Asset price bubbles and dynamic super-replication under transaction costs

Dissertation
an der Fakultät für Mathematik, Informatik und Statistik der
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2. Gutachter/in: **Prof. Dr. Luciano Campi**

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Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

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Zusammenfassung

Die Arbeit gliedert sich in zwei Hauptteile. Der erste Teil ist eine theoretische Untersuchung von Superhedging-Preisen und Finanzblasen in Marktmodellen mit proportionalen Transaktionskosten. Im zweiten Teil entwickeln wir eine Methode des maschinellen Lernens, um den Superhedging-Preis Prozess numerisch zu bestimmen.
Für den ersten Teil betrachten wir ein Finanzmarktmodell mit einem risikolosen und einem riskikobehafteten Vermögenswert unter proportionalen Transaktionskosten $\lambda \in (0,1)$ *auf einem endlichen Zeithorizont* T*. Wir liefern dynamische Versionen der Superhedging-Theoreme von [85]. Die Theoreme sind unterteilt in eine numéraire-freie Version, die gleichmäßig integrierbare Martingale als konsistente Preissysteme verwendet, und eine numéraire-basierte Version, die lokalen Martingalen als konsistente (lokale) Preissysteme entspricht. Die Superhedging-Theoreme garantieren, dass es keine Dualitätslücke zwischen dem ursprünglichen Problem des Superhedgens eines Contingent Claims unter proportionalen Transaktionskosten und dem entsprechenden dualen Problem gibt. Zu diesem Zweck erweitern wir den Begriff der zulässigen Strategien im numéraire-freien und im numéraire-basierten Sinne von Strategien auf* $[0,T]$ *auf Strategien auf* $[t,T]$*. In diesem Zusammenhang zeigen wir auch die Zeitunabhängigkeit der konsistenten (lokalen) Preissysteme in der dualen Formulierung. Inbesondere ist der Superhedging-Preis Prozess wohldefiniert. Unter weiteren Regularitätsannahmen beweisen wir Rechtsstetigkeit des Superhedging-Preis Prozesses.*
Wir schließen den ersten Teil mit der Untersuchung von Finanzblasen in dem Marktmodell mit proportionalen Transaktionskosten ab. In Anlehnung an [52] definieren wir den Fundamentalwert F *des risikobehafteten Vermögenswertes* S *als den Preis eines Superhedging Portfolios des Claims* $X_T = (0,1)$*, das heißt der Position, die zu einem Anteil des risikobehafteten Vermögenswertes und Null Bargeld führt. Unter Verwendung der Ergebnisse aus dem ersten Teil erhalten wir eine duale Darstellung des Fundamentalwerts. Der Finanzblasen-Prozess* β *is definiert als die Differenz aus dem Briefkurs* $(1+\lambda)S$ *und dem Fundamentalwert. Wir sagen, dass es eine Finanzblase im Marktmodell gibt, wenn* β *strikt positiv mit positiver Wahrscheinlichkeit für eine* $[0,T]$*-wertige Stoppzeit ist. Die Entstehung einer Finanzblase ist in unserem Modell direkt enthalten. Schließlich untersuchen wir den Einfluss von proportionalen Transaktionskosten auf die Entstehung und Größe von Finanzblasen. Diese Studie beweist, dass die Einführung von proportionalen Transaktionskosten die Bildung von Finanzblasen teilweise verhindern kann.*
Im zweiten Teil untersuchen wir eine Approximation basierend auf neuronalen Netzen für den Superhedging-Preis Prozess eines Contingent Claims in einem Marktmodell in diskreter Zeit von [40]. Die Approximation des Superhedging-Preis Prozesses ist in mehrere Schritte unterteilt. Zunächst beweisen wir, dass der α*-Quantil-Hedging-Preis für eine gegebene Erfolgswahrscheinlichkeit* $\alpha \in (0,1)$*, siehe [38], gegen den Superhedging-Preis konvergiert, wenn* α *gegen 1 geht. Die Berechnung des Superhedging-Preis Prozesses für* $t > 0$ *reduziert sich auf die Approximation des steigenden Prozesses* B *aus der gleichmäßigen Doob-Zerlegung, siehe [40], welcher manchmal auch als Konsumprozess bezeichnet wird. Anschließend zeigen wir, dass der* α*-Quantil-Hedging-Preis durch Long-Short-Term Memory neuronale Netze approximiert werden kann, indem wir die Superhedging-*

Strategien des α-Quantil-Hedging-Preises durch neuronale Netze approximieren, siehe [21]. Für $t > 0$ kann B_t durch ein essentielles Supremum über eine Menge von Zufallsvariablen auf der Basis von neuronalen Netzen approximiert werden. Schließlich präsentieren wir numerische Ergebnisse.

Abstract

The thesis is divided in two main parts. The first part is a theoretical study of super-replication prices and asset price bubbles in market models with proportional transaction costs. In the second part we develop a machine learning method to determine the super-replication price process numerically.

For the first part, we consider a financial market model with one risk-less and one risky asset under proportional transaction cost $\lambda \in (0,1)$ *on a finite time horizon* T*. We provide dynamic versions of the super-replication theorems of [85]. The theorems are divided in a numéraire-free version, which relates to uniformly integrable martingales as consistent price systems and a numéraire-based version, corresponding to local martingales as consistent (local) price systems. The super-replication theorems guarantee that there is no duality gap of the original problem of super-replicating a contingent claim under proportional transaction costs and the corresponding dual problem. For this purpose, we extend the notion of admissible strategies, in the numéraire-free and the numéraire-based sense, of [84] from strategies on* $[0,T]$*, to strategies on* $[t,T]$*. In this context we show time independence of the consistent (local) price systems in the dual formulation. In particular, the super-replication price process is well-defined. Under further regularity assumptions we prove right-continuity of the super-replication price process.*

We conclude the first part by the study of asset price bubbles in the market model with proportional transaction costs. By following [52], we define the fundamental value F*, of the risky asset* S*, as the price of a super-replicating portfolio of the claim* $X_T = (0,1)$*, i.e., the position resulting in one share of the risky asset and zero cash. Using the results from the first part we obtain a dual representation of the fundamental value. The bubble process* β *is defined as the difference of the ask-price* $(1+\lambda)S$ *and the fundamental value. We say that there is a bubble in the market model if* β *is strictly positive with positive probability for some* $[0,T]$*-valued stopping time. The birth of a bubble is directly included in our model. Finally, we investigate the impact of proportional transaction costs on the formation and size of asset price bubbles. This study proves that the introduction of proportional transaction costs can possible prevent bubbles' formation.*

In the second part we study neural network-based approximations for the super-replication price process of a contingent claim in a frictionless, discrete time market model of [40]. The approximation of the super-replication price is divided in several steps. First, we prove that the α*-quantile hedging price for a given probability of success* $\alpha \in (0,1)$*, see [38], converges to the super-replication price for* α *tending to* 1*. The calculation of the super-replication price process for* $t > 0$ *is reduced to the approximation of the increasing process* B *obtained from the uniform Doob decomposition, see [40], which is sometimes called process of consumption. Then, by approximating the superhedging strategies of the* α*-quantile hedging price by long short-term memory neural networks, see [21], we show that the* α*-quantile hedging price can be approximated by neural networks. For* $t > 0$*,* B_t *can be approximated by an essential supremums over a set of random variables based on neural networks. Finally, we present numerical results.*

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Contents

Introduction

This thesis is divided in two parts. In the first part, we consider a market model with proportional transaction costs on finite time horizon and provide a dynamic version of the super-replication duality. Further, we study asset price bubbles under proportional transaction costs, where the fundamental value is given by the super-replication price of the asset. In the second part, we establish a method to approximate the superhedging price of a contingent claim by neural networks using the quantile hedging price.
In complete markets all contingent claims can be perfectly hedged by definition and thus the price of a contingent claim is always unique. It can be determined either by the price of a hedging strategy or by taking the expectation of the option with respect to the unique equivalent martingale measure. On the other side, hedging is also used to secure the payout of a contingent claim. The classical option pricing model by Black, Schloes and Merton is just one example for a complete market model. A market model which fails to be complete is called incomplete, see [34] for continuous time or [40] for discrete time. Hence, there are contingent claims which are not attainable in incomplete markets and thus the price of a claim may be not unique but given by an open interval of arbitrage-free prices. In order to secure an option a trader can superhedge (resp. super-replicate) the option. The idea is to find a self-financing trading strategy with minimal initial investments which dominates the payout of a contingent claim. Similar to hedging, superhedging completely reduces the risk associated to the option but also reduces the opportunity to profit from the option. Different than the hedging price, the superhedging price is higher than any arbitrage-free price, and thus does not define an arbitrage-free price for the option. Although, one could consider the superhedging price too high by latter arguments, superhedging is a powerful and useful tool and there are also situations when the superhedging price is small, as in the case of portfolio constraints with not so tight bounds, see [20], [93]. We address the issue that the superhedging price may be considered as too high below. For this reason, super-replication, and particularly its dual representation, have been thoroughly studied in various model settings. It is impossible to cover the complete literature on superhedging here but we name a few. In continuous time, starting with [37] for continuous processes, extended to general càdlàg proceesses, [71], there are approaches for robust superhedging, [76], [98], pathwise superhedging on prediction sets, [6], [7], or superhedging under proportional transaction costs, [23], [29], [67], [85], [92]. Also in discrete time there exist various approaches in the literature like robust superhedging, [25], [78], superhedging under volatility uncertainty, [77], or model-free superhedging, [22].
Clearly, standard mathematical models for financial markets are an idealization of the real world. Assuming that trading orders can be given and executed in continuous without

delay or any additional costs is not realistic. At least in some cases, like currency markets or securities it seems reasonable not to consider a single price but a bid and an ask price. For simplicity one may assume that the transaction cost leading to the bid-ask spread are proportional. Market models with proportional transaction costs have attracted a lot of attention over the years. Under proportional transaction costs $\lambda \in (0,1)$ an agent has to pay $(1+\lambda)S_t$ to buy one share of the asset S at time t but the other agent only receives $(1-\lambda)S_t$ for selling one share of the asset. The interval $[(1-\lambda)S,(1+\lambda)S]$ is then called bid-ask-spread. On the one hand, it is natural to study similar problems as in frictionless markets. For instance, arbitrage theory, option pricing, super-replication dualities, portfolio optimization or asset price bubbles. On the other hand, the impact of proportional transaction costs on the subject of interest can be compared to the frictionless case. In particular, this issue also encounters the possibility to use (proportional) transaction costs as an instrument for regulation. It is well-known that in frictionless market models the price process must admit an equivalent local martingale measure to guarantee that the model is arbitrage-free in the sense that there is *no free lunch with vanishing risk* (NFLVR), see [33]. In particular, the geometric fractional Brownian motion does not describe an arbitrage-free price process, see [14]. In contrast, in models with proportional transaction costs the concept of consistent local price systems, see Definition 1.1, replaces equivalent local martingale measures and so the geometric fractional Brownian motion describes an arbitrage-free market model here, see [47]. A consistent local price system is a pair of a probability measure, which is equivalent to the objective measure and a process which lies in the bid-ask spread and is a local martingale under the associated measure and can be thought as a parallel frictionless market providing better conditions for trading. The existence of consistent local price systems for each $\lambda > 0$ guarantees the absence of arbitrage in the sense of Definition 1.12. The idea here is simple - the parallel market model offers better conditions for both, the buyer and the seller, and is by standard arguments arbitrage-free as the parallel price process is a local martingale. Obviously, the model with proportional transaction costs having poorer trading conditions must also be arbitrage-free. In [48], equivalence of the existence of continuous consistent price systems and the absence of arbitrage is established. Furthermore, in [9], the authors prove an equivalence between a weaker notion of strictly consistent local martingale systems and the NUPBR[1] and the NLABPs[2] conditions in the robust sense.

For market models with proportional transaction costs there exists a wide literature on super-replication. In [92] it is shown that there is no perfect hedging strategy under transaction costs and the least expensive strategy to dominate a contingent claim is the buy and hold strategy. Then, in [29] a martingale approach is presented, followed by an application in [30], showing that also here the cheapest super-replication strategy is to buy and hold the underlying asset. In [66] multi-dimensional currency markets are considered, see also [65]. In this model, a super-replication duality for the initial time is proved in [23]. Finally, in [85] also a local version of the super-replication duality at $t = 0$ is presented, using consistent local price systems, i.e., local martingales for the parallel market.

The phenomena of bubbles has been observed for a long time. Some of the first well-

[1] no unbounded profit with bounded risk

[2] no local arbitrage with bounded portfolios

documented bubbles are for example the so-called "Durch Tulipmania" (1634 – 1637), "Mississippi Bubble" (1719 – 1720), and the "South Sea Bubble" (1720), see [41]. Asset price bubbles have been extensively studied in the existing economic, as well as, mathematical literature. There is broad agreement that an asset price bubble occurs if the market price exceeds its intrinsic or fundamental value. In particular, a bubble is defined as the difference of the market price and its fundamental value. On the other hand, there is little agreement on the driving forces, see [96]. In the economic literature, there are approaches like asymmetric information, see [2], [3], heterogenous beliefs, see [51], [87], and noise trading such as positive feedback activity [32], [89], [94], in combination with limits to arbitrage, see [1], [31], [90], [91]. We note that bubbles may also appear in markets with transaction costs, see [5], [42] and also [43], [72], [88] for the specific case of the real estate market. Although, transaction costs can not fully prevent the occurrence of bubbles, they can still have positive impact on the formation and behavior of bubbles. For instance, in [87], the authors include transaction costs in an equilibrium model with heterogeneous beliefs. It is shown that even small transaction costs may reduce speculative trading preventing bubble's formation. On the other hand, the size of the bubble and the price volatility is not efficiently affected. For an overview of heterogeneous beliefs, we refer to [100]. The positive effect of transaction costs was also illustrated in an agent-based simulation in [95], where the market model was stabilized in the long run.
From a mathematical point of view, there is the popular martingale theory of asset price bubbles, [27], [64], [63], [74], with bubbles' birth included, [17], the approach via the super-replication price, [52], [53], [75], where bubbles' formation is caused by market failure, see [86]. Other models explicitly describe the impact of microeconomic interactions on asset price formation, see [17] and [62], where the fundamental value is exogenously given and asset price bubbles are endogenously determined by the impact of liquidity risk. In the approach of R. Jarrow, P. Protter, and K. Shimbo, [64] and [63], the authors define the fundamental value by the expectation of future cash flows with respect to an equivalent local martingale measure $\mathbf{Q} \in \mathcal{M}_{\text{loc}}(W)$, where $\mathcal{M}_{\text{loc}}(W)$ denotes the set of equivalent local martingale measure for the wealth process W. Then there is a bubble in the market model if and only if the wealth process is a strict local martingale under the measure $\mathbf{Q}$. In a complete market model the equivalent local martingale measure $\mathbf{Q}$ is uniquely given and hence the notion of $\mathbf{Q}$-bubble is distinct. On the other hand, in an incomplete market model, it is not clear which measure $\mathbf{Q}$ is the best or natural choice to define a $\mathbf{Q}$-bubble. Furthermore, a $\mathbf{Q}$-bubble exists in the market model either from the beginning or there is no $\mathbf{Q}$-bubble at all. In [11], F. Biagini et al. admit bubbles' birth by considering a flow in the space of equivalent local martingale measures. P. Guasoni and M. Rásonyi criticize the approach of $\mathbf{Q}$-bubbles in [46] as this notion of $\mathbf{Q}$-bubbles is very sensitive to the choice of the model. The argument of the authors is that in common diffusion models there exists semimartingale with uniformly close paths that is a martingale under an equivalent probability measure. The authors also provide a robust definition of a bubble in [46] which could also be interpreted as a bubble under proportional transaction costs. P. Protter replies to this critic of the "fragility" in [79]. Morally, P. Protter argues that the model is chosen through economic and probabilistic reasoning instead of fitting a curve to data. But he also agrees that a model should satisfy some robustness properties. In [52], M.

Herdegen and M. Schweizer define so-called strong bubbles by the super-replication price of the asset, which provides a robust definition of bubbles. Under sufficient assumptions on the market model, one can apply the well-known duality for super-replication prices of [71]. In this scenario, one can easily see that a strong bubble implies a **Q**-bubble for any choice of **Q** in a market model. Further references on asset price bubbles are [15], [16], [57], [60], [61]. We refer the interested reader to [79], [52] and the entry "Bubbles and Crashes" of [68].

In Part I, we provide a dynamic version of the super-replication dualities for the numéraire-based and numéraire-free case, which corresponds to a local and a non-local setting, extending the results of [23] and [85]. Further, we introduce the notion of asset price bubbles in the presence of proportional transaction costs using its super-replication price as fundamental value and study the impact of proportional transaction costs on the behavior and occurrence of bubbles. In [46], the authors introduce a robust bubble which can also be considered as a bubble under proportional transaction cost. To the best of our knowledge, however, there has been no thorough study of asset price bubbles under proportional transaction cost. Also, the setting and the definition provided in [46] is different to the one of the present thesis. Considering a market model with one risk-less and one risky asset, we specify both components of the trading strategies, the holdings in the bank account and in the risky asset. Thereby, we have the flexibility to elaborate the difference of holding the capital to buy one share of the asset, holding one share, and the liquidation value of one share. Following the approach of [52], the fundamental value is defined as the super-replication price of the position such that the trader holds one share of the asset at the terminal time. In particular, this follows also the idea of [58], the fundamental value should coincide with the price a trader is willing to pay if she had to hold the asset forever. The asset price bubble is then defined as the difference between the ask price of the asset and the fundamental value. Thus, the bubble is always non-negative. From the dynamic version of the super-replication duality we obtain a dual representation for the fundamental value, which is convenient for the study of further properties of the fundamental value, as well as, to the asset price bubble itself. Part I is concluded by the investigation of the impact of proportional transaction costs on bubbles' formation and examples illustrating our findings. Consistent to the economic literature, e.g. [87], we show that the introduction of transaction costs can prevent the appearance of asset price bubbles but that there is no reducing effect on the size of the bubble. Also, transaction costs cannot cause bubbles' formation. In Part I, we also provide a short motivation, Section 1.1, presenting our contributing results in more detail.

In certain situations it is possible to calculate explicitly or recursively the superhedging price, see e.g. [24]. [26], but in general incomplete markets it may be complicated. The calculation of the superhedging price process for $t > 0$ may even be more complex. In Part II of this thesis, we study this problem, using modern machine learning techniques to develop a neural network-based approximation of the superhedging and quantile hedging price. More recently, applications of machine learning methods have attracted a lot of attention due to the success of neural network-based methods in financial mathematics. There are applications for hedging an option, [21], determining stopping times, [10], asset pricing under transaction costs, [45], calibration of stochastic volatility models, [8], [28],

finding Markovian Nash Equililibriums, [49], solving PDEs, [44], [50] and many more. For the application of machine learning on hedging and option pricing we refer the interested reader to [82] and the references therein.
In [21], determining the hedging price and strategy of a call option in the continuous time Black-Scholes model there is the well-known Black-Scholes price and the corresponding Delta hedge as benchmark to compare the method. Furthermore, it is proved in [21] that the hedging strategy and the hedging price can be approximated by neural networks. A common method to prove that an approximation by neural networks is possible is based on the universal approximation theorem, see [55], and works as follows. Consider a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where $\mathcal{F} = \sigma(Y)$ for some real-valued random variable and another real-valued, integrable, continuous random variable Z. Then there exists a measurable function $f : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $f(Y) = Z$. Now, f can be approximated by neural networks by the universal theorem of approximation, [55]. In [21], it was proved that the approximation of the hedging price and the corresponding strategy by neural networks is feasible by applying the universal approximation theorem for each time step of the trading strategy.
We aim to approximate the superhedging price by neural networks. As mentioned above, the superhedging price may be too high. This is a well-known problem and it was addressed by [38]. Here, the authors proposed quantile hedging to reduce the price by increasing the risk. There are two different approaches of quantile hedging. In the first one, a trader can fix a some initial capital (less than the superhedging price) she is willing to spend to secure an option and use it to maximize the probability of superhedging. For the second approach, the trader fixes the probability of superhedging and minimizes the required capital. We may call the latter approach α-quantile hedging, where $\alpha \in (0,1)$ denotes the probability of success. In both cases, a trader can balance the trade off between security and costs based on personal preferences. In this respect, the α-quantile price can be considered as a dynamic version of the value at risk.
The standard superhedging dualities, [37], [71], guarantee that there exists a superhedging strategy with initial capital equal to the superhedging price. In the case of quantile hedging, however, it is in general not true that there exists a strategy starting with the quantile hedging price. Such a strategy exists only in special situations, see [38]. Therefore, in [38], the authors extended the problem, following the Neyman-Pearson lemma, to so-called success-ratios. In the extended formulation both optimization problems admit explicit solutions.
In Part II, we establish a method to approximate the superhedging price process in a discrete time financial market model by neural networks. This includes several steps. First, we prove that the α-quantile hedging price converges to the superhedging price as α tends to 1. In particular, the superhedging price at $t = 0$ can be approximated by the α-quantile hedging price for α sufficiently large. For $t > 0$ we assume that the superhedging price and the superhedging strategy is known from the first step. By the uniform Doob decomposition, see [40], it is now sufficient to calculate the so-called process of consumption, in order to determine the superhedging price process. We show that the process of consumption, which is a non-negative, increasing process given by the uniform Doob decomposition, can be represented by essential supremums. By relying on the universal approximation

theorem, [55], we prove that for all $\alpha \in (0,1)$ the α-quantile hedging price and the corresponding strategy and thus also the superhedging price at $t = 0$ can approximated by neural networks. Further, we express the approximated process of consumption by essential supremums of sets of neural networks and prove convergence to the theoretical process of consumption. These results show that quantile- and superhedging prices and the superhedging price process can be approximated arbitrarily well by neural networks. Finally, we present numerical results for this method. The superhedging probability can be implicitly adjusted via the loss function and then be calculated on the test set. For sample-based and finite models we obtain very reasonable numerical results. In general models, in which the price has unbounded support, our numerical results indicate that the additional error that arises from the discretization of the probability space is non-negligible and decreases very slowly in the number of employed samples.

The thesis is divided in Part I and Part II. Although the terms superhedging and super-replication are equivalent, it depends on the subject which of the terms is commonly used in the literature. To be consistent with the literature, we may use the term super-replication in Part I and the term superhedging in Part II.

Part I is organized as follows. In Chapter 1, we provide a short motivation before we introduce our market model with proportional transaction costs, Section 1.2. In Section 1.3, we provide the basics of consistent local price systems. The notion of admissible trading strategies, including detailed explanation of random initial endowments and related results of [84], are presented in Section 1.4. In Section 1.5 we provide results of [85] with minor modifications, which are required for the proof of the super-replication theorems. In particular, in Section 2.1, we extend the bipolar theorem of [67] to our setting, see Theorem 2.5. In Section 2.2, we prove the dynamic versions of the super-replication theorems in the numéraire-free, Theorem 2.6, and numéraire-based, Theorem 2.7, setting. We conclude Chapter 2 by some further properties of the super-replication price process, see Section 2.3. Finally, in Chapter 3, we introduce the notion asset price bubbles under proportional transaction costs. More precisely, in Section 3.1, we define the fundamental value and the asset price bubble, see Definition 3.1. Using results from Chapter 2, we obtain a dual representation of the fundamental value, which is convenient to derive further properties of the asset price bubble in Section 3.2. In Section 3.3, we illustrate the notion of bubbles and the impact of proportional transaction costs in several examples. Section 3.4 completes Part I by an investigation of the impact of proportional transaction costs on bubbles' formation.

Part II is of the following structure. In Chapter 4, we provide a short motivation and build the theoretical basis for Part II. We present the discrete time market model of [40] in Section 4.2. In Section 4.3, we introduce the notion of quantile hedging and prove that the α-quantile hedging price converges to the superhedging price as α tends to 1, see Theorem 4.9. We also provide a short digression, presenting the budget constraint approach of quantile hedging and the extended formulation of quantile hedging in terms of success ratios. In Section 4.4, we explain that by the uniform Doob decomposition it is sufficient to calculate the process of consumption, assuming that the superhedging price and the corresponding strategy is known. In Proposition 4.16, we prove a representation of the process of consumption by essential supremums. In Chapter 5 we show that a neural network-based approximation of the superhedging price process is feasible. In Section 5.1, we provide a mathematical definition of neural networks and prove a version of the universal approximation theorem of [55], which is mentioned without proof in Section 3 of [55]. In Section 5.2, we prove in Theorem 5.5 that the superhedging price can be approximated by neural networks. In Section 5.3, we then prove, Proposition 5.6, Theorem 5.7, that the neural network-based approximation of the process of consumption is possible and thus by Chapter 4 also the superhedging price process can be approximated by neural networks. Finally, in Chapter 6, we present numerical results. More precisely, we explain the implementation in Python including the loss function, architecture of the neural network and hyper-parameters. We also illustrate the relation of the superhedging probability $\alpha(\lambda) \in (0,1)$ and the $\alpha(\lambda)$-quantile hedging price. The parameter λ is used in the loss function to balance the price and the superhedging probability. In the Appendix A, we summarize some essential results on superhedging.

Contributing Manuscripts

This thesis is based on the following manuscripts, which were developed by the thesis' author T. Reitsam, in cooperation with coauthors:

i) F. Biagini, T. Reitsam, [18]: Asset price bubbles in market models with proportional transaction costs. *LMU Mathematics Institute, Preprint, 2019.*
Available at: https://www.fm.mathematik.uni-muenchen.de/publications_new/index.html

This paper emerged by a collaboration with Prof. Dr. F. Biagini and was written at the LMU Munich. The idea of the paper developed during a discussion of T. Reitsam and F. Biagini on fractional Brownian motion and asset price bubbles. F. Biagini suggested to follow ideas of [52] to find a suitable model for asset price bubbles under proportional transaction costs. In joint discussions the details of the definition of an asset price bubble evolved and the setting of Section 1.2 was built. In particular, the model presented in Section 3.1 is a result of close cooperation of F. Biagini and T. Reitsam. The results in Section 3.2 were mainly derived by T. Reitsam and reviewed in regular meetings by F. Biagini. Theorem 3.9 was found in joint work of F. Biagini and T. Reitsam. Section 3.3 was elaborated by T. Reitsam, as proposed by F. Biagini. Finally, the investigation in Section 3.4 was suggested by F. Biagini and carried out by T. Reitsam. Through all sections and steps, discussions and detailed feedback by F. Biagini enhanced the quality of the paper.

ii) F. Biagini, T. Reitsam, [19]: A dynamic version of the super-replication theorem under proportional transaction costs. *LMU Mathematics Institute, Preprint, 2021.*
Available at: https://www.fm.mathematik.uni-muenchen.de/publications_new/index.html

This paper extends the simple version of the dynamic super-replication duality which was used in [18]. The article was developed at LMU Munich. The results in Section 2.2 were obtained independently by T. Reitsam. The constant feedback by F. Biagini helped finalizing the proofs. Section 2.3 was mainly developed by T. Reitsam. Here, Theorem 2.19 and 2.21 were derived due to several discussions and in cooperation of F. Biagini and T. Reitsam. Through all sections and steps, discussions and detailed feedback by F. Biagini enhanced the quality of the paper.

iii) F. Biagini, L. Gonon, T. Reitsam, [13]: Learning superhedging prices. *LMU Mathematics Institute, Preprint, 2021.*
Available at: https://www.fm.mathematik.uni-muenchen.de/publications_new/index.html

The paper is the product of a joint work of T. Reitsam with two coauthors, Prof. Dr. F. Biagini and Prof. Dr. L. Gonon. It was developed at LMU Munich. The formulation of the question to approximate the superhedging price price process was proposed by F. Biagini. The idea to approximate the superhedging price at $t = 0$ by the quantile hedging price was then suggested by L. Gonon. The steps and guideline of the paper emerged in several discussion of all authors, F. Biagini, L. Gonon and T. Reitsam. The results in Section 4.3 were mainly derived by T. Reitsam and reviewed in regular meetings by F. Biagini and L. Gonon. The approach of Section 4.4 was proposed by L. Gonon. Then, Section 4.4, including the proof of Proposition 4.16, was elaborated by T. Reitsam. Section 5.2 and Section 5.3 were developed in close cooperation of L. Gonon and T. Reitsam. In particular, Theorem 5.5, Proposition 5.6 and Theorem 5.7 emerged from the close collaboration. Finally, the implementation in Section 6 was carried out by T. Reitsam with the help of constant feedback by L. Gonon. Through all sections and steps, discussions and detailed feedback by F. Biagini enhanced the quality of the paper.

