Asymptotic symmetries in FLRW and deformations of gravitational symmetry algebras

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"Not only is the Universe stranger than we think, it is stranger than we can think."

\- Werner Heisenberg

Erstgutachter: Prof. Dr. Ivo Sachs
Zweitgutachter: Prof. Dr. Dieter Lüst
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Zusammenfassung

Die vorliegende Arbeit untersucht Randeffekte der Gravitation in zweierlei Hinsicht. Teil I hat den Infrarotbereich von kosmologischen Raumzeiten, nämlich FLRW-Universen, zum Gegenstand. In Teil II beschreiben wir zwei Klassen von Algebren, die allgegenwärtig als Symmetriealgebren an Raumzeiträndern auftreten, und erforschen ihre Deformationen.

In Teil I erweitern wir die Analyse asymptotisch flacher Raumzeiten an der zukünftigen Nullunendlichkeit auf räumlich flache FLRW-Modelle mit sich verlangsamender Ausdehnung. Neben ihrer phänomenologischen Relevanz für die materie- und die strahlungsdominierte Stadien der kosmischen Geschichte dienen sie als Modell zur Beschreibung der asymptotischen Regionen realistischerer inflationärer Szenarien. Rein geometrisch betrachtet, definieren wir zunächst die Klasse von Raumzeiten als asymptotische, räumlich flache FLRW-Modelle mit sich verlangsamender Ausdehnung. Anschließend ermitteln wir diejenigen Transformationen, welche diese definierenden Eigenschaften erhalten, und berechnen deren Wirkung auf die asymptotischen Expansionskoeffizienten. Diese Analyse ist quasi unabhängig von der Dynamik und gilt für generische Gravitationstheorien. Daraufhin berechnen wir die asymptotischen Einstein-Gleichungen und stellen fest, dass die zeitliche Entwicklung der asymptotischen Expansionskoeffizienten durch Quellterme beschränkt wird. Dies unterscheidet sich vom asymptotisch flachen Szenario, in dem die tensorartigen Freiheitsgrade propagieren. Gleichwohl finden wir asymptotische Ladungen, die mit den kosmologischen Supertranslationen zusammenhängen und deren Evolutionsgleichung einen Hubble-Term enthält.

In Teil II untersuchen wir die Deformationen der Algebra der Vektorfelder auf einer Kugel und von Heisenberg-Randalgebren. Erstere spielt eine wichtige Rolle bei der Beschreibung scheinbar nicht verwandter Themen, etwa der relativistischen bosonischen Membran und der asymptotischen Algebren von flachen und FLRW-Raumzeiten. Letztere tauchen in der Analyse von Raumzeiträndern auf, insbesondere an Ereignishorizonten. Deformationen geben Aufschluss über die Starrheit von Algebren und ihre Nähebeziehungen sowie über die Darstellungstheorie und die Eigenschaften der durch die Algebren beschriebenen physikalischen Systeme. Dabei stellen wir fest, dass die erste Algebra unter linearen Deformationen starr ist. Hingegen zeigt unsere Analyse der zweiten Algebra explizit auf, dass wir Algebren, welche die Symmetrien unterschiedlicher Raumzeitregionen mit verschiedenen Randbedingungen erhalten, durch Deformationen in Beziehung setzen können.

Abstract

This thesis conducts a two-fold study of gravity at the boundaries. In the first part, we explore the infrared regime of cosmological spacetimes, namely FLRW universes. In the second part, we describe two classes of algebras which appear ubiquitously as symmetry algebras in gravitational boundaries and investigate their deformations.

In part I, we extend the asymptotic analysis in flat spacetimes at future null infinity to decelerating and spatially flat FLRW cosmologies. Besides their phenomenological relevance, depicting matter- and radiation-domination stages, they serve as a model to describe the asymptotia of more realistic inflationary scenarios. From a geometrical perspective, we define the spacetimes to be considered asymptotically decelerating and spatially flat FLRW at future null infinity, obtain the residual transformations which preserve this class of metrics and their effect on the asymptotic data. This analysis has little input from the dynamics and applies to generic gravity theories. Next, we compute the asymptotic Einstein equations, observing that the time evolution of the asymptotic data is constrained by the sources. This situation differs from the asymptotically flat case where the tensor degrees of freedom are propagating. Nonetheless, we find asymptotic charges associated with the cosmological supertranslations and whose evolution equation includes a Hubble term.

In part II, we explore the deformations of the algebra of vector fields on the sphere and of Heisenberg boundary algebras. The former plays a major role in the description of apparently unrelated fields, such as the relativistic bosonic membrane and the asymptotic algebras in flat and FLRW spacetimes. The latter arise in gravitational boundary analysis, particularly at event horizons. Deformations inform us about rigidity of algebras and closeness relationships between them. Besides, they provide valuable information about representation theory and properties of the physical systems described by the algebras. In this regard, we find that the first algebra is rigid under linear deformations, whereas our analysis of the second explicitly shows that we can relate symmetry algebras obtained by imposing diverse boundary conditions at different spacetime loci via deformation procedure.

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The research within this dissertation has been performed in its major part during the global coronavirus pandemic and the author sadly witnessed the first war in Europe since several decades. I would like to conclude by thanking all those who have lost their health, time and lives on behalf of freedom and a brighter future. You will never be forgotten!

General introduction

Motivation

In the end of the 19th century, it was widely considered that nothing fundamentally new was awaiting to be discovered in Physics. Indeed, gravitation was described by Newton's classical mechanics [1], thermodynamics was well established and electromagnetism was unified by Maxwell [2]. Few small discrepancies with experiments remained, among which the description of the "anomalous" precession of Mercury's perihelion and Hydrogen's atom spectrum stood up. At a theoretical level, concepts like action at a distance and an artificial medium like ether were not satisfactory.

These apparent caveats fell into an unprecedented transformation of Physics during the 20th century. Our understanding of fundamental concepts like observers, space, time, mass and energy changed dramatically with Einstein's relativity theory [3, 4]. Right after, the collective work of Planck, Bohr, Heisenberg, Schrödinger, Pauli, Dirac and many others led to quantum mechanics, which instated the realm of probabilities, challenged determinism and whose interpretation still remains unclear [5–9]. These theories set the basis and paved the path of modern physics until our days, with high energy physics and gravitation being the closest fields to this dissertation. Other successful fields, highly influenced by these developments, are quantum optics, condense matter and solid state physics, information field theory, biophysics and econophysics.

It is fair to stand out the research in the ultraviolet regime. The first stage of this path has been extremely successful both at a theoretical and experimental level. It began with the fusion of special relativity and quantum mechanics in the common framework of quantum field theory [10–13], leading ultimately to the establishment of the Standard Model of particle physics, which describes three out of the four fundamental interactions, namely electromagnetism and the nuclear weak and strong forces [12–21]. Despite its impressive experimental success, the Standard Model fails, among others, at the description of neutrino masses, the incorporation of gravity and presents renormalizability issues such as the running coupling of the Higgs boson. These problems motivated the so-called Beyond the Standard Model physics, which attempts in various different ways to find solutions for all or some of the arising issues [22, 23]. Conservative approaches include slight modifications of the Standard Model, the seesaw mechanism [24] and effective field theory analysis. A more ambitious line tries to merge both gravity and quantum mechanics at the most fundamental level into a quantum gravity theory which should solve all the problems and describe our universe in all its complexity. In this regard, the most prominent is String Theory [25, 26], which extends the works of Kaluza-Klein in extra-dimensions [10, 27] and applies them for extended objects, together with supersymmetry [28] and advanced mathematics. Another popular path follows the canonical quantization procedure leading to the Wheeler-DeWitt equation [29] and whose most popular branch is Loop Quantum Gravity [30].

In parallel and highly correlated, the investigation of spacetime asymptotia and gravitational radiation, black holes, astrophysics and cosmology have probably been the major research areas in gravity. The existence of gravitational waves was a major prediction of General Relativity whose detection by the LIGO and VIRGO scientific collaboration had to wait until February 2016 [31]. These are disturbances in the curvature of spacetime that propagate as waves outward from their source at the speed of light. It turns out that, early in the days, it was not completely clear whether gravitational waves were just a mere mathematical artifice or had a real physical significance. In the sixties, the work of Bondi, van der Burg, Metzner and Sachs (BMS) shed light on this regard by investigating the flow of energy at infinity due to propagating gravitational waves [32–34]. For this reason, they performed an asymptotic analysis of spacetimes which looked like flat at future null infinity, independently of their appearance at other regions, also called asymptotically flat spacetimes. Their work showed that gravitational waves do carry radiation to infinity and that the symmetries of the spacetime perceived by observers located far away from all sources of the gravitational field form an infinite-dimensional extension of the Poincaré group. In the recent years, these symmetries have been related to soft theorems [35] and, in principle measurable, memory effects [36] into the infrared triangle [37, 38].

Another central prediction of General Relativity is the existence of spacetime regions from which nothing can escape, the so-called black holes. The first hint came from the hand of Schwarzschild [39], who found the most general spherically symmetric vacuum solution of the Einstein's field equations according to Birkhoff's theorem [40]. Such solution presented a region, nowadays called event horizon, behind which everything is trapped, even the light. Experimentally, the detection of the presence of a black hole in the center of our galaxy led to the shared Nobel prize by Genzel and Ghez in 2020, and the first direct picture of the shadow of a black hole was taken by the Event Horizon Telescope collaboration in April 2019. These objects were theoretically investigated in great detail by Penrose, leading to the singularity theorems [41–43], and several works completed a thermodynamic analogy of black holes with Hawking radiation [44–46]. Two interconnected apparent paradoxes were raised by Bekenstein and Hawking, one related to the entropy of these objects [47] and the other to its information during the process of evaporation [45, 48]. In this regard, extrapolations of the BMS analysis to the event horizon of certain black hole solutions reveal that the event horizons present infinitely many boundary symmetries which could account for the microstate structure of black holes and their associated entropy [49–53] in a way compatible with the membrane paradigm [54,55] and highly inspired by holography [56–59]. This path might lead to solving both puzzles without the explicit use of a quantum gravity candidate theory such as String Theory, where a more refined analysis previously pointed towards a similar resolution [60]. An important caveat to understand in this context is which of the symmetries are actually physically relevant. This question is far from clear due to the variety of outcomes from symmetry analysis performed at various boundaries under different boundary conditions.

A third crucial direction is cosmology, which attempts to describe the physical origins and evolution of the Universe. Its modern stage began with Einstein's 1917 static model of the universe [61] and was developed in its early days particularly through the work of Lemaître [62]. In the sixties, cosmology transitioned to a mainstream area of physics' research. Current cosmology is based on two fundamental assumptions: General Relativity as the correct theory of gravity and the Cosmological Principle, namely that the universe is spatially isotropic and homogeneous at large scales. Such assumptions have lead to the ΛCDM model based on spatially flat FLRW Universe [63]. In the current paradigm, also called Standard Model of cosmology, the picture of our universe from past to present in a nutshell is the following: a first stage when quantum effects are important, classical gravity is expected to fail and quantum gravity becomes indispensable for any reliable description. A posterior inflationary period of exponential, almost purely de Sitter, expansion leading to an uniform, almost flat universe with linear Gaussian and nearly scale invariant density perturbations. At a time of about one second after the beginning, the constituents of the universe included neutrons, protons, electrons, photons, and neutrinos, tightly coupled and in local thermal equilibrium. Nucleosynthesis of light elements took place during an explosion of nuclear interactions. As the temperature dropped below approximately 4000 Kelvin, electrons became bound in stable atoms, and photons decoupled from the matter with a black-body spectrum. With the expansion of the universe, the photons cooled down adiabatically, leading to the dark ages and retaining a black-body spectrum with a temperature inverse to the cosmic radius. This cosmic background radiation carries important information about the state of the universe at decoupling. After decoupling, baryonic matter consisted almost entirely of neutral hydrogen and helium. Once the first generation of stars formed, the dark ages came to an end with light from the stars, re-ionizying the universe. Cold dark matter dominated the early stages of the structure formation. The first generation of stars aggregated into galaxies, and galaxies into clusters. Massive stars end their lives in supernova explosions and spread through space heavy elements that have been created in their interiors, enabling formation of second generation stars surrounded by planets. Dark energy eventually came to dominate the expansion of the universe, leading to accelerated expansion.

Even though this model has been recently called into question, due to the apparent tension between the determination of the Hubble parameter from the early and late Universe [64–67], the experimental accuracy of the ΛCDM model to describe features like properties of the cosmic microwave background (CMB), Type Ia supernovae luminosity distance, large-scale structure, horizon problem, flatness profile or the absence of magnetic monopoles is impressive, considering how simple the model is.

In view of the above, we find astonishing that the asymptotic structure and symmetries of cosmological spacetimes had not been carefully addressed yet. From a phenomenological viewpoint, it is questionable to analyze the asymptotic structure of flat spacetimes, considering that at large radius we enter the cosmological regime. The first part of this dissertation attempts to extend the analysis that BMS performed at future null infinity of flat spacetimes to FLRW. This will serve to verify that BMS-like analysis are possible at cosmological settings and raise important differences at both technical and conceptual level. We will observe that these spacetimes present as asymptotic algebras non-central extensions of Diff(S^2), a feature that they share with asymptotically flat spacetimes and with other spacetime null boundaries in four-spacetime dimensions such as event horizons. The latter present also Heisenberg-like algebras as boundary symmetry algebras. The microstate description of these boundaries is expected to be closely related to the representations of the associated symmetry algebras. As a consequence, we devote the second part of this thesis to the study of Diff(S^2), boundary Heisenberg algebras and their deformations. The deformations serve us as a tool to quantify the relationships between different algebras obtained from various boundary conditions at diverse spacetime loci. Long term, this mathematical approach can provide us with a better understanding of the physical boundary degrees of freedom and hint to universal properties of their representations.

List of published papers

Parts of this thesis are reproductions of the author's publications. Some of the results presented here have been published in the following papers:

- [A] M. Enriquez-Rojo and T. Heckelbacher, *Asymptotic symmetries in spatially flat FRW spacetimes*, Phys. Rev. D **103** (2021) 064009 [2011.01960].

- [B] M. Enriquez-Rojo and T. Heckelbacher, *Holography and black holes in asymptotically flat FLRW*, Phys. Rev. D **103** (2021) 104035 [2102.02234].

- [C] M. Enriquez-Rojo, T. Procházka and I. Sachs, *On deformations and extensions of Diff(S^2)*, JHEP **10** (2021) 133 [2105.13375].

- [D] M. Enriquez-Rojo and H.R. Safari, *Boundary Heisenberg algebras and their deformations*, JHEP **03** (2022) 089 [2111.13225].

- [E] M. Enriquez-Rojo, T. Heckelbacher and R. Oliveri, *Asymptotic dynamics and charges for FLRW spacetimes*, [2201.07600].

In addition, the author has performed research in the field of string phenomenology during the period of his dissertation, leading to the following publication:

- [F] M. Enríquez Rojo and E. Plauschinn, *Swampland conjectures for type IIB orientifolds with closed-string U(1)s*, *JHEP* **07** (2020) 026 [2002.04050].

Structure of the thesis

In order to make the exposition clearer, this dissertation is divided in two parts. Part I deals with the asymptotic symmetry corner of FLRW spacetimes. Part II studies gravitational symmetry algebras within the mathematical framework of deformation theory. Each part contains its own introductory section into the subject as well as its own summary of results and conclusions. The final part of this doctoral thesis comprises the general conclusion, appendices and the corresponding bibliography.

Part I

Asymptotic symmetries in FLRW spacetimes

Chapter 1

Introduction of part I

In the first part of this dissertation, our goal is to investigate the infrared structure of asymptotically decelerating and spatially flat FLRW spacetimes. Due to the fact that FLRW spacetimes contain asymptotically flat spacetimes, which historically preceded our work, the reader will find our technical treatment self-contained without the need of an extensive review of the flat case. Nonetheless, we begin by shortly revisiting the infrared structure of asymptotically flat spacetimes, which serves as a guidance on the steps to perform and as a model to compare our results. Then, we focus on the asymptotic symmetry corner of these cosmological universes, studying both their geometry and their dynamics in General Relativity.

1.1 Motivation

Since its discovery in 1915, General Relativity [4] has been extensively explored. Nevertheless, the asymptotic structure of the theory was not investigated until the seminal work of Bondi, van der Burg, Metzner and Sachs (BMS) [33, 34]. Contrary to the intuitive idea that one should only recover the Poincaré group at future null infinity of asymptotically flat spacetimes, they unveiled a much richer set of asymptotic transformations which also included the so-called supertranslations. In those days, the literature around this topic was surrounded by mathematical formality (see e.g. [68–76]).

A couple of decades after BMS, Brown and Henneaux [56] applied a similar approach to three-dimensional anti-de-Sitter (AdS_3), noticing that the algebra of asymptotic diffeomorphisms (and their charges) corresponded to a two-dimensional conformal field theory (CFT). Their paper was followed by successful attempts to roughly estimate a microscopic description for the BTZ black hole entropy [49, 50, 77], and it was intimately related to the holographic current [57, 58, 78] falling into Maldacena's AdS/CFT correspondence [59].

The modern era of asymptotic symmetries started with the Kerr/CFT correspondence [79] and the inclusion of superrotations [80–82]. They were merged with memory effects and soft theorems into infrared triangles [37, 38]. It diversified into a wide variety of topics, among which we would like to highlight flat holography [83–91], black hole entropy [51, 92–100], algebraic oriented studies [101–106], extension to timelike [107] and spatial flat infinity [108], to dS_4 and AdS_4 [109], to string theory [110–113] and the swampland [114], to higher dimensions [115–117] and Kaluza-Klein [118], to the membrane paradigm [55, 119, 120] and to alternative gravity theories [121–124].

The infrared triangle

Very recently, a connection between several a priori unrelated research areas in gravity and gauge theories: asymptotic symmetries, soft theorems and memory effects, has been established (see [38] for a review). These fields are usually referred to as the three corners of the infrared triangle.

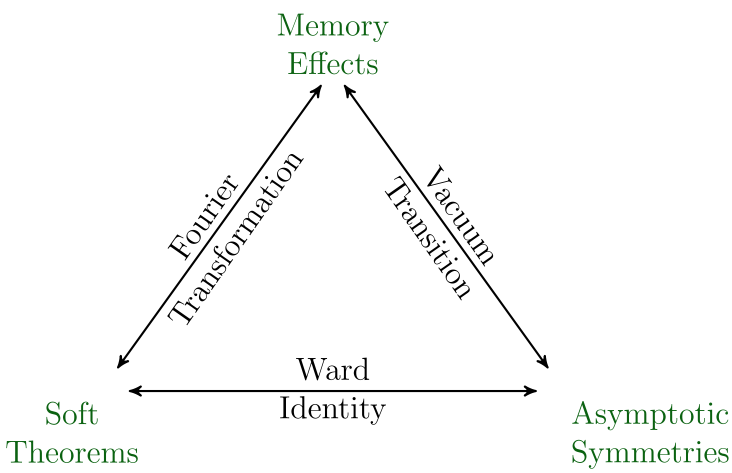

Figure 1.1: The infrared triangle.

The asymptotic symmetries corner is extensively studied in the first part of this thesis. Schematically, it comprises the residual gauge symmetries obtained at the boundaries of the spacetimes for gravity and gauge theories. The first step consists of fixing the gauge and boundary conditions, computing the non-trivial diffeomorphisms which preserve them and their algebra. Next, one obtains charges associated to these transformations following different methods and imposing multiple conditions. Upon application of the S-matrix formalism and Ward identities on these charges, one can derive their associated soft theorems [35,125]. These theorems state that any $(n+1)$-particles scattering amplitude involving a massless soft particle, namely a particle with momentum $q \to 0$ (that may be a photon, a gluon, a graviton ...), is equal to the n-particles scattering amplitude without the soft particle, multiplied by the soft factor, plus corrections of order $\mathcal{O}(q^0)$. The third corner of the triangle are the memory effects [36,126]. These measurable effects correspond to the permanent effect caused in physical systems by the passage of radiation associated to the propagation of degrees of freedom in gauge theories and gravity (e.g. gravitational waves in gravity or electromagnetic radiation in electrodynamics). Memory effects can be obtained from Fourier transforming soft theorems and have been found to be equivalent to performing asymptotic transformations which mediate between different "boundary vacua" [37].

We will review the infrared triangle for gravity in asymptotically flat spacetimes in more detail in chapter 2.

What about cosmology?

Rather astoundingly, the literature regarding the infrared structure of cosmological spacetimes is very limited. The first attempt belongs to Hawking who proposed that the asymptotic symmetry group of asymptotically FLRW spacetimes reduces to its global symmetry group [127] [1]. In the last years, several related studies have been performed in various directions: from the study of FLRW at timelike infinity [128] to the asymptotic symmetries with non-vanishing cosmological constant [109, 129–131]; from the relation between adiabatic modes and soft theorems [132–136] to memory effects in de-Sitter and ΛCDM cosmologies [137–143].

Our work is framed in this context with delving into the yet not well-understood asymptotic symmetry corner of the cosmological infrared triangle being our main objective. More concretely, we study the geometry and dynamics of asymptotically decelerating and spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes at future null infinity $\mathcal{I}^+$.

There are several reasons to perform and deepen into these studies. From a phenomenological point of view, our universe is not asymptotically flat, therefore, it is essential to transition from asymptotically flat towards cosmological spacetimes, and FLRW is the most natural candidate to begin with. Even though experimental data suggests that we live in a FLRW with accelerating expansion [144], we choose a decelerating FLRW in order to have a (conformal) boundary at null infinity. As our universe went through a phase of decelerated expansion, describing radiation and matter domination, we can imagine ourselves as observers looking at a decelerated universe from null infinity. This is of course only true as an approximation and we would have to extend our analysis to accelerated FLRW spacetimes to obtain more realistic results.

It also proves rewarding to investigate whether the increasingly refined technical tools and relations, introduced in the context of asymptotically flat spacetimes, hold in more realistic scenarios. A prominent example is to discern whether the infrared triangle [37, 38] connecting asymptotic symmetries, soft theorems and memory effects in asymptotically flat spacetimes survives in cosmological spacetimes and, either way, which are the possible modifications and interpretation.

[1] Nevertheless, only very specific dust-filled universes with negative spatial curvature were considered, while the treatment performed in this thesis deals with spatially flat universes allowing for general matter content.

In this context, we focus on the asymptotic symmetry corner and succeed to describe the class of spacetimes which asymptote to decelerating and spatially flat FLRW at future null infinity. From a geometrical perspective, we obtain consistent asymptotic transformations and their algebra which corresponds to a one-parameter deformation of the asymptotic algebra in flat spacetimes for diverse boundary conditions. Afterwards, we develop a dynamical analysis and realize that, in contrast to the asymptotically flat case, the tensor degrees of freedom are determined at the infinity, such that the dynamics is completely frozen. Moreover, we find an expression for the asymptotic charges which are associated with the cosmological supertranslations and incorporate a novel Hubble term.

1.2 Outline and notation

This first part is formed by four chapters in addition to this introduction. Let us briefly outline their contents:

- In chapter 2, we briefly revise the aspects of asymptotically flat spacetimes that will be of use in the next chapters. We dedicate most of this chapter to section 2.1, where we review the different possibilities for asymptotic symmetries in flat spacetimes, comprising supertranslations, several possibilities for superrotations and local Weyl symmetry. In section 2.2, we shortly comment on the other two corners of the infrared triangle, namely soft theorems and memory effects, particularly focusing on their connection with supertranslations. At last, we revise in section 2.3 how the connection between the membrane paradigm and BMS symmetries selects Diff(S^2) as the privileged transformations in the superrotation sector.

- In chapter 3, we study the geometry of asymptotically FLRW spacetimes. We begin by describing the asymptotia of spatially flat FLRW in section 3.1, which leads us to restrict ourselves to the decelerating case due to the presence of a future null infinity $\mathcal{I}^+$. In section 3.2, we specify the conditions that a spacetime has to fulfill in order to be considered asymptotically decelerating spatially flat FLRW at $\mathcal{I}^+$, leading to our ansatz metrics. A major price of going from asymptotically flat to asymptotically FLRW is that now we have to consider a time-dependent boundary metric instead of the simpler time-independent Minkowski. Nevertheless, in section 3.3, we manage to obtain consistent asymptotic transformations, whose algebra(s) is found to be a one-parameter deformation of its flat counterpart(s) for several boundary conditions. The deformation parameter is directly related to the equation of state of the fluid, pointing towards a cosmological holographic flow which connects the asymptotic algebra of flat spacetimes with that of FLRW. Nevertheless, when we allow for the more general boundary conditions, this deformation becomes trivial and leads to an isomorphism. At last, we investigate in section 3.4 if our ansatz includes several cosmological black hole solutions and conclude that we have to enlarge it and allow for logarithmic terms. The resultant ansatz does not satisfy the peeling property but preserves intact the asymptotic algebra. The required asymptotic Lie derivatives for this analysis and the computation of the Weyl scalars are relegated to the appendices A and B.

- In chapter 4, we delve into the dynamics of asymptotically decelerating and spatially flat FLRW universes at $\mathcal{I}^+$. In section 4.1, we calculate the asymptotic Einstein equations for our metrics with finite fluxes and explicitly show that the dynamics is totally constrained by the stress-tensor of the sources, contrary to the asymptotically flat case where the tensor degrees of freedom are not determined at the boundary. Finally, in section 4.2, we postulate an expression for the asymptotic charges which are associated with the cosmological supertranslations and whose evolution equation presents a novel contribution arising from the Hubble-Lemaître flow.

- In chapter 5, we gather our results and present our conclusions.

Notation: We generally use "mathfrak" font for the algebras, e.g. $\mathfrak{bms}$ for the BMS algebra. The term *boundary* is widely used as a replacement of "limiting causally defined spacetime hypersurface", these can be either located at finite distance (e.g. black hole event horizons) or at infinite distance (e.g. future null infinity). By GKV we denote global Killing vectors and by CKV we refer to conformal Killing vectors. Along this part I and following a practice extended in the literature, we utilize the terms Diff(S^2) and $\mathfrak{vect}(S^2)$ interchangeably. Indices on the sphere are denoted by capital latin letters $A, B, C, \ldots$. These indices are raised and lowered with the leading term q_{AB} of the expansion of the metric on the sphere. D_A denotes the covariant derivative with respect to q_{AB}. The Ricci scalar on the 2-sphere is denoted by $\mathcal{R}$, while R_{flat} and R^{FLRW} denote the Ricci scalar on the 4-manifold of asymptotically flat and exact FLRW spacetime. $\triangle G_{\mu\nu} \equiv G_{\mu\nu} - G^{\text{FLRW}}_{\mu\nu}$ stands for the difference between the Einstein tensor of asymptotically FLRW and exact FLRW. We use δ for the variations along the phase space, e.g. δ_f denotes the action on the phase space of a vector field generated by f. The Hubble scale is given by $H = \partial_u a$, where a is the conformal expansion scale factor of FLRW.

Chapter 2

Review of asymptotically flat spacetimes

In this chapter, we give an overview of the essential aspects of asymptotically flat spacetimes which will play a role in the upcoming chapters. We focus mostly on the asymptotic symmetry corner of the infrared triangle (fig. 1.1), where we introduce the different possibilities for fixing the asymptotic diffeomorphisms and the schematic structure of the associated charges. Next, we briefly discuss the other two corners, namely the memory effects and soft theorems. Finally, we depict a fascinating connection between the set of symmetries and conserved charges of the BMS group and those of a fluid membrane at future null infinity.

This chapter is largely inspired by the reviews [38, 145, 146], as well as the works [55, 142, 147–149].

2.1 Asymptotic symmetries

In this section, we briefly review the modern approach to asymptotically flat spacetimes at future null infinity $\mathcal{I}^+$, paving the way for the analysis in the next chapters. The following discussion is mostly based on [38, 142, 145, 147–149].

Before starting, let us provide the reader with the conformal diagram of Minkowski spacetime, where the red and green lines represent, respectively, lightlike and timelike radial null geodesics, $i^\pm$ denote future/past timelike infinity, i^0 corresponds to spatial infinity and $\mathcal{I}^\pm$ indicate future/past null infinity. For more details we refer the reader to [38, 63, 145, 146].

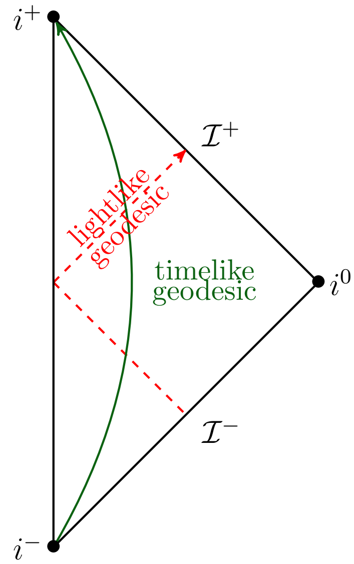

Figure 2.1: Conformal diagram of Minkowski spacetime.

General procedure

Since the pioneer works of Bondi, van der Burg, Metzner [33] and Sachs [34] (BMS), many studies have been performed allowing for different falloff conditions on the metric and on the diffeomorphisms generating the asymptotic transformations. A common feature most of these approaches share is the use of Bondi coordinates

$$u = t - \sqrt{x^i x_i}\ , \quad r = \sqrt{x^i x_i}\ , \quad z = \frac{x^1 + i x^2}{x^3 + \sqrt{x^i x_i}}\ , \quad \bar{z} = \frac{x^1 - i x^2}{x^3 + \sqrt{x^i x_i}}\ , \tag{2.1.1}$$

adequate to describe the asymptotic metrics near $\mathcal{I}^+$, together with the Bondi gauge

$$g_{rr} = g_{rA} = 0\ , \quad \partial_r \det\left(\frac{g_{AB}}{r^2}\right) = 0\ , \tag{2.1.2}$$

which completely fixes the local diffeomorphism invariance.

Nevertheless, we still need to specify what we consider by asymptotic flatness. This is accomplished through a choice of falloff conditions on the metric components at large r. The asymptotic symmetries are generated by those diffeomorphisms that preserve the Bondi gauge (2.1.2), as well as the selected boundary conditions. Therefore, a final ingredient are the large-r falloff conditions on the diffeomorphisms. The relationships between these diffeomorphisms can be investigated at $\mathcal{I}^+$, defining the so-called asymptotic algebra of diffeomorphisms.

So far, we only described the geometry of these spacetimes and did not make use of the equations of motion, such that this procedure remains valid for general gravitational theories. The next step is to select a concrete theory, which along this work will always be General Relativity, and to study the specific asymptotic dynamics.

Once the equations of motion, residual diffeomorphisms and their algebra are derived, one can compute the associated charges whose algebra is usually denoted as asymptotic symmetry algebra. Obtaining consistent asymptotic charges is, in general, a complicated task that has been performed in many increasingly difficult and formal ways. A first approach is to try to "guess" the form of the charges and check that they are in the adequate representation of the asymptotic algebra (e.g. [38, 51]). Another path is the usage of the covariant phase space formalism to derive the charges directly from the action in a Lagrangian [150] or Hamiltonian [151,152] framework. This method usually leads to ambiguities which can be fixed in some cases by adding boundary terms according to the holographic renormalization procedure (e.g. [149, 153–155]). We will not delve into more details in this thesis, as we will only follow the first method for obtaining supertranslation charges for asymptotically FLRW.

Supertranslations

The first possibility, already explored by BMS [33,34] are the so-called supertranslation diffeomorphisms. Supertranslations are derived from a rather restrictive choice of boundary conditions, which still allows for interesting physical solutions, given by

$$\begin{aligned} g_{uu} = -1 + \mathcal{O}(r^{-1}) \ , \quad g_{ur} = -1 + \mathcal{O}(r^{-2}) \ , \quad g_{uz} = \mathcal{O}(1) \ , \\ g_{zz} = \mathcal{O}(r) \ , \quad g_{z\bar{z}} = r^2\gamma_{z\bar{z}} + \mathcal{O}(1) \ , \quad g_{rr} = g_{rz} = 0 \ , \end{aligned} \tag{2.1.3}$$

where $\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$ is the round metric in the sphere, and

$$\xi^u, \xi^r \sim \mathcal{O}(1) \ , \quad \xi^z, \xi^{\bar{z}} \sim \mathcal{O}(r^{-1}) \ . \tag{2.1.4}$$

At large r, the structure of the metric is constrained to be of the form [1]

$$\begin{aligned} \mathrm{d}s^2 = &-\left(1 - \frac{2m}{r}\right)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + D^z C_{zz}\mathrm{d}u\mathrm{d}z + D^{\bar{z}} C_{\bar{z}\bar{z}}\mathrm{d}u\mathrm{d}\bar{z} + \frac{4r^2}{(1+z\bar{z})^2}\mathrm{d}z\mathrm{d}\bar{z} \\ &+ rC_{zz}\mathrm{d}z^2 + rC_{\bar{z}\bar{z}}\mathrm{d}\bar{z}^2 + \frac{1}{r}\left(\frac{4}{3}(N_z + u\partial_z m) - \frac{1}{4}\partial_z(C_{zz}C^{zz})\right)\mathrm{d}u\mathrm{d}z + \text{c.c.} + \ldots \end{aligned} \tag{2.1.5}$$

and the asymptotic Killing vectors are given by

$$\xi(f(z,\bar{z})) = f\partial_u + D^A D_A f \partial_r - \frac{1}{r} D^A f \partial_A + \ldots \qquad A = z, \bar{z} \ . \tag{2.1.6}$$

[1] Although this treatment is purely geometrical, one way to introduce this ansatz is to adopt the falloff conditions (2.6) in [147] for the stress energy tensor and to use the Einstein equations. Such assumptions are motivated by the behaviour of radiative scalar field solutions in Minkowski spacetime.

m, N_{AB} and C_{AB} denote, respectively, the Bondi mass aspect, angular momentum aspect and tensor degrees of freedom. Note from (2.1.6) that the asymptotic Killing vectors are fully determined by smooth functions on the sphere $f(z,\bar{z}) \in \mathcal{C}^\infty(S^2) \equiv \mathfrak{s}$. Expanding f in spherical harmonics, one can prove that the modes $l = 0, 1$ correspond (to leading order) respectively to time and space translation generators [38], while the remaining modes can be roughly interpreted as "angle-dependent translations associated to the conservation of energy flux at every angle" [38,55,149].

The associated asymptotic charges have the generic structure

$$Q_f = \frac{1}{4\pi}\int_{S^2} \sqrt{\gamma} f(x^A) m + \dots \ , \tag{2.1.7}$$

where γ is the determinant of the round metric on S^2, $x^A = \{z, \bar{z}\}$ and the $\dots$ represent terms which can be added to take care of conventions, integrability of charges or other issues (see e.g. [156]).

Superrotations

The next option, initiated by [80–82], comprises the superrotation diffeomorphisms. There are several different possibilities which we shortly summarize.

The transformations can be enlarged relaxing some of the previous conditions, the solutions being still physically acceptable. We consider now diffeomorphisms with the following large r behaviour

$$\xi^u \sim \mathcal{O}(1) \ , \quad \xi^r \sim \mathcal{O}(r) \ , \quad \xi^z, \xi^{\bar{z}} \sim \mathcal{O}(1) \ . \tag{2.1.8}$$

The novel $\mathcal{O}(1)$ modes in ξ^A allow for extra asymptotic symmetries naturally linked to rotations and boosts:

$$\begin{aligned} \xi(V^A(z,\bar{z})) =& V^A\partial_A + \frac{u}{2} D_A V^A \partial_u - \frac{r}{2} D_A V^A \partial_r \\ & - \frac{u}{2r} D^A D_B V^B \partial_A + \frac{u}{4} D_B D^B D_A V^A \partial_r + \dots \end{aligned} \tag{2.1.9}$$

Therefore, the enhanced asymptotic symmetries are given by:

$$\xi(f, V^A) = \xi(f) + \xi(V^A) \ . \tag{2.1.10}$$

Nevertheless, the previous falloff conditions for the metric (2.1.3) are not generally preserved under (2.1.9). Concretely, terms with $\delta g_{AB} \sim \mathcal{O}(r^2)$ and $\delta g_{uu} \sim \mathcal{O}(1)$ arise. This leads us to the following possibilities analyzed in the literature:

- *Original BMS*: $\mathfrak{b} \simeq \mathfrak{so}(1,3) \ltimes \mathfrak{s}$ [33,34]

  If the only allowed superrotations are $V^A = 1, z, z^2, i, iz, iz^2$, that is, the six global CKV on S^2, one can show to leading order in r that (2.1.9) generates the Lorentz transformations [38]. Therefore, together with the $l = 0, 1$ modes of f in (2.1.6), we recover the Poincaré algebra from (2.1.10). Terms with $\delta g_{AB} \sim \mathcal{O}(r^2)$ and $\delta g_{uu} \sim \mathcal{O}(1)$ do not show up.

- *BMS*: $\mathfrak{bms} \simeq (\mathfrak{witt} \oplus \mathfrak{witt}) \ltimes \mathfrak{s}$ [80–82]

  Another natural possibility is to admit V^A which are also locally defined CKV on S^2. Terms with $\delta g_{AB} \sim \mathcal{O}(r^2)$ arise only at isolated points corresponding to the singularities of the meromorphic CKV in a similar fashion as found long ago in 2d CFT [157].

- *Generalized BMS*: $\mathfrak{gbms} \simeq \mathfrak{vect}(S^2) \ltimes \mathfrak{s}$ [148, 158]

  A broader possibility is to consider for V^A all the smooth diffeomorphisms on S^2 [148]. Terms with $\delta g_{AB} \sim \mathcal{O}(r^2)$ and $\delta g_{uu} \sim \mathcal{O}(1)$ are present. Expanding $V^A \in \mathfrak{vect}(S^2)$ in vector spherical harmonics, one can prove that the modes $l = 1$ correspond (to leading order) respectively to the six global CKV generators on S^2 [38], while the remaining modes can be roughly interpreted as "angle-dependent rotations associated to the conservation of momentum flux at every angle" [38, 55, 149].

Let us remark that the Poincaré generators, which are the GKV of Minkowski, arise naturally as the global modes of the asymptotic diffeomorphisms in asymptotically flat spacetimes for the three cases.

The broad structure of the charges in the three cases is

$$Q_{\{f,V^A\}} = \frac{1}{8\pi}\int_{S^2} \sqrt{q}[2f(x^A)m + V^A(x^A)N_A] + \dots \ , \tag{2.1.11}$$

where q is the determinant of the metric on S^2, V^A represents global CKVs on S^2, local CKVs on S^2 or Diff(S^2)-rotations and the $\dots$ denote terms which can be added to take care of conventions, finiteness and integrability of charges or other issues (see e.g. [154, 156]).

Weyl transformations

A third option, very recently proposed in [149], adds a third free parameter accounting for local Weyl diffeomorphisms $\xi^{r(V)} \in \mathcal{C}^\infty(S^2) \equiv \mathfrak{w}$. In this case, although the fall-offs are still given by $g_{uu} = \mathcal{O}(1)$, $g_{ur} = \mathcal{O}(r^{-2})$, $g_{uA} = \mathcal{O}(1)$, the boundary conditions are relaxed by allowing for residual diffeomorphisms which do not preserve the determinant of the metric on S^2. In this case, the leading order asymptotic diffeomorphisms are given by:

$$\xi(f, V^A, \xi^{r(V)}) = [f - u\xi^{r(V)}]\partial_u + V^A\partial_A + \xi^{r(V)} r\partial_r \ . \tag{2.1.12}$$

The generic form of the charges in this case is

$$Q_{\{f,V^A,\xi^{r(V)}\}} = \frac{1}{8\pi}\int_{S^2} \sqrt{q}[2f(x^A)m + V^A(x^A)N_A - 2u\xi^{r(V)}(x^A)m] + \dots \ , \tag{2.1.13}$$

where the three parameters $f, V^A, \xi^{r(V)}$ depend only on x^A and, respectively, denote supertranslations, Diff(S^2)-rotations and local Weyl-transformations. The latter can be understood as "angle-dependent Weyl rescalings associated to the conservation of Weyl flux at every angle" and the algebra of diffeomorphisms at $\mathcal{I}^+$ is given by the Weyl-BMS algebra $\mathfrak{bmsw} \simeq (\mathfrak{vect}(S^2) \ltimes \mathfrak{w}) \ltimes \mathfrak{s}$ [149]. It is straightforward to observe that we recover the case $\mathfrak{gbms} \simeq \mathfrak{vect}(S^2) \ltimes \mathfrak{s}$ when $\xi^{r(V)} = -\frac{1}{2} D_A V^A$, which is equivalent to impose $\delta(\sqrt{q}) = 0$. The structure of the charges in this case is much more complicated and receives many corrections in order to account for finiteness, integrablility and more aspects [149].

2.2 Soft theorems and memory effects

Besides asymptotic symmetries, the infrared triangle comprises other two corners in asymptotically flat spacetimes: soft theorems and memory effects. Following [38], we will very briefly review how all these are interconnected in the case of supertranslations.

Asymptotic symmetries and soft theorems

Let us start with the supertranslation charges at $\mathcal{I}^+_-$ and $\mathcal{I}^-_+$, which correspond respectively to the "boundary spheres" at far past of $\mathcal{I}^+$ and far future of $\mathcal{I}^-$, given by (2.1.7)

$$Q^+_f = \frac{1}{4\pi} \int_{\mathcal{I}^+_-} dz^2 \gamma_{z\bar{z}} f m \ , \quad \tilde{Q}^-_f = \frac{1}{4\pi} \int_{\mathcal{I}^-_+} dz^2 \gamma_{z\bar{z}} f m \ . \tag{2.2.1}$$

In [83], it has been proposed that, in order to have a well defined scattering problem, the antipodal matching condition linking $\mathcal{I}^+_-$ and $\mathcal{I}^-_+$ has to hold. This is equivalent to:

$$m(z,\bar{z})|_{\mathcal{I}^+_-} = m(z,\bar{z})|_{\mathcal{I}^-_+} \ , \ f(z,\bar{z})|_{\mathcal{I}^+_-} = f(z,\bar{z})|_{\mathcal{I}^-_+} \Rightarrow Q^+_f = Q^-_f \ . \tag{2.2.2}$$

Using the mass loss equation of motion, integrating by parts and assuming that m decays to zero at very large times, the charge conservation equation can be recast in a more manageable form which, together with the fact that conserved charges commute with the $\mathcal{S}$-matrix $Q^+_f \mathcal{S} - \mathcal{S} Q^-_f = 0$, leads to Weinberg's soft graviton theorem [35] via Ward identity and after sandwitching $Q^+_f \mathcal{S} - \mathcal{S} Q^-_f = 0$ between in- and out-states [125].

An identical construction for conservation of superrotation charges (2.1.11) leads to subleading soft theorems [84, 159].

Memory effect

Let us begin with a pair of inertial detectors close to $\mathcal{I}^+$ in a region with no Bondi news, $N_{AB} = \partial_u C_{AB}$, at both late and early times. At intermediate times, gravitational waves may pass through and cause distorsions in their relative separations

$(s^z, s^{\bar{z}})$. In this case, the geodesic deviation equation implies

$$r^2\gamma_{z\bar{z}}\partial_u^2 s^{\bar{z}} = -R_{uzuz}s^z, \quad \text{with } R_{uzuz} = -\frac{r}{2}\partial_u^2 C_{zz} \; . \tag{2.2.3}$$

After integrating this equation, the initial and final separations differ by

$$\triangle s^{\bar{z}} = \frac{\gamma^{z\bar{z}}}{2r}\triangle C_{zz}s^z \; . \tag{2.2.4}$$

This is the so-called gravitational displacement memory effect [36, 126]. Clearly, this effect is more difficult to observe than gravitational waves themselves but numerous works have proposed methods to measure it (e.g. [160, 161]).

The equivalence between the gravitational memory effect and soft graviton theorem was noticed in [37] by comparing the original formulas in the literature and realizing that both can be brought to each other after few simple replacements and acting with a Fourier transform on Weinberg's momentum-space [35] formula to directly obtain Braginski-Thorne displacement shift memory [126].

Finally, the connection between supertranslations and memory effect is extremely intuitive. Indeed, it is simple to show that in the absence of energy flux and u-dependence of the asymptotic data in (2.1.5), then C_{zz} is fully generated by a supertranslation $\delta C_{zz} = -2D_z D_z f(z, \bar{z})$, meaning that we can think of a pulse of radiation through $\mathcal{I}^+$ as the causant of a transition between inequivalent BMS vacua. Taking a quick look at (2.2.4), it is trivial to realize that we can then write the displacement in the detector separations $\triangle s^{\bar{z}}$ in terms of a supertranslation $f(z, \bar{z})$ [2].

Altogether asymptotic symmetries, memory effects and soft theorems constitute the three legs of the infrared triangle (fig. 1.1), where we can regard the memory as the observable corner. In fact, we have noticed that memory is connected to the soft theorem via Fourier transform and measures transitions between inequivalent BMS vacua. The story we have recapitulated here just accounts for supertranslations. However, in the case of superrotations the Fourier transform of the subleading soft theorem, associated to superrotations, also leads to the so-called spin memory effect [162]. In the case of Weyl-BMS symmetry, it is still an open question to obtain the correspondence to the memory and soft theorem corners.

2.3 Membrane paradigm

We devote the last section of this chapter to present the connection between BMS symmetries and the membrane paradigm [54] for stationary asymptotically flat spacetimes in four dimensions [55]. We closely follow [55] to show how this relationship singles Diff(S^2) out of the different possibilities reviewed for superrotations in section 2.1.

[2] If one is interested in the concrete form of the supertranslation, one has to solve the mass-loss equation $D_z^2\triangle C^{zz} = 2\triangle m + 2\int du T_{uu}$ by means of a Green's function [37].

The membrane paradigm at $\mathcal{I}^+$

The membrane paradigm attaches a $2+1$ dimensional fluid stress-energy tensor to null surfaces. In this way, an observer in the exterior of an asymptotically flat black hole assigns fluid membranes to the event horizon and future null infinity. Herein, we review the definition of the membrane stress-energy tensor following [54, 55].

The membrane is a timelike cutoff surface placed slightly outside the null surface. At $\mathcal{I}^+$, the cutoff surface is a large but finite sphere called "stretched infinity". Let n be the unit normal of the membrane. The projection tensor $h_{ab} = g_{ab} - n_a n_b$ is the metric induced on the membrane by the $3+1$ dimensional spacetime metric g_{ab}. Let U^a be the worldlines of a family of fiducial observers. The metric on constant time slices of the membrane is $\gamma_{ab} = h_{ab} + U_a U_b$ [3].

The extrinsic curvature and membrane stress-tensor at $\mathcal{I}^+$ of the membrane are defined as follows

$$K^a_b = h^c_b \nabla_c n^a \ , \quad t_{ab} = -\frac{1}{8\pi}(K h_{ab} - K_{ab}) \ . \tag{2.3.1}$$

The energy density, momentum density and stress tensors are

$$\Sigma = t_{ab} U^a U^b = -\frac{\theta}{8\pi} \ , \ \pi_A = t_{aA} U^a \ , \ t_{AB} = p\gamma_{AB} - 2\eta\sigma_{AB} - \zeta\theta\gamma_{AB} \ , \tag{2.3.2}$$

where θ is the expansion scalar, η and ζ the shear and bulk viscosity coefficients, p the pressure and σ_{AB} is the shear tensor, which are given by the following expressions at $\mathcal{I}^+$

$$p = \frac{\kappa}{8\pi} \ , \ \theta = -K^A_A \ , \ \sigma_{AB} = -K_{AB} - \frac{1}{2}\theta\gamma_{AB} \ , \ \eta = \frac{1}{16\pi} \ , \ \zeta = -\frac{1}{16\pi} \ , \tag{2.3.3}$$

being κ the surface gravity of the membrane. Remarkably, the membrane has vanishing rotation $\omega_{AB} \simeq K_{[A,B]} = 0$ because n_a is hypersurface orthogonal.

Stationary spacetimes at $\mathcal{I}^+$

Imposing the Einstein equations, the membrane obeys the Damour-Navier-Stokes equations [163]

$$\mathcal{L}_U \pi_A + \nabla_A p - \zeta\nabla_A\theta - 2\eta\sigma^B_{A||B} + T^M_{nA} = 0 \ , \tag{2.3.4}$$

where T^M_{nA} denotes non-gravitational sources of momentum. In stationary spacetimes, it is possible to choose a slicing for which $\theta = \sigma_{AB} = 0$ and $p =$ constant. Moreover,

[3]Note that we use greek indices for tensors on $3+1$ dimensional spacetime, lowercase roman indices for tensors on the $2+1$ dimensional membrane, and uppercase roman indices for tensors on constant time slices of the membrane. We also denote the 4-covariant derivative by ∇_μ, the 3-covariant derivative by $_{|a}$, and the 2-covariant derivative by $_{||A}$.

we will assume $T^M_{nA} = 0$ along this section. In this case, the previous equations reduce to

$$\mathcal{L}_U \pi_A = 0 \ , \tag{2.3.5}$$

which implies that the momentum density π_A is conserved. As a consequence, we obtain an infinite set of conserved charges [4]

$$Q_{\{f,Y^A\}} = \int d^2x \sqrt{\gamma}(fp - Y^A \pi_A) \ , \tag{2.3.6}$$

where f and Y^A are arbitrary functions and the integral is over constant time slices of the membrane. Clearly, setting $f = \delta^2(x^P - \hat{x}^P)$ and $Y^A = 0$ gives a set of charges corresponding to "energy at every angle" and setting $f = 0$ and $Y^A = \delta^2(x^P - \hat{x}^P)\delta^A_B$ a set of charges corresponding to "momentum at every angle". Even though we focus on $\mathcal{I}^+$, these charges can be computed for any null surfaces.

