beta. \tag{A.6.11}$$

A useful list of identities follows from these:

$$\begin{aligned} (\gamma^\mu)_{\alpha\beta}(\gamma_\mu)_{\gamma\delta} &= \epsilon_{\alpha\gamma}\epsilon_{\delta\beta} + \epsilon_{\alpha\delta}\epsilon_{\gamma\beta}, \\ (\gamma^\mu\gamma^\rho\gamma_\mu)_{\alpha\beta} &= -(\gamma_\rho)_{\alpha\beta}, \\ A_\alpha{}^\beta &= \tfrac{1}{2}(\operatorname{tr} A)\delta_\alpha{}^\beta + \tfrac{1}{2}\operatorname{tr}(\gamma_\mu A)(\gamma^\mu)_\alpha{}^\beta. \end{aligned} \tag{A.6.12}$$

We may map vectors to bispinors and vice versa by

$$v_{\alpha\beta} = -2\gamma^{\mu}_{\alpha\beta} v_{\mu}, \quad v_{\mu} = \tfrac{1}{4}\gamma_{\mu}^{\alpha\beta} v_{\alpha\beta}. \tag{A.6.13}$$

These imply in particular that

$$\begin{aligned} v_{\pm} &= \tfrac{1}{4} v_{\pm\pm}, \\ v^{\pm} &= -\tfrac{1}{2} v^{\pm\pm}, \\ v^{\perp} &= v_{\perp} = -\tfrac{1}{2} v_{+-} = -\tfrac{1}{2} v^{-+}. \end{aligned} \tag{A.6.14}$$

Integration We fix the integration order/constants by

$$\int \mathrm{d}^2\theta\, \theta^2 = 1, \quad \int \mathrm{d}^2\overline{\theta}\, \overline{\theta}^2 = -1 \quad \text{and} \quad \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^+ \overline{\theta}^+ \theta^+ = 1 \tag{A.6.15}$$

i.e. adjacent symbols cancel in $(0,2)$-integration. These together imply

$$\int \mathrm{d}^2\theta = \frac{1}{2} \int \mathrm{d}\theta^- \mathrm{d}\theta^+, \quad \int \mathrm{d}^2\overline{\theta} = \frac{1}{2} \int \mathrm{d}\overline{\theta}^+ \mathrm{d}\overline{\theta}^-, \tag{A.6.16}$$

and

$$\int \mathrm{d}^4\theta = \frac{1}{4} \int \mathrm{d}^2\theta^+ \mathrm{d}^2\theta^-, \tag{A.6.17}$$

where we have defined

$$\int \mathrm{d}^2\theta^{\pm} = \int \mathrm{d}\theta^{\pm} \mathrm{d}\overline{\theta}^{\pm}. \tag{A.6.18}$$

A.6.2 Delta distributions on the boundary

The commutators of quantised operators involve some subtleties with respect to delta distributions and boundaries. We define

$$\int\limits_{(-\infty,0]} f(x)\delta(x) := f(0), \tag{A.6.19}$$

with the reasoning that we want the entire boundary to be part of the theory, thus anything on the boundary is fully part of the system, and that if the δ-distribution is to be understood as a limit of functions $g_n(x) \to \delta(x)$ which fulfil $\int_{(-\infty,0]} g_n(x) = 1$, then the above will follow automatically. This results in one important subtlety: Switching from $\frac{\partial}{\partial y}\delta(x-y)$ to $-\frac{\partial}{\partial x}\delta(x-y)$ introduces a boundary term, specifically

$$\int\limits_{(-\infty,0]} \mathrm{d}y \int\limits_{(-\infty,0]} \mathrm{d}x f(x) g(y) \left(\tfrac{\partial}{\partial x}\delta(x-y) + \tfrac{\partial}{\partial y}\delta(x-y)\right) = f(0)g(0). \tag{A.6.20}$$

This, in turn, implies

$$-\int\limits_{(-\infty,0]} \mathrm{d}y f(x) g(y) \tfrac{\partial}{\partial y}\delta(x-y) = -f(0)g(0) + \int\limits_{(-\infty,0]} \mathrm{d}y f(y) g'(y), \tag{A.6.21}$$

$$\int\limits_{(-\infty,0]} \mathrm{d}y f(x) g(y) \tfrac{\partial}{\partial x}\delta(x-y) = \int\limits_{(-\infty,0]} \mathrm{d}y f(y) g'(y), \tag{A.6.22}$$

as both boundary terms cancel in the second equation.

