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The average distance in Sierpiński triangle graphs and some remarks on the Linear Tower of Hanoi

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Abstract

In this thesis, several open questions related to Sierpiński graphs [12] will be addressed.

Chapter 1 describes the Linear Tower of Hanoi problem and summarises its history and its connection to Hanoi graphs which in turn are closely related to Sierpiński graphs. It turns out that this problem is an especially demanding version of the general Tower of Hanoi, because the Dudeney-Stockmeyer conjecture [13] cannot be applied. A common structure of optimal solutions for the 4-peg Tower of Hanoi, observed for small numbers of discs, does not hold indefinitely. Another conjecture regarding cases with more pegs than discs will be disproved by the construction of a counterexample. Some closing words in Chapter 1 address the importance of the Linear Tower of Hanoi problem, as the Dudeney-Stockmeyer conjecture makes use of optimal solutions of the Linear Tower of Hanoi in some cases.

In Chapter 2, Sierpiński triangle graphs of general base $p \geq 2$ and general exponent $n \geq 0$ (cf. [17]) are analysed and by distinguishing different types of pairs of subgraphs and describing how they relate to each other, a recursive formula for the Wiener index and therefore for the average distance is developed. Several examples are given, showing that this formula leads to the expected results. Some limits for $n \to \infty$ are calculated. It is demonstrated how a closed form solution can be generated from this formula by fixing either parameter. A closed form solution with both parameters variable is incredibly complicated and therefore omitted. However, an expression for the limit of $n \to \infty$ for arbitrary p is found. This expression is the same as in the case of Sierpiński graphs proper, giving strong evidence that these two types of graphs share the same asymptotic behaviour.

Zusammenfassung

Die vorliegende Arbeit behandelt verschiedene offene Fragen, die mit Sierpińskigraphen [12] im Zusammenhang stehen.

Kapitel 1 beschreibt die Problematik des Linearen Turms von Hanoi und fasst ihre Geschichte zusammen. Dabei wird auf den Bezug zu Hanoigraphen und deren Verbindung zu Sierpińskigraphen eingegangen. Es zeigt sich, dass der Lineare Turm von Hanoi eine außergewöhnlich herausfordernde Variante des allgemeinen Turms von Hanoi darstellt, da die Dudeney-Stockmeyer-Vermutung [13] nicht angewendet werden kann. Eine algorithmische Struktur, die viele optimale Lösungen für den Linearen Turm von Hanoi mit 4 Stäben und wenigen Scheiben hervorbringt, ist nicht im Allgemeinen optimal. Eine weitere Vermutung zu Fällen mit mehr Stäben als Scheiben wird durch Angabe eines Gegenbeispiels widerlegt. Am Ende von Kapitel 1 wird die Wichtigkeit der Problematik des Linearen Turms von Hanoi aufgezeigt, da die Dudeney-Stockmeyer-Vermutung in einigen Fällen die optimalen Lösungen des Linearen Turms von Hanoi verwendet.

In Kapitel 2 werden Sierpińskidreiecksgraphen mit verallgemeinerter Basis $p \geq 2$ und einem Exponenten $n \geq 0$ (vgl. [17]) analysiert und eine rekursive Formel für den Wiener-Index und damit den durchschnittlichen Abstand dieser Graphen entwickelt, indem Paare von Subgraphen in verschiedene Fälle eingeteilt und ihre Beziehungen zueinander beschrieben werden. Die Formel wird an mehreren Beispielen erfolgreich überprüft. Einige Grenzwerte für $n \to \infty$ werden berechnet. Es wird aufgezeigt, wie durch Fixierung eines Parameters eine geschlossene Form aus der rekursiven Formel abgeleitet werden kann. Eine geschlossene Form mit beiden freien Parametern ist unfassbar kompliziert und wird daher nicht angegeben. Jedoch wird ein Ausdruck für den Grenzwert für $n \to \infty$ und beliebiges p gefunden. Dieser Ausdruck gleicht dem entsprechenden Wert für eigentliche Sierpińskigraphen. Dies ist ein starker Hinweis darauf, dass diese beiden Arten von Graphen das gleiche Grenzwertverhalten aufweisen.

Erklärung an Eides statt
Statutory Declaration

Hiermit versichere ich, Simon Rolke, an Eides statt, die vorliegende Arbeit selbständig erarbeitet und keine anderen als nur die angegebenen Hilfsmittel verwendet zu haben. Grafiken und Formulierungen aus anderen Arbeiten habe ich als Zitate kenntlich gemacht. Diese Arbeit lag zuvor noch keinem anderen Prüfungsamt vor.

München, den 30. November 2021

I, Simon Rolke, hereby declare that I have authored this thesis independently and have not used other than the declared sources or resources. I have explicitly marked all material which has been quoted from the used sources. This thesis has not been submitted to any other examination authority before.

Munich, 30th of November 2021

Simon Rolke

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Introduction

This thesis will consist of two chapters, concerning topics that might seem unrelated at first. In the first chapter, the Linear Tower of Hanoi problem will be analysed and its difficulties explained. The second chapter will be about metric properties of Sierpiński triangle graphs. While at first sight these fields of research seem to be quite far apart, there is a strong connection linking them together. The Linear Tower of Hanoi with an arbitrary number of pegs leads to a type of graphs whose vertices represent the configurations of discs on the pegs. These graphs, the Linear Hanoi graphs ($H^n_{p,\mathrm{lin}}$), are similar in structure to proper Hanoi graphs (H^n_p), only with some edges removed. The Hanoi graphs representing the Tower of Hanoi game with three pegs are isomorphic to a family of graphs known as Sierpiński graphs (denoted by S^n_p) with $p = 3$, i.e. $H^n_3 \cong S^n_3$, and for odd p , the Sierpiński graphs S^n_p can always be embedded in the Hanoi graphs H^n_p. The Sierpiński graphs in turn have countless connections to the Sierpiński triangle graphs ($\widehat{S}^n_p$). For example, at the end of Chapter 2, it will be shown that the normalised average distance for $n \to \infty$ is the same for Sierpiński graphs and Sierpiński triangle graphs. As the name would suggest, the Sierpiński triangle graphs are the graph representation of the Sierpiński triangle fractal, and therefore the average geodesic distance in the Sierpiński triangle with sides of length 1 equals the normalised average distance of the $\widehat{S}^n_3$.

The two parts of this thesis do not directly border each other. Rather, they are close to the two ends of a connection spanning across a vast complex of topics within discrete mathematics. They also demonstrate that, when it comes to mathematics and abstract thinking, connections are to be found in rather unexpected places.

1 The Linear Tower of Hanoi

As the legend goes, in the Temple of Benares on the river Ganges, beneath a dome that marks the middle of the world, the monks of the Hindu god of creation, Brahma, work hard to solve a sacred puzzle. They move a set of sixty-four golden discs that are placed on three needles made of diamond, obeying the rules passed down to them by their god. In the beginning, all discs were on the same needle, and they will be again in the end, albeit on a different needle, as this is what they strive to achieve and when they succeed, the temple, the dome and all the world surrounding it will collapse and dissipate into oblivion (cf. Hinz, Klavžar, and Petr [11], pp. 1–2, quoting Claus).

Since Lucas (1842–1891), under the name of Claus, first published the Tower of Hanoi game in 1883, a lot of work has been done on its mathematical analysis and many problems related to it. Adaptions and variations of the original puzzle began to arise soon after. The basic idea however has been unchanged throughout. There is a number of pegs, and a set of discs strictly decreasing in diameter, all with a hole in the middle so they can be stacked on the pegs. Almost all variants of the game maintain the rule that a disc may never be placed atop a smaller one. All discs on the same peg form what is called the tower, likely for its resemblance of an Asian pagoda. (Therefore, the often used plural *Towers of Hanoi* is misleading, as there is only one tower.) The goal of the standard game is to transfer the tower from one peg, called *start peg*, to another one, called *destination peg*, by only moving one disc at a time, transferring the uppermost disc on one peg to the top of another one. The puzzle was originally posted with three pegs, and it could be seen quickly that the solution turns out to be fairly easy. The expansion to four pegs on the other hand has become famous for the difficulty in finding minimal solutions. The optimality of some solutions generated by algorithms was not proved up until 2014, when the verification was done by Bousch [4]. The same problem with an arbitrary number of pegs higher than three was found to be even more complicated. Though a certain algorithm was conjectured to be optimal since 1941, called the *Frame–Stewart conjecture* (cf. Frame [7] and Stewart [21]), it is not yet proven to be.

Among the many variations of the classical Tower of Hanoi problem, the *Linear Tower of Hanoi* and the questions relating to it seem to be especially complex. This problem was found among others in a paper by STOCKMEYER in 1994 [22]. Some research on different factors of this game was done since then, for example by EMERT, NELSON, and OWENS in 2007 [6] as well as by BEREND, SAPIR, and SOLOMON in 2012 [3]. A summary of known information about the topic is to be found in a yet unpublished paper by HINZ and PETR [15].

The difference from the basic puzzle is that the pegs are arranged in a row and a disc may only be transferred from the peg it is on to a peg directly adjacent to it. Again, the objective of the game is to transfer the tower from one peg to another, especially from one of the two outside pegs to the other one. While the case with three pegs is easily solvable, the difficulty of the problem with more than three pegs seems to be even higher than for the regular puzzle. After the formal introduction to the problem, the known concepts about it will be explained. It will then be shown that even for seemingly simple special cases, there are unexpected results.

1.1 The Dudeney–Stockmeyer conjecture

Different variations of the Tower of Hanoi have been addressed in literature since the original problem was published. One way to generalise these problems is to look at a so called *move graph* D, a connected graph with p vertices. One can think of these vertices as the pegs of a Tower of Hanoi setup. In addition to the *divine rule* that at no point in time a disc may be placed upon a smaller one, it is only allowed for a disc to be moved from one peg to another, if there is an edge between the corresponding vertices in the move graph D. For a given number of n discs, a *state graph* H_D^n can then be constructed. It consists of the legal configurations of the n discs as vertices and the legal moves between these configurations as edges. This is a great generalisation, albeit it does not embrace all Tower of Hanoi puzzles. First, one assumes every move to be reversible, limiting the move graphs to undirected graphs. This is maybe the biggest limitation imposed, considering that the problems relating to the circular directed move graph are quite demanding and central aspects of them remain yet unsolved (cf. STOCKMEYER [22], pp. 6–7). Second, special cases like the colour division variant of the Tower of Hanoi are neglected, though this has not yet gotten any reasonable amount of scientific interest.

As a side note, the state graph of the original problem posed by LUCAS is denoted by H_3^n, though it could be denoted following the notation above as $H_{K_3}^n$, with K_3 being the complete graph of order 3. This state graph is isomorphic to the Sierpiński graph S_3^n (cf. Hinz, Klavžar, and Petr [11, Proposition 5.42], p. 258). A generalised version of this graph are the so-called Hanoi graphs $H_p^n = H_{K_p}^n$. The aforementioned connection to the Sierpiński graphs does not work as well for $p \geq 4$, though for every odd p, the Sierpiński graph S_p^n can be isomorphically embedded in the Hanoi graph H_p^n (cf. Hinz, Klavžar, and Petr [11, Proposition 5.46], p. 260). Hanoi graphs with $p \geq 4$ are exceedingly complicated, even more so than the Sierpiński triangle graphs, to which the Sierpiński graphs in turn connect in many ways, one of which will be described in Chapter 2.

The huge advantage of the move graph generalisation mentioned above is, however, that one can conjecture that a certain type of solutions leads to optimal results in almost every case, where computational methods verify the expected results. This strategy has arisen from a transportation of the *Frame–Stewart algorithm* for the case of the classical K_4 Tower of Hanoi to different move graphs. The resulting strategy, which is conjectured to be optimal, is the *Dudeney–Stockmeyer strategy.*

Conjecture 1 (Dudeney–Stockmeyer). *Let D be a move graph with $p \geq 4$ vertices, corresponding to pegs, $n \geq 2$ an arbitrary number of discs. The problem is to move the tower of discs from state i^n, with all discs on peg i, to state j^n. Save for the move graph being the path graph and the task to move the tower from one outer peg to the other, the optimal solution strategy is the following.*

1. *Choose a path of at least length 2 containing both i and j, but not a certain peg $k \neq i, j$.*
2. *Move the smallest m discs from peg i to peg k; all pegs can be used to do so.*
3. *Move the remaining $n - m$ discs from peg i to the destination peg j, avoiding the now unavailable peg k.*
4. *Move the discs on peg k to peg j, for which all pegs may be used, thereby completing the task.*

There is always an optimal strategy of this form, therefore

5. *Minimise the number of moves with regard to the path chosen in step 1, the peg k and the number m.*

This conjecture is not yet proven. However, computational evidence supports it (cf. Hinz, Lužar, and Petr [13]).

Looking at the algorithm given, one can easily see why the case of the path graph with i and j being the outer vertices had to be ruled out. The first step of the algorithm is simply impossible to do, as the only path containing both i and j is the whole graph, thereby leaving no peg k to stow the upper part of the tower on. In fact, no optimal algorithm for this special case has yet been found, and no one to fit together with the computational optimal solutions found by breadth-first search. This case seems to prove especially difficult. Therefore, this chapter will address some phenomena regarding this *Linear Tower of Hanoi*.

1.2 Notation and the three-peg case

The Linear Tower of Hanoi problem for a given number of pegs p, called *base*, and a given number of discs n, called *exponent*, shall be addressed as L_p^n.

Let the pegs be denoted by $i \in [p]_0 = \{0, 1, 2, \dots, p-1\}$, where 0 is the leftmost peg, followed by 1, and so on until $p - 1$ denotes the rightmost peg. A configuration of the Tower of Hanoi game can then be represented as $c = c_n c_{n-1} \dots c_2 c_1$, where $c_k \in [p]_0, k \in [n]$, denotes the peg on which the k-th disc rests (disc 1 being the smallest one). If consecutive entries are identical, they may be written with exponential notation. For example, the constellation 211102 of the L_3^6 problem can also be written $21^3 02$. The problem of transferring the set of all discs, called the *tower*, from the *start peg* s to the *destination peg* d is denoted by $L_p^n(s, d)$, and the minimum number of moves required to solve this problem by $M_p^n(s, d)$. It can be observed that $M_p^n(s, d) = M_p^n(d, s)$ for reasons of symmetry and that $L_p^n(s, d)$ with $s = d$ is trivial with $M_p^n(s, d) = 0$. Furthermore, one can easily see that $M_p^n(p - 1 - s, p - 1 - d) = M_p^n(s, d)$, as one just needs to invert the order of the pegs to transform these two problems into one another.

The goal would be to give a closed form solution to $M_4^n(s, d)$ for any n, s and d. Afterwards, higher p could be considered.

In the case $p = 3$, the problem of optimal solutions is quite easily solvable. First, notice the following Lemma 2 that in fact holds true for any p.

Lemma 2. *The largest disc only moves in the direction of the destination peg. Therefore, it moves exactly $p - 1$ times if the goal is to transfer the tower from one extreme peg to the other.*

Proof. Whenever one would move the largest disc in the direction of the start peg, one can instead leave it where it is, as it does not inhibit moving the smaller discs. Doing so will save moves. □

Looking at the Linear Tower of Hanoi puzzle with three pegs, one can now see the unique optimal solution algorithm as well as the number of required moves.

Theorem 3. *Consider the Linear Tower of Hanoi puzzle with $p = 3$ pegs. Then the optimal solution for moving the tower from one peg to another is unique and takes $M_3^n(0,2) = 3^n - 1$ moves in the case of the two extreme pegs and $M_3^n(0,1) = \frac{3^n-1}{2}$ moves in the case of one extreme peg and the middle peg.* (Cf., e.g., Emert, Nelson, and Owens [6], p. 60.)

Proof. First, consider the case of two extreme pegs. It is known from Lemma 2 that the largest disc never moves in the opposite direction of the destination peg. It has to be transferred there, so it must move from the start peg to the middle peg at some point in time and from there to the destination peg. To move it from the start peg to the middle peg, all other discs need to be on the destination peg. To move it from the middle peg to the destination peg, all other discs must be on the start peg. So the algorithm for the optimal solution is:

First, move the $n-1$ smallest discs (all but the largest one) to the destination peg, taking $M_3^{n-1}(0,2)$ moves. Move the largest disc to the middle peg. Then move all the other discs back to the start peg, again taking $M_3^{n-1}(0,2)$ moves. Now move the largest disc to the destination peg and, finally, all the other discs there, too, again using $M_3^{n-1}(0,2)$ moves. Therefore, $M_p^n(0,2) = M_3^{n-1}(0,2) + 1 + M_3^{n-1}(0,2) + 1 + M_3^{n-1}(0,2) = 3M_3^{n-1}(0,2) + 2$. One can now show by induction that this leads to the aforementioned formula.

Base case $n = 1$: One needs two moves to transfer one disc from one extreme peg to the other. So $M_3^1(0,2) = 3^1 - 1 = 2$ holds true.

Induction step $n \mapsto n+1$: Let the formula hold for some n. Then $M_3^{n+1}(0,2) = 3M_3^n(0,2) + 2 = 3(3^n - 1) + 2 = 3^{n+1} - 1$. So the statement is true for all $n \in \mathbb{N}$.

Moving on to the case of one extreme peg and the middle peg, one can observe that for moving the largest disc to the middle peg, all other discs have to be on the non-start extreme peg, taking $M_3^{n-1}(0,2) = 3^{n-1} - 1$ moves to get there. The largest disc is then moved to the middle peg in one single turn. Finally, all the other discs are moved there, too, taking $M_3^{n-1}(0,1)$ moves. So a recursive formula is given by $M_3^n(0,1) = M_3^{n-1}(0,2) + 1 + M_3^{n-1}(0,1) = 3^{n-1} + M_3^{n-1}(0,1)$. Again, the closed form solution is shown by induction.

Base case $n = 1$: One needs 1 move and therefore $M_3^1 = \frac{3^1-1}{2} = 1$ holds.

Induction step $n \mapsto n+1$: By using the recursive formula, one gets

$$M_3^{n+1}(0,1) = 3^n + M_3^n(0,1) = \frac{2 \cdot 3^n}{2} + \frac{3^n - 1}{2} = \frac{3^{n+1} - 1}{2}.$$

And thus the closed form solutions mentioned above are proven.

The uniqueness of the optimal solutions is inherited from cases with lower n. For $n = 1$, it is obvious that the optimal solution is unique. The given algorithms are in fact forced.

They give the only optimal way for solving the problems. As they use the solutions for the $n-1$ case in the case for n, every solution inherits uniqueness from its predecessor. □

This proof can actually be done a lot quicker and more elegantly, by using the state graph $H^n_{3,\mathrm{lin}}$, with its vertices labelled like the state they represent. This is obviously a path graph, as every state allows for two moves, except for the situation where all discs are on the same extreme peg. Therefore, these perfect states form the end vertices of the path graph, which contains vertices corresponding to all legal states of the discs. Knowing that the graph is connected, it is necessary to pass through all vertices of the graph, corresponding to states of the Tower of Hanoi puzzle. Therefore, the number of moves for the extreme-to-extreme problem is the number of vertices in $H^n_{3,\mathrm{lin}}$ minus one, which is $3^n - 1$. For the extreme-to-middle problem, the perfect state with all discs on the middle peg has to correspond to the middle vertex of the path graph due to reasons of symmetry. Therefore, the problem takes half the number of edges in $H^n_{3,\mathrm{lin}}$, namely $\frac{3^n-1}{2}$ steps. Anyway, the solution is unique, because its states form a path graph.

It is important to keep the 3-peg case in mind, as its results will be used for the more complex cases.

1.3 The higher cases

Again, using Lemma 2, one can see that the largest disc takes exactly $p-1$ moves in every optimal solution, moving towards d with each of those moves. Something similar holds true for the second largest disc. However, it is not that strict.

Proposition 4. *The second largest disc has to move at least $p+3$ times in transferring the tower from one extreme peg to the other one.*

Proof. The second largest disc has to make at least 2 moves backwards to cross over the largest disc. This leads to 4 moves of the second largest disc in addition to $p-1$ moves from the start peg to the destination peg, therefore giving a minimum of $p+3$ moves. □

Conjecture 5. *For any p and n there are optimal solutions where the second largest disc moves no more than $p+3$ times.*

Looking at the state graph $H^n_{4,\mathrm{lin}}$, one can see that a recursive structure occurs. When vertices the labels of which only differ in their last entry (which signifies the position of the smallest disc) are identified with each other and treated as one vertex, the lower exponent state graph $H^{n-1}_{4,\mathrm{lin}}$ appears. In the same way, the state graph can be reduced to any lower degree by contracting vertices which only differ from each other in the last k entries. This holds true for any base p, not only $p = 4$. The reason is that, no matter how the smaller discs move, the larger ones only have the ways to go they would have when alone. So, by acting like the smaller discs were not there (i.e. contracting vertices only differing in their last entries), the graph is reduced to the state graph of only the remaining larger discs.

A common structure for optimal extreme-to-extreme solutions can be identified for $p = 4$ and some cases with higher p. This algorithm can be described in the following way.

Let $p \geq 4$ and $n \geq p - 1$.

1. Choose $n_1, n_2, \dots, n_{p-2} \geq 1$ and $m_1, m_2, \dots, m_{p-2} \geq 1$ such that

$$\sum_{k=1}^{p-2} n_k = \sum_{k=1}^{p-2} m_k = n - 1.$$

2. First, move the smallest n_1 discs to the destination peg $p - 1$.
3. Next, move the next smallest n_2 discs to the peg $p-2$. Continue in the same fashion until you move the n_{p-2} discs to peg 2.
4. Now, only the largest disc rests on peg 0 and is moved to peg 1.
5. Next, move the smallest m_1 discs to peg 0. If $m_1 > n_1$, begin with the discs from the peg with the smallest number, continuing until peg $p - 1$ is emptied.
6. Right-shift the remaining subtowers, such that the rightmost subtower rests on peg $p - 1$, the next on peg $p - 2$ and so on. Peg 1 is now empty.
7. Move the m_2 smallest discs that are not on peg 0 to peg 1, again beginning left, if they rest on multiple towers.
8. Continue until the smallest m_1 discs are on peg 0, the next-smallest m_2 on peg 1 and so on and m_{p-2} on peg $p - 3$. The largest disc can now be moved from peg $p - 2$ to $p - 1$.
9. Accumulate the subtowers on peg $p - 1$, beginning with the largest discs and continuing with progressively smaller ones.
10. Optimise over $n_1, n_2, \dots, n_{p-2}, m_1, m_2, \dots, m_{p-2}$.

While this seems quite complicated, it reduces the number of possibly optimal solutions to be compared significantly. A drawback is that it is recursive with respect to both parameters, as solutions with fewer pegs or discs or both are used.

For example take $p = 4$ and $n = 7$. The algorithm leads to an optimal solution for $n_1 = 3$, $n_2 = 3$, $m_1 = 4$ and $m_2 = 2$:

First, move three discs from the starting peg 0 to the destination peg 3. This is the problem $L_4^3(0,3)$, which takes 19 moves. Now, move another three discs to peg 2. This is the $L_3^3(0,2)$ problem, which takes 26 moves. The largest disc is now moved from peg 0 to peg 1. Next, the smallest disc on peg 2 is moved to peg 0, as are all discs on peg 3, taking $2 + 19 = 21$ moves. The remaining two discs on peg 2 are moved to peg 3, which is an $L_3^2(0,1)$ problem, taking 4 moves to solve. The largest disc is now moved to peg 2. The two discs on peg 3 are moved to peg 1, taking 8 moves ($L_3^2(0,2)$) and the largest disc is moved to the destination peg. Now, the two discs on peg 1 are also moved there, again taking 8 moves. Finally, the four discs on peg 0 are transferred to the destination peg. This is the problem $L_4^4(0,3)$, its optimal solution being 34 moves long. This gives a solution with

$$19 + 26 + 1 + 21 + 4 + 1 + 8 + 1 + 8 + 34 = 123$$

moves. Computations (cf. Hinz and Petr [15], p. 4) show that this is in fact an optimal solution. A table with most of the results from Hinz and Petr [15] is given below on page 15 (Tab. 1.2).

However, for $n = 10$ or $n \geq 13$, this algorithm does not lead to an optimal solution. The case of $n = 10$ shall be analysed further.

Consulting Table 1.1, one can see that the optimal result using the algorithm is 348. However, this is not optimal. The computational results found in Hinz and Petr [15] show that an optimal solution for the $L_4^{10}(0,3)$ problem is only 342 moves long. This number of moves can be achieved for example by the following strategy and sequence of moves.