Let us now show that (2.3.6) are indeed equivalent to the BMS charges (2.1.11). The first step is to write the metric near $\mathcal{I}^+$. It turns out that for the stationary case (2.1.5) can be written as follows [147]:

$$ds^2 = -\left(1 - \frac{2m}{r}\right) du^2 - 2dudr + \frac{4}{3}\frac{N_A}{r} dudx^A + r^2 q_{AB} dx^A dx^B \ , \tag{2.3.7}$$

where m is constant and N_A only depends on x^A. The membrane's unit normal is

$$n = \alpha^{-1} dr \ , \quad \alpha = \sqrt{1 - 2m/r} \ . \tag{2.3.8}$$

The surface gravity is given by $\kappa = m/r^2$, which means that $\sqrt{\gamma}\kappa$ is finite at $r \to \infty$ and so is the first term in (2.3.6). The momentum density is

$$\pi_A = \alpha t^u_A = -\frac{N_A}{8\pi r^2} + \mathcal{O}(r^{-3}) \ . \tag{2.3.9}$$

Again $\sqrt{\gamma}\pi_A$ is finite at $r \to \infty$ and so is the second term in (2.3.6). The conserved charges (2.3.6) are then

$$Q_{\{f,Y^A\}} = \frac{1}{8\pi} \int_{S^2} \sqrt{q}[2fm + Y^A N_A] \ . \tag{2.3.10}$$

This set of charges is clearly the same as the infinite set of BMS charges for stationary asymptotically flat spacetimes at $\mathcal{I}^+$ (2.1.11), when $V_A \in \mathfrak{vect}(S^2)$. Exactly the same situation is encountered at the event horizon [55]. As a consequence, we conclude that the relationship between the membrane paradigm and BMS points directly towards Diff(S^2)-rotations as the privileged superrotation generators.

As a final note, let us emphasize that the stationary model we discussed here naturally cannot exhibit local Weyl symmetry leading to the charges (2.1.13). It is still an open question how exactly the membrane paradigm could relate to Weyl-BMS.

[4] Let us note that there is still some ambiguity related to the normalization of these charges.

Chapter 3

Asymptotically FLRW spacetimes

This chapter is based on our works [164–166]. Here, we restrict to the geometrical or kinematical analysis of asymptotically Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes, while the dynamical aspects will be discussed in chapter 4.

First, we briefly review the asymptotia of spatially flat FLRW spacetimes. Next, we describe the notion of asymptotically decelerating and spatially flat FLRW spacetimes at future null infinity $\mathcal{I}^+$. Afterwards, we obtain the asymptotic diffeomorphisms compatible with these universes in Bondi gauge, as well as study their asymptotic algebra, which unveils a one-parameter deformation of the asymptotically flat algebra interpreted as a cosmological holographic flow for certain boundary conditions, and their effect on the asymptotic data. At last, we explicitly show that, in order to include several cosmological black hole solutions, we have to enlarge our ansatz by allowing for logarithms in the asymptotic expansion.

3.1 FLRW spacetimes and their asymptotia

The metric of spatially flat FLRW spacetimes sourced by a fluid with an equation-of-state parameter $\omega = \frac{p}{\rho}$ is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{S^2}\right) \ , \quad a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3(\omega+1)}} \ . \tag{3.1.1}$$

These metrics are related to the Minkowski metric by a Weyl transformation. Indeed, using the conformal time $\mathrm{d}\eta = \frac{\mathrm{d}t}{a(t)}$ and Bondi coordinates

$$u = \eta - \sqrt{x^i x_i} \ , \quad r = \sqrt{x^i x_i} \ , \quad z = \frac{x^1 + ix^2}{x^3 + \sqrt{x^i x_i}} \ , \quad \bar{z} = \frac{x^1 - ix^2}{x^3 + \sqrt{x^i x_i}} \ , \tag{3.1.2}$$

the spatially flat FLRW metric reads as

$$\mathrm{d}s^2 = a^2(u,r)\left(-\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + \frac{4r^2}{(1+z\bar{z})^2}\mathrm{d}z\mathrm{d}\bar{z}\right) \ , \quad a(u,r) = \left(\frac{r+u}{L}\right)^k \ , \tag{3.1.3}$$

where L is a length scale and $k = 2/(3\omega + 1)$.

These spacetimes can be divided into decelerating ($k > 0$) and accelerating ($k < 0$). The corresponding Penrose diagrams (see *e.g.*, [63,167]) are shown in figure 3.1.

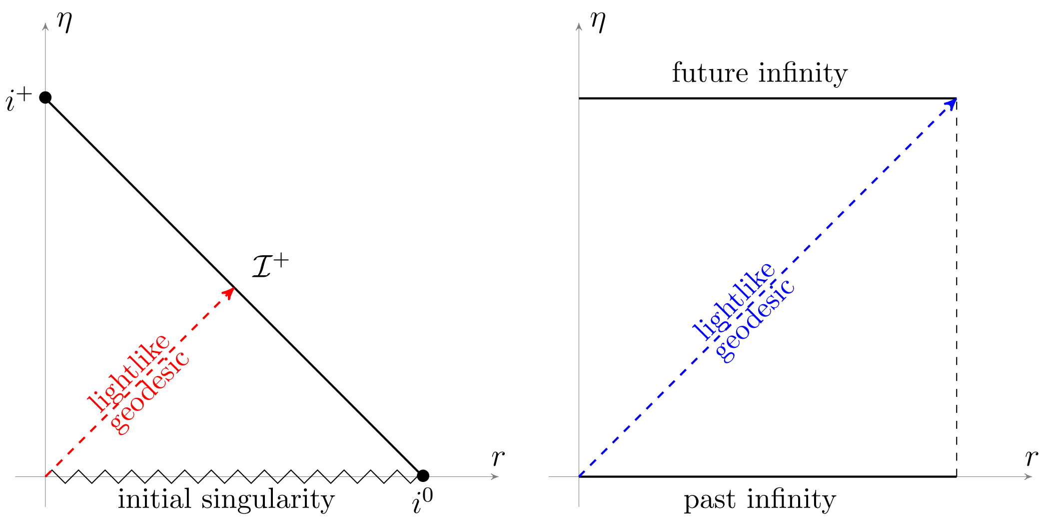

Figure 3.1: Penrose diagram of spatially flat decelerating FLRW (left) and accelerating FLRW (right).

Comparing the asymptotic regions of a lightlike geodesic, it is clear that only decelerating FLRW spacetimes have a future null infinity $\mathcal{I}^+$. For this reason, we will restrict ourselves to decelerating universes in this thesis, leaving the investigation of accelerating FLRW spacetimes for future work.

Finally, the non-vanishing components of the Einstein tensor of the exact FLRW background (3.1.3) are given by

$$G_{uu}^{\mathrm{FLRW}} = \frac{3k^2}{r^2} + \mathcal{O}(r^{-3}) \ , \tag{3.1.4}$$

$$G_{ur}^{\mathrm{FLRW}} = \frac{3k^2}{r^2} + \mathcal{O}(r^{-3}) \ , \tag{3.1.5}$$

$$G_{rr}^{\mathrm{FLRW}} = \frac{2k(k+1)}{r^2} + \mathcal{O}(r^{-3}) \ , \tag{3.1.6}$$

$$G_{z\bar{z}}^{\mathrm{FLRW}} = -\gamma_{z\bar{z}} k(k-2) + \mathcal{O}(r^{-1}) \ . \tag{3.1.7}$$

3.2 Asymptotically decelerating and spatially flat FLRW spacetimes

In order to define a class of spacetimes which asymptote to decelerating and spatially flat FLRW at $\mathcal{I}^+$, we impose the following conditions:

1. The background metric, that is the metric in which all the asymptotic expansion coefficients vanish, is the exact FLRW in (3.1.3).

2. The Bondi gauge is satisfied, meaning that

$$g_{rr} = 0, \qquad g_{rA} = 0, \qquad \partial_r \det\left(\frac{g_{AB}}{a^2 r^2}\right) = 0\ , \tag{3.2.1}$$

   where the indices $A, B \in \{z, \bar{z}\}$ label the angular coordinates.

3. Allowance of cosmological perturbations which preserve (to leading order) homogeneity, isotropy and spatial flatness and leave the equation of state of the background fluid invariant in the limit $r \to \infty$.

4. Closure of the r-expansion upon application of asymptotic diffeomorphisms, meaning that such large gauge transformations do not generate higher order terms in the r-expansion.

5. Trace and components of the Einstein tensor cannot diverge in the limit $r \to \infty$, when dimensionally scaled. Assuming General Relativity, these conditions translate directly into requirements for the energy momentum tensor.

These considerations lead to the following class of metrics [1]

$$\mathrm{d}s^2 = \left(\frac{r+u}{L}\right)^{2k} \Bigg\{ -\left(1 - \Phi - \frac{2m}{r}\right)\mathrm{d}u^2 - 2\left(1 - \frac{K}{r}\right)\mathrm{d}u\mathrm{d}r - 2\left(r\Theta_A + U_A\right.$$
$$\left. + \frac{1}{r}N_A\right)\mathrm{d}u\mathrm{d}x^A + \left(r^2 q_{AB} + rC_{AB} + \mathcal{D}_{AB} + \frac{1}{2}C_{AC}C^C_B\right)\mathrm{d}x^A\mathrm{d}x^B + \ldots \Bigg\}. \tag{3.2.2}$$

It represents an expansion in powers of $1/r$ for $r \to \infty$, where all the expansion coefficients are functions of u, z and $\bar{z}$, except for q_{AB} which only depends on the angular coordinates z and $\bar{z}$.

Before continuing, let us point out that the ansatz (3.2.2), as well as the asymptotic diffeomorphisms preserving it, naturally give the correct flat limit when k, K, $\Theta_A \to 0$. Furthermore, Φ, m and K transform as scalars under spatial rotations while Θ_A, U_A and N_A transform as vectors, and q_{AB}, C_{AB} and $\mathcal{D}_{AB}$ as tensors. The determinant condition in (3.2.1) implies C_{AB} and $\mathcal{D}_{AB}$ to be traceless. By comparing the expansion (3.2.2) to the asymptotically flat expansion (2.1.5), we expect the parameter m to be related to the mass of a central inhomogeneity, C_{AB} to the gravitational radiation and N_A to the angular momentum aspect of the spacetime. However, it is important to stress that we did not yet impose any equations of motion, such that the treatment so far has been off-shell and, therefore, the different coefficients do not have yet a sharp physical interpretation.

[1] Note that the sign of the coefficients in the $dudx^A$ part of the metric follows the opposite sign convention than in asymptotically flat spacetimes (e.g. in [38, 145]).

In section 3.4, we will find that the ansatz (3.2.2) naturally includes white holes but, in order to include simple cosmological black hole metrics like Sultana-Dyer, Thakurta and Vaidya, the expansion in $1/r$ has to be augmented with logarithmic terms. As expected, the logarithmic ansatz does not generally satisfy the peeling property but preserves the asymptotic algebra. In addition, we would like to comment that a u-dependent metric on the sphere q_{AB} would imply $a^{-2}g_{uu} \propto \mathcal{O}(r)$ because of the closure of the metric under the action of the asymptotic diffeomorphisms. However, this term is not compatible with the third condition leading to our ansatz.

We will come back to these observations in section 4.1 of the next chapter, where they will play an important role in the on-shell analysis.

3.3 Asymptotic diffeomorphisms

In this section, we study the residual diffeomorphisms preserving our metrics (3.2.2). These diffeomorphisms are shown to be consistent with the global Killing vectors of pure FLRW and their algebra at $\mathcal{I}^+$ is investigated, corresponding to a one-parameter deformation of the BMS and generalized BMS transformations in asymptotically flat FLRW spacetimes. The deformation parameter is intimately related to the equation of state of the fluid, unveiling a cosmological holographic flow at the level of asymptotic algebras. However, if we allow for local Weyl diffeomorphisms, this deformation becomes trivial and leads to an isomorphism between both asymptotic algebras. At last, we describe the effect of these transformations on the asymptotic data.

3.3.1 Residual transformations in Bondi gauge

We analyze the residual diffeomorphisms for the on-shell metrics (3.2.2) [2] starting from

$$\xi = \xi^u(u,z,\bar{z})\partial_u + \left[r\xi^{r(V)}(z,\bar{z}) + \xi^{r(0)} + \frac{\xi^{r(1)}}{r}\right]\partial_r + \left[V^B(z,\bar{z}) + \frac{\xi^{B(1)}}{r} + \frac{\xi^{B(2)}}{r^2}\right]\partial_B \ . \tag{3.3.1}$$

We look for the most general diffeomorphisms, meaning that we do not require the determinant of the metric on the sphere to be fixed. Instead of the strong Bondi gauge, we follow [149] and use the Bondi gauge

$$g_{rr} = 0, \quad g_{rA} = 0, \quad \partial_r \det\left(\frac{g_{AB}}{a^2 r^2}\right) = 0 \ . \tag{3.3.2}$$

[2] In the rest of this section, we will assume that the leading asymptotic coefficients Φ, Θ_A and q_{AB} are u-independent. This choice implies finite fluxes through the boundary and will be motivated by our on-shell treatment in section 4.1.1.

The condition on $\mathcal{L}_\xi g_{rr}$ is already verified by the ansatz. The vanishing of $\mathcal{L}_\xi g_{rA}$ leads to the following restrictions:

$$\xi_A^{(1)} = -D_A \xi^u \ , \tag{3.3.3}$$
$$\xi_A^{(2)} = \frac{1}{2}\left(K D_A \xi^u - C_{AB}\xi^{B(1)}\right) \ . \tag{3.3.4}$$

In order to satisfy the determinant condition, we have to demand that $q^{AB}C_{AB} = 0$, $q^{AB}S_{AB} = C^{AB}F_{AB}$ and that $q^{AB}K_{AB} = C^{AB}S_{AB} - C^A_C C^{CB}F_{AB} + (\mathcal{D}^{AB} + \frac{1}{2}C^A_C C^{CB})F_{AB}$, where K_{AB}, S_{AB} and F_{AB} are defined in (A.1.6). This leaves the leading order contribution to the spherical metric arbitrary, which means that the coefficient $\xi^{r(V)}$ in the expansion (3.3.1) joins f and V^A as a free parameter. Besides, we obtain:

$$\xi^{r(0)} = \frac{1}{1+k}\left[-\frac{1}{2}D_A\xi^{A(1)} - \frac{1}{2}\Theta^A D_A \xi^u + ku\xi^{r(V)} - k\xi^u\right] \ , \tag{3.3.5}$$
$$\xi^{r(1)} = \frac{1}{2(1+k)}\left[C^A_B \Theta_A D^B \xi^u - 2k\left(u^2\xi^{r(V)} - u\xi^{r(0)} - u\xi^u\right) - D_A\xi^{A(2)} + U^A D_A \xi^u\right] \ . \tag{3.3.6}$$

The remaining requirements come from $\mathcal{L}_\xi g_{uA} = \mathcal{O}(r)$, $\mathcal{L}_\xi g_{uu} = \mathcal{O}(1)$ and $\mathcal{L}_\xi g_{ur} = \mathcal{O}(r^{-1})$. Altogether they translate into

$$\partial_u V^A = \partial_u \xi^{r(V)} = 0 \ , \tag{3.3.7}$$
$$\partial_u \xi^u = -(1+2k)\xi^{r(V)} \qquad \Rightarrow \qquad \xi^u = f(z,\bar{z}) - u(1+2k)\xi^{r(V)}(z,\bar{z}) \ . \tag{3.3.8}$$

These are the most general residual diffeomorphisms which satisfy the five requirements leading to our metrics (3.2.2) and whose asymptotic algebra will be investigated in the next subsection. Nevertheless, there exist more restrictive possibilities which are closely related to their asymptotically flat counterparts reviewed in chapter 2. Let us briefly recall the alternatives which we can take:

1. **We do not impose further restrictions.** In this case, there are three free parameters in (3.3.1) given by $f \in \mathcal{C}^\infty(S^2)$, $V^A \in \mathfrak{vect}(S^2)$ and $\xi^{r(V)} \in \mathcal{C}^\infty(S^2)$ which, respectively, represent supertranslations, Diff(S^2)-rotations and local Weyl-transformations.

2. **We impose the strong Bondi gauge.** This is tantamount to impose an extra boundary condition on the celestial sphere $\delta\sqrt{\det(g_{AB})} = 0$, which effectively fixes the determinant of the metric on S^2. This requirement leads to

$$0 = q^{AB}F_{AB} = 4(1+k)\xi^{r(V)} + 2D_A V^A \Rightarrow \xi^{r(V)} = -\frac{1}{2(1+k)}D_A V^A \ , \tag{3.3.9}$$

meaning that now there are only two free parameters in (3.3.1) given by $f \in \mathcal{C}^\infty(S^2)$ and $V^A \in \mathfrak{vect}(S^2)$ which, respectively, represent supertranslations and Diff(S^2)-rotations.

3. **Local CKV on S^2.** Another option is to add on top that the leading order contribution to the metric on S^2 is given by the round metric $\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$, $\gamma_{zz} = \gamma_{\bar{z}\bar{z}} = 0$, such that the only terms with $\delta q_{AB} \neq 0$ arise only at isolated points corresponding to the singularities of meromorphic CKV. This leads to the conformal Killing equation

$$D_A V_B + D_B V_A = \gamma_{AB} D_C V^C \ . \tag{3.3.10}$$

As a consequence, we again have two free parameters in (3.3.1) but given by $f \in \mathcal{C}^\infty(S^2)$ and $V^A \in \mathfrak{witt} \oplus \mathfrak{witt}$ which, respectively, represent supertranslations and superrotations.

4. **Global CKV on S^2.** The last possibility is to further avoid these singularities, such that terms with $\delta g_{AB} \simeq \mathcal{O}(r^2)$ are absent. In this case, we still have supertranslations $f \in \mathcal{C}^\infty(S^2)$ but the only allowed superrotations are only the six globally defined CKV on the sphere.

Before continue, let us show how we consistently recover the global Killing vectors of pure FLRW from these residual diffeomorphisms.

Global Killing vectors - spatial translations

Let us show how we recover the global Killing vectors (GKV) associated to translations consistently from the supertranslation diffeomorphisms corresponding to (3.3.1) with $V^A = 0$ and $\xi^{r(V)} = 0$.

The global Killing vectors (GKV) are the solutions of the equation

$$\mathcal{L}_\xi g_{\mu\nu} = \xi^\lambda \partial_\lambda g_{\mu\nu} + g_{\nu\lambda}\partial_\mu \xi^\lambda + g_{\mu\lambda}\partial_\nu \xi^\lambda \stackrel{!}{=} 0 \ . \tag{3.3.11}$$

Maximally symmetric spaces have the maximum number of GKV given by $d(d+1)/2$. In flat space, we obtain ten GKV, four associated to translations and six associated to rotations and boosts. Unperturbed FLRW spaces are homogeneous and isotropic in the spatial components and, therefore, we obtain six GKV associated to the three spatial translations and three rotations. On the other hand, ∂_t is no more a GKV but ∂_η is a conformal Killing Vector (CKV).

The large r-limit of (3.3.1) when $V^A = 0$, $\xi^{r(V)} = 0$ and $\Phi, K, \Theta_A \to 0$ is given by

$$\xi = f(z,\bar{z})\partial_u + \frac{1}{2(1+k)}\left[D_A D^A f(z,\bar{z}) - 2kf(z,\bar{z}) + \mathcal{O}(r^{-1})\right]\partial_r \tag{3.3.12}$$

$$+ \left[-\frac{1}{r}D^B f(z,\bar{z}) + \mathcal{O}(r^{-2})\right]\partial_B + \mathcal{O}(\Phi, K, \Theta_A) \ , \tag{3.3.13}$$

whose action is equivalent to the following coordinate transformations:

$$u \to u + f \ , \quad r \to r + \frac{1}{2(1+k)}\left(D^A D_A f - 2kf\right) \ , \tag{3.3.14}$$

$$z \to z - \frac{1}{r}D^z f \ , \quad \bar{z} \to \bar{z} - \frac{1}{r}D^{\bar{z}} f \ . \tag{3.3.15}$$

Using the following convention [38] for the $l = 0, 1$ spherical armonics

$$Y_0^0 = 1 \ , \quad Y_1^1 = \frac{z}{1+z\bar{z}} \ , \quad Y_1^0 = \frac{1-z\bar{z}}{1+z\bar{z}} \ , \quad Y_1^{-1} = \frac{\bar{z}}{1+z\bar{z}} \ , \tag{3.3.16}$$

we aim to recover the unperturbed FLRW GKVs generating the spatial translations as some linear combinations of $\xi(Y_0^0)$, $\xi(Y_1^1)$, $\xi(Y_1^0)$ and $\xi(Y_1^{-1})$

$$\xi(Y_0^0) = \partial_u - \frac{k}{(1+k)}\partial_r = \partial_\eta - \frac{(1+2k)}{(1+k)}\partial_r \ , \tag{3.3.17}$$

$$\xi(Y_1^1) = \frac{z}{1+z\bar{z}}\left(\partial_u - \partial_r\right) + \frac{1}{r}\left(\frac{z^2}{2}\partial_z - \frac{1}{2}\partial_{\bar{z}}\right) \ , \tag{3.3.18}$$

$$\xi(Y_1^0) = \frac{1-z\bar{z}}{1+z\bar{z}}(\partial_u - \partial_r) + \frac{1}{r}(z\partial_z + \bar{z}\partial_{\bar{z}}) \ , \tag{3.3.19}$$

$$\xi(Y_1^{-1}) = \frac{\bar{z}}{1+z\bar{z}}(\partial_u - \partial_r) + \frac{1}{r}\left(-\frac{1}{2}\partial_z + \frac{\bar{z}^2}{2}\partial_{\bar{z}}\right) \ . \tag{3.3.20}$$

We can write them in terms of the Cartesian generators $X_i = \partial_{x^i}$ as:

$$\xi(Y_1^0) = -X_3 \ , \quad \xi(Y_1^1) = -\frac{1}{2}(X^1 + iX^2) \ , \quad \xi(Y_1^{-1}) = -\frac{1}{2}(X^1 - iX^2) \ , \tag{3.3.21}$$

obtaining exactly the same result as in flat space [38].

Consistently, we do not obtain the time translation generator from $\xi(Y_0^0)$ due to the fact that it is not a GKV in spatially flat FLRW but a CKV. Therefore, there is a linearly independent (and so unavoidable) correction term which corresponds to a spatial dilatation D pondered by the inverse of the radius

$$\xi(Y_0^0) = \frac{1}{(1+k)}\left[\partial_\eta - \frac{k}{r}x_i\partial_{x_i}\right] = \frac{1}{(1+k)}\left[T_{\text{Conf}} - \frac{k}{r}D\right] \ . \tag{3.3.22}$$

Remarks

- In case of (wrongly) considering flat BMS supertranslations (as in [168]), instead of the asymptotically spatially flat FLRW supertranslations that we study, the relations in (3.3.21) would be analogously verified, but (3.3.22) would be replaced by $\xi(Y_0^0) = T_{\text{Conf}}$ being in line with the fact that pure spatially flat FLRW is conformal to Minkowski after replacing t by η. However, we observe from the above discussion that we do not recover the correct global isometry group of FLRW from flat BMS supertranslations. Therefore, flat BMS transformations are not consistent asymptotic symmetries in our cosmological setting.

- If the coefficients Φ, K, Θ_A are non zero, they survive at infinity and we should recover the GKV of the corresponding perturbed spatially flat FLRW. Nevertheless, such spaces generally do not have GKV. Although, if $\Phi, K, \Theta_A << 1$ [3], then one can expand them in series and we obtain the previous results as a first approximation.

[3] As physically expected, otherwise the perturbations would spoil the observed homogeneity and isotropy at large r scales in our universe.

Global Killing vector - rotations

Next, we consistently obtain the global Killing vectors, associated to rotations, from the superrotation diffeomorphisms (3.3.1) with $f=0$, $\xi^{r(V)}$ given by (3.3.9) and V^A being globally defined CKV on S^2. In contrast to the asymptotically flat case, the boosts are no longer global Killing vectors.

The first step is to derive ξ for the global CKV on S^2 in the limit $\Phi, K, \Theta_A \to 0$. In this case we have

$$\begin{aligned}\xi =&\frac{u}{2}\frac{(1+2k)}{(1+k)}D_AV^A\partial_u - \frac{1}{2(1+k)^2}[r(1+k)+u(1+4k+2k^2)]D_AV^A\partial_r\\ &+\left(V^A-\frac{u}{2r}\frac{(1+2k)}{(1+k)}D^AD_BV^B\right)\partial_A \ . \end{aligned} \tag{3.3.23}$$

It is straightforward to check that the choices

$$\begin{aligned} V^z &= iz \ , & V^{\bar z} &= -i\bar z \ ,\\ V^z &= \frac{i}{2}(z^2-1) \ , & V^{\bar z} &= \frac{i}{2}(1-\bar z^2) \ ,\\ V^z &= \frac{1}{2}(1+z^2) \ , & V^{\bar z} &= \frac{1}{2}(1+\bar z^2) \end{aligned} \tag{3.3.24}$$

verify $D_AV^A=0$, which means $\xi^{r(V)}=0$, and correspond respectively to the rotation generators J_{12}, J_{23} and J_{31}, where $J_{ij}=x^i\partial_j - x^j\partial_i$ in Cartesian coordinates.

Moving on to CKVs with $D_AV^A\neq 0$, we can consider the choices

$$V^z = \frac{1}{2}(1-z^2) \ , \qquad V^{\bar z} = \frac{1}{2}(1-\bar z^2) \ , \tag{3.3.25}$$

$$V^z = \frac{i}{2}(1+z^2) \ , \qquad V^{\bar z} = -\frac{i}{2}(1+\bar z^2) \ , \tag{3.3.26}$$

$$V^z = -z \ , \qquad V^{\bar z} = -\bar z \ . \tag{3.3.27}$$

In the asymptotically flat case these correspond to the boosts in x, y and z direction respectively. To see how a flat boost would look like in our case we plug them into (3.3.23). After transforming the result into cartesian coordinates we discover that these transformations can be written in terms of a conformal boost term perturbed by a superposition of deformed conformal transformations:

$$\begin{aligned}\xi^{(i)} =&\frac{1}{(1+k)^2}\left\{B_i+\frac{k}{r}\left[\left(1+k-(3+2k)\frac{\eta}{r}\right)K_i\right.\right.\\ &\left.\left.+\left((6+5k)\frac{\eta}{r}-(5k+4)\right)x_iD+x_i\eta T_c\right]\right\} \ , \end{aligned} \tag{3.3.28}$$

where the boosts B_i, special conformal transformations K_i, dilatation D and conformal time translation T_c are given by:

$$B_i = \eta\partial_i + x_i\partial_\eta \ , \qquad K_i = 2x_ix^j\partial_j - r^2\partial_i \ , \qquad D = x^i\partial_i \ , \qquad T_c=\partial_\eta \ . \tag{3.3.29}$$

Consistently, we observe that we do not find pure boosts as in flat spacetimes but these receive unavoidable corrections, being the main reason that boosts are not GKV in spatially flat FLRW. Analogous remarks to those below (3.3.22) apply here.

3.3.2 Asymptotic algebra

Once we analyzed the more general residual diffeomorphisms and the more restrictive alternative possibilities, we are ready to explore the algebra they describe at future null infinity $\mathcal{I}^+$, also known as the asymptotic algebra of diffeomorphisms.

At $r \to \infty$, $r =$ constant, the asymptotic diffeomorphisms become

$$\xi[f(z,\bar{z}), \xi^{r(V)}(z,\bar{z}), V^A(z,\bar{z})] = \left[f - u(1+2k)\xi^{r(V)}\right]\partial_u + V^A\partial_A \ , \tag{3.3.30}$$

leading to the asymptotic algebra

$$V_{12} := [V_1, V_2]_{\text{Lie}} \ , \tag{3.3.31}$$
$$\xi_{12}^{r(V)} = V_1[\xi_2^{r(V)}] - V_2[\xi_1^{r(V)}] \ , \tag{3.3.32}$$
$$f_{12} = V_1[f_2] - V_2[f_1] - (1+2k)(f_1\xi_2^{r(V)} - f_2\xi_1^{r(V)}) \ . \tag{3.3.33}$$

We see that f and $\xi^{r(V)}$ transform as scalars under Diff(S^2), while f also transforms as a weight-$(1+2k)$ section of the scale bundle. An alternative way to visualize the algebra is to compute

$$\xi[\hat{f}, \hat{\xi}^{r(V)}, \hat{V}^A] = \Big[\xi[f, \xi^{r(V)}, V^A], \xi[f', \xi^{r(V)\prime}, V'^A]\Big] \,, \tag{3.3.34}$$

where

$$\begin{aligned} \hat{f} &= V^A D_A f' - V'^A D_A f - (1+2k)[f\xi^{r(V)\prime} - f'\xi^{r(V)}] \ , \\ \hat{V}^A &= V^B D_B V'^A - V'^B D_B V^A \ , \\ \hat{\xi}^{r(V)} &= V^A D_A \xi^{r(V)\prime} - V'^A D_A \xi^{r(V)} \ . \end{aligned} \tag{3.3.35}$$

Thus, we obtained the algebra $\mathfrak{bmsw}_k \simeq (\mathfrak{vect}(S^2) \ltimes \mathfrak{w}) \ltimes_k \mathfrak{s}_k$ which one would naively regard as a deformation of $\mathfrak{bmsw}$ obtained in [149]. Nevertheless, the fact that the Weyl-generators $\xi^{r(V)}$ are independent of V^A allows us to rescale the former such that the algebra $\mathfrak{bmsw}_k$ is isomorphic to the Weyl-BMS algebra $\mathfrak{bmsw}$.

Nevertheless, we can impose further boundary conditions. In fact, by means of (3.3.9) and the usage of the parameter $(1+s) \equiv (1+2k)/(1+k)$, with $0 \le s = \frac{k}{1+k} < 1$, we find that our asymptotic diffeomorphisms at $r \to \infty$, $r =$ constant reduce to

$$\xi[f(z,\bar{z}), V^A(z,\bar{z})] = \left(f + \frac{u}{2}(1+s)D_A V^A\right)\partial_u + V^A\partial_A \ . \tag{3.3.36}$$

Their Lie bracket gives

$$\xi[\hat{f}, \hat{V}^A] = \Big[\xi[f, V^A], \xi[f', V'^A]\Big], \tag{3.3.37}$$

where the hatted gauge parameters read as

$$\begin{aligned} \hat{f} &= V^A D_A f' - V'^A D_A f + \frac{(1+s)}{2}\left(f D_A V'^A - f' D_A V^A\right) \ , \\ \hat{V}^A &= V^B D_B V'^A - V'^B D_B V^A \ . \end{aligned} \tag{3.3.38}$$

We obtain a one-parameter deformation of the extended BMS algebra [82, 169] denoted as $\mathfrak{bms}_s \simeq (\mathfrak{witt} \oplus \mathfrak{witt}) \ltimes_s \mathfrak{s}_s$, where the vectors V^A are local CKV on S^2, and a deformation of the generalized BMS algebra [148, 158] denoted as $\mathfrak{gbms}_s \simeq \mathfrak{vect}(S^2) \ltimes_s \mathfrak{s}_s$, where the vectors V^A are smooth diffeomorphisms on the sphere. Both reduce to a one-parameter deformation of the original BMS algebra denoted as $\mathfrak{b}_s \simeq \mathfrak{so}(1,3) \ltimes_s \mathfrak{s}_s$, found in [170], when restricting to the six V^A that are globally defined CKV on S^2. [4]

As a consequence, we observe a universal structure. On the one hand, the most general diffeomorphism algebra $\mathfrak{bmsw}_k$ is really isomorphic to the Weyl-BMS $\mathfrak{bmsw}$ algebra, such that the latter describes the asymptotics of both asymptotically flat and decelerating flat FLRW universes. As a consequence, we notice that the $\mathfrak{bmsw}$ algebra is more universal because it is more rigid towards deformations than $\mathfrak{bms}$ and $\mathfrak{gbms}$. On the other hand, taking into account that the deformation parameter s is directly linked to the matter content of our cosmological background, we have found a non-trivial deformation connecting the boundary algebras of asymptotically flat ($s = 0$) with that of asymptotically decelerating FLRW ($0 \leq s < 1$) spacetimes for stricter boundary conditions. This provides us with a *cosmological holographic flow* of asymptotic algebras which we will now explore in detail for each of the three cases in a more illuminating basis.

Cosmological holographic flow - $\mathfrak{bms}_s$

Let us expand the algebra $\mathfrak{bms}_s$ in terms of the basis of $z, \bar{z}$ monomials on S^2. Our objectives are to express the algebra in terms of a more suited basis and to use it to relate it to the family of deformations of $\mathfrak{bms}$, $W(a, b; \bar{a}, \bar{b})$, discovered in [104].

Taking into account that $\mathfrak{bms}$ is proposed to govern flat holography and $\mathfrak{bms}_s$ appears to play a similar role in decelerating spatially flat FLRW holography, we wonder whether there exists a deformation relating both which could be interpreted as a s-cosmological holographic flow. Besides, the family of deformations $W(a, b, \bar{a}, \bar{b})$ has been found to interpolate between $\mathfrak{bms}$ ($W(-\frac{1}{2}, -\frac{1}{2}; -\frac{1}{2}, -\frac{1}{2})$) and near-horizon symmetries ($W(a, a; a, a)$, [93]) which are expected to play a major role in the description of black hole microstates. As a consequence, it is interesting to develop a similar analysis for $\mathfrak{bms}_s$, which could eventually lead to the near horizon symmetry algebra for cosmological black holes.

Let us firstly define the basis of $z, \bar{z}$ monomials on S^2:

$$f_{mn} = \frac{z^m \bar{z}^n}{1 + z\bar{z}} \ , \quad V^z_m = -z^{m+1} \ , \quad V^{\bar{z}}_m = -\bar{z}^{m+1} \ , \tag{3.3.39}$$

and the basis vectors $T_{m,n} = \xi(f_{mn}, 0)$ $\mathcal{L}_m = \xi(0, V^z_m)$ and $\hat{\mathcal{L}}_m = \xi(0, V^{\bar{z}}_m)$. In terms

[4] For a comparison between our results and those of [170], we refer the reader to [164, 165].

of them, the non-vanishing commutators of (3.3.38) become

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} \ , \quad [\hat{\mathcal{L}}_m, \hat{\mathcal{L}}_n] = (m-n)\hat{\mathcal{L}}_{m+n} \ , \quad [\mathcal{L}_m, \hat{\mathcal{L}}_n] = 0 \ , \tag{3.3.40}$$

$$[\mathcal{L}_m, T_{p,q}] = \left[\frac{(m+1)}{2}(1+s) - p\right] T_{m+p,q} - s\frac{1}{1+z\bar{z}} T_{m+p+1,q+1} \ , \tag{3.3.41}$$

$$[\hat{\mathcal{L}}_n, T_{p,q}] = \left[\frac{(n+1)}{2}(1+s) - q\right] T_{p,q+n} - s\frac{1}{1+z\bar{z}} T_{m+1,q+n+1} \ . \tag{3.3.42}$$

From the first commutators we obtain a $\mathfrak{witt} \oplus \mathfrak{witt}$ algebra. However, the last two commutators are more difficult to interpret. In fact, expanding $\frac{1}{1+z\bar{z}}$, we observe that for $s \neq 0$ the commutator does not finitely close, in the sense that we obtain infinitely many generators involving $T_{m+p+r,q+r}$ and $T_{p+r,q+n+r}$ with $r \in \mathbb{N}$. This already points to $s = 0$ being a critical point of a flow.

Nevertheless, if we instead use the basis of conformally weighted smooth functions on S^2

$$\tilde{f}_{mn} = \frac{z^m \bar{z}^n}{(1+z\bar{z})^{(1+s)}} \ \Rightarrow \ \tilde{T}_{p,q} = \xi(\tilde{f}_{mn}, 0) \ , \tag{3.3.43}$$

we find

$$[\mathcal{L}_m, \tilde{T}_{p,q}] = \left[\frac{(m+1)}{2}(1+s) - p\right] \tilde{T}_{m+p,q} \tag{3.3.44}$$

$$[\hat{\mathcal{L}}_n, \tilde{T}_{p,q}] = \left[\frac{(n+1)}{2}(1+s) - q\right] \tilde{T}_{p,q+n} \ , \tag{3.3.45}$$

that is $\mathfrak{bms}_s \simeq (\mathfrak{witt} \oplus \mathfrak{witt}) \ltimes_s \mathfrak{s}_s$. The $\mathcal{L}_m$ act on S^2 as conformal Killing vectors and the operators $\tilde{T}_{pq}$ correspond to functions on S^2 with conformal weight $1+s$, that is an ideal of conformally weighted supertranslations which non-centrally extend the conformal algebra spanned by the $\mathcal{L}_m$.

It is clear that this algebra corresponds to a one-parameter deformation of $\mathfrak{bms}$ in the generators $\tilde{T}_{p,q}$ and the $[\mathcal{L}_m, \tilde{T}_{p,q}]$, $[\hat{\mathcal{L}}_n, \tilde{T}_{p,q}]$ commutators.

It turns out that the non-trivial deformations of $\mathfrak{bms}$ have been studied in [104, 171] and denoted by $W(a, b; \bar{a}, \bar{b})$ with arbitrary $a, b \in \mathbb{R}$:

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} \ , \quad [\hat{\mathcal{L}}_m, \hat{\mathcal{L}}_n] = (m-n)\hat{\mathcal{L}}_{m+n} \ , \quad [\mathcal{L}_m, \hat{\mathcal{L}}_n] = 0 \tag{3.3.46a}$$

$$[\mathcal{L}_m, \tilde{T}_{p,q}] = -\left[p + bm + a\right] \tilde{T}_{m+p,q} \tag{3.3.46b}$$

$$[\hat{\mathcal{L}}_n, \tilde{T}_{p,q}] = -\left[q + \bar{b}n + \bar{a}\right] \tilde{T}_{p,q+n} \ . \tag{3.3.46c}$$

One can quickly realize that $\mathfrak{bms}$ is given by $W(-\frac{1}{2}, -\frac{1}{2}; -\frac{1}{2}, -\frac{1}{2})$ and $\mathfrak{bms}_s$ is given by $W(-\frac{1+s}{2}, -\frac{1+s}{2}; -\frac{1+s}{2}, -\frac{1+s}{2})$. Two concrete physically interesting cases correspond to radiation ($k = 1 \leftrightarrow s = \frac{1}{2}$, $W(-\frac{3}{4}, -\frac{3}{4}; -\frac{3}{4}, -\frac{3}{4})$) and dust ($k = 2 \leftrightarrow s = \frac{2}{3}$, $W(-\frac{5}{6}, -\frac{5}{6}; -\frac{5}{6}, -\frac{5}{6})$). For the decelerating range of s, these algebras are generic deformations of $\mathfrak{bms}$ [171], meaning that their deformations also lie in $W(a, b; \bar{a}, \bar{b})$.

As a final comment, note that due to symmetries shifting $a \leftrightarrow -a$ in $W(a, b; \bar{a}, \bar{b})$, it might be indeed possible to relate $\mathfrak{bms}_s$ to the algebra of accelerating spatially flat FLRW. Future studies of this algebra and their deformations shall be performed in order to unveil the exciting secrets of cosmological holography and its relation to flat holography [87], near horizon symmetries for cosmological black holes [172], fluid-gravity duality and membrane paradigm [55], Virasoro extension $\hat{W}(-\frac{1+s}{2},-\frac{1+s}{2};-\frac{1+s}{2},-\frac{1+s}{2})$ and its deformations [104, 171], among others.

Cosmological holographic flow - $\mathfrak{gbms}_s$

Next, we consider $\mathfrak{gbms}_s \simeq \mathfrak{vect}(S^2) \ltimes_s \mathfrak{s}_s$. Besides being expected to play a major role in holography for asymptotically spatially flat FLRW, this algebra is, to our knowledge, the first deformation of $\mathfrak{gbms}$ in the literature. Both algebras constitute non-central extensions of $\mathfrak{vect}(S^2)$, which appears ubiquitously in several physical systems like fluids on the sphere [173], membranes [174–176], flat holography [87, 148, 158] and black hole entropy [98] [5]. As a consequence, it is of ultimate relevance to study this algebra.

We will work in a more tractable basis by embedding $\mathfrak{vect}(S^2)$ into $\mathfrak{vect}(\mathbb{C}^\star)$, changing the topology to admit two punctures at the poles. This basis turns out to be over-complete for $\mathfrak{vect}(S^2)$ and singular at the poles. The situation is analogous to that of local superrotations where it was argued that the singularities could be understood in terms of cosmic string punctures [177] [6]. Remarkably, the two-punctured Riemann sphere has been argued to be the relevant one for celestial scattering amplitudes and soft theorems in the context of $\mathfrak{bms}$ [86, 177–179]. We analogously expect the same to take place for $\mathfrak{gbms}$.

Let us firstly define the basis of conformally weighted $z, \bar{z}$ monomials on S^2:

$$f_{mn} = \frac{z^m \bar{z}^n}{(1+z\bar{z})^{(1+s)}} \ , \quad V^z_{m,n} = -z^{m+1}\bar{z}^n \ , \quad V^{\bar{z}}_{m,n} = -z^m \bar{z}^{n+1} \ , \tag{3.3.47}$$

and the basis vectors $T_{m,n} = \xi(f_{mn}, 0)$ $\mathcal{L}_{m,n} = \xi(0, V^z_{m,n})$ and $\hat{\mathcal{L}}_{m,n} = \xi(0, V^{\bar{z}}_{m,n})$. In terms of them, the non-vanishing commutators of (3.3.38) become

$$[\mathcal{L}_{m,n}, \mathcal{L}_{r,s}] = (m-r)\mathcal{L}_{m+r,n+s} \ , \quad [\hat{\mathcal{L}}_{m,n}, \hat{\mathcal{L}}_{r,s}] = (n-s)\hat{\mathcal{L}}_{m+r,n+s} \ , \tag{3.3.48a}$$

$$[\mathcal{L}_{m,n}, \hat{\mathcal{L}}_{r,s}] = -r\hat{\mathcal{L}}_{m+r,n+s} + n\mathcal{L}_{m+r,n+s} \ , \tag{3.3.48b}$$

$$[\mathcal{L}_{m,n}, T_{p,q}] = \left[\frac{(m+1)}{2}(1+s) - p\right] T_{p+m,q+n} \ , \tag{3.3.48c}$$

$$[\hat{\mathcal{L}}_{m,n}, T_{p,q}] = \left[\frac{(n+1)}{2}(1+s) - q\right] T_{p+m,q+n} \ , \tag{3.3.48d}$$

that is $\mathfrak{gbms}_s \simeq \mathfrak{vect}(\mathbb{C}^*) \ltimes_s \mathfrak{s}_s$.

It is clear that this algebra corresponds to a one-parameter deformation of $\mathfrak{gbms}$ in the $[\mathcal{L}_{m,n}, T_{p,q}]$, $[\hat{\mathcal{L}}_{m,n}, T_{p,q}]$ commutators. The same comments below equation (3.3.46) apply here accordingly.

[5] In chapter 7, we will deepen into $\mathfrak{vect}(S^2)$, its central extensions and deformations.

[6] We are not aware of a similar interpretation for the basis we use in this section but it should be related because it still contains the $\mathfrak{witt}$ generators as a subalgebra.

At last, we would like to emphasize that the results herein point towards the existence of a similar family of deformations of $\mathfrak{gbms}$ as the family $W(a,b;\bar{a},\bar{b})$ for $\mathfrak{bms}$. Such a family indeed exists and will be discussed as one of the main results in chapter 7.

Universality of Weyl-BMS - isomorphism $\mathfrak{bmsw}_k \simeq \mathfrak{bmsw}$

Finally, let us explore the, a priori, deformed Weyl-BMS algebra $\mathfrak{bmsw}_k$ in a different basis by embedding $\mathfrak{vect}(S^2)$ into $\mathfrak{vect}(\mathbb{C}^\star)$, changing the topology to admit two punctures at the poles. In this case, the vector fields in (3.3.30) can be expressed as

$$\xi(f_{pq},0,0) := T_{p,q} = z^p \bar{z}^q \partial_u \ , \tag{3.3.49}$$

$$\xi(0,\xi^{r(V)}_{pq},0) := W_{p,q} = -(1+2k) z^p \bar{z}^q \ u\partial_u \ , \tag{3.3.50}$$

$$\xi(0,0,V^z_{mn}) := \mathcal{L}_{m,n} = -z^{m+1}\bar{z}^n \partial_z \ , \tag{3.3.51}$$

$$\xi(0,0,V^{\bar{z}}_{mn}) := \hat{\mathcal{L}}_{m,n} = -z^{m}\bar{z}^{n+1} \partial_{\bar{z}} \ . \tag{3.3.52}$$

In terms of this basis, we obtain the following non-vanishing commutators

$$[\mathcal{L}_{m,n},\mathcal{L}_{r,s}] = (m-r)\mathcal{L}_{m+r,n+s} \ , \tag{3.3.53a}$$

$$[\hat{\mathcal{L}}_{m,n},\hat{\mathcal{L}}_{r,s}] = (n-s)\hat{\mathcal{L}}_{m+r,n+s} \ , \tag{3.3.53b}$$

$$[\mathcal{L}_{m,n},\hat{\mathcal{L}}_{r,s}] = -r\hat{\mathcal{L}}_{m+r,n+s} + n\mathcal{L}_{m+r,n+s} \ , \tag{3.3.53c}$$

$$[\mathcal{L}_{m,n},W_{p,q}] = -pW_{p+m,q+n} \ , \tag{3.3.53d}$$

$$[\hat{\mathcal{L}}_{m,n},W_{p,q}] = -qW_{p+m,q+n} \ , \tag{3.3.53e}$$

$$[\mathcal{L}_{m,n},T_{p,q}] = -pT_{p+m,q+n} \ , \tag{3.3.53f}$$

$$[\hat{\mathcal{L}}_{m,n},T_{p,q}] = -qT_{p+m,q+n} \ , \tag{3.3.53g}$$

$$[W_{m,n},T_{p,q}] = (1+2k)T_{p+m,q+n} \ . \tag{3.3.53h}$$

It is now evident that the factor $(1+2k)$ in the last commutator can be easily removed by a rescaling of $W_{m,n}$, leading to the isomorphism $\mathfrak{bmsw}_k \simeq \mathfrak{bmsw}$. [7]

As a final comment, let us note that a very similar algebra to (3.3.53) with $\mathfrak{witt}$-superrotations instead of $\mathfrak{vect}(\mathbb{C}^\star)$ has been uncovered in equation (2.31) of [180]. Therein, the authors performed a near-horizon analysis where the surface gravity κ plays exactly the same role as the factor $(1+2k)$ in equation (3.3.53h). It is hard to believe that this could be a coincidence. Nevertheless, a major difference is that in their case κ cannot be reabsorbed due to the fact that the value $\kappa = 0$ is included, whereas in our case $(1+2k) \neq 0$.

[7] It would be very interesting to explore the family of linear deformations of $\mathfrak{bmsw}$, similarly to $W(a,b;\bar{a},\bar{b})$ for $\mathfrak{bms}$ and $gW(a,b;\bar{a},\bar{b})$ for $\mathfrak{gbms}$.

3.3.3 Action on the asymptotic data

For completion and posterior use, we give the explicit variations of the asymptotic coefficients under the asymptotic diffeomorphisms (3.3.1):

$$
\begin{aligned}
\delta\Phi &= V^A D_A\Phi - 2\partial_u\xi^{r(0)} - 2k(1-\Phi)\xi^{r(V)} - 2(1-\Phi)\partial_u\xi^u \\
&\quad + 2\Theta_A\partial_u\xi^{A(1)}, && (3.3.54)\\
\delta m &= \xi^u\partial_u m + V^A D_A m - k(1-\Phi)\xi^u - \left[(1-2k)m - ku(1-\Phi)\right]\xi^{r(V)} \\
&\quad - k(1-\Phi)\xi^{r(0)} + K\partial_u\xi^{r(0)} - \partial_u\xi^{r(1)} + m\partial_u\xi^u + U_A\partial_u\xi^{A(1)} \\
&\quad + \frac{1}{2}\xi^{A(1)}D_A\Phi + \Theta_A\partial_u\xi^{A(2)}, && (3.3.55)\\
\delta K &= \xi^u\partial_u K + V^A D_A K + K\partial_u\xi^u - \Theta_A\xi^{A(1)} + 2k\left(u\xi^{r(V)} - \xi^u - \xi^{r(0)}\right) \\
&\quad + 2kK\xi^{r(V)}, && (3.3.56)\\
\delta q_{AB} &= 2(1+k)\xi^{r(V)}q_{AB} + \mathcal{L}_V q_{AB}, && (3.3.57)\\
\delta C_{AB} &= \xi^u\partial_u C_{AB} + \mathcal{L}_V C_{AB} + (1+2k)C_{AB}\xi^{r(V)} + \mathcal{L}_{\xi^{A(1)}}q_{AB} + \Theta_A D_B\xi^u + \Theta_B D_A\xi^u \\
&\quad + 2q_{AB}\left[(1+k)\xi^{r(0)} - ku\xi^{r(V)} + k\xi^u\right], && (3.3.58)\\
\delta\Theta_A &= \mathcal{L}_V\Theta_A + (1+2k)\Theta_A\xi^{r(V)} - \partial_A\xi^{r(V)} + \Theta_A\partial_u\xi^u + q_{AB}\partial_u\xi^{B(1)}, && (3.3.59)\\
\delta U_A &= \xi^u\partial_u U_A + \mathcal{L}_V U_A + \mathcal{L}_{\xi^{C(1)}}\Theta_A + 2k\Theta_A(\xi^u + \xi^{r(0)} - u\xi^{r(V)}) - D_A\xi^{r(0)} \\
&\quad + KD_A\xi^{r(V)} - (1-\Phi)D_A\xi^u + U_A\partial_u\xi^u + C_{AB}\partial_u\xi^{B(1)} + 2kU_A\xi^{r(V)} \\
&\quad + \Theta_A\xi^{r(0)} + q_{AB}\partial_u\xi^{(2)}_B, && (3.3.60)\\
\delta N_A &= \xi^u\partial_u N_A + \mathcal{L}_V N_A - (1-2k)N_A\xi^{r(V)} + N_A\partial_u\xi^u + \mathcal{L}_{\xi^{C(1)}}U_A + \mathcal{L}_{\xi^{C(2)}}\Theta_A \\
&\quad + KD_A\xi^{r(0)} - D_A\xi^{r(1)} + 2mD_A\xi^u + 2kU_A\left(\xi^{r(0)} + \xi^u - u\xi^{r(V)}\right) \\
&\quad + 2k\Theta_A\left[u^2\xi^{r(V)} - u(\xi^{r(0)} + \xi^u) + \xi^{r(1)}\right] + \Theta_A\xi^{r(1)} \\
&\quad + \left(\mathcal{D}_{AB} + \frac{1}{2}C_{AC}C^C_B\right)\partial_u\xi^{B(1)} + C_{AB}\partial_u\xi^{B(2)}\ . && (3.3.61)
\end{aligned}
$$

These present significant differences with respect to the asymptotically flat case which we studied in detail in [164–166]. The two distinct features with major impact are the following:

- Terms with $g_{uu} \simeq \mathcal{O}(r^{-1})$ and $g_{ur} \simeq \mathcal{O}(r^{-1})$ are unavoidably generated for $k \neq 0$.
- For $k \neq 0$, $\xi^{r(V)}$ generates an inevitable term in $\delta\Theta_A$ and contributions for all the modes except C_{AB}. Remarkably, m and K acquire u-dependence through the term $ku\xi^{r(V)}$.