Supercharges and superderivatives The supercharges of 3D $\mathcal{N}=2$ supersymmetry in terms of $(x^\mu, \theta, \bar\theta)$-coordinates are

$$\begin{aligned} Q_\alpha &= \frac{\partial}{\partial\theta^\alpha} + i(\gamma^\mu\bar\theta)_\alpha\partial_\mu = \begin{pmatrix} \frac{\partial}{\partial\theta^-} + 2i\bar\theta^-\partial_- - i\bar\theta^+\partial_\perp \\ \frac{\partial}{\partial\theta^+} + 2i\bar\theta^+\partial_+ - i\bar\theta^-\partial_\perp \end{pmatrix}, \\ \bar Q_\alpha &= -\frac{\partial}{\partial\bar\theta^\alpha} - i(\gamma^\mu\theta)_\alpha\partial_\mu = \begin{pmatrix} -\frac{\partial}{\partial\bar\theta^-} - 2i\theta^-\partial_- + i\theta^+\partial_\perp \\ -\frac{\partial}{\partial\bar\theta^+} - 2i\theta^+\partial_+ + i\theta^-\partial_\perp \end{pmatrix}. \end{aligned} \tag{A.6.23}$$

The covariant derivatives are

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial\theta^\alpha} - i(\gamma^\mu\bar\theta)_\alpha\partial_\mu = \begin{pmatrix} \frac{\partial}{\partial\theta^-} - 2i\bar\theta^-\partial_- + i\bar\theta^+\partial_\perp \\ \frac{\partial}{\partial\theta^+} - 2i\bar\theta^+\partial_+ + i\bar\theta^-\partial_\perp \end{pmatrix}, \\ \bar D_\alpha &= -\frac{\partial}{\partial\bar\theta^\alpha} + i(\gamma^\mu\theta)_\alpha\partial_\mu = \begin{pmatrix} -\frac{\partial}{\partial\bar\theta^-} + 2i\theta^-\partial_- - i\theta^+\partial_\perp \\ -\frac{\partial}{\partial\bar\theta^+} + 2i\theta^+\partial_+ - i\theta^-\partial_\perp \end{pmatrix}. \end{aligned} \tag{A.6.24}$$

They satisfy the known algebra

$$\{Q_\alpha, \bar Q_\beta\} = 2i\gamma^\mu_{\alpha\beta}\partial_\mu, \quad \{D_\alpha, \bar D_\beta\} = -2i\gamma^\mu_{\alpha\beta}\partial_\mu. \tag{A.6.25}$$

A.7 Decomposition of 3D $\mathcal{N}=2$ to 2D $\mathcal{N}=(0,2)$

A.7.1 Superspace and branching coordinates

We may constructively decompose 3D $\mathcal{N}=2$ superfields and operators into their $(0,2)$-components. To do so, we use the *branching coordinate* ξ.[19] It is chosen such that in superspace with coordinates $(\xi^\mu, \theta^+, \theta^-)$, the representations of the preserved supercharge operators Q_+ and $\bar Q_+$ commute with θ^- and $\bar\theta^-$. Then Q_+ and $\bar Q_+$ can be restricted to the sub-superspace without θ^-. Another property of the branching coordinate is that the preserved supercharge operators do not contain or generate $P_\perp$.

If we want to preserve the $(0,2)$-subalgebra generated by $Q_+, \bar Q_+$, one can easily check that we need

$$\xi^\mu = (x^+, x^-, x^\perp + i(\theta^+\bar\theta^- - \theta^-\bar\theta^+)). \tag{A.7.1}$$