First, transfer the smallest 6 discs from the start peg 0 to the destination peg 3. This is an $L_4^6(0,3)$ problem, which takes 88 moves. (For these numbers cf. Hinz and Petr [15], p. 4, or the table below.) The resulting position is $0^4 3^6$. Next, transfer the smallest three discs on peg 0 to peg 2. As peg 3 is blocked, this is an $L_3^3(0,2)$ problem, taking 26 moves. Now the largest disc can be moved from peg 0 to peg 1. This results in the position $12^3 3^6$ after 115 moves. Until now, this solution is perfectly in accordance with the algorithm for $n_1 = 6$ (and therefore $n_2 = 3$). Now the three discs on peg 2 are moved back to peg 0

(n_1, m_1)	moves	(n_1, m_1)	moves	(n_1, m_1)	moves	(n_1, m_1)	moves
$(1,1)$	22972	$(2,3)$	4050	$(3,6)$	948	$(5,6)$	352
$(1,2)$	12046	$(2,4)$	2856	$(3,7)$	992	$(5,7)$	348
$(1,3)$	8416	$(2,5)$	2492	$(3,8)$	1192	$(5,8)$	404
$(1,4)$	7234	$(2,6)$	2442	$(4,4)$	952	$(6,6)$	358
$(1,5)$	6906	$(2,7)$	2594	$(4,5)$	572	$(6,7)$	350
$(1,6)$	6964	$(2,8)$	3118	$(4,6)$	474	$(6,8)$	394
$(1,7)$	7440	$(3,3)$	2608	$(4,7)$	482	$(7,7)$	400
$(1,8)$	8936	$(3,4)$	1410	$(4,8)$	574	$(7,8)$	440
$(2,2)$	7684	$(3,5)$	1034	$(5,5)$	454	$(8,8)$	538

Table 1.1: Numbers of moves that the algorithm takes for $p = 4$ and $n = 10$ and for certain n_1 and m_1. As $n_1 + n_2 = m_1 + m_2 = n - 1$, n_2 and m_2 are determined by n_1 and m_1 respectively. One can assume $m_1 \geq n_1$, because the order of the moves can always be inverted and rearranged to match the given algorithm with m_1 and n_1 switched. The optimal solution using the algorithm is 348 moves long and uses $n_1 = 5$ and $m_1 = 7$.

in another 26 moves. This could still be an equivalent transformation of the algorithm in the case $n_2 = m_2$. The largest disc is moved to peg 2 and the resulting position is $20^3 3^6$, achieved in 142 moves. At this point, however, things start to become a little strange. The three discs on peg 0 are transferred to peg 1, corresponding to an $L_3^3(0,1)$ problem, but the last move is omitted, ending in a slightly awkward 21103^6-position (this taking $13-1 = 12$ moves). Next, five of the six discs on peg 3 are transferred to peg 0. This takes 57 moves, as it is an $L_4^5(0,3)$ problem. Now the remaining disc on peg 3 is moved to peg 1 in two moves and the largest disc does its last move to peg 3, resulting in a 311010^5-position after 214 moves. In the following, the three discs now on peg 1, though they are not of consecutive size, are treated as a size-3 subtower and are moved to peg 3 in 26 moves, posing an $L_3^3(0,2)$ problem. Now the smallest three discs are also moved to peg 3, which is an $L_4^3(0,3)$ problem that can be solved in 19 moves. One thereby arrives in a $3^3 03003^3$ position in a total of 259 moves. The series of moves that follows next is quite unorthodox. While it is completely done with the three discs that are not on peg 3, it ends somewhere in between perfect states. The move sequence is given below. (If more than one move is done, the number of moves is given above the arrow, but only if consecutive moves are made with the same disc.)

$$\begin{aligned} &3330300333 \xrightarrow{2} 3330302333 \to 3330312333 \xrightarrow{2} 3330310333 \\ \to\ &3330320333 \xrightarrow{2} 3330322333 \to 3331322333 \xrightarrow{2} 3331320333 \end{aligned}$$

Now, the smallest three discs resting on peg 3 are moved to peg 0 in another 19 moves (an $L_4^3(0,3)$ problem again), ending in the state $3^3 1320^4$ with 289 moves done. The three scattered middle discs, discs 5, 6 and 7, are now accumulated on peg 3 with the following move sequence.

$$\begin{aligned} &3331320000 \to 3331330000 \to 3332330000 \xrightarrow{2} 3332310000 \\ \to\ &3332210000 \xrightarrow{2} 3332230000 \to 3332130000 \xrightarrow{2} 3332110000 \\ \to\ &3333110000 \xrightarrow{2} 3333130000 \to 3333230000 \xrightarrow{2} 3333210000 \\ \to\ &3333310000 \xrightarrow{2} 3333330000 \end{aligned}$$

Note that the move sequences given step-by-step are unique optimal solutions, if the set of discs that may be moved is given, as they are sections of the solution to an $L_3^3(0,2)$ problem.

The configuration is now $3^6 0^4$ after a total of 308 moves. Finally, the remaining 4 discs on peg 0 are moved to peg 3, which is an $L_4^4(0,3)$ problem, taking 34 moves to solve. Therefore, the problem L_4^{10} can be solved in 342 moves. Computations verify that this is an optimal solution.

As one can see, the nice looking solution strategy using subtowers of variable sizes does not work for every number of discs. In fact, it gives optimal solutions again for $n = 11$ and $n = 12$, but fails completely for $n \geq 13$ as far as computation has been done by now. The Linear Tower of Hanoi seems to elude simple strategies and require separate analysis for every case.

Yet another algorithm that was thought to be correct fails under thorough scrutiny: for a long time it was hypothesised there be a simple optimal strategy for $p > n$, working similarly to the one described above. One would lay out the discs in a row so that the smallest one is on the destination peg $p-1$ and the largest one on the peg $p-n$. Now one moves the smallest disc back to peg $p-n-1$ and shifts all other discs one peg to the right. Then the second smallest disc is moved to peg $p-n$, crossing over all larger discs, and the remaining $n-2$ discs are right-shifted again. This process is repeated until the second largest disc has crossed over the largest one. Now the smallest disc is on peg $p-n-1$ and the largest disc is on peg $p-1$. One can now accumulate all discs on peg $p-1$, beginning with the largest and finishing with the smallest. This is a trivial task. Using this solution, every disc k moves backward $1+n-k$ times. This disc therefore makes $p-1+2(1+n-k)$ moves. The single exception is the largest disc, which does not move backwards at all, respecting Lemma 2. Accordingly, it makes $p-1$ moves. This would lead to the following result.

Conjecture 6 (False Conjecture). *Let $p > n$. Then*

$$\begin{aligned} M_p^n &= p-1+\sum_{k=1}^{n-1}(p-1+2(1+n-k)) \\ &= n(p-1)+2(n-1)(n+1)-2\sum_{k=1}^{n-1}k \\ &= n(p-1)+2(n^2-1)-2\frac{n(n-1)}{2} \\ &= n(p-1)+2n^2-2-n^2+n \\ &= n(p+n)-2. \end{aligned}$$

However, this reasonable sounding conjecture turns out to be false. The first counter-example is $p=10$ and $n=9$, where the optimal solution does not require $9 \cdot 19 - 2 = 169$ moves, but only 167. This is the computer-generated, optimal move sequence:

In the beginning, all discs rest on the first peg. This position is denoted by 0^9. First, the discs are laid out as the conjecture would suggest. This takes up 45 moves. The discs are then in the position 123456789. Next, the five largest discs are rearranged as with the conjecture for $n=5$ and $p=6$ and right-shifted another time. This takes 28 moves. The move sequence is given below. (If more then one move is done, the number of moves is given above the arrow, but only if consecutive moves are made with the same disc.)

$$\begin{aligned} & 123456789 \xrightarrow{5} 123406789 \rightarrow 123506789 \rightarrow 124506789 \\ \rightarrow\ & 134506789 \rightarrow 234506789 \xrightarrow{4} 234106789 \rightarrow 235106789 \\ \rightarrow\ & 245106789 \rightarrow 345106789 \xrightarrow{3} 342106789 \rightarrow 352106789 \\ \rightarrow\ & 452106789 \xrightarrow{2} 432106789 \rightarrow 532106789 \rightarrow 542106789 \\ \rightarrow\ & 543106789 \rightarrow 543206789 \rightarrow 543216789 \end{aligned}$$

Next, the largest disc is moved to the third to last peg as quickly as possible, using the following 8 moves.

$$\begin{aligned} 543216789 &\xrightarrow{3} 543216489 \rightarrow 543217489 \\ \rightarrow 643217489 &\xrightarrow{2} 643215489 \rightarrow 743215489 \end{aligned}$$

The next 35 moves aim to create a dense subtower constellation allowing the largest disc to do its last two moves.

$$\begin{aligned} & 743215489 \xrightarrow{2} 743217489 \xrightarrow{3} 743217789 \xrightarrow{2} 763217789 \\ \xrightarrow{3}\ & 766217789 \xrightarrow{3} 766517789 \xrightarrow{3} 766547789 \xrightarrow{6} 766547783 \\ \rightarrow\ & 766647783 \rightarrow 766657783 \xrightarrow{3} 766657483 \xrightarrow{4} 766657443 \\ \xrightarrow{2}\ & 766655443 \rightarrow 866655443 \rightarrow 966655443 \end{aligned}$$

Next, the subtowers are moved to the destination peg 9 in order, beginning with the three discs on peg 6, using 19 moves to do so.

$$\begin{aligned} & 966655443 \xrightarrow{3} 966955443 \xrightarrow{2} 968955443 \rightarrow 978955443 \\ \xrightarrow{3}\ & 978655443 \rightarrow 979655443 \rightarrow 989655443 \xrightarrow{2} 987655443 \\ \rightarrow\ & 997655443 \xrightarrow{2} 999655443 \xrightarrow{3} 999955443 \end{aligned}$$

Now the other two subtowers are accumulated on the destination peg, finishing the task in another 32 moves.

$$\begin{aligned} & 999955443 \xrightarrow{4} 999959443 \xrightarrow{3} 999989443 \xrightarrow{2} 999987443 \\ \rightarrow\ & 999997443 \xrightarrow{2} 999999443 \xrightarrow{5} 999999493 \xrightarrow{4} 999999893 \\ \xrightarrow{2}\ & 999999873 \rightarrow 999999973 \xrightarrow{2} 999999993 \xrightarrow{6} 999999999 = 9^9 \end{aligned}$$

As one can see, this is a solution in $45 + 28 + 8 + 35 + 19 + 32 = 167$ moves. Note that the start peg 0 is used during the first 73 moves, but never again afterwards, and that the pegs 1 and 2 are totally unused while the largest disc is on the destination peg.

As one can see, even the most elementary problems in the Linear Tower of Hanoi can require quite elaborate tricks for an optimal solution.

Considering now that the case $p = 1$ does not even allow moves at all and for $p = 2$ one can only move the smallest disc from one peg to the other, the length of the optimal solutions for the extreme-to-extreme Linear Tower of Hanoi problem are summarised in Table 1.2 (cf. [15, Table 2], p. 4).

p \ n	1	2	3	4	5	6	7	8	9	10	11
1	0	0	0	0	0	0	0	0	0	0	0
2	1	—	—	—	—	—	—	—	—	—	—
3	2	8	26	80	242	728	2186	6560	19682	59048	177146
4	3	10	19	*34	*57	*88	*123	*176	*253	*342	*449
5	4	12	22	34	*52	*70	*96	*124	*156	*194	*236
6	5	14	25	38	53	*72	*93	*144	*139	*168	*199
7	6	16	28	42	58	76	*98	*120	*144	*168	*196
8	7	18	31	46	63	82	103	*126	*149	*174	*201
9	8	20	34	50	68	88	110	134	*158	*182	?
10	9	22	37	54	73	94	117	142	*167	*192	?
11	10	24	40	58	78	100	124	150	*176	*206	?

Table 1.2: Optimal solutions for the Linear Tower of Hanoi for small p and n.

The numbers in with asterisk before are known as a result of computational work by Hinz and Petr [15], more precisely by breadth-first search, a very inefficient brute force method. The underlying principles of these solutions are not known. Winning insight on them is the ultimate goal of research on the Linear Tower of Hanoi. The third row in the table is the $p = 3$ case, therefore it consists of the numbers $3^n - 1$ in the n-th column. The numbers of the 'lower triangle' are created by the aforementioned $n(n+p)-2$ formula up until $n = 9$. The first two rows do not really make much sense, but are given for reasons of completeness. (The fourth row continues 572, 749, 980, 1261, 1560, 1903, 2328, 2889, 3562.)

It can be observed that in many cases the number of moves of every disc is constant in the optimal solution for constant p and n. Sadly, this does not hold true in general, as the case of $p = 5 = n$ shows. However, there is not yet a counterexample known for $p = 4$.

disc	number of disks						
	1	2	3	4	5	6	7
1	3	3	3	3	3	3	3
2	–	7	7	7	7	7	7
3	–	–	9	11	11	21	21
4	–	–	–	13	13	9	23
5	–	–	–	–	23	21	13
6	–	–	–	–	–	27	25
7	–	–	–	–	–	–	31

Table 1.3: The numbers of moves for certain discs depending on the total number of discs for $p = 4$ fixed.

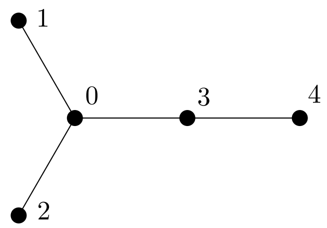

Figure 1.1: The F_5 fork graph. (Taken from Hinz, Lužar, and Petr [13, Figure 2], p. 10.)

The numbers behave erratically, and no theories on their evolution for increasing n have been presented until now.

The numbers of moves per disc for $p = 4$ are given in Table 1.3. Similar tables, though they describe the number of moves of a certain disc in terms of moves backward, can be found in Hinz and Petr [15], p. 5. One can see that the first row is constantly $p - 1$, as warranted by Lemma 2, and the next row is $p + 3$. This holds true for all yet calculated cases and may be a constant element among all Linear Tower of Hanoi problems, so this might be the point to start further work at.

The importance of the Linear Tower of Hanoi problem in the context of the generalized Tower of Hanoi should not be underestimated. Solutions for the Linear Tower of Hanoi are sometimes used in the Dudeney–Stockmeyer conjecture even when non-linear variants are regarded. Consider for example the fork graph F_5 (Figure 1.1) originally found in Hinz, Lužar, and Petr [13, Figure 2]. If one tries to solve the problem of transferring the Tower from peg 1 to peg 4, the only available peg to stow the upper part of the Tower that is required by the conjecture would be peg 2. Now, the remaining free pegs would form a

Linear Tower of Hanoi problem, so one could never be sure to have found a truly optimal solution for high n, as there is no optimal strategy known for the inherently linear part of the solution, nor is there a formula for the minimal number of moves. In fact, the same would hold true for every problem in the same graph, as there would always be an option in the Dudeney–Stockmeyer conjecture that would use a Linear Tower of Hanoi problem. This phenomenon now continues to the move graphs of order 6. For high n, no optimal solutions can be known for sure if the move graph contains the above-mentioned fork graph or a $p = 5$ Linear Tower of Hanoi.

The uncertainty of the Linear Tower of Hanoi spreads quickly throughout the move graphs and their associated problems as the order increases, especially if the size of the move graph is low. But in fact it suffices that there is a single subgraph in the move graph that is a path graph of order 4 or higher when all other vertices are removed. In these cases, one potentially cannot be sure if a solution is optimal for high n, as the Dudeney–Stockmeyer conjecture could lead to the best result when it uses this subgraph, for which one does not know the length of minimal solutions. It would in fact help in some cases to know lower bounds for the Linear Tower of Hanoi, since one could then maybe eliminate it as a potential optimal solution. Upper bounds might also give the information that a solution via a path subgraph is optimal, but one would still not know the exact result. The Dudeney–Stockmeyer conjecture is most likely correct,[1] but without a deeper insight into the Linear Tower of Hanoi, its use is severely limited.

And after all, maybe the most important fact to take away from this chapter is that the Linear Tower of Hanoi problem is exceedingly demanding and there is still a long way to go before one truly understands all aspects of Brahma's capricious creation.

[1] The author firmly believes it is.

2 Sierpiński triangle graphs

In 1915, the famous Polish mathematician Wacław Franciszek Sierpiński (1882–1969) constructed a fractal object [20], that would be known throughout the world: the Sierpiński triangle. Beginning with a closed triangle, an upside-down triangle was removed from the middle, dividing the original triangle into three smaller ones. Again, they would be divided into three smaller ones by removal of a triangle in their middle. This would be iterated to infinity.

Since then, the Sierpiński triangle was studied from many different perspectives. Observations from the point of fractal geometry have been done for example by Cristea and Steinsky in 2013 [5]. Another especially vital perspective has been graph theory. Interpreting the corners of all involved triangles as vertices and the sides as edges, one may arrive at a family of graphs depending on the step of iteration in the consecutive removal of upside-down triangles. For a long time, confusion of terminology prevailed, as these graphs were called Sierpiński graphs. The same name, however, denominates another class of graphs with slightly different properties. A 2017 paper by Hinz, Klavžar, and Zemljič [12] tried to establish an unambiguous and unified use of terminology by categorising different types of so-called Sierpiński-type graphs. The above-mentioned graphs are named *Sierpiński triangle graphs*, in contrast to (plain) *Sierpiński graphs.* Connections between the two types of graphs have been observed and are of great use for proving many elementary metric properties of Sierpiński triangle graphs, as the properties of Sierpiński graphs are often distinctively easier to calculate.

The family of the Sierpiński triangle graphs was generalised by Jakovac in 2014, by starting the process of iteration with a complete graph K_p instead of a triangle (which is a K_3) [17]. (Even more complicated generalisations are possible, cf. Alizadeh, Estaji, Klavžar, and Petkovšek [1], but are not covered here.) These generalised Sierpiński triangle graphs are denoted by $\widehat{S}_p^n$ as opposed to the Sierpiński graphs S_p^n. For a formal definition of S_p^n and $\widehat{S}_p^n$, see [10, Definitions 2 and 3]. Many interesting properties have been proven for the Sierpiński triangle graphs, for example that they are Hamiltonian (cf. [17, 23]).

However, some characteristic parameters have proven difficult to find for Sierpiński triangle graphs. The corresponding properties of Sierpiński graphs are often known. Many metric properties of Sierpiński graphs have been found by Hinz and Parisse in 2012 [14], especially their average eccentricity, while this parameter was unknown for Sierpiński triangle graphs until 2019, when it was calculated by Rolke [18]. For Sierpiński graphs S_3^n, an automaton was designed by Romik in 2006 [19] that could find shortest paths between any two vertices. This was extended in 2014 by Hinz and Holz auf der Heide [9] to general S_p^n. They also found a way to use a modified version of the same automaton for Sierpiński triangle graphs $\widehat{S}_p^n$ [16]. A 2021 paper by Hinz, Holz auf der Heide, and Zemljič [10] summarises the known metric properties of Sierpiński triangle graphs.

Nevertheless, the average distance in Sierpiński triangle graphs is yet an open problem. In this thesis, a recursive formula for the average distance in Sierpiński triangle graphs $\widehat{S}_p^n$ is developed. It is shown how closed form solutions can be created from there for fixed parameters. Limit observations then show that the average distance of Sierpiński triangle graphs equals the average distance for Sierpiński graphs as $n \to \infty$, leading to a formula also found in the book by Hinz, Klavžar, and Petr [11] and originally developed by Bandt and Kuschel in 1992 [2].

As opposed to the vertex set $V_p^n = [p]_0^n$ of S_p^n, the vertex set of a Sierpiński triangle graph $\widehat{S}_p^n$ is defined as

$$\widehat{V}_p^n = \left\{ \widehat{i} \;\middle|\; i \in [p]_0 \right\} \cup \left\{ s\widehat{ij} \;\middle|\; s \in [p]_0^k, k \in [n]_0, \{i,j\} \in \binom{[p]_0}{2} \right\}.$$

The vertices can be divided into three categories.

Vertices from the first subset of the vertex set are called *primitive vertices.* They are of the form $\widehat{i}$ with $i \in [p]_0$. In graph drawings of an $\widehat{S}_3^n$ they denominate the vertices in the corners of the triangle. The degree of a primitive vertex, i.e. the number of edges connected to the vertex, is $p-1$, as opposed to all other vertices in the graph, which have a degree of $2(p-1)$.

Vertices from the second subset of the vertex set with $k = 0$ are called *critical vertices.* They are of the form $\widehat{ij}$ with $\{i,j\} \in \binom{[p]_0}{2}$. They connect subgraphs of the $\widehat{S}_p^n$.

All other vertices are called *generic vertices.* (For a graphical depiction of the three types of vertices, cf. Fig. 2.1.)

2.1 The Wiener index and the average distance of Sierpiński triangle graphs

Let $G = (V, E)$ with $|V| \geq 2$ be a simple connected graph. The total distance of a vertex $s \in V$ is defined as

$$\mathrm{d}(s) = \sum_{t \in V} \mathrm{d}(s,t) \geq |V| - 1.$$

The total distance of the graph G itself is given by

$$\mathrm{d}(G) = \sum_{s \in V} \mathrm{d}(s) = \sum_{s,t \in V} \mathrm{d}(s,t) \geq |V|(|V| - 1).$$

The average distances in both cases can be defined in different ways, as the distance between a vertex and itself, always being 0, can be ruled out or accounted for. Ruling it out leads to

$$\tilde{\mathrm{d}}(s) = \frac{1}{|V| - 1}\mathrm{d}(s),$$

whereas treating it normally gives

$$\bar{\mathrm{d}}(s) = \frac{1}{|V|}\mathrm{d}(s).$$

Accordingly, ruling out distances between two identical vertices, the average distance of a graph is

$$\tilde{\mathrm{d}}(G) = \frac{1}{|V|(|V| - 1)}\mathrm{d}(G) \geq 1.$$

If these distances are taken into account, the average distance of a graph is given by

$$\bar{\mathrm{d}}(G) = \frac{1}{|V|^2}\mathrm{d}(G) \geq \frac{|V| - 1}{|V|}.$$

It is known from previous research, e.g., [10], p. 6, that the number of vertices in a Sierpiński triangle graph $\widehat{S}_p^n$, called the *order* of the graph, $n \geq 0$ and $p \geq 2$, is given by

$$\left|\widehat{S}_p^n\right| = \left|\widehat{V}_p^n\right| = \frac{p}{2}(p^n + 1).$$

The Wiener index of a graph $G = (V, E)$ is defined as

$$\mathrm{W}(G) = \sum_{\{s,t\} \in \binom{V}{2}} \mathrm{d}(s,t) = \frac{1}{2}\mathrm{d}(G),$$

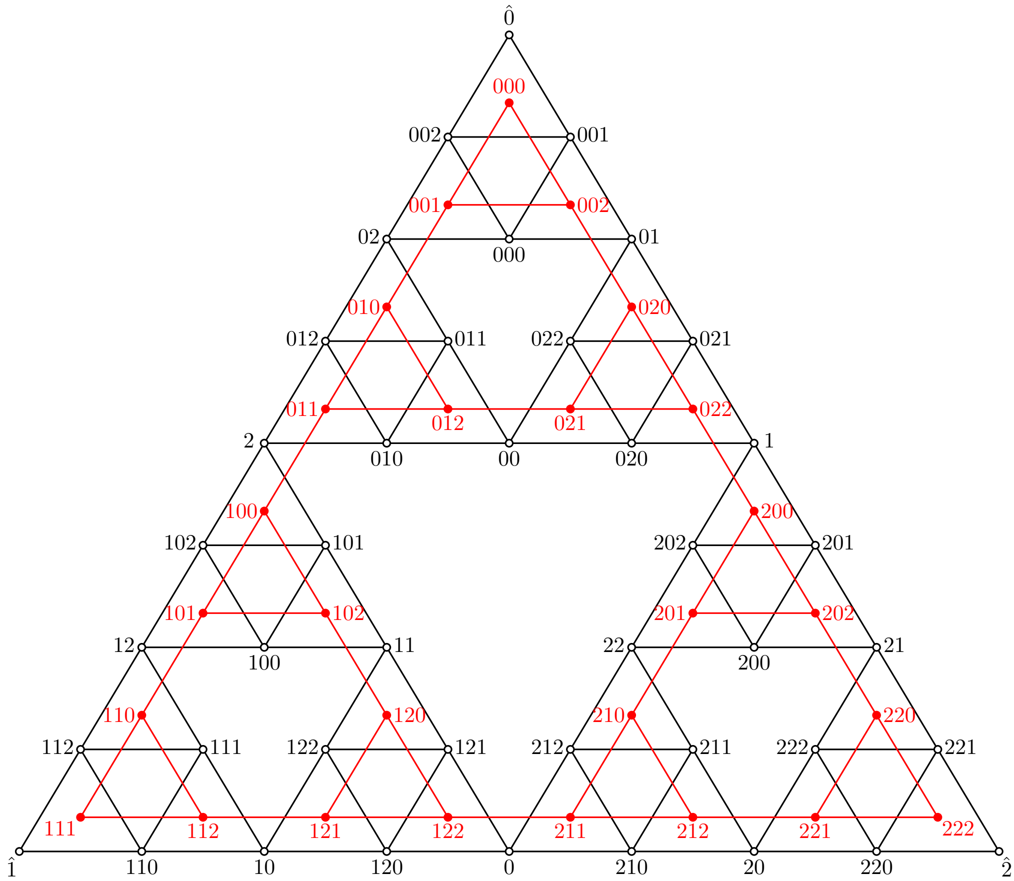

Figure 2.1: A Sierpiński triangle graph $\widehat{S}_3^3$ (black) and a Sierpiński graph S_3^3 (red). The Sierpiński triangle graph can be obtained by contracting certain edges (those that are not part of a K_3) in the Sierpiński graph of the same base and an exponent higher by one. The figure is taken from [12], p. 571, where a different labelling is used. The last entry in the figure is 0 for $\widehat{12}$, 1 for $\widehat{02}$ and 2 for $\widehat{01}$ for all vertices except $\widehat{0}$, $\widehat{1}$ and $\widehat{2}$. (This transformation, too, is found in [12], p. 571, where the $\widehat{12}$-labelling is called *contraction labelling* and the 0-labelling is called *idle peg labelling*, derived from the connection to the Tower of Hanoi.) The three vertices that form the corners of the triangle are called *primitive vertices*. The vertices that connect the largest sub-triangles ($\widehat{12}$, $\widehat{02}$ and $\widehat{01}$; or 0, 1, and 2, depending on the labelling) are called *critical vertices*. The remaining vertices are called *generic*.

i.e. the sum of the distances between every pair of two vertices of G. To find the average distance, it suffices to give a formula for the Wiener index, as the average distance may then be calculated by the relations above.

