Asymptotic Enumeration, Limit Laws, and Cluster Statistics for Combinatorial Structures

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Abstract

For a given combinatorial class $\mathcal{C}$ we study two types of combinatorial structures: the class $\mathcal{G} = \text{MSET}(\mathcal{C})$ satisfying the multiset construction, that is, every object in $\mathcal{G}$ is uniquely specified by a set of distinct $\mathcal{C}$-objects each paired with a multiplicity from $\mathbb{N}$ indicating the number of occurrences of that object in the multiset (e.g. unlabelled forests are multisets of unlabelled trees). And, secondly, the class $\mathcal{S} = \text{SET}(\mathcal{C})$ containing sets of labelled $\mathcal{C}$-objects, such that no label appears twice in the set and any two $\mathcal{C}$-objects are treated as distinct if their set of labels is (e.g. labelled forests are sets of labelled trees). In both settings, *clusters* (=components) are the $\mathcal{C}$-objects a (multi-)set is composed of. The focus of this work is to investigate several research questions related to these combinatorial constructions, such as asymptotic enumeration as well as limit laws for the statistics of the number of clusters and the smallest/largest cluster of random (multi-)sets. We consider the two broad cases that the counting sequence of $\mathcal{C}$ is either *subexponential* or *expansive*.

The enumerative results of this work are concerned with the following specific setting. We want to asymptotically determine the number of (multi-)sets of total size $n \in \mathbb{N}$ and with $N \in \mathbb{N}$ clusters as n *and* N both grow large. Apart from being mathematically challenging, this is interesting from another perspective. Knowing the answer to this counting problem is equivalent to knowing the distribution of the number of clusters of a (multi-)set of size n drawn uniformly at random from all (multi-)sets of that size. We complete this task for multisets in the subexponential setting, and for both sets and multisets in the expansive setting for the entire range of N. In the resulting formulas we see that these settings are inherently different: whereas the number of multisets in the subexponential case is basically given by the number of $\mathcal{C}$-objects of a certain size uniformly in N, the expansive case exposes a phase transition depending on N/n where the asymptotic order flips.

In the subexponential case we are additionally able to say much more about the structure of multisets of size n and with N clusters. It is possible to completely describe the multiset drawn uniformly at random from all such multisets in terms of an explicit distribution. Namely, after removing the largest cluster and all clusters of smallest possible size, the remainder converges in distribution to a limit given by the well-known Boltzmann model. We baptise this phenomenon *extreme condensation* as virtually all components in that multiset are of smallest possible size, a bounded number of clusters is bounded and there is one huge cluster receiving almost the entire possible size. In the expansive setting such a neat description is not possible and we will only be able to make (very) educated guesses about the structure.

The last batch of results is concerned with the cluster statistics from random (multi-)sets drawn uniformly at random from all (multi-)sets of size n in the expansive case, and in some instances in a slightly more general version called oscillating expansive. For that purpose, we show that a wide class related to the respective generating series is H-admissible, a property allowing us to compare different coefficients of a power series asymptotically. With this at hand, we determine all moments of the number of clusters. Further, assisted by our counting results about multisets of a particular size and number of clusters, we establish a local limit theorem for the number of clusters. Finally, we determine the scaling under which the size of the largest cluster in a uniform (multi-)set converges in distribution to the extreme value distribution and show that the size of the smallest cluster converges in distribution without scaling.

Our methods are based on reformulating the problem at hand into a probability involving iid random variables via the Pólya-Boltzmann model. This gives access to probability theory's large toolbox in the subexponential setting, where we make efficient use of existing results regarding subexponential distributions. In the counting problem of the expansive case we are initially confronted with the complex problem of determining a two-dimensional Cauchy integral. By well-known results such as Chernoff bounds and estimates for Poisson variables, the aforementioned probability can then be simplified to a one-dimensional integral which is tackled via the saddle-point method. Lastly, for the problems related to cluster statistics we develop a simple and unified approach using the framework of H-admissibility and the elementary inclusion/exclusion principle.

Zusammenfassung

Für eine kombinatorische Klasse $\mathcal{C}$ untersuchen wir zwei kombinatorische Konstruktionen: die Klasse $\mathcal{G} = \textsc{Mset}(\mathcal{C})$ der Multimengen, d.h. jedes Objekt in $\mathcal{G}$ ist eindeutig durch die darin enthaltenen $\mathcal{C}$-Objekte und deren Vielfachheit spezifiziert (z.B. sind unmarkierte Wälder Multimengen von unmarkierten Bäumen). Und, zweitens, die Klasse $\mathcal{S} = \textsc{Set}(\mathcal{C})$, die Mengen von markierten $\mathcal{C}$-Objekten enthält, so dass keine Markierung zweimal in der Menge vorkommt und zwei beliebige $\mathcal{C}$-Objekte verschieden sind, wenn ihre Markierungen verschieden sind (z.B. sind markierte Wälder Mengen von markierten Bäumen). In beiden Fällen nennen wir die $\mathcal{C}$-Objekte, aus denen eine (Multi-)Menge zusammengesetzt ist, *Komponenten*. Der Schwerpunkt dieser Arbeit liegt auf der Untersuchung verschiedener Forschungsfragen im Zusammenhang mit diesen kombinatorischen Konstruktionen, wie z.B. die asymptotische Aufzählung sowie Grenzwertsätze für die Anzahl von Komponenten und der kleinsten/größten Komponente in zufälligen (Multi-)Mengen. Wir betrachten die beiden allgemeinen Fälle, dass die Zählfolge von $\mathcal{C}$ entweder *subexponentiell* oder *expansiv* ist.

In den Zähl-Ergebnissen dieser Arbeit wollen wir asymptotisch die Anzahl der (Multi-)Mengen der Gesamtgröße $n \in \mathbb{N}$ und mit $N \in \mathbb{N}$ Komponenten bestimmen, wenn n *und* N beide groß werden. Abgesehen davon, dass dies eine mathematische Herausforderung ist, liefert uns die Antwort auf dieses Zählproblem die Verteilung der Anzahl der Komponenten einer (Multi-)Menge der Größe n, die gleichverteilt aus allen (Multi-)Mengen dieser Größe gezogen wird. Wir lösen diese Aufgabe für Multimengen im subexponentiellen Fall, und sowohl für Mengen als auch für Multimengen im expansiven Fall für alle N. In den Ergebnissen sehen wir, dass diese Fälle grundlegend unterschiedlich sind: Während die Anzahl der Multimengen im subexponentiellen Fall im Wesentlichen durch die Anzahl der $\mathcal{C}$-Objekte einer bestimmten Größe gleichmäßig in N gegeben ist, zeigt sich im expansiven Fall ein von N/n abhängiger Phasenübergang, bei dem sich die asymptotische Ordnung stark verändert.

Im subexponentiellen Fall untersuchen wir zusätzlich die Struktur von Multimengen der Größe n und mit N Komponenten. Es ist möglich, die Multimenge, die gleichverteilt aus allen solchen Multimengen gezogen wird, vollständig durch eine explizite Verteilung zu beschreiben. Nach dem Entfernen der größten Komponente und aller Komponenten der kleinstmöglichen Größe konvergiert die verbleibende Multimenge in Verteilung gegen einen Grenzwert, der durch das bekannte Boltzmann-Modell gegeben ist. Wir nennen dieses Phänomen *extreme Kondensation*, da praktisch alle Komponenten in dieser Multimenge die kleinstmögliche Größe haben, eine beschränkte Anzahl von Komponenten beschränkt ist und es eine große Komponente gibt, die fast die gesamte mögliche Masse erhält. Im expansiven Fall ist eine solche genaue Beschreibung nicht möglich, und wir werden nur Vermutungen über die Struktur anstellen können.

Die abschließenden Ergebnisse befassen sich mit der Komponenten-Struktur von zufälligen (Multi-)Mengen, die gleichverteilt aus allen (Multi-)Mengen der Größe n gezogen werden, im expansiven Fall und in einigen Ausnahmen in einer etwas allgemeineren Version namens oszillierend expansiv. Wir zeigen, dass eine breite Klasse an Erzeugendenfunktionen H-zulässig ist, was eine Eigenschaft ist, die es erlaubt, verschiedene Koeffizienten einer Potenzreihe asymptotisch zu vergleichen. Auf dieser Grundlage bestimmen wir die Skalierung, unter der die Größe der größten Komponente gegen die Extremwertverteilung konvergiert und zeigen, dass die Größe der kleinsten Komponente in Verteilung ohne Skalierung konvergiert. Schließlich bestimmen wir alle Momente der Anzahl der Komponenten und stellen mithilfe unserer Zählergebnisse einen lokalen Grenzwertsatz für die Anzahl der Komponenten auf.

Unsere Methoden beruhen auf der Umformulierung des jeweiligen Problems in eine Wahrscheinlichkeit über unabhängige gleichverteilte Zufallsvariablen über das Pólya-Boltzmann-Modell. Dies ermöglicht den Zugriff auf die vielseitigen Werkzeuge der Wahrscheinlichkeitstheorie im subexponentiellen Fall, in dem wir die bestehenden Ergebnisse zu subexponentiellen Verteilungen effizient nutzen. Bei dem Zählproblem im expansiven Fall sind wir zunächst mit der komplexen Aufgabe der Bestimmung eines zweidimensionalen

Cauchy-Integrals konfrontiert. Durch bekannte Ergebnisse wie Chernoff-Grenzen und Abschätzungen für Poisson-Variablen lässt sich die oben genannte Wahrscheinlichkeit dann auf ein eindimensionales Integral vereinfachen, das mit der Sattelpunktmethode gelöst wird. Abschließend entwickeln wir für die Probleme im Zusammenhang mit der Komponenten-Struktur einen einfachen und einheitlichen Ansatz unter Verwendung der H-Zulässigkeit und des elementaren Inklusions-/Exklusionsprinzips.

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Contributing Manuscripts

This thesis is based on the following manuscripts, which were developed by me, the thesis' author Leon Ramzews, in collaboration with my PhD supervisor Prof. Dr. Konstantinos Panagiotou:

(I) K. Panagiotou, L. Ramzews. Asymptotic Enumeration and Limit Laws for Multisets: the Subexponential Case, 2020. Available at https://arxiv.org/abs/2007.08274.
The results in Section 2.1 and the corresponding proofs in Section 5 are based on this.

(II) K. Panagiotou, L. Ramzews. Expansive Multisets: Asymptotic Enumeration, 2022. Available at https://arxiv.org/abs/2203.15543.
The results in Section 2.2 and the corresponding proofs in Section 6 are based on this.

(III) K. Panagiotou, L. Ramzews. Cluster Statistics in Combinatorial Structures, 2022. Available at https://arxiv.org/abs/2208.00925.
The results in Section 2.3 and the corresponding proofs in Section 7 are based on this.

All the results and proofs in (I)–(III) emerged in joint discussions with Konstantinos Panagiotou, where I contributed substantially. In particular, I developed key ideas, filled in many missing details and wrote substantial parts of the papers. All of this includes continual improvements by Konstantinos Panagiotou.

Contents

1 Introduction

In how many different ways can a natural number be written as a sum of other natural numbers? This seemingly simple question ignited a multifaceted branch of research combining various fields of mathematics such as analysis, probability theory and combinatorics. The answer to this question was given by Hardy and Ramanujan in their celebrated work [43] in which they discovered the beautiful asymptotic formula

$$|\mathcal{P}_n| \sim \frac{1}{4\sqrt{3}n} \exp\left\{\pi\sqrt{\frac{2}{3}n}\right\}, \quad \text{as } n \to \infty,$$

where $\mathcal{P}_n$ contains all number partitions of $n \in \mathbb{N}$. In their paper they made the simple, though substantial, observation that this kind of enumeration problem can be solved by applying Cauchy's formula and performing an appropriate analysis of the resulting complex integral. All that is needed for this approach is a description of the problem at hand in terms of a *generating series*, which for number partitions is well known to be

$$P(x) := \sum_{k\geq 1} |\mathcal{P}_k| x^k = \prod_{k\geq 1} (1 - x^k)^{-1}. \tag{1.1}$$

Then $|\mathcal{P}_n|$ is equal to $[x^n]P(x)$, the coefficient of x^n in $P(x)$. A vast amount of results in the theory of partitions followed, in particular treating the enumeration of other specific but related models. One of the first systematic approaches in that field was performed by Meinardus [57] who figured out a scheme to apply the analytical approach of [43] to a broader class of models, namely, the ubiquitous construction of *multisets*. This construction needs an underlying set $\mathcal{C}$ equipped with a size function mapping objects in $\mathcal{C}$ to $\mathbb{N}$. Then the class $\mathcal{G} = \textsc{Mset}(\mathcal{C})$ of $\mathcal{C}$-*multisets* or *(number) partitions weighted by* $\mathcal{C}$ contains all unordered collections of elements from $\mathcal{C}$ such that identical elements may appear multiple times. For instance, summands of a number partition may also appear several times so that the class of number partitions is given by $\mathcal{P} = \textsc{Mset}(\mathbb{N})$. The size of an element in $\mathcal{G}$ is given by the compound size of the $\mathcal{C}$-elements it is composed of. With this at hand, we let $\mathcal{C}_n$ and $\mathcal{G}_n$ contain all objects in $\mathcal{C}$ and $\mathcal{G}$, respectively, of size n. Then the problem of enumerating the number of elements in $\mathcal{G}_n$ is encoded in the fundamental relation between the generating series $G(x)$ of $\mathcal{G}$ and the counting sequence $(|\mathcal{C}_n|)_{n\in\mathbb{N}}$, see for example [30, 10],

$$G(x) := \sum_{k\geq 1} |\mathcal{G}_k| x^k = \prod_{k\geq 1} (1 - x^k)^{-|\mathcal{C}_k|}. \tag{1.2}$$

In light of this, Meinardus was one of the first to set up a general analytical scheme of conditions on the sequence $(|\mathcal{C}_n|)_{n\in\mathbb{N}}$ upon the fulfilment of which the first asymptotic order of $[x^n]G(x)$ can be systematically computed. Similar to [43] the principle idea was to find out what is needed in order for the following informal procedure to be successful. First apply Cauchy's integral formula and a change to polar coordinates to obtain for some arbitrary x_0 at which $G(x_0) < \infty$

$$[x^n]G(x) = \frac{x_0^{-n}}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \exp\{g(\theta)\}\, d\theta, \qquad g(\theta) = \ln G(x_0 e^{i\theta}) - ni\theta. \tag{1.3}$$

Subsequently, find an appropriate split of the integral into an asymptotically dominating and negligible part. In the dominating part an expansion of $e^{g(\theta)}$ into a "nice" (preferably Gaussian) form must be possible, which typically requires to choose the value of the *free parameter* x_0 such that $g'(x_0)$ is close to zero leading to the cancellation of the term involving n. To finish, evaluate the integral over the nice function. This rather generic idea is nowadays well-known as the *saddle-point method* or *method of steepest descent* and is, in fact, much older than [43] or [57], and has seen widespread applications in various fields inside and outside mathematics, particularly in physics. Of course, the execution of this method heavily depends on the specifics of the underlying problem. Nevertheless, besides [57], there were other systematic approaches to identify the most general possible conditions under which the saddle-point method is applicable, the most important for this thesis being the seminal paper by Hayman [44] where the concept of H-admissibility is introduced, see also Section 7.1.2. The field is far too broad for a comprehensive review; we refer to the classical textbook [18], the modern exposition [58], and to the excellent treatments in [30, 67], which also put a particular emphasis on the combinatorial perspective.

Another predominant idea in this area is to reformulate the counting problem as a probabilistic one, so as to profit from probability theory's huge toolbox. A central result was established by Arratia and Tavaré [4], who showed that many random combinatorial objects, multisets among others, possess a component structure whose joint distribution is equal to that of random variables that are conditioned to have a specific (weighted) sum and are otherwise *independent*. With this description at hand, the quantity $[x^n]G(x)$ can be expressed in terms of a probability that a sum of independent random variables equals n. Now the crucial feature of these random variables is that they depend on a *free parameter* which is to be chosen such that the probability is maximised, if possible making the expectation of the sum equal to n, in turn leading to a local limit theorem in many cases. This approach lies at the core of *Khinchin's probabilistic method* that originated in [51], see [34] for a historical perspective, and that makes the relation of counting and probability a general and guiding principle. Separate from that, a particularly prominent way to decompose random compound objects in dependence of a free parameter, though this time not in terms of the component structure, is the famous *Boltzmann model* having its roots in the pioneering papers [31, 24]. Both probabilistic descriptions have their merits as explained in the detailed exposition in Section 3.2.

The study of combinatorial objects, and in particular multisets, is an active and thriving area of research, in which these fundamentally different methods have proven themselves time and again. Since its existence, Meinardus' approach has been simplified and extended to various directions, see for example [69, 41, 60, 59, 42, 46]. The probabilistic method due to Khinchin has led to great advancements in [39, 40, 6, 34, 33] and the representation via the Boltzmann model was elegantly used in [74, 71]. We will highlight some of these publications later on, and refer the reader to the encompassing works [30, 3] and references therein for a plethora of examples and applications.

1.1 Setup and Objective of the Thesis

In this thesis we will show that there is a fruitful interplay between the analytical and probabilistic approach described above in the asymptotic study of multisets; in particular in the enumeration and in setting up central or local limit theorems for properties of random multisets. Additionally, a minor part of this thesis is concerned with another combinatorial construction, namely the labelled counterpart of multisets called *sets* or *assemblies*, which will be examined under the same aspects as multisets. In order to get more precise, it is necessary to recall and introduce some definitions. Assume that we are given a *combinatorial class* $\mathcal{C}$, that is, a set equipped with a size function $|\cdot| : \mathcal{C} \to \mathbb{N}$ such that $\mathcal{C}_n := \{C \in \mathcal{C} : |C| = n\}$ contains only finitely many elements for each $n \in \mathbb{N}$. We refer to the elements in $\mathcal{C}$ as *clusters* or *components*. It is clear that any "compound" object which is, according to some rule, assembled by clusters from $\mathcal{C}$ can be described by its *cluster structure* in the set of partitions

$$\Omega_n := \left\{(N_1, \dots, N_n) \in \mathbb{N}_0^n : \sum_{1 \le k \le n} kN_k = n\right\}; \tag{1.4}$$

that is, we say that any such compound object with cluster structure $(N_1, \dots, N_n) \in \Omega_n$ is composed of N_k clusters from $\mathcal{C}_k$ for $1 \le k \le n$ and its total size is $n \in \mathbb{N}$. Note that by this description one information gets lost: namely, *which* N_k objects from $\mathcal{C}_k$ are contained in the compound object. In many of the results this is not important, for example when investigating the number of clusters or the smallest and largest cluster. If we want to retrieve results on the structure of a compound object on the $\mathcal{C}$-object level, on the other hand, the description via Ω_n is not sufficient.

In what follows, for given power series $F(x)$ and $F(x,y)$ we write $[x^n]F(x)$ for the coefficient of x^n in $F(x)$ and $[x^n y^N]F(x,y)$ for the coefficient of $x^n y^N$ in $F(x,y)$.

Multisets. We have already encountered the first broad class which admits such a description and where the rule of assembling clusters from $\mathcal{C}$ is simply taking an arbitrary number of clusters with repetition from $\mathcal{C}$: the class of multisets $\mathcal{G} = \textsc{Mset}(\mathcal{C})$. In other words, $\mathcal{G}$ contains all elements of the form

$$\{(C_1, d_1), \dots, (C_k, d_k)\}, \quad C_i \text{ pairwise distinct}, \ \ d_i \in \mathbb{N}, \ \ 1 \le i \le k \in \mathbb{N}_0,$$

where $(C,d) \in G \in \mathcal{G}$ can be interpreted as C appearing d times in G. As mentioned, the probably most prominent example of multisets are number partitions where the clusters are natural numbers. Taking as clusters unlabelled connected graphs with some property we obtain the class of all unlabelled graphs with that property, see [30] for many more examples in the combinatorial setting. To each multiset we naturally associate a size and a number of clusters, or equivalently components, by

$$|G| := \sum_{1 \le i \le k} d_i |C_i| \quad \text{and} \quad \kappa(G) := \sum_{1 \le i \le k} d_i, \qquad \text{where } G = \{(C_1, d_1), \dots, (C_k, d_k)\} \in \mathcal{G}.$$

We further need the sets of all $\mathcal{C}$-multisets of size n as well as of size n and being comprised of N components

$$\mathcal{G}_n = \{G \in \mathcal{G} : |G| = n\} \quad \text{and} \quad \mathcal{G}_{n,N} := \{G \in \mathcal{G}_n : \kappa(G) = N\}, \qquad n, N \in \mathbb{N}.$$

Letting $c_n := |\mathcal{C}_n|$ for $n \in \mathbb{N}$ we write $C(x) := \sum_{k \ge 1} c_k x^k$ for the (ordinary) generating series of $\mathcal{C}$. Then, compare to [30, 10], the bivariate extension of the generating series (1.2) of the class $\mathcal{G}$ is known to fulfil

$$G(x,y) := \sum_{k,\ell \ge 0} |\mathcal{G}_{k,\ell}| x^k y^\ell = \prod_{k \ge 1} (1 - x^k y)^{-c_k} = \exp\Big\{ \sum_{j \ge 1} C(x^j) y^j / j \Big\}. \tag{1.5}$$

A substantial part of this thesis is devoted to the asymptotic enumeration of multisets, or equivalently to determining $[x^n]G(x,1) = [x^n]G(x)$ and $[x^n y^N]G(x,y)$ as $n, N \to \infty$. Knowing the answer to this counting problem is equivalent to knowing the point probability that the multiset G_n drawn uniformly at random from $\mathcal{G}_n$ has N components. Thus being able to determine $[x^n y^N]G(x,y)$ for a wide enough range of N may account for computing the tails of $\kappa(\mathsf{G}_n)$ or determining the scaling under which a local limit theorem holds true when N is close to the expected number of components. We will actually go further and also investigate properties of $\mathsf{G}_{n,N}$ drawn uniformly at random from $\mathcal{G}_{n,N}$ on a $\mathcal{C}$-object level under certain assumptions.

We note that by viewing $(c_k)_{k \in \mathbb{N}}$ in the definitions of $G(x,y)$ and $G(x) = G(x,1)$ as an arbitrary non-negative real-valued sequence without any combinatorial interpretation, it is still possible to compute $[x^n]G(x)$ and $[x^n y^N]G(x,y)$. But more importantly, it makes sense as there are applications beyond combinatorics. Before we reason this, let us further explore the combinatorial setting. The number of multisets in $\mathcal{G}$ with a given cluster structure $(N_1, \dots, N_n) \in \Omega_n$ is

$$\prod_{1 \le k \le n} \binom{c_k + N_k - 1}{N_k};$$

here the binomial coefficient counts the number of ways to choose N_k clusters with repetition from $\mathcal{C}_k$ for $1 \le k \le n$. Summing up the term in the previous display for all elements in Ω_n gives us $[x^n]G(x) = |\mathcal{G}_n|$, the number of $\mathcal{C}$-multisets of total size n. Then we are able to define the random cluster structure $\mathsf{G}^{(n)} \in \Omega_n$ by

$$\Pr\left[\mathsf{G}^{(n)} = (N_1, \dots, N_n)\right] := \frac{1}{[x^n]G(x)} \cdot \prod_{1 \le k \le n} \binom{c_k + N_k - 1}{N_k}, \qquad (N_1, \dots, N_n) \in \Omega_n, \tag{1.6}$$

for all n such that $[x^n]G(x) > 0$. Evidently, this is the cluster structure of the multiset G_n drawn uniformly at random from $\mathcal{G}_n$. As a matter of fact, this distribution is well-defined for *any* non-negative $(c_k)_{k\in\mathbb{N}}$ such that $[x^n]G(x) > 0$ and can be seen in the broader context of multiplicative [76] or Gibbsian [68] measures. In this particular form there are well-known applications from statistical physics. In the Bose-Einstein model of ideal gas [76] the clusters are particles and c_n counts the different positions a particle at energy level $n \in \mathbb{N}$ can be in; and in coagulation-fragmentation processes [25], where $(c_k)_{k\in\mathbb{N}}$ is related to the rate at which particles split from and merge into clusters, the measure of $\mathsf{G}^{(n)}$ is the measure at equilibrium. In this context $[x^n]G(x)$ is often referred to as *partition function*. We recommend [39] for an excellent overview and detailed examples. We define the size of the smallest and largest cluster of $\mathsf{G}^{(n)} = (\mathsf{G}_1^{(n)}, \dots, \mathsf{G}_n^{(n)})$ to be

$$\mathcal{M}(\mathsf{G}^{(n)}) := \min\{1 \le k \le n : \mathsf{G}_k^{(n)} > 0\} \quad \text{and} \quad \mathcal{L}(\mathsf{G}^{(n)}) = \max\{1 \le k \le n : \mathsf{G}_k^{(n)} > 0\}.$$

As mentioned, there will be results on the size-level of objects where it is sufficient to investigate $\mathsf{G}^{(n)}$ to stay in the more general setting, and there will be more detailed results concerning the actual $\mathcal{C}$-clusters of G_n and $\mathsf{G}_{n,N}$.

Sets. The next broad construction we consider and whose elements can be represented by (1.4) are sets, also called assemblies in the literature. First, consider the collection Π_n of all set partitions of $\{1, \dots, n\}$, that is, all elements of the form $\{\pi_1, \dots, \pi_k\}$ where $k \in \mathbb{N}$ and $(\pi_i)_{1\le i\le k}$ are pairwise non-intersecting such that $\bigcup_{1\le i\le k} \pi_i = \{1, \dots, n\}$. Then assume we are given a combinatorial class $\mathcal{C}$; but as opposed to the multiset case we need a notion of *labelling* for this class, meaning that any object in $\mathcal{C}_n$ is defined on the label set $\{1, \dots, n\}$. For example, graphs with n vertices numbered from 1 to n, cycles of n distinct elements from $\{1, \dots, n\}$ or simply the set $\{1, \dots, n\}$ itself. With this at hand, define for $C \in \mathcal{C}_n$ and some arbitrary set $U \subseteq \mathbb{N}$ with $|U| = n$ by $C[U]$ the object which is obtained by replacing the labels in $\{1, \dots, n\}$ by the ones in U in some canonical way. Then the class $\mathcal{S} = \textsc{Set}(\mathcal{C})$ of $\mathcal{C}$*-sets* or *set partitions weighted by* $\mathcal{C}$ is the union of all elements

$$\{C_1[\pi_1], \dots, C_k[\pi_k]\}, \quad C_i \in \mathcal{C}, \ |C_i| = |\pi_i|, \ 1 \le i \le k, \ \{\pi_1, \dots, \pi_k\} \in \bigcup_{n\in\mathbb{N}} \Pi_n.$$

Consequently, sets of labelled connected graphs with a certain property are labelled graphs whose connected components have this property, sets of cycles are permutations and sets of sets are set partitions. Again we refer to [30] for a variety of further examples. To each set we naturally associate a size and a number of clusters, or equivalently components, by

$$|S| := \sum_{1\le i\le k} |C_i| \quad \text{and} \quad \kappa(S) := k, \qquad \text{where } S = \{C_1[\pi_1], \dots, C_k[\pi_k]\} \in \mathcal{S}.$$

As before, we define the sets of all $\mathcal{C}$-sets of size n as well as of size n and being comprised of N components

$$\mathcal{S}_n = \{S \in \mathcal{S} : |S| = n\} \quad \text{and} \quad \mathcal{S}_{n,N} := \{S \in \mathcal{S}_n : \kappa(S) = N\}, \qquad n, N \in \mathbb{N}.$$

In this setting we define $c_n := |\mathcal{C}_n|/n!$ for $n \in \mathbb{N}$, the reason being that the labelling leads to the effect that each object of size n may, in the worst case, appear in $n!$ different relabelled forms. But then the generating series over the actual counting sequence could have radius of convergence zero, a rather undesirable situation to work with. Denoting by $C(x) := \sum_{k\geq 1} c_k x^k$ the (exponential) generating series of $\mathcal{C}$, the bivariate generating series of $\mathcal{S}$ is given by the beautifully simple relation, see once again [30, 10],

$$S(x,y) := \sum_{k,\ell\geq 0} \frac{|\mathcal{S}_{k,\ell}|}{k!} x^k y^\ell = \exp\{yC(x)\}. \tag{1.7}$$

The exact same remarks as for multisets are in place. Setting $S(x) := S(x,1)$ it is of interest to compute the number $|\mathcal{S}_n| = n! \cdot [x^n]S(x)$ and $|\mathcal{S}_{n,N}| = n! \cdot [x^n y^N]S(x,y)$ as $n, N \to \infty$. With this at hand, in certain situations it is possible to compute the tails and/or local limit theorems for the number of components of S_n, drawn uniformly at random from $\mathcal{S}_n$. We will also have a look at $\mathsf{S}_{n,N}$ drawn uniformly at random from $\mathcal{S}_{n,N}$.

Again, this setting can be viewed in a non-combinatorial context, for which we need the following preparations. Given a cluster structure $(N_1, \ldots, N_n) \in \Omega_n$ we determine the number of sets in $\mathcal{S}$ with that structure to be

$$\frac{n!}{\prod_{1\leq k\leq n} k!^{N_k}} \cdot \prod_{1\leq k\leq n} \frac{|\mathcal{C}_k|^{N_k}}{N_k!} = n! \cdot \prod_{1\leq k\leq n} \frac{c_k^{N_k}}{N_k!};$$

the first term is a multinomial coefficient that counts the number of partitions of $\{1, \ldots, n\}$ into N_k sets of size k for $1 \leq k \leq n$, and the second terms selects N_k objects from $\mathcal{C}_k$ where the factorial accounts for killing the ordering in order to obtain a set. Summing this up for all elements in Ω_n yields exactly the number of elements in $\mathcal{S}_n$ given by $n! \cdot [x^n]S(x)$. We define the random cluster structure $\mathsf{S}^{(n)} \in \Omega_n$, which also here is the cluster structure of the uniform set S_n, by

$$\Pr\left[\mathsf{S}^{(n)} = (N_1, \ldots, N_n)\right] := \frac{1}{[x^n]S(x)} \cdot \prod_{1\leq k\leq n} \frac{c_k^{N_k}}{N_k!}, \qquad (N_1, \ldots, N_n) \in \Omega_n, \tag{1.8}$$

for all n such that $[x^n]S(x) > 0$. This distribution, which is again a special instance of multiplicative or Gibbsian measures, has the following interpretations from statistical physics when $(c_k)_{k\in\mathbb{N}}$ is taken as some arbitrary non-negative real-valued sequence. The Maxwell-Boltzmann model of ideal gas [76] where the clusters are distinguishable particles and there are $n!c_n$ different positions for a particle at energy level $n \in \mathbb{N}$; and another version of coagulation-fragmentation processes [25] with rate $(c_k)_{k\in\mathbb{N}}$. A comprehensive source for these topics is [39]. A final definition before we move on to the research questions is the size of the smallest and largest cluster of $\mathsf{S}^{(n)} = (\mathsf{S}_1^{(n)}, \ldots, \mathsf{S}_n^{(n)})$ given by

$$\mathcal{M}(\mathsf{S}^{(n)}) := \min\{1 \leq k \leq n : \mathsf{S}_k^{(n)} > 0\} \quad \text{and} \quad \mathcal{L}(\mathsf{S}^{(n)}) = \max\{1 \leq k \leq n : \mathsf{S}_k^{(n)} > 0\}.$$

Research Questions. With these definitions at hand, we are able to formulate the questions which this thesis is driven by. First, there is the (well-studied) problem of asymptotically counting compound objects of total size n and the (not so well-studied) bivariate counting problem of finding the number of compound objects of total size n being comprised of N clusters.

(Q1) What is the value of $[x^n]G(x)$, $[x^n]S(x)$, $[x^n y^N]G(x,y)$ and $[x^n y^N]S(x,y)$ as $n, N \to \infty$?

In the process of investigating this question, it is possible to gain insights into the structure of the random compound objects at hand on a $\mathcal{C}$-cluster level, that is *which* objects from $\mathcal{C}$ typically appear. Additionally, phenomena such as condensation are of interest.

(Q2) What are the structural properties of $\mathsf{G}_n, \mathsf{S}_n, \mathsf{G}_{n,N}$ and $\mathsf{S}_{n,N}$ with high probability as $n, N \to \infty$?

Related to this question, we want to determine the distribution of the size of the smallest and largest component of a random compound object. Note that in the next question we change the notation to the Ω_n-valued random variables so as to account for the applications where $(c_k)_{k\in\mathbb{N}}$ is arbitrary.

(Q3) What is the scaling to obtain central limit theorems for the size of the smallest or largest clusters in $\mathsf{G}^{(n)}$ and $\mathsf{S}^{(n)}$?

As explained, the answer to *(Q1)* gives us the point probabilities for the distribution of the number of clusters of uniform compound objects. This always yields the tail point probabilities; and with some luck we may obtain a central/local limit theorem which gives a detailed description of what happens near the expected number of clusters. This brings up the final question.

(Q4) What is the scaling to obtain central/local limit theorems for $\kappa(\mathsf{G}^{(n)})$ and $\kappa(\mathsf{S}^{(n)})$?

1.2 State of the Art and Contribution

Within the scope of the questions *(Q1)–(Q4)*, we give an overview of the literature and explain where we extend or generalise existing results. A common and rather general assumption on the underlying sequence $(c_k)_{k\in\mathbb{N}}$ is the following. Let $h : [1,\infty) \to [0,\infty)$ be an eventually positive, continuous and slowly varying function, which means that

$$\lim_{x\to\infty} \frac{h(\lambda x)}{h(x)} = 1, \qquad \text{for } \lambda > 0.$$

Then

$$c_n := h(n) \cdot n^{\alpha-1} \cdot \rho^{-n}, \quad \alpha \in \mathbb{R},\ 0 < \rho \leq 1,\ n \in \mathbb{N}. \tag{1.9}$$

As it turns out, depending on the value of α the models have fairly different features. Let us gain a quick intuition for the influence of α before we continue. Setting $h(x)$ equal to 1, it is immediately clear that ρ is the radius of convergence of $C(x)$ and that $C(\rho)$ converges if $\alpha < 0$. Whenever $\alpha \geq 0$, however, we readily obtain that $C(\rho)$ is not defined. Here it is interesting what happens if x gets close to ρ: by elementary computations it is possible to show that $C(\rho e^{-\chi}) \sim -\ln\chi$ if $\alpha = 0$ and $C(\rho e^{-\chi}) = \Theta(\chi^{-\alpha})$ as $\chi \to 0$. In alignment with that, the three cases are called *convergent* ($\alpha < 0$), *logarithmic* ($\alpha = 0$) and *expansive* ($\alpha > 0$). A fourth case already discussed in the introduction is when $(c_k)_{k\in\mathbb{N}}$ fulfils Meinardus scheme of conditions [57]. The authors of [42] mention that the models under these conditions behave similarly to the expansive ones and thus suggest to call this case *quasi-expansive*. Moreover, there are two more cases relevant for this thesis, which generalise the convergent and expansive case, respectively. Namely, we call $(c_k)_{k\in\mathbb{N}}$ *subexponential* if for some $\rho > 0$

$$\frac{c_{n-1}}{c_n} \sim \rho \quad \text{and} \quad \frac{1}{c_n}\sum_{1\leq k\leq n-1} c_k c_{n-k} \sim 2C(\rho) < \infty, \quad \text{as } n\to\infty \tag{1.10}$$

and *oscillating expansive* if there are $\alpha > 0, 0 < \varepsilon < \alpha/3$ and $0 < \rho \leq 1$ such that for some $0 < A_1 < A_2$ and all n sufficiently large

$$A_1 \cdot n^{2\alpha/3+\varepsilon-1} \cdot \rho^{-n} \leq c_n \leq A_2 \cdot n^{\alpha-1} \cdot \rho^{-n}. \tag{1.11}$$

In the following we will consider the convergent and subexponential case solely for $0 < \rho < 1$. This is not a restriction in the combinatorial setting as $C(\rho) < \infty$ implies that $c_n = o(\rho^{-n}) = o(1)$ for $\rho \geq 1$ which is not reasonable for a counting sequence.

State of the art for Question *(Q1)*. A vast amount of literature is dedicated to and very well answers Question *(Q1)* in the univariate case. The encompassing state of the art works [40, 33] determine $[x^n]G(x)$ and $[x^n]S(x)$ in the oscillating expansive case, [5] in the logarithmic case, [74, 71] in the subexponential case and [42] in the quasi-expansive case. Their methods range from analytic to probabilistic as outlined in the introduction. More details will follow in Section 3.2, but we already anticipate that in the proofs for the oscillating expansive case in [40, 33] it is possible to choose the free parameter in the probabilistic method such that the corresponding probability follows a local limit theorem. As the introduction of the free parameter introduces an additional level of dependency and there are no sufficiently strong standard results for genuine triangle arrays, this limit theorem is computed "from scratch". This challenging task, however, is essentially performed by applying the saddle-point method, an interesting fact we will revisit in the context of *(Q3)* and *(Q4)*. On the other side, Stufler [74, 71] stays entirely in the probabilistic realm and makes efficient use of existing probabilistic results.

While the univariate case is fairly well-studied, there are many open questions in the bivariate setting. Nevertheless, in the special context of number partitions, the prime example for expansive multisets where $h, \alpha, \rho = 1$ and $\mathcal{P} = \textsc{Mset}(\mathbb{N})$, the picture is quite clear. In [52] the asymptotic order of $|\mathcal{P}_{n,N}|$ for $n, N, n - N \to \infty$ is determined and a phase transition, depending on whether N is $\mathcal{O}(n^{1/2})$ or $\omega(n^{1/2})$, in the structure of the counting sequence is observed. For $N \geq d\sqrt{n}\ln n$ and some $d > 0$ it is even true that $|\mathcal{P}_{n,N}| \sim |\mathcal{P}_{n-N}|$, as shown in [45]. And also set partitions, the prime example for expansive sets where $h, \alpha, \rho = 1$ and $\Pi = \textsc{Set}(\bigcup_{n\in\mathbb{N}}\{\{1, \ldots, n\}\})$, are fully investigated in this respect. Here [75] determines an asymptotic formula for $|\Pi_{n,N}|$, also known as Stirling number's of the second kind, holding as $n \to \infty$ and uniformly in $0 < N < n$.

Under general assumptions the problem is far less understood. Bell et al. [8] investigate the subexponential case and determine the limit of $[x^n y^N]G(x, y)/[x^n]G(x)$ and $[x^n y^N]S(x, y)/[x^n]S(x)$ for *fixed* $N \in \mathbb{N}$ as $n \to \infty$. The problem of determining $[x^n y^N]S(x, y) = [x^n]C(x)^N/N!$ can be seen in the broader context of extracting coefficients of large powers of power series. There are plenty of results, but either the assumptions are not on a counting sequence level ([19, 20], [30, Thm. IX.16]) or $n/N = \Theta(1)$ ([30, Thms. VIII.8 and 9], [62]). For instance, [62] treats the convergent case and finds that there is a phase transition depending on $\lim N/n = \lambda > 0$ at which the asymptotic formula for $[x^n y^N]S(x, y)$ switches from describing a condensation phenomenon to a Gaussian form.

In the quasi-expansive case under the additional stronger assumption $c_n \sim cn^{\alpha-1}$ for $c, \alpha > 0$, Stark [69] determines $[x^n y^N]G(x, y)$ asymptotically for $N = \omega(\ln^3 n)$ and $N = o(n^{\alpha/(\alpha+1)})$. Actually, in that paper the author accomplishes the herculean task of performing a bivariate saddle-point integration, though only for a (quite) limited range of the parameters. In any case, the results of [69] give reason to conjecture that $[x^n y^N]G(x, y)$, as in the case of number partitions, undergoes a phase transition depending on the ratio of N and $n^{\alpha/(\alpha+1)}$ in this general setting as well. Our results will confirm that such transitions are prototypical for the considered counting problems in the expansive setting. We devoted Section 3.2 to further discuss the similarities and disparities of some of the aforementioned publications related to *(Q1)* to our work.

Our contribution to Question *(Q1)*. In the subexponential case we are able to fully describe the asymptotic behaviour of $[x^n y^N]G(x, y)$ as $n \to \infty$ and uniformly for $N, n - mN \to \infty$ in Theorem 2.1, where m is the first index such that $c_m > 0$. The result suggests that this bivariate problem is in essence a univariate one: $[x^n y^N]G(x, y)$ is proportional to some constant, times the number of possibilities to build a multiset of $N - \mathcal{O}(1)$ clusters of smallest possible size m, times c_{n-mN}. Indeed, in the proof we extend the probabilistic reformulation via the Boltzmann model used in [74] to the bivariate setting. But in a next step a careful analysis makes it possible to reduce the problem to the probability that a sum of a bounded number of iid subexponential random variables hits $n - mN$. According to the well-known "single big jump" property this sum is dominated by one huge summand.

As opposed to the subexponential setting, a phase transition depending on the ratio N/n becomes visible in the expansive case. In Theorems 2.8(I) and 2.8(II) we determine $[x^n y^N]G(x,y)$ under the additional assumption that $0 < \rho < 1$ in the two different emerging regimes covering all $N \to \infty$ as $n \to \infty$ except the value at the phase transition itself. We will later find that the expected number of components of G_n lies in the first regime, so that Theorem 2.8(I) covers the bulk of the mass. Regarding the proof, the following challenging problem had to be overcome: both the saddle-point method and the probabilistic method hit a barrier. In the saddle-point method, we are faced with an integral over $\mathbb{C}^2$ and obtaining control over the integration contour is – as the orders of magnitude of n and N may be vastly incompatible – extremely challenging from a technical viewpoint in this generality; in the probabilistic method, on the other hand, we have to deal with bivariate local limit theorems for random variables with a specific dependency structure, which in addition are dependent on two free parameters. Although there are many general and notable results that address the multivariate setting in both the analytic and probabilistic settings, see for example the extensive treatment in [67], they are not sufficient for the desired level of generality considered here. We demonstrate that a *combination* of both methods is very effective for determining $[x^n y^N]G(x,y)$. Indeed, we first set up an appropriate probabilistic framework via the Boltzmann model in two parameters. Having achieved this, we use probabilistic methods to reduce the determination of $[x^n y^N]G(x,y)$ to the one dimensional problem of extracting coefficients of large powers of $C(x)$, which then, in turn, is tackled with the saddle-point method. At this point it seems unavoidable to resort to the saddle-point method, as the problem involves a genuine triangle array of random variables for which we need a local limit theorem.

As mentioned, computing $[x^n]C(x)^N$ directly accounts for obtaining $[x^n y^N]S(x,y)$. So, as a neat by-product we obtain Theorem 2.9 in the expansive case for the parameter range $0 < \rho \leq 1$ and $n/N \to \infty$ complementing the existing results for $n/N = \Theta(1)$.

State of the art for Question *(Q2)*. Investigating random compound objects on the granular level of $\mathcal{C}$-objects contained therein as suggested in *(Q2)* is very hard to achieve. The setting is only well-understood in the subexponential case. Here [74, 71] worked out a very detailed description of both G_n and S_n drawn uniformly at random from $\mathcal{G}_n$ and $\mathcal{S}_n$, respectively. Namely, after removing the largest cluster, the collection of remaining clusters converges in distribution to a limit given by the Pólya-Boltzmann model. That is, with high probability, the mass *condenses* into one huge component of size $n - \mathcal{O}(1)$ and the total size of the remaining clusters stays bounded.

Our contribution to Question *(Q2)*. We extend these results to $\mathsf{G}_{n,N}$ drawn uniformly at random from $\mathcal{G}_{n,N}$ in Theorems 2.3 and 2.4 where we discover a phenomenon we baptise *extreme condensation*: removing the largest component and all components of the smallest possible size, leaves us with an object which converges in distribution to a limit given by the Pólya-Boltzmann model as $n, N \to \infty$. Again, this implies that the largest component is with high probability virtually of the largest possible size any multiset in $\mathcal{G}_{n,N}$ can be. On a technical level, the (size of the) huge cluster is basically the huge summand which appears due to the single big jump principle mentioned before. This is really surprising as the behaviour of $\mathsf{S}_{n,N}$ drawn uniformly at random from $\mathcal{S}_{n,N}$ is completely different although G_n and S_n behave the same way, as explained in Section 3.1.

State of the art for Question *(Q3)*. Moving away from the cluster to the size level, the literature grows increasingly generous. First, let us mention that the results presented in the context of Question *(Q2)* imply that in the subexponential setting, the distribution of the size of the normalised largest clusters $\mathcal{L}(\mathsf{G}^{(n)}) - n$ and $\mathcal{L}(\mathsf{G}^{(n)}) - n$ is explicitly given by knowing the distribution of the size of the remainder. See also [6] who proved this fact earlier but for the slightly less general convergent case.

In the logarithmic case under some mild extra assumptions the $k \in \mathbb{N}$ largest cluster sizes $(L_1, \dots, L_k)$ of $\mathsf{G}^{(n)}$ or $\mathsf{S}^{(n)}$ scaled by n^{-1} have a limiting distribution that is Poisson-Dirichlet [5, Thm. 3.3] implying that G_n is composed of several "large" objects. The smallest cluster sizes are shown to be asymptotically independent and have a limit given in terms of negative binomial or Poisson random variables [5, Thm. 3.2].

As presented by [60] there is a scaling such that $\mathcal{L}(\mathsf{G}^{(n)})$ converges to the extreme value distribution in the quasi-expansive case. Far less is known with respect to the expansive case. Here [34] established that there is a threshold $n^{1/(\alpha+1)}$ for $\mathcal{L}(\mathsf{S}^{(n)})$, meaning that the probability of the event $\{\mathcal{L}(\mathsf{S}^{(n)}) \leq n^{\beta}\}$ tends to 0 if $\beta < 1/(\alpha+1)$ and to 1 if $\beta > 1/(\alpha+1)$. In their work also the limiting distribution of the minimal cluster size is determined which converges without scaling.

Our contribution to Question *(Q3)*. We show that the results of [60] are universal for multisets and sets in the expansive case and thereby extend [34]. In Theorem 2.11 it is stated that both $\mathcal{L}(\mathsf{G}^{(n)})$ and $\mathcal{L}(\mathsf{S}^{(n)})$ converge in distribution to the extreme value distribution after scaling properly. Corollary 2.13 determines the limiting distributions of the smallest clusters for both $\mathsf{G}^{(n)}$ and $\mathsf{S}^{(n)}$ for the oscillating expansive case, generalising [34].

Our findings are based on a simple yet far-reaching observation: the proofs in the state-of-the-art works [33, 40] encountered in the discussion about *(Q1)* are conducted by reformulating $[x^n]S(x)$ and $[x^n]G(x)$ in terms of Khinchin's probabilistic method to reduce the analytical problem to a probabilistic one; but in essence the authors apply the saddle-point method to the Cauchy integral representing the coefficients of the series at hand. For that they basically prove that $S(x)$ and $G(x)$ possess a property called H-admissible, see Section 7.1.2, without calling it that way. Now, the great advantage in realising that a series F is H-admissible is that by existing results one is able to compute/estimate $[x^n]F(x)$ and $[x^{n-k}]F(x)/[x^n]F(x)$ systematically for virtually *any* $0 < k \leq n$ as $n \to \infty$. This has powerful consequences as the proofs for the cluster statistics always rely at some point on determining such a fraction of coefficients of power series. In Lemma 7.3 we show that the respective generating series and many more are H-admissible. By combining the elementary methods of [27] with the H-admissibility discovery we are then able to find the proper scaling and prove the claimed convergence for the largest clusters. The limiting distribution for the smallest clusters is a straightforward consequence of H-admissibility.

State of the art for Question *(Q4)*. This last question is naturally intertwined with Question *(Q1)* since the distribution of the number of components $\kappa(\mathsf{G}^{(n)})$ and $\kappa(\mathsf{S}^{(n)})$ is given by $([x^n y^N]G(x,y)/[x^n]G(x))_{N\in\mathbb{N}}$ and $([x^n y^N]S(x,y)/[x^n]S(x))_{N\in\mathbb{N}}$, respectively. As mentioned, both these random variables hence converge in distribution without scaling due to [8] in the subexponential case.

The distribution of the number of components of $\mathsf{G}^{(n)}$ and $\mathsf{S}^{(n)}$ in the logarithmic case under some mild extra assumptions is typically of order $\lambda \ln n$ as presented in [5, Thm. 8.21] for some $\lambda > 0$; much more can actually be said, namely that the total variation distance between $\kappa(\mathsf{G}^{(n)})$ or $\kappa(\mathsf{S}^{(n)})$ and $\mathrm{Po}\,(\lambda \ln n)$ tends to zero [5, Thm. 8.15].

Again, the expansive case stands in stark contrast with the subexponential one: with proper scaling a local limit theorem for $\kappa(\mathsf{S}^{(n)})$ is established in [28] when $h(n)$ is constant and [59] shows a central limit theorem for $\kappa(\mathsf{G}^{(n)})$ in the quasi-expansive case. Interestingly, the results from [28] imply a Gaussian central limit theorem for any $\alpha > 0$ whereas the limit is only Gaussian for parts of the parameter range in the quasi-expansive case. This is in particular consistent with the results from the classical work [27] where it is shown that the scaled number of clusters converges to the extreme value distribution for the special case of integer partitions where $\alpha = 1$.

Our contribution to Question *(Q4)*. First of all, we determine all moments of $\kappa(\mathsf{S}^{(n)})$ and $\kappa(\mathsf{G}^{(n)})$ asymptotically in the oscillating expansive case in Corollary 2.14. With this at hand, we find that the expected number of components is covered by the results in Theorem 2.8(I). This, in turn, allows us to establish the proper scaling under which $\kappa(\mathsf{G}^{(n)})$ (for $0 < \rho < 1$) and $\kappa(\mathsf{S}^{(n)})$ (for all $0 < \rho \leq 1$) admit a local limit theorem in the expansive case. The proof heavily depends on Theorem 2.8(I) and the fact that we are able to determine the coefficients of H-admissible functions via the saddle-point method with respect to any free parameter.

1.3 Plan of the Thesis

In Section 1.4 we introduce the notation used in this thesis. All the main results are contained in Section 2 where, based on the manuscripts (I)–(III), we divided the statements thematically into three subsections. The findings about subexponential multisets are gathered in Section 2.1, the counting results for expansive multisets and sets in Section 2.2, and the statements about cluster statistics of expansive multisets and sets in Section 2.3. A discussion putting the results into a broader context is the objective of Section 3. In Section 3.1 we start by comparing the unlabelled multiset and the labelled set case with respect to the counting results we obtained. Then, in Section 3.2 we compare the classical proofs based on Khinchin's probabilistic method with our novel approach. Subsequently, Section 4 contains some auxiliary results which are needed for all the proofs. Sections 5–7 contain the proofs for the main results, where each of the Sections 2.1–2.3 is treated in a separate section. The proof sections all have a similar structure; we first establish the required preliminaries and then turn to the discussion of the proof of the respective main result. Finally, the self-contained Appendix A features textbook results about slowly varying functions. The influence of the manuscripts (I)–(III) is mentioned at the appropriate places.

1.4 Notation

We make plenty use of the Landau symbols, that is, for a real-valued function $f : [0, \infty) \mapsto \mathbb{R}$ define the sets

$$\begin{aligned}
o(f(x)) &:= \{g : [0, \infty) \mapsto \mathbb{R} : g(x)/f(x) \to 0 \text{ as } x \to \infty\},\\
\mathcal{O}(f(x)) &:= \{g : [0, \infty) \mapsto \mathbb{R} : \exists A, x_0 > 0 : |g(x)/f(x)| \leq A \text{ for all } x \geq x_0\},\\
\Theta(f(x)) &:= \{g : [0, \infty) \mapsto \mathbb{R} : \exists A_1, A_2, x_0 > 0 : A_1 \leq |g(x)/f(x)| \leq A_2 \text{ for all } x \geq x_0\},\\
\Omega(f(x)) &:= \{g : [0, \infty) \mapsto \mathbb{R} : \exists A, x_0 > 0 : |g(x)/f(x)| \geq A \text{ for all } x \geq x_0\} \quad \text{and}\\
\omega(f(x)) &:= \{g : [0, \infty) \mapsto \mathbb{R} : |g(x)/f(x)| \to \infty \text{ as } x \to \infty\}.
\end{aligned}$$

As usual this notation is abused by writing $g(x) = X(f(x))$ for some $g(x) \in X(f(x))$ as well as $X(f(x)) = Y(g(x))$ if $X(f(x)) \subseteq Y(g(x))$ for some $X, Y \in \{o, \mathcal{O}, \Theta, \Omega, \omega\}$ and real valued functions f, g fitting in the definitions above. When we put a minus sign in front of a Landau symbol the entire expression is meant to be negative, so that for example $g(x) = -\omega(1)$ is a function which tends to $-\infty$ as $x \to \infty$.

Given a real-valued sequence $(a_k)_{k\in\mathbb{N}}$ and a sequence $(b_k)_{k\in\mathbb{N}}$ which is non-zero for all $k \geq k_0$ and some $k_0 > 0$, we write that $a_n \sim b_n$ as $n \to \infty$ if $\lim_{n\to\infty} a_n/b_n = 1$. We further say that "$a_n$ is asymptotically equal to b_n" if $a_n \sim b_n$, "a_n is asymptotically negligible (compared) to b_n" if $a_n = o(b_n)$, "a_n is asymptotically proportional to b_n" if $a_n = \Theta(b_n)$ and "a_n dominates b_n asymptotically" if $a_n = \omega(b_n)$ as $n \to \infty$.

For a sequence of real-valued random variables $(X_k)_{k\in\mathbb{N}}$ and a non-negative sequence $(a_k)_{k\in\mathbb{N}}$ we write $X_n = \mathcal{O}_p(a_n)$ ("X_n is stochastically bounded by a_n") if for all $\varepsilon > 0$ there exists $K > 0$ such that $\limsup_{n\to\infty} \Pr\left[|X_n| \geq K a_n\right] \leq \varepsilon$. In the case $a_k \equiv 1$ we simply say "X_n is stochastically bounded".

For some real-valued sequence $(a_n)_{n\geq 1}$ and a property of real-valued sequences $\mathcal{E}$ we say that $(a_n)_{n\geq 1}$ fulfils $\mathcal{E}$ *eventually* or for *sufficiently large* n if there exists (a potentially large) $n_0 \in \mathbb{N}$ such that $(a_n)_{n\geq n_0}$ fulfils $\mathcal{E}$. For example, if $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are two positive sequences such that $a_n \sim b_n$ as $n \to \infty$, for any $\varepsilon > 0$ we *eventually* have that $a_n \leq (1 + \varepsilon)b_n$.

Given a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ in a discrete space $\mathcal{X}$ and a sequence of properties $(\mathcal{E}_n)_{n\in\mathbb{N}}$ in the powerset of $\mathcal{X}$ we say that X_n fulfils $\mathcal{E}_n$ *with high probability* (whp) if $\Pr[X_n \in \mathcal{E}_n] \to 1$ as $n \to \infty$.

We will use the following notation for formal power series. For a k-dimensional vector of formal variables $\mathbf{x} = (x_1, \ldots, x_k)$ and $\mathbf{d} = (d_1, \ldots, d_k) \in \mathbb{N}_0^k$ we write $\mathbf{x}^{\mathbf{d}}$ for the monomial $x_1^{d_1} \cdots x_k^{d_k}$. A multivariate power series with real-valued coefficients is given by $A(\mathbf{x}) = \sum_{\mathbf{d}\in\mathbb{N}_0^k} a_{\mathbf{d}}\mathbf{x}^{\mathbf{d}}$, where the $a_{\mathbf{d}}$'s are in $\mathbb{R}$. For $\mathbf{d} \in \mathbb{N}_0^k$ we write $[\mathbf{x}^{\mathbf{d}}]A(\mathbf{x}) = a_{\mathbf{d}}$ for the coefficient of $\mathbf{x}^{\mathbf{d}}$.

2 Main Results

This section contains all the main results, where Section 2.1 is based on Section 1 of Manuscript (I), Section 2.2 on Section 1 of Manuscript (II) and Section 2.3 on Section 1 of Manuscript (III). There is one exception: originally, Theorem 2.9 in Section 2.2 appears as Theorem 1.6 in Manuscript (III).

Recap of definitions. For better readability we briefly repeat some definitions already introduced in Section 1.1. Let $\mathcal{C}$ be a combinatorial class, that is, a set equipped with a size function $|\cdot| : \mathcal{C} \to \mathbb{N}$ such that

$$\mathcal{C}_n := \{C \in \mathcal{C} : |C| = n\}$$

is finite for all $n \in \mathbb{N}$. With this at hand, we define the classes of $\mathcal{C}$-multisets by

$$\mathcal{G} = \textsc{Mset}(\mathcal{C})$$

and, if objects in $\mathcal{C}$ are labelled, the class of $\mathcal{C}$-sets by

$$\mathcal{S} = \textsc{Set}(\mathcal{C}).$$

Denoting by $|\cdot|$ the size and by $\kappa(\cdot)$ the number of clusters an element in $\mathcal{G}$ or $\mathcal{S}$ is composed of, we write for $n, N \in \mathbb{N}$

$$\mathcal{G}_n := \{G \in \mathcal{G} : |G| = n\} \quad \text{and} \quad \mathcal{G}_{n,N} := \{G \in \mathcal{G}_n : \kappa(G) = N\}$$

as well as

$$\mathcal{S}_n := \{S \in \mathcal{S} : |S| = n\} \quad \text{and} \quad \mathcal{S}_{n,N} := \{S \in \mathcal{S}_n : \kappa(S) = N\}.$$

Setting $c_n = |\mathcal{C}_n|$ in the multiset case and $c_n = |\mathcal{C}_n|/n!$ in the set case for $n \in \mathbb{N}$, we obtain the bivariate generating series

$$G(x,y) := \exp\left\{\sum_{j\geq 1} \frac{C(x^j)y^j}{j}\right\}, \quad S(x,y) := \exp\{yC(x)\} \quad \text{where } C(x) = \sum_{k\geq 1} c_k x^k. \tag{2.1}$$

That is, $[x^n y^N]G(x,y) = |\mathcal{G}_{n,N}|$ and $[x^n y^N]S(x,y) = |\mathcal{S}_{n,N}|/n!$. The generating series only taking account of the size are then

$$G(x) := G(x,1) \quad \text{and} \quad S(x) := S(x,1).$$

We introduce the random variables G_n and $\mathsf{G}_{n,N}$ drawn uniformly at random from $\mathcal{G}_n$ and $\mathcal{G}_{n,N}$, respectively, for n, N such that these sets are non-empty. Likewise, S_n and $\mathsf{S}_{n,N}$ are drawn uniformly at random from $\mathcal{S}_n$ and $\mathcal{S}_{n,N}$. The random cluster structure from the set

$$\Omega_n := \left\{(N_1, \dots, N_n) : \sum_{k\geq 1} kN_k = n\right\} \tag{2.2}$$

of these random variables is then given by

$$\Pr\left[\mathsf{G}^{(n)} = (N_1, \dots, N_n)\right] := \frac{1}{[x^n]G(x,1)} \cdot \prod_{1\leq k\leq n} \binom{c_k + N_k - 1}{N_k}, \quad (N_1, \dots, N_n) \in \Omega_n. \tag{2.3}$$

and

$$\Pr\left[\mathsf{S}^{(n)} = (N_1, \dots, N_n)\right] := \frac{1}{[x^n]S(x,1)} \cdot \prod_{1\leq k\leq n} \frac{c_k^{N_k}}{N_k!}, \quad (N_1, \dots, N_n) \in \Omega_n. \tag{2.4}$$

Further we need the size of the smallest and largest cluster for $\mathrm{N} = (N_1, \dots, N_n) \in \Omega_n$ which are defined by

$$\mathcal{M}(\mathrm{N}) := \min\{1 \leq k \leq n : N_k > 0\} \quad \text{and} \quad \mathcal{L}(\mathrm{N}) := \max\{1 \leq k \leq n : N_k > 0\}. \tag{2.5}$$

Abusing this notation we also write $\mathcal{M}(G)$ and $\mathcal{L}(G)$ for the size of the smallest and largest cluster of $G \in \mathcal{G}$ and note that $\mathcal{M}(\mathsf{G}_n) \overset{(d)}{=} \mathcal{M}(\mathsf{G}^{(n)})$ as well as $\mathcal{L}(\mathsf{S}_n) \overset{(d)}{=} \mathcal{L}(\mathsf{S}^{(n)})$.

As mentioned in Section 1.1 the expressions (2.1)–(2.5) still make sense in the non-combinatorial setting when $(c_k)_{k\in\mathbb{N}}$ is some arbitrary non-negative real-valued sequence. In alignment with that, all results concerning $\mathsf{G}_n, \mathsf{G}_{n,N}, \mathsf{S}_n$ and $\mathsf{S}_{n,N}$ are to be seen in a strictly combinatorial setting, and the results about coefficients of $G(x,y)$ and $S(x,y)$ as well as properties of $\mathsf{G}^{(n)}$ and $\mathsf{S}^{(n)}$ hold for any sequence $(c_k)_{k\in\mathbb{N}}$ with the additional specific assumptions imposed in the statements.

2.1 Subexponential Multisets with Many Components

Following [32] we call $(c_k)_{k\in\mathbb{N}}$ ($C(x)$ and $\mathcal{C}$, respectively) *subexponential* with radius of convergence $\rho > 0$ if

$$\frac{c_{n-1}}{c_n} \sim \rho \quad \text{and} \quad \frac{1}{c_n} \sum_{1\leq k\leq n-1} c_k c_{n-k} \sim 2C(\rho) < \infty, \quad \text{as } n \to \infty. \tag{2.6}$$

For example sequences of the form $c_k = h(k) \cdot k^{\alpha-1} \cdot \rho^{-k}$ for $k \in \mathbb{N}$ where h is an eventually positive slowly varying function and $\alpha < 0$ are subexponential. Counting sequences of that form are omnipresent in combinatorical settings since many classes have generating series which allow for a singular expansion of order $-\alpha > 0$ which translates to a polynomial term $k^{\alpha-1}$ in c_k. Prominent examples are subcritical block-stable classes ($\alpha = -3/2$) such as (unlabelled and connected) Cacti graphs, outerplanar graphs, series-parallel graphs [21] or trees ($\alpha = -3/2$, see [61]).

2.1.1 Enumeration

In general, the problem of counting the number of $\mathcal{C}$-multisets of size n without a restriction on the number of components is well understood in the subexponential setting. By methods which inspired those in this thesis, [74] establishes (among much more general things) the counting result

$$g_n := [x^n]G(x) \sim G(\rho) \cdot c_n \qquad \text{as } n \to \infty. \tag{2.7}$$

What can we say about $g_{n,N} := [x^n y^N]G(x,y)$ in the subexponential setting? A directly relevant result is given by Bell et al. [8] where the authors show that $\kappa(\mathsf{G}_n)$ has a limiting distribution given by $1 + \sum_{j\geq 1} jP_j$ for independent Poisson random variables P_j with parameters $C(\rho^j)/j$ for $j \in \mathbb{N}$. Equivalently, this means that $g_{n,N}$ is known for *fixed* values of N. Letting $m \in \mathbb{N}$ be the size of the smallest possible object in $\mathcal{C}$, that is,

$$m = \min\{k \in \mathbb{N} : c_k > 0\},$$

the other end of the spectrum is the range of N for which $n - mN = \mathcal{O}(1)$. But then the structure of any multiset $G \in \mathcal{G}_{n,N}$ is rather simple: all but a bounded number of objects in G are of smallest possible size m and a bounded number of objects is of size $m + \mathcal{O}(1)$. Note that concerning the quantity $g_{n,N}$ in this case, there are

$$\binom{c_m + N - \mathcal{O}(1)}{c_m - 1} \sim \frac{N^{c_m - 1}}{\Gamma(c_m)} \tag{2.8}$$

possibilities to build a multiset of $N - \mathcal{O}(1)$ objects from $\mathcal{C}_m$. As there is only a bounded number of objects of bounded size left with which the multiset can be completed we deduce $g_{n,N} = \Theta(N^{c_m-1})$. In particular, if $c_m = 1$, then $g_{n,N}$ is bounded.

We conclude that $g_{n,N}$ is well understood when N is close to the boundary of its range. Our first contribution addresses the enumeration problem in all other remaining cases, namely when n, N and $n - mN$ tend to infinity. For the presentation of the next result define

$$G_{>m}(x) := \exp\left\{\sum_{j\geq 1} \frac{C(x^j) - c_m x^{jm}}{j x^{jm}}\right\}. \tag{2.9}$$

Theorem 2.1. *Suppose that $C(x)$ is subexponential and $0 < \rho < 1$. Then, as $n, N, n - mN \to \infty$,*

$$[x^n y^N]G(x,y) \sim G_{>m}(\rho) \cdot \frac{N^{c_m-1}}{\Gamma(c_m)} \cdot c_{n-m(N-1)}. \tag{2.10}$$

The proof can be found in Section 5.2.3. Some remarks are in place. First, knowing $g_{n,N}$ and g_n implies knowing the point probabilities of $\kappa(\mathsf{G}_n)$. Hence a straightforward consequence of Theorem 2.1 and the fact that $g_{n,N} = \mathcal{O}\left(N^{c_m-1}\right)$ if $n - mN = \mathcal{O}(1)$ combined with (2.7) are tail estimates which hold uniformly in N.

Corollary 2.2. *Suppose that $C(x)$ is subexponential and $0 < \rho < 1$. Let $\varepsilon > 0$. Then for n, N sufficiently large*

$$\Pr\left[\kappa(\mathsf{G}_n) \geq N\right] \leq ((1+\varepsilon)\rho)^{mN}.$$

Having more information available about the class $\mathcal{C}$, our results allow for explicit tail point probabilities as in the next example.

Example. *Let $\mathcal{T}$ be the class of unlabelled trees, that is, connected acyclic unlabelled graphs. Then the class of unlabelled forests is $\mathcal{F} = \textsc{MSet}(\mathcal{T})$. Due to [61] there are constants $A > 0$ and $0 < \rho < 1$ such that $t_n := |\mathcal{T}_n| \sim A \cdot n^{-5/2} \cdot \rho^{-n}$ as $n \to \infty$. With (2.7) and (2.10) at hand, we thus obtain the explicit tail point probabilities of the distribution of the number of components of a forest F_n drawn uniformly at random* *from $\mathcal{F}_n$. Concretely, denoting by $T(x)$ the generating series of trees, $F(x)$ the one for forests and letting $F_{>1}(x) = \exp\left\{\sum_{j\geq 1}(T(x^j) - x^j)/(jx^j)\right\}$ be as in (2.9), as $n, N, n-N \to \infty$,*

$$\Pr\left[\kappa(\mathsf{F}_n) = N\right] \sim \frac{F_{>1}(\rho)}{F(\rho)}\left(1 - \frac{N}{n}\right)^{-5/2}\rho^{N-1}.$$

As a last remark and as a motivation for the upcoming results let us gain a better understanding of the terms that $[x^n y^N]G(x,y)$ is asymptotically composed of. Looking at the right-hand side of (2.10) we detect the following unexpected fact. The quantity $[x^n y^N]G(x,y)$ is proportional to $N^{c_m-1}/\Gamma(c_m) \sim \binom{c_m+N-1}{c_m-1}$ times $c_{n-m(N-1)}$, the number of $n-m(N-1)$-sized $\mathcal{C}$-objects. Just as in (2.8) the first term $\binom{c_m+N-1}{c_m-1}$ counts the number of multisets composed of N objects from $\mathcal{C}_m$. A possible interpretation is that a "typical" object from $\mathcal{G}_{n,N}$ consists mostly of components of the smallest possible size m and one extremely large object from $\mathcal{C}_{n-m(N-1)}$ which is as large as a component of a multiset in $\mathcal{G}_{n,N}$ can be.