In the following we indicate how the three manuscripts above contribute to each part of the present thesis. In this thesis the formulation of the statements of propositions, lemmas, theorems, definitions, etc. is identical as in the three articles.

i) The Introduction was developed independently by T. Reitsam to present a brief summary of the literature of super-replication/superhedging and asset price bubbles. In particular, it connects Part I and Part II.

ii) Chapter 1 is based on F. Biagini, T. Reitsam [18] and [13]. The setting and preliminaries for Part I, which are shared between [18] and [19], are presented. In Section 1.4 and 1.5 some results of [84] and [85] are presented with small modifications which are not provided in the articles above.

iii) Chapter 2 is based on F. Biagini, T. Reitsam [19].

iv) Chapter 3 is based on F. Biagini, T. Reitsam [18].

v) Part II, which includes Chapter 4 - Chapter 6, is based on F. Biagini, L. Gonon, T. Reitsam [13].

Part I

Dynamic super-replication and asset price bubbles under proportional transaction costs

Chapter 1

Setting and preliminaries

This chapter is based on Section 2 of [18] and Section 2 of [19]. After giving a short motivation of Part I, we introduce the setting for market models with proportional transaction costs. This includes the notion of admissible strategies in the numéraire-free and numéraire-based sense. The details of admissible strategies are crucial for the super-replication theorems later. In this context we also introduce consistent price systems in the local and in the non-local sense. In the next part we recall the duality representation of consistent (local) price systems. In a second part, we present some important results of [84] and [85] and adapt them to our more general setting.

1.1 Motivation Part I

In the economic literature there are various studies, discussing the impact of transaction costs on the behavior of bubbles. In mathematics there exists also a wide literature for bubbles, however, there is no thorough study of asset price bubbles in the presence of transaction costs. In [46], the authors briefly present the notion of a robust bubble, which can also be interpreted as a bubble under proportional transaction cost. In contrast, we provide a different notion of asset price bubble for market models under proportional transaction costs such that we can also study the impact of transaction costs on the occurrence of bubbles. For this purpose, we admit trading strategies on subintervals with random initial endowments based on the available information which generalizes the setting of [84] and [85] and ensures more flexibility. In particular, both components, holdings in the bank account and in the risky asset, of a trading strategy $\varphi = (\varphi_t^1, \varphi_t^2)_{t\in[0,T]}$ are specified. This gives us the required flexibility to define the fundamental value as the super-replication price of the position $X_T = (0,1)$, i.e., of the position of holding one share of the asset at the terminal time $T > 0$. This definition follows ideas of [52] and [58]. Then, at time $t \in [0,T]$ the bubble is defined as the difference of the ask price $(1+\lambda)S_t$ and of the fundamental value F_t.
For this approach, we first prove a dynamic version of the super-replication theorems of [23] and [85]. To prove the super-replication theorem in the numéraire-free setting, Theorem 2.6, we extend a bipolar theorem of [67], see Theorem 2.5, allowing us to use similar ideas as in [23], [85]. Then, following [85], we can also prove the numéraire-based version of the

super-replication theorem, see Theorem 2.7.
Next, we introduce the definition of the fundamental value and of the asset price bubble by its super-replication price as described above, see Definition 3.1 and use the duality result of Theorem 2.7 to obtain a representation of the fundamental value via consistent local price systems.
We provide several examples to illustrate our setting and the behavior of asset price bubbles. To conclude Part I, we discuss the impact of transaction costs on the appearance and the size of bubbles. Our results are consistent with the economic literature, e.g. [87], and show that the introduction of proportional transaction costs can prevent bubbles' formation but has no effect on the size of the bubble. We also show that transaction costs cannot be the reason for the occurrence of bubbles.

1.2 Setting

Let $T > 0$ describe a finite time horizon and let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, \mathbf{P})$ be a filtered probability space where the filtration $\mathbb{F} := (\mathcal{F}_t)_{0 \le t \le T}$ satisfies the usual conditions of right-continuity and saturatedness, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. We consider a financial market model consisting of a risk-free asset B, normalized to $B \equiv 1$, and a risky asset S. For Part I of the thesis we assume that $S = (S_t)_{0 \le t \le T}$ is an $\mathbb{F}$-adapted stochastic process, with càdlàg and strictly positive paths. For trading the risky asset in the market model, proportional transaction costs $0 < \lambda < 1$ are charged, i.e., to buy one share of S at time t the trader has to pay $(1 + \lambda)S_t$ and for selling one share of S at time t the trader receives $(1 - \lambda)S_t$. The interval $[(1 - \lambda)S_t, (1 + \lambda)S_t]$ is called *bid-ask-spread*. Let $\lambda \in (0, 1)$ be fixed. Further, we assume that $S_t \in L^1_+(\mathcal{F}_t, \mathbf{P})$ for all $t \in [0, T]$. If not stated explicitly, all equalities and inequalities of random variables have to be understood $\mathbf{P}$ almost surely, throughout the thesis.

1.3 Consistent price systems

Definition 1.1. For $0 \le s < t \le T$, we call $\mathrm{CPS}(s, t)$ (resp. $\mathrm{CPS}_{\mathrm{loc}}(s, t)$) the family of pairs $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}})$ such that $\mathbf{Q}$ is a probability measure on $\mathcal{F}_t$, $\mathbf{Q} \sim \mathbf{P}|_{\mathcal{F}_t}$, $\tilde{S}^{\mathbf{Q}}$ is a martingale (resp. local martingale) under $\mathbf{Q}$ on $[s, t]$, and

$$(1 - \lambda)S_u \le \tilde{S}^{\mathbf{Q}}_u \le (1 + \lambda)S_u, \quad \text{for } s \le u \le t. \tag{1.1}$$

A pair $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}})$ in $\mathrm{CPS}(s, t)$ (resp. $\mathrm{CPS}_{\mathrm{loc}}(s, t)$) is called a *consistent price system* (resp. *consistent local price system*). If (1.1) holds strictly we say that $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}})$ is a strictly consistent (local) price systems and denote the corresponding set by $\mathrm{SCPS}(s, T)$ (resp. $\mathrm{SCPS}_{\mathrm{loc}}(s, T)$). By $\mathcal{Q}(s, T)$ (resp. $\mathcal{Q}_{\mathrm{loc}}(s, T)$) we denote the set of measures $\mathbf{Q}$ such that there exists a pair $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}(s, T)$ (resp. $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}_{\mathrm{loc}}(s, T)$). Further, we write $L^p(\mathcal{F}_s, \mathcal{Q}) := \bigcap_{\mathbf{Q} \in \mathcal{Q}(s,T)} L^p(\mathcal{F}_s, \mathbf{Q})$ and $L^p(\mathcal{F}_s, \mathcal{Q}_{\mathrm{loc}}) := \bigcap_{\mathbf{Q} \in \mathcal{Q}_{\mathrm{loc}}(s,T)} L^p(\mathcal{F}_s, \mathbf{Q})$. By $L^p_+(\mathcal{F}_s, \mathcal{Q})$ (resp. $L^p_+(\mathcal{F}_s, \mathcal{Q}_{\mathrm{loc}})$) we denote the space of $[0, \infty)$-valued random variables $X \in L^p(\mathcal{F}_s, \mathcal{Q})$ (resp. $X \in L^p(\mathcal{F}_s, \mathcal{Q}_{\mathrm{loc}})$).

A consistent (local) price systems $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}})$ can be imagined as a parallel frictionless market with better terms at all times for the traders. More precisely, in this parallel market the risky asset can be bought for $\tilde{S}^{\mathbf{Q}}_t \le (1+\lambda)S_t$ and the seller receives $\tilde{S}^{\mathbf{Q}}_t \ge (1-\lambda)S_t$ at time $t \in [0,T]$. Therefore, if the parallel market $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}})$ is arbitrage-free, then also the corresponding market with proportional transaction costs is arbitrage-free in the sense of Definition 1.12 (see also Definition 4 of [48]). In particular, the existence of a consistent (local) price systems guarantees the absence of arbitrage, see [47], [48].
Furthermore, if a contingent claim X can be hedged (resp. super-replicated) with some capital x in the market with proportional transaction costs, x is sufficient to hedge (resp. super-replicate) X in the frictionless market $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}})$. This observation is the key to the super-replication theorems, see Theorem 2.6 and 2.7.
Following [48], [66], [83], we introduce a dual theory for consistent (local) price systems. For fixed $\lambda > 0$ we denote by K_t the solvency cone at time t, defined as

$$K_t(\omega) = \operatorname{cone}\left\{(1+\lambda)S_t(\omega)e_1 - e_2, -e_1 + \frac{1}{(1-\lambda)S_t(\omega)}e_2\right\}, \tag{1.2}$$

where $e_1 = (1,0), e_2 = (0,1)$ are the unit vectors in $\mathbb{R}^2$, and by $K^*_t = (-K_t)^\circ$ the corresponding polar cone, given by

$$\begin{aligned} K^*_t(\omega) = (-K_t)^\circ(\omega) &= \left\{(y_1,y_2) \in \mathbb{R}^2_+ : (1-\lambda)S_t(\omega) \le \frac{y_2}{y_1} \le (1+\lambda)S_t(\omega)\right\} \\ &= \left\{y \in \mathbb{R}^2 : \langle x, y\rangle \le 0, \forall x \in (-K_t(\omega))\right\} \\ &= \left\{y \in \mathbb{R}^2 : \langle x, y\rangle \ge 0, \ \forall x \in K_t(\omega)\right\}. \end{aligned} \tag{1.3}$$

Definition 1.2. We define $\mathcal{Z}(s,T)$ (resp. $\mathcal{Z}_{\text{loc}}(s,T)$) as the set of processes $Z = (Z^1_t, Z^2_t)_{s\le t\le T}$ such that Z^1 is a $\mathbf{P}$-martingale and Z^2 is a $\mathbf{P}$-martingale (resp. local $\mathbf{P}$-martingale) and such that $Z_t \in K^*_t \setminus \{0\}$ a.s. for all $t \in [s,T]$.

The following proposition from [48] provides a useful representation of consistent (local) price systems by elements in $\mathcal{Z}$ (resp. $\mathcal{Z}_{\text{loc}}$) and follows directly from the definition of K^*_t in (1.3).

Proposition 1.3 (Proposition 3, [48])**.** *Let $Z = (Z^1_t, Z^2_t)_{s\le t\le T}$ be a 2-dimensional stochastic process with $Z^1_T \in L^1(\mathcal{F}_T, \boldsymbol{P})$. Define the measure $\boldsymbol{Q}(Z) \ll \boldsymbol{P}$ by $d\boldsymbol{Q}(Z)/d\boldsymbol{P} := Z^1_T/\mathbb{E}[Z^1_T]$. Then $Z \in \mathcal{Z}(s,T)$ (resp. $Z \in \mathcal{Z}_{loc}(s,T)$) if and only if $(\boldsymbol{Q}(Z), (Z^2/Z^1))$ is a consistent price system (resp. consistent local price system) on $[s,T]$.*

The representation of consistent (local) price systems given by Proposition 1.3 can be easily extended to higher dimensions. Assume we have $d > 1$ risky assets. Then $Z = (Z^1_t, \dots, Z^d_t)_{0\le t\le T}$ is a called a consistent price system if Z is an adapted $\mathbb{R}^d_+\setminus\{0\}$-valued, càdlàg $\mathbf{P}$-martingale and $Z_t \in K^*_t$ for all $t \in [0,T]$, see e.g. Definition 2.3 of [23]. For the convenience of the reader we summarize the assumptions that we use through out the paper.

Assumption 1.4. We assume that S admits a consistent *local* price system for every $0 < \lambda' \le \lambda$.

Assumption 1.5. We assume that S admits a consistent price system for every $0 < \lambda' \le \lambda$.

Lemma 1.6 and Corollary 1.8 will later be used to extend consistent price systems from a sub-interval of $[0,T]$ to the complete interval.

Lemma 1.6. *Let Assumption 1.4 hold. For each stopping time $0 \le \sigma \le T$ and each random variable $f \in L^1(\mathcal{F}_\sigma, \boldsymbol{P})$ such that*

$$(1-\lambda)S_\sigma < f < (1+\lambda)S_\sigma, \tag{1.4}$$

and for each $\bar{\lambda} > \lambda$ there is an $\bar{\lambda}$-consistent local price system $(\check{\boldsymbol{Q}}, \check{S}) \in CPS_{loc}(0,T,\bar{\lambda})$ with $\check{S}_\sigma = f$.

Proof. The proof is partially based[1] on the proof of Lemma 6 of [48]. Consider the sequence of stopping time $(\tau_n)_{n\in\mathbb{N}}$, where

$$\tau_n(\omega) := \inf\{t \ge 0 \mid S_t(\omega) \ge n\} \wedge T.$$

By $(\tau_n)_{n\in\mathbb{N}}$ we have a localizing sequence for all λ-consistent local price systems on $[0,T]$. Indeed, for $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}_{\mathrm{loc}}(0,T,\lambda)$ we have

$$\tilde{S}^{\mathbf{Q}}_t \le (1+\lambda)S_t \le (1+\lambda)n, \tag{1.5}$$

for all $0 \le t < \tau_n$, which by Proposition 6.1 of [85] implies that $(\tilde{S}^{\mathbf{Q}})^{\tau_n}$ is true $\mathbf{Q}$-martingale and clearly $\tau_n \uparrow T$ $\mathbf{P}$-a.s. Fix $\bar{\lambda} > \lambda$ and consider the interval $[\![0,\sigma]\!]$. Define $\delta \le \lambda$ such that

$$\delta + (1+\delta)(\lambda+\delta)/(1-\delta) < \bar{\lambda}. \tag{1.6}$$

Assumption 1.4 guarantees the existence of a δ-consistent local price system $(\mathbf{Q}(\delta), \tilde{S}(\delta)) \in \mathrm{CPS}_{\mathrm{loc}}(0,T,\delta)$ on the interval $[\![0,\sigma]\!]$, which satisfies

$$(1-\delta)S_{\tau_n\wedge\sigma} \le \tilde{S}_{\tau_n\wedge\sigma}(\delta) \le (1+\delta)S_{\tau_n\wedge\sigma}. \tag{1.7}$$

We define the sequence $(f_n)_{n\in\mathbb{N}}$ by

$$f_n := \begin{cases} f & \text{on } \{\tau_n \ge \sigma\}, \\ \tilde{S}(\delta)_{\tau_n} & \text{on } \{\tau_n < \sigma\}, \end{cases}$$

such that $f_n \xrightarrow{\mathbf{P}\text{-a.s.}} f$ as $\tau_n \uparrow T$ $\mathbf{P}$-a.s. By (1.5) we get $f_n \in L^1(\mathcal{F}_{\tau_n\wedge\sigma}, \mathbf{P})$ and that

$$(1-\lambda)S_{\tau_n\wedge\sigma} < f_n < (1+\lambda)S_{\tau_n\wedge\sigma}. \tag{1.8}$$

Moreover, by (1.6) we have

$$\left|\tilde{S}_{\tau_n\wedge\sigma}(\delta) - f_n\right| < (\lambda+\delta)S_{\tau_n\wedge\sigma} \le \frac{\lambda+\delta}{1-\delta}\tilde{S}_{\tau_n\wedge\sigma}(\delta). \tag{1.9}$$

[1] The main difference with respect to the proof of Lemma 6 of [48] is that we cannot use the martingale property of consistent price systems as in [48], because we are now in the local setting. Hence we need some further technicalities.

Therefore, $f_n \in L^1(\mathcal{F}_{\tau_n \wedge \sigma}, \mathbf{Q}(\delta))$. Further, $f \in L^1(\mathcal{F}_\sigma, \mathbf{Q}(\delta))$ by (1.4) and the fact that

$$f \leq (1+\lambda) S_\sigma \leq \frac{1+\lambda}{1-\lambda} \tilde{S}_\sigma(\delta).$$

Let ρ be a stopping time with $0 \leq \rho \leq (\tau_n \wedge \sigma)$ and define $\bar{S}^n_\rho := \mathbb{E}_{\mathbf{Q}(\delta)}[f_n \mid \mathcal{F}_\rho]$. Then, by (1.9) we have

$$\left|\mathbb{E}_{\mathbf{Q}(\delta)}\left[f_n \mid \mathcal{F}_\rho\right] - \mathbb{E}_{\mathbf{Q}(\delta)}\left[\tilde{S}_{\tau_n \wedge \sigma}(\delta) \mid \mathcal{F}_\rho\right]\right| < \tilde{S}_\rho(\delta) \frac{\lambda+\delta}{1-\delta} \leq S_\rho \frac{(\lambda+\delta)(1+\delta)}{1-\delta}, \; n \in \mathbb{N}.$$

In particular, (1.7) implies that

$$(1-\bar{\lambda}) S_\rho < \bar{S}^n_\rho < (1+\bar{\lambda}) S_\rho, \tag{1.10}$$

and thus $(\mathbf{Q}(\delta), \bar{S}^n) \in \mathrm{CPS}(0, (\tau_n \wedge \sigma), \bar{\lambda})$ is a $\bar{\lambda}$-consistent price system in the non-local sense for $S^{\tau_n \wedge \sigma}$.
We show that $\bar{S}^n_\rho$ converges $\mathbf{P}$-almost surely to a random variable $\bar{S}^{\mathbf{Q}(\delta)}_\rho$ for all $0 \leq \rho \leq \sigma$. We rewrite $\bar{S}^n_\rho$ by

$$\mathbb{E}_{\mathbf{Q}(\delta)}\left[f_n \mid \mathcal{F}_\rho\right] = \mathbb{E}_{\mathbf{Q}(\delta)}\left[\mathbb{1}_{\{\tau_n \geq \sigma\}} f \mid \mathcal{F}_\rho\right] + \mathbb{E}_{\mathbf{Q}(\delta)}\left[\mathbb{1}_{\{\tau_n < \sigma\}} \tilde{S}_{\tau_n}(\delta) \mid \mathcal{F}_\rho\right]. \tag{1.11}$$

For the first term of (1.11) the Theorem of Monotone Convergence implies that

$$\mathbb{E}_{\mathbf{Q}(\delta)}\left[\mathbb{1}_{\{\tau_n \geq \sigma\}} f \mid \mathcal{F}_\rho\right] \overset{\mathbf{P}\text{-a.s.}}{\longrightarrow} \mathbb{E}_{\mathbf{Q}(\delta)}\left[f \mid \mathcal{F}_\rho\right], \quad \text{as } n \to \infty. \tag{1.12}$$

On the other hand, we obtain for the second term of (1.11) that

$$\begin{aligned}\mathbb{E}_{\mathbf{Q}(\delta)}\left[\mathbb{1}_{\{\tau_n < \sigma\}} \tilde{S}_{\tau_n}(\delta) \mid \mathcal{F}_\rho\right] &= \mathbb{E}_{\mathbf{Q}(\delta)}\left[\tilde{S}_{\tau_n \wedge \sigma}(\delta) \mid \mathcal{F}_\rho\right] - \mathbb{E}_{\mathbf{Q}(\delta)}\left[\mathbb{1}_{\{\tau_n \geq \sigma\}} \tilde{S}_{\tau_n \wedge \sigma}(\delta) \mid \mathcal{F}_\rho\right] \\ &= \tilde{S}_{\tau_n \wedge \rho}(\delta) - \mathbb{E}_{\mathbf{Q}(\delta)}\left[\mathbb{1}_{\{\tau_n \geq \sigma\}} \tilde{S}_\sigma(\delta) \mid \mathcal{F}_\rho\right].\end{aligned}$$

Clearly,

$$\tilde{S}_{\tau_n \wedge \rho}(\delta) \xrightarrow{\mathbf{P}\text{-a.s.}} \tilde{S}_\rho(\delta), \quad \text{as } n \to \infty, \tag{1.13}$$

since $\tau_n \uparrow T$. Further, it holds that $\mathbb{1}_{\{\tau_n \geq \sigma\}} \leq \mathbb{1}_{\{\tau_{n+1} \geq \sigma\}}$ for all $n \in \mathbb{N}$. Thus, the Theorem of Monotone Convergence yields

$$\mathbb{E}_{\mathbf{Q}(\delta)}\left[\mathbb{1}_{\{\tau_n \geq \sigma\}} \tilde{S}_\sigma(\delta) \mid \mathcal{F}_\rho\right] \xrightarrow{\mathbf{P}\text{-a.s.}} \mathbb{E}_{\mathbf{Q}(\delta)}\left[\tilde{S}_\sigma(\delta) \mid \mathcal{F}_\rho\right] \quad \text{as } n \to \infty. \tag{1.14}$$

Then, (1.12), (1.13), and (1.14) yield

$$\bar{S}^n_\rho \overset{\mathbf{P}\text{-a.s.}}{\longrightarrow} \mathbb{E}_{\mathbf{Q}(\delta)}\left[f \mid \mathcal{F}_\rho\right] + \tilde{S}_\rho(\delta) - \mathbb{E}_{\mathbf{Q}(\delta)}\left[\tilde{S}_\sigma(\delta) \mid \mathcal{F}_\rho\right] =: \bar{S}^{\mathbf{Q}(\delta)}_\rho, \quad \text{as } n \to \infty. \tag{1.15}$$

We define the process $\bar{S}^{\mathbf{Q}(\delta)} = (\bar{S}^{\mathbf{Q}(\delta)}_t)_{0 \leq t \leq \sigma}$ by (1.15). Therefore, $\bar{S}^{\mathbf{Q}(\delta)}$ is a well-defined local $\mathbf{Q}(\delta)$-martingale, which admits a càdlàg modification. By (1.10) $\bar{S}^{\mathbf{Q}(\delta)}$ lies in the bid-ask spread for $\bar{\lambda}$ and thus $(\mathbf{Q}(\delta), \bar{S}^{\mathbf{Q}(\delta)})$ defines $\bar{\lambda}$-consistent local price system on $[\![0, \sigma]\!]$ satisfying $\bar{S}^{\mathbf{Q}(\delta)}_\sigma = f$.

For $[\![\sigma, T]\!]$ we follow the construction of Lemma 6 of [48]. Let $(\mathbf{Q}(\varepsilon), \bar{S}(\varepsilon))$ be a $\min\{\varepsilon, \delta\}$-consistent local price system on $[0, T]$ with a variable $\varepsilon \in (0,1)$ which will vary later. By construction we have

$$1 - \varepsilon \leq \frac{\bar{S}_\sigma(\varepsilon)}{S_\sigma} \leq 1 + \varepsilon.$$

For $k \geq 1$, we define

$$A_k^+ := \left\{ \left(1 + \frac{k\lambda}{k+1}\right) S_\sigma > f \geq \left(1 + \frac{(k-1)\lambda}{k}\right) S_\sigma \right\} \in \mathcal{F}_\sigma, \tag{1.16}$$

$$A_k^- := \left\{ \left(1 - \frac{(k-1)\lambda}{k}\right) S_\sigma > f \geq \left(1 - \frac{k\lambda}{k+1}\right) S_\sigma \right\} \in \mathcal{F}_\sigma. \tag{1.17}$$

Now set $\widehat{\mathbf{Q}} := \sum_{k=1}^\infty \mathbb{1}_{A_k^+ \cup A_k^-} \mathbf{Q}(\lambda/(9k+3))$. Further,

$$\widehat{S}_u^{\widehat{\mathbf{Q}}} := \sum_{k=1}^\infty \mathbb{1}_{A_k^+ \cup A_k^-} \frac{f}{\bar{S}_\sigma\left(\frac{\lambda}{9k+3}\right)} \bar{S}_u\left(\frac{\lambda}{9k+3}\right), \quad \sigma \leq u \leq T. \tag{1.18}$$

For $u \in [\![\sigma, T]\!]$, $\widehat{S}_u^{\widehat{\mathbf{Q}}}$ is a.s. finite as $(A_k^+ \cup A_k^-)_{k\in\mathbb{N}}$ defines a partition of $\Omega \backslash N$ where $N \in \mathcal{F}$ is some null-set. Moreover, $\widehat{S}_u^{\widehat{\mathbf{Q}}}$ is in $L^1(\mathcal{F}_u, \mathbf{P})$ as it is bounded by $(1+\lambda)S_u$. The fact that $(\widehat{S}_u^{\widehat{\mathbf{Q}}})_{\sigma \leq u \leq T}$ is a local $\widehat{\mathbf{Q}}$-martingale on the interval $[\![\sigma, T]\!]$ follows immediately since $\mathbb{1}_{A_k^+}, \mathbb{1}_{A_k^-}$, f and $\bar{S}_\sigma$ are $\mathcal{F}_\sigma$-measurable. Next, we show that $\widehat{S}^{\widehat{\mathbf{Q}}}$ lies in the λ-bid-ask spread on $[\![\sigma, T]\!]$. Let $\omega \in A_k^+$ for some $k \in \mathbb{N}$. Then we have for $t \in [\![\sigma, T]\!]$ that

$$\begin{aligned}
(1-\lambda)S_t(\omega) &< \frac{1 - \frac{\lambda}{9k+1}}{1 + \frac{\lambda}{9k+3}} S_t(\omega) \leq \frac{f(\omega)}{\bar{S}_\sigma\left(\frac{\lambda}{9k+3}\right)(\omega)} \left(1 - \frac{\lambda}{9k+3}\right) S_t(\omega) \\
&\leq \widehat{S}_t^{\widehat{\mathbf{Q}}}(\omega) \leq \frac{f(\omega)}{\bar{S}_\sigma\left(\frac{\lambda}{9k+3}\right)(\omega)} \left(1 + \frac{\lambda}{9k+3}\right) S_t(\omega) \\
&\leq \left(1 + \frac{k\lambda}{k+1}\right) \frac{1}{1 - \frac{\lambda}{9k+3}} \left(1 + \frac{\lambda}{9k+3}\right) S_t(\omega) \\
&\leq (1+\lambda) S_t(\omega).
\end{aligned}$$

Conversely, if $\omega \in A_k^-$ we get that,

$$\begin{aligned}
(1+\lambda)S_t(\omega) &> \frac{1 + \frac{\lambda}{9k+3}}{1 - \frac{\lambda}{9k+3}} S_t(\omega) \geq \frac{f(\omega}{\bar{S}_\sigma\left(\frac{\lambda}{9k+3}\right)(\omega)} \left(1 + \frac{\lambda}{9k+3}\right) S_t(\omega) \\
&\geq \widehat{S}_t^{\widehat{\mathbf{Q}}}(\omega) \geq \frac{f(\omega)}{\bar{S}_\sigma\left(\frac{\lambda}{9k+3}\right)} \left(1 - \frac{\lambda}{9k+3}\right) S_t(\omega) \\
&\geq \left(1 - \frac{k\lambda}{k+1}\right) \frac{1}{1 + \frac{\lambda}{9k+3}} \left(1 - \frac{\lambda}{9k+3}\right) S_t(\omega) \\
&> (1-\lambda) S_t(\omega).
\end{aligned}$$

Therefore, $(\widehat{\mathbf{Q}}, \widehat{S}^{\widehat{\mathbf{Q}}})$ is a λ-consistent local price system on $[\![\sigma, T]\!]$.
We now define $(\check{\mathbf{Q}}, \check{S}^{\check{\mathbf{Q}}}) \in \mathrm{CPS}_{\mathrm{loc}}(0, T, \bar{\lambda}))$ which satisfies $\check{S}_\sigma^{\check{\mathbf{Q}}} = f$. Define

$$\frac{d\check{\mathbf{Q}}}{d\mathbf{P}} := \frac{\frac{d\mathbf{Q}(\delta)}{d\mathbf{P}}}{\mathbb{E}_\mathbf{P}\left[\frac{d\widehat{\mathbf{Q}}}{d\mathbf{P}} \mid \mathcal{F}_\sigma\right]} \frac{d\widehat{\mathbf{Q}}}{d\mathbf{P}},$$

and

$$\check{S}_t^{\check{\mathbf{Q}}} := \begin{cases} \bar{S}_t^{\mathbf{Q}(\delta)}, & \text{for } 0 \le t \le \sigma \\ \widehat{S}_t^{\widehat{\mathbf{Q}}}, & \text{for } \sigma \le t \le T. \end{cases}$$

Then $(\check{\mathbf{Q}}, \check{S}^{\check{\mathbf{Q}}}) \in \mathrm{CPS}_{\mathrm{loc}}(0,T,\bar{\lambda})$ and $\check{S}_\sigma^{\check{\mathbf{Q}}} = \widehat{S}_\sigma^{\widehat{\mathbf{Q}}} = \bar{S}_\sigma^{\mathbf{Q}(\delta)} = f$. □

Remark 1.7. *Note that in the case of a consistent price system in the non-local sense, Lemma 1.6 coincides with Lemma 6 of [48].*

Corollary 1.8. *Let Assumption 1.4 hold. For any stopping time $0 \le \sigma \le T$, probability measure $\overline{\boldsymbol{Q}} \sim \boldsymbol{P}|_{\mathcal{F}_\sigma}$ on $\mathcal{F}_\sigma$ and random variable $f \in L^1(\mathcal{F}_\sigma, \boldsymbol{P})$ with*

$$(1-\lambda)S_\sigma \le f \le (1+\lambda)S_\sigma,$$

there exists a (strictly) λ-consistent local price system $(\boldsymbol{Q}, \tilde{S}^{\boldsymbol{Q}}) \in SCPS_{loc}(\sigma, T, \lambda)$ such that $\tilde{S}_\sigma^{\boldsymbol{Q}} = f$ and $\boldsymbol{Q}|_{\mathcal{F}_\sigma} = \overline{\boldsymbol{Q}}$.