To do so, the unordered pairs of vertices will be grouped together to form sets with similar attributes. First of all, the pairs of two identical vertices may just be ignored, as their distance is 0. Now look at pairs in which one is a primitive vertex, i.e. of the form $\widehat{i}$. A result that can be found, e.g., in [18, Theorem 19], p. 56, is used.

Lemma 7 (Primitive distance). *The sum of the distances of all vertices to a fixed primitive vertex is given by*

$$\tau_p^n = \mathrm{d}(\widehat{i}) = 2^{n-1}(p-1)(p^n+1).$$

This shall henceforth be called the primitive distance *of* $\widehat{S}_p^n$.

As there are p primitive vertices in a Sierpiński triangle graph, for the sum of all distances in $\widehat{S}_p^n$ with a primitive vertex involved, one gets

$$\sum_{\{s,t\}\in\binom{\widehat{V}_p^n}{2},s\in\widehat{P},t\in\widehat{V}_p^n} \mathrm{d}(s,t) = 2^{n-1}p(p-1)(p^n+1) - 2^{n-1}p(p-1) = 2^{n-1}p^{n+1}(p-1)$$

$$\text{using} \quad P = [p]_0 \quad \text{and} \quad \widehat{P} = \left\{\widehat{p} \,\middle|\, p \in P\right\}.$$

The subtraction takes place because the distances between two different primitive vertices were counted two times, once in each primitive distance, and the subtraction term corrects this issue. (There are $\frac{1}{2}p(p-1)$ pairs of different primitive vertices, whose distance is 2^n.)

2.2 Decomposition of the Wiener index

Now all distances in which a primitive vertex is involved have been taken care off. The Wiener index of a Sierpiński triangle graph that is stripped of its primitive vertices shall be called T_p^n. It can be decomposed as follows, using $P^* = \widehat{P} \cup \left\{\widehat{kl} \,\middle|\, \{k,l\} \in \binom{P}{2}\right\}$:

$$\begin{aligned} T_p^n &= \sum_{\{s,t\}\in\binom{\widehat{V}_p^n\setminus\widehat{P}}{2}} \mathrm{d}(s,t) \\ &= \frac{1}{2}\sum_{\{i,j\},\{k,l\}\in\binom{P}{2}} \mathrm{d}(\widehat{ij},\widehat{kl}) + \sum_{\{i,j\}\in\binom{P}{2},s\in\widehat{V}_p^n\setminus P^*} \mathrm{d}(\widehat{ij},s) + \sum_{i\in P;\{s,t\}\in\binom{\widehat{V}_p^{n-1}\setminus\widehat{P}}{2}} \mathrm{d}(is,it) + \sum_{i,j\in P;i<j;s,t\in\widehat{V}_p^{n-1}\setminus\widehat{P}} \mathrm{d}(is,jt) \end{aligned}$$

It is easy to see that

$$\sum_{i\in P;\{s,t\}\in\binom{\widehat{V}_p^{n-1}\setminus\widehat{P}}{2}} \mathrm{d}(is,it) \quad = \quad p\sum_{\{s,t\}\in\binom{\widehat{V}_p^{n-1}\setminus\widehat{P}}{2}} \mathrm{d}(s,t) = pT_p^{n-1}.$$

Furthermore, basic distance formulas found e.g. in [10], p. 9, lead to (with use of the Iverson convention)

$$\sum_{\{i,j\},\{k,l\}\in\binom{P}{2}} \mathrm{d}(\widehat{ij},\widehat{kl}) \quad = \quad \sum_{\{i,j\},\{k,l\}\in\binom{P}{2}} 2^{n-1}\big((|\{i,j,k,l\}|\geq 3)+(|\{i,j,k,l\}|=4)\big).$$

There are obviously $\frac{1}{8}p(p-1)(p-2)(p-3)$ unordered pairs of unordered pairs $\{\{i,j\},\{k,l\}\}$ fulfilling $|\{i,j,k,l\}|=4$, as there are $\frac{1}{2}p(p-1)$ possibilities for $\{i,j\}$ and then k can be chosen from a set of size $(p-2)$ and l from a set of size $(p-3)$. This needs to be halved since $\{k,l\}=\{l,k\}$, and halved again as $\{i,j\}$ and $\{k,l\}$ are also interchangeable.

To fulfil $|\{i,j,k,l\}|\geq 3$ it suffices that $\{i,j\}\neq\{k,l\}$. So there are $\frac{1}{2}p(p-1)$ possibilities for $\{i,j\}$ and $\frac{1}{2}p(p-1)-1$ possibilities for $\{k,l\}$. The product must be halved, as the sets are interchangeable, so there are $\frac{1}{8}p(p-1)(p(p-1)-2)$ choices for $\{\{i,j\},\{k,l\}\}$ with $|\{i,j,k,l\}|\geq 3$. Thus

$$\begin{aligned}\frac{1}{2}\sum_{\{i,j\},\{k,l\}\in\binom{P}{2}} \mathrm{d}(\widehat{ij},\widehat{kl}) &= 2^{n-1}\cdot\frac{1}{8}\big(p(p-1)(p(p-1)-2)+p(p-1)(p-2)(p-3)\big)\big)\\ &= 2^{n-4}\big(2p^4-8p^3+10p^2-4p\big) = 2^{n-3}p\big(p^3-4p^2+5p-2\big)\\ &= 2^{n-3}p(p-1)^2(p-2).\end{aligned}$$

In the second line, it can be seen that 1 is a zero of the last term. The rest is done by polynomial long division and use of the formula for reduced quadratic equations.

The formula for the sum of distances between a critical vertex and a non-primitive non-critical (so called *generic*) vertex is more complicated, but still has a nice, albeit long, closed form. Some thoughts are needed beforehand about which ways are shortest:

Let therefore $\widehat{ij}$ be fixed. Now all vertices in the i-subgraph or the j-subgraph are reached directly from $\widehat{ij}$ in its role as the primitive vertex $\widehat{i}$ of the j-subgraph or as the primitive vertex $\widehat{j}$ of the i-subgraph. This means that the primitive distance can be used for these subgraphs. However, corrections need to be done to avoid counting the distances to the other critical vertices or the true primitive vertex of the respective subgraph, since they have already been counted. As the primitive distance of a subgraph is used, the critical vertices of the graph as a whole are primitive vertices with respect to this subgraph.

Subtracting these, the result for the sum of distances between the critical edge and the non-primitive edges of the subgraph is $\tau_p^{n-1} - 2^{n-1}(p-1)$. For all vertices that do not have i or j as their name's first entry, the shortest path leads through the critical vertex between the subgraph they are in and either the i- or the j-subgraph. For these, the second entry decides which way is shortest, if, and only if, said entry is (with $k, k_1, k_2 \in P\backslash\{i,j\}$) (a) i or $\widehat{ik}$ or (b) j or $\widehat{jk}$ or (c) $\widehat{ij}$ or $\widehat{k_1k_2}$. For the first case, the shortest path leads through the i-subgraph, and likewise it leads through the j-subgraph in the second case, whereas in the third case both ways are of equal length. If the second entry still contains neither i nor j, the decision is further postponed or, in the case $\widehat{k_1k_2}$, will be treated in a special manner explained later. For every entry that does not decide which path is the shortest, the shortest path leads through another part of the graph half as big as the part associated with the previous entry. The formula can therefore be constructed with paths of increasing size and primitive distances of decreasingly sized subgraphs, until only vertices are left whose names do not contain i or j at all. These all have the same distance to $\widehat{ij}$, namely 2^n, and they form a Sierpiński triangle graph $\widehat{S}_{p-2}^n$ (where $\widehat{S}_1^n$ is a single edge and $\widehat{S}_0^n$ has an empty edge set), which only needs to be stripped of its primitive and critical vertices. Finally, this leads to the following formula with fixed $\widehat{ij}$ (use $P^* = \widehat{P} \cup \left\{ \widehat{kl} \,\middle|\, \{k,l\} \in \binom{P}{2} \right\}$):

$$
\begin{aligned}
\sum_{s \in \widehat{V}_p^n \backslash P^*} \mathrm{d}(\widehat{ij}, s) &= 2\left(\tau_p^{n-1} - 2^{n-1}(p-1)\right) \\
&\quad + 2\sum_{a=1}^{n-1} (p-2)^a \Big(\tau_p^{n-1-a} - 2^{n-2-a} \\
&\qquad + \left(\left|\widehat{V}_p^{n-1-a}\right| - \frac{3}{2} \right)(2^n - 2^{n-a}) \Big) \\
&\quad + 2^n \left(\left|\widehat{V}_{p-2}^n\right| - (p-2) - \frac{1}{2}(p-2)(p-3) \right) \\
&= 2\left(2^{n-2}(p-1)(p^{n-1}+1) - 2^{n-1}(p-1)\right) \\
&\quad + 2\sum_{a=1}^{n-1} (p-2)^a \Big(2^{n-2-a}(p-1)(p^{n-1-a}+1) - 2^{n-2-a} \\
&\qquad + \left(\frac{p}{2}(p^{n-1-a}+1) - \frac{3}{2} \right)(2^n - 2^{n-a}) \Big) \\
&\quad + 2^n \left(\frac{p-2}{2}((p-2)^n + 1) - \frac{1}{2}(p-1)(p-2) \right) \\
&= 2^{n-1}\Big((p-1)(p^{n-1}-1) + (p-2)^2\big((p-2)^{n-1} - 1\big)\Big) \\
&\quad + \sum_{a=1}^{n-1} (p-2)^a \Big(2^{n-1-a}\big((p-1)(p^{n-1-a}+1) - 1\big)
\end{aligned}
$$

$$+ \big(p(p^{n-1-a}+1)-3\big)(2^n-2^{n-a})\Big)$$

There are $\frac{1}{2}p(p-1)$ critical vertices in $\widehat{S}_p^n$. Therefore,

$$\begin{aligned}\sum_{\{i,j\}\in\binom{P}{2},s\in\widehat{V}_p^n\setminus P^*} \mathrm{d}(\widehat{ij},s) = \frac{1}{2}p(p-1)\Bigg(& 2^{n-1}\Big((p-1)(p^{n-1}-1) \\ &+(p-2)^2\big((p-2)^{n-1}-1\big)\Big) \\ &+\sum_{a=1}^{n-1}(p-2)^a\Big(2^{n-1-a}\big((p-1)(p^{n-1-a}+1)-1\big) \\ &+\big(p(p^{n-1-a}+1)-3\big)(2^n-2^{n-a})\Big)\Bigg).\end{aligned}$$

In the case of $p=2$, this simplifies to

$$2^{n-1}\big(2^{n-1}-1\big),$$

as all terms in the sum are zero.

Now the only part of T_p^n left to calculate is

$$U_p^n = \sum_{i,j\in P;i<j;s,t\in\widehat{V}_p^{n-1}\setminus\widehat{P}} \mathrm{d}(is,jt).$$

This turns out to be pretty difficult. As a helpful tool, the following result can be used.

Lemma 8 (Cross distance of graphs). *Let $\widehat{S}_p^{n_1} = \big(\widehat{V}_p^{n_1},\widehat{E}_p^{n_1}\big)$ and $\widehat{S}_p^{n_2} = \big(\widehat{V}_p^{n_2},\widehat{E}_p^{n_2}\big)$ be two Sierpiński triangle subgraphs in a Sierpiński triangle graph $\widehat{S}_p^n$, $n_1,n_2<n$, such that for all pairs (s_1,s_2), $s_1\in\widehat{V}_p^{n_1}$, $s_2\in\widehat{V}_p^{n_2}$ the shortest path between s_1 and s_2 leads through the primitive vertex $\widehat{i_1}$ of $\widehat{S}_p^{n_1}$ and the primitive vertex $\widehat{i_2}$ of $\widehat{S}_p^{n_2}$. The exponent of $\widehat{S}_p^{n_1}$ is n_1, the exponent of $\widehat{S}_p^{n_2}$ is n_2 and their base is p. Then the sum of distances between one vertex in $\widehat{S}_p^{n_1}$ and another in $\widehat{S}_p^{n_2}$ is given by*

$$\sum_{s_1\in\widehat{V}_p^{n_1},s_2\in\widehat{V}_p^{n_2}} \mathrm{d}(s_1,s_2) = \tau_p^{n_1}\big|\widehat{V}_p^{n_2}\big| + \tau_p^{n_2}\big|\widehat{V}_p^{n_1}\big| + \mathrm{d}\big(\widehat{i_1},\widehat{i_2}\big)\big|\widehat{V}_p^{n_1}\big|\big|\widehat{V}_p^{n_2}\big|.$$

This sum of distances shall be called cross distance *of the two graphs.*

Proof. Under the following sum, s_1 and s_2 are understood to be $s_1 \in \widehat{V}_p^{n_1}$ and $s_2 \in \widehat{V}_p^{n_2}$; we calculate

$$\begin{aligned}\sum_{s_1,s_2} \mathrm{d}(s_1, s_2) &= \sum_{s_1,s_2}\Big(\mathrm{d}\big(s_1,\widehat{i_1}\big) + \mathrm{d}\big(\widehat{i_1},\widehat{i_2}\big) + \mathrm{d}\big(\widehat{i_2}, s_2\big)\Big)\\ &= \sum_{s_1,s_2} \mathrm{d}\big(s_1,\widehat{i_1}\big) + \sum_{s_1,s_2} \mathrm{d}\big(s_2,\widehat{i_2}\big) + \sum_{s_1,s_2} \mathrm{d}\big(\widehat{i_1},\widehat{i_2}\big)\\ &= |\widehat{V}_p^{n_2}| \sum_{s_1} \mathrm{d}\big(s_1,\widehat{i_1}\big) + |\widehat{V}_p^{n_1}| \sum_{s_2} \mathrm{d}\big(s_2,\widehat{i_2}\big) + |\widehat{V}_p^{n_1}||\widehat{V}_p^{n_2}|\mathrm{d}\big(\widehat{i_1},\widehat{i_2}\big)\\ &= |\widehat{V}_p^{n_2}|\tau_p^{n_1} + |\widehat{V}_p^{n_1}|\tau_p^{n_2} + |\widehat{V}_p^{n_1}||\widehat{V}_p^{n_2}|\mathrm{d}\big(\widehat{i_1},\widehat{i_2}\big).\end{aligned}$$

□

As, in the following, sums of distances between subgraphs in a Sierpiński triangle graph will be regarded, the following definition is very useful.

Definition 9 (Universal distance). Let $\widehat{S}_p^{n_1} = \big(\widehat{V}_p^{n_1}, \widehat{E}_p^{n_1}\big)$ and $\widehat{S}_p^{n_2} = \big(\widehat{V}_p^{n_2}, \widehat{E}_p^{n_2}\big)$ be two Sierpiński triangle graphs that are subgraphs[1] of a Sierpiński triangle graph $\widehat{S}_p^{n} = \big(\widehat{V}_p^{n}, \widehat{E}_p^{n}\big)$, in the sense that $\widehat{S}_p^{n_1}$ and $\widehat{S}_p^{n_2}$ are both Sierpiński triangle graphs, $n_1, n_2 < n$, and $\widehat{V}_p^{n_1}, \widehat{V}_p^{n_2} \subseteq \widehat{V}_p^{n}$. Then

$$\min_{s\in \widehat{S}_p^{n_1}, t\in \widehat{S}_p^{n_2}} \mathrm{d}(s,t)$$

is called the *universal distance* of the two subgraphs.

2.3 Recursions between subgraphs

Now for reasons of symmetry, one only needs to calculate the sum of distances where one vertex is a non-primitive vertex of the i-subgraph (meaning it is not $\widehat{i}$ or $\widehat{ik}$) and the other is a non-primitive vertex of the j-subgraph. It is helpful to differentiate between the critical vertices of the subgraph (i. e. those of the form $i\widehat{xy}$ or $j\widehat{xy}$, respectively) and the generic ones, which can be unambiguously identified with one subgraph of the respective subgraph. Momentarily ignoring the critical vertices, one can calculate the sum of distances between the vertices of the two subgraphs as follows.

[1] In all other places throughout this dissertation, subgraph means a subgraph with an exponent smaller by exactly one. However, in this definition it means any Sierpiński triangle graph whose vertices are contained in $\widehat{S}_p^n$.

For every pair of subgraphs k and l of the i- and j-subgraph respectively, the universal distance is given by

$$\min_{s \in ik\widehat{S}_p^{n-2}, t \in jl\widehat{S}_p^{n-2}} \mathrm{d}(s,t) = 2^{n-2}((k \neq j) + (l \neq i)).$$

This is due to the fact that the universal distance can always be achieved via the vertex $\widehat{ij}$ of $\widehat{S}_p^n$. For the subgraph k of the i-subgraph look at the vertex $i\widehat{jk} \in ik\widehat{S}_p^n$ and for the l-subgraph of the j-subgraph, look at $j\widehat{il} \in \widehat{S}_p^n$. The distance between these two vertices in $\widehat{S}_p^n$ is

$$\mathrm{d}\big(i\widehat{jk}, j\widehat{il}\big) = 2^{n-1}.$$

(For $k = j$ or $i = l$, the vertices $i\widehat{jk}$ or $j\widehat{il}$ make no sense. However, in these cases $\widehat{ij}$ is part of the subgraph in question leading to an even shorter distance.) On any indirect path, a subgraph other than the i- or j-subgraph has to be traversed, taking at least 2^{n-1}, which is the diameter of this subgraph.

For every pair (s,t) of generic vertices of the i- and j-subgraph, the universal distance between their subgraphs is part of the distance between them. The sum of universal distances for all pairs of generic vertices can be constructed as

$$\begin{aligned}
&2^{n-2}\big(|\widehat{V}_p^{n-2}| - p\big)^2 \sum_{k,l \in P}\big((k \neq j) + (l \neq i)\big) \\
&= 2^{n-2}\Big(\frac{p}{2}(p^{n-2} - 1)\Big)^2 \big(2(p-1)^2 + 2(p-1)\big) \\
&= 2^{n-3} p^3 (p-1)(p^{n-2} - 1)^2.
\end{aligned}$$

Now, with the restriction to generic vertices, every pair of subgraphs k of the i-subgraph and l of the j-subgraph can be treated as if they shared a common primitive vertex. These subgraphs now fall in four categories.

1. Vertices from the j-subgraph of the i-subgraph in combination with vertices from the whole j-subgraph, or vertices from the whole i-subgraph in combination with vertices from the i-subgraph of the j-subgraph, or vertices from the i-subgraph of the i-subgraph in combination with vertices from the j-subgraph of the j-subgraph; shortest distances can only be achieved on a shortest path through $\widehat{ij}$ (the *direct connection*), allowing for the use of the cross distance formula.

2. Vertices from the i-subgraph of the i-subgraph combined with vertices of the k-subgraph ($k \in P \backslash \{i,j\}$) of the j-subgraph, or vice versa; the shortest paths pass either through $\widehat{ij}$ or $\widehat{ik}$ and $\widehat{jk}$, but the direct path is advantageous as of now. As a result of

the universal distance correction above, the subgraphs can be treated as sharing a common vertex (the vertex $\widehat{iij}$ associated with $\widehat{jik}$, or the vertex $\widehat{ijk}$ associated with $\widehat{jij}$) and the possible indirect path also shortened by the universal distance (which is 2^{n-1} in this case). The corresponding sum of distances for such a pair of subgraphs, both with exponent n and base p, universal distance not included, shall be denoted by A_p^n.

3. Vertices from the k-subgraph of the i-subgraph combined with vertices of the k-subgraph of the j-subgraph; the shortest paths pass through $\widehat{ij}$ or $\widehat{ik}$ and $\widehat{jk}$; both possibilities leading to equal universal distance as of now. As in the previous case, the subgraphs can be treated as sharing a common vertex ($\widehat{ijk}$ associated with $\widehat{jik}$) and the other possible path is also shortened by the universal distance 2^{n-1}. This leads to the association of two other vertices, $\widehat{ik}$ and $\widehat{jk}$, as their distance is reduced to zero. The sum of distances between two such subgraphs of exponent n and base p, universal distance not included, shall be denoted by B_p^n.

4. Vertices of the k_1-subgraph of the i-subgraph combined with vertices of the k_2-subgraph of the j-subgraph; there are now three possibilities for the shortest path: passing through $\widehat{ij}$, or through $\widehat{ik_1}$ and $\widehat{jk_1}$, or through $\widehat{ik_2}$ and $\widehat{jk_2}$, with the direct path advantageous so far. Again, the subgraphs can be treated as sharing a common vertex ($\widehat{ijk_1}$ associated with $\widehat{jik_2}$) and the possible indirect paths are shortened by the universal distance of 2^{n-1}. In this case the sum of distances between two such subgraphs of exponent n and base p, universal distance not included, shall be called F_p^n.

Still paying no attention to the critical vertices, it can be seen that the total sum of distances between one vertex in the i-subgraph and one vertex in the j-subgraph has:

1. $2p$ pairs of the first type. The common attribute is that shortest paths include the vertex $\widehat{ij}$. The sum of distances, not including universal distance, is given by the cross distance of the two subgraphs of subgraphs without their primitive vertices.

$$2 \cdot 2p\Big(\big|\widehat{V}_p^{n-2}\big| - p\Big)\Big(\tau_p^{n-2} - (p-1)2^{n-2}\Big) = 2^{n-2}p^2(p-1)(p^{n-2}-1)^2.$$

2. $2(p-2)$ pairs of the second type. This contributes

$$2(p-2)A_p^{n-2}$$

to the aforementioned total sum.

3. $(p-2)$ pairs of the third type, contributing
$$(p-2)B_p^{n-2}$$
to said sum.

4. $(p-2)(p-3)$ pairs of type four, contributing
$$(p-2)(p-3)F_p^{n-2}$$
to the total sum.

The numbers of pairs sum up to
$$2p + 2(p-2) + (p-2) + (p-2)(p-3) = 5p - 6 + p^2 - 5p + 6 = p^2,$$
and there are obviously p^2 ordered pairs of subgraphs.

Together, the sum of distances of all pairs of a generic vertex in the i-subgraph and a generic vertex in the j-subgraph is
$$\begin{aligned} &2^{n-3}p^3(p-1)(p^{n-2}-1)^2 + 2^{n-2}p^2(p-1)(p^{n-2}-1)^2 \\ &\quad + 2(p-2)A_p^{n-2} + (p-2)B_p^{n-2} + (p-2)(p-3)F_p^{n-2} \\ &= 2^{n-3}p^2(p-1)(p+2)(p^{n-2}-1)^2 \\ &\quad + 2(p-2)A_p^{n-2} + (p-2)B_p^{n-2} + (p-2)(p-3)F_p^{n-2}. \end{aligned}$$

Now the critical vertices shall be taken care of. Making use of symmetries, it suffices to calculate the sum of distances between the critical vertices in the i-subgraph (i. e. of the form $i\widehat{xy}$) and all the non-primitive vertices in the j-subgraph. This number can then be doubled, and finally the distances between critical vertices of the two subgraphs can be subtracted once, as they are taken into account twice until then.

The shortest path from all vertices of the form $i\widehat{jx}$ with $x \in P\backslash\{j\}$ to vertices in the j-subgraph is the direct path. As they are all 2^{n-2} away from the critical vertex $\widehat{ij}$ and there is $p-1$ of these vertices, this adds
$$\begin{aligned} &(p-1)\Big(2^{n-2}\big(|\widehat{V}_p^{n-1}| - p\big) + \tau_p^{n-1} - (p-1)2^{n-1}\Big) \\ &= (p-1)\big(2^{n-3}p(p^{n-1}-1) + 2^{n-2}(p-1)(p^{n-1}-1)\big) \\ &= 2^{n-3}(p-1)(3p-2)(p^{n-1}-1) \end{aligned}$$
to the total sum.

For the vertices of the form $\widehat{iik}$ with $k \in P\backslash\{i,j\}$, the direct path is optimal for all subgraphs of the j-subgraph but the k-subgraph of the j-subgraph. The k-subgraph behaves in a way that will keep occurring later on when critical vertices are to be considered, and seen similarly above with the critical vertices of the graph as a whole. The formula for the sum of distances between such a vertex, of which there are $(p-2)$, and an arbitrary one is given by

$$\begin{aligned}
&2^{n-1}\big(|\widehat{V}_p^{n-1}| - p\big) + \tau_p^{n-1} - (p-1)2^{n-1} \\
&- \big(\tau_p^{n-2} - (p-1)2^{n-2}\big) + 2^{n-2}\big(|\widehat{V}_{p-2}^{n-2}| - (p-2)\big) \\
&+ 2\sum_{a=1}^{n-2}(p-2)^{a-1}\cdot\Big(\tau_p^{n-2-a} - 2^{n-3-a} + \big(|\widehat{V}_p^{n-2-a}| - \frac{3}{2}\big)(2^{n-2} - 2^{n-1-a})\Big) \\
=\ &2^{n-2}p(p^{n-1}-1) + 2^{n-2}(p-1)(p^{n-1}-1) \\
&- 2^{n-3}(p-1)(p^{n-2}-1) + 2^{n-3}(p-2)\big((p-2)^{n-2}-1\big) \\
&+ \sum_{a=1}^{n-2}(p-2)^{a-1}\Big(2^{n-2-a}((p-1)(p^{n-2-a}+1)-1\big) \\
&\quad + \big(2^{n-2} - 2^{n-1-a}\big)\big(p(p^{n-2-a}+1)-3\big)\Big) \\
=\ &2^{n-2}(2p-1)\big(p^{n-1}-1\big) - 2^{n-3}(p-1)\big(p^{n-2}-1\big) + \Lambda_p^{n-2},
\end{aligned}$$

where, for the sake of simplicity, the definition

$$\begin{aligned}
\Lambda_p^n = &\sum_{a=1}^{n}(p-2)^{a-1}(2^{n-a}((p-1)(p^{n-a}+1)-1) \\
&+ \big(2^n - 2^{n+1-a}\big)(p(p^{n-a}+1)-3)\big) + 2^{n-1}(p-2)((p-2)^n-1)
\end{aligned}$$

is introduced. It will be used again later.