2.1.2 The Largest Component

Our next main result formalises this intuition. We show that, except for a stochastically bounded term $\mathcal{O}_p(1)$, the largest component in $\mathsf{G}_{n,N}$ is indeed of the largest possible size $n - m(N-1)$.

Theorem 2.3. *Suppose that $C(x)$ is subexponential and $0 < \rho < 1$. Then, as $n, N, n - mN \to \infty$,*

$$\mathcal{L}(\mathsf{G}_{n,N}) = n - mN + \mathcal{O}_p(1).$$

The proof can be found in Section 5.2.4. We baptise the phenomenon established in Theorem 2.3 *extreme condensation*: $\mathsf{G}_{n,N}$ typically has a giant component that is *essentially as large as possible*; its size is close to the largest possible size $n - m(N-1)$ and virtually all other components are as small as possible. This behaviour is rather unique in the literature as far as we are aware of, at least in the analytical ($\rho > 0$) setting considered here.[1] Moreover, this behaviour is surprising for one more reason: if we consider the labelled counterparts of our unlabelled objects, then the typical structure is well known to undergo various phase transitions (from subcritical to condensation) depending on the number of components, but the condensation in the labelled case still leaves room for other objects to be large. See [47, 62] and Section 3.1 for a more detailed discussion.

2.1.3 The Remainder

In our final result about subexponential multisets we complete the picture of the typical structure of $\mathsf{G}_{n,N}$. Theorem 2.3 implies that after removing the largest component, which is of size $n - mN + \mathcal{O}_p(1)$, there is only a size of $mN + \mathcal{O}_p(1)$ left to be distributed over the remaining $N-1$ components. As discussed, this leads to the effect that almost all other components are of smallest possible size m. But what can we say about the remaining $\mathcal{O}_p(1)$ components which are of size $m + \mathcal{O}_p(1)$? The answer is encoded in the term $G_{>m}(\rho)$ in the enumeration formula of Theorem 2.1. Define the class $\mathcal{C}_{>m} = \bigcup_{k>m}\mathcal{C}_k$ containing all objects from $\mathcal{C}$ of size larger than m and equip $\mathcal{C}_{>m}$ with the modified size function $|C|_{>m} := |C| - m$. Then the generating series of $\mathcal{C}_{>m}$ is given by $C_{>m}(x) = (C(x) - c_m x^m)/x^m$. Here the term $c_m x^m$ accounts for removing all objects of size m and dividing by x^m results in assigning objects in $\mathcal{C}_k$, $k > m$, the size $k - m$. Analogous to (2.1) the generating series of $\mathcal{C}_{>m}$-multisets $\mathcal{G}_{>m} = \textsc{Mset}(\mathcal{C}_{>m})$ is hence given by $G_{>m}(x)$. Further, the size of an object G in $\mathcal{G}_{>m}$ is $|G|_{>m} = |G| - m\kappa(G)$. Since the coefficients of $C_{>m}(x)$ are $(c_{k+m})_{k\in\mathbb{N}}$ we

[1] For example, it is known that a factorial weight sequence induces extreme condensation in the balls-in-boxes model, see [47, Example 19.36]. In such situations the respective generating series has radius of convergence 0.

deduce that $C_{>m}(x)$ is also subexponential with radius of convergence ρ such that $G_{>m}(\rho) < \infty$. Define the random variable $\Gamma G_{>m}(\rho)$ on $\mathcal{G}_{>m}$ by

$$\Pr\left[\Gamma G_{>m}(\rho) = G\right] = \frac{\rho^{|G|_{>m}}}{G_{>m}(\rho)} = \exp\left\{-\sum_{j\geq 1} \frac{C(\rho^j) - c_m \rho^{jm}}{j\rho^{jm}}\right\} \rho^{|G|-m\kappa(G)}, \qquad G \in \mathcal{G}_{>m}. \quad (2.11)$$

We note in passing that this is the well-known Boltzmann distribution (on the class $\mathcal{G}_{>m}$) about which we talk later in more detail. For a multiset $G \in \mathcal{G}$ define the remainder $\mathcal{R}(G)$ to be the multiset that is obtained after removing all tuples $(C, d) \in G$ with $C \in \mathcal{C}_m$ and one largest component from G (which can be selected canonically by numbering all objects in $\mathcal{C}$). Formally, the last step means that if the object of largest size occurs with multiplicity $d > 1$ then replace d by $d - 1$ and otherwise remove the object and its multiplicity 1 entirely from G. With this at hand, we prove that the remainder $\mathcal{R}(\mathsf{G}_{n,N})$ has a limiting distribution which turns out to be $\Gamma G_{>m}(\rho)$.

Theorem 2.4. *Suppose that $C(x)$ is subexponential and $0 < \rho < 1$. Then, as $n, N, n - mN \to \infty$, in distribution $\mathcal{R}(\mathsf{G}_{n,N}) \to \Gamma G_{>m}(\rho)$.*

The proof can be found in Section 5.2.5. We close this section and the presentation of the main results in the subexponential setting by catching up with our previous example regarding the class $\mathcal{T}$ of unlabelled trees and $\mathcal{F} = \textsc{MSet}(\mathcal{T})$ of unlabelled forests.

Example (continued). *Let $\mathsf{F}_{n,N}$ be the random forest drawn uniformly from all forest of size n and composed of N trees from $\mathcal{T}$. Then Theorems 2.3 and 2.4 tell us that, with high probability, $\mathsf{F}_{n,N}$ consist of one huge tree of size $n - mN - \mathcal{O}(1)$, $N - \mathcal{O}(1)$ "trivial" trees that are singletons and a bounded forest following the distribution $\Gamma F_{>m}(\rho)$. This is in stark contrast to the known behaviour of random labelled forests, see Section 3.1 for a detailed discussion, but also from unlabelled models such as random unrooted ordered forests, cf. [12].*

We proceed with an application of our results to Benjamini-Schramm convergence of unlabelled graphs with many components. The Benjamini-Schramm limit of a sequence of graphs describes what a uniformly at random chosen vertex typically sees in its neighbourhood and is a special instance of local weak convergence, see also [2, 9]. Given a graph $G = (V, E)$ we form the *rooted* graph (G, o) by distinguishing a vertex $o \in V$. Let $\mathcal{B}$ be the collection of all these rooted graphs. Then two graphs (G, o) and (G', o') in $\mathcal{B}$ are called isomorphic, $(G, o) \simeq (G', o')$, if there exists an edge-preserving bijection Φ on the vertex sets of G and G' such that $\Phi(o) = o'$. Hence, the collection $\mathcal{B}_* = \mathcal{B}/\simeq$ of equivalence classes in $\mathcal{B}$ under the relation $\simeq$ contains all unlabelled rooted graphs.

Set $B_k(G, o)$ to be the induced subgraph of $(G, o) \in \mathcal{B}_*$ containing all vertices within graph distance k from the root o. Then we say that a sequence of (labelled or unlabelled) simple connected locally finite graphs $(\mathsf{G}_n)_{n\geq 1}$ (possibly random) converges in the *Benjamini-Schramm* (BS) sense to a limiting object $(\mathbb{G}, \mathbb{o}) \in \mathcal{B}_*$ if for a vertex o_n being selected uniformly at random from G_n

$$\lim_{n\to\infty} \Pr\left[B_k(\mathsf{G}_n, o_n) \simeq (G, o)\right] = \Pr\left[B_k(\mathbb{G}, \mathbb{o}) \simeq (G, o)\right], \quad k \in \mathbb{N}, (G, o) \in \mathcal{B}_*. \quad (2.12)$$

Back to our setting, we consider $\mathcal{C}$ to be a class of unlabelled finite connected graphs (with subexponential counting sequence and $m \in \mathbb{N}$ denotes the size of the smallest possible graph in $\mathcal{C}$) such that $\mathcal{G} = \textsc{MSet}(\mathcal{C})$ is the class of unlabelled graphs with connected components in $\mathcal{C}$. In order to adapt to the setting above we let $(\mathsf{G}_{n,N}, o_n)$ denote the connected component around a uniformly at random chosen root o_n in $\mathsf{G}_{n,N}$. Let C_n

be drawn uniformly at random from $\mathcal{C}_n$. With this at hand, the extension of BS convergence to non-connected graphs is evident and we obtain the following result.

Proposition 2.5. *Suppose that $C(x)$ is subexponential. Assume that $mN/n \to \lambda \in [0,1)$ as $n, N \to \infty$. If the sequence $(\mathsf{C}_n)_{n\geq 1}$ converges to a limit object $(\mathbb{C}, o)$ in the BS sense, then $\mathsf{G}_{n,N}$ converges as $n, N \to \infty$ to a limit object $(\mathbb{G}, o)$ in the BS sense given by the law*

$$(1-\lambda)\delta_{(\mathbb{C},o)} + \lambda\delta_{(\mathsf{C}_m,o_m)},$$

where o_m is a vertex chosen uniformly at random among the m vertices in C_m. In particular, if $N = o(n)$ we have that $(\mathbb{G}, o) = (\mathbb{C}, o)$.

The proof is found in Section 5.2.6. The authors of [37] show that any subcritical class $\mathcal{C}$ of connected unlabelled graphs fulfils the conditions of Proposition 2.5. In the subcritical setting the BS limit of connected unlabelled rooted graphs is also the BS limit of the respective unrooted graphs as shown in [70]. In particular, prior to these works it was shown in [73, 72] that the BS limits of unlabelled unrooted trees and of unlabelled rooted trees, also called Pólya trees, both exist and coincide. Additionally, this limit, say $(\mathbb{T}, o)$, is made explicit in these publications.

Example (further continued) *We obtain with Proposition 2.5 that the BS limit $(\mathbb{F}, o)$ of $\mathsf{F}_{n,N}$, assuming that $N/n \to \lambda \in [0,1)$ as $n, N \to \infty$, has law*

$$(1-\lambda)\delta_{(\mathbb{T},o)} + \lambda\delta_{\mathcal{X}},$$

where $\mathcal{X}$ is a single rooted vertex. In other words, with probability $1-\lambda$ the neighbourhood of a uniformly at random chosen vertex from $\mathsf{F}_{n,N}$ looks like the infinite tree $\mathbb{T}$ and with probability λ the neighbourhood is empty.

2.2 Expansive (Multi-)sets with Many Components

Let us fix the setting that we consider. We call a function $h : [1,\infty) \to [0,\infty)$ eventually positive and slowly varying if $h(x) > 0$ for all x sufficiently large and

$$\lim_{x\to\infty} \frac{h(\lambda x)}{h(x)} = 1, \quad \lambda > 0.$$

Following [40] we call $(c_k)_{k\in\mathbb{N}}$ (the corresponding series $C(x)$ and class $\mathcal{C}$, respectively) *expansive* if

$$c_n = h(n)\cdot n^{\alpha-1}\cdot\rho^{-n}, \quad h \text{ is eventually positive and slowly varying}, \ \alpha > 0,\ 0 < \rho \leq 1 \text{ and } n \in \mathbb{N}. \tag{2.13}$$

Further, define $m = m(C) \in \mathbb{N}$ to be the smallest integer such that $c_m \neq 0$, that is,

$$m = \min\{k \in \mathbb{N} : c_k > 0\}.$$

We will need the auxiliary power series

$$G^{\geq 2}(x,y) := \exp\left\{\sum_{j\geq 2} \frac{C(x^j)}{j} y^j\right\}.$$

As already mentioned, the quantity $g_n = [x^n]G(x,1)$ is a well-researched object. The authors of [40] investigated, among other cases, g_n in the expansive case. For comparison with our results later on, we present the following theorem that is a straightforward consequence from the results and their proofs in [40, Thm. 1 and Cor. 1]. However, as we use a different notation and the connection to [40] is not immediately obvious, we will give a short self-contained two-page proof that also demonstrates our methodology quite well in Section 6.2.3. We remark that the first part of Section 2.3 will be concerned with obtaining coefficient extraction results as in the next statement in greater generality.

Theorem 2.6. *Suppose that $C(x)$ is expansive and $0 < \rho < 1$. Let z_n be the unique solution to $z_n C'(z_n) = n$. Then, as $n \to \infty$,*

$$z_n \sim \rho \quad \text{and} \quad [x^n]G(x,1) \sim G^{\geq 2}(\rho,1) \cdot \frac{\exp\{C(z_n)\}}{\sqrt{2\pi z_n^2 C''(z_n)}} \cdot z_n^{-n}. \tag{LLT}$$

The form of the enumeration result (LLT) is prototypical: there is a saddle-point (z_n), an exponential term (z_n^{-n}), a term in which the power series is evaluated $(G^{\geq 2}(\rho,1)\exp C(z_n) \sim G(z_n,1))$, and a polynomial term $(2\pi z_n^2 C''(z_n))^{-1/2}$. The latter is the result of an appropriate integration around the saddle-point, or, in a terminology that we prefer here, the result of a local limit theorem, i.e., the probability that a sum of specific independent and identically distributed random variables equals its mean. We will see much more of that later. Let us remark, however, that in the generality considered here, we cannot expect to be able to say much more than (LLT): in general, it is not possible to derive a more explicit asymptotic expression for $\exp\{C(z_n)\}$ (though it is possible to do so for $C(z_n)$ and for its derivatives), and the actual order of magnitude depends very much on the micro-structure of h.

We now move on the main mission of this section, namely the study of $g_{n,N} = [x^n y^N]G(x,y)$. We need some preparations. For $(n,N) \in \mathbb{N}^2$ consider the system of equations in the variables x, y

$$xyC'(x) + mc_m \frac{x^m y}{1 - x^m y} = n, \quad yC(x) + c_m \frac{x^m y}{1 - x^m y} = N, \quad x, y > 0 \text{ and } x^m y < 1. \tag{2.14}$$

We will explain later in detail where these equations come from and only tease for now that the left-hand sides are (more or less) the expected values of the size and the number of components, which we "tune" to n and N, respectively, in a specifically designed random multiset. Equations (2.14) are – in some sense – the bivariate equivalent of the saddle-point equation in Theorem 2.6. Further, for $v \in \mathbb{R}^+$ consider the equation in the single variable u

$$u \cdot h(u)^{1/(\alpha+1)} = v^{1/(\alpha+1)} \quad \text{and} \quad 1 \leq u \leq v. \tag{2.15}$$

Our first auxiliary result is that the previous systems of equations have unique solutions.

Lemma 2.7. *The following statements are true.*

(i) Suppose that $C(x)$ is expansive. For n, N and $n - mN$ sufficiently large there is a unique solution $(x_{n,N}, y_{n,N})$ to (2.14).

(ii) For v sufficiently large there is a unique solution u_v to (2.15) given by $u_v = v^{1/(\alpha+1)}/g(v)$, where $g : \mathbb{R}^+ \to \mathbb{R}^+$ is slowly varying.

With the slowly varying function g from Lemma 2.7(ii) at hand, define the "magic" value

$$N_n^* := C_0 \cdot g(n) \cdot n^{\alpha/(\alpha+1)}, \quad \text{where} \quad C_0 := \alpha^{-1}(\rho^{-m}\Gamma(\alpha+1))^{1/(\alpha+1)}. \tag{2.16}$$

As we will see shortly, the quantity N_n^* marks a phase transition in the structure of the counting sequence $[x^n y^N]G(x,y)$ (and much more ...) depending on whether $N/N_n^* < 1$ or $N/N_n^* > 1$. More precisely, assume that we are given a real-valued positive sequence $(\lambda_n)_{n\in\mathbb{N}}$ such that, as $n \to \infty$,

$$N_n = \lambda_n N_n^* \in \mathbb{N}, \quad N_n \to \infty \quad \text{and} \quad n - mN_n \to \infty. \tag{2.17}$$

Note that by these definitions N_n becomes a function of n and the solutions $(x_n, y_n) \equiv (x_{n,N_n}, y_{n,N_n})$ to (2.14) and N_n^* from (2.16) are all well defined for sufficiently large n due to Lemma 2.7. We will from now on distinguish the two cases

$$\text{(I)} : \limsup_{n\to\infty} \lambda_n < 1 \qquad \text{and} \qquad \text{(II)} : \liminf_{n\to\infty} \lambda_n > 1.$$

2.2.1 Enumeration of Multisets

Our first main result determines the asymptotic growth (in several equivalent forms that have their own merits) of $[x^n y^{N_n}]G(x,y)$ as $n \to \infty$ in case (I).

Theorem 2.8(I). *Suppose that $C(x)$ is expansive. In case (I), as $n \to \infty$,*

$$[x^n y^{N_n}]G(x,y) \sim G^{\geq 2}(\rho, y_n) \cdot \frac{\exp\{y_n C(x_n)\}}{2\pi\sqrt{N_n y_n x_n^2 C''(x_n)/(\alpha+1)}} \cdot x_n^{-n} y_n^{-N_n} \tag{LLT-I}$$

$$\sim G^{\geq 2}(\rho, y_n) \cdot \frac{\sqrt{\alpha}}{2\pi} \cdot \exp\left\{ -\frac{c_m \rho^m y_n}{1-\rho^m y_n} \right\} \cdot \frac{1}{n} \cdot x_n^{-n} (y_n/\mathrm{e})^{-N_n} \tag{Explicit-I}$$

$$\sim G^{\geq 2}(\rho, y_n) \cdot \frac{1}{N_n!}[x^n]C(x)^{N_n}. \tag{Comb-I}$$

The proof can be found in Section 6.2.2. Some remarks are in place. First, (LLT-I) is the prototypical form of the result that very much resembles (LLT) in a bivariate setting. The second form (Explicit-I) is handy and most convenient to work with, as it includes no evaluation of derivatives and – crucially – powers of C at the saddle-point. We feel lucky that we were able to derive such a form of the sequence, and this is mostly owed to the structure of the Equations (2.14) that have a very special property in case (I) (just to look ahead a bit, in that case $x^m y$ stays bounded away from one, so that $1/(1-x^m y)$ remains bounded). The last identity (Comb-I) hints at an interesting fact when $G(x,y)$ is viewed as the generating series of multisets of a class $\mathcal{C}$. Indeed, $C(x)^N$ is the generating series for *sequences* $(C_1, \ldots, C_N)$ that are composed of N objects from $\mathcal{C}$. Then $C(x)^N/N!$ enumerates *sets* $\{C_1, \ldots, C_N\}$, *provided that all elements are distinct*; otherwise there is no reasonable interpretation. So, (Comb-I) may let us speculate that a typical $\mathcal{C}$-multiset in $\mathcal{G}_{n,N_n}$ has only distinct components, and moreover, that we can accurately describe a typical/random element in $\mathcal{G}_{n,N_n}$ by a sequence of N_n objects from $\mathcal{C}$ that are conditioned to have total size n and are otherwise independent.

Let us finish the discussion about Theorem 2.8(I) with the following remark. If $\lambda := \lim_{n\to\infty} \lambda_n \in [0,1)$ exists, then, as we shall see in Lemma 6.16 below, $\lim_{n\to\infty} y_n = \rho^{-m}\lambda^{\alpha+1}$. Consequently, by writing $d(\lambda) = G^{\geq 2}(\rho, \rho^{-m}\lambda^{\alpha+1})$, the counting sequence has the simpler form

$$[x^n y^{N_n}]G(x,y) \sim d(\lambda) \cdot \frac{1}{N_n!}[x^n]C(x)^{N_n} \sim d(\lambda) \cdot \frac{\sqrt{\alpha}}{2\pi} \cdot \exp\left\{ -\frac{c_m \lambda^{\alpha+1}}{1-\lambda^{\alpha+1}} \right\} \cdot \frac{1}{n} \cdot x_n^{-n} (y_n/\mathrm{e})^{-N_n}.$$

In order to treat $g_{n,N}$ in the second case (II) we define

$$G^{\geq 2}_{>m}(x,y) := \exp\left\{ \sum_{j\geq 2} \frac{C(x^j) - c_m x^{jm}}{j x^{jm}} y^j \right\}.$$

Let us give a quick explanation for the choice of notation. Consider the class $\mathcal{C}_{>m}$ of all elements in $\mathcal{C}$ of size greater than m together with the modified size function $|C|_{>m} := |C| - m > 0$ for all $C \in \mathcal{C}_{>m}$. We obtain that the generating series of $\mathcal{C}_{>m}$ is given by $C_{>m}(x) = (C(x) - c_m x^m)/x^m$. Then the generating series of $\mathcal{G}_{>m} := \textsc{MSet}(\mathcal{C}_{>m})$ is given by $G_{>m}(x) = \exp\{\sum_{j\geq 1} C_{>m}(x^j)/j\}$. The superscript in $G_{>m}^{\geq 2}$ accounts for the fact that we only sum up starting at $j = 2$.

Theorem 2.8(II). *Suppose that $C(x)$ is expansive. In case (II) there is a non-negative sequence $(a_n)_{n\in\mathbb{N}}$ given by*

$$a_n := \lambda_n^{-1} \cdot \frac{g(n - mN_n)}{g(n)} \cdot \left(\frac{n - mN_n}{n}\right)^{\alpha/(\alpha+1)}$$

such that, as $n \to \infty$,

$$[x^n y^{N_n}]G(x,y) \sim G_{>m}^{\geq 2}(\rho) \cdot \frac{\exp\{C_{>m}(x_n)\}}{\sqrt{2\pi\rho^{-m} x_n^2 C''(x_n)}} \cdot x_n^{-(n-mN_n)} \tag{LLT-II}$$

$$\sim G_{>m}^{\geq 2}(\rho) \cdot \frac{\big((1-a_n)N_n\big)^{c_m - 1}}{\Gamma(c_m)} \cdot [x^{n-mN_n}]\mathrm{e}^{C_{>m}(x)}. \tag{Comb-II}$$

The proof can be found in Section 6.2.2. We have again remarks. First, (LLT-II) is the classical form that looks like (LLT-I) and (LLT). Note that, however, (LLT-II) looks much more like (LLT) in the sense that it resembles a *uni*variate local limit theorem – y_n does not appear in the formulation at all! This simplification is quite surprising, as we would expect a bivariate law like in Theorem 2.8(I). Here, the alternative form (Comb-II) comes to help and gives – as before – a hint about what may be going on. The factor

$$\frac{((1-a_n)N)^{c_m-1}}{\Gamma(c_m)} \sim \binom{(1-a_n)N + c_m - 1}{c_m - 1}$$

in (Comb-II) counts the number of ways to create a multiset with $\sim (1-a_n)N$ objects from $\mathcal{C}_m$. Where are the remaining $a_n N$ components? For the other term, note that $\exp\{C_{>m}(x)\} = \sum_{k\geq 0} C_{>m}(x)^k/k!$. Then, as before, $C_{>m}(x)^k$ is the generating series of sequences of $\mathcal{C}_{>m}$-objects of length k and, provided all elements in the sequence are distinct, $C_{>m}(x)^k/k!$ counts sets of k objects. Then $[x^{n-mN_n}]\exp\{C_{>m}(x)\}$ enumerates *sets* of distinct $\mathcal{C}_{>m}$-objects with a varying number of components, which, however, concentrates around $a_n N_n$, and thus with (actual) size $\sim n - mN_n(1-a_n)$. Hence, we may speculate that a typical $\mathcal{C}$-multiset in $\mathcal{G}_{n,N_n}$ contains a random set of N' (that typically is $\sim a_n N_n$) components with size $> m$, and the remaining $N_n - N'$ components are of size m. So, since there is no restriction for the component count for objects of size $> m$, we arrive at a univariate limit law as in (LLT-II). Let us note that in case (II) an explicit form as in (Explicit-I) is in general out of reach due to the reasons outlined after Theorem 2.6 – although it is possible to establish the first order of $C_{>m}(x_n)$ (and its derivatives), achieving a similar statement for $\exp\{C_{>m}(x_n)\}$ seems intractable in the general setting considered here.

As we will see in Lemma 6.17 below, if the limit $\lambda := \lim_{n\to\infty}\lambda_n$ exists, then $a_n \sim \lambda^{-1}$ so that defining $d(\lambda) := G_{>m}^{\geq 2}(\rho)(1-\lambda^{-1})^{c_m-1}$, Theorem 2.8(II) yields the slightly simpler form

$$[x^n y^{N_n}]G(x,y) \sim d(\lambda) \cdot \frac{N^{c_m-1}}{\Gamma(c_m)} \cdot [x^{n-mN_n}]\mathrm{e}^{C_{>m}(x)}.$$

We close this section with a final remark. We, unfortunately, cannot offer a Theorem (I ½) that describes what happens when $\lambda_n \to 1$. The point is that the answer actually depends on how this limit is approached and at what speed. We leave it as an open problem to describe the scaling window and the exact behaviour in- and outside of it. However, as a cliffhanger for Section 2.3, the expected number of components in G_n is fortunately bounded away from N_n^* so that we cover all "relevant" cases.

2.2.2 Enumeration of Sets

In the proofs of Theorems 2.8(I) and 2.8(II) we basically reduce the determination of $[x^n y^N]G(x, y)$ to finding $[x^n]C(x)^N/N!$ which is exactly $[x^n y^N]S(x, y)$. Obtaining $[x^n]C(x)^N$ for $n/N \in \Theta(1)$ is a solved problem, see for example [30, Thms. VIII.8 and 9]. So, as a neat by-product of our proofs we are able to state the following theorem, which extends the range to all n, N such that $n/N \to \infty$. Note that $\rho = 1$ is possible in the next theorem as opposed to the other results in Section 2.2.

Theorem 2.9. *Suppose that $C(x)$ is expansive. Let $N = N_n$ be such that $N, n/N \to \infty$ as $n \to \infty$. Set r_n to be the solution to $r_n C'(r_n)/C(r_n) = n/N$. Then as $n \to \infty$,*

$$[x^n y^N]S(x, y) \sim \frac{1}{N!} \cdot \frac{C(r_n)^N}{\sqrt{2\pi N r_n^2 C''(r_n)/((\alpha + 1)C(r_n))}} \cdot r_n^{-n}.$$

2.3 Cluster Statistics of Expansive (Multi-)sets

We first fix our setting again. Recall that $(c_k)_{k\in\mathbb{N}}$ (the corresponding series $C(x)$ and class $\mathcal{C}$, respectively) is called expansive if (2.13) is fulfilled. We further call these objects *oscillating expansive* if for $\alpha > 0, 0 < \varepsilon < \alpha/3$ and $0 < \rho \leq 1$ there are constants $0 < A_1 < A_2$ such that for all n sufficiently large

$$A_1 \cdot n^{\alpha_1 - 1} \cdot \rho^{-n} \leq c_n \leq A_2 \cdot n^{\alpha_2 - 1} \cdot \rho^{-n}.$$

2.3.1 Coefficient Extraction and Counting

In our first main result we asymptotically determine the coefficients of S, G and a wide class of related power series in the oscillating expansive setting. In fact, these results are the consequence of a much more general statement, which is deferred to Lemma 7.3 in the proof section. This result establishes that the corresponding series are H-admissible, see Section 7.1.2 for more information on that property.

Theorem 2.10. *Suppose that $C(x)$ is oscillating expansive. Let z_n be the solution to $z_n C'(z_n) = n$. Then for $\ell \in \mathbb{N}_0$ and as $n \to \infty$*

$$[x^n]S(x) \cdot C(x)^\ell \sim \frac{S(z_n)C(z_n)^\ell}{\sqrt{2\pi z_n^2 C''(z_n)}} \cdot z_n^{-n}. \tag{2.18}$$

Further, if $0 < \rho < 1$, then for $\ell \in \mathbb{N}_0, (p_1, \ldots, p_\ell) \in \mathbb{N}_0^\ell$ and as $n \to \infty$

$$[x^n]G(x) \prod_{1\leq i\leq \ell} \sum\nolimits_{j\geq 1} j^{p_i} C(x^j) \sim \exp\left\{\sum\nolimits_{j\geq 2} C(\rho^j)/j\right\} \cdot \frac{S(z_n)C(z_n)^\ell}{\sqrt{2\pi z_n^2 C''(z_n)}} \cdot z_n^{-n}. \tag{2.19}$$

If $\rho = 1$, let q_n be the solution to $\sum_{j\geq 1} q_n^j C'(q_n^j) = n$. Then for $\ell \in \mathbb{N}_0, (p_1, \ldots, p_\ell) \in \mathbb{N}_0^\ell$ and as $n \to \infty$

$$[x^n]G(x) \prod_{1\leq i\leq \ell} \sum\nolimits_{j\geq 1} j^{p_i} C(x^j) \sim \frac{G(q_n) \prod_{1\leq i\leq \ell} \sum_{j\geq 1} j^{p_i} C(q_n^j)}{\sqrt{2\pi \sum_{j\geq 1} j q_n^{2j} C''(q_n^j)}} \cdot q_n^{-n}. \tag{2.20}$$

The proof can be found in Section 7.2.2. The natural application of Theorem 2.10 is determining the numbers $s_n/n! := [x^n]S(x)$ and $g_n = [x^n]G(x)$ of $\mathcal{C}$-sets and $\mathcal{C}$-multisets of size $n \to \infty$ asymptotically.

That is, if $C(x)$ is oscillating expansive we recover the results from [33, 40] (though slightly different which is mainly due to notation) by

$$s_n \sim \frac{S(z_n)}{\sqrt{2\pi z_n^2 C''(z_n)}} \cdot z_n^{-n} \cdot n!, \quad z_n \text{ solves } z_n C'(z_n) = n$$

and

$$g_n \sim \begin{cases} \exp\left\{\sum_{j\geq 2} C(\rho^j)/j\right\} \frac{\exp\{C(z_n)\}}{\sqrt{2\pi z_n^2 C''(z_n)}} \cdot z_n^{-n}, & 0<\rho<1 \text{ and } z_n \text{ solves } z_n C'(z_n) = n \\ \frac{G(q_n)}{\sqrt{2\pi \sum_{j\geq 1} j q_n^{2j} C''(q_n^j)}} \cdot q_n^{-n}, & \rho = 1 \text{ and } q_n \text{ solves } \sum_{j\geq 1} q_n^j C'(q_n^j) = n \end{cases}.$$

2.3.2 The Distribution of the Largest and Smallest Clusters

Theorem 2.10, or rather the underlying Lemma 7.3 this theorem is based on, is not only helpful to obtain counting results, but can also be applied for fine grained cluster statistics of random (multi-)sets. Let $\mathsf{S}^{(n)}$ and $\mathsf{G}^{(n)}$ be the random cluster structures from (2.4) and (2.3), respectively. The next statements are concerned with the distribution of the extreme (in both directions of the spectrum) cluster sizes in $\mathsf{S}^{(n)}$ and $\mathsf{G}^{(n)}$. Recall that the size of one of the smallest and largest clusters of $\mathsf{F}^{(n)} \in \{\mathsf{S}^{(n)}, \mathsf{G}^{(n)}\}$ is defined by

$$\mathcal{M}(\mathsf{F}^{(n)}) := \min\{1 \leq k \leq n : \mathsf{F}_k^{(n)} > 0\} \quad \text{and} \quad \mathcal{S}(\mathsf{F}^{(n)}) := \max\{1 \leq k \leq n : \mathsf{F}_k^{(n)} > 0\}.$$

First we treat the largest clusters. Freiman and Granovsky [34] show that, if $C(x)$ is expansive, the threshold for the size of the largest cluster in $\mathsf{S}^{(n)}$ is given by $n^{1/(\alpha+1)}$, in the sense that

$$\lim_{n\to\infty} \Pr\left[\mathcal{L}(\mathsf{S}^{(n)}) \leq n^\beta\right] = \begin{cases} 0, & \beta < 1/(\alpha+1) \\ 1, & \beta > 1/(\alpha+1) \end{cases}.$$

However, the question about the actual order of magnitude, and even more, the limiting distribution, remained an open problem. Mutafchiev [60] treated $\mathsf{G}^{(n)}$ in the quasi-expansive setting – under the so-called Meinardus scheme of conditions – that in particular requires $\rho = 1$. He establishes that, by an appropriate scaling, the size of the largest cluster of $\mathsf{G}^{(n)}$ converges to the standard Gumbel distribution. That is, he determines functions $f(n), g(n)$ which tend to infinity as $n \to \infty$ such that

$$\lim_{n\to\infty} \Pr\left[\mathcal{L}(\mathsf{G}^{(n)})/f(n) - g(n) \leq t\right] = \mathrm{e}^{-\mathrm{e}^{-t}}, \quad t \in \mathbb{R};$$

a fact which was proven for integer partitions in the classical work [27] long ago. In our next theorem we state that this is a universal behaviour of (multi-)sets and extend the aforementioned works.

Theorem 2.11. *Suppose that $C(x)$ is expansive. Let $z_n = \rho \mathrm{e}^{-\eta_n}$ be the solution to $z_n C'(z_n) = n$ and $q_n = \rho\mathrm{e}^{-\xi_n}$ the solution to $\sum_{j\geq 1} q_n^j C'(q_n^j) = n$. For $t \in \mathbb{R}$ and $\beta > 0$ set*

$$s(t,\beta) := \beta^{-1}\big(\ln X + t\big), \quad \textit{where } X = \Gamma(\alpha)^{-1} C(\rho \mathrm{e}^{-\beta})(\ln C(\rho\mathrm{e}^{-\beta}))^{\alpha-1} \frac{h(\beta^{-1}\ln C(\rho\mathrm{e}^{-\beta}))}{h(\beta^{-1})}.$$

Then for $t \in \mathbb{R}$

$$\lim_{n\to\infty} \Pr\left[\mathcal{L}(\mathsf{F}^{(n)}) \leq s(t,\beta_n)\right] = \mathrm{e}^{-\mathrm{e}^{-t}}, \quad \textit{where } \beta_n = \begin{cases} \eta_n, & \mathsf{F}^{(n)} = \mathsf{S}^{(n)} \textit{ or } \mathsf{F}^{(n)} = \mathsf{G}^{(n)}, 0<\rho<1 \\ \xi_n, & \mathsf{F}^{(n)} = \mathsf{G}^{(n)}, \rho = 1 \end{cases}.$$

The proof in Section 7.2.3 is based on a combination of the formalisation via H-admissibility together with the inclusion/exclusion principle introduced for this kind of problem by [27]. In [28, Lem. 4.2] a very precise formula determining $C(\rho e^{-\beta})$ and $\rho e^{-\beta} C'(\rho e^{-\beta})$ up to $o(1)$-orders is derived when $c_n = n^{\alpha-1}\rho^{-n}$. We utilise this to get the exact value for $s(t,\beta)$ in the following example, the proof of which is presented after the proof of Theorem 2.11.

Example 2.12. *Let $c_n = n^{\alpha-1}\rho^{-n}$ for $\alpha > 0, 0 < \rho \le 1$ and $n \in \mathbb{N}$. Define*

$$f(n) = (n/\Gamma(\alpha+1))^{1/(\alpha+1)}.$$

Then, if $z_n = \rho e^{-\eta_n}$ is the solution to $z_n C'(z_n) = n$,

$$\eta_n = f(n)^{-1} + o(n^{-1}) \quad \text{and} \quad \ln X = \alpha \ln f(n) + (\alpha-1)\ln\ln f(n) + (\alpha-1)\ln\alpha + o(1).$$

This gives the exact scaling for the distribution of the largest cluster in $\mathsf{S}^{(n)}$ for all $0 < \rho \le 1$ and in $\mathsf{G}^{(n)}$ for $0 < \rho < 1$. For $\rho = 1$ we need to compute the solution q_n to $\sum_{j\ge1} q_n^j C'(q_n^j)$ up to order $\mathcal{O}\left(n^{-1}\right)$ which is not covered by the methods of [28]. This, in fact, is possible under the aforementioned Meinardus scheme of conditions or for special cases such as $\alpha = 1$ (integer partitions).

Next we consider the smallest clusters. In the next result we determine the distribution of $\mathcal{M}(\mathsf{S}^{(n)})$ and $\mathcal{M}(\mathsf{G}^{(n)})$. Freiman and Granovsky [34] determine the limiting distribution of $\mathcal{M}(\mathsf{S}^{(n)})$. We extend their result to the oscillating expansive case and further to $\mathcal{M}(\mathsf{G}^{(n)})$.

Corollary 2.13. *For $\alpha_2 > 0$, $2\alpha_2/3 < \alpha_1 < \alpha_2$ and $0 < \rho \le 1$ let $(c_k)_{k\in\mathbb{N}} \in \mathcal{F}(\alpha_1,\alpha_2,\rho)$. Let q_n be the solution to $\sum_{j\ge1} q_n^j C'(q_n^j) = n$. Then, for $s \in \mathbb{N}$,*

$$\Pr\left[\mathcal{M}(\mathsf{F}^{(n)}) > s\right] = (1+o(1)) \begin{cases} \exp\Big\{-\sum\limits_{1\le k\le s} c_k\rho^k\Big\}, & \mathsf{F}^{(n)} = \mathsf{S}^{(n)} \\ \exp\Big\{-\sum\limits_{j\ge1}\sum\limits_{1\le k\le s} c_k q_n^{jk}/j\Big\}, & \mathsf{F}^{(n)} = \mathsf{G}^{(n)} \end{cases}.$$

Moreover, in any case $\lim_{n\to\infty} \Pr\left[\mathcal{M}(\mathsf{F}^{(n)}) > s_n\right] = 0$ for any $s_n \to \infty$ and $\mathsf{F}^{(n)} \in \{\mathsf{S}^{(n)}, \mathsf{G}^{(n)}\}$.

The proof can be found in Section 7.2.3.

2.3.3 The Cluster Distribution

In this section we are interested in the distribution of the number of clusters in $\mathsf{S}^{(n)}$ and $\mathsf{G}^{(n)}$, which is given by

$$\Pr\left[\kappa(\mathsf{S}^{(n)}) = k\right] = \frac{[x^n y^k]S(x,y)}{[x^n]S(x)} \quad \text{and} \quad \Pr\left[\kappa(\mathsf{G}^{(n)}) = k\right] = \frac{[x^n y^k]G(x,y)}{[x^n]G(x)}, \quad k \in \mathbb{N}.$$

For $n,\ell \in \mathbb{N}$ let $n^{\underline{\ell}}$ denote the falling factorial $n^{\underline{\ell}} := n(n-1)\cdots(n-\ell+1)$. By a straightforward computation we obtain the well-known relation for $\ell \in \mathbb{N}$

$$\mathbb{E}\left[\kappa(\mathsf{S}^{(n)})^{\underline{\ell}}\right] = \frac{[x^n]\mathrm{d}^\ell/(\mathrm{d}y^\ell)S(x,y)\big|_{y=1}}{[x^n]S(x)} \quad \text{and} \quad \mathbb{E}\left[\kappa(\mathsf{G}^{(n)})^{\underline{\ell}}\right] = \frac{[x^n]\mathrm{d}^\ell/(\mathrm{d}y^\ell)G(x,y)\big|_{y=1}}{[x^n]G(x)}. \tag{2.21}$$

With this at hand we are able to compute the moments of $\kappa(\mathsf{S}^{(n)})$ and $\kappa(\mathsf{G}^{(n)})$. For the case $c_n \sim cn^{\alpha-1}\rho^{-n}$ and $c,\alpha > 0, 0 < \rho \le 1$ the results in Erlihson and Granovsky [29] imply arbitrary moments of $\kappa(\mathsf{S}^{(n)})$; for $\kappa(\mathsf{G}^{(n)})$ there are no comparable statements as far as we are aware of. The next corollary completes the picture.

Corollary 2.14. *Suppose that $C(x)$ is oscillating expansive. Let z_n be the solution to $z_n C'(z_n) = n$. Then*

$$\mathbb{E}\left[\kappa(\mathsf{F}^{(n)})^\ell\right] \sim C(z_n)^\ell \quad \textit{for } \mathsf{F}^{(n)} = \begin{cases} \mathsf{S}^{(n)}, & 0 < \rho \le 1 \\ \mathsf{G}^{(n)}, & 0 < \rho < 1 \end{cases}.$$

If $\rho = 1$, let q_n be the solution to $\sum_{j\ge1} q_n^j C'(q_n^j) = n$. Then

$$\mathbb{E}\left[\kappa(\mathsf{G}^{(n)})\right] \sim \sum_{j\ge1} C(q_n^j) \quad \textit{and} \quad \mathbb{E}\left[\kappa(\mathsf{G}^{(n)})^2\right] \sim \Big(\sum_{j\ge1} C(q_n^j)\Big)^2 + \sum_{j\ge1} jC(q_n^j).$$

The proof can be found in Section 7.2.4. Note that we may compute *any* moment of $\kappa(\mathsf{G}^{(n)})$ for $\rho = 1$ by carefully determining the derivatives in (2.21) and then making use of Theorem 2.10. However, we remark that already for the case $\ell = 2$ the second term in the asymptotic formula for $\mathbb{E}\left[\kappa(\mathsf{G}^{(n)})^2\right]$ can be of the same order as the first term. To see this, consider following example. Let $c_n \sim n^{\alpha-1}$ for some $0 < \alpha < 1$. Set $q_n = \mathrm{e}^{-\xi_n}$. Then, as we will show later in Lemma 7.1, we have that $\sum_{j\ge1} C(q_n^j) = \Theta(\xi_n^{-1})$ and $\sum_{j\ge1} jC(q_n^j) = \Theta(\xi_n^{-2})$. For this reason it seems out of reach to get a "nice" formula for $\mathbb{E}\left[\kappa(\mathsf{G}^{(n)})^\ell\right]$ for general ℓ. So, we are content with only presenting the cases $\ell = 1, 2$. As a second remark, we want to mention that we cannot compute the variance for the remaining ρ, as we do not know (and in general, cannot obtain) the second asymptotic order of the expressions at hand.

Under the slightly stronger assumption that $C(x)$ is expansive, we can say much more. For such $C(x)$ we determined the asymptotic number of $\mathcal{C}$-(multi-)sets of total size n and with N clusters in Theorems 2.8(I), 2.8(II) and 2.9, which is nothing else than $[x^n y^N]G(x, y)$ and $[x^n y^N]S(x, y)$, as $n, N, n - mN \to \infty$. Equipped with this we proceed to investigating local limit theorems for $\kappa(\mathsf{S}^{(n)})$ and $\kappa(\mathsf{G}^{(n)})$. Erlihson and Granovsky [28] derived a local limit theorem (and a central limit theorem) for $\kappa(\mathsf{S}^{(n)})$ under the condition $c_n = cn^{\alpha-1}\rho^{-n}$ for $c, \alpha > 0, 0 < \rho \le 1$ and $n \in \mathbb{N}$; note that this is a very restricted choice of $h(n)$ in Equation (2.13). They remark that it is necessary to know the second order of $C(\rho \mathrm{e}^{-\chi})$ as $\chi \to 0$ in order to successfully treat this problem. In our next result we generalize [28] and we do so, to our own surprise, without knowing the second order of the related generating series. The second order of $C(\rho\mathrm{e}^{-\chi})$ as $\chi \to 0$ is in the generality considered here not known (or even not possible to derive). In addition, we state a local limit theorem for $\kappa(\mathsf{G}^{(n)})$, which is a new result standing on the shoulders of Theorem 2.8(I).

Theorem 2.15. *Suppose that $C(x)$ is expansive. Then for any $K > 0$, as $n \to \infty$*

$$\Pr\left[\kappa(\mathsf{S}^{(n)}) = \lfloor C(z_n) + t\sqrt{C(z_n)/(\alpha+1)}\rfloor\right] \sim \mathrm{e}^{-t^2/2} \cdot \frac{1}{\sqrt{2\pi C(z_n)/(\alpha+1)}}, \quad t \in [-K, K].$$

If $0 < \rho < 1$ we further get that for any $t \in \mathbb{R}$, as $n \to \infty$,

$$\Pr\left[\kappa(\mathsf{G}^{(n)}) = \lfloor C(z_n) + t\sqrt{C(z_n)/(\alpha+1)}\rfloor\right] \sim \mathrm{e}^{-t^2/2} \cdot \frac{1}{\sqrt{2\pi C(z_n)/(\alpha+1)}}.$$

The proof can be found in Section 7.2.4. This of course strongly suggests that the variance of $\kappa(\mathsf{S}^{(n)})$ and $\kappa(\mathsf{G}^{(n)})$ is asymptotically given by $C(z_n)/(\alpha+1)$; a fact which we leave as an open problem. Moreover, note that in Theorem 2.15 the statement about $\kappa(\mathsf{S}^{(n)})$ implies a central limit theorem, whereas the statement about $\kappa(\mathsf{G}^{(n)})$ is weaker and not sufficient to obtain a central limit theorem.

3 Discussion

3.1 The Unlabelled vs. the Labelled Setting

A direct connection between the unlabelled multiset and the labelled set case can be established by the so-called star transformation for a series $C(x) = \sum_{k\geq 1} c_k x^k$ given by

$$C^*(x) := \sum_{k\geq 1} c_k^* x^k, \quad \text{where } c_n^* := \sum_{jk=n} c_j/k, \quad n \in \mathbb{N}.$$

Then, by noting that $\sum_{n\geq 1}\sum_{jk=n} = \sum_{j,k\geq 1}$,

$$\exp\{C^*(x)\} = G(x).$$

Hence multisets with parameters $(c_k)_{k\geq 1}$ are basically sets with parameters $(c_k^*)_{k\geq 1}$ although there might not be a combinatorial interpretation for this sequence, which is not relevant if we want to compare for instance $[x^n]\exp\{C(x)\}$ to $[x^n]\exp\{C^*(x)\}$. The following lemma says that under a condition which in particular holds in our subexponential and expansive setting whenever $0 < \rho < 1$ multisets and sets are very much alike.

Lemma 3.1 ([7, Lem. 5.1])**.** *If $c_n/c_{n-1} \to \rho$ and $0 < \rho < 1$ then $c_n^* \sim c_n$ as $n \to \infty$.*

Subexponential Case. So, one might be inclined to say that the discussion is closed for the subexponential case where necessarily $0 < \rho < 1$. Indeed, the results from [71, 74] confirm this claim by implying

$$[x^n]G(x) \sim G(\rho)\cdot c_n \quad \text{and} \quad [x^n]S(x) \sim S(\rho)\cdot c_n.$$

Also the global structure of the random variables G_n and S_n is in both cases governed by the same condensation effect, see [71, 74]. However, the situation changes dramatically by additionally taking into account N components as $N \to \infty$. The works [62, 47] treat this topic extensively[2]: under the condition that $c_n \sim bn^{-(1+\beta)}\rho^{-n}$ for $b > 0$ and $\beta > 1$ as $n \to \infty$ there emerges a "trichotomy" ($1 < \beta \leq 2$) and in some cases a "dichotomy" ($\beta > 2$) depending on the asymptotic regime of N. To illustrate the nature of these results, let us consider the class of labelled trees $\mathcal{T}^\iota$ such that $\mathcal{F}^\iota = \textsc{Set}(\mathcal{T}^\iota)$ is the class of labelled forests. The well-known formula by Cayley states that $t_n^\iota = n^{n-2} \sim (2\pi)^{-1/2} n^{-5/2} \mathrm{e}^n n!$, so that $\beta = 3/2$. Abbreviating by $f_{n,N}^\iota$ the number of forests on n nodes and N trees, the following detailed result exposing a phase transition is known. Let $N := \lfloor \lambda n \rfloor$, then

$$\frac{N!}{n!}\cdot f_{n,N}^\iota \sim \begin{cases} c_-(\lambda)\cdot n^{-3/2}\cdot \mathrm{e}^n 2^{-N}, & \lambda \in (0,1/2) \\ c\cdot n^{-2/3}\cdot \mathrm{e}^n 2^{-N}, & \lambda = 1/2 \\ c_+(\lambda)\cdot n^{-1/2}\cdot f(\lambda)^n, & \lambda \in (1/2,1) \end{cases}, \quad n \to \infty,$$

for positive real-valued continuous functions $c_{-/+}(\lambda), f(\lambda)$ and a constant c; note that the critical exponent jumps form $3/2$ to $2/3$ and then to $1/2$. All in all, the main results from Section 2.1 reveal substantial differences between the labelled and the unlabelled case already at the level of the counting sequences: as we stated before in our example, the number of unlabelled forests of size n with $N = \lfloor \lambda n \rfloor$ components is asymptotically equal to $A\cdot(1-\lambda)^{-5/2} n^{-5/2} \rho^{n-N}$; in particular, the critical exponent does not vary.

The aforementioned variation in the critical exponent has also important consequences for the global structure of a labelled forest $\mathsf{F}_{n,N}^\iota$ drawn uniformly at random from the set of labelled forests of size n composed of N trees. Three different cases emerge as n approaches infinity:

[2]In particular we want to highlight [47, Theorems 18.12, 18.14, 19.34, 19.49].

1. In the case where there are "few" components ($0 < \lambda < 1/2$), most of the mass is concentrated in one large tree containing a linear fraction (that is $1 - 2\lambda$) of all nodes and the remaining $N - 1$ trees all have size $\mathcal{O}_p(n^{2/3})$.

2. In the case where the ratio between components and total size is "balanced" ($\lambda = 1/2$) all trees have size $\mathcal{O}_p(n^{2/3})$.

3. Whenever there are "many" components with respect to the total number of nodes ($1/2 < \lambda < 1$), all trees are small in the sense that their size is stochastically bounded by $\ln n$.

For a detailed discussion for what happens near the critical point $\lambda = 1/2$ see also [55]. This is again substantially different to the unlabelled case, where we showed that extreme condensation dominates the picture for all values of N.

Expansive Case. The expansive case is concomitant with similar effects. We see in Section 2.3 that whenever $0 < \rho < 1$ the unlabelled multiset and labelled set case are treated the same in the sense that the point z_n solving $z_n C'(z_n) = n$ is chosen for both multisets and sets. In particular, Theorem 2.10 gives us that

$$[x^n]G(x) \sim \frac{G(z_n)}{\sqrt{2\pi C''(z_n)}} \cdot z_n^{-n} \quad \text{and} \quad [x^n]S(x) \sim \frac{S(z_n)}{\sqrt{2\pi C''(z_n)}} \cdot z_n^{-n}.$$

Again, these similarities vanish when bringing many components into play as can be directly seen from the different natures of Theorems 2.8(I), 2.8(II) and 2.9. But, ever more surprising as oddly opposed to the subexponential case, this time there is a phase transition in the counting sequence of the multisets.

For $\rho = 1$ the labelled and unlabelled settings differ already in the univariate case. The limit theorems for $\kappa(\mathsf{G}^{(n)})$ in [59] and the ones for $\kappa(\mathsf{S}^{(n)})$ in [28] are of complete different nature for $0 < \alpha < 2$. The different cases appearing in [59] depending on α give the hint that extending Theorems 2.8(I) and 2.8(II) to $\rho = 1$ may need this distinction as well. This complicates the analysis considerably since, and we will see that in the next section, $C(\rho^j)$ diverges for any $j \geq 1$ in that case.

3.2 Notes on the Proofs

Let $(x_0, y_0) \in (\mathbb{R}^+)^2$ be such that $G(x_0, y_0) < \infty$. As in (1.3) the straightforward approach to determine $g_{n,N} = [x^n y^N]G(x, y)$ is to reformulate $g_{n,N}$ by Cauchy's integral formula and then apply the saddle-point method to

$$[x^n y^N]G(x, y) = \frac{x_0^{-n} y_0^{-N}}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \exp\{f(\theta, \phi)\}\, d\theta d\phi, \quad f(\theta, \phi) = \ln G(x_0 \mathrm{e}^{\mathrm{i}\theta}, y_0 \mathrm{e}^{\mathrm{i}\phi}) - n\mathrm{i}\theta - N\mathrm{i}\phi. \tag{3.1}$$

However, a two-dimensional saddle-point integration is deeply technical and hard to achieve. In what follows we describe alternative ways to tackle this problem used both in this thesis and by other authors. Namely, the main idea of the proofs in Sections 2.1 and 2.2 is to translate the task of determining $g_{n,N}$ into a probabilistic problem by considering the multiset G_n drawn uniformly at random from $\mathcal{G}_n$. Let us first discuss the quantity $g_n = [x^n]G(x, 1)$ – the *total* number of multisets of size n with no restrictions on the number of components – for which this kind of analysis has already been carried out successfully in several cases. Assume that we have at our disposal a randomised algorithm/stochastic process G that outputs elements from $\mathcal{G}$ with *a priori* no control on the size or the number of components, but with the guarantee that all objects of the same size are equiprobable. Then

$$\frac{1}{g_n} = \Pr[\mathsf{G}_n = G] = \frac{\Pr[\mathsf{G} = G]}{\Pr[|\mathsf{G}| = n]} \quad \text{for any } G \in \mathcal{G}_n. \tag{3.2}$$

The goal is to find an algorithm such that we can determine/estimate the terms in the latter expression, namely $\Pr[\mathsf{G} = G]$ (that is usually easy, by design of the algorithm) and $\Pr[|\mathsf{G}| = n]$ (the hard one). As it turns out, such algorithms exist and two different approaches stand out in the literature.

The first and classical approach, based on Khinchin's probabilistic method [51] and also referred to as *conditioning relation*, generates G by (first) sampling for each $k \in \mathbb{N}$ independently a random number X_k of components of size k; thus $|\mathsf{G}| = \sum_{k\geq 1} kX_k$. The "right" choice for X_k is a negative binomial distribution with parameters (c_k, x_n^k) for a control value x_n.[3] This is related to the representation of the generating function (1.5) for $\mathcal{C}$-multisets given by

$$G(x, y) = \prod_{k\geq 1}(1 - yx^k)^{-c_k} \tag{3.3}$$

for $y = 1$. Then, by "tuning" x_n such that $\mathbb{E}[|\mathsf{G}|] = n$ a local limit theorem for $\Pr[|\mathsf{G}| = n]$ can be shown to be true in certain cases. Note that introducing this additional dependence on n yields a genuine triangle array of independent random variables, and in this general situation there are often no standardised local limit theorems to resort to. In the (oscillating) expansive case, however, the authors of [40] accomplish this difficult task and proceed as previously described to determine g_n asymptotically. This tuning procedure is in general a feasible method for expansive multisets, as $\mathbb{E}[|\mathsf{G}|]$ is proportional to $C'(x_n)/C(x_n)$ and $\lim_{x_n\to\rho} C'(x_n) = \infty$ for $c_n = h(n) \cdot n^{\alpha-1} \cdot \rho^{-n}$ such that $\alpha > 0$; in other cases, in particular when $\alpha < 0$, tuning is not possible. The conditioning relation is also used by other authors for solving various related problems, see for example the works also mentioned in the introduction [41, 59, 42, 6, 28, 5, 35]. Some historical remarks about Khinchin's probabilistic method are made in [34]. Apart from multisets at least two other broad combinatorial constructions admit this representation of their component structure in terms of independent random variables, namely sets and selections, see [4]. Publications treating these models using the tuning procedure are for example [40, 33].

On the other hand, and this is the method that we choose and develop further in this thesis, the *Pólya-Boltzmann model* [14, Prop. 38] is used to find a decomposition of G into random $\mathcal{C}$-objects attached to cycles of a random permutation, which is helpful to get rid of cumbersome appearances of symmetries and that gives rise to Poisson distributions instead of negative binomials; this difference is reflected by the two different representations (2.1) and (3.3). Again we obtain independent random variables tuned by a control parameter x_n that can be used to describe $|\mathsf{G}|$ in a better-to-handle manner. For a precise description of the involved random variables have a peek at Sections 5.2.1 and 6.2.1, where we present the univariate framework, and a bivariate extension, respectively, in full detail. The crucial property of the corresponding variables is that all objects attached to cycles of length $j \geq 2$ in the permutation are drawn according to a probability distribution with mean proportional to $C(x_n^j)$. Since $0 < x_n \leq \rho$ we infer that $C(x_n^j)$ is decreasing exponentially fast in j for $0 < \rho < 1$. Hence, objects associated to fixpoints, that is, for $j = 1$, are dominant in the structure of G; this essentially reduces the analysis to the study of a sum of iid random variables, rendering this method particularly useful whenever $0 < \rho < 1$. This is basically what happens in the proof of Theorem 2.6 where we recover parts of the results in [40] as a demonstration of our methodology. In the subexponential setting, where tuning is not possible and thus $x_n = \rho$ is the generic choice, [74] determines (among other more general things) g_n with this

[3] Note that this is exactly the distribution such that $\mathsf{G}^{(n)} = \left(X_1, \ldots, X_n \mid \sum_{1\leq k\leq n} kX_k = n\right)$ for $\mathsf{G}^{(n)}$ defined in (2.3), see also [4].

technique. Here the power of translating the enumeration problem into a probabilistic one becomes visible as the author efficiently uses existing results about subexponential distributions to treat the iid random variables. For other applications of the Pólya-Boltzmann model regarding random multisets, see for instance [65, 63].

Let us return to our actual problem from Sections 2.1 and 2.2, where we want to find an asymptotic expression for $g_{n,N}$ as $n \to \infty$ and for essentially all $N \to \infty$. As in (3.2) we obtain

$$\frac{g_{n,N}}{g_n} = \Pr\left[\kappa(\mathsf{G}_n) = N\right] = \frac{\Pr\left[|\mathsf{G}| = n, \kappa(\mathsf{G}) = N\right]}{\Pr\left[|\mathsf{G}| = n\right]}. \tag{3.4}$$

In the subexponential case where tuning is anyhow not possible, we proceed with the one-parametric Boltzmann model and set $x_n = \rho$. Then, completely in spirit of [74], we reduce the problem to a sum of iid random variables hitting a certain value which can be efficiently computed by existing probabilistic results. In the expansive case, however, we see a major difficulty that suddenly appears: the one-parametric models have the property $\mathbb{E}\left[|\mathsf{G}|\right] = n$ that we obtained by tuning x_n, but at the same time $\kappa(\mathsf{G})$ can possibly be far away from N. Consequently, a two-parametric description of G must be found in order to tune the expectation of $\kappa(\mathsf{G})$ to be (close to) N. Then the problem of determining $g_{n,N}$ boils down to finding a two-dimensional local limit law for $K_{n,N} := \Pr\left[|\mathsf{G}| = n, \kappa(\mathsf{G}) = N\right]$. Achieving this is a challenging problem, since there is a significant interplay between $|\mathsf{G}|$ and $\kappa(\mathsf{G})$, and the involved random variables now depend on two additional parameters. The author of [69] solves this problem under Meinardus scheme of conditions and the assumption that $c_n \sim Cn^{\alpha-1}$ for $C, \alpha > 0$. Similar to the one-parametric case of the conditioning relation the author reformulates $|\mathsf{G}| = \sum_{k\ge1} kX_k$ and $\kappa(\mathsf{G}) = \sum_{k\ge1} X_k$ for X_k having negative binomial distribution with parameters $(c_k, y_{n,N}x_{n,N}^k)$ for two control parameters $x_{n,N}, y_{n,N}$ and $k \ge 1$. Then by choosing carefully $x_{n,N}$ and $y_{n,N}$ such that $\mathbb{E}\left[|\mathsf{G}|\right] = n$ and $\mathbb{E}\left[\kappa(\mathsf{G})\right] = N$, a local limit theorem for $K_{n,N}$ is proven by a bivariate saddle-point integration of an integral similar to (3.1). This, as mentioned, is a tough problem so that the results in [69] hold only in the limited parameter range $N = o(n^{\alpha/(\alpha+1)})$ and $N = \omega(\ln^3 n)$. Other parameter ranges have not been studied as far as we are aware of.

As opposed to that approach, in the expansive case for $0 < \rho < 1$ we get a grip on $K_{n,N}$ by conducting a novel application of the bivariate Boltzmann model with the parameters $x_{n,N}$ and $y_{n,N}$ solving (2.14). This solution asserts (more or less) that $\mathbb{E}\left[|\mathsf{G}|\right] = n$ and $\mathbb{E}\left[\kappa(\mathsf{G})\right] = N$. The first phenomena we observe in our proofs is that depending on the ratio of n and N there is a sharp phase transition at which the main contribution to $|\mathsf{G}|$ or $\kappa(\mathsf{G})$ is not given by only by the fixpoints anymore. This leads to the different natures of Theorems 2.8(I) and 2.8(II). Nevertheless, we are able to quantify the number of fixpoints in both regimes, and since objects not stemming from fixpoints are typically of smallest possible size, we obtain the desired simplification. So, at the core of the problem lies again the probability that a given number of iid random variables hits a certain value. This can be expressed in terms of the coefficients of a large power of the corresponding probability generating function – a one-dimensional problem, which is still complex but way more accessible than (3.1). Finally, we solve this by a detailed saddle-point analysis in one variable.

4 General Preliminaries

In this section we gather some general statements which will be needed for all the different proofs in Sections 5–7. We begin with estimates and asymptotics (of coefficients) of power series in Section 4.1. Subsequently, Section 4.2 contains the well-known Euler-Maclaurin summation formula.

4.1 Estimates of (Power) Series

We will often be given a product of two power series, whose coefficients need to be retrieved. The next classical lemma gives very general conditions under which this task can be performed.

Lemma 4.1. *[15, Thm. 3.42] Let $A(x), R(x)$ be power series with radii of convergence $\rho_A, \rho_R > 0$. Suppose that $[x^{n-1}]A(x)/[x^n]A(x) \sim \rho_A^{-1}$. Moreover, assume that $\rho_R > \rho_A$ and $R(\rho_A) \neq 0$. Then*

$$[x^n]A(x)R(x) \sim R(\rho_A) \cdot [x^n]A(x), \quad n \to \infty.$$

Note that Lemma 4.1 does not require R to have non-negative coefficients only. We will later apply the lemma with (powers of)

$$A(x) := (1 - \rho_A^{-1}x)^{-1} = \sum_{k \geq 0} (\rho_A^{-1}x)^k$$

for some $\rho_A > 0$. The next statement tells us more about the coefficients of A in this particular form.

Lemma 4.2. *Let $\alpha, \beta \in \mathbb{R}_+$. Then*

$$[x^n](1 - \beta x)^{-\alpha} \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}\beta^n, \quad n \to \infty.$$

We remark that this is straightforward by writing with Newton's generalised binomial theorem

$$(1 - x)^{-\alpha} = \sum_{k \geq 1} \binom{\alpha + k - 1}{k} x^k \quad \text{and} \quad \binom{\alpha + n - 1}{n} \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}, \quad n \to \infty.$$

The following simple and well-known saddle-point estimate for the coefficients of a power series will be useful several times.

Lemma 4.3. *Let $F(x) = \sum_{k \geq 0} f_k x^k$ be a power series with non-negative coefficients. Then, for any $n \in \mathbb{N}$ and $z > 0$*

$$f_n = [x^n]F(x) \leq F(z)z^{-n}. \tag{4.1}$$

Next, we present an estimate which holds for any power series with non-negative coefficients and positive radius of convergence.

Lemma 4.4. *Let $m \in \mathbb{N}$. Let $F(x) = \sum_{k \geq m} f_k x^k$ be a power series with non-negative coefficients such that $f_m > 0$ and radius of convergence $0 < \rho < 1$. Let $0 < \varepsilon < 1$. Then there exists $A > 0$ such that uniformly in $0 \leq z \leq (1 - \varepsilon)\rho$*

$$1 \leq \frac{F(z)}{f_m z^m} \leq 1 + Az.$$

Proof. The first inequality follows directly from the definition of F and m. Abbreviate $a := (1 - \varepsilon)\rho$ and note that since $\rho < 1$ also $a < \rho < 1$. Then

$$\frac{F(z)}{f_m z^m} \leq 1 + z\frac{1}{f_m}\sum_{k > m} f_k a^{k-m-1} = 1 + z\frac{a^{-m-1}}{f_m}\sum_{k > m} f_k a^k. \tag{4.2}$$

Thus, as $a < \rho < 1$, we obtain from (4.2) the claimed bound with $A = F(a)a^{-m-1}/f_m$. □

The next statement tells us that knowing the asymptotic equality of two sequences, we always find an upper bound holding uniformly in n. This basic lemma, that follows directly from the definition of convergence, will be used mostly without further reference.

Lemma 4.5. *Let $(a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0}$ be real-valued sequences such that $a_n \sim b_n$. Suppose that $b_n \neq 0$ for all $n \in \mathbb{N}_0$. Then there exists $A > 0$ such that uniformly in $n \in \mathbb{N}_0$*

$$a_n \leq A \cdot |b_n|.$$

We close this section with an asymptotic identity which will be applied numerous times for the specific expansive setting considered in Sections 2.2 and 2.3. The proof follows straightforwardly from Theorem A.5.

Lemma 4.6. *Let $(c_k)_{k\in\mathbb{N}}$ be expansive, that is, $c_n = h(n) \cdot n^{\alpha-1} \cdot \rho^{-n}$ for some eventually positive, slowly varying h, $\alpha > 0$, $0 < \rho \leq 1$ and $n \in \mathbb{N}$. Set $C(x) := \sum_{k\geq 1} c_k x^k$. Then*

$$(\rho e^{-\chi})^{\ell} C^{(\ell)}(\rho e^{-\chi}) \sim \Gamma(\alpha+\ell) \cdot h(\chi^{-1}) \cdot \chi^{-(\alpha+\ell)} \quad \text{for } \ell \in \mathbb{N} \text{ and as } \chi \to 0.$$

4.2 Euler-Maclaurin Summation

For a function g we will need to compute the sum $\sum_{k=a}^{b} g(k)$ for some $a < b$. The Euler-Maclaurin summation formula, discussed for example in [38, Ch. 9.5], relates the sum to an integral, which is sometimes easier to determine.

Lemma 4.7. *Let $g : \mathbb{R} \to \mathbb{R}$ be differentiable and $P_1(x) := x - \lfloor x \rfloor - 1/2$. Then for any $a < b$*

$$\sum_{k=a}^{b} g(k) = \int_a^b g(x)dx + \frac{g(a)+g(b)}{2} + \int_a^b g'(x)P_1(x)dx.$$

Among other things, we will apply this to functions of the form $g(x) = e^{-dx^2}$ for some $d > 0$. Note that $|P_1(x)| \leq 1$ and $|g'(x)|$ equals $g'(x)$ for $x < 0$ and $-g'(x)$ for $x > 0$. Accordingly,

$$\sum_{k=a}^{b} g(k) = \int_a^b g(x)dx + Q, \quad \text{where} \quad |Q| \leq \frac{g(a)+g(b)}{2} + \int_{-\infty}^{\infty} |g'(x)|dx \leq 3. \tag{4.3}$$

This will simplify the computation of $\sum g(k)$ considerably, as it will turn out that the remainder Q is negligible compared to $\int g(x)dx$ which, in turn, is well-studied.

5 Subexponential Multisets With Many Components

This section contains the proofs of the main results in Section 2.1 and is based on Sections 2 and 3 of the contributing Manuscript (I).

Plan of the Section. We start by summarising results about subexponential power series/distributions in Section 5.1. Subsequently, we conduct the proofs of the main results from Section 2.1 in Section 5.2. Here, we first introduce the univariate Boltzmann model in Section 5.2.1 which lies at the heart of the probabilistic approach taken in this thesis. In Section 5.2.2 we show how this model can be adapted to the non-combinatorial setting. This is followed by Sections 5.2.3–5.2.5 containing the proofs for the main Theorems 2.1–2.4. Lastly, the proof of Proposition 2.5 is located in Section 5.2.6.

5.1 Subexponential Power Series

In this section we collect (and prove) some properties of subexponential power series that will be quite handy in the proofs later on. Many of the definitions and statements shown here are taken from [26] or [32] and adapted to the discrete case, see also [74].