Proof. The assertion follows by the construction of the second part of the proof of Lemma 1.6. □

1.4 Trading strategies

We follow the approach of [18] and define admissible trading strategies as follows.

Definition 1.9. A *self-financing trading strategy* starting with initial endowment $(X_s^1, X_s^2) \in L_+^0(\mathcal{F}_s, \mathbf{P}) \times L_+^0(\mathcal{F}_s, \mathbf{P})$ is a pair of $\mathbb{F}$-predictable finite variation processes $(\varphi_t^1, \varphi_t^2)_{s \le t \le T}$ on $[s,T]$ such that

i) $\varphi_s^1 = X_s^1$ and $\varphi_s^2 = X_s^2$,

ii) denoting by $\varphi_t^1 = \varphi_s^1 + \varphi_t^{1,\uparrow} - \varphi_t^{1,\downarrow}$ and $\varphi_t^2 = \varphi_t^{2,\uparrow} - \varphi_t^{2,\downarrow}$, the Jordan-Hahn decomposition of φ^1 and φ^2 into the difference of two non-decreasing processes, starting at $\varphi_s^{1,\uparrow} = \varphi_s^{1,\downarrow} = \varphi_s^{2,\uparrow} = \varphi_s^{2,\downarrow} = 0$, these processes satisfy

$$\mathrm{d}\varphi_t^{1,\uparrow} \le (1-\lambda)S_t \mathrm{d}\varphi_t^{2,\downarrow}, \quad \mathrm{d}\varphi_t^{1,\downarrow} \ge (1+\lambda)S_t \mathrm{d}\varphi_t^{2,\uparrow}, \quad s \le t \le T. \tag{1.19}$$

The processes $(\varphi_t^1)_{0 \le t \le T}$ and $(\varphi_t^2)_{0 \le t \le T}$ describe the holdings units of bond and stock, respectively at time t. Let us give more details on the differential form of 1.19, see also [84], [85]. If $\varphi = (\varphi_t^1, \varphi_t^2)_{0 \le t \le T}$ is continuous, then (1.19) is understood as the integral requirement, i.e.,

$$\int_\sigma^\tau \left((1-\lambda)S_t \mathrm{d}\varphi_t^{2,\downarrow} - \mathrm{d}\varphi_t^{1,\uparrow}\right) \ge 0, \quad \int_\sigma^\tau \left(\mathrm{d}\varphi_t^{1,\downarrow} - (1+\lambda)S_t \mathrm{d}\varphi_t^{2,\uparrow}\right) \ge 0, \tag{1.20}$$

for all stopping times $0 \le \sigma \le \tau \le T$. As φ is continuous and of finite variation and S is càdlàg, the integrals in (1.20) are pathwise well-defined as Riemann-Stieltjes integral.

If φ may have jumps (1.19) requires special attention. For every stopping time τ the left and right limits $\varphi_{\tau-}$ and $\varphi_{\tau+}$ exist because φ is of bounded variation. However, the values $\varphi_{\tau-}$, φ_τ and $\varphi_{\tau+}$ do not necessarily coincide. Following [23], [84], we denote the increments by

$$\Delta\varphi_\tau := \varphi_\tau - \varphi_{\tau-}, \quad \Delta_+\varphi_t := \varphi_{\tau+} - \varphi_\tau.$$

We decompose $\varphi^\uparrow$ and $\varphi^\downarrow$ in a continuous part given by

$$\varphi_t^{\uparrow,c} = \varphi_t^\uparrow - \sum_{s<t}\Delta_+\varphi_t^\uparrow - \sum_{s\leq t}\Delta\varphi_s^\uparrow,$$
$$\varphi_t^{\downarrow,c} = \varphi_t^\downarrow - \sum_{s<t}\Delta_+\varphi_t^\downarrow - \sum_{s\leq t}\Delta\varphi_s^\downarrow,$$

and a part with jumps.
The continuous part must fulfill (1.20). To complete the requirement of (1.19) we add the condition for the left and right jumps, i.e., for all $[0,T]$-valued stopping times τ we have for left jumps that

$$\Delta\varphi_\tau^{1,\uparrow} \leq (1-\lambda)S_{\tau-}\Delta\varphi_\tau^{2,\downarrow}, \quad \Delta\varphi_\tau^{1,\downarrow} \geq (1+\lambda)S_{\tau-}\Delta\varphi_\tau^{2,\uparrow}, \tag{1.21}$$

and in the case of right jumps that

$$\Delta_+\varphi_\tau^{1,\uparrow} \leq (1-\lambda S_{\tau+}\Delta_+\varphi_\tau^{2,\downarrow}, \quad \Delta_+\varphi_\tau^{1,\downarrow} \geq (1+\lambda)S_{\tau+}\Delta\varphi_\tau^{2,\uparrow}. \tag{1.22}$$

Following [84], we explicitly specify the holdings in the bond φ^1 and the holdings in the risky asset φ^2. Typically, in the frictionless theory only one process, which describes the holdings in the risky asset, is specified. By requiring equality in (1.19) we could define φ^1 by

$$d\varphi_t^1 = (1-\lambda)S_t \mathrm{d}\varphi_t^{2,\downarrow} - (1+\lambda)S_t \mathrm{d}\varphi_t^{2,\uparrow}.$$

In particular, any pair (φ^1,φ^2) satisfying (1.19) can be dominated by a pair $(\tilde{\varphi}^1,\tilde{\varphi}^2)$ where equality holds. However, for the theory of proportional transaction costs it may be reasonable to specify both accounts separately to stress out the different values of buying, holding and selling a stock. Note that, the definition in (1.19) allows to "throw money away".

Definition 1.10. i) Let $X_s \in L^1_+(\mathcal{F}_s,\mathcal{Q}_{\text{loc}})$. A self-financing trading strategy $\varphi = (\varphi^1,\varphi^2)$ is called *admissible in the numéraire-based* sense on $[s,T]$ starting with initial endowment $\varphi_s = (X_s,0)$ if there is $M_s \in L^1_+(\mathcal{F}_s,\mathcal{Q}_{\text{loc}})$ such that the liquidation value V_τ^{liq} satisfies

$$V_\tau^{liq}(\varphi^1,\varphi^2) := \varphi_\tau^1 + \left(\varphi_\tau^2\right)^+(1-\lambda)S_\tau - \left(\varphi_\tau^2\right)^-(1+\lambda)S_\tau \geq -M_s, \tag{1.23}$$

for all $[s,T]$-valued stopping times τ.

ii) Let $(X_s^1,X_s^2) \in L^1_+(\mathcal{F}_s,\mathcal{Q}) \times L^\infty_+(\mathcal{F}_s,\mathcal{Q})$. A self-financing trading strategy $\varphi = (\varphi^1,\varphi^2)$ is called *admissible in the numéraire-free* sense on $[s,T]$ starting with initial endowment $\varphi_s = (X_s^1,X_s^2)$ if there is $M_s := (M_s^1,M_s^2) \in L^1_+(\mathcal{F}_s,\mathcal{Q}) \times L^\infty_+(\mathcal{F}_s,\mathcal{Q})$ such that

$$V_\tau^{liq}(\varphi^1,\varphi^2) := \varphi_\tau^1 + \left(\varphi_\tau^2\right)^+(1-\lambda)S_\tau - \left(\varphi_\tau^2\right)^-(1+\lambda)S_\tau \geq -M_s^1 - M_s^2 S_\tau, \tag{1.24}$$

for all $[s,T]$-valued stopping times τ.

If M_s is given, we call a strategy satisfying (1.23) or (1.24) M_s-admissible in the numéraire-based or numéraire-free sense, respectively. We denote by $\mathcal{V}_{s,T}(X_s,\lambda)$ (resp. $\mathcal{V}^{\mathrm{loc}}_{s,T}(X_s,\lambda)$) the set of all strategies which are M_s-admissible for some M_s.

In order to clarify the difference of admissibility in Definition 1.10 we consider the frictionless case. In the frictionless case no arbitrage can be characterized by true martingales or local martingales. The subtle difference here lies in the choice of admissible trading strategies. If we use local martingales and fix a numéraire the portfolio is controlled in units of the numéraire. In particular, short sales are not allowed. On the other hand, if there is no natural numéraire and we consider true martingales, the portfolio can be compared with a position, which may be short in each asset. Models with proportional transaction costs are often considered in the context of currency markets, where no natural numéraire exists. See also [99], [101] for more details.

Analogously, in the presence of proportional transaction costs consistent local price systems correspond to strategies which are admissible in the numéraire-based sense. In (1.23) the portfolio is bounded from below in units of the numéraire, i.e., can be hedged in units of the numéraire. In particular, no short positions in the risky asset are admissible.

Strategies which are admissible in the numéraire-free sense are used in the context of non-local consistent price systems. In (1.24), the portfolio is bounded from below by a position, which depends on each of the assets. Thus, also short positions in the risky asset are admissible. See also Chapter 5 of [48].

Remark 1.11. *We now compare the definition of admissible strategies, Definition 1.10, to Definition 3 and 5 of [84]. We consider the numéraire-based case here. However, the argument for the numéraire-free case is similar. In [84] strategies are only defined for the complete interval* $[0,T]$*. In opposite, in the present setting strategies are allowed to start at any time* $0 \le s \le T$*. In order to define strategies with non-zero initial endowment rigorously, we need to extend Definition 3 and 5 of [84].*

First, we discuss the case of zero initial endowments. Let $\varphi = (\varphi^1_t,\varphi^2_t)_{s\le t\le T}$ *be an admissible strategy on* $[s,T]$ *with* $\varphi_s = (0,0)$*, i.e.,* $V^{liq}_\tau(\varphi) \ge -M$ *for all* $[s,T]$*-valued stopping times* τ *and a constant* $M > 0$*. Then* φ *can be identified with an admissible strategy* $\psi = (\psi^1_t,\psi^2_t)_{0\le t\le T}$ *on* $[0,T]$*, where* $\psi_t = (0,0)$ *for all* $0 \le t \le s$ *and* $\psi_t = \varphi_t$ *for all* $s \le t \le T$*. On the other hand, any strategy* $\psi = (\psi^1_t,\psi^2_t)_{0\le t\le T}$ *on* $[0,T]$ *with* $\psi_t = (0,0)$ *for all* $0 \le t \le s$*, which is admissible in the numéraire-based sense in the sense of Definition 3 and 5 of [84], also satisfies Definition 1.10. Admissible strategies on* $[0,T]$ *with non-zero initial endowments can be defined by translation. Normalizing the initial value to zero has no impact on the admissibility of the strategy.*

For strategies on $[s,T]$ *an analogous normalization is more delicate. Let* $\varphi = (\varphi^1_t,\varphi^2_t)_{s\le t\le T}$ *be a strategy on* $[s,T]$ *with* $\varphi_s = (X_s,0)$ *for some* $X_s \in L^1_+(\mathcal{F}_s,\mathcal{Q}_{loc})$ *with* $V^{liq}_\tau(\varphi) \ge -M$ *for all* $[s,T]$*-valued stopping time* τ *and a constant* $M > 0$*. We normalize the strategy to zero initial endowment and obtain* $\tilde{\varphi}_t = (\tilde{\varphi}^1_t,\tilde{\varphi}^2_t) := (\varphi_t - X_\sigma,\varphi_t)$ *for all* $s \le t \le T$*. Then* $V^{liq}_\tau(\tilde{\varphi}) = V^{liq}_\tau(\varphi) - X_\sigma \ge -M - X_\sigma =: -M_\sigma$*. Thus, for a one-to-one correspondence of admissible strategies with and without endowments on* $[s,T]$ *it is too restrictive to require that the liquidation value is bounded from below by a constant.*

Definition 1.10 allows to obtain from any admissible strategy ψ on $[0,T]$ an admissible strategy $\varphi := \psi|_{[s,T]}$ on $[s,T]$. For $s = 0$ Definition 1.10 and Definition 3 of [84] coincide. Let us briefly motivate Definition 1.10 from an economical perspective. The role of the lower bound of the liquidation value is to avoid unbounded loss. In particular, the lower bound can be seen as the required capital to superhedge the portfolio in units of the bonds, see [84]. Naturally, it seems reasonable to include the information which are available up to time s to superhedge a portfolio on $[s,T]$.

For the notion of *arbitrage* we follow Definition 4 of [48].

Definition 1.12. The market model given by (B,S) admits *arbitrage* with λ-transaction costs if there is a strategy φ admissible in the numéraire-free (resp. numéraire-based) sense such that $V_T^{liq}(\varphi) \geq 0$ and $\mathbf{P}\left(V_T^{liq}(\varphi) > 0\right) > 0$.

As explained in Remark 1.11 we wish to extend the definitions of [84] to include admissible strategies on an arbitrary interval with arbitrary initial endowment. For this purpose, we need to impose condition (1.23) (resp. (1.24)). It must be guaranteed that the market model is arbitrage-free. Let $\varphi = (\varphi_t^1, \varphi_t^2)_{0\leq t\leq T}$ be an admissible strategy in the sense of Definition 3 (resp. Definition 5) of [84]. Then, the process $(\varphi_t^1 + \varphi_t^2 \tilde{S}_t^{\mathbf{Q}})_{0\leq t\leq T}$ is an optional strong $\mathbf{Q}$-supermartingale for all $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}_{\mathrm{loc}}(0,T)$ (resp. $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}(0,T)$), see Proposition 2 (resp. Proposition 3) of [84]. This property is needed to prove that the existence of a consistent (local) price system guarantees the absence of arbitrage.
In Definition 1.10 we require integrability conditions on the lower bound M_s. We prove that these integrability conditions are sufficient to ensure that $(\varphi_t^1 + \varphi_t^2 \tilde{S}_t^{\mathbf{Q}})_{s\leq t\leq T}$ is an optional strong $\mathbf{Q}$-supermartingale for all $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}_{\mathrm{loc}}(s,T)$ (resp. $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}(s,T)$) and admissible strategies φ in the sense of Definition 1.10. Only a few modifications of the original proofs of Proposition 2 and 3 of [84] are needed. For sake of completeness, we reproduce the complete proof of Proposition 2 of [84]. Note that, in [84] the author uses one-sided transaction costs. One-sided transaction costs are equivalent to symmetric transaction costs, see [48].

Proposition 1.13 (Proposition 2, [84])**.** *Let $\varphi = (\varphi_t^1, \varphi_t^2)_{s\leq t\leq T}$ be an admissible strategy in the numéraire-based sense. Suppose $(\boldsymbol{Q}, \tilde{S}^{\boldsymbol{Q}})$ is a consistent local price system. Then the process*

$$\tilde{V}_t := \varphi_t^1 + \varphi_t^2 \tilde{S}_t^{\boldsymbol{Q}}, \quad s \leq t \leq T,$$

satisfies $\tilde{V} \geq V^{liq}(\varphi)$ and is an optional strong $\boldsymbol{Q}$-supermartingale.

Proof. We directly observe that $\tilde{V} \geq V^{liq}(\varphi)$ follows from the fact that $\tilde{S}_t^{\mathbf{Q}} \in [(1-\lambda)S_t, (1+\lambda)S_t]$, for all $t \in [s,T]$.
In order to prove that $\tilde{V}$ is an optional strong $\mathbf{Q}$-supermartingale, we show that $\tilde{V}$ admits a Doob-Meyer or Mertens decomposition, i.e.,

$$\tilde{V} = M - A, \tag{1.25}$$

where M is a (càdlàg) local $\mathbf{Q}$-martingale as well as a supermartingale and A is an increasing predictable process. Note that A is not necessarily càdlàg.

Let φ be a strategy admissible in the numéraire-based sense and assume that the total variation of φ is uniformly bounded. We decompose φ in a continuous and purely discontinuous part with jumps

$$\varphi = \varphi^c + \varphi^j. \tag{1.26}$$

We consider the continuous part and the purely discontinous part separately. Further, we distinguish between right and left jumps. For the continuous part we have, that φ^c is a semimartingale of finite variation. Thus, we can apply Itô calculus to $\tilde{V}$. By the product rule we obtain

$$d\tilde{V}_t = \left(d\varphi_t^{c,1} + \tilde{S}_t^{\mathbf{Q}} d\varphi_t^{c,2}\right) + \varphi_t^{c,2} d\tilde{S}_t^{\mathbf{Q}},$$

which translates to

$$\tilde{V}_t = \int_0^t \left(d\varphi_u^{c,1} + \tilde{S}_u^{\mathbf{Q}} d\varphi_u^{c,2}\right) + \int_0^t \varphi_u^{c,2} d\tilde{S}_u^{\mathbf{Q}}. \tag{1.27}$$

The first term of (1.27) is decreasing by (1.19) and the fact that $\tilde{S}^{\mathbf{Q}} \in [(1-\lambda)S, (1+\lambda)S]$. The second term of (1.27) is a local $\mathbf{Q}$-martingale.
In the case when φ admits jumps the process $\tilde{V}$ is not necessarily càdlàg but still optional. For the right jumps, assume first that ψ^r is of the form

$$\psi_t^r = \Delta_+(\psi_\tau^{r,1}, \psi_\tau^{r,2}) \mathbb{1}_{]\!]\tau,T]\!]}(t), \tag{1.28}$$

where τ is a $[s,T]$-valued stopping time and $\Delta_+(\psi_\tau^{j,1}, \psi_\tau^{r,2})$ are $\mathcal{F}_\tau$-measurable bounded random variables such that (1.19) holds. Then, we get that

$$\begin{aligned} \tilde{V}_t &= \left(\Delta_+\psi_\tau^{r,1} + \left(\Delta_+\psi_\tau^{r,2}\right)\tilde{S}_t^{\mathbf{Q}}\right) \mathbb{1}_{]\!]\tau,T]\!]}(t) \\ &= \left(\Delta_+\psi_\tau^{r,1} + \left(\Delta_+\psi_\tau^{r,2}\right)\tilde{S}_\tau^{\mathbf{Q}}\right) \mathbb{1}_{]\!]\tau,T]\!]}(t) + \left(\Delta_+\psi_\tau^{r,2}\right)\left(\tilde{S}_t^{\mathbf{Q}} - \tilde{S}_\tau^{\mathbf{Q}}\right) \mathbb{1}_{]\!]\tau,T]\!]}(t). \end{aligned} \tag{1.29}$$

The first term of (1.29) is a decreasing process by (1.19) and the second term is a local $\mathbf{Q}$-martingale.
For the left jumps, assume that ψ^l is of the form

$$\psi_t^l = \Delta(\psi_\tau^{l,1}, \psi_\tau^{l,2}) \mathbb{1}_{[\![\tau,T]\!]}(t), \tag{1.30}$$

where τ is a $[s,T]$-valued stopping time and $\Delta(\psi_\tau^{l,1}, \psi_\tau^{l,2})$ are $\mathcal{F}_\tau$-measurable bounded random variables such that (1.19) holds. Then, we get that

$$\begin{aligned} \tilde{V}_t &= \left(\Delta\psi_\tau^{l,1} + \left(\Delta\psi_\tau^{l,2}\right)\tilde{S}_t^{\mathbf{Q}}\right) \mathbb{1}_{[\![\tau,T]\!]}(t) \\ &= \left(\Delta\psi_\tau^{l,1} + \left(\Delta\psi_\tau^{l,2}\right)\tilde{S}_\tau^{\mathbf{Q}}\right) \mathbb{1}_{[\![\tau,T]\!]}(t) + \left(\Delta\psi_\tau^{l,2}\right)\left(\tilde{S}_t^{\mathbf{Q}} - \tilde{S}_\tau^{\mathbf{Q}}\right) \mathbb{1}_{[\![\tau,T]\!]}(t). \end{aligned} \tag{1.31}$$

The first term of (1.31) is a decreasing process by (1.19) and the second term is a local $\mathbf{Q}$-martingale.
We may find sequences of $(\tau_n^r)_{n\in\mathbb{N}}$ and $(\tau_n^l)_{n\in\mathbb{N}}$ of $[s,T] \cup \{\infty\}$-valued stopping times such that the supports of each sequence $([\![\tau_n^r]\!])_{n\in\mathbb{N}}$ and $([\![\tau_n^l]\!])_{n\in\mathbb{N}}$ are mutually disjoint and that the occurrence of right jumps is covered by $\bigcup_{n\in\mathbb{N}}[\![\tau_n^r]\!]$ and the occurrence of left jumps is covered by $\bigcup_{n\in\mathbb{N}}[\![\tau_n^l]\!]$. With the decomposition from (1.26) of φ we can sum up over all stopping times $(\tau_n^r)_{n\in\mathbb{N}}$ and $(\tau_n^l)_{n\in\mathbb{N}}$ and apply (1.27), (1.29) and (1.31). This sum converges to $\tilde{V} = M - A$, where M is a local $\mathbf{Q}$-martingale and A is an increasing process.

Thus, we obtained the desired representation under the assumption that the total variation of φ is uniformly bounded. Because φ is admissible in the numéraire-based sense, the local $\mathbf{Q}$-martingale part is bounded from below by some $-M_s$ where $M_s \in L^1_+(\mathcal{F}_s, \mathcal{Q}_{\text{loc}})$. Hence by Proposition 3.3 of [4], respectively Theorem 1 of [97], $\left(\int_0^t \varphi_u^{c,2} d\tilde{S}_u^{\mathbf{Q}}\right)_{s\le t\le T}$ is a supermartingale under $\mathbf{Q}$. Therefore, $\tilde{V}$ is an optional strong $\mathbf{Q}$-supermarginale.
Now we drop the assumption that the total variation of φ is uniformly bounded. Since φ has finite total variation and is predictable we can find a localizing sequence $(\sigma_n)_{n\in\mathbb{N}}$ such that the stopped process φ^{σ_n} has uniformly bounded variation for each $n \in \mathbb{N}$. We apply the above argument to each φ^{σ_n} and obtain that φ admits the decomposition given in (1.25). Thus $\tilde{V}$ is an optional strong $\mathbf{Q}$-supermartingale. □

Proposition 1.14 (Proposition 3, [84])**.** *Let $\varphi = (\varphi_t^1, \varphi_t^2)_{s\le t\le T}$ be an admissible strategy in the numéraire-free sense. Suppose $(\boldsymbol{Q}, \tilde{S}^{\boldsymbol{Q}})$ is a non-local consistent price system. Then the process*

$$\tilde{V}_t := \varphi_t^1 + \varphi_t^2 \tilde{S}_t^{\boldsymbol{Q}}, \quad s \le t \le T,$$

satisfies $\tilde{V} \ge V^{liq}(\varphi)$ is an optional strong $\boldsymbol{Q}$-supermartingale.

Proof. Analogously to the proof of 1.13 we obtain a decomposition of $\tilde{V}$ into an increasing process and a local $\mathbf{Q}$-martingale. The only difference lies in the lower bound given by (1.24). At this point we can only conclude that $\tilde{V}$ is a local optional strong $\mathbf{Q}$-supermartingale. We now show that it is also an optional strong $\mathbf{Q}$-supermartingale (in the non-local sense).
We apply the following conditional version of Fatou's lemma. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real-valued random variables on $(\Omega, \mathcal{F}, \mathbf{Q})$ converging almost surely to X and such that the negative parts $(X_n^-)_{n\in\mathbb{N}}$ are uniformly $\mathbf{Q}$-integrable. Then

$$\mathbb{E}_{\mathbf{Q}}\left[\liminf_{n\to\infty} X_n \mid \mathcal{G}\right] \le \liminf_{n\to\infty} \mathbb{E}_{\mathbf{Q}}\left[X_n \mid \mathcal{G}\right].$$

The family $\{(\varphi_\tau^1 + \varphi_\tau^2 \tilde{S}_\tau^{\mathbf{Q}})^- : \sigma \le \tau \le T\}$ is uniformly $\mathbf{Q}$-integrable with respect to $\mathbf{Q}$ for all $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \text{CPS}(\sigma, T)$, as we have for $\sigma \le \tau \le T$

$$\varphi_\tau^1 + \varphi_\tau^2 \tilde{S}_\tau^{\mathbf{Q}} \ge V_\tau^{liq}(\varphi^1, \varphi^2) \ge -M_\sigma^1 - M_\sigma^2 S_\tau,$$

because $S_\tau \le \frac{1}{1-\lambda}\tilde{S}_\tau^{\mathbf{Q}}$ for any $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \text{CPS}(\sigma, T)$ and $\tilde{S}^{\mathbf{Q}}$ is a $\mathbf{Q}$-martingale, and $(M_\sigma^1, M_\sigma^2) \in L^1_+(\mathcal{F}_\sigma, \mathcal{Q}) \times L^\infty_+(\mathcal{F}_\sigma, \mathcal{Q})$ by assumption. Therefore, $(\varphi_t^1 + \varphi_t^2 \tilde{S}_t^{\mathbf{Q}})_{\sigma\le t\le T}$ is an optional strong $\mathbf{Q}$-supermartingale on $[\![\sigma, T]\!]$ for all $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \text{CPS}(\sigma, T)$ and all trading strategies $\varphi = (\varphi_t^1, \varphi_t^2)_{\sigma\le t\le T}$ are admissible in the numéraire-free sense. □

Remark 1.15. *The proofs of Proposition 1.13 and 1.14 are provided for the sake of completeness. In comparison to the original proofs of Proposition 2 and 3 of [84] the only changes in the proofs of Proposition 1.13 and 1.14 are in respect of the lower bound, which is used to show that $\tilde{V}$ is not only a local optional strong $\boldsymbol{Q}$-supermartingale but also in a non-local sense. In the case of Proposition 1.13 it is still possible to apply Proposition 3.3 of [4], respectively Theorem 1 of [97]. Also the use of the lower bound in Proposition 1.14 is very similar to the original version, Proposition 3 of [84].*

We conclude the section with another useful result from [84]. Theorem 1 of [84] also holds true in our more general setting. Again, we only need a few small modifications in the proof.