In [164], we worked out many concrete examples upon application of these diffeomorphims. To be concise, here we just present the simplest case of transformations over a pure FLRW background. Next, we show the non-vanishing contributions of some simple subsets of transformations to the asymptotic data.

- $\xi^{r(V)} = V^A = 0$ *and* $\xi^u(u,z,\bar{z}) = f =$ *constant*

$$\delta m = -\frac{k(k+2)}{(1+k)^2} f; \qquad \delta K = -\frac{2k}{1+k} f. \tag{3.3.62}$$

- $\xi^{r(V)} = V^A = 0$ *and* $\xi^u(u,z,\bar{z}) = f(z,\bar{z})$

$$\delta m = -\frac{k(k+2)}{2(1+k)^2}[D_A D^A + 2] f(z,\bar{z}); \qquad \delta C_{zz} = -2 D_z D_z f(z,\bar{z}); \tag{3.3.63}$$

$$\delta K = -\frac{k}{1+k}[D_A D^A + 2] f(z,\bar{z}); \qquad \delta C_{\bar{z}\bar{z}} = -2 D_{\bar{z}} D_{\bar{z}} f(z,\bar{z}); \tag{3.3.64}$$

$$\delta U_A = -\frac{1}{2(1+k)} D_A [D_C D^C + 2] f(z,\bar{z}); \tag{3.3.65}$$

$$\delta N_A = -\frac{ku}{2(1+k)^2} D_A [D_C D^C + 2] f(z,\bar{z}) \; . \tag{3.3.66}$$

Notice that all variations vanish in the case $f(z,\bar{z}) \propto Y^1_m$, consistently corresponding to the spatial translations analyzed in section 3.3.1. Besides, δm, δK and δN_A are generated for $k \neq 0$.

- $\xi^u = \xi^{r(V)} = 0$ *and* $V^A(z,\bar{z}) \neq 0$

$$\delta q_{AB} = \mathcal{L}_V q_{AB} \; . \tag{3.3.67}$$

It is zero for rotations but non-zero for general vector fields on S^2.

3.4 Cosmological black holes and logarithmic expansion

So far, we have not wondered about the presence of explicit exact solutions within the ansatz (3.2.2) other than pure FLRW. In this section, we transform several cosmological black and white hole solutions into Bondi coordinates. This serves us to become aware that the ansatz (3.2.2) describes white holes but requires a logarithmic extension in order to include their black hole counterparts. Afterwards, we propose a logarithmic ansatz incorporating these solutions and explore its properties.

3.4.1 Cosmological black holes

Our aim is to transform three inequivalent representatives of asymptotically spatially flat FLRW central inhomogeneities to Bondi coordinates. This will permit us to uncover a pattern for this class of solutions, which will motivate the logarithmic ansatz of section 3.4.2.

Firstly, we consider the Thakurta solution [181] which represents the late time attractor of a larger class of solutions, the so-called Faraoni-Jacques or generalized McVittie [182, 183]. Besides, this metric was used in [184] to describe a potential model for primordial black holes. Next, we move on to Sultana-Dyer black and white holes [185], which have been studied in more detail in [186] and also try to set the basis for describing primordial black holes which expand with the universe flow [186]. Finally, we turn to Vaidya black and white holes [187], representing inhomogeneities decoupled from the cosmological flow which aim to be a simplified model of astrophysical black holes.

Some studies on the physical feasibility of these metrics have been performed [183, 186, 188], uncovering possible pathologies, like near horizon superluminality, or advocating doubts on whether or not they really represent black hole solutions. Nevertheless, they constitute the building blocks of potentially more realistic solutions (e.g. Lemaître-Tolman-Bondi [189]). Therefore, we proceed to explore if these metrics are included in our ansatz in a similar way that the Schwarzschild solution belongs to asymptotically flat spacetimes.

Thakurta black hole

The non-rotating Thakurta black hole [181] corresponds to superimposing a FLRW background over a Schwarzschild black hole in areal coordinates [8]:

$$\begin{aligned} \mathrm{d}s^2 &= -\left(1-\frac{2m}{r}\right)\mathrm{d}t^2 + a(t)^2\left[\frac{\mathrm{d}r^2}{1-\frac{2m}{r}} + r^2\mathrm{d}\Omega^2\right] \\ &= a^2(\eta)\left[-\left(1-\frac{2m}{r}\right)\mathrm{d}\eta^2 + \frac{\mathrm{d}r^2}{1-\frac{2m}{r}} + r^2\mathrm{d}\Omega^2\right] . \end{aligned} \tag{3.4.1}$$

Using $\eta = u + r + 2m\log\left(\frac{r}{2m}-1\right)$, we can write the previous metric in Bondi coordinates:

$$\mathrm{d}s^2 = \left(\frac{u + r + 2m\log\left(\frac{r}{2m}-1\right)}{L}\right)^{2k}\left[-\left(1-\frac{2m}{r}\right)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + 2r^2\gamma_{z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z}\right] . \tag{3.4.2}$$

Before we continue, let us note that the metrics of the form

$$\mathrm{d}s^2 = \left(A\eta + B^2\eta^2\right)^2\left[-\left(1-\frac{2m}{r}\right)\mathrm{d}\eta^2 + \frac{4m}{r}\mathrm{d}\eta\mathrm{d}r + \left(1+\frac{2m}{r}\right)\mathrm{d}r^2 + 2r^2\gamma_{z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z}\right] \tag{3.4.3}$$

have been proposed to describe primordial black holes in [184] and have the same transformation to Bondi coordinates in the large r regime, being clearly included in our logarithmic expansion of section 3.4.2 for $k = 2$ (matter domination), after expanding the scale factor in series.

[8] This conformal time η agrees with the one used in pure FLRW, being therefore the appropriate one to be compared with our asymptotic expansion.

Sultana-Dyer black hole

The Sultana-Dyer solution [185] consists of a time-dependent Kerr-Schild transformation of Minkowski. The resulting metric is given by:

$$\mathrm{d}s^2 = a^2(\eta)\left[-\mathrm{d}\eta^2 + \mathrm{d}r^2 + r^2 d\Omega + \frac{2m}{r}(\mathrm{d}\eta \pm \mathrm{d}r)^2\right] , \tag{3.4.4}$$

where $\pm$ correspond respectively to black hole and white hole solutions [9]. The Sultana-Dyer black hole solution is equivalently written as:

$$\mathrm{d}s^2 = \left(\frac{\eta}{L}\right)^{2k}\left[-\left(1-\frac{2m}{r}\right)\mathrm{d}\eta^2 + \frac{4m}{r}\mathrm{d}\eta\mathrm{d}r + \left(1+\frac{2m}{r}\right)\mathrm{d}r^2 + 2r^2\gamma_{z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z}\right] . \tag{3.4.5}$$

In order to convert into conformal Schwarzschild, we have to reverse the Kerr-Schild transformation such that $d\bar{\eta} = d\eta - \frac{2m}{r-2m}dr$, $\bar{\eta} = \eta - 2m\log\left(\frac{r}{2m} - 1\right)$. Finally, to transform to Bondi coordinates, we use $\bar{\eta} \to u + r + 2m\log\left(\frac{r}{2m} - 1\right)$. As a result we obtain

$$\mathrm{d}s^2 = \left(\frac{u + r + 4m\log\left(\frac{r}{2m} - 1\right)}{L}\right)^{2k}\left[-\left(1-\frac{2m}{r}\right)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + 2r^2\gamma_{z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z}\right] . \tag{3.4.6}$$

A similar analysis for the white hole reveals that the logarithms in the changes of coordinates cancel each other and we find:

$$\mathrm{d}s^2 = \left(\frac{u+r}{L}\right)^{2k}\left[-\left(1-\frac{2m}{r}\right)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + 2r^2\gamma_{z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z}\right] , \tag{3.4.7}$$

which is a solution in our expansion (3.2.2).

Let us finally comment that the metrics of this form, replacing the scale factor by a combination of radiation phase pasted to matter dominated phase, have been proposed to describe primordial black holes in [186] and have an identical transformation to Bondi coordinates in the large r regime, being clearly included in our logarithmic expansion of section 3.4.2 for $k = 2$, after expanding the scale factor in series.

Vaidya black hole

Vaidya's cosmological black and white holes [187] are obtained from application of a conformal transformation over Minkowski, getting spatially flat FLRW in conformal coordinates, and subsequently performing a time-independent Kerr-Schild transformation over it:

$$ds^2 = a^2(\eta)\left[-\mathrm{d}\eta^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega + \frac{2m}{ra^2(\eta)}(\mathrm{d}\eta \pm \mathrm{d}r)^2\right] . \tag{3.4.8}$$

[9] Note that this metric equals Vaidya when $ma^2(\eta) \to m$.

The black hole solution is then written as

$$\mathrm{d}s^2 = \left(\frac{\eta}{L}\right)^{2k}\left[-\left(1-\frac{2m}{r\left(\frac{\eta}{L}\right)^{2k}}\right)\mathrm{d}\eta^2 + \frac{4m}{r\left(\frac{\eta}{L}\right)^{2k}}\mathrm{d}\eta\mathrm{d}r \right.$$
$$\left. + \left(1+\frac{2m}{r\left(\frac{\eta}{L}\right)^{2k}}\right)\mathrm{d}r^2 + 2r^2\gamma_{z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z}\right] . \tag{3.4.9}$$

The exact transformation of this equation to Bondi coordinates is far from obvious to us. Nevertheless, in the limit $r \to \infty$, $\eta \sim r$ we obtain

$$\mathrm{d}s^2 = a^2(r)\left[-\left(1-\frac{2m}{a^2(r)r}\right)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + 2r^2\gamma_{z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z}\right] , \tag{3.4.10}$$

upon solving the differential equation

$$\mathrm{d}\eta = \mathrm{d}r + \frac{4m}{\left(\frac{\eta(r)}{L}\right)^{2k} r - 2m}\mathrm{d}r , \tag{3.4.11}$$

in order to find $\eta(r)$ such that $a^2(r) = \left(\frac{\eta(r)}{L}\right)^{2k}$. Expanding around $r \to \infty$

$$\mathrm{d}\eta = \mathrm{d}r + \frac{4m}{\left(\frac{\eta}{L}\right)^{2k} r}\left(1 + \frac{2m}{\left(\frac{\eta}{L}\right)^{2k} r} + \frac{4m^2}{\left(\frac{\eta}{L}\right)^{4k} r^2} + ...\right)\mathrm{d}r , \tag{3.4.12}$$

we solve the first order expansion in the limit $\eta \sim r \to \infty$:

$$\mathrm{d}\eta = \mathrm{d}r + \frac{4m}{\left(\frac{\eta}{L}\right)^{2k} r}\mathrm{d}r + ... \Rightarrow \eta \sim r - \frac{2mL^{2k}}{k} r^{-2k} \quad k \neq 0$$
$$\eta \sim r + 4m\log(r) \quad k = 0 . \tag{3.4.13}$$

This permits us to check that Vaidya black hole is expressible in terms of logarithms and k-powers of $1/r$ in the region determined by $\eta \sim r \to \infty$, while it is not clear whether this metric is analytically expressible in terms of our ansatz in general [10].

An exact analysis for the white hole reveals that:

$$\mathrm{d}s^2 = \left(\frac{u+r}{L}\right)^{2k}\left[-\left(1-\frac{2m}{\left(\frac{u+r}{L}\right)^{2k} r}\right)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + 2r^2\gamma_{z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z}\right] , \tag{3.4.14}$$

which turns out to be much simpler than the black hole but still presents subtleties because k can be fractional and our $1/r$ expansion would not contain this example

[10] Note that one could try to solve (3.4.11) in the limit $\eta \sim r \to \infty$. Although the solutions turn out to be very complicated hypergeometric functions, in the large r regime one can observe logarithmic and polynomial behaviour in $\frac{1}{r}$.

(exactly as in the black hole case). Nevertheless, the physically relevant cases of radiation and dust correspond to $k = 1$ and $k = 2$ respectively, so both do not require any fractional expansion and are included in our ansatz (3.2.2).

Before moving on, we would like to comment on the possibility of primordial Vaidya black hole solutions. Metrics of this form, replacing the scale factor by a combination of radiation phase pasted to matter dominated phase, have been explored in [186] and have a similar transformation to Bondi coordinates in the large r regime as the one explored in this section for $k = 2$.

Physical interpretation These coordinate transformations highlight the fact that the ansatz for asymptotically spatially flat FLRW spacetimes (3.2.2) covers only white hole solutions, whereas we observe more involved scale factors for the black holes which need a logarithmic ansatz (section 3.4.2). This might be indeed related to the fact that white holes can be qualitatively regarded as an inversion of the arrow of time in black hole solutions, meaning that the black hole horizon is distinguished from $\mathcal{I}^+$ and its coupling to the cosmological flow manifests in the scale factor as a growing portion of spacetime, from which nothing can reach anymore $\mathcal{I}^+$. On the contrary, the white hole horizon has no effect on $\mathcal{I}^+$ more than its shared "topological" m contribution due to the singularity at $r = 0$.

In fact, we realize that the cases of Thakurta and Sultana-Dyer are similar, the only difference being the conformal time used to build them. In both cases the inhomogeneity expands with the universe, while Vaidya differs because it detaches from the expansion of the universe, possessing a shrinking event horizon and leading to complicated analytical dependence from the perspective of $\mathcal{I}^+$.

It is also worth to note that Vaidya's metric points out the special role of $2k \in \mathbb{N}$ backgrounds which do not require a fractional $1/r$ expansion. Precisely the physically favoured radiation ($k = 1$) and matter ($k = 2$) dominated universes present this distinguished feature.

3.4.2 Logarithmic expansion

As we noticed in the analysis of subsection 3.4.1, cosmological black hole models are not covered by the ansatz (3.2.2), since the expansion at $r \to \infty$ involves logarithmic terms. These logarithmic terms in the scale factor diverge towards $\mathcal{I}^+$. Since the retarded time u is finite at null infinity, the log term will always dominate over the u term in the scale factor and we, therefore, cannot write the metric at null infinity in the form of a time-dependent scale factor $a^2 \propto (u + r)^{2k}$ times an asymptotically flat part.

Let us take as an example the Sultana-Dyer black hole (3.4.6) and factorize the scale factor in the following way:

$$a^2 = \left(\frac{u + r + 4m\log\left(\frac{r}{2m} - 1\right)}{L}\right)^{2k} = \left(\frac{r}{L}\right)^{2k}\left(1 + \frac{u + 4m\log\left(\frac{r}{2m} - 1\right)}{r}\right)^{2k} . \tag{3.4.15}$$

The first term is divergent at $\mathcal{I}^+$, so we will extract it to be the asymptotic scale factor. The second part of (3.4.15) is finite and differentiable at null infinity so we can expand it in terms of $\frac{\log^m r}{r^n}$:

$$\left(1 + \frac{u + 4m\log\left(\frac{r}{2m} - 1\right)}{r}\right)^{2k} = 1 + \frac{2k\left(u + 4m\log\frac{r}{2m}\right)}{r} + \dots \tag{3.4.16}$$

In order to generalize our ansatz to include cosmological black hole solutions, we conclude that we have to choose a time independent asymptote and expansion which includes logarithmic terms. This approach is similar to [170], apart from the essential fact that they did not include logarithmic terms.
The most general ansatz we can write down which includes logarithmic terms in the expansion and is finite and differentiable at null infinity, apart from an r-dependent scale factor, is given by [11]:

$$\begin{aligned} \mathrm{d}s^2 = \left(\frac{r}{L}\right)^{2k}\Bigg\{&-\left(1 - \Phi - \frac{2m + A\log\frac{r}{B}}{r}\right)\mathrm{d}u^2 - 2\left(1 - \frac{K + E\log\frac{r}{F}}{r}\right)\mathrm{d}u\mathrm{d}r \\ &+ 2\left(r\Theta_A + U_A + G_A\log\frac{r}{H_A}\right)\mathrm{d}u\mathrm{d}x^A \\ &+ \left(r^2 q_{AB} + rC_{AB} + rM_{AB}\log\frac{r}{N_{AB}}\right)\mathrm{d}x^A\mathrm{d}x^B\Bigg\} , \end{aligned} \tag{3.4.17}$$

where q_{AB} is a general metric on the sphere. In order to preserve the Bondi gauge, we now have to demand that

$$q^{AB}C_{AB} = q^{AB}M_{AB}\log\frac{r}{N_{AB}} = 0 . \tag{3.4.18}$$

To compare the asymptotic symmetry algebra of the logarithmic ansatz (3.4.17) to the previous ansatz with a time dependent scale factor (3.2.2), we calculate the Lie derivatives of the above metric with respect to the diffeomorphisms (3.3.1) in appendix A.2. Repeating the analysis performed in section 3.3 together with $\partial_u q_{AB} = 0$, we discover that the asymptotic diffeomorphisms at leading order are exactly the same as the ones from section 3.3. The algebra at future null infinity is not spoiled by introducing logarithmic terms in the expansion and, therefore, applies to cosmological black holes as well.

[11] Logarithmic ansatz have also been explored in asymptotically flat spacetimes long time ago by [190].

As a final point, we comment on the form of the Weyl tensor for the metrics (3.4.17). For technical reasons, we consider as an example (3.4.17) with all the logarithmic terms given by $4m\log(r/2m)$ and the only other non-zero coefficient being $m(u,z,\bar{z})$. This corresponds to the asymptotic expansion of the Sultana-Dyer black hole with a time dependent mass.
For this case some non-vanishing components of the Weyl tensor are given by [12]:

$$W_{ruru} = \left(\frac{r}{L}\right)^{2k}\left(-\frac{2m}{r^3} - \frac{1}{r^4}\left(4km(u + 4m\log(r/2m))\right) + ...\right) \ , \tag{3.4.19}$$

$$W_{ruAu} = \left(\frac{r}{L}\right)^{2k}\left(\frac{3\partial_u m}{2r^2} + \frac{3kD_A m(u + 4m\log(r/2m))}{r^3} + ...\right) \ . \tag{3.4.20}$$

We can observe that the appearence of the logarithmic terms in the expansion spoils the peeling property of the Weyl tensor, building, therefore, a major difference with respect to the Schwarzschild solution within the asymptotically flat case.

[12] We choose the same index convention as in [76].

Chapter 4

Asymptotic dynamics and charges in FLRW spacetimes

In the previous chapter, we have undergone a purely geometrical analysis of asymptotically spatially flat and decelerating FLRW spacetimes which is valid for general gravity theories. However, we have to adopt a concrete theory in order to evaluate the actual dynamics for these spacetimes. In this chapter, we select General Relativity as a gravity theory and evaluate the equations of motion for asymptotically FLRW spacetimes with finite fluxes, showing that the dynamics is fully constrained by the stress-tensor of the sources. This situation is in sharp contrast to asymptotically flat spacetimes where the tensor modes constitute free data. Next, we propose an expression for the charges which are associated with the cosmological supertranslations and whose evolution equation features a novel contribution from the Hubble-Lemaître flow. This chapter is based on our work [166].

4.1 Equations of motion

In this section, we adopt General Relativity as our gravity theory and perform an on-shell analysis. This means that we analyse the Einstein tensor as an expansion in r^{-1}, such that the expansion coefficients $G^{(i)}_{\mu\nu}$ are defined by:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \sum_i \frac{G^{(i)}_{\mu\nu}}{r^i}, \tag{4.1.1}$$

and the Ricci scalar is expanded as

$$R = \left(\frac{r+u}{L}\right)^{-2k} \sum_{i=0}^{\infty} \frac{R^{(i)}}{r^i} \ . \tag{4.1.2}$$

4.1.1 Metrics with finite fluxes

We begin by introducing the leading uu and uA components of the Einstein tensor obtained from the ansatz (3.2.2):

$$G^{(1)}_{uu} = -(1+k)\partial_u\Phi - q_{AB}(D^B\partial_u\Theta^A + (1+2k)\Theta^A\partial_u\Theta^B)\ ,\ G^{(0)}_{uA} = -\frac{1}{2}\partial_u\Theta_A\ .$$

It can be easily observed that these components lead to linearly divergent fluxes at large r.[1] [2]

As a consequence, we restrict ourselves to the solutions where these components vanish, which is equivalent to imposing $\partial_u\Phi = \partial_u\Theta_A = 0$. This choice is consistent because the variations $\delta\Phi$ and $\delta\Theta_A$ generated by means of asymptotic transformations are u-independent if we start with Φ and Θ_A which do not depend on u, as can be quickly noticed from (A.1.1) and (A.1.4).

The resulting metrics satisfy a series of properties that make them suited for a Bondi analysis. First, it is easy to notice that all the leading terms are u-independent, such that only the subleading terms can be dynamical. This is equivalent to taking as a boundary the equivalence class of unperturbed FLRW metrics allowed by the $\mathfrak{bmsw}_k$ asymptotic transformations described in the previous chapter, while the potential dynamics is restricted to the subleading terms m, K, U_A and C_{AB}. The latter transform respectively as scalars, vector and tensor encoding (up to combinations) a maximum of six degrees of freedom, which can be reduced after imposing the remaining equations of motion. Secondly, one can check that the resulting $G_{\mu\nu}$ components are of the same order in r as the perfect fluid background, which is a reminiscence of the analysis performed in [170]. This guarantees that not only the $G_{\mu\nu}$ components but also their fluxes through future null infinity $\mathcal{I}^+$ are finite.

4.1.2 Asymptotic Einstein equations and degrees of freedom

Following the analysis of the previous subsection, we analyze the equations of motion and corresponding degrees of freedom for the on-shell ansatz (3.2.2) with $\partial_u\Phi = \partial_u\Theta_A = \partial_u q_{AB} = 0$.

General case

Let us present the leading Einstein equations and classify them in scalar, vector and tensor equations for general settings.

[1] The presence of u-dependent leading terms, such as $\Phi(u)$, $\Theta_A(u)$ and $q_{AB}(u)$, would be necessary if one wanted to describe dynamical perturbations of the FLRW boundary among our boundary metrics.

[2] We remark that the logarithmic terms described in section 3.4 enter at subleading order and, therefore, should be included in this on-shell analysis and adequately treated. Such an analysis is beyond the scope of this thesis, but we expect that it will not distort the essence of the results contained herein.

Scalar equations We start with the leading expression of G_{uu}

$$
\begin{aligned}
G^{(2)}_{uu} = & \frac{1}{4}\partial_u C_{AB}\partial_u C^{AB} + D_A\partial_u U^A + 2(1+k)\partial_u\left(m+\Phi K\right) \\
& + \frac{1}{2}(q_{AC}q_{BD} - q_{AD}q_{BC})D^B\Theta^A D^D\Theta^C + \partial_u K\left(2 + 2k\Theta^A\Theta_A + D_A\Theta^A\right) \\
& + \frac{1}{2}(1-2k)\Theta^A D_A\Phi - \frac{1}{2}D_A D^A\Phi + \Theta^A\left(D_A\partial_u K + 2k\partial_u U_A\right) \\
& + (\Phi - 1)\left[-\frac{1}{2}\mathcal{R} + \frac{1}{4}(1+8k+4k^2)\Theta_A\Theta^A + 2(k+1)D_A\Theta^A\right] \\
& - (\Phi-1)\left[(2k+1)(\Phi-1) + k^2(\Phi+1)\right] + 2k(k+1).
\end{aligned} \tag{4.1.3}
$$

This equation corresponds to the Bondi mass-loss equation in the asymptotically flat limit.

The constraint equation for the parameter K reads as

$$
G^{(3)}_{rr} = -2(1+k)(2ku - K)\ . \tag{4.1.4}
$$

Note that K is completely fixed by the corresponding term in the expansion of the energy-momentum tensor.

Besides, we also have

$$
G^{(2)}_{ur} = \frac{1}{2}(\mathcal{R} - 2) + 3k^2 + (1+k)^2\Phi - \frac{1}{4}(1+2k)^2\Theta_A\Theta^A + \frac{1}{2}(3+4k)D_A\Theta^A\ , \tag{4.1.5}
$$

which does not generally impose any extra condition on the parameters.

Vector equations At leading order a novel constraint for the parameter Θ_A appears. It is given by

$$
G^{(1)}_{rA} = (1+k)\Theta_A\ . \tag{4.1.6}
$$

The function Θ_A, just as K, is now completely determined by the corresponding expansion coefficient of the energy momentum tensor.

At subleading orders, we obtain the generalized version of the well-known constraint for U_A in flat spacetimes

$$
G^{(2)}_{rA} = \frac{1}{2}\left[2ku\Theta_A + (1+2k)\left(C_{AB}\Theta^B - 2U_A - K\Theta_A\right) - (3+2k)D_A K - D_B C^B_A\right] \tag{4.1.7}
$$

and

$$
\begin{aligned}
G^{(1)}_{uA} = & \Theta_A\left[\frac{1}{2}\mathcal{R} - 1 + \Phi - k(2-\Phi) + k^2(1+\Phi)\right] \\
& + \frac{\Theta_A}{4}\left[-(1+2k)^2\Theta_B\Theta^B + 2(3+4k)D_B\Theta^B + 6\partial_u K\right] \\
& + \frac{1}{2}\Big[-2kD_A\Phi + \partial_u D^B C_{AB} - D_B D_A\Theta^B + D_A D^A\Theta_A \\
& - \Theta^B(-2k(D_A\Theta_B - D_B\Theta_A) + (1+2k)\partial_u C_{AB}) + 2\partial_u U_A + \partial_u D_A K\Big]\ ,
\end{aligned} \tag{4.1.8}
$$

which do not generally impose any new condition on the parameters.

Tensor equations The leading order tensor components are given by

$$\begin{aligned} G^{(0)}_{AB} = &-\frac{1}{2}\left[\Theta_A\Theta_B - (1+2k)(D_A\Theta_B + D_B\Theta_A) - 2k\partial_u C_{AB}\right] \\ &+ \frac{1}{4}q_{AB}\left[(4k^2-1)\Theta_C\Theta^C - 4(k(k(1+\Phi)-2) + (1+2k)D_C\Theta^C + \partial_u K)\right] \ , \end{aligned} \tag{4.1.9}$$

which constitutes a novel constraint for the time evolution of C_{AB} that is absent in asymptotically flat spacetimes ($k=0$). Interestingly, this condition, only present for $k \neq 0$, is associated with the presence of a Hubble scale in expanding universes from which all the modes stop being oscillating and are frozen [63].

Ricci scalar Finally, let us analyze the value of the leading order Ricci scalar for our metrics

$$\begin{aligned} R^{(2)} &= \mathcal{R} - \frac{3}{2}(1+2k)^2\Theta_A\Theta^A - 2\left((1+3k)(1-\Phi) - 3k^2(1+\Phi) - (2+3k)D_A\Theta^A - \partial_u K\right) \\ &= R^{(2)}_{\text{FLRW}} + \left[\mathcal{R} - 2 - \frac{3}{2}(1+2k)^2\Theta_A\Theta^A + 2\left((1+3k+3k^2)\Phi + (2+3k)D_A\Theta^A + \partial_u K\right)\right] . \end{aligned} \tag{4.1.10}$$

We would like to recall that in the flat limit, i.e. $k \to 0$, $\Theta_A \to 0$ and $K \to 0$, this equation becomes

$$R^{(2)}_{flat} = 2\Phi + \mathcal{R} - 2 \ . \tag{4.1.11}$$

In fact, the condition $R^{(2)}_{flat} \overset{!}{=} 0$ is imposed as a flatness condition, leading to a constraint on Φ; see [149]. Following the same logic, we can impose $R^{(2)} \overset{!}{=} R^{(2)}_{\text{FLRW}}$, which again constrains Φ in terms of $\mathcal{R}$, $T^{(1)}_{rA}$ and $T^{(3)}_{rr}$, determining a balance equation which ensures that the spacetimes under analysis still have a FLRW profile.

Before continuing, it is instructive to have a closer look at the values of the variations (3.3.54), (3.3.56) and (3.3.59) in our setting. In fact, we observe that they can be expressed as:

$$\begin{aligned} \delta\Theta_A &= \mathcal{L}_V\Theta_A + 2k\partial_A\xi^{r(V)} \ , \\ \delta\Phi &= V^A D_A\Phi + \left[2(1-\Phi)(1+k) - 4k + \frac{1+2k}{1+k}\left(D_A D^A + (1+2k)\Theta^A D_A\right)\right]\xi^{r(V)}, \\ \delta K &= \xi^u\partial_u K + V^A D_A K - K\xi^{r(V)} + \frac{2k}{(1+k)}[u\xi^{r(V)} - D_A D^A\xi^u - \xi^u] \\ &\quad + \frac{(1+3k)}{(1+k)}\Theta^A D_A\xi^u \ . \end{aligned}$$

These Lie derivatives confirm explicitly our previous statement that the choice $\partial_u\Phi = \partial_u\Theta_A = 0$ is consistent because the variations $\delta\Phi$ and $\delta\Theta_A$ generated by means of asymptotic transformations are u-independent if we start with Φ and Θ_A not depending on u. Moreover, we observe that Θ_A is unavoidably generated by Weyl transformations, while in the presence of only supertranslations this component is not necessarily present. The same statement holds true for Φ, whereas K is generated in any case. Remarkably, in the absence of Weyl-transformations, K does not need to be u-dependent.

Absence of Weyl transformations

We have noticed how complicated the analytical treatment becomes in general settings. Nevertheless, the physical picture and the role of the different coefficients, as well as the nature of the different degrees of freedom, are exactly the same as in simpler backgrounds. [3] Therefore, we will now restrict ourselves to a simple setting with $\Theta_A = \Phi = 0$, which is consistent with supertranslations and the absence of Weyl diffeomorphisms (*i.e.* $\xi^{r(V)} = 0$), and analyze it in more detail, solving the Einstein equations explicitly.

Let us start by writing down the relevant Einstein equations (4.1.3)-(4.1.10) in our simplified setting.

$$\begin{aligned}
G^{(2)}_{uu} &= \frac{1}{2}(\mathcal{R}-2) + 3k^2 + 2\partial_u K - \partial_u(D_A U^A) + \frac{1}{4}\partial_u C_{AB}\partial_u C^{AB} \\
&\quad - 2(1+k)\partial_u m\ , && (4.1.12)\\
G^{(3)}_{rr} &= -2(1+k)(2ku - K), && (4.1.13)\\
G^{(2)}_{ur} &= \frac{1}{2}(\mathcal{R}-2) + 3k^2, && (4.1.14)\\
G^{(1)}_{uA} &= +\frac{1}{2}\left(\partial_u D^B C_{AB} + 2\partial_u U_A + \partial_u D_A K\right), && (4.1.15)\\
G^{(2)}_{rA} &= \frac{1}{2}\left[-2(1+2k)U_A - (3+2k)D_A K - D_B C^B_A\right], && (4.1.16)\\
G^{(0)}_{AB} &= k\partial_u C_{AB} - q_{AB}[k(k-2) + \partial_u K], && (4.1.17)
\end{aligned}$$

together with

$$R^{(2)} = R^{(2)}_{\text{FLRW}} + [\mathcal{R} - 2 + 2\partial_u K] \overset{!}{=} R^{(2)}_{\text{FLRW}}\ . \qquad (4.1.18)$$

[3] Note that the backgrounds are encoded in the u-independent coefficients Θ_A and Φ.

From these equations we find the constraints

$$K = \frac{8\pi G}{2(1+k)} T_{rr}^{(3)} + 2ku = \frac{8\pi G}{2(1+k)} (T_{rr}^{(3)} - T_{rr\ \mathrm{FLRW}}^{(3)}) = \frac{8\pi G}{2(1+k)} \triangle T_{rr}^{(3)} \ , \tag{4.1.19}$$

$$\partial_u K = \frac{1}{2}(2 - \mathcal{R}) = \frac{8\pi G}{2(1+k)} \partial_u (\triangle T_{rr}^{(3)}) \ , \tag{4.1.20}$$

$$U_A = -\frac{8\pi G}{(1+k)} T_{rA}^{(2)} - \frac{q^{BM}}{2(1+k)} (D_M C_{AB}) - \frac{3+2k}{2(1+k)} D_A K \ , \tag{4.1.21}$$

which indicate that K and U_A do not propagate and are completely determined in terms of the sources and other fields.

Next, we examine in detail (4.1.12) and (4.1.17). Let us begin by decomposing (4.1.17) into trace and traceless components

$$q^{AB} G_{AB}^{(0)} = -2[k(k-2) + \partial_u K] \ , \tag{4.1.22}$$

$$G_{AB}^{(0)} - \frac{1}{2} q_{AB} q^{CD} G_{CD}^{(0)} = k \partial_u C_{AB} \ . \tag{4.1.23}$$

The former equation does not convey special information but the latter tells us that, for $k \neq 0$, the time evolution of C_{AB} is constrained by the sources and it is not anymore a field carrying dynamical degrees of freedom at future null infinity $\mathcal{I}^+$. This is a crucial difference with respect to asymptotically flat spacetimes, where C_{AB} only enters the Bondi mass-loss formula (4.1.12) and is unconstrained. Finally, looking in detail at equation (4.1.12), we observe that m enters the mass loss equation with only one time derivative, which would define its evolution as Cauchy data in terms of energy momentum components.

Furthermore, after a lengthy computation, it can be shown that N_A also enters the equations of motion for $G_{uA}^{(2)}$ with only one time derivative and is constrained by the energy momentum tensor. The subleading coefficient $\mathcal{E}$ in $g_{ur} \simeq \mathcal{O}(r^{-2})$ enters as $q_{AB}\partial_u \mathcal{E}$ in $G_{AB}^{(1)}$ but in $G_{rr}^{(4)}$ it appears linearly without derivative and is fully constrained as can be seen from

$$G_{rr}^{(4)} = -\frac{1}{4} C_{AB} C^{AB} + 2\left[3k^2 u^2 + 2\mathcal{E} + K^2 + k\left(3u^2 + 2\mathcal{E} - uK + K^2\right)\right] \ . \tag{4.1.24}$$

Short summary We observe that for asymptotically decelerating FLRW spacetimes, the dynamics at future null infinity $\mathcal{I}^+$ is completely constrained. This could have been expected taking into account that, in an expanding universe, there is a Hubble scale from which all the modes stop being oscillating and simply become frozen [63].

Let us conclude this section with two brief comments. In the background $\Theta_A = \Phi = 0$, the coefficients $K, \mathcal{E}, U_A$ are fully constrained, while m, N_A, C_{AB} are non-propagating and their evolution equations are determined by the sources. These coefficients represent frozen scalar, vector and tensor modes which stop being dynamical at the Hubble scale due to the appearance of well-known friction terms [63].

In the most general case with u-dependent Φ, Θ_A and q_{AB}, we point out that these four coefficients and/or their time evolution are also completely constrained in terms of the energy momentum tensor, such that they are again non-propagating.

The results derived in this section raise the question whether non-trivial infrared structure can be expected in more realistic cosmological settings where expansion and the Hubble scale are present.

4.2 Asymptotic charges for supertranslations

In this section, we will propose asymptotic charges for supertranslations in the absence of Weyl transformations. In fact, this is the setting we explored in detail in the last part of the previous section 4.1.2, for which $\Theta_A = \Phi = 0$.

A naive ansatz, inspired by the asymptotically flat supertranslation charges (2.1.7), to start with is

$$\tilde{Q}_f = \int_{S^2} \sqrt{q} f(x^A)(a^2 m) \ . \tag{4.2.1}$$

We compute the algebra of charges using the definition for integrable charges in [156]

$$\left\{\tilde{Q}_{f_1}, \tilde{Q}_{f_2}\right\} = -\delta_{f_2}\tilde{Q}_{f_1} = -\int_{S^2} \sqrt{q}[f_1 \delta_{f_2}(a^2 m)] \ . \tag{4.2.2}$$

The required variation reads as

$$\begin{aligned}\delta_{f_2}(a^2 m) = a^2 \Bigg\{ & f_2 \partial_u m - \frac{k(k+2)}{2(1+k)^2}\left(D_A D^A + 2\right) f_2 \\ & - \frac{1}{2(1+k)}\left[(\partial_u U^A)(D_A f_2) - \frac{1}{2} D_A(\partial_u C^{AB} D_B f_2) - \frac{1}{2} D_A(\partial_u K D^A f_2)\right]\Bigg\}\end{aligned} \tag{4.2.3}$$

and, plugging in the equations of motion, it leads to

$$\begin{aligned}\delta_{f_2}(a^2 m) = a^2 \Bigg\{ & -\frac{f_2 \triangle G^{(2)}_{uu} + D^A(\triangle G^{(1)}_{uA} f_2) - [(\mathcal{R}-2) + 2\partial_u K]}{2(1+k)} \\ & - \frac{k(k+2)}{2(1+k)^2}\left(D_A D^A + 2\right) f_2 - \frac{\partial_u C^{AB} \partial_u C_{AB}}{8(1+k)} f_2 \\ & + \frac{1}{4(1+k)}\left[D_A D_B (f_2 \partial_u C^{AB}) + D_A D^A (f_2 \partial_u K)\right]\Bigg\}.\end{aligned} \tag{4.2.4}$$

Let us now recall that we only consider supertranslations, which means $\mathcal{R} = 2$. As a consequence, equation (4.1.18) tells us that $\partial_u K = 0$. This reduces the previous expression to:

$$\delta_{f_2}(a^2 m) = a^2 \Bigg\{ -\frac{f_2 \triangle G^{(2)}_{uu} + D^A(\triangle G^{(1)}_{uA} f_2)}{2(1+k)} - \frac{k(k+2)}{2(1+k)^2}\left(D_A D^A + 2\right) f_2 - \frac{\partial_u C^{AB}\partial_u C_{AB}}{8(1+k)} f_2 + \frac{1}{4(1+k)} D_A D_B (f_2 \partial_u C^{AB}) \Bigg\}. \tag{4.2.5}$$

The second term in the first line can be reabsorbed by a redefinition of the charge as:

$$\boxed{Q_f := \tilde{Q}_f - \frac{(k+2)}{2(1+k)} \int_{S^2} \sqrt{q} a^2 f(x^A) K = \int_{S^2} \sqrt{q} a^2 f(x^A) \left[m - \frac{(k+2)}{2(1+k)} K \right]}\,. \tag{4.2.6}$$

In this way, we obtain

$$\begin{aligned} \{Q_{f_1}, Q_{f_2}\} &= -\delta_{f_2} Q_{f_1} \\ &= \int_{S^2} a^2 \sqrt{g_{S^2}} \frac{1}{(1+k)} \Bigg[\frac{1}{8} f_1 f_2 \partial_u C^{AB} \partial_u C_{AB} - \frac{1}{4} f_2 \partial_u C^{AB} D_A D_B f_1 \\ &\qquad + \frac{f_1 f_2 \triangle G^{(2)}_{uu} - \triangle G^{(1)}_{uA} f_2 D^A f_1}{2} \Bigg]\,. \end{aligned} \tag{4.2.7}$$

The terms in the first line can be absorbed by a modification of the bracket derived in [156] for asymptotically flat spacetimes, as follows:

$$\boxed{\{Q_{f_1}, Q_{f_2}\} = -\delta_{f_2} Q_{f_1} + \int_{S^2} \sqrt{q} \frac{a^2}{8(1+k)} \partial_u C^{BC} f_2 (-\delta_{f_1} C_{BC})}\,. \tag{4.2.8}$$

The remaining terms are fluxes and non-integrable terms which can either be added to the definition of the charge, making it non-integrable, or cured by redefinition of the bracket. In the case in which $\triangle G^{(2)}_{uu} = \triangle G^{(1)}_{uA} = 0$, we have a well-defined charge given by Eq. (4.2.6) and the charge bracket in Eq. (4.2.8). The algebra is abelian and the charges are non-integrable only when $\partial_u C_{AB} \neq 0$.

In order to study the non-conservation of the charges, we use the evolution equation [156]

$$\frac{d}{du} Q_f = \frac{\partial}{\partial u} Q_f + \delta_1 Q_f. \tag{4.2.9}$$

Contrary to the analysis in flat spacetimes, $\partial Q_f / \partial u$ does not vanish because of the presence of the u-dependent scale factor. As a result, for the setting with $\triangle G^{(2)}_{uu} = \triangle G^{(1)}_{uA} = 0$, we obtain:

$$\boxed{\frac{d}{du} Q_f = 2\frac{H}{a} Q_f - \frac{1}{(1+k)} \int_{S^2} \sqrt{q} a^2 \left(\frac{1}{8} f \partial_u C^{AB} \partial_u C_{AB} - \frac{1}{4} \partial_u C^{AB} D_A D_B f \right)}\,, \tag{4.2.10}$$

where $H = \partial_u a$ denotes the Hubble parameter.

The first term is new with respect to asymptotically flat spacetimes and can be interpreted as a Hubble flow of the evolution of the charge. For the concrete case of $f = 1$, the first term is positive and the second is negative. As a consequence, the charge $Q_{f=1}$ is not guaranteed to be monotonically decreasing in time, in contrast to the Bondi mass in asymptotically flat spacetimes.

Let us conclude this section and chapter with some relevant comments:

- Using the charge (4.2.6) and the bracket (4.2.8), we have obtained $\{Q_{f_1}, Q_{f_2}\} = Q_{[f_1,f_2]=0} = 0$ for a subset of metrics compatible with supertranslations, in which $\Phi = \Theta_A = \partial_u K = \triangle G^{(2)}_{uu} = \triangle G^{(1)}_{uA} = 0$.

- It is of utmost importance to emphasize that, contrary to asymptotically flat spacetimes, $\partial_u C_{AB}$ can be expressed in terms of the energy momentum tensor components $T^{(0)}_{AB}$ following Eq. (4.1.23). This means that the notion of Bondi news associated to propagating degrees of freedom is absent. Instead, a matter flux through the boundary takes the place of the Bondi news. When it vanishes, it renders the charges integrable.

- In general, due to the fact that the evolution of all the metric coefficients is determined by the energy momentum tensor components, we point out that the interpretation of these charges might differ substantially from that in asymptotically flat spacetimes. An important point to consider in this regard is that there is no preferred translation subalgebra preserved under supertranslation-like transformations [4], contrary to the asymptotically flat case.

- Although the charges we presented are well motivated, we remark that it should be possible to derive them from first principles, e.g. using the Barnich-Brandt method [191] upon linearizing over the FLRW background. We leave this for future studies.

[4]In fact, there is no finite-dimensional Lie ideal of the asymptotic algebras in FLRW, as firstly appreciated in [170] for $\mathfrak{b}_s \simeq \mathfrak{so}(1,3) \ltimes_s \mathfrak{s}_s$.

Chapter 5

Summary and conclusions of part I

In the first part of this dissertation, we initiated the investigation of the asymptotic symmetry corner of a cosmological infrared triangle, namely the FLRW infrared triangle. To accomplish such endeavour, we began by exploring the geometry of asymptotically decelerating and spatially flat FLRW spacetimes at $\mathcal{I}^+$ in a broad fashion, valid for generic gravity theories. Afterwards, we delved into the specific dynamics of these spacetimes for General Relativity with the analysis of the equations of motion and the procurement of asymptotic charges.

Summary

We started this part I with an overview of asymptotically flat spacetimes in chapter 2, focusing on those aspects which served us as inspiration for the cosmological extension. In particular, we revised the residual diffeomorphisms of asymptotically flat spacetimes after fixing the Bondi gauge, reviewing the diverse possibilities corresponding to a variety of boundary conditions. Depending on the latter, we found supertranslations, three possibilities for superrotations and local Weyl transformations. To each one of them, there are associated asymptotic algebra and charges, which we briefly described. For completion, we introduced the reader to the other corners of the infrared triangle and their interconnections, as well as to the relation between the membrane paradigm and BMS symmetries, which promoted Diff(S^2)-rotations as the "canonical choice" for the superrotation sector.

This paved the way to our original research, describing the geometry and dynamics of asymptotically decelerating and spatially flat FLRW spacetimes in the following two chapters.

- In chapter 3, we built the metrics to be considered as "asymptotically decelerating and spatially flat FLRW at $\mathcal{I}^+$" from reasonable assumptions following an off-shell path, which makes this construction very generic and valid for general gravity theories. Next, we obtained the residual asymptotic diffeomorphisms in Bondi gauge for a variety of boundary conditions which lead (from more restrictive to more relaxed) to the following asymptotic algebras: $\mathfrak{bms}_s \simeq$

$(\mathfrak{witt} \oplus \mathfrak{witt}) \ltimes_s \mathfrak{s}_s$, $\mathfrak{gbms}_s \simeq \mathfrak{vect}(S^2) \ltimes_s \mathfrak{s}_s$ and $\mathfrak{bmsw}_k \simeq (\mathfrak{vect}(S^2) \ltimes \mathfrak{w}) \ltimes_k \mathfrak{s}$, where $\mathfrak{s}$ and $\mathfrak{w}$ denote, respectively, supertranslations and local Weyl diffeomorphisms [1]. The first two algebras correspond to one-parameter deformations of their asymptotically flat counterparts, where s is directly related to the fluid content of the universe. Therefore, we observe a general pattern where this parameter connects the asymptotic algebras at $\mathcal{I}^+$ of asymptotically flat and FLRW spacetimes, unveiling a cosmological holographic flow at the level of boundary algebras. Nevertheless, when we allowed for local Weyl diffeomorphisms, the algebra $\mathfrak{bmsw}_k$ was found to be isomorphic to its flat counterpart $\mathfrak{bmsw}$, pointing towards the fact that the Weyl-BMS algebra is more rigid and universal. Moreover, we computed the action of these asymptotic diffeomorphisms on the asymptotic data, consistently recovering the flat limit when $k \to 0$, and calculated the associated Weyl scalars in appendix B, indicating that our metrics do not generally satisfy the peeling property of asymptotically flat spacetimes. At last, we showed that our ansatz contains solutions like cosmological white holes but has to be extended with logarithmic terms in order to include several cosmological black hole solutions in the literature. We presented such a logarithmically extended ansatz and noticed that it preserves the asymptotic algebra and breaks the peeling property in a more drastic manner than before.

- In chapter 4, we performed an on-shell analysis of asymptotically decelerating spatially flat FLRW spacetimes at future null infinity $\mathcal{I}^+$ by computing and analyzing the asymptotic Einstein equations. The general pattern and constraints on the metric coefficients are clear. Nonetheless, for the sake of technical simplicity, we explicitly solved the equations for a subclass of metrics compatible with the supertranslation-like sector. Strikingly, we observed that the boundary dynamics is completely constrained by the sources, such that not even the tensor degrees of freedom propagate in contrast to asymptotically flat spacetimes. From a cosmological perspective, this result is consistent with the presence of a Hubble scale in the expanding universes beyond which all dynamics is frozen. Making use of the on-shell treatment, we obtained well-defined candidates for supertranslation-like charges in some concrete settings. Interestingly, their evolution equation involves a new Hubble term and they are not guaranteed to be monotonically decreasing, which differs from the Bondi mass aspect in asymptotically flat spacetimes.

Discussion and future directions

After many decades searching for a quantum gravity theory and looking at the ultraviolet regime of physics, an increasing part of the high energy physics and gravity community is turning its interest towards the infrared structure of gravity and gauge

[1] Recall that $\mathfrak{vect}(S^2)$ can be replaced by $\mathfrak{vect}(\mathbb{C}^*)$ depending on the topology under consideration.

theories. This field has experienced major breakthroughs in the last decade with the extension of the original BMS symmetries and the realization of the infrared triangle, linking three a priori independent research areas, namely asymptotic symmetries, soft theorems and memory effects. Nonetheless, the research previous to this dissertation in the asymptotic symmetry corner has been clearly focused on non-cosmological settings.