Definition 5.1. *A power series* $C(x) = \sum_{k\geq 0} c_k x^k$ *with non-negative coefficients and radius of convergence* $0 < \rho < \infty$ *is called* subexponential *if*

$$\frac{1}{c_n} \sum_{0\leq k\leq n} c_{n-k}c_k \sim 2C(\rho) < \infty \quad \textit{and} \tag{S_1}$$

$$\frac{c_{n-1}}{c_n} \sim \rho, \quad n \to \infty. \tag{S_2}$$

Note that the radius of convergence of a power series $C(x)$ satisfying (S_2) (in particular of any subexponential power series) is ρ and that eventually $[x^n]C(x) > 0$, where as usual, $[x^n]C(x) = c_n$ denotes the coefficient of x^n in $C(x)$. Any arbitrary subexponential power series $C(x)$ with radius of convergence ρ induces the probability generating series of a $\mathbb{N}_0$-valued random variable by setting

$$d_k := \frac{c_k\rho^k}{C(\rho)}, \quad n \in \mathbb{N}_0.$$

Then $D(x) = \sum_{k\geq 0} d_k x^k$ is subexponential with $\rho = 1$ and $D(\rho) = 1$. There are several results about the asymptotic behaviour of sums of random variables with such a subexponential generating series. Here we will need Lemma 5.2(i) below, which corresponds to determining the probability that a randomly stopped sum of random variables with distribution $(d_k)_{k\geq 0}$ attains a large value. Moreover, Lemma 5.2(ii) will be particularly useful, since it provides bounds holding uniformly in the given parameters. In Lemma 5.2(iii) we present and prove a statement often referred to – with various interpretations – as "principle of a single big jump". The dominant contribution to a large sum of subexponential random variables stems typically from one single summand.

Lemma 5.2. *Let* $(D_i)_{i\in\mathbb{N}}$ *be iid* $\mathbb{N}_0$*-valued random variables with probability generating function* $D(x)$*. Assume that* $D(x)$ *is subexponential with radius of convergence* 1*. For* $p \in \mathbb{N}$ *let* $S_p := \sum_{1\leq i\leq p} D_i$ *and* $M_p := \max\{D_1, \dots, D_p\}$*. Then the following statements are true.*

(i) *[32, Theorem 4.30] Let* τ *be a* $\mathbb{N}_0$*-valued random variable independent of* $(D_i)_{i\in\mathbb{N}}$*. Further, assume that the probability generating function of* τ *is analytic at* 1*. Then*

$$\Pr[S_\tau = n] \sim \mathbb{E}[\tau] \Pr[D_1 = n], \quad n \to \infty.$$

(ii) *[32, Theorem 4.11] For every* $\delta > 0$ *there exists* $n_0 \in \mathbb{N}$ *and a* $C > 0$ *such that*

$$\Pr[S_p = n] \leq C(1+\delta)^p \Pr[D_1 = n], \quad \textit{for all } n \geq n_0,\ p \in \mathbb{N}.$$

(iii) *For any* $p \geq 2$

$$(M_p \mid S_p = n) = n + \mathcal{O}_p(1), \quad n \to \infty.$$

Proof of Lemma 5.2(iii). Let $\varepsilon > 0$ be arbitrary. To prove the claim we will establish the existence of $K \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \Pr\left[|M_p - n| \geq K \mid S_p = n\right] < \varepsilon.$$

Clearly under the condition $S_p = n$ we have that $n/p \leq M_p \leq n$. Thus, for any $K \in \mathbb{N}$

$$\Pr\left[|M_p - n| \geq K \mid S_p = n\right] = \sum_{K\leq k\leq(1-p^{-1})n} \Pr\left[M_p = n-k \mid S_p = n\right]. \tag{5.1}$$

Since $D_1, \ldots, D_p$ are iid we obtain for any $k \geq K$

$$\Pr\left[M_p = n-k \mid S_p = n\right] \leq \Pr\left[\bigcup_{1\leq i\leq p} \{D_i = n-k\} \mid S_p = n\right] \leq p\frac{\Pr\left[D_1 = n-k\right]\Pr\left[S_{p-1} = k\right]}{\Pr\left[S_p = n\right]}.$$

Together with Lemma 5.2(ii) we find some constant $C > 0$ such that for $k \geq K$ sufficiently large

$$\Pr\left[S_{p-1} = k\right] \leq C(1+\varepsilon)^{p-1}\Pr\left[D_1 = k\right].$$

Part (i) asserts for n sufficiently large that $\Pr\left[S_p = n\right] \geq (1-\varepsilon)p\Pr\left[D_1 = n\right]$. All in all, for a suitably chosen constant $C' = C'(p)$ the expression in (5.1) can be estimated by

$$\Pr\left[|M_p - n| \geq K \mid S_p = n\right] \leq C' \sum_{K\leq k\leq(1-p^{-1})n} \frac{\Pr\left[D_1 = n-k\right]\Pr\left[D_1 = k\right]}{\Pr\left[D_1 = n\right]}.$$

Property (S_1) and $p \geq 2$ then imply that this is smaller than ε by choosing K large enough. This finishes the proof. □

5.2 Proofs

We briefly (re-)collect all assumptions and fix the notation needed in this section. Note that Theorem 2.1 is valid for real-valued sequences $(c_k)_{k\in\mathbb{N}}$, whereas the remaining results are only reasonable in a combinatorial setting. In this section we will begin with the combinatorial setting in order to build the proof framework; then we will show in Section 5.2.2 how the concepts generalise rather easily to the general real-valued setting.

For a combinatorial class $\mathcal{C}$ let $C(x) = \sum_{k\geq 1} c_k x^k$ denote the power series with coefficients $c_k := |\{C \in \mathcal{C} : |C| = k\}|$, $k \in \mathbb{N}$. Further, let

$$m = m_C := \min\{k \in \mathbb{N} \ : \ c_k > 0\} \tag{5.2}$$

be the index of the first coefficient that does not equal zero. We also assume that $C(x)$ is subexponential, which implies that the radius of convergence fulfils $0 < \rho < 1$. However, the subexponentiality feature is only needed in the very last step of the proof, cf. Lemma 5.10; all other statements preceding this lemma are valid even without this assumption as long as $0 < \rho < 1$ and $C(\rho) < \infty$. Further we define $G(x,y) := \exp\left\{\sum_{j\geq 1} C(x^j)y^j/j\right\}$ and $G(x) := G(x,1)$. We begin with two auxiliary statements. The first one is about the radius of convergence of $G(x)$.

Lemma 5.3. *Assume that $C(x)$ is a power series with non-negative real-valued coefficients and radius of convergence $0 < \rho < 1$ and $C(0) = 0$, $C(\rho) < \infty$. Then $G(x)$ has radius of convergence ρ and $G(\rho) < \infty$.*

Proof. From the definition of G we obtain that $G(x) = e^{C(x)}H(x)$, where $\ln H(x) = \sum_{j\geq 2} C(x^j)/j$. Since $\rho \in (0,1)$ we obtain for any $\varepsilon > 0$ such that $(1+\varepsilon)^j\rho^j \leq \rho$ for all $j \geq 2$ and $(1+\varepsilon)\rho < 1$ that for $j \geq 2$

$$C\big((1+\varepsilon)^j\rho^j\big) = \sum_{k\geq 1} c_k(1+\varepsilon)^{jk}\rho^{jk} = (1+\varepsilon)^j\rho^j \sum_{k\geq 1} c_k((1+\varepsilon)^j\rho^j)^{k-1} \leq (1+\varepsilon)^j\rho^{j-1}C(\rho).$$

In particular, $H((1+\varepsilon)\rho) < \infty$ and the radius of convergence of H is larger than ρ. Thus, the radius of convergence of G is ρ, and $G(\rho) = e^{C(\rho)}H(\rho) < \infty$. □

5.2.1 The Univariate Boltzmann Model

In this section we will introduce the *Boltzmann model* from the pioneering paper [24], which has found various applications in the study of the typical shape of combinatorial structures, see for example [23, 1, 71, 66, 22, 16, 11, 64]. With the help of this model we translate the initial problem of extracting coefficients of the multiset generating function into a probabilistic question. This gives us the proper idea for the general approach for arbitrary functions of the form (2.1), i.e. when the coefficients are not necessarily integers. Further, the formalisation via this model will allow us to prove the extreme condensation phenomenon.

Assume that $z \in \mathbb{R}_+$ is chosen such that $C(z) > 0$ is finite. The unlabelled Boltzmann model defines a random variable $\Gamma C(z)$ taking values in the entire space $\mathcal{C}$ through

$$\Pr\left[\Gamma C(z) = C\right] = \frac{z^{|C|}}{C(z)}, \quad C \in \mathcal{C}.$$

In complete analogy the random variable $\Gamma G(z)$ is defined on $\mathcal{G} = \textsc{Mset}(\mathcal{C})$, where in this case the parameter $z > 0$ is such that $G(z) < \infty$. In the rest of this section we fix $z = \rho$ recalling that $0 < \rho < 1$ is the radius of convergence of C. Then, in virtue of Lemma 5.3, G has radius of convergence ρ and $G(\rho) < \infty$, so that both $\Gamma C(\rho), \Gamma G(\rho)$ are well-defined, and we just write $\Gamma C, \Gamma G$.

Let g_n be the number of objects of size n in $\mathcal{G}$ and $g_{n,N}$ those of size n comprised of N components. By using Bayes' Theorem and that the Boltzmann model induces a uniform distribution on objects of the same size, we immediately obtain

$$\frac{g_{n,N}}{g_n} = \Pr\left[\kappa(\Gamma G) = N \mid |\Gamma G| = n\right] = \Pr\left[|\Gamma G| = n \mid \kappa(\Gamma G) = N\right] \frac{\Pr\left[\kappa(\Gamma G) = N\right]}{\Pr\left[|\Gamma G| = n\right]}, \quad n, N \in \mathbb{N}. \quad (5.3)$$

To get a handle on this expression we exploit a powerful description of the distribution of $\Gamma G(z)$ in terms of $\Gamma C(\cdot)$, derived in [31]. In the next steps, the notation $\bigsqcup_{j\in J} A_j$ is used to denote a multiset of elements A_j from a set $\mathcal{A}$, $j \in J$ being indices in some countable set J. That is, multiple occurrences of identical elements are allowed and $\bigsqcup_{j\in J} A_j$ is completely determined by the different elements it contains and their multiplicities.

(1) Let $(P_j)_{j\geq 1}$ be independent random variables, where $P_j \sim \mathrm{Po}\left(C(\rho^j)/j\right)$.

(2) Let $(\gamma_{j,i})_{j,i\geq 1}$ be independent random variables with $\gamma_{j,i} \sim \Gamma C(\rho^j)$ for $j, i \geq 1$.

(3) For $j, i \geq 1$ and $1 \leq k \leq j$ set $\gamma_{j,i}^{(k)} = \gamma_{j,i}$, that is, make j copies of $\gamma_{j,i}$. Let

$$\Lambda G := \bigsqcup_{j\geq 1} \bigsqcup_{1\leq i\leq P_j} \bigsqcup_{1\leq k\leq j} \gamma_{j,i}^{(k)}.$$

Intuitively, we interpret P_j as the number of j-cycles in some not further specified permutation and to each cycle of length j we attach j times an identical copy of a $\Gamma C(\rho^j)$-distributed $\mathcal{C}$-object. Afterwards we discard the permutation and the cycles and keep the multiset of the generated $\mathcal{C}$-objects. This construction is also made explicit in [14, Prop. 37].

Lemma 5.4. *[31, Prop. 2.1] The distributions of* ΓG *and* ΛG *are identical.*

This statement paves the way to study ΓG. In particular, if we write $C_{j,i} = |\gamma_{j,i}|$, note that the definition of ΛG guarantees that in distribution

$$\kappa(\Gamma G) = \sum_{j\ge1} jP_j \quad \text{and} \quad |\Gamma G| = \sum_{j\ge1} j \sum_{1\le i\le P_j} C_{j,i}.$$

So, let us for $n, N \in \mathbb{N}$ define the events

$$\mathcal{P}_N := \left\{ \sum_{j\ge1} jP_j = N \right\} \quad \text{and} \quad \mathcal{E}_n := \left\{ \sum_{j\ge1} j \sum_{1\le i\le P_j} C_{j,i} = n \right\}. \tag{5.4}$$

With $\Pr[\mathcal{E}_n] = \Pr[|\Lambda G| = n] = g_n\rho^n/G(\rho)$ at hand, Lemma 5.4 and (5.3) then guarantee that

$$g_{n,N} = G(\rho)\rho^{-n}\Pr[\mathcal{E}_n \mid \mathcal{P}_N]\Pr[\mathcal{P}_N]. \tag{5.5}$$

Note that for all $1 \le i \le P_j$ and $j \in \mathbb{N}$, we have

$$\Pr[C_{j,i} = k] = \frac{c_k\rho^{jk}}{C(\rho^j)}, \quad k \in \mathbb{N}. \tag{5.6}$$

Equation (5.5) enables us to reduce the problem of determining $g_{n,N} = [x^n y^N]G(x,y)$ to the problem of determining the probability of the events $\mathcal{P}_N$ and $\mathcal{E}_n$ conditioned on $\mathcal{P}_N$.

5.2.2 Real-valued Sequences in Theorem 2.1

In Theorem 2.1 we consider $(c_k)_{k\in\mathbb{N}}$ to be a real-valued non-negative sequence and assume $0 < \rho < 1$. In complete analogy to the discussion prior to this subsection let $P_j \sim \mathrm{Po}\left(C(\rho^j)/j\right)$ for $j \in \mathbb{N}$, $(C_{j,1}, \ldots, C_{j,P_j})_{j\in\mathbb{N}}$ be as in (5.6), and assume that all these variables are independent. As a matter of fact, also in this (more general) case we obtain exactly the same representation of $[x^n y^N]G(x,y)$ in terms of $\mathcal{E}_n$ and $\mathcal{P}_N$ defined in (5.4) without using the combinatorial Boltzmann model.

Lemma 5.5. *Let* $C(x)$ *be a power series with non-negative real-valued coefficients and radius of convergence* $0 < \rho < 1$ *at which* $C(\rho) < \infty$. *Then*

$$[x^n y^N]G(x,y) = G(\rho)\rho^{-n}\Pr[\mathcal{E}_n \mid \mathcal{P}_N]\Pr[\mathcal{P}_N], \quad n, N \in \mathbb{N}.$$

Proof. We begin with the simple observation

$$\begin{aligned}\Pr[\mathcal{P}_N, \mathcal{E}_n] &= [x^n y^N] \sum_{k\ge0}\sum_{\ell\ge0} \Pr[\mathcal{P}_k, \mathcal{E}_\ell]\, x^\ell y^k \\ &= [x^n y^N] \sum_{k\ge0} y^k \sum_{\sum_{j\ge1} jp_j = k} \prod_{j\ge1} \Pr[P_j = p_j] \sum_{\ell\ge0} \Pr\left[\sum_{j\ge1} j \sum_{1\le i\le p_j} C_{j,i} = \ell\right] x^\ell.\end{aligned} \tag{5.7}$$

We will study this expression by first simplifying the sum over ℓ, then the sum over all p_j's, and eventually the sum over k. We begin with the sum over ℓ. For a $\mathbb{N}_0$-valued random variable A let $A(x) := \sum_{\ell\geq 0}\Pr[A_j=\ell]\,x^\ell$ denote its probability generating series. Then, if $(A_j)_{j\in\mathbb{N}}$ is a sequence of independent $\mathbb{N}_0$-valued random variables,

$$(A_1+\cdots+A_m)(x) = \prod_{1\leq j\leq m} A_j(x), \quad m\in\mathbb{N}. \tag{5.8}$$

Let us write $C_j(x)$ for the probability generating series of $jC_{j,i}$; note that the actual value of i is not important, since the $(C_{j,i})_{i\in\mathbb{N}}$ are iid. Then, whenever $\sum_{j\geq 1} p_j$ is finite, (5.8) implies

$$\sum_{\ell\geq 0}\Pr\left[\sum_{j\geq 1} j \sum_{1\leq i\leq p_j} C_{j,i}=\ell\right] x^\ell = \prod_{j\geq 1} C_j(x)^{p_j}.$$

Noting that $jC_{j,1}$ takes only values in the lattice $j\mathbb{N}_0$, we obtain

$$C_j(x) = \sum_{\ell\geq 0}\Pr[jC_{j,1}=\ell]\,x^\ell = \sum_{\ell\geq 0}\Pr[C_{j,1}=\ell]\,x^{j\ell} = \frac{1}{C(\rho^j)}\sum_{\ell\geq 0} c_\ell \rho^{j\ell}x^{j\ell} = \frac{C((\rho x)^j)}{C(\rho^j)}.$$

We deduce

$$\sum_{\ell\geq 0}\Pr\left[\sum_{j\geq 1} j \sum_{1\leq i\leq p_j} C_{j,i}=\ell\right] x^\ell = \prod_{j\geq 1}\left(\frac{C((\rho x)^j)}{C(\rho^j)}\right)^{p_j}.$$

This puts the sum over ℓ in (5.7) in compact form. To simplify the sum over the p_j's in (5.7) define independent random variables $(H_j)_{j\geq 1}$ with $H_j \sim \mathrm{Po}\left(C((\rho x)^j)/j\right)$. Then

$$\sum_{\sum_{j\geq 1} jp_j = k}\prod_{j\geq 1}\Pr[P_j=p_j]\left(\frac{C((\rho x)^j)}{C(\rho^j)}\right)^{p_j} = \frac{G(\rho x,1)}{G(\rho,1)}\Pr\left[\sum_{j\geq 1} jH_j = k\right].$$

By similar reasoning as before the probability generating function of jH_j is given by

$$\sum_{\ell\geq 0}\Pr[H_j=\ell]\,y^{j\ell} = \exp\left\{-C((\rho x)^j)/j\right\}\sum_{\ell\geq 0}\frac{(C((\rho x)^j)y^j/j)^\ell}{\ell!} = \frac{\exp\left\{C((\rho x)^j)y^j/j\right\}}{\exp\left\{C((\rho x)^j)/j\right\}}.$$

Applying (5.8), where we set $A_j := jH_j$, in combination with this identity and plugging everything into (5.7) yields

$$\sum_{k\geq 0}\Pr\left[\sum_{j\geq 1} jH_j = k\right] y^k = \frac{G(\rho x,y)}{G(\rho x,1)}.$$

All in all, we have shown that $\Pr[\mathcal{P}_N,\mathcal{E}_n] = G(\rho)^{-1}[x^ny^N]G(\rho x,y)$. With $[x^n]F(ax) = a^n[x^n]F(x)$ for any power series F and $a\in\mathbb{R}$ we finish the proof. $\square$

5.2.3 Enumeration

In this section we show that Theorem 2.1 is true. Let $\mathcal{P}_N$, $\mathcal{E}_n$ be as in the previous section, see (5.4), where $P_j \sim \mathrm{Po}\left(C(\rho^j)/j\right)$ and $C_{j,1}, \dots, C_{j,P_j}$ for $j \in \mathbb{N}$ have the distribution specified in (5.6). Moreover, we assume that all these random variables are independent. Equipped with Lemma 5.5 from the previous section, the proof of Theorem 2.1 boils down to estimating $\Pr[\mathcal{E}_n \mid \mathcal{P}_N]$ and $\Pr[\mathcal{P}_N]$. Before we actually do so, let us introduce some more auxiliary quantities. Set

$$P := \sum_{j\geq 1} jP_j \quad \text{and} \quad P^{(\ell)} := \sum_{j>\ell} jP_j, \quad \ell \in \mathbb{N}_0.$$

With this notation, $\mathcal{P}_N$ is the same as $\{P = N\}$ and $\{P^{(0)} = N\}$. Moreover, recall (5.2) and set

$$L := \sum_{1\leq i\leq P_1} (C_{1,i} - m) \quad \text{and} \quad R := \sum_{j\geq 2} j \sum_{1\leq i\leq P_j} (C_{j,i} - m). \tag{5.9}$$

With this notation

$$\Pr[\mathcal{E}_n \mid \mathcal{P}_N] = \Pr[L + R = n - mN \mid \mathcal{P}_N]. \tag{5.10}$$

The driving idea behind these definitions is that the random variables $C_{j,i} - m$, for $j \geq 2$, have exponential tails, and these tails get thinner as we increase j; in particular, the probability that $C_{j,i} - m = 0$ approaches one exponentially fast as we increase j. However, things are not so easy, since we always condition on $\mathcal{P}_N$, and in this space some of the P_j's might be large. This brings us to our general proof strategy. First of all, we will study our probability space conditioned on $\mathcal{P}_N$; in particular, in Corollary 5.7 and Lemma 5.8 below we describe the joint distribution of $P_1, \dots, P_N$ given $\mathcal{P}_N$. More specifically, these results show that the P_j's are (more or less) distributed like Poisson random variables with bounded expectations. This will allow us then in Lemma 5.9 to show that L dominates the sum $L+R$ in the sense that $\Pr[L + R = n - mN \mid \mathcal{P}_N] \sim \Pr[L = n - mN \mid \mathcal{P}_N]$ as $n, N, n - N \to \infty$. Subsequently, in Lemma 5.10 we exploit the subexponentiality and establish that this last probability is essentially a multiple of $\Pr[C_{1,1} = n - mN]$. Just as a side remark and so as to make the notation more accessible: it is instructive to think of the random variable L as something (that will turn out to be) large, and R as some remainder (that will turn out to be small with exponential tails).

Our first aim is to study the distribution – in particular the tails – of P and $P^{(\ell)}$, that is, we want to estimate the probability of $\mathcal{P}_N$. To this end, consider the probability generating series $F(x)$ and $F^{(\ell)}(x)$ of P and $P^{(\ell)}$, respectively, that is

$$F^{(\ell)}(x) = \frac{1}{G^{(\ell)}(\rho)} \cdot \exp\left\{\sum_{j>\ell} \frac{C(\rho^j)}{j} x^j\right\}, \quad \text{where} \quad G^{(\ell)}(\rho) := \exp\left\{\sum_{j>\ell} \frac{C(\rho^j)}{j}\right\}$$

and $F(x) = F^{(0)}(x)$, $G^{(0)}(\rho) = G(\rho)$. Hence, the distribution of $P^{(\ell)}$ (and P) is given by $(\Pr[P^{(\ell)} = N])_{N\geq 0} = ([x^N]F^{(\ell)}(x))_{N\geq 0}$. In Lemma 5.6 we determine the precise asymptotic behaviour of these probabilities.

Lemma 5.6. *There exist constants* $(B^{(\ell)})_{\ell\in\mathbb{N}_0} > 0$ *such that, as* $N \to \infty$

$$\Pr[\mathcal{P}_N] = [x^N]F(x) \sim B^{(0)} \cdot N^{c_m - 1}\rho^{mN} \quad \text{and} \quad [x^N]F^{(\ell)}(x) \sim B^{(\ell)} \cdot N^{c_m-1}\rho^{mN}, \quad \ell \in \mathbb{N},$$

where

$$B^{(0)} = \frac{\exp\left\{\sum_{j\geq 1} \frac{C(\rho^j) - c_m\rho^{jm}}{j\rho^{jm}}\right\}}{G(\rho)\Gamma(c_m)} \quad \text{and} \quad \frac{B^{(\ell)}}{B^{(0)}} = \exp\left\{\sum_{1\leq j\leq \ell} \frac{C(\rho^j)}{j}\right\} \exp\left\{-\sum_{1\leq j\leq \ell} \frac{C(\rho^j)}{j}\rho^{-jm}\right\}.$$

Proof. We split up

$$F(x) = \frac{1}{G(\rho)} \cdot \exp\left\{\sum_{j\geq 1} \frac{c_m \rho^{jm}}{j} x^j\right\} \cdot \exp\left\{\sum_{j\geq 1} \frac{C(\rho^j) - c_m\rho^{jm}}{j} x^j\right\} =: \frac{1}{G(\rho)} \cdot A(x) \cdot B(x).$$

Lemma 4.4 asserts that $B(x)$ has radius of convergence $\rho_B \geq \rho^{-(m+1)}$. Further,

$$A(x) = (1-\rho^m x)^{-c_m},$$

and the radius of convergence of $A(x)$ is $\rho_A = \rho^{-m} < \rho^{-(m+1)} \leq \rho_B$ (since $\rho < 1$). Using Lemma 4.2 we obtain that $[x^N]A(x) \sim N^{c_m-1}\rho^{mN}/\Gamma(c_m)$ and thus $A(x)$ has property (S_2). From Lemma 4.1 we then obtain that

$$\Pr[P = N] = [x^N]F(x) \sim \frac{1}{G(\rho)} B(\rho_A)[x^N]A(x) \sim \frac{B(\rho^{-m})}{G(\rho)\Gamma(c_m)} \cdot N^{c_m-1}\rho^{mN}, \quad N \to \infty, \tag{5.11}$$

Similarly, for $\ell \in \mathbb{N}$

$$F^{(\ell)}(x) = \frac{A(x)}{G^{(\ell)}(\rho)} \cdot \exp\left\{\sum_{j\geq 1} \frac{C(\rho^j) - c_m\rho^{jm}}{j} x^j\right\} \exp\left\{-\sum_{1\leq j\leq \ell} \frac{C(\rho^j)}{j} x^j\right\} =: \frac{A(x)}{G^{(\ell)}(\rho)} \cdot B^{(\ell)}(x).$$

Since the radius of convergence of $B^{(\ell)}(x)$ is again (at least) $\rho^{-(m+1)}$

$$[x^N]F^{(\ell)}(x) \sim \frac{B^{(\ell)}(\rho^{-m})}{G^{(\ell)}(\rho)\Gamma(c_m)} \cdot N^{c_m-1}\rho^{mN}.$$

□

As an immediate consequence of Lemma 5.6 we establish the asymptotic distribution of the random vector $(P_1, \ldots, P_\ell)$ conditioned on the event $\mathcal{P}_N$ for fixed $\ell \in \mathbb{N}$; this will be useful later when we consider the distribution of L, cf. (5.9). Clearly, the condition $\mathcal{P}_N$ makes $P_1, \ldots, P_\ell$ dependent, but the corollary says that this effect vanishes for large N. Moreover, we study the moments of P_1 given $\mathcal{P}_N$.

Corollary 5.7. *Let $\ell \in \mathbb{N}$ and $(p_1, \ldots, p_\ell) \in \mathbb{N}_0^\ell$. Then*

$$\Pr\left[\bigcap_{1\leq j\leq \ell} \{P_j = p_j\} \mid \mathcal{P}_N\right] \to \prod_{1\leq j\leq \ell} \Pr\left[\mathrm{Po}\left(\frac{C(\rho^j)}{j\rho^{jm}}\right) = p_j\right], \quad N \to \infty. \tag{5.12}$$

Moreover, for any $z \in \mathbb{R}$, as $N \to \infty$

$$\mathbb{E}\left[z^{P_1} \mid \mathcal{P}_N\right] \to \mathbb{E}\left[z^{\mathrm{Po}\left(\frac{C(\rho)}{\rho^m}\right)}\right] = \mathrm{e}^{\frac{C(\rho)}{\rho^m}(z-1)}, \quad \mathbb{E}\left[P_1 \mid \mathcal{P}_N\right] \to \mathbb{E}\left[\mathrm{Po}\left(\frac{C(\rho)}{\rho^m}\right)\right] = C(\rho)\rho^{-m}.$$

Proof. Let $s = \sum_{1\leq j\leq \ell} jp_j$. Using the definition of conditional probability we obtain readily

$$\Pr\left[\bigcap_{1\leq j\leq \ell} \{P_j = p_j\} \mid \mathcal{P}_N\right] = \frac{\Pr\left[\bigcap_{1\leq j\leq \ell} \{P_j = p_j\} \cap \{P^{(\ell)} = N - s\}\right]}{\Pr[P = N]}.$$

Since $P_1, \dots, P_\ell, P^{(\ell)}$ are independent, the right-hand size equals

$$\prod_{1\le j\le \ell} \Pr\left[P_j = p_j\right] \cdot [x^{N-s}]F^{(\ell)}(x)/[x^N]F(x), \tag{5.13}$$

and (5.12) follows by applying Lemma 5.6. We will next show P_1 given $\mathcal{P}_N$ has exponential moments. Abbreviate $B := C(\rho)\rho^{-m}$. Note that (5.12) (where we use $\ell = 1$) yields for any fixed $K \in \mathbb{N}$

$$\sum_{0\le k\le K} z^k \Pr\left[P_1 = k \mid \mathcal{P}_N\right] \to \sum_{0\le k\le K} z^k \Pr\left[\mathrm{Po}\left(B\right) = k\right], \quad N \to \infty.$$

Let $\varepsilon > 0$. Note that we can choose K large enough such that the right hand side differs at most ε from $\mathbb{E}\left[z^{\mathrm{Po}(B)}\right] = e^{B(z-1)}$. In order finish the proof we will argue that if K and N are large enough, then $\sum_{K\le k\le N} z^k \Pr\left[P_1 = k \mid \mathcal{P}_N\right] < \varepsilon$ as well. First, by Lemma 5.6

$$z^N \Pr\left[P_1 = N \mid \mathcal{P}_N\right] \le z^N \frac{\Pr\left[P_1 = N\right]}{\Pr\left[\mathcal{P}_N\right]} \sim \frac{\mathrm{e}^{-C(\rho)}}{B^{(0)}} N^{-c_m+1} \frac{(zB)^N}{N!} \to 0, \quad N \to \infty.$$

Moreover, according to Lemmas 5.6 and 4.5 there exists a constant $A_1 > 0$ such that we are able to estimate $[x^{N-k}]F^{(1)}(x)/[x^N]F(x) \le A_1 \cdot (1 - k/N)^{c_m - 1}\rho^{-mk}$ for all $0 \le k \le N-1$. Then with (5.13) we obtain

$$\sum_{K\le k\le N-1} z^k \Pr\left[P_1 = k \mid \mathcal{P}_N\right] \le A_1 \sum_{K\le k\le N-1} t_k, \quad \text{where } t_k := (1 - k/N)^{c_m-1}\frac{(zB)^k}{k!}. \tag{5.14}$$

Note that we can choose K large enough such that, say, $t_{k+1} \le t_k/2$ for all $K \le k < N-1$. Then the sum is bounded by $2t_K$, and choosing K once more large enough gives $2t_K < \varepsilon$. □

Note that Corollary 5.7 (only) holds for a fixed $\ell \in \mathbb{N}$; it does not tell us anything about $(P_1, \dots, P_\ell)$ in the case where ℓ is not fixed, or, more importantly, when $\ell = N$ (note that $P_{N'} = 0$ for all $N' > N$ if we condition on $\mathcal{P}_N$). Regarding this general case, the following statement gives an upper bound for the probability of the event $\bigcap_{1\le j\le N}\{P_j = p_j\}$ that is not too far from the right-hand side in Corollary 5.7. For the remainder of this section it is convenient to define

$$\Omega_N := \Big\{(p_1, \dots, p_N) \in \mathbb{N}_0^N \;:\; \sum_{1\le j\le N} j p_j = N\Big\}, \quad N \ge 2.$$

In what follows we derive a stochastic upper bound for the distribution of $(P_1, \dots, P_N)$ conditioned on $\mathcal{P}_N$.

Lemma 5.8. *There exists an $A > 0$ such that for all N and all $(p_1, \dots, p_N) \in \Omega_N$*

$$\Pr\left[\bigcap_{1\le j\le N}\{P_j = p_j\} \mid \mathcal{P}_N\right] \le A \cdot N \cdot \prod_{1\le j\le N} \Pr\left[\mathrm{Po}\left(\frac{C(\rho^j)}{j\rho^{jm}}\right) = p_j\right].$$

Proof. Using the definition of conditional probability and recalling that the P_j's are independent and $P_j \sim \mathrm{Po}\left(C(\rho^j)/j\right)$

$$\begin{aligned}\Pr\left[\bigcap_{1\le j\le N}\{P_j = p_j\} \mid \mathcal{P}_N\right] &\le \frac{1}{\Pr\left[\mathcal{P}_N\right]} \cdot \prod_{1\le j\le N} \left(\frac{C(\rho^j)}{j}\right)^{p_j} \frac{1}{p_j!} \\ &= \frac{1}{\Pr\left[\mathcal{P}_N\right]} \cdot \exp\left\{\sum_{1\le j\le N} \frac{C(\rho^j)}{j\rho^{jm}}\right\} \rho^{mN} \cdot \prod_{1\le j\le N} \Pr\left[\mathrm{Po}\left(\frac{C(\rho^j)}{j\rho^{jm}}\right) = p_j\right].\end{aligned}$$

With Lemma 5.6 we obtain the existence of $B_1 > 0$ such that for N large enough

$$\Pr[\mathcal{P}_N]^{-1} \le B_1 \rho^{-mN} N^{1-c_m}.$$

By Lemma 4.4 there exists a constant $B_2 > 0$ such that $C(\rho^j)/\rho^{jm} \le c_m + B_2 c_m \rho^j$. Consequently, since $\rho \in (0,1)$ there exists $B_3 > 0$ such that

$$\exp\left\{ \sum_{1\le j\le N} \frac{C(\rho^j)}{j\rho^{jm}} \right\} \le B_3 N^{c_m},$$

which concludes the proof. □

With this result at hand we are ready to study the distribution of R, cf. (5.9). As it will be necessary later, we show uniform tails bounds that hold for the joint distribution of P_1 and R conditioned on $\mathcal{P}_N$.

Lemma 5.9. *There exist $A > 0$ and $0 < a < 1$ such that*

$$\Pr[P_1 = p, R = r \mid \mathcal{P}_N] \le A \cdot a^{p+r}, \quad p, r, N \in \mathbb{N}.$$

Proof. We will prove the claimed bound by showing appropriate bounds for the moment generating function $\mathbb{E}\left[e^{\lambda R} \mid \mathcal{P}_N\right]$. Let us fix any $0 < \lambda < -\ln(\rho)/2$ such that $\rho e^{\lambda} < 1$. Then $\rho^j \mathrm{e}^{\lambda j} < \rho$ for all $j \ge 2$. Recall that $\Pr[C_{j,i} = k] = c_k \rho^{jk}/C(\rho^j), k \in \mathbb{N}, j \ge 2, i \ge 1$, see (5.6). We obtain that

$$\mathbb{E}\left[\mathrm{e}^{\lambda(j(C_{j,i}-m))}\right] = \sum_{s\ge 0} \Pr[C_{j,i} = s+m]\, \mathrm{e}^{\lambda j s} = \mathrm{e}^{-\lambda j m} \frac{C(\rho^j \mathrm{e}^{\lambda j})}{C(\rho^j)}, \quad i \ge 1, j \ge 2.$$

Let $\Omega_{N,p}$ be the set of all $\mathrm{p} = (p_2, \ldots, p_N) \in \mathbb{N}_0^{N-1}$ such that $(p, p_2, \ldots, p_N) \in \Omega_N$, i.e. $p = N - \sum_{2\le j\le N} j p_j$, and let $\mathcal{E}_{\mathrm{p}}$ be the event

$$\mathcal{E}_{\mathrm{p}} := \{P_1 = p\} \cap \bigcap_{2\le j\le N} \{P_j = p_j\}.$$

Then by Markov's inequality and the independence of the $C_{j,i}$'s and the P_j's, for any $\mathrm{p} \in \Omega_{N,p}$

$$\Pr[P_1 = p, R \ge r \mid \mathcal{E}_{\mathrm{p}}] = \Pr\left[\mathrm{e}^{\lambda R} \ge \mathrm{e}^{\lambda r} \mid \mathcal{E}_{\mathrm{p}}\right] \le \mathrm{e}^{-\lambda r} \mathbb{E}\left[\mathrm{e}^{\lambda R} \mid \mathcal{E}_{\mathrm{p}}\right] = \mathrm{e}^{-\lambda r} \prod_{j=2}^{N} \left(\frac{C(\rho^j \mathrm{e}^{\lambda j})}{C(\rho^j)\mathrm{e}^{\lambda j m}} \right)^{p_j}.$$

Abbreviate $\tau_j := C((\rho \mathrm{e}^{\lambda})^j)/(\rho \mathrm{e}^{\lambda})^{jm}$ for $j \in \mathbb{N}$. By Lemma 5.8 there exists $A_1 > 0$ such that

$$\begin{aligned} \Pr[P_1 = p, R \ge r \mid \mathcal{P}_N] &= \sum_{\mathrm{p}\in\Omega_{N,p}} \Pr[R \ge r \mid \mathcal{E}_{\mathrm{p}}] \Pr[\mathcal{E}_{\mathrm{p}} \mid \mathcal{P}_N] \\ &\le A_1 \mathrm{e}^{-\lambda r} N \exp\left\{ - \sum_{2\le j\le N} \frac{C(\rho^j)}{j\rho^{jm}} \right\} \frac{(C(\rho)/\rho^m)^p}{p!} \sum_{\mathrm{p}\in\Omega_{N,p}} \prod_{2\le j\le N} \frac{(\tau_j/j)^{p_j}}{p_j!}. \end{aligned} \tag{5.15}$$

With Lemma 4.4 we find $A_2 > 0$ with

$$\exp\left\{ - \sum_{2\le j\le N} \frac{C(\rho^j)}{j\rho^{jm}} \right\} \le \exp\left\{ -c_m \sum_{2\le j\le N} \frac{1}{j} \right\} \le A_2 N^{-c_m}.$$

Let $H_j \sim \text{Po}(\tau_j/j)$ be independent for $j = 2, \ldots, N$ and set $\tau := \exp(\sum_{2 \le j \le N} \tau_j/j)$. Moreover, abbreviate $B := C(\rho)/\rho^m$. From (5.15) we obtain that there is an $A_3 > 0$ such that

$$\Pr[P_1 = p, R \ge r \mid \mathcal{P}_N] \le A_3 \mathrm{e}^{-\lambda r} N^{1-c_m} \cdot \tau \cdot \frac{B^p}{p!} \sum_{\mathrm{p} \in \Omega_{N,p}} \prod_{j=2}^{N} \Pr[H_j = p_j]. \tag{5.16}$$

Note that

$$\sum_{\mathrm{p} \in \Omega_{N,p}} \prod_{j=2}^{N} \Pr[H_j = p_j] = \Pr\left[\sum_{j=2}^{N} jH_j = N - p\right] = \tau^{-1} \cdot [x^{N-p}] \exp\left\{\sum_{j \ge 2} \frac{\tau_j}{j} x^j\right\}.$$

Observe that in the last expression we actually have to restrict the summation to the interval $2 \le j \le N$; however, $[x^M]\exp(\sum_{j \ge 2} \tau_j x^j/j) = [x^M]\exp(\sum_{2 \le j \le M} \tau_j x^j/j)$ for all $M \in \mathbb{N}$. Then

$$\exp\left\{\sum_{j \ge 2} \frac{\tau_j}{j} x^j\right\} = \exp\left\{c_m \sum_{j \ge 1} \frac{x^j}{j}\right\} \cdot \exp\left\{-c_m x + \sum_{j \ge 2} \frac{x^j}{j}(\tau_j - c_m)\right\} =: G(x) \cdot H(x).$$

By Lemma 4.4 there exists a constant $A_4 > 0$ such that $\tau_j \le c_m(1 + A_4(\rho \mathrm{e}^\lambda)^j)$. With this at hand we deduce that $H(x)$ has radius of convergence (at least) $(\rho \mathrm{e}^\lambda)^{-1}$, which by our choice of λ is > 1. Note that $G(x) = (1-x)^{-c_m}$, which shows together with Lemma 4.2 that G has property (S_2) with radius of convergence 1. As $G(x)$ only has positive coefficients, by Lemmas 4.1 and 4.5 there is an $A_5 > 0$ such that

$$[x^{N-p}]G(x)H(x) \le A_5(N-p)^{c_m-1}, \quad p = 0, \ldots, N-1.$$

All in all,

$$\Pr\left[\sum_{2 \le j \le N} jH_j = N - p\right] \le A_5 \tau^{-1}(N-p)^{c_m-1}, \quad p = 0, \ldots, N-1. \tag{5.17}$$

For the case $p = N$ note that the probability that $\sum_{2 \le j \le N} jH_j = 0$ equals τ^{-1}. Putting the pieces together, we get from (5.16) that there is an $A_6 > 0$ such that

$$\Pr[P_1 = p, R \ge r \mid \mathcal{P}_N] \le A_6 \mathrm{e}^{-\lambda r} N^{1-c_m} \left(\frac{B^N}{N!} + \frac{B^p}{p!}(N-p)^{c_m-1} \cdot \mathbf{1}[p \ne N]\right). \tag{5.18}$$

Observe that $N^{1-c_m} B^N/N! \le e^{-\lambda N}$ for N large enough. Additionally, if $N/2 \le p < N$, then $(1 - p/N)^{c_m-1} \le \max\{2^{1-c_m}, N\}$ so that for N large enough

$$N^{1-c_m} \frac{(e^\lambda B)^p}{p!}(N-p)^{c_m-1} = (1-p/N)^{c_m-1} \frac{(e^\lambda B)^p}{p!} \le 2p \cdot \frac{(e^\lambda B)^p}{p!} \le 1$$

and for $0 \le p \le N/2$

$$N^{1-c_m} \frac{(e^\lambda B)^p}{p!}(N-p)^{c_m-1} \le \max\{2^{1-c_m}, 1\} \cdot e^{e^\lambda B} \Pr\left[\text{Po}\left(e^\lambda B\right) = p\right] \le \max\{2^{1-c_m}, 1\} \cdot e^{e^\lambda B}$$

is also bounded. Plugging these bounds into (5.18) completes the proof. □

We have just proven that P_1, R have (joint) exponential tails when conditioned on $\mathcal{P}_N$. The next lemma is the last essential step towards the proof of Theorem 2.1, where we estimate $\Pr[\mathcal{E}_n \mid \mathcal{P}_N]$. Recall from (5.10) that

$$\Pr[\mathcal{E}_n \mid \mathcal{P}_N] = \Pr[L + R = n - mN \mid \mathcal{P}_N], \quad \text{where} \quad L = \sum_{1 \le i \le P_1} (C_{1,i} - m).$$

Lemma 5.10. *Let $C(x)$ be subexponential. Then*

$$\Pr[\mathcal{E}_n \mid \mathcal{P}_N] \sim c_{n-m(N-1)}\rho^{n-mN}, \quad n, N, n - mN \to \infty.$$

Proof. For the entire proof we abbreviate $\widetilde{N} := n - mN$. Then

$$\Pr[\mathcal{E}_n \mid \mathcal{P}_N] = \sum_{p \ge 0} \sum_{r \ge 0} \Pr\left[L = \widetilde{N} - r \mid \mathcal{P}_N, P_1 = p, R = r\right] \Pr[P_1 = p, R = r \mid \mathcal{P}_N]. \tag{5.19}$$

For brevity, let us write in the remainder

$$\mathcal{D}_{N,p,r} = \mathcal{P}_N \cap \{P_1 = p\} \cap \{R = r\} \quad \text{and} \quad Q_{\widetilde{N}} := \Pr\left[C_{1,1} = \widetilde{N} + m\right] = \frac{c_{n-m(N-1)}\rho^{n-m(N-1)}}{C(\rho)}.$$

We will show that

$$\Pr\left[L = \widetilde{N} - r \mid \mathcal{D}_{N,p,r}\right] \sim p \cdot Q_{\widetilde{N}} \quad \text{for} \quad p, r \in \mathbb{N}_0, \text{ as } \widetilde{N} \to \infty. \tag{5.20}$$

Let $a \in (0, 1)$ be the constant guaranteed to exist from Lemma 5.9, and choose $\delta > 0$ such that $(1 + \delta)a < 1$. We will also show that there are $C > 0, N_0 \in \mathbb{N}$ such that

$$\Pr\left[L = \widetilde{N} - r \mid \mathcal{D}_{N,p,r}\right] \le C(1 + \delta)^{p+r} \cdot Q_{\widetilde{N}} \text{ for all } p, r \in \mathbb{N}_0, \widetilde{N} \ge N_0. \tag{5.21}$$

From the two facts (5.20) and (5.21) the statement in the lemma can be obtained as follows. We will assume throughout that δ is fixed as described above, say for concreteness $\delta = (a^{-1} - 1)/2$, and choose an $0 < \varepsilon < 1$ arbitrarily. Moreover, we will fix $K \in \mathbb{N}$ in dependence of ε only, and we will split the double sum in (5.19) in three (overlapping) parts with (p, r) in the sets

$$B_{\le} = \{(p, r) : 0 \le p, r \le K\}, \quad B_{>,\cdot} = \{(p, r) : p > K, r \in \mathbb{N}_0\}, \quad B_{\cdot,>} = \{(p, r) : p \in \mathbb{N}_0, r > K\}.$$

We will show that the main contribution to $\Pr[\mathcal{E}_n \mid \mathcal{P}_N]$ stems from $B_{\le}$, while the other two parts contribute rather insignificantly. Let us begin with treating the latter parts. Observe that using Lemma 5.9 and (5.21) we obtain that there is a constant $C' > 0$ such that for all $r \in \mathbb{N}_0$ and $K \ge K_0(\varepsilon)$

$$\begin{aligned}\sum_{p \ge K} \Pr\left[L = \widetilde{N} - r \mid \mathcal{D}_{N,p,r}\right] \Pr[P_1 = p, R = r \mid \mathcal{P}_N] &\le C' \sum_{p \ge K} (1 + \delta)^{p+r} \cdot a^{p+r} \cdot Q_{\widetilde{N}} \\ &\le \varepsilon \cdot ((1 + \delta)a)^r \cdot Q_{\widetilde{N}}.\end{aligned}$$

Since $(1 + \delta)a < 1$, summing this over all r readily yields for $c = (1 - (1 + \delta)a)^{-1}$ that

$$\sum_{(p,r) \in B_{>,\cdot}} \Pr\left[L = \widetilde{N} - r \mid \mathcal{D}_{N,p,r}\right] \Pr[P_1 = p, R = r \mid \mathcal{P}_N] \le c\varepsilon \cdot Q_{\widetilde{N}}. \tag{5.22}$$

Completely analogously with the roles of p, r interchanged we obtain that also

$$\sum_{(p,r)\in B_{\cdot,>}} \Pr\left[L = \widetilde{N} - r \mid \mathcal{D}_{N,p,r}\right] \Pr\left[P_1 = p, R = r \mid \mathcal{P}_N\right] \le c\varepsilon \cdot Q_{\widetilde{N}}. \tag{5.23}$$

It remains to handle the part of the sum in (5.19) with $p, r \in B_{\le}$. Using (5.20) we infer that

$$\sum_{(p,r)\in B_{\le}} \Pr\left[L = \widetilde{N} - r \mid \mathcal{D}_{N,p,r}\right] \Pr\left[P_1 = p, R = r \mid \mathcal{P}_N\right] \sim \sum_{(p,r)\in B_{\le}} p\Pr\left[P_1 = p, R = r \mid \mathcal{P}_N\right] \cdot Q_{\widetilde{N}}.$$

Using Lemma 5.9 once again note that we can choose K large enough such that

$$\sum_{0\le p\le K} \sum_{r\ge K} p\Pr\left[P_1 = p, R = r \mid \mathcal{P}_N\right] \le A \sum_{0\le p\le K} \sum_{r\ge K} pa^{p+r} \le \varepsilon$$

and that

$$\left|\sum_{p\ge 0} p\Pr\left[P_1 = p \mid \mathcal{P}_N\right] - \sum_{0\le p\le K} p\Pr\left[P_1 = p \mid \mathcal{P}_N\right]\right| = \left|\sum_{p>K} p\Pr\left[P_1 = p \mid \mathcal{P}_N\right]\right| \le \varepsilon.$$

Altogether this establishes that

$$\left|\sum_{(p,r)\in B_{\le}} \Pr\left[L = \widetilde{N} - r \mid \mathcal{D}_{N,p,r}\right] \Pr\left[P_1 = p, R = r \mid \mathcal{P}_N\right] - \mathbb{E}\left[P_1 \mid \mathcal{P}_N\right] Q_{\widetilde{N}}\right| \le 2\varepsilon Q_{\widetilde{N}}.$$

Corollary 5.7 asserts that $\mathbb{E}\left[P_1 \mid \mathcal{P}_N\right] \to C(\rho)\rho^{-m}$. Since $\varepsilon > 0$ was arbitrary, combining this with (5.22) and (5.23) we obtain from (5.19) that $\Pr\left[\mathcal{E}_n \mid \mathcal{P}_N\right] \sim C(\rho)\rho^{-m} \cdot Q_{\widetilde{N}}$, which is the claim of the lemma.

In order to complete the proof it remains to show the two claims (5.20) and (5.21). We begin with (5.20). Note that for $p, r \in \mathbb{N}_0$

$$\Pr\left[L = \widetilde{N} - r \mid \mathcal{P}_N, P_1 = p, R = r\right] = \Pr\left[\sum_{1\le i\le p} C_{1,i} = \widetilde{N} - r + pm\right]. \tag{5.24}$$

Recall that $\Pr[C_{1,1} = k] = c_k\rho^k/C(\rho)$, where ρ is the radius of convergence of C. Since C is subexponential, $c_{k-1} \sim \rho c_k$ and thus the distribution of the $C_{1,i}$'s is also subexponential with $\Pr[C_{1,1} = k-1] \sim \Pr[C_{1,1} = k]$. We obtain with Lemma 5.2(i) that the probability (5.24) is $\sim p\Pr[C_{1,1} = \widetilde{N} - r + pm]$, as $\widetilde{N} \to \infty$. Moreover, as $\widetilde{N} \to \infty$, $\Pr[C_{1,1} = \widetilde{N} - r + pm] \sim Q_{\widetilde{N}}$, and (5.20) is established.

We finally show (5.21). Our starting point is again (5.24). Note that with Lemma 5.2(ii) there are $C > 0$ and $N_0 \in \mathbb{N}$ such that the sought probability is at most $C(1+\delta)^p\Pr[C_{1,1} = \widetilde{N} - r + pm]$ for all $\widetilde{N} - r + pm \ge N_0$. Moreover, as we have argued in the previous paragraph, the distribution of $C_{1,1}$ is subexponential with $\Pr[C_{1,1} = k-1] \sim \Pr[C_{1,1} = k]$; we thus may choose C and N_0 large enough such that in addition $\Pr[C_{1,1} = \widetilde{N} - r + pm] \le C(1+\delta)^r Q_{\widetilde{N}}$. This establishes (5.21) if $\widetilde{N} - r + pm \ge N_0$. To treat the remaining cases, note that in this situation we have $r > \widetilde{N} - N_0$. Since $(c_k)_{k\in\mathbb{N}}$ is subexponential we obtain that for any $\varepsilon > 0$ and $\tilde{N}$ sufficiently large

$$Q_{\tilde{N}} = \frac{c_{\tilde{N}+m}\rho^{\tilde{N}+m}}{C(\rho)} \ge \frac{(1-\varepsilon)^m}{C(\rho)}(1-\varepsilon)^{\tilde{N}}.$$

Choosing ε such that $(1+\delta)(1-\varepsilon) > 1$ we obtain that $C(1+\delta)^r Q_{\widetilde{N}} > 1$ for sufficiently large $\widetilde{N}$; thus (5.21) is trivially true in this case. □

With all these facts at hand the proof of Theorem 2.1 is straightforward. With Lemma 5.5 and 5.6 we obtain as $n, N, n - mN \to \infty$,

$$[x^n y^N]G(x,y) = G(\rho)\rho^{-n}\Pr[\mathcal{E}_n \mid \mathcal{P}_N]\Pr[\mathcal{P}_N]$$
$$\sim \frac{1}{\Gamma(c_m)} \exp\left\{\sum_{j\geq 1} \frac{C(\rho^j) - c_m\rho^{jm}}{j}\rho^{-jm}\right\} N^{c_m - 1} c_{n-m(N-1)}.$$

5.2.4 The Largest Component

This section is devoted to the proof of Theorem 2.3. Let us begin with (re-)collecting all basic definitions that will be needed in the proof. Suppose that $C(x)$ is subexponential with radius of convergence $0 < \rho < 1$ and set $m := \min\{k \in \mathbb{N} : c_k > 0\}$, see also (5.2). Moreover, let $P_j \sim \text{Po}\left(C(\rho^j)/j\right), j \in \mathbb{N}$ and $C_{j,1}, \ldots, C_{j,P_j}, j \in \mathbb{N}$ have the distribution specified in (5.6), that is, $\Pr[C_{j,i} = k] = c_k\rho^{jk}/C(\rho^j), k, i, j \in \mathbb{N}$. We assume that all these random variables are independent. Let $\mathcal{P}_N, \mathcal{E}_n$ be as in (5.4), that is, with

$$P = \sum_{j\geq 1} jP_j, \quad L = \sum_{1\leq i\leq P_1} (C_{j,i} - m), \quad R = \sum_{j\geq 2} j \sum_{1\leq i\leq P_j} (C_{j,i} - m)$$

we have that $\mathcal{P}_N = \{P = N\}$ and $\mathcal{E}_n = \{L + R = n - mP\}$.

With this notation at hand, let $\mathsf{G}_{n,N}$ be a uniformly drawn random object from $\mathcal{G}_{n,N}$, meaning that the number of atoms is n and the number of components N. According to Lemma 5.4 and using that the Boltzmann model induces the uniform distribution on objects of the same size, we infer that

$$\Pr[\mathsf{G}_{n,N} = G] = \frac{1}{|\mathcal{G}_{n,N}|} = \frac{\rho^n/C(\rho)}{|\mathcal{G}_{n,N}|\rho^n/C(\rho)} = \frac{\Pr[\Lambda G = G]}{\Pr[\mathcal{P}_N, \mathcal{E}_n]} = \Pr[\Lambda G = G \mid \mathcal{P}_N, \mathcal{E}_n], \quad G \in \mathcal{G}_{n,N},$$

that is, studying the distribution of $\mathsf{G}_{n,N}$ boils down to considering the distribution of ΛG conditional on both $\mathcal{P}_N, \mathcal{E}_n$. This is the starting point of our investigations. In particular, $\mathsf{G}_{n,N}$ has N components with sizes given by the vector $(C_{j,i} : 1 \leq j \leq N, 1 \leq i \leq P_j)$. Our aim is here to study the properties of that vector in the conditional space given by $\mathcal{P}_N, \mathcal{E}_n$. To this end, set

$$M^* := \max_{j\geq 1, 1\leq i\leq P_j} C_{j,i} \quad \text{and} \quad C_p^* := \max\{C_{1,1}, \ldots, C_{1,p}\} \quad \text{for } p \in \mathbb{N}. \tag{5.25}$$

Then the statement of the theorem is that, conditional on $\mathcal{P}_N, \mathcal{E}_n$, we have that $M^* = n - mN + \mathcal{O}_p(1)$; since the total number of atoms is n, the number of components is N, and the smallest component contains m atoms, this immediately implies that there are $N + \mathcal{O}_p(1)$ components with exactly m atoms, and all remaining components have a total size of $\mathcal{O}_p(1)$ as well.

The general proof strategy in the remaining section is as follows. We first show in Lemma 5.11 that both P_1, R are "small" in the conditioned space; this makes sure that only a bounded number of entries in the vector $(C_{j,i})_{j\geq 2, 1\leq i\leq P_j}$ are larger than m, and that this total excess is bounded. Hence, the remaining number of $n - (N - P_1)m + \mathcal{O}_p(1)$ atoms is to be found in the components with sizes in $(C_{1,i})_{1\leq i\leq P_1}$. In Lemma 5.11 we exclude that P_1 grows too large conditioned on $\mathcal{E}_n, \mathcal{P}_N$; indeed, we show that it is stochastically bounded. Then the property of subexponentiality guarantees that only the maximum of the $C_{1,i}$'s dominates the entire sum, cf. Lemma 5.2(iii), and Theorem 2.3 follows.

Let us now fill this overview with details. Recall Lemma 5.9, which says that P_1, R have (joint) exponential tails given $\mathcal{P}_N$. We show that conditioning in addition to $\mathcal{E}_n$ does not change the behaviour qualitatively. The proof can be found at the end of the section.

Lemma 5.11. *There exist $A > 0$ and $0 < a < 1$ such that for all sufficiently large $n - mN$*

$$\Pr\left[P_1 = p, R = r \mid \mathcal{E}_n, \mathcal{P}_N\right] \le A \cdot a^{p+r}, \quad p, r \in \mathbb{N}.$$

With this lemma the proof of the theorem can be completed as follows. Let $\varepsilon > 0$ be arbitrary. Abbreviate $\widetilde{N} = n - mN$. With M^* as in (5.25) we will show that there is $K \in \mathbb{N}$ such that

$$\Pr\left[|M^* - \widetilde{N}| \ge K \mid \mathcal{E}_n, \mathcal{P}_N\right] < \varepsilon$$

for $n, N, \widetilde{N}$ sufficiently large, which is the statement of the theorem. According to Lemma 5.11 there exist constants $C_R, C_P \in \mathbb{N}$ such that

$$\Pr\left[\{R \ge C_R\} \cup \{P_1 \ge C_P\} \mid \mathcal{E}_n, \mathcal{P}_N\right] < \varepsilon/2, \quad \widetilde{N} \text{ sufficiently large.}$$

We deduce

$$\Pr\left[|M^* - \widetilde{N}| \ge K \mid \mathcal{E}_n, \mathcal{P}_N\right] \le \frac{\varepsilon}{2} + \sum_{0 \le r \le C_R} \sum_{1 \le p \le C_P} \Pr\left[|M^* - \widetilde{N}| \ge K \mid \mathcal{E}_n, \mathcal{P}_N, R = r, P_1 = p\right]. \quad (5.26)$$

This allows us to view p, r as fixed. Further note that we only need to consider values of p which are larger than 1 as $p = 0$ excludes $R = r \le C_R < \widetilde{N}$. The event "$\mathcal{E}_n, \mathcal{P}_N, R = r, P_1 = p$" implies that $|C_{j,i}| \le m + r$ for all $j \ge 2, 1 \le i \le P_j$, and $S_p := \sum_{1 \le i \le p} C_{1,i} = \widetilde{N} - r + pm$. Recall the definition of C^* from (5.25). Assume that $C_p^* \le m + r$, then we get the contradiction $\widetilde{N} - r + pm = S_p \le p(m + r) < \widetilde{N} - r + pm$ for $\widetilde{N}$ large enough. It follows that $C_p^* > m + r$ and hence $C_p^* = M^*$ in this conditioned space. That yields

$$\Pr\left[|M^* - \widetilde{N}| \ge K \mid \mathcal{E}_n, \mathcal{P}_N, R = r, P_1 = p\right] = \Pr\left[|C_p^* - \widetilde{N}| \ge K \mid S_p = \widetilde{N} - r + pm\right],$$

for $1 \le p \le C_P, 0 \le r \le C_R$. As C_p^* is at most $\widetilde{N} - r + pm$ under this condition, we particularly obtain that $\{C_p^* \ge \widetilde{N} + K\} = \emptyset$ for $K \ge mC_P$ as long as $0 \le p \le C_P$ and $r \ge 0$. Consequently, for $1 \le p \le C_P, 0 \le r \le C_R$,

$$\Pr\left[|C_p^* - \widetilde{N}| \ge K \mid S_p = \widetilde{N} - r + pm\right] = \Pr\left[C_p^* \le \widetilde{N} - K \mid S_p = \widetilde{N} - r + pm\right].$$

Now Lemma 5.2(iii) is applicable as $C_{1,i}$ has subexponential distribution for $1 \le i \le p$ and hence for $1 \le p \le C_P, 0 \le r \le C_R$ we have $(C_p^* \mid S_p = \widetilde{N} - r + pm) = \widetilde{N} - r + pm + \mathcal{O}_p(1)$ as $\widetilde{N} \to \infty$. Consequently, choosing K large enough,

$$\Pr\left[C_p^* \le \widetilde{N} - K \mid S_p = \widetilde{N} - r + pm\right] < \frac{\varepsilon}{2C_R C_P}, \quad 1 \le p \le C_P, 0 \le r \le C_R.$$

We conclude from (5.26)

$$\Pr\left[|M^* - \widetilde{N}| \ge K \mid \mathcal{E}_n, \mathcal{P}_N\right] \le \frac{\varepsilon}{2} + \sum_{0 \le r \le C_R} \sum_{1 \le p \le C_P} \Pr\left[C_p^* \le \widetilde{N} - K \mid S_p = \widetilde{N} - r + pm\right] < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we have just proven that the largest component satisfies $(M^* \mid \mathcal{E}_n, \mathcal{P}_N) = \widetilde{N} + \mathcal{O}_p(1)$, and the proof is completed.

Proof of Lemma 5.11. First note according to Lemma 5.5 we obtain

$$\Pr\left[\mathcal{E}_n, \mathcal{P}_N\right] = [x^n y^N] G(x,y) \rho^n G(\rho)^{-1}.$$

For n and $n - mN$ sufficiently large this expression is strictly greater than zero because of $[x^n y^N]G(x,y) \geq c_m^{N-1} c_{n-m(N-1)}$. Hence $\Pr\left[\mathcal{E}_n \mid \mathcal{P}_N\right] > 0$. We start with the observation

$$\Pr\left[P_1 = p, R = r \mid \mathcal{E}_n, \mathcal{P}_N\right] = \Pr\left[\mathcal{E}_n \mid P_1 = p, R = r, \mathcal{P}_N\right] \Pr\left[P_1 = p, R = r \mid \mathcal{P}_N\right] \Pr\left[\mathcal{E}_n \mid \mathcal{P}_N\right]^{-1}. \tag{5.27}$$

Set $\widetilde{N} := n - mN$ and $L_p := \sum_{1 \leq i \leq p} (C_{1,i} - m)$ for $p \in \mathbb{N}_0$ as well as $Q_{\widetilde{N}} = \Pr[C_{1,1} - m = \widetilde{N}]$. Let $0 < a < 1$ be the constant from Lemma 5.9 and let $\delta > 0$ be such that $(1+\delta)a < 1$. From (5.21) we obtain that there exists $A_1 > 0$ such that for sufficiently large $\widetilde{N}$

$$\Pr\left[\mathcal{E}_n \mid P_1 = p, R = r, \mathcal{P}_N\right] = \Pr\left[L = \widetilde{N} - r \mid \mathcal{D}_{N,p,r}\right] \leq A_1 (1+\delta)^{p+r} Q_{\widetilde{N}}, \quad p, r, n, N \in \mathbb{N}.$$

Lemma 5.9 tells us that we find $A_2 > 0$ with

$$\Pr\left[P_1 = p, R = r \mid \mathcal{P}_N\right] \leq A_2 a^{p+r}, \quad p, r, N \in \mathbb{N}.$$

Finally, according to Lemma 5.10 there is a $A_3 > 0$ such that for all sufficiently large $\widetilde{N}$

$$\Pr\left[\mathcal{E}_n \mid \mathcal{P}_N\right] \geq A_3 Q_{\widetilde{N}}, \quad n, N \in \mathbb{N},$$

and the claim follows with a replaced by $(1+\delta)a < 1$ by plugging everything into (5.27). □

5.2.5 The Remainder

In this section we prove Theorem 2.4. We begin with a simple definition. We define the family of multiplicity counting functions $(d_C(\cdot))_{C \in \mathcal{C}}$, where $d_C(G)$ is the multiplicity of $C \in \mathcal{C}$ in $G \in \mathcal{G}$. Note that for any G we have that $d_C(G) = 0$ for all but finitely many $C \in \mathcal{C}$. Assume that $N(n) \equiv N$ is such that $N(n), n - mN(n) \to \infty$ as $n \to \infty$. Let us write $R_{n,N}$ for the object obtained after removing all objects of size m and a largest component from $\mathsf{G}_{n,N}$. The statement of the theorem is equivalent to showing that for any fixed $G \in \mathcal{G}_{>m}$

$$\Pr\left[R_{n,N} = G\right] \to G_{>m}(\rho)^{-1} \rho^{|G| - m\kappa(G)}, \quad \text{as } n \to \infty,$$

see also (2.11). We immediately obtain that

$$\Pr\left[R_{n,N} = G\right] = \Pr\left[\forall C \in \mathcal{C}_{>m} : d_C(R_{n,N}) = d_C(G)\right].$$

In the remainder we write $d_C = d_C(G)$ for short. Let $S > \max\{m, |G|\}$ be some arbitrary integer to be specified later. We infer that

$$\Pr\left[R_{n,N} = G\right] \leq \Pr\left[\forall C \in \mathcal{C}_{m+1,S} : d_C(R_{n,N}) = d_C\right].$$

To obtain a lower bound, since $S > |G|$, we observe that $\{\forall C \in \mathcal{C}_{>m} : d_C(R_{n,N}) = d_C\}$ is the same as $\{\forall C \in \mathcal{C}_{m+1,S} : d_C(R_{n,N}) = d_C\} \cap \{\forall C \in \mathcal{C}_{>S} : d_C(R_{n,N}) = 0\}$. Moreover, note that $|R_{n,N}| \leq S$ implies $d_C(R_{n,N}) = 0$ for all $C \in \mathcal{C}_{>S}$. Thus

$$\begin{aligned}\Pr\left[R_{n,N} = G\right] &\geq \Pr\left[\forall C \in \mathcal{C}_{m+1,S} : d_C(R_{n,N}) = d_C, |R_{n,N}| \leq S\right] \\ &\geq \Pr\left[\forall C \in \mathcal{C}_{m+1,S} : d_C(R_{n,N}) = d_C\right] - \Pr\left[|R_{n,N}| > S\right].\end{aligned}$$

Let $\varepsilon > 0$. According to Theorem 2.3 there is $S_1 > \max\{m, |G|\}$ so that $\Pr[|R_{n,N}| > S_1] < \varepsilon$. Hence $\Pr[R_{n,N} = G]$ differs by at most ε from $\Pr[\forall C \in \mathcal{C}_{m+1,S} : d_C(R_{n,N}) = d_C]$ for all $S > S_1$. Let us write $L_{n,N}$ for the size of a largest component in $\mathsf{G}_{n,N}$. Theorem 2.3 guarantees that $L_{n,N}$ is unbounded whp, that is $\Pr[L_{n,N} = \mathcal{O}(1)] = o(1)$, and so we obtain for any $S \in \mathbb{N}$

$$\Pr[\forall C \in \mathcal{C}_{m+1,S} : d_C(R_{n,N}) = d_C] = \Pr[\forall C \in \mathcal{C}_{m+1,S} : d_C(R_{n,N}) = d_C, |L_{n,N}| > S] + o(1).$$

However, the event $\{\forall C \in \mathcal{C}_{m+1,S} : d_C(R_{n,N}) = d_C, |L_{n,N}| > S\}$ is equivalent to the event $\{\forall C \in \mathcal{C}_{m+1,S} : d_C(\mathsf{G}_{n,N}) = d_C, |L_{n,N}| > S\}$, since we obtain $R_{n,N}$ by removing all components with size m and a largest component (of size $> S$) from $\mathsf{G}_{n,N}$. Now we add and subtract

$$\Pr[\forall C \in \mathcal{C}_{m+1,s} : d_C(\mathsf{G}_{n,N}) = d_C, |L_{n,N}| \le S] = o(1)$$

in order to get rid of the event $|L_{n,N}| > S$ and arrive at the fact

$$\Pr[\forall C \in \mathcal{C}_{m+1,S} : d_C(R_{n,N}) = d_C] = \Pr[\forall C \in \mathcal{C}_{m+1,S} : d_C(\mathsf{G}_{n,N}) = d_C] + o(1).$$

Combining all previous facts yields that for n sufficiently large

$$\big|\Pr[R_{n,N} = G] - \Pr[\forall C \in \mathcal{C}_{m+1,S} : d_C(\mathsf{G}_{n,N}) = d_C]\big| \le 2\varepsilon \tag{5.28}$$

and thus we are left with estimating $\Pr[\forall C \in \mathcal{C}_{m+1,S} : d_C(\mathsf{G}_{n,N}) = d_C]$. For $\mathbf{v}_S := (v_C)_{C\in\mathcal{C}_{m+1,S}}$ denote by $G(x, y, \mathbf{v}_S)$ the generating series of $\mathcal{G}$ such that x marks the size, y the number of components and $\mathbf{v}_S$ the multiplicities of $(C)_{C\in\mathcal{C}_{m+1,S}}$. In other words, for $\ell, k \in \mathbb{N}_0, \mathbf{t}_S := (t_C)_{C\in\mathcal{C}_{m+1,S}} \in \mathbb{N}_0^{|\mathcal{C}_{m+1,S}|}$

$$g_{\ell,k,\mathbf{t}_S} = [x^\ell y^k \mathbf{v}_S^{\mathbf{t}_S}]G(x, y, \mathbf{v}_S) = |\{G \in \mathcal{G} : |G| = \ell, \kappa(G) = k, \forall C \in \mathcal{C}_{m+1,S} : d_C(G) = t_C\}|.$$

Setting $v_C = 1$ for all $C \in \mathcal{C}_{m+1,S}$ we obtain the generating series $G(x, y)$ counting only size and number of components by x and y respectively. As $\mathsf{G}_{n,N}$ is drawn uniformly at random from $\mathcal{G}_{n,N}$ the proof reduces to determining

$$\Pr[\forall C \in \mathcal{C}_{m+1,S} : d_C(\mathsf{G}_{n,N}) = d_C] = \frac{[x^n y^N \mathbf{v}_s^{\mathbf{d}_S}]G(x, y, \mathbf{v}_S)}{[x^n y^N]G(x, y)}.$$

The following lemma, whose proof is shifted to the end of this section, accomplishes this task.