Corollary 1.16 (Theorem 2, [84]). *Suppose Assumption 1.5 is satisfied. Let* $\varphi = (\varphi_t^1, \varphi_t^2)_{s\le t\le T}$ *be a strategy admissible in the numéraire-free sense starting with zero endowment, and suppose that there is* $M_s \in L_+^1(\mathcal{F}_s, \mathcal{Q}_{loc})$ *such that for the terminal liquidation value* V_T^{liq} *we have*

$$V_T^{liq}(\varphi^1,\varphi^2) = \varphi_T^1 + (\varphi_T^2)^+(1-\lambda)S_T - (\varphi_T^2)^-(1+\lambda)S_T \ge -M_s. \tag{1.32}$$

We then also have that

$$V_\tau^{liq}(\varphi^1,\varphi^2) = \varphi_\tau^1 + (\varphi_\tau^2)^+(1-\lambda)S_\tau - (\varphi_\tau^2)^-(1+\lambda)S_\tau \ge -M_s, \tag{1.33}$$

a.s. for every stopping time $0 \le \tau \le T$.

Proof. Suppose that (1.33) does not hold for some $[s,T]$-valued stopping time τ. For $\alpha \in (0,1)$, define

$$A_+(\alpha) = \left\{\varphi_\tau^2 \ge 0, \varphi_\tau^1 + \varphi_\tau^2 \frac{1-\lambda}{1-\alpha} S_\tau < -M_s\right\}, \tag{1.34}$$

$$A_-(\alpha) = \left\{\varphi_\tau^2 \le 0, \varphi_\tau^1 + \varphi_\tau^2 (1+\lambda)(1-\alpha)^2 S_\tau < -M_s\right\}. \tag{1.35}$$

Then, we have

$$\bigcup_{\alpha>0} A_+(\alpha) = \{\varphi_\tau^2 \ge 0, \varphi_\tau^1 + \varphi_\tau^2(1-\lambda)S_\tau < -M_s\},$$
$$\bigcup_{\alpha>0} A_-(\alpha) = \{\varphi_\tau^2 \le 0, \varphi_\tau^1 + \varphi_\tau^2(1+\lambda)S_\tau < -M_s\}.$$

If (1.33) does not hold, there exists $\frac{\lambda}{2} > \alpha > 0$, such that either $\mathbf{P}(A_+(\alpha)) > 0$ or $\mathbf{P}(A_-(\alpha)) > 0$. Let $0 < \lambda' < \alpha$ such that $\frac{2\lambda'}{1+\lambda'} < \alpha$ and fix a λ'-consistent price system $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}})$ on $[s,T]$. As $\tilde{S}^{\mathbf{Q}}$ takes values in $[(1-\lambda')S, (1+\lambda')S]$, we have that $(1-\alpha)\tilde{S}^{\mathbf{Q}}$ as well as $\frac{1-\lambda}{(1-\alpha)(1+\lambda')}\tilde{S}^{\mathbf{Q}}$ takes values in $[(1-\lambda)S, (1+\lambda)S]$ because

$$\frac{1-\lambda}{1-\lambda'} < (1-\alpha) < \frac{1+\lambda}{1+\lambda'} \quad \text{and} \quad \frac{1-\lambda}{1-\lambda'} < \frac{1-\lambda}{(1-\alpha)(1+\lambda')} < \frac{1+\lambda}{1+\lambda'}.$$

It follows that $(\mathbf{Q}, (1-\alpha)\tilde{S}^{\mathbf{Q}})$ as well as $(\mathbf{Q}, \frac{1-\lambda}{(1-\alpha)(1+\lambda')}\tilde{S}^{\mathbf{Q}})$ are consistent price systems under transaction costs λ. By Proposition 1.14 we obtain that

$$\left(\varphi_t^1 + \varphi_t^2(1-\alpha)\tilde{S}_t^{\mathbf{Q}}\right)_{s\le t\le T} \quad \text{and} \quad \left(\varphi_t^1 + \varphi_t^2 \frac{1-\lambda}{(1-\alpha)(1+\lambda')}\tilde{S}_t^{\mathbf{Q}}\right)_{s\le t\le T} \tag{1.36}$$

are optional strong $\mathbf{Q}$-supermartingales. Note that $\tilde{S}^{\mathbf{Q}} \le (1+\lambda)S$. Assume now that $\mathbf{P}(A_+(\alpha)) > 0$ for some $\frac{\lambda}{2} > \alpha > 0$. By equivalence of the measures this implies that

$\mathbf{Q}(A_+(\alpha)) > 0$. By (1.34) we obtain with the second process of (1.36) that

$$\begin{aligned}
\mathbb{E}_{\mathbf{Q}}\left[V_T^{liq} \mid A_+(\alpha)\right] &\leq \mathbb{E}_{\mathbf{Q}}\left[\left(\varphi_T^1 + \varphi_T^2 \frac{1-\lambda}{(1-\alpha)(1+\lambda')}\tilde{S}_T^{\mathbf{Q}}\right) \mid A_+(\alpha)\right] \\
&\leq \mathbb{E}_{\mathbf{Q}}\left[\mathbb{E}_{\mathbf{Q}}\left[\left(\varphi_T^1 + \varphi_T^2 \frac{1-\lambda}{(1-\alpha)(1+\lambda')}\tilde{S}_T^{\mathbf{Q}}\right) \mid \mathcal{F}_\tau\right] \mid A_+(\alpha)\right] \\
&\leq \mathbb{E}_{\mathbf{Q}}\left[\mathbb{E}_{\mathbf{Q}}\left[\left(\varphi_T^1 + \varphi_T^2 \frac{1-\lambda}{(1-\alpha)(1+\lambda')}\tilde{S}_T^{\mathbf{Q}}\right) \mid \mathcal{F}_\tau\right] \mid A_+(\alpha)\right] \\
&\leq \mathbb{E}_{\mathbf{Q}}\left[\left(\varphi_\tau^1 + \varphi_\tau^2 \frac{1-\lambda}{(1-\alpha)(1+\lambda')}\tilde{S}_\tau^{\mathbf{Q}}\right) \mid A_+(\alpha)\right] \\
&\leq \mathbb{E}_{\mathbf{Q}}\left[\left(\varphi_\tau^1 + \varphi_\tau^2 \frac{1-\lambda}{1-\alpha} S_\tau\right) \mid A_+(\alpha)\right] \\
&< \mathbb{E}_{\mathbf{Q}}\left[-M_s \mid A_+(\alpha)\right]
\end{aligned}$$

This implies that $\mathbf{Q}(V_T^{liq} < -M_s) > 0$, and again by equivalence we get $\mathbf{P}(V_T^{liq} < -M_s) > 0$, which contradicts (1.32).
Conversely, assuming that $\mathbf{P}(A_-(\alpha)) > 0$ for some $\frac{\lambda}{2} > \alpha > 0$, analogously implies that $\mathbf{Q}(A_-(\alpha)) > 0$. In fact,

$$\tilde{S}^{\mathbf{Q}} \geq (1-\lambda')S \geq (1-\alpha)S$$

and thus

$$\varphi_\tau^2(1-\alpha)\tilde{S}_\tau^{\mathbf{Q}} \leq \varphi_\tau^2(1-\alpha)^2 S_\tau \text{ on } A_-(\alpha).$$

From (1.35) and with the first process of (1.36) we obtain that

$$\begin{aligned}
\mathbb{E}_{\mathbf{Q}}\left[V_T^{liq} \mid A_-(\alpha)\right] &\leq \mathbb{E}_{\mathbf{Q}}\left[\varphi_T^1 + \varphi_T^2(1-\alpha)\tilde{S}_T^{\mathbf{Q}} \mid A_-(\alpha)\right] \\
&\leq \mathbb{E}_{\mathbf{Q}}\left[\varphi_\tau^1 + \varphi_\tau^2(1-\alpha)\tilde{S}_\tau^{\mathbf{Q}} \mid A_-(\alpha)\right] \\
&\leq \mathbb{E}_{\mathbf{Q}}\left[\varphi_\tau^1 + \varphi_\tau^2(1-\alpha)^2 S_\tau \mid A_-(\alpha)\right] \\
&< -\mathbb{E}_{\mathbf{Q}}[M_s \mid A_-(\alpha)].
\end{aligned}$$

With the same arguments as above $\mathbf{P}(V_T^{liq} < -M_s) > 0$, which is a contradiction to (1.32). To conclude, we note that if (1.33) fails, either $A_+(\alpha)$ or $A_-(\alpha)$ has positive probability for some α. In both cases we obtain a contradiction to (1.32). Therefore, (1.33) must hold. □

1.5 Closedness of the cone of attainable claims

We reproduce some results of [84] and [85], which are required for Section 2.1, see also [23]. More precisely, we extend Lemma 3.1, Theorem 3.4 and Theorem 3.6 of [85] to the present setting. Instead of only allowing trading strategies on $[0,T]$ as in [85], we allow trading strategies on any interval $[s,T]$ as in Definition 1.10. In particular, the bounds for the liquidation value in (1.23) and (1.24) may be random. This leads to some minor modifications of the proofs. Note that, in [85] the author uses one-sided transaction costs.

Definition 1.17. A contingent claim $X_T = (X_T^1, X_T^2)$ is an $\mathcal{F}_T$-measurable random variable in $L^0(\mathcal{F}_T, \mathbf{P}; \mathbb{R}^2)$ which pays X_T^1 units of the bond and X_T^2 units of the risky asset at time T.

Note that by Definition 1.17 a contingent claim is not assumed to be strictly positive. However, in the sequel we will require some lower bound properties depending on (1.23) and (1.24).

Definition 1.18. For $M_s := (M_s^1, M_s^2) \in L_+^1(\mathcal{F}_s, \mathcal{Q}) \times L_+^\infty(\mathcal{F}_s, \mathcal{Q})$ (resp. $M_s \in L_+^1(\mathcal{F}_s, \mathcal{Q}_{\text{loc}})$) we denote by $\mathcal{A}_{s,T}^{M_s}$ (resp. $\mathcal{A}_{s,T}^{M_s,\text{loc}}$) the set of pairs $(\varphi_T^1, \varphi_T^1) \in L^0(\mathcal{F}_T, \mathbf{P}; \mathbb{R}^2)$ of terminal values of self-financing trading strategies φ, starting at $\varphi_s = (0,0)$, which are M_s-admissible in the numéraire-free sense (resp. numéraire-based sense). Further, we define

$$\mathcal{A}_{s,T} := \left\{\varphi_T : \varphi_T \in \mathcal{A}_{s,T}^{M_s} \text{ for some } M_s = (M_s^1, M_s^2) \in L_+^1(\mathcal{F}_s, \mathcal{Q}) \times L_+^\infty(\mathcal{F}_s, \mathcal{Q})\right\}. \qquad (1.37)$$

Lemma 1.19 (Lemma 3.1, [85]). *Suppose that there exists a local price system for some* $0 < \lambda' < \lambda$. *Then for* $M_s = (M_s^1, M_s^2) \in L_+^1(\mathcal{F}_s, \mathcal{Q}) \times L_+^\infty(\mathcal{F}_s, \mathcal{Q})$ *and* $\varepsilon > 0$ *there exists* $C > 0$ *such that, for all* M_s*-admissible,* λ*-self-financing strategies* φ*, starting at* $(\varphi_s^1, \varphi_s^2) = (0,0)$*, and for all increasing sequences* $s = \tau_0 < \tau_1 < \cdots < \tau_k = T$ *of stopping times we have*

$$\boldsymbol{P}\left(\sum_{k=1}^{K} \left|\varphi_{\tau_k}^1 - \varphi_{\tau_{k-1}}^1\right| \geq C\right) < \varepsilon, \qquad (1.38)$$

$$\boldsymbol{P}\left(\sum_{k=1}^{K} \left|\varphi_{\tau_k}^2 - \varphi_{\tau_{k-1}}^2\right| \geq C\right) < \varepsilon. \qquad (1.39)$$

Proof. Fix $0 < \lambda' < \lambda$ as above and let $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}})$ be a λ'-consistent local price system for the interval $[s,T]$. There exists a localizing sequence $(\sigma_n)_{n\in\mathbb{N}}$ for $\tilde{S}^{\mathbf{Q}}$ such that $(\tilde{S}^{\mathbf{Q}})^{\sigma_n}$ is a true $\mathbf{Q}$-martingale for all $n \in \mathbb{N}$ and $\sigma_n \to \infty$ $\mathbf{P}$-a.s. as n tends to infinity. In particular, for given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\mathbf{P}(\sigma_n < T) < \frac{\varepsilon}{2}, \quad \text{for all } n \geq N. \qquad (1.40)$$

Thus, we may assume that $\tilde{S}^{\mathbf{Q}}$ is true $\mathbf{Q}$-martingale by stopping or we can consider $(\tilde{S}^{\mathbf{Q}})^{\sigma_N}$ on $\{\sigma_N \geq T\}$. For sake of notational simplicity we consider $\tilde{S}^{\mathbf{Q}}$ instead of $(\tilde{S}^{\mathbf{Q}})^{\sigma_N}$. Note also that $\mathbf{Q} \sim \mathbf{P}$. Further, we assume without loss of generality that the stock position is liquidated at time T, i.e., $\varphi_T^2 = 0$.
Fix $M_s = (M_s^1, M_s^2) \in L_+^1(\mathcal{F}_s, \mathcal{Q}) \times L_+^\infty(\mathcal{F}_s, \mathcal{Q})$, and a λ-self-financing strategy $\varphi = (\varphi_t^1, \varphi_t^2)_{s\leq t\leq T}$ with $(\varphi_s^1, \varphi_s^2) = (0,0)$, which is M_s-admissible in the numéraire-free sense. We use the Jordan-Hahn decomposition to get $\varphi^1 = \varphi^{1,\uparrow} - \varphi^{1,\downarrow}$ and $\varphi^2 = \varphi^{2,\uparrow} - \varphi^{2,\downarrow}$ as the canonical differences of increasing processes. Define the process $\varphi' = ((\varphi^1)', (\varphi^2)')$ by

$$\varphi_t' = ((\varphi^1)_t', (\varphi^2)_t') = \left(\varphi_t^1 + \frac{\lambda - \lambda'}{1 - \lambda}\varphi_t^{1,\uparrow}, \varphi_t^2\right), \quad s \leq t \leq T.$$

This is a self-financing process under transaction costs λ'. To see this, let $d\varphi_t^1 > 0$. Then $d\varphi_t^1 = d\varphi_t^{1,\uparrow}$, i.e., the agent receives money on her bank-account by selling a stock. Under λ transaction costs, the agent receives $d\varphi_t^{1,\uparrow} = (1-\lambda)S_t d\varphi_t^{2,\downarrow}$ many bonds. Under λ' transaction costs, the agent receives $(1-\lambda')S_t d\varphi_t^{2,\downarrow} = \frac{1-\lambda'}{1-\lambda} d\varphi_T^{1,\uparrow}$ many bonds. Thus, the agent does better under λ' transaction costs by $\frac{\lambda-\lambda'}{1-\lambda} d\varphi_t^{1,\uparrow}$. On the other hand, when $d\varphi_t^1 < 0$ so that $d\varphi_t^1 = -d\varphi_t^{1,\downarrow} \leq -(1+\lambda)S_t d\varphi_t^{2,\uparrow} \leq -(1+\lambda')S_t d\varphi_t^{2,\uparrow}$. Therefore, $\varphi' = ((\varphi^1)', (\varphi^2)')$

is also self-financing under transaction costs λ'. Clearly, φ' is still M_s-admissible in the numéraire-free sense.
By Proposition 1.13 the process given by

$$((\varphi^1)'_t + (\varphi^2)'_t \tilde{S}^{\mathbf{Q}}_t)_{s\le t\le T} = \left((\varphi^1)'_t + \varphi^2_t \tilde{S}^{\mathbf{Q}}_t\right)_{s\le t\le T} = \left(\varphi^1_t + \frac{\lambda-\lambda'}{1-\lambda}\varphi^{1,\uparrow}_t + \varphi^2_t \tilde{S}^{\mathbf{Q}}_t\right)_{s\le t\le T}$$

is an optional strong $\mathbf{Q}$-supermartingale on $[s,T]$. By the supermartingale property we obtain

$$\mathbb{E}_{\mathbf{Q}}\left[\varphi^1_T + \varphi^2_T \tilde{S}^{\mathbf{Q}}_T\right] + \frac{\lambda-\lambda'}{1-\lambda}\mathbb{E}_{\mathbf{Q}}\left[\varphi^{1,\uparrow}_T\right] \le 0.$$

Thus we get

$$\mathbb{E}_{\mathbf{Q}}\left[\varphi^{1,\uparrow}_T\right] \le \frac{1-\lambda}{\lambda-\lambda'}\left(-\mathbb{E}_{\mathbf{Q}}\left[\varphi^1_T + \varphi^2_T \tilde{S}^{\mathbf{Q}}_T\right]\right) \le \frac{\mathbb{E}_{\mathbf{Q}}\left[M^1_s + M^2_s S_T\right]}{\lambda-\lambda'}, \tag{1.41}$$

where we used that by admissibility and Proposition 1.13

$$\varphi^1_T + \varphi^2_T \tilde{S}^{\mathbf{Q}}_T \ge -M^1_s - M^2_s S_T.$$

Recall that $\varphi^2_T = 0$ and hence

$$\varphi^1_T = \varphi^1_T + \varphi^2_T \tilde{S}^{\mathbf{Q}}_T \ge -M^1_s - M^2_s S_T.$$

Therefore, we obtain that

$$\varphi^{1,\downarrow}_T \le \varphi^{1,\uparrow}_T + M^1_s + M^2_s S_T. \tag{1.42}$$

By (1.41) and (1.42) we obtain for the total variation $\varphi^{1,\uparrow}_T + \varphi^{1,\downarrow}_T$ of φ^1 that

$$\mathbb{E}_{\mathbf{Q}}\left[\varphi^{1,\uparrow}_T + \varphi^{1,\downarrow}_T\right] \le \mathbb{E}_{\mathbf{Q}}\left[2\varphi^{1,\uparrow}_T + M^1_s + M^2_s S_T\right] \le \left(\frac{1}{\lambda-\lambda'}+1\right)\mathbb{E}_{\mathbf{Q}}\left[M^1_s + M^2_s S_T\right] =: \tilde{C}. \tag{1.43}$$

To conclude (1.38) we have to derive the required $L^0(\mathcal{F}_T,\mathbf{P})$ estimate from the $L^1(\mathcal{F}_T,\mathbf{Q})$-bound in (1.43). Recall that an arbitrary $\varepsilon > 0$ was given above. By equivalence of $\mathbf{Q}$ and $\mathbf{P}$ there exists $\delta > 0$ such that for any $A \in \mathcal{F}_T$ with $\mathbf{Q}(A) < \delta$ we get $\mathbf{P}(A) < \frac{\varepsilon}{2}$.
Define C by

$$C := \frac{\tilde{C}}{\delta},$$

where $\tilde{C}$ is defined in (1.43). By Tschebyscheff we get from (1.43) that

$$\mathbf{P}\left(\varphi^{1,\uparrow}_T + \varphi^{1,\downarrow}_T \ge C\right) < \frac{\varepsilon}{2}. \tag{1.44}$$

Since we assumed that $\tilde{S}^{\mathbf{Q}}$ is a true $\mathbf{Q}$-martingale we need (1.40) and (1.44) to obtain

$$\mathbf{P}\left(\sum_{k=1}^{K}\left|\varphi^1_{\tau_k} - \varphi^1_{\tau_{k-1}}\right| \ge C\right) \le \mathbf{P}\left(\left\{\sum_{k=1}^{K}\left|\varphi^1_{\tau_k} - \varphi^1_{\tau_{k-1}}\right| \ge C\right\} \cap \{\sigma_N \ge T\}\right) + \mathbf{P}\left(\sigma_N < T\right) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Consider (1.39), now. By (1.19) we obtain

$$d\varphi^{2,\uparrow}_t \le \frac{d\varphi^{1,\downarrow}_t}{(1+\lambda)S_t}, \tag{1.45}$$

where we used that S is strictly positive by assumption. Equation (1.45) must be understood in the sense of (1.20), (1.21) and (1.22). Since we assumed that $\tilde{S}^{\mathbf{Q}}$ is a true $\mathbf{Q}$-martingale satisfying $\mathbf{P}(\tilde{S}^{\mathbf{Q}}_T > 0) = 1$ we get that $\mathbf{P}(\tilde{S}^{\mathbf{Q}}_t > 0) = 1$. Summing up, for $\varepsilon > 0$, we may find $\delta > 0$ such that

$$\mathbf{P}\left(\inf_{s\le t\le T} S_t < \delta\right) < \frac{\varepsilon}{3}.$$

We control $\varphi^{2,\uparrow}_T$ by (1.45). In particular, we control $\varphi^{1,\downarrow}_T$ by (1.44). Finally, for $\varphi^{2,\downarrow}_T$ we observe that $\varphi^{2,\uparrow}_T - \varphi^{2,\downarrow}_T = \varphi^2_T - \varphi^2_0 = 0$. This concludes (1.39). $\square$

Remark 1.20. *Note that the assumption in Lemma 1.19 are weaker than Assumption 1.4 because it is only required to have a consistent local price system for one particular $0 < \lambda' < \lambda$ and not for all $0 < \lambda' < \lambda$ as in Assumption 1.4.*
At first sight it may be confusing that in Lemma 1.19 we consider consistent local price systems but strategies which are admissible in the numéraire-free sense. However, this gives us a more general result which is valid for both scenarios:

- *consistent local price systems and strategies which are admissible in the numéraire-based sense;*
- *consistent price systems in the non-local sense and strategies which are admissible in the numéraire-free sense.*

Remark 1.21 (Remark 3.2, [85]). *Note that, the proof also shows that convex combinations of each of $\varphi^{1,\uparrow}_T$, $\varphi^{1,\downarrow}_T$, $\varphi^{2,\uparrow}_T$, $\varphi^{2,\downarrow}_T$ are bounded in $L^0(\mathcal{F}_T, \boldsymbol{P})$. Equation* (1.45) *shows that the convex hull of the functions $\varphi^{1,\uparrow}_T$ is bounded in $L^1(\mathcal{F}_T, \boldsymbol{Q})$ and* (1.43) *yields the same for $\varphi^{1,\downarrow}_T$. For $\varphi^{2,\uparrow}_T$ and $\varphi^{2,\downarrow}_T$ the argument is similar.*

Corollary 1.22 (Theorem 3.4 (numéraire-based), [85])**.** *Suppose that there exists a consistent local price system for some $0 < \lambda' < \lambda$. Fix $M_s \in L^1_+(\mathcal{F}_s, \mathcal{Q}_{loc})$. The convex set $\mathcal{A}^{M_s,loc}_{s,T} \subset L^0(\mathcal{F}_T, \boldsymbol{P}; \mathbb{R}^2)$ is closed with respect to the topology of convergence in measure.*

Proof. The proof follows as in the proof of Theorem 3.4 of [85] using Lemma 1.19 instead of Lemma 3.1 of [85]. $\square$

Corollary 1.23 (Theorem 3.6 (numéraire-free), [85])**.** *Suppose that there exists a consistent price system in the non-local sense for some $0 < \lambda' < \lambda$. Fix $M_s = (M^1_s, M^2_s) \in L^1_+(\mathcal{F}_s, \mathcal{Q}) \times L^\infty_+(\mathcal{F}_s, \mathcal{Q})$. The convex set $\mathcal{A}^{M_s,loc}_{s,T} \subset L^0(\mathcal{F}_T, \boldsymbol{P}; \mathbb{R}^2)$ is closed with respect to the topology of convergence in measure.*

Proof. The proof follows as in the proof of Theorem 3.6 of [85] using Lemma 1.19 instead of Lemma 3.1 of [85]. $\square$

Chapter 2

Dynamic super-replication

This chapter is based on [19]. In this chapter we present two of our main results, Theorems 2.6, 2.7. We prove dynamic super-replication dualities in the context of local and non-local consistent price systems. For this purpose, we extend the bipolar theorem of [67]. First, we prove the non-local version, which is then used for the proof of the local version. Furthermore, we derive a duality representation for super-replication prices given via consistent (local) price systems. We conclude the section with some further properties of the super-replication price process. For instance we prove that the consistent (local) price systems, which are used for the dual representation are independent of the time, see Theorems 2.11, 2.12. We also establish sufficient condition such that the super-replication price process is càdlàg, see Theorem 2.19, 2.21.

2.1 A Bipolar Theorem

In this section, we complete the technical basis to prove the dynamic super-replication theorems, Theorem 2.7, 2.6. The main theorem of this section is a bipolar theorem, Theorem 2.5. It can be seen as an extension of Theorem 4.3 of [67] (see also Theorem 5.5.3 of [65]), which is adapted to our setting.
With this bipolar theorem we can apply similar techniques as in [23] and [85] to prove the dynamic super-replication theorems.

Definition 2.1. We define the partial order $\succeq$ on $L^0(\mathcal{F}_T, \mathbf{P}; \mathbb{R}^2)$ by letting $\varphi \succeq \psi$ if and only if $V_T^{liq}(\varphi^1 - \psi^1, \varphi^2 - \psi^2) \geq 0$, i.e. if the portfolio $\varphi - \psi$ can be liquidated to the zero portfolio.

Definition 2.2. We say a set $\Phi \subset L^0(\mathcal{F}_T, \mathbf{P}; \mathbb{R}^2)$ is directed upwards if for any $\psi_1, \psi_2 \in \Phi$ there exists $\psi \in \Phi$ with $\psi \succeq \psi_1 \vee \psi_2$.

Let $L^0_{1,\infty}$ be the cone in $L^0(\mathcal{F}_T, \mathbf{P}; \mathbb{R}^2)$ given by the random variables $\xi = (\xi^1, \xi^2)$ such that $(\xi^1, \xi^2) \succeq (-M_s^1, -M_s^2)$ for some $(M_s^1, M_s^2) \in L^1_+(\mathcal{F}_s, \mathcal{Q}) \times L^\infty_+(\mathcal{F}_s, \mathcal{Q})$.
Further, we denote by $L^0_b \subset L^0(\mathcal{F}_T, \mathbf{P}; \mathbb{R}^2)$ the cone formed by random variables ξ such that $(\xi^1, \xi^2) \succeq (-M, -M)$ for some $M > 0$, see Section 5.5. of [65].