The remaining critical vertices are of the form $\widehat{ik_1k_2}$ with $(k_1,k_2) \in \binom{P\backslash\{i,j\}}{2}$. For these, the situation is similar to the one above, with the exception of two subgraphs of the j-subgraph containing vertices with possibly an indirect shortest path. The formula for the sum of distances between such a vertex, of which there are $\frac{1}{2}(p-2)(p-3)$, and all the other vertices is therefore given by

$$\begin{aligned}
&2^{n-1}\big(|\widehat{V}_p^{n-1}| - p\big) + \tau_p^{n-1} - (p-1)2^{n-1} - 2\Big(\tau_p^{n-2} - (p-1)2^{n-2}\Big) \\
&+ 4\sum_{i=1}^{n-2}(p-2)^{a-1}\cdot\Big(\tau_p^{n-2-a} - 2^{n-3-a} + \big(|\widehat{V}_p^{n-2-a}| - \frac{3}{2}\big)(2^{n-2} - 2^{n-1-a})\Big)
\end{aligned}$$

p	Formula for given p	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
3	$\frac{1}{10}(7\cdot 6^n - 5\cdot 2^n - 2)$	3	23	147	899	5427	32627
4	$\frac{7}{6}(8^n - 2^n)$	7	70	588	4760	38192	305706
5	$\frac{1}{14}(23\cdot 10^n - 2\cdot 3^n - 21\cdot 2^n)$	13	157	1627	16393	164203	1642657
6	$\frac{1}{8}(17\cdot 12^n - 4^n - 16\cdot 2^n)$	21	296	3648	44000	528576	6344576

Table 2.1: Values of Λ_p^n for $n \in \{1,2,3,4,5,6\}$ and $p \in \{3,4,5,6\}$. $\Lambda_p^0 = 0$ for every $p \geq 3$, should it be needed, and $\Lambda_2^n = 0$.

$$
\begin{aligned}
&\quad + 2^{n-1}\big(\big|\widehat{V}_{p-2}^{n-2}\big| - (p-2)\big)\\
&= 2^{n-2}p(p^{n-1}-1) + 2^{n-2}(p-1)(p^{n-1}-1)\\
&\quad - 2^{n-2}(p-1)(p^{n-2}-1) + 2^{n-2}(p-2)\big((p-2)^{n-2}-1\big)\\
&\quad + 2\sum_{a=1}^{n-2}(p-2)^{a-1}\Big(2^{n-2-a}\big((p-1)\big(p^{n-2-a}+1\big)-1\big)\\
&\qquad + \big(2^{n-2}-2^{n-1-a}\big)\big(p(p^{n-2-a}+1)-3\big)\Big)\\
&= 2^{n-2}(2p-1)(p^{n-1}-1) - 2^{n-2}(p-1)(p^{n-2}-1) + 2\Lambda_p^{n-2}.
\end{aligned}
$$

Together there are

$$\frac{1}{2}(p-2)(p-3) + (p-2) + (p-1) = \frac{1}{2}\big(p^2 - 5p + 6 + 2p - 4 + 2p - 2\big) = \frac{1}{2}p(p-1)$$

critical vertices.

These formulas can now be summed up, the ones for critical vertices doubled and the distances between critical vertices subtracted once. The sum of distances between a critical vertex of one graph and another of the other can be found by calculating the sum of distances between the critical vertices of a graph and the common vertex of the two, multiplying it by the number of critical vertices in the other and doubling, mimicking the cross distance formula; or calculating the cross distance for two $\widehat{S}_p^1$ connected in one vertex, as in this case the direct path is always optimal between critical vertices, and scaling up with the exponent. This leads to $2^{n-2}p(p-1)^3$. Therefore,

$$
\begin{aligned}
&2^{n-2}(p-1)(3p-2)(p^{n-1}-1)\\
&+ 2(p-2)\Big(2^{n-2}(2p-1)(p^{n-1}-1) - 2^{n-3}(p-1)(p^{n-2}-1) + \Lambda_p^{n-2}\Big)
\end{aligned}
$$

p	Formula for set p
2	$\frac{1}{8}(8^n - 4 \cdot 4^n + 4 \cdot 2^n)$
3	$\frac{1}{12}(5 \cdot 18^n - 8 \cdot 6^n - 3 \cdot 2^n) + 6\Lambda_3^{n-2}$
4	$\frac{9}{32}(3 \cdot 32^n - 4 \cdot 8^n - 128 \cdot 2^n) + 48\Lambda_4^{n-2}$
5	$\frac{1}{5}(7 \cdot 50^n + 32 \cdot 10^n - 1175 \cdot 2^n) + 180\Lambda_5^{n-2}$
6	$\frac{25}{12}(72^n + 14 \cdot 12^n - 450 \cdot 2^n) + 480\Lambda_6^{n-2}$

p \ n	2	3	4	5	6
2	2	36	392	3600	30752
3	72	2082	41706	775218	14099346
4	648	27120	882912	28301760	905900928
5	3200	181860	8838500	438425340	$\approx 22 \cdot 10^9$
6	11250	830580	56719080	4040057040	$\approx 290 \cdot 10^9$

Table 2.2: Values of v_p^n for $n \in \{2,3,4,5,6\}$ and $p \in \{2,3,4,5,6\}$.

$$
\begin{aligned}
&+ (p-2)(p-3)\Big(2^{n-2}(2p-1)(p^{n-1}-1) - 2^{n-2}(p-1)(p^{n-2}-1) + 2\Lambda_p^{n-2}\Big)\\
&- 2^{n-2}p(p-1)^3 + 2^{n-3}p^2(p-1)(p+2)(p^{n-2}-1)^2\\
&+ 2(p-2)A_p^{n-2} + (p-2)B_p^{n-2} + (p-2)(p-3)F_p^{n-2}\\
= {}& 2^{n-1}p(p-1)^2(p^{n-1}-1) - 2^{n-2}(p-1)(p-2)^2(p^{n-2}-1)\\
&- 2^{n-2}p(p-1)^3 + 2^{n-3}p^2(p-1)(p+2)(p^{n-2}-1)^2\\
&+ 2(p-2)A_p^{n-2} + (p-2)B_p^{n-2} + (p-2)(p-3)F_p^{n-2} + 2(p-2)^2\Lambda_p^{n-2}.
\end{aligned}
$$

As there are $\frac{1}{2}p(p-1)$ pairs of subgraphs in $\widehat{S}_p^n$ with $n \geq 2$, this leads to

$$
\begin{aligned}
U_p^n &= \sum_{i,j\in P;i<j;s,t\in \widehat{V}_p^{n-1}\setminus\widehat{P}} \mathrm{d}(is,jt)\\
&= \frac{1}{2}p(p-1)\Bigg(2^{n-1}p(p-1)^2(p^{n-1}-1) - 2^{n-2}(p-1)(p-2)^2(p^{n-2}-1)\\
&\quad - \frac{1}{1}2^{n-2}p(p-1)^3 + 2^{n-3}p^2(p-1)(p+2)(p^{n-2}-1)^2 + 2(p-2)^2\Lambda_p^{n-2}\Bigg)
\end{aligned}
$$

$$+ p(p-1)(p-2)A_p^{n-2} + \frac{1}{2}p(p-1)(p-2)B_p^{n-2} + \frac{1}{2}p(p-1)(p-2)(p-3)F_p^{n-2}$$
$$= v_p^n + p(p-1)(p-2)A_p^{n-2} + \frac{1}{2}p(p-1)(p-2)B_p^{n-2} + \frac{1}{2}p(p-1)(p-2)(p-3)F_p^{n-2},$$

where v_p^n is defined as

$$\begin{aligned} v_p^n = &\frac{1}{2}p(p-1)\Big(2^{n-1}p(p-1)^2(p^{n-1}-1) - 2^{n-2}(p-1)(p-2)^2(p^{n-2}-1) \\ &- 2^{n-2}p(p-1)^3 + 2^{n-3}p^2(p-1)(p+2)(p^{n-2}-1)^2 + 2(p-2)^2\Lambda_p^{n-2}\Big) \\ = &\, 2^{n-4}p(p-1)^2\Big(4p(p-1)(p^{n-1}-1) - 2(p-2)^2(p^{n-2}-1) - 2p(p-1)^2 \\ &+ p^2(p+2)(p^{n-2}-1)^2\Big) + p(p-1)(p-2)^2\Lambda_p^{n-2}. \end{aligned}$$

It is now necessary to analyse A_p^n, B_p^n and F_p^n, which will recursively link back to each other with lower n. One can see that F_p^n is a special case here, because, while it uses A_p^{n-1} and B_p^{n-1}, A_p^n and B_p^n clearly make no use of F_p^{n-1}. For A- and B-cases, there is already only one other subgraph through which shortest paths may pass, so they cannot consist of smaller F-cases. Furthermore, for $p = 2$ one can see that $U_p^n = v_p^n$, so that A_2^n, B_2^n and F_2^n are not needed.

2.3.1 *A*-cases

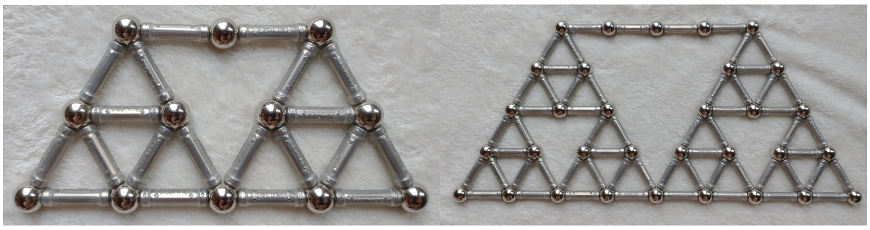

Figure 2.2: The A_3^1-case (left) and A_3^2-case (right). The line above is the long connection, while the shared vertex between the two triangles is the direct connection.

First, A_p^n shall be considered. It is modelled on considerations about two Sierpiński triangle graphs of exponent n joined in a common primitive vertex *(direct connection)*, where a path of length 2^n connects two other primitive vertices *(long connection)*, one of each graph. The universal distance between the two graphs has already been calculated separately above. Now the goal is to find the sum of distances between one vertex in the first and the other in the second of these graphs, neither being primitive vertices of their graphs. At first, critical vertices shall be ignored. The universal distance of a pair of the graphs' subgraphs is 0, if both subgraphs border the direct connection, 2^{n-1}, if only one of them does, or 2^n if neither borders the direct connection. Therefore, the sum of universal distances is

$$\begin{aligned} &2^{n-1}\left(\left|\widehat{V}_p^{n-1}\right| - p\right)^2\left(2(p-1) + 2(p-1)^2\right) \\ =\; &2^{n-2}p^3(p-1)(p^{n-1}-1)^2. \end{aligned}$$

Now the subgraphs can be treated as if they were connected in one vertex. Obviously, when any vertex in the subgraph bordering the direct connection of the two graphs participates in the distance, the direct path is optimal. The same is true, if neither vertex is in a subgraph bordering the long connection. Therefore, there are

$$p + p - 1 + (p-2)^2 = p^2 - 2p + 3$$

pairs of subgraphs which use the direct connection for every pair of vertices. Using the cross distance formula, this leads to

$$\begin{aligned} &2\left(p^2 - 2p + 3\right)\left(\left|\widehat{V}_p^{n-1}\right| - p\right)\left(\tau_p^{n-1} - 2^{n-1}(p-1)\right) \\ =\; &2^{n-2}p(p-1)(p^2-2p+3)(p^{n-1}-1)^2. \end{aligned}$$

If both vertices lie in the subgraphs bordering the long connection, this is a B-case, as these subgraphs are equally distanced by the long connection and the subgraphs leading to the direct connection (both of diameter 2^{n-1}). This is a universal distance and is therefore already handled, so only

$$B_p^{n-1}$$

has to be added to the formula.

Cases where one vertex is in a subgraph bordering the long connection, while the other is in a 'neutral' subgraph (i. e. bordering neither the long connection nor the direct connection), lead to another A-case, just of smaller exponent. There are $2(p-2)$ such pairs of subgraphs. As the universal distance is already included in the respective formula above, one only needs to add

$$2(p-2)A_p^{n-1}.$$

Next, the critical vertices shall be considered. It is helpful to use the same trick as above, calculating the sum of distances between the critical vertices of one graph and any vertex of the other, then doubling the result and subtracting $2^{n-1}p(p-1)^3$, the sum of distances between a critical vertex in one and one in the other graph, as this sum was counted twice. The only critical vertices for which the shortest path does not lead through the long connection for all vertices of the other graph are the critical vertices between the subgraph bordering the long connection and another subgraph that does not border the direct connection. There are $p-2$ such critical vertices, thus adding

$$\begin{aligned}&(p-2)\Big(2^n\big(|\widehat{V}_p^n|-p\big)+\tau_p^n-2^n(p-1)-\big(\tau_p^{n-1}-2^{n-1}(p-1)\big)+\Lambda_p^{n-1}\Big)\\&=(p-2)\Big(2^{n-1}(2p-1)(p^n-1)-2^{n-2}(p-1)(p^{n-1}-1)+\Lambda_p^{n-1}\Big).\end{aligned}$$

For the remaining critical vertices, the direct path is optimal for all vertices of the other graph. There are $\frac{1}{2}\big(p^2-3p+4\big)$ critical vertices remaining. The distance between the critical vertex and the common vertex of the two graphs is 2^n for $\frac{1}{2}(p-2)(p-3)$ of the critical vertices and 2^{n-1} for the $(p-1)$ others (the ones between the subgraph bordering the direct connection and any other subgraph). Considering all this, one has to add

$$\begin{aligned}&\frac{1}{2}(p^2-3p+4)\big(\tau_p^n-2^n(p-1)\big)+2^{n-1}(p^2-4p+5)\big(|\widehat{V}_p^n|-p\big)\\&=2^{n-2}(p-1)(p^2-3p+4)(p^n-1)+2^{n-2}p(p^2-4p+5)(p^n-1)\\&=2^{n-1}(p^3-4p^2+6p-2)(p^n-1).\end{aligned}$$

Putting together all the parts, one gets

$$\begin{aligned}A_p^n&=2^{n-2}p^3(p-1)(p^{n-1}-1)^2+2^{n-2}p(p-1)(p^2-2p+3)(p^{n-1}-1)^2\\&\quad+2(p-2)\Big(2^{n-1}(2p-1)(p^n-1)-2^{n-2}(p-1)(p^{n-1}-1)+\Lambda_p^{n-1}\Big)\\&\quad+2^n(p^3-4p^2+6p-2)(p^n-1)-2^{n-1}p(p-1)^3+2(p-2)A_p^{n-1}+B_p^{n-1}\\&=\alpha_p^n+2(p-2)A_p^{n-1}+B_p^{n-1},\end{aligned}$$

where

$$\begin{aligned}\alpha_p^n&=2^{n-2}p^3(p-1)(p^{n-1}-1)^2+2^{n-2}p(p-1)(p^2-2p+3)(p^{n-1}-1)^2\\&\quad+2(p-2)\big(2^{n-1}(2p-1)(p^n-1)-2^{n-2}(p-1)(p^{n-1}-1)\big)\\&\quad+2^n(p^3-4p^2+6p-2)(p^n-1)-2^{n-1}p(p-1)^3+2(p-2)\Lambda_p^{n-1}\\&=2^{n-2}p(p-1)(2p^2-2p+3)(p^{n-1}-1)^2+2^np(p-1)^2(p^n-1)\\&\quad-2^{n-1}(p-1)(p-2)(p^{n-1}-1)-2^{n-1}p(p-1)^3+2(p-2)\Lambda_p^{n-1}.\end{aligned}$$

For A_p^0, the formula above does not make too much sense. As the two connected graphs only consist of primitive vertices in the case of A_p^0, and these are ruled out from the distance calculation and handled otherwise, the only sensible thing to do is to set $A_p^0=0$

p	Formula for set p
3	$\frac{1}{6}(15 \cdot 18^n - 20 \cdot 6^n - 3 \cdot 2^n) + 2\Lambda_3^{n-1}$
4	$\frac{3}{16}(27 \cdot 32^n - 28 \cdot 8^n - 32 \cdot 2^n) + 4\Lambda_4^{n-1}$
5	$\frac{1}{5}(43 \cdot 50^n - 36 \cdot 10^n - 95 \cdot 2^n) + 6\Lambda_5^{n-1}$
6	$\frac{5}{24}(63 \cdot 72^n - 44 \cdot 12^n - 204 \cdot 2^n) + 8\Lambda_6^{n-1}$

p \ n	1	2	3	4	5	6
3	24	694	13902	258406	4699782	84885862
4	108	4852	163432	5289168	169716128	5434594112
5	320	20782	1068590	53687458	2686877750	$\approx 134 \cdot 10^9$
6	750	66718	4885068	352557784	$\approx 25 \cdot 10^9$	$\approx 1828 \cdot 10^9$

Table 2.3: Values of α_p^n for $n \in \{1, 2, 3, 4, 5, 6\}$ and $p \in \{3, 4, 5, 6\}$. Notice that $A_p^0 = B_p^0 = F_p^0 = 0$ makes sense, since empty sets of vertices are regarded. Therefore, it should hold that $A_p^1 = \alpha_p^1$.

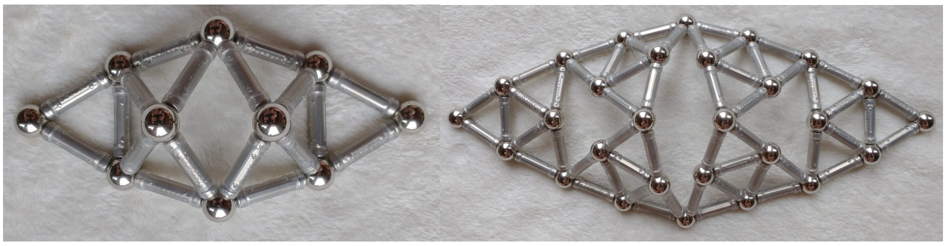

Figure 2.3: The B_3^1-case (left) and the B_3^2-case (right). The two graphs are treated as being connected in two common primitive vertices.

by definition. (Along the same lines, $B_p^0 = 0$ and $F_p^0 = 0$ make sense.) Assuming this definition, one finds that

$$A_p^1 = \alpha_p^n = p(p-1)^3,$$

which is exactly the sum of distances between a non-primitive vertex in one and a non-primitive vertex in the other subgraph. This was already used twice as the correction term for critical vertices above.

2.3.2 *B*-cases

B_p^n can be calculated in a similar fashion to A_p^n. As a way of visualising how B_p^n is constructed, one may imagine two Sierpiński triangle graphs of base p and exponent n, which share two primitive vertices (called *connections*). Again, critical vertices are ignored at first. As opposed to the A-cases, universal distances of pairs of subgraphs do not all necessarily pass through the same connection (which, in the A-case, is the direct connection). The universal distance is 0, if both subgraphs in the pair border the same connection, 2^{n-1}, if one borders a connection the other does not, and 2^n, if neither borders any connection. This means that there are 2 pairs with a universal distance of 0, $4p-6$ with a universal distance[2] of 2^{n-1} and $(p-2)^2$ pairs, whose universal distance is 2^n. Together, these are

$$2 + 4p - 6 + (p-2)^2 = p^2,$$

[2]The $4p-6$ pairs consist of the following groups: a) The subgraph of the first graph bordering the one connection, together with all subgraphs of the other graph but the one at the same long connection. These are $p-1$ pairs. b) The same for the other connection, thus another $(p-1)$. c) The subgraph of the other graph bordering the one connection, together with all subgraphs of the other graph not bordering a connection. (The one subgraph bordering a connection leads to a universal distance of 0 and the other has already been handled under a).) These are $p-2$ pairs. d) The same as c) for the other connection, which are $p-2$ pairs again.
Together, there are $4p-6$ pairs with a universal distance of 2^{n-1}.

so all pairs of subgraphs are taken care of with regard to universal distances, and the universal distances sum up to

$$\begin{aligned} &2^{n-1}\big(|\widehat{V}_p^{n-1}| - p\big)^2\big(4p - 6 + 2(p-2)^2\big) \\ =\ &2^{n-2}p^2(p-1)^2(p^{n-1}-1)^2. \end{aligned}$$

The pairs of subgraphs can now be treated as being connected in at least one primitive vertex. They fall into the following categories:

If both subgraphs border the same connection, the optimal path is the direct one for all pairs of vertices in the two subgraphs. Using the cross distance formula, this leads to

$$\begin{aligned} &2 \cdot 2\big(|\widehat{V}_p^{n-1}| - p\big)\big(\tau_p^{n-1} - 2^{n-1}(p-1)\big) \\ =\ &2^{n-1}p(p-1)(p^{n-1}-1)^2. \end{aligned}$$

If only one subgraph of the pair borders a connection, while the other does not, this leads to an A-case. As a result of the universal distances being handled above, the subgraphs can be treated as being connected in one primitive vertex, the indirect distance shortened by the universal distance 2^{n-1}, thus leading to a direct and a long connection. There are $4(p-2)$ such pairs, adding to the term

$$4(p-2)A_p^{n-1}.$$

If neither subgraph of a pair borders a connection or the two subgraphs border different connections, they can be treated as sharing two primitive vertices, as soon as universal distances are calculated. They thereby recur to a B-case of lower exponent. As there are

$$(p-2)^2 + 2 = p^2 - 4p + 6$$

such pairs, this leads to

$$(p^2 - 4p + 6)B_p^{n-1}.$$

Until this point, the critical vertices of the two graphs were omitted. Now they shall be taken care of. Again, the method from above is used, calculating the distances for the critical vertices of one graph, doubling the sum, and finally subtracting the distances between two critical vertices once.