Lemma 5.12. *Let* $\mathbf{d} = (d_C)_{C\in\mathcal{C}_{m+1,S}}$ *with* $D := \sum_{C\in\mathcal{C}_{m+1,S}} |C| d_C$ *and* $D' := \sum_{C\in\mathcal{C}_{m+1,S}} d_C$. *Then*

$$\frac{[x^n y^N \mathbf{v}_s^{\mathbf{d}_S}]G(x, y, \mathbf{v}_S)}{[x^n y^N]G(x, y)} \to \rho^{D-mD'} \prod_{C\in\mathcal{C}_{m+1,S}} (1 - \rho^{|C|-m}), \quad n \to \infty.$$

Lemma 5.12 yields directly for sufficiently large n

$$\left|\Pr[\forall C \in \mathcal{C}_{m+1,S} : d_C(\mathsf{G}_{n,N}) = d_C] - \rho^{|G|-m\kappa(G)} \prod_{C\in\mathcal{C}_{m+1,S}} (1 - \rho^{|C|-m})\right| < \varepsilon.$$

Now observe that with defining $C_{m+1,S}(x) := \sum_{m<\ell\le S} |\mathcal{C}_\ell| x^\ell$ we obtain

$$\lim_{S\to\infty} \prod_{C\in\mathcal{C}_{m+1,S}} (1 - \rho^{|C|-m}) = \lim_{S\to\infty} \prod_{m<\ell\le S} \exp\left\{|\mathcal{C}_\ell| \ln(1 - \rho^{\ell-m})\right\} = \lim_{S\to\infty} \exp\left\{-\sum_{j\ge1} \frac{C_{m+1,S}(\rho^j)}{j\rho^{jm}}\right\}.$$

By the continuity of $\exp\{\cdot\}$ and monotone convergence this equals $G_{>m}(\rho)^{-1}$. Choose $S_2 > \max\{m, |G|\}$ large enough that $\prod_{C\in\mathcal{C}_{m+1,S}}(1-\rho^{|C|-m})$ differs at most by ε from $G_{>m}(\rho)^{-1}$ for all $S > S_2$. Summarising, fixing $S \geq \max\{S_1, S_2\}$ we obtain for sufficiently large n

$$\left|\Pr\left[\forall C \in \mathcal{C}_{m+1,S} : d_C(\mathsf{G}_{n,N}) = d_C\right] - \rho^{|G|-m\kappa(G)} G_{>m}(\rho)^{-1}\right| \leq 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary the proof of the theorem is finished with (5.28).

Proof of Lemma 5.12. First we determine $G(x, y, \mathbf{v}_S)$ explicitly. Define the multivariate generating series

$$C(x, y, \mathbf{v}_S) = y\left(C(x) + \sum_{C\in\mathcal{C}_{m+1,S}} (v_C - 1)x^{|C|}\right),$$

where as usual x marks the size, y the number of components (which by convention is always 1 for $C \in \mathcal{C}$) and $\mathbf{v}_S$ objects in $\mathcal{C}_{m+1,S}$. Note that these parameters are clearly additive when forming multisets. Hence, according to [30, Theorem III.1] the formula (2.1) extends to the multivariate version

$$G(x, y, \mathbf{v}_S) = \exp\left\{\sum_{j\geq 1} \frac{C(x^j, y^j, \mathbf{v}_S^j)}{j}\right\}, \tag{5.29}$$

where $\mathbf{v}_S^j = (v_C^j)_{C\in\mathcal{C}_{m+1,S}}$. Setting $v_C = 1$ for all $C \in \mathcal{C}_{m+1,S}$ we see that $G(x, y, \mathbf{1}) \equiv G(x, y)$ such that $[x^n y^N]G(x, y) = |\mathcal{G}_{n,N}|$. By elementary algebraic manipulations we reformulate (5.29) to

$$\begin{aligned} G(x, y, \mathbf{v}_S) &= G(x, y) \exp\left\{\sum_{C\in\mathcal{C}_{m+1,S}} \left(\sum_{j\geq 1} \frac{(x^{|C|} y v_C)^j}{j} - \sum_{j\geq 1} \frac{(x^{|C|} y)^j}{j}\right)\right\} \\ &= G(x, y) \prod_{C\in\mathcal{C}_{m+1,S}} \frac{1 - x^{|C|} y}{1 - x^{|C|} y v_C}. \end{aligned} \tag{5.30}$$

Let us now turn to the initial claim in Lemma 5.12. We obtain that

$$\begin{aligned} [x^n y^N \mathbf{v}_S^{\mathbf{d}_S}]G(x, y, \mathbf{v}_S) &= [x^n y^N]G(x, y) \prod_{C\in\mathcal{C}_{m+1,S}} [v_C^{d_C}] \frac{1 - x^{|C|} y}{1 - x^{|C|} v_C y} \\ &= [x^{n-D} y^{N-D'}]G(x, y) \prod_{C\in\mathcal{C}_{m+1,S}} (1 - x^{|C|} y). \end{aligned}$$

Since $\mathcal{C}_{m+1,S}$ does only have finitely many elements, there exist $L, K \in \mathbb{N}$ such that $[x^\ell y^k] \prod_{C\in\mathcal{C}_{m+1,S}}(1 - x^{|C|} y) = 0$ for all $\ell \geq L, k \geq K$. Recall that, using Theorem 2.1,

$$[x^n y^N]G(x, y) \sim \exp\left\{\sum_{j\geq 1} \frac{C(\rho^j) - c_m \rho^{jm}}{j\rho^{jm}}\right\} \frac{N^{c_m - 1}}{\Gamma(c_m)} |\mathcal{C}_{n-m(N-1)}|, \quad n \to \infty,$$

and so $[x^{n-a}y^{N-b}]G(x,y) \sim [x^n y^N]G(x,y)\rho^{a-mb}$ for fixed $a, b \in \mathbb{N}$ as $\mathcal{C}$ is subexponential. Hence, as $n \to \infty$,

$$\begin{aligned}[x^n y^N \mathbf{v}_S^{\mathbf{d}_S}]G(x,y,\mathbf{v}_S) &= \sum_{\ell\in[L],k\in[K]} [x^{n-D-\ell}y^{N-D'-k}]G(x,y)[x^\ell y^k] \prod_{C\in\mathcal{C}_{m+1,S}} (1-x^{|C|}y) \\ &\sim [x^n y^N]G(x,y)\cdot\rho^{D-mD'} \sum_{\ell\in[L],k\in[K]} \rho^{\ell-mk}[x^\ell y^k] \prod_{C\in\mathcal{C}_{m+1,S}} (1-x^{|C|}y) \\ &= [x^n y^N]G(x,y)\cdot\rho^{D-mD'} \prod_{C\in\mathcal{C}_{m+1,S}} (1-\rho^{|C|-m}),\end{aligned}$$

which finishes the proof. □

5.2.6 Proof of Benjamini-Schramm Convergence

Proof of Proposition 2.5. It is a well-known fact that the weak convergence of (G_n, o_n) to $(\mathbb{G}, \mathbb{o})$ in (2.12) is equivalent to showing that for any bounded and continuous function $f : \mathcal{B}_* \to \mathbb{R}$

$$\lim_{n\to\infty} \mathbb{E}\left[f(\mathsf{G}_n, o_n)\right] = \mathbb{E}\left[f(\mathbb{G}, \mathbb{o})\right].$$

For any finite graph G denote by o_G a vertex chosen uniformly at random from its vertex set. Let $\mathcal{M}(\mathsf{G}_{n,N})$ denote a (canonically chosen) largest component of $\mathsf{G}_{n,N}$ and $\mathcal{R}(\mathsf{G}_{n,N})$ the remainder after removing all objects of size m and $\mathcal{M}(\mathsf{G}_{n,N})$. Let $f : \mathcal{B}_* \to \mathbb{R}$ be an arbitrary bounded and continuous function. Then

$$\begin{aligned}\mathbb{E}\left[f(\mathsf{G}_{n,N}, o_n)\right] =& \mathbb{E}\left[f(\mathcal{M}(\mathsf{G}_{n,N}), o_{\mathcal{M}(\mathsf{G}_{n,N})})\right] \Pr\left[o_n \in \mathcal{M}(\mathsf{G}_{n,N})\right] \\ &+\mathbb{E}\left[f(\mathcal{R}(\mathsf{G}_{n,N}), o_{\mathcal{R}(\mathsf{G}_{n,N})})\right] \Pr\left[o_n \in \mathcal{R}(\mathsf{G}_{n,N})\right] \\ &+\mathbb{E}\left[f(\mathsf{C}_m, o_m)\right] \Pr\left[o_n \notin \mathcal{R}(\mathsf{G}_{n,N}) \cup \mathcal{M}(\mathsf{G}_{n,N})\right].\end{aligned}$$

According to Theorem 2.3 we have that $|\mathcal{M}(\mathsf{G}_{n,N})| = n - mN + \mathcal{O}_p(1)$ implying $\Pr\left[o_n \in \mathcal{M}(\mathsf{G}_{n,N})\right] \sim (n-mN)/n \to 1-\lambda$. As the size of $\mathcal{M}(\mathsf{G}_{n,N}) \in \mathcal{C}$ tends to infinity and $(\mathsf{C}_n)_{n\geq 1}$ converges in the BS sense to $(\mathbb{C}, \mathbb{o})$ we have that

$$\mathbb{E}\left[f(\mathcal{M}(\mathsf{G}_{n,N}), o_{\mathcal{M}(\mathsf{G}_{n,N})})\right] \Pr\left[o_n \in \mathcal{M}(\mathsf{G}_{n,N})\right] \to (1-\lambda)\mathbb{E}\left[f(\mathbb{C}, \mathbb{o})\right], \quad n, N \to \infty.$$

Theorem 2.3 entails that $\mathcal{R}(\mathsf{G}_{n,N})$ has a limiting distribution and hence $\Pr\left[o_n \in \mathcal{R}(\mathsf{G}_{n,N})\right] \to 0$. As f is bounded

$$\mathbb{E}\left[f(\mathcal{R}(\mathsf{G}_{n,N}), o_{\mathcal{R}(\mathsf{G}_{n,N})})\right] \Pr\left[o_n \in \mathcal{R}(\mathsf{G}_{n,N})\right] \to 0, \quad n \to \infty.$$

Finally, we obtain by combining Theorems 2.3 and 2.3 that $n - |\mathcal{R}(\mathsf{G}_{n,N}) \cup \mathcal{M}(\mathsf{G}_{n,N})| = mN + \mathcal{O}_p(1)$ and consequently $\Pr\left[o_n \notin \mathcal{R}(\mathsf{G}_{n,N}) \cup \mathcal{M}(\mathsf{G}_{n,N})\right] \sim mN/n \to \lambda$. Thus,

$$\lim_{n,N\to\infty} \mathbb{E}\left[f(\mathsf{G}_{n,N}, o_n)\right] = (1-\lambda)\mathbb{E}\left[f(\mathbb{C}, \mathbb{o})\right] + \lambda\mathbb{E}\left[f(\mathsf{C}_m, o_m)\right].$$

□

6 Expansive (Multi-)sets with Many Components

This section contains the proofs of the main results in Section 2.2 and is based on Sections 2–4 of the contributing Manuscript (II). The only exception is the proof of Theorem 2.9 in Section 6.2.4 which is based on the proof of Theorem 1.6 in Manuscript (III).

Plan of the Section. The preparations for the proofs of the main theorems are contained in Section 6.1, which is a collection of many auxiliary results and technical statements. In particular, properties of x_n, y_n, N_n^* and more general results regarding coefficients of products of certain power series tailored to our needs are derived. Section 6.2 containing the proofs begins with Section 6.2.1 where we introduce the probabilistic reduction via the bivariate Boltzmann model. Subsequently, we present the proof of our main theorems in a layered structure starting with the statement of more general important lemmas in Section 6.2.2, where we also describe how they can be combined such as to arrive at the assertions of Theorems 2.8(I) and 2.8(II). The respective proofs of these lemmas can be then found in Section 6.2.3. Finally, we prove Theorem 2.9 in Section 6.2.4. Note that we make heavy use of the (textbook) results about slowly varying functions which are placed in the self-contained Appendix A.

6.1 Preliminaries

6.1.1 Existence, Uniqueness and Properties of N_n^* and (x_n, y_n)

In this section we prove Lemma 2.7 and find some further properties of x_n, y_n and N_n^*.

Proof of Lemma 2.7(i). Define the power series $S : \mathbb{R}_+ \to \mathbb{R} \cup \{\infty\}$, $z \mapsto \sum_{k\geq 1} z^k$ and consider the system of equations

$$xyC'(x) + mc_m S(x^m y) = n, \qquad (6.1)$$
$$yC(x) + c_m S(x^m y) = N. \qquad (6.2)$$

If we find a pair $(x_{n,N}, y_{n,N}) \in \mathbb{R}_+^2$ satisfying (6.1) and (6.2), then this pair is also a solution to our system (2.14). $S(x_{n,N}^m y_{n,N})$ diverges for $x_{n,N}^m y_{n,N} \geq 1$, so that $x_{n,N}^m y_{n,N} < 1$ and $S(x_{n,N}^m y_{n,N}) = x_{n,N}^m y_{n,N}/(1-x_{n,N}^m y_{n,N})$. Vice versa, any solution to (2.14) satisfies (6.1), (6.2). Thus it suffices to find a pair $(x_{n,N}, y_{n,N}) \in \mathbb{R}_+^2$ satisfying (6.1), (6.2). Subtracting m times (6.2) from (6.1) we obtain

$$y = (n - mN)\left(\frac{xC'(x)}{C(x)} - m\right)^{-1} C(x)^{-1} =: a(x) \cdot C(x)^{-1}. \qquad (6.3)$$

Plugging this into (6.2) and reformulating yields the one-variable equation

$$f(x) := a(x) + c_m S\left(b(x)\right) = N, \quad \text{where} \quad b(x) := x^m \cdot a(x)C(x)^{-1}. \qquad (6.4)$$

Before we solve this equation let us have a closer look at the expression $xC'(x)/C(x)$. Since $(c_k)_{k\geq 1}$ is a non-negative and non-zero sequence and since C has radius of convergence ρ we know that $xC'(x)/C(x)$ is continuous and strictly increasing on the (open) interval $(0, \rho)$. Moreover, as m is the first index such that $c_m > 0$ and as $c_n = h(n)n^{\alpha-1}\rho^{-n}$ for some $\alpha > 0$

$$\lim_{x\to 0} \frac{xC'(x)}{C(x)} = m \quad \text{and} \quad \lim_{x\to \rho} \frac{xC'(x)}{C(x)} = \infty.$$

With these facts at hand we study a and b. The monotonicity properties of $xC'(x)/C(x)$ imply that $a(x)$ is strictly decreasing in $(0, \rho)$ with

$$\lim_{x\to 0} a(x) = \infty \quad \text{and} \quad \lim_{x\to\rho} a(x) = 0.$$

Moreover, $b(x)$ is also strictly decreasing on $(0, \rho)$ as the product of two strictly decreasing positive functions $a(x)$ and $x^m/C(x) = 1/\sum_{k\geq m} c_k x^{k-m}$ and satisfies

$$\lim_{x\to 0} b(x) = \infty \quad \text{and} \quad \lim_{x\to\rho} b(x) = 0.$$

Let $\delta \in (0, \rho)$ be the unique number such that $b(\rho - \delta) = 1$ and so $b(x) \in (0, 1)$ for $x \in (\rho - \delta, \rho)$. We immediately obtain that any solution to (6.4), that is, any x solving $f(x) = N$ must be in $(\rho - \delta, \rho)$ and we look only in this interval for solutions.

Note that if there is any solution, then it must be unique. Indeed, a, b are both strictly decreasing, and $S(b(x))$ too, since S has only non-negative coefficients. Thus, f is strictly decreasing in $(\rho - \delta, \rho)$.

Finally, we argue that there is a solution to $f(x) = N$. From our considerations we obtain that the function $S(b(x))$, defined on $(\rho - \varepsilon, \rho)$, takes any value in $(0, \infty)$. On the other hand, we know that $a(x) \to 0$ as $x \to \rho$. We conclude that $f(x) \to 0$ as $x \to \rho$ and $f(x) \to \infty$ as $x \to \rho - \delta$. This implies the existence of (a unique) $x_{n,N}$ such that $f(x_{n,N}) = N$. With this we determine $y_{n,N}$ from (6.3), and the proof is completed. □

Proof of Lemma 2.7(ii). Let $0 < \varepsilon < 1/(\alpha + 1)$. We first show that there is a solution in the interval (u_-, u_+), where $u_- := v^{1/(\alpha+1)-\varepsilon}$ and $u_+ := v^{1/(\alpha+1)+\varepsilon}$, and that there are no solutions in $[1, v] \setminus (u_-, u_+)$. Set

$$F_v(x) := xh(x)^{1/(\alpha+1)} - v^{1/(\alpha+1)}.$$

Then $F_v(u_-) = v^{1/(\alpha+1)-\varepsilon} h(v^{1/(\alpha+1)-\varepsilon}) - v^{1/(\alpha+1)} < 0$ for v sufficiently large, since h being slowly varying guarantees that $h(t) = t^{o(1)}$ as $t \to \infty$, see also (A.2). Analogously we obtain that $F_v(u_+) > 0$ for large v. Since h is continuous by assumption, also F_v is continuous and there is $u_v \in (u_-, u_+)$ with $F_v(u_v) = 0$ for v large enough.

Now consider $1 \leq u \leq u_-$. Let $\delta > 0$ be such that $(1/(\alpha + 1) - \varepsilon)(1 + \delta) < 1/(\alpha + 1)$. Due to (A.2), for v large enough

$$F_v(u) = uh(u)^{1/(\alpha+1)} - v^{1/(\alpha+1)} \leq u_-^{1+\delta} - v^{1/(\alpha+1)} = v^{(1/(\alpha+1)-\varepsilon)(1+\delta)} - v^{1/(\alpha+1)} < 0.$$

So, there is no solution in $[1, u_-]$. Next consider $u_+ \leq u \leq v$. Then (A.2) guarantees with room to spare that $h(u)^{1/(\alpha+1)} > u^{-\varepsilon/2} \geq v^{-\varepsilon/2}$ for sufficiently large v. Hence

$$F_v(u) = uh(u)^{1/(\alpha+1)} - v^{1/(\alpha+1)} > v^{1/(\alpha+1)+\varepsilon} v^{-\varepsilon/2} - v^{1/(\alpha+1)} > 0.$$

which proves that there is no solution in $[u_+, v]$.

Next, let us show that the solution $u_v \in (u_-, u_+)$ is unique. Assume that $u_- < u_-^* < u_+^* < u_+$ are two distinct solutions. Then due to (A.2) we obtain for $0 < \delta < 1$ and v sufficiently large

$$\frac{u_-^*}{u_+^*} = \frac{h(u_+^*)^{1/(\alpha+1)}}{h(u_-^*)^{1/(\alpha+1)}} \geq \left(\frac{u_-^*}{u_+^*}\right)^{\delta}$$

implying that $u_-^*/u_+^* \geq 1$, a contradiction.

Finally, we show that $g(v) = h(u_v)^{1/(\alpha+1)}$ is slowly varying; since $u_v = g(v)^{-1}v^{1/(\alpha+1)}$ the proof is finished. Let $\lambda > 0$ be arbitrary. Denote by $u_{\lambda v}$ the solution to $u_{\lambda v}h(u_{\lambda v})^{1/(\alpha+1)} = (\lambda v)^{1/(\alpha+1)}$ which, as we have just shown, exists and is unique for λv sufficiently large. Then

$$\left(\frac{g(v)}{g(\lambda v)}\right)^{\alpha+1} = \frac{h(u_v)}{h(u_{\lambda v})} = \frac{h\big(h(u_v)^{-1/(\alpha+1)}v^{1/(\alpha+1)}\big)}{h\big(h(u_{\lambda v})^{-1/(\alpha+1)}(\lambda v)^{1/(\alpha+1)}\big)}.$$

Abbreviate $t = h(u_v)/h(u_{\lambda v})$. By applying (A.2) to the last expression in the previous display we obtain for any $0 < \delta < 1$ and sufficiently large v

$$\min\left\{(\lambda t)^{\delta}, (\lambda t)^{-\delta}\right\} \le t \le \max\left\{(\lambda t)^{\delta}, (\lambda t)^{-\delta}\right\}.$$

Note that $\lambda^{\delta}, \lambda^{-\delta}$ get arbitrarily close to 1 if we let $\delta \to 0$. So, since $\delta < 1$ we have proven that g is slowly varying, i.e., $g(v)/g(\lambda v) \to 1$ as $n \to \infty$. □

In the rest of the section we retrieve useful asymptotic properties of $(x_n, y_n) \equiv (x_{n,N_n}, y_{n,N_n})$, where it is instructive to write $x_n = \rho e^{-\chi_n}$.

Lemma 6.1. *Let $N_n = \lambda_n N_n^*$, $n \in \mathbb{N}$ be a sequence as in (2.17) and set $\chi_n = \ln(\rho/x_n)$. Let g be the slowly varying function from Lemma 2.7(ii). Then, as $n \to \infty$,*

$$\chi_n \sim g\left(\frac{n - mN_n}{y_n}\right) \cdot \left(\frac{n - mN_n}{\Gamma(\alpha+1)y_n}\right)^{-1/(\alpha+1)} \sim o(1), \tag{6.5}$$

and further, for $k \in \mathbb{N}_0$ as $n \to \infty$,

$$x_n^k C^{(k)}(x_n) \sim \Gamma(\alpha+k)\cdot h(\chi_n^{-1})\cdot \chi_n^{-\alpha-k} \sim \Gamma(\alpha+k)\cdot g\left(\frac{n - mN_n}{y_n}\right)^{1-k} \cdot \left(\frac{n - mN_n}{\Gamma(\alpha+1)y_n}\right)^{\frac{\alpha+k}{\alpha+1}} \tag{6.6}$$

and moreover $x_n y_n C'(x_n) \sim n - mN_n$.

Proof. First we show that y_n is bounded from above. For the sake of contradiction, assume that there is an increasing $\mathbb{N}$-valued sequence $(n_\ell)_{\ell\in\mathbb{N}}$ such that $y_{n_\ell} \to \infty$. Since $x_{n_\ell}^m y_{n_\ell} < 1$ due to (2.14) we have $x_{n_\ell} \to 0$ as $\ell \to \infty$. As $C(x) \sim c_m x^m$ and $C'(x) \sim mc_n x^{m-1}$ as $x \to 0$ we get as $\ell \to \infty$

$$y_{n_\ell}C(x_{n_\ell}) = \mathcal{O}(1) \quad \text{and} \quad x_{n_\ell}y_{n_\ell}C'(x_{n_\ell}) = \mathcal{O}(1).$$

But $x_{n_\ell}y_{n_\ell}C'(x_{n_\ell}) - my_{n_\ell}C(x_{n_\ell}) = n_\ell - mN_{n_\ell} \to \infty$, contradicting (2.14).

Next we show that $\chi_n \to 0$. For the sake of contradiction, assume that there is a strictly increasing $\mathbb{N}$-valued sequence $(n_\ell)_{\ell\in\mathbb{N}}$ and an $\varepsilon > 0$ such that $\chi_{n_\ell} := \ln(\rho/x_{n_\ell}) \ge \varepsilon$ for all $\ell \in \mathbb{N}$. Since both C, C' have radius of convergence ρ we infer that there is a $\Delta > 0$ such that $C(x_{n_\ell}), x_{n_\ell}C'(x_{n_\ell}) \le \Delta$ for all $\ell \in \mathbb{N}$. From (2.14) and the fact that y_n is bounded we then obtain that, as $\ell \to \infty$,

$$c_m S_{n_\ell} := c_m x_{n_\ell}^m y_{n_\ell}/(1 - x_{n_\ell}^m y_{n_\ell}) = N_{n_\ell} + \mathcal{O}(1) \quad \text{and} \quad mc_m S_{n_\ell} = n_\ell + \mathcal{O}(1).$$

Since $n_\ell - mN_{n_\ell} \to \infty$ this is a contradiction, and we have established that $\chi_n \to 0$.

The proof can now be finished rather routinely. By applying a well-known result for slowly varying functions, see [13, Thm. 1.7.1] and also Theorem A.5, we immediately obtain

$$x_n^k C^{(k)}(x_n) \sim \Gamma(\alpha+k)h(\chi_n^{-1})\chi_n^{-\alpha-k}, \quad k \in \mathbb{N}_0, n \to \infty. \tag{6.7}$$

From this we readily obtain that $x_nC'(x_n) = \omega(C(x_n))$, and then, since $x_ny_nC'(x_n) - my_nC(x_n) = n - mN_n$ according to (2.14) we infer that $x_ny_nC'(x_n) \sim n - mN_n$. Plugging (6.7) into $x_ny_nC'(x_n) \sim n - mN_n$ and rearranging the terms yields

$$\chi_n^{-1}h(\chi_n^{-1})^{1/(\alpha+1)} = \left(\frac{n - mN_n}{\Gamma(\alpha+1)y_n}\right)^{1/(\alpha+1)}(1 + o(1)). \tag{6.8}$$

As $y_n > 0$ is bounded from above it follows that $t := (n - mN_n)/(\Gamma(\alpha+1)y_n) \cdot (1 + o(1)) \to \infty$. Consequently, Lemma 2.7(ii) asserts that there is a unique solution $\chi_n^{-1} = t^{1/(\alpha+1)}/g(t)$. As g is slowly varying we obtain (6.5). Further, plugging (6.5) into (6.7) yields the last remaining statement (6.6). □

6.1.2 Probabilistic Estimates

The following well-known statement gives estimates for Poisson distributed random variables. A proof can be easily conducted by using Stirling's formula.

Proposition 6.2. *Let $\lambda > 0$ and let X be $\mathrm{Po}(\lambda)$-distributed. Then there is an $a > 0$ such that*

$$\Pr\left[|X - \lambda| \geq x\sqrt{\lambda}\right] \leq \mathrm{e}^{-ax\min\{x,\sqrt{\lambda}\}}, \quad x \geq 0. \tag{6.9}$$

Further, as $\lambda \to \infty$

$$\Pr\left[X = \lfloor \lambda + x\sqrt{\lambda} \rfloor\right] \sim (2\pi\lambda)^{-1/2}\mathrm{e}^{-x^2/2}, \qquad x = o(\lambda^{1/6}). \tag{6.10}$$

6.1.3 Estimates of (Power) Series

In the proofs of our main results we will often find ourselves in the situation where we have to retrieve coefficients of the product of two power series A and R. In the basic setting encountered in Lemma 4.1 the coefficient of the product is proportional to the coefficient of the series with the smaller radius of convergence. In our forthcoming arguments the involved series A, R will also depend on n; there we will use the following (rather technical) statement, which we tried to simplify as much as possible; it seems that the conditions cannot be weakened to obtain the desired conclusion that mimics Lemma 4.1.

Lemma 6.3. *Let $\big(A_n(x)\big)_{n\in\mathbb{N}}$ be a sequence of power series and $(\rho_n)_{n\in\mathbb{N}}$ a real-valued positive sequence with $\overline{\rho} = \limsup_{n\to\infty}\rho_n \in \mathbb{R}$ such that*

$$\frac{[x^{n-k}]A_n(x)}{[x^n]A_n(x)} \sim \rho_n^k \quad \text{for } k \in \mathbb{N}_0 \text{ and as } n \to \infty. \tag{6.11}$$

Moreover, assume that there exist $\varepsilon > 0$ and $n_0, k_0 \in \mathbb{N}$ such that

$$\left|\frac{[x^{n-k}]A_n(x)}{[x^n]A_n(x)}\right| \leq (1+\varepsilon)^k\rho_n^k \quad \text{for } k_0 \leq k \leq n \text{ and } n \geq n_0. \tag{6.12}$$

Let $(R_n(x))_{n\in\mathbb{N}}$ be a sequence of power series with radii of convergence at least $a := (1+\varepsilon)\overline{\rho}$. Moreover, suppose that there is a sequence $(d_k)_{k\in\mathbb{N}}$ such that $|[x^k]R_n(x)| \leq d_k$ for all $k, n \in \mathbb{N}$ and $\sum_{k\geq1} d_ka^k < \infty$. In addition, let $Q : [0, a] \to \mathbb{R}$ be such that R_n converges uniformly to Q on $[0, a]$. If $\inf_{n\geq n_0} Q(\rho_n) > 0$, then

$$[x^n]A_n(x)R_n(x) \sim Q(\rho_n) \cdot [x^n]A_n(x) \quad \text{as } n \to \infty.$$

Proof. We abbreviate $a_{k,n} := [x^k]A_n(x)$ and $r_{k,n} := [x^k]R_n(x)$ for $k, n \in \mathbb{N}$. Write for $K \in \mathbb{N}$

$$\frac{1}{a_{n,n}}[x^n]A_n(x)R_n(x) = I(0,K) + I(K,n), \quad \text{where} \quad I(i,j) := \sum_{i \le k < j} \frac{a_{n-k,n}}{a_{n,n}} r_{k,n}.$$

Let $\varepsilon' > 0$. Let K_0 be such that for all $K \geq K_0$

$$\Big| \sum_{0 \le k < K} r_{k,n}\rho_n^k - Q(\rho_n) \Big| \le |R_n(\rho_n) - Q(\rho_n)| + \Big| \sum_{k > K} r_{k,n}\rho_n^k \Big| < \varepsilon';$$

such a K_0 exists since $R_n \to Q$ uniformly on $[0, a]$, $\rho_n \le a$ for all n, and since $|\sum_{k>K} r_{k,n}\rho_n^k| \le \sum_{k>K} d_k a^k$ gets arbitrarily small by the assumption that $\sum_{k \ge 1} d_k a^k$ converges.

Note further that for any $K \ge K_0$ the property (6.11) entails for sufficiently large n

$$\Big| I(0,K) - \sum_{0 \le k < K} r_{k,n}\rho_n^k \Big| < \varepsilon'.$$

The triangle inequality readily implies that $|I(0,K) - Q(\rho_n)| < 2\varepsilon'$ for all $K \ge K_0$ and n sufficiently large. Further, (6.12) guarantees that there is $\varepsilon > 0$ such that $0 < a_{n-k,n}/a_{n,n} \le (1+\varepsilon)^k \rho_n^k$ for all k, n sufficiently large. From that we conclude that there is a $K_1 \in \mathbb{N}$ such that for all $K \ge K_1$

$$|I(K,n)| \le \sum_{k \ge K} |r_{k,n}| (1+\varepsilon)^k \rho_n^k \le \sum_{k \ge K} d_k (1+\varepsilon)^k \overline{\rho}^k < \varepsilon'.$$

All together, fixing $K \ge \max\{K_0, K_1\}$, we proved that there is an error term $|E| < 3\varepsilon'$ such that $a_{n,n}^{-1} \cdot [x^n]A_n(x)R_n(x) = Q(\rho_n) + E$ for n sufficiently large. Since $\varepsilon' > 0$ was arbitrary and $Q(\rho_n)$ is bounded away from zero, the claim follows. □

The next statement applies Lemma 6.3 to the special case $A_n(x) = \exp\{h_n x\}$ and $Q(x) = R_n(x) = (1-x)^{-\gamma}$, where $\gamma > 0$ and h_n is approaching infinity. In particular, we observe what the effect of Q is on $[x^k]A_n(x) = h_n^k/k!$.

Lemma 6.4. *Let $\gamma > 0$ and for a (eventually) positive sequence $(\alpha_n)_{n \in \mathbb{N}}$ define $h_n := \alpha_n n$.*

(i) For $k \in \mathbb{N}$ and α_n such that $h_n \to \infty$

$$[x^k]\frac{1}{(1-x)^\gamma} e^{h_n x} \sim [x^k] e^{h_n x} \sim \frac{h_n^k}{k!} \qquad n \to \infty. \tag{6.13}$$

(ii) If $\liminf_{n\to\infty} \alpha_n > 1$

$$[x^n]\frac{1}{(1-x)^\gamma} e^{h_n x} \sim \left(\frac{1}{1-\alpha_n^{-1}}\right)^\gamma \cdot \frac{h_n^n}{n!}, \qquad n \to \infty. \tag{6.14}$$

(iii) If $\limsup_{n\to\infty} \alpha_n < 1$ and $h_n \to \infty$ then

$$[x^n]\frac{1}{(1-\gamma)^\gamma} e^{h_n x} \sim \frac{\big((1-\alpha_n)n\big)^{\gamma-1}}{\Gamma(\gamma)} e^{h_n}, \qquad n \to \infty. \tag{6.15}$$

Proof. Statement (i) is easily verified, as for *fixed* k

$$[x^k]\frac{1}{(1-x)^\gamma}e^{h_n x} = \sum_{0\le \ell\le k}\binom{k-\ell+\gamma-1}{\gamma-1}\frac{h_n^\ell}{\ell!} \sim \frac{h_n^k}{k!}.$$

We proceed to Part (ii). Set $R_n(x) := Q(x) := (1-x)^{-\gamma}$ for all $n\in\mathbb{N}$ and $A_n(x) := e^{h_n\cdot x}$. We want to apply Lemma 6.3 and verify its conditions one by one. First,

$$\frac{[x^{n-k-1}]A_n(x)}{[x^{n-k}]A_n(x)} = \frac{n-k}{h_n} \sim \frac{1}{\alpha_n} := \rho_n, \qquad k\in\mathbb{N}_0, n\to\infty$$

that is, (6.11) is established. Next we see that (6.12) is valid since

$$\frac{[x^{N_n-k}]A_n(x)}{[x^n]A_n(x)} = \frac{n(n-1)\cdots(n-k-1)}{h_n^k} \le \rho_n^k, \qquad 0\le k\le N_n\in\mathbb{N}.$$

Since $\liminf_{n\to\infty}\alpha_n > 1$ we have that $\overline{\rho} = \limsup_{n\to\infty}\rho_n < 1$. Let $\varepsilon > 0$ be such that $a := (1+\varepsilon)\overline{\rho} < 1$. Establishing uniform convergence of $R_n \to Q$ on $[0,a]$ and estimating the coefficients of $R_n(x)$ is trivial, as $R_n = Q$ for all $n\in\mathbb{N}$ and Q is absolute convergent on $[0,a]$. Lastly, $Q(\rho_n)$ is bounded from below away from zero as $0 < \rho_n \le \overline{\rho} < 1$ for all $n\in\mathbb{N}$. Hence Lemma 6.3 entails

$$[x^n]\frac{1}{(1-x)^\gamma}e^{h_n x} \sim \frac{1}{(1-\alpha_n^{-1})^\gamma}[x^n]e^{h_n x} = \frac{1}{(1-\alpha_n^{-1})^\gamma}\frac{h_n^n}{n!}.$$

Finally, we show (iii). Let $X\sim \mathrm{Po}(h_n)$. We split up the sum

$$\begin{aligned}
[x^n]\frac{1}{(1-x)^\gamma}e^{h_n x} &= \sum_{0\le k\le n}\binom{n-k+\gamma-1}{\gamma-1}\frac{h_n^k}{k!}\\
&= e^{h_n}\left(\sum_{|k-h_n|\le\sqrt{h_n}\ln n} + \sum_{|k-h_n|>\sqrt{h_n}\ln n}\right)\binom{n-k+\gamma-1}{\gamma-1}\Pr\left[X=k\right]\\
&=: e^{h_n}(S_1+S_2).
\end{aligned}$$

For $h_n = \alpha_n n$ we have $n-k\sim(1-\alpha_n)n = \omega(1)$ for all $|k-h_n|\le\sqrt{h_n}\ln n$. Hence

$$S_1 \sim \frac{\big((1-\alpha_n)n\big)^{\gamma-1}}{\Gamma(k)}\Pr\left[|X-h_n|\le\sqrt{h_n}\ln n\right].$$

By applying (6.9) we obtain for some $d>0$ that $\Pr[|X-h_n| > \sqrt{h_n}\ln n] \le e^{-d\ln^2 n}$, so that $|X-h_n|\le \sqrt{h_n}\ln n$ with probability ~ 1. Moreover, it is easy to see that S_2 is negligible: note that

$$S_2 \le e^{-d\ln^2 n}\sum_{0\le k\le n}\binom{n-k+\gamma-1}{\gamma-1} = e^{-d\ln^2 n}\binom{n+\gamma}{\gamma} = \Theta\left(e^{-d\ln^2 n}n^\gamma\right) = o(S_1).$$

□

6.2 Proofs

6.2.1 The Bivariate Boltzmann Model

We begin with the definition of the Boltzmann model that will be central in the forthcoming considerations. We have already encountered the univariate Boltzmann model in Section 5.2.1 and here we present the bivariate extension "on a size and component count level". That is, as opposed to the model introduced in Section 5.2.1, we will define random variables with values in the space $\mathbb{N}^2$ representing the tuple $(|\mathsf{G}|, \kappa(\mathsf{G}))$ of some random multiset G. Nevertheless, we will use the same notation as in Section 5.2.1 as the clear separation of Sections 5 and 6 allows no ambiguity.

Given a non-negative real-valued sequence $(c_k)_{k\in\mathbb{N}}$, the associated power series $C(x)$, and a positive real x_0 such that $C(x_0) < \infty$, we define the random variable $\Gamma C(x_0)$ by

$$\Pr\left[\Gamma C(x_0) = n\right] = c_n \frac{x_0^n}{C(x_0)}, \qquad n \in \mathbb{N}.$$

We will refer to this distribution as the *Boltzmann distribution* or the *Boltzmann model* for $C(x)$ or for $(c_k)_{k\in\mathbb{N}}$ at x_0. The Boltzmann model has a natural interpretation, if $(c_k)_{k\in\mathbb{N}}$ is the counting sequence of some combinatorial class $\mathcal{C}$; this is actually where this terminology originates, see the seminal paper [24]. Indeed, imagine in that case that we put a "weight" of x_0^n to any object of size n in $\mathcal{C}$, so that the total weight $C(x_0)$ is finite. If we then draw an object from $\mathcal{C}$ with a probability that is proportional to its weight, then we get precisely the Boltzmann distribution. Since we are going to need that later several times without explicitly referencing it, note that

$$\mathbb{E}[\Gamma C(x_0)] = \frac{x_0 C'(x_0)}{C(x_0)} \quad \text{and} \quad \mathbb{E}\left[\Gamma C(x_0)^2\right] = \frac{x_0^2 C''(x_0)}{C(x_0)} + \frac{x_0 C'(x_0)}{C(x_0)}.$$

In a completely analogous way we can define a bivariate variant of the Boltzmann model. Suppose that we are given a sequence of non-negative sequences $((g_{n,N})_{N\in\mathbb{N}})_{n\in\mathbb{N}}$ with associated power series $G(x,y)$ and positive reals x_0, y_0 such that $G(x_0, y_0) < \infty$. Then we consider the $\mathbb{N}^2$-valued random variable $\Gamma G(x_0, y_0)$ with distribution

$$\Pr\left[\Gamma G(x_0, y_0) = (n, N)\right] = g_{n,N} \frac{x_0^n y_0^N}{G(x_0, y_0)}, \qquad n, N \in \mathbb{N}, \tag{6.16}$$

that we also call a Boltzmann distribution/model for G at (x_0, y_0).

The crucial point behind the previous definitions is that we can exploit them to actually determine c_n and $g_{n,N}$. Indeed, if we could determine the probability on the left-hand side of (6.16), then would also know $g_{n,N}$. This consideration is of course only useful if we had an appropriate hands-on description of $\Gamma G(x_0, y_0)$ that we could study appropriately. Here the Boltzmann models come into play: a particularly useful property of them – and one that made them so successful in combinatorics – is that they *compose well*, see also [24, 14]. For example, suppose that $C(x) = A(x)B(x)$ and x_0 be such that $A(x_0), B(x_0) < \infty$. Then we can relate the Boltzmann models $\Gamma C(x_0)$ and $\Gamma A(x_0), \Gamma B(x_0)$ as follows. Consider the following simple random process, that first draws independently from $\Gamma A(x_0), \Gamma B(x_0)$ and then creates the sum:

(P) Let independently $A = \Gamma A(x_0)$ and $B = \Gamma B(x_0)$ and set $C = A + B$.

Then it is quite easy to see that C is distributed like $\Gamma C(x_0)$; we do not show that here, since we do not need it, but we keep the guiding principle in mind: the *product of power series corresponds to independent components in the Boltzmann model*. Let us study a second example. Suppose that we are given $C(x)$ and $D(x) = \exp\{C(x)\}$. Assume that x_0 is such that $C(x_0) < \infty$, so that $D(x_0) < \infty$ as well. Then we can relate the models $\Gamma C(x_0)$ and $\Gamma D(x_0)$ by first drawing a Poisson random variable and then summing up independent $\Gamma C(x_0)$'s:

(S1) Let P be a Poisson distributed random variable with parameter $C(x_0)$.

(S2) Let $C_1, \dots, C_P$ independent random variables distributed like $\Gamma C(x_0)$.

(S3) Set $D = C_1 + \cdots + C_P$.

Then it is quite easy to see that D is distributed like $\Gamma D(x_0)$; again, we do not show that, but keep in mind: *exponentiation on the power series level corresponds to a Poisson distribution on the Boltzmann level.* Moreover, we can say that $\Gamma D(x_0)$ has a number of $\mathcal{C}$-components distributed like $\mathrm{Po}(C(x_0))$. Let us look at a last example. Suppose that $H(x) = C(x^j)$ for some $j \in \mathbb{N}$ and x_0 be such that $H(x_0) < \infty$. Consider the process

(E) Let $C = \Gamma C(x_0^j)$ and set $H = jC$.

Then we obtain that H is distributed like $\Gamma H(x_0)$, so that *potentiation of the argument on the power series level corresponds to multiplication on the Boltzmann level*, and we can say that $\Gamma H(x_0)$ has j components.

Here we are interested in the Boltzmann model on

$$G(x,y) = \exp\left\{\sum_{j\geq 1} C(x^j)y^j/j\right\} = \prod_{j\geq 1} \exp\left\{C(x^j)y^j/j\right\}, \quad \text{where } [x^n]C(x) = c_n \text{ satisfies (2.13)}$$

at some $(x_0, y_0) \in (\mathbb{R}^+)^2$ such that $G(x_0, y_0) < \infty$. Guided by the general principles (product $\to$ independent components, exponentiation $\to$ Poisson, potentiation $\to$ multiplication) we consider the following process:

1. Let $(P_j)_{j\geq 1}$ be independent Poisson random variables with parameters $(C(x_0^j)y_0^j/j)_{j\geq 1}$.
2. Let $(C_{j,i})_{j,i\geq 1}$ be independent random variables with $C_{j,i} \sim \Gamma C(x_0^j)$ for $j, i \geq 1$.
3. Set $\Lambda(x_0, y_0) := \left(\sum_{j\geq 1} j \sum_{1\leq i\leq P_j} C_{j,i},\ \sum_{j\geq 1} jP_j\right)$.

Then, rather unsurprisingly, we obtain the following statement, whose proof is in Section 6.2.3.

Lemma 6.5. *The distributions of* $\Gamma G(x_0, y_0)$ *and* $\Lambda(x_0, y_0)$ *are identical.*

However, we can extract more from the aforementioned description of Λ. Let us write – motivated from the combinatorial background – for short for a pair $P = (n, N)$ (like $\Lambda(x_0, y_0)$) $\kappa(P) = N$ for the "number of components" and $|P| = n$ for the "size". Define the events

$$\mathcal{P}_N := \{\kappa(\Lambda(x_0, y_0)) = N\} = \left\{\sum_{j\geq 1} jP_j = N\right\}, \ \mathcal{E}_n := \{|\Lambda(x_0, y_0)| = n\} = \left\{\sum_{j\geq 1} j \sum_{1\leq i\leq P_j} C_{j,i} = n\right\}. \tag{6.17}$$

Then Lemma 6.5 reveals that

$$[x^n y^N]G(x,y)\frac{x_0^n y_0^N}{G(x_0, y_0)} = \Pr\left[\Gamma G(x_0, y_0) = (n, N)\right] = \Pr\left[\Lambda(x_0, y_0) = (n, N)\right] = \Pr\left[\mathcal{E}_n, \mathcal{P}_N\right].$$

Rewriting this yields an alternative representation of $[x^n y^N]G(x,y)$ in terms of iid random variables.

Corollary 6.6. *Let* $x_0, y_0 > 0$ *be such that* $G(x_0, y_0) < \infty$. *For any* $n, N \in \mathbb{N}$ *such that* $[x^n y^N]G(x,y) > 0$

$$[x^n y^N]G(x,y) = x_0^{-n} y_0^{-N} G(x_0, y_0) \Pr\left[\mathcal{P}_N\right] \Pr\left[\mathcal{E}_n \mid \mathcal{P}_N\right].$$

In other words, if we can compute $G(x_0, y_0)$, $\Pr\left[\mathcal{P}_N\right]$ and $\Pr\left[\mathcal{E}_n \mid \mathcal{P}_N\right]$ then we also obtain the desired quantity $[x^n y^N]G(x,y)$ and we are done. How this can achieved is the topic of the next section.

6.2.2 General Proof Strategy

In order to prove Theorems 2.8(I) and 2.8(II) we will apply Corollary 6.6. Let us begin with a remark. Although Corollary 6.6 is true for all $n, N \in \mathbb{N}$ and $x_0, y_0 > 0$ such that $G(x_0, y_0) < \infty$, we certainly cannot expect that it is *useful* for all choices of the parameters. Here, where we want to prove Theorems 2.8(I) and 2.8(II), we consider (large) sizes $n \in \mathbb{N}$ and a corresponding sequence N_n satisfying (2.17) and (2.16), that is,

$$N_n = \lambda_n N_n^* \quad \text{such that} \quad N_n, n - mN_n \to \infty.$$

Then x_0, y_0 should be chosen is such a way that the events $\mathcal{P}_{N_n}$ and $\mathcal{E}_n$ are "typical". Note that the expectations satisfy

$$\mathbb{E}\left[|\Lambda(x_0, y_0)|\right] = \mathbb{E}\left[\sum_{j\geq 1} j \sum_{1\leq i\leq P_j} C_{j,i}\right] = \sum_{j\geq 1} \mathbb{E}\left[P_j\right] j \mathbb{E}\left[C_{j,i}\right] = \sum_{j\geq 1} x_0^j y_0^j C'(x_0^j).$$

and

$$\mathbb{E}\left[\kappa(\Lambda(x_0, y_0))\right] = \mathbb{E}\left[\sum_{j\geq 1} j \mathbb{E}\left[P_j\right]\right] = \sum_{j\geq 1} y_0^j C(x_0^j).$$

So, in order to get the most out of Corollary 6.6, it seems reasonable to choose x_0, y_0 such that

$$\sum_{j\geq 1} x_0^j y_0^j C'(x_0^j) = n \quad \text{and} \quad \sum_{j\geq 1} y_0^j C(x_0^j) = N_n. \tag{6.18}$$

Note, however, that actually we will *not* (quite) do that. Instead, we will choose (x_0, y_0) to be the unique solution (x_n, y_n) of (2.14); to wit, (2.14) reads here

$$x_n y_n C'(x_n) + mc_m \frac{x_n^m y_n}{1 - x_n^m y_n} = n, \quad y_n C(x_n) + c_m \frac{x_n^m y_n}{1 - x_n^m y_n} = N_n, \quad x_n, y_n > 0, \quad x_n^m y_n < 1. \tag{6.19}$$

To justify – informally, at this point – the "switch" to the set of simpler equations let us look closer at the second equation in (6.18):

$$N_n = \sum_{j\geq 1} y_0^j C(x_0^j) = y_0 C(x_0) + c_m \sum_{j\geq 1} (x_0^m y_0)^j + \sum_{j\geq 2} y_0^j (C(x_0^j) - c_m x_0^{jm}) - c_m x_0^m y_0.$$

(Note that, somehow arbitrarily, we pulled out the term $c_m x_0^m y_0$ so that the we got a geometric series starting at 1. This will turn out convenient later, but actually it makes no difference.) Then we must certainly have that $x_0 < \rho$ and $0 < x_0^m y_0 < 1$, and so the first sum on the right-hand side equals $(x_0^m y_0)/(1 - x_0^m y_m)$ and the second one is bounded (since uniformly $C(x_0^j) - c_m x_0^{jm} = \mathcal{O}(x_0^{j(m+1)})$ by Lemma 4.4 and $\rho < 1$). So,

$$N_n = y_0 C(x_0) + c_m \frac{x_0^m y_0}{1 - x_0^m y_0} + \mathcal{O}\left(y_0\right) \tag{6.20}$$

and by ignoring the additive error term we arrive at the second equation in (6.19). Similarly we can justify the switch to the first equation.

To wrap up, in all of the following we will work with $\Lambda(x_n, y_n)$, where (x_n, y_n) is the unique solution to (6.19) and $N_n = \lambda_n N_n^*$ satisfies (2.16) and (2.17). In particular, we will consider independent random variables $(P_j)_{j\geq 1}$ and $(C_{j,i})_{j,i\geq 1}$ such that

$$P_j \sim \mathrm{Po}\left(C(x_n^j)y_n^j/j\right), \quad j \in \mathbb{N}, \quad \text{and} \quad \Pr\left[C_{j,i} = k\right] = c_k \frac{x_n^{jk}}{C(x_n^j)}, \quad j,i,k \in \mathbb{N}. \tag{6.21}$$

By applying Corollary 6.6 we see that for the proof of Theorems 2.8(I) and 2.8(II) it suffices to determine

$$G(x_n, y_n), \quad \Pr\left[\mathcal{P}_{N_n}\right] \quad \text{and} \quad \Pr\left[\mathcal{E}_n \mid \mathcal{P}_{N_n}\right]$$

for $\mathcal{P}_{N_n}$ and $\mathcal{E}_n$ defined in (6.17). This will be performed in versions (I) and (II) of Lemmas 6.8, 6.9 and 6.14. Along the way some more (intermediate) statements will be needed. We will also abbreviate throughout without further reference

$$S_n := \frac{x_n^m y_n}{1 - x_n^m y_n}.$$

Up to this point there is nothing special about the relation of n and N_n. However, towards the proof of Theorems 2.8(I) and 2.8(II) we establish in the next lemma the key role of the value N_n^*. In particular, if we write $x_n = \rho e^{-\chi_n}$, then Lemma 6.1 reveals that $\chi_n = o(1)$ so that $x_n \sim \rho$. If we now consider $S_n = x_n^m y_n/(1 - x_n^m y_n)$, then it is obvious that y_n plays a crucial role: if y_n stays well below ρ^{-m}, then S_n is bounded, otherwise it becomes large. This transition happens precisely at N_n^*, as established in the following lemma, and it has far reaching consequences in the remainder; depending on whether $\limsup_{n\to\infty} < \rho^{-m}$ or $y_n \sim \rho^{-m}$ there isn't/there is a crucial interplay between the terms $x_n y_n C'(x_n)$ (and $y_n C(x_n)$) and S_n in (6.19).

Lemma 6.7(I). *In case (I), that is, when $\limsup_{n\to\infty} \lambda_n < 1$,*

$$\limsup_{n\to\infty} y_n < \rho^{-m} \qquad \text{and consequently} \qquad y_n C(x_n) = N_n + \Theta(y_n), \quad S_n = \Theta(y_n).$$

Lemma 6.7(II). *In case (II), that is, when $\liminf_{n\to\infty} \lambda_n > 1$,*

$$y_n \sim \rho^{-m} \qquad \text{and consequently} \qquad y_n C(x_n) \sim a_n \cdot N_n, \quad S_n \sim \frac{1 - a_n}{c_m} \cdot N_n$$

for some non-negative sequence $(a_n)_{n\in\mathbb{N}}$ such that $\limsup_{n\to\infty} a_n < 1$ and

$$a_n := \lambda_n^{-1} \cdot \frac{g(n - mN_n)}{g(n)} \left(\frac{n - mN_n}{n}\right)^{\alpha/(\alpha+1)}.$$

The proofs can be found in Section 6.2.3. These statements have an decisive impact on the quantities discussed in this section. Recall the definitions of $G^{\geq 2}$ and $G^{\geq 2}_{>m}$ from Section 2. We start with $G(x_n, y_n)$, where we already observe a qualitative difference in the asymptotic behaviour.

Lemma 6.8(I). *In case (I)*

$$G(x_n, y_n) \sim G^{\geq 2}(\rho, y_n) \cdot \exp\left\{y_n C(x_n)\right\} \sim G^{\geq 2}(\rho, y_n) \cdot \exp\left\{-c_m \frac{\rho^m y_n}{1 - \rho^m y_n}\right\} \cdot \exp\left\{N_n\right\}.$$

Lemma 6.8(II). *Let $(a_n)_{n\in\mathbb{N}}$ be the sequence from Lemma 6.7(II). In case (II)*

$$G(x_n, y_n) \sim (ec_m)^{-c_m} \cdot G^{\geq 2}_{>m}(\rho) \cdot ((1-a_n)\, N_n)^{c_m} \cdot \exp\{y_n C(x_n)\}\,.$$

The proofs are in Section 6.2.3. Next we consider $\Pr[\mathcal{P}_{N_n}]$. Recall that $P_j \sim \mathrm{Po}(y_n^j C(x_n^j)/j)$ for $j \in \mathbb{N}$. We saw in the discussion around (6.18)-(6.20) that $\sum_{j\geq 2} y_n^j C(x_n^j)$ is comparable to $S_n + \mathcal{O}(y_n)$. Hence, by Lemma 6.7(I), in case (I) P_1 has mean $N_n + \mathcal{O}(1)$, and moreover, the mean of the sum $\sum_{j\geq 2} jP_j$ is $\mathcal{O}(1)$. Thus, we suspect that $\Pr[\mathcal{P}_{N_n}] \approx \Pr[P_1 = N_n]$, that is, the whole "mass" condenses into P_1. This is established in the following lemma.

Lemma 6.9(I). *In case (I)*

$$\Pr[\mathcal{P}_{N_n}] \sim \Pr[P_1 = N_n] \sim (2\pi N_n)^{-1/2}.$$

Further, there exist $A > 0, 0 < a < 1$ such that

$$\Pr\left[\sum_{j\geq 2} jP_j = K\right] \leq A \cdot \min\{a, y_n\}^K, \qquad K \in \mathbb{N}. \tag{6.22}$$

The proof is in Section 6.2.3. In case (II) the behaviour is quite different from that. We observe that $y_n^j C(x_n^j)$ is essentially $c_m (x_n^m y_n)^j$ as j grows bigger, so that in a first approximation $\sum_{j\geq 1} jP_j$ should behave like

$$P_1 + \sum_{j\geq 1} j\mathrm{Po}\left(c_m (x_n^m y_n)^j / j\right).$$

Here Lemma 6.7(II) reveals that the mean of this sum is large, actually linear in N_n. By comparing the characteristic functions by an exp-ln-transformation we have for any $k \in \mathbb{N}$ and $0 < \beta < 1$ that the sum of independent Poisson random variables $\sum_{j\geq 1} j\mathrm{Po}\left(k\beta^j/j\right)$ is equal in distribution to the sum of iid geometric distributed random variables $\sum_{1\leq i\leq k} \mathrm{Geom}_i(1-\beta)$. But this is nothing else than a multinomial distribution with parameters $1-\beta$ and k, so that

$$\Pr\left[\sum_{1\leq i\leq k} \mathrm{Geom}_i(1-\beta) = N_n\right] = \binom{N_n - 1}{k-1} (1-\beta)^k \beta^{N_n - k}.$$

Plugging back $\beta = x_n^m y_n$ and $k = c_m$ as well as using $(1 - x_n^m y_n) \sim S_n^{-1} \sim c_m((1-a_n)N_n)^{-1}$ due to Lemma 6.7(II) we obtain that

$$\Pr\left[\sum_{j\geq 1} j\mathrm{Po}\left(c_m \frac{(x_n^m y_n)^j}{j}\right) = N_n\right] \sim \frac{c_m^{c_m}}{\Gamma(c_m)(1-a_n)^{c_m}} \frac{(x_n^m y_n)^{N_n}}{N_n}.$$

Since P_1 is either negligible compared to $\sum_{j\geq 2} jP_j$ or at most of the same order ($a_n N_n$ vs. $(1-a_n)N_n$ by Lemma 6.7(II)), this should be qualitatively the actual result. It turns out that this is true.

Lemma 6.9(II). *Let $(a_n)_{n\in\mathbb{N}}$ be the sequence from Lemma 6.7(II). In case (II),*

$$\Pr[\mathcal{P}_{N_n}] \sim \frac{c_m^{c_m} \exp\left\{c_m \frac{a_n}{1-a_n}\right\}}{\Gamma(c_m)} \cdot \frac{(x_n^m y_n)^{N_n}}{(1-a_n)N_n}.$$

Let $(K_n)_{n\in\mathbb{N}}$ *be sequence in* $\mathbb{N}$ *such that* $K_n \to \infty$ *as* $n \to \infty$*. Then, for all* $\ell \in \mathbb{N}$ *as* $n \to \infty$

$$\Pr\left[\sum_{j>\ell} jP_j = K_n\right] \sim \frac{c_m^{c_m}}{(1-a_n)^{c_m}\Gamma(c_m)} \cdot \frac{(x_n^m y_n)^{K_n}}{K_n} \cdot \left(\frac{K_n}{N_n}\right)^{c_m}. \tag{6.23}$$

The proof is in Section 6.2.3. Having studied the event $\mathcal{P}_{N_n}$ itself we continue by investigating the effect on the probability space when conditioning on $\mathcal{P}_{N_n}$ in order to determine $\Pr[\mathcal{E}_n \mid \mathcal{P}_{N_n}]$. For this purpose, we introduce some auxiliary notation. Define

$$L_p := \sum_{1\le i\le p} (C_{1,i} - m), \ \ p \in \mathbb{N}, \quad \text{and} \quad L := L_{P_1}, \quad \text{and} \quad R := \sum_{j\ge 2} j \sum_{1\le i\le P_j} (C_{j,i} - m).$$

With this at hand we reformulate

$$\Pr[\mathcal{E}_n \mid \mathcal{P}_{N_n}] = \Pr[L + R = n - mN_n \mid \mathcal{P}_{N_n}]. \tag{6.24}$$

The driving idea behind these definitions is to split up $\sum_{j\ge 1} j \sum_{1\le i\le P_j}(C_{j,i} - m)$ into a "dominant" large part L and "negligible" remainder R. We observe that the random variables $C_{j,i}$ have exponential tails for $j \ge 2$ since $x_n \le \rho < 1$. In addition, $\mathbb{E}[C_{j,i}] = x_n C'(x_n^j)/C(x_n^j)$ tends to m exponentially fast in j so that the probability of $\{C_{j,i} - m = 0\}$ should tend exponentially fast to 1 in j. However, as we are conditioning on the event $\mathcal{P}_{N_n}$, where some of the P_j's might be large, it is not obvious that R will be also small. The next lemma clarifies the picture.

Lemma 6.10. *In both cases* (I) *and* (II) *there are* $0 < a < 1, A > 0$ *such that*

$$\Pr[R = r \mid \mathcal{P}_{N_n}] \le A \cdot a^r, \qquad r, n \in \mathbb{N}.$$

The proof is in Section 6.2.3. With this at hand, we try to get a handle on (6.24) by conditioning on R and P_1 having certain values, i.e.

$$\Pr[\mathcal{E}_n \mid \mathcal{P}_{N_n}] = \sum_{p,r\ge 0} \Pr[L_p = n - mN_n - r] \Pr[P_1 = p, R = r \mid \mathcal{P}_{N_n}].$$

Then the exponential tails of R conditioned on $\mathcal{P}_{N_n}$ established in Lemma 6.10 guarantee that we can omit all terms where r is large; further, all terms where p deviates "too much" from $\mathbb{E}[P_1] = y_n C(x_n)$ should be negligible as well, since P_1 is very much concentrated around its mean. This leads to

$$\Pr[\mathcal{E}_n \mid \mathcal{P}_{N_n}] \approx \sum_{r \text{ "small"}, \, p \text{ "close to" } \mathbb{E}[P_1]} \Pr[L_p = n - mN_n - r] \Pr[P_1 = p, R = r \mid \mathcal{P}_{N_n}]. \tag{6.25}$$

To finish we will use the fact that the mean of L_p is close to $n - mN_n - r$ for small r and p in the vicinity of $\mathbb{E}[P_1]$. In the next lemma we actually show that L_p follows a local central limit theorem, which will allow us to obtain a very fine-grained understanding of (6.25). For the sake of generality, we will prove this lemma for $L_p(\chi)$, the version of L_p where χ_n is replaced by some general $\chi \to 0$ in the underlying random variables (6.21). More precisely, for $\chi > 0$ set $q := \rho e^{-\chi}$. With this at hand, let $(C_{1,i}(\chi))_{i\ge 1}$ be iid random variables with distribution $\Pr[C_{1,i} = k] = c_k q^k / C(q)$ for $k \in \mathbb{N}$, compare to (6.21). Set $L_p(\chi) := \sum_{1\le i\le p}(C_{1,i}(\chi) - m)$. Note that in this notation $L_p = L_p(\chi_n)$. The mean and variance of $L_p(\chi)$ are given by

$$\mu_p(\chi) := \mathbb{E}[L_p(\chi)] = p\left(\frac{qC'(q)}{C(q)} - m\right) \quad \text{and} \tag{6.26}$$

$$\sigma_p(\chi)^2 := \mathrm{Var}(L_p(\chi)) = p\left(\frac{q^2C''(q) + qC'(q)}{C(q)} - \left(\frac{qC'(q)}{C(q)}\right)^2\right). \tag{6.27}$$

According to Lemma 4.6 we then get the asymptotic expressions

$$\mu_p(\chi) \sim \alpha p \chi^{-1} \qquad \text{and} \qquad \sigma_p(\chi)^2 \sim \alpha p \chi^{-2} \qquad \text{as } \chi \to 0. \tag{6.28}$$

To prove the local limit theorem we will reformulate $\Pr[L_p(\chi) = s] = C(q)^{-p}[x^s]C(qx)^p$ for $s = \mu_p(\chi) + t\sigma_p(\chi)$ and $t \in \mathbb{R}$. Although there are many results in the literature on how to determine large coefficients of $H(x)^p$/how to obtain local limit theorems, none of these are applicable in the generality considered here. To wit, in [19, 20] or [30, Thm. IX.16] the function H is assumed to be either logarithmic or to allow for a singular expansion; and in [30, Thms. VIII.8 and 9] as well as several applications in [67] the ratio n/p needs to be in $\Theta(1)$ to be able to determine $[x^n]H(x)^p$. Clearly this is not the case here as $\mu_p(\chi)/p = \omega(1)$ as $\chi \to 0$. In [17] a local limit theorem is derived, provided that a central limit theorem and additional assumptions, that in particular imply $\sigma_p(\chi)^2/p = \mathcal{O}(1)$, are true; but note that $\sigma_p(\chi)^2/p = \omega(1)$ as $\chi \to 0$ in our setting. Indeed, we have to deal here with a *genuine* triangle array of independent random variables and thus we conduct a detailed saddle-point analysis from scratch on our own.

Lemma 6.11. *Let $p = p(\chi) \in \mathbb{N}$ be such that $p \to \infty$ as $\chi \to 0$. Then for $t = o(p^{1/6})$, as $\chi \to 0$,*

$$\Pr[L_p(\chi) = \mu_p(\chi) + t\sigma_p(\chi)] \sim \mathrm{e}^{-t^2/2} \cdot \Pr[L_p(\chi) = \mu_p(\chi)] \sim \mathrm{e}^{-t^2/2} \frac{1}{\sqrt{2\pi}\,\sigma_p(\chi)} \sim \mathrm{e}^{-t^2/2} \frac{1}{\sqrt{2\pi}} \frac{\chi}{\sqrt{\alpha p}}.$$

The proof is in Section 6.2.3. Assisted by this lemma we obtain from (6.25)

$$\Pr[\mathcal{E}_n \mid \mathcal{P}_{N_n}] \approx \sum_{p \text{ “close to” } \mathbb{E}[P_1]} \Pr[L_p = n - mN_n] \Pr[P_1 = p \mid \mathcal{P}_{N_n}].$$

In the final step of the proof we determine $\Pr[L_p = n - mN_n]$. Here we observe for a last time the effect of cases (I)/(II). In particular, in case (I) it seems reasonable that it is enough to consider the event $P_1 = N_n$, see also Lemma 6.9(I); we obtain the following statements as direct consequences of Lemma 6.11 (for $\chi = \chi_n$).

Corollary 6.12. *In case (I)*

$$\Pr[L_{N_n} = n - mN_n] = \Pr\left[\sum_{1 \le i \le N_n} C_{1,i} = n\right] \sim \sqrt{\frac{\alpha + 1}{2\pi y_n x_n^2 C''(x_n)}} \sim \sqrt{\frac{\alpha}{2\pi}} \frac{\sqrt{N_n}}{n}.$$

The proof is in Section 6.2.3. In case (II) we expect in light of $\mathbb{E}[P_1] \sim a_n N_n$ is (much) smaller than N_n a different behaviour; here we do not stick to a particular value of P_1.

Corollary 6.13. *In case (II)*

$$\Pr[L = n - mN_n] \sim \frac{1}{\sqrt{2\pi\rho^{-m} x_n^2 C''(x_n)}} \sim \sqrt{\frac{\alpha C_0}{2\pi(\alpha + 1)} \cdot g(n - mN_n) \cdot (n - mN_n)^{-(\alpha+2)/(\alpha+1)}}.$$

Backed by this this groundwork we are able to determine $\Pr[\mathcal{E}_n \mid \mathcal{P}_{N_n}]$. The proofs are in Section 6.2.3.

Lemma 6.14(I). *In case (I)*

$$\Pr[\mathcal{E}_n \mid \mathcal{P}_{N_n}] \sim \Pr\left[\sum_{1 \le i \le N_n} C_{1,i} = n\right].$$

Lemma 6.14(II). *In case* (II)

$$\Pr\left[\mathcal{E}_n \mid \mathcal{P}_{N_n}\right] \sim \Pr\left[L = n - mN_n\right].$$

The proofs are almost completed. It is straightforward to obtain the asymptotic order of $[x^n y^{N_n}]G(x,y)$ in Theorems 2.8(I) and 2.8(II) by combining the respective versions (I) and (II) of Lemmas 6.8, 6.9 and 6.14 and applying Corollary 6.6. We conclude this section by summarising all the auxiliary statements that allow us to infer the "combinatorial" forms in Theorems 2.8(I) and 2.8(II). The proofs are in Section 6.2.3.