Remark 2.3. *Note that the conditional expectation*

$$\mathbb{E}_{\boldsymbol{P}}[\xi \cdot Z_T \mid \mathcal{F}_s], \quad Z_T \in L^1(K_T^*),\ \xi \in L^0_{1,\infty} \tag{2.1}$$

is well-defined. In fact, by the definition of $L^0_{1,\infty}$ *there exists* $M_s = (M_s^1, M_s^2) \in L^1(\mathbb{R}_+; \mathcal{F}_s, \mathcal{Q}) \times L^\infty(\mathbb{R}_+; \mathcal{F}_s, \mathcal{Q})$ *such that* $(\xi + M_s) \in K_T$ *and hence* $(\xi + M_s) \cdot Z_T \geq 0$. *In particular, we get*

$$\mathbb{E}_{\boldsymbol{P}}[\xi \cdot Z_T \mid \mathcal{F}_s] = \mathbb{E}_{\boldsymbol{P}}[(\xi + M_s) \cdot Z_T - M_s \cdot Z_T \mid \mathcal{F}_s].$$

For non-negative random variables the conditional expectation is always well-defined, although it might be infinity. Thus,

$$\mathbb{E}_{\boldsymbol{P}}[(\xi + M_s) \cdot Z_T \mid \mathcal{F}_s]$$

is well-defined. Furthermore, we need

$$\mathbb{E}_{\boldsymbol{P}}[M_s \cdot Z_T \mid \mathcal{F}_s] < \infty. \tag{2.2}$$

First, note that $M_s \cdot Z_T \geq 0$ *and hence* (2.2) *is well-defined. Following Section 27 of [73], we use that* M_s *is* $\mathcal{F}_s$*-measurable and* $Z_T \in L^1(K_T^*, \mathcal{F}_T, \boldsymbol{P})$ *to conclude*

$$\mathbb{E}_{\boldsymbol{P}}[M_s \cdot Z_T \mid \mathcal{F}_s] = M_s \mathbb{E}_{\boldsymbol{P}}[Z_T \mid \mathcal{F}_s] < \infty.$$

Therefore,

$$\mathbb{E}_{\boldsymbol{P}}[\xi \cdot Z_T \mid \mathcal{F}_s] \geq -M_s \mathbb{E}_{\boldsymbol{P}}[Z_T \mid \mathcal{F}_s] > -\infty$$

is well-defined. However, for $s > 0$

$$\mathbb{E}_{\boldsymbol{P}}[M_s \cdot Z_T]$$

is in general not well-defined. In contrast, for $\eta \in L^0_b$ *and* $Z_T \in L^1(K_T^*; \mathcal{F}_T, \boldsymbol{P})$ *as in the bipolar theorem of [67] there exists* $M > 0$ *such that*

$$\mathbb{E}_{\boldsymbol{P}}[\eta \cdot Z_T] \geq -M \mathbb{E}_{\boldsymbol{P}}[Z_T] > -\infty$$

is well-defined.

We now extend the definition of Fatou convergence.

Definition 2.4. Consider a sequence $(X_n)_{n\in\mathbb{N}} = (X_n^1, X_n^2)_{n\in\mathbb{N}} \subset L^0(\mathcal{F}_T, \mathbf{P}; \mathbb{R}^2)$. We say that $(X_n)_{n\in\mathbb{N}}$ is $L^0(\mathcal{F}_s)$-Fatou converging to $X = (X^1, X^2)$ if $X_n \xrightarrow{\mathbf{P}\text{-a.s.}} X$ and $(X_n^1, X_n^2) \geq (-M_s^1, -M_s^2)$ for all $n \in \mathbb{N}$ and some $(M_s^1, M_s^2) \in L^1_+(\mathcal{F}_s, \mathcal{Q}) \times L^\infty_+(\mathcal{F}_s, \mathcal{Q})$.

If $(-M_s^1, -M_s^2) = (-M, -M)$ for some $M \in \mathbb{R}_+$, $L^0(\mathcal{F}_s)$-Fatou convergence coincides with the Fatou convergence[1] as defined in [23], [67], [85].
Consider $\mathcal{A}_{s,T}$, defined in (1.37), which is a convex subset of $L^0_{1,\infty}$.
For $Z_T = (Z_T^1, Z_T^2) \in L^1(\mathcal{F}_T, \mathbf{P}; K_T^*)$ and $\xi = (\xi^1, \xi^2) \in L_{1,\infty}$ we set $\xi \cdot Z_T = \xi^1 Z_T^1 + \xi^2 Z_T^2$.

[1]Following [67], [85], let $(X_n)_{n\in\mathbb{N}} \subset L^0(\mathcal{F}_T, \mathbf{P}; \mathbb{R}^2)$. We say that $(X_n)_{n\in\mathbb{N}}$ is Fatou converging to $X = (X^1, X^2)$ if $X_n \xrightarrow{\mathbf{P}\text{-a.s.}} X$ and $(X_n^1, X_n^2) \geq (-M, -M)$ for all $n \in \mathbb{N}$ and some $M > 0$.

Theorem 2.5. *Let $s \in [0,T]$. It holds that*

$$\mathcal{A}_{s,T} = \left\{ \xi \in L^0_{1,\infty} : \mathbb{E}_{\boldsymbol{P}}\left[\xi \cdot Z_T \mid \mathcal{F}_s\right] \le \operatorname*{ess\,sup}_{\eta \in \mathcal{A}_{s,T}} \mathbb{E}_{\boldsymbol{P}}\left[\eta \cdot Z_T \mid \mathcal{F}_s\right] \;\; \forall Z_T \in L^1(\mathcal{F}_T, \boldsymbol{P}; K_T^*) \right\}, \quad (2.3)$$

where $\mathcal{A}_{s,T}$ is defined in (1.37)

Proof. The inclusion " $\subseteq$ " is trivial.
For the reverse inclusion we make use of the bipolar theorem of [67], Theorem 4.2, see also Theorem 5.5.3 of [65]. Let $Z_T = (Z_T^1, Z_T^2) \in L^1(\mathcal{F}_T, \mathbf{P}; K_T^*)$ and $\xi = (\xi^1, \xi^2) \in L_{1,\infty}$. As noted in Remark 2.3, for $Z_T \in L^1(\mathcal{F}_T, \mathbf{P}; K_T^*)$ and $\xi \in L^0_{1,\infty}$ the (conditional) expectation of $\xi \cdot Z_T$ is well-defined.
If the conditions of Theorem 4.2 of [67] are satisfied for $\mathcal{A}_{s,T} \cap L^0_b$, then we obtain

$$\mathcal{A}_{s,T} \cap L^0_b = \left\{ \xi \in L^0_b : \mathbb{E}_{\mathbf{P}}\left[\xi \cdot Z_T\right] \le \sup_{\eta \in \mathcal{A}_{s,T} \cap L^0_b} \mathbb{E}_{\mathbf{P}}\left[\eta \cdot Z_T\right] \;\; \forall Z_T \in L^1(\mathcal{F}_T, \mathbf{P}; K_T^*) \right\}. \quad (2.4)$$

First, we prove that the assumptions of Theorem 4.2 of [67] are indeed fulfilled for $\mathcal{A}_{s,T} \cap L^0_b$. Compare also the proofs of Theorem 4.1 of [23] and of Theorem 1.5 of [85]. Corollary 1.23 (resp. Theorem 3.6 of [85]) implies that $\mathcal{A}_{s,T} \cap L^0_b$ is Fatou-closed. In fact, consider a sequence $(\varphi_n)_{n \in \mathbb{N}} = (\varphi_n^1, \varphi_n^2)_{n \in \mathbb{N}} \subset \mathcal{A}_{s,T} \cap L^0_b$ such that $\varphi_n \ge (-M, -M)$ for all $n \in \mathbb{N}$ and some $M > 0$ and $\varphi_n \xrightarrow{\mathbf{P}\text{-a.s.}} \varphi$ for some $\varphi \in L^0(\mathcal{F}_T, \mathbf{P}; \mathbb{R}^2)$ by the definition of Fatou convergence. Clearly $\varphi \ge (-M, -M)$ and $\varphi \in \mathcal{A}_{s,T}$, because Corollary 1.23 (resp. Theorem 3.6 of [85]) guarantees that $\mathcal{A}^M_{s,T} \subset \mathcal{A}_{s,T} \cap L^0_b$ is closed with respect to the topology of measure and thus also with respect to almost sure convergence. Next, we show that $\mathcal{A}_{s,T} \cap L^\infty(\mathcal{F}_T, \mathbf{P})$ is dense in $\mathcal{A}_{s,T} \cap L^0_b$ with respect to Fatou-convergence. Let $\varphi = (\varphi^1, \varphi^2) \in \mathcal{A}_{s,T} \cap L^0_b$, i.e. $\varphi \ge (-M, -M)$ for some $M > 0$. We define the sequence $\varphi_n := \varphi \mathbb{1}_{\{|\varphi| \le n\}} - (M, M) \mathbb{1}_{\{|\varphi| > n\}}$. Then $(\varphi_n)_{n \in \mathbb{N}} \subset (\mathcal{A}_{s,T} \cap L^0_b) \cap L^\infty(\mathcal{F}_T, \mathbf{P})$ and $\varphi_n \xrightarrow{\mathbf{P}\text{-a.s.}} \varphi$. Furthermore, $\varphi_n \ge (-M, -M)$ for all $n \in \mathbb{N}$ which guarantees that φ_n Fatou-converges to φ. Therefore, $(\mathcal{A}_{s,T} \cap L^0_b) \cap L^\infty(\mathcal{F}_T, \mathbf{P})$ is dense with respect to Fatou-convergence in $\mathcal{A}_{s,T} \cap L^0_b$. It is left to show that $-L^\infty(\mathcal{F}_T, \mathbf{P}; K_T) \subset \mathcal{A}_{s,T} \cap L^0_b$. For this purpose, let $\psi \in -L^\infty(\mathcal{F}_T, \mathbf{P}; K_T)$ be arbitrary. Then $\|\psi^i\|_\infty \le M$ for $i = 1, 2$ and some $M > 0$. In particular, $\psi \ge (-M, -M)$ and thus $\psi \in \mathcal{A}_{s,T} \cap L^0_b$.
In order to make use of Theorem 4.2 of [67], we show that

$$\mathbb{E}_{\mathbf{P}}\left[\operatorname*{ess\,sup}_{\eta \in \mathcal{A}_{s,T} \cap L^0_b} \mathbb{E}_{\mathbf{P}}\left[\eta \cdot Z_T \mid \mathcal{F}_s\right] \right] = \sup_{\eta \in \mathcal{A}_{s,T} \cap L^0_b} \mathbb{E}_{\mathbf{P}}\left[\eta \cdot Z_T\right]. \quad (2.5)$$

By monotonicity we obtain that

$$\mathbb{E}_{\mathbf{P}}\left[\operatorname*{ess\,sup}_{\eta \in \mathcal{A}_{s,T} \cap L^0_b} \mathbb{E}_{\mathbf{P}}\left[\eta \cdot Z_T \mid \mathcal{F}_s\right] \right] \ge \sup_{\eta \in \mathcal{A}_{s,T} \cap L^0_b} \mathbb{E}_{\mathbf{P}}\left[\eta \cdot Z_T\right].$$

We define $\Phi_{Z_T} := \{\mathbb{E}_{\mathbf{P}}[\eta Z_T \mid \mathcal{F}_s] : \eta \in L^0_b\}$ and observe that Φ_{Z_T} is directed upwards (see Definition 2.2). Indeed, let $\eta, \tilde{\eta} \in L^0_b$ and define

$$D_s := \{\mathbb{E}_{\mathbf{P}}\left[\eta \cdot Z_T \mid \mathcal{F}_s\right] \ge \mathbb{E}_{\mathbf{P}}\left[\tilde{\eta} \cdot Z_T \mid \mathcal{F}_s\right]\} \in \mathcal{F}_s,$$

and

$$\psi := \eta \mathbb{1}_{D_s} + \tilde{\eta} \mathbb{1}_{D_s^c} \in L_b^0.$$

Then, we obtain by linearity that

$$\begin{aligned}\mathbb{E}_{\mathbf{P}}[\psi \cdot Z_T \mid \mathcal{F}_s] &= \mathbb{E}_{\mathbf{P}}[\eta \cdot Z_T \mid \mathcal{F}_s] \mathbb{1}_{D_s} + \mathbb{E}_{\mathbf{P}}[\tilde{\eta} \cdot Z_T \mid \mathcal{F}_s] \mathbb{1}_{D_s^c} \\ &\geq \mathbb{E}_{\mathbf{P}}[\eta \cdot Z_T \mid \mathcal{F}_s] \vee \mathbb{E}_{\mathbf{P}}[\tilde{\eta} \cdot Z_T \mid \mathcal{F}_s].\end{aligned}$$

Thus, Φ_{Z_T} is directed upwards. Therefore, Theorem A.33 of [40] guarantees the existence of a sequence $(\eta_n)_{n\in\mathbb{N}} = (\eta_n(Z_T))_{n\in\mathbb{N}} \subset L_b^0$ such that

$$\mathbb{E}_{\mathbf{P}}[\eta_n \cdot Z_T \mid \mathcal{F}_s] \uparrow \operatorname*{ess\,sup}_{\eta\in\mathcal{A}_{s,T}\cap L_b^0} \mathbb{E}_{\mathbf{P}}[\eta \cdot Z_T \mid \mathcal{F}_s], \ \mathbf{P}\text{-a.s., as } n \to \infty.$$

Hence, we get

$$\mathbb{E}_{\mathbf{P}}\left[\operatorname*{ess\,sup}_{\eta\in\mathcal{A}_{s,T}\cap L_b^0} \mathbb{E}_{\mathbf{P}}[\eta \cdot Z_T \mid \mathcal{F}_s]\right] = \mathbb{E}_{\mathbf{P}}\left[\lim_{n\to\infty} \mathbb{E}_{\mathbf{P}}[\eta_n \cdot Z_T \mid \mathcal{F}_s]\right].$$

We can assume without loss of generality that $\mathbb{E}_{\mathbf{P}}[\eta_n \cdot Z_T \mid \mathcal{F}_s] \geq 0$ for all $n \in \mathbb{N}$ as $0 \in \mathcal{A}_{s,T} \cap L_b^0$. By monotone convergence we obtain

$$\mathbb{E}_{\mathbf{P}}\left[\lim_{n\to\infty} \mathbb{E}_{\mathbf{P}}[\eta_n \cdot Z_T \mid \mathcal{F}_s]\right] = \lim_{n\to\infty} \mathbb{E}_{\mathbf{P}}[\eta_n \cdot Z_T] \leq \sup_{\eta\in\mathcal{A}_{s,T}\cap L_b^0} \mathbb{E}_{\mathbf{P}}[\eta \cdot Z_T],$$

and thus (2.5) is fulfilled. By linearity of the expectation and (2.5) we obtain that

$$\begin{aligned}&\left\{\xi \in L_b^0 : \mathbb{E}_{\mathbf{P}}[\xi \cdot Z_T \mid \mathcal{F}_s] \leq \operatorname*{ess\,sup}_{\eta\in\mathcal{A}_{s,T}\cap L_b^0} \mathbb{E}_{\mathbf{P}}[\eta \cdot Z_T \mid \mathcal{F}_s], \ \forall Z_T \in L^1(\mathcal{F}_T, \mathbf{P}; K_T^*)\right\} \\ &\subseteq \left\{\xi \in L_b^0 : \mathbb{E}_{\mathbf{P}}[\xi \cdot Z_T] \leq \sup_{\eta\in\mathcal{A}_{s,T}\cap L_b^0} \mathbb{E}_{\mathbf{P}}[\eta \cdot Z_T], \ \forall Z_T \in L^1(\mathcal{F}_T, \mathbf{P}; K_T^*)\right\}. \end{aligned} \tag{2.6}$$

Now, we apply Theorem 4.2 of [67] and (2.6) to get that

$$I := \left\{\xi \in L_b^0 : \mathbb{E}_{\mathbf{P}}[\xi \cdot Z_T \mid \mathcal{F}_s] \leq \operatorname*{ess\,sup}_{\eta\in\mathcal{A}_{s,T}\cap L_b^0} \mathbb{E}_{\mathbf{P}}[\eta \cdot Z_T \mid \mathcal{F}_s] \ \ \forall Z_T \in L^1(\mathcal{F}_T, \mathbf{P}; K_T^*)\right\} \subseteq \mathcal{A}_{s,T} \cap L_b^0. \tag{2.7}$$

It is left to show that (2.7) is also valid on $\mathcal{A}_{s,T} = \mathcal{A}_{s,T} \cap L_{1,\infty}^0$ and not only on $\mathcal{A}_{s,T} \cap L_b^0$. For this purpose, we show that L_b^0 is dense in $L_{1,\infty}^0$ with respect to $L^0(\mathcal{F}_s)$-Fatou-convergence. First, we note that $\mathcal{A}_{s,T}$ is $L^0(\mathcal{F}_s)$-Fatou closed by Corollary 1.23 (resp. Theorem 3.6 of [85]). Let $\xi \in L_{1,\infty}^0$, then $\xi \geq (-M_s^1, -M_s^2)$ for some $(M_s^1, M_s^2) \in L_+^1(\mathcal{F}_s, \mathcal{Q}) \times L_+^\infty(\mathcal{F}_s, \mathcal{Q})$. For $n \in \mathbb{N}$, we define $\xi_n := \xi^+ - (\xi^- \wedge n)$. Then, for all $n \in \mathbb{N}$ it holds that $\xi_n \in L_b^0$, $\xi_n \geq (-M_s^1, -M_s^2)$ and $\xi_n \xrightarrow{\mathbf{P}\text{-a.s.}} \xi$. In particular, $\xi_n \subset \mathcal{A}_{s,T} \cap L_b^0$ $L^0(\mathcal{F}_s)$-Fatou converges to $\xi \in \mathcal{A}_{s,T}$. Thus, we get

$$\operatorname*{ess\,sup}_{\eta\in\mathcal{A}_{s,T}\cap L_b^0} \mathbb{E}_{\mathbf{P}}[\eta Z_T \mid \mathcal{F}_s] = \operatorname*{ess\,sup}_{\eta\in\mathcal{A}_{s,T}} \mathbb{E}_{\mathbf{P}}[\eta Z_T \mid \mathcal{F}_s]. \tag{2.8}$$

Indeed, with similar arguments as above and by Theorem A.33 [40] there exist sequences $(\xi_n)_{n\in\mathbb{N}} \subset \mathcal{A}_{s,T}$ and $(\eta_n)_{n\in\mathbb{N}} \subset \mathcal{A}_{s,T} \cap L^0_b$ such that

$$\mathbb{E}_{\mathbf{P}}\left[\xi_n \cdot Z_T \mid \mathcal{F}_s\right] \uparrow \operatorname*{ess\,sup}_{\xi\in\mathcal{A}_{s,T}} \mathbb{E}_{\mathbf{P}}\left[\xi \cdot Z_T \mid \mathcal{F}_s\right], \ \mathbf{P}\text{-a.s., as } n \to \infty, \tag{2.9}$$

$$\mathbb{E}_{\mathbf{P}}\left[\eta_n \cdot Z_T \mid \mathcal{F}_s\right] \uparrow \operatorname*{ess\,sup}_{\eta\in\mathcal{A}_{s,T}\cap L^0_b} \mathbb{E}_{\mathbf{P}}\left[\eta \cdot Z_T \mid \mathcal{F}_s\right], \ \mathbf{P}\text{-a.s., as } n \to \infty. \tag{2.10}$$

$$\tag{2.11}$$

Hence (2.9) and (2.10) and the fact that $\mathcal{A}_{s,T} \cap L^0_b$ is dense in $\mathcal{A}_{s,T}$ with respect to $L^0(\mathcal{F}_s)$-Fatou convergence imply (2.8). Furthermore,

$$\bar{I} = \overline{\left\{\xi \in L^0_{1,\infty} : \mathbb{E}_{\mathbf{P}}\left[\xi \cdot Z_T \mid \mathcal{F}_s\right] \le \operatorname*{ess\,sup}_{\eta\in\mathcal{A}_{s,T}\cap L^0_b} \mathbb{E}_{\mathbf{P}}\left[\eta Z_T \mid \mathcal{F}_s\right] \ \forall Z_T \in L^1(\mathcal{F}_T, \mathbf{P}; K^*_T)\right\}}, \tag{2.12}$$

where the closure is taken with respect to $L^0(\mathcal{F}_s)$-Fatou convergence. We conclude the proof by

$$\bar{I} \subset \overline{\mathcal{A}_{s,T} \cap L^0_b} = \mathcal{A}_{s,T},$$

where the closure is taken with respect to the $L^0(\mathcal{F}_s)$-Fatou convergence. $\square$

2.2 Dynamic super-replication theorems

We start with the super-replication theorems for the numéraire-free setting. The numéraire-free version of the super-replication theorem, Theorem 2.6, is then used to prove the numéraire-based version, Theorem 2.7.

Theorem 2.6. *Let Assumption 1.5 hold and $s \in [0,T]$. Let $X_T = (X^1_T, X^2_T)$ be a contingent claim such that*

$$X^1_T + (X^2_T)^+(1-\lambda)S_T - (X^2_T)^-(1+\lambda)S_T \ge -M^1_s - M^2_s S_T, \tag{2.13}$$

for some $(M^1_s, M^2_s) \in L^1_+(\mathcal{F}_s, \mathcal{Q}) \times L^\infty_+(\mathcal{F}_s, \mathcal{Q})$. For a random variable $X_s = (X^1_s, X^2_s) \in L^1_+(\mathcal{F}_s, \mathcal{Q}) \times L^\infty_+(\mathcal{F}_s, \mathcal{Q})$ the following assertions are equivalent:

i) There is a self-financing trading strategy $\varphi = (\varphi^1_t, \varphi^2_t)_{s\le t\le T}$ with $\varphi_s = (X^1_s, X^2_s)$ and $\varphi_T = (X^1_T, X^2_T)$ which is admissible in the numéraire-free sense on the interval $[s,T]$, see (1.24).

ii) For every consistent price system $(\boldsymbol{Q}, \tilde{S}^{\boldsymbol{Q}}) \in CPS(s,T)$ we have

$$\mathbb{E}_{\boldsymbol{Q}}\left[X^1_T - X^1_s + (X^2_T - X^2_s)\tilde{S}^{\boldsymbol{Q}}_T \mid \mathcal{F}_s\right] \le 0. \tag{2.14}$$

Proof. $i) \Rightarrow ii)$: Let $\varphi = (\varphi_t^1, \varphi_t^2)_{s\le t\le T}$ be a strategy which is admissible in the numéraire-free sense such that $\varphi_s = (X_s^1, X_s^2)$ and $\varphi_T = (X_T^1, X_T^2)$. By Proposition 1.14, $(\varphi_t^1 + \varphi_t^2 \tilde{S}_t^{\mathbf{Q}})_{s\le t\le T}$ is an optional strong supermartingale for all $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}(s,T)$. Therefore, we obtain

$$\mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s\right] = \mathbb{E}_{\mathbf{Q}}\left[\varphi_T^1 + \varphi_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s\right] \le \varphi_s^1 + \varphi_s^2 \tilde{S}_s^{\mathbf{Q}} = X_s^1 + X_s^2 \tilde{S}_s^{\mathbf{Q}}.$$

By the $\mathbf{Q}$-martingale property of $\tilde{S}^{\mathbf{Q}}$ and measurability we have

$$X_s^1 + X_s^2 \tilde{S}_s^{\mathbf{Q}} = X_s^1 + X_s^2 \mathbb{E}_{\mathbf{Q}}\left[\tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s\right] = \mathbb{E}_{\mathbf{Q}}\left[X_s^1 + X_s^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s\right].$$

Hence, we get

$$\mathbb{E}_{\mathbf{Q}}\left[X_T^1 - X_s^1 + (X_T^2 - X_s^2)\tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s\right] \le 0$$

for all $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}(s,T)$.

$ii) \Rightarrow i)$: It is sufficient to prove the assertion for strategies with zero initial endowments. Indeed, for any contingent claim X_T, there is an admissible strategy in the numéraire-free sense φ with $\varphi_s = (X_s^1, X_s^2)$ and $\varphi_T = (X_T^1, X_T^2)$ if and only if $\tilde{X}_T := (X_T^1 - X_s^1, X_T^2 - X_s^2) \in \mathcal{A}_{s,T}$, i.e., if there is an admissible strategy in the numéraire-free sense $\tilde{\varphi}$ with $\tilde{\varphi}_s = (0,0)$ and $\tilde{\varphi}_T = (\tilde{X}_T^1, \tilde{X}_T^2)$. Instead of directly proving that "$ii) \Rightarrow i)$" we prove that "$\neg i) \Rightarrow \neg ii)$". More precisely, if $\tilde{X}_T \notin \mathcal{A}_{s,T}$, then there exists a consistent price system in the non-local sense $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}(s,T)$ such that

$$\mathbf{P}(\mathbb{E}_{\mathbf{Q}}[\tilde{X}_T^1 + \tilde{X}_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s] > 0) > 0. \tag{2.15}$$

By Theorem 2.5 there exists $Y = (Y^1, Y^2) \in L^1(\mathcal{F}_T, \mathbf{P}; K_T^*)$ such that

$$\mathbf{P}\left(B^Y\right) > 0, \tag{2.16}$$

where

$$B^Y := \left\{\omega \in \Omega : \ \mathbb{E}_{\mathbf{P}}\left[Y^1 \tilde{X}_T^1 + Y^2 \tilde{X}_T^2 \mid \mathcal{F}_s\right](\omega) > \operatorname*{ess\,sup}_{\eta \in \mathcal{A}_{s,T}} \mathbb{E}_{\mathbf{P}}\left[\eta^1 Y^1 + \eta^2 Y^2 \mid \mathcal{F}_s\right](\omega)\right\} \in \mathcal{F}_s. \tag{2.17}$$

We now construct a consistent price system $(\widehat{\mathbf{Q}}, \widehat{S}^{\widehat{\mathbf{Q}}}) \in \mathrm{CPS}(s,T)$ such that (2.15) is fulfilled. Using Proposition 1.3 we represent $(\widehat{\mathbf{Q}}, \widehat{S}^{\widehat{\mathbf{Q}}})$ by $\hat{Z} = (\hat{Z}_t^1, \hat{Z}_t^2)_{s\le t\le T}$, where $\hat{Z}^i$ is a $\mathbf{P}$-martingale on $[s,T]$ for $i = 1,2$ and $\hat{Z}_t^2/\hat{Z}_t^1$ takes values in the bid-ask spread. Recall, that taking values in the bid-ask spread is equivalent for (Z_t^1, Z_t^2) taking almost surely values in $K_t^* \backslash \{0\}$, where $K_t^* = K_t^*(\omega)$ depends on $\omega \in \Omega$.