Indeed, one can check that in terms of ξ^μ the operators (A.6.23), (A.6.24) take the following form

$$\begin{aligned} Q_+ &= \frac{\partial}{\partial\theta^+} + 2i\bar\theta^+\frac{\partial}{\partial\xi^+}, & Q_- &= \frac{\partial}{\partial\theta^-} + 2i\bar\theta^-\frac{\partial}{\partial\xi^-} - 2i\bar\theta^+\frac{\partial}{\partial\xi^\perp}, \\ \bar Q_+ &= -\frac{\partial}{\partial\bar\theta^+} - 2i\theta^+\frac{\partial}{\partial\xi^+}, & \bar Q_- &= -\frac{\partial}{\partial\bar\theta^-} - 2i\theta^-\frac{\partial}{\partial\xi^-} + 2i\theta^+\frac{\partial}{\partial\xi^\perp}, \\ D_+ &= \frac{\partial}{\partial\theta^+} - 2i\bar\theta^+\frac{\partial}{\partial\xi^+} + 2i\bar\theta^-\frac{\partial}{\partial\xi^\perp}, & D_- &= \frac{\partial}{\partial\theta^-} - 2i\bar\theta^-\frac{\partial}{\partial\xi^-}, \\ \bar D_+ &= -\frac{\partial}{\partial\bar\theta^+} + 2i\theta^+\frac{\partial}{\partial\xi^+} - 2i\theta^-\frac{\partial}{\partial\xi^\perp}, & \bar D_- &= -\frac{\partial}{\partial\bar\theta^-} + 2i\theta^-\frac{\partial}{\partial\xi^-}. \end{aligned} \tag{A.7.2}$$

In particular, the $Q_+, \bar Q_+$ do not contain any $\partial_\perp$ terms. The $(0,2)$-covariant derivatives are defined by

$$\begin{aligned} D_+^{(0,2)} &= \frac{\partial}{\partial\theta^+} - 2i\bar\theta^+\frac{\partial}{\partial\xi^+}, \\ \bar D_+^{(0,2)} &= -\frac{\partial}{\partial\bar\theta^+} + 2i\theta^+\frac{\partial}{\partial\xi^+}, \end{aligned} \tag{A.7.3}$$

[19] $\xi^\perp$ is also called "invariant coordinate" in [37, 36].

so we have that

$$\begin{aligned} D_+ &= D_+^{(0,2)} + 2i\bar{\theta}^-\partial_\perp, \\ \bar{D}_+ &= \bar{D}_+^{(0,2)} - 2i\theta^-\partial_\perp. \end{aligned} \tag{A.7.4}$$

We often drop the (0, 2)-label when the covariant derivative type is clear from context. We may now simply perform a Taylor expansion of an $\mathcal{N}=2$ superfield $\mathcal{X}$:

$$\begin{aligned} \mathcal{X}(x,\theta,\bar{\theta}) = X^{(0)}(\xi,\theta^+,\bar{\theta}^+) + \theta^- X^{(1a)}(\xi,\theta^+,\bar{\theta}^+) \\ + X^{(1b)}(\xi,\theta^+,\bar{\theta}^+)\,\bar{\theta}^- + \theta^-\bar{\theta}^- X^{(2)}(\xi,\theta^+,\bar{\theta}^+). \end{aligned} \tag{A.7.5}$$

It is clear that the (0, 2)-operators $\{Q_+, \bar{Q}_+, D_+^{(0,2)}, \bar{D}_+^{(0,2)}\}$ do not "mix" the coefficients of different orders in $\theta^-, \bar{\theta}^-$. In other words, the coefficients are exactly the (0, 2)-subrepresentations of $\mathcal{X}(x,\theta,\bar{\theta})$.

As an example, let us decompose the 3D chiral field (4.4.2) to its (0, 2)-submultiplets. In terms of ξ we find that

$$(y^+, y^-, y^\perp) = (\xi^+ - 2i\theta^+\bar{\theta}^-, \xi^- - 2i\theta^-\bar{\theta}^-, \xi^\perp + 2i\theta^-\bar{\theta}^+). \tag{A.7.6}$$

The expansion of Φ_{3D} gives then

$$\Phi_{3D}(x,\theta,\bar{\theta}) = \Phi(\xi,\theta^+,\bar{\theta}^+) - 2i\theta^-\bar{\theta}^-\partial_-\Phi(\xi,\theta^+,\bar{\theta}^+) + \sqrt{2}\theta^-\Psi(\xi,\theta^+,\bar{\theta}^+), \tag{A.7.7}$$

where the chiral and Fermi multiplets are

$$\begin{aligned} \Phi &= \phi + \sqrt{2}\theta^+\psi_+ - 2i\theta^+\bar{\theta}^+\partial_+\phi, \\ \Psi &= \psi_- - \sqrt{2}\theta^+F - 2i\theta^+\bar{\theta}^+\partial_+\psi_- + \sqrt{2}i\bar{\theta}^+\partial_\perp\phi - 2i\theta^+\bar{\theta}^+\partial_\perp\psi_+. \end{aligned} \tag{A.7.8}$$

in agreement with [92]. These satisfy

$$\bar{D}_+\Phi = 0, \qquad \bar{D}_+\Psi = -i\sqrt{2}\partial_\perp\Phi. \tag{A.7.9}$$