For the critical vertices between two subgraphs bordering neither connection as well as the critical vertex between the two subgraphs at the connections, the other graph is completely symmetrical. In the first case, this leads to

$$2^n\left(|\widehat{V}_p^n| - p\right) + \Lambda_p^n$$

and in the second to

$$2^{n-1}\left(|\widehat{V}_p^n| - p\right) + \Lambda_p^n.$$

While the critical vertex between the subgraphs bordering the connection is unique, there are $\frac{1}{2}(p-2)(p-3)$ critical vertices of the other type, so this adds up to

$$\begin{aligned} &2^{n-1}\big((p-2)(p-3)+1\big)\left(|\widehat{V}_p^n| - p\right) + \left(\frac{1}{2}(p-2)(p-3)+1\right)\Lambda_p^n \\ &= 2^{n-2}p(p^2-5p+7)(p^n-1) + \frac{1}{2}(p^2-5p+8)\Lambda_p^n. \end{aligned}$$

The remaining critical vertices are between a subgraph bordering a connection and one that does not. There are $2(p-2)$ of these pairs of subgraphs. For the vertices between the subgraph at one connection and a neutral one, the path through said connection is optimal for all vertices of the other graph but these in the subgraph bordering the other connection. This subgraph is reached equally fast through both connections. Altogether, this leads to

$$\begin{aligned} &2\cdot(p-2)\Big(2^{n-1}\left(|\widehat{V}_p^n| - p\right) + \tau_p^n - 2^n(p-1) - \left(\tau_p^{n-1} - 2^{n-1}(p-1)\right) + \Lambda_p^{n-1}\Big) \\ &= 2^{n-1}(p-2)(3p-2)(p^n-1) - 2^{n-1}(p-1)(p-2)(p^{n-1}-1) + 2(p-2)\Lambda_p^{n-1}. \end{aligned}$$

The distances between the critical vertices in such a case are $2^{n-1}\cdot B_p^1$, as in the B_p^1-case there are only critical vertices. The distance between vertices in B_p^1 is 2, if both vertices link to the respective subgraphs that border the same connection. There are

$$2p^2 - 4p + 1$$

such pairs of vertices. It is 4, if both vertices lie between subgraphs bordering neither connection. Of these vertices,

$$\left(\frac{1}{2}(p-2)(p-3)\right)^2$$

exist. In all other cases, the distance is 3. This covers the

$$2p^3 - 11p^2 + 19p - 10$$

pairs of vertices remaining. Therefore,

$$\begin{aligned}B_p^1 &= 2(2p^2 - 4p + 1) + 3(2p^3 - 11p^2 + 19p - 10) + 4\Big(\frac{1}{2}(p-2)(p-3)\Big)^2 \\ &= p^4 - 4p^3 + 8p^2 - 11p + 8 \\ &= (p-1)(p-2)(p^2 - p + 3) + 2.\end{aligned}$$

Altogether the formula for B_p^n is

$$\begin{aligned}B_p^n &= 2^{n-2}p^2(p-1)^2(p^{n-1}-1)^2 + 2^{n-1}p(p-1)(p^{n-1}-1)^2 \\ &\quad + 2^{n-1}p(p^2-5p+7)(p^n-1) + 2^n(p-2)(3p-2)(p^n-1) \\ &\quad - 2^n(p-1)(p-2)(p^{n-1}-1) - 2^{n-1}(p-1)(p-2)(p^2-p+3) - 2^n \\ &\quad + 4(p-2)A_p^{n-1} + (p^2-4p+6)B_p^{n-1} + (p^2-5p+8)\Lambda_p^n + 4(p-2)\Lambda_p^{n-1} \\ &= 2^{n-2}p(p-1)(p^2-p+2)(p^{n-1}-1)^2 + 2^{n-1}(p^3+p^2-9p+8)(p^n-1) \\ &\quad - 2^{n-1}(p-1)(p-2)\big(2(p^{n-1}-1) + (p^2-p+3)\big) - 2^n \\ &\quad + 4(p-2)A_p^{n-1} + (p^2-4p+6)B_p^{n-1} + (p^2-5p+8)\Lambda_p^n + 4(p-2)\Lambda_p^{n-1} \\ &= \beta_p^n + 4(p-2)A_p^{n-1} + (p^2-4p+6)B_p^{n-1},\end{aligned}$$

where

$$\begin{aligned}\beta_p^n &= 2^{n-2}p(p-1)(p^2-p+2)(p^{n-1}-1)^2 + 2^{n-1}(p^3+p^2-9p+8)(p^n-1) \\ &\quad - 2^{n-1}(p-1)(p-2)\big(2(p^{n-1}-1) + (p^2-p+3)\big) - 2^n \\ &\quad + (p^2-5p+8)\Lambda_p^n + 4(p-2)\Lambda_p^{n-1}.\end{aligned}$$

The value of B_p^1, where the recursion does not work, is given by

$$B_p^1 = (p-1)(p-2)(p^2-p+3) + 2.$$

2.3.3 *F*-cases

The final component is F_p^n. With the universal distance subtracted, it is modelled after two Sierpiński triangle graphs of exponent n (and base p) that share a common primitive vertex (again called *direct connection*), and there would be two pairs of primitive vertices, each containing one vertex of each graph, such that between the two vertices in a pair, there is a path of length 2^n to connect them (so called *long connections*). As usual, critical vertices are ignored at first. As in the A-case, the universal distance between a subgraph

p	Formula for set p
3	$\frac{1}{6}(8 \cdot 18^n - 6^n - 27 \cdot 2^n) + 2\Lambda_3^n + 4\Lambda_3^{n-1}$
4	$\frac{1}{8}(21 \cdot 32^n + 28 \cdot 8^n - 192 \cdot 2^n) + 4\Lambda_4^n + 8\Lambda_4^{n-1}$
5	$\frac{1}{10}(44 \cdot 50^n + 101 \cdot 10^n - 735 \cdot 2^n) + 8\Lambda_5^n + 12\Lambda_5^{n-1}$
6	$\frac{1}{3}(20 \cdot 72^n + 59 \cdot 12^n - 522 \cdot 2^n) + 14\Lambda_6^n + 16\Lambda_6^{n-1}$

p \ n	1	2	3	4	5	6
3	20	466	8090	142066	2532434	45428530
4	92	3152	90528	2790208	88385152	2821016616
5	278	13128	574412	27750492	1377517988	$\approx 68 \cdot 10^9$
6	662	41176	2576720	180238432	$\approx 13 \cdot 10^9$	$\approx 929 \cdot 10^9$

Table 2.4: Values of β_p^n for $n \in \{1, 2, 3, 4, 5, 6\}$ and $p \in \{3, 4, 5, 6\}$.

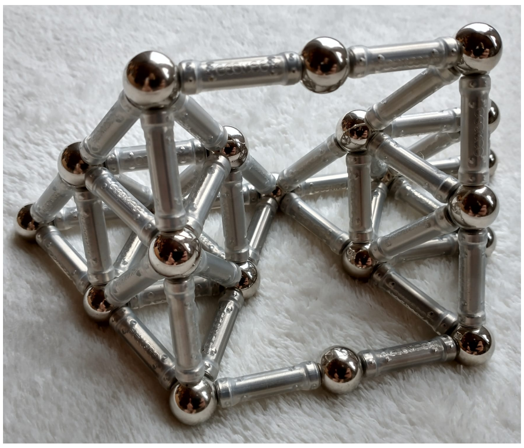

Figure 2.4: The F_4^1-case. There is a shared vertex and two long connections.

of the one and a subgraph of the other graph is always achieved via the direct connection, and therefore, the sum of universal distances is the same as for the A-case:

$$2^{n-2}p^3(p-1)(p^{n-1}-1)^2.$$

If one of the two subgraphs borders the direct connection, or if neither subgraph borders any connection, the direct path is optimal, and the cross distance formula can be used. There are

$$2p-1+(p-3)^2 = p^2-4p+8$$

such pairs of subgraphs, adding to the formula

$$\begin{aligned}&(p^2-4p+8)\cdot 2\big(|\widehat{V}_p^{n-1}|-p\big)\big(\tau_p^{n-1}-2^{n-1}(p-1)\big)\\ =\ &2^{n-2}p(p-1)(p^2-4p+8)(p^{n-1}-1)^2.\end{aligned}$$

Cases where one subgraph is bordering a long connection, while the other does not border any connection at all lead to an A case, where only one of the two long connections is still valid, namely the one which the subgraph included. There are $4(p-3)$ such cases. This adds to the formula

$$4(p-3)A_p^{n-1}.$$

Now the only cases left are the ones with two subgraphs at a long connection involved. However, these must be further differentiated depending on whether the long connection is the same or not. If it is the same, this leads to a B case, and the other long connection is now invalid. On the other hand, if they are different long connections, this leads back to another F case, with two valid long connections, but a still more efficient direct connection. There are two cases for both scenarios, leading to

$$2B_p^{n-1}+2F_p^{n-1}.$$

The subgraphs are therefore handled. Now the critical vertices of the two graphs are to be taken care of. This will again be done using the same method as in the above calculations, i. e. calculating the sum of distances where the first vertex is critical, doubling it for symmetry and then correcting the double counting for critical-critical distances, which are obviously the same as in the A-case. For the critical vertices bordering the subgraph at the direct connection, the direct path is optimal for the whole other graph, and there are $(p-1)$ such vertices, adding

$$\begin{aligned}&(p-1)\Big(2^{n-1}\big(\big|\widehat{V}_p^n\big|-p\big)+\big(\tau_p^n-(p-1)2^n\big)\Big)\\&=2^{n-2}p(p-1)(p^n-1)+2^{n-1}(p-1)^2(p^n-1)\\&=2^{n-2}(p-1)(3p-2)(p^n-1).\end{aligned}$$

Critical vertices between subgraphs bordering neither connection, of which there are $\frac{1}{2}(p-3)(p-4)$, also take the direct path. This leads to

$$\begin{aligned}&\frac{1}{2}(p-3)(p-4)\Big(2^n\big(\big|\widehat{V}_p^n\big|-p\big)+\big(\tau_p^n-(p-1)2^n\big)\Big)\\&=2^{n-2}(p-3)(p-4)(p(p^n-1)+(p-1)(p^n-1))\\&=2^{n-2}(p-3)(p-4)(2p-1)(p^n-1).\end{aligned}$$

There are $2(p-3)$ vertices between a subgraph bordering a connection and a subgraph not doing so. For them, only the subgraph bordering the same connection could have indirect shortest paths. All other vertices are accessed via the direct connection. Therefore,

$$\begin{aligned}&2(p-3)\Big(2^n\big(\big|\widehat{V}_p^n\big|-p\big)+\big(\tau_p^n-(p-1)2^n\big)-\big(\tau_p^{n-1}+2^{n-1}(p-1)\big)+\Lambda_p^{n-1}\Big)\\&=2(p-3)\Big(2^{n-1}p(p^n-1)+2^{n-1}(p-1)(p^n-1)-2^{n-2}(p-1)(p^{n-1}-1)+\Lambda_p^{n-1}\Big)\\&=2^{n-1}(p-3)\big(2(2p-1)(p^n-1)-(p-1)(p^{n-1}-1)\big)+2(p-3)\Lambda_p^{n-1}.\end{aligned}$$

The final vertex to consider is the one between the two subgraphs bordering the connections. Here, both subgraphs of the other graph bordering connections could have shortest indirect paths. For all other subgraphs, the direct path is optimal. This is expressed by the formula below.

$$\begin{aligned}&2^n\big(\big|\widehat{V}_p^n\big|-p\big)+\big(\tau_p^n-2^n(p-1)\big)-2\Big(\tau_p^{n-1}+2^{n-1}(p-1)\Big)+2\Lambda_p^{n-1}\\&=2^{n-1}p(p^n-1)+2^{n-1}(p-1)(p^n-1)-2^{n-1}(p-1)(p^{n-1}-1)+2\Lambda_p^{n-1}\\&=2^{n-1}\big((2p-1)(p^n-1)-(p-1)(p^{n-1}-1)\big)+2\Lambda_p^{n-1}.\end{aligned}$$

As already mentioned above, one can easily see that

$$F_p^1=A_p^1=p(p-1)^3.$$

p	Formula for set p
3	$\frac{1}{3}(7 \cdot 18^n - 8 \cdot 6^n - 3 \cdot 2^n) + 4\Lambda_3^{n-1}$
4	$\frac{1}{2}(9 \cdot 32^n - 3 \cdot 8^n - 24 \cdot 2^n) + 8\Lambda_4^{n-1}$
5	$\frac{2}{5}(19 \cdot 50^n + 4 \cdot 10^n - 95 \cdot 2^n) + 12\Lambda_5^{n-1}$
6	$\frac{5}{3}(7 \cdot 72^n + 4 \cdot 12^n - 51 \cdot 2^n) + 16\Lambda_6^{n-1}$

p \ n	1	2	3	4	5	6
3	24	668	13116	242060	4391820	79259084
4	108	4520	147152	4716960	150983488	4831596992
5	320	19164	953180	47534916	2375355500	$\approx 119 \cdot 10^9$
6	750	61436	4370136	313723568	$\approx 23 \cdot 10^9$	$\approx 1625 \cdot 10^9$

Table 2.5: Values of ϕ_p^n for $n \in \{1, 2, 3, 4, 5, 6\}$ and $p \in \{3, 4, 5, 6\}$. The values for $p = 3$ are somewhat theoretical, as no F-cases appear for $p = 3$.

Combining the thoughts given above, the overall formula for F_p^n is given by

$$\begin{aligned}
F_p^n &= 2^{n-2}p^3(p-1)(p^{n-1}-1)^2 + 2^{n-2}p(p-1)(p^2-4p+8)(p^{n-1}-1)^2 \\
&\quad + 2^{n-1}(p-1)(3p-2)(p^n-1) + 2^{n-1}(p-3)(p-4)(2p-1)(p^n-1) \\
&\quad + 2^n(p-3)\big(2(2p-1)(p^n-1) - (p-1)(p^{n-1}-1)\big) \\
&\quad + 2^n\big((2p-1)(p^n-1) + (p-1)(p^{n-1}-1)\big) - 2^{n-1}p(p-1)^3 \\
&\quad + 4(p-2)\Lambda_p^{n-1} + 4(p-3)A_p^{n-1} + 2B_p^{n-1} + 2F_p^{n-1} \\
&= 2^{n-1}p(p-1)(p^2-2p+4)(p^{n-1}-1)^2 + 2^n p(p-1)^2(p^n-1) \\
&\quad - 2^n(p-1)(p-2)(p^{n-1}-1) - 2^{n-1}p(p-1)^3 + 4(p-2)\Lambda^{n-1} \\
&\quad + 4(p-3)A_p^{n-1} + 2B_p^{n-1} + 2F_p^{n-1} \\
&= \phi_p^n + 4(p-3)A_p^{n-1} + 2B_p^{n-1} + 2F_p^{n-1},
\end{aligned}$$

where

$$\begin{aligned}
\phi_p^n &= 2^{n-1}p(p-1)(p^2-2p+4)(p^{n-1}-1)^2 + 2^n p(p-1)^2(p^n-1) \\
&\quad - 2^n(p-1)(p-2)(p^{n-1}-1) - 2^{n-1}p(p-1)^3 + 4(p-2)\Lambda_p^{n-1}.
\end{aligned}$$

It can be seen that a long connection only allows shorter paths for the subgraphs that border it. Therefore, when compared to the cross distance, two long connections should save twice the distance than one, leading to

$$2\left(|\widehat{V}_p^n| - p\right)\left(\tau_p^n - 2^n(p-1)\right) - F_p^n = 2\Big(2\left(|\widehat{V}_p^n| - p\right)\left(\tau_p^n - 2^n(p-1)\right) - A_p^n\Big)$$
$$\iff 2A_p^n - F_p^n = 2^{n-1}p(p-1)(p^n-1)^2.$$

Choosing $p = 4$ and $n = 3$ as an example,

$$2A_4^3 - F_4^3 = 2 \cdot 189504 - 188496 = 190512 = 2^2 \cdot 4 \cdot 3 \cdot (4^3-1)^2.$$

2.4 The recursion equation

All the formulas above form a set of mutually recursive terms to give a definition of the average distance and the Wiener index.

$$\begin{aligned}
\bar{\mathrm{d}}\left(\widehat{S}_p^n\right) &= \frac{2}{|\widehat{V}_p^n|^2} \cdot \mathrm{W}\left(\widehat{S}_p^n\right) = \frac{8}{p^2(p^n+1)^2} \cdot \mathrm{W}\left(\widehat{S}_p^n\right), \\
\mathrm{W}\left(\widehat{S}_p^n\right) &= T_p^n + 2^{n-1}p^{n+1}(p-1), \\
T_p^n &= \theta_p^n + pT_p^{n-1} + U_p^n \\
&= \theta_p^n + \upsilon_p^n + pT_p^{n-1} + p(p-1)(p-2)A_p^{n-2} \\
&\quad + \frac{1}{2}p(p-1)(p-2)B_p^{n-2} + \frac{1}{2}p(p-1)(p-2)(p-3)F_p^{n-2},
\end{aligned}$$

with the definition

$$\begin{aligned}
\theta_p^n &= 2^{n-3}p(p-1)^2(p-2) + 2^{n-2}p(p-1)^2(p^{n-1}-1) \\
&\quad + 2^{n-2}p(p-1)(p-2)^2\left((p-2)^{n-1}-1\right) \\
&\quad + p(p-1)\sum_{a=1}^{n-1}(p-2)^a\Big(2^{n-2-a}\left((p-1)(p^{n-1-a}+1)-1\right) \\
&\qquad + \left(p(p^{n-1-a}+1)-3\right)(2^{n-1}-2^{n-1-a})\Big).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\upsilon_p^n = 2^{n-4}p(p-1)^2\Big(&4p(p-1)\left(p^{n-1}-1\right) - 2(p-2)^2\left(p^{n-2}-1\right) - 2p(p-1)^2 \\
&+ p^2(p+2)\left(p^{n-2}-1\right)^2\Big) + p(p-1)(p-2)^2\Lambda_p^{n-2},
\end{aligned}$$

p	Formula for set p
2	$\frac{1}{4}(4^n - 2 \cdot 2^n)$
3	$\frac{1}{10}(21 \cdot 6^n - 45 \cdot 2^n - 6)$
4	$7 \cdot 8^n - 19 \cdot 2^n$
5	$\frac{1}{7}(115 \cdot 10^n - 10 \cdot 3^n - 350 \cdot 2^n)$
6	$\frac{1}{8}(255 \cdot 12^n - 15 \cdot 4^n - 900 \cdot 2^n)$
any	$\frac{p(p-1)}{8(p+2)}\left(2\left(p^2 - 2\right)(2p)^n - 4(p-2)^n - p(p-1)(p+2)2^n\right)$

p \ n	1	2	3	4	5	6
2	0	2	12	56	240	992
3	3	57	417	2649	16185	97689
4	18	372	3432	28368	228768	1833792
5	60	1430	15990	163370	1640910	16424330
6	150	4110	54060	658680	7926000	95163360

Table 2.6: Values of θ_p^n for $n \in \{1, 2, 3, 4, 5, 6\}$ and $p \in \{2, 3, 4, 5, 6\}$.

$$
\begin{aligned}
A_p^n &= \alpha_p^n + 2(p-2)A_p^{n-1} + B_p^{n-1},\\
B_p^n &= \beta_p^n + 4(p-2)A_p^{n-1} + (p^2-4p+6)B_p^{n-1},\\
F_p^n &= \phi_p^n + 4(p-3)A_p^{n-1} + 2B_p^{n-1} + 2F_p^{n-1},\\
\alpha_p^n &= 2^{n-2}p(p-1)(2p^2-2p+3)(p^{n-1}-1)^2 + 2^n p(p-1)^2(p^n-1)\\
&\quad - 2^{n-1}(p-1)(p-2)(p^{n-1}-1) - 2^{n-1}p(p-1)^3 + 2(p-2)\Lambda_p^{n-1},\\
\beta_p^n &= 2^{n-2}p(p-1)(p^2-p+2)(p^{n-1}-1)^2 + 2^{n-1}(p^3+p^2-9p+8)(p^n-1)\\
&\quad - 2^n(p-1)(p-2)(p^{n-1}-1) - 2^{n-1}(p-1)(p-2)(p^2-p+3) - 2^n\\
&\quad + (p^2-5p+8)\Lambda_p^n + 4(p-2)\Lambda_p^{n-1},\\
\phi_p^n &= 2^{n-1}p(p-1)(p^2-2p+4)(p^{n-1}-1)^2 + 2^n p(p-1)^2(p^n-1)\\
&\quad - 2^n(p-1)(p-2)(p^{n-1}-1) - 2^{n-1}p(p-1)^3 + 4(p-2)\Lambda_p^{n-1},\\
\Lambda_p^n &= \sum_{a=1}^{n}(p-2)^{a-1}\big(2^{n-1-a}((p-1)(p^{n-a}+1)-1) + \big(2^{n-1}-2^{n-a}\big)(p(p^{n-a}+1)-3)\big),\\
A_p^1 &= p(p-1)^3 = F_p^1,\\
B_p^1 &= (p-1)(p-2)\big(p^2-p+3\big) + 2.
\end{aligned}
$$

One can see that A_p^n and B_p^n form a recursive system on their own. In matrix representation

$$
\begin{pmatrix} A_p^n \\ B_p^n \end{pmatrix} = \begin{pmatrix} \alpha_p^n \\ \beta_p^n \end{pmatrix} + \begin{pmatrix} 2(p-2) & 1 \\ 4(p-2) & p^2-4p+6 \end{pmatrix}\begin{pmatrix} A_p^{n-1} \\ B_p^{n-1} \end{pmatrix}.
$$

The closed form solution of this formula is given by

$$
\begin{aligned}
\begin{pmatrix} A_p^n \\ B_p^n \end{pmatrix} &= \begin{pmatrix} 2p-3 & 1 \\ 4(p-2) & p^2-4p+6 \end{pmatrix}^{n-2}\begin{pmatrix} A_p^1 \\ B_p^1 \end{pmatrix}\\
&\quad + \sum_{k=0}^{n-1}\begin{pmatrix} 2p-3 & 1 \\ 4(p-2) & p^2-4p+6 \end{pmatrix}^{k}\begin{pmatrix} \alpha_p^{n-k} \\ \beta_p^{n-k} \end{pmatrix}\\
&= \begin{pmatrix} 2p-3 & 1 \\ 4(p-2) & p^2-4p+6 \end{pmatrix}^{n-2}\begin{pmatrix} p(p-1)^3 \\ 2+(p-1)(p-2)(p^2-p+3) \end{pmatrix}\\
&\quad + \sum_{k=0}^{n-1}\begin{pmatrix} 2p-3 & 1 \\ 4(p-2) & p^2-4p+6 \end{pmatrix}^{k}\begin{pmatrix} \alpha_p^{n-k} \\ \beta_p^{n-k} \end{pmatrix}.
\end{aligned}
$$

The situation complicates even further, when F_p^n is considered:

$$\begin{pmatrix} A_p^n \\ B_p^n \\ F_p^n \end{pmatrix} = M_{3,p}^{n-2} \begin{pmatrix} A_p^1 \\ B_p^1 \\ F_p^1 \end{pmatrix} + \sum_{k=0}^{n-1} M_{3,p}^k \begin{pmatrix} \alpha_p^{n-k} \\ \beta_p^{n-k} \\ \phi_p^{n-k} \end{pmatrix}$$
$$= M_{3,p}^{n-2} \begin{pmatrix} p(p-1)^3 \\ 2+(p-1)(p-2)(p^2-p+3) \\ p(p-1)^3 \end{pmatrix} + \sum_{k=0}^{n-1} M_{3,p}^k \begin{pmatrix} \alpha_p^{n-k} \\ \beta_p^{n-k} \\ \phi_p^{n-k} \end{pmatrix}$$

$$\text{using} \quad M_{3,p} = \begin{pmatrix} 2(p-2) & 1 & 0 \\ 4(p-2) & p^2-4p+6 & 0 \\ 4(p-3) & 2 & 2 \end{pmatrix}.$$

And finally, even T_p^n itself is recursive, so it is necessary to consider

$$\begin{pmatrix} T_p^n \\ A_p^{n-1} \\ B_p^{n-1} \\ F_p^{n-1} \end{pmatrix} = M_{4,p}^{n-3} \begin{pmatrix} T_p^2 \\ p(p-1)^3 \\ 2+(p-1)(p-2)(p^2-p+3) \\ p(p-1)^3 \end{pmatrix} + \sum_{k=0}^{n-2} M_{4,p}^k \begin{pmatrix} \theta_p^{n-k} + \upsilon_p^{n-k} \\ \alpha_p^{n-1-k} \\ \beta_p^{n-1-k} \\ \phi_p^{n-1-k} \end{pmatrix}$$

$$\text{using} \quad M_{4,p} = \begin{pmatrix} p & p(p-1)(p-2) & \frac{1}{2}p(p-1)(p-2) & \frac{1}{2}p(p-1)(p-2)(p-3) \\ 0 & 2(p-2) & 1 & 0 \\ 0 & 4(p-2) & p^2-4p+6 & 0 \\ 0 & 4(p-3) & 2 & 2 \end{pmatrix}.$$

As θ_p^k, υ_p^k, α_p^k, β_p^k and ϕ_p^k are already explicit, the things to do are to calculate T_p^2 for any $p \geq 3$ and maybe to give a nicer looking version of the matrix powers. The matrix power calculation, however, would require a diagonalisation to be done for the general case, and it seems easier to do it for a fixed p when it is required, especially as matrix multiplication can be done quite easily by a computer. Once the powers are calculated, it suffices to consider the first row, as only the uppermost entry of the final vector is of true interest. All else is a matter of form.

Now for a final step, T_p^2 shall be calculated. But the necessary formulas have already been found. The sum of distances with at least one critical vertices involved is given by

$$\begin{aligned}\theta_p^2 &= \frac{p(p-1)}{8(p+2)}\left(2(p^2-2)(2p)^2 - 4(p-2)^2 - p(p-1)(p+2)2^2\right)\\ &= \frac{1}{2}p(p-1)(2p^3-5p^2+4p-2).\end{aligned}$$

The sum of distances between two generic vertices in a fixed subgraph is given by $T_p^1 = \theta_p^1$, as the exponent of the subgraphs of $\widehat{S}_p^2$ is 1. Since there are p subgraphs, this adds up to

$$\begin{aligned}p\theta_p^1 &= p\left(\frac{p(p-1)}{8(p+2)}\left(4p(p^2-2) - 4p + 8 - 2p(p^2+p-2)\right)\right)\\ &= \frac{p^2(p-1)}{4(p+2)}(p^3-p^2-4p+4)\\ &= \frac{1}{4}p^2(p-1)^2(p-2).\end{aligned}$$

This also shows that $T_p^1 = \frac{1}{4}p(p-1)^2(p-2)$. The sum of distances between generic vertices of two different subgraphs in $\widehat{S}_p^2$ is given by

$$\frac{1}{2}p(p-1)A_p^1 = \frac{1}{2}p^2(p-1)^4,$$

and adding up these three parts, one gets

$$\begin{aligned}T_p^2 &= \frac{1}{2}p^2(p-1)^4 + \frac{1}{4}p^2(p-1)^2(p-2) + \frac{1}{2}p(p-1)(2p^3-5p^2+4p-2)\\ &= \frac{1}{4}p(p-1)(2p^4-p^3-7p^2+8p-4).\end{aligned}$$

For $p \in \{2, 3, 4, 5, 6\}$, this leads to the following values.

p	2	3	4	5	6
T_p^2	4	138	1092	4930	16260

Now everything necessary for the formula to work has been done.