For this purpose, we define the process $Z = (Z_t^1, Z_t^2)_{s\le t\le T}$ by $Z_t^i := \mathbb{E}_{\mathbf{P}}\left[Y^i \mathbb{1}_{B^Y} \mid \mathcal{F}_t\right]$, $s \le t \le T$, $i = 1,2$, where $Y = (Y^1, Y^2)$ is given by (2.16) and (2.17). Since $0 \in \mathcal{A}_{s,T}$, we know that

$$\operatorname*{ess\,sup}_{\eta \in \mathcal{A}_{s,T}} \mathbb{E}_{\mathbf{P}}\left[\eta^1 Y^1 + \eta^2 Y^2 \mid \mathcal{F}_s\right] \ge 0.$$

Thus, we follow from (2.16) and (2.17), that

$$\mathbf{P}\left(\mathbb{E}_{\mathbf{P}}\left[Z_T^1 \tilde{X}_T^1 + Z_T^2 \tilde{X}_T^2 \mid \mathcal{F}_s\right] > 0\right) \ge \mathbf{P}\left(B^Y\right) > 0.$$

To this end, we show that $Z_t \in K_t^*$ a.s. for all $t \in [s,T]$. In particular, if Z was $\mathbb{R}_+ \backslash \{0\}$-valued it is a consistent price system in the non-local sense. This detail will be encountered at the end of the proof.
Consider the process

$$\psi_u := -\nu\gamma \mathbb{1}_{]t,T]}(u), \quad u \in [s,T], \tag{2.18}$$

for some $t \in [s,T]$ and arbitrary random variables $\nu \in L_+^\infty(\mathcal{F}_t, \mathbf{P})$ and $\gamma \in L^\infty(\mathcal{F}_t, \mathbf{P}; K_T)$. As $\psi \in L^\infty(\mathcal{F}_T, \mathbf{P})$, ψ is an admissible strategy in the numéraire-free sense. Because $\mathbb{F}$ is right-continuous by the usual hypothesis, ψ_T is $\mathcal{F}_t$-measurable by definition. By the tower-property we obtain that

$$\begin{aligned} \mathbb{E}_\mathbf{P}\left[\psi_T \cdot Y \mathbb{1}_{B^Y} \mid \mathcal{F}_s\right] &= \mathbb{E}_\mathbf{P}\left[\left(-\nu\gamma\mathbb{1}_{]t,T]}(T)\right) \cdot Z_T \mid \mathcal{F}_s\right] \\ &= \mathbb{E}_\mathbf{P}\left[\mathbb{E}_\mathbf{P}\left[\left(-\nu\gamma\mathbb{1}_{]t,T]}(T)\right) \cdot Z_T \mid \mathcal{F}_t\right] \mid \mathcal{F}_s\right] \\ &= -\mathbb{E}_\mathbf{P}\left[\nu\gamma \cdot \mathbb{E}_\mathbf{P}\left[Z_T \mid \mathcal{F}_t\right] \mid \mathcal{F}_s\right] \\ &= -\mathbb{E}_\mathbf{P}\left[\nu\gamma \cdot Z_t \mid \mathcal{F}_s\right]. \end{aligned} \tag{2.19}$$

Since $\psi_T \in \mathcal{A}_{s,T}$, (2.17) and (2.19) yield for $\omega \in B^Y$ that

$$\mathbb{E}_\mathbf{P}\left[\nu\gamma \cdot Z_t \mid \mathcal{F}_s\right](\omega) > -\mathbb{E}_\mathbf{P}\left[\left(Y \cdot \tilde{X}_T\right)\mathbb{1}_{B^Y} \mid \mathcal{F}_s\right](\omega) = -\left(\mathbb{E}_\mathbf{P}\left[\left(Y \cdot \tilde{X}_T\right) \mid \mathcal{F}_s\right]\mathbb{1}_{B^Y}\right)(\omega), \tag{2.20}$$

where we used that $B^Y \in \mathcal{F}_s$. Further, for $\omega \in (B^Y)^c$ we get

$$\mathbb{E}_\mathbf{P}\left[\nu\gamma \cdot Z_t \mid \mathcal{F}_s\right](\omega) = -\left(\mathbb{E}_\mathbf{P}\left[\psi_T \cdot Y \mid \mathcal{F}_s\right]\mathbb{1}_{B^Y}\right)(\omega) = 0.$$

Because ν was arbitrary, we can deduce that $Z_t \cdot \gamma \geq 0$ for all $\gamma \in L^\infty(\mathcal{F}_t, \mathbf{P}; K_t)$. This implies that $Z_t \in K_t^*$ a.s. In particular, Z is a $\mathbf{P}$-martingale satisfying $Z_t \in K_t^*$ for all $t \in [s,T]$.
It is still possible that $\mathbf{P}(Z_t = 0) > 0$ for some $t \in [s,T]$ and thus Z is not necessarily a consistent price systems. We now construct the desired consistent price system $\hat{Z} = (\hat{Z}_t^1, \hat{Z}_t^2)_{t\in[s,T]}$ as follows. For this purpose, take any consistent price system in the non-local sense $\tilde{Z} = (\tilde{Z}_t^1, \tilde{Z}_t^2)_{s\leq t\leq T}$. Then for suitable $\beta \in L^\infty(\mathcal{F}_s, \mathbf{P}; (0,1])$ we have that $\hat{Z}_t^i := (\beta\tilde{Z}_t^i + (1-\beta)Z_t^i) \in K_t^* \backslash \{0\}$, $t \in [s,T]$, $i = 1,2$ and satisfies

$$\mathbf{P}\left(\mathbb{E}_\mathbf{P}\left[\hat{Z}_T^1\tilde{X}_T^1 + \hat{Z}_T^2\tilde{X}_T^2 \mid \mathcal{F}_s\right] > 0\right) > 0. \tag{2.21}$$

For instance, we can define β by

$$\beta(\omega) := \begin{cases} 1, & \omega \in (B^Y)^c, \\ \frac{\mathbb{E}_\mathbf{P}\left[Z_T \cdot \tilde{X}_T \mid \mathcal{F}_s\right]}{\left|\mathbb{E}_\mathbf{P}\left[\tilde{Z}_T \cdot \tilde{X}_T \mid \mathcal{F}_s\right]\right| + \mathbb{E}_\mathbf{P}\left[Z_T \cdot \tilde{X}_T \mid \mathcal{F}_s\right]}, & \omega \in B^Y. \end{cases}$$

Then, $\beta \in L^\infty(\mathcal{F}_s, \mathbf{P}; (0,1])$ is well-defined, because $\mathbb{E}_\mathbf{P}\left[X_T Z_T \mid \mathcal{F}_s\right](\omega) > 0$ for $\omega \in B^Y$. Clearly, $\hat{Z}$ is still a consistent price system in the non-local sense on $[s,T]$.
For $X_T \notin \mathcal{A}_{s,T}$ we have constructed a consistent price system in the non-local sense $\hat{Z} = (\hat{Z}_t^1, \hat{Z}_t^2)_{s\leq t\leq T}$ satisfying $\mathbf{P}(\mathbb{E}_\mathbf{P}[\hat{Z}_T^1 X_T^1 + \hat{Z}_T^2 X_T^2 \mid \mathcal{F}_s] > 0) > 0$. This concludes the proof. □

Now we can also prove the local or numéraire-based version of the super-replication theorem. The arguments in the proof of Theorem 2.7 follow closely the proof of Theorem 1.4 of [85].

Theorem 2.7. *Let Assumption 1.4 hold and $s \in [0,T]$. Let $X_T = (X_T^1, X_T^2)$ be a contingent claim such that*

$$X_T^1 + (X_T^2)^+(1-\lambda)S_T - (X_T^2)^-(1+\lambda)S_T \geq -M_s \tag{2.22}$$

for some $M_s \in L^1_+(\mathcal{F}_s, \mathcal{Q}_{loc})$. For a random variable $X_s = (X_s^1, 0) \in L^1(\mathcal{F}_s, \mathcal{Q}_{loc})$ the following assertions are equivalent:

i) There is a self-financing trading strategy $\varphi = (\varphi_t^1, \varphi_t^2)_{s \le t \le T}$ with $\varphi_s = (X_s^1, 0)$ and $\varphi_T = (X_T^1, X_T^2)$ which is admissible in the numéraire-based sense on the interval $[s,T]$, see (1.23).

ii) For every consistent price system $(\boldsymbol{Q}, \tilde{S}^{\boldsymbol{Q}}) \in CPS_{loc}(s,T)$ we have

$$\mathbb{E}_{\boldsymbol{Q}}\left[X_T^1 - X_s^1 + X_T^2 \tilde{S}_T^{\boldsymbol{Q}} \mid \mathcal{F}_s\right] \leq 0. \tag{2.23}$$

Proof. For sake of notational convenience, we write $X = (X^1, X^2) := (X_T^1, X_T^2)$ and assume that X satisfies (2.22).
$i) \Rightarrow ii)$: Let $\varphi = (\varphi_t^1, \varphi_t^2)_{s \le t \le T}$ be a strategy, which is admissible in the numéraire-based sense such that $\varphi_s = (X_s, 0)$ and $\varphi_T = (X^1, X^2)$. Then Proposition 1.13 guarantees that $(\varphi_t^1 + \varphi_t^2 \tilde{S}_t^{\mathbf{Q}})_{s \le t \le T}$ is an optional strong supermartingale under $\mathbf{Q}$ for all $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in$ $\mathrm{CPS}_{\mathrm{loc}}(s,T)$. Therefore, we obtain

$$\mathbb{E}_{\mathbf{Q}}\left[X^1 + X^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s\right] = \mathbb{E}_{\mathbf{Q}}\left[\varphi_T^1 + \varphi_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s\right] \leq \varphi_s^1 + \varphi_s^2 \tilde{S}_s^{\mathbf{Q}} = X_s^1, \tag{2.24}$$

for all $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}_{\mathrm{loc}}(s,T)$ and hence

$$\mathbb{E}_{\mathbf{Q}}\left[X_T^1 - X_s^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s\right] \leq 0, \quad \text{for all } (\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}_{\mathrm{loc}}(s,T).$$

$ii) \Rightarrow i)$: Suppose now that $X = (X^1, X^2)$ fulfills (2.22). Assume without loss of generality that $ii)$ holds for $X_s^1 = 0$, i.e.,

$$\mathbb{E}_{\mathbf{Q}}\left[X^1 + X^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s\right] \leq 0, \quad \text{for all } (\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}_{\mathrm{loc}}(s,T). \tag{2.25}$$

Indeed, this is no restriction as by shifting, we could simply consider $(X^1 - X_s^1, X^2)$ auch that (2.22) is still fulfilled. We construct a strategy $\varphi = (\varphi_t^1, \varphi_t^2)_{t \in [s,T]}$, admissible in the numéraire-based sense, with $\varphi_s = (0,0)$ and $\varphi_T = (X^1, X^2)$. Define the $[s,T]$-valued stopping time τ_n by

$$\tau_n := \inf\{t \geq s \mid S_t \geq n\} \wedge T.$$

Because X is an $\mathcal{F}_T$-measurable claim, we define

$$X_n = (X_n^1, X_n^2) = \begin{cases} (X^1, X^2), & \text{on } \{\tau_n = T\}, \\ (-M_s, 0), & \text{on } \{\tau_n < T\}, \end{cases} \tag{2.26}$$

so that X_n is $\mathcal{F}_{\tau_n}$-measurable. Equation (2.22) and the definition of X_n in (2.26) imply that

$$X_n^1 + \left(X_n^2\right)^+ (1-\lambda)S_T - \left(X_n^2\right)^- (1+\lambda)S_T \geq -M_s. \tag{2.27}$$

Further, we have that $(X_n^1 + X_n^2 \tilde{S}_{\tau_n}^{\mathbf{Q}})$ is increasing and converges $\mathbf{P}$-a.s. to $(X^1 + X^2 \tilde{S}_T^{\mathbf{Q}})$, as n tends to infinity.

By Assumption 1.4 there exists a λ'-consistent local price system for S for every $0 < \lambda' < \lambda$. For a consistent local price system $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}_{\mathrm{loc}}(s,T)$ of S, Proposition 6.1 of [85] guarantees that $(\mathbf{Q}, (\tilde{S}^{\mathbf{Q}})^{\tau_n}) \in \mathrm{CPS}_{\mathrm{loc}}(s,\tau_n)$ is a consistent price system in the non-local sense for S^{τ_n}, $n \in \mathbb{N}$. Thus, Assumption 1.5 is fulfilled for S^{τ_n} for each $n \in \mathbb{N}$. Note that, by (2.27) X_n also fulfills (2.13) for every $n \in \mathbb{N}$. Thus, we can apply Theorem 2.6 for S^{τ_n} and the claim X_n.

By Lemma 1.6 and Corollary 1.8, we get that for every $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}(s,\tau_n,\lambda)$ there exists $(\widehat{\mathbf{Q}}, \widehat{S}^{\widehat{\mathbf{Q}}}) \in \mathrm{CPS}_{\mathrm{loc}}(\tau_n, T, \lambda)$ such that $\mathbf{Q}|_{\mathcal{F}_{\tau_n}} = \widehat{\mathbf{Q}}|_{\mathcal{F}_{\tau_n}}$ and $\tilde{S}_{\tau_n}^{\mathbf{Q}} = \widehat{S}_{\tau_n}^{\widehat{\mathbf{Q}}}$. Concatenating $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}})$ and $(\widehat{\mathbf{Q}}, \widehat{S}^{\widehat{\mathbf{Q}}})$ yields a consistent local price system $(\widehat{\mathbf{Q}}, \widehat{S}^{\widehat{\mathbf{Q}}}) \in \mathrm{CPS}_{\mathrm{loc}}(s,T,\lambda)$ satisfying $\mathbf{Q}|_{\mathcal{F}_{\tau_n}} = \overline{\mathbf{Q}}|_{\mathcal{F}_{\tau_n}}$ and $\tilde{S}_t^{\mathbf{Q}} = \bar{S}_t^{\bar{\mathbf{Q}}}$ for all $t \in [\![s, \tau_n]\!]$. More precisely, we define

$$\bar{S}_t^{\bar{\mathbf{Q}}} := \begin{cases} \tilde{S}_t^{\mathbf{Q}}, & t \in [\![s, \tau_n]\!], \\ \widehat{S}_t^{\widehat{\mathbf{Q}}}, & t \in [\![\tau_n, T]\!], \end{cases}$$

and

$$\frac{d\overline{\mathbf{Q}}}{d\mathbf{P}} := \frac{\frac{d\mathbf{Q}}{d\mathbf{P}}}{\mathbb{E}_{\mathbf{P}}\left[\frac{d\widehat{\mathbf{Q}}}{d\mathbf{P}} \mid \mathcal{F}_{\tau_n}\right]} \frac{d\widehat{\mathbf{Q}}}{d\mathbf{P}}.$$

By the construction of $(\overline{\mathbf{Q}}, \bar{S}^{\bar{\mathbf{Q}}})$ and X_n in (2.26) and (2.25) (resp. (2.23)), we obtain

$$\mathbb{E}_{\mathbf{Q}}\left[X_n^1 + X_n^2 \tilde{S}_{\tau_n}^{\mathbf{Q}} \mid \mathcal{F}_s\right] = \mathbb{E}_{\overline{\mathbf{Q}}}\left[X_n^1 + X_n^2 \bar{S}_{\tau_n}^{\bar{\mathbf{Q}}} \mid \mathcal{F}_s\right] \leq \mathbb{E}_{\overline{\mathbf{Q}}}\left[X^1 + X^2 \bar{S}_T^{\bar{\mathbf{Q}}} \mid \mathcal{F}_s\right] \leq 0 \quad n \in \mathbb{N}. \tag{2.28}$$

Since $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}(s,\tau_n)$ was arbitrary, by Theorem 2.6 there exists a λ-self-financing strategy $(\tilde{\varphi}_t^n)_{s \leq t \leq \tau_n}$ for S^{τ_n} such that $\tilde{\varphi}_{\tau_n}^n = X_n$, $n \in \mathbb{N}$, and which is admissible in the numéraire-free sense for some $\tilde{M}_s = (\tilde{M}_s^1, \tilde{M}_s^2) \in L_+^1(\mathcal{F}_s, \mathcal{Q}) \times L_+^\infty(\mathcal{F}_s, \mathcal{Q})$. We define the strategy for the desired time interval $[s,T]$ with no trading after time τ_n given by $\varphi^n = (\varphi_t^n)_{s \leq t \leq T}$ with $\varphi_t^n = \tilde{\varphi}_t^n$ for all $t \in [\![s, \tau_n]\!]$ and $\varphi_t^n = \tilde{\varphi}_{\tau_n}^n$ for all $t \in [\![\tau_n, T]\!]$. Thus, φ^n is a self-financing strategy for S which is $(\tilde{M}_s^1, \tilde{M}_s^2)$-admissible in the numéraire-free sense for all $n \in \mathbb{N}$. By Corollary 1.16 (see also Theorem 1 of [84]), we get that each φ^n is also M_s-admissible in the numéraire-based sense. To conclude the proof, we apply Corollary 1.22 (see also Theorem 3.4 of [85]) and Remark 3.5 of [85] to obtain desired self-financing strategy φ as a limit of $(\varphi^n)_{n=1}^\infty$. Recall that Corollary 1.22 and Remark 3.5 of [85] guarantees, that the set $A_{s,T}^{M_s,\mathrm{loc}}$ is not only closed in the topology of convergence in measure but also fulfills a convex compactness property. This strategy φ has the required properties. □

The dualities of the Theorems 2.6 and 2.7 can be formulated in the following way.

Proposition 2.8. *Let Assumption 1.4 hold and $s \in [0,T]$. Let $X_T = (X_T^1, X_T^2)$ be a contingent claim such that*

$$X_T^1 + (X_T^2)^+(1-\lambda)S_T - (X_T^2)^-(1+\lambda)S_T \geq -M_s \tag{2.29}$$

for some $M_s \in L^1_+(\mathcal{F}_s, \mathcal{Q}_{loc})$. *If*

$$\operatorname*{ess\,sup}_{(\boldsymbol{Q},\tilde{S}^{\boldsymbol{Q}})\in CPS_{loc}(s,T)} \mathbb{E}_{\boldsymbol{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\boldsymbol{Q}} \mid \mathcal{F}_s\right] \in L^1(\mathcal{F}_s, \mathcal{Q}_{loc}),$$

then we have

$$\begin{aligned}&\operatorname{ess\,inf}\left\{\xi_s^1 \in L^1_+(\mathcal{F}_s, \mathcal{Q}_{loc}) : \exists \varphi \in \mathcal{V}^{loc}_{s,T}(\xi_s, \lambda) \text{ with } \varphi_s = (\xi_s, 0),\ \varphi_T = X_T\right\}\\ =& \operatorname*{ess\,sup}_{(\boldsymbol{Q},\tilde{S}^{\boldsymbol{Q}})\in CPS_{loc}(s,T)} \mathbb{E}_{\boldsymbol{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\boldsymbol{Q}} \mid \mathcal{F}_s\right].\end{aligned} \tag{2.30}$$

Proof. We can apply Theorem 2.7 because X_T satisfies (2.29). Hence, we obtain

$$\begin{aligned}&\left\{X_s \in L^1(\mathcal{F}_s, \mathcal{Q}_{\text{loc}}) :\ \exists \varphi \in \mathcal{V}^{\text{loc}}_{s,T}(X_s, \lambda) \text{ with } \varphi_s = (X_s, 0) \text{ and } \varphi_T = (X_T^1, X_T^2)\right\}\\ =& \underbrace{\left\{X_s \in L^1(\mathcal{F}_s, \mathcal{Q}_{\text{loc}}) :\ \mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s\right] \le X_s,\ \forall (\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \text{CPS}_{\text{loc}}(s,T)\right\}}_{=:D_s}\end{aligned}$$

It is left to show that

$$\operatorname{ess\,inf} D_s = \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in \text{CPS}_{\text{loc}}(s,T)} \mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s\right].$$

For the first direction "≤", we note that $\operatorname{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in \text{CPS}_{\text{loc}}(s,T)} \mathbb{E}_{\mathbf{Q}}[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s] \in D_s$, because $\operatorname{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in \text{CPS}_{\text{loc}}(s,T)} \mathbb{E}_{\mathbf{Q}}[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s] \in L^1(\mathcal{F}_s, \mathcal{Q}_{\text{loc}})$.
For the reverse direction "≥", we have that $\operatorname{ess\,inf} D_s \ge \mathbb{E}_{\mathbf{Q}}[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s]$ for all $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \text{CPS}_{\text{loc}}(s,T)$ which implies by the definition of the essential supremum that

$$\operatorname{ess\,inf} D_s \ge \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in \text{CPS}_{\text{loc}}(s,T)} \mathbb{E}_{\mathbf{Q}}[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s].$$

□

Proposition 2.9. *Let Assumption 1.5 hold* $s \in [0,T]$. *Let* $X_T = (X_T^1, X_T^2)$ *be a contingent claim such that*

$$X_T^1 + (X_T^2)^+(1-\lambda)S_T - (X_T^2)^-(1+\lambda)S_T \ge -M_s^1 - M_s^2 S_\tau \tag{2.31}$$

for some $(M_s^1, M_s^2) \in L^1_+(\mathcal{F}_s, \mathcal{Q}) \times L^\infty_+(\mathcal{F}_s, \mathcal{Q})$. *If*

$$\operatorname*{ess\,sup}_{(\boldsymbol{Q},\tilde{S}^{\boldsymbol{Q}})\in CPS_{loc}(s,T)} \mathbb{E}_{\boldsymbol{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\boldsymbol{Q}} \mid \mathcal{F}_s\right] \in L^1(\mathcal{F}_s, \mathcal{Q}),$$

then we have

$$\begin{aligned}&\operatorname{ess\,inf}\{\xi_s \in L_+(\mathcal{F}_s, \mathcal{Q}) : \exists \varphi \in \mathcal{V}_{s,T}(\xi, \lambda) \text{ with } \varphi_s = (\xi_s, 0)\ \varphi_T = X_T\}\\ =& \operatorname*{ess\,sup}_{(\boldsymbol{Q},\tilde{S}^{\boldsymbol{Q}})\in CPS(s,T)} \mathbb{E}_{\boldsymbol{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\boldsymbol{Q}} \mid \mathcal{F}_s\right].\end{aligned} \tag{2.32}$$

Proof. We obtain (2.32) with the same arguments as in the proof of Proposition 2.8. □

2.3 Further properties

From now on we assume that $\mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2\tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s\right] \in L^1(\mathcal{F}_s, \mathcal{Q}_{\text{loc}})$ for all $s \in [0,T]$ in the local setting, i.e., under Assumption 1.4, and $\mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2\tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s\right] \in L^1(\mathcal{F}_s, \mathcal{Q})$ for all $s \in [0,T]$ in the non-local setting, i.e., under Assumption 1.5.
In this section we prove what we like to call *time independence* of the consistent (local) price systems in the dual representation, see Theorems 2.11 and 2.12. Further, we prove that the super-replication process is a well-defined process. Afterwards, we provide sufficient conditions such that the super-replication process is càdlàg, see Theorems 2.19 and 2.21. We start with a preparatory lemma for the proof of the Theorems 2.11 and 2.12.

Lemma 2.10. *Let Assumption 1.4 hold and $s \in [0,T]$. Let $\bar{\lambda} < \lambda$ and $(\boldsymbol{Q}, \tilde{S}^{\boldsymbol{Q}}) \in SCPS_{loc}(s,T,\lambda)$ be a strictly consistent local price system for S on $[s,T]$ which satisfies*

$$(1-\bar{\lambda})S_s \leq \tilde{S}_s^{\boldsymbol{Q}} \leq (1+\bar{\lambda})S_s. \tag{2.33}$$

Let $X_T = (X_T^1, X_T^2)$ be a contingent claim such that

$$X_T^1 + (X_T^2)^+(1-\lambda)S_T - (X_T^2)^-(1+\lambda)S_T \geq -M_s$$

for some $M_s \in L_+^1(\mathcal{F}_s, \mathcal{Q}_{loc})$. Then there exists a consistent local price system $(\overline{\boldsymbol{Q}}, \bar{S}^{\bar{\boldsymbol{Q}}}) \in CPS_{loc}(0,T,\lambda)$ such that

$$\mathbb{E}_{\boldsymbol{Q}}\left[X_T^1 + X_T^2\tilde{S}_T^{\boldsymbol{Q}} \mid \mathcal{F}_s\right] = \mathbb{E}_{\overline{\boldsymbol{Q}}}\left[X_T^1 + X_T^2\bar{S}_T^{\bar{\boldsymbol{Q}}} \mid \mathcal{F}_s\right].$$

Proof. Let $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \text{SCPS}_{\text{loc}}(s,T,\lambda)$ with associated (local) $\mathbf{P}$-martingales Z^1, Z^2 as in Proposition 1.3 and assume $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}})$ satisfies (2.33). By Lemma 1.6 and Corollary 1.8 there exists $(\widehat{\mathbf{Q}}, \widehat{S}^{\widehat{\mathbf{Q}}}) \in \text{CPS}_{\text{loc}}(0,T,\lambda)$ such that $\widehat{S}_s^{\widehat{\mathbf{Q}}} = \tilde{S}_s^{\mathbf{Q}}$. Let $\widehat{Z}^1$, $\widehat{Z}^2$ be the associated (local) $\mathbf{P}$-martingales of $(\widehat{\mathbf{Q}}, \widehat{S}^{\widehat{\mathbf{Q}}})$. We construct a new λ-consistent price system $(\overline{\mathbf{Q}}, \bar{S}^{\bar{\mathbf{Q}}}) \in \text{CPS}_{\text{loc}}(0,T,\lambda)$ by its associated (local) $\mathbf{P}$-martingales $\bar{Z}^1$, $\bar{Z}^2$ by

$$\bar{Z}_t^i := \begin{cases} \widehat{Z}_t^i, & 0 \leq t \leq s, \\ Z_t^i \frac{\widehat{Z}_s^i}{Z_s^i}, & s \leq t \leq T, \end{cases}$$

for $i = 1,2$. First, we show that $\bar{Z}^1$, $\bar{Z}^2$ fulfill the desired properties of Proposition 1.3 to ensure that $\bar{Z}^1$, $\bar{Z}^2$ define a consistent local price systems. Let $0 \leq t \leq u \leq T$, then

$$\mathbb{E}_{\mathbf{P}}\left[\bar{Z}_u^1 \mid \mathcal{F}_t\right] = \begin{cases} \mathbb{E}_{\mathbf{P}}[\widehat{Z}_u^1 \mid \mathcal{F}_t] = \widehat{Z}_t^1 = Z_t^1, & 0 \leq u \leq s, \\ \mathbb{E}_{\mathbf{P}}\left[Z_u^1 \frac{\widehat{Z}_s^1}{Z_s^1} \mid \mathcal{F}_t\right] = \frac{\widehat{Z}_s^1}{Z_s^1}\mathbb{E}_{\mathbf{P}}[Z_u^1 \mid \mathcal{F}_t] = \frac{\widehat{Z}_s^1}{Z_s^1} Z_t^1 = Z_t^1, & s \leq t \leq u, \\ \mathbb{E}_{\mathbf{P}}\left[Z_u^1 \frac{\widehat{Z}_s^1}{Z_s^1} \mid \mathcal{F}_t\right] = \mathbb{E}_{\mathbf{P}}\left[\frac{\widehat{Z}_s^1}{Z_s^1}\mathbb{E}_{\mathbf{P}}[Z_u^1 \mid \mathcal{F}_s] \mid \mathcal{F}_t\right] = \mathbb{E}_{\mathbf{P}}[\widehat{Z}_s \mid \mathcal{F}_t] = Z_t^1, & t \leq s \leq u. \end{cases}$$

Note that integrability with respect to $\mathbf{P}$ of $\widehat{Z}^1$ follows in the same way since $\widehat{Z}^1 \geq 0$. Thus, $\bar{Z}^1$ is a $\mathbf{P}$-martingale on $[0,T]$. Let $(\tau_n)_{n\in\mathbb{N}}$, $(\widehat{\tau}_n)_{n\in\mathbb{N}}$ denote localizing sequences for Z^2, $\widehat{Z}^2$, respectively. Define $(\bar{\tau}_n)_{n\in\in\mathbb{N}}$ by $\bar{\tau}_n = \tau_n \wedge \widehat{\tau}_n$, $n \in \mathbb{N}$. Note that $(\bar{\tau}_n)_{n\in\mathbb{N}}$ is also a localizing sequence for Z^2 and $\widehat{Z}^2$. We claim that $(\bar{\tau}_n)_{n\in\mathbb{N}}$ is a localizing sequence for $\bar{Z}^2$ which makes $\bar{Z}^2$ a local $\mathbf{P}$-martingale. With similar arguments as above we distinguish

three different cases.
Case 1: Let $t \le (u \wedge \bar\tau_n) \le s$, then

$$\mathbb{E}_{\mathbf{P}}\left[(\bar Z_u^2)^{\bar\tau_n} \mid \mathcal{F}_t\right] = \mathbb{E}_{\mathbf{P}}[(\widehat Z_u^2)^{\bar\tau_n} \mid \mathcal{F}_t] = (\widehat Z_t^2)^{\bar\tau_n} = (Z_t^2)^{\bar\tau_n}.$$