We can obtain the full expansion using (A.7.1) on the right-hand side of (A.7.7)

$$\begin{aligned} \Phi_{3D} = \Phi + \sqrt{2}\theta^-\Psi + i(\theta^+\bar{\theta}^- - \theta^-\bar{\theta}^+)\partial_\perp\Phi - 2i\theta^-\bar{\theta}^-\partial_-\Phi \\ - \sqrt{2}i\theta^+\theta^-\bar{\theta}^-\partial_\perp\Psi - \theta^+\bar{\theta}^+\theta^-\bar{\theta}^-\partial_\perp^2\Phi, \end{aligned} \tag{A.7.10}$$

where now all (super-)functions depend on x.

A.7.2 Decomposition of supercurrent multiplets

We decompose the bulk multiplet $\mathcal{S}_{\alpha\beta} = -2\gamma^\mu_{\alpha\beta}\mathcal{S}_\mu$ using the branching coordinate ξ according to

$$\mathcal{S}_{\alpha\beta}(x,\theta,\bar{\theta}) = \mathcal{S}^{(0)}_{\alpha\beta}(\xi,\theta^+,\bar{\theta}^+) + \theta^-\mathcal{S}^{(1)}_{\alpha\beta}(\xi,\theta^+,\bar{\theta}^+) - \bar{\theta}^-\bar{\mathcal{S}}^{(1)}_{\alpha\beta}(\xi,\theta^+,\bar{\theta}^+) + \theta^-\bar{\theta}^-\mathcal{S}^{(2)}_{\alpha\beta}(\xi,\theta^+,\bar{\theta}^+). \tag{A.7.11}$$

We obtain:

(i) The $+$-direction

$$\mathcal{S}^{(0)}_{++} = 4j_+ - 4i\theta^+(S_+)_+ - 4i\overline{\theta}^+(\overline{S}_+)_+ - 16\theta^+\overline{\theta}^+T_{++}, \tag{A.7.12a}$$

$$\begin{aligned}\mathcal{S}^{(1)}_{++} &= -4i(S_+)_- - 2\sqrt{2}\bar{\omega}_+ + \overline{\theta}^+(4i\partial_\perp j_+ + 4K_{+\perp} + 4iL_+) - 4i\theta^+\bar{Y}_+ \\ &\quad +8\theta^+\overline{\theta}^+\partial_+(S_+)_-,\end{aligned} \tag{A.7.12b}$$

$$\begin{aligned}\mathcal{S}^{(2)}_{++} &= -8K_{+-} + 8\theta^+\partial_\perp(S_+)_- - 8\overline{\theta}^+\partial_\perp(\overline{S}_+)_- + 8\theta^+\partial_-(S_+)_+ - 8\overline{\theta}^+\partial_-(\overline{S}_+)_+ \\ &\quad - 4\sqrt{2}i\theta^+\partial_+\bar{\omega}_- - 4\sqrt{2}i\overline{\theta}^+\partial_+\omega_- - 4\sqrt{2}i\theta^+\partial_\perp\omega_+ - 4\sqrt{2}i\overline{\theta}^+\partial_\perp\bar{\omega}_+ \\ &\quad - 4\theta^+\overline{\theta}^+\partial_\perp^2 j_+ - 8\theta^+\overline{\theta}^+\partial_\perp L_+ - 4\theta^+\overline{\theta}^+(-2\partial_+\partial_\nu j^\nu + \partial^2 j_+).\end{aligned} \tag{A.7.12c}$$