2.5 The special case $p = 2$

Now consider $p = 2$, though this case was mostly given for free in the above. The graph $\widehat{S}_2^0$ is the complete graph K_2. Its Wiener index is obviously given by $\mathrm{W}\left(\widehat{S}_2^0\right) = 1$. For the average distance, the formula is

$$\begin{aligned}\bar{\mathrm{d}}\big(\widehat{S}_2^0\big) &= \mathrm{W}\big(\widehat{S}_2^0\big) \cdot \frac{8}{p^2(p^n+1)^2} \\ &= 1 \cdot \frac{8}{4 \cdot 4} = \frac{1}{2}.\end{aligned}$$

The graph $\widehat{S}_2^1$ is now a path graph of order 3, $\widehat{S}_2^2$ a path graph of order 5, and so on. Generally speaking, $\widehat{S}_2^n$ is a path graph of order $2^n + 1$. For a path graph of order L, the Wiener index is given by

$$\sum_{l=1}^{L-1} \sum_{k=1}^{l} k = \frac{1}{6}(L-1) \cdot L \cdot (L+1).$$

Thus, the Wiener index of $\widehat{S}_2^n$ is

$$\mathrm{W}\big(\widehat{S}_2^n\big) = \frac{1}{6} 2^n (2^n+1)(2^n+2) = \frac{1}{6}(8^n + 3 \cdot 4^n + 2 \cdot 2^n).$$

Alternatively, the formulas above can be used to find a recursive expression:

$$\begin{aligned}\theta_2^n + v_2^n &= \frac{1}{4}(4^n - 2 \cdot 2^n) + \frac{1}{8}(8^n - 4 \cdot 4^n + 4 \cdot 2^n) \\ &= \frac{1}{8}(8^n - 2 \cdot 4^n), \\ T_2^n &= 2T_2^{n-1} + \frac{1}{8}(8^n - 2 \cdot 4^n),\end{aligned}$$

with a starting value of $T_2^1 = 0$ or, alternatively, $T_2^2 = 4$. Using this formula to create a closed form solution, one gets

$$T_2^n = \frac{1}{6}(8^n - 3 \cdot 4^n + 2 \cdot 2^n),$$

as

$$\begin{aligned}&2 \cdot \frac{1}{6}(8^{n-1} - 3 \cdot 4^{n-1} + 2 \cdot 2^{n-1}) + \frac{1}{8}(8^n - 2 \cdot 4^n) \\ &= \frac{1}{6}(2 \cdot 8^{n-1} - 6 \cdot 4^{n-1} + 2 \cdot 2^n) + 8^{n-1} - 4^{n-1} \\ &= \frac{1}{6}\big((2+6) \cdot 8^{n-1} - (6+6)4^{n-1} + 2 \cdot 2^n\big) \\ &= \frac{1}{6}(8^n - 3 \cdot 4^n + 2 \cdot 2^n)\end{aligned}$$

and

$$T_2^1 = \frac{1}{6}(8 - 12 + 4) = 0 \qquad \text{or} \qquad T_2^2 = \frac{1}{6}(64 - 48 + 8) = 4.$$

Now,

$$W(\widehat{S}_2^n) = \frac{1}{6}(8^n - 3\cdot 4^n + 2\cdot 2^n) + 2^{n-1}2^{n+1} = \frac{1}{6}(8^n + 3\cdot 4^n + 2\cdot 2^n),$$

to no surprise.

Finally, the normalised average distance of $\widehat{S}_2^n$ is given by

$$\frac{1}{2^n}\bar{\mathrm{d}}(\widehat{S}_2^n) = W(\widehat{S}_2^n)\cdot\frac{2}{2^n(2^n+1)^2} = \frac{2^n(2^n+1)(2^n+2)}{3\cdot 2^n(2^n+1)^2} = \frac{1}{3}\cdot\frac{2^n+2}{2^n+1},$$

which converges to $\frac{1}{3}$ for $n\to\infty$.

2.6 Further examples

First, consider $p = 3$. This is a special case where there are no F-cases. The recursion simplifies to

$$\begin{pmatrix} T_p^n \\ A_p^{n-1} \\ B_p^{n-1} \end{pmatrix} = \begin{pmatrix} \theta_p^n + \upsilon_p^n \\ \alpha_p^{n-1} \\ \beta_p^{n-1} \end{pmatrix} + \begin{pmatrix} p & p(p-1)(p-2) & \frac{1}{2}p(p-1)(p-2) \\ 0 & 2(p-2) & 1 \\ 0 & 4(p-2) & p^2-4p+6 \end{pmatrix}\begin{pmatrix} T_p^{n-1} \\ A_p^{n-2} \\ B_p^{n-2} \end{pmatrix}$$

and since $p = 3$, one gets

$$\begin{pmatrix} T_3^n \\ A_3^{n-1} \\ B_3^{n-1} \end{pmatrix} = \begin{pmatrix} \theta_3^n + \upsilon_3^n \\ \alpha_3^{n-1} \\ \beta_3^{n-1} \end{pmatrix} + \begin{pmatrix} 3 & 6 & 3 \\ 0 & 2 & 1 \\ 0 & 4 & 3 \end{pmatrix}\begin{pmatrix} T_3^{n-1} \\ A_3^{n-2} \\ B_3^{n-2} \end{pmatrix}.$$

From the tables of this chapter, it can be seen that

$$\begin{aligned} T_3^2 &= 138, \\ A_3^1 &= 24, \text{ and} \\ B_3^1 &= 20. \end{aligned}$$

Likewise, the values for α_3^n, β_3^n, υ_3^n and θ_3^n can be found up to $n = 6$. Hence,

$$\begin{pmatrix} T_3^3 \\ A_3^2 \\ B_3^2 \end{pmatrix} = \begin{pmatrix} \theta_3^3 + \upsilon_3^3 \\ \alpha_3^2 \\ \beta_3^2 \end{pmatrix} + \begin{pmatrix} 3 & 6 & 3 \\ 0 & 2 & 1 \\ 0 & 4 & 3 \end{pmatrix}\begin{pmatrix} 138 \\ 24 \\ 20 \end{pmatrix} = \begin{pmatrix} 417+2082 \\ 694 \\ 466 \end{pmatrix} + \begin{pmatrix} 618 \\ 68 \\ 156 \end{pmatrix} = \begin{pmatrix} 3117 \\ 762 \\ 622 \end{pmatrix},$$

$$\begin{pmatrix} T_3^4 \\ A_3^3 \\ B_3^3 \end{pmatrix} = \begin{pmatrix} \theta_3^4 + \upsilon_3^4 \\ \alpha_3^3 \\ \beta_3^3 \end{pmatrix} + \begin{pmatrix} 3 & 6 & 3 \\ 0 & 2 & 1 \\ 0 & 4 & 3 \end{pmatrix} \begin{pmatrix} 3117 \\ 762 \\ 622 \end{pmatrix} = \begin{pmatrix} 2649 + 41706 \\ 13902 \\ 8090 \end{pmatrix} + \begin{pmatrix} 15789 \\ 2146 \\ 4914 \end{pmatrix} = \begin{pmatrix} 60144 \\ 16048 \\ 13004 \end{pmatrix},$$

$$\begin{pmatrix} T_3^5 \\ A_3^4 \\ B_3^4 \end{pmatrix} = \begin{pmatrix} \theta_3^5 + \upsilon_3^5 \\ \alpha_3^4 \\ \beta_3^4 \end{pmatrix} + \begin{pmatrix} 3 & 6 & 3 \\ 0 & 2 & 1 \\ 0 & 4 & 3 \end{pmatrix} \begin{pmatrix} 60144 \\ 16048 \\ 13004 \end{pmatrix} = \begin{pmatrix} 16185 + 775218 + 315732 \\ 258406 \quad + \quad 45100 \\ 142066 \quad + 103204 \end{pmatrix} = \begin{pmatrix} 1107135 \\ 303506 \\ 245270 \end{pmatrix},$$

$$\begin{pmatrix} T_3^6 \\ A_3^5 \\ B_3^5 \end{pmatrix} = \begin{pmatrix} \theta_3^6 + \upsilon_3^6 \\ \alpha_3^5 \\ \beta_3^5 \end{pmatrix} + \begin{pmatrix} 3 & 6 & 3 \\ 0 & 2 & 1 \\ 0 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1107135 \\ 303506 \\ 245270 \end{pmatrix} = \begin{pmatrix} 97689 + 14099346 + 5878251 \\ 4699782 \quad + \quad 852282 \\ 2532434 \quad + 1949834 \end{pmatrix} = \begin{pmatrix} 20075286 \\ 5552064 \\ 4482268 \end{pmatrix}.$$

Using these results, one can see that

$$\mathrm{W}(\widehat{S}_3^2) = T_3^2 + 2^{n-1}p^{n+1}(p-1) = 138 + 3 \cdot 6^n = 246$$

and

$$\bar{\mathrm{d}}(\widehat{S}_3^2) = \mathrm{W}(\widehat{S}_3^2) \cdot \frac{8}{9(3^n+1)^2} = \frac{8 \cdot 246}{900} = \frac{164}{75} = \frac{41}{75} \cdot 2^2;$$

the other exponents work the same:

$$\begin{aligned} \mathrm{W}(\widehat{S}_3^3) &= T_3^3 + 3 \cdot 6^3 = 3117 + 648 = 3765, \\ \bar{\mathrm{d}}(\widehat{S}_3^3) &= 3765 \cdot \frac{8}{9 \cdot 28^2} = \frac{1255}{294} = \frac{1255}{2352} \cdot 2^3; \\ \mathrm{W}(\widehat{S}_3^4) &= T_3^4 + 3 \cdot 6^4 = 60144 + 3888 = 64032, \\ \bar{\mathrm{d}}(\widehat{S}_3^4) &= 64032 \cdot \frac{8}{9 \cdot 82^2} = \frac{42688}{5043} = \frac{2668}{5043} \cdot 2^4; \\ \mathrm{W}(\widehat{S}_3^5) &= T_3^5 + 3 \cdot 6^5 = 1107135 + 23328 = 1130463, \\ \bar{\mathrm{d}}(\widehat{S}_3^5) &= 1130463 \cdot \frac{8}{9 \cdot 244^2} = \frac{1004856}{59536} = \frac{125607}{238144} \cdot 2^5. \end{aligned}$$

The same can be done for any p, but for $p > 3$ it is necessary to regard the F-cases. Consider for example $p = 5$.

$$\begin{aligned} T_5^2 &= 4930, \\ A_5^1 = F_5^1 &= 320, \text{ and} \\ B_5^1 &= 278. \end{aligned}$$

Now,

$$\begin{pmatrix} T_5^3 \\ A_5^2 \\ B_5^2 \\ F_5^2 \end{pmatrix} = \begin{pmatrix} \theta_5^3 + \upsilon_5^3 \\ \alpha_5^2 \\ \beta_5^2 \\ \phi_5^2 \end{pmatrix} + \begin{pmatrix} 5 & 60 & 30 & 60 \\ 0 & 6 & 1 & 0 \\ 0 & 12 & 11 & 0 \\ 0 & 8 & 2 & 2 \end{pmatrix} \begin{pmatrix} 4930 \\ 320 \\ 278 \\ 320 \end{pmatrix} = \begin{pmatrix} 15990 + 181860 + 71390 \\ 20782 \quad + 2198 \\ 13128 \quad + 6898 \\ 19164 \quad + 3756 \end{pmatrix} = \begin{pmatrix} 269240 \\ 22980 \\ 20026 \\ 22920 \end{pmatrix},$$

$$\mathrm{W}(\widehat{S}_5^3) = T_5^3 + 2^2 \cdot 5^4 \cdot 4 = 269240 + 10000 = 279240,$$

$$\bar{\mathrm{d}}(\widehat{S}_5^3) = 279240 \cdot \frac{8}{25 \cdot 126^2} = \frac{37232}{6615} = \frac{4654}{6615} \cdot 2^3.$$

2.7 Limit observations

In accordance with the results from the previous section, one can define

$$c_p^n = 2^{-n} \cdot \bar{\mathrm{d}}(\widehat{S}_p^n)$$

with $c_p^n < 1$. This makes sense, as c_p^n is the average distance of $\widehat{S}_p^n$ with the diameter defined as 1 instead of 2^n. As there are distances smaller than the diameter, but by definition no larger ones, $0 < c_p^n < 1$ clearly holds true.

As the Sierpiński triangle graphs are modelled after the famous Sierpiński fractal, it seems intuitive to consider $n \to \infty$, as the graphs then approach the Sierpiński fractal. It is of particular interest to find

$$c_p = \lim_{n \to \infty} c_p^n,$$

as this is the geodesic distance of the fractal, like it has already been calculated for $p = 3$. The result $c_3 = \frac{466}{885}$ is found for example by Hinz [8, Corollary 3], p. 307. One might wonder how to imagine the graphs and the fractal for $p > 3$. For $p = 4$ the graphs look like tetrahedra with four Sierpiński triangles as sides. This object is sometimes called the Sierpiński tetrahedron. In the same manner as the $\widehat{S}_4^n$ is a 3-dimensional version of the Sierpiński triangle, the $\widehat{S}_5^n$ is a 4-dimensional Sierpiński triangle or generally the $\widehat{S}_p^n$ is a $(p-1)$-dimensional Sierpiński triangle.

The following calculation will confirm the known result for c_3 and give a result for c_4 as well as the method to calculate c_p for any $p \geq 4$.

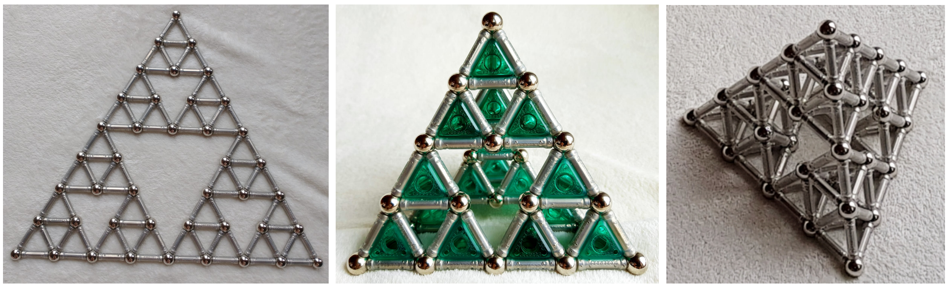

Figure 2.5: A Sierpiński triangle graph $\widehat{S}_3^3$ (left); together with an $\widehat{S}_4^2$, an approximation of a Sierpiński tetrahedron, viewed from two angles (middle and right).

From the results above it is known that

$$\begin{pmatrix} T_3^n \\ A_3^{n-1} \\ B_3^{n-1} \end{pmatrix} = \begin{pmatrix} \theta_3^n + \upsilon_3^n \\ \alpha_3^{n-1} \\ \beta_3^{n-1} \end{pmatrix} + M_{3,3} \begin{pmatrix} T_3^{n-1} \\ A_3^{n-2} \\ B_3^{n-2} \end{pmatrix}, \quad M_{3,3} = \begin{pmatrix} 3 & 6 & 3 \\ 0 & 2 & 1 \\ 0 & 4 & 3 \end{pmatrix},$$

and by iteration, we get

$$\begin{aligned}
\begin{pmatrix} T_3^n \\ A_3^{n-1} \\ B_3^{n-1} \end{pmatrix} &= \sum_{k=0}^{n-3} M_{3,3}^k \begin{pmatrix} \theta_3^{n-k} + \upsilon_3^{n-k} \\ \alpha_3^{n-1-k} \\ \beta_3^{n-1-k} \end{pmatrix} + M_{3,3}^{n-2} \begin{pmatrix} T_3^2 \\ A_3^1 \\ B_3^1 \end{pmatrix} \\
&= M_{3,3}^{n-2} \begin{pmatrix} 138 \\ 24 \\ 20 \end{pmatrix} + \sum_{k=0}^{n-3} M_{3,3}^k \\
&\quad \left(\begin{pmatrix} \frac{1}{10}(21 \cdot 6^{n-k} - 45 \cdot 2^{n-k} - 6) + \frac{1}{12}(5 \cdot 18^{n-k} - 20 \cdot 6^{n-k} - 9 \cdot 2^{n-k}) \\ \frac{1}{6}(15 \cdot 18^{n-1-k} - 20 \cdot 6^{n-1-k} - 3 \cdot 2^{n-1-k}) \\ \frac{1}{6}(8 \cdot 18^{n-1-k} - 6^{n-1-k} - 27 \cdot 2^{n-1-k}) + \frac{1}{5}(7 \cdot 6^{n-1-k} - 5 \cdot 2^{n-1-k} - 2) \end{pmatrix} \right. \\
&\qquad \left. + \begin{pmatrix} \frac{3}{5}(7 \cdot 6^{n-2-k} - 5 \cdot 2^{n-2-k} - 2) \\ \frac{1}{5}(7 \cdot 6^{n-2-k} - 5 \cdot 2^{n-2-k} - 2) \\ \frac{2}{5}(7 \cdot 6^{n-2-k} - 5 \cdot 2^{n-2-k} - 2) \end{pmatrix} \right) \\
&= M_{3,3}^{n-2} \begin{pmatrix} 138 \\ 24 \\ 20 \end{pmatrix} + \sum_{k=0}^{n-3} M_{3,3}^k \begin{pmatrix} \frac{1}{60}(25 \cdot 18^{n-k} + 33 \cdot 6^{n-k} - 360 \cdot 2^{n-k} - 108) \\ \frac{1}{180}(25 \cdot 18^{n-k} - 93 \cdot 6^{n-k} - 90 \cdot 2^{n-k} - 72) \\ \frac{1}{540}(40 \cdot 18^{n-k} + 153 \cdot 6^{n-k} - 1755 \cdot 2^{n-k} - 648) \end{pmatrix}.
\end{aligned}$$

By diagonalization it can be calculated that, using $r = 5+\sqrt{17}$ and $\bar{r} = 5-\sqrt{17}$, $s = r^k+\bar{r}^k$ and $\bar{s} = r^k - \bar{r}^k$ as well as $h = 3s - 6^{k+1}$:

$$M_{3,3}^k = \frac{1}{2^{k+1}\sqrt{17}}\begin{pmatrix} 2\cdot 6^k\sqrt{17} & h\sqrt{17}+5\bar{s} & \frac{3}{4}h\sqrt{17}+\frac{33}{4}\bar{s} \\ 0 & s\sqrt{17}-\bar{s} & 2\bar{s} \\ 0 & 8\bar{s} & \sqrt{17}s+\bar{s} \end{pmatrix}.$$

Since only the first row is of interest for T_p^n, using $s_k = r^k + \bar{r}^k$ and $\bar{s}_k = r^k - \bar{r}^k$, $v(x,z) = x + \frac{z}{\sqrt{17}}$ and $\bar{v}(x,z) = x - \frac{z}{\sqrt{17}}$, as well as $w_k(x,z) = v(x,z)r^k + \bar{v}(x,z)\bar{r}^k$ and $\bar{w}_k(x,z) = \bar{v}(x,z)r^k + v(x,z)\bar{r}^k$ the equation above can be simplified to

$$\begin{aligned}
T_3^n = \frac{46}{3}&\cdot 3^n - 8\cdot 3^n - 5\cdot 3^n \\
&+ 144\cdot 2^{-n}\cdot v(s_{n-2}, 5\bar{s}_{n-2}) + 30\cdot 2^{-n}\cdot v(3s_{n-2}, 11\bar{s}_{n-2}) \\
&+ \sum_{k=0}^{n-3}\Bigg(\frac{3^k}{60}(25\cdot 18^{n-k} + 33\cdot 6^{n-k} - 360\cdot 2^{n-k} - 108) \\
&\quad + \frac{2^{-k}}{120}(25\cdot 18^{n-k} - 93\cdot 6^{n-k} - 90\cdot 2^{n-k} - 72)\cdot v(s_k, 5\bar{s}_k) \\
&\quad - \frac{3^k}{60}(25\cdot 18^{n-k} - 93\cdot 6^{n-k} - 90\cdot 2^{n-k} - 72) \\
&\quad + \frac{2^{-k}}{1440}(40\cdot 18^{n-k} + 153\cdot 6^{n-k} - 1755\cdot 2^{n-k} - 648)\cdot v(3s_k, 11\bar{s}_k) \\
&\quad - \frac{3^k}{240}(40\cdot 18^{n-k} + 153\cdot 6^{n-k} - 1755\cdot 2^{n-k} - 648)\Bigg) \\
=\ & 2^{-n}r^{n-2}\big(v(234,1050) + \bar{v}(234,1050)\big) + \frac{7}{3}3^n \\
&+ \sum_{k=0}^{n-3}\Bigg(-\frac{1}{6}18^n 6^{-k} + \frac{117}{80}6^n 2^{-k} + \frac{45}{16}2^{n-k}3^k + \frac{21}{10}3^k \\
&\quad + \frac{1}{72}18^n 36^{-k}\cdot w_k(21,97) - \frac{1}{160}6^n 12^{-k}\cdot w_k(73,433) \\
&\quad - \frac{3}{32}2^n 4^{-k}\cdot w_k(47,183) - \frac{3}{20}2^{-k}\cdot w_k(13,53)\Bigg) \\
=\ & \frac{699}{1180}18^n - \frac{3}{2}6^n + \frac{9}{20}3^n + \frac{3}{4}2^n - \frac{3}{2}2^{-n}\cdot\bar{w}_n(7,45) + \frac{1}{472}2^{-n}\cdot\bar{w}_n(11259,71523) \\
&- \frac{27}{10}2^{-n}\cdot\bar{w}_n(6,37) + \frac{3}{8}2^{-n}\cdot\bar{w}_n(7,43) + \frac{3}{40}2^{-n}\cdot\bar{w}_n(1,7) \\
=\ & \frac{699}{1180}18^n - \frac{3}{2}6^n + \frac{9}{20}3^n + \frac{3}{4}2^n - \frac{1}{472}2^{-n}\cdot w_n(69,369).
\end{aligned}$$

Adding the distances with primitive vertices included leads to

$$\begin{aligned}\mathrm{W}\big(\widehat{S}_3^n\big) &= \frac{699}{1180}18^n - \frac{3}{2}6^n + \frac{9}{20}3^n + \frac{3}{4}2^n + 3\cdot 6^n - \frac{2^{-n}}{472}\cdot w_n(69,369)\\ &= \frac{699}{1180}18^n + \frac{3}{2}6^n + \frac{9}{20}3^n + \frac{3}{4}2^n - \frac{2^{-n}}{472}\cdot w_n(69,369).\end{aligned}$$

For the average distance, this leads to

$$\bar{\mathrm{d}}\big(\widehat{S}_3^n\big) = \frac{8\mathrm{W}\big(\widehat{S}_3^n\big)}{9(3^n+1)^2} = \frac{1}{(3^n+1)^2}\left(\frac{466}{885}18^n + \frac{4}{3}6^n + \frac{2}{5}3^n + \frac{2}{3}2^n - \frac{1}{177}(2^{-n}w_n(23,123))\right)$$

Isolating the increase of the diameter of 2^n,

$$2^{-n}\cdot\bar{\mathrm{d}}\big(\widehat{S}_3^n\big) = \frac{466}{885}\frac{9^n}{(3^n+1)^2}+\frac{4}{3}\frac{3^n}{(3^n+1)^2}+\frac{2}{5}\frac{3^n}{2^n(3^n+1)^2}+\frac{2}{3}-\frac{1}{177}\left(\frac{w_n(23,123)}{4^n(3^n+1)^2}+\frac{\bar{w}_n(23,123)}{4^n(3^n+1)^2}\right).$$

Now considering

$$c_3 = \lim_{n\to\infty} c_3^n = \lim_{n\to\infty}\Big(2^{-n}\cdot\bar{\mathrm{d}}\big(\widehat{S}_3^n\big)\Big),$$

one can see, for the first term,

$$\frac{466}{885}\frac{9^n}{(3^n+1)^2} \overset{n\to\infty}{\longrightarrow} \frac{466}{885},$$

while all other terms go to 0. Therefore,

$$c_3 = \frac{466}{885}.$$

This result is already known to hold for Sierpiński graphs (cf. Hinz [8]). However, neither is c_4 for Sierpiński triangle graphs known, nor a closed form solution for $\mathrm{W}\big(\widehat{S}_4^n\big)$ or $\bar{\mathrm{d}}\big(\widehat{S}_4^n\big)$. They can now be found with reasonable effort:

First, it is helpful to calculate the following, using the formulas from Tab. 2.1 to Tab. 2.6.

$$\begin{aligned}\theta_4^{n-k} + v_4^{n-k} &= 7\cdot 8^{n-k} - 19\cdot 2^{n-k} + \frac{9}{32}(3\cdot 32^{n-k} - 4\cdot 8^{n-k} - 128\cdot 2^{n-k}) + 48\Lambda_4^{n-k-2}\\ &= 7\cdot 8^{n-k} - 19\cdot 2^{n-k} + \frac{27}{32}32^{n-k} - \frac{9}{8}8^{n-k} - 36\cdot 2^{n-k} + 56(8^{n-2-k} - 2^{n-2-k})\\ &= \frac{27}{32}32^{n-k} + \frac{27}{4}8^{n-k} - 69\cdot 2^{n-k},\\ \alpha_4^{n-1-k} &= \frac{3}{16}(27\cdot 32^{n-1-k} - 28\cdot 8^{n-1-k} - 32\cdot 2^{n-1-k}) + 4\Lambda_4^{n-2-k}\\ &= \frac{81}{512}32^{n-k} - \frac{21}{32}8^{n-k} - 3\cdot 2^{n-k} + \frac{14}{3}(8^{n-2-k} - 2^{n-2-k})\\ &= \frac{81}{512}32^{n-k} - \frac{7}{12}8^{n-k} - \frac{25}{6}2^{n-k},\end{aligned}$$

$$\begin{aligned}
\beta_4^{n-1-k} &= \frac{1}{8}(21 \cdot 32^{n-1-k} + 28 \cdot 8^{n-1-k} - 192 \cdot 2^{n-1-k}) + 4\Lambda_4^{n-1-k} + 8\Lambda_4^{n-2-k} \\
&= \frac{21}{256}32^{n-k} + \frac{7}{16}8^{n-k} - 12 \cdot 2^{n-k} + \frac{14}{3}(8^{n-1-k} - 2^{n-1-k}) + \frac{28}{3}(8^{n-2-k} - 2^{n-2-k}) \\
&= \frac{21}{256}32^{n-k} + \frac{7}{6}8^{n-k} - \frac{50}{3}2^{n-k}, \\
\phi_4^{n-1-k} &= \frac{1}{2}(9 \cdot 32^{n-1-k} - 3 \cdot 8^{n-1-k} - 24 \cdot 2^{n-1-k}) + 8\Lambda_4^{n-2-k} \\
&= \frac{9}{64}32^{n-k} - \frac{3}{16}8^{n-k} - 6 \cdot 2^{n-k} + 8 \cdot \frac{7}{6}(8^{n-2-k} - 2^{n-2-k}) \\
&= \frac{9}{64}32^{n-k} - \frac{1}{24}8^{n-k} - \frac{25}{3}2^{n-k}.
\end{aligned}$$