After all, we do not live in flat spacetimes at large scales and, therefore, studying the asymptotics of those is of little phenomenological relevance. The long-term goal of the part I of this dissertation is to push forward the connection to cosmological settings and initiate investigation of the infrared triangle in those universes. In fact, we have shown that such endeavour is technically subtle but possible by studying the geometry and dynamics of asymptotically FLRW spacetimes. Herein, we investigated the decelerating spatially flat FLRW case due to the presence of a future null infinity boundary, which was crucial at a technical level. Nevertheless, a very intuitive path to follow is extending our machinery to other types of FLRW universes and boundaries, with a special emphasis on accelerating spatially flat ones [2], and comparing to the results obtained in this thesis.

Along this path, we have recursively found a rich mathematical structure where the asymptotic algebras of flat and FLRW spacetimes are connected via one-parameter deformations, being the deformation parameter linked to the matter content of the universe. Furthermore, we have noticed that the extended BMS algebra $\mathfrak{bms}$ and the corresponding deformation $\mathfrak{bms}_s$ are members of the wider family of deformations $W(a,b;\bar{a},\bar{b})$, which connects symmetry algebras coming from seemingly unrelated boundary conditions and spacetime loci (e.g. near event horizon). This observation points to the question of whether the same holds true for $\mathfrak{gbms}_s$ and $\mathfrak{bmsw}_k \simeq \mathfrak{bmsw}$, which are non-central extensions of $\mathfrak{vect}(S^2)$ (or $\mathfrak{vect}(\mathbb{C}^*)$). It would be indeed very appealing to delve into these more mathematical aspects, which serves as essential motivation for the research in the upcoming part II of this thesis.

When we started this project, we expected to benefit from the richer structure of FLRW spacetimes and, therefore, to explore not only tensor modes (as in asymptotically flat spacetimes) but also scalar and vector modes and their corresponding memories. Nevertheless, our investigation of the Einstein equations revealed the opposite conclusion: all the modes at future null infinity $\mathcal{I}^+$ are constrained by the sources. There are, nonetheless, two caveats worth to be explored. Firstly, we have used General Relativity as gravity theory, while alternative gravity theories (see e.g. [124]) might permit richer dynamics for these cosmological spacetimes at $\mathcal{I}^+$. Secondly, we should have allowed for logarithmic terms in r in the metrics (3.2.2) to incorporate solutions like cosmological black holes and, therefore, to explore the dynamics of metrics like (3.4.17). The reason for not including such terms is purely technical, based on the high difficulty of performing their on-shell analysis. How-

[2]Let us point out that accelerating and spatially flat FLRW universes can be expressed as conformally de-Sitter, such that a direct application of the techniques used in this dissertation to the works [109, 130, 192] should pave the way for describing their asymptotia.

ever, it might be that including those terms would lead to less restrictive equations of motion and allow for free radiative data at $\mathcal{I}^+$.

We followed a very intuitive procedure to obtain supertranslation-like charges. Nevertheless, we expect that it should be possible to derive them explicitly from the Barnich-Brandt method [191] by linearizing over a FLRW background. This technical step is worth pursuing in future studies. Besides, it would be desirable to obtain charges for the superrotation-like and local Weyl sectors. It is a challenging task, even for the global Killing vectors in S^2, because it would involve the next order in the $1/r$ expansion of the Einstein equations, which determines the evolution of the angular momentum aspect N_A. We expect that a refinement of the techniques of holographic renormalization developed for asymptotically flat spacetimes [149, 154] will be very useful in such attempt.

Finally, we cannot leave these conclusions without reflecting about the other corners of the infrared triangle. Indeed, we did not comment much about memory effects because all the modes are non-propagating. A naive geodesic deviation equation similar to that in asymptotically flat spacetimes would lead to an expression close to (2.2.3)-(2.2.4) multiplied by the universe scale factor $a(u, r)$ in (3.1.3). Nevertheless, in our case, the tensor components C_{AB} are not propagating and such an equation could not be considered as a memory caused by the passage of radiation but it just becomes a balance equation involving the stress-tensor of the sources. Furthermore, the asymptotic structure of decelerating FLRW depicted in picture 3.1 shows the absence of a past null infinity $\mathcal{I}^-$ which is replaced by a Big Bang singularity. As a consequence, it is not clear to us whether it is meaningful to discuss scattering and soft theorems in a similar fashion that in flat spacetimes where $\mathcal{I}^-$ is present and not pathological. Nonetheless, we would like to draw the attention of the reader to [135, 193–196] for progress in this regard in the accelerating case.

There is still much work left in order to properly understand the infrared structure of cosmological spacetimes. It is certainly a more challenging task than in the anti-de-Sitter and flat cases but it promises to be much more rewarding for actual physics.

Part II

Deformations of gravitational symmetry algebras

Chapter 6

Introduction of part II

In the second part of this dissertation, we aim to endow the previous studies of the infrared structure of gravity with a mathematical perspective. On the one hand, we look closer at the algebra of vector fields on the sphere, $\mathfrak{vect}(S^2)$, due to its ubiquitous appearance and central role in the asymptotic algebras encountered in asymptotically flat and FLRW spacetimes, as well as various seemingly unrelated fields to be discussed below. On the other hand, we investigate boundary Heisenberg-like algebras with the objective of establishing a better understanding of the relationship between symmetry algebras which show up by taking diverse boundary conditions at various spacetime loci.

6.1 Motivation

$\mathfrak{vect}(S^2)$ as a harbinger of four-dimensional physics

The algebra of vector fields on the circle, $\mathfrak{vect}(S^1)$, and its Virasoro central extension have played a major role in quantum gravity and theoretical high energy physics for the last half century. It is the symmetry algebra of two-dimensional conformal field theories (CFT) [157], plays a fundamental role in string theory and appears constantly in a wide range of applications, among which we would like to highlight black hole microstate counting [49,77], fluid-gravity duality [120,197], and asymptotic symmetries in three dimensions [56, 198, 199].

Nonetheless, it is the algebra of vector fields on the sphere, $\mathfrak{vect}(S^2)$, that naturally arises when trying to describe two-dimensional membranes, instead of strings, and to approach black hole microstate counting, fluid-gravity duality and asymptotic symmetries in four dimensions.

In the part I of this doctoral thesis, we encountered non-central extensions of $\mathfrak{vect}(S^2)$ as asymptotic symmetry algebras of asymptotically flat and asymptotically decelerating spatially flat FLRW spacetimes at future null infinity in four spacetime dimensions. Non-central extensions of this algebra have also been discussed in the context of asymptotically (anti) de-Sitter [130] and asymptotic symmetries in null hypersurfaces (including event horizons) [200–203]. Besides, by means of the membrane paradigm [54, 204], a connection between these asymptotic symmetries at null hypersurfaces and fluids on the sphere has been elucidated in [55, 119] and briefly reviewed in chapter 2.3.

Furthermore, it has been known for a long time that the spherical two-dimensional membrane in light-cone gauge is invariant under area preserving diffeomorphisms on the sphere SDiff(S^2), whose algebra of smooth vector fields is denoted by $\mathfrak{svect}(S^2)$, and that their large N $SU(N)$ discretization is presently the most viable path to membrane quantization [174–176]. More precisely, a one-parameter family of algebras, known as $hs[\lambda]$, reduces to $SU(N)$ for integer λ and becomes $\mathfrak{svect}(S^2)$ in the limit $\lambda \to \infty$ [205, 206] [1].

Therefore, the study of the algebra of vector fields on the sphere is required in order to deepen into the aforementioned physical research fields. Rather surprisingly, few studies have been performed trying to investigate the structure and properties of this algebra, as far as we are aware. In [211], it has been shown that $\mathfrak{svect}(S^2)$ does not admit central extensions, while [212] studied generalized Kac-Moody algebras as loop algebras for $\mathfrak{vect}(S^2)$, and [213] investigated harmonic distributions on the sphere and related them to Diff(S^2). More recently, the representation theory of SDiff(S^2) has been explored using the method of coadjoint orbits in [100].

In chapter 7 of this thesis, we further investigate this algebra following two main paths. First, we analyze the structure and deformations of the algebra of globally defined vector fields on the sphere, $\mathfrak{vect}(S^2)$ as well as its "chiral" subalgebras generated by holomorphic and anti-holomorphic vector fields respectively. Next, we embed $\mathfrak{vect}(S^2)$ in the algebra of vector fields on the two-punctured sphere, or punctured complex plane, $\mathfrak{vect}(\mathbb{C}^*)$, in order to investigate some of its physically relevant non-central extensions and devising simple free field realizations for them.

Boundary Heisenberg algebras - deformations as a connector of symmetry algebras

So far, we have mostly focused on the asymptotic structure of the spacetimes at future null infinity. Nevertheless, the analysis of asymptotically flat spacetimes performed by Bondi, van der Burg, Metzner and Sachs (BMS) [33,34,74] has been refined and extended to many other dimensions, spacetimes and boundaries. Notably, the structure of spacetime near generic null surfaces (including event and cosmological horizons), not only but mostly in three spacetime dimensions, has been intensively investigated in the last few years [93, 214–216]. In this context, boundary Heisenberg algebras have played a predominant role all the way through and constitute a fundamental piece behind the different symmetry algebras popping up in a variety of boundary symmetry analysis.

[1] It is worth noting that SDiff(S^2) and its deformation $hs[\lambda]$ also play an important role in the context of higher spin $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence [207, 208] and, particularly, in the structure and properties of W_∞-algebras [209, 210].

Contemporaneously and linked to these developments, a new research area exploring deformations of these infinite-dimensional symmetry algebras has emerged [104, 171, 217]. A major reason behind it is that different symmetry algebras arise from different boundary conditions imposed at the same loci. From a physical viewpoint, it would be desirable to have a better understanding of this. Several attempts have been carried out in this regard. For example, a thermodynamical interpretation for the different boundary conditions has been pursued in [218], the idea of connecting different symmetry algebras through changes of slicing and the existence of a fundamental slicing, where the null boundary symmetry algebra is Heisenberg$\oplus$Diff$(d-2)$, have been proposed and explored in [202, 203], and several claims on the closure and lack of central extensions of these algebras have been made in [155, 219]. However, another possible approach is that boundary algebras associated to different boundary conditions should be connected via deformations, constituting families of deformations which might unveil and help to discern properties coming from possible diverse choices of boundary conditions. Ultimately, it is expected that the deformation analysis of the corresponding algebras can shed light on how to better select and understand choices of boundary conditions. Examples validating such an approach can be found in [171] and new ones will be encountered along this thesis.

In addition, different symmetry algebras come from the analysis at various boundaries. These are also expected to be related in a unique framework. For example, asymptotic and near horizon symmetry algebras have been interpolated in [94]. The deformation analysis has also been fruitful in relating such algebras. In fact, it has been shown that certain near horizon and asymptotic symmetry algebras of three-dimensional and four-dimensional asymptotically flat and FLRW spacetimes form part of the same multi-parametric families of deformation algebras, denoted as W-algebras [2] [104, 165, 171, 221].

Another motivation for studying deformations of infinite dimensional algebras is that they provide us with a path to construct new algebras, which can possibly be realized as symmetry algebras under new boundary conditions and in different geometric settings. In fact, some examples of this procedure will be illustrated along this work.

Bearing this in mind, we explore deformations of boundary Heisenberg algebras in chapter 8 of this dissertation. For technical simplicity, we restrict ourselves to the three-dimensional case, although we expect the main features of this analysis to follow in higher dimensions. In particular, we explore deformations of the infinite dimensional Heisenberg and Heisenberg $\oplus$ $\mathfrak{witt}$ algebras.

6.2 Outline and notation

This second part is formed by three chapters in addition to this introduction. Let us briefly outline their contents:

[2] Not to be confused with $\mathcal{W}$-algebras present in the context of conformal field theories [220].

- In chapter 7, we investigate the algebra of vector fields on the sphere. In section 7.1, we describe $\mathfrak{vect}(S^2)$ and review the appearance of its area-preserving subalgebra as a residual gauge symmetry of the relativistic spherical membrane. We then investigate its deformations in section 7.2 and find that linear deformations of this algebra are obstructed under reasonable conditions. In section 7.3, we study some non-central extensions through the embedding of $\mathfrak{vect}(S^2)$ into $\mathfrak{vect}(\mathbb{C}^*)$. For the latter, we discuss a three-parameter family of non-central extensions which contains the residual diffeomorphism algebras of asymptotically flat and asymptotically FLRW spacetimes at future null infinity under certain boundary conditions, admitting a simple free field realization.

- In chapter 8, we discuss boundary Heisenberg algebras and examine their deformations. In section 8.1, we briefly review the spacetime structure near generic null surfaces and the role that boundary Heisenberg algebras play therein, as well as list the boundary and asymptotic symmetry algebras directly involved in our analysis. Section 8.2 contains a review of deformation theory and its relation to cohomology of Lie algebras where we describe the deformations we investigate and the methodology we follow in this chapter. In sections 8.3 and 8.4, we respectively consider infinitesimal and formal deformations of the infinite dimensional Heisenberg and Heisenberg $\oplus\, \mathfrak{witt}$ algebras, supporting that symmetry algebras associated to diverse boundary conditions and spacetime loci are algebraically interconnected through the deformation procedure. The computational details concerning the analysis of Jacobi identities, necessary to obtain the allowed deformations, and the relevant study of the inverse procedure of deformations, the so-called contractions, are relegated to the appendices C and D.

- In chapter 9, we gather our results and present our conclusions.

Notation. Along this part II and following a practice extended in the literature, we utilize interchangeably the terms Diff(S^2) and $\mathfrak{vect}(S^2)$. The former denotes the (group of) diffeomorphisms on the sphere and the latter its smooth algebra of vector fields. We denote their area-preserving restrictions by, respectively, SDiff(S^2) and $\mathfrak{svect}(S^2)$. The algebra of vector fields on the two-punctured sphere, or punctured complex plane, is denoted by $\mathfrak{vect}(\mathbb{C}^*)$. We use generally "mathfrak" font for the algebras, e.g. $\mathfrak{vir}, \mathfrak{witt}$ and $\mathfrak{H}_3$ for the Virasoro, Witt and infinite dimensional Heisenberg algebras respectively. Their centrally extended versions will be denoted by a hat, e.g. $\mathfrak{vir} = \widehat{\mathfrak{witt}}$. We will use sub-indices for distinguished deformations of main algebras, e.g. $\mathfrak{H}_{3\nu\alpha}$ corresponds to a two-parameter (ν and α) deformation of $\mathfrak{H}_3$. We will be using "$W(a,b)$ family" of algebras to denote a set of algebras for different values of the a, b parameters. Similarly, we make use of the terms $W(a,b;\bar{a},\bar{b})$ and $gW(a,b;\bar{a})$ to denote another relevant sets of multi-parametric algebras.

Chapter 7

Diffeomorphisms on S^2

This chapter is mostly based on our work [221]. Herein, we describe the algebra of vector fields on the sphere and review the striking connection between its area-preserving subalgebra and the relativistic bosonic spherical membrane [174–176]. The latter admits a large N discretization carried by a one-parameter deformation of $\mathfrak{svect}(S^2)$, called $hs[\lambda]$ [174, 205, 206]. Then, we investigate linear deformations of $\mathfrak{vect}(S^2)$ and its chiral subalgebras, showing explicitly that $hs[\lambda]$ does not extend to a deformation of the entire algebra. Finally, we study non-central extensions via the embedding into the algebra of vector fields on the punctured complex plane $\mathfrak{vect}(\mathbb{C}^*)$. For the latter, we encounter a three-parameter family of non-central extensions which contains residual diffeomorphism algebras of asymptotically flat ($\mathfrak{gbms}$) and Friedmann ($\mathfrak{gbms}_s$) spacetimes at future null infinity.

7.1 Vector fields on S^2

We begin by reviewing some basic properties of the algebra of smooth vector fields on the sphere $\mathfrak{vect}(S^2)$. In the following subsection, we then revise the appearance of the area-preserving subalgebra $\mathfrak{svect}(S^2)$ in the context of the spherical membrane.

7.1.1 Description of the algebra

The generators of $\mathfrak{vect}(S^2)$ fall naturally into two classes: [1]

- Area-preserving

$$T^\ell_m \quad = i\sqrt{\tfrac{4\pi}{3}}\epsilon^{ab}(\partial_b Y^\ell_m)\partial_a = \tfrac{i}{\sin\theta}\sqrt{\tfrac{4\pi}{3}}\left((\partial_\varphi Y^\ell_m)\partial_\theta - (\partial_\theta Y^\ell_m)\partial_\varphi\right) \tag{7.1.1}$$

- Non area-preserving

$$S^\ell_m \quad = i\sqrt{\tfrac{4\pi}{3}}g^{ab}(\partial_a Y^\ell_m)\partial_b = i\sqrt{\tfrac{4\pi}{3}}\left((\partial_\theta Y^\ell_m)\partial_\theta + \tfrac{1}{\sin^2\theta}(\partial_\varphi Y^\ell_m)\partial_\varphi\right)\ , \tag{7.1.2}$$

[1] We use the conventions for the spherical harmonics Y^ℓ_m in Mathematica [222] up to a global prefactor in order to have standard normalization for the $so(1,3)$ subalgebra.

where $l > 0$ and $-l \leq m \leq l$ denote the orbital and magnetic quantum numbers respectively, while $\epsilon^{\theta\varphi} = \frac{1}{\sin(\theta)}$ and g^{ab} are the inverse volume form and metric of the round sphere respectively. Let us now summarize some features of this algebra that will be useful in the sequel:

1. The area preserving vector fields T^l_m form a closed subalgebra called $\mathfrak{svect}(S^2)$ and $\{T^l_0\}$ form an abelian closed subalgebra of the latter [2]. On the other hand, the non-area preserving vector fields S^l_m do not close on themselves.

2. The generators with $l = 1$ form a subalgebra. In particular,

$$L_3 := -T^1_0 \ , \quad L_1 := \frac{1}{\sqrt{2}}(T^1_1 - T^1_{-1}) \ , \quad L_2 := \frac{1}{\sqrt{2}i}(T^1_1 + T^1_{-1}) \ , \tag{7.1.3}$$

with

$$[L_i, L_j] = i\epsilon_{ijk} L_k \quad i, j = 1, 2, 3 \ , \tag{7.1.4}$$

generate the $so(3)$ subalgebra of rotations. Together with

$$S_3 := -S^1_0 \ , \quad S_1 := \frac{1}{\sqrt{2}}(S^1_1 - S^1_{-1}) \ , \quad S_2 := \frac{1}{\sqrt{2}i}(S^1_1 + S^1_{-1}) \tag{7.1.5}$$

and the commutation relations

$$\begin{aligned} [L_i, S_j] &= [S_i, L_j] = i\epsilon_{ijk} S_k \\ [S_i, S_j] &= -i\epsilon_{ijk} L_k \end{aligned} \tag{7.1.6}$$

they generate the subalgebra $so(1, 3)$ of conformal diffeomorphisms on the sphere.

3. The Lie algebra isomorphism $so(1, 3) \simeq sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})$ is made manifest by the complex linear combinations, $A^{\pm}_i := \frac{1}{2}(L_i \pm iS_i)$, with

$$\left[A^+_i, A^+_j\right] = i\epsilon_{ijk} A^+_k \ , \qquad \left[A^-_i, A^-_j\right] = i\epsilon_{ijk} A^-_k \ , \qquad \left[A^+_i, A^-_j\right] = 0 \ . \tag{7.1.7}$$

4. The generators with $\ell > 1$ transform as vectors under the $so(3)$ subalgebra of rotations, that is ($B^l_m \in \{T^l_m, S^l_m\}$)

$$\begin{aligned} [T^1_0, B^l_m] &= -m B^l_m \ , \\ [T^1_1, B^l_m] &= \frac{\sqrt{(l+m+1)(l-m)}}{\sqrt{2}} B^l_{m+1} \ , \\ [T^1_{-1}, B^l_m] &= -\frac{\sqrt{(l-m+1)(l+m)}}{\sqrt{2}} B^l_{m-1} \ . \end{aligned} \tag{7.1.8}$$

In addition, they transform in a representation of $so(1, 3)$ but, since the latter is infinite dimensional, this will not be of use here.

[2] There are infinitely many subalgebras of this form corresponding to different choices of z axis.

5. They have a definite transformation under parity, $\mathcal{P}: \quad \theta \to \pi - \theta,\ \phi \to \pi + \phi$, with $\mathcal{P}(T^l_m) = (-1)^{l+1}$, $\mathcal{P}(S^l_m) = (-1)^l$. The commutation relations are compatible with parity.

Chiral subalgebras

The decomposition of the algebra as in (7.1.7) does not generalize to $\ell > 1$. However, there are subalgebras $A^\pm$ for $\ell > 1$. This is more easily seen in stereographic coordinates

$$z = e^{i\varphi}\cot(\theta/2) \ , \quad \bar{z} = z^* \ . \tag{7.1.9}$$

In these coordinates we have

$$T^l_m = \sqrt{\frac{4\pi}{3}}[\frac{(1+z\bar{z})^2}{2}[(\partial_z Y^l_m)\partial_{\bar{z}} - (\partial_{\bar{z}} Y^l_m)\partial_z] \ , \tag{7.1.10}$$

$$S^l_m = i\sqrt{\frac{4\pi}{3}}\frac{(1+z\bar{z})^2}{2}[(\partial_z Y^l_m)\partial_{\bar{z}} + (\partial_{\bar{z}} Y^l_m)\partial_z] \ , \tag{7.1.11}$$

which makes the decomposition of $\mathfrak{vect}(S^2)$ into holomorphic- and anti-holomorphic vector fields manifest, that is

$$(A^l_m)^+ = -\sqrt{\frac{4\pi}{3}}\frac{(1+z\bar{z})^2}{2}(\partial_{\bar{z}} Y^l_m)\partial_z \ , \quad (A^l_m)^- = \sqrt{\frac{4\pi}{3}}\frac{(1+z\bar{z})^2}{2}(\partial_z Y^l_m)\partial_{\bar{z}} \ . \tag{7.1.12}$$

This reveals further subalgebras. Here we list some of their features:

1. The $l = 1$ subalgebra (7.1.7) is recovered with

$$\begin{aligned} (A^1_0)^+ &= -z\partial_z \ , \quad (A^1_1)^+ = -\frac{1}{\sqrt{2}}z^2\partial_z \ , \quad (A^1_{-1})^+ = -\frac{1}{\sqrt{2}}\partial_z \ , \\ (A^1_0)^- &= \bar{z}\partial_{\bar{z}} \ , \quad (A^1_1)^- = -\frac{1}{\sqrt{2}}\partial_{\bar{z}} \ , \quad (A^1_{-1})^- = -\frac{1}{\sqrt{2}}\bar{z}^2\partial_{\bar{z}} \ . \end{aligned} \tag{7.1.13}$$

   Furthermore, $(A^l_m)^\pm$ transform as vectors under the $so(3)$ subalgebra of rotations

$$\begin{aligned} [T^1_0, (A^l_m)^\pm] &= -m(A^l_m)^\pm \\ [T^1_1, (A^l_m)^\pm] &= \frac{\sqrt{(l+m+1)(l-m)}}{\sqrt{2}}(A^l_{m+1})^\pm \\ [T^1_{-1}, (A^l_m)^\pm] &= -\frac{\sqrt{(l-m+1)(l+m)}}{\sqrt{2}}(A^l_{m-1})^\pm \ . \end{aligned} \tag{7.1.14}$$

2. The chiral algebras $A^\pm = \{(A^l_m)^\pm\}$ form subalgebras which are mapped to each other by parity and do not commute $[(A^l_m)^+, (A^{l'}_{m'})^-] \neq 0$ for $l, l' > 1$. They

can be extended further by $A^\pm \oplus (A^1_m)^\mp$ to maximal subalgebras [3]. In addition to $\{(A^1_m)^\mp\}$, both of the latter admit $\{A^l_0\}$ as non-abelian subalgebras. Further subalgebras are generated by $\{(A^\ell_{\pm\ell})^+\}$, $\{(A^\ell_{\pm\ell})^-\}$, as well as $\{(A^l_l)^+\}\cup\{(A^{l'}_{l'})^-\}$, $\{(A^l_{-l})^+\}\cup\{(A^{l'}_{-l'})^-\}$ and $\{(A^l_0)^+\}\cup\{(A^{l'}_0)^-\}$.

The subalgebras generated by $\{(A^\ell_{\pm\ell})^+\}$ (and similarly by $\{(A^\ell_{\pm\ell})^-\}$) are isomorphic to the subalgebra of the Witt algebra, $[L_n, L_m] = (m-n)L_{m+n}$, generated by L_n with $n > 0$, usually called half-Witt algebra. This can be seen by a change of normalization, for instance

$$L_1 = (A^1_1)^+ \ , \ L_2 = \frac{1}{2\sqrt{10}}(A^2_2)^+ \ , \ L_3 = \frac{1}{2\sqrt{210}}(A^3_3)^+ \ , \ L_4 = \frac{1}{8\sqrt{210}}(A^4_4)^+ \ , \ L_5 = \frac{1}{20\sqrt{462}}(A^5_5)^+ \ .$$

These half-Witt subalgebras miss a lowering operator and, contrary to the usual two-dimensional conformal field theory on the sphere, the corresponding vector fields are regular everywhere.[4]

3. More generally, the generators of the chiral subalgebra A^+ (and similarly A^-) can be constructed from a single generator $(A^2_{-2})^+$ with a raising operator, A^1_{-1}, and a horizontal operator, $T^1_{\pm1}$, as described in fig. 7.1.

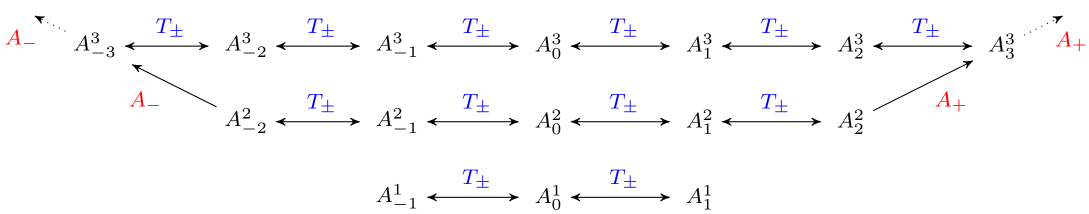

Figure 7.1: Generation of chiral generators starting from $A^2_{\mp2}$ and then first acting repeatedly with $A_\mp = A^1_{\mp1}$ to obtain $A^\ell_{\mp\ell}$ and then acting with $T_\pm = T^1_{\pm1}$ to obtain A^ℓ_m.

Schematically, we can envision $\mathfrak{vect}(S^2)$ as $(so(3) \hookrightarrow hW) \cup (\overline{so(3) \hookrightarrow hW})$, where $so(3) \hookrightarrow hW$ denotes the action of $so(3)$ on half-Witt and the bar on top signals parity conjugation. This structure resembles and contains $so(1,3) \simeq sl(2,\mathbb{R}) \oplus \overline{sl(2,\mathbb{R})}$.

7.1.2 SDiff(S^2), $hs[\lambda]$ and the bosonic membrane

The first direct application of higher-dimensional diffeomorphism algebras in modern theoretical physics traces back to the relativistic bosonic membrane [174, 175, 223]. More concretely, area-preserving diffeomorphisms of its spatial worldvolume topology arise as residual transformations in the light-cone gauge. The most promising route to quantize these objects exploits the discretization of the classical generators. In the

[3] Orthogonality is with respect to the canonical inner product on S^2.
[4] Singular vector fields arise e.g. on the celestial sphere as in e.g. [86, 87].

particular case of S^2, the one-parameter family of algebras $hs[\lambda]$ reduces to $SU(N)$ for integer λ and becomes $\mathfrak{svect}(S^2)$ in the limit $\lambda \to \infty$ [205, 206, 223]. Here, we briefly review these facts closely following [176, 205, 206, 223–225].

The membrane action and area-preserving diffeomorphisms

The classical action for a relativistic bosonic membrane moving in D-dimensional Minkowski spacetime takes the well-known Nambu-Goto form

$$S = -T \int d^3\sigma \sqrt{-\det h_{\alpha\beta}} \;, \quad h_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu \;, \tag{7.1.15}$$

where $\sigma^\alpha = (\tau, \sigma^a)$, with $\alpha \in \{0,1,2\}$ and $a \in \{1,2\}$, are the worldvolume coordinates of the membrane. The set of D functions $X^\mu(\sigma^\alpha)$ describe the motion of the membrane through the spacetime and $T = 1/(2\pi)^2 l_p^3$ is a constant which can be regarded as the membrane tension.

Due to the presence of the square root, it is very inconvenient to analyze the membrane using directly Nambu-Goto's action. Thus, we make use of the analogue of the Polyakov action for the bosonic string. Introducing an auxiliary metric $\gamma_{\alpha\beta}$, it can be shown that the action

$$S = -\frac{T}{2} \int d^3\sigma \sqrt{-\gamma} (\gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu - 1) \tag{7.1.16}$$

is equivalent (at a classical level) to (7.1.15). In order to simplify the analysis, we use the symmetries of the theory to gauge fix $\gamma^{\alpha\beta}$. We have three diffeomorphism symmetries and six independent metric components. We make use of these symmetries to fix the components $\gamma_{0\alpha}$ to be

$$\gamma_{0a} = 0 \;, \gamma_{00} = -\frac{4}{\nu^2}\bar{h} \equiv -\frac{4}{\nu^2}\det h_{ab} \;, \tag{7.1.17}$$

where ν is an arbitrary constant. Let us note that this gauge can only be adopted when the membrane worldvolume is of the form $\Sigma \times \mathbb{R}$, with Σ a Riemann surface of fixed topology. In this gauge, the membrane action becomes

$$S = \frac{T\nu}{4} \int d^3\sigma \left(\dot{X}^\mu \dot{X}_\mu - \frac{4}{\nu^2}\bar{h} \right) \;. \tag{7.1.18}$$

Remarkably, we can rewrite this action in terms of a canonical Poisson bracket on the membrane $\{f, g\} \equiv \epsilon^{ab}\partial_a f \partial_b g$ with $\epsilon^{12} = 1$, such that we obtain

$$S = \frac{T\nu}{4} \int d^3\sigma \left(\dot{X}^\mu \dot{X}_\mu - \frac{2}{\nu^2}\{X^\mu, X^\nu\}\{X_\mu, X_\nu\} \right) \;. \tag{7.1.19}$$

The equations of motion for the fields X^μ are

$$\ddot{X}^\mu = \frac{4}{\nu^2}\{\{X^\mu, X^\nu\}, X_\nu\} \tag{7.1.20}$$

and the constraints on the system are given by

$$\dot{X}^\mu \dot{X}_\mu = -\frac{2}{\nu^2}\{X^\mu, X^\nu\}\{X_\mu, X_\nu\} \quad \text{and} \quad \dot{X}^\mu \partial_a X_\mu = 0 \; . \tag{7.1.21}$$

This theory is still covariant but extremely difficult to quantize because of the non-linearity of the equations of motion and the constraints. Therefore, we now consider the membrane in light-cone coordinates

$$X^\pm = (X^0 \pm X^{D-1})/\sqrt{2} \tag{7.1.22}$$

and explicitly solve the constraints by taking $X^+(\tau, \sigma^a) = \tau$. One can then show that the Hamiltonian of the theory is given by

$$H = \frac{T\nu}{4} \int d^2\sigma \left(\dot{X}^i \dot{X}^i + \frac{2}{\nu^2}\{X^i, X^j\}\{X^i, X^j\} \right) \tag{7.1.23}$$

with the only remaining constraint for the transverse degrees of freedom

$$\{\dot{X}^i, X^i\} = 0 \; . \tag{7.1.24}$$

It is now straightforward to notice that this theory has residual invariance under (time-independent) area-preserving diffeomorphisms SDiff(Σ). These preserve the symplectic form, leaving the Hamiltonian manifestly invariant.

Matrix regularization on S^2 leads to $SU(N)$

Unfortunately, this theory is still difficult to quantize. The most promising path until the date is the so-called matrix regularization procedure, where the functions on the membrane surface are mapped to finite sized matrices and the Poisson brackets are replaced by matrix commutators. This procedure has been originally proposed and applied to the case $\Sigma = S^2$ by Goldstone and Hoppe in 1982 [174].

The advantage of spherical topology is that we have a rotational $SO(3)$ symmetry and any function on it can be expanded as a sum of spherical harmonics. Without getting into too many details, the correspondence between continuum and matrix-regularized quantities is achieved by

$$\xi_A \leftrightarrow \frac{2}{N} J_A \; , \quad \{\cdot,\cdot\} \leftrightarrow \frac{-iN}{2}[\cdot,\cdot] \; , \quad \frac{1}{4\pi}\int d^2\sigma \leftrightarrow \frac{1}{N}\mathrm{Tr} \; , \tag{7.1.25}$$

where ξ_A are the coordinates on S^2, J_A are the generators of the N-dimensional matrix representation of $SU(2)$ with $N = \nu$ and Tr denotes the trace.

We can always express function on the sphere in terms of spherical harmonics $f(\xi_A) = \sum_{l,m} c_{lm} Y_{lm}(\xi_A)$ and these can be written as sums of monomials in the coordinate functions $Y_{lm}(\xi_A) = \sum_k t^{(lm)}_{A_1 \ldots A_l} \xi_{A_1} \ldots \xi_{A_l}$, where the coefficients $t^{(lm)}_{A_1 \ldots A_l}$ have to be symmetric and traceless in order to satisfy $\xi_A^2 = 1$. As a consequence, we can use the correspondence (7.1.25) to obtain matrix approximations for the spherical harmonics $\mathbf{Y}_{lm}$ with $l < N$. It is clear that for a fixed value of N only spherical harmonics with $l < N$ can be constructed because higher order monomials in the generators J_A will not generate linearly independent matrices. As a consequence, the matrix approximation of an arbitrary function on S^2 is given by $f(\xi_A) \to F = \sum_{l<N,m} c_{lm}\mathbf{Y}_{lm}$.

The indispensable consistency requirement for this matrix regularization is that structure constants arising in the commutator algebra of matrix spherical harmonics $[\mathbf{Y}_{lm}, \mathbf{Y}_{l'm'}] = C^{l''m''}_{lm,l'm'}\mathbf{Y}_{l''m''}$ coincide with those of SDiff(S^2) in the "classical" large N limit. The proof of this statement is intricated and not precisely illuminating. As a consequence, we shall omit it here, referring the reader to the original computation [174].

The matrix regularized Hamiltonian for the membrane is then given by

$$H = \frac{1}{2\pi l_p^3}\mathrm{Tr}\left(\frac{1}{2}\dot{\mathbf{X}}^i\dot{\mathbf{X}}^i - \frac{1}{4}[\mathbf{X}^i, \mathbf{X}^j][\mathbf{X}^i, \mathbf{X}^j]\right) \tag{7.1.26}$$

and leads the following matrix equations of motion

$$\ddot{\mathbf{X}}^i + [[\mathbf{X}^i, \mathbf{X}^j], \mathbf{X}^j] = 0 \ , \tag{7.1.27}$$

which must be supplemented with the constraint $[\dot{\mathbf{X}}^i, \mathbf{X}^i] = 0$.

This regularized model has $N \times N$ matrix degrees of freedom and a symmetry group $SU(N)$ with respect to which the matrices $\mathbf{X}^i$ are in the adjoint representation [5]. The discretized equations of motion (7.1.27) are substantially easier to solve than their counterpart in the continuum, routed in the fact that the non-linear products of derivatives coming from $\{\{X^i, X^j\}, X^j\}$ are replaced by an algebraic cubic potential. This cubic potential allows for a quantum mechanical formulation in terms of a Schrödinger-like equation, supplemented by quantum constraints, which scales with the spacetime dimensionality D and with N [225] [6]. Although the quantization of this system is simpler by far, numerous subtleties arise when trying to interpret such quantization and solving the actual theory can be very tricky. For example, simplified analyses using the $SU(N)$ regularized theory lead to the discovery of a discrete

[5] Nevertheless, it is well known that the $N \to \infty$ limit is not unique. In fact, a similar regularization procedure for two-dimensional toroidal topology $\Sigma = \mathbb{T}^2$ leads to $SU(N)$ matrix theory. To be precise, the limit $N \to \infty$ for the torus leads to $SU(\infty) \simeq$ SDiff$(\mathbb{T}^2)$ and for the sphere conducts to $SU(\infty)_+ \simeq$ SDiff(S^2) [223].

[6] We would like to remark that a direct quantization of the classical Hamiltonian (7.1.23) in terms of canonical commutation relationships describing an infinite-dimensional version of the Heisenberg algebra would lead to an a priori "ill-defined" infinite-dimensional Schrödinger-like operator [226]. Such algebra is very close to the D-dimensional version of the infinite-dimensional Heisenberg algebra (8.1.10) which plays a prominent role in near-horizon symmetry analysis and whose deformations we will study in the upcoming chapter 8. This indicates that there is a deep connection worth to be further explored between the Heisenberg boundary algebras, the relativistic membrane and SDiff(S^2).

quantum spectrum for the bosonic membrane, whereas the addition of supersymmetry, necessary for renormalizability of the theory, implies a continuous spectrum, pointing towards the "supermembrane" being a second quantized theory from the beginning [176, 227]. We will not delve into these details here but highly encourage the interested reader to continue with [174, 176, 224, 225].

$Hs[\lambda]$ connects $SU(N)$ and SDiff(S^2)

There exists a one-parameter deformation of SDiff(S^2), the so-called higher spin algebra denoted by $hs[\lambda]$, which contains $SU(N)$ and SDiff(S^2) for concrete values of λ [205, 206]. This continuous deformation, which is linear in the generators, is obtained from the so-called lone-star product of area-preserving generators [205]:

$$\begin{aligned} T^{j_1}_{m_1} \star T^{j_2}_{m_2} = & \sum_{j=|j_1-j_2|}^{j_1+j_2} T^{j}_{m_1+m_2} \frac{1}{4^{j_1+j_2-j}(j_1+j_2-j)!}\, {}_4F_3\begin{pmatrix} \frac{1}{2}+\lambda, \frac{1}{2}-\lambda, \frac{1+j-j_1-j_2}{2}, \frac{j-j_1-j_2}{2} \\ \frac{1}{2}-j_1, \frac{1}{2}-j_2, \frac{3}{2}+j \end{pmatrix} \\ \times & \sum_{m=0}^{j_1+j_2-j} \binom{j_1+j_2-j}{m} [j_1-m_1]_m [j_1+m_1]_{j_1+j_2-j-m} [j_2-m_2]_{j_1+j_2-j-m} [j_2+m_2]_m \ , \end{aligned} \tag{7.1.28}$$

where the generalized hypergeometric function ${}_4F_3$ is evaluated at $z=1$. Using this product, one obtains the deformed commutator as

$$[T^{j_1}_{m_1}, T^{j_2}_{m_2}] = T^{j_1}_{m_1} \star T^{j_2}_{m_2} - T^{j_2}_{m_2} \star T^{j_1}_{m_1} \, . \tag{7.1.29}$$

By means of a Mathematica code similar to the one we attached to [221], it is possible to explicitly check that, if $|\lambda| \in \mathbb{N}_+$, the lone-star product corresponds to associative matrix multiplication compatible with $SU(N)$, being $N = |\lambda|$, and the limit $\lambda \to \infty$ leads to SDiff(S^2).

7.2 Linear deformations

In this section, we investigate the linear deformations of $\mathfrak{vect}(S^2)$ and its chiral subalgebras $A^{\pm}$. By linear we mean that the commutation relations do not generate higher powers of the generators. This problem is hard to tackle analytically because of the complicated form of the structure constants. Here, we reformulate the problem in a way that can be analyzed level by level with the help of the computer software Mathematica [222].

7.2.1 Deformations of $\mathfrak{vect}(S^2)$

To study deformations, we first have to specify the conditions that we impose on any consistent deformation. Concretely we demand:

1. The Jacobi identities have to be satisfied.

2. The generators $(T^j_m,\, S^j_m)$ have to transform as spherical tensors under T^1_m (i.e. the isometry group of S^2 is not deformed).

3. The possible deformations have to include the classical algebra $\mathfrak{vect}(S^2)$ as a limit in the deformation parameter.

4. The generators are required to have a definite transformation under parity.

The general ansatz for the commutators, imposing covariance under the rotation group (condition 2) reads as follows:

$$[T^{j_1}_{m_1}, T^{j_2}_{m_2}] = \sum_{j=|j_1-j_2|}^{j_1+j_2} [A(j_1,j_2,j) T^j_{m_1+m_2} + B(j_1,j_2,j) S^j_{m_1+m_2}] \times C^{j_1 j_2 j}_{m_1 m_2 m} \tag{7.2.1}$$

$$[T^{j_1}_{m_1}, S^{j_2}_{m_2}] = \sum_{j=|j_1-j_2|}^{j_1+j_2} [C(j_1,j_2,j) T^j_{m_1+m_2} + D(j_1,j_2,j) S^j_{m_1+m_2}] \times C^{j_1 j_2 j}_{m_1 m_2 m} \tag{7.2.2}$$

$$[S^{j_1}_{m_1}, S^{j_2}_{m_2}] = \sum_{j=|j_1-j_2|}^{j_1+j_2} [E(j_1,j_2,j) T^j_{m_1+m_2} + F(j_1,j_2,j) S^j_{m_1+m_2}] \times C^{j_1 j_2 j}_{m_1 m_2 m} \tag{7.2.3}$$

with the m-dependence of the commutators completely fixed by the $so(3)$ subalgebra (second condition) together with the Wigner-Eckart theorem such that, using the conventions of [205]:

$$C^{j_1 j_2 j}_{m_1 m_2 m} = \sum_{m=0}^{j_1+j_2-j} \binom{j_1+j_2-j}{m} [j_1-m_1]_m [j_1+m_1]_{j_1+j_2-j-m} [j_2-m_2]_{j_1+j_2-j-m} [j_2+m_2]_m,$$

where the combinatorial factor $[a]_n = a(a-1)...(a-n+1)$ is a Pochhammer symbol.

There are infinitely many free coefficients in the above commutators. These will be reduced by imposing parity invariance and the Jacobi identity while taking care at the same time that the solutions remain in the classical branch. Furthermore, for each j there is a rescaling freedom of the generators $(T^j_m,\, S^j_m)$ which we fix by choosing coefficients that do not vanish in $\mathfrak{vect}(S^2)$ and assign them a value. The practical way to perform this analysis at the computational level is to solve the Jacobi identities order by order in $j_{\max}$ [7], replacing the coefficients of lower j in terms of the ones with higher j and analyze the resultant algebra at every level to observe if there are free parameters left.

In the Mathematica notebook VECTS2DEFORMATIONS.NB, attached to [221], we carry out this analysis up to $j_{\max} = 7$ and we observe that the algebra of commutators with $j \leq 3$ is completely determined. In fact, it is clear from the computations that the number of independent equations grows faster than the number of free coefficients. It is not possible for us to continue this analysis to large $j_{\max}$ due to

[7] For given value of $j_{\max}$, we solve the Jacobi identities $[B^{j_1}_{m_1}, [B^{j_2}_{m_2}, B^{j_3}_{m_3}]]$+cyclic permutations $=0$ for $0 \leq j_1 \leq j_{\max}$, $0 \leq j_2 \leq j_{\max} - j_1$ and $0 \leq j_3 \leq j_{\max} - j_1 - j_2$, where B^j_m represents both T^j_m and S^j_m.

lack of computational power. Nevertheless, from the computational perspective it is rather evident that the algebra will not admit deformations at higher j. On this ground, we arrive to the following claim:

No linear deformations of $\mathfrak{vect}(S^2)$.

The algebra of smooth diffeomorphisms on the sphere, $\mathfrak{vect}(S^2)$, does not admit linear deformations satisfying the Jacobi identities, parity and vector representation of the generators under rotations [8].

Let us contrast this result to the well known one-parameter linear deformation of $\mathfrak{svect}(S^2)$ which is also known as higher spin algebra or $hs[\lambda]$ [205, 206]. The latter is obtained from the deformed commutator (7.1.29) and the lone-star product of area-preserving generators (7.1.28) [205, 206]. Replacing (7.2.1) by (7.1.28) and (7.1.29), combined with (7.2.2) and (7.2.3), we investigate whether $hs[\lambda]$ extends to a linear deformation of the entire $\mathfrak{vect}(S^2)$ algebra. Imposing the same conditions and following an analogous procedure as before, we find that the Jacobi identity $[T^2_{-2}, T^3_0, T^1_{-1}]$ cannot be verified, leading to the conclusion:

Hs$[\lambda]$ is not compatible with $\mathfrak{vect}(S^2)$.

The one-parameter deformation of the algebra of area-preserving diffeomorphisms on the sphere, $hs[\lambda]$, does not extend to the full algebra of smooth diffeomorphisms on the sphere, $\mathfrak{vect}(S^2)$, if the Jacobi identities, parity and vector representation of the generators under rotations are satisfied [8].

This result agrees with and provides further support for the previous proposal on the absence of linear deformations of $\mathfrak{vect}(S^2)$.

7.2.2 Deformations of the chiral subalgebra

While $\mathfrak{vect}(S^2)$ appears to admit no linear deformations, this might still leave room to the possibility that, in addition to $\mathfrak{svect}(S^2)$, other subalgebras admit linear deformations. Bearing this in mind, we now consider possible linear deformations of the chiral subalgebras $A^{\pm}$. In this case, the assumptions we adopt are:

1. The Jacobi identities have to be satisfied.
2. The generators $(A^j_m)^{\pm}$ have to transform as spherical tensors under T^1_m.
3. The possible deformations have to include the non-deformed classical chiral subalgebras.

The second requirement leads to the ansatz

$$[A^{j_1}_{m_1}, A^{j_2}_{m_2}] = \sum_{j=|j_1-j_2|}^{j_1+j_2} G(j_1, j_2, j) A^j_{m_1+m_2} \times C^{j_1 j_2 j}_{m_1 m_2 m} \tag{7.2.4}$$

[8] Note that parity seems to follow naturally from the Jacobi identities, so it might actually not be a necessary requirement.

with

$$C^{j_1 j_2 j}_{m_1 m_2 m} = \sum_{m=0}^{j_1+j_2-j} \binom{j_1+j_2-j}{m} [j_1-m_1]_m [j_1+m_1]_{j_1+j_2-j-m} [j_2-m_2]_{j_1+j_2-j-m} [j_2+m_2]_m .$$

Performing an analysis identical to the previous cases up to $j_{\max} = 7$, we observe that the algebra of commutators with $j \leq 3$ is completely determined. It is again clearly noticeable that the number of independent equations grows faster than the number of free coefficients, and from the computational perspective it seems clear that the algebra will not admit deformations at higher j. Thus, we collect strong evidence in favor of:

No linear deformations of chiral $A^{\pm}$ subalgebras.

The chiral subalgebras of smooth diffeomorphisms on the sphere, $A^{\pm}$, do not admit linear deformations satisfying the Jacobi identities and vector representation of the generators under rotations.

Summary and implications

Let us briefly analyze the main result obtained in this section and its implications. We have gathered strong evidence in favor of a no-go theorem for deformations of $\mathfrak{vect}(S^2)$ under the following assumptions: the deformation is linear, the Jacobi identities have to be satisfied and the generators have to transform as spherical tensors under T^1_m. The last two conditions are necessary if we aim to obtain deformations which are Lie algebras and whose isometry group is still that of the sphere, allowing us to use the Wigner-Eckart theorem in the ansatz (7.2.1)-(7.2.3). We cannot exclude non-linear deformations akin to W-algebras [228] in the case of $\mathfrak{vect}(S^1)$, the Witt algebra. To our knowledge, such non-linear deformations have not been explored even for the area preserving subalgebra $\mathfrak{svect}(S^2)$. Linearity could be relaxed at the cost of the ansatz (7.2.1)-(7.2.3) having to be modified in order to include terms non-linear in the generators in the right hand side. This analysis would be certainly more involved as we would have to figure out how to efficiently use the Wigner-Eckart theorem and the conservation of angular momentum to constrain the allowed non-linearities. Besides, the computational power required to perform such an analysis is beyond our present capacities. Nevertheless, such an exploration is definitely worth to be pursued in future studies.

Most of the implications of this result are surely yet to be unveiled, even though we can already notice some important consequences. Firstly, the rigidity under linear deformations of $\mathfrak{vect}(S^2)$ is in sharp contrast to the well known $hs[\lambda]$ deformation of $\mathfrak{svect}(S^2)$. The latter reduces to $SU(N)$ for integer λ and defines a large N discretization of $\mathfrak{svect}(S^2)$ [205, 206] which has been linked to membrane quantization [174–176]. The possibility of a similar discretization for $\mathfrak{vect}(S^2)$ is ruled out by our analysis, at least at a linear level, which points towards a fundamental difference between algebras of diffeomorphism and their area-preserving subalgebras.

Furthermore, we expect the rigidity of $\mathfrak{vect}(S^2)$ to play an essential role in the understanding of its potential representations and (quantum) deformations. For instance, our result might well pose constraints to generalize the quantum deformations of the $\mathfrak{bms}$ algebra, studied in [105], to $\mathfrak{gbms}$, $\mathfrak{gbms}_s$ and $\mathfrak{bmsw} \simeq \mathfrak{bmsw}_k$, which are non-central extensions of $\mathfrak{vect}(S^2)$ arising in the study of asymptotically flat and FLRW spacetimes and described in detail in the first part of this dissertation.

It would certainly be interesting to explore whether the rigidity of $\mathfrak{vect}(S^2)$ extends to other two-dimensional surfaces like the plane or the torus. As far as we are aware, such analyses have not been performed in the literature so far. Unfortunately, our algorithm does not straightforwardly extend to these spaces. The main obstacle is the lack of spherical symmetry organizing the generators and the subsequent loss of the Wigner-Eckart theorem, which severely constrains the free coefficients to be determined in (7.2.1)-(7.2.3). As a consequence, the number of free coefficients grows substantially making it very challenging to constrain them efficiently.