Lemma 6.15(I). *In case* (I)

$$\Pr\left[\sum_{1\le i\le N_n} C_{1,i} = n\right] \sim \sqrt{2\pi N_n}\cdot x_n^n y_n^{N_n}\cdot \exp\left\{c_m \frac{\rho^m y_n}{1-\rho^m y_n}\right\}\cdot \mathrm{e}^{-N_n}\cdot \frac{1}{N_n!}[x^n]C(x)^{N_n}.$$

Lemma 6.15(II). *Let* $(a_n)_{n\in\mathbb{N}}$ *be the sequence from Lemma 6.7(II). In case* (II)

$$\Pr\left[L = n - mN_n\right] \sim \mathrm{e}^{c_m}\cdot x_n^{n-mN_n}\cdot \exp\left\{-\frac{C(x_n)}{x_n^m}\right\}\cdot [x^{n-mN_n}]\exp\left\{\frac{C(x)-c_m x^m}{x^m}\right\}.$$

Further

$$y_n C(x_n) - \frac{C(x_n)}{x_n^m} \sim -c_m \frac{a_n}{1-a_n}.$$

6.2.3 Proofs of the Supporting Results

In this section we prove all lemmas and corollaries from Section 6.2.2, together with some auxiliary statements.

Proof of Lemma 6.5

Proof of Lemma 6.5. Let n, N be such that $[x^n y^N]G(x,y) > 0$ and $x_0, y_0 > 0$ such that $G(x_0,y_0) < \infty$. For $k\in\mathbb{N}$ define the set $\Omega_k := \{(p_1,p_2,\dots)\in\mathbb{N}_0^\infty : \sum_{j\ge1} jp_j = k\}$. Then per definition

$$\begin{aligned}\Pr\left[\Lambda(x_0,y_0) = (n,N)\right] &= \Pr\left[\sum_{j\ge1} j \sum_{1\le i\le P_j} C_{j,i} = n, \sum_{j\ge1} jP_j = N\right] \\ &= \frac{y_0^N}{G(x_0,y_0)} \sum_{\mathrm{p}\in\Omega_N} \Pr\left[\sum_{j\ge1} j\sum_{1\le i\le p_j} C_{j,i} = n\right] \prod_{j\ge1} \frac{(C(x_0^j)/j)^{p_j}}{p_j!}. \end{aligned} \tag{6.29}$$

Next we reformulate

$$\Pr\left[\sum_{j\ge1} j\sum_{1\le i\le p_j} C_{j,i} = n\right] = [x^n]\mathbb{E}\left[x^{\sum_{j\ge1} j\sum_{1\le i\le p_j} C_{j,i}}\right] = x_0^n [x^n]\prod_{j\ge1}\left(\frac{C(x^j)}{C(x_0^j)}\right)^{p_j}.$$

Plugging this back into (6.29) yields

$$\begin{aligned}\Pr\left[\Lambda(x_0,y_0) = (n,N)\right] &= \frac{x_0^n y_0^N}{G(x_0,y_0)}\cdot [x^n]\sum_{\mathrm{p}\in\Omega_N}\prod_{j\ge1}\frac{(C(x^j)/j)^{p_j}}{p_j!} \\ &= \frac{x_0^n y_0^N}{G(x_0,y_0)}[x^n y^N]\sum_{k\ge0}\sum_{\mathrm{p}\in\Omega_k}\prod_{j\ge1}\frac{(C(x^j)y^j/j)^{p_j}}{p_j!}. \end{aligned} \tag{6.30}$$

Define by $(P_j(x,y))_{j\ge1}$ independent Poisson variables with parameters $(C(x^j)y^j/j)_{j\ge1}$. Then

$$[x^ny^N]\sum_{k\ge0}\sum_{p\in\Omega_k}\prod_{j\ge1}\frac{(C(x^j)y^j/j)^{p_j}}{p_j!}=[x^ny^N]G(x,y)\sum_{k\ge0}\Pr\left[\sum_{j\ge1}jP_j(x,y)=k\right]=[x^ny^N]G(x,y)$$

and inserting this into (6.30) yields $\Pr\left[\Lambda(x_0,y_0)=(n,N)\right]=[x^ny^N]G(x,y)x_0^ny_0^N/G(x_0,y_0)$, which equals $\Pr\left[\Gamma G(x_0,y_0)=(n,N)\right]$, as claimed. □

Proof of Lemma 6.7

Proof of Lemma 6.7(I). As we will need that later, we prove the following more general statement, form which Lemma 6.7(I) follows immediately.

Lemma 6.16. *Let $\chi_n=\ln(\rho/x_n)$. In case (I)*

$$\chi_n\sim\alpha\frac{N_n}{n},\qquad y_n\sim\rho^{-m}\cdot\frac{h(n/N_n^*)}{h(n/N_n)}\cdot\lambda_n^{\alpha+1}\qquad\textit{and}\qquad\limsup_{n\to\infty}y_n<\rho^{-m}.$$

Further, if $\lambda_n=o(1)$ then $y_n=o(1)$ and if $\lambda_n=\Theta(1)$ then $y_n\sim\rho^{-m}\lambda_n^{\alpha+1}=\Theta(1)$. Moreover,

$$y_nC(x_n)=N_n+c_mS_n\qquad\textit{and}\qquad S_n=\Theta(y_n).$$

Proof. We first show that $y_nC(x_n)=\Theta(N_n)$. Since $S_n=x_n^my_n/(1-x_n^my_n)>0$ we infer from (6.19) that $y_nC(x_n)<N_n$. For the sake of contradiction assume that there is a sequence $(n_\ell)_{\ell\in\mathbb{N}}$ such that $y_{n_\ell}C(x_{n_\ell})=o(N_{n_\ell})$. This implies that $S_{n_\ell}\sim N_{n_\ell}$ due to (6.19), which is only possible if $y_n\sim\rho^{-m}$, as we know that $x_n\sim\rho^m$ from Lemma 6.1. By applying (6.6) for $k=0$ and since $n_\ell-mN_{n_\ell}\sim n_\ell$ in case (I) we thus infer that $y_{n_\ell}C(x_{n_\ell})=\Theta(g(n_\ell)n_\ell^{\alpha/(\alpha+1)})=\Theta(N^*_{n_\ell})=\Omega(N_{n_\ell})$, the desired contradiction. We showed that $y_nC(x_n)=\Theta(N_n)$. We also immediately obtain from this fact that $\chi_n=\Theta(n/N_n)$, since $\chi_n^{-1}=\Theta(x_nC'(x_n)/C(x_n))$ due to Lemma 6.1.

Next we show that $S_n=\mathcal{O}(1)$, again by contradiction. Assume that there is a sequence $(n_\ell)_{\ell\in\mathbb{N}}$ such that $S_{n_\ell}=\omega(1)$. Then again $y_{n_\ell}\sim\rho^{-m}$ so that we get with Lemma 6.1 and the definition of $N^*_{n_\ell}$ from (2.16) that, as $\ell\to\infty$,

$$y_{n_\ell}C(x_{n_\ell})\sim C_0g(n_\ell)n_\ell^{\alpha/(\alpha+1)}=N^*_{n_\ell}.\tag{6.31}$$

But in case (I) we have that $\limsup_{\ell\to\infty}y_{n_\ell}C(x_{n_\ell})/N^*_{n_\ell}\le\limsup_{\ell\to\infty}N_{n_\ell}/N^*_{n_\ell}<1$, a contradiction. We just showed that $S_n=\mathcal{O}(1)$. It immediately follows that $y_nC(x_n)\sim N_n$ and $\alpha\chi_n^{-1}\sim x_nC'(x_n)/C(x_n)\sim n/N_n$ from (6.19).

From $S_n=\mathcal{O}(1)$ we also conclude that $\limsup y_n<\rho^{-m}$, as $x_n\sim\rho^m$ due to Lemma 6.1. This, in turn, also implies $S_n=\Theta(y_n)$, as claimed. It remains to show the validity of the asymptotic expression of y_n. By applying Lemma 6.1, in particular (6.8),

$$\chi_n^{-1}h(\chi_n^{-1})^{1/(\alpha+1)}\sim\left(\frac{n}{\Gamma(\alpha+1)y_n}\right)^{1/(\alpha+1)}.$$

Solving for y_n yields $y_n\sim n\chi_n^{\alpha+1}/(h(\chi^{-1})\Gamma(\alpha+1))$. We plug in $\chi_n\sim\alpha N_n/n$ as well as the definitions of $C_0=\alpha^{-1}(\rho^{-m}\Gamma(\alpha+1))^{1/(\alpha+1)}$ and N_n^* from (2.16) and $N=\lambda_nN_n^*=\lambda_ng(n)n^{\alpha/(\alpha+1)}$ to obtain that

$$y_n\sim\rho^{-m}\cdot\frac{g(n)^{\alpha+1}}{h(n/N_n)}\cdot\lambda_n^{\alpha+1}.$$

Since $g(n) = h(n/N_n^*)^{1/(\alpha+1)}$, which can be seen directly from (2.15) for $u = n/N_n^*$ and $v = n$, we are done. If $\lambda_n = o(1)$ we have with (A.2) that there is a $0 < \delta < \alpha + 1$ such that $\lambda_n^{\alpha+1} \cdot h(n/N_n^*)/h(n/N_n) \leq (N_n^*/N_n)^\delta = \lambda_n^{\alpha+1-\delta} = o(1)$. If $\lambda_n = \Theta(1)$ then $h(n/N_n^*)/h(n/N_n) \sim 1$ finishing the proof. □

Proof of Lemma 6.7(II). We later need the following statement which contains Lemma 6.7(II) together with asymptotic properties of the solution (x_n, y_n) to (6.19) in case (II).

Lemma 6.17. *Let* $\chi_n = \ln(\rho/x_n)$. *Consider the non-negative sequence*

$$a_n := \lambda_n^{-1} \cdot \frac{g(n - mN_n)}{g(n)} \cdot \left(\frac{n - mN_n}{n}\right)^{\alpha/(\alpha+1)}, \quad n \in \mathbb{N} \tag{6.32}$$

that fulfils

$$a_n \leq \lambda_n^{-1} \text{ for } n \in \mathbb{N} \text{ sufficiently large} \quad \text{and} \quad a_n \sim \lambda_n^{-1} \text{ for } N_n = o(n).$$

Then, in case (II),

$$y_n \sim \rho^{-m} \quad \text{and} \quad \chi_n \sim \alpha \cdot a_n \cdot \frac{N_n}{n - mN_n} \sim \alpha C_0 \cdot g(n - mN_n) \cdot (n - mN_n)^{-1/(\alpha+1)},$$
$$y_n C(x_n) \sim a_n \cdot N_n \sim C_0 \cdot g(n - mN_n) \cdot (n - mN_n)^{\alpha/(\alpha+1)} \quad \text{and} \quad S_n \sim \frac{1 - a_n}{c_m} \cdot N_n.$$

Proof. Since $x_n^m y_n < 1$ and $x_n \sim \rho$ according to Lemma 6.1, it is clear that $y_n \leq (1 + \varepsilon)\rho^{-m}$ for all $\varepsilon > 0$ and all sufficiently large n. First of all, we show that $y_n \sim \rho^{-m}$ by establishing that even $S_n = \Theta(N_n)$. We know that $S_n \leq N_n$, as $y_n C(x_n) \geq 0$ in (6.19). We show $S_n = \Omega(N_n)$ by contradiction. Assume there is a sequence $(n_\ell)_{\ell \in \mathbb{N}}$ such that $S_{n_\ell} = o(N_{n_\ell})$. From (6.19) and by applying Lemma 6.1 and $N_{n_\ell} = \lambda_{n_\ell} N_{n_\ell}^* = \lambda_{n_\ell} C_0 g(n_\ell) n_\ell^{\alpha/(\alpha+1)}$, where $C_0 = \alpha^{-1}(\rho^{-m}\Gamma(\alpha+1))^{1/(\alpha+1)}$, we obtain

$$1 \sim \frac{y_{n_\ell} C(x_{n_\ell})}{N_{n_\ell}} \sim \rho^{m/(\alpha+1)} y_{n_\ell} \lambda_{n_\ell}^{-1} \frac{g\big((n_\ell - mN_{n_\ell})/y_{n_\ell}\big)}{g(n_\ell)} \left(\frac{n_\ell - mN_{n_\ell}}{n_\ell y_{n_\ell}}\right)^{\alpha/(\alpha+1)}. \tag{6.33}$$

Since g is slowly varying we obtain from (A.2) for $0 < \delta < 1/(\alpha+1)$ and sufficiently large ℓ

$$\frac{g\big((n_\ell - mN_{n_\ell})/y_{n_\ell}\big)}{g(n_\ell)} \leq \max\left\{\left(\frac{n_\ell - mN_{n_\ell}}{n_\ell y_{n_\ell}}\right)^\delta, \left(\frac{n_\ell - mN_{n_\ell}}{n_\ell y_{n_\ell}}\right)^{-\delta}\right\}.$$

Together with (6.33) and the simple fact $(n_\ell - mN_{n_\ell})/(n_\ell y_{n_\ell}) \leq y_{n_\ell}^{-1}$ we obtain

$$\frac{y_{n_\ell} C(x_{n_\ell})}{N_{n_\ell}} \leq \rho^{m/(\alpha+1)} y_{n_\ell} \lambda_{n_\ell}^{-1} \max\{y_{n_\ell}^{-\alpha/(\alpha+1)-\delta}, y_{n_\ell}^{-\alpha/(\alpha+1)+\delta}\} \leq (1+\varepsilon)^{1/(\alpha+1)} \lambda_{n_\ell}^{-1}((1+\varepsilon)\rho)^\delta.$$

As $\liminf_{\ell\to\infty} \lambda_{n_\ell} > 1$, this can be made < 1 by choosing $\varepsilon, \delta > 0$ sufficiently small. This contradicts (6.33) so that we have shown that $S_n = \Theta(N_n)$ also implying $y_n \sim \rho^{-m}$.

With this at hand and plugging $y_n \sim \rho^{-m}$ into the expressions for $C(x_n)$ and χ_n from Lemma 6.1 as well as recalling $C_0 := \alpha^{-1}(\rho^{-m}\Gamma(\alpha+1))^{1/(\alpha+1)}$ we obtain

$$y_n C(x_n) \sim C_0 \cdot g(n - mN_n) \cdot (n - mN_n)^{\alpha/(\alpha+1)} \text{ and } \chi_n \sim \alpha C_0 \cdot g(n - mN_n) \cdot (n - mN)^{-1/(\alpha+1)}.$$

Multiplying both right-hand sides by $1 = N_n^*/N_n^* = \lambda_n^{-1} N_n/(C_0 g(n) n^{\alpha/(\alpha+1)})$ we obtain the claimed representations $y_n C(x_n) \sim a_n \cdot N_n$ and $\chi_n \sim \alpha \cdot a_n \cdot N_n/(n - mN_n)$ for

$$a_n := \lambda_n^{-1} \cdot \frac{g(n - mN_n)}{g(n)} \cdot \left(\frac{n - mN_n}{n}\right)^{\alpha/(\alpha+1)}.$$

The properties of a_n are readily obtained by noting that g is slowly varying and the simple fact $(n - mN_n)/n \le 1$. Finally, since a_n is bounded away from one, we also obtain from (6.19) that $S_n = (N_n - y_n C(x_n))/c_m \sim (1 - a_n)/c_m \cdot N_n$. □

Proof of Lemma 6.8

Proof of Lemma 6.8(I). Due to (6.19) and Lemma 6.16 we know that $y_n C(x_n) = N_n - c_m x_n^m y_n/(1 - x_n^m y_n) = N_n - c_m \rho^m y_n/(1 - \rho^m y_n) + o(1)$. Further, Lemma 4.4 entails that there is $A > 0$ such that

$$\sum_{j\ge2} \frac{C(x_n^j)}{j} y_n^j = \sum_{j\ge2} (x_n^m y_n)^j \frac{C(x_n^j)}{j x_n^{jm}} \le c_m \sum_{j\ge2} (x_n^m y_n)^j (1 + A x_n^j).$$

This is bounded from above: due to Lemma 6.16 we know that $\limsup y_n < \rho^{-m}$ and $x_n \le \rho$, which in turn implies that $x_n^m y_n$ is bounded from above by something strictly smaller than 1. Hence, using that $G(x, y)$ is continuous in x and $x_n \sim \rho$ we obtain by dominated convergence

$$G(x_n, y_n) = \exp\left\{ y_n C(x_n) + \sum_{j\ge2} \frac{C(x_n^j)}{j} y_n^j \right\} \sim \exp\left\{ N_n - c_m \frac{\rho^m y_n}{1 - \rho^m y_n} \right\} \exp\left\{ \sum_{j\ge2} \frac{C(\rho^j)}{j} y_n^j \right\}.$$

□

Proof of Lemma 6.8(II). Let $(a_n)_{n\in\mathbb{N}}$ be the sequence from Lemma 6.17. Since $x_n^m y_n \to 1$ and $(1 - x_n^m y_n)^{-1} \sim (1 - a_n)/c_m \cdot N_n$ due to Lemma 6.17 we obtain that

$$\begin{aligned} G(x_n, y_n) &= \exp\left\{ y_n C(x_n) + c_m \sum_{j\ge1} \frac{(x_n^m y_n)^j}{j} - c_m x_n^m y_n + \sum_{j\ge2} \frac{C(x_n^j) y_n^j - c_m x_n^{jm} y_n^j}{j} \right\} \\ &\sim \exp\{y_n C(x_n)\} \cdot \frac{1}{c_m^{c_m}} \cdot ((1 - a_n) N_n)^{c_m} \cdot \exp\left\{ -c_m + \sum_{j\ge2} \frac{C(x_n^j) y_n^j - c_m x_n^{jm} y_n^j}{j} \right\}. \end{aligned}$$

With Lemma 4.4 there is some $A > 0$ such that for all $j \ge 2$

$$\left(C(x_n^j) y_n^j - c_m x_n^{jm} y_n^j\right)/j \le A (x_n^m y_n)^j x_n^j.$$

Since $\limsup x_n^{m+1} y_n = \rho < 1$ the expression in the previous display is summable and by dominated convergence we obtain that

$$\exp\left\{ \sum_{j\ge2} (C(x_n^j) y_n^j - c_m x_n^{jm} y_n^j)/j \right\} \to \exp\left\{ \sum_{j\ge2} \frac{C(\rho^j) - c_m \rho^{jm}}{j \rho^{jm}} \right\} = G_{>m}^{\ge2}(\rho).$$

□

Proof of Lemma 6.9

We will use the probability generating function (pgf) $F^{(\ell)}(x) = \mathbb{E}[x^{P^{(\ell)}}]$ of the sum of Poisson random variables $P^{(\ell)} := \sum_{j>\ell} jP_j$ for $\ell \in \mathbb{N}_0$. Define the auxiliary series

$$G^{(\ell)}(x,y) := \exp\left\{\sum_{j>\ell} \frac{C(x^j)}{j} y^j\right\} \text{ and } R^{(\ell)}(x) := \exp\left\{-c_m \sum_{1\le j\le \ell} \frac{x^j}{j} + \sum_{j>\ell} \frac{C(x_n^j) - c_m x_n^{jm}}{j x_n^{jm}} x^j\right\}.$$

The pgf of a $\mathrm{Po}(\lambda)$ random variable equals $\mathrm{e}^{\lambda(x-1)}$. By the independence of the P_j's,

$$F^{(\ell)}(x) = \frac{1}{G^{(\ell)}(x_n, y_n)} \exp\left\{\sum_{j>\ell} \frac{C(x_n^j) y_n^j}{j} x^j\right\}.$$

We want to determine $[x^{N_n}]F^{(0)}(x)$ that is precisely the probability of $\mathcal{P}_{N_n} = \{P^{(0)} = N_n\}$. In general, when computing coefficients of these pgfs we will factorise $F^{(\ell)}(x)$ in such a way that Lemma 6.4 is applicable. More precisely, we split up for any $K \in \mathbb{N}$ and $\ell \in \mathbb{N}_0$

$$[x^K]F^{(\ell)}(x) = \frac{(x_n^m y_n)^K}{G^{(\ell)}(x_n, y_n)} \cdot [x^K] \frac{1}{(1-x)^{c_m}} \cdot R^{(\ell)}(x). \tag{6.34}$$

When investigating (6.34) great care has to be taken. In $R^{(0)}(x)$ the term $C(x_n)/x_n^m$, which tends to infinity due to Lemma 6.1, appears. Setting $R(x) := R^{(1)}(x)$ and $F(x) := F^{(0)}(x)$ we rewrite

$$[x^{N_n}]F(x) = \frac{(x_n^m y_n)^{N_n}}{G(x_n, y_n)} \cdot [x^{N_n}] \frac{1}{(1-x)^{c_m}} \cdot \exp\left\{\frac{C(x_n)}{x_n^m} x\right\} \cdot R(x). \tag{6.35}$$

In order to apply Lemma 6.4 we get rid of the term $R(x)$. This is achieved by using Lemma 6.3, where we need to establish uniform convergence of R as n gets large. Note that the coefficients of $F^{(\ell)}(x)$ are much easier to compute for $\ell > 1$ as the term involving $C(x_n)/x_n^m$ is absent. Nevertheless, we need uniform convergence for $R^{(\ell)}$ in this case as well.

Lemma 6.18. *Let $\ell \in \mathbb{N}_0$ and set*

$$Q^{(\ell)}(x) := \exp\left\{-c_m \sum_{1\le j\le \ell} \frac{x^j}{j} + \sum_{j>\ell} \frac{C(\rho^j) - c_m \rho^{jm}}{j\rho^{jm}} x^j\right\}.$$

Let $0 < \varepsilon < \rho^{-1}$. Then $R^{(\ell)}$ converges uniformly to $Q^{(\ell)}$ on $[0, \rho^{-1} - \varepsilon]$ and the radius of convergence of $Q^{(\ell)}$ is at least $\rho^{-1} > 1$. Further, there exists a sequence $(d_n)_{n\in\mathbb{N}_0}$ such that $|[x^k]R^{(\ell)}(x)| \le d_k$ for all $k, n \in \mathbb{N}_0$ such that $\sum_{k\ge 0} d_k x^k < \infty$ for any $x \in [0, \rho^{-1} - \varepsilon]$.

Proof. We will use the following standard result in real analysis. Let $(f_n)_{n\in\mathbb{N}}$ be real-valued functions all defined on some closed interval $[a, b]$. Suppose that f_n is strictly monotone for each $n \in \mathbb{N}$ and that $(f_n)_{n\in\mathbb{N}}$ converges pointwise to a continuous function f. Then the convergence is uniform. In our setting, we have that

$$R^{(\ell)}(x) = \exp\left\{-c_m \sum_{1\le j\le \ell} \frac{x^j}{j}\right\} \cdot \exp\left\{\sum_{j>\ell} \frac{C(x_n^j) - c_m x_n^{jm}}{j x_n^{jm}} \cdot x^j\right\} =: e(x) \cdot f_n(x).$$

Clearly, if we show that $f_n(x)$ converges uniformly to some $f(x)$, then $R^{(\ell)}(x)$ converges uniformly to $e(x)f(x)$. A direct application of Lemma 4.4 yields that there exists some $A > 0$ such that

$$\ln f_n(x) \le A \sum_{j>\ell} \frac{x_n^j}{j} \le A \sum_{j>\ell} \frac{\rho^j}{j}. \tag{6.36}$$

Hence the radius of convergence of $f_n(x)$ is at least ρ^{-1}, implying that $f_n(x)$ is defined on $[0, \rho^{-1} - \varepsilon]$ for any $0 < \varepsilon < \rho^{-1}$ and $n \in \mathbb{N}$. Moreover, f_n has only non-negative coefficients, so it is strictly increasing. Finally, (6.36) shows that by dominated convergence f_n converges pointwise to $f(x) := Q^{(\ell)}(x)/e(x)$. Since f is continuous we have proven that the convergence is uniform and the claim that $R^{(\ell)}(x)$ converges uniformly to $e(x)f(x) = Q^{(\ell)}(x)$ follows immediately.

We proceed with estimating $|[x^k]R^{(\ell)}(x)|$. Let us first note that $|[x^k]\mathrm{e}^{-px^q}| = [x^k]\mathrm{e}^{px^q}$ for any $p > 0$ and $k, q \in \mathbb{N}$. Further the second sum in $\ln R^{(\ell)}(x)$ involves only non-negative terms, so that

$$|[x^k]R^{(\ell)}(x)| = [x^k]R_+^{(\ell)}(x), \text{ where } R_+^{(\ell)} := \exp\Big\{c_m \sum_{1 \le j \le \ell} \frac{x^j}{j} + \sum_{j>\ell} \frac{C(x_n^j) - c_m x_n^{jm}}{j x_n^{jm}} x^j\Big\}.$$

Since $R_+^{(\ell)}(x)$ has only non-negative coefficients, we deduce from (4.1) that for all $0 < k \in \mathbb{N}_0$ and setting $y = \rho^{-1} - \varepsilon/2$

$$|[x^k]R^{(\ell)}(x)| \le R_+^{(\ell)}(y) \cdot y^{-k}.$$

Since $x_n \sim \rho$ due to Lemma 6.1 we obtain analogous to (6.36) that there is some $A_3 > 0$ such that $R_+^{(\ell)}(y) \le A_3$. Setting $d_k := A_3 y^{-k}$ for $k \in \mathbb{N}_0$ the claim is verified since $d_k(\rho^{-1} - \varepsilon)^k = A_3(\rho^{-1} - \varepsilon)^k/(\rho^{-1} - \varepsilon/2)^k$ is summable. □

With these ingredients at hand we are able to prove Lemmas 6.9(I) and 6.9(II).

Proof of Lemma 6.9(I). Abbreviate $h_n := C(x_n)/x_n^m$ and set

$$A(x) := \frac{1}{(1-x)^{c_m}} \cdot \exp\{h_n x\} \quad \text{and} \quad R(x) := \exp\left\{-c_m x + \sum_{j \ge 2} \frac{C(x_n^j) - c_m x_n^{jm}}{j x_n^{jm}} x^j\right\}.$$

With this at hand, we use (6.35) to reformulate

$$\Pr[\mathcal{P}_{N_n}] = \frac{(x_n^m y_n)^{N_n}}{G(x_n, y_n)} \cdot [x^{N_n}]A(x)R(x).$$

We will apply Lemma 6.3 to $A(x)R(x)$ and verify conditions (6.11) and (6.12) first. Set $\alpha_n := h_n/N_n$ and $\alpha_{n,k} := h_n/(N_n - k)$ for $0 \le k < N_n$. We obtain from Lemma 6.16

$$\alpha_n = \frac{y_n C(x_n)}{y_n x_n^m} \frac{1}{N_n} \sim \frac{1}{y_n x_n^m} \sim \frac{h(n/N_n)}{h(n/N_n^*)} \cdot \lambda_n^{-(\alpha+1)}, \tag{6.37}$$

so that, again by Lemma 6.16, $\liminf \alpha_n = \rho^{-m}/\limsup y_n > 1$ and $\liminf \alpha_{n,k} > 1$, too, uniformly for $0 \le k < N_n$. Hence, Lemma 6.4(ii) is applicable and we obtain for all k such that $N_n - k \to \infty$

$$[x^{N_n-k}]A(x) = [x^{N_n-k}]\frac{1}{(1-x)^{c_m}}\mathrm{e}^{\alpha_{n,k}(N_n-k)x} \sim \left(\frac{1}{1-\alpha_{n,k}^{-1}}\right)^{c_m} \cdot \frac{h_n^{N_n-k}}{(N_n-k)!}. \tag{6.38}$$

This implies (6.11) together with (6.37), i.e., for $k \in \mathbb{N}_0$ and as $n \to \infty$

$$\frac{[x^{N_n-k}]A(x)}{[x^{N_n-k+1}]A(x)} \sim \left(\frac{1-\alpha_{n,k}^{-1}}{1-\alpha_{n,k-1}^{-1}}\right)^{c_m} \frac{N_n}{h_n} \sim \frac{h(n/N_n^*)}{h(n/N_n)} \cdot \lambda_n^{\alpha+1} =: \rho_n.$$

Set $\overline{\rho} := \limsup \rho_n < 1$. Due to (6.38) we have for any k such that $N_n - k \to \infty$

$$\frac{[x^{N_n-k}]A(x)}{[x^{N_n}]A(x)} \sim \left(\frac{1-\alpha_n^{-1}}{1-\alpha_{n,k}^{-1}}\right)^{c_m} \frac{N_n!}{(N_n-k)!} N_n^{-k} \left(\frac{N_n}{h_n}\right)^k \sim \left(\frac{1-\alpha_n^{-1}}{1-\alpha_{n,k}^{-1}}\right)^{c_m} \frac{N_n!}{(N_n-k)!} N_n^{-k} \rho_n^k.$$

Since $\alpha_{n,k} \geq \alpha_n$ and $N_n!/(N_n-k)!N_n^{-k} \leq 1$ we obtain for any $\varepsilon > 0$

$$\frac{[x^{N_n-k}]A(x)}{[x^{N_n}]A(x)} \leq (1+\varepsilon)^k \rho_n^k, \quad N_n - k \text{ sufficiently large.} \tag{6.39}$$

Lemma 6.4(i) and (6.38) imply that for any k such that $N_n - k = \mathcal{O}(1)$

$$\frac{[x^{N_n-k}]A(x)}{[x^{N_n}]A(x)} \sim (1-\alpha_n^{-1})^{c_m} \frac{N_n!}{(N_n-k)!} h_n^{-k} \sim (1-\alpha_n^{-1})^{c_m} \frac{N_n!}{(N_n-k)!} N_n^{-k} \rho_n^k \leq \rho_n^k.$$

Again this entails that for any $\varepsilon > 0$ and N_n sufficiently large

$$\frac{[x^{N_n-k}]A(x)}{[x^{N_n}]A(x)} \leq (1+\varepsilon)^k \rho_n^k \quad \text{for } N_n - k = \mathcal{O}(1). \tag{6.40}$$

Together with (6.39) this shows that condition (6.12) is fulfilled for any $\varepsilon > 0$ as long as n is sufficiently large. Let us proceed with checking the remaining conditions of Lemma 6.3. Choose $\varepsilon > 0$ in (6.39) and (6.40) such that $a := (1+\varepsilon)\overline{\rho} < \rho^{-1}$; this is possible, since $\overline{\rho} < 1$ and $\rho > 1$. Then the series $R(x)$ are all analytic at a by Lemma 6.18. Moreover, by the same lemma, $R(x)$ converges uniformly to

$$Q(x) := \exp\left\{-c_m x + \sum_{j\geq 2} \frac{C(\rho^j) - c_m \rho^{jm}}{j\rho^{jm}} \cdot x^j\right\}$$

in any closed subinterval of $[0, \rho^{-1})$. In particular, as $\rho^{-1} > 1$, we obtain uniform convergence on $[0, a]$. Moreover, Lemma 6.18 guarantees the existence of a sequence $(d_n)_{n\in\mathbb{N}_0}$ such that $|[x^k]R(x)| \leq d_k$ and $\sum_{k\geq 1} d_k a^k < \infty$. Finally $Q(\rho_n) \geq \exp\{-c_m\overline{\rho}\} > 0$ for all $n \in \mathbb{N}$ so that all conditions of Lemma 6.3 are met. We deduce with a combination of (6.37) and (6.38)

$$[x^{N_n}]A(x) \cdot R(x) \sim Q(\rho_n) \cdot [x^{N_n}]A(x) \sim Q(\rho_n) \cdot \left(\frac{1}{1-\rho_n}\right)^{c_m} \cdot \frac{h_n^{N_n}}{N_n!}.$$

Recalling (6.35) we have shown so far

$$\Pr[\mathcal{P}_{N_n}] \sim Q(\rho_n) \cdot \frac{(x_n^m y_n)^{N_n}}{G(x_n, y_n)} \left(\frac{1}{1-\rho_n}\right)^{c_m} \cdot \frac{h_n^{N_n}}{N_n!}.$$

In what follows, we simplify this term. First note that $\rho^{-m}\rho_n \sim y_n$ by Lemma 6.16. Thus

$$\left(\frac{1}{1-\rho_n}\right)^{c_m} \sim \exp\left\{c_m \sum_{j\geq 1} \frac{(\rho^m y_n)^j}{j}\right\}.$$

Further, recalling that $h_n := C(x_n)/x_n^m$, we obtain

$$(x_n^m y_n)^{N_n} h_n^{N_n} = (y_n C(x_n))^{N_n} \overset{(6.19)}{=} \left(N_n - c_m \frac{x_n^m y_n}{1 - x_n^m y_n}\right)^{N_n} \sim N_n^{N_n} \exp\left\{-c_m \frac{\rho^m y_n}{1 - \rho^m y_n}\right\}.$$

Together with Lemma 6.8(I) this establishes the limit law

$$\Pr\left[\mathcal{P}_{N_n}\right] \sim Q\big(\rho^m y_n\big) \cdot \exp\left\{c_m \sum_{j\geq 1} \frac{(\rho^m y_n)^j}{j}\right\} \cdot \exp\left\{-\sum_{j\geq 2} \frac{C(\rho^j)}{j} y_n^j\right\} \cdot \mathrm{e}^{-N_n} \frac{N_n^{N_n}}{N_n!} \sim \frac{1}{\sqrt{2\pi N_n}},$$

as claimed. To show the second statement (6.22) we first estimate

$$\Pr\left[\sum_{j\geq 2} jP_j = K\right] \leq [x^K] \exp\left\{\sum_{j\geq 2} \frac{C(x_n^j) y_n^j}{j} x^j\right\}.$$

Applying (4.1) yields

$$\Pr\left[\sum_{j\geq 2} jP_j = K\right] \leq \exp\left\{\sum_{j\geq 2} \frac{C(x_n^j)}{j}\right\} y_n^K. \tag{6.41}$$

Further $\sum_{j\geq 2} C(x_n^j)/j \leq \sum_{j\geq 2} C(\rho^j)/j < \infty$. At the same time we observe with Lemma 4.4 that there is some $A > 0$ such that

$$\sum_{j\geq 2} \frac{C(x_n^j) y_n^j}{j} x^j \leq A \cdot \sum_{j\geq 2} \frac{(x_n^m y_n)^j}{j} x^j.$$

Since according to Lemma 6.16 we have that $\limsup x_n^m y_n \leq 1 - \varepsilon$ for some $0 < \varepsilon < 1$ we deduce that there is some $a > 1$ with $(1-\varepsilon)a < 1$ such that by (4.1) we obtain

$$\Pr\left[\sum_{j\geq 2} jP_j = K\right] \leq \exp\left\{A \cdot \sum_{j\geq 2} \frac{((1-\varepsilon)a)^j}{j}\right\} a^{-K},$$

finishing the proof together with (6.41). □

Proof of Lemma 6.9(II). Abbreviate $h_n := C(x_n)/x_n^m$ and set

$$A(x) := \frac{1}{(1-x)^{c_m}} \cdot \exp\{h_n x\} \quad \text{and} \quad R(x) := \exp\left\{-c_m x + \sum_{j\geq 2} \frac{C(x_n^j) - c_m x_n^{jm}}{j x_n^{jm}} x^j\right\}.$$

Then, from (6.35) we obtain that

$$\Pr\left[\mathcal{P}_{N_n}\right] = \frac{(x_n^m y_n)^{N_n}}{G(x_n, y_n)} \cdot [x^{N_n}] A(x) \cdot R(x).$$

We will apply Lemma 6.3 to $A(x)R(x)$. Let us first verify the conditions (6.11), (6.12). Set $\alpha_n := h_n/N_n$ and $\alpha_{n,k} := h_n/(N_n - k)$. Let $(a_n)_{n\in\mathbb{N}}$ be the sequence from (6.32). Then, as stated in Lemma 6.17,

$$\alpha_n = \frac{y_n C(x_n)}{x_n^m y_n} \frac{1}{N_n} \sim a_n$$

and

$$[x^{N_n-k}]A(x) = [x^{N_n-k}]\frac{1}{(1-x)^{c_m}}\mathrm{e}^{\alpha_{n,k}(N_n-k)x}, \quad p \in \mathbb{N}_0. \tag{6.42}$$

For (fixed) $k \in \mathbb{N}_0$ and using (6.32) we obtain that $\limsup \alpha_{n,k} = \limsup \alpha_n \leq \limsup \lambda_n^{-1} < 1$. Thus Lemma 6.4(iii) yields

$$[x^{N_n-p}]A(x) \sim \frac{\big((1-\alpha_n)N_n\big)^{c_m-1}}{\Gamma(c_m)}\mathrm{e}^{h_n}, \quad k \in \mathbb{N}_0, n \to \infty. \tag{6.43}$$

Hence condition (6.11) is fulfilled with $\rho_n \sim 1$ and $\overline{\rho} = \limsup \rho_n = 1$. We verify condition (6.12) as well. Let $\delta > 0$ be such that $(1+\delta)\limsup h_n/N_n < 1$. Such a δ exists, since $\limsup h_n/N_n = \limsup \alpha_n < 1$, as we already saw before. With this at hand we split up $\{0, \ldots, N_n\}$ into

$$B_1 := \{0, \ldots, N_n - (1+\delta)h_n - 1\} \quad \text{and} \quad B_2 := \{N_n - (1+\delta)h_n, \ldots, N_n\}.$$

Consider $k \in B_1$. Then $\limsup \alpha_{n,k} \leq \limsup \alpha_n N_n/((1+\delta)h_n) = (1+\delta)^{-1} < 1$. Further $N_n - k \to \infty$ in this case, so that we obtain from (6.42) and Lemma 6.4(iii)

$$[x^{N_n-k}]A(x) \sim \frac{\big((1-\alpha_{n,k})(N_n-k)\big)^{c_m-1}}{\Gamma(c_m)}\mathrm{e}^{h_n}.$$

This leads to

$$\frac{[x^{N_n-k}]A(x)}{[x^{N_n}]A(x)} \sim \left(\frac{1-\alpha_{n,N_n-k}}{1-\alpha_n}\right)^{c_m-1}\left(1-\frac{k}{N_n}\right)^{c_m-1} = \left(1-\frac{k}{N_n(1-\alpha_n)}\right)^{c_m-1}, \quad k \in B_1.$$

If $c_m - 1 \geq 0$, then this is at most 1. If $c_m - 1 \in (-1, 0)$, then it is at most $(1 - k/N_n(1-\alpha_n))^{-1} = 1 + k/(N_n(1-\alpha_n) - k) \leq e^{k/(N_n(1-\alpha_n)-k)}$. By assumption $N_n(1-\alpha_n) - k \to \infty$, and so for any $\varepsilon > 0$,

$$\frac{[x^{N_n-k}]A(x)}{[x^{N_n}]A(x)} \leq (1+\varepsilon)^k, \quad k \in B_1. \tag{6.44}$$

Next consider $k \in B_2$. Set $z := 1 - h_n^{-1/2}$. Then the bound (4.1) together with (6.43) yield

$$\frac{[x^{N_n-k}]A(x)}{[x^{N_n}]A(x)} \leq \frac{(1-z)^{-c_m}\exp\{h_n z\}\, z^{-(N_n-k)}}{[x^{N_n}]A(x)} = \mathcal{O}\left(N_n\left(\frac{\sqrt{h_n}}{N_n}\right)^{c_m}\exp\{-\sqrt{h_n}\}z^{k-N_n}\right). \tag{6.45}$$

We further get that for any $k \in B_2$ that there exists $-h_n \leq t \leq \delta h_n$ with $N_n - k = h_n + t$ so that

$$z^{k-N_n} \leq \exp\left\{-h_n^{-1/2}(k-N_n)\right\} = \exp\left\{\sqrt{h_n} + t/\sqrt{h_n}\right\}.$$

Since $\sqrt{h_n} = o(N_n)$ we get from (6.45)

$$\frac{[x^{N_n-k}]A(x)}{[x^{N_n}]A(x)} = o\left(N_n\exp\left\{t/\sqrt{h_n}\right\}\right) = o\left(N_n\exp\left\{\delta\sqrt{N_n}\right\}\right) \quad \text{for all } k \in B_2.$$

However, since any $k \in B_2$ satisfies $k = \Omega(N_n)$, this bound is also $\le (1+\varepsilon)^k$ for any $\varepsilon > 0$ and n sufficiently large. Together with (6.44) this finally verifies condition (6.12) of Lemma 6.3 (recall that $\rho_n \sim 1$).

We show that the remaining conditions of Lemma 6.3 are satisfied as well. By applying Lemma 6.18 we obtain that $R(x)$ converges uniformly to

$$Q(x) := \exp\left\{ -c_m x + \sum_{j\ge 2} \frac{C(\rho^j) - c_m\rho^{jm}}{j\rho^{jm}} \cdot x^j \right\}$$

on any interval $[0, a]$ with $a < \rho^{-1}$. Since $\rho^{-1} > 1$ we may even choose $\varepsilon > 0$ such that $a = (1+\varepsilon)\overline{\rho}$ and $1 = \overline{\rho} < a < \rho^{-1}$. Then Lemma 6.18 gives us the existence of the sequence $(d_n)_{n\in\mathbb{N}_0}$ with $|[x^k]R(x)| \le d_k$ and $\sum_{k\ge 1} d_k a^k < \infty$. Finally $Q(\rho_n) \sim Q(1) > 0$ for all $n \in \mathbb{N}$ so that all conditions of Lemma 6.3 are met and we deduce from (6.43)

$$\Pr[\mathcal{P}_{N_n}] \sim \frac{(x_n^m y_n)^{N_n}}{G(x_n, y_n)} \cdot Q(1) \cdot [x^{N_n}]A(x) \sim \frac{(x_n^m y_n)^{N_n}}{G(x_n, y_n)} \cdot Q(1) \cdot \frac{((1-a_n)\,N_n)^{c_m - 1}}{\Gamma(c_m)} \mathrm{e}^{h_n}.$$

Lemma 6.8(II) gives us (the asymptotics of) $G(x_n, y_n)$ and we obtain

$$\Pr[\mathcal{P}_{N_n}] \sim \frac{c_m^{c_m}}{(1-a_n)\Gamma(c_m)} \cdot \frac{(x_n^m y_n)^{N_n}}{N_n} \cdot \mathrm{e}^{h_n - y_n C(x_n)}.$$

To conclude we observe by applying Lemma 6.17

$$h_n - y_n C(x_n) = \frac{C(x_n)}{x_n^m} - y_n C(x_n) = y_n C(x_n) \cdot \frac{1 - x_n^m y_n}{x_n^m y_n} = \frac{y_n C(x_n)}{S_n} \sim c_m \frac{a_n}{1-a_n}. \tag{6.46}$$

This shows the first statement of the lemma. Next we prove the second statement (6.23). Recall that $K \equiv K_n$ is such that $K_n \to \infty$ as $n \to \infty$. From (6.34) we obtain

$$\Pr\left[\sum_{j>\ell} jP_j = K\right] = \frac{(x_n^m y_n)^K}{G^{(\ell)}(x_n, y_n)} \cdot [x^K]\frac{1}{(1-x)^{c_m}} \cdot R^{(\ell)}(x).$$

Similar to the case $\ell = 0$ we want to apply Lemma 6.3 to $(1-x)^{-c_m} R^{(\ell)}(x)$, but this time it is much easier as $(1-x)^{-c_m}$ does not depend on n. It is elementary to verify that

$$[x^K]\frac{1}{(1-x)^{c_m}} = \binom{K + c_m - 1}{K} \sim \frac{K^{c_m - 1}}{\Gamma(c_m)}, \quad \text{as } K \to \infty. \tag{6.47}$$

Hence (6.11) is fulfilled with $\rho_n = 1$ and $\overline{\rho} := \limsup \rho_n = 1$, and the explicit form readily allows us to establish condition (6.12) as well. Moreover, Lemma 6.18 asserts that $R^{(\ell)}(x)$ converges to

$$Q^{(\ell)}(x) := \exp\left\{ -c_m \sum_{1\le i\le \ell} \frac{x^j}{j} + \sum_{j>\ell} \frac{C(\rho^j) - c_m\rho^{jm}}{j\rho^{jm}} x^j \right\}$$

uniformly on any closed interval in $[0, \rho^{-m})$. Completely analogous to the case $\ell = 0$ all the remaining conditions of Lemma 6.3 are also fulfilled and we obtain that

$$\Pr\left[\sum_{j>\ell} jP_j = K\right] \sim \frac{(x_n^m y_n)^K}{G^{(\ell)}(x_n, y_n)} \cdot Q^{(\ell)}(1) \cdot \frac{K^{c_m - 1}}{\Gamma(c_m)}. \tag{6.48}$$

By a similar reformulation as in the proof of Lemma 6.8(II) we derive with Lemma 6.18

$$G^{(\ell)}(x_n, y_n) \sim c_m^{-c_m} \cdot ((1-a_n)\,N_n)^{c_m} \cdot Q^{(\ell)}(1),$$

and plugging this into (6.48) finally establishes (6.23). □

Proof of Lemma 6.10

In order to prove that $\Pr[R \geq r \mid \mathcal{P}_{N_n}]$ gets exponentially small in r as claimed in Lemma 6.10 we show as a preparation the following estimate (that is nothing else than a Chernoff-type bound).

Lemma 6.19. *Set* $\Omega_n := \{(p_1, \dots, p_{N_n}) \in \mathbb{N}_0^{N_n} : p_1 + 2p_2 \cdots + N_n p_{N_n} = N_n\}$ *and let* $\lambda > 0$ *be such that* $0 < \lambda < -\ln \rho/2$. *Then for* $(\tau_j)_{j\geq 2} := (C(x_n^j \mathrm{e}^{\lambda j}) y_n^j / \mathrm{e}^{\lambda jm})_{j \geq 2}$

$$\Pr[R \geq r \mid \mathcal{P}_{N_n}] \leq \mathrm{e}^{-\lambda r} \cdot \mathrm{e}^{-\sum_{2\leq j\leq N_n} C(x_n^j) y_n^j/j} \cdot \sum_{\mathrm{p}\in\Omega_n} \frac{\Pr[P_1 = p_1]}{\Pr[\mathcal{P}_{N_n}]} \cdot \prod_{2\leq j\leq N_n} \frac{(\tau_j/j)^{p_j}}{p_j!}, \qquad r \in \mathbb{N}.$$

Proof. The choice of λ guarantees that $\mathrm{e}^{j\lambda} x_n^j < \rho$ for all $j \geq 2$. Hence

$$\mathbb{E}\left[\mathrm{e}^{\lambda j (C_{j,i} - m)}\right] = \mathrm{e}^{-\lambda jm} \frac{\sum_{k\geq 1} c_k (x_n^j \mathrm{e}^{\lambda j})^k}{C(x_n^j)} = \frac{C(x_n^j \mathrm{e}^{\lambda j})}{C(x_n^j)\mathrm{e}^{\lambda jm}}, \qquad i \geq 1, j \geq 2.$$

Then by Bayes and the independence of the P_j's

$$\begin{aligned}
\Pr[R \geq r \mid \mathcal{P}_{N_n}] &= \sum_{\mathrm{p}\in\Omega_n} \Pr\left[R \geq r \,\middle|\, \bigcap_{1\leq j\leq N} \{P_j = p_j\}\right] \Pr\left[\bigcap_{1\leq j\leq N} \{P_j = p_j\} \,\middle|\, \mathcal{P}_{N_n}\right] \\
&= \Pr[\mathcal{P}_{N_n}]^{-1} \cdot \sum_{\mathrm{p}\in\Omega_n} \Pr\left[\sum_{j\geq 2} j \sum_{1\leq i\leq p_j} (C_{j,i} - m) \geq r\right] \prod_{1\leq j\leq N_n} \Pr[P_j = p_j]\,.
\end{aligned} \tag{6.49}$$

With Markov's inequality and the independence of the $(C_{j,i})_{j,i\geq 1}$ and $(P_j)_{j\geq 1}$ we obtain

$$\Pr\left[\sum_{j\geq 2} j \sum_{1\leq i\leq p_j} (C_{j,i} - m) \geq r\right] \leq \mathrm{e}^{-\lambda r} \prod_{2\leq j\leq N_n} \mathbb{E}\left[\mathrm{e}^{\lambda j (C_{j,i} - m)}\right]^{p_j} = \mathrm{e}^{-\lambda r} \prod_{2\leq j\leq N_n} \left(\frac{C(x_n^j \mathrm{e}^{\lambda j})}{C(x_n^j)\mathrm{e}^{\lambda jm}}\right)^{p_j}.$$

By plugging this into (6.49) and using that $P_j \sim \mathrm{Po}(C(x_n^j) y_n^j / j)$ we obtain the claimed statement. □

Proof of Lemma 6.10 in case (I). By applying Lemma 6.19 and using that the Poisson distribution is maximised at its mean, so that $\Pr[P_1 = p_1] / \Pr[\mathcal{P}_{N_n}] \leq 1$, we obtain for some $\lambda > 0$

$$\Pr[R \geq r \mid \mathcal{P}_{N_n}] \leq \mathrm{e}^{-\lambda r} \cdot \sum_{\mathrm{p}\in\Omega_n} \prod_{2\leq j\leq N_n} \frac{(\tau_j/j)^{p_j}}{p_j!}.$$

Lemma 4.4 yields that for some $A > 0$

$$\tau_j \leq c_m (x_n^m y_n)^j (1 + A x_n^j \mathrm{e}^{\lambda j}), \quad j \geq 2.$$

Lemma 6.16 states that there is a $0 < \varepsilon < 1$ such that $x_n^m y_n < 1 - \varepsilon$ for all sufficiently large n. We deduce that $\exp\{\sum_{2\leq j\leq N_n} \tau_j / j\} = \exp\{\mathcal{O}(\sum_{j\geq 2} (1-\varepsilon)^j / j)\}$ is bounded. Consequently, setting $H = \sum_{2\leq j\leq N_n} j H_j$ with $H_j \sim \mathrm{Po}(\tau_j / j)$ independent, we obtain for some (other) $A > 0$

$$\Pr[R \geq r \mid \mathcal{P}_{N_n}] \leq A \cdot \mathrm{e}^{-\lambda r} \cdot \sum_{0\leq p\leq N_n} \Pr[H = N_n - p] \leq A \cdot \mathrm{e}^{-\lambda r}.$$

□

Proof of Lemma 6.10 in case (II). Our starting point is Lemma 6.19 so that

$$\Pr[R \geq r \mid \mathcal{P}_{N_n}] \leq \mathrm{e}^{-\lambda r} \cdot \exp\left\{-\sum_{2\leq j\leq N_n} \frac{C(x_n^j)y_n^j}{j}\right\} \sum_{\mathrm{p}\in\Omega_n} \frac{\Pr[P_1 = p_1]}{\Pr[\mathcal{P}_{N_n}]} \cdot \sum_{2\leq j\leq N_n} \frac{(\tau_j/j)^{p_j}}{p_j!}$$

for $0 < \lambda < -\ln\rho/2$. Using Lemmas 6.17 and 4.4 we obtain the estimate

$$\exp\left\{\sum_{j\geq 2} \frac{C(x_n^j)y_n^j}{j}\right\} = \frac{1}{(1 - x_n^m y_n)^{c_m}} \exp\left\{-c_m x_n^m y_n + \sum_{j\geq 2} \frac{C(x_n^j)y_n^j - c_m x_n^{jm} y_n^j}{j}\right\} = \mathcal{O}(N_n^{c_m}).$$

Next let $H_j \sim \mathrm{Po}(\tau_j/j)$ be independent random variables for $j \geq 2$ and set $H := \sum_{2\leq j\leq N_n} jH_j$. Abbreviate $\Upsilon := \exp\{\sum_{2\leq j\leq N_n} \tau_j/j\}$. We obtain for some $A_1 > 0$

$$\Pr[R = r \mid \mathcal{P}_{N_n}] \leq A_1 \cdot \mathrm{e}^{-\lambda r} \cdot N^{-c_m} \cdot \Upsilon \cdot \sum_{0\leq p\leq N_n} \frac{\Pr[P_1 = p]}{\Pr[\mathcal{P}_{N_n}]} \Pr[H = N_n - p]. \tag{6.50}$$

Since $(H_j)_{j\geq 2}$ are independent

$$\Pr[H = N_n - p] = \Upsilon^{-1}[x^{N_n - p}] \exp\Big\{\sum_{j\geq 2} \tau_j x^j/j\Big\}. \tag{6.51}$$

In the last expression we actually have to restrict to $2 \leq j \leq N_n$; however, for all $0 \leq M \leq N_n$, $[x^M]\exp\{\sum_{2\leq j\leq N_n} \tau_j x^j/j\} = [x^M]\exp\{\sum_{j\geq 2} \tau_j x^j/j\}$. Then

$$\exp\Big\{\sum_{j\geq 2} \tau_j x^j/j\Big\} = \frac{1}{(1 - x_n^m y_n x)^{c_m}} \cdot \exp\left\{-c_m x_n^m y_n x + \sum_{j\geq 2} (\tau_j - c_m (x_n^m y_n)^j) \frac{x^j}{j}\right\}$$

and so, for any $0 \leq M \leq N_n$

$$[x^M]\exp\Big\{\sum_{j\geq 2} \tau_j x^j/j\Big\} = (x_n^m y_n)^M \cdot [x^M] \frac{1}{(1-x)^{c_m}} \cdot \exp\left\{-c_m x + \sum_{j\geq 2}\left(\frac{C(x_n^j \mathrm{e}^{\lambda j})}{x_n^{jm} \mathrm{e}^{\lambda jm}} - c_m\right)\frac{x^j}{j}\right\}.$$

With Lemma 4.4 we get a bound which holds uniformly for some $A > 0$ and all $j \geq 2$

$$0 < a_{n,j} := \frac{C(x_n^j \mathrm{e}^{\lambda j})}{x_n^{jm} \mathrm{e}^{\lambda jm}} - c_m \leq \frac{C(\rho^j \mathrm{e}^{\lambda j})}{\rho^{jm} \mathrm{e}^{\lambda jm}} - c_m \leq A(\rho \mathrm{e}^{\lambda})^j =: a_j.$$

Using the fact that for a power series f with non-negative coefficients the coefficients of $\mathrm{e}^{f(x)}$ can get only larger if we make the coefficients of f larger we get that, summing over $(m_1, m_2, m_3) \in \mathbb{N}_0^3$ such that $m_1 + m_2 + m_3 = M$,

$$\begin{aligned}
(x_n^m y_n)^{-M}[x^M]\exp\Big\{\sum_{j\geq 2} \tau_j x^j/j\Big\} &\leq \sum [x^{m_1}] \frac{1}{(1-x)^{c_m}} \cdot |[x^{m_2}]\mathrm{e}^{-c_m x}| \cdot \Big|[x^{m_3}]\exp\Big\{\sum_{j\geq 2} a_{n,j} x^j/j\Big\}\Big| \\
&\leq \sum [x^{m_1}] \frac{1}{(1-x)^{c_m}} \cdot [x^{m_2}]\mathrm{e}^{c_m x} \cdot [x^{m_3}]\exp\Big\{\sum_{j\geq 2} a_j x^j/j\Big\} \\
&= [x^M] \frac{1}{(1-x)^{c_m}} \exp\Big\{c_m x + \sum_{j\geq 2} a_j x^j/j\Big\} =: [x^M]A(x)\tilde{R}(x).
\end{aligned}$$

Inserting this into (6.51) we obtain $\Pr[H = N_n - p] \le \Upsilon^{-1}(x_n^m y_n)^{N_n - p}[x^{N_n-p}]A(x)\tilde{R}(x)$. The advantage of this estimate is that $\tilde{R}$ does not depend on n anymore. We have that, compare to (6.47), $[x^n]A(x) \sim n^{c_m-1}/\Gamma(c_m)$ implying that $[x^{n-1}]A(x)/[x^n]A(x) \sim 1$. In addition, $\tilde{R}(1) < \infty$, as the radius of convergence of $\tilde{R}$ is at least $(\rho e^{\lambda})^{-1} > 1$. With Lemma 4.1 we consequently obtain $[x^n]A(x)\tilde{R}(x) \sim \tilde{R}(1)n^{c_m-1}/\Gamma(c_m)$. This, on the other hand, implies that we can find a $A_2 > 0$ such that uniformly in n and $0 \le p < N_n$

$$[x^{N_n-p}]A(x)\tilde{R}(x) \le A_2(N_n - p)^{c_m-1}.$$

All in all, noting that $(x_n^m y_n)^{N_n-p} \le 1$, we get uniformly in n and $0 \le p < N_n$

$$\Pr[H = N_n - p] = \mathcal{O}\left(\Upsilon^{-1}(N_n - p)^{c_m-1}\right).$$

For the case $p = N_n$ note that the probability that H equals 0 is Υ^{-1}. Putting these pieces together into (6.50) we obtain that

$$\Pr[R \ge r \mid \mathcal{P}_{N_n}] = \mathcal{O}\left(e^{-\lambda r} \cdot N^{-c_m} \cdot \left(\sum_{0\le p<N_n} \frac{\Pr[P_1 = p]}{\Pr[\mathcal{P}_{N_n}]}(N_n - p)^{c_m-1} + \frac{\Pr[P_1 = N_n]}{\Pr[\mathcal{P}_{N_n}]}\right)\right).$$

Since according to Lemma 6.9(II) we have that $\Pr[\mathcal{P}_{N_n}] = \Theta\big((x_n^m y_n)^{N_n}/N_n\big)$ and $x_n^m y_n = 1 - \Theta(1/N_n)$ due to Lemma 6.17, we have $\Pr[\mathcal{P}_{N_n}] = \Theta(1/N_n)$. Further Lemma 6.17 yields that $\limsup y_n C(x_n)/N_n \le \limsup \lambda_n^{-1} < 1$ so that we can estimate $\Pr[P_1 = N_n] = \exp\{-\Omega(N_n)\}$ by (6.9). Hence

$$N^{-c_m}\Pr[P_1 = N_n]/\Pr[\mathcal{P}_{N_n}] = \mathcal{O}\left(N_n^{1-c_m}e^{-\Omega(N_n)}\right) = o(1).$$

With the estimates for $\Pr[\mathcal{P}_{N_n}]$ we further obtain

$$N_n^{-c_m} \cdot \sum_{0\le p<N_n} \frac{\Pr[P_1 = p]}{\Pr[\mathcal{P}_{N_n}]}(N_n - p)^{c_m-1} = \mathcal{O}\left(\sum_{0\le p<N_n} \Pr[P_1 = p]\left(1 - \frac{p}{N_n}\right)^{c_m-1}\right).$$

To finish the proof we show that the latter expression is $\mathcal{O}(1)$. This is clear for $c_m \ge 1$. If $0 < c_m < 1$ we note that there is some $\delta > 0$ such that $(1+\delta)\limsup \mathbb{E}[P_1]/N_n = (1+\delta)\limsup y_n C(x_n)/N_n \le (1+\delta)\lambda_n^{-1} < 1$ due to Lemma 6.17. Then $\sum_{0\le p\le N_n/(1+\delta)} \Pr[P_1 = p](1 - p/N_n)^{c_m-1} \le (1+\delta^{-1})^{1-c_m}$. For $N/(1+\delta) < p < N_n$, on the other hand, we get that $\sum_{N_n/(1+\delta)<p<N_n} \Pr[P_1 = p](1 - p/N_n)^{c_m-1} \le N_n^{1-c_m}\Pr[P_1 > N_n/(1+\delta)]$. This is $e^{-\Omega(N_n)}$ by (6.9) and the proof is completed. □

Proof of Lemma 6.11

This entire section is devoted to the proof of Lemma 6.11. Recall that $q = \rho e^{-\chi}$. First of all note that the probability generating function of $C_{1,1}(\chi)$ is given by $H(x) = C(qx)/C(q)$, that is, $\Pr[C_{1,1}(\chi) = k] = [x^k]H(x)$ for $k \in \mathbb{N}$. Define $K_p = K_p(\chi) := \sum_{1\le i\le p} C_{1,i}(\chi)$ and $\nu = \nu(\chi) := zC'(z)/C(z)$. Then

$$\Pr[L_p(\chi) = \mu_p(\chi) + t\sigma_p(\chi)] = \Pr[K_p = p\nu + t\sigma_p(\chi)] = [x^{p\nu + t\sigma_p(\chi)}]H(x)^p.$$

The tool of our choice for tackling this problem is the saddle-point method. Therefore we need appropriate bounds for H in $\mathbb{C}$ on a circle centred at the origin with radius close to ρ; the next lemma shows a rather diverse picture.

Lemma 6.20. *Let $\alpha > 1, 0 < \rho < 1$. Then there exist $\eta_0 > 0, c < 1, A > 0$ such that the following is true. Let $0 < \eta < \eta_0$ and set $G(x) := C(\omega x)/C(\omega)$, where $\omega = \rho \mathrm{e}^{-\eta}$. Then*

$$|G(\mathrm{e}^{\mathrm{i}\theta})| \le 1 - \frac{\alpha}{2}\left(\frac{\theta}{\eta}\right)^2 \quad \text{for any} \quad |\theta| \le \eta/(24\alpha^2). \tag{6.52}$$

Moreover,

$$|G(\mathrm{e}^{\mathrm{i}\theta})| \le c \quad \text{for any} \quad \eta/(24\alpha^2) \le |\theta| \le \pi \tag{6.53}$$

and

$$|G(\mathrm{e}^{\mathrm{i}\theta})| \le A \cdot \max\{\eta/|\theta|, \eta\} \quad \text{for any} \quad \eta \le |\theta| \le \pi. \tag{6.54}$$

Proof. We start with showing the first bound. Recall the basic inequalities

$$\cos(x) \le 1 - x^2/2 + x^4/24 \quad \text{and} \quad \sin(x) \le |x|, \qquad x \in \mathbb{R},$$

that we will use more than once. Let $\delta_1 \in (0,1)$. Let R denote the real part of $G(\mathrm{e}^{\mathrm{i}\theta})$. Then, using (A.6), for $\eta > 0$ sufficiently small,

$$R = \frac{1}{C(\omega)} \sum_{k\ge1} c_k \rho^k \mathrm{e}^{-\eta k} \cos(\theta k) \le 1 - (1-\delta_1)\frac{(\alpha+1)\alpha}{2}\left(\frac{\theta}{\eta}\right)^2 + \frac{(\alpha+3)(\alpha+2)(\alpha+1)\alpha}{12}\left(\frac{\theta}{\eta}\right)^4.$$

Further, if $|\theta| \le ((\alpha+3)(\alpha+2))^{-1/2}\sqrt{\delta_1}\eta$,

$$R \le 1 - (1-2\delta_1)\frac{(\alpha+1)\alpha}{2}\left(\frac{\theta}{\eta}\right)^2.$$

Moreover, consider the imaginary part I of $G(\mathrm{e}^{\mathrm{i}\theta})$. Then we obtain with (A.6) that for $\eta > 0$ sufficiently small

$$I^2 = \frac{1}{C(\omega)^2}\left(\sum_{k\ge1} c_k \rho^k \mathrm{e}^{-\eta k} \sin(\theta k)\right)^2 \le (1+\delta_1)\alpha^2 \cdot \left(\frac{\theta}{\eta}\right)^2.$$

Combining these bounds yields for $\delta_1 > 0$ and $|\theta| \le ((\alpha+3)(\alpha+2))^{-1/2}\sqrt{\delta_1}\eta$ that

$$|G(\mathrm{e}^{\mathrm{i}\theta})|^2 = R^2 + I^2 \le 1 - \left((1-2\delta_1)\alpha - 3\delta_1\alpha^2\right)\left(\frac{\theta}{\eta}\right)^2 + (\alpha+1)^2\alpha^2\left(\frac{\theta}{\eta}\right)^4.$$

Then, as $|\theta| \le ((\alpha+3)(\alpha+2))^{-1/2}\sqrt{\delta_1}\eta \le \sqrt{\delta_1}\eta/(\alpha+1)$, we obtain

$$|G(\mathrm{e}^{\mathrm{i}\theta})|^2 \le 1 - \left((1-2\delta_1)\alpha - 4\delta_1\alpha^2\right)\left(\frac{\theta}{\eta}\right)^2.$$

Choosing $\delta_1 = 1/(4+8\alpha)$ yields $|G(\mathrm{e}^{\mathrm{i}\theta})| \le 1 - \alpha(\theta/\eta)^2/2$, as claimed and being very generous in the bound for θ. Note that for $\alpha > 1$ we have $1/(24\alpha^2) \le \sqrt{\delta_1}/\sqrt{(\alpha+2)(\alpha+3)} \le \sqrt{\delta_1}/(\alpha+1)$ so that (6.52) follows.

We continue with the proof of (6.53). Here we will use a basic trick that was used in similar forms already long ago, see [36], where $|G(\mathrm{e}^{\mathrm{i}\theta})|$ is related to the sum of the differences of consecutive terms. Here we use the following construction. Note that

$$(1-\mathrm{e}^{-\eta}\mathrm{e}^{\mathrm{i}\theta})G(\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{C(\omega)}\sum_{k\geq 1}\mathrm{e}^{\mathrm{i}\theta k-\eta k}(\rho^k c_k - \rho^{k-1}c_{k-1}).$$

Note that $\rho^{-1}c_{k-1}/c_k = h(k-1)/h(k)\cdot(1-k^{-1})^{\alpha-1} \leq (1-k^{-1})^{\alpha-1+o(1)}$ due to (A.2). Accordingly, since $\alpha > 1$, we have for sufficiently large k that $\rho^{-1}c_{k-1} \leq c_k$. Let $\delta > 0$. Then, using (A.2), whenever k is sufficiently large,

$$\begin{aligned}|\rho^k c_k - \rho^{k-1}c_{k-1}| = \rho^k c_k\left(1-\frac{\rho^{-1}c_{k-1}}{c_k}\right) &= \rho^k c_k\left(1-\frac{h(k-1)}{h(k)}\left(1-\frac{1}{k}\right)^{\alpha-1}\right)\\ &\leq \rho^k c_k\left(1-\left(1-\frac{1}{k}\right)^{\alpha-1+\delta}\right).\end{aligned}$$

Note that $(1-x)^a \geq 1-(1+\delta)ax$ for any $a,\delta > 0$ and x sufficiently small. We thus obtain for sufficiently large k that

$$|\rho^k c_k - \rho^{k-1}c_{k-1}| \leq \frac{\alpha-1+\delta(\alpha+\delta)}{k}\rho^k c_k.$$

Using this and the triangle inequality we obtain that for any $\delta > 0$ there is some $K \in \mathbb{N}$ and $d > 0$ such that

$$|G(\mathrm{e}^{\mathrm{i}\theta})| \leq \frac{1}{C(\omega)|1-\mathrm{e}^{-\eta}\mathrm{e}^{\mathrm{i}\theta}|}\left((\alpha-1+\delta(\alpha+\delta))\sum_{k\geq K}\mathrm{e}^{-\eta k}\frac{\rho^k c_k}{k}+d\right). \tag{6.55}$$

A simple calculation reveals that

$$|1-\mathrm{e}^{-\eta}\mathrm{e}^{\mathrm{i}\theta}|^2 = 1+\mathrm{e}^{-2\eta}-2\mathrm{e}^{-\eta}\cos(\theta).$$

Since $|\theta| \in [\eta/24\alpha^2, \pi]$ the cos is maximized for $|\theta| = \eta/24\alpha^2$; using $\mathrm{e}^{-x} = 1-x+x^2/2+\mathcal{O}\left(x^3\right)$ and $\cos(x) = 1-x^2/2+\mathcal{O}\left(x^4\right)$ for $x \to 0$ and abbreviating $a = 2(24\alpha^2)^2$ we get for sufficiently small η

$$\begin{aligned}|1-\mathrm{e}^{-\eta}\mathrm{e}^{\mathrm{i}\theta}|^2 &\geq 1+\mathrm{e}^{-2\eta}-2\mathrm{e}^{-\eta}\cos(\eta/24\alpha^2)\\ &= 1+(1-2\eta+2\eta^2)-2(1-\eta+\eta^2/2)(1-\eta^2/a)+\mathcal{O}\left(\eta^3\right)\\ &= (1+2/a)\eta^2+\mathcal{O}\left(\eta^3\right)\\ &\geq (1+1/a)\eta^2.\end{aligned} \tag{6.56}$$

Moreover, since $\alpha > 1$ and using (A.6) as $\eta \to 0$

$$\sum_{k\geq K}\mathrm{e}^{-\eta k}\frac{\rho^k c_k}{k} \leq \sum_{k\geq 1}\mathrm{e}^{-\eta k}\frac{\rho^k c_k}{k} \sim \Gamma(\alpha-1)h(\eta^{-1})\eta^{-\alpha+1}.$$

All in all, if $\alpha > 1$ and using the latter inequality as well as $C(\omega) \sim \Gamma(\alpha)h(\eta^{-1})\eta^{-\alpha}$ we obtain by plugging in (6.56) into (6.55) that for any $\delta > 0$ and η sufficiently small

$$|G(\mathrm{e}^{\mathrm{i}\theta})| \leq (1+\delta)\frac{\alpha-1+\delta(\alpha+\delta)}{\Gamma(\alpha)h(\eta^{-1})\eta^{-\alpha}}\cdot\frac{\Gamma(\alpha-1)h(\eta^{-1})\eta^{-\alpha+1}}{(1+1/a)^{1/2}\eta} \leq \frac{1+\delta(1+(\alpha+\delta)/(\alpha-1))}{(1+1/a)^{1/2}}. \tag{6.57}$$

Since $\delta > 0$ was arbitrary and $a > 0$ the claim in (6.53) is established by choosing δ sufficiently small.