(ii) The $-$-direction

$$\begin{aligned}\mathcal{S}^{(0)}_{--} &= 4j_- - 4i\theta^+(S_-)_+ - 4i\overline{\theta}^+(\overline{S}_-)_+ + 2\sqrt{2}\theta^+\bar{\omega}_- \\ &\quad -2\sqrt{2}\overline{\theta}^+\omega_- - 8\theta^+\overline{\theta}^+K_{-+},\end{aligned} \tag{A.7.13a}$$

$$\begin{aligned}\mathcal{S}^{(1)}_{--} &= -4i(S_-)_- - 4i\theta^+\bar{Y}_- + 4\overline{\theta}^+(K_{-\perp} + i\partial_\perp j_- + iL_-) \\ &\quad +8\theta^+\overline{\theta}^+\partial_+(S_-)_- + 4\sqrt{2}i\theta^+\overline{\theta}^+\partial_-\bar{\omega}_+,\end{aligned} \tag{A.7.13b}$$

$$\begin{aligned}\mathcal{S}^{(2)}_{--} &= -16T_{--} + 8\theta^+\partial_\perp(S_-)_- + 8\theta^+\partial_-(S_-)_+ - 8\overline{\theta}^+\partial_-(\overline{S}_-)_+ - 8\overline{\theta}^+\partial_\perp(\overline{S}_-)_- \\ &\quad + 4\theta^+\overline{\theta}^+\partial_\perp^2 j_- - 8\theta^+\overline{\theta}^+\partial_\perp L_- - 4\theta^+\overline{\theta}^+(-2\partial_-\partial^\nu j_\nu + \partial^2 j_-).\end{aligned} \tag{A.7.13c}$$

(iii) The $\perp$-direction

$$\begin{aligned}\mathcal{S}^{(0)}_{-+} &= -2j_\perp + 2i\theta^+(S_\perp)_+ + 2i\overline{\theta}^+(\overline{S}_\perp)_+ + \sqrt{2}\theta^+\bar{\omega}_+ \\ &\quad -\sqrt{2}\overline{\theta}^+\omega_+ + 4\theta^+\overline{\theta}^+K_{\perp+},\end{aligned} \tag{A.7.14a}$$

$$\begin{aligned}\mathcal{S}^{(1)}_{-+} &= +2i(S_\perp)_- - \sqrt{2}\bar{\omega}_- - 2\overline{\theta}^+(K_{\perp\perp} + i\partial_\perp j_\perp + iL_\perp) + 2i\theta^+\bar{Y}_\perp \\ &\quad - 4\theta^+\overline{\theta}^+\partial_+(S_\perp)_- - 2\sqrt{2}i\theta^+\overline{\theta}^+\partial_\perp\bar{\omega}_+ - 2\sqrt{2}i\theta^+\overline{\theta}^+\partial_+\bar{\omega}_-,\end{aligned} \tag{A.7.14b}$$

$$\begin{aligned}\mathcal{S}^{(2)}_{-+} &= +4K_{\perp-} - 4\theta^+\partial_\perp(S_\perp)_- + 4\overline{\theta}^+\partial_\perp(\overline{S}_\perp)_- - 4\theta^+\partial_-(S_\perp)_+ \\ &\quad +4\overline{\theta}^+\partial_-(\overline{S}_\perp)_+ + \sqrt{2}i\theta^+\partial_-\bar{\omega}_+ + \sqrt{2}i\overline{\theta}^+\partial_-\omega_+ + 4\theta^+\overline{\theta}^+\partial_\perp L_\perp \\ &\quad +2\theta^+\overline{\theta}^+\partial_\perp^2 j_\perp - 4\theta^+\overline{\theta}^+(\partial_\perp\partial^\nu j_\nu - \tfrac{1}{2}\partial^2 j_\perp).\end{aligned} \tag{A.7.14c}$$

where

$$\begin{aligned}K_{\mu\nu} &= 2T_{\nu\mu} - \eta_{\mu\nu}A - \tfrac{1}{4}\epsilon_{\mu\nu\rho}H^\rho &&= 2T_{\nu\mu} - \eta_{\mu\nu}A - \tfrac{1}{4}C_{\mu\nu}, \\ L_\mu &= \tfrac{1}{4}\epsilon_{\mu\nu\rho}F^{\nu\rho} + \epsilon_{\mu\nu\rho}\partial^\nu j^\rho &&= \tfrac{1}{4}C_\mu + \epsilon_{\mu\nu\rho}\partial^\nu j^\rho.\end{aligned} \tag{A.7.15}$$

where we have also defined the brane currents $C_{\mu\nu} = \epsilon_{\mu\nu\rho}H^\rho$ and $C_\mu = \epsilon_{\mu\nu\rho}F^{\nu\rho}$. The decomposition for the $\mathcal{R}$-multiplet is found simply by setting the multiplet $\mathcal{Y}_\alpha \ni (\omega_\alpha, A, Y_\mu)$ to zero.