As a final comment, let us note that, while $\mathfrak{vect}(S^1)$ admits a central extension, the Virasoro algebra, it has been known for a long time [211] that $\mathfrak{vect}(S^2)$ does not admit central extensions [9]. On the other hand, non-central extensions do exist but their description is cumbersome due to the complicated form of the structure constants of $\mathfrak{vect}(S^2)$. In the next section, we will discuss some extensions by embedding $\mathfrak{vect}(S^2)$ in $\mathfrak{vect}(\mathbb{C}^*)$.

7.3 Embedding in $\mathfrak{vect}(\mathbb{C}^*)$

We can embed $\mathfrak{vect}(S^2)$ in $\mathfrak{vect}(\mathbb{C})$ simply by replacing (7.1.12) by arbitrary smooth holomorphic and anti-holomorphic vector fields on $\mathbb{C}$. More generally, if we allow the vector fields to be singular at the origin, we can choose the following basis of $\mathfrak{vect}(\mathbb{C}^*)$

$$\mathcal{L}_{m,n} = -z^{m+1}\bar{z}^n\partial_z \ , \quad \hat{\mathcal{L}}_{m,n} = -z^m\bar{z}^{n+1}\partial_{\bar{z}} \ , \tag{7.3.1}$$

with $m, n \in \mathbb{Z}$ and non-vanishing commutators

$$\begin{aligned} [\mathcal{L}_{m,n}, \mathcal{L}_{r,s}] = (m-r)\mathcal{L}_{m+r,n+s} \ , \quad [\hat{\mathcal{L}}_{m,n}, \hat{\mathcal{L}}_{r,s}] = (n-s)\hat{\mathcal{L}}_{m+r,n+s} \ , \\ [\mathcal{L}_{m,n}, \hat{\mathcal{L}}_{r,s}] = -r\hat{\mathcal{L}}_{m+r,n+s} + n\mathcal{L}_{m+r,n+s} \ . \end{aligned} \tag{7.3.2}$$

In fact, (7.3.2) makes it clear that (7.3.1) is isomorphic to $\mathfrak{vect}(\mathbb{C}^*)$ and to $\mathfrak{vect}(\mathbb{T}^2)$ (see [229–235] for a detailed analysis and some representations). This is not surprising since both can be obtained from the two-punctured sphere with suitable identifications. Hence, we actually have

$$\mathfrak{vect}(S^2) \hookrightarrow \mathfrak{vect}(\mathbb{C}^*) \hookleftarrow \mathfrak{vect}(\mathbb{T}^2) \ , \tag{7.3.3}$$

[9] More precisely, there the absence of central extensions for area-preserving diffeomorphisms was shown. But that is sufficient to imply the result.

which is compatible with the geometric picture of the cylinder being an open subset either to S^2 or $\mathbb{T}^2$.

Unlike $(A^l_m)^\pm$, the basis (7.3.1) does not diagonalize the $so(3)$ Casimir [10] but, instead, simultaneously diagonalizes

$$(A^1_0)^\pm \text{ and } (A^\pm)^2 \ . \tag{7.3.4}$$

As already mentioned, these vector fields are generally singular on S^2, for $z, \bar{z} \to 0$ and $z, \bar{z} \to \infty$. In fact, they form an over-complete basis for the global vector fields in $\mathfrak{vect}(S^2)$. This can be seen by noticing that the global vector fields on S^2 have the form

$$\frac{1}{(1+z\bar{z})^{l-1}} P(z^a \bar{z}^b)\partial_z \quad \text{and} \quad \frac{1}{(1+z\bar{z})^{l-1}} P(z^a \bar{z}^b)\partial_{\bar{z}} \ , \tag{7.3.5}$$

where $P(z^a \bar{z}^b)$ is a polynomial. Thus, expanding around the south pole ($z\bar{z} \to 0$)

$$\frac{1}{1+z\bar{z}} = 1 - z\bar{z} + (z\bar{z})^2 - ... = \sum_{p\geq 0} (-1)^p (z\bar{z})^p$$

or, around the north pole ($z\bar{z} \to \infty$)

$$\frac{1}{1+z\bar{z}} = (z\bar{z})^{-1}(1 - (z\bar{z})^{-1} + (z\bar{z})^{-2}) = \sum_{p\geq 0} (-1)^p (z\bar{z})^{-(p+1)} \ ,$$

they are clearly infinite linear combinations of (7.3.1).

We see that this gives two different ways of representing a smooth vector field on S^2 as an infinite linear combination of the elements (7.3.1). This is analogous to the fact that the Taylor expansion of a rational function of two variables z and w in two different regions ($|z| \gg |w|$ and $|w| \gg |z|$) leads to different formal power series representing the function [236].

7.3.1 Extensions of $\mathfrak{vect}(\mathbb{C}^*)$

As shown in [230, 232], $\mathfrak{vect}(\mathbb{C}^*)$ and $\mathfrak{vect}(\mathbb{T}^2)$ do not admit (non-topological) central extensions either. However, there are non-central extensions which reduce to the Virasoro central extension when viewed as a subalgebra [230, 232]. They can be described as

$$\begin{aligned}
&[\mathcal{L}_{m,n}, \mathcal{L}_{r,s}] = (m-r)\mathcal{L}_{m+r,n+s} - mr(c_1+c_2)(m\mathcal{S}_{m+r,n+s} + n\hat{\mathcal{S}}_{m+r,n+s}) \ , \\
&[\hat{\mathcal{L}}_{m,n}, \hat{\mathcal{L}}_{r,s}] = (n-s)\hat{\mathcal{L}}_{m+r,n+s} - ns(c_1+c_2)(m\mathcal{S}_{m+r,n+s} + n\hat{\mathcal{S}}_{m+r,n+s}) \ , \\
&[\mathcal{L}_{m,n}, \hat{\mathcal{L}}_{r,s}] = -r\hat{\mathcal{L}}_{m+r,n+s} + n\mathcal{L}_{m+r,n+s} - (c_1 nr + c_2 ms)(m\mathcal{S}_{m+r,n+s} + n\hat{\mathcal{S}}_{m+r,n+s}) \ , \\
&[\mathcal{L}_{m,n}, \mathcal{S}_{r,s}] = s\hat{\mathcal{S}}_{m+r,n+s} \ , \quad [\mathcal{L}_{m,n}, \hat{\mathcal{S}}_{r,s}] = -r\hat{\mathcal{S}}_{m+r,n+s} \ , \\
&[\hat{\mathcal{L}}_{m,n}, \mathcal{S}_{r,s}] = -s\mathcal{S}_{m+r,n+s} \ , \quad [\hat{\mathcal{L}}_{m,n}, \hat{\mathcal{S}}_{r,s}] = r\mathcal{S}_{m+r,n+s} \ ,
\end{aligned} \tag{7.3.6}$$

[10] As it is well known from textbook literature on the Runge-Lenz vector.

subject to $m\mathcal{S}_{m,n} + n\hat{\mathcal{S}}_{m,n} = 0$. It is not hard to see that the non-central extensions parametrized by c_1 and c_2 are not compatible with regularity at the origin. Comparing (7.3.5) with (7.3.1), we see that $\mathfrak{vect}(\mathbb{C})$ contains the elements

$$\mathcal{L}_{m\geq -1, n\geq 0} \quad \text{and} \quad \hat{\mathcal{L}}_{m\geq 0, n\geq -1} \ . \tag{7.3.7}$$

Then, for non-vanishing c_1 and c_2, (7.3.6) contains non-central extensions $\mathcal{S}_{r,s}$ and $\hat{\mathcal{S}}_{r,s}$ with arbitrary negative values of r and s.

$gW(a, b, \bar{a})$ - a family of non-central extensions

Another class of non-central extensions is obtained by considering representations of $\mathfrak{vect}(\mathbb{C}^*)$ on tensors. For instance, the extension (3.3.48)

$$\begin{aligned}
[\mathcal{L}_{m,n}, \mathcal{L}_{r,s}] = (m-r)\mathcal{L}_{m+r,n+s} \ , \quad [\hat{\mathcal{L}}_{m,n}, \hat{\mathcal{L}}_{r,s}] &= (n-s)\hat{\mathcal{L}}_{m+r,n+s} \ , \\
[\mathcal{L}_{m,n}, \hat{\mathcal{L}}_{r,s}] &= -r\hat{\mathcal{L}}_{m+r,n+s} + n\mathcal{L}_{m+r,n+s} \ , \\
[\mathcal{L}_{m,n}, T_{pq}] &= \left[\frac{(m+1)}{2}(1+s) - p\right] T_{p+m,q+n} \ , \\
[\hat{\mathcal{L}}_{m,n}, T_{pq}] &= \left[\frac{(n+1)}{2}(1+s) - q\right] T_{p+m,q+n} \ ,
\end{aligned} \tag{7.3.8}$$

known as $\mathfrak{gbms}_s \simeq \mathfrak{vect}(\mathbb{C}^*) \ltimes_s \mathfrak{s}_s$ algebras[11] [165] has recently played a role as an asymptotic symmetry algebra of decelerating asymptotically spatially flat Friedmann spacetimes at $\mathcal{I}^+$, as we discussed in part I of this thesis. It turns out that this algebra forms part of a bigger family of deformations of $\mathfrak{gbms}$, analogously to the $W(a, b; \bar{a}, \bar{b})$ deformations for $\mathfrak{bms}$ (3.3.46) [104, 171] [12]. Let us denote these algebras by generalized $W(a, b; \bar{a}, \bar{b})$, or $gW(a, b; \bar{a}, \bar{b})$, and postulate the commutation relations as

$$[\mathcal{L}_{m,n}, \mathcal{L}_{r,s}] = (m-r)\mathcal{L}_{m+r,n+s} \ , \quad [\hat{\mathcal{L}}_{m,n}, \hat{\mathcal{L}}_{r,s}] = (n-s)\hat{\mathcal{L}}_{m+r,n+s} \ , \tag{7.3.9}$$

$$[\mathcal{L}_{m,n}, \hat{\mathcal{L}}_{r,s}] = -r\hat{\mathcal{L}}_{m+r,n+s} + n\mathcal{L}_{m+r,n+s} \ , \tag{7.3.10}$$

$$[\mathcal{L}_{m,n}, T_{pq}] = -\left[p + bm + a\right] T_{p+m,q+n} \ , \tag{7.3.11}$$

$$[\hat{\mathcal{L}}_{m,n}, T_{pq}] = -\left[q + \bar{b}n + \bar{a}\right] T_{p+m,q+n} \ . \tag{7.3.12}$$

With this general ansatz, these algebras are actually inconsistent due to the Jacobi identity

$$[\mathcal{L}_{m,n}, [\hat{\mathcal{L}}_{r,s}, T_{pq}]] + \text{cyclic permutations} = nr(b - \bar{b}) \overset{!}{=} 0 \ . \tag{7.3.13}$$

Consequently, we find that, unless $n = 0$ and/or $r = 0$, which correspond to the Witt subalgebras, we are forced to set $b = \bar{b}$. Therefore, the family of algebras is

[11] That is generalised $\mathfrak{bms}_s$, where $\mathfrak{s}_s$ stands for conformally weighted supertranslations.

[12] In fact, the $W(a, b; \bar{a}, \bar{b})$ algebras are given by (7.3.9)-(7.3.12) if we restrict to $\mathcal{L}_{m,0}$ and $\hat{\mathcal{L}}_{0,n}$.

actually $gW(a,b;\bar{a})$. Some examples of algebras in this family are given by $\mathfrak{gbms} \simeq gW(-\frac{1}{2},-\frac{1}{2};-\frac{1}{2})$ and $\mathfrak{gbms}_s \simeq gW(-\frac{1+s}{2},-\frac{1+s}{2};-\frac{1+s}{2})$. We note in passing that if the superrotation-like vector fields appearing in the near horizon symmetry algebras described in [98, 172] and [93] are not constrained to satisfy the conformal Killing equation, these algebras are described by $gW(0,0;0)$ and $gW(a,a;a)$, respectively.

Free field realization

It might seem difficult to find a representation of these complicated algebras. Nevertheless, it turns out that there exists a Heisenberg-like construction which provides us with a free field realization for the family $gW(a,b;\bar{a})$ [13]. This is given by

$$\mathcal{L}_{m,n} = \sum_{\alpha,\beta}(\alpha+(b-1)m+a)\bar{a}_{m-\alpha,n-\beta}a_{\alpha,\beta}\ , \tag{7.3.14}$$

$$\hat{\mathcal{L}}_{m,n} = \sum_{\alpha,\beta}(\beta+(\bar{b}-1)n+\bar{a})\bar{a}_{m-\alpha,n-\beta}a_{\alpha,\beta}\ , \tag{7.3.15}$$

$$T_{p,q} = a_{p,q}\ , \tag{7.3.16}$$

with

$$[a_{\alpha,\beta},\bar{a}_{\gamma,\delta}] = \delta_{\alpha+\gamma,0}\delta_{\beta+\delta,0} \tag{7.3.17}$$

and $b=\bar{b}$.

This free field realization helps us to visualize the physical meaning of the uniparametric family of deformations $\mathfrak{gbms}_s \simeq gW(-\frac{1+s}{2},-\frac{1+s}{2};-\frac{1+s}{2})$, being s related to the weight in the lattice. Besides, it is evident that this representation describes also the subfamily $W(a,b;\bar{a},\bar{b})$ with $n=0$ in (7.3.14) and $m=0$ in (7.3.15) and sheds light on the symmetries of the coefficients $a,\bar{a},b,\bar{b}$ described in section 5.3 of [171].

$g\hat{W}(a,b,\bar{a})$ - two compatible non-central extensions

Guided by the fact that $W(a,b;\bar{a},\bar{b})$ admits a central extension, $\hat{W}(a,b;\bar{a},\bar{b})$, obtained by adding central extensions to both Witt subalgebras, we expect to find an equivalent extension for $gW(a,b,\bar{a})$, which we will call $g\hat{W}(a,b,\bar{a})$, as the addition of both non-central extensions of $\mathfrak{vect}(\mathbb{C}^*)$

$$\begin{aligned}
&[\mathcal{L}_{m,n},\mathcal{L}_{r,s}] = (m-r)\mathcal{L}_{m+r,n+s} - mr(c_1+c_2)(m\mathcal{S}_{m+r,n+s}+n\hat{\mathcal{S}}_{m+r,n+s})\ ,\\
&[\hat{\mathcal{L}}_{m,n},\hat{\mathcal{L}}_{r,s}] = (n-s)\hat{\mathcal{L}}_{m+r,n+s} - ns(c_1+c_2)(m\mathcal{S}_{m+r,n+s}+n\hat{\mathcal{S}}_{m+r,n+s})\ ,\\
&[\mathcal{L}_{m,n},\hat{\mathcal{L}}_{r,s}] = -r\hat{\mathcal{L}}_{m+r,n+s} + n\mathcal{L}_{m+r,n+s} - (c_1nr+c_2ms)(m\mathcal{S}_{m+r,n+s}+n\hat{\mathcal{S}}_{m+r,n+s})\ ,\\
&[\mathcal{L}_{m,n},\mathcal{S}_{r,s}] = s\hat{\mathcal{S}}_{m+r,n+s}\ , \quad [\mathcal{L}_{m,n},\hat{\mathcal{S}}_{r,s}] = -r\hat{\mathcal{S}}_{m+r,n+s}\ ,\\
&[\hat{\mathcal{L}}_{m,n},\mathcal{S}_{r,s}] = -s\mathcal{S}_{m+r,n+s}\ , \quad [\hat{\mathcal{L}}_{m,n},\hat{\mathcal{S}}_{r,s}] = r\mathcal{S}_{m+r,n+s}\ ,\\
&[\mathcal{L}_{m,n},T_{pq}] = -\left[p+bm+a\right]T_{p+m,q+n}\ , \quad [\hat{\mathcal{L}}_{m,n},T_{pq}] = -\left[q+bn+\bar{a}\right]T_{p+m,q+n}\ .
\end{aligned} \tag{7.3.18}$$

[13]This technique is in fact very close to the Sugawara construction in section 8.3.4.

It turns out that both non-central extensions are compatible, in terms of Jacobi identities, if the commutators among their generators vanish, $[T_{pq}, \mathcal{S}_{r,s}] = [T_{pq}, \hat{\mathcal{S}}_{r,s}] = 0$. Of course, we cannot interpret this algebra as an extension of $\mathfrak{vect}(S^2)$ since, as mentioned above, $c_1 = c_2 = 0$ for the latter [211]. The same happens to the centrally extended $\hat{W}(a, b; \bar{a}, \bar{b})$ algebras, although one can define them abstractly using the punctured complex plane as suggested by the works of [86,104,171,178,179]. In fact, $\hat{W}(a, b; \bar{a}, \bar{b})$ is a subfamily of (7.3.18) after using the conditions

$$\begin{aligned} m\mathcal{S}_{m,n} + n\hat{\mathcal{S}}_{m,n} = 0 \ , \quad \mathcal{S}_{m,0} = \mathcal{S}_{0,0}\delta_{m0} \ , \quad \hat{\mathcal{S}}_{0,n} = \hat{\mathcal{S}}_{0,0}\delta_{n0} \ , \\ (c_1 + c_2)\mathcal{S}_{0,0} = \frac{c}{12} \ , \quad (c_1 + c_2)\hat{\mathcal{S}}_{0,0} = \frac{\bar{c}}{12} \ , \end{aligned} \tag{7.3.19}$$

which allow to recover Virasoro central extensions in the one-dimensional limit.

As a final comment, let us point out that the supersymmetric version of the family $W(a, b; \bar{a}, \bar{b})$ has been recently introduced in [237] and it would be appealing to investigate such a supersymmetric extension for $gW(a, b, \bar{a})$ and $g\hat{W}(a, b, \bar{a})$, which should contain the "super-$\mathfrak{gbms}_s$" algebra. We expect the latter to naturally arise as the asymptotic algebra of decelerating and spatially flat FLRW spacetimes at $\mathcal{I}^+$ in supergravity [238].

Chapter 8

Boundary Heisenberg algebras

This chapter is based on our work [239]. We begin by revisiting the spacetime structure near generic null surfaces and, in particular, how boundary Heisenberg algebras naturally emerge as gravitational symmetry algebras. Within this chapter, we focus on the three-dimensional case for technical simplicity and briefly describe the main algebras involved in our analysis. Next, we provide the reader with a primer in deformation theory of Lie algebras and define the concrete deformations we investigate.

At this point, we are prepared to perform a thorough analysis of the deformations of the infinite dimensional Heisenberg and Heisenberg $\oplus\,\mathfrak{witt}$ algebras. As a result of the deformation procedure, we find a large class of algebras, some of which are new while others have already been obtained as asymptotic and boundary symmetry algebras. This supports the idea that symmetry algebras associated to diverse boundary conditions and spacetime loci are algebraically interconnected via deformation of algebras. We further explore the deformation and contraction relationships between the novel algebras and explicitly show that the deformation procedure reaches new algebras inaccessible to the celebrated Sugawara construction.

8.1 Symmetry algebras

In this section, we start by revising how boundary Heisenberg algebras arise as symmetry algebras of generic null surfaces for arbitrary spacetime dimensions. We then particularize for the case of three-dimensional manifolds and show how the infinite dimensional Heisenberg and Heisenberg $\oplus\,\mathfrak{witt}$ algebras are obtained. Afterwards, we briefly list relevant three-dimensional symmetry algebras obtained at asymptotic boundaries.

8.1.1 Spacetime structure near generic null surfaces

Heisenberg algebras have a ubiquitous appearance in the boundary/surface charge algebras, especially those computed on a null surface or on the event horizon [93,202, 203,214,216,218]. Let us parametrize the null surface $\mathcal{N}$ by v, x^A, $A = 1, 2, \ldots, D-2$ and assume it has the topology of $\mathbb{R} \ltimes \mathcal{N}_v$ (the topology of $\mathcal{N}_v$ is not fixed) where $\mathcal{N}_v$ is the codimension-two compact spacelike surface and denotes constant v slice on $\mathcal{N}$. It has been argued in [202, 203, 218] that the algebra of surface charges depends on the phase space slicing used. In particular, for two such slicings the maximal boundary algebra for a null surface in D-dimensional pure Einstein gravity takes the following forms:

Null boundary algebra in the "thermodynamic slicing".

$$
\begin{aligned}
\{\mathcal{T}(v,x),\mathcal{T}(v',x')\} &= \left(\mathcal{T}(v,x)\partial_{v'} - \mathcal{T}(v',x')\partial_v\right)\delta(v-v')\delta^{D-2}(x-x'), && \text{(8.1.1a)}\\
\{\mathcal{W}(v,x),\mathcal{W}(v',x')\} &= 0, && \text{(8.1.1b)}\\
\{\mathcal{J}_A(v,x),\mathcal{J}_B(v',x')\} &= \left(\mathcal{J}_A(v,x')\partial_B - \mathcal{J}_B(v,x)\partial'_A\right)\delta^{D-2}(x-x')\delta(v-v'), && \text{(8.1.1c)}\\
\{\mathcal{T}(v,x),\mathcal{W}(v',x')\} &= \mathcal{W}(v,x)\partial_v\delta(v-v')\delta^{D-2}(x-x'), && \text{(8.1.1d)}\\
\{\mathcal{T}(v,x),\mathcal{J}_A(v',x')\} &= \left(-\mathcal{T}(v,x)\partial_A + \mathcal{J}_A(v,x)\partial_v\right)\delta^{D-2}(x-x')\delta(v-v'), && \text{(8.1.1e)}\\
\{\mathcal{W}(v,x),\mathcal{J}_A(v',x')\} &= \mathcal{W}(v,x)\partial_A\delta^{D-2}(x-x')\delta(v-v'). && \text{(8.1.1f)}
\end{aligned}
$$

The above algebra is WDiff($\mathcal{N}$), that is Diff($\mathcal{N}$), generated by $\mathcal{T}(v,x)$ and $\mathcal{J}_A(v,x)$, extended by Weyl scalings $\mathcal{W}(v,x)$. Since topologically $\mathcal{N} \sim \mathbb{R}\ltimes\mathcal{N}_v$, the $\mathcal{T}$ generator forms a Witt algebra at each slice of $\mathcal{N}_v$ (8.1.1a). Moreover, the subalgebra spanned by $\mathcal{T}$ and $\mathcal{W}$ is in fact $U(1)$ Kac-Moody algebra (at each slice of $\mathcal{N}_v$) which can be deformed into the $\mathfrak{bms}_3$ algebra. Therefore, the algebra may also be viewed as $(\mathfrak{bms}_3)_{\mathcal{N}_v} \ltimes$ Diff($\mathcal{N}_v$).

Null boundary algebra in "Heisenberg-Direct sum slicing". The algebra of charges in this slicing takes the form of Heisenberg $\oplus$ Diff($\mathcal{N}_v$), i.e.

$$
\begin{aligned}
&\{\mathcal{Q}(v,x),\mathcal{Q}(v,x')\} = \{\mathcal{P}(v,x),\mathcal{P}(v,x')\} = 0, && \text{(8.1.2a)}\\
&\{\mathcal{Q}(v,x),\mathcal{P}(v,x')\} = \delta^{D-2}\left(x-x'\right), && \text{(8.1.2b)}\\
&\{\mathcal{J}_A(v,x),\mathcal{Q}(v,x')\} = \{\mathcal{J}_A(v,x),\mathcal{P}(v,x')\} = 0, && \text{(8.1.2c)}\\
&\{\mathcal{J}_A(v,x),\mathcal{J}_B(v,x')\} = \left(\mathcal{J}_A(v,x')\partial_B - \mathcal{J}_B(v,x)\partial'_A\right)\delta^{D-2}\left(x-x'\right). && \text{(8.1.2d)}
\end{aligned}
$$

The Diff($\mathcal{N}_v$) part may also admit central extensions. In particular, for the $D = 3$ case in the Topologically Massive Gravity (TMG) theories, it has been shown that such a central extension exists there [240]. Moreover, in special cases where $\mathcal{N}_v$ has toroidal topology, Diff($\mathcal{N}_v$) may be replaced by other Heisenberg algebras.

Remarkably, while the charges have arbitrary v dependence, the right hand side of the commutators, the structure constants of the algebra, are v independent. This means that we have the same algebra at any constant v slice, which is in contrast to the case in (8.1.1).

Non-expanding null surface algebra. The above charges and algebra hold for generic null surfaces. However, in the interesting and important special case of non-expanding surfaces (i.e. when the expansion parameter of the surface vanishes), one loses the $\mathcal{P}$ tower of charges and, importantly, the v dependence of the $\tilde{\mathcal{J}}_A$, while $\mathcal{Q}$ is fixed by gravity equations of motion. In this case, and in an appropriate slicing of the solution phase space, we remain with the algebra

$$\{\mathcal{Q}(x), \mathcal{Q}(x')\} = 0, \tag{8.1.3a}$$
$$\{\tilde{\mathcal{J}}_A(x), \mathcal{Q}(x')\} = -\mathcal{Q}(x)\partial'_A\delta^{D-2}(x-x'), \tag{8.1.3b}$$
$$\{\tilde{\mathcal{J}}_A(x), \tilde{\mathcal{J}}_B(x')\} = \left(\tilde{\mathcal{J}}_A(x')\partial_B - \tilde{\mathcal{J}}_B(x)\partial'_A\right)\delta^{D-2}(x-x'). \tag{8.1.3c}$$

The above algebra is WDiff($\mathcal{N}_v$), where the Diff($\mathcal{N}_v$) part is generated by $\tilde{\mathcal{J}}_A(x)$ and the Weyl scaling by $\mathcal{Q}(x)$.

Non-expanding null surface algebra in Heisenberg slicing. Upon redefinition of the generator $\tilde{\mathcal{J}}_A \to \tilde{\mathcal{J}}_A/\mathcal{Q}$, (8.1.3) takes the form

$$\begin{aligned}
&\{\mathcal{Q}(x), \mathcal{Q}(y)\} = 0,\\
&\{\tilde{\mathcal{J}}_A(x), \mathcal{Q}(y)\} = \frac{\partial}{\partial x^A}\delta(x-y),\\
&\{\tilde{\mathcal{J}}_A(x), \tilde{\mathcal{J}}_B(y)\} = \mathcal{Q}^{-1}(x)\tilde{F}_{BA}(x)\delta(x-y),
\end{aligned} \tag{8.1.4}$$

where $\tilde{F}_{AB} = \partial_A\tilde{\mathcal{J}}_B - \partial_B\tilde{\mathcal{J}}_A$. The $\tilde{\mathcal{J}}_A(x)$ charge is a one-form on $\mathcal{N}_v$ and may be decomposed into exact and coexact parts using Hodge decomposition:

$$\tilde{\mathcal{J}}_A = 8\pi G\partial_A\Pi + \nabla^B\mathcal{J}_{AB}\ , \tag{8.1.5}$$

where $\mathcal{J}_{AB}$ is a two-form and ∇^B is the covariant derivative on $\mathcal{N}_v$. $\tilde{F}_{AB}$ is, therefore, only depending on the coexact part $\mathcal{J}_{AB}$. We then observe that the second commutator becomes

$$\begin{aligned}
&\{\mathcal{Q}(x), \Pi(y)\} = \frac{1}{8\pi G}\delta(x-y)\ ,\\
&\{\mathcal{Q}(x), \mathcal{J}_{AB}\} = 0\ .
\end{aligned} \tag{8.1.6}$$

Notoriously, we have a Heisenberg part in the algebra again.

We stress that the maximal null boundary algebras are not limited to the two cases (8.1.1) and (8.1.2) and there are many more such algebras obtained through change of slicing on the null boundary phase space. Nonetheless, it is believed that all these algebras can be obtained from deformations of these two (for further discussions, we refer the reader to [202, 203, 218]). The deformations obtained throughout this work provide further evidence in this regard.

Three-dimensional case

So far, we discussed four different algebras, (8.1.1), (8.1.2), (8.1.3) and (8.1.4). In the three-dimensional case, they take simpler forms [202,240]. Let us briefly list the boundary symmetry algebras which will play a major role in our analysis [1].

Heisenberg $\oplus$ Diff(S^1). Since the structure constants are independent of v, for the sake of simplicity one may suppress the v-dependence of the charges in (8.1.2). Moreover, in three spacetime dimensions, one may Fourier expand the three towers of charges:

$$\mathcal{J}(x) = \sum_{-\infty}^{+\infty} \mathcal{J}_m e^{imx}, \qquad \mathcal{Q}(y) = \sum_{-\infty}^{+\infty} \mathcal{Q}_n e^{inx}, \qquad \mathcal{P}(x) = \sum_{-\infty}^{+\infty} \mathcal{P}_n e^{inx}. \tag{8.1.7}$$

Then the algebra of charges in the "Heisenberg-Direct sum slicing" (8.1.2), also called "fundamental slicing", takes the form [202]

$$[\mathcal{Q}_m, \mathcal{Q}_n] = [\mathcal{P}_m, \mathcal{P}_n] = 0 \ , \tag{8.1.8a}$$
$$[\mathcal{Q}_m, \mathcal{P}_n] = i\hbar\delta_{m+n,0} \ , \tag{8.1.8b}$$
$$[\mathcal{J}_m, \mathcal{Q}_n] = [\mathcal{J}_m, \mathcal{P}_n] = 0 \ , \tag{8.1.8c}$$
$$[\mathcal{J}_m, \mathcal{J}_n] = (m-n)\mathcal{J}_{m+n} + \frac{c}{12} m^3 \delta_{m+n,0} \ , \tag{8.1.8d}$$

where we have also added a central term c which arises in the TMG case [240].

Virasoro-Kac-Moody algebra. The "non-expanding null surface algebra" (8.1.3), in terms of Fourier modes and after inclusion of the central terms, takes the form

$$[\mathcal{Q}_m, \mathcal{Q}_n] = \tilde{c}\, m\, \delta_{m+n,0} \ , \tag{8.1.9a}$$
$$[\mathcal{J}_m, \mathcal{Q}_n] = -n\mathcal{Q}_{m+n} + \bar{c}\, m^2\, \delta_{m+n,0} \ , \tag{8.1.9b}$$
$$[\mathcal{J}_m, \mathcal{J}_n] = (m-n)\mathcal{J}_{m+n} + \frac{c}{12}\, m^3 \delta_{m+n,0} \ . \tag{8.1.9c}$$

This algebra is the member $\widehat{W}(0,0)$ of a wider family of near-horizon symmetry algebras $\widehat{W}(0,b)$ [93,217] which will be described in the next subsection. Furthermore, this algebra can be also realized as asymptotic symmetry algebra in (A)dS$_3$ by imposing Compère-Song-Strominger (CSS) boundary conditions [241] or as a near-horizon symmetry algebra in [242]. In the latter cases, the $\mathcal{Q}$ generators are interpreted as supertranslations, while in (8.1.9) they correspond to Weyl generators. Besides, it is worth to point out that only c and $\tilde{c}$ show up as a central extension in [241], while the three central terms arise in [242].

[1] In all cases we "quantize" the algebra by replacing Poisson bracket with Dirac bracket as $\{,\} \to i[,]$.

Heisenberg-like algebra. In three dimensions, $\tilde{F}_{AB} = 0$ and the "non-expanding null surface algebra in Heisenberg slicing" (8.1.4) turns out to become

$$\begin{aligned} &[\mathcal{J}_m, \mathcal{J}_n] = 0 \ , \\ &[\mathcal{J}_m, \mathcal{P}_n] = i\hbar\, m\delta_{m+n,0} \ , \\ &[\mathcal{P}_m, \mathcal{P}_n] = 0 \ . \end{aligned} \tag{8.1.10}$$

By means of a redefinition $\mathcal{J}_m \to m\,\mathcal{J}_m$, one gets the infinite dimensional Heisenberg algebra in [214,216], denoted by $\mathfrak{H}_3$ [2]. Remarkably, this algebra can also be obtained in a two-dimensional analysis as discussed in [202].

In this doctoral thesis, we analyze deformations and stability of the infinite dimensional algebras (8.1.10) and (8.1.8) which involve a Heisenberg part [3].

8.1.2 Asymptotic symmetry algebras in three dimensions

In the part I of this thesis, we discussed that the infrared structure of several spacetimes has been revisited and further explored during the last years, inspired by the earlier work of Bondi, van der Burg, Metzner and Sachs (BMS) [33, 34, 74]. As a consequence, many new asymptotic symmetry algebras of diffeomorphisms and charges have arisen in different spacetime dimensions and diverse background spacetimes (prominently including flat [33,34,74,81,82,148,149,156,158], (Anti-) de Sitter [56,109,130,243] and FLRW [164–166,170]). Not aiming here for an exhaustive review, we shall only mention those three-dimensional cases which will be connected to our deformation analysis, besides the well-known $\mathfrak{witt}$ and $\mathfrak{vir}$ algebras coming from standard analysis of asymptotically (A)dS$_3$ spacetimes [4].

$\mathfrak{bms}_3$ and $W(a,b)$. The algebra $\mathfrak{bms}_3$ arises from the study of asymptotically flat spacetimes in three dimensions when the "supertranslation" and "superrotation" sectors are present but local Weyl scalings are not allowed [82]. $\mathfrak{bms}_3$ admits a

[2]For the sake of clarity and computational simplicity, we rescale the generators such that the factor $i\hbar \to 1$, leading to $[\mathcal{J}_m, \mathcal{P}_n] = m\delta_{m+n,0}$.

[3]Let us point out that the rescaling $\mathcal{J}_m \to m\,\mathcal{J}_m$ affects the zero mode of the infinite dimensional Heisenberg algebra. We consider deformations of (8.1.10), which does not include the zero mode, along this work. Nonetheless, we have noticed that there are significant changes if we instead deform $[\mathcal{J}_m, \mathcal{P}_n] = \delta_{m+n,0}$. Indeed, the new algebras reached via deformation include (8.3.8), (8.3.16) for $\beta = 0$, (8.3.17) for $\eta = 0$, (8.3.35) and (8.3.39), all of them without the coefficient m in the second commutator. Nevertheless, the deformation (8.3.26), without the coefficient m in the second commutator, is only allowed for $b = 1$, which is a very important difference. An exhaustive treatment of such deformations is beyond the scope of this thesis, although we highly encourage it.

[4]To be more accurate, by imposing the Brown-Henneaux boundary condition [56] in (A)dS$_3$, one obtains two copies of Virasoro algebras $\mathfrak{vir} \oplus \mathfrak{vir}$.

two-parametric family of deformations [171, 217] denoted as $W(a,b)$ and given by:

$$[\mathcal{J}_m, \mathcal{J}_n] = (m-n)\mathcal{J}_{m+n} \ , \tag{8.1.11a}$$
$$[\mathcal{J}_m, \mathcal{P}_n] = -(n+bm+a)\mathcal{P}_{m+n} \ , \tag{8.1.11b}$$
$$[\mathcal{P}_m, \mathcal{P}_n] = 0 \ , \tag{8.1.11c}$$

where $\mathcal{J}_m$ and $\mathcal{P}_m$ represent, respectively, the superrotation and supertranslation generators and $\mathfrak{bms}_3 \simeq W(0,-1)$. The $W(0,b)$ algebras are also realized as near-horizon symmetry algebras of three-dimensional black holes [93]. Concretely, the Virasoro-Kac-Moody algebra (8.1.9) corresponds to $\widehat{W}(0,0)$ and can also be obtained as asymptotic symmetry algebra in AdS_3 by imposing CSS boundary conditions [241]. Central extensions of both algebras are denoted by $\widehat{\mathfrak{bms}}_3$ and $\widehat{W}(a,b)$.

Weyl-BMS. It was recently pointed out that, besides "supertranslations" and "superrotations", one can admit also local Weyl/scaling transformations [88, 149, 202, 244]. In the three-dimensional case, the algebra of asymptotically flat spacetimes, called 3D Weyl-BMS and denoted as $\mathfrak{bmsw}_3$, is augmented to

$$[\mathcal{J}_m, \mathcal{J}_n] = (m-n)\mathcal{J}_{m+n} \ , \tag{8.1.12a}$$
$$[\mathcal{J}_m, \mathcal{P}_n] = (m-n)\mathcal{P}_{m+n} \ , \tag{8.1.12b}$$
$$[\mathcal{J}_m, \mathcal{D}_n] = -n\mathcal{D}_{m+n} \ , \tag{8.1.12c}$$
$$[\mathcal{P}_m, \mathcal{P}_n] = 0 \ , \tag{8.1.12d}$$
$$[\mathcal{D}_m, \mathcal{P}_n] = \mathcal{P}_{m+n} \ , \tag{8.1.12e}$$
$$[\mathcal{D}_m, \mathcal{D}_n] = 0 \ , \tag{8.1.12f}$$

where $\mathcal{D}_m$ are the Weyl generators. This algebra admits non-trivial central as well as non-central extensions which can be found in [245]. It is worth highlighting that, although at first sight the Weyl symmetry leads to get $\mathfrak{bmsw}_3$, this is not the only way to realize such symmetry. In fact, it has been shown that the Weyl symmetry leads to a larger symmetry algebra $\mathfrak{vir} \oplus \mathfrak{vir} \oplus \mathfrak{u}(1)$ [246, 247].

At last, we would like to point out that $\widehat{\mathfrak{bms}}_3 \oplus \mathfrak{vir}$ and $\mathfrak{vir} \oplus \mathfrak{vir} \oplus \mathfrak{vir}$ have been realized as the asymptotic symmetry algebras of three-dimensional Maxwell Chern-Simons gravity theories invariant under $\mathfrak{iso}(2) \oplus \mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{so}(2,2) \oplus \mathfrak{sl}(2,\mathbb{R})$, respectively, by considering certain boundary conditions [248, 249].

8.2 Deformation theory of Lie algebras

In this section, we briefly review the concept of deformation of Lie algebras, their relation to the cohomology of Lie algebras and define the terminology used in the following sections, indicating the precise deformations we analyze in this chapter. We refer the reader to [171, 250] for more details.

A *deformation* of a certain Lie algebra $\mathfrak{g}$ is a modification of its structure constants. Some of such deformations could just be a change of basis and are called trivial deformations. Non-trivial deformations modify/deform a Lie algebra $\mathfrak{g}$ to another Lie algebra with the same vector space structure [5]. The concept of deformation of rings and algebras was firstly introduced in a series of papers by Gerstenhaber [251–254] and by Nijenhuis and Richardson for Lie algebras in [255]. A Lie algebra $\mathfrak{g}$ is called *rigid* or stable if it does not admit any deformation or, equivalently, if the algebras obtained from it via deformation $\mathfrak{g}_\varepsilon$, ε being the possible deformation parameters, are isomorphic to the initial algebra $\mathfrak{g}$. Conversely, given an algebra $\mathfrak{g}$, one can take the limit $\varepsilon \to 0$ and obtain $\mathfrak{g}_0$. This procedure is known as *contraction* of Lie algebras [6]. The contraction and deformation procedures are hence inverse of each other.

8.2.1 Formal deformation of Lie algebras

We denote by $(\mathfrak{g}, [,])$ a Lie algebra in which $\mathfrak{g}$ is a vector space over a field $\mathbb{F}$ with characteristic zero (e.g. $\mathbb{R}$) equipped with a Lie bracket, that is a bilinear and antisymmetric product function

$$[,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g} \ , \tag{8.2.1}$$

which must also satisfy the Jacobi identities

$$[g_i, [g_j, g_k]] + [g_j, [g_k, g_i]] + [g_k, [g_i, g_j]] = 0, \qquad \forall g_i \in \mathfrak{g} \ . \tag{8.2.2}$$

A *formal one-parameter deformation* of a Lie algebra $(\mathfrak{g}, [,]_0)$, abbreviated as $\mathfrak{g}$, is defined as a skew symmetric bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}[[\varepsilon]]$ which satisfies the Jacobi identities to all orders of ε, where $\mathfrak{g}[[\varepsilon]]$ is the space of formal power series in ε with coefficients in $\mathfrak{g}$ [258]. This means that the commutation relations of $\mathfrak{g}$ are modified as follows:

$$[g_i, g_j]_\varepsilon := \Psi(g_i, g_j; \varepsilon) = \Psi(g_i, g_j; \varepsilon = 0) + \psi_1(g_i, g_j)\varepsilon + \psi_2(g_i, g_j)\varepsilon^2 + \dots \ , \tag{8.2.3}$$

where $\Psi(g_i, g_j; \varepsilon = 0) = [g_i, g_j]_0$, g_i and g_j are basis elements of $\mathfrak{g}$, $\varepsilon \in \mathbb{F}$ is the *deformation parameter* and $\psi_i : \ \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ are bilinear antisymmetric functions, the so-called 2-*cochains*. For every ε, the new Lie algebra $(\mathfrak{g}, [,]_\varepsilon)$ should satisfy the Jacobi identity

$$[g_i, [g_j, g_k]_\varepsilon]_\varepsilon + \text{cyclic permutations of } (g_i, g_j, g_k) = 0 \tag{8.2.4}$$

[5] In the case of finite dimensional Lie algebras, the latter implies that the deformation does not change the dimension of the algebra.

[6] Let us note that the deformation/contraction procedure differs from the semigroup expansion method in that one can preserve the contraction parameter and use it as an expansion parameter rather than just approaching zero [256,257]. This procedure leads to an extension of the contracted algebra. For instance, the semigroup expansion of a finite algebra at each level in the expansion parameter leads to a larger algebra with more generators.

order by order in ε, which leads to an infinite sequence of equations among ψ_i.

For small ε, the leading deformation is given by the $\psi_1(g_i, g_j)$-term and the associated Jacobi identities lead to

$$[g_i, \psi_1(g_j, g_k)]_0 + \psi_1(g_i, [g_j, g_k]_0) + \text{cyclic permutations of } (g_i, g_j, g_k) = 0. \qquad (8.2.5)$$

This relation is known as the 2-cocycle condition. Its solution, the 2-cocycle ψ_1, specifies an *infinitesimal deformation* of Lie algebra $\mathfrak{g}$. The Jacobi identity for higher orders of ε should also be checked as integrability conditions of ψ_1 and may lead to obstructions, which will be discussed later in this section. From now on, we denote the deformed algebra $(\mathfrak{g}, [,]_\varepsilon)$ simply by $\mathfrak{g}_\varepsilon$.

One can readily check that the relation

$$\psi_1(g_i, g_j) = \varphi_1([g_i, g_j])_0 - [\varphi_1(g_i), g_j]_0 - [g_i, \varphi_1(g_j)]_0 \qquad (8.2.6)$$

satisfies the 2-cocycle condition (8.2.5). In fact, (8.2.6) shows that ψ_1 is a 2-coboundary if φ_1 is a 1-cochain. When ψ_1 is a 2-coboundary, the deformation (8.2.3) is called *trivial*, meaning that the deformation is just a redefinition of the basis elements.

8.2.2 Deformation theory and cohomology of Lie algebras

We start with the definition of the Chevalley-Eilenberg complex and differential. A vector space $\mathbb{V}$ is called a $\mathfrak{g}$-module if there exists a bilinear map $\omega: \mathfrak{g} \times \mathbb{V} \longrightarrow \mathbb{V}$ for all $x \in \mathbb{V}$ and $g_1, g_2 \in \mathfrak{g}$ with the property $\omega([g_1, g_2], x) = \omega(g_1, \omega(g_2, x)) - \omega(g_2, \omega(g_1, x))$ [250]. In this setting, the Jacobi identities of the Lie bracket imply that a Lie algebra $\mathfrak{g}$ with the adjoint action is a $\mathfrak{g}$-module. A p-cochain ψ is a $\mathbb{V}$-valued (as $\mathfrak{g}$-module), bilinear and completely antisymmetric function which is defined as:

$$\begin{aligned} \psi : \underbrace{\mathfrak{g} \times \cdots \times \mathfrak{g}}_{p\ \ times} &\longrightarrow \mathbb{V} \\ (g_1, \cdots, g_p) &\longmapsto \psi(g_1, \cdots, g_p); \qquad 0 \leq p \leq dim(\mathfrak{g}). \end{aligned}$$

Suppose $\mathcal{C}^p(\mathfrak{g}; \mathbb{V})$ is the space of $\mathbb{V}$-valued p-cochains on $\mathfrak{g}$. One can then define the cochain complex $\mathcal{C}^*(\mathfrak{g}; \mathbb{V}) = \oplus_{p=0}^{dim(\mathfrak{g})} \mathcal{C}^p(\mathfrak{g}; \mathbb{V})$, which is known as the *Chevalley-Eilenberg complex.*

The *Chevalley-Eilenberg differential* or, equivalently, *coboundary operator* "d" is a linear map defined as [259, 260]

$$\begin{aligned} d_p : \ \mathcal{C}^p(\mathfrak{g}; \mathfrak{g}) &\longrightarrow \mathcal{C}^{p+1}(\mathfrak{g}; \mathfrak{g}), \\ \psi &\longmapsto d_{\mathrm{p}}\psi, \end{aligned}$$

and the $p+1$-cochain $d_p\psi$ is given by

$$\begin{aligned}(d_p\psi)(g_0,\ldots,g_p) \equiv & \sum_{0\leq i<j\leq p} (1)^{i+j-1}\psi\left([g_i,g_j],g_0,\ldots,\widehat{g_i},\ldots,\widehat{g_j},\ldots,g_{p+1}\right)\\ &+\sum_{1\leq i\leq p+1}(-1)^i\left[g_i,\psi\left(g_0,\ldots,\widehat{g_i},\ldots,g_p\right)\right],\end{aligned} \tag{8.2.7}$$

where the hat denotes omission. One can check that $d_p\circ d_{p-1}=0$. A p-cochain ψ is called *p-cocycle* if $d_p\psi=0$, and *p-coboundary* if $\psi=d_{p-1}\varphi$.

By means of the property $d_p\circ d_{p-1}=0$, one concludes that every p-coboundary is also a p-cocycle. With this definition, one can check that 2-cocycle condition (8.2.5) is just $d_2\psi_1=0$, where ψ is a $\mathfrak{g}$-valued 2-cochain and d_2 given in (8.2.7), and the relation (8.2.6) is a 2-coboundary condition $\psi_1=d_1\varphi_1$, where φ_1 is a $\mathfrak{g}$-valued 1-cochain.

We define $Z^p(\mathfrak{g};\mathbb{V})$ as a space of p-cocycles which is the kernel of the differential d as

$$Z^p(\mathfrak{g};\mathbb{V})=\{\psi\in\mathcal{C}^p(\mathfrak{g};\mathbb{V})|d_p\psi=0\}. \tag{8.2.8}$$

$Z^2(\mathfrak{g};\mathfrak{g})$ is hence the space of all $\mathfrak{g}$-valued 2-cocycles which satisfy the relation (8.2.5). One also define $B^p(\mathfrak{g};\mathbb{V})$ as the space of p-coboundaries in the following way

$$B^p(\mathfrak{g};\mathbb{V})=\{\psi\in\mathcal{C}^p(\mathfrak{g};\mathbb{V})|\psi=d_{p-1}\varphi\ \text{ for some }\ \varphi\ \text{in}\ \ \mathcal{C}^{p-1}(\mathfrak{g};\mathbb{V})\}. \tag{8.2.9}$$

$B^2(\mathfrak{g};\mathfrak{g})$ is, therefore, the space of all $\mathfrak{g}$-valued 2-cocycles which are also 2-coboundaries, meaning that its elements satisfy both relations (8.2.5) and (8.2.6). A p^{th} *cohomology* space of $\mathfrak{g}$ with coefficients in $\mathbb{V}$ is then defined as the quotient of the space of p-cocycles $Z^p(\mathfrak{g};\mathbb{V})$ to the space of p-coboundaries $B^p(\mathfrak{g};\mathbb{V})$ as

$$\mathcal{H}^p(\mathfrak{g};\mathbb{V}):=Z^p(\mathfrak{g};\mathbb{V})/B^p(\mathfrak{g};\mathbb{V})=\mathrm{Ker}(\ d_p)/\mathrm{Im}(\ d_{p-1}). \tag{8.2.10}$$

It is worth to highlight that isomorphic Lie algebras have the same cohomology spaces and that $\mathcal{H}^2(\mathfrak{g};\mathfrak{g})$, the second adjoint cohomology, classifies all infinitesimal deformations of the algebra $\mathfrak{g}$. Let us emphasize that not all infinitesimal deformations integrate to a formal (finite) deformation, there could be obstructions.

Integrability conditions and obstructions. As pointed out earlier, in order to have a formal deformation (8.2.3), we need the corresponding non-trivial infinitesimal deformation to be integrable, i.e. to be valid to all orders in the deformation parameter. To the first few orders in ε, (8.2.4) leads to

$$[g_i,[g_j,g_k]_0]_0+\text{cyclic permutation of }(g_i,g_j,g_k)=0\ , \tag{8.2.11a}$$
$$d_2\psi_1=0\ , \tag{8.2.11b}$$
$$d_2\psi_2=-\frac{1}{2}[\![\psi_1,\psi_1]\!]\ , \tag{8.2.11c}$$
$$d_2\psi_3=-[\![\psi_1,\psi_2]\!]\ , \tag{8.2.11d}$$
$$\ldots$$

where we used the definition of the Chevalley-Eilenberg differential d_2 in (8.2.7) and the double-bracket is the Nijenhuis and Richardson bracket [255] defined as

$$\frac{1}{2}[\![\psi_r, \psi_s]\!](g_i, g_j, g_k) := \psi_r(g_i, \psi_s(g_j, g_k)) + \text{cyclic permutation of } (g_i, g_j, g_k).$$

The zeroth order in ε, (8.2.11a), is nothing but the Jacobi relation for the undeformed algebra and is, therefore, satisfied by definition. The second equation (8.2.11b) is the 2-cocycle condition (8.2.5) for ψ_1 and its solutions provide non-trivial *infinitesimal deformations*. Equation (8.2.11c) would then guarantee that there are no obstructions in viewing $\psi_1(g_i, g_j)$ as the first order term of a formal deformation $\Psi(g_i, g_j; \varepsilon)$ which admits a power series expansion in ε. Naturally, one should follow the same reasoning in higher orders of ε. For example, it is readily observed that for order ε^3 one should satisfy (8.2.11d). The sequence of relations will stop if there is an obstruction. From a cohomological point of view, one can verify that all obstructions are in the space $\mathcal{H}^3(\mathfrak{g}; \mathfrak{g})$ such that if $\mathcal{H}^3(\mathfrak{g}; \mathfrak{g}) = 0$, then there are no obstructions [255].