Case 2: Let $s \le t \le (u \wedge \bar\tau_n)$, then

$$\mathbb{E}_{\mathbf{P}}\left[(\bar Z_u^2)^{\bar\tau_n} \mid \mathcal{F}_t\right] = \mathbb{E}_{\mathbf{P}}\left[\left(Z_u^2\right)^{\bar\tau_n}\frac{\widehat Z_s^2}{Z_s^2} \mid \mathcal{F}_t\right] = \frac{\widehat Z_s^2}{Z_s^2}\mathbb{E}_{\mathbf{P}}[(Z_u^2)^{\bar\tau_n} \mid \mathcal{F}_t] = \frac{\widehat Z_s^2}{Z_s^2}\left(Z_t^2\right)^{\bar\tau_n} = (\bar Z_t^2)^{\bar\tau_n}.$$

Case 3: Let $t \le s \le (u \wedge \bar\tau_n)$, then

$$\begin{aligned}\mathbb{E}_{\mathbf{P}}\left[(\bar Z_u^2)^{\bar\tau_n} \mid \mathcal{F}_t\right] &= \mathbf{P}\left[\left(Z_u^2\right)^{\bar\tau_n}\frac{\widehat Z_s^2}{Z_s^2} \mid \mathcal{F}_t\right] = \mathbb{E}_{\mathbf{P}}\left[\frac{\widehat Z_s^2}{Z_s^2}\mathbb{E}_{\mathbf{P}}[(Z_u^2)^{\bar\tau_n} \mid \mathcal{F}_s] \mid \mathcal{F}_t\right]\\ &= \mathbb{E}_{\mathbf{P}}[(\widehat Z_s^2)^{\bar\tau_n} \mid \mathcal{F}_t] = (\widehat Z_t^2)^{\bar\tau_n} = (Z_t^2)^{\bar\tau_n}.\end{aligned}$$

In particular, $\bar Z^2$ is a local $\mathbf{P}$-martingale with localizing sequence $(\bar\tau_n)_{n\in\mathbb{N}}$. Furthermore, $(\bar Z_t^2/\bar Z_t^1)_{t\in[0,T]}$ lies in the bid-ask spread. For $t \le s$ the assertion is clear. For $s \le t$ we get

$$\frac{\bar Z_t^2}{\bar Z_t^1} = \frac{Z_t^2}{Z_t^1}\cdot\frac{\widehat Z_s^2}{\widehat Z_s^1}\cdot\frac{Z_s^1}{Z_s^2} = \tilde S_t^{\mathbf{Q}}\cdot\frac{\widehat S_s^{\widehat{\mathbf{Q}}}}{\tilde S_s^{\mathbf{Q}}} = \tilde S_t^{\mathbf{Q}} \in [(1-\lambda)S_t, (1+\lambda)S_t],$$

where we used that $\tilde S_s^{\mathbf{Q}} = \widehat S_s^{\widehat{\mathbf{Q}}}$. Therefore, $(\bar Z^1, \bar Z^2)$ define a consistent local price system $(\overline{\mathbf{Q}}, \bar S^{\bar{\mathbf{Q}}}) \in \mathrm{CPS}_{\mathrm{loc}}(0,T,\lambda)$ by Proposition 1.3. It is left to show that

$$\mathbb{E}_{\overline{\mathbf{Q}}}\left[X_T^1 + X_T^2\bar S_T^{\bar{\mathbf{Q}}} \mid \mathcal{F}_s\right] = \mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2\tilde S_T^{\mathbf{Q}} \mid \mathcal{F}_s\right].$$

Indeed, we get

$$\begin{aligned}\mathbb{E}_{\overline{\mathbf{Q}}}\left[X_T^1 + X_T^2\bar S^{\bar{\mathbf{Q}}} \mid \mathcal{F}_s\right] &= \mathbb{E}_{\mathbf{P}}\left[X_T^1\bar Z_T^1 + X_T^2\bar Z_T^2 \mid \mathcal{F}_s\right](\bar Z_s^1)^{-1}\\ &= \mathbb{E}_{\mathbf{P}}\left[X_T^1 Z_T^1\frac{\widehat Z_s^1}{Z_s^1} + X_T^2 Z_T^2\frac{\widehat Z_s^2}{Z_s^2} \mid \mathcal{F}_s\right](\widehat Z_s^1)^{-1}\\ &= \mathbb{E}_{\mathbf{P}}\left[X_T^1 Z_T^1 + X_T^2 Z_T^2\frac{\widehat Z_s^2}{Z_s^2}\frac{Z_s^1}{\widehat Z_s^1} \mid \mathcal{F}_s\right](Z_s^1)^{-1}\\ &= \mathbb{E}_{\mathbf{P}}\left[X_T^1 Z_T^1 + X_T^2 Z_T^2 \mid \mathcal{F}_s\right](Z_s^1)^{-1}\\ &= \mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2\tilde S_T^{\mathbf{Q}} \mid \mathcal{F}_s\right],\end{aligned}$$

where we used that

$$\frac{\widehat Z_s^2}{Z_s^2}\cdot\frac{Z_s^1}{\widehat Z_s^1} = \frac{\widehat S_s^{\widehat{\mathbf{Q}}}}{\tilde S_s^{\mathbf{Q}}} = 1.$$

This concludes the proof. □

Theorem 2.11. *Let Assumption 1.4 hold and $s \in [0,T]$. Let $X_T = (X_T^1, X_T^2)$ be a contingent claim such that*

$$X_T^1 + (X_T^2)^+(1-\lambda)S_T - (X_T^2)^-(1+\lambda)S_T \ge -M_s$$

for some $M_s \in L_+^1(\mathcal{F}_s, \mathcal{Q}_{loc})$. Then, the following identity holds:

$$\operatorname*{ess\,sup}_{(\boldsymbol{Q},\tilde S^{\boldsymbol{Q}})\in CPS_{loc}(s,T)} \mathbb{E}_{\boldsymbol{Q}}\left[X_T^1 + X_T^2\tilde S_T^{\boldsymbol{Q}} \mid \mathcal{F}_s\right] = \operatorname*{ess\,sup}_{(\boldsymbol{Q},\tilde S^{\boldsymbol{Q}})\in CPS_{loc}(0,T)} \mathbb{E}_{\boldsymbol{Q}}\left[X_T^1 + X_T^2\tilde S_T^{\boldsymbol{Q}} \mid \mathcal{F}_s\right]. \quad (2.34)$$

Proof. For the first direction we observe we observe that $\mathrm{CPS}_{\mathrm{loc}}(0,T) \subseteq \mathrm{CPS}_{\mathrm{loc}}(s,T)$ in the sense that, if $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}_{\mathrm{loc}}(0,T)$, then $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}|_{[s,T]}) \in \mathrm{CPS}_{\mathrm{loc}}(s,T)$. So we obtain

$$\operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in \mathrm{CPS}_{\mathrm{loc}}(s,T)} \mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s\right] \geq \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in \mathrm{CPS}_{\mathrm{loc}}(0,T)} \mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s\right].$$

For the converse direction let $(\lambda_n)_{n\in\mathbb{N}} \subseteq (0,1)$ be a sequence such that $\lambda_n \uparrow \lambda$ as n tends to infinity. Fix an arbitrary $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{SCPS}_{\mathrm{loc}}(s,T,\lambda)$ with associated (local) $\mathbf{P}$-martingales Z^1, Z^2. Then, we can approximate $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{SCPS}_{\mathrm{loc}}(s,T,\lambda)$ by a sequence $(\widehat{\mathbf{Q}}^n, \hat{S}^n)_{n\in\mathbb{N}} \subset \mathrm{SCPS}_{\mathrm{loc}}(s,T,\lambda)$ satisfying

$$(1-\lambda_n)S_s \leq \hat{S}_s^n \leq (1+\lambda_n)S_s, \tag{2.35}$$

for each $n \in \mathbb{N}$. More specifically, for $n \in \mathbb{N}$ define the set $C_s^n \in \mathcal{F}_s$ by

$$C_s^n := \left\{\omega \in \Omega : (1-\lambda_n)S_s(\omega) \leq \tilde{S}_s^{\mathbf{Q}}(\omega) \leq (1+\lambda_n)S_s(\omega)\right\}, \quad n \in \mathbb{N}. \tag{2.36}$$

Corollary 1.8 guarantees that there exists a consistent local price system

$$(\mathbf{Q}^n, \tilde{S}^n) \in \mathrm{CPS}_{\mathrm{loc}}(s,T,\lambda_n)$$

for all $n \in \mathbb{N}$ such that $\tilde{S}_s^n = (1+\lambda_n)S_s$. By $Z^{1,n} = (Z_t^{1,n})_{t\in[s,T]}$ and $Z^{2,n} = (Z_t^{2,n})_{t\in[s,T]}$ we denote the associated (local) $\mathbf{P}$-martingales. Then we construct a sequence of consistent local price system $(\widehat{\mathbf{Q}}^n, \hat{S}^n)_{n\in\mathbb{N}} \subset \mathrm{CPS}_{\mathrm{loc}}(0,T,\lambda)$ by its associated (local) $\mathbf{P}$-martingales $\widehat{Z}^1 = (\widehat{Z}_t^{1,n})_{t\in[s,T]}$ and $\widehat{Z}^{2,n} = (\widehat{Z}_t^{2,n})_{t\in[s,T]}$ by

$$\widehat{Z}_t^i := \mathbb{1}_{C_s^n} Z_t^i + \mathbb{1}_{(C_s^n)^c} Z_t^{i,n}, \quad t \in [s,T],\ i = 1,2,\ n \in \mathbb{N}. \tag{2.37}$$

By this construction $\widehat{Z}^1$, $\widehat{Z}^2$) are (local) $\mathbf{P}$-martingales on $[s,T]$. To show this, we use that $C_s^n \in \mathcal{F}_s$. Thus, for $\widehat{Z}^{1,n}$ we have for $s \leq t \leq T$ that

$$\mathbb{E}_{\mathbf{P}}\left[\widehat{Z}_T^{1,n} \mid \mathcal{F}_t\right] = \mathbb{1}_{C_s^n}\mathbb{E}_{\mathbf{P}}[Z_T^1 \mid \mathcal{F}_t] + \mathbb{1}_{(C_s^n)^c}\mathbb{E}_{\mathbf{P}}\left[Z_T^{1,n} \mid \mathcal{F}_t\right] = \mathbb{1}_{C_s^n} Z_t^1 + \mathbb{1}_{(C_s^n)^c} Z_t^{1,n} = \widehat{Z}_t^1.$$

Let $(\tau_k)_{k\in\mathbb{N}}$ and $(\tau_k^n)_{k\in\mathbb{N}}$ be localizing sequences on $[s,T]$ of Z^2, $Z^{2,n}$, respectively, and define $(\widehat{\tau}_k^n)_{k\in\mathbb{N}}$ by

$$\widehat{\tau}_k^n := \tau_k \wedge \tau_k^n, \quad k,n \in \mathbb{N}.$$

Then, we have for $\widehat{Z}^2$ and $s \leq t \leq T$ that

$$\begin{aligned}\mathbb{E}_{\mathbf{P}}\left[\left(\widehat{Z}_T^{2,n}\right)^{\widehat{\tau}_k^n} \mid \mathcal{F}_t\right] &= \mathbb{1}_{C_s^n}\mathbb{E}_{\mathbf{P}}[\left(Z_T^2\right)^{\widehat{\tau}_k^n} \mid \mathcal{F}_t] + \mathbb{1}_{(C_s^n)^c}\mathbb{E}_{\mathbf{P}}\left[\left(Z_T^{1,n}\right)^{\widehat{\tau}_k^n} \mid \mathcal{F}_t\right] \\ &= \mathbb{1}_{C_s^n}\left(Z_t^1\right)^{\widehat{\tau}_k^n} + \mathbb{1}_{(C_s^n)^c}\left(Z_t^{2,n}\right)^{\widehat{\tau}_k^n} = \left(\widehat{Z}_t^2\right)^{\widehat{\tau}_k^n},\end{aligned}$$

where we used that $(\widehat{\tau}_k^n)_{k\in\mathbb{N}}$ is again a localizing sequence for Z^2 and $Z^{2,n}$. Clearly, $\widehat{Z}_t^n \in K_t^* \backslash \{0\}$ for all $t \in [s,T]$ and $n \in \mathbb{N}$ because

$$\frac{\widehat{Z}_t^{2,n}}{\widehat{Z}_t^{1,n}} = \mathbb{1}_{C_s^n}\frac{Z_t^2}{Z_t^1} + \mathbb{1}_{(C_s^n)^c}\frac{Z_t^{2,n}}{Z_t^{1,n}} \in [(1-\lambda)S_t, (1+\lambda)S_t],$$

where we used that $\mathbf{P}(C_s^n \cup (C_s^n)^c) = 1$, and Z, Z^n are consistent local price systems by definition. Therefore, $(\widehat{Z}^n)_{n\in\mathbb{N}} = (\widehat{Z}^{1,n}, \widehat{Z}^{2,n})_{n\in\mathbb{N}}$ defines a sequence of consistent local price systems on $[s,T]$ and satisfies (2.35) for all $n \in \mathbb{N}$. In particular, the conditions of Lemma 2.10 are satisfied by $\widehat{Z}^n$. By Lemma 2.10 there exists $(\overline{\mathbf{Q}}^n, \bar{S}^n) \in \mathrm{CPS}_{\mathrm{loc}}(0,T,\lambda)$, for $n \in \mathbb{N}$, such that

$$\mathbb{E}_{\widehat{\mathbf{Q}}^n}\left[X_T^1 + X_T^2 \hat{S}_T^n \mid \mathcal{F}_s\right] = \mathbb{E}_{\overline{\mathbf{Q}}^n}\left[X_T^1 + X_T^2 \bar{S}^n \mid \mathcal{F}_s\right]. \tag{2.38}$$

Further, $\widehat{Z}_t^n \xrightarrow{\mathbf{P}\text{-a.s.}} Z_t$ for all $t \in [s,T]$ because $C_s^n \subseteq C_s^{n+1}$ and $\mathbf{P}(C_s^n) \uparrow 1$ which implies that $\mathbb{1}_{C_s^n} \xrightarrow{\mathbf{P}\text{-a.s.}} 1$. For this fact it is important that $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}})$ is a strictly consistent local price system on $[s,T]$. By (2.38) and because $\widehat{Z}^{i,n}$ converges to Z^i, $i = 1,2$, as n tends to infinity, we obtain that

$$\begin{aligned} \operatorname*{ess\,sup}_{(\mathbf{Q}_0, \tilde{S}^{\mathbf{Q}_0}) \in \mathrm{CPS}_{\mathrm{loc}}(0,T,\lambda)} \mathbb{E}_{\mathbf{Q}_0}\left[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}_0} \mid \mathcal{F}_s\right] &\geq \operatorname*{ess\,sup}_{n \in \mathbb{N}} \mathbb{E}_{\overline{\mathbf{Q}}^n}\left[X_T^1 + X_T^2 \bar{S}_T^n \mid \mathcal{F}_s\right] \\ &\geq \mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s\right]. \end{aligned} \tag{2.39}$$

Because $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{SCPS}_{\mathrm{loc}}(s,T,\lambda)$ was arbitrary, we can take the essential supremum over $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{SCPS}_{\mathrm{loc}}(s,T,\lambda)$ (resp. $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}_{\mathrm{loc}}(s,T,\lambda)$) on the right-hand side of (2.39) to follow that

$$\operatorname*{ess\,sup}_{(\mathbf{Q}_0, \tilde{S}^{\mathbf{Q}_0}) \in \mathrm{CPS}_{\mathrm{loc}}(0,T,\lambda)} \mathbb{E}_{\mathbf{Q}_0}\left[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}_0} \mid \mathcal{F}_s\right] \geq \operatorname*{ess\,sup}_{(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}_{\mathrm{loc}}(s,T,\lambda)} \mathbb{E}_{\mathbf{Q}}\left[X_t^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s\right].$$

Note that the essential supremum over $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}_{\mathrm{loc}}(s,T,\lambda)$ is equal to the essential supremum over $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{SCPS}_{\mathrm{loc}}(s,T,\lambda)$. In fact, on the one hand, we have

$$\operatorname*{ess\,sup}_{(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{SCPS}_{\mathrm{loc}}(\sigma,T,\lambda)} \mathbb{E}_{\mathbf{Q}}\left[\tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_\sigma\right] \leq \operatorname*{ess\,sup}_{(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}_{\mathrm{loc}}(\sigma,T,\lambda)} \mathbb{E}_{\mathbf{Q}}\left[\tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_\sigma\right],$$

because $\mathrm{SCPS}_{\mathrm{loc}}(\sigma,T,\lambda) \subseteq \mathrm{CPS}_{\mathrm{loc}}(\sigma,T,\lambda)$. On the other hand, let $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}_{\mathrm{loc}}(\sigma,T,\lambda)$ and $(\widehat{\mathbf{Q}}, \widehat{S}^{\widehat{\mathbf{Q}}}) \in \mathrm{SCPS}_{\mathrm{loc}}(\sigma,T,\lambda)$ be arbitrary with associated (local) $\mathbf{P}$-martingales Z^1, Z^2, $\widehat{Z}^1, \widehat{Z}^2$, respectively. Define $(\overline{\mathbf{Q}}^\epsilon, \bar{S}^{\bar{\mathbf{Q}}}(\epsilon))$ by its associated (local) $\mathbf{P}$-martingales $\bar{Z}^{1,\epsilon}, \bar{Z}^{2,\epsilon}$ by

$$\bar{Z}_t^{i,\epsilon} := (1-\epsilon) Z_t^i + \epsilon \widehat{Z}_t^i, \quad t \in [\![\sigma, T]\!], \; i \in \{1,2\},$$

for some $\epsilon \in (0,1)$. Then $(\overline{\mathbf{Q}}^\epsilon, \bar{S}^{\bar{\mathbf{Q}}}(\epsilon)) \in \mathrm{SCPS}_{\mathrm{loc}}(\sigma,T,\lambda)$ by Proposition 1.3. First, we note that $\bar{Z}^{1,\epsilon}$, $\bar{Z}^{2,\epsilon}$ are (local) $\mathbf{P}$-martingales as a convex combination of (local) $\mathbf{P}$-martingales and $\frac{\bar{Z}_t^{2,\epsilon}}{\bar{Z}_t^{1,\epsilon}} \in \mathrm{int}(-K_t^\lambda)^\circ$, where int denotes the interior, by linearity of the inner product. Because $\epsilon \in (0,1)$ was arbitrary, we get the desired result for ϵ tending to 0. This concludes the proof. □

Theorem 2.12. *Let Assumption 1.5 hold and $s \in [0,T]$. Let $X_T = (X_T^1, X_T^2)$ be a contingent claim such that*

$$X_T^1 + (X_T^2)^+(1-\lambda)S_T - (X_T^2)^-(1+\lambda)S_T \geq -M_s^1 + -M_s^2 \S_T,$$

for some $(M_s^1, M_s^2) \in L_+^1(\mathcal{F}_s, \mathcal{Q}) \times L_+^\infty(\mathcal{F}_s, \mathcal{Q})$. Then, the following identity holds:

$$\operatorname*{ess\,sup}_{(\boldsymbol{Q}, \tilde{S}^{\boldsymbol{Q}}) \in CPS_(s,T)} \mathbb{E}_{\boldsymbol{Q}}\Big[X_T^1 + X_T^2 \tilde{S}_T^{\boldsymbol{Q}} \mid \mathcal{F}_s\Big] = \operatorname*{ess\,sup}_{(\boldsymbol{Q}, \tilde{S}^{\boldsymbol{Q}}) \in CPS(0,T)} \mathbb{E}_{\boldsymbol{Q}}\Big[X_T^1 + X_T^2 \tilde{S}_T^{\boldsymbol{Q}} \mid \mathcal{F}_s\Big]. \tag{2.40}$$

Proof. The proof of Theorem 2.11 carries over verbatim to the present setting. $\square$

From now on, we set $\mathrm{CPS}_{\mathrm{loc}} := \mathrm{CPS}_{\mathrm{loc}}(0,T,\lambda)$ (resp. $\mathrm{CPS} := \mathrm{CPS}(0,T,\lambda)$). Further, we define $V = (V_t)_{t\in[0,T]}$ by

$$V_t := \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\mathrm{CPS}_{\mathrm{loc}}} \mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2\tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_t\right], \quad \text{for all } t\in[0,T]. \tag{2.41}$$

Lemma 2.13. *Let Assumption 1.4 hold and $s\in[0,T]$. Let $X_T = (X_T^1, X_T^2)$ be a contingent claim such that*

$$X_T^1 + (X_T^2)^+(1-\lambda)S_T - (X_T^2)^-(1+\lambda)S_T \ge -M_s,$$

for some $M_s^1 \in L_+^1(\mathcal{F}_s, \mathcal{Q}_{loc})$. Then, $V = (V_t)_{t\in[s,T]}$ defined in (2.41) *is an adapted stochastic process, which is unique up to an evanescent set.*

Proof. Theorem A.33 of [40] guarantees that V_t is $\mathcal{F}_t$-measurable for all $t\in[t,T]$. It is important that $\mathbf{Q}\sim\mathbf{P}$ for all $\mathbf{Q}\in\mathcal{Q}_{\mathrm{loc}}$ such that

$$\left\{\mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2\tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_t\right] : (\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\mathrm{CPS}_{\mathrm{loc}}(0,T)\right\} \subset L^0(\mathcal{F}_t,\mathbf{P}).$$

As Theorem 2.7 and Theorem 2.11 hold for all stopping times $0\le\sigma\le T$, V is unique to within an evanescent set because of the Optional Cross-Section Theorem, see Theorem 86 in Chapter IV of [35]. Therefore, V is a well-defined adapted process on $[s,T]$. $\square$

Lemma 2.14. *Let Assumption 1.5 hold and $s\in[0,T]$. Let $X_T = (X_T^1, X_T^2)$ be a contingent claim such that*

$$X_T^1 + (X_T^2)^+(1-\lambda)S_T - (X_T^2)^-(1+\lambda)S_T \ge -M_s^1 + -M_s^2 \S_T,$$

for some $(M_s^1, M_s^2)\in L_+^1(\mathcal{F}_s,\mathcal{Q})\times L_+^\infty(\mathcal{F}_s,\mathcal{Q})$. Then,

$$\left(\operatorname*{ess\,sup}_{(\boldsymbol{Q},\tilde{S}^{\boldsymbol{Q}})\in CPS} \mathbb{E}_{\boldsymbol{Q}}\left[X_T^1 + X_T^2\tilde{S}_T^{\boldsymbol{Q}} \mid \mathcal{F}_t\right]\right)_{t\in[s,T]},$$

is an adapted stochastic process, which is unique to within evanescent processes.

Proof. The proof follows with the same arguments as Lemma 2.13, using Theorem 2.6 and Theorem 2.12. $\square$

Lemma 2.15. *Let Assumption 1.4 hold and $s\in[0,T]$. Let $X_T = (X_T^1, X_T^2)\in L^0(\mathcal{F}_T,\boldsymbol{P};\mathbb{R}^2)$ be a contingent claim such that*

$$X_T^1 + (X_T^2)^+(1-\lambda)S_T - (X_T^2)^-(1+\lambda)S_T \ge -M_s \tag{2.42}$$

for some $M_s\in L_+^1(\mathcal{F}_s,\mathcal{Q}_{loc})$. Then for any $\boldsymbol{Q}_0\in\mathcal{Q}_{loc}$ and V the following identity holds

$$\mathbb{E}_{\boldsymbol{Q}_0}[V_t] = \sup_{(\boldsymbol{Q},\tilde{S}^{\boldsymbol{Q}})\in CPS_{loc}} \mathbb{E}_{\boldsymbol{Q}_0}\left[\mathbb{E}_{\boldsymbol{Q}}\left[X_T^1 + X_T^2\tilde{S}_T^{\boldsymbol{Q}} \mid \mathcal{F}_t\right]\right], \tag{2.43}$$

for $t\in[s,T]$.