Explicit bulk components of $\mathcal{S}_\mu$ for the 3D LG model

We may compute the components of the supercurrent multiplets for the 3D Landau–Ginzburg model where

$$\begin{aligned}\mathcal{S}_{\alpha\beta} &= D_\alpha\Phi_{3D}\bar{D}_\beta\bar{\Phi}_{3D} + D_\beta\Phi_{3D}\bar{D}_\alpha\bar{\Phi}_{3D},\\ \chi_\alpha &= -\tfrac{1}{2}\bar{D}^2\mathcal{D}_\beta(\bar{\Phi}_{3D}\Phi_{3D}),\\ \mathcal{Y}_\alpha &= -\bar{D}^2\bar{\Phi}_{3D}D_\alpha\Phi_{3D}.\end{aligned} \tag{A.7.16}$$

according to the expansions (4.3.2). In other words, we are in the $\mathcal{S}$-frame (cf. Section 4.3) and we obtain:

$$\begin{aligned}
&R\text{-“current”}:[20] && j^\mu = (\bar{\psi}\gamma^\mu\psi), && \text{(A.7.17a)}\\
&\text{supercurrent}: && S_{\mu\alpha} = \sqrt{2}(\gamma^\nu\gamma_\mu\psi)_\alpha\partial_\nu\bar{\phi} - \sqrt{2}i(\gamma_\mu\bar{\psi})_\alpha\bar{W}', && \text{(A.7.17b)}\\
&\text{lowest in } \chi_\alpha: && \lambda_\alpha = 2\sqrt{2}(\gamma^\mu\bar{\psi})_\alpha\partial_\mu\phi + 2\sqrt{2}iW'\psi_\alpha, && \text{(A.7.17c)}\\
&\text{lowest in } \mathcal{Y}_\alpha: && \omega_\alpha = 4W'\psi_\alpha, && \text{(A.7.17d)}\\
&\text{EM tensor}: && T_{\nu\rho} = (\partial_\rho\phi\partial_\nu\bar{\phi} + \partial_\nu\phi\partial_\rho\bar{\phi}) - \eta_{\nu\rho}(|\partial\phi|^2 + |W'|^2) && \\
& && \quad + \tfrac{i}{2}(\partial_{(\rho}\bar{\psi}\gamma_{\nu)}\psi) - \tfrac{i}{2}(\bar{\psi}\gamma_{(\nu}\partial_{\rho)}\psi), && \text{(A.7.17e)}\\
&\text{irrelevant auxiliary}: && A = -4|W'|^2 + i\partial_\mu\bar{\psi}\gamma^\mu\psi - i\bar{\psi}\gamma^\mu\partial_\mu\psi, && \text{(A.7.17f)}\\
&\{Q, S\}\text{ 1-brane charge}: && Y_\mu = 4\partial_\mu W, && \text{(A.7.17g)}\\
&\{\bar{Q}, S\}\text{ 1-brane charge}: && H^\mu = -2i\partial^\mu(\bar{\psi}\psi), && \text{(A.7.17h)}\\
&\{\bar{Q}, S\}\text{ 0-brane charge}: && \epsilon_{\rho\mu\nu}F^{\mu\nu} = -4\epsilon_{\rho\mu\nu}\partial^\mu j^\nu - 8i\epsilon_{\rho\mu\nu}\partial^\mu\phi\partial^\nu\bar{\phi}. && \text{(A.7.17i)}
\end{aligned}$$

Note that all (Hodge duals to) brane currents are *exact* forms. This is to be expected, since we are working on a trivial space, and it only shows local triviality in general backgrounds. For example, if W is not a properly defined function, then Y^μ is not globally exact.

[20] This particular R-“current” is not conserved for most superpotentials. If we improve the multiplet to an $\mathcal{R}$-multiplet, this component of the multiplet is the conserved R-current.

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