We complete the proof by showing (6.54). Note that there is a constant $b > 0$ such that for all $0 \le x \le \pi$

$$|\mathrm{e}^{-x} - (1-x)| \le bx^2 \quad \text{and} \quad |\cos(x) - (1 - x^2/2)| \le bx^4.$$

Applying this to any occurrence of e^x and $\cos(x)$ below we obtain that there is a $b > 0$ such that for all $|\theta| \le \pi$ and η sufficiently small

$$\big||1 - \mathrm{e}^{-\eta}\mathrm{e}^{\mathrm{i}\theta}|^2 - \theta^2\big| = \big|\big(1 + \mathrm{e}^{-2\eta} - 2\mathrm{e}^{-\eta}\cos(\theta)\big) - \theta^2\big| \le b(\eta^2 + \eta\theta^2 + \theta^4).$$

In particular, if $|\theta| \ge \eta$ we obtain that

$$\big||1 - \mathrm{e}^{-\eta}\mathrm{e}^{\mathrm{i}\theta}|^2 - \theta^2\big| \le 3b\theta^4.$$

Especially, if $|\theta| \le (6b)^{-1/2}$, then $|1-\mathrm{e}^{-\eta}\mathrm{e}^{\mathrm{i}\theta}|^2 \ge \theta^2/2$. Moreover, since $|1-\mathrm{e}^{-\eta}\mathrm{e}^{\mathrm{i}\theta}|^2$ is monotone increasing for $\theta \in [0,\pi]$ we obtain $|1-\mathrm{e}^{-\eta}\mathrm{e}^{\mathrm{i}\theta}|^2 \ge (12b)^{-1}$ for all $|\theta| \ge (6b)^{-1/2}$. Using these two last statements instead of (6.56) in (6.55) we arrive similarly to (6.57) at the desired estimate (6.54). □

The previous statement applies only when $\alpha > 1$. To handle the case $0 < \alpha \le 1$ we show the following property, which establishes that summing up iid random variables with probability generating function $G(x)$ sufficiently often we obtain a random variable based on the modified sequence $\tilde{c}_n = \tilde{h}(n)n^{\tilde{\alpha}-1}\rho^n$ for $n \in \mathbb{N}$ where $\tilde{h}$ is slowly varying and $\tilde{\alpha} > 1$. With this trick we will be able to apply Lemma 6.20 even in the case $0 < \alpha \le 1$.

Lemma 6.21. *Let $\alpha > 0$, $0 < \rho < 1$. Let $s \in \mathbb{N}$ and $Y_s := \sum_{1 \le i \le s} X_i$ where $X_1, X_2, \dots$ are iid with probability generating function $G(x) = C(\omega x)/C(\omega)$, where $\omega = \rho e^{-\eta}$. Then there exists a eventually positive, continuous and slowly varying function $\tilde{h}$ such that*

$$\Pr[Y_s = n] = \frac{\tilde{h}(n)n^{s\alpha-1}\mathrm{e}^{-\eta n}}{C(\omega)^k}, \qquad n \in \mathbb{N}.$$

Proof. Let $\alpha' = \ell\alpha$ for some $\ell \in \mathbb{N}$ and set $\beta = \alpha - 1$ as well as $\beta' = \alpha' - 1$. Let f be an eventually positive slowly varying function. We will show that

$$S := \sum_{k=1}^{n-1} f(k)k^{\beta'} \cdot h(n-k)(n-k)^{\beta} = \tilde{h}(n)n^{\beta'+\beta+1} \tag{6.58}$$

for some slowly varying $\tilde{h}$ and $n \in \mathbb{N}$ such that $\tilde{h}(n) \sim I(\beta') \cdot f(n)h(n)$ for $I(\beta') = \int_0^1 x^{\beta'}(1-x)^{\beta}dx$. Then the claimed statement follows readily by induction. In order to see (6.58) let $\varepsilon > 0$ be arbitrary and consider the (*middle*) sum

$$M := \sum_{k=\varepsilon n}^{(1-\varepsilon)n} f(k)k^{\beta'} \cdot h(n-k)(n-k)^{\beta} = f(n)h(n)n^{\beta'+\beta} \sum_{k=\varepsilon n}^{(1-\varepsilon)n} \frac{f(k)}{f(n)}\frac{h(n-k)}{h(n)}\left(\frac{k}{n}\right)^{\beta'}\left(1-\frac{k}{n}\right)^{\beta}.$$

By applying the Uniform Convergence Theorem A.1 we obtain that $f(k)/f(n)$ and $h(n-k)/h(n)$ both tend to 1 for any $k \in \{\varepsilon n, \dots, (1-\varepsilon)n\}$ as $n \to \infty$. Further note that

$$\frac{1}{(1-2\varepsilon)n+2}\sum_{k=\varepsilon n}^{(1-\varepsilon)n}\left(\frac{k}{n}\right)^{\beta'}\left(1-\frac{k}{n}\right)^{\beta}$$

is a Riemann sum of $I_\varepsilon := \int_\varepsilon^{1-\varepsilon} x^{\beta'}(1-x)^\beta dx \in \mathbb{R}$. As $\varepsilon > 0$ was arbitrary and $I_0 < \infty$

$$M \sim I_\varepsilon \cdot f(n)h(n)n^{\beta'+\beta+1}, \quad n \to \infty. \tag{6.59}$$

Next consider the (*tail*) sum. With (A.3), Corollary A.3 and $h(\varepsilon n) \sim h(n)$ as well as $f(\varepsilon n) \sim f(n)$ we obtain

$$T := \sum_{k=1}^{\varepsilon n} f(k)k^{\beta'} \cdot h(n-k)(n-k)^\beta \le \sup_{(1-\varepsilon)n \le k \le n} h(k)k^\beta \cdot \sum_{k=1}^{\varepsilon n} f(k)k^{\beta'}$$
$$= \varepsilon^{\beta'+1} \cdot \mathcal{O}\left(f(n)h(n)n^{\beta'+\beta+1}\right).$$

Analogously, consider the remaining terms

$$T' := \sum_{t=1}^{\varepsilon n} f(n-k)(n-k)^{\beta'} \cdot h(k)k^\beta = \varepsilon^{\beta+1} \cdot \mathcal{O}\left(f(n)h(n)n^{\beta'+\beta+1}\right).$$

Comparing the estimates for T and T' with (6.59) and letting $\varepsilon \to 0$ we see that $S \sim I_0 f(n)h(n)n^{\beta'+\beta+1}$. Moreover, we readily see that $\tilde{h}(n) = I_0 \cdot f(n)h(n) + o(f(n)h(n))$, which is obviously slowly varying as f, h are. □

Combining all these statements we are finally able to prove Lemma 6.11.

Proof of Lemma 6.11. For the ease of reading we repeat some definitions. Let $q = \rho e^{-\chi}$ for $\chi > 0$. We will let χ tend 0 and all the forthcoming limits are with respect to $\chi \to 0$. For $p \in \mathbb{N}$ let $K_p := \sum_{1 \le i \le p} C_{1,i}(\chi)$ and define $\nu = \nu(\chi) := \mathbb{E}[C_{1,i}(\chi)] = qC'(q)/C(q)$. Further we need $\sigma_p^2 = \sigma_p(\chi)^2 := \mathrm{Var}[K_p] = q(z^2C''(q)/C(q) + \nu - \nu^2)$. Then (6.27),(6.28) assert that $\nu \sim \alpha\chi^{-1}$ and $\sigma_p(\chi)^2 \sim \alpha p \chi^{-2}$. We will use these properties several times without explicitly referencing them. Set $M := p\nu + t\sigma_p(\chi)$. Then

$$\Pr\left[L_p(\chi) = \mu_p(\chi) + t\sigma_p(\chi)\right] = \Pr\left[K_p = M\right] = [z^M]H(z)^p, \qquad H(z) = \frac{C(qz)}{C(q)}.$$

With Cauchy's integral formula, where we integrate over an arbitrary closed curve encircling the origin,

$$\Pr\left[L_p(\chi) = \mu_p(\chi) + t\sigma_p(\chi)\right] = \frac{C(q)^{-p}}{2\pi i} \oint e^{f(z)} \frac{dz}{z}, \qquad f(z) = p \ln C(qz) - M \ln z. \tag{6.60}$$

Since we will need that several times, let us note that

$$f'(z) = p\frac{qC'(qz)}{C(qz)} - \frac{M}{z}, \quad f''(z) = p\left(\frac{q^2C''(qz)}{C(qz)} - \left(\frac{qC'(qz)}{C(qz)}\right)^2\right) + \frac{M}{z^2} \quad \text{and}$$
$$f'''(z) = p\left(\frac{q^3C'''(qz)}{C(qz)} - 3\frac{q^3C''(qz)C'(qz)}{C(qz)^2} + \left(\frac{qC'(qz)}{C(qz)}\right)^3\right) - 2\frac{M}{z^3}. \tag{6.61}$$

The subsequent proof follows a very clear route that is strewn with several technical statements (as it is typical in this area). These statements, which have self-contained proofs, are clearly marked, and the proofs are presented at the end of the section. We have in mind to apply the saddle-point method, that is, we will split the integral in (6.60) up into one dominating part, where a quadratic expansion of $f(z)$ is valid, and one negligible part. To this end, consider the saddle-point equation

$$f'(z) = 0 \iff p\frac{qzC'(qz)}{C(qz)} = M. \tag{6.62}$$

This equation has obviously a unique solution that we call $w \equiv w(\chi, p, t)$, the saddle-point. Note that $w = 1$ if $t = 0$. We claim that in general w satisfies

$$w = w(\chi) = \mathrm{e}^{\xi}, \quad \text{where} \quad \xi = \xi(\chi) = t\sigma_p^{-1} + \xi_1, \quad \xi_1 = \xi_1(\chi) = \mathcal{O}\left(t^2 p^{-1} \chi\right) \tag{SaddleAsym}$$

uniformly for all $t = o(p^{1/6})$; the self-contained proof is at the end of the section. We proceed by specifying in (6.60) the curve over which integrate. We choose the simplest curve that passes the saddle-point, i.e., the circle with radius w. By switching to polar coordinates

$$\frac{C(q)^{-p}}{2\pi\mathrm{i}} \oint \mathrm{e}^{f(z)} \frac{dz}{z} = \frac{C(q)^{-p}}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{f(w\mathrm{e}^{\mathrm{i}\theta})} d\theta. \tag{6.63}$$

In the next step we study $f(w\mathrm{e}^{\mathrm{i}\theta})$ by considering the Taylor series around $\theta = 0$. Then $f'(w) = 0$ guarantees that $f(w\mathrm{e}^{\mathrm{i}\theta}) = f(w) - w^2 f''(w)\theta^2/2 + R$ for some remainder R that should be negligible in an appropriate interval around 0. Let us substantiate this. Set for the remainder of the proof

$$\eta = \eta(\chi) := \chi - \xi \sim \chi, \qquad \theta_0 = \theta_0(\chi) := p^{-1/2+\varepsilon}\eta \text{ for some } 0 < \varepsilon < 1/6.$$

By applying Lemma 4.6 to (6.61) we obtain

$$f''(w) \sim \alpha p \eta^{-2} = \Theta(p\eta^{-2}) \qquad \text{and} \qquad f'''(w) = \mathcal{O}\left(p\eta^{-3}\right), \tag{6.64}$$

so that the saddle-point heuristic $f''(w)\theta_0^2 = \omega(1)$ and $f'''(w)\theta_0^3 = o(1)$ is fulfilled. We claim that a Gaussian expansion holds for $f(w\mathrm{e}^{\mathrm{i}\theta})$ as $\eta \to 0$ uniformly in $|\theta| < \theta_0$, that is, for $t = o(p^{1/6})$,

$$f(w\mathrm{e}^{\mathrm{i}\theta}) = f(w) - w^2 f''(w)\theta^2/2 + o(1), \tag{GaussExp}$$

the proof of which is at the end of this section. With this at hand, we split up (6.63) into the integral over the – as we will establish, dominant – arc $\{w\mathrm{e}^{\theta} : -\theta_0 \le \theta \le \theta_0\}$ and the remainder, i.e.,

$$\Pr\left[L_p = \mu_p + t\sigma_p\right] = \frac{C(q)^{-p}}{2\pi\mathrm{i}} \oint \mathrm{e}^{f(z)} \frac{dz}{z} = \frac{C(q)^{-p}}{2\pi} \left(\int_{-\theta_0}^{\theta_0} + \int_{\theta_0 \le |\theta| \le \pi} \right) \mathrm{e}^{f(w\mathrm{e}^{\mathrm{i}\theta})} d\theta =: I_1 + I_2. \tag{6.65}$$

We start with I_1. Since θ_0 is such that $\theta_0^2 f''(w) \to \infty$ and $\int_{\mathbb{R}} \mathrm{e}^{-y^2/2} dy = \sqrt{2\pi}$ we obtain due to (GaussExp)

$$I_1 \sim \frac{C(q)^{-p}}{2\pi} \int_{-\theta_0}^{\theta_0} \mathrm{e}^{f(w) - w^2 f''(w)\frac{\theta^2}{2}} d\theta \sim \frac{1}{\sqrt{2\pi}} \left(\frac{C(qw)}{C(q)}\right)^p w^{-M} \frac{1}{\sqrt{w^2 f''(w)}}. \tag{6.66}$$

We claim that for $t = o(p^{1/6})$ the first order of I_1 – the alleged dominating integral – satisfies

$$I_1 \sim \mathrm{e}^{-t^2/2} (2\pi w^2 f''(w))^{-1/2}. \tag{DomInt}$$

Due to (6.64) and using that $w \sim 1$ as well as $\eta \sim \chi$ we obtain that $w^2 f''(w) \sim \alpha p \chi^{-2} \sim \sigma_p^2$. Plugging this into (DomInt) we conclude that

$$I_1 \sim \mathrm{e}^{-t^2/2} \frac{1}{\sqrt{2\pi}\sigma_p} \sim \mathrm{e}^{-t^2/2} \frac{1}{\sqrt{2\pi}} \frac{\chi}{\sqrt{p\alpha}}.$$

This establishes the claimed first order asymptotic of $\Pr[K_p = M] = \Pr[L_p(\chi) = \mu_p(\chi) + t\sigma_p(\chi)]$. Hence, to finish the proof, we need to show that I_2 in (6.65) is negligible compared to I_1. We first reformulate

$$I_2 = \left(\frac{C(qw)}{C(q)}\right)^p w^{-M} \frac{1}{2\pi} \int_{\theta_0 \le |\theta| \le \pi} G(\mathrm{e}^{\mathrm{i}\theta})^p \mathrm{e}^{-\mathrm{i}\theta M} d\theta, \qquad G(y) = \frac{C(qwy)}{C(qw)}. \tag{6.67}$$

In light of (6.66) and (6.64), in order to show that $I_2 = o(I_1)$ we just need to show that

$$\left| \int_{\theta_0}^{\pi} G(\mathrm{e}^{\mathrm{i}\theta})^p \mathrm{e}^{-\mathrm{i}\theta M} d\theta \right| = o\left(\eta p^{-1/2}\right). \tag{6.68}$$

We first consider the case $\alpha > 1$. With Lemma 6.20 we obtain that there are $c < 1, A, b > 0$ such that for η sufficiently small

$$|G(\mathrm{e}^{\mathrm{i}\theta})| \le \begin{cases} 1 - \alpha\left(\theta/\eta\right)^2/2, & \theta \le b\eta \\ c, & \theta \ge b\eta \\ A\max\{\eta/|\theta|, \eta\}, & \theta \ge \eta \end{cases}. \tag{6.69}$$

Applying the triangle inequality yields

$$\left| \int_{\theta_0}^{\pi} G(\mathrm{e}^{\mathrm{i}\theta})^p \mathrm{e}^{-\mathrm{i}\theta M} d\theta \right| \le \left(\int_{\theta_0}^{b\eta} + \int_{b\eta}^{\min\{\pi, \eta\ln p\}} + \int_{\min\{\pi,\eta\ln p\}}^{\pi} \right) |G(\mathrm{e}^{\mathrm{i}\theta})|^p d\theta =: R_1 + R_2 + R_3.$$

Due to (6.68) all that is left to show in order to obtain $I_2 = o(I_1)$ is

$$R_i = o\left(\eta p^{-1/2}\right), \qquad i = 1, 2, 3. \tag{6.70}$$

Applying the first inequality of (6.69) and $1 - x \le \mathrm{e}^{-x}$ for sufficiently small $x > 0$ entails that

$$R_1 \le \int_{\theta_0}^{b\eta} \exp\left\{-\frac{p\alpha}{2}\left(\frac{\theta}{\eta}\right)^2\right\} d\theta = \frac{\eta}{\sqrt{p\alpha}} \int_{\sqrt{\alpha}p^{\varepsilon}}^{b\sqrt{\alpha}p^{1/2}} \mathrm{e}^{-t^2/2} dx = o\left(\eta p^{-1/2}\right)$$

showing (6.70) for $i = 1$. We proceed to R_2, where we apply the second inequality in (6.69) to obtain

$$R_2 \le \eta \ln p \cdot c^p = (\eta p^{-1/2}) \cdot (p^{1/2} \ln p \cdot c^p).$$

Since $0 < c < 1$ this expression is $o(\eta p^{-1/2})$. For the remaining case we apply the third inequality in (6.69) to obtain that

$$R_3 \le \left(\int_{\min\{1,\eta\ln p\}}^{1} + \int_{1}^{\pi} \right) |G(\mathrm{e}^{\mathrm{i}\theta})|^p d\theta \le (A\eta)^p \int_{\min\{1,\eta\ln p\}}^{1} \theta^{-p} d\theta + \pi \cdot (C\eta)^p. \tag{6.71}$$

Then, as $p \to \infty$,

$$0 \le (A\eta)^p \int_{\eta\ln p}^{1} \theta^{-p} d\theta = (A\eta)^p \left[-\frac{\theta^{-p+1}}{p}\right]_{\eta\ln p}^{1} \le \frac{A^p \eta (\ln p)^{-p+1}}{p} \le \frac{\eta}{\sqrt{p}} \cdot \left(\frac{A}{\ln p}\right)^p = o\left(\eta p^{-1/2}\right).$$

Hence all the terms on the right hand side of (6.71) are in $o(\eta p^{-1/2})$ validating (6.70) for $i = 3$. We have just demonstrated the validity of (6.70) for $i = 1, 2, 3$ and thus the assertion of Lemma 6.11 is fully proven for $\alpha > 1$.

In the case $0 < \alpha \leq 1$ we need to apply a trick in order to be able to make use of Lemma 6.20. Instead of (6.67) we write for some $s \in \mathbb{N}$ such that $s\alpha > 1$

$$I_2 = \left(\frac{C(qw)}{C(q)}\right)^p w^{-M} \frac{1}{2\pi} \int_{\theta_0}^{\pi} G(\mathrm{e}^{\mathrm{i}\theta})^{p/s} \mathrm{e}^{-\mathrm{i}\theta M} d\theta, \qquad G(y) = \left(\frac{C(qwy)}{C(qw)}\right)^s.$$

Now $G(y)$ is the generating function of a sum Y_s of s iid random variables with probability generating function $C(qwy)/C(qw)$. Hence we know from Lemma 6.21 that Y_s has probability generating function $\tilde{C}(qwy)/\tilde{C}(qw)$, where the coefficients of $\tilde{C}$ are given by

$$\tilde{c}_n = \tilde{h}(n) n^{s\alpha-1}\rho^{-n}, \qquad \tilde{h} \text{ eventually positive, continuous and slowly varying.}$$

As $s\alpha > 1$ this is exactly the setting considered in Lemma 6.20 so that adapting the proof after (6.68) is straightforward.

Proof of (SaddleAsym). Define for some (large) constant $A > 0$ the quantities $\xi_\pm = \xi_\pm(\chi) = t\sigma_p^{-1} \pm A \cdot t^2 p^{-1}\chi$ and note that $qe^{\xi_\pm} = \rho e^{-\chi+\mathcal{O}(t\chi/\sqrt{p})} < \rho$. We will show that $f'(e^{\xi_-}) < 0 < f'(e^{\xi_+})$, from which (SaddleAsym) follows immediately from the continuity of f'. To this end we compute the Taylor series of $qzC'(qz)/C(qz)$ around $z = 1$. Let $\beta > 0$ such that $q\mathrm{e}^\beta < \rho$. Then we obtain that there is some $|\delta| \in [0, \beta]$ such that

$$\frac{q\mathrm{e}^\beta C'(q\mathrm{e}^\beta)}{C(q\mathrm{e}^\beta)} = \nu + \frac{\sigma_p(\chi)^2}{p}(\mathrm{e}^\beta - 1) + R(\delta, \mathrm{e}^\beta), \qquad R(\delta, \mathrm{e}^\beta) := \frac{d^2}{dz^2}\frac{qzC'(qz)}{C(qz)}\bigg|_{z=\mathrm{e}^\delta} \frac{(\mathrm{e}^\beta-1)^2}{2}.$$

Hence, there are $|\delta_\pm| \in [0, \xi_\pm]$ such that plugging $\beta = \xi_\pm$ into the previous equation yields

$$p\frac{q\mathrm{e}^{\xi_\pm} C'(q\mathrm{e}^{\xi_\pm})}{C(q\mathrm{e}^{\xi_\pm})} = M \pm At^2\sigma_p^2 p^{-1}\chi + \mathcal{O}\left(t^2\right) + pR(\delta_\pm, \mathrm{e}^{\xi_\pm}). \tag{6.72}$$

Moreover,

$$R(\delta_\pm, \mathrm{e}^{\xi_\pm}) = \Theta\left(\frac{C'''(q\mathrm{e}^{\delta_\pm})}{C(q\mathrm{e}^{\delta_\pm})} + \frac{C''(q\mathrm{e}^{\delta_\pm})C'(q\mathrm{e}^{\delta_\pm})}{C(q\mathrm{e}^{\delta_\pm})^2} + \frac{C'(q\mathrm{e}^{\delta_\pm})^3}{C(q\mathrm{e}^{\delta_\pm})^3}\right) \cdot \xi_\pm^2,$$

where, crucially, the constants implicit in Θ do not depend on A. Using Lemma 4.6 and since $|\delta_\pm| \leq |\xi_\pm| = o(\chi_n)$ we get that

$$pR(\delta_\pm, \mathrm{e}^{\xi_\pm}) = \mathcal{O}\left(p\chi^{-3} \cdot \xi_\pm^2\right) = \mathcal{O}\left(p\chi^{-3} \cdot t^2\sigma_p(\chi)^{-2}\right) = \mathcal{O}\left(\chi^{-1}t^2\right).$$

Moreover, since $1 = o(\sigma_p^2 p^{-1}\chi)$ we obtain from (6.72)

$$p\frac{q\mathrm{e}^{\xi_\pm} C'(q\mathrm{e}^{\xi_\pm})}{C(q\mathrm{e}^{\xi_\pm})} = M \pm At^2\chi^{-1} + \mathcal{O}\left(\chi^{-1}t^2\right),$$

where, again, the constant in $\mathcal{O}$ does not depend on A. So, we may choose A large enough such that for ξ_+ this expression is $> M$ and for ξ_- it is $< M$. Together with $f'(z) = pqC'(qz)/C(qz) - M/z$ we have thus established $f'(e^{\xi_-}) < 0 < f'(e^{\xi_+})$, as claimed.

Proof of (GaussExp). By applying Taylor's theorem we obtain that for every $|\theta| \leq \theta_0$ there is a $|\zeta| \leq |\theta|$ such that

$$f(w\mathrm{e}^{\mathrm{i}\theta}) = f(w) - w^2 f''(w)\frac{\theta^2}{2} + (w^3 f'''(w\mathrm{e}^{\mathrm{i}\zeta}) + 2w^2 f''(w\mathrm{e}^{\mathrm{i}\zeta}) + wf'(w(\mathrm{e}^{\mathrm{i}\zeta}))\frac{(\mathrm{i}\theta)^3}{6}. \tag{6.73}$$

To show that the terms involving ζ are $o(1)$, we claim that

$$|C(qwe^{i\zeta})| \sim C(qw). \tag{6.74}$$

With this at hand, and as C and all its derivatives have only non-negative coefficients,

$$|\theta|^3|w^3f'''(we^{i\zeta})+2w^2f''(we^{i\zeta})+wf'(w(e^{i\zeta})| = |\theta|^3p\cdot\mathcal{O}\left(\frac{C'''(qw)}{C(qw)}+\frac{C''(qw)C'(qw)}{C(qw)^2}+\frac{C'(qw)^3}{C(qw)^3}\right).$$

Applying Lemma 4.6 and using that $|\zeta| \le |\theta| \le \theta_0 = p^{-1/2+\varepsilon}\eta$ we obtain that the last term is $\mathcal{O}\left(\theta_0^3 p\cdot\eta^{-3}\right) = \mathcal{O}(p^{-1/2+3\varepsilon})$ and, since $\varepsilon < 1/6$, we are done with the proof of (GaussExp). All is left to show is (6.74). Clearly, $|C(qwe^{i\zeta})| \le C(qw)$ since C has only non-negative coefficients. Let $A > 0$ be a (large) constant. For $1 \le k \le A\eta^{-1}$ we obtain that $k\zeta = o(1)$ for $|\zeta| \le \theta_0 = o(\eta)$ so that in this regime $\cos(\zeta k) \sim 1$. Thus

$$|C(qwe^{i\zeta})| \ge \left|\sum_{k\ge1} c_k(qw)^k\cos(\zeta k)\right| \sim |C(qw)-R_0+R_1|, \quad R_i = \sum_{k>A\eta^{-1}} c_k(qw)^k\cos(\zeta k)^i, i=0,1.$$

We estimate for $i = 0, 1$ and assisted by (A.3)

$$\begin{aligned}|R_i| &\le \sum_{k\ge A\eta^{-1}} h(k)k^{\alpha-1}e^{-\eta k} \le \sum_{k\ge A\eta^{-1}} \sup_{\ell\ge A\eta^{-1}} h(\ell)\ell^{-1}\cdot k^\alpha e^{-\eta k}\\ &\sim \frac{h(A\eta^{-1})}{A\eta^{-1}}\cdot\eta^{-\alpha}\cdot\sum_{k\ge A\eta^{-1}}(k\eta)^\alpha e^{-\eta k} \sim A^{-1}h(\eta^{-1})\eta^{-\alpha}\int_A^\infty x^\alpha e^{-x}dx.\end{aligned}$$

The integral in the previous display is finite and so, letting $A \to \infty$, we obtain that $|R_i| = o(h(\eta^{-1})\eta^{-\alpha}) = o(C(qw))$. It follows that $|C(qwe^{i\zeta})| \sim C(qw)$, that is, (6.74) is valid which finishes the proof.

Proof of (DomInt). We need to show that

$$\left(\frac{C(qw)}{C(q)}\right)^p w^{-M} \sim e^{-t^2/2}.$$

Keeping in mind that $w = e^\xi$ we have due to a standard Taylor expansion of $C(qz)/C(q)$ around $z = 1$ that there is $\delta_1 \in [0, \xi]$ such that

$$\frac{C(qw)}{C(q)} = 1+\frac{qC'(q)}{C(q)}(e^\xi-1)+\frac{q^2C''(q)}{C(q)}\frac{(e^\xi-1)^2}{2}+R', \qquad R' := \frac{q^3C'''(qe^{\delta_1})}{C(q)}\frac{(e^\xi-1)^3}{6}. \tag{6.75}$$

Since $|\delta_1| \le |\xi| = o(\chi)$, we obtain by applying Lemma 6.1 that $R' = \mathcal{O}\left(\xi^3\chi^{-3}\right) = o(p^{-1})$ for $t = o(p^{1/6})$. Further, we obtain with Lemma 6.1 that for $t = o(p^{1/6})$

$$\frac{qC'(q)}{C(q)}\xi^3 = o(p^{-1}) \quad \text{and} \quad \frac{q^2C''(q)}{C(q)}\xi^3 = o(p^{-1}).$$

Hence we may rewrite (6.75) to

$$\frac{C(qw)}{C(q)} = 1+\xi\frac{qC'(q)}{C(q)}+\frac{\xi^2}{2}\frac{qC'(q)}{C(q)}+\frac{\xi^2}{2}\frac{q^2C''(q)}{C(q)}+o(p^{-1}).$$

All the terms involving ξ in the latter equation are $o(1)$ for $t = o(p^{1/6})$. With this at hand, we obtain for such t

$$\left(\frac{C(qw)}{C(q)}\right)^p = \exp\left\{p \ln\left(1 + \xi\frac{qC'(q)}{C(q)} + \frac{\xi^2}{2}\frac{qC'(q)}{C(q)} + \frac{\xi^2}{2}\frac{q^2C''(q)}{C(q)} + o(p^{-1})\right)\right\}$$
$$\sim \exp\left\{p\xi\frac{qC'(q)}{C(q)} + p\frac{\xi^2}{2}\left(\frac{q^2C''(q)}{C(q)} - \left(\frac{qC'(q)}{C(q)}\right)^2\right)\right\} = \exp\left\{\xi p\nu + \xi^2\sigma_p^2/2\right\}.$$

Hence, recalling that $M = p\nu + t\sigma_p(\chi)$,

$$\left(\frac{C(qw)}{C(q)}\right)^p w^{-M} \sim \exp\left\{\xi p\nu + \xi^2\sigma_p^2/2 - \xi(p\nu + t\sigma_p)\right\} = \exp\left\{\xi^2\sigma_p^2/2 - \xi t\sigma_p\right\}.$$

Next we replace ξ by the expressions given in (SaddleAsym) to obtain

$$\xi^2\sigma_p^2/2 = t^2/2 + \mathcal{O}(t^4p^{-1}) \quad \text{and} \quad \xi t\sigma_p = t^2 + \mathcal{O}(t^3p^{-1/2}).$$

For $t = o(p^{1/6})$ both $\mathcal{O}$-terms vanish and (DomInt) follows. □

Proof of Corollaries 6.12 and 6.13

Proof of Corollary 6.12. We have that $x_ny_nC'(x_n) = n + \mathcal{O}(1)$ and $y_nC(x_n) = n + \mathcal{O}(1)$ due to (6.19) and Lemma 6.7(I). Hence $\mu_{N_n} = \mathbb{E}[\sum_{1\le i\le N_n} C_{1,i}] = N_nx_nC'(x_n)/C(x_n) = n + \mathcal{O}(n/N_n)$ and further $\sigma_{N_n}^2 = \text{Var}(\sum_{1\le i\le N_n} C_{1,i}) \sim n^2/(\alpha N_n)$ due to (6.28) and Lemma 6.16. We conclude that

$$\Pr\left[\sum_{1\le i\le N_n} C_{1,i} = n\right] = \Pr\left[\sum_{1\le i\le N_n} C_{1,i} = \mu_{N_n} + \sigma_{N_n}\cdot\mathcal{O}\left(N_n^{-1/2}\right)\right].$$

Then apply Lemma 6.11 for $\chi = \chi_n$ and some properly chosen $t = \mathcal{O}(N_n^{-1/2})$. Since $\sigma_{N_n}^2 \sim N_n(x_n^2C''(x_n)/C(x_n) - (x_nC'(x_n)/C(x_n))^2) \sim y_n(x_n^2C''(x_n) - (x_nC'(x_n))^2/C(x_n)) \sim y_nx_n^2C''(x_n)/(\alpha+1)$ due to Lemmas 6.1 and 6.16 the claim follows. □

Proof of Corollary 6.13. Set $\tau := \mathbb{E}[P_1] = y_nC(x_n)$ and $\tilde{n} = n - mN_n$. Let

$$B_\le := \{p \in \mathbb{N}_0 : |\tau - p| \le \sqrt{\tau}\ln\tau\} \quad \text{and} \quad B_> := \{p \in \mathbb{N}_0 : |\tau - p| > \sqrt{\tau}\ln\tau\}.$$

Then we split up

$$\Pr[L = \tilde{n}] = \left(\sum_{p\in B_\le} + \sum_{p\in B_>}\right)\Pr[L_p = \tilde{n}]\Pr[P_1 = p] =: I_1 + I_2.$$

Recall that according to (6.28) the mean and variance of L_p are such that $\mu_p = p(x_nC'(x_n)/C(x_n) - m) \sim \alpha p\chi_n^{-1}$ and $\sigma_p^2 \sim \alpha p\chi_n^{-2}$, respectively. Due to (6.19) we see

$$\mathbb{E}[L] = \tau(\mathbb{E}[C_{1,1} - m]) = \tau\left(\frac{x_nC'(x_n)}{C(x_n)} - m\right) = \tilde{n}. \tag{6.76}$$

With this at hand we reformulate for $p \in \mathbb{N}$

$$\Pr\left[L_p = \tilde{n}\right] = \Pr\left[L_p = \mu_p + t\sigma_p\right], \qquad t = t(p) = \frac{\tilde{n} - \mu_p}{\sigma_p} = (\tau - p)\frac{x_n C'(x_n)/C(x_n) - m}{\sigma_p}. \quad (6.77)$$

Let us first study I_1. For $p \in B_\leq$ we obtain from Lemma 6.1, the asymptotics for σ_p, and $p \sim \tau$ that $t = \mathcal{O}(\ln \tau)$. Since $\ln \tau = o(p^{1/6})$ we can apply Lemma 6.11 with $\chi = \chi_n$ to (6.77). We obtain for all $p \in B_\leq$

$$\Pr\left[L_p = \tilde{n}\right] \sim \mathrm{e}^{-t^2/2}(2\pi\sigma_p^2)^{-1/2} \sim \mathrm{e}^{-t^2/2}(2\pi\sigma_\tau^2)^{-1/2}$$

Further (6.10) yields for such p

$$\Pr\left[P_1 = p\right] = \Pr\left[P_1 = \tau + y\sqrt{\tau}\right] \sim \mathrm{e}^{-y^2/2}(2\pi\tau)^{-1/2}, \qquad y = y(p) = \frac{p - \tau}{\sqrt{\tau}}.$$

With this at hand we obtain

$$I_1 \sim \frac{1}{2\pi\sqrt{\tau\sigma_\tau^2}} \sum_{p \in B_\leq} \mathrm{e}^{-(t^2+y^2)/2}.$$

Set

$$\Delta := \frac{(x_n C'(x_n)/C(x_n) - m)^2}{\sigma_\tau^2} + \frac{1}{\tau} \sim \frac{\alpha + 1}{\tau}. \quad (6.78)$$

Since for all $p \in B_\leq$ we have due to (6.27) and (6.28)

$$|t^2 + y^2 - (\tau - p)^2\Delta| = (\tau - p)^2 \left(\frac{x_n C'(x_n)}{C(x_n)} - m\right)^2 \cdot \left|\frac{1}{\sigma_p^2} - \frac{1}{\sigma_\tau^2}\right| = \mathcal{O}\left((\tau - p)^3\tau^{-2}\right) = o(1).$$

Accordingly, we obtain

$$I_1 \sim \frac{1}{2\pi\sqrt{\tau\sigma_\tau^2}} \sum_{|p| \leq \sqrt{\tau}\ln\tau} \mathrm{e}^{-p^2\Delta/2}. \quad (6.79)$$

Applying (4.3) there exists Q with $|Q| \leq 3$ such that

$$\sum_{|p| \leq \sqrt{\tau}\ln\tau} \mathrm{e}^{-p^2\Delta/2} = \int_{-\sqrt{\tau}\ln\tau}^{\sqrt{\tau}\ln\tau} \mathrm{e}^{-x^2\Delta/2} dx + Q.$$

Since (6.78) guarantees that $\sqrt{\tau}\ln\tau\sqrt{\Delta} = \Theta(\ln\tau) = \omega(1)$ we compute

$$\int_{-\sqrt{\tau}\ln\tau}^{\sqrt{\tau}\ln\tau} \mathrm{e}^{-x^2\Delta/2} dx = \Delta^{-1/2} \int_{-\sqrt{\tau}\ln\tau\sqrt{\Delta}}^{\sqrt{\tau}\ln\tau\sqrt{\Delta}} \mathrm{e}^{-x^2/2} dx \sim \sqrt{\frac{2\pi\tau}{\alpha + 1}} = \omega(1).$$

Hence (6.79) yields together with the expressions for the asymptotic behaviour of χ_n and τ in Lemma 6.17 as well as $\sigma_\tau^2 \sim \alpha p \chi_n^{-2}$

$$I_1 \sim \frac{1}{\sqrt{2\pi(\alpha + 1)}\sigma_\tau} \sim \sqrt{\frac{\alpha C_0}{2\pi(\alpha + 1)} \cdot g(n - mN_n) \cdot (n - mN_n)^{-(\alpha+2)/(\alpha+1)}}.$$

By plugging in the asymptotics from Lemma 6.1 we also obtain $(\alpha + 1)\sigma_\tau^2 \sim \rho^{-m}(\alpha + 1)(x_n C''(x_n) - (x_n C'(x_n))^2/C(x_n)) \sim \rho^{-m} x_n^2 C''(x_n)$. To show that I_2 is negligible compared to I_1 we apply (6.9) to obtain the existence of $d > 0$ such that $I_2 \leq \mathrm{e}^{-d(\ln\tau)^2}$. Moreover, by applying Lemma 6.17 we obtain that for some $\delta > 0$ eventually $\tau \geq (n - mN_n)^\delta$ and in addition $I_1 \geq (n - mN_n)^{-\delta}$. From this we infer that $I_2 \leq \mathrm{e}^{-d(\ln\tau)^2} = o(1)$ and the proof is finished. □

Proof of Lemma 6.14

Proof of Lemma 6.14(I). We write $\tilde{n} := n - mN_n$. Assisted by (6.24) and using the independence of L, R we obtain

$$\Pr\left[\mathcal{E}_n \mid \mathcal{P}_{N_n}\right] = \sum_{p,r\geq 0} \Pr\left[L_p = \tilde{n} - r\right] \Pr\left[P_1 = p, R = r \mid \mathcal{P}_{N_n}\right].$$

We partition the summation into three parts. For some $b > 0$ (that we will choose appropriately) set

$$\begin{aligned} B_{\leq} &:= \{(p,r) \in \mathbb{N}_0^2 : |N_n - p| \leq \sqrt{N_n} \ln N_n, r \leq b \ln n\}, \\ B_{\cdot,>} &:= \{(p,r) \in \mathbb{N}_0^2 : r > b \ln n\} \qquad \text{and} \\ B_{>,\leq} &:= \{(p,r) \in \mathbb{N}_0^2 : |N_n - p| > \sqrt{N_n} \ln N_n, r \leq b \ln n\}. \end{aligned}$$

Then we obtain the three partial sums

$$\begin{aligned} \Pr\left[\mathcal{E}_n \mid \mathcal{P}_{N_n}\right] &= \Bigg(\sum_{(p,r)\in B_{\leq}} + \sum_{(p,r)\in B_{\cdot,>}} + \sum_{(p,r)\in B_{>,\leq}} \Bigg) \Pr\left[L_p = \tilde{n} - r\right] \Pr\left[P_1 = p, R = r \mid \mathcal{P}_{N_n}\right] \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

It will turn out that the sum over $B_{\leq}$ is essentially the whole sum. We will argue that

$$I_1 \sim \Pr\left[L_{N_n} = n\right], \quad I_2 = o(I_1) \quad \text{and} \quad I_3 = o(I_1). \tag{6.80}$$

Let us start by showing that $I_1 \sim \Pr\left[L_{N_n} = n\right]$. By (6.19) and (6.26)

$$\tilde{n} = y_n C(x_n) \left(\frac{x_n C'(x_n)}{C(x_n)} - m \right) \quad \text{and} \quad \mu_p = \mathbb{E}\left[L_p\right] = p \left(\frac{x_n C'(x_n)}{C(x_n)} - m \right). \tag{6.81}$$

Also recall (6.28) which says

$$\mu_p \sim \alpha p \chi_n^{-1} \quad \text{and} \quad \sigma_p^2 := \mathrm{Var}\left(L_p\right) \sim \alpha p \chi_n^{-2}. \tag{6.82}$$

Then

$$\Pr\left[L_p = \tilde{n} - r\right] = \Pr\left[L_p = \mu_p + (t - \tilde{r})\sigma_p\right], \qquad t = t(p) := \frac{\tilde{n} - \mu_p}{\sigma_p}, \;\; \tilde{r} = \tilde{r}(p) := \frac{r}{\sigma_p}.$$

According to Lemma 6.16 $y_n C(x_n) = N_n + \mathcal{O}(1)$. Hence with (6.81), (6.82) we obtain for $(p, r) \in B_{\leq}$ that

$$t = (y_n C(x_n) - p)(x_n C'(x_n)/C(x_n) - m)\sigma_p{}^{-1} \sim (N_n - p)\sqrt{\alpha/p} = \mathcal{O}\left(\ln N_n\right).$$

Further, as $0 \leq r \leq b \ln n$ and $\chi_n \sim \alpha n / N_n$ according to Lemma 6.16,

$$\tilde{r} = \mathcal{O}\left(r/\sqrt{p}\chi_n^{-1}\right) = o(1) \quad \text{and} \quad t\tilde{r} = o(1).$$

Since $t - \tilde{r} = o(p^{1/6})$ we may apply Lemma 6.11 with $\chi = \chi_n$ and get for all $(p, r) \in B_{\leq}$

$$\Pr\left[L_p = \tilde{n} - r\right] \sim \mathrm{e}^{-(t-\tilde{r})^2/2} \Pr\left[L_p = \mu_p\right] \sim \mathrm{e}^{-t^2/2} \Pr\left[L_p = \mu_p\right].$$

We also deduce from Lemma 6.11 and by plugging in $\chi_n \sim \alpha N_n/n$ from Lemma 6.16 that $\Pr\left[L_p = \mu_p\right] \sim \sqrt{\alpha N_n/(2\pi))}/n$ whenever $p \sim N_n$. Then Corollary 6.12 reveals that $\Pr\left[L_p = \mu_p\right] \sim \Pr\left[L_{N_n} = \tilde{n}\right]$. Hence for all $(p, r) \in B_{\leq}$

$$\Pr\left[L_p = \tilde{n} - r\right] \sim \mathrm{e}^{-t^2/2} \Pr\left[L_{N_n} = \tilde{n}\right]. \tag{6.83}$$

With this at hand, we simplify

$$\Pr\left[L_{N_n} = \tilde{n}\right]^{-1} \cdot I_1 = \sum_{|N_n - p| \leq \sqrt{N_n} \ln N_n} \mathrm{e}^{-t^2/2} \Pr\left[P_1 = p \mid \mathcal{P}_{N_n}\right] + \mathcal{O}\left(\Pr\left[R \geq b \ln n \mid \mathcal{P}_{N_n}\right]\right).$$

Due to Lemma 6.10 there is some $0 < a < 1$ such that $\Pr\left[P_1 = p, R = r \mid \mathcal{P}_{N_n}\right] \leq a^{b \ln n}$ for all $r > b \ln n$, so that the $\mathcal{O}$-term is $\mathcal{O}\left(n^{-1}\right)$ for b sufficiently large. Hence

$$I_1 \sim \Pr\left[L_{N_n} = \tilde{n}\right] \sum_{|N_n - p| \leq \sqrt{N_n} \ln N_n} \mathrm{e}^{-t^2/2} \Pr\left[P_1 = p \mid \mathcal{P}_{N_n}\right] + o\left(\frac{\sqrt{N_n}}{n}\right). \tag{6.84}$$

Next we show that the main contribution to the sum in (6.84) is given by a very small range, namely $|N_n - p| \leq \ln N_n$. For that recall (6.81) and set

$$\Delta_p := \frac{t^2}{(\tau - p)^2} = \frac{\left(x_n C'(x_n)/C(x_n) - m\right)^2}{\sigma_p^2}.$$

It is true that $\Pr\left[P_1 = p \mid \mathcal{P}_{N_n}\right] = 0$ for $p > N_n$. Hence we can rewrite the sum in (6.84) as

$$\sum_{|N_n - p| \leq \sqrt{N_n} \ln N_n} \mathrm{e}^{-t^2/2} \Pr\left[P_1 = p \mid \mathcal{P}_{N_n}\right] = \sum_{q=0}^{\sqrt{N_n} \ln N_n} \mathrm{e}^{-(\tau - N_n + q)^2 \Delta_{N_n - q}/2} \Pr\left[P_1 = N_n - q \mid \mathcal{P}_{N_n}\right]. \tag{6.85}$$

Disassemble

$$\Pr\left[P_1 = N_n - q \mid \mathcal{P}_{N_n}\right] = \frac{1}{\Pr\left[\mathcal{P}_{N_n}\right]} \Pr\left[P_1 = N_n - q\right] \Pr\left[\sum_{j \geq 2} j P_j = q\right].$$

We known that the density of a Poisson random variable P_1 is maximised at its mean, and so $\Pr\left[P_1 = N_n - q\right] \leq \Pr\left[P_1 = \tau\right]$ for any $q \in \mathbb{N}$ and further $\Pr\left[P_1 = \tau\right] \sim \Pr\left[\mathcal{P}_{N_n}\right]$ with Lemma 6.9(I). With Lemma 6.9(I) it also follows that $\Pr\left[\sum_{j \geq 2} j P_j = q\right] = \mathcal{O}\left(a^q\right)$ for some $0 < a < 1$ as $q \to \infty$. Thus, we obtain

$$\sum_{\ln N_n < q < \sqrt{N_n} \ln N_n} \mathrm{e}^{-(\tau - N_n + q)^2 \Delta_{N_n - q}/2} \Pr\left[P_1 = N_n - q \mid \mathcal{P}_{N_n}\right] \leq \sum_{q > \ln N_n} \Pr\left[\sum_{j \geq 2} j P_j = q\right] = o(1). \tag{6.86}$$

Next we consider the range $0 \leq q \leq \ln N_n$. Here we have that $\Pr\left[P_1 = N_n - q\right] \sim \Pr\left[P_1 = \tau\right]$ since $\tau = N_n + \mathcal{O}(1)$ and by applying (6.10). Further note that $\Delta_{N_n - q} \sim \alpha/N_n$ for $|N_n - p| \leq \ln N_n$ according to (6.82) for that range of q such that $(\tau - N_n + q)^2 \cdot \Delta_{N_n - q} \sim \alpha(\ln N_n)^2/N_n = o(1)$. Consequently, with the same $0 < a < 1$ from (6.86),

$$\sum_{0 \le q \le \ln N_n} \mathrm{e}^{-(\tau - N_n + q)^2 \Delta_{N-q}/2} \Pr\left[P_1 = N_n - q \mid \mathcal{P}_{N_n}\right] \sim \sum_{0 \le q \le \ln N_n} \Pr\left[\sum_{j \ge 2} jP_j = q\right]$$
$$= 1 - \mathcal{O}\left(a^{\ln N_n}\right) \sim 1. \tag{6.87}$$

Plugging (6.86) and (6.87) into (6.85) and then into (6.84) yields $I_1 \sim \Pr[L_{N_n} = \tilde{n}] + o(\sqrt{N_n}/n)$. Since $\Pr[L_{N_n} = n] = \Theta(\sqrt{N_n}/n)$ due to Corollary 6.12 the first part in (6.80) follows.

Next we dedicate ourselves to showing the remaining claims in (6.80). Since $I_1 = \Theta(\sqrt{N_n}/n)$ we need to prove that I_2 and I_3 are in $o(\sqrt{N_n}/n)$. Start with I_2. According to Lemma 6.10 we obtain that there exists $0 < a < 1$ yielding for b sufficiently large

$$I_2 \le \sum_{(p,r) \in B_{\cdot,>}} \Pr\left[P_1 = p, R = r \mid \mathcal{P}_{N_n}\right] \le \Pr\left[R \ge r \mid \mathcal{P}_{N_n}\right] \le a^{b \ln n} = o\left(\frac{\sqrt{N_n}}{n}\right). \tag{6.88}$$

Next we treat I_3. We observe that $\Pr\left[|N_n - P_1| > \sqrt{N_n} \ln N_n\right] / \mathcal{P}_{N_n} = o(1)$ according to Lemma 6.9(I) and (6.9). Then Lemma 6.9(I) entails that there is some $0 < a < 1$ such that

$$I_3 \le \sum_{|N_n - p| > \sqrt{N_n} \ln N_n} \Pr\left[P_1 = p \mid \mathcal{P}_n\right] = o\big(\min\{a, y_n\}^{\sqrt{N_n} \ln N_n}\big). \tag{6.89}$$

If $N_n \ge (\ln n)^3$, then this is certainly in $o(\sqrt{N_n}/n)$. If $N_n \le (\ln n)^3$ and $N_n \to \infty$, then $\lambda_n \le (\ln n)^3/N_n^*$ and from Lemma 6.16 we obtain that $y_n \le n^{-c}$ for some $c > 0$ and all sufficiently large n. Plugging this into (6.89) yields $I_3 = n^{-\omega(1)}$ and the proof is finished. □

Proof of Lemma 6.14(II). Let $\tilde{n} = n - mN_n$. With (6.24) we get

$$\Pr\left[\mathcal{E}_n \mid \mathcal{P}_{N_n}\right] = \sum_{p,r \ge 0} \Pr\left[L_p = \tilde{n} - r\right] \Pr\left[P_1 = p, R = r \mid \mathcal{P}_{N_n}\right].$$

Set $\tau := \mathbb{E}[P_1] = y_n C(x_n)$. We partition the summation regime into three parts, namely, for some constant $b > 0$ (which we need to choose sufficiently large later) we define

$$B_{\le} := \{(p,r) \in \mathbb{N}_0^2 : |p - \tau| \le \sqrt{\tau} \ln \tau, r \le b(\ln \tilde{n})^2\},$$
$$B_{\cdot,>} := \{(p,r) \in \mathbb{N}_0^2 : r > b(\ln \tilde{n})^2\} \quad \text{and}$$
$$B_{>,\le} := \{(p,r) \in \mathbb{N}_0^2 : |p - \tau| > \sqrt{\tau} \ln \tau, r \le b(\ln \tilde{n})^2\}.$$

Then

$$\Pr\left[\mathcal{E}_n \mid \mathcal{P}_{N_n}\right] = \left(\sum_{(p,r) \in B_{\le}} + \sum_{(p,r) \in B_{\cdot,>}} + \sum_{(p,r) \in B_{>,\le}}\right) \Pr\left[L_p = \tilde{n} - r\right] \Pr\left[P_1 = p, R = r \mid \mathcal{P}_{N_n}\right]$$
$$=: I_1 + I_2 + I_3.$$

We will show that the sum over $B_{\le}$ is dominant compared to the negligible sums over $B_{\cdot,>}$ and $B_{>,\le}$ as n tends to infinity, that is,

$$I_1 \sim \Pr\left[L = \tilde{n}\right], \quad I_2 = o(I_1) \quad \text{and} \quad I_3 = o(I_1). \tag{6.90}$$

Let us first determine I_1. Set $\mu_p := \mathbb{E}[L_p] = p(x_n C'(x_n)/C(x_n) - m)$ and $\sigma_p^2 := \mathrm{Var}(L_p)$. Due to (6.28) the asymptotics of these expressions are $\mu_p \sim \alpha p \chi_n^{-1}$ and $\sigma_p^2 \sim \alpha p \chi_n^{-2}$. We observe that for any p, r, compare also to (6.76),

$$\Pr[L_p = \tilde{n} - r] = \Pr[L_p = \mu_p + (t - \tilde{r})\sigma_p], \ t := t(p) = (\tau - p)\frac{x_n C'(x_n)/C(x_n) - m}{\sigma_p}, \ \tilde{r} := \tilde{r}(p) = \frac{r}{\sigma_p}.$$

We want to apply Lemma 6.11 for $\chi = \chi_n$. Using (6.6), for $(p, r) \in B_{\leq}$ we obtain that $t = \mathcal{O}(\ln \tau) = o(p^{1/6})$. Moreover, plugging in the asymptotics of τ and χ_n from Lemma 6.17 and noting that g grows slower than any polynomial we obtain

$$\tilde{r} = \mathcal{O}\left((\ln \tilde{n})^2 \tau^{-1/2} \chi_n\right) = \mathcal{O}\left((\ln \tilde{n})^2 \sqrt{g(\tilde{n})}\tilde{n}^{-(\alpha+1)/(2(\alpha+1))}\right) = o(1) \tag{6.91}$$

implying $t - \tilde{r} = o(p^{1/6})$. So, by Lemma 6.11 we obtain for all $(p, r) \in B_{\leq}$

$$\frac{\Pr[L_p = \tilde{n} - r]}{\Pr[L_p = \tilde{n}]} = \frac{\Pr[L_p = \mu_p + (t - \tilde{r})\sigma_p]}{\Pr[L_p = \mu_p + t\sigma_p]} \sim \mathrm{e}^{-((t-\tilde{r})^2 + t^2)/2}. \tag{6.92}$$

Since $t = \mathcal{O}(\ln \tau) = \mathcal{O}(\ln \tilde{n})$ due to Lemma 6.17, Equation (6.91) also implies that $t\tilde{r} = o(1)$. Thus we get due to (6.92) for all $(p, r) \in B_{\leq}$

$$\Pr[L_p = \tilde{n} - r] \sim \Pr[L_p = \tilde{n}].$$

Accordingly,

$$I_1 \sim \sum_{|p-\tau| \leq \sqrt{\tau}\ln\tau} \Pr[L_p = \tilde{n}] \sum_{0 \leq r \leq b(\ln\tilde{n})^2} \Pr[P_1 = p, R = r \mid \mathcal{P}_{N_n}]. \tag{6.93}$$

Next we claim that the sum over r equals asymptotically $\Pr[P_1 = p]$. For any p we have

$$\sum_{0 \leq r \leq b(\ln\tilde{n})^2} \Pr[P_1 = p, R = r \mid \mathcal{P}_{N_n}] = \Pr[P_1 = p \mid \mathcal{P}_{N_n}] - \sum_{r > b(\ln\tilde{n})^2} \Pr[P_1 = p, R = r \mid \mathcal{P}_{N_n}]. \tag{6.94}$$

According to Lemma 6.10 there is some $0 < a < 1$ such that

$$\sum_{r > b(\ln\tilde{n})^2} \Pr[P_1 = p, R = r \mid \mathcal{P}_{N_n}] = \mathcal{O}\left(a^{b(\ln\tilde{n})^2}\right).$$

We further obtain with Lemma 6.9(II) for any $(p, r) \in B_{\leq}$

$$\begin{aligned}\Pr[P_1 = p \mid \mathcal{P}_{N_n}] &= \frac{\Pr\left[\sum_{j\geq 2} jP_j = N_n - p\right]}{\Pr[\mathcal{P}_{N_n}]}\Pr[P_1 = p] \\ &\sim \frac{\exp\left\{-c_m \frac{a_n}{1-a_n}\right\}}{(1-a_n)^{c_m - 1}} \cdot (x_n^m y_n)^{-p} \cdot \left(\frac{N_n - p}{N_n}\right)^{c_m - 1} \cdot \Pr[P_1 = p].\end{aligned} \tag{6.95}$$

Since according to Lemma 6.17 we have $p \sim \tau = y_n C(x_n) \sim a_n \cdot N_n$ we get that

$$\left(\frac{N_n - p}{(1 - a_n)N_n}\right)^{c_m - 1} \sim 1.$$

Further, Lemma 6.17 reveals that $\limsup a_n \leq \limsup \lambda_n^{-1} < 1$ and $x_n^m y_n/(1-x_n^m y_n) \sim (1-a_n)N_n/c_m$ implying $x_n^m y_n = 1 - c_m(1-a_n)^{-1}N_n^{-1} + o((1-a_n)^{-1}N_n^{-1})$, so that

$$(x_n^m y_n)^{-p} = \exp\{-p\ln(x_n^m y_n)\} = \exp\left\{c_m \frac{a_n}{1-a_n} + \mathcal{O}\left(\frac{a_n}{(1-a_n)^2}N_n^{-1}\right)\right\} \sim \exp\left\{c_m \frac{a_n}{1-a_n}\right\}.$$

Plugging the asymptotic identities in the previous two displays into (6.95) we deduce for $(p,r) \in B_{\leq}$ that $\Pr[P_1 = p \mid \mathcal{P}_{N_n}] \sim \Pr[P_1 = p]$. Since p differs at most by $\sqrt{\tau}\ln\tau$ from the mean τ of P_1 we further obtain by (6.10) that $\Pr[P_1 = p] \sim 1/\sqrt{2\pi\tau}\exp\left\{-\mathcal{O}\left((\ln\tau)^2\right)\right\}$. From (A.2) and Lemma 6.17 we deduce that for any $\delta > 0$ eventually $\tau \leq \tilde{n}^{\alpha/(\alpha+1)+\delta}$. Hence $\Pr[P_1 = p] = \exp\left\{-\mathcal{O}\left((\ln\tilde{n})^2\right)\right\} = \omega(a^{b(\ln\tilde{n})^2})$ for b sufficiently large so that the expression in (6.94) is asymptotically given by $\Pr[P_1 = p]$ for $|\tau - p| \leq \sqrt{\tau}\ln\tau$. So far we have shown that (6.93) can be asymptotically simplified to

$$I_1 \sim \sum_{|p-\tau|\leq\sqrt{\tau}\ln\tau} \Pr[L_p = \tilde{n}]\Pr[P_1 = p],$$

where this sum in turn equals

$$I_1 \sim \Pr[L = \tilde{n}] - \sum_{|p-\tau|>\sqrt{\tau}\ln\tau} \Pr[L_p = \tilde{n}]\Pr[P_1 = p]. \tag{6.96}$$

With (6.9) and again using that $\ln\tau = \Omega(\ln\tilde{n})$ we obtain for some $d > 0$

$$\sum_{|p-\tau|>\sqrt{\tau}\ln\tau} \Pr[L_p = \tilde{n}]\Pr[P_1 = p] \leq \Pr\left[|P_1 - \tau| > \sqrt{\tau}\ln\tau\right] \leq \mathrm{e}^{-d(\ln\tau)^2} = \mathrm{e}^{-\Omega((\ln\tilde{n})^2)}.$$

Applying Corollary 6.13 and once again (A.2) we get for some $c > 0$ that

$$\Pr[L = \tilde{n}] = \omega(\tilde{n}^{-c}) = \omega(\mathrm{e}^{-\Omega((\ln\tilde{n})^2)}),$$

delivering the first part of (6.90) in light of (6.96). We continue by applying Lemma 6.10 to I_2, i.e. with $0 < a < 1$ from before we have

$$I_2 \leq \sum_{r>b(\ln\tilde{n})^2} \Pr[R = r \mid \mathcal{P}_{N_n}] = \mathcal{O}\left(a^{b(\ln\tilde{n})^2}\right) = o(\tilde{n}^{-c}) = o\left(\Pr[L = \tilde{n}]\right)$$

showing the second part of (6.90). Next we show that $I_3 = o(I_1)$. Let $\varepsilon > 0$ be such that $(1+\varepsilon)\limsup \tau/N_n < 1$. We can find such an ε because $\limsup \tau/N_n \leq \limsup \lambda_n^{-1} < 1$ due to Lemma 6.17. Then

$$I_3 \leq \left(\sum_{|p-\tau|>\sqrt{\tau}\ln\tau, p<N_n/(1+\varepsilon)} + \sum_{N_n/(1+\varepsilon)\leq p\leq N_n}\right) \Pr[P_1 = p \mid \mathcal{P}_{N_n}] =: R_1 + R_2.$$

Since $\liminf(1-a_n) > 0$, see Lemma 6.17, we get analogous to (6.95) that

$$R_1 = \mathcal{O}\left(\sum_{|p-\tau|>\sqrt{\tau}\ln\tau, p<N_n/(1+\varepsilon)} \Pr[P_1 = p]\,(x_n^m y_n)^{-p}\left(1 - \frac{p}{N_n}\right)^{c_m-1}\right).$$

Further $x_n^m y_n = 1 - \Theta(N_n^{-1})$ due to Lemma 6.17 by which we conclude for $p < N_n/(1+\varepsilon)$ that $(x_n^m y_n)^{-p} = \mathcal{O}(1)$. For that range of p we also have that $1 - p/N_n = \Theta(1)$ so that with (6.9) there is some $d > 0$ yielding

$$R_1 = \mathcal{O}\left(\Pr\left[|P_1 - \tau| > \sqrt{\tau}\ln\tau\right]\right) = \mathcal{O}\left(\mathrm{e}^{-d(\ln\tau)^2}\right) = o\left(\Pr[L = \tilde{n}]\right).$$

We proceed by treating R_2. Here we estimate $\Pr[P_1 = p, R = r \mid \mathcal{P}_{N_n}] \le \Pr[P_1 = p]/\Pr[\mathcal{P}_{N_n}]$. With Lemma 6.9(II) we further obtain $\Pr[\mathcal{P}_{N_n}] = \Theta(N_n^{-1})$. Setting $s = s(n) := (N_n/(1+\varepsilon) - \tau)/\sqrt{\tau}$ an application of (6.9) gives for some $d > 0$

$$R_2 \le \frac{\Pr[P_1 > N_n/(1+\varepsilon)]}{\Pr[\mathcal{P}_{N_n}]} = \mathcal{O}\left(N_n \Pr\left[P_1 > \tau + s\sqrt{\tau}\right]\right) = \mathcal{O}\left(N_n \mathrm{e}^{-ds\min\{s,\sqrt{\tau}\}}\right).$$

By the choice of ε we have that $s = N_n/\sqrt{\tau}((1+\varepsilon)^{-1} - \tau/N_n) = \Theta(N_n/\sqrt{\tau})$. This lets us conclude that $s\min\{s,\sqrt{\tau}\} = \Theta(\min\{N_n^2/\tau, N_n\}) = \Omega(N_n) = \Omega(N_n^*)$. Due to (2.16) and (A.2) we know that $N_n^* = n^{\alpha/(\alpha+1)+o(1)}$ leading to $R_2 = n^{-\omega(1)}$ and the proof is finished. □

Proof of Lemma 6.15

Proof of Lemma 6.15(I). Together with the basic fact that $[x^n]A(ax) = a^n[x^n]A(x)$ for any power series A we obtain with Lemma 6.14(I)

$$\Pr\left[\sum_{1\le i\le N_n} C_{1,i} = n\right] = [x^n]\frac{C(x_n x)^{N_n}}{C(x_n)^{N_n}} = \frac{x_n^n}{C(x_n)^{N_n}}[x^n]C(x)^{N_n}. \tag{6.97}$$

Further as $S_n = \mathcal{O}(1)$ according to Lemma 6.16

$$(y_n C(x_n))^{N_n} = (N_n - c_m S)^{N_n} = N_n^{N_n}\left(1 - c_m\frac{S_n}{N_n}\right)^{N_n} \sim N_n^{N_n}\exp\{-c_m S_n\}. \tag{6.98}$$

Lemma 6.15(I) follows since Lemma 6.16 implies that $\mathrm{e}^{c_m S} \sim \mathrm{e}^{c_m\rho^m y_n/(1-\rho^m y_n)}$ and by applying Stirling's formula to $N_n^{N_n}$. □

Proof of Lemma 6.15(II). First we argue that $\tau = y_n C(x_n)$ and $\tilde{\tau} := C(x_n)/x_n^m$ differ by $-c_m a_n/(1-a_n) + o(1) = \mathcal{O}(1)$. This is true since $\tau - \tilde{\tau} = y_n C(x_n)(1 - (x_n^m y_n)^{-1})$ and Lemma 6.17 gives $y_n C(x_n) \sim a_n N_n$ as well as $1 - (x_n^m y_n)^{-1} = -S_n^{-1} \sim -c_m/((1-a_n)N_n)$. Thus by replacing τ by $\tilde{\tau}$, P_1 by $\tilde{P}_1 \sim \mathrm{Po}(\tilde{\tau})$ and L by $\tilde{L} := \sum_{1\le i\le \tilde{\tau}}(C_{1,i} - m)$ in the proof of Corollary 6.13 we obtain

$$\Pr\left[\tilde{L} = n - mN_n\right] \sim \frac{1}{\sqrt{2\pi\rho^{-m}x_n^2 C''(x_n)}} \sim \Pr[L = n - mN_n].$$

As the probability generating function of $C_{1,1} - m$ is given by $x^{-m}C(x_n x)/C(x_n)$ we obtain

$$\Pr[L = n - mN_n] \sim \Pr\left[\tilde{L} = n - mN_n\right] = [x^{n-mN_n}]\exp\left\{\frac{C(x_n)}{x_n^m}\left(x^{-m}\frac{C(x_n x)}{C(x_n)} - 1\right)\right\}.$$

Next we use the basic fact that $[x^n]bF(ax) = a^n b[x^n]F(x)$ for any series F and $a, b \in \mathbb{R}$. Hence

$$\Pr[L = n - mN_n] \sim x_n^{n-mN_n}\exp\left\{-\frac{C(x_n)}{x_n^m}\right\}[x^{n-mN_n}]\exp\left\{\frac{C(x)}{x^m}\right\}.$$

□

Proof of Theorem 2.6

Proof of Theorem 2.6. Let z_n be the solution to $z_n C'(z_n) = n$. Clearly $z_n = \rho e^{-\eta_n}$ with $\eta_n \to 0$ as $n \to \infty$, so that by Lemma 4.6

$$\eta_n \sim (\Gamma(\alpha+1) h(\eta_n^{-1}))^{1/(\alpha+1)} n^{-1/(\alpha+1)}.$$

Since (2.13) implies that $c_n/c_{n-1} \sim \rho^{-1}$ we have due to [7, Cor. 4.3] that

$$\frac{[z^n] \exp\{C(z)\}}{[z^{n-1}] \exp\{C(z)\}} \sim \rho^{-1}.$$

Further $0 < \rho < 1$ implies that the radius of convergence of $\sum_{j \geq 2} C(z^j)/j$ is greater than ρ. Since the radius of convergence of $\exp\{C(z)\}$ is ρ we deduce from Lemma 4.1

$$g_n = [z^n] \exp\left\{ C(z) + \sum_{j \geq 2} \frac{C(z^j)}{j} \right\} \sim \exp\left\{ \sum_{j \geq 2} \frac{C(\rho^j)}{j} \right\} [z^n] \exp\{C(z)\}, \qquad \text{as } n \to \infty. \tag{6.99}$$

By virtue of this the task of determining g_n reduces to computing the coefficient of $\exp\{C(z)\}$. In what follows we consider the one-parametric Boltzmann model with the parameter z_n. We need the following notation. Let P be a $\mathrm{Po}(C(z_n))$ distributed random variable and $C_1, C_2, \ldots$ iid copies of $\Gamma C(z_n)$, that is,

$$\Pr[C_1 = k] = \frac{c_k z_n^k}{C(z_n)}, \qquad k \in \mathbb{N}.$$

Further we need the sum of these random variables $K_p := \sum_{1 \leq i \leq p} C_i$ for $p \in \mathbb{N}_0$ and its randomly stopped version $K := K_P$. Noting that the probability generating functions of P and C_1 are given by $\exp\{C(z_n)(z-1)\}$ and $C(z_n z)/C(z_n)$, respectively, we reformulate

$$[z^n] e^{C(z)} = z_n^{-n} e^{C(z_n)} [z^n] \exp\left\{ C(z_n) \left(\frac{C(z_n z)}{C(z_n)} - 1 \right) \right\} = z_n^{-n} e^{C(z_n)} \Pr[K = n]. \tag{6.100}$$

Write for short $\tau := C(z_n)$ and consider the sets

$$B_{\leq} := \{p \in \mathbb{N}_0 : |p - \tau| \leq \sqrt{\tau} \ln n\} \quad \text{and} \quad B_{>} := \{p \in \mathbb{N}_0 : |p - \tau| > \sqrt{\tau} \ln n\}.$$

With these definitions at hand we split up

$$\Pr[K = n] = \left(\sum_{p \in B_{\leq}} + \sum_{p \in B_{>}} \right) \Pr[K_p = n] \Pr[P = p] =: I_1 + I_2. \tag{6.101}$$

We begin with estimating I_1. With the help of Lemma 4.6 we establish the identities

$$\nu_p := \mathbb{E}[K_p] = p \frac{z_n C'(z_n)}{C(z_n)} \sim \alpha p \eta_n^{-1} \quad \text{and}$$

$$\sigma_p^2 := \mathrm{Var}(K_p) = p \left(\frac{z_n^2 C''(z_n) + z_n C'(z_n)}{C(z_n)} - \left(\frac{z_n C'(z_n)}{C(z_n)} \right)^2 \right) \sim \alpha p \eta_n^{-2}.$$

Define $L_p = K_p - mp$ and $\mu_p = \mathbb{E}[L_p] = \nu_p - mp$ implying $\{L_p = \mu_p + d\} = \{K_p = \nu_p + d\}$ for any $d \in \mathbb{R}$. Let

$$t = t(p) := \frac{(\tau - p) z_n C'(z_n)/C(z_n)}{\sigma_p}.$$

With this definition of t we have $n = \tau z_n C'(z_n)/C(z_n) = \nu_p + t\sigma_p$. Further $t = \mathcal{O}(\ln n)$ for $p \in B_\leq$ so that Lemma 6.11 is applicable with $\chi = \eta_n$ and we obtain

$$\Pr[K_p = n] = \Pr[L_p = \mu_p + t\sigma_p] \sim \mathrm{e}^{-t^2/2}(2\pi\sigma_\tau^2)^{-1/2}. \tag{6.102}$$

Note that we used $\sigma_p \sim \sigma_\tau$ for $p \sim \tau$ in the latter display. Next we observe due to (6.10) that for $p \in B_\leq$ and $s = s(p) := (p-\tau)/\sqrt{\tau} = \mathcal{O}(\ln n)$

$$\Pr[P = p] = \Pr\left[P = \tau + s\sqrt{\tau}\right] \sim (2\pi\tau)^{-1/2}\mathrm{e}^{-s^2/2}.$$

Plugging this and (6.102) into I_1 defined in (6.101) yields

$$I_1 \sim \frac{1}{2\pi}(\tau\sigma_\tau^2)^{-1/2} \sum_{p\in B_\leq} \mathrm{e}^{-(s^2+t^2)/2}. \tag{6.103}$$

Set

$$\Delta := \frac{(z_n C'(z_n)/C(z_n))^2}{\sigma_\tau^2} + \frac{1}{\tau}.$$

Note that

$$|s^2 + t^2 - (\tau-p)^2\Delta| = (\tau-p)^2\left(\frac{z_n C'(z_n)}{C(z_n)}\right)^2\left|\frac{1}{\sigma_p^2} - \frac{1}{\sigma_\tau^2}\right| = \mathcal{O}\left((\ln n)^3\frac{\tau}{\tau^{3/2}}\right) = o(1).$$

From (6.103) we then obtain the asymptotic identity

$$I_1 \sim \frac{1}{2\pi}(\tau\sigma_\tau^2)^{-1/2}\sum_{p\in B_\leq}\mathrm{e}^{-(\tau-p)^2\Delta/2} = \frac{1}{2\pi}(\tau\sigma_\tau^2)^{-1/2}\sum_{|p|\leq\sqrt{\tau}\ln n}\mathrm{e}^{-p^2\Delta/2}. \tag{6.104}$$

Using (4.3) we obtain that there exists Q with $|Q| \leq 3$ such that

$$\sum_{|p|\leq\sqrt{\tau}\ln n}\mathrm{e}^{-p^2\Delta/2} = \int_{-\sqrt{\tau}\ln n}^{\sqrt{\tau}\ln n}\mathrm{e}^{-x^2\Delta/2}dx + Q. \tag{6.105}$$

By a change of variables and since $\sqrt{\Delta\tau}\ln n = \Theta(\ln n) = \omega(1)$

$$\int_{-\sqrt{\tau}\ln n}^{\sqrt{\tau}\ln n}\mathrm{e}^{-x^2\Delta/2}dx = \Delta^{-1/2}\int_{-\sqrt{\Delta\tau}\ln n}^{\sqrt{\Delta\tau}\ln n}\mathrm{e}^{-x^2/2}dx \sim \Delta^{-1/2}\sqrt{2\pi}.$$

Combined with Equations (6.104) and (6.105) we readily obtain that $I_1 \sim (2\pi\tau\sigma_\tau^2\Delta)^{-1/2}$. Moreover, since $\tau\sigma_\tau^2\Delta = \tau\cdot(\sigma_\tau^2\Delta) = z_n C''(z_n) + z_n C'(z_n) \sim z_n C''(z_n)$, all is left to show in order to finish the proof is $I_2 = o(I_1)$. From (6.9) we obtain for some $d > 0$ that $I_2 \leq \Pr[|P-\tau| > \sqrt{\tau}\ln n] \leq \mathrm{e}^{-d(\ln n)^2}$ and the proof is completed. □

6.2.4 Enumeration of Sets

Proof of Theorem 2.9. Let $C_1, C_2, \dots$ be iid with probability generating function $C(r_n x)/C(r_n)$. Further let $S_p := \sum_{1\le i\le p} C_i$ for $p \in \mathbb{N}$ and set

$$\nu_p := \mathbb{E}\left[S_p\right] = p\frac{r_n C'(r_n)}{C(r_n)} \quad \text{and} \quad \sigma_p^2 := \mathrm{Var}\left(S_p\right) = p\left(\frac{{r_n}^2 C''(r_n) + r_n C'(r_n)}{C(r_n)} - \left(\frac{r_n C'(r_n)}{C(r_n)}\right)^2\right).$$

Recalling $r_n C'(r_n)/C(r_n) = n/N$ we obtain that $\nu_N = n$. Summarising,

$$[x^n y^N]S(x,y) = \frac{1}{N!}[x^n]C(x)^N = \frac{{r_n}^{-n}C(r_n)^N}{N!}\cdot \Pr\left[S_N = n\right] = \frac{{r_n}^{-n}C(r_n)^N}{N!}\cdot \Pr\left[S_N = \nu_N\right].$$

Let $r_n = \rho \mathrm{e}^{-\varphi_n}$. Since $r_n C'(r_n)/C(r_n) = n/N \to \infty$ we necessarily have that $\varphi_n \to 0$ as $n \to \infty$ so that we are allowed to apply Lemma 6.11 and obtain

$$\Pr\left[S_N = \nu_N\right] \sim \frac{1}{\sqrt{2\pi}\sigma_N}.$$

Making use of Lemma 4.6 we finish the proof by computing

$$\sigma_N^2 = \frac{N}{C(r_n)}\left(r_n C''(r_n) + r_n C'(r_n) - (r_n C'(r_n))^2/C(r_n)\right) \sim N\frac{r_n^2 C''(r_n)}{(\alpha+1)C(r_n)}.$$

□

7 Cluster Statistics for Expansive (Multi-)sets

This section contains the proofs of the main results in Section 2.3 and is based on Sections 2 and 3 of the contributing Manuscript (III). As mentioned at the beginning of Section 6, for reasons of coherence Theorem 1.6 from (III) is presented as Theorem 2.9, the proof of which is contained in Section 6.