There are three different approaches to check the integrability conditions [7]:

1. Sequential method: One can consider the entire infinite sequence of relations (8.2.11) and directly verify their solutions or probable obstructions.

2. $\mathcal{H}^3$ method: We mentioned that all obstructions are located in $\mathcal{H}^3(\mathfrak{g}; \mathfrak{g})$. If $\mathcal{H}^3(\mathfrak{g}; \mathfrak{g})$ vanishes, there is no obstruction. For further discussions, we refer the reader to [171].

3. Direct method: One may examine if an infinitesimal deformation is indeed formal by promoting the linear (infinitesimal) deformation $\psi_1(g_i, g_j)$ to $\Psi(g_i, g_j; \varepsilon)$ and check whether they satisfy the Jacobi identities or not. If one finds that the linear term in the Taylor expansion of $\Psi(g_i, g_j; \varepsilon)$ satisfies the Jacobi identities (8.2.2), one concludes that it is also a formal deformation of algebra.

Deformations investigated in this work

Throughout this chapter, we tackle with infinitesimal and formal deformations of Heisenberg-like algebras. First, we mention that the deformation parameters are considered as a part of the 2-cocycle functions definition, which are introduced as deformation terms. Looking for infinitesimal deformation is equivalent to keeping only the linear term of the functions in the Jacobi identities. Once we classify all infinitesimal deformations, which is equivalent to obtain the second adjoint cohomology $\mathcal{H}^2(\mathfrak{g}; \mathfrak{g})$, we explore which of these infinitesimal deformations can be enhanced to formal deformations using the direct method to evaluate their integrability conditions. Finally, we consider which of these deformations are non-trivial. In this way,

[7] It should be emphasized that other methods than direct calculations are very complex and time consuming, especially in the case of infinite dimensional algebras. For this reason, we restrict ourselves to the third method within this chapter.

we determine the formal deformations of Heisenberg-like algebras in sections 8.3 and 8.4.

8.3 Deformations of the Heisenberg algebra

In this section, we consider infinitesimal and formal deformations of the infinite dimensional Heisenberg algebra $\mathfrak{H}_3$ (8.1.10) as described in section 8.2. First, we investigate deformations of each commutator separately and then we study the deformations of $\mathfrak{H}_3$ in all commutators at once. In this analysis, we rescale the generators to set $i\hbar \to 1$ for convenience.

8.3.1 Deformation of separate commutators

In the following, we deform individually each of the three commutators present in (8.1.10) and find several new algebras as a byproduct.

Deformations of $[\mathcal{J}, \mathcal{J}]$

We proceed to analyze the possible deformations of two $\mathcal{J}$

$$\begin{aligned} &[\mathcal{J}_m, \mathcal{J}_n] = (m-n)F(m,n)\mathcal{J}_{m+n} + (m-n)G(m,n)\mathcal{P}_{m+n} + (m-n)A(m,n)\ , \\ &[\mathcal{J}_m, \mathcal{P}_n] = m\delta_{m+n,0}\ , \\ &[\mathcal{P}_m, \mathcal{P}_n] = 0\ , \end{aligned} \tag{8.3.1}$$

where $F(m,n)$, $G(m,n)$ and $A(m,n)$ are symmetric functions. The Jacobi identities $[\mathcal{J}_m, [\mathcal{J}_n, \mathcal{J}_l]] + \text{cyclic permutations} = 0$ lead to independent relations

$$\begin{aligned} (n-l)(m-n-l)F(n,l)F(m,n+l) + (l-m)(n-l-m)F(l,m)F(n,l+m)+ \\ (m-n)(l-m-n)F(m,n)F(l,m+n) = 0\ , \end{aligned} \tag{8.3.2}$$

$$\begin{aligned} (n-l)(m-n-l)F(n,l)G(m,n+l) + (l-m)(n-l-m)F(l,m)G(n,l+m)+ \\ (m-n)(l-m-n)F(m,n)G(l,m+n) = 0\ , \end{aligned} \tag{8.3.3}$$

and

$$\begin{aligned} &\{(n-l)(m-n-l)F(n,l)A(m,n+l) + (l-m)(n-l-m)F(l,m)A(n,l+m) \\ &\qquad + (m-n)(l-m-n)F(m,n)A(l,m+n)\} \\ &+ \{m(n-l)G(n,l) + n(l-m)G(l,m) + l(m-n)G(m,n)\}\delta_{m+n+l,0} = 0\ . \end{aligned} \tag{8.3.4}$$

We now consider the infinitesimal deformations corresponding to relations including only the first order in the functions

$$(m(n-l)G(n,l) + n(l-m)G(l,m) + l(m-n)G(m,n))\delta_{m+n+l,0} = 0\ , \tag{8.3.5}$$

which is solved by $G(m,n) = constant$. [8]

The Jacobi identities $[\mathcal{J}_m, [\mathcal{J}_n, \mathcal{P}_l]] +$ cyclic permutations $= 0$ lead to the relation

$$(m+n)(m-n)F(m,n)\delta_{m+n+l,0} = 0 \ , \tag{8.3.6}$$

which is solved by $F(m,n) = 0$. Besides, it is easy to check that there is no constraint on $A(m,n)$ and that the deformation induced by $G(m,n) = constant$ is a formal deformation. Therefore, we have the new deformed algebra

$$\begin{aligned}
&[\mathcal{J}_m, \mathcal{J}_n] = \nu(m-n)\mathcal{P}_{m+n} + (m-n)A(m,n) \ , \\
&[\mathcal{J}_m, \mathcal{P}_n] = m\delta_{m+n,0} \ , \\
&[\mathcal{P}_m, \mathcal{P}_n] = 0 \ ,
\end{aligned} \tag{8.3.7}$$

where ν is an arbitrary constant. This is the formal deformation of $\mathfrak{H}_3$ in its commutator $[\mathcal{J}, \mathcal{J}]$. For $\nu \neq 0$, (8.3.7) can be simplified by the redefinition $\mathcal{P}_m \to \mathcal{P}_m - \frac{A(m,n)}{\nu}$, leading to

$$\boxed{\begin{aligned}
&[\mathcal{J}_m, \mathcal{J}_n] = \nu(m-n)\mathcal{P}_{m+n} \ , \\
&[\mathcal{J}_m, \mathcal{P}_n] = m\delta_{m+n,0} \ , \\
&[\mathcal{P}_m, \mathcal{P}_n] = 0 \ ,
\end{aligned}} \tag{8.3.8}$$

which we called $\mathfrak{H}_{3\nu}$.

Let us note that the algebra (8.3.8) can be obtained as contraction of two Virasoro algebras, as we discuss in detail in Appendix D. Besides, (8.3.8) can be obtained from contraction of the $W(0,b)$ algebra (8.3.31) when the linear central term in the commutator $[\mathcal{J}, \mathcal{P}]$ is considered.

Deformations of $[\mathcal{J}, \mathcal{P}]$

Next, we consider deformations of the commutator $[\mathcal{J}, \mathcal{P}]$

$$\begin{aligned}
&[\mathcal{J}_m, \mathcal{J}_n] = 0 \ , \\
&[\mathcal{J}_m, \mathcal{P}_n] = m\delta_{m+n,0} + \bar{F}(m,n)\mathcal{J}_{m+n} + \bar{G}(m,n)\mathcal{P}_{m+n} + \bar{A}(m,n) \ , \\
&[\mathcal{P}_m, \mathcal{P}_n] = 0 \ .
\end{aligned} \tag{8.3.9}$$

The Jacobi identities $[\mathcal{J}_m, [\mathcal{J}_n, \mathcal{P}_l]] +$ cyclic permutations $= 0$ lead to the relations

$$\bar{G}(n,l)\bar{F}(m,n+l) - \bar{G}(m,l)\bar{F}(n,l+m) = 0 \ , \tag{8.3.10}$$

$$\bar{G}(n,l)\bar{G}(m,n+l) - \bar{G}(m,l)\bar{G}(n,l+m) = 0 \ , \tag{8.3.11}$$

[8] We would like to note that $G(m,n) = G(n,m)$, what makes $(m(n-l)G(n,l) + n(l-m)G(l,m) + l(m-n)G(m,n))$ antisymmetric under the exchange of either of m, n and l, as well as symmetric under the simultaneous exchange $m \to n$, $n \to l$ and $l \to m$. Using this information, we could not find a general argument against the presence of other polynomial solutions. Nevertheless, we discarded linear and quadratic ansatz for $G(m,n)$ by explicit computation and are firmly convinced that this equation has no other solution than $G(m,n) = constant$.

$$(m\bar{G}(n,l)-n\bar{G}(m,l))\delta_{m+n+l,0}+\bar{G}(n,l)\bar{A}(m,n+l)-\bar{G}(m,l)\bar{A}(n,l+m)=0\ . \quad (8.3.12)$$

On the other hand, the Jacobi identities $[\mathcal{P}_m,[\mathcal{P}_n,\mathcal{J}_l]]+$cyclic permutations $=0$ lead to

$$\bar{F}(l,n)\bar{F}(n+l,m)-\bar{F}(l,m)\bar{F}(l+m,n)=0\ , \quad (8.3.13)$$

$$\bar{F}(l,n)\bar{G}(n+l,m)-\bar{F}(l,m)\bar{G}(l+m,n)=0\ , \quad (8.3.14)$$

$$\begin{aligned}&(\bar{F}(l,n)(n+l)-\bar{F}(l,m)(l+m))\delta_{m+n+l,0}\\&+\bar{F}(l,n)\bar{A}(n+l,m)-\bar{F}(l,m)\bar{A}(l+m,n)=0\ .\end{aligned} \quad (8.3.15)$$

Focusing on infinitesimal deformation means that we should consider separately relations with first order in functions, which are $(m\bar{G}(n,l)-n\bar{G}(m,l))\delta_{m+n+l,0}=0$ and $(\bar{F}(l,n)(n+l)-\bar{F}(l,m)(l+m))\delta_{m+n+l,0}=0$. The former relation is solved by $\bar{G}(m,n)=\alpha m+\beta f(m)\delta_{m+n,0}$, while the latter leads to $\bar{F}(m,n)=\alpha n+\beta g(n)\delta_{m+n,0}+\gamma(m-n)$. By plugging these infinitesimal solutions into (8.3.10)-(8.3.12) and (8.3.13)-(8.3.15), one finds that they are solved only by $\bar{G}(m,n)=\alpha m$, $\bar{G}(m,n)=\beta m^k\delta_{m+n,0}$, $\bar{F}(m,n)=\gamma n$ or $\bar{F}(m,n)=\eta n^k\delta_{m+n,0}$, where $k\in\mathbb{Z}_+$ is a positive integer. In fact, each of the solutions of $\bar{G}(m,n)$ leads to an independent formal deformation. The same takes place for solutions of $\bar{F}(m,n)$, such that we find four independent formal deformations [9]. Thus, we find new algebras through deformation procedure by $\bar{G}(m,n)$

$$\boxed{\begin{aligned}&[\mathcal{J}_m,\mathcal{J}_n]=0\ ,\\&[\mathcal{J}_m,\mathcal{P}_n]=m\,\delta_{m+n,0}+(\alpha\,m\,+\beta\,m^k\,\delta_{m+n,0})\mathcal{P}_{m+n}\ ,\\&[\mathcal{P}_m,\mathcal{P}_n]=0\ ,\end{aligned}} \quad (8.3.16)$$

and new algebras induced by $\bar{F}(m,n)$

$$\boxed{\begin{aligned}&[\mathcal{J}_m,\mathcal{J}_n]=0\ ,\\&[\mathcal{J}_m,\mathcal{P}_n]=m\delta_{m+n,0}+(\gamma\,n\,+\eta\,n^k\,\delta_{m+n,0})\mathcal{J}_{m+n}\ ,\\&[\mathcal{P}_m,\mathcal{P}_n]=0\ ,\end{aligned}} \quad (8.3.17)$$

where α, β, γ and η are four independent parameters, corresponding to four independent formal deformations which cannot be generally turned on simultaneously. For future use, we denote the algebra (8.3.16) by $\mathfrak{H}_{3\alpha}$ when $\beta=0$.

For the deformation induced by $\beta=0$, one can easily check that, by rescaling $\mathcal{J}_m\to m\,\mathcal{J}_m$ in (8.3.16), a new commutator $[\mathcal{J}_m,\mathcal{P}_n]=\delta_{m+n,0}+\alpha\mathcal{P}_{m+n}$ is obtained.

[9] We also realize that the relation (8.3.12) leads to $\bar{A}(m,n)=\bar{\varepsilon}m\delta_{m+n,0}$, which just modifies the value of central term and will not be considered as a new algebra.

One can further introduce another redefinition $\mathcal{P}_m \to \mathcal{P}_m - \frac{\delta_{m,0}}{\alpha}$ [10] to obtain a new algebra

$$
\begin{aligned}
&[\mathcal{J}_m, \mathcal{J}_n] = 0 \ , \\
&[\mathcal{J}_m, \mathcal{P}_n] = \alpha\, \mathcal{P}_{m+n} \ , \\
&[\mathcal{P}_m, \mathcal{P}_n] = 0 \ .
\end{aligned} \tag{8.3.18}
$$

In fact, we can rescale $\mathcal{J}_m$ again such that $\alpha = 1$, which unveils that we are dealing with a discrete deformation. This means that all $\alpha \neq 0$ are really equivalent, so the only inequivalent algebras are $\alpha = 0$ (undeformed Heisenberg) and $\alpha \neq 0$ (8.3.18). It is also worth to point out that the algebra (8.3.16) can also be obtained as a contraction of $W(0,b)$ algebra if $\beta = 0$.

In section 8.4, we will discuss that (8.3.18) is part of the larger $\mathfrak{bmsw}_3$ algebra (8.1.12) and will play an important role connecting the latter to $\mathfrak{witt} \oplus \mathfrak{H}_3$ (8.1.8).

Deformations of $[\mathcal{P}, \mathcal{P}]$

Finally, we investigate the deformations of the commutator $[\mathcal{P}, \mathcal{P}]$

$$
\begin{aligned}
&[\mathcal{J}_m, \mathcal{J}_n] = 0 \ , \\
&[\mathcal{J}_m, \mathcal{P}_n] = m\delta_{m+n,0} \ , \\
&[\mathcal{P}_m, \mathcal{P}_n] = (m-n)\tilde{F}(m,n)\mathcal{J}_{m+n} + (m-n)\tilde{G}(m,n)\mathcal{P}_{m+n} + (m-n)\tilde{A}(m,n) \ ,
\end{aligned} \tag{8.3.19}
$$

where $\tilde{F}(m,n)$, $\tilde{G}(m,n)$ and $\tilde{A}(m,n)$ are symmetric functions. The Jacobi identities $[\mathcal{P}_m, [\mathcal{P}_n, \mathcal{P}_l]]$ + cyclic permutations $= 0$ yield

$$
\begin{aligned}
(n-l)(m-n-l)\tilde{G}(n,l)\tilde{G}(m,n+l) + (l-m)(n-l-m)\tilde{G}(l,m)\tilde{G}(n,l+m) + \\
(m-n)(l-m-n)\tilde{G}(m,n)\tilde{G}(l,m+n) = 0 \ ,
\end{aligned} \tag{8.3.20}
$$

$$
\begin{aligned}
(n-l)(m-n-l)\tilde{G}(n,l)\tilde{F}(m,n+l) + (l-m)(n-l-m)\tilde{G}(l,m)\tilde{F}(n,l+m) + \\
(m-n)(l-m-n)\tilde{G}(m,n)\tilde{F}(l,m+n) = 0 \ ,
\end{aligned} \tag{8.3.21}
$$

and

$$
\begin{aligned}
&\{(n+l)(n-l)\tilde{F}(n,l) + (l+m)(l-m)\tilde{F}(l,m) + (m+n)(m-n)\tilde{F}(m,n)\}\delta_{m+n+l,0} \\
&+ (n+l-m)(n-l)\tilde{G}(n,l)\tilde{A}(l+n,m) + (l+m-n)(l-m)\tilde{G}(l,m)\tilde{A}(l+m,n) \\
&\qquad + (m+n-l)(m-n)\tilde{G}(m,n)\tilde{A}(m+n,l) = 0 \ .
\end{aligned} \tag{8.3.22}
$$

[10] Note that for $\alpha = 0$ this redefinition is singular and we really just have the Heisenberg algebra. If we naively take $\alpha = 0$ in (8.3.18), then we clearly we do not lie in the Heisenberg branch anymore. As a consequence, the value $\alpha = 0$ cannot be taken in (8.3.18).

Then we consider the Jacobi identities $[\mathcal{P}_m,[\mathcal{P}_n,\mathcal{J}_l]]$+cyclic permutations $=0$, which lead to

$$(m+n)(m-n)\tilde{G}(m,n)=0\ , \tag{8.3.23}$$

implying $\tilde{G}(m,n)=0$. Following the same argument mentioned before, the relation $\{(n+l)(n-l)\tilde{F}(n,l)+(l+m)(l-m)\tilde{F}(l,m)+(m+n)(m-n)\tilde{F}(m,n)\}\delta_{m+n+l,0}=0$, which is first order in functions, leads to $\tilde{F}(m,n)=\mathit{constant}$. Besides, it is easy to check that there is no constraint on $\tilde{A}(m,n)$ even though it can be absorbed by a redefinition (when $\eta\neq 0$), and that $\tilde{F}(m,n)=\mathit{constant}\equiv\eta$ induces a formal deformation

$$\boxed{\begin{aligned}&[\mathcal{J}_m,\mathcal{J}_n]=0\ ,\\&[\mathcal{J}_m,\mathcal{P}_n]=m\delta_{m+n,0}\ ,\\&[\mathcal{P}_m,\mathcal{P}_n]=\eta(m-n)\mathcal{J}_{m+n}\ .\end{aligned}} \tag{8.3.24}$$

We note that, by exchanging $\mathcal{J}_m$ and $\mathcal{P}_m$ in (8.3.24), (8.3.8) can be obtained. As a consequence, both algebras are isomorphic.

8.3.2 General deformations of $\mathfrak{H}_3$

We proceed to investigate the deformations of (8.1.10) in all the commutators simultaneously

$$\begin{aligned}&[\mathcal{J}_m,\mathcal{J}_n]=(m-n)F(m,n)\mathcal{J}_{m+n}+(m-n)G(m,n)\mathcal{P}_{m+n}+(m-n)A(m,n)\ ,\\&[\mathcal{J}_m,\mathcal{P}_n]=m\delta_{m+n,0}+\bar{F}(m,n)\mathcal{J}_{m+n}+\bar{G}(m,n)\mathcal{P}_{m+n}+\bar{A}(m,n)\ ,\\&[\mathcal{P}_m,\mathcal{P}_n]=(m-n)\tilde{F}(m,n)\mathcal{J}_{m+n}+(m-n)\tilde{G}(m,n)\mathcal{P}_{m+n}+(m-n)\tilde{A}(m,n)\ ,\end{aligned} \tag{8.3.25}$$

where $F(m,n)$, $G(m,n)$, $A(m,n)$, $\tilde{F}(m,n)$, $\tilde{G}(m,n)$ and $\tilde{A}(m,n)$ are symmetric functions.

The detailed analysis of the constraints coming from the Jacobi identities is cumbersome and not very instructive. Therefore, we relegate it to appendix C.1. From this analysis, we obtain the following possibilities for the deformations:

1. The deformation induced by $G(m,n)=\mathit{constant}$ and $\tilde{F}(m,n)=\mathit{constant}$, when the other deformations are turned off, does not lead to a formal deformation since the last term in both relations (C.1.4) and (C.1.8) implies that G or $\tilde{F}$ have to be zero. The corresponding algebras are either (8.3.7) or (8.3.24).

2. The deformation induced by $\bar{G}(m,n)=\alpha m-\beta n$ and $F(m,n)=\mathit{constant}=\beta$, when the other deformations are turned off, leads to a formal deformation. Relations (C.1.1) and (C.1.5) are satisfied with the mentioned solutions. The corresponding algebra reads

$$\boxed{\begin{aligned}&[\mathcal{J}_m,\mathcal{J}_n]=\beta(m-n)\mathcal{J}_{m+n}\ ,\\&[\mathcal{J}_m,\mathcal{P}_n]=m\delta_{m+n,0}-\beta(bm+n)\mathcal{P}_{m+n}\ ,\\&[\mathcal{P}_m,\mathcal{P}_n]=0\ ,\end{aligned}} \tag{8.3.26}$$

where $b = -\frac{\alpha}{\beta}$. The same deformation will be obtained when one considers $\tilde{G}(m,n) = constant = \alpha$ and $\bar{F}(m,n) = \alpha m - \beta n$, using the fact that the algebra is symmetric under the exchange $\mathcal{J} \leftrightarrow \mathcal{P}$.

Relations (C.1.3) and (C.1.6) lead to obtain the central extension of $W(0,b)$, denoted as $\widehat{W}(0,b)$, for generic value of b

$$\boxed{\begin{aligned}
&[\mathcal{J}_m, \mathcal{J}_n] = \beta(m-n)\mathcal{J}_{m+n} + (\alpha m^3 - \alpha' m)\delta_{m+n,0} \ , \\
&[\mathcal{J}_m, \mathcal{P}_n] = m\delta_{m+n,0} - \beta(bm+n)\mathcal{P}_{m+n} \ , \\
&[\mathcal{P}_m, \mathcal{P}_n] = 0 \ .
\end{aligned}} \tag{8.3.27}$$

On the other hand, for specific values $b = \{-1, 0, 1\}$, one can find new central terms in other commutators. The new central terms as solutions of (C.1.3), (C.1.6) and (C.1.9) give rise, respectively, to the algebras

$$\begin{aligned}
&[\mathcal{J}_m, \mathcal{J}_n] = \beta(m-n)\mathcal{J}_{m+n} + (\alpha m^3 - \alpha' m)\delta_{m+n,0} \ , \\
&[\mathcal{J}_m, \mathcal{P}_n] = m\delta_{m+n,0} + \beta(m-n)\mathcal{P}_{m+n} + (\bar{\alpha} m^3 - \bar{\alpha}' m)\delta_{m+n,0} \ , \\
&[\mathcal{P}_m, \mathcal{P}_n] = 0 \ ,
\end{aligned} \tag{8.3.28}$$

$$\begin{aligned}
&[\mathcal{J}_m, \mathcal{J}_n] = \beta(m-n)\mathcal{J}_{m+n} + (\alpha m^3 - \alpha' m)\delta_{m+n,0} \ , \\
&[\mathcal{J}_m, \mathcal{P}_n] = m\delta_{m+n,0} + \beta(-n)\mathcal{P}_{m+n} + (\bar{\alpha} m^2 + \bar{\alpha}' m)\delta_{m+n,0} \ , \\
&[\mathcal{P}_m, \mathcal{P}_n] = \tilde{\alpha} m\delta_{m+n,0} \ ,
\end{aligned} \tag{8.3.29}$$

and

$$\begin{aligned}
&[\mathcal{J}_m, \mathcal{J}_n] = \beta(m-n)\mathcal{J}_{m+n} + (\alpha m^3 - \alpha' m)\delta_{m+n,0} \ , \\
&[\mathcal{J}_m, \mathcal{P}_n] = m\delta_{m+n,0} - \beta(m+n)\mathcal{P}_{m+n} + (\bar{\alpha} m + \bar{\alpha}')\delta_{m+n,0} \ , \\
&[\mathcal{P}_m, \mathcal{P}_n] = 0 \ ,
\end{aligned} \tag{8.3.30}$$

which is in agreement with the theorem 1.2 of [261].

Let us emphasize that when $b \neq 1$, by means of the redefinition $P_m := \mathcal{P}_m - \frac{\delta_{m,0}}{\beta(b-1)}$, the linear central term can be absorbed in (8.3.26). A further rescaling of the generators $\mathcal{J}_m \to \beta\mathcal{J}_m$ and $P_m \to \frac{P_m}{\beta}$ shows that indeed (8.3.26) is isomorphic to $W(0;b)$ for $b \neq 1$. Furthermore, rescaling $\mathcal{J}_m \to \alpha\mathcal{J}_m$, it is clear that (8.3.28), (8.3.29) and (8.3.30) correspond to $\widehat{W}(0;-1)$, $\widehat{W}(0;0)$ and $\widehat{W}(0;1)$, respectively.

It is noteworthy to point out that, although we start with the Heisenberg non-trivial central extension, new algebras are obtained after deformation in which central terms that are trivial and can be absorbed by redefinition pop up. This shows that the deformation procedure can change the role of a non-trivial central term to a trivial one.

3. The deformation induced by $G(m,n) = constant = \nu$ and $\bar{G}(m,n) = \alpha m - \beta n$ and $F(m,n) = constant = \beta$, when other deformations are turned off, leads to a formal deformation. Relations (C.1.1), (C.1.2) and (C.1.5) are satisfied with the mentioned solutions. The corresponding algebra reads

$$\boxed{\begin{aligned}&[\mathcal{J}_m, \mathcal{J}_n] = \beta(m-n)\mathcal{J}_{m+n} + \nu(m-n)\mathcal{P}_{m+n} \ ,\\ &[\mathcal{J}_m, \mathcal{P}_n] = m\delta_{m+n,0} - \beta(bm+n)\mathcal{P}_{m+n} \ ,\\ &[\mathcal{P}_m, \mathcal{P}_n] = 0 \ ,\end{aligned}} \tag{8.3.31}$$

where again we have $b = -\frac{\alpha}{\beta}$. The new algebra is denoted by $\widehat{W}_\nu(0,b)$ and it is one of the two deformation mother algebras of $\mathfrak{H}_3$. This is a three-parametric family of algebras which can be centrally extended to become four-parametric with a central charge in the $[\mathcal{J},\mathcal{J}]$ commutator. The algebras (8.3.8), (8.3.16) and (8.3.39) can be obtained from various choices of the parameters in $\widehat{W}_\nu(0,b)$.

The same deformation will be obtained when one considers $\tilde{G}(m,n) = constant = \beta$ and $\bar{F}(m,n) = \alpha m - \beta n$ and $\tilde{F}(m,n) = constant = \nu$, using the fact that the algebra is symmetric under the exchange $\mathcal{J} \leftrightarrow \mathcal{P}$.

In the algebra (8.3.31) one can make use of the redefinition $\mathcal{P}_m \to P_m + \frac{\delta_{m,0}}{\beta(b-1)}$ to obtain

$$\begin{aligned}&[\mathcal{J}_m, \mathcal{J}_n] = \beta(m-n)\mathcal{J}_{m+n} + \nu(m-n)P_{m+n} + \nu(m-n)\left(\frac{1}{\beta(b-1)}\delta_{m+n,0}\right) \ ,\\ &[\mathcal{J}_m, P_n] = -\beta(bm+n)P_{m+n} \ ,\\ &[P_m, P_n] = 0 \ .\end{aligned} \tag{8.3.32}$$

Another redefinition $\mathcal{J}_m \to J_m + \frac{1}{\beta}(\frac{-\nu}{\beta(b-1)}\delta_{m,0})$, followed by the rescaling $J_m \to \beta J_m$, leads to

$$\begin{aligned}&[J_m, J_n] = (m-n)J_{m+n} + \frac{\nu}{\beta^2}(m-n)P_{m+n} \ ,\\ &[J_m, P_n] = -(bm+n)P_{m+n} \ ,\\ &[P_m, P_n] = 0 \ ,\end{aligned} \tag{8.3.33}$$

where the procedure is only valid for $b \neq 1$. Next, we can make use of the redefinition $J_m \to \bar{J}_m - \frac{\nu}{\beta^2 b}P_m$ to obtain

$$\begin{aligned}&[\bar{J}_m, \bar{J}_n] = (m-n)\bar{J}_{m+n} \ ,\\ &[\bar{J}_m, P_n] = -(bm+n)P_{m+n} \ ,\\ &[P_m, P_n] = 0 \ ,\end{aligned} \tag{8.3.34}$$

which is only valid for $b \neq 0$. On the other hand, one finds that the ν-term in (8.3.31), when $b = 1$, can be absorbed by a redefinition $\mathcal{J}_m \to J_m - \frac{\nu}{\beta}P_m - \frac{\nu}{\beta^2}\delta_{m,0}$

and $\mathcal{P}_m \to P_m$. The resultant algebra is given by

$$\begin{aligned}
&[J_m, J_n] = (m-n)J_{m+n} \ , \\
&[J_m, P_n] = -(m+n)P_{m+n} + m\delta_{m+n,0} \ , \\
&[P_m, P_n] = 0 \ .
\end{aligned} \tag{8.3.35}$$

In this way, we have found three independent algebras: (8.3.35) when $b = 1$, (8.3.33) when $b = 0$ and (8.3.34) for other values of b. Note that the algebra (8.3.34) corresponds to $W(0; b \neq \{0, 1\})$, while the algebras (8.3.33) (for $b = 0$) and (8.3.35) (for $b = 1$) can be understood, respectively, as a deformation of $W(0; 0)$ and a central extension of $W(0; 1)$.

The central terms as solutions of (C.1.6), (C.1.3) and (C.1.9) for (8.3.31) when $b = \{-1, 1\}$ give us, respectively, the algebras

$$\begin{aligned}
&[\mathcal{J}_m, \mathcal{J}_n] = \alpha(m-n)\mathcal{J}_{m+n} + \nu(m-n)\mathcal{P}_{m+n} + (\eta\, m^3 - \beta\, m)\delta_{m+n,0} \ , \\
&[\mathcal{J}_m, \mathcal{P}_n] = m\delta_{m+n,0} + \alpha(m-n)\mathcal{P}_{m+n} + (\bar{\eta}\, m^3 - \bar{\beta}\, m)\delta_{m+n,0} \ , \\
&[\mathcal{P}_m, \mathcal{P}_n] = 0 \ ,
\end{aligned} \tag{8.3.36}$$

and

$$\begin{aligned}
&[\mathcal{J}_m, \mathcal{J}_n] = \alpha(m-n)\mathcal{J}_{m+n} + \nu(m-n)\mathcal{P}_{m+n} + (\eta\, m^3 - \beta\, m)\delta_{m+n,0} \ , \\
&[\mathcal{J}_m, \mathcal{P}_n] = m\delta_{m+n,0} - \alpha(m+n)\mathcal{P}_{m+n} + (\bar{\eta}\, m + \bar{\beta})\delta_{m+n,0} \ , \\
&[\mathcal{P}_m, \mathcal{P}_n] = 0 \ .
\end{aligned} \tag{8.3.37}$$

From the last term of (C.1.6), we observe that the ν-term cannot appear when $b = 0$, since it admits a central term in its last commutator [11].

One can easily notice that, in the algebra (8.3.36), the linear central terms can be absorbed by a redefinition of generators in a similar line as described before. Besides, the ν-term can be absorbed by a redefinition as $\mathcal{J}_m \to J_m + \frac{\nu}{\alpha}P_m$ and $\mathcal{P}_m \to P_m$. This does not change $\bar{\eta}$ while shifting η to $\tilde{\eta} = \eta - \frac{2\nu}{\alpha}\bar{\eta}$. The new algebra has the form

$$\begin{aligned}
&[J_m, J_n] = \alpha(m-n)J_{m+n} + (\tilde{\eta}\, m^3)\delta_{m+n,0} \ , \\
&[J_m, P_n] = \alpha(m-n)P_{m+n} + (\bar{\eta}\, m^3)\delta_{m+n,0} \ , \\
&[P_m, P_n] = 0 \ .
\end{aligned} \tag{8.3.38}$$

In this way, the ν-term, which changes the value of central term, can be interpreted as a quantum correction in asymptotically flat spacetime analysis [262].

It turns out that the ν-term and the linear central term in the first commutator of (8.3.37) can also be reabsorbed by the redefinition $\mathcal{J}_m \to J_m - \frac{\nu}{\alpha}P_m - 2\frac{\nu}{\alpha^2}(1+\bar{\eta})\delta_{m,0} + \frac{\beta}{2\alpha}\delta_{m,0}$.

[11] One may consider the case $\tilde{\alpha} = 0$ in (8.3.29) to obtain a ν-term as a non-trivial deformation of $\widehat{W}(0, 0)$ which cannot be absorbed by a change of the basis.

Remark 1. Relation (C.1.14) is also solved by $F(m,n) = 0$ and $\bar{G}(m,n) = \alpha m$. The only relation that should be checked is (C.1.2) which is satisfied by $\bar{G}(m,n) = \alpha m$ and $G(m,n) = constant = \nu$, leading to the new algebra

$$\boxed{\begin{aligned}&[\mathcal{J}_m, \mathcal{J}_n] = \nu(m-n)\mathcal{P}_{m+n} \ , \\ &[\mathcal{J}_m, \mathcal{P}_n] = m\delta_{m+n,0} + \alpha m\mathcal{P}_{m+n} \ , \\ &[\mathcal{P}_m, \mathcal{P}_n] = 0 \ ,\end{aligned}} \tag{8.3.39}$$

which we denote by $\mathfrak{H}_{3\nu\alpha}$.

For $\alpha \neq 0$, and using the redefinition as $\mathcal{J} \to \mathcal{J} + \frac{\nu}{\alpha}\mathcal{P}$ and $\mathcal{P} \to \mathcal{P}$, we obtain

$$\begin{aligned}&[\mathcal{J}_m, \mathcal{J}_n] = \beta\, m\delta_{m+n} \ , \\ &[\mathcal{J}_m, \mathcal{P}_n] = m\delta_{m+n,0} + \alpha m\mathcal{P}_{m+n} \ , \\ &[\mathcal{P}_m, \mathcal{P}_n] = 0 \ ,\end{aligned} \tag{8.3.40}$$

where $\beta = -\frac{2\nu}{\alpha}$. It is obvious that the algebra (8.3.40) can be obtained as a specific contraction of $\widehat{W}(0,b)$. A further redefinition $\mathcal{P}_m \to \mathcal{P}_m - \frac{\delta_{m,0}}{\alpha}$ leads to

$$\begin{aligned}&[\mathcal{J}_m, \mathcal{J}_n] = \beta m\delta_{m+n.0} \ , \\ &[\mathcal{J}_m, \mathcal{P}_n] = \alpha m\mathcal{P}_{m+n} \ , \\ &[\mathcal{P}_m, \mathcal{P}_n] = 0 \ .\end{aligned} \tag{8.3.41}$$

This algebra can be understood as a central extension of (8.3.18) in the $[\mathcal{J}, \mathcal{J}]$ commutator.

Remark 2. Another formal deformation is generated by $F(m,n) = 0$ and $\bar{G}(m,n) = \alpha\, m^k\, \delta_{m+n,0}$ for the values of $k = 1, 3$. In the case $k = 1$, the algebra reads as follows after suitable redefinition

$$\begin{aligned}&[\mathcal{J}_m, \mathcal{J}_n] = \frac{\nu}{1+\alpha}(m-n)\mathcal{P}_{m+n} \ , \\ &[\mathcal{J}_m, \mathcal{P}_n] = m\, \mathcal{P}_0\, \delta_{m+n,0} \ , \\ &[\mathcal{P}_m, \mathcal{P}_n] = 0 \ .\end{aligned} \tag{8.3.42}$$

This algebra can be viewed as a combination of (8.3.8) and (8.3.16) when $\alpha = 0$ and $k = 1$.

4. The deformation induced by $G(m,n) = constant$ and $\bar{F}(m,n) = \alpha(m-n)$ and $\tilde{G}(m,n) = constant$, when other deformations are turned off, leads to a formal deformation. Relations (C.1.1), (C.1.5) and (C.1.7) are satisfied with the mentioned solutions. The other solutions of $\bar{F}(m,n)$, like $\bar{F}(m,n) = -\alpha n$

and $\bar{F}(m,n) = -\alpha(m+n)$, cannot satisfy relation (C.1.5) so they do not lead to any new algebra. The corresponding algebra reads

$$\boxed{\begin{aligned}&[\mathcal{J}_m,\mathcal{J}_n] = \nu(m-n)\mathcal{P}_{m+n}\ ,\\ &[\mathcal{J}_m,\mathcal{P}_n] = m\delta_{m+n,0} + \alpha(m-n)\mathcal{J}_{m+n}\ ,\\ &[\mathcal{P}_m,\mathcal{P}_n] = \alpha(m-n)\mathcal{P}_{m+n}\ ,\end{aligned}} \tag{8.3.43}$$

which consists of just two copies of the Witt algebra when the linear central term can be absorbed by redefinition of generator $\mathcal{J}$. The same deformation can be obtained when considering $\tilde{F}(m,n) = constant = \nu$, $\bar{G}(m,n) = \alpha(m-n)$ and $F(m,n) = constant = \alpha$, using the fact that the algebra is symmetric under the exchange $\mathcal{J} \leftrightarrow \mathcal{P}$.

The central terms as solutions of (C.1.6), (C.1.3), (C.1.9) and (C.1.12) give rise to the algebra

$$\boxed{\begin{aligned}&[\mathcal{J}_m,\mathcal{J}_n] = \nu(m-n)\mathcal{P}_{m+n} + (\alpha' m^3 - \beta m)\delta_{m+n,0}\ ,\\ &[\mathcal{J}_m,\mathcal{P}_n] = m\delta_{m+n,0} + \alpha(m-n)\mathcal{J}_{m+n} + (\bar{\alpha} m^3 - \bar{\beta} m)\delta_{m+n,0}\ ,\\ &[\mathcal{P}_m,\mathcal{P}_n] = \alpha(m-n)\mathcal{P}_{m+n} + (\tilde{\alpha} m^3 - \tilde{\beta} m)\delta_{m+n,0}\ ,\end{aligned}} \tag{8.3.44}$$

with the constraints $-\nu\tilde{\beta} + \alpha\beta = 0$ and $-\nu\tilde{\alpha} + \alpha\alpha' = 0$. These constraints, together with the possibility of absorbing two linear central terms via redefinition of $\mathcal{J}_m$ and $\mathcal{P}_m$, translate into the known fact that there are just two independent central terms for two copies of Virasoro $\mathfrak{vir} \oplus \mathfrak{vir}$.

Remark. It is worth to point out that the linear central term in (8.3.43) is a trivial central term, in the sense that it can be absorbed by an appropriate redefinition of generators, while the central term in $\mathfrak{H}_3$ we started with is a non-trivial central term. Indeed, deformation and contraction procedures can change the role of the generators. This is analogous to the deformation/contraction relation between Poincaré and Bargmann algebras. In fact, to obtain the Bargmann algebra through contraction of the Poincaré algebra a new $\mathfrak{u}(1)$ generator, as trivial central term, has to be added to the Poincaré algebra [263]. This trivial central term becomes after contraction a non-trivial central term in the Bargmann algebra. The same takes place for the ν-term in (8.3.8) and, for instance, (8.3.31). In fact, in (8.3.8) the ν-term is a non-trivial part of the algebra but, after deformation to (8.3.31), it can be absorbed by a redefinition for $b \neq \{0,1\}$.

5. The deformation induced by $F(m,n) = constant = \alpha$, $\bar{G}(m,n) = \alpha(m-n)$, $\tilde{G}(m,n) = constant = \beta$ and $\bar{F}(m,n) = \beta(m-n)$, when other deformations are turned off, does not lead to a formal deformation since relations (C.1.8) and (C.1.4) are not satisfied with the mentioned solutions turned on simultaneously.

6. The deformation induced by $F(m,n) = constant = \alpha$, $G(m,n) = constant$, $\bar{F}(m,n) = \eta(m-n)$, $\bar{G}(m,n) = \alpha(m-n)$, $\tilde{F}(m,n) = constant$ and $\tilde{G}(m,n) = constant = \eta$, when they are turned on simultaneously, yields a formal deformation, since all relations obtained through Jacobi identities are satisfied. The corresponding algebra reads

$$\boxed{\begin{aligned}&[\mathcal{J}_m,\mathcal{J}_n] = \alpha(m-n)\mathcal{J}_{m+n} + \nu(m-n)\mathcal{P}_{m+n}\ ,\\&[\mathcal{J}_m,\mathcal{P}_n] = m\delta_{m+n,0} + \eta(m-n)\mathcal{J}_{m+n} + \alpha(m-n)\mathcal{P}_{m+n}\ ,\\&[\mathcal{P}_m,\mathcal{P}_n] = \zeta(m-n)\mathcal{J}_{m+n} + \eta(m-n)\mathcal{P}_{m+n}\ ,\end{aligned}} \tag{8.3.45}$$

with the constraint $\alpha\eta-\nu\zeta = 0$. This algebra turns out to be a three-parametric deformation mother algebra which we will denote by $\mathcal{H}_3(\alpha,\eta,\nu)$ and which contains $\mathfrak{bms}_3$, $\mathfrak{witt}\oplus\mathfrak{witt}$ and $\mathfrak{witt}\oplus\mathfrak{u}(1)$ for several choices of the parameters.

For example, let us explore the simple case $\alpha = \eta = \nu = \zeta$ where we denote them as ε. Making use of the redefinitions $\mathcal{J}_m \to P_m - J_m$ and $\mathcal{P}_m \to P_m + J_m$, we obtain

$$\boxed{\begin{aligned}&[J_m,J_n] = -\frac{m}{2}\delta_{m+n,0}\ ,\\&[P_m,J_n] = 0\ ,\\&[P_m,P_n] = 2\varepsilon(m-n)P_{m+n} + \frac{m}{2}\delta_{m+n,0}\ .\end{aligned}} \tag{8.3.46}$$

This corresponds to a direct sum of the Witt algebra with a current algebra, which can be also obtained from contraction of two Virasoro algebras, as shown in the appendix D.

The central terms arising as solutions of (C.1.6), (C.1.3), (C.1.9) and (C.1.12) give rise to the algebra

$$\boxed{\begin{aligned}&[\mathcal{J}_m,\mathcal{J}_n] = \alpha(m-n)\mathcal{J}_{m+n} + \nu(m-n)\mathcal{P}_{m+n} + (\alpha'' m^3 - \beta'' m)\delta_{m+n,0}\ ,\\&[\mathcal{J}_m,\mathcal{P}_n] = m\delta_{m+n,0} + \eta(m-n)\mathcal{J}_{m+n} + \alpha(m-n)\mathcal{P}_{m+n} + (\bar{\alpha} m^3 - \bar{\beta} m)\delta_{m+n,0}\ ,\\&[\mathcal{P}_m,\mathcal{P}_n] = \zeta(m-n)\mathcal{J}_{m+n} + \eta(m-n)\mathcal{P}_{m+n} + (\tilde{\alpha} m^3 - \tilde{\beta} m)\delta_{m+n,0}\ ,\end{aligned}} \tag{8.3.47}$$

with the constraints $\alpha\eta - \nu\zeta = 0$, $-\eta\beta'' + \nu\tilde{\beta} = 0$, $-\eta\alpha'' + \nu\tilde{\alpha} = 0$, $-\zeta\beta'' + \alpha\tilde{\beta} = 0$ and $-\zeta\alpha'' + \alpha\tilde{\alpha} = 0$. This algebra turns out to be a five-parametric deformation mother algebra which we will denote by $\widehat{\mathcal{H}_3}(\alpha,\eta,\nu)$ and which contains $\widehat{\mathfrak{bms}_3}$, $\mathfrak{vir}\oplus\mathfrak{vir}$ and $\mathfrak{vir}\oplus\mathfrak{u}(1)$ for several choices of the parameters. In fact, it corresponds to the centrally extended $\mathcal{H}_3(\alpha,\eta,\nu)$ mother algebra (8.3.45).

For example, let us investigate the simple case $\alpha = \nu = \eta = \zeta = \alpha'' = \beta'' = \bar{\alpha} = \bar{\beta} = \tilde{\alpha} = \tilde{\beta} = \sigma$. By means of the redefinitions $\mathcal{J}_m \to P_m - J_m$ and $\mathcal{P}_m \to P_m + J_m$, we learn that the right hand side is equivalent to that of

(8.3.45) together with the addition of a central term $\sigma(m^3-m)\delta_{m+n,0}$:

$$\boxed{\begin{aligned}&[J_m,J_n]=-\frac{m}{2}\delta_{m+n,0}\ ,\\&[P_m,J_n]=0\ ,\\&[P_m,P_n]=2\sigma(m-n)P_{m+n}+\frac{m}{2}\delta_{m+n,0}+\sigma(m^3-m)\delta_{m+n,0}\ .\end{aligned}} \tag{8.3.48}$$

The linear central terms in the last commutator can also be absorbed such that (8.3.48) is just a direct sum of Virasoro and a current algebra $\mathfrak{vir}\oplus\mathfrak{u}(1)$.

8.3.3 Summary of new algebras and disjoint families

We have shown that $\mathfrak{H}_3$ can be deformed into various algebras. Some of these algebras are connected to each other through deformation/contraction procedures. The algebras which cannot be related in this way are called "*disjoint*" algebras. In figure 8.1, we summarize the main algebras obtained through deformations of $\mathfrak{H}_3$ and specify their connections [12]. Each line in the diagram indicates a deformation/contraction relation between two algebras [13]. For instance, the algebra (8.3.39) can be deformed neither into $\mathfrak{bms}_3$ nor into two Virasoro algebras. On the other hand, it can be deformed into (8.3.27) and contracted to (8.3.8) and (8.3.16) (when $\beta=0$).

It has been shown in [217] that two copies of the Virasoro algebra, $\mathfrak{vir}\oplus\mathfrak{vir}$, are a rigid algebra, in the sense that it cannot be deformed into any non-trivial algebra. According to a notion of rigidity for family algebras introduced in [171], which states that the $\widehat{W}(0,b)$ family algebra can be deformed into the $\widehat{W}(0,\bar{b})$ algebra with shifted parameters [14], one can check that the two algebras $\widehat{W}_\nu(0,b)$ and $\mathfrak{vir}\oplus\mathfrak{vir}$ are disjoint algebras for generic values of b. It is also clear that $\mathfrak{vir}\oplus\mathfrak{u}(1)$ can just be deformed into $\mathfrak{vir}\oplus\mathfrak{vir}$ and contracted to $\mathfrak{H}_3$ such that it is disjoint from the other algebras in the middle column of diagram. Moreover, $\mathfrak{bms}_3$ and $\mathfrak{H}_{3\nu\alpha}$ cannot be connected through deformation/contraction relation, being disjoint algebras.

Furthermore, it is worth pointing out that, among the various algebras in figure 8.1, $\mathfrak{H}_{3\nu}$, $\mathfrak{H}_{3\alpha}$, $\mathfrak{H}_{3\nu\alpha}$ and $\mathfrak{vir}\oplus\mathfrak{u}(1)$ are algebras which have not yet been obtained as asymptotic/near horizon symmetry algebras.

[12] Let us note that we obtained more algebras through deformation of $\mathfrak{H}_3$, as described in sections 8.3.1 and 8.3.2, whose deformation/contraction connections are worth to study in future works.

[13] We made use of appendix D for investigating contractions.

[14] In [171], it was shown that $\widehat{W}(0,b)$ cannot be deformed into $\widehat{W}(a,b)$ because of the presence of a linear central term.

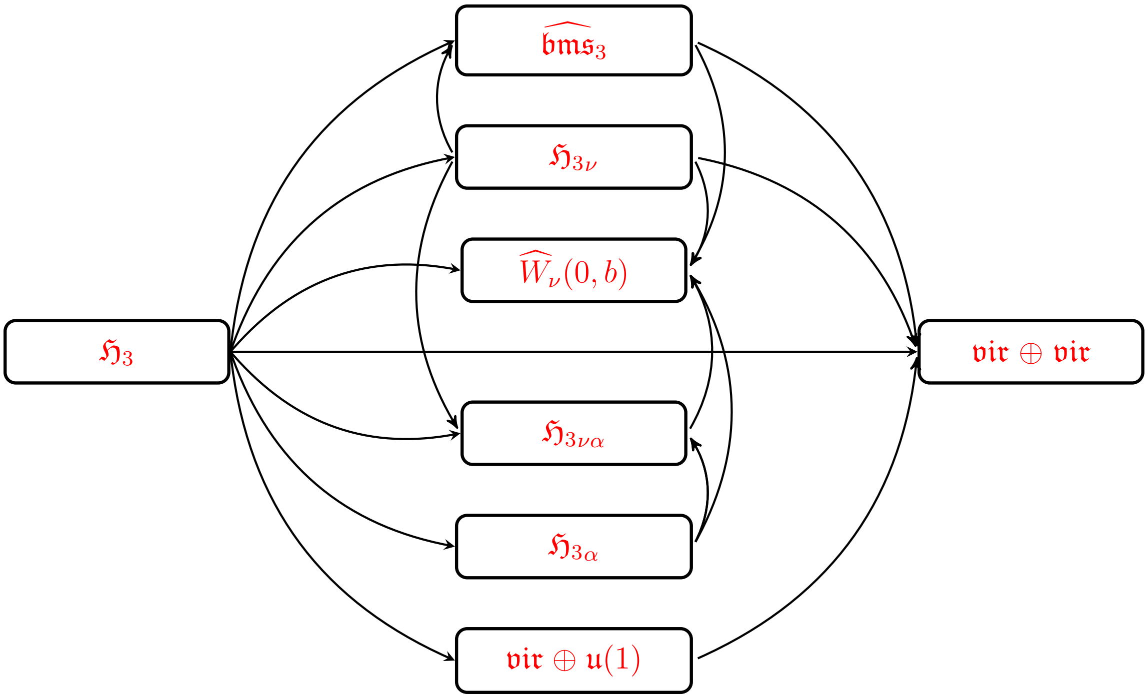

Figure 8.1: Various algebras obtained as formal deformations of $\mathfrak{H}_3$ and their connections. Each line indicates a deformation/contraction relationship. The arrows signal the direction of deformation, while the contractions follow the inverse direction.