Using these formulas, one gets

$$\begin{aligned}
\begin{pmatrix} T_4^n \\ A_4^{n-1} \\ B_4^{n-1} \\ F_4^{n-1} \end{pmatrix} &= \sum_{k=0}^{n-3} M_{4,4}^k \begin{pmatrix} \theta_4^{n-k} + \upsilon_4^{n-k} \\ \alpha_4^{n-1-k} \\ \beta_4^{n-1-k} \\ \phi_4^{n-1-k} \end{pmatrix} + M_{4,4}^{n-2} \begin{pmatrix} T_4^2 \\ A_4^1 \\ B_4^1 \\ F_4^1 \end{pmatrix} \\
&= M_{4,4}^{n-2} \begin{pmatrix} 1092 \\ 108 \\ 92 \\ 108 \end{pmatrix} + \sum_{k=0}^{n-3} M_{4,4}^k \begin{pmatrix} \frac{27}{32}32^{n-k} + \frac{27}{4}8^{n-k} - 69 \cdot 2^{n-k} \\ \frac{81}{512}32^{n-k} - \frac{7}{12}8^{n-k} - \frac{25}{6}2^{n-k} \\ \frac{21}{256}32^{n-k} + \frac{7}{6}8^{n-k} - \frac{50}{3}2^{n-k} \\ \frac{9}{64}32^{n-k} - \frac{1}{24}8^{n-k} - \frac{25}{3}2^{n-k} \end{pmatrix} \\
&= M_{4,4}^{n-2} \begin{pmatrix} 1092 \\ 108 \\ 92 \\ 108 \end{pmatrix} + \sum_{k=0}^{n-3} M_{4,4}^k \left(\frac{32^{n-k}}{512} \begin{pmatrix} 432 \\ 81 \\ 42 \\ 72 \end{pmatrix} + \frac{8^{n-k}}{24} \begin{pmatrix} 162 \\ -14 \\ 28 \\ -1 \end{pmatrix} - \frac{2^{n-k}}{6} \begin{pmatrix} 414 \\ 25 \\ 100 \\ 50 \end{pmatrix} \right).
\end{aligned}$$

Matrix powers can be calculated as usual by diagonalization. The result is

$$M_{4,4}^k = \begin{pmatrix} 4^k & 4(2 \cdot 8^k - 3 \cdot 4^k + 2^k) & 2(2 \cdot 8^k - 3 \cdot 4^k + 2^k) & 6(4^k - 2^k) \\ 0 & \frac{1}{3}(8^k + 2 \cdot 2^k) & \frac{1}{6}(8^k - 2^k) & 0 \\ 0 & \frac{4}{3}(8^k - 2^k) & \frac{1}{3}(2 \cdot 8^k + 2^k) & 0 \\ 0 & \frac{2}{3}(8^k - 2^k) & \frac{1}{3}(8^k - 2^k) & 2^k \end{pmatrix}.$$

Again, only the first row is of interest.

$$\begin{aligned}
T_4^n = {} & 1092 \cdot 4^{n-2} + 108 \cdot 4\big(2 \cdot 8^{n-2} - 3 \cdot 4^{n-2} + 2^{n-2}\big) \\
& + 92 \cdot 2\big(2 \cdot 8^{n-2} - 3 \cdot 4^{n-2} + 2^{n-2}\big) + 108 \cdot 6\big(4^{n-2} - 2^{n-2}\big)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{k=0}^{n-3}\Big(\frac{27}{32}32^n 8^{-k} + \frac{27}{4}8^n 2^{-k} - 69 \cdot 2^n \cdot 2^k \\
&\quad + \frac{51}{64}32^n(2 \cdot 4^{-k} - 3 \cdot 8^{-k} + 16^{-k}) + \frac{27}{32}32^n(8^{-k} - 16^{-k}) \\
&\quad - \frac{1}{4}8^n(2^{-k} - 4^{-k}) - 50 \cdot 2^n(2^k - 1) - 50 \cdot 2^n(2 \cdot 4^k - 3 \cdot 2^k + 1)\Big) \\
&= \frac{77}{4}8^n - \frac{27}{4}4^n - 8 \cdot 2^n \\
&+ \sum_{k=0}^{n-3}\Big(\frac{1}{64}32^n(102 \cdot 4^{-k} - 45 \cdot 8^{-k} - 3 \cdot 16^{-k}) \\
&\quad + \frac{1}{4}8^n(26 \cdot 2^{-k} + 4^{-k}) - 2^n(100 \cdot 4^k - 31 \cdot 2^k)\Big) \\
&= \frac{89}{70}32^n - \frac{7}{2}8^n + \frac{3}{7}4^n + \frac{9}{5}2^n.
\end{aligned}
$$

Adding the distances with involvement of primitive vertices results in

$$
\mathrm{W}\big(\widehat{S}_4^n\big) = 6 \cdot 8^n + T_4^n = \frac{89}{70}32^n + \frac{5}{2}8^n + \frac{3}{7}4^n + \frac{9}{5}2^n.
$$

The first values of the Wiener index are:

n	2	3	4	5
$\mathrm{W}\big(\widehat{S}_4^n\big)$	1476	42984	1343568	42744480

The average distance of Sierpiński triangle graphs with base 4 is given by

$$
\begin{aligned}
\bar{\mathrm{d}}\big(\widehat{S}_4^n\big) &= \frac{1}{2(4^n+1)^2}\mathrm{W}\big(\widehat{S}_4^n\big) \\
&= \frac{89}{140}\frac{32^n}{(4^n+1)^2} + \frac{5}{4}\frac{8^n}{(4^n+1)^2} + \frac{3}{14}\frac{4^n}{(4^n+1)^2} + \frac{9}{10}\frac{2^n}{(4^n+1)^2}.
\end{aligned}
$$

Now consider

$$
c_4 = \lim_{n\to\infty} c_4^n = \lim_{n\to\infty} 2^{-n} \cdot \bar{\mathrm{d}}\big(\widehat{S}_4^n\big).
$$

Then

$$
\frac{89}{140}\frac{16^n}{(4^n+1)^2} \overset{n\to\infty}{\longrightarrow} \frac{89}{140},
$$

as all higher terms tend to 0. Therefore,

$$
c_4 = \frac{89}{140}.
$$

If the only interest is in calculating c_p, the calculations can be simplified by ignoring everything that does not contribute to the coefficient of the highest-base term. This shall be demonstrated for $p = 5$. The first row $([M_{4,5}^k]_{11}, [M_{4,5}^k]_{12}, [M_{4,5}^k]_{13}, [M_{4,5}^k]_{14})$ of

$$M_{4,5}^k = \begin{pmatrix} 5 & 60 & 30 & 60 \\ 0 & 6 & 1 & 0 \\ 0 & 12 & 11 & 0 \\ 0 & 8 & 2 & 2 \end{pmatrix}^k$$

is, using $r = 17 + \sqrt{73}$ and $\bar{r} = 17 - \sqrt{73}$ as well as $s_k = r^k + \bar{r}^k$ and $\bar{s}_k = r^k - \bar{r}^k$, given by

$$\begin{aligned}
[M_{4,5}^k]_{11} &= 5^k, \\
[M_{4,5}^k]_{12} &= 80 \cdot 5^k + 40 \cdot 2^k - 60 \cdot 2^{-k} s_k + \frac{600}{\sqrt{73}} 2^{-k} \bar{s}_k, \\
[M_{4,5}^k]_{13} &= -25 \cdot 5^k + \frac{25}{2} 2^{-k} s_k - \frac{115}{2\sqrt{73}} 2^{-k} \bar{s}_k, \\
[M_{4,5}^k]_{14} &= 20 \cdot (5^k - 2^k).
\end{aligned}$$

As there are no instances of $50^k = 2^k 5^{2k}$, the product $M_{4,5}^{n-2}(T_5^2, A_5^1, B_5^1, F_5^1)^{\mathrm{T}}$ can be ignored for the calculation of c_5, as it would vanish in the average distance for $n \to \infty$ anyway.

Calculating the other vector leads to

$$\begin{pmatrix} \theta_5^{n-k} + \upsilon_5^{n-k} \\ \alpha_5^{n-1-k} \\ \beta_5^{n-1-k} \\ \phi_5^{n-1-k} \end{pmatrix} \approx \begin{pmatrix} \frac{7}{5} \cdot 50^{n-k} \\ \frac{43}{250} \cdot 50^{n-k} \\ \frac{22}{250} \cdot 50^{n-k} \\ \frac{38}{250} \cdot 50^{n-k} \end{pmatrix} = \frac{1}{250} 50^{n-k} \begin{pmatrix} 350 \\ 43 \\ 22 \\ 38 \end{pmatrix},$$

where the first relation (denoted by $\approx$) is not an equation, but already cleansed of terms irrelevant to c_5. Now calculate

$$\begin{aligned}
&\frac{1}{250} 50^n \sum_{k=0}^{n-3} \Big(350 \cdot 10^{-k} + 3440 \cdot 10^{-k} - 550 \cdot 10^{-k} + 760 \cdot 10^{-k} \\
&\qquad + 1720 \cdot 25^{-k} - 760 \cdot 25^{-k} - 2305 \cdot 100^{-k} s_k + \frac{24535}{\sqrt{73}} \cdot 100^{-k} \bar{s}_k \Big) \\
&= \frac{1}{50} 50^n \sum_{k=0}^{n-3} \Big(800 \cdot 10^{-k} + 192 \cdot 25^{-k} - 461 \cdot 100^{-k} s_k + \frac{4907}{\sqrt{73}} \cdot 100^{-k} \bar{s}_k \Big)
\end{aligned}$$

$$
\begin{aligned}
&\approx \frac{1}{50}50^n\left(\frac{8000}{9} + 200 - 461\left(\frac{9050+850\sqrt{73}}{6891+615\sqrt{73}} + \frac{9050-850\sqrt{73}}{6891-615\sqrt{73}}\right)\right.\\
&\qquad \left. + \frac{4907}{\sqrt{73}}\left(\frac{9050+850\sqrt{73}}{6891+615\sqrt{73}} - \frac{9050-850\sqrt{73}}{6891-615\sqrt{73}}\right)\right)\\
&= \frac{1}{18}50^n\left(\frac{111328}{284} - \frac{114789}{284} + \frac{14721}{284}\right) = \frac{2815}{1278}50^n.
\end{aligned}
$$

And hence

$$c_5 = \lim_{n\to\infty} 2^{-n}\cdot \bar{\mathrm{d}}\left(\widehat{S}_5^n\right) = \lim_{n\to\infty} \frac{2\cdot 2815\cdot 25^n}{1278\left(\frac{5}{2}\right)^2(5^n+1)^2} = \frac{2252}{3195}.$$

Comparing this to the exact result for

$$c_5^3 = 2^{-3}\cdot \bar{\mathrm{d}}\left(\widehat{S}_5^3\right) = \frac{4654}{6615}$$

from above, one can see that

$$\left|c_5^3 - c_5\right| = \left|\frac{4654}{6615} - \frac{2252}{3195}\right| \approx 0.0013,$$

wherefore $c_5 = \frac{2252}{3195}$ is plausible.

2.8 The formula for $n \to \infty$ and arbitrary p

Using the same method as above, ignoring all terms growing slower than $2^n p^{2n}$, it is possible to create a formula for arbitrary p. The matrix product

$$\begin{pmatrix} p & p(p-1)(p-2) & \frac{1}{2}p(p-1)(p-2) & \frac{1}{2}p(p-1)(p-2)(p-3) \\ 0 & 2(p-2) & 1 & 0 \\ 0 & 4(p-2) & p^2-4p+6 & 0 \\ 0 & 4(p-3) & 2 & 2 \end{pmatrix}^{n-1} \begin{pmatrix} T_p^2 \\ A_p^1 \\ B_p^1 \\ F_p^1 \end{pmatrix}$$

does not contain any terms growing sufficiently fast. Therefore, it can be ignored. The same holds true for all the Λ_p^n-terms used in the calculation of θ, υ, α, β and ϕ. In fact, θ can be ignored completely, as its fastest growing term contains only $(2p)^n$. Calculating the remaining variables and ignoring factors of insufficient growth leads to

$$
\begin{aligned}
\upsilon_p^n &= 2^{n-4} p(p-1)^2 \Big(4p(p-1)(p^{n-1}-1) - 2(p-2)^2(p^{n-2}-1) - 2p(p-1)^2 \\
&\qquad + p^2(p+2)(p^{n-2}-1)^2 \Big) \\
&\quad + p(p-1)(p-2)^2 \Lambda_p^{n-2} \\
&\approx 2^{n-4} p(p-1)^2 p^2 (p+2) p^{2n-4} \\
&= \frac{(p-1)^2(p+2)}{16p} (2^n p^{2n}), \\
\alpha_p^n &= 2^{n-2} p(p-1)(2p^2-2p+3)(p^{n-1}-1)^2 + 2^n p(p-1)^2(p^n-1) \\
&\quad - 2^{n-1}(p-1)(p-2)(p^{n-1}-1) - 2^{n-1} p(p-1)^3 + 2(p-2)\Lambda_p^{n-1} \\
&\approx 2^{n-2} p(p-1)(2p^2-2p+3) p^{2n-2} \\
&= \frac{(p-1)\big(2p^2-2p+3\big)}{4p} (2^n p^{2n}), \\
\beta_p^n &= 2^{n-2} p(p-1)(p^2-p+2)(p^{n-1}-1)^2 + 2^{n-1}(p^3+p^2-9p+8)(p^n-1) \\
&\quad - 2^n(p-1)(p-2)(p^{n-1}-1) - 2^{n-1}(p-1)(p-2)(p^2-p+3) - 2^n \\
&\quad + (p^2-5p+8)\Lambda_p^n + 4(p-2)\Lambda_p^{n-1} \\
&\approx 2^{n-2} p(p-1)(p^2-p+2) p^{2n-2} \\
&= \frac{(p-1)(p^2-p+2)}{4p} (2^n p^{2n}), \\
\phi_p^n &= 2^{n-1} p(p-1)(p^2-2p+4)(p^{n-1}-1)^2 + 2^n p(p-1)^2(p^n-1) \\
&\quad - 2^n(p-1)(p-2)(p^{n-1}-1) - 2^{n-1} p(p-1)^3 + 4(p-2)\Lambda_p^{n-1} \\
&\approx 2^{n-1} p(p-1)(p^2-2p+4) p^{2n-2} \\
&= \frac{(p-1)(p^2-2p+4)}{2p} (2^n p^{2n}).
\end{aligned}
$$

The latter three of these formulas come with an exponent lower by 1 than the first, leading to a factor of $2p^2$ in the divisor. Therefore, the relevant vector can be given by

$$\begin{pmatrix} v_p^{n-k} \\ \alpha_p^{n-k-1} \\ \beta_p^{n-k-1} \\ \phi_p^{n-k-1} \end{pmatrix} = 2^{n-k} p^{2(n-k)} \begin{pmatrix} \frac{(p-1)^2(p+2)}{16p} \\ \frac{(p-1)(2p^2-2p+3)}{8p^3} \\ \frac{(p-1)(p^2-p+2)}{8p^3} \\ \frac{(p-1)(p^2-2p+4)}{4p^3} \end{pmatrix} = 2^{n-k} p^{2(n-k)} \frac{p-1}{16p^3} \begin{pmatrix} p^2(p-1)(p+2) \\ 4p^2-4p+6 \\ 2p^2-2p+4 \\ 4p^2-8p+16 \end{pmatrix}.$$

Next, the first row of the following matrix is needed:

$$M_{4,p}^k = \begin{pmatrix} p & p(p-1)(p-2) & \frac{1}{2}p(p-1)(p-2) & \frac{1}{2}p(p-1)(p-2)(p-3) \\ 0 & 2(p-2) & 1 & 0 \\ 0 & 4(p-2) & p^2-4p+6 & 0 \\ 0 & 4(p-3) & 2 & 2 \end{pmatrix}^k.$$

The first entry is obviously p^k, but the other three do not look particularly nice. As the first step, the characteristic polynomial[3] and thereby the eigenvalues have to be calculated.

$$\chi_{M_{4,p}}(\lambda) = (2-\lambda)(p-\lambda)\big(\lambda^2 - (p^2-2p+2)\lambda + 2p^3 - 12p^2 + 24p - 16\big).$$

Immediately, we get

$$\lambda_1 = 2 \quad \text{and}$$
$$\lambda_2 = p,$$

and solving the remaining quadratic equation yields

$$\begin{aligned} \lambda_3 &= \frac{1}{2}p^2 - p + 1 + \sqrt{\left(\frac{1}{2}p^2 - p + 1\right)^2 - 2p^3 + 12p^2 - 24p + 16} \\ &= \frac{1}{2}p^2 - p + 1 + \sqrt{\frac{1}{4}p^4 - 3p^3 + 14p^2 - 26p + 17} \quad \text{and} \\ \lambda_4 &= \frac{1}{2}p^2 - p + 1 - \sqrt{\frac{1}{4}p^4 - 3p^3 + 14p^2 - 26p + 17} \end{aligned}$$

[3] The characteristic polynomial was calculated using Wolfram|Alpha, as were the more complicated eigenvectors and the inverted transformation matrix.

as the eigenvalues. For somewhat obvious reasons, in the following, the latter two eigenvalues will be referred as λ_+ and λ_- (read 'lambda plus' and 'lambda minus'). Notice that there are four eigenvalues which implies that the eigenspaces are 1-dimensional.

The eigenvalues lead to the following eigenspaces:

$$\begin{aligned}
\mathrm{Eig}_2(M_{4,p}) &= \ker(M_{4,p} - 2\mathbb{1}_4) \\
&= \ker\begin{pmatrix} p-2 & p(p-1)(p-2) & \frac{1}{2}p(p-1)(p-2) & \frac{1}{2}p(p-1)(p-2)(p-3) \\ 0 & 2(p-3) & 1 & 0 \\ 0 & 4(p-2) & p^2-4p+4 & 0 \\ 0 & 4(p-3) & 2 & 0 \end{pmatrix} \\
&= \left\langle -\frac{1}{2}p(p-1)(p-3)\cdot \mathbf{e}_1 + \mathbf{e}_4 \right\rangle,
\end{aligned}$$

where $\mathbb{1}_n$ denotes the $n\times n$ identity matrix, Eig_λ denotes the eigenspace for the eigenvalue λ, $\langle\,\rangle$ denotes the span of some vector(s), and $\mathbf{e}_i$ denote the i-th unit vectors. For the second eigenspace, we get

$$\begin{aligned}
\mathrm{Eig}_p(M_{4,p}) &= \ker(M_{4,p} - p\mathbb{1}_4) \\
&= \ker\begin{pmatrix} 0 & p(p-1)(p-2) & \frac{1}{2}p(p-1)(p-2) & \frac{1}{2}p(p-1)(p-2)(p-3) \\ 0 & p-4 & 1 & 0 \\ 0 & 4(p-2) & p^2-5p+6 & 0 \\ 0 & 4(p-3) & 2 & 2-p \end{pmatrix} \\
&= \langle \mathbf{e}_1 \rangle.
\end{aligned}$$

Unfortunately, the remaining two eigenspaces do not look that nice. Using

$$\xi = \sqrt{p^4 - 12p^3 + 56p^2 - 104p + 68}$$

one gets

$$\begin{aligned}
\mathrm{Eig}_{\lambda_+}(M_{4,p}) &= \ker\Big(M_{4,p} - \Big(\frac{1}{2}p^2 - p + 1 + \frac{1}{2}\xi\Big)\mathbb{1}_4\Big) \\
&= \Big\langle t_{13}\mathbf{e}_1 + \tfrac{1}{2}\mathbf{e}_2 + t_{33}\mathbf{e}_3 + 1\mathbf{e}_4 \Big\rangle, \\
\text{with } t_{13} &= \frac{p(p-1)(p-2)^3\big(p^4 - 9p^3 + 32p^2 - 38p + 8 + (p^2-3p+4)\xi\big)}{2(p^2-4p+2+\xi)\big(p^4 - 9p^3 + 32p^2 - 50p + 32 + p(p-3)\xi\big)} \text{ and} \\
t_{33} &= \frac{(p-2)\big(p^5 - 13p^4 + 68p^3 - 166p^2 + 184p - 80 + (p^3 - 7p^2 + 16p - 8)\xi\big)}{2\big(p^4 - 9p^3 + 32p^2 - 50p + 32 + p(p-3)\xi\big)},
\end{aligned}$$

and finally,

$$\begin{aligned}
\mathrm{Eig}_{\lambda_-}(M_{4,p}) &= \ker\Big(M_{4,p} - \Big(\frac{1}{2}p^2 - p + 1 - \frac{1}{2}\xi\Big)\mathbb{1}_4\Big) \\
&= \Big\langle t_{14}\mathbf{e}_1 + \tfrac{1}{2}\mathbf{e}_2 + t_{34}\mathbf{e}_3 + 1\mathbf{e}_4\Big\rangle, \\
\text{with } t_{14} &= \frac{p(p-1)(p-2)^3\big(p^4 - 9p^3 + 32p^2 - 38p + 8 - (p^2 - 3p + 4)\xi\big)}{2\big(p^2 - 4p + 2 + \xi\big)\big(p^4 - 9p^3 + 32p^2 - 50p + 32 - p(p-3)\xi\big)} \text{ and} \\
t_{34} &= \frac{(p-2)\big(p^5 - 13p^4 + 68p^3 - 166p^2 + 184p - 80 - (p^3 - 7p^2 + 16p - 8)\xi\big)}{2\big(p^4 - 9p^3 + 32p^2 - 50p + 32 - p(p-3)\xi\big)}.
\end{aligned}$$

Then the matrix $M_{4,p}$ can be written as $T \cdot D \cdot T^{-1}$, where D is a diagonal matrix with the eigenvalues on the diagonal and T has eigenvectors as columns. Then, using $t_{11} = -\frac{1}{2}p(p-1)(p-3)$,

$$T = \begin{pmatrix} t_{11} & 1 & t_{13} & t_{14} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & t_{33} & t_{34} \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

The diagonal matrix D and the inverse matrix T^{-1} of T are

$$D = \begin{pmatrix} 2 & & & \\ & p & & \\ & & \lambda_+ & \\ & & & \lambda_- \end{pmatrix} \quad \text{and} \quad T^{-1} = \begin{pmatrix} 0 & -2 & 0 & 1 \\ 1 & t_{22}^{(-1)} & t_{23}^{(-1)} & t_{24}^{(-1)} \\ 0 & t_{32}^{(-1)} & \frac{2}{\xi} & 0 \\ 0 & t_{42}^{(-1)} & -\frac{2}{\xi} & 0 \end{pmatrix}.$$

$$\begin{aligned}
\text{with} \quad \lambda_\pm &= \frac{1}{2}p^2 - p + 1 \pm \frac{1}{2}\xi, \\
t_{22}^{(-1)} &= -p\big(2p^4 - 19p^3 + 60p^2 - 75p + 32\big)d^{-1}, \\
t_{23}^{(-1)} &= \frac{1}{2}p^2(p-1)d^{-1}, \\
t_{24}^{(-1)} &= \frac{1}{2}p(p-1)(p-3), \\
t_{32}^{(-1)} &= a_+b_-c^{-1}, \\
t_{42}^{(-1)} &= a_-b_+c^{-1}, \\
a_\pm &= \mp\big(p^4 - 9p^3 + 32p^2 - 50p + 32 \pm p(p-3)\xi\big), \\
b_\pm &= \big(p^5 - 13p^4 + 68p^3 - 166p^2 + 184p - 80 \pm (p^3 - 7p^2 + 16p - 8)\xi\big), \\
c &= 4(p-1)(p-2)(p-4)\big(p^2 - 7p + 8\big)\xi, \\
d &= p^2 - 7p + 8.
\end{aligned}$$

Now the powers of the original matrix can be calculated as follows.

$$\begin{aligned} M_{4,p}^k &= (T \cdot D \cdot T^{-1})^k = T \cdot D^k \cdot T^{-1} \\ &= T \cdot \operatorname{diag}(2^k, p^k, \lambda_+^k, \lambda_-^k) \cdot T^{-1} \\ &= \begin{pmatrix} p^k & m_{12}^{(k)} & m_{13}^{(k)} & t_{24}^{(-1)}(p^k - 2^k) \\ 0 & m_{22}^{(k)} & \frac{1}{\xi}(\lambda_+^k - \lambda_-^k) & 0 \\ 0 & m_{32}^{(k)} & m_{33}^{(k)} & 0 \\ 0 & m_{42}^{(k)} & \frac{2}{\xi}(\lambda_+^k - \lambda_-^k) & 2^k \end{pmatrix}, \end{aligned}$$

with

$$\begin{aligned} m_{12}^{(k)} &= t_{24}^{(-1)} 2^{k+1} - (2p^4 - 19p^3 + 60p^2 - 75p + 32)p^{k+1} d^{-1} - \lambda_+^k a_+ b_+^{(2)} c_+^{-1} - \lambda_-^k a_- b_{(2,-)} c_-^{-1}, \\ m_{13}^{(k)} &= \frac{1}{2} p^2 (p-1) p^k d^{-1} + \frac{1}{2} \lambda_+^k b_+^{(3)} d_+^{-1} + \frac{1}{2} \lambda_-^k b_-^{(3)} d_-^{-1}, \quad \text{using} \\ a_\pm &= p^5 - 13p^4 + 68p^3 - 166p^2 + 184p - 80 \mp (p^3 - 7p^2 + 16p - 8)\xi, \\ b_\pm^{(i)} &= p(p-1)(p-2)^i \big(p^4 - 9p^3 + 32p^2 - 38p + 8 \pm (p^2 - 3p + 4)\xi\big), \\ c_\pm &= 8(p-1)(p-4)d\big(\xi^2 \pm (p^2 - 4p + 2)\xi\big), \\ d_\pm &= (p^4 - 8p^3 + 23p^2 - 28p + 16)\xi^2 \pm (p^6 - 14p^5 + 81p^4 - 234p^3 + 338p^2 - 216p + 32)\xi, \\ d &= p^2 - 7p + 8. \end{aligned}$$

Notice how some of the abbreviations or parts of them are present as subterms in t_{13}, t_{33}, t_{14}, and t_{34} above.