Plan of the Section. The preparations for the proofs of the main theorems are contained in Section 7.1, in which we determine the asymtotics of a sum related to derivatives of $\ln G(x)$ and present the concept of H-admissibility. Then, in Section 7.2 detailed proofs for the results in Section 2.3 are given. Here we first show that the underlying generating series are all H-admissible, a crucial fact the other proofs depend on. Subsequently, we move on with proving all statements from Sections 2.3.1–2.3.3 in Sections 7.2.2–7.2.4. We further remark that for the proofs of the local limit theorem for the cluster distribution in Theorem 2.15 we will need to make use of results from the previous Section 6 and also Theorems 2.8(I) as well as 2.9.

7.1 Preliminaries

7.1.1 An Asymptotic Expression

First we state the following auxiliary lemma which will help us to compute asymptotic bounds for the sum $\sum_{j\ge1} j^{\beta-1} r^{\gamma j} C^{(\gamma)}((\mathrm{e}^{-\chi})^j)$ for $\beta \in \mathbb{N}, \gamma \in \mathbb{N}_0$ and as $\chi \to 0$. In the proof we use the Euler-MacLaurin summation formula (and the computations are inspired by [56, Appendix A]).

Lemma 7.1. *Let $\beta, \gamma \in \mathbb{R}_0^+$. Then, as $\chi \to 0$,*

$$\sum_{k\geq 1} \frac{k^\gamma e^{-\chi k}}{(1-e^{-\chi k})^\beta} \sim \begin{cases} d_1 \cdot \chi^{-(\gamma+1)}, & \beta < 1+\gamma \\ \chi^{-(\gamma+1)} \ln(\chi^{-1}), & \beta = 1+\gamma \\ d_2 \cdot \chi^{-\beta}, & \beta > 1+\gamma \end{cases},$$

where, letting ζ denote the Zeta-function,

$$d_1 = \int_0^\infty \frac{t^\gamma e^{-t}}{(1-e^{-t})^\beta} dt \quad \text{and} \quad d_2 = \zeta(\beta-\gamma).$$

Proof. Define $g(t) := t^\gamma e^{-\chi t}/(1-e^{-\chi t})^\beta$. For $\beta - \gamma < 1$ the integral $\int_0^\infty t^\gamma e^{-t}/(1-e^{-t})^\beta dt$ exists (since the integrand is asymptotically $t^{-(\beta-\gamma)}$ as $t \to 0$) so that by convergence of the Riemann sum to the corresponding integral we obtain

$$\sum_{k\geq 1} g(k) = \chi^{-(\gamma+1)} \cdot \chi \cdot \sum_{k\geq 1} \frac{(\chi k)^\gamma e^{-\chi k}}{(1-e^{-\chi k})^\beta} \sim \chi^{-(\gamma+1)} \int_0^\infty \frac{t^\gamma e^{-t}}{(1-e^{-t})^\beta} dt.$$

Next we consider the case $\beta - \gamma > 1$. Let $P_1(x) = x - \lfloor x \rfloor - 1/2$. Then the Euler-Maclaurin formula, see for example [38, Ch. 9.5], gives us

$$\sum_{k\geq 1} g(k) = \int_1^\infty g(t)dt + \frac{g(1)}{2} + \int_1^\infty g'(x) P_1(x) dx. \tag{7.1}$$

We will determine the first integral by using dominated convergence. Note that for $t \geq 1$

$$\chi^\beta g(t) t^{\beta-\gamma} = (\chi t)^\beta e^{-\chi t}/(1-e^{-\chi t})^\beta.$$

Thus

$$\lim_{\chi\to 0} \chi^\beta g(t) = t^{\gamma-\beta}, \quad t \geq 1. \tag{7.2}$$

Further, the continuous function $z^\beta e^{-z}/(1-e^{-z})^\beta$ tends to 1 as $z \to 0$ and to 0 as $z \to \infty$. Hence there is some $A > 0$ such that

$$\chi^\beta g(t) \leq A t^{\gamma-\beta}, \quad t \geq 1. \tag{7.3}$$

Thus, by dominated convergence, (7.2), and the fact that $\int_1^\infty t^{-(\beta-\gamma)} dt$ exists

$$\int_1^\infty g(t)dt = \chi^{-\beta} \int_1^\infty \chi^\beta g(t) dt \sim \chi^{-\beta} \int_1^\infty t^{-(\beta-\gamma)} dt = \chi^{-\beta} \frac{1}{\beta-\gamma-1}. \tag{7.4}$$

The next term in (7.1) is $g(1)/2 \sim \chi^{-\beta}/2$. Moreover,

$$\int_1^\infty g'(t) P_1(t) dt = \int_1^\infty \left(\gamma \frac{t^{\gamma-1} e^{-\chi t}}{(1-e^{-\chi t})^\beta} - \chi \frac{t^\gamma e^{-\chi t}}{(1-e^{-\chi t})^\beta} - \chi\beta \frac{t^\gamma e^{-2\chi t}}{(1-e^{-\chi t})^{\beta+1}} \right) P_1(t) dt. \tag{7.5}$$

Note that $\beta - \gamma > 1$ implies that $\beta - (\gamma - 1) > 1$ and $(\beta + 1) - \gamma > 1$ so that as before

$$\int_1^\infty g'(t) P_1(t) dt \sim \chi^{-\beta} (\gamma - \beta) \int_1^\infty t^{-(\beta-\gamma+1)} P_1(t) dt - \chi^{-\beta+1} \int_1^\infty t^{-(\beta-\gamma)} P_1(t) dt.$$

As $\beta-\gamma>1$, the last term is $\mathcal{O}\left(\chi^{-\beta+1}\right)=o(\chi^{-\beta})$. Moreover, by using again Euler-Maclaurin summation

$$\sum_{k\geq 1}\frac{k^{-(\beta-\gamma)}}{\gamma-\beta}=\int_1^\infty\frac{t^{-(\beta-\gamma)}}{\gamma-\beta}dt+\frac{1}{2(\gamma-\beta)}+\int_1^\infty t^{-(\beta-\gamma+1)}P_1(t)dt.$$

Computing the integral and rearranging the terms yields

$$\int_1^\infty t^{-(\beta-\gamma+1)}P_1(t)dt=\frac{\zeta(\beta-\gamma)}{\gamma-\beta}+\frac{1}{(\gamma-\beta)(\gamma-\beta+1)}-\frac{1}{2(\gamma-\beta)}.$$

The claim follows for $\beta-\gamma>1$. Finally, let us consider the case $\beta=\gamma+1$. Set $a=\ln(\chi^{-1})^{-1/(2(\gamma+1))}=o(1)$. We have that

$$\chi^{\gamma+1}\int_1^\infty g(t)dt=\int_\chi^\infty\frac{t^\gamma\mathrm{e}^{-t}}{(1-\mathrm{e}^{-t})^{\gamma+1}}dt=\left(\int_\chi^a+\int_a^1+\int_1^\infty\right)\frac{t^\gamma\mathrm{e}^{-t}}{(1-\mathrm{e}^{-t})^{\gamma+1}}dt=:I_1+I_2+\mathcal{O}\left(1\right).$$

In I_1 we use that $t=o(1)$ to obtain that $I_1\sim\int_\chi^a t^{-1}dt=\ln a-\ln\chi\sim\ln(\chi^{-1})$. In I_2 we estimate $I_2\leq(1-\mathrm{e}^{-a})^{-(\gamma+1)}\int_a^1 t^\gamma\mathrm{e}^{-t}=\Theta(\ln(\chi^{-1})^{1/2})=o(\ln(\chi^{-1}))$. Hence

$$\int_1^\infty g(t)dt\sim\chi^{-(\gamma+1)}\ln(\chi^{-1}).\tag{7.6}$$

Further $g(1)\sim\chi^{-\beta}=\chi^{-(\gamma+1)}$ and by estimating $|P_1(x)|\leq 1$ in (7.5) we get

$$\left|\int_1^\infty g'(t)P_1(t)dt\right|\leq\int_1^\infty\left(\gamma\frac{t^{\gamma-1}\mathrm{e}^{-\chi t}}{(1-\mathrm{e}^{-\chi})^\beta}+\chi\frac{t^\gamma\mathrm{e}^{-\chi t}}{(1-\mathrm{e}^{-\chi t})^\beta}+\chi\beta\frac{t^\gamma\mathrm{e}^{-2\chi t}}{(1-\mathrm{e}^{-\chi t})^{\beta+1}}\right)dt=:J_1+J_2+J_3.$$

Since $\beta=\gamma+1$ we have that $\gamma-1-\beta=-2$ and the integral $\int_1^\infty t^{-2}dt$ exists. Hence, analogous to (7.2)–(7.4) we obtain by dominated convergence that $J_i=\mathcal{O}\left(\chi^{-(\gamma+1)}\right)=o(\chi^{-(\gamma+1)}\ln(\chi^{-1}))$ for $i=1,3$. Analogous to (7.6) we obtain that

$$\chi^{-1}J_2=\int_1^\infty g(t)dt\sim\chi^{-(\gamma+1)}\ln(\chi^{-1})$$

implying that $J_2=o(\chi^{-(\gamma+1)}\ln(\chi^{-1}))$. This finishes the proof. □

7.1.2 H-admissibility

We start by reviewing the concept of H-admissibility, which is a general set of conditions on a function $F(x)$ with radius of convergence $0<\rho\leq\infty$ under which $[x^n]F(x)$ can be computed asymptotically. This theory was initiated in the seminal paper [44] and has seen numerous extensions and applications. As a general reference we recommend [30, Ch. VIII.5].

Set $F(x)=\mathrm{e}^{f(x)}$. By applying Cauchy's coefficient formula and switching to polar coordinates we obtain for some $0<r<\rho$

$$[x^n]F(x)=\frac{r^{-n}}{2\pi}\int_{-\pi}^{\pi}F(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-n\mathrm{i}\theta}d\theta.\tag{7.7}$$

To get a grip on this expression we expand $F(re^{i\theta})$ at $\theta = 0$, so that for some $|\xi| \le |\theta| \le \pi$

$$F(re^{i\theta})e^{-ni\theta} = F(r) \cdot \exp\left\{i\theta(a(r)-n) - \frac{\theta^2}{2}b(r) + \frac{(i\theta)^3}{6}c(re^{i\xi})\right\} \tag{7.8}$$

for functions a, b and c given by

$$ia(x) := \frac{\partial}{\partial\theta} f(xe^{i\theta})\Big|_{\theta=0}, \quad -b(x) := \frac{\partial^2}{\partial\theta^2} f(xe^{i\theta})\Big|_{\theta=0} \quad \text{and} \quad i^3c(x) := \frac{\partial^3}{\partial\theta^3} f(xe^{i\theta})\Big|_{\theta=0}.$$

In particular,

$$a(x) = xf'(x), \quad b(x) = x^2f''(x) + xf'(x) \quad \text{and} \quad c(x) = x^3f'''(x) + 3x^2f''(x) + xf'(x). \tag{7.9}$$

With these definitions at hand we (informally) say that $F(x)$ is H-admissible, if it is possible to split up (7.7) into a dominant part, where $f(re^{i\theta}) - ni\theta = f(r) + i\theta(a(r)-n) - \theta^2 b(r)/2 + o(1)$, and another integral that is negligible. Then by choosing $a(r)$ to be (close to) n the asymptotic value of the dominant integral can be retrieved, as it is of "Gaussian" type. The following three conditions formalise this idea, where F is assumed to be a function with radius of convergence $0 < \rho \le \infty$ which is positive on some interval $(R_0, \rho) \subseteq (0, \rho)$.

(H_1) [Capture Condition] $a(r)$ and $b(r)$ tend to infinity as $r \to \rho$.

(H_2) [Locality Condition] For some function $\theta_0 : (R_0, \rho) \to \mathbb{R}^+$ we have as $r \to \rho$ uniformly in $|\theta| \le \theta_0(r)$

$$F(re^{i\theta}) \sim F(r) \cdot \exp\left\{i\theta a(r) - \theta^2 b(r)/2\right\}.$$

(H_3) [Decay Condition] As $r \to \rho$ uniformly in $\theta_0(r) \le |\theta| < \pi$

$$F(re^{i\theta}) = o\left(b(r)^{-1/2}F(r)\right).$$

We call $F(x)$ H-admissible if it has the three proprties (H_1)–(H_3). The following statement, which originates in [44, Thm. 1], provides a useful tool for determining the n-th coefficient of a H-admissible function, see also [30, Prop. VIII.5].

Lemma 7.2. *Suppose that $F(x)$ is H-admissible. Then as $r \to \rho$*

$$[x^n]F(x) = \frac{F(r)}{\sqrt{2\pi b(r)}} \cdot r^{-n} \cdot \left(\exp\left\{-\frac{(a(r)-n)^2}{2b(r)}\right\} + \varepsilon_n\right),$$

where $\lim_{r\to\rho}\sup_{n\in\mathbb{N}}|\varepsilon_n| = 0$.

In particular, by choosing any r such that $(a(r)-n)^2/b(r) = \mathcal{O}(1)$ we get the first asymptotic order of $[x^n]F(x)$. This allows us to compare *different* coefficients $[x^{n-k}]F(x)$ and $[x^n]F(x)$ using the *identical* saddle-point r, a simple yet impactful fact we will use numerous times later.

7.2 Proofs

7.2.1 H-admissibility of the Related Generating Series

In this section we prove the following lemma, which is the backbone of the other forthcoming proofs in this section, and on the way some properties of functions related to (7.9).

Lemma 7.3. *Suppose that $C(x)$ is oscillating expansive. Then*

$$S(x)C(x)^\ell \quad \textit{and} \quad G(x)\prod_{1\le i\le \ell}\sum\nolimits_{j\ge 1} j^{p_i}C(x^j)$$

are H-admissible for $\ell \in \mathbb{N}_0$ and $(p_1,\dots,p_\ell)\in\mathbb{N}_0^\ell$.

The proof is in Section 7.2.1.

Asymptotic Properties of the Functions in (H_1)–(H_3)

Recall that if $C(x)$ is oscillating expansive, there are $\alpha > 0$, $0 < \varepsilon < \alpha/3$ and $0 < A_1 < A_2$ such that for all n sufficiently large

$$A_1 \cdot n^{2\alpha/3+\varepsilon-1}\cdot \rho^{-n} \le c_n \le A_2 \cdot n^{\alpha-1}\cdot\rho^{-n}.$$

If the parameters at hand are important, we will say that $C(x)$ is oscillating expansive with parameters $\alpha, \varepsilon, \rho$.

The set case. We will first investigate the functions a_ℓ, b_ℓ and c_ℓ from (7.9) for $S(x)C(x)^\ell$ and $\ell\in\mathbb{N}_0$. To simplify the notation later on we introduce

$$A_s(x) := \sum_{k\ge 1} k^s c_k x^k, \qquad s\in\mathbb{N}_0. \tag{7.10}$$

The idea behind these definitions is that we are able to abbreviate

$$A_0(x) = C(x), \quad A_1(x) = xC'(x), \quad A_2(x) = x^2C''(x) + xC'(x) \quad \text{and}$$
$$A_3(x) = x^3C'''(x) + 3x^2C''(x) + xC'(x).$$

Then we obtain

$$\begin{aligned} a_\ell(x) &= A_1(x) + \ell\frac{A_1(x)}{A_0(x)}, \quad b_\ell(x) = A_2(x) + \ell\left(\frac{A_2(x)}{A_0(x)} - \frac{A_1(x)^2}{A_0(x)^2}\right) \quad \text{and} \\ c_\ell(x) &= A_3(x) + \ell\left(\frac{A_3(x)}{A_0(x)} - 3\frac{A_1(x)A_2(x)}{A_0(x)^2} + 2\frac{A_1(x)^3}{A_0(x)^3}\right). \end{aligned} \tag{7.11}$$

In the next statement we show that the functions in (7.11) are asymptotically equal to versions of (7.10) when their argument gets close the radius of convergence ρ.

Lemma 7.4. *Suppose that $C(x)$ is oscillating expansive. Set $r = \rho e^{-\chi}$ for $\chi > 0$. Then the functions (7.11) fulfil for any $\ell\in\mathbb{N}_0$*

$$a_\ell(r) \sim A_1(r) \quad \textit{and} \quad b_\ell(r)\sim A_2(r), \quad \textit{as } \chi\to 0. \tag{7.12}$$

Before we prove this lemma we note that an immediate consequence of Lemma 4.6 is the following corollary.

Corollary 7.5. *Suppose that $C(x)$ is oscillating expansive with parameters $\alpha > 0, 0 < \varepsilon < \alpha/3$ and $0 < \rho \le 1$. Set $r = \rho e^{-\chi}$ for $\chi > 0$. Then for any $s \in \mathbb{N}_0$*

$$A_s(r) = \mathcal{O}\big(\chi^{-(\alpha+s)}\big) \quad \text{and} \quad A_s(r) = \Omega(\chi^{-(2\alpha/3+\varepsilon+s)}), \qquad \text{as } \chi \to 0. \tag{7.13}$$

Proof of Lemma 7.4. Since $A_0(r) \to \infty$ according to (7.13) it immediately follows that $a_\ell(r) = A_1(r) + \mathcal{O}\left(A_1(r)/A_0(r)\right) \sim A_1(r)$ as $\chi \to 0$. Further, Hölder's inequality gives us

$$A_1(r)^2 = \left(\sum_{k\ge1} k\sqrt{c_k r^k} \cdot \sqrt{c_k r^k}\right)^2 \le A_2(r)A_0(r). \tag{7.14}$$

Hence $(A_1(r)/A_0(r))^2 \le A_2(r)/A_0(r)$. Since $A_0(r) \to \infty$ according to (7.13) it readily follows $b_\ell(r) = A_2(r) + \mathcal{O}\left(A_2(r)/A_0(r)\right) \sim A_2(r)$ as $\chi \to 0$. □

The multiset case. Next we consider the functions $\tilde{a}_\ell, \tilde{b}_\ell$ and $\tilde{c}_\ell$ from (7.9) for $G(x)\prod_{1\le i\le \ell} j^{p_i}C(x^j)$ and $\ell \in \mathbb{N}_0$, $(p_1, \dots, p_\ell) \in \mathbb{N}_0^\ell$. To simplify the notation later on we introduce

$$A_{s,t}(x) := \sum_{j\ge1} j^{t-1} \sum_{k\ge1} k^s c_k x^{jk}, \qquad s,t \in \mathbb{N}_0. \tag{7.15}$$

This definition allows us to get the much shorter expressions for

$$\begin{aligned}
A_{0,t}(x) &= \sum_{j\ge1} j^{t-1} C(x^j), \quad A_{1,t}(x) = \sum_{j\ge1} j^{t-1} x^j C'(x^j),\\
A_{2,t}(x) &= \sum_{j\ge1} j^{t-1}(x^{2j}C''(x^j) + x^jC'(x^j)) \quad \text{and}\\
A_{3,t}(x) &= \sum_{j\ge1} j^{t-1}(x^{3j}C'''(x^j) + 3x^{2j}C''(x^j) + x^jC'(x^j)).
\end{aligned}$$

With this at hand, we can write

$$\begin{aligned}
\tilde{a}_\ell(x) &= A_{1,1}(x) + \sum_{1\le i\le \ell} \frac{A_{1,2+p_i}(x)}{A_{0,1+p_i}(x)}, \quad \tilde{b}_\ell(x) = A_{2,2}(x) + \sum_{1\le i\le \ell}\left(\frac{A_{2,3+p_i}(x)}{A_{0,1+p_i}(x)} - \frac{A_{1,2+p_i}(x)^2}{A_{0,1+p_i}(x)^2}\right) \quad \text{and}\\
\tilde{c}_\ell(x) &= A_{3,3}(x) + \sum_{1\le i\le \ell}\left(\frac{A_{3,4+p_i}(x)}{A_{0,1+p_i}(x)} - 3\frac{A_{1,2+p_i}(x)A_{2,3+p_i}(x)}{A_{0,1+p_i}(x)^2} + 2\frac{A_{1,2+p_i}(x)^3}{A_{0,1+p_i}(x)^3}\right).
\end{aligned} \tag{7.16}$$

In the next statement we show that the functions in (7.16) are asymptotically equal to versions of (7.15) when their argument gets close the radius of convergence ρ.

Lemma 7.6. *Suppose that $C(x)$ is oscillating expansive. Then the functions (7.11) fulfil for all $\ell \in \mathbb{N}_0, (p_1, \dots, p_\ell) \in \mathbb{N}_0^\ell$ that*

$$\tilde{a}_\ell(r) \sim A_{1,1}(r) \quad \text{and} \quad \tilde{b}_\ell(r) \sim A_{2,2}(r), \qquad \text{as } \chi \to 0. \tag{7.17}$$

Before we prove this lemma we state some asymptotic properties of (7.15) based on Lemma 7.1. Recall the the definition of $A_s(x)$ from (7.10).

Lemma 7.7. *Suppose that $C(x)$ is oscillating expansive with parameters $\alpha > 0, 0 < \varepsilon < \alpha/3$ and $0 < \rho \leq 1$. Set $r = \rho e^{-\chi}$ for $\chi > 0$. Then for any $s, t \in \mathbb{N}_0$*

$$A_{s,t}(r) = A_s(r) + \sum_{j\geq 2} j^{t-1} \sum_{k\geq 1} k^s c_k \rho^{js} + o(1) = A_s(r) + \mathcal{O}(1), \qquad \textit{for } 0 < \rho < 1 \textit{ as } \chi \to 0. \tag{7.18}$$

Moreover,

$$A_{s,t}(r) = \begin{cases} \mathcal{O}\left(\chi^{-(\alpha+s)}\right), & t < \alpha + s \\ \mathcal{O}\left(\chi^{-(\alpha+s)} \ln(\chi^{-1})\right), & t = \alpha + s \\ \mathcal{O}\left(\chi^{-t}\right), & t > \alpha + s \end{cases}, \qquad \textit{for } \rho = 1 \textit{ as } \chi \to 0, \tag{7.19}$$

as well as

$$A_{s,t}(r) = \begin{cases} \Omega(\chi^{-(2\alpha/3+\varepsilon+s)}), & t < 2\alpha/3 + \varepsilon + s \\ \Omega(\chi^{-(2\alpha/3+\varepsilon+s)} \ln(\chi^{-1})), & t = 2\alpha/3 + \varepsilon + s \\ \Omega(\chi^{-t}), & t > 2\alpha/3 + \varepsilon + s \end{cases}, \qquad \textit{for } \rho = 1 \textit{ as } \chi \to 0. \tag{7.20}$$

Finally, for any $s, t \in \mathbb{N}_0$

$$\frac{A_{s,s+t}(r)}{A_{s,s}(r)A_{0,t}(r)} = o(1), \quad \textit{as } \chi \to 0. \tag{7.21}$$

Proof of Lemma 7.7. To show (7.18) we first note that $A_{s,t}(r) = A_s(r) + \sum_{j\geq 2} j^{t-1} \sum_{k\geq 1} k^s c_k r^{jk}$. Applying Lemma 4.4 shows that for $0 < \rho < 1$

$$\sum_{j\geq 2} j^{t-1} \sum_{k\geq 1} k^s c_k r^{jk} = \mathcal{O}\left(\sum_{j\geq 2} j^{t-1} \rho^j\right) = \mathcal{O}(1).$$

By dominated convergence we can let $r \to \rho$. In addition, if $0 < \rho < 1$, then due to (7.18) we obtain $A_{s,s+t}(r)/(A_{s,s}(r)A_{0,t}(r)) \sim A_{0,t}(r)^{-1} = o(1)$. This shows all the statements for $0 < \rho < 1$.

For the remaining proof assume $\rho = 1$. Then for $s, t \in \mathbb{N}_0$ as $\chi \to 0$

$$A_{s,t}(r) = \sum_{k\geq 1} k^s c_k \sum_{j\geq 1} j^{t-1} e^{-\chi kj} = \Theta\left(\sum_{k\geq 1} \frac{k^s c_k e^{-\chi k}}{(1 - e^{-\chi k})^t}\right).$$

With this at hand, Lemma 7.1 reveals (7.19) and (7.20). In turn, with (7.19) and (7.20) we compute for $s, t \in \mathbb{N}$

$$\frac{A_{s,s+t}(r)}{A_{s,s}(r)A_{0,t}(r)} = \begin{cases} \mathcal{O}\left(\chi^{2\alpha/3+\varepsilon} \ln(\chi^{-1})\right), & 2\alpha/3 + \varepsilon < \alpha \leq t \\ \mathcal{O}\left(\chi^{-\alpha/3+\varepsilon+t}\right), & 2\alpha/3 + \varepsilon \leq t < \alpha \\ \mathcal{O}\left(\chi^{\alpha/3+2\varepsilon}\right), & t < 2\alpha/3 + \varepsilon < \alpha \end{cases}.$$

The only term which is not readily in $o(1)$ is $\chi^{-\alpha/3+\varepsilon+t}$. But $\alpha < 3t/2$ in order for $2\alpha/3 + \varepsilon < t$ to hold so that $-\alpha/3 + \varepsilon + t > \varepsilon + t/2 > 0$ and it follows that $\chi^{-\alpha/3+\varepsilon+t} = o(1)$. This delivers (7.21) and we are done. □

With this at hand, we prove Lemma 7.6.

Proof of Lemma 7.6. For any $p \in \mathbb{N}_0$ we have that $A_{1,2+p}(r)/(A_{1,1}(r)A_{0,1+p}(r)) = o(1)$ due to (7.21) giving us $\tilde{a}_\ell(r) \sim A_{1,1}(r)$. And with Hölder's inequality we obtain for any $p \in \mathbb{N}_0$

$$A_{1,2+p}(r)^2 = \left(\sum_{j\geq 1} \sum_{k\geq 1} j^{(p+2)/2} k \sqrt{c_k r^{jk}} \cdot j^{p/2} \sqrt{c_k r^{jk}} \right)^2 \leq A_{2,3+p}(r) A_{0,1+p}(r). \tag{7.22}$$

Hence (7.21) delivers that $A_{2,3+p}(r)/(A_{2,2}(r)A_{0,1+p}(r)) = o(1)$ for any $p \in \mathbb{N}_0$ giving $\tilde{b}_\ell(r) \sim A_{2,2}(r)$ as $\chi \to 0$. □

Proof of Lemma 7.3

Proof of Lemma 7.3. Recall that $C(x)$ is oscillating expansive with parameters $\alpha > 0, 0 < \varepsilon < \alpha/3$ and $0 < \rho \leq 1$.

We first show that $D(x) = \exp\{C(x)\} C(x)^\ell$ is H-admissible for any $\ell \in \mathbb{N}_0$. The functions (7.9) for which we need to verify properties (H_1)–(H_3) are given by $a_\ell(x), b_\ell(x)$ and $c_\ell(x)$ from (7.11). Set $r = \rho e^{-\chi}$. From (7.12) we obtain that $a_\ell(r) \sim A_1(r)$ and $b_\ell(r) \sim A_2(r)$. Then (7.13) implies that $a_\ell(r)$ and $b_\ell(r)$ both tend to infinity as $\chi \to 0$, thus establishing (H_1).

We continue by proving (H_2). For some $\alpha/3 < \delta < \alpha/3 + \varepsilon/2$ set

$$\theta_0 := \chi^{1+\delta}. \tag{7.23}$$

By applying Taylor's expansion we obtain that for $|\theta| \leq \theta_0$ there is a $\xi = \xi(\theta) \in (-\theta, \theta)$ such that

$$S(re^{i\theta}) = S(r) \cdot \exp\left\{ i\theta a_\ell(r) - \frac{\theta^2}{2} b_\ell(r) + i^3 \frac{\theta^3}{6} c_\ell(re^{i\xi}) \right\}.$$

Defining

$$d(x) := \frac{A_3(x)}{A_0(x)} - 3\frac{A_1(x)A_2(x)}{A_0(x)^2} + 2\frac{A_1(x)^3}{A_0(x)^3}$$

we get in view of (7.11) that $|\theta^3 c_\ell(re^{i\xi})| \leq \theta_0^3 |A_3(re^{i\xi})| + \theta_0^3 |d(re^{i\xi})|$ uniformly in $|\theta| \leq \theta_0$. Since $A_3(r)$ has only non-negative coefficients we obtain with the triangle inequality that $|A_3(re^{i\xi})| \leq A_3(r)$. With (7.13) we further get that $A_3(r) = \mathcal{O}\left(\chi^{-(\alpha+3)}\right)$ as $\chi \to 0$. We conclude that $\theta_0^3 |A_3(re^{i\xi})| \leq \theta_0 A_3(r) = \mathcal{O}\left(\chi^{-\alpha+3\delta}\right) = o(1)$ as $\chi \to 0$ uniformly in $|\theta| \leq \theta_0$. It remains to show that $\theta_0^3 |d(re^{i\xi})| = o(1)$ in order to get (H_2). Since the functions appearing in d are not necessarily power series with non-negative coefficients anymore (since powers of A_0 appear in the denominator), we cannot use the triangle inequality to get rid of $e^{i\xi}$. So we first have to find a lower bound for $|A_0(re^{i\xi})|$ holding uniformly in $|\theta| \leq \theta_0$; in fact we are going to show that $|A_0(re^{i\xi})| \sim A_0(r)$. From the definition of the absolute value of complex numbers

$$|A_0(re^{i\xi})|^2 = \left(\sum_{k\geq 1} c_k r^k \cos(\xi k) \right)^2 + \left(\sum_{k\geq 1} c_k r^k \sin(\xi k) \right)^2 \geq \left(\sum_{k\geq 1} c_k r^k \cos(\xi k) \right)^2. \tag{7.24}$$

For $1 \leq k \leq \chi^{-1-\delta/2}$ we have $k\xi = o(1)$ since $|\xi| \leq |\theta| \leq \theta_0 = \chi^{1+\delta}$. Hence

$$\sum_{k\geq 1} c_k r^k \cos(\xi k) \sim A_0(r) + R_0 + R_1, \quad R_i = \sum_{k > \chi^{-1-\delta/2}} c_k r^k \cos(\xi k)^i, \; i = 0, 1.$$

Then for $i = 0, 1$ and recalling that $C(x)$ is oscillating expansive

$$|R_i| = \mathcal{O}\Bigg(\sum_{k > \chi^{-1-\delta/2}} k^{\alpha-1} \mathrm{e}^{-\chi k} \Bigg) = \Theta\left(\chi^{-\alpha} \int_{\chi^{-\delta/2}} t^{\alpha-1} \mathrm{e}^{-t} dt \right) = o(1).$$

Together with $|A_0(r\mathrm{e}^{\mathrm{i}\xi})| \leq A_0(r)$ we thus deduce $|A_0(r\mathrm{e}^{\mathrm{i}\xi})| \sim A_0(r)$ as $\chi \to 0$ uniformly in $|\xi| \leq |\theta| \leq \theta_0$. Using that A_s has only non-negative coefficients for $s \geq 0$ we then get the estimate

$$|d(r\mathrm{e}^{\mathrm{i}\xi})| \leq \frac{A_3(r)}{|A_0(r\mathrm{e}^{\mathrm{i}\xi})|} + 3\frac{A_1(r)A_2(r)}{|A_0(r\mathrm{e}^{\mathrm{i}\xi})|^2} + 2\frac{A_1(r)^3}{|A_0(r\mathrm{e}^{\mathrm{i}\xi})|^3} \sim \frac{A_3(r)}{A_0(r)} + 3\frac{A_1(r)A_2(r)}{A_0(r)^2} + 2\frac{A_1(r)^3}{A_0(r)^3}.$$

With Hölder's inequality we obtain

$$A_2(r)^2 = \left(\sum_{k\geq 1} k^{3/2}\sqrt{c_k r^k} \cdot \sqrt{k c_k r^k} \right)^2 \leq A_3(r)A_1(r) \quad \text{and}$$
$$A_1(r)^3 = \left(\sum_{k\geq 1} k(c_k r^k)^{1/3} \cdot (c_k r^k)^{2/3} \right)^3 \leq A_3(r)A_0(r)^2.$$

In (7.14) we showed that $A_1(r)^2 \leq A_2(r)A_0(r)$. With this at hand, we get that $A_1(r)A_2(r)/A_0(r)^2 \leq A_2(r)^2/(A_0(r)A_1(r)) \leq A_3(r)/A_0(r)$ and $A_1(r)^3/A_0(r)^3 \leq A_3(r)/A_0(r)$. Since $A_0(r) \to \infty$ according to (7.13) this implies with (7.13) that, as $\chi \to 0$ and uniformly in $|\theta| \leq \theta_0$,

$$\theta_0^3 |d(r\mathrm{e}^{\mathrm{i}\xi})| = \mathcal{O}\left(\theta_0^3 \frac{A_3(r)}{A_0(r)} \right) = o(\theta_0^3 A_3(r)) = o(1).$$

This delivers (H_2).

The hard part of the proof is to show (H_3), but [33, Lem. 7] solves the problem in an almost identical setting. Since $C(x)$ has only non-negative coefficients we compute

$$\left| \frac{D(r\mathrm{e}^{\mathrm{i}\theta})}{D(r)} \right| \leq \left| \exp\left\{ C(r\mathrm{e}^{\mathrm{i}\theta}) - C(r) \right\} \right| \leq \exp\left\{ \sum_{k\geq 1} c_k r^k (\cos(\theta k) - 1) \right\}$$
$$= \exp\left\{ -2\sum_{k\geq 1} c_k r^k \sin^2(\theta k/2) \right\}. \tag{7.25}$$

First of all, since $\sin^2(x)$ is symmetric, it is no restriction to consider $\theta > 0$ from now on. Denote by $\|x\|$ the distance from $x \in \mathbb{R}$ to its nearest integer. Then, see [33],

$$\sin^2(\pi x) \geq 4\|x\|.$$

Consequently, setting $\alpha_1 := 2\alpha/3 + \varepsilon$ and letting $A_1 > 0$ be such that $c_k \geq A_2 k^{\alpha_1 - 1}\rho^{-k}$ for $k \in \mathbb{N}$, we get that

$$(4A_1)^{-1} \sum_{k\geq 1} c_k r^k \sin^2(\theta k/2) \geq \sum_{k\geq 1} k^{\alpha_1 - 1} \mathrm{e}^{-\chi k} \|\theta k/(2\pi)\| =: V(\theta/(2\pi)).$$

We claim that there exists $f > 0$ such that, as $\chi \to 0$ and uniformly in $\theta_0/(2\pi) \leq \theta < 1/2$,

$$V(\theta) \geq \chi^{-f}. \tag{7.26}$$

Since $b_\ell(r) \sim A_2(r) = \mathcal{O}\left(\chi^{-(\alpha+2)}\right)$ due to (7.13) we obtain that $b(r)^{-1/2} = \Omega(\chi^{-(\alpha/2+1)})$ implying by (7.25) that as $\chi \to 0$ and uniformly in $\theta_0 \le \theta < \pi$

$$\left|\frac{D(re^{i\theta})}{D(r)}\right| \le \exp\{-8A_1V(\theta)\} \le \exp\left\{-8A_1\chi^{-f}\right\} = o(b(r)^{-1/2}),$$

which is Condition (H_3). So, if we show (7.26) we are done with showing that D is H-admissible.

From here we basically copy the proof of [33, Lem. 7] with minor adaptions. We partition $[\theta_0/2\pi, 1/2)$ into $I_1 := [\theta_0/(2\pi), \chi]$ and $I_2 := (\chi, 1/2)$. Consider $\theta \in I_1$. For such θ we have that

$$\|\theta k\| = \theta k, \quad k \le (2\chi)^{-1},$$

implying

$$V(\theta) \ge \theta_0^2/(2\pi)^2 \sum_{1\le k\le(2\chi)^{-1}} k^{\alpha_1+1}e^{-\chi k} = \Theta(\chi^{2\delta-\alpha_1}). \tag{7.27}$$

Since δ is chosen such that $2\delta - \alpha_1 = 2\delta - 2\alpha/3 - \varepsilon < -\alpha/3 - \varepsilon/2 < 0$ the claim (7.26) follows as $\chi \to 0$ and uniformly in $\theta \in I_1$.

Let us now consider $\theta \in I_2$. Define the sets

$$Q(\theta) := \{k \ge 1 : \|\theta k\| \ge 1/4\} = \bigcup_{j\ge 0} Q_j(\theta), \quad Q_j(\theta) := \{k \ge 1 : j + 1/4 \le \theta k \le j + 3/4\}.$$

Then

$$16 \cdot V(\theta) \ge \sum_{k\in Q(\theta)} k^{\alpha_1-1}e^{-\chi k} = \sum_{j\ge 0}\sum_{k\in Q_j(\theta)} k^{\alpha_1-1}e^{-\chi k}.$$

The intuition behind the choice of these sets is that for any $\theta \in I_2$ and $j \ge 0$ there is a least one element of order j/θ in $Q_j(\theta)$ because $(j+3/4)/\theta - (j+1/4)/\theta = (2\theta)^{-1} \ge 1$. In particular for j close to $\chi^{-1}\theta$ we sum over at least one k with magnitude χ^{-1}, the range where $k^{\alpha_1-1}e^{-\chi k}$ contributes the most to the entire sum so that the asymptotic order of $\sum_{k\ge1} k^{\alpha_1-1}e^{-\chi k} = \Theta(\chi^{-\alpha_1})$ can be recovered even in the limited range $k \in Q(\theta)$. At the same time the term $\|\theta k\|$ is bounded from below uniformly in $k \in Q(\theta)$ and $\theta \in I_2$.

To substantiate this, as a next step we estimate the sum by an integral. Since for $j \ge 0$ we have that $(j+1/4)/\theta \le k \le (j+3/4)/\theta$ we deduce for $\chi j/\theta$ sufficiently large, say $\ge s_0 > 0$, that $(\chi k)^{\alpha-1}e^{-\chi k}$ is monotonic decreasing for $k \in \bigcup_{j\ge s_0\theta\chi^{-1}} Q_j(\theta)$. Hence we find the following lower bound for the sum which holds as $\chi \to 0$ uniformly in $\theta \in I_2$

$$\begin{aligned}16 \cdot V(\theta) &\ge \int_{s_0\theta\chi^{-1}}^{\infty}\int_{(u+1/4)/\theta}^{(u+3/4)/\theta} v^{\alpha_1-1}e^{-\chi v}dvdu = \chi^{-\alpha_1}\int_{s_0\theta\chi^{-1}}^{\infty}\int_{\chi(u+1/4)/\theta}^{\chi(u+3/4)/\theta} t^{\alpha_1-1}e^{-t}dtdv \\ &= \chi^{-\alpha_1}\frac{\theta}{\chi}\int_{s_0}^{\infty}\int_{s+\chi/(4\theta)}^{s+3\chi/(4\theta)} t^{\alpha_1-1}e^{-t}dtds,\end{aligned}$$

where we first applied the change of variables $t = \chi v$ and then $s = \chi/\theta \cdot u$. Estimating $3\chi/(4\theta) \le 1$ and using again that the involved functions are decreasing in the range we are integrating over we get as $\chi \to 0$ uniformly in $\theta \in I_2$

$$16 \cdot V(\theta) \ge \chi^{-\alpha_1}\frac{\theta}{\chi}\int_{s_0}^{\infty}\int_{s+\chi/(4\theta)}^{s+3\chi/(4\theta)} t^{\alpha_1-1}e^{-t}dtds \ge \chi^{-\alpha_1}/2\int_{t_0}^{\infty}(s+1)^{\alpha-1}e^{-(s+1)}ds = \Theta(\chi^{-\alpha_1}).$$

This shows together with (7.27) that the claim (7.26) is valid as $\chi \to 0$ uniformly in $\theta \in I_1 \cup I_2$, which gives us (H_3). Accordingly, having proven (H_1)–(H_3), the function $D(x) = \exp\{C(x)\}\, C(x)^\ell$ is H-admissible for any $\ell \in \mathbb{N}_0$.

Next we prove that $E(x) = \exp\left\{\sum_{j\geq 1} C(x^j)\right\}\prod_{1\leq i\leq \ell}\sum_{j\geq 1} j^{p_i}C(x^j)$ **is** H**-admissible for any** $\ell \in \mathbb{N}_0, (p_1,\dots,p_\ell)\in\mathbb{N}_0^\ell$**.** The functions (7.9) we need to show (H_1)–(H_3) for are given by $\tilde{a}_\ell(x), \tilde{b}_\ell(x)$ and $\tilde{c}_\ell(x)$ in (7.16). Set $r = \rho e^{-\chi}$ for $\chi > 0$. Then (7.17) shows that $a_\ell(r) \sim A_{1,1}(r)$ and $\tilde{b}_\ell(r) \sim A_{2,2}(r)$. Equations (7.13)–(7.18) for $0<\rho<1$ and (7.20) for $\rho = 1$ show that $\tilde{a}_\ell(r)$ and $\tilde{b}_\ell(r)$ both tend to infinity as $\chi \to 0$ showing (H_1).

As in (7.23) we define $\theta_0 = \chi^{1+\delta}$ for some $\alpha/3 < \delta < \alpha/3 + \varepsilon/2$. The Taylor expansion of $E(re^{i\theta})$ yields that for $|\theta| \leq \theta_0$ there is some $\eta = \eta(\theta) \in (-\theta,\theta)$ such that

$$E(re^{i\theta}) = E(r)\cdot\exp\left\{i\theta\tilde{a}_\ell(r) - \frac{\theta^2}{2}\tilde{b}_\ell(r) + i^3\frac{\theta^3}{6}\tilde{c}_\ell(re^{i\eta})\right\}.$$

Define

$$\tilde{d}(x) := \sum_{1\leq i\leq \ell}\left(\frac{A_{3,4+p_i}(x)}{A_{0,1+p_i}(x)} - 3\frac{A_{1,2+p_i}(x)A_{2,3+p_i}(x)}{A_{0,1+p_i}(x)^2} + 2\frac{A_{1,2+p_i}(x)^3}{A_{0,1+p_i}(x)^3}\right).$$

In view of (7.16) we obtain for $|\theta|\leq\theta_0$ that $|\theta\tilde{c}_\ell(re^{i\eta})| \leq \theta_0^3|A_{3,3}(re^{i\eta})| + \theta_0^3|\tilde{d}(re^{i\eta})|$. Since $A_{3,3}(x)$ does only have non-negative coefficients we obtain that $|A_{3,3}(re^{i\eta})| \leq A_{3,3}(r)$ uniformly in η so that by (7.13), (7.18) (for $0<\rho<1$) and (7.19) (for $\rho=1$) we get $\theta_0^3|A_{3,3}(re^{i\eta})| \leq \theta_0^3 A_{3,3}(r) = \mathcal{O}\left(\chi^{3\delta-\alpha}\right) = o(1)$ as $\chi\to 0$ and uniformly in $|\theta|\leq\theta_0$. Thus, it is left to show that $\theta_0^3|\tilde{d}(re^{i\eta})| = o(1)$ to get (H_2). We cannot simply apply the triangle inequality to get rid of $e^{i\eta}$ as powers of $A_{0,1+p_i}$, $1\leq i\leq\ell$, are appearing in the denominators in the sum representing $\tilde{d}(re^{i\eta})$. So, we first establish that $|A_{0,1+p}(re^{i\eta})| \sim A_{0,1+p}(r)$ as $\chi\to 0$ uniformly in $|\theta|\leq\theta_0$ and for any $p\in\mathbb{N}_0$. We proceed similar to (7.24) and the subsequent text. For $j\cdot k \leq \chi^{-1-\delta/2}$ we get that $\eta jk = o(1)$ since $|\eta| \leq \theta_0 = \chi^{-1-\delta}$ implying

$$|A_{0,1+p}(re^{i\eta})| \geq \left|\sum_{j\geq 1} j^p\sum_{k\geq 1} c_k r^{jk}\cos(\eta jk)\right| \sim \left|\sum_{\substack{j,k\geq 1,\\ jk<\chi^{-1-\delta/2}}} j^p c_k r^{jk} + \sum_{\substack{j,k\geq 1,\\ jk\geq\chi^{-1-\delta/2}}} j^p c_k r^{jk}\cos(\eta jk)\right|.$$

Hence, as $\chi\to 0$ uniformly in $|\eta|\leq|\theta|\leq\theta_0 = \chi^{-1-\delta/2}$

$$\sum_{j\geq 1} j^p\sum_{k\geq 1} c_k r^{jk}\cos(\eta jk) \sim A_{0,1+p}(r) - R_0 + R_1, \quad \text{where } R_i := \sum_{j,k\geq 1, jk\geq\chi^{-1-\delta/2}} j^p c_k r^{jk}\cos(\eta jk)^i,\ i = 0,1.$$

Note that we have that $c_k r^{jk} = \mathcal{O}\left(k^{\alpha-1}\rho^{-k}\rho^{jk}e^{-\chi jk}\right) = \mathcal{O}\left(k^{\alpha-1}e^{-\chi jk}\right)$ since $C(x)$ is oscillating expansive. Further observe that if $1\leq k\leq\chi^{-1-\delta/2}$ then $\max\{1,\chi^{-1-\delta/2}/k\} = \chi^{-1-\delta/2}/k$ and 1 otherwise. Thus we obtain for $i = 0,1$

$$\begin{aligned}|R_i| &= \mathcal{O}\left(\sum_{1\leq k\leq\chi^{-1-\delta/2}} k^{\alpha-1}\sum_{j\geq\chi^{-1-\delta/2}/k} j^p e^{-\chi jk}\right) + \mathcal{O}\left(\sum_{k>\chi^{-1-\delta/2}} k^{\alpha-1}\sum_{j\geq 1} j^p e^{-\chi jk}\right)\\ &= \mathcal{O}\left(e^{-\chi^{-\delta/2}}\sum_{1\leq k\leq\chi^{-1-\delta/2}} k^{\alpha-1}\frac{1}{(1-e^{-\chi k})^{p+1}}\right) + \mathcal{O}\left(\sum_{k>\chi^{-1-\delta/2}} k^{\alpha-1}\frac{e^{-\chi k}}{(1-e^{-\chi k})^{p+1}}\right).\end{aligned}$$

The first term is bounded by

$$e^{-\chi^{-\delta/2}}\sum_{1\leq k\leq\chi^{-1-\delta/2}} c_k\frac{1}{(1-e^{-\chi k})^{p+1}} = \mathcal{O}\left(e^{-\chi^{-\delta/2}}\chi^{-1-\delta/2}(\chi^{-1-\delta/2})^{\alpha-1}\chi^{-(p+1)}\right) = o(1).$$

For the second term we get

$$\sum_{k>\chi^{-1-\delta/2}} c_k \frac{\mathrm{e}^{-\chi k}}{(1-\mathrm{e}^{-\chi k})^{p+1}} \sim \sum_{k>\chi^{-1-\delta/2}} c_k \mathrm{e}^{-\chi k} = \mathcal{O}\left(\int_{\chi^{-1-\delta/2}}^{\infty} x^{\alpha-1}\mathrm{e}^{-\chi x}dx\right) = o(1).$$

All in all, $|R_i| = o(1)$ for $i = 0,1$. This implies that $|A_{0,1+p}(r\mathrm{e}^{\mathrm{i}\eta})| \geq A_{0,1+p}(r) + o(1)$. As additionally $A_{0,1+p}$ has only non-negative coefficients we obtain $|A_{0,1+p}(r\mathrm{e}^{\mathrm{i}\eta})| \leq A_{0,1+p}(r)$. Thus, for any $p \in \mathbb{N}_0$, we conclude that $A_{0,1+p}(r\mathrm{e}^{\mathrm{i}\eta}) \sim A_{0,1+p}(r)$ as $\chi \to 0$ and uniformly in $|\eta| \leq |\theta| \leq \theta_0$. Using that $A_{s,t}$ has only non-negative coefficients for any $s,t \in \mathbb{N}_0$ we then get the estimate

$$\begin{aligned} |\tilde{d}(r\mathrm{e}^{\mathrm{i}\eta})| &\leq \sum_{1\leq i\leq \ell}\left(\frac{A_{3,4+p_i}(r)}{|A_{0,1+p_i}(r\mathrm{e}^{\mathrm{i}\eta})|} + 3\frac{A_{1,2+p_i}(r)A_{2,3+p_i}(r)}{|A_{0,1+p_i}(r\mathrm{e}^{\mathrm{i}\eta})|^2} + 2\frac{A_{1,2+p_i}(r)^3}{|A_{0,1+p_i}(r\mathrm{e}^{\mathrm{i}\eta})|^3}\right) \\ &\sim \sum_{1\leq i\leq \ell}\left(\frac{A_{3,4+p_i}(r)}{A_{0,1+p_i}(r)} + 3\frac{A_{1,2+p_i}(r)A_{2,3+p_i}(r)}{A_{0,1+p_i}(r)^2} + 2\frac{A_{1,2+p_i}(r)^3}{A_{0,1+p_i}(r)^3}\right). \end{aligned}$$

With Hölder's inequality we obtain for any $p \in \mathbb{N}_0$

$$A_{2,3+p}(r)^2 = \left(\sum_{j\geq 1}\sum_{k\geq 1} j^{p/2+3/2}k^{3/2}\sqrt{c_k r^{jk}} \cdot j^{p/2+1/2}\sqrt{kc_k r^{jk}}\right)^2 \leq A_{3,4+p}(r)A_{1,2+p}(r) \quad \text{and}$$

$$A_{1,2+p}(r)^3 = \left(\sum_{k\geq 1} j^{p/3+1}k(c_k r^{jk})^{1/3} \cdot j^{2p/3}(c_k r^{jk})^{2/3}\right)^3 \leq A_{3,4+p}(r)A_{0,1+p}(r)^2.$$

The bounds in the previous display together with $A_{1,2+p}(r)^2 \leq A_{2,3+p}(r)A_{0,1+p}(r)$, which was showed in (7.22), entail the estimates

$$\frac{A_{1,2+p}(r)A_{2,3+p}(r)}{A_{0,1+p}(r)^2} \leq \frac{A_{2,3+p}(r)^2}{A_{0,1+p}(r)A_{1,2+p}(r)} \leq \frac{A_{3,4+p}(r)}{A_{0,1+p}(r)} \quad \text{and} \quad \frac{A_{1,2+p}(r)^3}{A_{0,1+p}(r)^3} \leq \frac{A_{3,4+p}(r)}{A_{0,1+p}(r)}$$

hold true. We conclude that

$$|\tilde{d}(r\mathrm{e}^{\mathrm{i}\eta})| = \mathcal{O}\left(\sum_{1\leq i\leq \ell}\frac{A_{3,4+p_i}(r)}{A_{0,1+p_i}(r)}\right).$$

Since according to (7.21) we have for any $p \in \mathbb{N}_0$ that $A_{3,4+p}(r)/(A_{0,1+p}(r)A_{3,3}(r)) = o(1)$ as $\chi \to 0$ we obtain in turn $\theta_0^3|\tilde{d}(r\mathrm{e}^{\mathrm{i}\eta})| = o(\theta_0^3 A_{3,3}(r)) = o(1)$ as $\chi \to 0$ and uniformly in $|\eta| \leq |\theta| \leq \theta_0$ finishing the proof for (H_2).

In order to show (H_3) we use that $\sum_{j\geq 1} j^{p_i}C(x^j)$ has only non-negative coefficients and obtain similar to (7.25)

$$\left|\frac{E(r\mathrm{e}^{\mathrm{i}\theta})}{E(r)}\right| \leq \left|\frac{G(r\mathrm{e}^{\mathrm{i}\theta})}{G(r)}\right| = \exp\left\{-2\sum_{j\geq 1} j^{-1}\sum_{k\geq 1} c_k r^{jk}\sin^2(\theta jk/2)\right\} \leq \exp\left\{-2\sum_{k\geq 1} c_k r^k \sin^2(\theta k/2)\right\}.$$

Noting that we chose the same θ_0 as in (7.23) we obtain as in (7.26) that there is some $f > 0$ yielding as $\chi \to 0$ and uniformly in $\theta_0 \leq |\theta| < \pi$

$$\left|\frac{E(r\mathrm{e}^{\mathrm{i}\theta})}{E(r)}\right| \leq \mathrm{e}^{-\chi^{-f}}.$$

Since $\tilde{b}_\ell(r) \sim A_{2,2}(r) = \Omega(\chi^{-2\alpha/3+\varepsilon+2})$ due to (7.17) as well as (7.13), (7.18) (for $0 < \rho < 1$) and (7.20) (for $\rho = 1$) this shows (H_3) and we are done. □

7.2.2 Coefficient Extraction and Counting

Proof of Theorem 2.10. Due to Lemma 7.3 we know that $S(x)C(x)^\ell$ is H-admissible for any $\ell \in \mathbb{N}_0$. Let $a_\ell(x)$ and $b_\ell(x)$ be the respective functions defined in (7.9) and given by (7.11) for $h(x) = C(x) + \ell \ln C(x)$. Further let z_n be such that $z_n C'(z_n) = n$. Clearly this implies that $z_n = \rho e^{-\eta_n}$ such that $\eta_n \to 0$ as $n \to \infty$. Then (7.12) gives us that $a_\ell(z_n) = A_1(z_n) + \mathcal{O}(A_1(z_n)/A(z_n)) = n + \mathcal{O}(A_1(z_n)/A(z_n))$ and $b_\ell(z_n) \sim A_2(z_n) \sim z_n^2 C''(z_n)$. Thus Lemma 7.2 reveals that

$$[x^n]S(x)C(x)^\ell = \frac{S(z_n)C(z_n)^\ell}{\sqrt{2\pi z_n^2 C''(z_n)}} z_n^{-n}\left(\exp\left\{ \mathcal{O}\left(\frac{A_1(z_n)^2}{A(z_n)^2 A_2(z_n)} \right) \right\} + o(1) \right).$$

We obtain with (7.13) that

$$\frac{A_1(z_n)^2}{A(z_n)^2 A_2(z_n)} = \mathcal{O}\left(\eta_n^{3\varepsilon}\right) = o(1)$$

and the claim (2.18) follows.