Let us emphasize a couple of points about the diagram. Firstly, $\widehat{\mathfrak{bms}}_3$ can be deformed into $\widehat{W}_\nu(0,b)$ when $c_{JP}=0$, even though we kept the trivial linear central term in (8.3.28). Secondly, it was discussed in the previous section that the ν-deformation in $\widehat{W}_\nu(0,b)$ is only non-trivial for $b=0$ and, for other values of b, can be absorbed by redefinition of generators.

From the diagram 8.1, we observe that all the deformations end in one of the two mother algebras $\widehat{W}_\nu(0,b)$ (8.3.31) and $\widehat{\mathcal{H}_3}(\alpha,\nu,\eta)$ (8.3.47) obtained in section 8.3.2. Interestingly, the algebra $\mathfrak{vir}\oplus\mathfrak{vir}$ is a rigid representative of $\widehat{\mathcal{H}_3}(\alpha,\nu,\eta)$. In addition, $\mathfrak{bms}_3$ is a representative of both $\widehat{\mathcal{H}_3}(\alpha,\nu,\eta)$ and $\widehat{W}_\nu(0,b)$.

8.3.4 Deformations vs Sugawara construction

An alternative to deformations which relates $\mathfrak{H}_3$ to other algebras is the celebrated Sugawara construction [264, 265]. It has been shown in [214] that $\mathfrak{vir}$, $\mathfrak{bms}_3$ and Virasoro-Kac-Moody algebras can be obtained from Heisenberg algebras through different twisted and untwisted Sugawara constructions. On the other hand, algebras like $W(0,b)$ (8.1.11) for some values of b cannot be found in this way. In fact, the Witt algebra can be obtained through a quadratic Sugawara-like term as

$$\mathcal{L}_m := \sum_k \mathcal{J}_{m-k}\mathcal{P}_k, \tag{8.3.49}$$

where $\mathcal{L}'s$ satisfy Witt algebra. Taking into account that the ideal part of $W(0,b)$ is abelian, its generators can be expressed in terms of generators of $\mathfrak{H}_3$ as

$$\mathcal{M}_m := a\sum_k \mathcal{J}_{m-k}\mathcal{J}_k + b\mathcal{J}_m + c\mathcal{P}_m. \tag{8.3.50}$$

One then finds that $\mathcal{L}'s$ and $\mathcal{M}'s$ satisfy $W(0,-1)$ when $b=c=0$, $a=1$, and $W(0,0)$ when $b=c=1$, $a=0$. However, it is not possible to obtain any other solution for $b\neq\{0,-1\}$.

A natural question to ask is whether we can obtain all the new algebras coming from $\mathfrak{H}_3$ deformations by means of a Sugawara construction. It turns out that some of the new algebras can be found in this way. For example, (8.3.8) can be obtained through the following Sugawara construction

$$J_m = \frac{\nu}{2}\sum_k \mathcal{P}_{m-k}\mathcal{P}_k + am\mathcal{P}_m + \mathcal{J}_m\ ,\quad P_m = \mathcal{P}_m\ . \tag{8.3.51}$$

Nevertheless, it does not seem to be always possible to obtain our new algebras using such constructions. For example, let us analyze a possible Sugawara construction connecting Heisenberg to (8.3.16). One can readily note that for a maximally quadratic ansatz of the form:

$$J_m = \sum_k (a_1\mathcal{J}_{m-k}\mathcal{J}_k + b_1\mathcal{J}_{m-k}\mathcal{P}_k + c_1\mathcal{P}_{m-k}\mathcal{P}_k) + \text{non-quadratic}\ , \tag{8.3.52}$$

$$P_m = \sum_k (a_2\mathcal{J}_{m-k}\mathcal{J}_k + b_2\mathcal{J}_{m-k}\mathcal{P}_k + c_2\mathcal{P}_{m-k}\mathcal{P}_k) + \text{non-quadratic}\ , \tag{8.3.53}$$

the commutators $[J,J]=[P,P]=0$ impose the conditions $a_ib_i=0$, $c_ib_i=0$ and $4a_ic_i+b_i^2=0$ for both $i=1,2$. As a consequence, the most general solution we can have is $b_i=0$ and $a_i\neq 0$ or $c_i\neq 0$. None of the four possibilities with both J_m and P_m having a quadratic term allows for the quadratic piece of a term $\alpha m P_m$ in the commutator $[J_m,P_n]$. Another two possibilities consist of setting any quadratic term in P_m to zero and allow for quadratic terms in J_m as $J_m\propto\sum_k a_1\mathcal{J}_{m-k}\mathcal{J}_k$ or $J_m\propto\sum_k c_1\mathcal{P}_{m-k}\mathcal{P}_k$. It is easy to check that such terms cannot generate the desired $\alpha m P_m$ independently of the form of the non-quadratic terms. Finally, it is straightforward to realize that when the quadratic terms in both generators are fixed to vanish, one cannot generate $\alpha m P_{m+n}$ in the commutator $[J_m,P_n]$. As a consequence, a conventional quadratic Sugawara construction cannot connect $\mathfrak{H}_3$ with (8.3.16).

Replicating the argument for the absence of Sugawara construction for $W(0,b\neq\{0,-1\})$, we easily realize that another example which cannot be obtained through a possible Sugawara construction is (8.3.26).

To sum up, the deformations belonging to the family $\mathcal{H}_3(\alpha,\nu,\eta)$ (8.3.47) can be Sugawara constructed due to the presence of $(m-n)$ terms, while those coming from the family $\widehat{W}_\nu(0,b)$ (8.3.31) cannot be obtained via a conventional quadratic Sugawara construction of $\mathfrak{H}_3$ for generic values of b.

8.4 Deformations of $\mathfrak{witt} \oplus$ Heisenberg

In the following, we will study the deformations of $\mathfrak{witt}\oplus\mathfrak{H}_3$ as described in section 8.2 and, specifically, its connection with the $\mathfrak{bmsw}_3$ algebra (8.1.12), recently discussed in various works [88, 202, 244].

8.4.1 Deformations of $\mathfrak{witt} \oplus \mathfrak{H}_3$ without central extensions

In this subsection, we would like to study the deformations of $\mathfrak{witt} \oplus \mathfrak{H}_3$, excluding central terms. To this end, we deform the commutators of $\mathfrak{witt} \oplus \mathfrak{H}_3$ as follows [15]:

$$
\begin{aligned}
[\mathcal{L}_m, \mathcal{L}_n] &= (m-n)\mathcal{L}_{m+n} + A_1(m,n)\mathcal{L}_{m+n} + B_1(m,n)\mathcal{J}_{m+n} + C_1(m,n)\mathcal{P}_{m+n} \ , \\
[\mathcal{L}_m, \mathcal{J}_n] &= A_2(m,n)\mathcal{L}_{m+n} + B_2(m,n)\mathcal{J}_{m+n} + C_2(m,n)\mathcal{P}_{m+n} \ , \\
[\mathcal{L}_m, \mathcal{P}_n] &= A_3(m,n)\mathcal{L}_{m+n} + B_3(m,n)\mathcal{J}_{m+n} + C_3(m,n)\mathcal{P}_{m+n} \ , \\
[\mathcal{P}_m, \mathcal{P}_n] &= A_4(m,n)\mathcal{L}_{m+n} + (m-n)\tilde{F}(m,n)\mathcal{J}_{m+n} + (m-n)\tilde{G}(m,n)\mathcal{P}_{m+n} \ , \\
[\mathcal{J}_m, \mathcal{J}_n] &= A_5(m,n)\mathcal{L}_{m+n} + (m-n)F(m,n)\mathcal{J}_{m+n} + (m-n)G(m,n)\mathcal{P}_{m+n} \ , \\
[\mathcal{J}_m, \mathcal{P}_n] &= m\delta_{m+n,0} + A_6(m,n)\mathcal{L}_{m+n} + \bar{F}(m,n)\mathcal{J}_{m+n} + \bar{G}(m,n)\mathcal{P}_{m+n} \ ,
\end{aligned}
\tag{8.4.1}
$$

where A_i, B_i and C_i are arbitrary functions to be determined by the Jacobi identities. First of all, we consider infinitesimal deformations such that we should only keep linear order in the functions. Besides, we can use the fact that $\mathfrak{witt}$ is a rigid subalgebra [266], which allows us to set $A_1(m,n) = 0$ in (8.4.1). The Jacobi identities $[\mathcal{L}_m, [\mathcal{L}_n, \mathcal{J}_l]] +$ cyclic permutations $= 0$ yield

$$
\begin{aligned}
&(m-n)B_2(m+n,l)\mathcal{J}_{l+m+n} + (m-n)C_2(m+n,l)\mathcal{P}_{l+m+n} - lC_1(m,n)\delta_{l+m+n,0} \\
&+ [-(l+m-n)A_2(m,l) + (l-m+n)A_2(n,l) + (m-n)A_2(m+n,l)]\mathcal{L}_{l+m+n} = 0 \ ,
\end{aligned}
\tag{8.4.2}
$$

which leads to $B_2(m,n) = C_2(m,n) = C_1(m,n) = 0$ and a constraint for $A_2(m,n) = \alpha(m-n)$.

The Jacobi identities $[\mathcal{L}_m, [\mathcal{L}_n, \mathcal{P}_l]] +$ cyclic permutations $= 0$ lead to

$$
\begin{aligned}
&(m-n)B_3(m+n,l)\mathcal{J}_{l+m+n} + (m-n)C_3(m+n,l)\mathcal{P}_{l+m+n} - lB_1(m,n)\delta_{l+m+n,0} \\
&+ [-(l+m-n)A_3(m,l) + (l-m+n)A_3(n,l) + (m-n)A_3(m+n,l)]\mathcal{L}_{l+m+n} = 0 \ ,
\end{aligned}
\tag{8.4.3}
$$

which forces $B_3(m,n) = C_3(m,n) = B_1(m,n) = 0$ and gives a constraint for $A_3(m,n) = \beta(m-n)$.

Similarly, the Jacobi identities $[\mathcal{L}_m, [\mathcal{P}_n, \mathcal{P}_l]] +$ cyclic permutations $= 0$ and $[\mathcal{L}_m, [\mathcal{J}_n, \mathcal{J}_l]] +$ cyclic permutations $= 0$ cause $A_4(m,n) = A_5(m,n) = 0$. Lastly, the Jacobi identities $[\mathcal{L}_m, [\mathcal{J}_n, \mathcal{P}_l]] +$ cyclic permutations $= 0$ lead to $A_6(m,n) = 0$. It

[15] Here we take back the notation used in (8.3.25) for the $\mathfrak{H}_3$ part without the central terms.

follows that the Jacobi identities for the $\mathfrak{H}_3$ sector decouple from $\mathcal{L}$ and are exactly the same as in section 8.3.

Therefore, the unique new infinitesimal deformations with respect to section 8.3 are those given by $A_2(m,n) = \alpha(m-n)$ and $A_3(m,n) = \beta(m-n)$. These new infinitesimal deformations can only be formal if they also satisfy the non-linear equations coming from the Jacobi identities $[\mathcal{L}_m, [\mathcal{J}_n, \mathcal{J}_l]]$ + cyclic permutations $= 0$, $[\mathcal{L}_m, [\mathcal{P}_n, \mathcal{P}_l]]$ + cyclic permutations $= 0$ and $[\mathcal{L}_m, [\mathcal{J}_n, \mathcal{P}_l]]$ + cyclic permutations $= 0$. It turns out that they are only involved in the following

$$\begin{aligned}
&(l-n)(l-m+n)[\alpha^2 - \alpha F(n,l) - \beta G(n,l)]\mathcal{L}_{l+m+n} = 0 \ , \\
&(l-n)(l-m+n)[\beta^2 - \beta \tilde{G}(n,l) - \alpha \tilde{F}(n,l)]\mathcal{L}_{l+m+n} = 0 \ , \\
&(l-m+n)[\alpha\beta(l-n) + \beta \bar{G}(n,l) + \alpha \bar{F}(n,l)]\mathcal{L}_{l+m+n} = 0 \ .
\end{aligned} \tag{8.4.4}$$

Thus, in order to find new formal deformations from $A_2(m,n) = \alpha(m-n)$ and/or $A_3(m,n) = \beta(m-n)$, we will have to check if any of the formal deformations found in section 8.3 satisfies also (8.4.4) for $\alpha \neq 0$ and/or $\beta \neq 0$. Let us analyze the different possibilities:

1. $\alpha \neq 0$ while $\beta = 0$. To obtain a non-trivial solution, the relation (8.4.4) implies that $\bar{F}(m,n) = \tilde{F}(m,n) = 0$ while $F(m,n) = \alpha$. So all the algebras obtained through deformations of $\mathfrak{H}_3$ in previous section which satisfy these constraints can be considered as deformations of $\mathfrak{witt} \oplus \mathfrak{H}_3$ induced by the function $A_2(m,n) = \alpha(m-n)$. For instance, we can obtain the algebra

$$\begin{aligned}
&[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} \ , \\
&[\mathcal{L}_m, \mathcal{J}_n] = \alpha\,(m-n)\mathcal{L}_{m+n} \ , \\
&[\mathcal{L}_m, \mathcal{P}_n] = 0 \ , \\
&[\mathcal{J}_m, \mathcal{J}_n] = \alpha\,(m-n)\mathcal{J}_{m+n} \ , \\
&[\mathcal{J}_m, \mathcal{P}_n] = m\delta_{m+n,0} - \alpha\,(bm+n)\mathcal{P}_{m+n} \ , \\
&[\mathcal{P}_m, \mathcal{P}_n] = 0.
\end{aligned} \tag{8.4.5}$$

   One can check that, after redefinition of the generators, this algebra is nothing but $\mathfrak{witt} \oplus W(0;b)$.

2. $\beta \neq 0$ while $\alpha = 0$. To obtain a non-trivial solution, the relation (8.4.4) implies that $\bar{G}(m,n) = \tilde{G}(m,n) = 0$ while $G(m,n) = \beta$. These constraints lead to obtaining the same algebras as in the previous case by using the replacement $\mathcal{J} \leftrightarrow \mathcal{P}$.

3. $\alpha = \beta \neq 0$. There are various options which look all equivalent after redefini-

tion. The most democratic one is given by:

$$
\begin{aligned}
&[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} \ , \\
&[\mathcal{L}_m, \mathcal{J}_n] = \alpha\,(m-n)\mathcal{L}_{m+n} \ , \\
&[\mathcal{L}_m, \mathcal{P}_n] = \alpha\,(m-n)\mathcal{L}_{m+n} \ , \\
&[\mathcal{J}_m, \mathcal{J}_n] = \frac{\alpha}{2}\,(m-n)\mathcal{J}_{m+n} + \frac{\alpha}{2}\,(m-n)\mathcal{P}_{m+n} \ , \\
&[\mathcal{J}_m, \mathcal{P}_n] = m\delta_{m+n,0} + \frac{\alpha}{2}\,(m-n)\mathcal{J}_{m+n} + \frac{\alpha}{2}\,(m-n)\mathcal{P}_{m+n} \ , \\
&[\mathcal{P}_m, \mathcal{P}_n] = \frac{\alpha}{2}\,(m-n)\mathcal{J}_{m+n} + \frac{\alpha}{2}\,(m-n)\mathcal{P}_{m+n}.
\end{aligned}
\tag{8.4.6}
$$

We note that, after redefinition of the generators, this algebra corresponds to the direct sum of two Witt algebras and a Kac-Moody current algebra $\mathfrak{witt} \oplus \mathfrak{witt} \oplus \mathfrak{u}(1)$.

4. $\alpha \neq 0 \neq \beta$. In order to satisfy (8.4.4), the unique reasonable ansatz is given by $\tilde{F}(m,n) = G(m,n) = 0$, $\tilde{G}(m,n) = \beta$, $F(m,n) = \alpha$. Three possibilities follow: $\{\bar{F}(m,n), \bar{G}(m,n)\} = \frac{(m-n)}{2}\{\beta, \alpha\}$, $\{\bar{F}(m,n), \bar{G}(m,n)\} = (m-n)\{\beta, 0\}$ and $\{\bar{F}(m,n), \bar{G}(m,n)\} = (m-n)\{0, \alpha\}$. Nevertheless, the $\mathfrak{H}_3$ Jacobi identities are not satisfied unless $\alpha = 0$ and/or $\beta = 0$ for the three options.

Interestingly, we observe that all the new algebras obtained via deformation of $\mathfrak{witt} \oplus \mathfrak{H}_3$, without involving central extensions, come from deformations on the $\mathfrak{H}_3$ part. The $\mathfrak{witt}$ part of the algebra is well known to be rigid [266] and all the studied mixed deformations can be redefined as deformations only in the $\mathfrak{H}_3$ part. In the next subsection, we will explore whether this pattern still holds if we allow for central extensions.

8.4.2 Deformations of $\mathfrak{witt} \oplus \mathfrak{H}_3$ with central extensions

In the following, we examine the effect of adding central terms. Henceforth, we study the deformations of $\mathfrak{witt} \oplus \mathfrak{H}_3$, including the addition of central terms. To this end, we deform the commutators of $\mathfrak{witt} \oplus \mathfrak{H}_3$ as follows:

$$
\begin{aligned}
&[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + A_1(m,n)\mathcal{L}_{m+n} + B_1(m,n)\mathcal{J}_{m+n} + C_1(m,n)\mathcal{P}_{m+n} + D_1(m,n) \ , \\
&[\mathcal{L}_m, \mathcal{J}_n] = A_2(m,n)\mathcal{L}_{m+n} + B_2(m,n)\mathcal{J}_{m+n} + C_2(m,n)\mathcal{P}_{m+n} + D_2(m,n) \ , \\
&[\mathcal{L}_m, \mathcal{P}_n] = A_3(m,n)\mathcal{L}_{m+n} + B_3(m,n)\mathcal{J}_{m+n} + C_3(m,n)\mathcal{P}_{m+n} + D_3(m,n) \ , \\
&[\mathcal{P}_m, \mathcal{P}_n] = A_4(m,n)\mathcal{L}_{m+n} + (m-n)\tilde{F}(m,n)\mathcal{J}_{m+n} + (m-n)\tilde{G}(m,n)\mathcal{P}_{m+n} + (m-n)\tilde{A}(m,n) \ , \\
&[\mathcal{J}_m, \mathcal{J}_n] = A_5(m,n)\mathcal{L}_{m+n} + (m-n)F(m,n)\mathcal{J}_{m+n} + (m-n)G(m,n)\mathcal{P}_{m+n} + (m-n)A(m,n) \ , \\
&[\mathcal{J}_m, \mathcal{P}_n] = m\delta_{m+n,0} + A_6(m,n)\mathcal{L}_{m+n} + \bar{F}(m,n)\mathcal{J}_{m+n} + \bar{G}(m,n)\mathcal{P}_{m+n} + \bar{A}(m,n) \ ,
\end{aligned}
\tag{8.4.7}
$$

where A_i, B_i, C_i and D_i $(i = 1-6)$ are arbitrary functions whose form is constrained by the Jacobi identities. Such an analysis is straightforward, although lengthy and tedious. As a consequence, we relegate the details to appendix C.2.

We do not aim for an exhaustive analysis of all possible deformations in this case. Nevertheless, we would like to discuss several interesting different possibilities for formal deformations:

1. When all the deformations in the commutators $[\mathcal{L},\mathcal{L}]$, $[\mathcal{L},\mathcal{J}]$ and $[\mathcal{L},\mathcal{P}]$ are turned off except for $D_1(m,n) = \sigma(m^3-m)\delta_{m+n,0}$ (as well as $A_4(m,n) = A_5(m,n) = A_6(m,n) = 0$), we obtain formal deformations. This means that all the deformations of $\mathfrak{H}_3$ studied in section 8.3 are compatible deformations of $\mathfrak{witt}\oplus\mathfrak{H}_3$, even when the $\mathfrak{witt}$ subalgebra is also centrally deformed. In this way, we can deform $\mathfrak{witt}\oplus\mathfrak{H}_3$ to, among others, the direct sum of three Virasoro algebras or to the direct sum of centrally extended $W(0,b)$ and the Virasoro algebra. The latter algebra, for $b=-1$, has been realized in [248] as the asymptotic symmetry algebra of Maxwell Chern-Simons gravity in 3D by considering certain boundary conditions, whereas the former has emerged in [249] as an asymptotic symmetry algebra of AdS-Lorentz Chern-Simons gravity.

2. For the case $\alpha\neq 0$ while $\beta=0$. A possibility that solves Jacobi identities is:
$$\begin{aligned}
&[\mathcal{L}_m,\mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + \alpha(m-n)\mathcal{P}_{m+n} + (c_1 m^3 - c_2 m)\delta_{m+n,0}\ ,\\
&[\mathcal{L}_m,\mathcal{J}_n] = \alpha(m-n)\mathcal{L}_{m+n} + \alpha n\delta_{m+n,0}\ ,\\
&[\mathcal{L}_m,\mathcal{P}_n] = 0\ ,\\
&[\mathcal{P}_m,\mathcal{P}_n] = 0\ ,\\
&[\mathcal{J}_m,\mathcal{J}_n] = \alpha(m-n)\mathcal{J}_{m+n}\ ,\\
&[\mathcal{J}_m,\mathcal{P}_n] = m\delta_{m+n,0} + \alpha(m-n)\mathcal{P}_{m+n} + (c_1 m^3 - c_2 m)\delta_{m+n,0}\ .
\end{aligned}\tag{8.4.8}$$
One can check that, after redefinition of the generators, this algebra is just $\mathfrak{vir}\oplus\widehat{\mathfrak{bms}}_3$. Similar algebras can be obtained for the case $\beta\neq 0$ and $\alpha=0$ by using the replacement $\mathcal{P}\leftrightarrow\mathcal{J}$.

3. For the case $\alpha=\beta\neq 0$. The following deformation solves the Jacobi identities:
$$\begin{aligned}
&[\mathcal{L}_m,\mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + \gamma(m-n)\mathcal{J}_{m+n} + \gamma(m-n)\mathcal{P}_{m+n}\ ,\\
&[\mathcal{L}_m,\mathcal{J}_n] = \alpha\,(m-n)\mathcal{L}_{m+n} + \gamma n\delta_{m+n,0}\ ,\\
&[\mathcal{L}_m,\mathcal{P}_n] = \alpha\,(m-n)\mathcal{L}_{m+n} + \gamma n\delta_{m+n,0}\ ,\\
&[\mathcal{J}_m,\mathcal{J}_n] = \frac{\alpha}{2}\,(m-n)\mathcal{J}_{m+n} + \frac{\alpha}{2}\,(m-n)\mathcal{P}_{m+n}\ ,\\
&[\mathcal{J}_m,\mathcal{P}_n] = m\delta_{m+n,0} + \frac{\alpha}{2}\,(m-n)\mathcal{J}_{m+n} + \frac{\alpha}{2}\,(m-n)\mathcal{P}_{m+n}\ ,\\
&[\mathcal{P}_m,\mathcal{P}_n] = \frac{\alpha}{2}\,(m-n)\mathcal{J}_{m+n} + \frac{\alpha}{2}\,(m-n)\mathcal{P}_{m+n}\ .
\end{aligned}\tag{8.4.9}$$
We note that, after redefinition of the generators, this algebra corresponds to the direct sum of two Witt algebras with a Kac-Moody current algebra $\mathfrak{witt}\oplus\mathfrak{witt}\oplus\mathfrak{u}(1)$.

Although there might be more possibilities leading to new non-trivial algebras, we do not aim to classify all of them here.

It is noteworthy to point out that, through deformation of $\mathfrak{witt} \oplus \mathfrak{H}_3$, we can obtain the direct sum of two Virasoro algebras with a Kac-Moody current algebra $\mathfrak{vir} \oplus \mathfrak{vir} \oplus \mathfrak{u}(1)$. This algebra has been realized as asymptotic symmetry algebra of AdS_3 when extra Weyl symmetry is considered [246, 247]. In fact, as it is discussed in [202], when the Weyl symmetry is taken into account, various algebras, which are related to each other through slicing procedure, can be obtained. Also, one can obtain $\mathfrak{vir} \oplus \mathfrak{vir} \oplus \mathfrak{u}(1)$ by the Sugawara construction of $\mathfrak{vir} \oplus \mathfrak{H}_3$.

Remark. We observe again that all the new algebras obtained via deformation of $\mathfrak{witt} \oplus \mathfrak{H}_3$, allowing for central extensions, come from deformations on the $\mathfrak{H}_3$ part. The $\mathfrak{witt}$ part of the algebra is well known to be rigid [266] and all the studied mixed deformations can be redefined as deformations only in the $\mathfrak{H}_3$ part. This example serves us to speculate that, for infinite-dimensional algebras $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ in which $\mathfrak{g}_1$ is rigid, all the deformations come from the $\mathfrak{g}_2$ part and, therefore, the mixed commutators remain untouched [16]. Examples in favor of this statement are the deformations obtained in this work for $\mathfrak{witt} \oplus \mathfrak{H}_3$ and the fact that $\mathfrak{vir} \oplus \mathfrak{vir}$ is rigid [217].

8.4.3 Relation between $\mathfrak{witt} \oplus \mathfrak{H}_3$ and $\mathfrak{bmsw}_3$

The results within this section rule out the possibility of relating $\mathfrak{witt} \oplus \mathfrak{H}_3$ and $\mathfrak{bmsw}_3$ through a direct continuous deformation. Another way to visualize this result is to recall the fact that there is no infinitesimal linear deformation connecting $\mathfrak{witt}$ plus abelian algebra with the $\mathfrak{bms}_3$ algebra. The same argumentation discards the possibility of relating $\mathfrak{witt} \oplus \mathfrak{H}_3$ and the algebra (3.34) in [202] through a direct continuous deformation.

Nevertheless, let us point out that we can relate $\mathfrak{witt} \oplus \mathfrak{H}_3$ and $\mathfrak{bmsw}_3$ by means of a contraction followed by a double deformation. The first step is to contract $\mathfrak{witt}$ to an Abelian subalgebra, which can be easily achieved by a redefinition $\mathcal{L} \to \varepsilon\mathcal{L}$ and taking the limit $\varepsilon \to \infty$. Next, we deform and redefine the $\mathfrak{H}_3$ subalgebra to (8.3.18). Finally, the new formal deformation

$$\begin{aligned}
&[\mathcal{L}_m, \mathcal{L}_n] = \beta(m-n)\mathcal{L}_{m+n} \ , \\
&[\mathcal{L}_m, \mathcal{P}_n] = \beta(m-n)\mathcal{P}_{m+n} \ , \\
&[\mathcal{L}_m, \mathcal{J}_n] = -\beta n\mathcal{J}_{m+n} \ , \\
&[\mathcal{J}_m, \mathcal{J}_n] = 0 \ , \\
&[\mathcal{J}_m, \mathcal{P}_n] = \alpha\mathcal{P}_{m+n} \ , \\
&[\mathcal{P}_m, \mathcal{P}_n] = 0
\end{aligned} \tag{8.4.10}$$

[16] We are not aware if this statement holds for finite-dimensional algebras and it would be definitely interesting to explore it.

contacts $\mathfrak{bms w}_3$ when α, β are reabsorbed by redefinition of the generators. Naturally, the inverse procedure from $\mathfrak{bmsw}_3$ to $\mathfrak{witt} \oplus \mathfrak{H}_3$ requires the opposite steps, meaning a contraction from (8.4.10) to (8.3.18), then a redefinition followed by a contraction to obtain $\mathfrak{H}_3$ plus abelian ideal and, finally, deforming the latter to $\mathfrak{witt} \oplus \mathfrak{H}_3$.

An alternative way to find the $\mathfrak{bmsw}_3$ algebra is to use a different basis as a starting point. One may start with the Heisenberg-like basis (twisted double Heisenberg algebra) [17]

$$\begin{aligned} &[\mathcal{L}_m, \mathcal{P}_n] = c\, m\, \delta_{m+n,0} \ , \\ &[\mathcal{J}_m, \mathcal{P}_n] = \bar{c}\delta_{m+n,0} \ , \end{aligned} \tag{8.4.11}$$

where the other commutators vanish. The algebra (8.4.11) can be deformed into

$$\begin{aligned} &[\mathcal{L}_m, \mathcal{L}_n] = \alpha(m-n)\mathcal{L}_{m+n} \ , \\ &[\mathcal{L}_m, \mathcal{P}_n] = \alpha(\beta\, m-n)\mathcal{P}_{m+n} + c\, m\, \delta_{m+n,0} \ , \\ &[\mathcal{L}_m, \mathcal{J}_n] = \alpha(-n)\mathcal{J}_{m+n} \ , \\ &[\mathcal{J}_m, \mathcal{J}_n] = 0 \ , \\ &[\mathcal{J}_m, \mathcal{P}_n] = \alpha\, \mathcal{P}_{m+n} + \bar{c}\, \delta_{m+n,0} \ , \\ &[\mathcal{P}_m, \mathcal{P}_n] = 0 \ , \end{aligned} \tag{8.4.12}$$

where α is deformation parameter and the Jacobi identities force $\beta = \frac{c}{\bar{c}} - 1$. This algebra is equivalent to (3.34) of [202] for $s \neq 0$. It should be highlighted that, for $\beta \neq -1$, the central terms can be absorbed by a redefinition of generators. For the specific value $\beta = 1$, which implies that $\bar{c} = \frac{c}{2}$, we find $\mathfrak{bmsw}_3$ as

$$\begin{aligned} &[\mathcal{L}_m, \mathcal{L}_n] = \alpha(m-n)\mathcal{L}_{m+n} \ , \\ &[\mathcal{L}_m, \mathcal{P}_n] = \alpha(m-n)\mathcal{P}_{m+n} + c\, m\, \delta_{m+n,0} \ , \\ &[\mathcal{L}_m, \mathcal{J}_n] = \alpha(-n)\mathcal{J}_{m+n} \ , \\ &[\mathcal{J}_m, \mathcal{J}_n] = 0 \ , \\ &[\mathcal{J}_m, \mathcal{P}_n] = \alpha\, \mathcal{P}_{m+n} + \frac{c}{2}\, \delta_{m+n,0} \ , \\ &[\mathcal{P}_m, \mathcal{P}_n] = 0 \ . \end{aligned} \tag{8.4.13}$$

One can show that the central terms can be absorbed by redefinition as $\mathcal{P}_m \to P_m - \frac{c}{2\alpha}\delta_{m,0}$. Here we are tackling with a similar situation as we discussed in the remark below equation (8.3.44) where non-trivial central terms become trivial after deformation.

[17] We are not aware of any discussions of this algebra in the asymptotic/boundary symmetries literature and it is intriguing to investigate its potential role in that context.

We also note that the algebra (8.4.11) can be deformed into

$$\begin{aligned}
&[\mathcal{L}_m, \mathcal{L}_n] = \alpha(m-n)\mathcal{L}_{m+n} \ , \\
&[\mathcal{L}_m, \mathcal{P}_n] = \alpha(-m-n)\mathcal{P}_{m+n} + c\, m\, \delta_{m+n,0} \ , \\
&[\mathcal{L}_m, \mathcal{J}_n] = \alpha(-n)\mathcal{J}_{m+n} \ , \\
&[\mathcal{J}_m, \mathcal{J}_n] = 0 \ , \\
&[\mathcal{J}_m, \mathcal{P}_n] = \bar{c}\, \delta_{m+n,0} \ , \\
&[\mathcal{P}_m, \mathcal{P}_n] = 0 \ ,
\end{aligned} \tag{8.4.14}$$

which is exactly the algebra (3.34) introduced in [202] when $s = 0$.

Another deformation of (8.4.11) (when $c = 0$) leads to get $\mathfrak{witt} \oplus \mathfrak{H}_3$ as

$$\begin{aligned}
&[\mathcal{L}_m, \mathcal{L}_n] = \alpha(m-n)\mathcal{L}_{m+n} \ , \\
&[\mathcal{J}_m, \mathcal{P}_n] = \bar{c}\, \delta_{m+n,0} \ ,
\end{aligned} \tag{8.4.15}$$

where the other commutators are zero.

Chapter 9

Summary and conclusions of part II

The second part of this doctoral thesis focused on the investigation of the algebra of vector fields on the sphere and the boundary Heisenberg-like algebras $\mathfrak{H}_3$ and $\mathfrak{witt} \oplus \mathfrak{H}_3$. First, we presented $\mathfrak{vect}(S^2)$, exploring its deformations and extensions. Subsequently, we introduced the boundary Heisenberg-like algebras, studying the deformations of $\mathfrak{H}_3$ and $\mathfrak{witt} \oplus \mathfrak{H}_3$.

Summary

- The object of study of chapter 7 is the algebra of vector fields on the sphere. Firstly, we restricted to smooth vector fields, which form the algebra $\mathfrak{vect}(S^2)$, describing its algebra in terms of the area preserving (T^l_m) and non-area preserving (S^l_m) vector fields. Next, with the help of stereographic coordinates, we found a more illuminating chiral basis that splits into vector fields with purely holomorphic $(A^l_m)^+$ and antiholomorphic components $(A^l_m)^-$. By means of the chiral basis, we observed that $(A^l_{\pm l})^+$ and $(A^l_{\pm l})^-$ describe half-Witt subalgebras generated by smooth vector fields on S^2. Moreover, both chiral subalgebras $A^\pm$ can be reconstructed from a half-Witt subalgebra and the action of rotation operators as described in the picture 7.1.

  Next, we investigated the linear deformations of $\mathfrak{vect}(S^2)$ and found that the Jacobi identities fix the structure constants for small values of j completely, which strongly suggests that $\mathfrak{vect}(S^2)$ does not admit linear deformations satisfying the Jacobi identities, being compatible with parity and transforming in given representations of the rotation group. In particular, we showed that the higher-spin one-parameter deformation of $\mathfrak{svect}(S^2)$, $hs[\lambda]$, does not extend to $\mathfrak{vect}(S^2)$ under these requirements. For the chiral subalgebras of $\mathfrak{vect}(S^2)$, $A^\pm$, we also found obstructions against linear deformations satisfying Jacobi identities and vector representation of the generators under rotations.

  Finally, we loosened the smoothness condition for the vector fields and embedded $\mathfrak{vect}(S^2)$ in $\mathfrak{vect}(\mathbb{C}^*)$, allowing for two punctures. In terms of the locally defined vector fields on the two-punctured sphere, we uncovered a three-parameter family of non-central extensions, $gW(a, b, \bar{a})$, which contains asymptotic symmetry algebras of asymptotically flat ($\mathfrak{gbms}$) and asymptotically decelerating spatially flat FLRW ($\mathfrak{gbms}_s$) spacetimes at future null infinity. It contains and generalizes the $W(a, b; \bar{a}, \bar{b})$ family of deformations for $\mathfrak{bms}$ and admits a simple free field realization. In addition, guided by the fact that $W(a, b; \bar{a}, \bar{b})$ admits a central extension, $\hat{W}(a, b; \bar{a}, \bar{b})$, obtained by centrally extending both Witt algebras, we found an equivalent extension for $gW(a, b, \bar{a})$, which we denoted by $g\hat{W}(a, b, \bar{a})$.

- In chapter 8, we delved into boundary Heisenberg algebras and their corresponding deformations. Foremost, we briefly described how these algebras appear predominantly as boundary symmetry algebras in diverse solution phase space slicings and play a major role in the description of the spacetime structure near generic null surfaces. Afterwards, we provided the reader with a primer on deformation theory and specified the methodology and deformations we studied.

We investigated the deformations of the infinite dimensional Heisenberg algebra $\mathfrak{H}_3$. Through the deformation procedure, we obtained various well known asymptotic and near horizon symmetry algebras which arise under diverse boundary conditions and in diverse gravitational theories, spacetimes and loci. In particular, we showed that the two near horizon algebras $\mathfrak{H}_3$ and $W(0, b)$ are related through deformation procedure and we found that $\mathfrak{H}_3$ can be deformed into two copies of the Virasoro algebra $\mathfrak{vir} \oplus \mathfrak{vir}$ which emerges both as near horizon and asymptotic symmetry algebra. Furthermore, we showed that the asymptotic symmetry algebras $\widehat{W}(0, 0)$, $\widehat{\mathfrak{bms}}_3$ and $\widehat{W}(0, 1)$ are connected via deformation to $\mathfrak{H}_3$. In this way, we have shown for explicit examples that, although there are various choices of boundary conditions in the gravitational context, their corresponding symmetry algebras are connected through deformations.

The new algebras can be organized in two three-parametric deformation families of algebras, the mother algebras $\widehat{W}_\nu(0, b)$ and $\mathcal{H}_3(\alpha, \nu, \eta)$, together with their corresponding central extensions. Their deformation/contraction relationships were collected in figure 8.1 and discussed in section 8.3.3.

We further noticed that, although some of the new algebras obtained through deformation can also be found by means of Sugawara constructions, this is not a general feature. Our results provide evidence that the deformation procedure reaches more algebras. Specifically, we gather evidence supporting the statement that those algebras belonging to the family $\widehat{W}_\nu(0, b)$ cannot be obtained via a Sugawara construction for arbitrary b, while those within $\mathcal{H}_3(\alpha, \nu, \eta)$ can be reached using this procedure.

At last, we investigated the deformations of $\mathfrak{witt} \oplus \mathfrak{H}_3$ with and without allowing central extensions. By deforming $\mathfrak{witt} \oplus \mathfrak{H}_3$, we found, among other algebras, the direct sum of three Virasoro algebras, the algebra $\mathfrak{vir} \oplus \mathfrak{vir} \oplus \mathfrak{u}(1)$ and the direct sum of centrally extended $W(0, b)$ and the Virasoro algebra, which have been identified as asymptotic symmetry algebras of gravitational theories. Remarkably, all the deformations we derived come uniquely from deforming the Heisenberg part of the algebra, which leads us to speculate that, for two infinite-dimensional algebras $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, in which $\mathfrak{g}_1$ is rigid, all the deformations come from the $\mathfrak{g}_2$ part. Examples supporting this conjecture are the deformations obtained within this work for $\mathfrak{witt} \oplus \mathfrak{H}_3$ and the fact that $\mathfrak{vir} \oplus \mathfrak{vir}$ is rigid. Our analysis of deformations discarded the possibility that $\mathfrak{witt} \oplus \mathfrak{H}_3$ could be connected through direct continuous deformation to the $\mathfrak{bmsw}_3$ algebra. Contrarily, we showed that the relation with $\mathfrak{bmsw}_3$ is more involved and consists of a contraction followed by a double deformation procedure. Instead, one can start from a twisted double Heisenberg algebra (8.4.11) and reach via deformation $\mathfrak{witt} \oplus \mathfrak{H}_3$, as well as $\mathfrak{bmsw}_3$.

Discussion and future directions

Besides being mathematically interesting per se and scarcely studied, the appearance in modern gravitational theory of the algebras studied in the part II of this dissertation is overwhelming. On the one hand, the algebra of vector fields on the sphere plays a central role in membrane theory, fluid-gravity duality and emerges from recent investigations in asymptotically flat and asymptotically spatially flat FLRW spacetimes, as detailed in the part I of this thesis. On the other hand, boundary Heisenberg algebras pop up ubiquitously as symmetry algebras of generic null surfaces, including event horizons. Strikingly, their direct sum $\mathrm{Diff}(S^2) \oplus \mathfrak{H}_3$ has been obtained as a boundary algebra for the four-dimensional case with spherical boundaries. Moreover, recent studies [51, 53, 95, 97] suggest that symmetry charges associated to diffeomorphisms in gravity may, at least partially, codify the microstrate structure of black holes. However, there exist multiple possible choices of boundary conditions, leading to seemingly unrelated algebras at the same and at different loci. Therefore, a clearer picture of this issue might turn out to be a fundamental step for a better understanding of the infrared structure of gravitational theories and the black hole information puzzle.

Having said that, we find well justified to delve into the mathematical structure of these algebras. We opted to focus on the study of deformations because they endow us with a way to quantify how related to each other algebra are and which degree of variability or uniqueness they possess. These are properties which can be interpreted from a physical viewpoint. For example, the absence of linear deformations of $\mathrm{Diff}(S^2)$ tells us that this algebra is indeed very special and cannot be discretized in the same way that its area-preserving subalgebra $\mathrm{SDiff}(S^2)$, which at the same time is providing us with a property of its potential representations and of its non-central extensions arising in the aforementioned gravitational context. We have also observed how deformations interpolate between multiple symmetry algebras obtained by imposition of different boundary conditions at diverse loci (e.g. future null infinity or event horizons), establishing closeness relationships which effectively tell us how close or far apart these algebras are. In the long term, we expect the study of deformations to unveil a deeper mathematical structure, guiding us in the selection of boundary conditions and degrees of freedom.

Regarding $\mathrm{Diff}(S^2)$, we consider especially interesting to find and explore explicit field realizations, to investigate the properties of its non-central extensions and to discern whether or not it admits non-linear deformations, as well as extending this research towards other surface diffeomorphism algebras like $\mathrm{Diff}(\mathbb{T}^2)$.

Concerning the boundary Heisenberg algebras, we think it would be certainly appealing to explore whether the new algebras obtained via deformation which have not yet appeared in a boundary symmetry analysis (e.g. (8.3.8), (8.3.16), (8.3.39) and (8.3.48)) can actually be realized under new choices of boundary conditions [1]. Following the approach of [218], one could investigate their corresponding thermodynamical interpretation and help to concretize the potential relation between the deformation parameters and thermodynamical or other physical quantities. From a holographic perspective, interpolation between the near horizon and asymptotic region may be interpreted as an RG flow in the dual field theory side. This idea was discussed in [171, 217] and supported, at the level of algebras, by the results within this work [2]. In addition, considering the relevance of the Heisenberg algebra in Physics, we expect that the content and results herein can have yet unknown applications and implications beyond the context of asymptotic and boundary symmetries.

To conclude, the results we obtained in both main chapters of this part II can be easily merged to produce new interesting outputs. For example, as we discussed above, $\mathrm{Diff}(S^2) \oplus \mathfrak{H}_3$ may play a main role in the microstate description of spherically symmetric four-dimensional black holes and, in this dissertation, we studied the deformations of both individual algebra components. Being $\mathrm{Diff}(S^2)$ rigid is already telling us that, most probably, all the deformations of such algebra come from the $\mathfrak{H}_3$ sector [3]. If we are right, this will certainly constrain the possible representations

[1] In fact, shortly after we published [239], one of our new algebras (8.3.30) was realized as asymptotic symmetry algebra of flat JT gravity. This is the so-called "BMS_2 algebra" arising in equations (1.1)-(1.3) and (3.8)-(3.10) of [267].

[2] For example, in the context of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, the near horizon and asymptotic regions of AdS_3 would correspond to IR and UV regions in its dual CFT_2. At the same time, imposing specific boundary conditions for asymptotically AdS_3 spacetimes, the near horizon symmetry algebra is given by $\mathfrak{H}_3$ while the asymptotic symmetry algebra corresponds to $\mathfrak{vir} \oplus \mathfrak{vir}$. In this work, we showed that these two algebras are related to each other through deformation/contraction procedure. As a consequence, deformation/contraction might be linked to a RG flow between UV and IR fixed points of the dual CFT_2.

[3] Recall our observation that for two infinite-dimensional algebras $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, in which $\mathfrak{g}_1$ is rigid, all the deformations seem to come from the $\mathfrak{g}_2$ part.

of $\mathrm{Diff}(S^2) \oplus \mathfrak{H}_3$ and their properties.

Part III

Closure

Chapter 10

General conclusion

During the last decade, we experienced a transition in the theoretical physics community, partially shifting the interest from ultraviolet affairs, like attempts to obtain a consistent quantum gravity theory, to the infrared structure of gravity and gauge theories. Many of us realized how much is still left to learn about classical gravitation and opted for turning our research into a better understanding of a priori more modest issues, with the hope that these will eventually help to solve major challenges.

Meanwhile, the standard model of cosmology, known as flat ΛCDM model, has been increasingly questioned [64–67]. This model is based on General Relativity as gravity theory and the cosmological principle, leading to homogeneous and isotropic FLRW backgrounds, and has been very successful to explain cosmological observations from the CMB, Type Ia supernovae luminosity distances and more phenomena. However, the determination of the Hubble constant from the early universe (CMB) and from the late universe (e.g. Type Ia supernovae) seems to differ substantially, resulting in the so-called Hubble tension. Even though these works are yet far from being definitive [268, 269], they pose a question which threatens FLRW cosmology and could break down the current paradigm in cosmology [268, 270].

In this context, the author of this thesis is convinced that merging both fields of study could be beneficial and provide new approaches, techniques and theoretical insight. Although still at a preliminary stage, we have shown that it is possible to study the infrared structure of certain decelerating cosmological spacetimes, from which we learnt that the dynamics at infinity is frozen, contrarily to flat spacetimes. Our work poses, therefore, major challenges for the practical application of flat techniques to cosmology but brings a technical construction which could be extrapolated to more realistic models with acceleration at different spacetime regions. It also shows that the presence of asymptotic charges is not restricted to maximally symmetric spacetimes but also applies to cosmology.

Furthermore, our studies in FLRW unveiled a cosmological holographic flow which connects the asymptotic algebras of flat and FLRW spacetimes at future null infinity by one-parameter deformations for certain boundary conditions, where the continuous parameter is connected to the fluid filling the universe. We expect this phenomenon to extend to other cosmological models.

By studying deformations of two distinguished classes of algebras, we have noticed that this feature is much more general. In fact, we have been able to relate, via families of deformations, multiple algebras, emerging in the symmetry analysis of gravitational theories and obtained from different boundary conditions at diverse spacetime loci. We expect that the deformations derived in this thesis can also shed light on the right selection of boundary conditions and degrees of freedom. In addition, we are strongly convinced that the fact that $\mathrm{Diff}(S^2)$ does not admit linear deformations will impact the properties of the physical realizations at gravitational boundaries in four-dimensional theories. A first consequence is the fact that a large N discretization, like the one of $\mathrm{SDiff}(S^2)$, used to "quantize" the spherical bosonic membrane, will most surely not be available. All these aspects may ultimately lead to a better understanding of the microstate boundary structure of entities like black holes.

On a final note, we expect the interplay between the symmetry analysis of gravitational boundaries, cosmology and the mathematics of deformations to be very fruitful in the coming decades. This is just the beginning of an exciting journey.