The remaining entries $m_{ij}^{(k)}$ with $i > 1$ are irrelevant to the following calculations, as they are not needed for the calculation of T_p^n anyway. The Wiener index cleansed of irrelevant terms is

$$\begin{aligned} \mathrm{W}(\widehat{S}_p^n) \approx 2^n p^{2n} \frac{p-1}{16p^3} \Big(& p^2(p-1)(p+2) - \frac{p(4p^2 - 4p + 6)(2p^4 - 19p^3 + 60p^2 - 75p + 32)}{p^2 - 7p + 8} \\ & + \frac{p^2(p-1)(2p^2 - 2p + 4)}{2(p^2 - 7p + 8)} + \frac{1}{2} p(p-1)(p-3)(4p^2 - 8p + 16) \Big) \sum_{k=0}^{n-3} (2p)^k \\ & + \Big(p(p-1)(p-3)\big(4p^2 - 4p + 6\big) - \frac{1}{2} p(p-1)(p-3)(4p^2 - 8p + 16) \Big) \sum_{k=0}^{n-3} p^{-2k} \\ & + \Big(\frac{p(p-1)(p-2)^3(p^2 - p + 2)\big(p^4 - 9p^3 + 32p^2 - 38p + 8 + (p^2 - 3p + 4)\xi\big)}{(p^4 - 8p^3 + 23p^2 - 28p + 16)\xi^2 + (p^6 - 14p^5 + 81p^4 - 234p^3 + 338p^2 - 216p + 32)\xi} \\ & - \frac{p(p-1)(p-2)^2(2p^2 - 2p + 3)\big(p^4 - 9p^3 + 32p^2 - 38p + 8 + (p^2 - 3p + 4)\xi\big)}{4(p^4 - 12p^3 + 47p^2 - 68p + 32)\big(p^4 - 12p^3 + 56p^2 - 104p + 68 + (p^2 - 4p + 2)\xi\big)} \\ & \cdot \big(p^5 - 13p^4 + 68p^3 - 166p^2 + 184p - 80 - (p^3 - 7p^2 + 16p - 8)\xi\big) \Big) \sum_{k=0}^{n-3} \Big(\frac{\frac{1}{2}p^2 - p + 1 + \frac{1}{2}\xi}{2p^2} \Big)^k \end{aligned}$$

$$+\Bigg(\frac{p(p-1)(p-2)^3(p^2-p+2)\big(p^4-9p^3+32p^2-38p+8-(p^2-3p+4)\xi\big)}{(p^4-8p^3+23p^2-28p+16)\xi^2-(p^6-14p^5+81p^4-234p^3+338p^2-216p+32)\xi}$$
$$-\frac{p(p-1)(p-2)^2(2p^2-2p+3)\big(p^4-9p^3+32p^2-38p+8-(p^2-3p+4)\xi\big)}{4(p^4-12p^3+47p^2-68p+32)\big(p^4-12p^3+56p^2-104p+68-(p^2-4p+2)\xi\big)}$$
$$\cdot\big(p^5-13p^4+68p^3-166p^2+184p-80+(p^3-7p^2+16p-8)\xi\big)\Bigg)\sum_{k=0}^{n-3}\left(\frac{\frac{1}{2}p^2-p+1-\frac{1}{2}\xi}{2p^2}\right)^k.$$

The general formula for the geometric sum is

$$\sum_{k=0}^{n} a^k = \frac{a^{n+1}-1}{a-1},$$

which converges for $|a| < 1$. Using this, one can show that

$$\sum_{k=0}^{n-3} a^{-k} = \frac{a^{2-n}-1}{a^{-1}-1} = \frac{a-a^{3-n}}{a-1}.$$

Therefore, the sums in the formula can be simplified to

$$\sum_{k=0}^{n-3}(2p)^{-k} = \frac{2p}{2p-1} - \frac{(2p)^3}{(2p-1)(2p)^n},$$
$$\sum_{k=0}^{n-3} p^{-2k} = \frac{p^2}{p^2-1} - \frac{p^6}{(p^2-1)p^{2n}},$$
$$\sum_{k=0}^{n-3}\left(\frac{\frac{1}{2}p^2-p+1+\frac{1}{2}\xi}{2p^2}\right)^k = \frac{\frac{4p^2}{p^2-2p+2+\xi} - \left(\frac{4p^2}{p^2-2p+2+\xi}\right)^{3-n}}{\frac{4p^2}{p^2-2p+2+\xi}-1}$$
$$= \frac{4p^2}{3p^2+2p-2-\xi}\left(1-\left(\frac{4p^2}{p^2-2p+2+\xi}\right)^{2-n}\right),$$
$$\sum_{k=0}^{n-3}\left(\frac{\frac{1}{2}p^2-p+1-\frac{1}{2}\xi}{2p^2}\right)^k = \frac{\frac{4p^2}{p^2-2p+2-\xi} - \left(\frac{4p^2}{p^2-2p+2-\xi}\right)^{3-n}}{\frac{4p^2}{p^2-2p+2-\xi}-1}$$
$$= \frac{4p^2}{3p^2+2p-2+\xi}\left(1-\left(\frac{4p^2}{p^2-2p+2-\xi}\right)^{2-n}\right).$$

Using these formulas to simplify the parts of the equation and ignoring the terms that will converge to 0 later on, the part with $\sum_{k=0}^{n-3} p^{-2k}$ leads to

$$\begin{aligned}
&2^n p^{2n} \frac{p-1}{16p^3} \big(p(p-1)(p-3)(4p^2-4p+6) - \tfrac{1}{2} p(p-1)(p-3)(4p^2-8p+16)\big) \frac{p^2}{p^2-1} \\
&= 2^n p^{2n} \frac{(p-1)^2(p-3)}{16(p^2-1)} (4p^2-4p+6-2p^2+4p-8) = 2^n p^{2n} \frac{(p-1)^2(p-3)}{8}.
\end{aligned}$$

The term that stems from the p^k entries and is therefore using the sum $\sum_{k=0}^{n-3} (2p)^{-k}$ gives us

$$\begin{aligned}
&2^n p^{2n} \frac{p-1}{16p^3} \bigg(p^2(p-1)(p+2) - \frac{p(4p^2-4p+6)(2p^4-19p^3+60p^2-75p+32)}{p^2-7p+8} \\
&\quad + \frac{p^2(p-1)(2p^2-2p+4)}{2(p^2-7p+8)} + \frac{1}{2} p(p-1)(p-3)(4p^2-8p+16) \bigg) \frac{2p}{2p-1} \\
&= 2^n p^{2n} \frac{(p-1)}{8p(2p-1)} \bigg(p(p-1)(p+2) + (p-1)(p-3)(2p^2-4p+8) \\
&\quad - \frac{(4p^2-4p+6)(p-1)(2p^3-17p^2+43p-32) - p(p-1)(p^2-p+2)}{p^2-7p+8} \bigg) \\
&= 2^n p^{2n} \frac{(p-1)^2}{8p(2p-1)} \bigg(-\frac{8p^5-76p^4+251p^3-401p^2+384p-192}{p^2-7p+8} + (p-2)(2p^2-5p+12) \bigg) \\
&= -2^n p^{2n} \frac{(p-1)^2(6p^4-53p^3+150p^2-151p+40)}{8(2p-1)(p^2-7p+8)}.
\end{aligned}$$

The remaining two expressions are a lot harder to calculate. The one belonging to the sum $\sum_{k=0}^{n-3} \left(\frac{\frac{1}{2}p^2-p+1+\frac{1}{2}\xi}{2p^2} \right)^k$ leads to the following:

$$\begin{aligned}
&2^n p^{2n} \frac{p-1}{16p^3} \frac{4p^2}{3p^2+2p-2-\xi} \bigg(\frac{p(p-1)(p-2)^3(2p^2-2p+4)\big(p^4-9p^3+32p^2-38p+8+(p^2-3p+4)\xi\big)}{2\big((p^4-8p^3+23p^2-28p+16)\xi^2+(p^6-14p^5+81p^4-234p^3+338p^2-216p+32)\xi\big)} \\
&\quad - \frac{p(p-1)(p-2)^2(4p^2-4p+6)\big(p^4-9p^3+32p^2-38p+8+(p^2-3p+4)\xi\big)}{8(p-1)(p-4)(p^2-7p+8)\big(\xi^2+(p^2-4p+2)\xi\big)} \\
&\quad \cdot \big(p^5-13p^4+68p^3-166p^2+184p-80-(p^3-7p^2+16p-8)\xi\big) \bigg) \\
&= 2^n p^{2n} \frac{(p-1)^2(p-2)^2}{4(3p^2+2p-2-\xi)} \big(p^4-9p^3+32p^2-38p+8+(p^2-3p+4)\xi\big) \\
&\quad \cdot \bigg(\frac{(p-2)(p^2-p+2)}{(p^4-8p^3+23p^2-28p+16)\xi^2+(p^6-14p^5+81p^4-234p^3+338p^2-216p+32)\xi} \\
&\quad - \frac{(2p^2-2p+3)\big(p^5-13p^4+68p^3-166p^2+184p-80-(p^3-7p^2+16p-8)\xi\big)}{4(p-1)(p-4)(p^2-7p+8)\big(\xi^2+(p^2-4p+2)\xi\big)} \bigg) \\
&= 2^n p^{2n} \frac{(p-1)^2(p-2)^2\big(p^4-9p^3+32p^2-38p+8+(p^2-3p+4)\xi\big)}{4(3p^2+2p-2-\xi)\xi^2} \\
&\quad \cdot \bigg(\frac{(p-2)(p^2-p+2)\big((p^4-8p^3+23p^2-28p+16)\xi^2-(p^6-14p^5+81p^4-234p^3+338p^2-216p+32)\xi\big)}{(p^4-8p^3+23p^2-28p+16)^2\xi^2-(p^6-14p^5+81p^4-234p^3+338p^2-216p+32)^2} \\
&\quad - \frac{(2p^2-2p+3)\big(p^5-13p^4+68p^3-166p^2+184p-80-(p^3-7p^2+16p-8)\xi\big)\big(\xi^2-(p^2-4p+2)\xi\big)}{4(p-1)(p-4)(p^2-7p+8)\big(\xi^2-(p^2-4p+2)^2\big)} \bigg)
\end{aligned}$$

$$
\begin{aligned}
&= 2^n p^{2n} \frac{(p-1)^2(p-2)^2\big(p^4-9p^3+32p^2-38p+8+(p^2-3p+4)\xi\big)}{4(3p^2+2p-2-\xi)\xi^2} \\
&\quad \cdot\Big(-\frac{2(p-2)(p^2-p+2)\big((p^4-8p^3+23p^2-28p+16)\xi^2-(p^6-14p^5+81p^4-234p^3+338p^2-216p+32)\xi\big)}{16(p-1)(p-2)^3(p-4)(p^2-7p+8)^2} \\
&\qquad + \frac{(2p^2-2p+3)\big(p^5-13p^4+68p^3-166p^2+184p-80-(p^3-7p^2+16p-8)\xi\big)\big(\xi^2-(p^2-4p+2)\xi\big)}{16(p-1)(p-2)(p-4)(p^2-7p+8)^2}\Big) \\
&= 2^n p^{2n} \frac{(p-1)\big(p^4-9p^3+32p^2-38p+8+(p^2-3p+4)\xi\big)}{64(p-4)(p^2-7p+8)^2(3p^2+2p-2-\xi)\xi^2} \\
&\quad \cdot\Big((p-2)(2p^2-2p+3)\big(p^5-13p^4+68p^3-166p^2+184p-80-(p^3-7p^2+16p-8)\xi\big)\big(\xi^2-(p^2-4p+2)\xi\big) \\
&\qquad - 2(p^2-p+2)\big((p^4-8p^3+23p^2-28p+16)\xi^2-(p^6-14p^5+81p^4-234p^3+338p^2-216p+32)\xi\big)\Big) \\
&= 2^n p^{2n} \frac{(p-1)\big(p^4-9p^3+32p^2-38p+8+(p^2-3p+4)\xi\big)(3p^2+2p-2+\xi)}{32(p-4)(p^2-7p+8)^2\big((3p^2+2p-2)^2-\xi^2\big)\xi^2} \\
&\quad \cdot\Big((2p^8-30p^7+192p^6-675p^5+1442p^4-1997p^3+1822p^2-1008p+256)\xi^2 \\
&\qquad - (2p^{10}-42p^9+392p^8-2111p^7+7236p^6-16603p^5+26264p^4-28898p^3+21524p^2-9824p+2048)\xi\Big) \\
&= 2^n p^{2n} \frac{(p-1)(p^5-9p^4+34p^3-54p^2+68p-64)+p(p^3-4p^2+9p-6)\xi}{64(p-4)(p^2-7p+8)^2(p^3+4p^2-4p+8)\xi^2} \\
&\quad \cdot\Big((2p^8-30p^7+192p^6-675p^5+1442p^4-1997p^3+1822p^2-1008p+256)\xi^2 \\
&\qquad - (2p^{10}-42p^9+392p^8-2111p^7+7236p^6-16603p^5+26264p^4-28898p^3+21524p^2-9824p+2048)\xi\Big) \\
&= 2^n p^{2n} \frac{p-1}{16(p^2-7p+8)(p^3+4p^2-4p+8)\xi^2} \cdot \Big((2p^7-8p^6-31p^5+201p^4-442p^3+552p^2-400p+128)\xi^2 \\
&\qquad - (2p^9-20p^8+35p^7+327p^6-1940p^5+4930p^4-7420p^3+7088p^2-4032p+1024)\xi\Big).
\end{aligned}
$$

The other formula, belonging to the sum $\sum_{k=0}^{n-3}\Big(\frac{\frac{1}{2}p^2-p+1+\frac{1}{2}\xi}{2p^2}\Big)^k$, behaves similarly, only different in the sign of certain terms.

$$
\begin{aligned}
&2^n p^{2n} \frac{p-1}{16p^3}\,\frac{4p^2}{3p^2+2p-2+\xi}\Big(\frac{p(p-1)(p-2)^3(2p^2-2p+4)\big(p^4-9p^3+32p^2-38p+8-(p^2-3p+4)\xi\big)}{2\big((p^4-8p^3+23p^2-28p+16)\xi^2-(p^6-14p^5+81p^4-234p^3+338p^2-216p+32)\xi\big)} \\
&\quad - \frac{p(p-1)(p-2)^2(4p^2-4p+6)\big(p^4-9p^3+32p^2-38p+8-(p^2-3p+4)\xi\big)}{8(p-1)(p-4)(p^2-7p+8)\big(\xi^2-(p^2-4p+2)\xi\big)} \\
&\quad \cdot\big(p^5-13p^4+68p^3-166p^2+184p-80+(p^3-7p^2+16p-8)\xi\big)\Big) \\
&= 2^n p^{2n} \frac{(p-1)^2(p-2)^2}{4(3p^2+2p-2+\xi)}\big(p^4-9p^3+32p^2-38p+8-(p^2-3p+4)\xi\big) \\
&\quad \cdot\Big(\frac{(p-2)(p^2-p+2)}{(p^4-8p^3+23p^2-28p+16)\xi^2-(p^6-14p^5+81p^4-234p^3+338p^2-216p+32)\xi} \\
&\qquad - \frac{(2p^2-2p+3)\big(p^5-13p^4+68p^3-166p^2+184p-80+(p^3-7p^2+16p-8)\xi\big)}{4(p-1)(p-4)(p^2-7p+8)\big(\xi^2-(p^2-4p+2)\xi\big)}\Big) \\
&= 2^n p^{2n} \frac{(p-1)^2(p-2)^2\big(p^4-9p^3+32p^2-38p+8-(p^2-3p+4)\xi\big)}{4(3p^2+2p-2+\xi)\xi^2} \\
&\quad \cdot\Big(\frac{(p-2)(p^2-p+2)\big((p^4-8p^3+23p^2-28p+16)\xi^2+(p^6-14p^5+81p^4-234p^3+338p^2-216p+32)\xi\big)}{(p^4-8p^3+23p^2-28p+16)^2\xi^2-(p^6-14p^5+81p^4-234p^3+338p^2-216p+32)^2} \\
&\qquad - \frac{(2p^2-2p+3)\big(p^5-13p^4+68p^3-166p^2+184p-80+(p^3-7p^2+16p-8)\xi\big)\big(\xi^2+(p^2-4p+2)\xi\big)}{4(p-1)(p-4)(p^2-7p+8)\big(\xi^2-(p^2-4p+2)^2\big)}\Big) \\
&= 2^n p^{2n} \frac{(p-1)^2(p-2)^2\big(p^4-9p^3+32p^2-38p+8-(p^2-3p+4)\xi\big)}{4(3p^2+2p-2+\xi)\xi^2}
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \Bigg(- \frac{2(p-2)(p^2-p+2)\big((p^4-8p^3+23p^2-28p+16)\xi^2+(p^6-14p^5+81p^4-234p^3+338p^2-216p+32)\xi\big)}{16(p-1)(p-2)^3(p-4)(p^2-7p+8)^2} \\
&\quad + \frac{(2p^2-2p+3)\big(p^5-13p^4+68p^3-166p^2+184p-80+(p^3-7p^2+16p-8)\xi\big)\big(\xi^2+(p^2-4p+2)\xi\big)}{16(p-1)(p-2)(p-4)(p^2-7p+8)^2}\Bigg) \\
&= 2^n p^{2n} \frac{(p-1)\big(p^4-9p^3+32p^2-38p+8-(p^2-3p+4)\xi\big)}{64(p-4)(p^2-7p+8)^2(3p^2+2p-2+\xi)\xi^2} \\
&\cdot \Big((p-2)(2p^2-2p+3)\big(p^5-13p^4+68p^3-166p^2+184p-80+(p^3-7p^2+16p-8)\xi\big)\big(\xi^2+(p^2-4p+2)\xi\big) \\
&\quad - 2(p^2-p+2)\big((p^4-8p^3+23p^2-28p+16)\xi^2+(p^6-14p^5+81p^4-234p^3+338p^2-216p+32)\xi\big)\Big) \\
&= 2^n p^{2n} \frac{(p-1)\big(p^4-9p^3+32p^2-38p+8-(p^2-3p+4)\xi\big)(3p^2+2p-2-\xi)}{32(p-4)(p^2-7p+8)^2\big((3p^2+2p-2)^2-\xi^2\big)\xi^2} \\
&\cdot \Big((2p^8-30p^7+192p^6-675p^5+1442p^4-1997p^3+1822p^2-1008p+256)\xi^2 \\
&\quad + (2p^{10}-42p^9+392p^8-2111p^7+7236p^6-16603p^5+26264p^4-28898p^3+21524p^2-9824p+2048)\xi\Big) \\
&= 2^n p^{2n} \frac{(p-1)(p^5-9p^4+34p^3-54p^2+68p-64)-p(p^3-4p^2+9p-6)\xi}{64(p-4)(p^2-7p+8)^2(p^3+4p^2-4p+8)\xi^2} \\
&\cdot \Big((2p^8-30p^7+192p^6-675p^5+1442p^4-1997p^3+1822p^2-1008p+256)\xi^2 \\
&\quad + (2p^{10}-42p^9+392p^8-2111p^7+7236p^6-16603p^5+26264p^4-28898p^3+21524p^2-9824p+2048)\xi\Big) \\
&= 2^n p^{2n} \frac{p-1}{16(p^2-7p+8)(p^3+4p^2-4p+8)\xi^2} \cdot \Big((2p^7-8p^6-31p^5+201p^4-442p^3+552p^2-400p+128)\xi^2 \\
&\quad + (2p^9-20p^8+35p^7+327p^6-1940p^5+4930p^4-7420p^3+7088p^2-4032p+1024)\xi\Big).
\end{aligned}
$$

Note that there occurs $p-4$ in the denominator in both formulas. This means that the calculations above cannot be used for the case $p = 4$ and the formula has to be checked separately in this case, but this has already been done in the previous section.

Now, the four parts can be added together:

$$
\begin{aligned}
&2^n p^{2n} \Bigg(\frac{(p-1)^2(p-3)}{8} - \frac{(p-1)^2(6p^4-53p^3+150p^2-151p+40)}{8(2p-1)(p^2-7p+8)} \\
&\quad + \frac{p-1}{16(p^2-7p+8)(p^3+4p^2-4p+8)\xi^2} \\
&\quad \cdot \Big(2(2p^7-8p^6-31p^5+201p^4-442p^3+552p^2-400p+128)\xi^2 \\
&\qquad - (2p^9-20p^8+35p^7+327p^6-1940p^5+4930p^4-7420p^3+7088p^2-4032p+1024)\xi \\
&\qquad + (2p^9-20p^8+35p^7+327p^6-1940p^5+4930p^4-7420p^3+7088p^2-4032p+1024)\xi\Big)\Bigg) \\
&= 2^n p^{2n} \frac{p-1}{8} \Bigg((p-1)(p-3) - \frac{(p-1)(6p^4-53p^3+150p^2-151p+40)}{(2p-1)(p^2-7p+8)} \\
&\quad + \frac{2p^7-8p^6-31p^5+201p^4-442p^3+552p^2-400p+128}{(p^2-7p+8)(p^3+4p^2-4p+8)}\Bigg)
\end{aligned}
$$

$$= 2^n p^{2n} \frac{p-1}{8} \left(\frac{(p-1)(-4p^4+32p^3-82p^2+74p-16)}{(2p-1)(p^2-7p+8)} \right.$$
$$\left. + \frac{2p^7-8p^6-31p^5+201p^4-442p^3+552p^2-400p+128}{(p^2-7p+8)(p^3+4p^2-4p+8)} \right)$$
$$= 2^n p^{2n} \cdot \frac{p(p-1)(2p^4+6p^3-17p^2+26p-16)}{8(2p-1)(p^3+4p^2-4p+8)}.$$

This is a formula for the limit for $n \to \infty$ of the Wiener index, cleared of all terms that do not grow fast enough to matter. As is known from above, the average distance is $\bar{\mathrm{d}}(\widehat{S}_p^n) = \mathrm{W}(\widehat{S}_p^n) \cdot \frac{8}{p^2(p^n+1)^2}$. As $\frac{p^{2n}}{(p^n+1)^2} \overset{n\to\infty}{\longrightarrow} 1$, its normalised limit is given by

$$\lim_{n\to\infty} \frac{1}{2^n} \cdot \bar{\mathrm{d}}(\widehat{S}_p^n) = \frac{(p-1)(2p^4+6p^3-17p^2+26p-16)}{p(2p-1)(p^3+4p^2-4p+8)}.$$

This is a well-known formula[4] for the geodesic distance of a Sierpiński triangle and normalised average distance of Sierpiński graphs, originally found by Bandt and Kuschel [2, Section 4], and quoted, e.g., by Hinz, Klavžar, and Petr [11], p. 197. This strongly supports the hypothesis that Sierpiński graphs and Sierpiński triangle graphs behave progressively similar with regard to their metric properties, as the exponent increases and both resemble the Sierpiński triangle more closely.

One may conclude that the most important metric properties of Sierpiński triangle graphs are found. However, there is still much more research to be done. Of the three major colourings, for example, two are not yet done for Sierpiński triangle graphs; only the chromatic number is known (cf. [17]). Domination numbers pose another problem that is still to be solved. It is interesting to consider that the corresponding results for Sierpiński graphs are known. Maybe these properties of the Sierpiński triangle graphs can be derived form the Sierpiński graphs.

[4]Note that the formula leads to $\frac{89}{140}$ for $p = 4$ and therefore the division by $p-4$ is no problem.

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