Next we investigate the coefficients of $E(x) = G(x)\prod_{1\le i\le \ell}\sum_{j\ge1} j^{p_i} C(x^j)$. Lemma 7.3 reveals that $E(x)$ is H-admissible for any $\ell \in \mathbb{N}_0$, $(p_1,\dots,p_\ell) \in \mathbb{N}_0^\ell$. Let $\tilde{a}_\ell, \tilde{b}_\ell$ be the functions (7.9) for $h(x) = \ln E(x)$ as defined in (7.16). Let $z_n = \rho e^{-\eta_n}$ be such that $z_n C'(z_n) = n$ and $q_n = \rho e^{-\xi_n}$ be the solution to $\sum_{j\ge1} C(q_n^j)/j = n$. In both cases necessarily $\eta_n, \xi_n \to 0$ as $n \to \infty$. Thus (7.17) yields $b_\ell(w_n) \sim A_{2,2}(w_n)$ for $w_n \in \{z_n, q_n\}$. Recalling (7.16) and using (7.18) for $0 < \rho < 1$ we further obtain for $0 < \rho \le 1$ that $\tilde{a}_\ell(w_n) = n + \mathcal{O}\big(\sum_{1\le i\le\ell} A_{1,2+p_i}(w_n)/A_{0,1+p_i}(w_n)\big)$. Then Lemma 7.2 delivers for $w_n \in \{z_n, q_n\}$

$$[x^n]E(x) = \frac{E(w_n)}{\sqrt{2\pi A_{2,2}(w_n)}} w_n^{-n}\left(\exp\left\{ \mathcal{O}\left(\left(\sum_{1\le i\le\ell} \frac{A_{1,2+p_i}(w_n)}{A_{0,1+p_i}(w_n)} \right)^2 \frac{1}{A_{2,2}(w_n)} \right) \right\} + o(1) \right).$$

According to (7.13), (7.18) for $0 < \rho < 1$ and (7.19), (7.20) for $\rho = 1$ we obtain for any $p \in \mathbb{N}_0$ that

$$\frac{A_{1,2+p}(w_n)^2}{A_{0,1+p}(w_n)^2 A_{2,2}(w_n)} = \mathcal{O}\left(\beta_n^{3\varepsilon}\right) = o(1), \quad \beta_n \in \{\xi_n, \eta_n\}.$$

Hence $[x^n]E(x) \sim E(w_n)/\sqrt{2\pi A_{2,2}(w_n)}\, w_n^{-n}$ for $w_n \in \{z_n, q_n\}$. Note that we can rewrite $E(x) = \exp\{A_{0,0}(x)\}\prod_{1\le i\le\ell} A_{0,p_i}(x)$. Hence, if $0 < \rho < 1$, we get according to (7.18) that asymptotically $E(z_n) \sim \exp\left\{\sum_{j\ge2} C(\rho^j)/j\right\} \exp\{C(z_n)\} C(z_n)^\ell$. In addition, if $\rho < 1$, (7.18) gives that $A_{2,2}(z_n) \sim z_n^2 C''(z_n)$. So, (2.19) follows. If $\rho = 1$ we use that $C''(q_n^j) = \omega(C'(q_n^j))$ for any $j \in \mathbb{N}$, which is true since C'', C' have non-negative coefficients, to obtain $A_{2,2}(q_n) \sim \sum_{j\ge1} j q_n^{2j} C''(q_n^j)$. This entails (2.20) and we are done. □

7.2.3 The Distribution of the Largest and Smallest Components

Proof of Theorem 2.11

Proof of Theorem 2.11. We conduct the proof for $\mathcal{L}(\mathsf{S}^{(n)})$ and $\mathcal{L}(\mathsf{G}^{(n)})$ simultaneously. Therefore let $\mathsf{F}^{(n)} \in \Omega_n$ be either $\mathsf{S}^{(n)}$ or $\mathsf{G}^{(n)}$ and we will specify when there is need to differentiate the two cases. We start with some statements about the values of $z_n = \rho e^{-\eta_n}$ and $q_n = \rho e^{-\xi_n}$ solving $z_n C'(z_n) = n$ and $\sum_{j\ge1} q_n^j C'(q_n^j) = n$, respectively. Set

$$w_n = \rho e^{-\beta_n} = \begin{cases} z_n, & \mathsf{F}^{(n)} = \mathsf{S}^{(n)} \text{ or } \mathsf{F}^{(n)} = \mathsf{G}^{(n)} \text{ and } 0 < \rho < 1 \\ q_n, & \mathsf{F}^{(n)} = \mathsf{G}^{(n)} \text{ and } \rho = 1 \end{cases} \quad \text{and} \quad \beta_n = \ln(\rho/w_n). \tag{7.28}$$

Since $n^{-\delta} \leq h(n) \leq n^{\delta}$ for n sufficiently large and any $0 < \delta < \alpha/5$ according to (A.2) we have that $C(x)$ is oscillating expansive with parameters $\alpha + \delta, \alpha/3 - 5\delta/3$ and $0 < \rho \leq 1$. Hence $n = z_n C'(z_n) = \mathcal{O}(\eta_n^{-(\alpha-\delta+1)}) \cap \Omega(\eta_n^{-(\alpha+\delta+1)})$ as well as $n = \sum_{j\geq 1} q_n^j C'(q_n^j) = \mathcal{O}(\xi_n^{-(\alpha-\delta+1)}) \cap \Omega(\xi_n^{-(\alpha+\delta+1)})$ according to (7.13), (7.18) and (7.19),(7.20), respectively. Thus

$$\beta_n = \mathcal{O}(n^{-1/(\alpha+\delta+1)}) \cap \Omega(n^{-1/(\alpha-\delta+1)}) \quad \text{and} \quad C(w_n) = \mathcal{O}(n^{(\alpha+\delta)/(\alpha-\delta+1)}) \cap \Omega(n^{(\alpha-\delta)/(\alpha+\delta+1)}). \tag{7.29}$$

Define the subset of $\Omega_n = \{(N_1, \ldots, N_n) \in \mathbb{N}_0^n : \sum_{1\leq k\leq n} kN_k = n\}$ which contains all cluster structures such that no cluster is larger than s by

$$\Omega_{n,\leq s} := \{(N_1, \ldots, N_n) \in \Omega_n : \forall s < i \leq n : N_i = 0\}.$$

Then

$$\Pr\left[\mathcal{L}(\mathsf{F}^{(n)}) \leq s\right] = \Pr\left[\mathsf{F}^{(n)} \in \Omega_{n,\leq s}\right]. \tag{7.30}$$

Further let the subset of Ω_n containing all cluster structures such that there are clusters of sizes $k_1, \ldots, k_\ell \in \mathbb{N}$ be given by

$$\Omega_{n,k_1,\ldots,k_\ell} := \{(N_1, \ldots, N_n) \in \Omega_n : \forall 1 \leq i \leq \ell : N_{k_i} \geq 1\}. \tag{7.31}$$

With this at hand, we obtain

$$\Omega_{n,\leq s} = \Omega_n \setminus \bigcup_{k>s} \Omega_{n,k}. \tag{7.32}$$

Define

$$A_\ell = A_\ell(n) := \sum_{\substack{s<k_1<\cdots<k_\ell \\ k_1+\cdots+k_\ell\leq n}} \Pr\left[\mathsf{F}^{(n)} \in \Omega_{n,k_1,\ldots,k_\ell}\right], \qquad \ell \in \mathbb{N}.$$

The inclusion/exclusion principle, which was also successfully employed for this kind of problem by [27], then yields in light of (7.30) and (7.32)

$$\Pr\left[\mathcal{L}(\mathsf{F}^{(n)}) \leq s\right] = \Pr\left[\mathsf{F}^{(n)} \in \Omega_n \setminus \bigcup_{k>s} \Omega_{n,k}\right] = \Pr\left[\mathsf{F}^{(n)} \in \Omega_n\right] + \sum_{\ell\geq 1}(-1)^\ell A_\ell = 1 + \sum_{\ell\geq 1}(-1)^\ell A_\ell. \tag{7.33}$$

Computing the union of events by the inclusion/exclusion principle entails the helpful "sandwich-property" that for any $M > 1$

$$1 + \sum_{1\leq\ell\leq 2M-1} (-1)^\ell A_\ell \leq \Pr\left[\mathcal{L}(\mathsf{F}^{(n)}) \leq s\right] \leq 1 + \sum_{1\leq\ell\leq 2M} (-1)^\ell A_\ell. \tag{7.34}$$

This has the great advantage that we can take large but fixed M when analysing $\mathcal{L}(\mathsf{F}^{(n)})$ and investigating $A_\ell = A_\ell(n)$ for fixed ℓ is much easier as we only need to let one parameter (namely n) tend to infinity. This reduction of complexity is as a matter of fact the foundation of this proof. Recall that for $t \in \mathbb{R}$ we defined

$$s_n = s(t, \beta_n) := \beta_n^{-1}\big(\ln X(\beta_n) + t\big), \quad X(\beta_n) := \frac{1}{\Gamma(\alpha)} C(w_n)(\ln C(w_n))^{\alpha-1} \frac{h(\beta_n^{-1}) \ln C(w_n)}{h(\beta_n^{-1})}.$$

Since $h(\beta_n^{-1}\ln C(w_n))/h(\beta_n^{-1}) = \mathcal{O}(\ln C(w_n))$ due to (A.2) we obtain

$$s_n \sim \beta_n^{-1}\ln C(w_n), \quad n\to\infty. \tag{7.35}$$

In order to get a grip on (7.33) we claim

$$\Pr\left[\mathsf{F}^{(n)} \in \Omega_{n,k_1,\dots,k_\ell}\right] \sim \prod_{1\le i\le \ell} c_{k_i} w_n^{k_i}, \quad \ell\in\mathbb{N},\ s_n < k_1 < \cdots < k_\ell < s_n + o(s_n) \tag{7.36}$$

and for any $\varepsilon > 0$ when n sufficiently large

$$\Pr\left[\mathsf{F}^{(n)} \in \Omega_{n,k_1,\dots,k_\ell}\right] \le (1+\varepsilon)\cdot \prod_{1\le i\le \ell} c_{k_i} w_n^{k_i}, \quad \ell\in\mathbb{N},\ 0<k_1,\dots,k_\ell. \tag{7.37}$$

The proof of (7.36) and (7.37), which is rather lengthy and relies heavily upon the underlying generating series to be H-admissible, is deferred to the end of this section for better readability. We start by showing that $A_1 \sim \mathrm{e}^{-x}$. Let $\nu \equiv \nu(\beta_n)$ be in $\omega(\beta_n^{-1}) \cap o(s_n)$, which is possible due to (7.35). Note further that $\nu = o(\beta_n^{-1}\ln C(w_n)) = o(n)$ as $n\to\infty$ according to (7.29). Then

$$A_1 = \left(\sum_{s_n<k\le s_n+\nu} + \sum_{k>s_n+\nu}\right)\Pr\left[\mathsf{F}^{(n)}\in\Omega_{n,k}\right] =: A_{1,1} + A_{1,2}.$$

We are first going to show that $A_{1,1} \sim \mathrm{e}^{-t}$. Due to (7.36) we obtain

$$A_{1,1} \sim \sum_{s_n<k\le s_n+\nu} c_k w_n^k = \sum_{s_n<k\le s_n+\nu} h(k)k^{\alpha-1}\mathrm{e}^{-\beta_n k} \sim h(s_n)s_n^{\alpha-1}\mathrm{e}^{-\beta_n s_n}\sum_{0<k\le\nu}\mathrm{e}^{-\beta_n k}$$
$$\sim h(s_n)s_n^{\alpha-1}\mathrm{e}^{-\beta_n s_n}\beta_n^{-1}. \tag{7.38}$$

Recalling $s_n \sim \beta_n^{-1}\ln C(w_n)$ from (7.35) and plugging in s_n into the expression in the previous display gives by Lemma 4.6

$$A_{1,1} \sim \frac{\Gamma(\alpha)h(\beta_n^{-1})\beta_n^{-\alpha}}{C(w_n)}\cdot \mathrm{e}^{-t} \sim \mathrm{e}^{-t}. \tag{7.39}$$

For $k > s_n+\nu$ we use the estimate (7.37) and obtain

$$\Pr\left[\mathsf{F}^{(n)}\in\Omega_{n,k}\right] = \mathcal{O}\left(c_k w_n^k\right).$$

With this at hand and since $\nu = \omega(\beta_n^{-1})$ Lemma A.4 reveals for $k > s_n+\nu$ that

$$A_{1,2} = \mathcal{O}\left(\sum_{k>s_n+\nu} c_k w_n^k\right) = \mathcal{O}\left(h(s_n)s_n^{\alpha-1}\mathrm{e}^{-\beta_n(s_n+\nu)}\beta_n^{-1}\right) = \mathcal{O}\left(\mathrm{e}^{-t}\mathrm{e}^{-\beta_n\nu}\right) = o(1). \tag{7.40}$$

It follows that $A_1 \sim \mathrm{e}^{-t}$. Next we show that

$$A_\ell \sim \frac{1}{\ell!}(\mathrm{e}^{-t})^\ell. \tag{7.41}$$

Then choosing $M \in \mathbb{N}$ sufficiently large in the upper and lower bound (7.34) we get that $\Pr\left[\mathcal{L}(\mathsf{F}^{(n)}) \le s_n\right]$ gets arbitrarily close to $1 + \sum_{\ell\ge1}(-1)^\ell(\mathrm{e}^{-t})^\ell/\ell! = \mathrm{e}^{-\mathrm{e}^{-t}}$ and the claim follows. Thus all is left to show is (7.41). Define $B := \{(k_1,\dots,k_\ell) \in \mathbb{N}^\ell : \forall 1 \le i \le \ell : k_i > s_n, k_1 + \dots + k_\ell \le n\}$ and

$$B_= := \{(k_1,\dots,k_\ell) \in B : 0 < k_1 \le \dots \le k_\ell, \exists 1 \le i < \ell : k_i = k_{i+1}\}.$$

In addition we need the sets

$$B_\le := \{(k_1,\dots,k_\ell) \in B : \forall 1 \le i \le \ell : k_i \le s_n + \nu\} \quad \text{and}$$
$$B_> := \{(k_1,\dots,k_\ell) \in B : \exists 1 \le i \le \ell : k_i > s_n + \nu\}.$$

Further set

$$f(k_1,\dots,k_\ell) := \prod_{1\le i\le k} c_{k_i} w_n^{k_i}, \quad (k_1,\dots,k_\ell) \in B.$$

With these definitions at hand we obtain by (7.36) and (7.37)

$$\begin{aligned}
A_\ell &= \Bigg(\sum_{\substack{(k_1,\dots,k_\ell)\in B_\le,\\ k_1<\dots<k_\ell}} + \sum_{\substack{(k_1,\dots,k_\ell)\in B_>,\\ k_1<\dots<k_\ell}} \Bigg) \Pr\left[\mathsf{F}^{(n)} \in B_{n,k_1,\dots,k_\ell}\right] \\
&\sim \sum_{\substack{(k_1,\dots,k_\ell)\in B_\le,\\ k_1<\dots<k_\ell}} f(k_1,\dots,k_\ell) + \mathcal{O}\Bigg(\sum_{\substack{(k_1,\dots,k_\ell)\in B_>,\\ k_1<\dots<k_\ell}} f(k_1,\dots,k_\ell) \Bigg) \\
&= \left(\frac{1}{\ell!} \sum_{B_\le} - \sum_{B_=} \right) f(k_1,\dots,k_\ell) + \mathcal{O}\left(\sum_{B_>} f(k_1,\dots,k_\ell) \right); \qquad (7.42)
\end{aligned}$$

where in the last line and in what follows we abuse the notation and write $\sum_{\mathcal{X}} = \sum_{x\in\mathcal{X}}$ for any set $\mathcal{X}$. We will prove

$$\frac{1}{\ell!} \sum_{B_\le} f(k_1,\dots,k_\ell) \sim \frac{1}{\ell!}(\mathrm{e}^{-t})^\ell, \quad \sum_{B_>} f(k_1,\dots,k_\ell) = o(1) \quad \text{and} \quad \sum_{B_=} f(k_1,\dots,k_\ell) = o(1), \quad (7.43)$$

from which (7.41) follows immediately in light of (7.42). We start with the first asymptotic identity in (7.43). The estimate $k_i \le s_n + \nu$ for all $1 \le i \le \ell$ implies that $k_1 + \dots + k_\ell \le \ell(s_n + \nu) = o(n)$ so that with (7.39)

$$\sum_{B_\le} f(k_1,\dots,k_\ell) = \sum_{B_\le} \prod_{1\le i\le \ell} c_{k_i} w_n^{k_i} = A_{1,1}^\ell \sim (\mathrm{e}^{-t})^\ell.$$

Next we show the second claim in (7.43). Let $(k_1,\dots,k_\ell) \in B_\ge$. Applying (7.40) we obtain

$$\sum_{B_>} f(k_1,\dots,k_\ell) = \sum_{B_>} \prod_{1\le i\le \ell} c_{k_i} w_n^{k_i} = \mathcal{O}\left(A_{1,2} \cdot A_1^{\ell-1}\right) = o(\mathrm{e}^{-t(\ell-1)}) = o(1).$$

Finally, we show the third asymptotic identity in (7.43). We get

$$\sum_{B_=} f(k_1,\dots,k_\ell) \le \binom{\ell}{2} \sum_{s_n<k_1,\dots,k_{\ell-1}\le n} f(k_1,k_1,k_2,\dots,k_\ell) = \mathcal{O}\left(\sum_{s_n<k\le n} c_k^2 w_n^{2k} \cdot A_{1,1}^{\ell-2} \right).$$

According to Lemma A.4 and analogous to (7.40) we find that

$$\sum_{k>s_n+\nu} c_k^2 w_n^{2k} = \mathcal{O}\left(h(s_n)^2 s_n^{2(\alpha-1)} \mathrm{e}^{-2\beta_n(s_n+\nu)} \beta_n^{-1}\right) = o(1).$$

And just as in (7.39) we compute that

$$\begin{aligned}\sum_{s_n<k\le s_n+\nu} c_k^2 w_n^{2k} \sim h(s_n)^2 s_n^{2(\alpha-1)} \mathrm{e}^{-2\beta_n s_n} \sum_{1\le k\le \nu} \mathrm{e}^{-2\beta_n k} &= \mathcal{O}\left((h(s_n) s_n^{\alpha-1} \mathrm{e}^{-\beta_n s_n} \beta_n^{-1})^2 \beta_n\right)\\ &= \mathcal{O}\left(\mathrm{e}^{-2t}\beta_n\right) = o(1).\end{aligned}$$

Since $A_{1,1}^{\ell-2} \sim \mathrm{e}^{-t(\ell-2)}$ this shows that $\sum_{B_=} f(k_1,\dots,k_\ell) = o(1)$ and we haven proven (7.43). In other words, we are done.

Proof of Equations (7.36) *and* (7.37). Let $\ell \in \mathbb{N}$, $\mathrm{k} := (k_1,\dots,k_\ell) \in \mathbb{N}^\ell$ and recall from (7.31)

$$\Omega_{n,\mathrm{k}} := \{(N_1,\dots,N_n) \in \Omega_n : \forall 1 \le i \le \ell : N_{k_i} \ge 1\}.$$

We further abbreviate

$$\sum_{\Omega_{n,\mathrm{k}}} = \sum_{(N_1,\dots,N_n)\in\Omega_{n,\mathrm{k}}}, \qquad \Sigma_\mathrm{k} := k_1 + \dots + k_\ell \qquad \text{and} \qquad [n]_{\neq \mathrm{k}} := \{1,\dots,n\} \setminus \{k_1,\dots,k_\ell\}.$$

From here we need to distinguish the two cases. Note that we call the cases “set” and “multiset” but as always in this paper we consider the general case where $(c_k)_{k\in\mathbb{N}}$ is an arbitrary non-negative real-valued sequence (which is of course expansive).

The set case. Due to (2.4) we obtain

$$\Pr\left[\mathsf{S}^{(n)} \in \Omega_{n,\mathrm{k}}\right] = \frac{1}{[x^n]S(x)} \sum_{\Omega_{n,\mathrm{k}}} \prod_{1\le i\le n} \frac{c_i^{N_i}}{N_i!} = \frac{1}{[x^n]S(x)} \cdot \prod_{1\le i\le \ell} c_{k_i} \cdot \sum_{\Omega_{n,\mathrm{k}}} \prod_{i\in[n]_{\neq \mathrm{k}}} \frac{c_i^{N_i}}{N_i!} \cdot \prod_{1\le i\le\ell} \frac{c_{k_i}^{N_{k_i}-1}}{N_{k_i}!}. \tag{7.44}$$

First observe that if $(N_1,\dots,N_n) \in \Omega_{n,\mathrm{k}}$ we necessarily have that $N_i = 0$ for $i > n - \Sigma_\mathrm{k}$. Since further for $N_{k_i} \ge 1$ the estimate $c_{k_i}^{N_{k_i}-1}/N_{k_i} \le c_{k_i}^{N_{k_i}-1}/(N_{k_i}-1)!$ holds for $1 \le i \le \ell$ we get that

$$\sum_{\Omega_{n,\mathrm{k}}} \prod_{i\in[n]_{\neq \mathrm{k}}} \frac{c_i^{N_i}}{N_i!} \cdot \prod_{1\le i\le\ell} \frac{c_{k_i}^{N_{k_i}-1}}{N_i!} \le \sum_{\Omega_{n-\Sigma_\mathrm{k}}} \prod_{i=1}^{n-\Sigma_\mathrm{k}} \frac{c_i^{N_i}}{N_i!} = [x^{n-\Sigma_\mathrm{k}}]S(x).$$

All in all, we have shown that

$$\Pr\left[\mathsf{F}^{(n)} \in \Omega_{n,\mathrm{k}}\right] \le \frac{[x^{n-\Sigma_\mathrm{k}}]S(x)}{[x^n]S(x)} \cdot \prod_{1\le i\le k} c_{k_i}. \tag{7.45}$$

Due to Lemma 7.3 we know that $S(x)$ is H-admissible. Recall the definition of $A_s(x)$ from (7.10) and note that the functions $a(x), b(x)$ from (7.9) for $f(x) = \ln S(x)$ are exactly $A_1(x), A_2(x)$, compare to (7.11) with

$\ell = 0$. We chose z_n such that $A_1(z_n) = z_n C'(z_n) = n$. Let $\varepsilon > 0$. Then Lemma 7.2 reveals that there is some (potentially large but fixed) $K = K(\varepsilon) > 0$ such that for n sufficiently large

$$\frac{[x^{n-\Sigma_\mathrm{k}}]S(x)}{[x^n]S(x)} \leq (1+\varepsilon)\cdot z_n^{\Sigma_\mathrm{k}}, \quad \text{uniformly in } \mathrm{k} \in \mathbb{N}^\ell \text{ with } 0 < \Sigma_\mathrm{k} \leq n - K. \tag{7.46}$$

For all $\mathrm{k} \in \mathbb{N}^\ell$ such that $n - K < \Sigma_\mathrm{k} \leq n$ we have that $[x^{n-\Sigma_\mathrm{k}}]S(x) = \mathcal{O}(1)$ and hence with Lemma 7.2, noting that $z_n \leq \rho \leq 1$,

$$\frac{[x^{n-\Sigma_\mathrm{k}}]S(x)}{[x^n]S(x)} = \mathcal{O}\left(z_n^n\sqrt{b(z_n)}/S(z_n)\right) = o\big(z_n^{\Sigma_\mathrm{k}}\big), \quad \text{uniformly in } n - K < \Sigma_\mathrm{k} \leq n. \tag{7.47}$$

In any case, combining (7.45) with (7.46)–(7.47), and since ℓ is fixed, we obtain for n sufficiently large

$$\Pr\left[\mathsf{S}^{(n)} \in \Omega_{n,\mathrm{k}}\right] \leq (1+\varepsilon)\cdot \prod_{1\leq i\leq \ell} c_{k_i} z_n^{k_i}, \quad \text{uniformly in } \mathrm{k} \in \mathbb{N}^\ell \text{ with } 0 < \Sigma_\mathrm{k} \leq n. \tag{7.48}$$

This shows (7.37). Let us next demonstrate that (7.36) is valid. In light of Equation (7.48) it is left to show that

$$\Pr\left[\mathsf{S}^{(n)} \in \Omega_{n,\mathrm{k}}\right] \geq (1+o(1))\cdot \prod_{1\leq i\leq \ell} c_{k_i} z_n^{k_i} \quad \text{for } s_n < k_1 < \cdots < k_\ell < s_n + o(s_n) \text{ and as } n \to \infty. \tag{7.49}$$

For that let $S_{\neq \mathrm{k}}(x)$ be the generating series of elements such that there are no clusters of sizes $k_1, \ldots, k_\ell$, that is,

$$S_{\neq \mathrm{k}}(x) = S(x)\cdot T_1(x), \quad \text{where } T_1(x) = \exp\Big\{-\sum_{1\leq i\leq \ell} c_{k_i} x^{k_i}\Big\}. \tag{7.50}$$

Note that for any $s_n < k < s_n + o(s_n)$ and by plugging in s_n we get analogous to (7.38) (where $\beta_n = \eta_n$ for $\mathsf{F}^{(n)} = \mathsf{S}^{(n)}$) that $c_k z_n^k = h(k)k^{\alpha-1}\mathrm{e}^{-\eta_n k} \leq h(s_n)s_n^{\alpha-1}\mathrm{e}^{-\eta_n s_n}(1+o(1)) \sim \eta_n = o(1)$. Hence

$$T_1(z_n) \sim 1. \tag{7.51}$$

Writing $\mathsf{S}^{(n)} = (\mathsf{S}_1^{(n)}, \ldots, \mathsf{S}_n^{(n)})$, we conclude

$$\Pr\left[\mathsf{S}^{(n)} \in \Omega_{n,\mathrm{k}}\right] \geq \Pr\left[\forall 1 \leq i \leq \ell : \mathsf{S}_{k_i}^{(n)} = 1\right] = \frac{[x^{n-\Sigma_\mathrm{k}}]S_{\neq \mathrm{k}}(x)}{[x^n]S(x)}\cdot \prod_{1\leq i\leq \ell} c_{k_i}. \tag{7.52}$$

We compute

$$\frac{[x^{n-\Sigma_\mathrm{k}}]S_{\neq \mathrm{k}}(x)}{[x^n]S(x)} = \frac{[x^{n-\Sigma_\mathrm{k}}]S(x)}{[x^n]S(x)} + \sum_{1\leq u\leq n} \frac{[x^{n-\Sigma_\mathrm{k}-u}]S(x)}{[x^n]S(x)}[x^u]T_1(x). \tag{7.53}$$

Analogous to (7.46) and (7.47) we obtain due to (7.51)

$$\sum_{1\leq u\leq n} \frac{[x^{n-\Sigma_\mathrm{k}-u}]S(x)}{[x^n]S(x)}[x^u]T_1(x) = \mathcal{O}\Big(z_n^{\Sigma_\mathrm{k}} \sum_{u\geq 1}[x^u]T_1(x)z_n^u\Big) = \mathcal{O}\big(z_n^{\Sigma_\mathrm{k}}T_1(z_n)\big) = o\big(z_n^{\Sigma_\mathrm{k}}\big). \tag{7.54}$$

Next we show that the first term on the right-hand side of (7.53) is asymtotically equal to $z_n^{\Sigma_{\mathrm{k}}}$. Since $\Sigma_{\mathrm{k}} = \mathcal{O}(\eta_n^{-1} \ln C(z_n)) = o(n)$ for $s_n < k_1 < \cdots < k_\ell < s_n + o(s_n)$ due to (7.29) and (7.35) we obtain by Lemma 7.2 as $n \to \infty$

$$\frac{[x^{n-\Sigma_{\mathrm{k}}}]S(x)}{[x^n]S(x)} = z_n^{\Sigma_{\mathrm{k}}} \left(\exp\left\{ - \frac{\big(A_1(z_n) - (n - \Sigma_{\mathrm{k}})\big)^2}{2A_2(z_n)} \right\} + o(1) \right).$$

Since $A_1(z_n) = n$ we just need to show that $\Sigma_{\mathrm{k}}^2 / b(z_n) = o(1)$. We compute, noting that $A_2(z_n) = \Theta(h(\eta_n^{-1})\eta_n^{-(\alpha+2)})$ according to Lemma 4.6,

$$\frac{\Sigma_{\mathrm{k}}^2}{A_2(z_n)} = \mathcal{O}\left(\eta_n^\alpha \cdot \ln^2 C(z_n) \cdot h(\eta_n^{-1})\right) = o(1). \tag{7.55}$$

Concluding, we obtain by plugging in (7.54) into (7.53) that

$$\frac{[x^{n-\Sigma_{\mathrm{k}}}]S_{\neq \mathrm{k}}(x)}{[x^n]S(x)} = (1 + o(1)) \cdot z_n^{\Sigma_{\mathrm{k}}}, \quad \text{for } s_n < k_1 < \cdots < k_\ell < s_n + o(s_n) \text{ and as } n \to \infty,$$

which in turn brings with (7.52) that (7.49) is true. This concludes the proof of (7.36). The set case is completed and we move on to the multiset case.

The multiset case. This case is proven almost analogously to the set case, which is why we will be sparing with details. Like in (7.44) we get due to (2.3)

$$\Pr\left[\mathsf{G}^{(n)} \in \Omega_{n,\mathrm{k}}\right] = \frac{1}{[x^n]G(x)} \cdot \prod_{1 \le i \le \ell} c_{k_i} \cdot \sum_{\Omega_{n,\mathrm{k}}} \prod_{i \in [n]_{\neq \mathrm{k}}} \binom{c_i + N_i - 1}{N_i} \prod_{1 \le i \le \ell} \binom{c_{k_i} + N_{k_i} - 1}{N_{k_i}} \frac{1}{c_{k_i}}.$$

It is easy to check that $\binom{a+b-1}{b}/a \le \binom{a+b-2}{b-1}$ for $a, b \in \mathbb{N}$. In addition $N_i = 0$ for any $i > n - \Sigma_{\mathrm{k}}$ if $(N_1, \dots, N_n) \in \Omega_{n,\mathrm{k}}$. Thus, since $c_{k_i}, N_{k_i} \in \mathbb{N}$ for $1 \le i \le \ell$ (otherwise the claims (7.36)–(7.37) are trivially true) we obtain that

$$\Pr\left[\mathsf{G}^{(n)} \in \Omega_{n,\mathrm{k}}\right] \le \frac{[x^{n-\Sigma_{\mathrm{k}}}]G(x)}{[x^n]G(x)} \cdot \prod_{1 \le i \le \ell} c_{k_i}.$$

Let w_n be given as in (7.28). Replacing S by G and z_n by w_n we obtain completely analogous to (7.48) that for any $\varepsilon > 0$ and sufficiently large n

$$\Pr\left[\mathsf{G}^{(n)} \in \Omega_{n,\mathrm{k}}\right] \le (1 + \varepsilon) \prod_{1 \le i \le \ell} c_{k_i} w_n^{k_i}, \quad \text{uniformly in } \mathrm{k} \in \mathbb{N}^\ell \text{ with } 0 < \Sigma_{\mathrm{k}} \le n,$$

proving (7.37). To finish the proof in the multiset case it suffices to show that

$$\Pr\left[\mathsf{G}^{(n)} \in \Omega_{n,\mathrm{k}}\right] \ge (1 + o(1)) \cdot \prod_{1 \le i \le \ell} c_{k_i} w_n^{k_i} \quad \text{for } s_n < k_1 < \cdots k_\ell < s_n + o(s_n) \text{ and as } n \to \infty. \tag{7.56}$$

Let $G_{\neq \mathrm{k}}(x)$ be the generating series of elements such that there are no clusters of sizes $k_1, \dots, k_\ell$, that is,

$$G_{\neq \mathrm{k}}(x) = G(x) \cdot T_2(x), \quad \text{where } T_2(x) = \exp\left\{ - \sum_{j \ge 1} \sum_{1 \le i \le \ell} c_{k_i} x^{j k_i} \right\}.$$

For any $s_n < k < s_n + o(s_n)$ and by plugging in s_n we get analogous to (7.38) that $c_k w_n^k = h(k)k^{\alpha-1}\mathrm{e}^{-\beta_n k} \leq h(s_n)s_n^{\alpha-1}\mathrm{e}^{-\beta_n s_n}(1+o(1)) \sim \beta_n = o(1)$. Hence

$$T_2(w_n) \sim 1. \tag{7.57}$$

Consequently, we are at the exact same starting point as in (7.50) and (7.51). Analogous to (7.52)–(7.54) we thus obtain as $n \to \infty$ and for $s_n < k_1 < \cdots < k_\ell < s_n + o(s_n)$ as $n \to \infty$

$$\Pr\left[\mathsf{G}^{(n)} \in \Omega_{n,\mathrm{k}}\right] \geq \prod_{1\leq i\leq \ell} c_{k_i} \cdot \left(\frac{[x^{n-\Sigma_\mathrm{k}}]S(x)}{[x^n]S(x)} + o\big(z_n^{\Sigma_\mathrm{k}}\big)\right). \tag{7.58}$$

Recall the definition of $A_{s,t}(x)$ from (7.15). The functions a, b from (7.9) for $f(x) = \ln G(x)$ are then given by $A_{1,1}(x), A_{2,2}(x)$, see also (7.16) with $\ell = 0$. Since $\Sigma_\mathrm{k} = o(n)$ for $s_n < k_1 < \cdots < k_\ell < s_n + o(s_n)$ we obtain by Lemma 7.2

$$\frac{[x^{n-\Sigma_\mathrm{k}}]S(x)}{[x^n]S(x)} = w_n^{\Sigma_\mathrm{k}}\left(\exp\left\{-\frac{\big(A_{1,1}(w_n) - (n-\Sigma_\mathrm{k})\big)^2}{2A_{2,2}(w_n)}\right\} + o(1)\right) \tag{7.59}$$

Start with $0 < \rho < 1$, i.e. $w_n = z_n$ in (7.28). Due to (7.18) we have in this setting $A_{1,1}(z_n) = A_1(z_n) + \mathcal{O}(1)$ and $A_{2,2}(z_n) \sim A_2(z_n)$. Hence analogous to (7.55) we get

$$\frac{\big(\tilde{a}(z_n) - (n-\Sigma_\mathrm{k})\big)^2}{2\tilde{b}(z_n)} \sim \frac{\Sigma_\mathrm{k}^2}{A_2(z_n)} = o(1). \tag{7.60}$$

Now consider $\rho = 1$ in which case $w_n = q_n$, chosen such that $A_{1,1}(q_n) = n$. Since $n^{-\delta} \leq h(n) \leq n^{\delta}$ for any $0 < \delta < \alpha/5$ and n sufficiently large due to (A.2) we have that $C(x)$ is oscillating expansive with parameters $\alpha + \delta, \alpha/3 - 5\delta/3$ and $0 < \rho \leq 1$. So, we compute with (7.20) that $A_{2,2}(q_n) = \Omega(\xi_n^{-(\alpha-\delta+2)})$ giving us with (7.29) and (7.35) that

$$\frac{\big(A_{1,1}(q_n) - (n-\Sigma_\mathrm{k})\big)^2}{A_{2,2}(q_n)} = \mathcal{O}\left(\frac{s_n^2}{A_{2,2}(q_n)}\right) = \mathcal{O}\left(\ln^2 C(z_n)\xi_n^{\alpha-\delta}\right) = o(1). \tag{7.61}$$

Plugging (7.59)–(7.61) into (7.58) yields (7.56) and we are done. □

Proof of Example 2.12. We get by [28, Lem. 4.2] that there is a constant $A(\alpha) > 0$ depending on α such that

$$C(z_n) = \Gamma(\alpha)(\eta_n^{-\alpha} + A(\alpha)) + \mathcal{O}(\eta_n) \quad \text{and} \quad z_n C'(z_n) = \Gamma(\alpha+1)(\eta_n^{-(\alpha+1)} + A(\alpha+1)) + \mathcal{O}(\eta_n).$$

This immediately gives us that $z_n C'(z_n) = n$ implies $\eta_n = \Gamma(\alpha+1)^{1/(\alpha+1)}n^{-1/(\alpha+1)} + o(n^{-1})$. Plugging this into $C(z_n)$ yields $C(z_n) = \Gamma(\alpha)\Gamma(\alpha+1)^{-\alpha/(\alpha+1)}n^{\alpha/(\alpha+1)} + \mathcal{O}(1)$. Hence, setting $f(n) = (n/\Gamma(\alpha+1))^{1/(\alpha+1)}$,

$$\ln X = \ln\left(\Gamma(\alpha)^{-1}C(z_n)(\ln C(z_n))^{\alpha-1}\right) = \alpha \ln f(n) + (\alpha-1)\ln\ln f(n) + (\alpha-1)\ln\alpha.$$

□

Proof of Corollary 2.13

Proof of Corollary 2.13. Let $\mathsf{F}^{(n)}$ be either $\mathsf{S}^{(n)}$ or $\mathsf{G}^{(n)}$. Set $F(x) = S(x)$ if $\mathsf{F}^{(n)} = \mathsf{S}^{(n)}$ and $F(x) = G(x)$ if $\mathsf{F}^{(n)} = \mathsf{G}^{(n)}$. Further we define the generating series for all elements such that the smallest object is of size greater than s by

$$F_{>s}(x) := \begin{cases} \exp\left\{\sum_{k>s} c_k x^k\right\}, & \mathsf{F}^{(n)} = \mathsf{S}^{(n)} \\ \exp\left\{\sum_{j\geq 1}\sum_{k>s} c_k x^{jk}/j\right\}, & \mathsf{F}^{(n)} = \mathsf{G}^{(n)} \end{cases}.$$

Then

$$\Pr\left[\mathcal{M}(\mathsf{F}^{(n)}) > s\right] = \frac{[x^n]F_{>s}(x)}{[x^n]F(x)}.$$

Since $(c_k)_{k\in\mathbb{N}}$ is oscillating expansive the same holds for $(c_k)_{k>s}$ for fixed $s \in \mathbb{N}$. Then Lemma 7.3 reveals that both $F_{>s}$ and F are H-admissible. Letting a, b and $a_{>s}, b_{>s}$ be the functions (7.9) we immediately see that $a(x) - a_{>s}(x)$ and $b(x) - b_{>s}(x)$ are bounded uniformly in $x < \rho$. Then Lemma 7.2 gives for any $w_n \to \rho$ as $n \to \infty$

$$\Pr\left[\mathcal{M}(\mathsf{F}^{(n)}) > s\right] = \frac{[x^n]F_{>s}(x)}{[x^n]F(x)} \sim \frac{F_{>s}(w_n)}{F(w_n)}\left(\exp\left\{-\frac{(a_{>s}(w_n)-n)^2}{2b_{>s}(w_n)}\right\}\exp\left\{\frac{(a(w_n)-n)^2}{2b(w_n)}\right\} + o(1)\right).$$

Choosing w_n as in Theorem 2.10 for the different cases depending on S, G and ρ as well as noting again that $a(w_n) - a_{>s}(w_n) = \mathcal{O}(1)$ we get that the exponents of the exponential functions in the previous display are $o(1)$ and so (as s is fixed and letting $w_n \to \rho$ if $\mathsf{F}^{(n)} = \mathsf{S}^{(n)}$)

$$\Pr\left[\mathcal{M}(\mathsf{F}_n) > s\right] \sim \frac{F_{>s}(w_n)}{F(w_n)} \sim \begin{cases} \exp\left\{-\sum_{1\leq k\leq s} c_k\rho^k\right\}, & \mathsf{F}^{(n)} = \mathsf{S}^{(n)} \\ \exp\left\{-\sum_{j\geq 1}\sum_{1\leq k\leq s} c_k w_n^{jk}/j\right\}, & \mathsf{F}^{(n)} = \mathsf{G}^{(n)} \end{cases}.$$

□

7.2.4 The Cluster Distribution

Proof of Corollary 2.14

Proof of Corollary 2.14. Due to (2.21) we have that for any $\ell \in \mathbb{N}$

$$\mathbb{E}\left[\kappa(\mathsf{S}^{(n)})^{\underline{\ell}}\right] = \frac{[x^n]\exp\{C(x)\}\,C(x)^\ell}{[x^n]\exp\{C(x)\}}.$$

An application of Theorem 2.10 delivers

$$\mathbb{E}\left[\kappa(\mathsf{S}^{(n)})^{\underline{\ell}}\right] \sim C(z_n)^\ell. \tag{7.62}$$

In particular $\mathbb{E}\left[\kappa(\mathsf{S}^{(n)})\right] \sim C(z_n)$, which is the starting point for our induction. Assume that $\mathbb{E}\left[\kappa(\mathsf{S}^{(n)})^{\ell}\right] \sim C(z_n)^{\ell}$ for $\ell \in \mathbb{N}$. There are constants $(d_1, \dots, d_\ell) = (d_1(\ell), \dots, d_\ell(\ell) \in \mathbb{R}^{\ell}$ such that

$$\begin{aligned}\mathbb{E}\left[\kappa(\mathsf{S}^{(n)})^{\underline{\ell+1}}\right] &= \mathbb{E}\left[\kappa(\mathsf{S}^{(n)})(\kappa(\mathsf{S}^{(n)})-1)\cdots(\kappa(\mathsf{S}^{(n)})-\ell+2)\right] \\ &= \mathbb{E}\left[(\kappa(\mathsf{S}^{(n)})^{\ell+1}\right] + \sum_{1\le i\le \ell} d_i \mathbb{E}\left[(\kappa(\mathsf{S}^{(n)})^{i}\right] \sim \mathbb{E}\left[(\kappa(\mathsf{S}^{(n)})^{\ell+1}\right] + \sum_{1\le i\le \ell} d_i C(z_n)^i,\end{aligned}$$

where we used the induction hypothesis in the last asymptotic identity of the previous display. Since (7.62) reveals that $\mathbb{E}\left[\kappa(\mathsf{S}^{(n)})^{\underline{\ell+1}}\right] \sim C(z_n)^{\ell+1}$ and $C(z_n)^{\ell+1} = \omega(C(z_n)^i)$ for all $1 \le i \le \ell$ the claim follows.

Next we want to compute $\mathbb{E}\left[\kappa(\mathsf{G}^{(n)})^{\ell}\right]$ for $0 < \rho < 1$. Setting $x^{\underline{0}} = 1$ for any $x \in \mathbb{R}$, define $B_k(x,y) := \sum_{j\ge1}(j-1)^{\underline{k-1}} C(x^j) y^{j-k}$. Then $\mathrm{d}/\mathrm{d}y G(x,y) = G(x,y)B_1(x,y)$ and $\mathrm{d}/\mathrm{d}y B_k(x,y) = B_{k+1}(x,y)$ for $k \in \mathbb{N}$. By a simple induction there exist real-valued constants $(d_{k_1,\dots,k_{\ell-1}})_{k_1,\dots,k_{\ell-1}\in\mathbb{N}_0} = (d_{k_1,\dots,k_{\ell-1}}(\ell))_{k_1,\dots,k_{\ell-1}\in\mathbb{N}_0}$ such that

$$\frac{\mathrm{d}^{\ell}}{\mathrm{d}y^{\ell}} G(x,y) = G(x,y)\Bigg(B_1(x,y)^{\ell} + \sum_{\substack{0\le k_1\le\cdots\le k_{\ell-1}\\ k_1+\cdots+k_{\ell-1}=\ell}} d_{k_1,\dots,k_{\ell-1}} \prod_{1\le i\le \ell-1} B_{k_i}(x,y)\Bigg). \tag{7.63}$$

Recall the definition of $A_{s,t}$ from (7.15). Clearly, for any $k \in \mathbb{N}$, we can rewrite $B_k(x,1) = A_{0,k}(x) + \sum_{1\le i\le k-1} b_i A_{0,i}(x)$ for some real-valued constants $(b_1, \dots, b_{\ell-1}) = (b_1(\ell), \dots, b_{\ell-1}(\ell))$. Hence together with (7.63) there are real-valued constants $(d'_{k_1,\dots,k_{\ell-1}})_{k_1,\dots,k_{\ell-1}\in\mathbb{N}_0} = (d'_{k_1,\dots,k_{\ell-1}}(\ell))_{k_1,\dots,k_{\ell-1}\in\mathbb{N}_0}$ such that

$$\frac{\mathrm{d}^{\ell}}{\mathrm{d}y^{\ell}} G(x,y)\bigg|_{y=1} = G(x,y)\Bigg(A_{0,1}(x)^{\ell} + \sum_{\substack{0\le k_1\le\cdots\le k_{\ell-1}\\ k_1+\cdots+k_{\ell-1}\le\ell}} d'_{k_1,\dots,k_{\ell-1}} \prod_{1\le i\le \ell-1} A_{0,k_i}(x)\Bigg). \tag{7.64}$$

Now (2.21) and Theorem 2.10 give us

$$\mathbb{E}\left[\kappa(\mathsf{G}^{(n)})^{\underline{\ell}}\right] = \frac{[x^n]\mathrm{d}^{\ell}/(\mathrm{d}y^{\ell})G(x,y)}{[x^n]G(x,y)} \sim A_{0,1}(z_n)^{\ell} + \sum_{\substack{0\le k_1\le\cdots\le k_{\ell-1}\\ k_1+\cdots+k_{\ell-1}=\ell}} d'_{k_1,\dots,k_{\ell-1}} \prod_{1\le i\le \ell-1} A_{0,k_i}(z_n).$$

Due to (7.18) we get that $A_{0,k}(z_n) \sim A_0(z_n) = C(z_n)$. Since for any $0 \le k_1 \le \cdots \le k_{\ell-1}$ we always have that $\prod_{1\le i\le\ell-1} A_{0,k_i}(z_n) \sim C(z_n)^{\ell'}$ for $\ell' < \ell$ and $C(z_n)^{\ell} = \omega(C(z_n)^{\ell'})$ we finally obtain that

$$\mathbb{E}\left[\kappa(\mathsf{G}^{(n)})^{\underline{\ell}}\right] \sim C(z_n)^{\ell}.$$

The claim $\mathbb{E}\left[\kappa(\mathsf{G}^{(n)})^{\ell}\right] \sim C(z_n)^{\ell}$ follows analogously to the induction after (7.62).

Let us now consider $\rho = 1$. Here we would also get (7.64) but we cannot simplify $A_{0,k}(z_n) \sim C(z_n)$ and thereby let all the terms but $A_{0,1}(z_n)^{\ell}$ asymptotically vanish; in fact, all the terms could (depending on $\alpha > 0$) play a role. This is why we are content with only computing $\mathbb{E}\left[\kappa(\mathsf{G}^{(n)})^{\ell}\right]$ for $\ell = 1, 2$ in this case. With (2.21) and Theorem 2.10 we have that

$$\mathbb{E}\left[\kappa(\mathsf{G}^{(n)})\right] = \frac{[x^n]G(x)\sum_{j\ge1} C(x^j)}{[x^n]G(x)} \sim \sum_{j\ge1} C(q_n^j).$$

Due to (2.21) and Theorem 2.10

$$\mathbb{E}\left[\kappa(\mathsf{G}^{(n)})^{\underline{2}}\right] = \frac{[x^n]G(x)\big(\sum_{j\ge1} C(x^j)\big)^2}{[x^n]G(x)} + \frac{[x^n]G(x)\sum_{j\ge1} jC(x^j)}{[x^n]G(x)} \sim \Bigg(\sum_{j\ge1} C(q_n^j)\Bigg)^2 + \sum_{j\ge1} jC(q_n^j).$$

Since $\mathbb{E}\left[\kappa(\mathsf{G}^{(n)})^{\underline{2}}\right] = \mathbb{E}\left[\kappa(\mathsf{G}^{(n)})^2\right] - \mathbb{E}\left[\kappa(\mathsf{G}^{(n)})\right]$ and $(\sum_{j\geq 1} C(q_n^j))^2 = \omega(\sum_{j\geq 1} C(q_n^j))$ the claim follows. □

Proof of Theorem 2.15

Proof of Theorem 2.15. **We start with the local limit theorem for $\kappa(\mathsf{S}^{(n)})$.** Set $N' = N'(n,t) := C(z_n) + L$ where $L = \lfloor C(z_n) + t\sqrt{C(z_n)/(\alpha+1)}\rfloor - C(z_n) = t\sqrt{C(z_n)/(\alpha+1)} + \mathcal{O}(1)$. We want to determine $\Pr\left[\kappa(\mathsf{S}^{(n)}) = N'\right] = [x^n y^{N'}]S(x,y)/[x^n]S(x)$. Let $z_n = \rho e^{-\eta_n}$ be such that $z_n C'(z_n) = n$ (implying that $\eta_n \to 0$ as $n \to \infty$). Let $C_1, C_2, \ldots$ be iid with probability generating function $C(z_n x)/C(z_n)$. Further let $S_p := \sum_{1\leq i\leq p} C_i$ for $p \in \mathbb{N}$ and set

$$\nu_p := \mathbb{E}\left[S_p\right] = p\frac{z_n C'(z_n)}{C(z_n)} \quad \text{and} \quad \sigma_p^2 := \mathrm{Var}\left(S_p\right) = p\left(\frac{{z_n}^2 C''(z_n) + z_n C'(z_n)}{C(z_n)} - \left(\frac{z_n C'(z_n)}{C(z_n)}\right)^2\right).$$

Then we obtain

$$[x^n y^{N'}]S(x,y) = \frac{z_n^{-n} C(z_n)^{N'}}{N'!}\Pr\left[S_{N'} = n\right]. \tag{7.65}$$

We have $\nu_{N'} = (C(z_n) + L)z_n C'(z_n)/C(z_n) = n + L z_n C'(z_n)/C(z_n)$. Further as $N' \sim C(z_n)$ we get with Lemma 4.6 $L z_n C'(z_n)/C(z_n) \sim t\cdot\sqrt{\alpha/(\alpha+1)}\cdot\sqrt{z_n^2 C''(z_n)/(\alpha+1)} \sim t\cdot\sqrt{\alpha/(\alpha+1)}\cdot\sigma_{N'}$. Hence Lemma 6.11 delivers for any $K > 0$ uniformly in $t \in [-K, K]$

$$\Pr\left[S_{N'} = n\right] = \Pr\left[S_{N'} = \nu_{N'} - (t\sqrt{\alpha/(\alpha+1)} + o(1))\sigma_{N'}\right] \sim e^{-t^2\alpha/(2(\alpha+1))}\frac{1}{\sqrt{2\pi z_n^2 C''(z_n)/(\alpha+1)}}.$$

We treat the remaining terms in (7.65) by Stirling's formula and using that $(1+a)^b = \exp\left\{b\ln(1+a)\right\} = \exp\left\{b(a - a^2/2 + a^3/3 - \cdots)\right\}$ for $b > 0, 0 < a < 1$ which gives us

$$\frac{C(z_n)^{N'}}{N'!} \sim \frac{e^{N'}}{\sqrt{2\pi C(z_n)}}\left(1 + \frac{L}{C(z_n)}\right)^{-N'} \sim \frac{e^{N'}}{\sqrt{2\pi C(z_n)}} e^{-L - t^2/(2(\alpha+1))}.$$

Plugging everything back together yields

$$[x^n y^{N'}]S(x,y) \sim e^{-t^2/2}\frac{\exp\left\{C(z_n)\right\}}{2\pi\sqrt{C(z_n) z_n^2 C''(z_n)/(\alpha+1)}}\cdot z_n^{-n}.$$

The claim follows by computing $[x^n]S(x) \sim \exp\left\{C(z_n)\right\}/\sqrt{2\pi z_n^2 C''(z_n)}\cdot z_n^{-n}$ due to Theorem 2.10 and dividing $[x^n y^{N'}]S(x,y)/[x^n]S(x)$.

Next we show the local limit theorem for $\kappa(\mathsf{G}^{(n)})$. We write $N' = N'(n,t) := C(z_n) + L$ where $L = \lfloor C(z_n) + t\sqrt{C(z_n)/(\alpha+1)}\rfloor - C(z_n) = t\sqrt{C(z_n)/(\alpha+1)} + \mathcal{O}(1)$. In what follows we want to determine the probability $\Pr\left[\kappa(\mathsf{G}^{(n)}) = N'\right] = [x^n y^{N'}]G(x,y)/[x^n]G(x)$. For that we need to repeat some notation from Section 2.2. Let m be the first index such that $c_m > 0$. For $n, N \in \mathbb{N}$ let x, y be the solution to the system of equations

$$xyC'(x) + mc_m\frac{x^m y}{1 - x^m y} = n, \quad yC(x) + c_m\frac{x^m y}{1 - x^m y} = N \quad \text{and} \quad x^m y < 1. \tag{7.66}$$

Further let u the solution to the system in the variable v

$$uh(u)^{1/(\alpha+1)} = v^{1/(\alpha+1)}. \tag{7.67}$$

Then Lemma 2.7 says that for $n, N, n-mN$ and v sufficiently large there are unique solutions $x_{n,N}, y_{n,N}$ and u_v solving (7.66) and (7.67), respectively. In particular, there is a slowly varying function $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that $u_v = v^{1/(\alpha+1)}/g(v)$. With this at hand, define

$$N_n^* = C_0 \cdot g(n) \cdot n^{\alpha/(\alpha+1)}, \quad \text{where } C_0 := \alpha^{-1}(\rho^{-m}\Gamma(\alpha+1))^{1/(\alpha+1)}.$$

First we are going to show that

$$C(z_n) \sim \alpha^{-1}\Gamma(\alpha+1)^{1/(\alpha+1)} \cdot g(n) \cdot n^{\alpha/(\alpha+1)}. \tag{7.68}$$

This implies that $C(z_n)/N_n^* \sim \rho^{m/(\alpha+1)} < 1$ so that we are able to use all the results for Case (I) from Sections 2.2 and 6, in particular Theorem 2.8(I) for the determination of $[x^n y^{N'}]G(x,y)$. Since $z_n C'(z_n) = n$ we necessarily have that $z_n = \rho \mathrm{e}^{-\eta_n}$ such that $\eta_n \to 0$ as $n \to \infty$ due to Lemma 4.6, which also gives us $z_n C'(z_n) \sim \Gamma(\alpha+1)h(\eta_n^{-1})\eta_n^{-(\alpha+1)}$. Hence

$$\eta_n^{-1}h(\eta_n^{-1})^{1/(\alpha+1)} = (n/\Gamma(\alpha+1))^{1/(\alpha+1)}(1+o(1)).$$

Since $n/\Gamma(\alpha+1) \to \infty$ it directly follows from (7.67) and the subsequent text that $\eta_n^{-1} \sim (n/\Gamma(\alpha+1))^{1/(\alpha+1)}/g(n)$. We also deduce that $g(n)^{\alpha+1} \sim h(\eta_n^{-1})$ by comparing the two representations of η_n^{-1}. Hence Lemma 4.6 yields

$$C(z_n) \sim \Gamma(\alpha)h(\eta_n^{-1})\eta_n^{-\alpha} \sim \alpha^{-1}\Gamma(\alpha+1)^{1/(\alpha+1)} \cdot g(n) \cdot n^{\alpha/(\alpha+1)}.$$

Consequently $\limsup N'/N_n^* < 1$ and $N'/N_n^* = \Theta(1)$. Let $(x_n, y_n) = (x_{n,N'}, y_{n,N'})$ be the solution to (7.66). Then Theorem 2.8(I) together with Theorem 2.10 reveal that

$$\frac{g_{n,N'}}{g_n} \sim \frac{\exp\left\{\sum_{j\ge 2} C(\rho^j)y_n^j/j\right\}}{\exp\left\{\sum_{j\ge 2} C(\rho^j)/j\right\}} \cdot \frac{\exp\{y_n C(x_n) - C(z_n)\}}{\sqrt{2\pi N' y_n x_n^2 C''(x_n)/((\alpha+1)z_n^2 C''(z_n))}} \cdot \left(\frac{z_n}{x_n}\right)^n \cdot y_n^{-N'}. \tag{7.69}$$

In the remaining proof we show that z_n/x_n and y_n are so close to 1 that the right-hand side of (7.69) is asymptotically $(2\pi N'/(\alpha+1))^{-1/2}\mathrm{e}^{-t^2/2}$. For that we first repeat some important properties of (x_n, y_n) from Lemma 6.16, that is,

$$x_n \sim \rho, \quad \limsup_{n\to\infty} y_n < \rho^{-m} \quad \text{and} \quad S_n := \frac{x_n^m y_n}{1 - x_n^m y_n} = \Theta(1). \tag{7.70}$$

Parameterise $x_n = z_n \mathrm{e}^{\delta_n}$ for an appropriate δ_n. We first show that

$$\delta_n = o(\eta_n). \tag{7.71}$$

By (7.66) and (7.70) we have that $y_n C(x_n) \sim C(z_n) + L \sim C(z_n)$ and $x_n y_n C'(x_n) \sim n = z_n C'(z_n)$. Plugging in $x_n = z_n \mathrm{e}^{\delta_n} = \rho \mathrm{e}^{-(\eta_n - \delta_n)}$ we obtain by Lemma 4.6

$$C(z_n) \sim y_n C(x_n) \sim y_n \Gamma(\alpha)h(\eta_n - \delta_n)(\eta_n - \delta_n)^{-\alpha} \sim y_n C(z_n)\frac{h(\eta_n - \delta_n)}{h(\eta_n)}\frac{(\eta_n - \delta_n)^{-\alpha}}{\eta_n^{-\alpha}}$$

implying that

$$y_n \frac{h(\eta_n - \delta_n)}{h(\eta_n)} \frac{(\eta_n - \delta_n)^{-\alpha}}{\eta_n^{-\alpha}} \sim 1. \tag{7.72}$$

Analogously

$$n \sim x_n y_n C'(x_n) \sim y_n z_n C'(z_n) \frac{h(\eta_n - \delta_n)}{h(\eta_n)} \frac{(\eta_n - \delta_n)^{-(\alpha+1)}}{\eta_n^{-(\alpha+1)}}$$

implying that

$$y_n \frac{h(\eta_n - \delta_n)}{h(\eta_n)} \frac{(\eta_n - \delta_n)^{-(\alpha+1)}}{\eta_n^{-(\alpha+1)}} \sim 1. \tag{7.73}$$

Combining (7.72) and (7.73) we obtain that $(\eta_n - \delta_n)/\eta_n \sim 1$ implying (7.71).

In what follows we use, without mentioning it every time, that $\delta_n = o(\eta_n)$ and Lemma 4.6 imply

$$z_n^k C^{(k)}(z_n)\delta_n = \Theta(h(\eta_n^{-1})\eta_n^{-(\alpha+k)}\delta_n) = o(h(\eta_n^{-1})\eta_n^{-(\alpha+k-1)}) = o(C^{(k-1)}(z_n)), \quad k \in \mathbb{N}.$$

Next we expand $C(z_n e^{\delta})$ at $\delta = 0$. Since $\delta_n = o(\eta_n)$ and $y_n = \mathcal{O}(1)$ we obtain that

$$\begin{aligned} y_n C(x_n) &= y_n C(z_n) + y_n z_n C'(z_n)\delta_n + y_n z_n^2 C''(z_n)\delta_n^2/2 + o(C''(z_n)\delta_n^2) \\ &= y_n C(z_n) + y_n z_n C'(z_n)\delta_n + o(y_n C'(z_n)\delta_n). \end{aligned} \tag{7.74}$$

Then the second identity in (7.66) gives

$$N' = y_n C(z_n) + y_n z_n C'(z_n)\delta_n + c_m S_n + o(C'(z_n)\delta_n).$$

Recalling that $N' = C(z_n) + L$ and dividing both sides by $C(z_n)$ entails that $y_n \sim 1$ and

$$y_n = 1 + \frac{L - c_m S}{C(z_n)} - \frac{z_n C'(z_n)}{C(z_n)}\delta_n + o\left(\frac{C'(z_n)}{C(z_n)}\delta_n\right). \tag{7.75}$$

We proceed similarly with the second identity in (7.66). Expanding $z_n e^{\delta} y_n C'(z_n e^{\delta})$ around $\delta = 0$ and using (7.71) yields

$$\begin{aligned} x_n y_n C'(x_n) &= y_n z_n C'(z_n) + y_n(z_n C'(z_n) + z_n^2 C''(z_n))\delta_n + \mathcal{O}\left(C'''(z_n)\delta_n^2\right) \\ &= y_n n + y_n z_n C''(z_n)\delta_n + o(C''(z_n)\delta_n). \end{aligned} \tag{7.76}$$

Note that Lemma 4.6 implies that $C''(z_n) = \Theta(C'(z_n)^2/C(z_n)) = \Theta(nC'(z_n)/C(z_n))$. Keeping this in mind, we plug in (7.76) and y_n from (7.75) into the first equation of (7.66) to obtain

$$\begin{aligned} n &= y_n n + y_n z_n^2 C''(z_n)\delta_n + o(C''(z_n)\delta_n) + m c_m S_n \\ &= n + n\frac{L - c_m S}{C(z_n)} + \delta_n\left(z_n^2 C''(z_n) - n\frac{z_n C'(z_n)}{C(z_n)}\right) + o(C''(z_n)\delta_n) + m c_m S_n. \end{aligned}$$

Since $S_n = \Theta(1)$ due to (7.70) this implies together with Lemma 4.6 that

$$\delta_n \sim -n\frac{L - c_m S}{C(z_n)}\left(z_n^2 C''(z_n) - n\frac{z_n C'(z_n)}{C(z_n)}\right)^{-1} \sim -\alpha\frac{L - c_m S}{n}.$$

This, in turn, implies with (7.75) that

$$y_n = 1 + (\alpha+1)\frac{L - c_m S_n}{C(z_n)} + o\left(\frac{L - c_m S_n}{C(z_n)}\right).$$

It follows that there are $\alpha_n = o(L/n)$ and $\beta_n = o(L/C(z_n))$ such that

$$\delta_n = -\alpha\frac{L - c_m S_n}{n} + \alpha_n \quad \text{and} \quad y_n = 1 + (\alpha+1)\frac{L - c_m S_n}{C(z_n)} + \beta_n. \tag{7.77}$$

Recall that $L \sim t\sqrt{C(z_n)/(\alpha+1)}$ and $S_n = \Theta(1)$ due to (7.70). Plugging this and the expressions for δ_n, y_n into (7.74) as well as using Lemma 4.6 for $C(z_n), z_nC'(z_n) = n, z_n^2C''(z_n)$ yields

$$\begin{aligned} y_nC(x_n) &= y_nC(z_n) + y_nz_nC'(z_n)\delta_n + y_nz_n^2C''(z_n)\delta_n^2/2 + o(C''(z_n)\delta_n^2) \\ &= C(z_n) + (\alpha+1)(L - c_mS_n) + \beta_nC(z_n) + -\alpha(L - c_mS) + \alpha_n n \\ &\quad - \alpha(\alpha+1)\frac{L^2}{C(z_n)} + z_n^2C''(z_n)\frac{\alpha^2}{2}\frac{L^2}{n^2} + o(1) \\ &= C(z_n) + L - c_mS_n + \beta_nC(z_n) + \alpha_n n - \frac{\alpha}{2}t^2 + o(1). \end{aligned}$$

Plugging this into the second identity of (7.66) entails

$$N' = y_nC(x_n) + c_mS_n \Rightarrow \beta_nC(z_n) + \alpha_n n = \frac{\alpha}{2}t^2 + o(1). \tag{7.78}$$

With (7.77) at hand we obtain that

$$\left(\frac{z_n}{x_n}\right)^n = \mathrm{e}^{-\delta_n n} = \mathrm{e}^{\alpha(L - c_mS_n) - \alpha_n n}$$

and

$$y_n^{-N'} \sim \left(1 + (\alpha+1)\frac{L - c_mS_n}{C(z_n)} + \beta_n\right)^{-C(z_n) + t\sqrt{C(z_n)/(\alpha+1)}} \sim \mathrm{e}^{-(\alpha+1)(L - c_mS_n) - t^2 - \beta_nC(z_n) + (\alpha+1)t^2/2}.$$

Combining the previous two displays with (7.78) delivers

$$\left(\frac{z_n}{x_n}\right)^n y_n^{-N'} = \mathrm{e}^{\alpha(L - c_mS_n) - \alpha_n n - (\alpha+1)(L - c_mS_n) - t^2 - \beta_nC(z_n) + (\alpha+1)t^2/2} \sim \mathrm{e}^{L - c_mS_n - t^2/2}.$$

From $N' = y_nC(x_n) + c_mS_n$ we directly get $y_nC(x_n) - C(z_n) = L - c_mS_n$ so that

$$\exp\{y_nC(x_n) - C(z_n)\} \cdot \left(\frac{z_n}{x_n}\right)^n \cdot y_n^{-N'} \sim \mathrm{e}^{-t^2/2}.$$

Since $\delta_n = o(\eta_n)$ as showed in (7.71) we obtain that $z_n^2C''(z_n) \sim x_n^2C''(x_n)$. Concluding, and plugging in $y_n \sim 1$, we obtain in (7.69)

$$\frac{g_{n,N'}}{g_n} \sim \frac{1}{\sqrt{2\pi N'/(\alpha+1)}}\mathrm{e}^{-t^2/2} \sim \frac{1}{\sqrt{2\pi C(z_n)/(\alpha+1)}}\mathrm{e}^{-t^2/2}$$

as claimed. □

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A Appendix: Slowly Varying Functions

Let $h : [1, \infty) \to (0, \infty)$ be a slowly varying function, that is, h is measurable and for any $\lambda > 0$

$$\lim_{x \to \infty} \frac{h(\lambda x)}{h(x)} = 1. \tag{A.1}$$

"Slowly varying" means essentially smaller than any polynomial, see also below in (A.2) for formal variants of this statement. All the results in this section for slowly varying strictly positive h do straightforwardly hold for $h[1, \infty) \to [0, \infty)$ such that h is eventually positive and (A.1) is valid.

Results on Slowly Varying Functions The theory presented in this chapter goes back to Jovan Karamata, who proved all the basic results in his works [48, 49, 50]. A thorough overview can be found in the comprehensive textbooks [13] or [53, Chapter IV]. Let us begin with the *Uniform Convergence Theorem* that will be useful later.

Theorem A.1 ([13, Thm. 1.2.1])**.** *The convergence in* (A.1) *is uniform for* λ *in any compact subset of* $(0, \infty)$.

Let us continue with the famous *Representation Theorem*, first obtained by [50] in the continuous setting and again for arbitrary measurable functions by [54], c.f. [13, Theorem 1.3.1]. It states that there exist bounded measurable functions $c(x)$ and $\varepsilon(x)$ such that

$$h(x) = c(x) \exp\left(\int_1^x \frac{\varepsilon(t)}{t} dt \right),$$

where, for some $c > 0$,

$$c(x) \to c \qquad \text{and} \qquad \varepsilon(x) \to 0 \quad \text{as } x \to \infty.$$

From this we immediately obtain for any $\delta > 0$ that there is an x_0 such that

$$x^{-\delta} \le h(x) \le x^{\delta} \quad \text{and} \quad \left(\frac{x}{x'}\right)^{\delta} \le \frac{h(x')}{h(x)} \le \left(\frac{x'}{x}\right)^{\delta} \quad \text{for all } x' \ge x \ge x_0. \tag{A.2}$$

Moreover, see [13, Theorem 1.5.3], we obtain that for any $\mu > 0$

$$\sup_{1 \le y \le x} h(y) y^{\mu} \sim h(x) x^{\mu} \quad \text{and} \quad \sup_{y \ge x} h(y) y^{-\mu} \sim h(x) x^{-\mu} \qquad \text{as } x \to \infty. \tag{A.3}$$

Let $\alpha > 0$. From here we consider the function

$$c(s) = h(s) s^{\alpha - 1}, \quad s \ge 1,$$

where h is continuous. Note that this is no restriction in our setting as c_n given in (2.13) is only defined for natural numbers, such that we can simply interpolate linearly to obtain continuity. We proceed with the following important result, known as *Karamata's Theorem*.

Theorem A.2 ([13, Prop. 1.5.8])**.** *Let* h *be slowly varying and* $\alpha > 0$*. Then for any* $a \ge 1$

$$\int_a^x c(t) dt \sim \alpha^{-1} h(x) x^{\alpha}, \quad \textit{as } x \to \infty.$$

We will be interested in sums rather than integrals. A simple trick and the "sub-polynomiality" of slowly varying functions will help us here. We doubt that the following statement was not known before, but we know of no reference.

Corollary A.3. *The previous theorem holds with $\int_a^x$ replaced by $\sum_a^{x-1}$ for $a \in \mathbb{N}$.*

Proof. The bounds in (A.2) guarantee that $c(s)^{-1}\sup_{0\le x\le 1} c(s+x) \le \sup_{0\le x\le 1}(1-x/s)^{\alpha} = 1$ and also $c(x)^{-1}\inf_{0\le x\le 1} c(s+x) \ge \inf_{0\le x\le 1}(1+x/s)^{-1} \sim 1-x/s$ for $s\to\infty$. Hence for any $\varepsilon > 0$ there is $s_0 \in \mathbb{N}$ such that

$$\left|\sup_{0\le x\le 1} c(s+x) - \inf_{0\le x\le 1} c(s+x)\right| \le c(s)\varepsilon \quad \text{for all} \quad s\in\mathbb{N}, s\ge s_0. \tag{A.4}$$

This helps us in proving the claimed statement as follows. Note that

$$\left|\sum_{s=s_0}^{x-1} c(s) - \int_{s_0}^{x} c(t)dt\right| \le \sum_{s=s_0}^{x-1}\left|c(s) - \int_{s}^{s+1} c(t)dt\right| \le \sum_{s=s_0}^{x-1}\left|\sup_{0\le x\le 1} c(s+x) - \inf_{0\le x\le 1} c(s+x)\right|.$$

By using (A.4) we infer that this sum is at most $\varepsilon\sum_{s=s_0}^{x-1} c(s)$. Hence

$$\left|1 - \frac{\sum_a^{x-1} c(s)}{\int_a^x c(t)dt}\right| \sim \left|1 - \frac{\sum_{s_0}^{x-1} c(s)}{\int_{s_0}^x c(t)dt}\right| \le \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, the proof is completed. □

Lemma A.4 ([34, Prop. 2])**.** *Let $\alpha > 0$ and h be an eventually positive and slowly varying function. Let $(b_n)_{n\in\mathbb{N}}, (t_n)_{n\in\mathbb{N}}$ be sequences such that $b_n \to b \in (0,\infty]$ and $b_n z_n \to \infty$ as $n\to\infty$. Then, as $n\to\infty$,*

$$\int_{b_n}^{\infty} h(xt_n)x^{\alpha}\mathrm{e}^{-x}dx \sim h(b_n t_n)\int_{b_n}^{\infty} x^{\alpha}\mathrm{e}^{-x}dx.$$

Another ingredient in our proofs is the following property of sums, where the terms depend on some slowly varying function. Let U be a non-decreasing right-continuous function on $\mathbb{R}$ such that $U(x) = 0$ for $x < 0$. Consider the Laplace-Stieltjes transform

$$\hat{U}(\chi) = \int_0^{\infty} \mathrm{e}^{-\chi x}\mathrm{d}U(x).$$

If U is a step function with jumps at the integers, that is, $U(s) = U(\lfloor s\rfloor)$ for all $s\in\mathbb{R}$, then

$$\hat{U}(\chi) = \sum_{s\ge 0} U(s)\mathrm{e}^{-\chi s}. \tag{A.5}$$

The next result is referenced to as *Karamata's Tauberian Theorem* and was derived in [49].

Theorem A.5 ([13, Thm. 1.7.1])**.** *Let $\alpha \ge 0, h$ slowly varying and $c > 0$. Then the following statements are equivalent.*

1. *$U(x) \sim \frac{c}{\Gamma(\alpha+1)} h(x)\, x^{\alpha}$ as $x\to\infty$.*
2. *$\hat{U}(\chi) \sim c\, h(\chi^{-1})\, \chi^{-(\alpha+1)}$ as $\chi \to 0$.*

A direct application of this theorem is that

$$\sum_{k\ge 1} h(k)k^{\alpha-1}\mathrm{e}^{-\chi k} \sim \Gamma(\alpha)h(\chi^{-1})\eta^{-\alpha} \quad \alpha > 0, \text{ as } \chi\to 0. \tag{A.6}$$