Appendix A

Asymptotic Lie derivatives

A.1 BMSW-like expansion

In this appendix, we calculate the Lie derivatives of the off-shell metric (3.2.2) with respect to the asymptotic diffeomorphisms (3.3.1). These are given by

$$\begin{aligned}
a^{-2}\mathcal{L}_\xi g_{uu} &= 2r\left(\Theta^A\partial_u V_A - \partial_u\xi^{r(V)}\right) \\
&\quad + \big[V^A D_A\Phi + \xi^u\partial_u\Phi + 2U_A\partial_u V^A - 2\partial_u\xi^{r(0)} - 2k(1-\Phi)\xi^{r(V)} \\
&\qquad +2K\partial_u\xi^{r(V)} - 2(1-\Phi)\partial_u\xi^u + 2\Theta_A\partial_u\xi^{A(1)}\big] \\
&\quad + \frac{2}{r}\big[\xi^u\partial_u m - k(1-\Phi)\xi^u - \left((1-2k)m - ku(1-\Phi)\right)\xi^{r(V)} \\
&\qquad - k(1-\Phi)\xi^{r(0)} + V^A D_A m + \frac{1}{2}\xi^{A(1)}D_A\Phi + K\partial_u\xi^{r(0)} - \partial_u\xi^{r(1)} \\
&\qquad +m\partial_u\xi^u + U_A\partial_u\xi^{A(1)} + \Theta_A\partial_u\xi^{A(2)} + N_A\partial_u V^A\big] + \mathcal{O}(r^{-2}), \qquad \text{(A.1.1)} \\
a^{-2}\mathcal{L}_\xi g_{ur} &= -\left[(1+2k)\xi^{r(V)} + \partial_u\xi^u\right] \\
&\quad + \frac{1}{r}\big[\xi^u\partial_u K + V^A D_A K + K\partial_u\xi^u - \Theta_A\xi^{A(1)} \\
&\qquad +2k\left(u\xi^{r(V)} - \xi^u - \xi^{r(0)}\right) + 2kK\xi^{r(V)}\big] + \mathcal{O}(r^{-2}), \qquad \text{(A.1.2)} \\
a^{-2}\mathcal{L}_\xi g_{rA} &= -q_{AB}\xi^{B(1)} - D_A\xi^u + \frac{1}{r}\left(KD_A\xi^u - C_{AB}\xi^{B(1)} - 2q_{AB}\xi^{B(2)}\right) \\
&\quad + \mathcal{O}(r^{-2}), \qquad \text{(A.1.3)} \\
a^{-2}\mathcal{L}_\xi g_{uA} &= q_{AB}\partial_u V^B r^2 + r\big[(1+2k)\Theta_A\xi^{r(V)} + \mathcal{L}_V\Theta_A \\
&\quad -\partial_A\xi^{r(V)} + C_{AB}\partial_u V^B + \xi^u\partial_u\Theta_A + \Theta_A\partial_u\xi^u + q_{AB}\partial_u\xi^{B(1)}\big] \\
&\quad + \Big[(2k\Theta_A + \partial_u U_A)\xi^u + (1+2k)\Theta_A\xi^{r(0)} + 2k\xi^{r(V)}(U_A - u\Theta_A) \\
&\qquad + \mathcal{L}_V U_A + \mathcal{L}_{\xi^{C(1)}}\Theta_A - D_A\xi^{r(0)} + KD_A\xi^{r(V)} - (1-\Phi)D_A\xi^u \\
&\qquad + \left(\mathcal{D}_{AB} + \frac{1}{2}C_{AC}C^C_B\right)\partial_u V^B + U_A\partial_u\xi^u + C_{AB}\partial_u\xi^{B(1)} + q_{AB}\partial_u\xi^{B(2)}\Big]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{r}\Big[\xi^u\partial_u N_A + N_A\partial_u\xi^u + \mathcal{L}_V N_A - (1-2k)N_A\xi^{r(V)} \\
&\quad + K D_A\xi^{r(0)} - D_A\xi^{r(1)} + 2m D_A\xi^u + 2kU_A(\xi^{r(0)} + \xi^u - u\xi^{r(V)}) \\
&\quad + 2k\Theta_A\left(u^2\xi^{r(V)} - u(\xi^{r(0)}+\xi^u) + \xi^{r(1)}\right) + \Theta_A\xi^{r(1)} + C_{AB}\partial_u\xi^{B(2)} \\
&\quad + \left(\mathcal{D}_{AB} + \frac{1}{2}C_{AC}C^C_B\right)\partial_u\xi^{B(1)} + \mathcal{L}_{\xi^{B(1)}}U_A + \mathcal{L}_{\xi^{B(2)}}\Theta_A\Big] \\
&+\mathcal{O}(r^{-2}),
\end{aligned}
\tag{A.1.4}
$$

$$
a^{-2}\mathcal{L}_\xi g_{AB} = r^2F_{AB} + rS_{AB} + K_{AB}, \tag{A.1.5}
$$

with

$$
\begin{aligned}
F_{AB} &= 2(1+k)\xi^{r(V)}q_{AB} + \xi^u\partial_u q_{AB} + \mathcal{L}_V q_{AB}\ , \\
S_{AB} &= 2q_{AB}((1+k)\xi^{r(0)} - ku\xi^{r(V)} + k\xi^u) + \mathcal{L}_{\xi^{A(1)}}q_{AB} \\
&\quad + \Theta_A D_B\xi^u + \Theta_B D_A\xi^u + (1+2k)C_{AB}\xi^{r(V)} + \mathcal{L}_V C_{AB} + \xi^u\partial_u C_{AB}\ , \\
K_{AB} &= 2kq_{AB}\left(u^2\xi^{r(V)} - u\xi^{r(0)} - u\xi^u\right) + 2(1+k)q_{AB}\xi^{r(1)} + \mathcal{L}_{\xi^{A(2)}}q_{AB} \\
&\quad + U_A D_B\xi^u + U_B D_A\xi^u + \mathcal{L}_{\xi^{A(1)}}C_{AB} + 2k\left(\mathcal{D}_{AB} + \frac{1}{2}C_{AC}C^C_B\right)\xi^{r(V)} \\
&\quad + \xi^u\partial_u\left(\mathcal{D}_{AB} + \frac{1}{2}C_{AC}C^C_B\right) + \mathcal{L}_V\left(\mathcal{D}_{AB} + \frac{1}{2}C_{AC}C^C_B\right)\ .
\end{aligned}
\tag{A.1.6}
$$

A.2 Logarithmic expansion

The Lie derivatives of (3.4.17) with respect to the asymptotic diffeomorphisms (3.3.1) are as follows:

$$
\mathcal{L}_\xi g_{uu} = \left(\frac{r}{L}\right)^{2k}\left\{2r(\Theta_A\partial_u V^A - \partial_u\xi^{r(V)}) + \mathcal{O}(r^0)\right\}\ , \tag{A.2.1}
$$

$$
\mathcal{L}_\xi g_{ur} = \left(\frac{r}{L}\right)^{2k}\left\{-(1+2k)\xi^{r(V)} - \partial_u\xi^u + \mathcal{O}(r^{-1})\right\}\ , \tag{A.2.2}
$$

$$
\mathcal{L}_\xi g_{rA} = \left(\frac{r}{L}\right)^{2k}\left\{-D_A\xi^u - q_{AB}\xi^{B(1)} + \mathcal{O}(r^{-1})\right\}\ , \tag{A.2.3}
$$

$$
\mathcal{L}_\xi g_{uA} = \left(\frac{r}{L}\right)^{2k}\left\{q_{AB}\partial_u V^B r^2 + \mathcal{O}(r)\right\}\ , \tag{A.2.4}
$$

$$
\mathcal{L}_\xi g_{AB} = \left(\frac{r}{L}\right)^{2k}\left\{r^2\left(\mathcal{L}_V q_{AB} + 2(1+k)q_{AB}\xi^{r(V)} + \xi^u\partial_u q_{AB}\right) + \mathcal{O}(r)\right\}\ . \tag{A.2.5}
$$

Appendix B

Weyl scalars

The Bondi gauge suggests a frame where one can compute the Weyl scalars. This computation has been useful to identify covariant quantities in asymptotically flat spacetimes [149] and we expect that it can also be useful for asymptotically FLRW. For completion, we compute in this appendix the Weyl scalars associated with the on-shell metric (3.2.2) with finite fluxes, that is when Θ_A and Φ are u-independent.

Our starting point is the historical Bondi-Sachs form of the metric

$$\mathrm{d}s^2 = -2\mathrm{e}^{2\beta}a^2\mathrm{d}u(\mathrm{d}r + F\mathrm{d}u) + g_{AB}(\mathrm{d}x^A - \tilde{U}^A\mathrm{d}u)(\mathrm{d}x^B - \tilde{U}^B\mathrm{d}u) \ . \tag{B.1}$$

The null tetrads are defined by $\eta_{ab}e^a \otimes e^b = g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$ with

$$\eta_{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \ . \tag{B.2}$$

For the metric in equation (B.1) they are given by:

$$e^0 = a\mathrm{e}^{2\beta}\mathrm{d}u \ , \quad e^1 = a\left(\mathrm{d}r + F\mathrm{d}u\right) \ , \quad e^i = arE^i_A\left(\mathrm{d}x^A - U^A\right) \ , \tag{B.3}$$

with $\eta_{ij}E^i_AE^j_B = \frac{1}{a^2r^2}g_{AB}$ for $i,j \in \{2,3\}$. The corresponding vectors are given by:

$$\hat{e}_0 = \frac{1}{a}\mathrm{e}^{-2\beta}\left(\partial_u - F\partial_r + U^A\partial_A\right) \ , \quad \hat{e}_1 = \frac{1}{a}\partial_r \ , \quad \hat{e}_i = \frac{1}{a}\frac{1}{r}\hat{E}^A_i\partial_A \ . \tag{B.4}$$

It can be checked easily that the vectors $\hat{e}_a$ are null. To obtain the metric in the previous form (3.2.2), we have to expand the parameters in (B.1) in the following way:

$$
\begin{aligned}
&\beta = -\frac{K}{2r} - \frac{\mathcal{E} + \frac{1}{2}K^2}{2r^2} + \mathcal{O}(r^{-3}) \ , \\
&F = F_0 + \frac{F_1}{r} + \frac{F_2}{r^2} + \mathcal{O}(r^{-3}) \ , \\
&F_0 = \frac{1}{2}(1 - \Phi + \Theta_A\Theta^A) \ , \\
&F_1 = (K(1-\Phi) - 2m + \Theta^A(K\Theta_A - C_{AB}\Theta^B + 2U_A)) \ , \\
&F_2 = \frac{1}{2}\left(\mathcal{E}(1-\Phi) - \mathcal{F} + K(K(1-\Phi) - 2m) + 2N^A\Theta_A + (\mathcal{E} + K^2)\Theta_A\Theta^A \right. \\
&\left. + \frac{1}{2}(C_A^M C_{BM} - 2\mathcal{D}_{AB})\Theta^A\Theta^B + U_A(2K\Theta^A + U^A) - C_{AB}\Theta^A(K\Theta^B + 2U^B)\right) \ , \\
&\frac{g_{AB}}{a^2} = r^2\left[q_{AB} + \frac{1}{r}C_{AB} + \frac{1}{r^2}\left(\mathcal{D}_{AB} + \frac{1}{2}C_{AC}C^C_B\right) + \frac{1}{r^3}E_{AB} + +\mathcal{O}(r^{-4})\right] \ , \\
&\frac{g_{AB}\tilde{U}^B}{a^2} = r\Theta_A + U_A + \frac{1}{r}N_A + \mathcal{O}(r^{-2}) \ ,
\end{aligned}
\tag{B.5}
$$

with $C^A_A = \mathcal{D}^A_A = 0$ to satisfy the determinant condition of the Bondi gauge. The tetrads on the sphere are expanded as:

$$
\begin{aligned}
E^i_A &= \bar{E}^i_A + \frac{1}{2r}\bar{E}^i_B C^B_A + \frac{1}{2r^2}\bar{E}^i_B\left(\mathcal{D}^B_A + \frac{1}{4}C_{AC}C^{BC}\right) \\
&+ \frac{1}{2r^3}\hat{\bar{E}}^i_B\left(E^B_A - \frac{1}{4}\left(\mathcal{D}^C_A C^B_C + C_{AC}\mathcal{D}^{CB}\right) - \frac{1}{8}C_{AC}C^B_D C^{CD}\right) + \mathcal{O}(r^{-4}) \ ,
\end{aligned}
\tag{B.6}
$$

$$
\begin{aligned}
\hat{E}^A_i &= \hat{\bar{E}}^A_i - \frac{1}{2r}\hat{\bar{E}}^B_i C^A_B - \frac{1}{2r^2}\hat{\bar{E}}^B_i\left(\mathcal{D}^A_B - \frac{1}{4}C^{AC}C_{BC}\right) \\
&+ \frac{1}{2r^3}\hat{\bar{E}}^B_i\left(-E^A_B + \frac{1}{8}C^D_B C_{CD}C^{AC} + \frac{3}{4}\left(\mathcal{D}^A_C C^C_B + \mathcal{D}_{BC}C^{CA}\right)\right) + \mathcal{O}(r^{-4}),
\end{aligned}
\tag{B.7}
$$

where $\hat{\bar{E}}^A_i$ are the tetrads of the leading term of the metric on the sphere, defined as $q^{AB} = \hat{\bar{E}}^A_i\hat{\bar{E}}^B_j\eta^{ij}$ and $\epsilon^{AB} = \hat{\bar{E}}^A_i\hat{\bar{E}}^B_j\epsilon^{ij}$.

With these tetrads, the Weyl scalars are given by:

$$\Psi_4 = C_{\mu\nu\gamma\delta}\hat{e}^\mu_0\hat{e}^\nu_3\hat{e}^\gamma_0\hat{e}^\delta_3 = C_{\hat{0}\hat{3}\hat{0}\hat{3}} = a^{-2}\hat{\bar{E}}^A_3\hat{\bar{E}}^B_3\left[\frac{1}{r}\psi^4_{AB} + \mathcal{O}(r^{-2})\right] \ , \tag{B.8}$$

$$\Psi_3 = C_{\hat{0}\hat{3}\hat{0}\hat{1}} = a^{-2}\hat{\bar{E}}^A_3\left[\frac{1}{r^2}\left(\psi^3_A\right) + \mathcal{O}(r^{-3})\right] \ , \tag{B.9}$$

$$\mathrm{Re}(\Psi_2) = C_{\hat{1}\hat{0}\hat{1}\hat{0}} = a^{-2}\left[\frac{1}{r^2}\psi^{2,1} + \frac{1}{r^3}\psi^{2,2} + \mathcal{O}(r^{-4})\right] \ , \tag{B.10}$$

$$\mathrm{Im}(\Psi_2) = C_{\hat{1}\hat{0}\hat{2}\hat{3}} = a^{-2}\left[\frac{1}{r^2}\tilde{\psi}^{2,1} + \frac{1}{r^3}\tilde{\psi}^{2,2} + \mathcal{O}(r^{-4})\right] \ , \tag{B.11}$$

$$\Psi_1 = C_{\hat{1}\hat{0}\hat{1}\hat{2}} = a^{-2}\hat{\hat{E}}_2^A\left[\frac{1}{r^3}\psi_A^{1,1} + \frac{1}{r^4}\psi_A^{1,2} + \mathcal{O}(r^{-5})\right] , \tag{B.12}$$

$$\Psi_0 = C_{\hat{1}\hat{2}\hat{1}\hat{2}} = a^{-2}\hat{\hat{E}}_2^A\hat{\hat{E}}_2^B\left[\frac{1}{r^4}\psi_{AB}^{0,1} + \frac{1}{r^5}\psi_{AB}^{0,2} + \mathcal{O}(r^{-6})\right] , \tag{B.13}$$

with

$$\psi^4_{AB} = -\frac{1}{2}\partial_u^2 C_{AB} , \tag{B.14}$$

$$\begin{aligned}\psi^3_A = \frac{1}{4}\big(&2D_A\Phi + 2\partial_u U_A - \partial_u D_B C_A^B + \Theta_A\mathcal{R} + 2\Theta^B(D_B\Theta_A - D_A\Theta_B)\\ &+\partial_u D_A K + \Theta_A\partial_u K - \Theta_A D_B\Theta^B + D_B D_A\Theta^B - D_B D^B\Theta_A\big) ,\end{aligned} \tag{B.15}$$

$$\psi^{2,1} = -\frac{1}{6}\left(2\Phi + \mathcal{R} - 2 + D_A\Theta^A + 2\partial_u K\right) , \tag{B.16}$$

$$\begin{aligned}\psi^{2,2} = &-2m - \frac{1}{6}C^{AB}\partial_u C_{AB} - \frac{2}{3}\partial_u\mathcal{E} - \frac{1}{6}D_A(2U^A + D_BC^{AB}) + \frac{1}{3}\Theta^A\left(2U_A + D_BC_A^B\right)\\ &+ K\left(1 - \Phi + \frac{1}{3}\Theta_A\Theta^A - \frac{1}{6}D_A\Theta^A - \partial_u K\right) - \frac{1}{2}\Theta^A D_A K + \frac{1}{6}D_A D^A K ,\end{aligned} \tag{B.17}$$

$$\begin{aligned}\tilde{\psi}^{2,1} &= \frac{1}{4}\epsilon^{AB}D_A\Theta_B ,\\ \tilde{\psi}^{2,2} &= \frac{1}{2}\epsilon^{AB}\bigg(D_AU_B - \frac{1}{4}C_A^C\partial_u C_{CB} - \frac{1}{2}C_{AC}D^C\Theta_B + \frac{1}{2}C_{AC}D_B\Theta^C\\ &\quad + \frac{1}{2}KD_A\Theta_B - \frac{1}{2}\Theta_A D_B K - \frac{1}{2}\Theta^C D_A C_{BC}\bigg) ,\end{aligned} \tag{B.18}$$

(B.19)

$$\psi_A^{1,1} = \frac{1}{4}\left(2U_A + D_BC_A^B + K\Theta_A - D_AK\right) , \tag{B.20}$$

$$\begin{aligned}\psi_A^{1,2} = \frac{1}{2}\bigg(&3N_A - \frac{1}{4}C_A^B(2U_B + D_CC_B^C) + D_B\mathcal{D}_A^B - \frac{1}{4}C^{BC}D_AC_{BC}\\ &-\Theta^B\Big(\mathcal{D}_{AB} + \frac{1}{2}C_{BC}C_A^C\Big) + \frac{3}{4}C_{AB}(D^BK - K\Theta^B) + \Theta_A(K^2 + \mathcal{E})\\ &+ 2K\Big(U_A - \frac{1}{2}D_AK\Big) - \frac{1}{2}D_A\mathcal{E}\bigg) ,\end{aligned} \tag{B.21}$$

$$\psi_{AB}^{0,1} = -\mathcal{D}_{AB} - \frac{1}{4}C_A^CC_{CB} - \frac{1}{2}C_{AB}K , \tag{B.22}$$

$$\psi_{AB}^{0,2} = -3E_{AB} + \frac{1}{2}C_A^CC_B^DC_{CD} + 2\mathcal{D}_{CA}C_B^C - C_{AB}\mathcal{E} - \mathcal{D}_{AB}K - \frac{1}{2}C_{AB}K^2 . \tag{B.23}$$

Therefore, we observe that the peeling property is not preserved by this metric ansatz, since the terms $\psi^{2,1}$, $\tilde{\psi}^{2,1}$, $\psi_A^{1,1}$ and $\psi_{AB}^{0,1}$ spoil it. Remarkably, the components K and Θ_A, which are directly determined in terms of the fluid stress tensor components (4.1.4) and (4.1.6), are the causant. However, we consistently recover the peeling property in the flat limit, where these four components vanish.

It is noteworthy that the Weyl scalars do not provide us with an evident candidate for covariant mass aspect directly entering the charges (4.2.6), contrarily to the asymptotically flat case [149] where $\mathrm{Re}(\Psi_2)$ straightforwardly gives the right combination.

Appendix C

Analysis of Jacobi identities

C.1 Heisenberg algebra

In this appendix, we develop the analysis of the Jacobi identities and associated constraints for the ansatz (8.3.25).

The Jacobi identities $[\mathcal{J}_m, [\mathcal{J}_n, \mathcal{J}_l]]$ + cyclic permutations $= 0$ lead to the independent relations:

$$\begin{aligned}(n-l)(m-n-l)F(n,l)F(m,n+l) + (l-m)(n-l-m)F(l,m)F(n,l+m)+ \\ (m-n)(l-m-n)F(m,n)F(l,m+n)+ \\ (n-l)G(n,l)\bar{F}(m,n+l) + (l-m)G(l,m)\bar{F}(n,l+m) + (m-n)G(m,n)\bar{F}(l,m+n) = 0\ ,\end{aligned} \tag{C.1.1}$$

$$\begin{aligned}(n-l)(m-n-l)F(n,l)G(m,n+l) + (l-m)(n-l-m)F(l,m)G(n,l+m)+ \\ (m-n)(l-m-n)F(m,n)G(l,m+n)+ \\ (n-l)G(n,l)\bar{G}(m,n+l) + (l-m)G(l,m)\bar{G}(n,l+m) + (m-n)G(m,n)\bar{G}(l,m+n) = 0\ ,\end{aligned} \tag{C.1.2}$$

and

$$\begin{aligned}(m-n-l)(n-l)F(n,l)A(l+n,m) + (n-l-m)(l-m)F(l,m)A(m+l,n) \\ + (l-m-n)(m-n)F(m,n)A(m+n,l)+ \\ (n-l)G(n,l)\bar{A}(m,n+l) + (l-m)G(l,m)\bar{A}(n,l+m) + (m-n)G(m,n)\bar{A}(l,m+n) = 0 \\ + \{m(n-l)G(n,l) + n(l-m)G(l,m) + l(m-n)G(m,n)\}\delta_{m+n+l,0}\ .\end{aligned} \tag{C.1.3}$$

The Jacobi identities $[\mathcal{J}_m, [\mathcal{J}_n, \mathcal{P}_l]]$ + cyclic permutations $= 0$ yield

$$\begin{aligned}(m-n-l)\bar{F}(n,l)F(m,n+l) + \bar{G}(n,l)\bar{F}(m,n+l) - (n-l-m)\bar{F}(m,l)F(n,l+m) \\ -\bar{G}(m,l)\bar{F}(n,l+m) - (m-n)F(m,n)\bar{F}(m+n,l) + (m-n)(l-m-n)G(m,n)\tilde{F}(l,m+n) = 0\ ,\end{aligned} \tag{C.1.4}$$

$$(m-n-l)\bar{F}(n,l)G(m,n+l)+\bar{G}(n,l)\bar{G}(m,n+l)-(n-l-m)\bar{F}(m,l)G(n,l+m)$$
$$-\bar{G}(m,l)\bar{G}(n,l+m)-(m-n)F(m,n)\bar{G}(m+n,l)+(m-n)(l-m-n)G(m,n)\tilde{G}(l,m+n)=0\ , \tag{C.1.5}$$

and

$$(m\bar{G}(n,l)-n\bar{G}(m,l)-(m+n)(m-n)F(m,n))\delta_{m+n+l,0}+\bar{G}(n,l)\bar{A}(m,n+l)$$
$$-\bar{G}(m,l)\bar{A}(n,l+m)-(m-n)F(m,n)\bar{A}(m+n,l)+(m-n-l)\bar{F}(n,l)A(m,l+n)$$
$$-(n-m-l)\bar{F}(m,l)A(n,l+m)+(l-m-n)(m-n)G(m,n)\tilde{A}(l,m+n)=0\ . \tag{C.1.6}$$

The Jacobi identities $[\mathcal{P}_m,[\mathcal{P}_n,\mathcal{J}_l]]$ + cyclic permutations $=0$ yield

$$\bar{F}(l,n)\bar{F}(n+l,m)-(m-n-l)\bar{G}(l,n)\tilde{F}(m,n+l)-\bar{F}(l,m)\bar{F}(l+m,n)$$
$$+(n-l-m)\bar{G}(l,m)\tilde{F}(n,l+m)+(m-n)(l-m-n)F(l,m+n)\tilde{F}(m,n)$$
$$+(m-n)\tilde{G}(m,n)\bar{F}(l,m+n)=0\ , \tag{C.1.7}$$

$$-(m-n-l)\bar{G}(l,n)\tilde{G}(m,n+l)+\bar{F}(l,n)\bar{G}(n+l,m)+(n-l-m)\bar{G}(l,m)\tilde{G}(n,l+m)$$
$$-\bar{F}(l,m)\bar{G}(l+m,n)+(m-n)\tilde{G}(m,n)\bar{G}(l,m+n)+(m-n)(l-m-n)G(l,m+n)\tilde{F}(m,n)=0\ , \tag{C.1.8}$$

and

$$((n+l)\bar{F}(l,n)-(m+l)\bar{F}(l,m)+(l)(m-n)\tilde{G}(m,n))\delta_{m+n+l,0}+\bar{F}(l,n)\bar{A}(n+l,m)$$
$$-\bar{F}(l,m)\bar{A}(l+m,n)+(m-n)\tilde{G}(m,n)\bar{A}(l,m+n)-(m-l-n)\bar{G}(l,n)\tilde{A}(m,l+n)$$
$$+(n-l-m)\bar{G}(l,m)\tilde{A}(n,m+l)+(l-m-n)(m-n)\tilde{F}(m,n)A(l,m+n)=0\ . \tag{C.1.9}$$

Finally, the Jacobi identities $[\mathcal{P}_m,[\mathcal{P}_n,\mathcal{P}_l]]$ + cyclic permutations $=0$ lead to

$$(n-l)(m-n-l)\tilde{G}(n,l)\tilde{G}(m,n+l)+(l-m)(n-l-m)\tilde{G}(l,m)\tilde{G}(n,l+m)+$$
$$(m-n)(l-m-n)\tilde{G}(m,n)\tilde{G}(l,m+n)$$
$$-(n-l)\tilde{F}(n,l)\bar{G}(n+l,m)-(l-m)\tilde{F}(l,m)\bar{G}(l+m,n)-(m-n)\tilde{F}(m,n)\bar{G}(m+n,l)=0\ , \tag{C.1.10}$$

$$(n-l)(m-n-l)\tilde{G}(n,l)\tilde{F}(m,n+l)+(l-m)(n-l-m)\tilde{G}(l,m)\tilde{F}(n,l+m)+$$
$$(m-n)(l-m-n)\tilde{G}(m,n)\tilde{F}(l,m+n)$$
$$-(n-l)\tilde{F}(n,l)\bar{F}(n+l,m)-(l-m)\tilde{F}(l,m)\bar{F}(l+m,n)-(m-n)\tilde{F}(m,n)\bar{F}(m+n,l)=0\ , \tag{C.1.11}$$

and

$$\begin{aligned}
&\{-(n+l)(n-l)\tilde{F}(n,l)-(l+m)(l-m)\tilde{F}(l,m)-(m+n)(m-n)\tilde{F}(m,n)\}\delta_{m+n+l,0} \\
&\quad +(n-l)(m-n-l)\tilde{G}(n,l)\tilde{A}(m,n+l)+(l-m)(n-m-l)\tilde{G}(l,m)\tilde{A}(n,m+l) \\
&\qquad +(m-n)(l-m-n)\tilde{G}(m,n)\tilde{A}(l,m+n) \\
&-(n-l)\tilde{F}(n,l)\bar{A}(n+l,m)-(l-m)\tilde{F}(l,m)\bar{A}(l+m,n)-(m-n)\tilde{F}(m,n)\bar{A}(m+n,l)=0\ .
\end{aligned} \tag{C.1.12}$$

First of all, we consider infinitesimal deformations such that we just study the relations which include first order of functions. These relations are given by

$$(m(n-l)G(n,l)+n(l-m)G(l,m)+l(m-n)G(m,n))\delta_{m+n+l,0}=0\ , \tag{C.1.13}$$

$$(m\bar{G}(n,l)-n\bar{G}(m,l)-(m+n)(m-n)F(m,n))\delta_{m+n+l,0}=0\ , \tag{C.1.14}$$

$$((n+l)\bar{F}(l,n)-(m+l)\bar{F}(l,m)+(l)(m-n)\tilde{G}(m,n))\delta_{m+n+l,0}=0\ , \tag{C.1.15}$$

$$(-(n+l)(n-l)\tilde{F}(n,l)-(l+m)(l-m)\tilde{F}(l,m)-(m+n)(m-n)\tilde{F}(m,n))\delta_{m+n+l,0}=0\ . \tag{C.1.16}$$

As discussed in the section 8.3.1, the first and last relations have as solutions $G(m,n) = constant$ and $\tilde{F}(m,n) = constant$. The second relation is solved by $F(m,n) = constant = \beta$ and $\bar{G}(m,n) = \alpha m - \beta n$ [1]. The same argument is true for $\tilde{G}(m,n)$ and $\bar{F}(m,n)$ in the third relation. So we can recognize four independent infinitesimal deformations by $G(m,n)$, $\{F(m,n), \bar{G}(m,n)\}$, $\{\bar{F}(m,n), \tilde{G}(m,n)\}$ and $\tilde{F}(m,n)$ with the mentioned solutions. We should also consider the possible combinations of these deformations.

C.2 $\mathfrak{witt} \oplus \mathfrak{H}_3$ algebra

In this appendix, we develop the analysis of the Jacobi identities and associated constraints for the ansatz (8.4.7).

First of all, we consider infinitesimal deformations such that we only keep linear order in the functions. Besides, we can use the fact that $\mathfrak{witt}$ is a rigid subalgebra [266], which allows us to set $A_1(m,n) = 0$ in (8.4.7). The Jacobi identities $[\mathcal{L}_m, [\mathcal{L}_n, \mathcal{L}_l]]$ + cyclic permutations = 0 give us three linear relations

$$(l-m)D_1(l+m,n)+(-l+n)D_1(l+n,m)+(m-n)D_1(m+n,l)=0\ , \tag{C.2.1}$$

$$[(l-m)C_1(l+m,n)+(-l+n)C_1(l+n,m)+(m-n)C_1(m+n,l)]\mathcal{P}_{l+m+n}=0\ , \tag{C.2.2}$$

[1] Another solution given by $\bar{G}(m,n) = \beta m^k \delta_{m+n,0}$ and $F(m,n) = 0$ is possible and will also be discussed.

and

$$[(l-m)B_1(l+m,n)+(-l+n)B_1(l+n,m)+(m-n)B_1(m+n,l)]\mathcal{J}_{l+m+n} = 0 \ . \tag{C.2.3}$$

These constraints lead to the following possibilities $D_1(m,n)=(m-n)\tilde{D}_1(m+n)$, $C_1(m,n) = (m-n)\tilde{C}_1(m+n)$ and $B_1(m,n) = (m-n)\tilde{B}_1(m+n)$. The central extension analysis of the Virasoro algebra showed that the solution of (C.2.1) is $D_1(m,n)=\sigma(m^3-m)\delta_{m+n,0}$.

Next, the Jacobi identities $[\mathcal{L}_m,[\mathcal{L}_n,\mathcal{J}_l]]$ + cyclic permutations = 0 yield

$$\begin{aligned}[-(l+m-n)A_2(m,l)+(l-m+n)A_2(n,l)+(m-n)A_2(m+n,l)]\mathcal{L}_{l+m+n}\\ +(m-n)B_2(m+n,l)\mathcal{J}_{l+m+n}+(m-n)C_2(m+n,l)\mathcal{P}_{l+m+n}\\ -lC_1(m,n)\delta_{l+m+n,0}+(m-n)D_2(m+n,l)=0 \ ,\end{aligned} \tag{C.2.4}$$

which leads to $B_2(m,n)=C_2(m,n)=0$ and $A_2=\alpha(m-n)$. In addition, we find that

$$-lC_1(m,n)\delta_{l+m+n,0}+(m-n)D_2(m+n,l)=0, \tag{C.2.5}$$

which is equivalent to $D_2(m+n,l)\overset{!}{=} l\tilde{C}_1(m+n)\delta_{l+m+n,0}$.

The Jacobi identities $[\mathcal{L}_m,[\mathcal{L}_n,\mathcal{P}_l]]$ + cyclic permutations = 0 give

$$\begin{aligned}[-(l+m-n)A_3(m,l)+(l-m+n)A_3(n,l)+(m-n)A_3(m+n,l)]\mathcal{L}_{l+m+n}\\ +(m-n)B_3(m+n,l)\mathcal{J}_{l+m+n}+(m-n)C_3(m+n,l)\mathcal{P}_{l+m+n}\\ -lB_1(m,n)\delta_{l+m+n,0}+(m-n)D_3(m+n,l)=0 \ ,\end{aligned} \tag{C.2.6}$$

which implies $B_3(m,n)=C_3(m,n)=0$ and $A_3=\beta(m-n)$. In addition, we find that

$$-lB_1(m,n)\delta_{l+m+n,0}+(m-n)D_3(m+n,l)=0, \tag{C.2.7}$$

which is equivalent to $D_3(m+n,l)\overset{!}{=} l\tilde{B}_1(m+n)\delta_{l+m+n,0}$.

Similarly, the Jacobi identities $[\mathcal{L}_m,[\mathcal{P}_n,\mathcal{P}_l]]$ + cyclic permutations = 0 and $[\mathcal{L}_m,[\mathcal{J}_n,\mathcal{J}_l]]$+cyclic permutations = 0 force $A_4(m,n)=A_5(m,n)=0$. Furthermore, the Jacobi identities $[\mathcal{L}_m,[\mathcal{J}_n,\mathcal{P}_l]]$ + cyclic permutations = 0 cause $A_6(m,n)=0$. It follows that the Jacobi identities for the $\mathfrak{H}_3$ sector decouple from $\mathcal{L}$ and are exactly the same as in section 8.3.

Now, we proceed to explore whether the new infinitesimal deformations we have found are indeed formal or not. For this, we explore the non-linear relations coming from the Jacobi identities.

The Jacobi identities $[\mathcal{L}_m,[\mathcal{L}_n,\mathcal{L}_l]]$ + cyclic permutations = 0 give us

$$\begin{aligned}[(m-n-l)(n-l)(\beta\tilde{C}_1(n,l)+\alpha\tilde{B}_1(n,l))+(n-m-l)(l-m)(\beta\tilde{C}_1(l,m)+\alpha\tilde{B}_1(l,m))\\ +(l-m-n)(m-n)(\beta\tilde{C}_1(m,n)+\alpha\tilde{B}_1(m,n))]\mathcal{L}_{m+n+l}=0 \ ,\end{aligned} \tag{C.2.8}$$

and

$$(n-l)\tilde{C}_1(n,l)D_3(m,l+n)+(l-m)\tilde{C}_1(l,m)D_3(n,l+m)+(m-n)\tilde{C}_1(m,n)D_3(l,m+n) \\ +(n-l)\tilde{B}_1(n,l)D_2(m,l+n)+(l-m)\tilde{B}_1(l,m)D_2(n,l+m) \\ +(m-n)\tilde{B}_1(m,n)D_2(l,m+n)=0\ . \quad \text{(C.2.9)}$$

Next, the Jacobi identities $[\mathcal{L}_m,[\mathcal{L}_n,\mathcal{J}_l]]$ + cyclic permutations = 0 yield

$$[\alpha(l-m)(l+m-n)\tilde{B}_1(l+m,n)-\alpha(l-n)(l-m+n)\tilde{B}_1(l+n,m)+ \\ +(m-n)((-l+m+n)\tilde{B}_1(m,n)F(m+n,l)-\tilde{C}_1(m,n)\bar{F}(l,m+n))]\mathcal{J}_{l+m+n}=0\ , \quad \text{(C.2.10)}$$

$$[\alpha(l-m)(l+m-n)\tilde{C}_1(l+m,n)-\alpha(l-n)(l-m+n)\tilde{C}_1(l+n,m) \\ +(m-n)((-l+m+n)\tilde{B}_1(m,n)G(m+n,l)-\tilde{C}_1(m,n)\bar{G}(l,m+n))]\mathcal{P}_{l+m+n}=0\ , \quad \text{(C.2.11)}$$

and

$$(m-n)(-l+m+n)A(m+n,l)\tilde{B}_1(m,n)+(-m+n)\bar{A}(l,m+n)\tilde{C}_1(m,n)+ \\ +\alpha[(l-m)(l+m-n)\tilde{D}_1(l+m,n)-(l-n)(l+n-m)\tilde{D}_1(l+n,m)]=0\ . \quad \text{(C.2.12)}$$

The Jacobi identities $[\mathcal{L}_m,[\mathcal{L}_n,\mathcal{P}_l]]$ + cyclic permutations = 0 give

$$[\beta(l-m)(l+m-n)\tilde{B}_1(l+m,n)-\beta(l-n)(l-m+n)\tilde{B}_1(l+n,m)+ \\ +(m-n)((-l+m+n)\tilde{C}_1(m,n)\tilde{F}(m+n,l)+\tilde{B}_1(m,n)\bar{F}(l,m+n))]\mathcal{J}_{l+m+n}=0\ , \quad \text{(C.2.13)}$$

$$[\beta(l-m)(l+m-n)\tilde{C}_1(l+m,n)-\beta(l-n)(l-m+n)\tilde{C}_1(l+n,m) \\ +(m-n)((-l+m+n)\tilde{C}_1(m,n)\tilde{G}(m+n,l)+\tilde{B}_1(m,n)\bar{G}(l,m+n))]\mathcal{P}_{l+m+n}=0\ , \quad \text{(C.2.14)}$$

and

$$(m-n)\bar{A}(m+n,l)\tilde{B}_1(m,n)+(m-n)(-l+m+n)\tilde{A}(m+n,l)\tilde{C}_1(m,n)+ \\ +\beta[(l-m)(l+m-n)\tilde{D}_1(l+m,n)-(l-n)(l+n-m)\tilde{D}_1(l+n,m)]=0\ . \quad \text{(C.2.15)}$$

The Jacobi identities $[\mathcal{L}_m,[\mathcal{J}_n,\mathcal{J}_l]]$ + cyclic permutations = 0 yield

$$\alpha(l-m)D_2(l+m,n)+\alpha(m-n)D_2(m+n,l) \\ +(l-n)[D_2(m,l+n)F(n,l)+D_3(m,l+n)G(n,l)]=0\ , \quad \text{(C.2.16)}$$

and

$$(l-n)(l-m+n)[\alpha^2-\alpha F(n,l)-\beta G(n,l)]\mathcal{L}_{l+m+n}=0\ . \quad \text{(C.2.17)}$$

The Jacobi identities $[\mathcal{L}_m,[\mathcal{P}_n,\mathcal{P}_l]]$ + cyclic permutations = 0 give rise to

$$\beta(l-m)D_3(l+m,n)+\beta(m-n)D_3(m+n,l) \\ +(l-n)[D_2(m,l+n)\tilde{F}(n,l)+D_3(m,l+n)\tilde{G}(n,l)]=0\ , \quad \text{(C.2.18)}$$

and

$$(l-n)(l-m+n)[\beta^2-\beta\tilde{G}(n,l)-\alpha\tilde{F}(n,l)]\mathcal{L}_{l+m+n}=0\ . \quad \text{(C.2.19)}$$

Finally, the Jacobi identities $[\mathcal{L}_m,[\mathcal{J}_n,\mathcal{P}_l]]$ + cyclic permutations = 0 lead to

$$\bar{F}(n,l)D_2(m,l+n)+\bar{G}(n,l)D_3(m,l+n) \\ +\alpha(n-m)D_3(m+n,l)+\beta(m-l)D_2(l+m,n)=0\ , \quad \text{(C.2.20)}$$

and

$$(l-m+n)[\alpha\beta(l-n)+\beta\bar{G}(n,l)+\alpha\bar{F}(n,l)]\mathcal{L}_{l+m+n}=0\ . \quad \text{(C.2.21)}$$

Appendix D

Heisenberg from contractions

In this appendix, we complement our analysis of deformations in chapter 8 with a discussion on the inverse procedure, the so-called contractions.

D.1 General contractions of $\mathfrak{vir} \oplus \mathfrak{vir}$

Here, we investigate the most general contractions of two Virasoro algebras

$$\begin{aligned}
&[\mathcal{L}_m, \mathcal{L}_n] = \frac{1}{\varrho}(m-n)\mathcal{L}_{m+n} + \frac{c}{12\varrho^2}(m^3 - \alpha m)\delta_{n+m,0} \ , \\
&[\mathcal{L}_m, \bar{\mathcal{L}}_n] = 0 \ , \\
&[\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n] = \frac{1}{\vartheta}(m-n)\bar{\mathcal{L}}_{m+n} + \frac{\bar{c}}{12\vartheta^2}(m^3 - \beta m)\delta_{n+m,0} \ ,
\end{aligned} \tag{D.1.1}$$

where ϱ and ϑ are contraction parameters. The various limits of these parameters lead to different algebras which are classified as follows:

1. $\vartheta \to \infty$ which leads to take the direct sum of one Virasoro with an Abelian ideal algebra

$$\begin{aligned}
&[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + \frac{c}{12}(m^3 - \alpha m)\delta_{n+m,0} \ , \\
&[\mathcal{L}_m, \bar{\mathcal{L}}_n] = 0 \ , \\
&[\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n] = 0 \ .
\end{aligned} \tag{D.1.2}$$

2. The next case is when we take the limit $\vartheta \to \infty$ while keeping $\frac{\bar{c}\beta}{12\vartheta^2}$ to be finite. This leads to the direct sum of a Virasoro algebra with a current algebra

$$\begin{aligned}
&[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + \frac{c}{12\varrho}(m^3 - \alpha m)\delta_{n+m,0} \ , \\
&[\mathcal{L}_m, \bar{\mathcal{L}}_n] = 0 \ , \\
&[\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n] = \frac{\bar{c}\beta}{12\vartheta} m\delta_{n+m,0} \ ,
\end{aligned} \tag{D.1.3}$$

Let us point out that one can also consider the algebra (D.1.1) without linear central term and get the same result. To this end, one starts with the Virasoro algebra

$$[\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n] = (m-n)\bar{\mathcal{L}}_{m+n} + \frac{\bar{c}\beta}{12\vartheta} m^3 \delta_{n+m,0} \ , \tag{D.1.4}$$

and uses the redefinition

$$\mathcal{J}_m := \frac{1}{m\varepsilon}\bar{\mathcal{L}}_m, \quad \varepsilon \to \infty, \qquad \frac{c}{12\varepsilon^2} = \hbar = \text{fixed} \ . \tag{D.1.5}$$

It is then immediate to see that in the limit $\varepsilon \to \infty$ and keeping $\mathcal{J}_n$, with $\hbar$ fixed, we obtain

$$[\mathcal{J}_m, \mathcal{J}_n] = \hbar \, n\delta_{m+n,0} \ . \tag{D.1.6}$$

3. The limit $\vartheta \to \infty$ and $\varrho \to \infty$ takes us to the direct sum of two current algebras

$$\begin{aligned}
[\mathcal{L}_m, \mathcal{L}_n] &= \hbar\, m\, \delta_{n+m,0} \ , \\
[\mathcal{L}_m, \bar{\mathcal{L}}_n] &= 0 \ , \\
[\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n] &= \bar{\hbar}\, m\, \delta_{n+m,0} \ ,
\end{aligned} \tag{D.1.7}$$

where two coefficients $\hbar = \frac{c\alpha}{12\varrho^2}$ and $\bar{\hbar} = \frac{\bar{c}\beta}{12\vartheta^2}$ are kept to be finite. This algebra is isomorphic to (8.1.10) (by a complex redefinition), which can be also obtained as another contraction of two Virasoro algebras in a different basis (see Appendix D.2).

4. The limit $\vartheta \to \infty$ and $\varrho \to \infty$ brings us to the direct sum of a current algebra and an Abelian ideal algebra

$$\begin{aligned}
[\mathcal{L}_m, \mathcal{L}_n] &= \hbar\, m\, \delta_{n+m,0} \ , \\
[\mathcal{L}_m, \bar{\mathcal{L}}_n] &= 0 \ , \\
[\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n] &= 0 \ ,
\end{aligned} \tag{D.1.8}$$

where the coefficient $\hbar = \frac{c\alpha}{12\varrho^2}$ is kept to be finite.

5. The limit $\vartheta \to \infty$ and $\varrho \to \infty$ yields a trivial contraction in which all the generators commute with each other.

Next, we explore contractions of two Virasoro algebras in a new basis:

$$\begin{aligned}
[\mathcal{J}_m, \mathcal{J}_n] &= \frac{1}{\varepsilon}(m-n)\mathcal{J}_{m+n} + \frac{c_{JJ}}{12\varepsilon^2}(m^3 - \alpha m)\delta_{n+m,0} \ , \\
[\mathcal{J}_m, \mathcal{P}_n] &= \frac{1}{\varepsilon}(m-n)\mathcal{P}_{m+n} + \frac{c_{JP}}{12\varepsilon\varsigma}(m^3 - \bar{\alpha} m)\delta_{n+m,0} \ , \\
[\mathcal{P}_m, \mathcal{P}_n] &= \frac{\varepsilon}{\varsigma^2}(m-n)\mathcal{J}_{m+n} + \frac{c_{JJ}}{12\varsigma^2}(m^3 - \alpha m)\delta_{n+m,0} \ ,
\end{aligned} \tag{D.1.9}$$

where ε and ς are new contractions parameters. Also, one easily checks that $c_{JJ} = c - \bar{c}$ and $c_{JP} = c + \bar{c}$. One can obtain (D.1.9) from (D.1.1) by using the redefinition of generators

$$\mathcal{L}_m = \frac{1}{2}(\mathcal{J}_m + \mathcal{P}_m). \quad \bar{\mathcal{L}}_{-m} = \frac{1}{2}(\mathcal{P}_m - \mathcal{J}_m) \ . \tag{D.1.10}$$

Now we consider different limits of contraction parameters ε and ς:

1. $\varsigma \to \infty$ while keeping ε to be finite. For this case, one can obtain the algebra

$$\begin{aligned}
&[\mathcal{J}_m, \mathcal{J}_n] = (m-n)\mathcal{J}_{m+n} + \frac{c_{JJ}}{12}(m^3 - \alpha m)\delta_{n+m,0} \ ,\\
&[\mathcal{J}_m, \mathcal{P}_n] = (m-n)\mathcal{P}_{m+n} + \frac{\bar{c}_{JP}}{12}(m^3 - \bar{\alpha} m)\delta_{n+m,0} \ ,\\
&[\mathcal{P}_m, \mathcal{P}_n] = 0 \ ,
\end{aligned} \tag{D.1.11}$$

   which is nothing but the central extension of the $\mathfrak{bms}_3$ algebra. Notice that we kept $\frac{\bar{c}_{JP}}{12} = \frac{c_{JP}}{12\varsigma}$ to be also finite.

2. $\varepsilon \to \infty$ as well as $\varsigma^2 \to \infty$ while keeping $\frac{\varepsilon}{\varsigma^2} = \nu$ to be finite. The final algebra takes the form

$$\begin{aligned}
&[\mathcal{J}_m, \mathcal{J}_n] = 0 \ ,\\
&[\mathcal{J}_m, \mathcal{P}_n] = \hbar m \delta_{n+m,0} \ ,\\
&[\mathcal{P}_m, \mathcal{P}_n] = \nu(m-n)\mathcal{J}_{m+n} \ ,
\end{aligned} \tag{D.1.12}$$

   in which we assumed that $\hbar = \frac{\bar{\alpha} c_{JP}}{12\varepsilon\varsigma}$ is finite. The algebra (D.1.12), which is obtained by contraction of two copies of Virasoro, is isomorphic to the algebras (8.3.8) and (8.3.24).

3. $\varepsilon \to \infty$ as well as $\varsigma^2 \to \infty$ while keeping $\hbar = \frac{\bar{\alpha} c_{JP}}{12\varepsilon\varsigma}$ to be finite. The final algebra takes the form

$$\begin{aligned}
&[\mathcal{J}_m, \mathcal{J}_n] = 0 \ ,\\
&[\mathcal{J}_m, \mathcal{P}_n] = \hbar\, m\, \delta_{n+m,0} \ ,\\
&[\mathcal{P}_m, \mathcal{P}_n] = 0 \ ,
\end{aligned} \tag{D.1.13}$$

   which is exactly the same as the Heisenberg algebra (8.1.10), whose deformations we considered in this work.

   - There are two other options which can be obtained as contraction of two Virasoro algebras in this basis:

$$\begin{aligned}
&[\mathcal{J}_m, \mathcal{J}_n] = \bar{\hbar}\, m\, \delta_{m+n,0} \ ,\\
&[\mathcal{J}_m, \mathcal{P}_n] = \hbar\, m\, \delta_{n+m,0} \ ,\\
&[\mathcal{P}_m, \mathcal{P}_n] = 0 \ ,
\end{aligned} \tag{D.1.14}$$

where $\bar{\hbar} = \frac{c_{JJ}\alpha}{12\varepsilon^2}$, and

$$\begin{aligned}
[\mathcal{J}_m, \mathcal{J}_n] &= \bar{\hbar}\, m\, \delta_{m+n,0} \ , \\
[\mathcal{J}_m, \mathcal{P}_n] &= \hbar\, m\, \delta_{n+m,0} \ , \\
[\mathcal{P}_m, \mathcal{P}_n] &= \tilde{\hbar}\, m\, \delta_{m+n,0} \ ,
\end{aligned} \tag{D.1.15}$$

where $\tilde{\hbar} = \frac{c_{JJ}\alpha}{12\varsigma^2}$. Then, it can be shown that, by an appropriate redefinition of the generators, both of them are equivalent to (D.1.13), as long as $\tilde{\hbar}$ is related to $\bar{\hbar}$.

D.2 $\mathfrak{H}_3$ from $\mathfrak{vir} \oplus \mathfrak{vir}$ via contraction

The contraction above constitutes the inverse procedure of the twisted Sugawara construction, the "twisted Sugawara contraction". To see it, let us start with

$$\mathcal{L}_n = \frac{1}{2\hbar} \sum_m \mathcal{J}_{n-m} \mathcal{J}_m + \varepsilon n \mathcal{J}_n \ . \tag{D.2.1}$$

Clearly, the $\mathcal{L}_n$ satisfy the Virasoro algebra with central charge $\frac{c}{12} = \varepsilon^2 \hbar$. In the limit $\varepsilon \to \infty$, $\hbar =$ fixed, (D.2.1) clearly shows how $\mathcal{L}_n$ can be reduced to $\mathcal{J}_n$ which satisfies (D.1.6).

This current algebra (D.1.6) is closely related to the Heisenberg algebra. Let

$$X_n := \frac{1}{\sqrt{2n}} (\mathcal{J}_n + i \mathcal{J}_{-n}) \ , \qquad P_n := \frac{i}{\sqrt{2n}} (\mathcal{J}_{-n} + i \mathcal{J}_n) \ , \quad n > 0 \ . \tag{D.2.2}$$

It is then immediate to realize that $[X_n, P_m] = i\hbar \delta_{m,n}$.

Similarly, starting with the direct sum of two current algebras obtained from the "twisted Sugawara contraction" (D.2.1) of two copies of the Virasoro algebra

$$\begin{aligned}
[\mathcal{J}^+_m, \mathcal{J}^+_n] &= \hbar\ n \delta_{m+n,0} \ , \\
[\mathcal{J}^-_m, \mathcal{J}^-_n] &= \hbar\ n \delta_{m+n,0} \ , \\
[\mathcal{J}^+_m, \mathcal{J}^-_n] &= 0 \ ,
\end{aligned} \tag{D.2.3}$$

one may construct

$$\begin{aligned}
X_n &:= \begin{cases} \frac{1}{\sqrt{2n}} (\mathcal{J}^+_n + i \mathcal{J}^+_{-n}) & n > 0 \\ \frac{1}{\sqrt{-2n}} (\mathcal{J}^-_n + i \mathcal{J}^-_{-n}) & n < 0 \ , \end{cases} \\
P_n &:= \begin{cases} \frac{i}{\sqrt{2n}} (\mathcal{J}^+_{-n} + i \mathcal{J}^+_n) & n > 0 \\ \frac{i}{\sqrt{-2n}} (\mathcal{J}^-_n + i \mathcal{J}^-_{-n}) & n < 0 \ , \end{cases}
\end{aligned} \tag{D.2.4}$$

such that $[X_n, P_m] = i\hbar \delta_{m,n}$. We note, however, that $\mathcal{J}^\pm_0$, which commute with all the other generators, do not enter the Heisenberg algebra of X_n, P_m and also that X_0, P_0 are not defined. One may add the latter to the algebra by hand.

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