Proof. Let $(\mathbf{Q}_0, \tilde{S}^{\mathbf{Q}_0}) \in \mathrm{CPS}_{\mathrm{loc}}(0,T)$. By monotonicity we immediately obtain

$$\mathbb{E}_{\mathbf{Q}_0}[V_t] \geq \mathbb{E}_{\mathbf{Q}_0}\left[\mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_t\right]\right], \tag{2.44}$$

for all $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}_{\mathrm{loc}}(0,T)$. Because (2.44) holds for all $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}_{\mathrm{loc}}(0,T)$ we get

$$\mathbb{E}_{\mathbf{Q}_0}[V_t] \geq \sup_{(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}_{\mathrm{loc}}} \mathbb{E}_{\mathbf{Q}_0}\left[\mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_t\right]\right].$$

For the reverse inequality we first show that the set

$$\Phi_t := \left\{\mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_t\right] : (\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}_{\mathrm{loc}}(t,T)\right\}$$

is directed upwards, see Definition 2.2. Let $\mathbb{E}_{\mathbf{Q}}[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_t], \mathbb{E}_{\bar{\mathbf{Q}}}[X_T^1 + X_T^2 \bar{S}_T^{\bar{\mathbf{Q}}} \mid \mathcal{F}_t] \in \Phi_t$. We construct $(\widehat{\mathbf{Q}}, \widehat{S}^{\widehat{\mathbf{Q}}}) \in \mathrm{CPS}_{\mathrm{loc}}(t,T)$ such that $\mathbb{E}_{\widehat{\mathbf{Q}}}[X_T^1 + X_T^2 \widehat{S}^{\widehat{\mathbf{Q}}} \mid \mathcal{F}_t] \in \Phi_t$ and

$$\mathbb{E}_{\widehat{\mathbf{Q}}}\left[X_T^1 + X_T^2 \widehat{S}_T^{\widehat{\mathbf{Q}}} \mid \mathcal{F}_t\right] \geq \mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_t\right] \vee \mathbb{E}_{\bar{\mathbf{Q}}}\left[X_T^1 + X_T^2 \bar{S}_T^{\bar{\mathbf{Q}}} \mid \mathcal{F}_t\right].$$

Define

$$A_t := \left\{\mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_t\right] \geq \mathbb{E}_{\bar{\mathbf{Q}}}\left[X_T^1 + X_T^2 \bar{S}_T^{\bar{\mathbf{Q}}} \mid \mathcal{F}_t\right]\right\} \in \mathcal{F}_t.$$

Let $Z = (Z^1, Z^2)$ and $\bar{Z} = (\bar{Z}^1, \bar{Z}^2)$ be the (local) $\mathbf{P}$-martingales associated to $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}})$ and $(\overline{\mathbf{Q}}, \bar{S}^{\bar{\mathbf{Q}}})$ respectively, as in Proposition 1.3. Then we define

$$\frac{d\widehat{\mathbf{Q}}}{d\mathbf{P}} = \frac{\widehat{Z}_T^1}{\mathbb{E}_{\mathbf{P}}\left[\widehat{Z}_T^1\right]} := \frac{\mathbb{1}_{A_t} Z_T^1 + \mathbb{1}_{A_t^c} \bar{Z}_T^1}{\mathbb{E}_{\mathbf{P}}\left[\mathbb{1}_{A_t} Z_T^1 + \mathbb{1}_{A_t^c} \bar{Z}_T^1\right]}, \tag{2.45}$$

and for $t \leq u \leq T$,

$$\widehat{Z}_u^2 := \mathbb{1}_{A_t} Z_u^2 + \mathbb{1}_{A_t^c} \bar{Z}_u^2 \tag{2.46}$$

with corresponding $\widehat{S}^{\widehat{\mathbf{Q}}}$ given by

$$\widehat{S}_u^{\widehat{\mathbf{Q}}} = \frac{\widehat{Z}_u^2}{\widehat{Z}_u^1}. \tag{2.47}$$

We now prove that $(\widehat{\mathbf{Q}}, \widehat{S}^{\widehat{\mathbf{Q}}}) \in \mathrm{CPS}_{\mathrm{loc}}(t,T)$. Clearly,

$$(1-\lambda) S_u \leq \widehat{S}_u^{\widehat{\mathbf{Q}}} \leq (1+\lambda) S_u \quad \text{for all } u \in [t,T],$$

as $\widehat{S}^{\widehat{\mathbf{Q}}}$ coincides with $\tilde{S}^{\mathbf{Q}}$ on A_t and with $\bar{S}^{\bar{\mathbf{Q}}}$ on A_t^c. For the local martingale property of $\widehat{S}^{\widehat{\mathbf{Q}}}$ let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence for $\tilde{S}^{\mathbf{Q}}$ and $\bar{S}^{\bar{\mathbf{Q}}}$. For instance, we can define $(\tau_n)_{n\in\mathbb{N}}$ as the minimum of the localizing sequences of $\tilde{S}^{\mathbf{Q}}$ and $\bar{S}^{\bar{\mathbf{Q}}}$. For $t \leq u \leq v \leq T$ we get

$$\begin{aligned}
\mathbb{E}_{\widehat{\mathbf{Q}}}\left[\left(\widehat{S}_v^{\widehat{\mathbf{Q}}}\right)^{\tau_n} \mid \mathcal{F}_u\right] &= \mathbb{E}_{\mathbf{P}}\left[\left(\frac{\widehat{Z}_v^2}{\widehat{Z}_v^1}\right)^{\tau_n} \frac{\widehat{Z}_T^1}{\mathbb{E}_{\mathbf{P}}\left[\widehat{Z}_T^1\right]} \mid \mathcal{F}_u\right] \frac{\mathbb{E}_{\mathbf{P}}\left[\widehat{Z}_T^1\right]}{\widehat{Z}_{u\wedge\tau_n}^1} \\
&= \mathbb{E}_{\mathbf{P}}\left[\left(\mathbb{1}_{A_t} Z_v^1 + \mathbb{1}_{A_t^c} \bar{Z}_v^2\right)^{\tau_n} \mid \mathcal{F}_u\right] \frac{1}{\widehat{Z}_{u\wedge\tau_n}^1} = \left(\mathbb{1}_{A_t} \mathbb{E}_{\mathbf{P}}\left[(Z_v^2)^{\tau_n} \mid \mathcal{F}_u\right] + \mathbb{1}_{A_t^c} \mathbb{E}_{\mathbf{P}}\left[(\bar{Z}_v^2)^{\tau_n} \mid \mathcal{F}_u\right]\right) \frac{1}{\widehat{Z}_{u\wedge\tau_n}^1} \\
&= \left(\mathbb{1}_{A_t} Z_{u\wedge\tau_n}^2 + \mathbb{1}_{A_t^c} \bar{Z}_{u\wedge\tau_n}^2\right) \frac{1}{\widehat{Z}_{u\wedge\tau_n}^1} = \left(\widehat{S}_u\right)^{\tau_n},
\end{aligned}$$

where we used that $\mathbb{1}_{A_t}, \mathbb{1}_{A_t^c}$ are measurable for $\mathcal{F}_t \subset \mathcal{F}_u$. Hence, Φ_t is directed upwards and by Theorem A.33 of [40], there exists an increasing sequence $(\mathbb{E}_{\mathbf{Q}^n}[X_T^1 + X_T^2 \tilde{S}_T^n \mid \mathcal{F}_t])_{n\in\mathbb{N}} \subset \Phi_t$ such that

$$\lim_{n\to\infty} \mathbb{E}_{\mathbf{Q}^n}\left[X_T^1 + X_T^2 \tilde{S}_T^n \mid \mathcal{F}_t\right] = \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in \mathrm{CPS}_{\mathrm{loc}}(t,T)} \mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_t\right] = V_t. \tag{2.48}$$

By the Theorem of Monotone Convergence we obtain

$$\begin{aligned}\mathbb{E}_{\mathbf{Q}_0}[V_t] &= \lim_{n\to\infty} \mathbb{E}_{\mathbf{Q}_0}\left[\mathbb{E}_{\mathbf{Q}^n}\left[X_T^1 + X_T^2 \tilde{S}_T^n \mid \mathcal{F}_t\right]\right] \\ &\le \sup_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in \mathrm{CPS}_{\mathrm{loc}}(t,T)} \mathbb{E}_{\mathbf{Q}_0}\left[\mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_t\right]\right].\end{aligned}$$

With the same arguments as in the proof of Theorem 2.12, we obtain that

$$\sup_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in \mathrm{CPS}_{\mathrm{loc}}(t,T)} \mathbb{E}_{\mathbf{Q}_0}\left[\mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_t\right]\right] = \sup_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in \mathrm{CPS}_{\mathrm{loc}}} \mathbb{E}_{\mathbf{Q}_0}\left[\mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_t\right]\right],$$

which completes the proof. □

Lemma 2.16. *Let Assumption 1.5 hold and $s \in [0,T]$. Let $X_T = (X_T^1, X_T^2) \in L^0(\mathcal{F}_T, \boldsymbol{P}; \mathbb{R}^2)$ such that*

$$X_T^1 + (X_T^2)^+(1-\lambda)S_T - (X_T^2)^-(1+\lambda)S_T \ge -M_s^1 - M_s^2 S_T \tag{2.49}$$

for some $(M_s^1, M_s^2) \in L_+^1(\mathcal{F}_s, \mathcal{Q}) \times L_+^\infty(\mathcal{F}_s, \mathcal{Q})$. Then for any $\boldsymbol{Q}_0 \in \mathcal{Q}$ the following identity holds

$$\mathbb{E}_{\boldsymbol{Q}_0}\left[\operatorname*{ess\,sup}_{(\boldsymbol{Q},\tilde{S}^{\boldsymbol{Q}})\in CPS} \mathbb{E}_{\boldsymbol{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\boldsymbol{Q}} \mid \mathcal{F}_t\right]\right] = \sup_{(\boldsymbol{Q},\tilde{S}^{\boldsymbol{Q}})\in CPS} \mathbb{E}_{\boldsymbol{Q}_0}\left[\mathbb{E}_{\boldsymbol{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\boldsymbol{Q}} \mid \mathcal{F}_t\right]\right], \tag{2.50}$$

for $t \in [s,T]$.

Proof. The arguments are identical to the proof of Lemma 2.15. □

Let $(\sigma_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of decreasing, $[0,T]$-valued stopping times with $\sigma_n \downarrow \sigma = \sigma_\infty$ as n tends to infinity. In the sequel we set $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$.

Remark 2.17. *Lemma 2.15 shows that for any $n \in \bar{\mathbb{N}}$ there exists an increasing sequence*

$$(\boldsymbol{Q}(m_k(n)), \tilde{S}^{\boldsymbol{Q}}(m_k(n)))_{k\in\mathbb{N}} \subset CPS_{loc}(0,T)$$

such that

$$\lim_{k\to\infty} \mathbb{E}_{\boldsymbol{Q}_0}\left[\mathbb{E}_{\boldsymbol{Q}(m_k(n))}\left[X_T^1 + X_T^2 \tilde{S}^{\boldsymbol{Q}(m_k(n))} \mid \mathcal{F}_{\sigma_n}\right]\right] = \mathbb{E}_{\boldsymbol{Q}_0}\left[V_{\sigma_n}\right].$$

Further, it is easy to see that these sequences can be taken uniformly over $n \in \bar{\mathbb{N}}$. Indeed, for $n \in \bar{\mathbb{N}}$ take the subsequence $(m_{k_l}(n))_{l\in\mathbb{N}} \subset (m_k(n))_{k\in\mathbb{N}}$ defined by

$$m_{k_1}(n) := \inf\left\{k \ge 1 : \left|\mathbb{E}_{\boldsymbol{Q}_0}\left[\mathbb{E}_{\boldsymbol{Q}(m_k(n))}\left[X_T^1 + X_T^2 \tilde{S}_T^{\boldsymbol{Q}(m_k(n))} \mid \mathcal{F}_{\sigma_n}\right]\right] - \mathbb{E}_{\boldsymbol{Q}_0}\left[V_{\sigma_n}\right]\right| < 1\right\}$$
$$m_{k_l}(n) := \inf\left\{k > m_{k_{l-1}}(n) : \left|\mathbb{E}_{\boldsymbol{Q}_0}\left[\mathbb{E}_{\boldsymbol{Q}(m_k(n))}\left[X_T^1 + X_T^2 \tilde{S}_T^{\boldsymbol{Q}(m_k(n))} \mid \mathcal{F}_{\sigma_n}\right]\right] - \mathbb{E}_{\boldsymbol{Q}_0}\left[V_{\sigma_n}\right]\right| < \frac{1}{l}\right\}.$$

For sake of notational convenience, we may use for $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}_{\mathrm{loc}}(0,T)$ (resp. $\mathrm{CPS}(0,T)$) the imprecise notation $V^{\mathbf{Q},\tilde{S}^{\mathbf{Q}}} = V^{\mathbf{Q}} = (V_t^{\mathbf{Q}})_{t\in[0,T]}$ defined by

$$V_t^{\mathbf{Q}} := \mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_t\right], \quad \text{for all } t \in [0,T]. \tag{2.51}$$

Assumption 2.18. We assume the existence of $\mathbf{Q}_0 \in \mathcal{Q}_{\mathrm{loc}}$ such that for any decreasing sequence of stopping times $0 \le (\sigma_n)_{n\in\mathbb{N}} \le T$ with $\sigma_n \downarrow \sigma$ as n tends to infinity, there exists a sequence

$$(\mathbf{Q}(m_k(n)), \tilde{S}^{\mathbf{Q}}(m_k(n))_{k\in\mathbb{N}} \subset \mathrm{CPS}_{\mathrm{loc}}(0,T),\ n \in \bar{\mathbb{N}},$$

such that

$$\left(\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right]\right)_{k\in\mathbb{N}}$$

converges uniformly over all $n \in \bar{\mathbb{N}}$ to $\mathbb{E}_{\mathbf{Q}_0}[V_{\sigma_n}]$, i.e., for all $\varepsilon > 0$ exists $K = K(\varepsilon) \in \mathbb{N}$ such that

$$\left|\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}\right]\right| < \varepsilon,\ \text{for all } k \ge K,\ \text{and for all } n \in \bar{\mathbb{N}}, \tag{2.52}$$

and that for all $k \in \mathbb{N}$, $\mathbf{Q}_0(\bigcup_{N\in\mathbb{N}} A_N^{\varepsilon,k}) = 1$, where

$$A_N^{\varepsilon,k} := \left\{\omega \in \Omega : |V_{\sigma_n}^{\mathbf{Q}(m_k(n_0))} - V_{\sigma}^{\mathbf{Q}(m_k(n_0))}|(\omega) < \varepsilon,\ \forall n \ge N,\ \forall n_0 \in \bar{\mathbb{N}}\right\}. \tag{2.53}$$

Let us give some intuition on Assumption 2.18. The assumption can be thought as equicontinuity in time at level k, of a family of approximating sequences of consistent local price systems. Note that (2.52) is always fulfilled by Lemma 2.15 and Remark 2.17. In general it is not true that $\mathbf{Q}_0(\bigcup_{N\in\mathbb{N}} A_N^{\varepsilon,k}) = 1$ for given $A_N^{\varepsilon,k}$ given in (2.53). This is the key feature of Assumption 2.18.

Theorem 2.19. *Let Assumption 1.4 and 2.18 hold and $s \in [0,T]$. Let $X_T = (X_T^1, X_T^2) \in L^\infty(\mathcal{F}_T, \boldsymbol{P}) \times L^\infty(\mathcal{F}_T, \boldsymbol{P})$ such that*

$$X_T^1 + (X_T^2)^+(1-\lambda)S_T - (X_T^2)^-(1+\lambda)S_T \ge -M_s \tag{2.54}$$

for some $M_s \in L_+^1(\mathcal{F}_s, \mathcal{Q}_{loc})$. Then V in (2.41) admits a right-continuous modification on $[s,T]$ with respect to $\boldsymbol{P}$.

Proof. Let $\mathbf{Q}_0 \in \mathcal{Q}_{\mathrm{loc}}$ be the measure given by Assumption 2.18. Since all measure $\mathbf{Q} \in \mathcal{Q}_{\mathrm{loc}}$ are equivalent to $\mathbf{P}$, it is equivalent to show that V admits a right-continuous modification with respect to $\mathbf{P}$.

By Theorem 48 in [36], the paths of V are right-continuous (outside an evanescent set), if $\lim_{n\to\infty} \mathbb{E}_{\mathbf{P}}\left[V_{\sigma_n}\right] = \mathbb{E}_{\mathbf{P}}\left[V_{\lim_{n\to\infty}\sigma_n}\right]$ for every decreasing sequence $(\sigma_n)_{n\in\mathbb{N}}$ of bounded stopping times.

Let $(\sigma_n)_{n\in\mathbb{N}}$ be a decreasing sequence of stopping times with values in $[s,T]$ such that $\sigma_n \downarrow \sigma$ as n tends to infinity. We now prove that

$$\lim_{n\to\infty} \mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}\right] = \mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma}\right].$$

For this purpose, we show that the family

$$\mathcal{G} := \left\{\left|V_{\sigma_n}^{\mathbf{Q}} - V_{\sigma}^{\mathbf{Q}}\right| : n \in \mathbb{N},\ (\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}_{\mathrm{loc}}\right\}$$

is uniformly integrable with respect to $\mathbf{Q}_0$. First note, that

$$\left|V_{\sigma_n}^{\mathbf{Q}} - V_{\sigma}^{\mathbf{Q}}\right| \leq \left|V_{\sigma_n}^{\mathbf{Q}}\right| + \left|V_{\sigma}^{\mathbf{Q}}\right| \leq |V_{\sigma_n}| + |V_{\sigma}|. \tag{2.55}$$

Because $(X_T^1, X_T^2) \in L^\infty(\mathcal{F}_T, \mathbf{P}) \times L^\infty(\mathcal{F}_T, \mathbf{P})$, there exists $C_1, C_2 \in \mathbb{R}$ such that $|X_T^1| \leq C_1$ and $|X_T^2| \leq C_2$. For any $[s,T]$-valued stopping time ρ we thus have,

$$\begin{aligned}
|V_\rho| &\leq \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in \mathrm{CPS}_{\mathrm{loc}}} \mathbb{E}_{\mathbf{Q}}\left[\left|X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}}\right| \mid \mathcal{F}_\rho\right] \\
&\leq \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in \mathrm{CPS}_{\mathrm{loc}}} \mathbb{E}_{\mathbf{Q}}\left[C_1 + C_2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_\rho\right] \\
&= C_1 + C_2 \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in \mathrm{CPS}_{\mathrm{loc}}} \mathbb{E}_{\mathbf{Q}}\left[\tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_\rho\right] \\
&\leq C_1 + C_2 \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in \mathrm{CPS}_{\mathrm{loc}}} \tilde{S}_\rho^{\mathbf{Q}} \\
&\leq C_1 + C_2 (1+\lambda) S_\rho \\
&\leq C_1 + C_2 \frac{1+\lambda}{1-\lambda} \tilde{S}_\rho^{\mathbf{Q}_0}.
\end{aligned} \tag{2.56}$$

Equation (2.56) implies that $V_\sigma \in L^1(\mathcal{F}_\sigma, \mathbf{Q}_0)$ and that

$$|V_{\sigma_n}| \leq C_1 + C_2 \frac{1+\lambda}{1-\lambda} \tilde{S}_{\sigma_n}^{\mathbf{Q}_0}, \quad \text{for all } n \in \bar{\mathbb{N}}.$$

Therefore, it is sufficient to prove that $\{\tilde{S}_{\sigma_n}^{\mathbf{Q}_0} : n \in \mathbb{N}\}$ is uniformly integrable with respect to $\mathbf{Q}_0$. For this purpose, we first show that $\tilde{S}_{\sigma_n}^{\mathbf{Q}_0} \xrightarrow{L^1} \tilde{S}_{\sigma}^{\mathbf{Q}_0}$. By definition of a consistent local price system, $\tilde{S}^{\mathbf{Q}_0}$ is a non-negative, càdlàg local $\mathbf{Q}_0$-martingale. In particular, $\tilde{S}^{\mathbf{Q}_0}$ is also a supermartingale under $\mathbf{Q}_0$ and hence we get by Theorem 9 of [80] that

$$\lim_{n\to\infty} \mathbb{E}_{\mathbf{Q}_0}[\tilde{S}_{\sigma_n}^{\mathbf{Q}_0}] = \mathbb{E}_{\mathbf{Q}_0}[\tilde{S}_{\sigma}^{\mathbf{Q}_0}]. \tag{2.57}$$

Thus, by (2.57) and because $\tilde{S}_{\sigma_n}^{\mathbf{Q}_0} \xrightarrow{\mathbf{P}\text{-a.s.}} \tilde{S}_{\sigma}^{\mathbf{Q}_0}$ Scheffè's Lemma guarantees that $\tilde{S}_{\sigma_n}^{\mathbf{Q}_0} \xrightarrow{L^1} \tilde{S}_{\sigma}^{\mathbf{Q}_0}$. The family $\{\tilde{S}_{\sigma_n}^{\mathbf{Q}_0} : n \in \mathbb{N}\}$ is uniformly integrable with respect ot $\mathbf{Q}_0$ by Theorem 6.25 of [70] as $\tilde{S}_{\sigma_n}^{\mathbf{Q}_0} \in L^1(\mathcal{F}_T, \mathbf{Q}_0)$ for all $n \in \mathbb{N}$ and $\tilde{S}_{\sigma_n}^{\mathbf{Q}_0} \xrightarrow{\mathbf{P}\text{-a.s.}} \tilde{S}_{\sigma}^{\mathbf{Q}_0}$ and $\tilde{S}_{\sigma_n}^{\mathbf{Q}_0} \xrightarrow{L^1} \tilde{S}_{\sigma}^{\mathbf{Q}_0}$. Therefore, $\mathcal{G}$ is uniformly integrable with respect to $\mathbf{Q}_0$. Fix $\varepsilon > 0$. By Assumption 2.18 there exists for each $n \in \bar{\mathbb{N}}$ a sequence

$$(\mathbf{Q}(m_k(n)), \tilde{S}^{\mathbf{Q}}(m_k(n)))_{k\in\mathbb{N}} \subset \mathrm{CPS}_{\mathrm{loc}}(0,T)$$

such that for suitable $K = K(\varepsilon) \in \mathbb{N}$

$$\left|\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}\right]\right| < \frac{\varepsilon}{8}, \text{ for all } k \geq K, \text{ and for all } n \in \bar{\mathbb{N}},$$

and for $N \in \mathbb{N}$ and $k \in \mathbb{N}$ the set $A_N^k = A_N^{\varepsilon/8,k}$ defined by

$$A_N^k = \left\{\omega \in \Omega : |V_{\sigma_n}^{\mathbf{Q}(m_k(n_0))} - V_{\sigma}^{\mathbf{Q}(m_k(n_0))}|(\omega) < \frac{\varepsilon}{8}, \ \forall n \geq N, \ \forall n_0 \in \bar{\mathbb{N}}\right\}, \tag{2.58}$$

satisfies $\mathbf{Q}_0(\bigcup_{N\in\mathbb{N}} A_N^k) = 1$ for all $k \in \mathbb{N}$. By $(\mathbb{E}_{\mathbf{Q}_0}[V_\sigma^{\mathbf{Q}(m_k(\infty))}])_{k\in\mathbb{N}}$ we denote the sequence converging to $\mathbb{E}_{\mathbf{Q}_0}[V_\sigma]$.
As for fixed $k \in \mathbb{N}$ we have $A_N^k \subset A_{N+1}^k$, we can conclude that $\mathbf{Q}_0(A_N^k) \uparrow 1$ as N tends to infinity. Fix $k \in \mathbb{N}$ such that

$$\left|\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}\right]\right| < \frac{\varepsilon}{8}, \text{ for all } n \in \bar{\mathbb{N}}. \tag{2.59}$$

By uniform integrability of $\mathcal{G}$, there exists $\delta = \delta(\varepsilon)$ such that for all $\Lambda \in \mathcal{F}_T$ satisfying $\mathbf{Q}_0(\Lambda) < \delta$, we get

$$\mathbb{E}_{\mathbf{Q}_0}\left[\left|V_{\sigma_n}^{\mathbf{Q}(m_k(n_0))} - V_\sigma^{\mathbf{Q}(m_k(n_0))}\right| \mathbb{1}_\Lambda\right] < \frac{\varepsilon}{8}, \tag{2.60}$$

for all $n, n_0 \in \bar{\mathbb{N}}$. Since $\mathbf{Q}_0(A_N^k) \uparrow 1$ as N tends to infinity, there exists $N_0 = N_0(\varepsilon, k(\varepsilon)) \in \mathbb{N}$ such that $\mathbf{Q}_0((A_N^k)^c) < \delta$ for all $N \geq N_0$. Fix $N \geq N_0$ and let $n \geq N$. Then we have

$$\begin{aligned}\mathbb{E}_{\mathbf{Q}_0}\left[\left|V_{\sigma_n}^{\mathbf{Q}(m_k(n_0))} - V_\sigma^{\mathbf{Q}(m_k(n_0))}\right|\right] &= \mathbb{E}_{\mathbf{Q}_0}\left[\left|V_{\sigma_n}^{\mathbf{Q}(m_k(n_0))} - V_\sigma^{\mathbf{Q}(m_k(n_0))}\right| \mathbb{1}_{A_N^k}\right] \\ &\quad + \mathbb{E}_{\mathbf{Q}_0}\left[\left|V_{\sigma_n}^{\mathbf{Q}(m_k(n_0))} - V_\sigma^{\mathbf{Q}(m_k(n_0))}\right| \mathbb{1}_{(A_N^k)^c}\right] \\ &< \frac{\varepsilon}{4}, \end{aligned} \tag{2.61}$$

because of (2.58) and (2.60), where we used that $\mathbf{Q}_0((A_N^k)^c) < \delta$. We consider three different cases.

Case 1: Assume

$$\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right] \leq \mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}^{\mathbf{Q}(m_k(\infty))}\right]. \tag{2.62}$$

Then we have

$$\begin{aligned}&\left|\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma\right]\right| \\ &\leq \left|\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}^{\mathbf{Q}(m_k(\infty))}\right]\right| + \left|\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}^{\mathbf{Q}(m_k(\infty))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma^{\mathbf{Q}(m_k(\infty))}\right]\right| \\ &+ \left|\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma^{\mathbf{Q}(m_k(\infty))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma\right]\right| \\ &< \left|\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right]\right| + \left|\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}^{\mathbf{Q}(m_k(\infty))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma^{\mathbf{Q}(m_k(\infty))}\right]\right| + \frac{\varepsilon}{8} \\ &< \frac{2\varepsilon}{8} + \mathbb{E}_{\mathbf{Q}_0}\left[\left|V_{\sigma_n}^{\mathbf{Q}(m_k(\infty))} - V_\sigma^{\mathbf{Q}(m_k(\infty))}\right| \mathbb{1}_{A_N^k}\right] + \mathbb{E}_{\mathbf{Q}_0}\left[\left|V_{\sigma_n}^{\mathbf{Q}(m_k(\infty))} - V_\sigma^{\mathbf{Q}(m_k(\infty))}\right| \mathbb{1}_{(A_N^k)^c}\right] \end{aligned} \tag{2.63}$$

$$< \frac{4\varepsilon}{8} < \varepsilon$$

The first part of the second inequality holds due to (2.62) and to the fact that V_{σ_n} is the essential supremum over all consistent price systems. The second part of the second inequality holds because of (2.59). Also (2.63) holds due to (2.59). In the last step we applied (2.61).

Case 2: Assume

$$\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma^{\mathbf{Q}(m_k(n))}\right] \geq \mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma^{\mathbf{Q}(m_k(\infty))}\right]. \tag{2.64}$$

Then we have

$$
\begin{aligned}
&\left|\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}\right]-\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma\right]\right| \\
\leq&\left|\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}\right]-\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right]\right|+\left|\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right]-\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma^{\mathbf{Q}(m_k(n))}\right]\right| \\
+&\left|\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma^{\mathbf{Q}(m_k(n))}\right]-\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma\right]\right| \\
<&\frac{\varepsilon}{8}+\left|\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right]-\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma^{\mathbf{Q}(m_k(n))}\right]\right|+\left|\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma^{\mathbf{Q}(m_k(\infty))}\right]-\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma\right]\right| \\
<&\frac{2\varepsilon}{8}+\mathbb{E}_{\mathbf{Q}_0}\left[\left|V_{\sigma_n}^{\mathbf{Q}(m_k(n))}-V_\sigma^{\mathbf{Q}(m_k(n))}\right|\mathbb{1}_{A_N^k}\right]+\mathbb{E}_{\mathbf{Q}_0}\left[\left|V_{\sigma_n}^{\mathbf{Q}(m_k(n))}-V_\sigma^{\mathbf{Q}(m_k(n))}\right|\mathbb{1}_{(A_N^k)^c}\right] \\
<&\frac{4\varepsilon}{8}<\varepsilon
\end{aligned}
$$

The steps in Case 2 are analogously to Case 1, replacing (2.62) by (2.64).

Case 3: Assume

$$
\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right]>\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}^{\mathbf{Q}(m_k(\infty))}\right] \text{ and } \mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma^{\mathbf{Q}(m_k(n))}\right]<\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma^{\mathbf{Q}(m_k(\infty))}\right]. \tag{2.65}
$$

Then we have

$$
\begin{aligned}
&\left|\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}\right]-\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma\right]\right| \\
\leq&\left|\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}\right]-\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right]\right|+\left|\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right]-\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma^{\mathbf{Q}(m_k(\infty))}\right]\right| \\
+&\left|\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma^{\mathbf{Q}(m_k(\infty))}\right]-\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma\right]\right| \\
<&\frac{\varepsilon}{8}+\left|\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right]-\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma^{\mathbf{Q}(m_k(\infty))}\right]\right|+\frac{\varepsilon}{8}
\end{aligned}
$$

The second inequality holds due to (2.59). For the remaining part, we obtain by (2.61) that

$$
\begin{aligned}
&\left|\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right]-\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma^{\mathbf{Q}(m_k(\infty))}\right]\right| \\
\leq&\left|\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right]-\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma^{\mathbf{Q}(m_k(n))}\right]\right|+\left|\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma^{\mathbf{Q}(m_k(n))}\right]-\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma^{\mathbf{Q}(m_k(\infty)}\right]\right| \\
<&\frac{2\varepsilon}{8}+\left|\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma^{\mathbf{Q}(m_k(\infty))}\right]-\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma^{\mathbf{Q}(m_k(n))}\right]\right|.
\end{aligned}
$$

Then (2.65) and (2.61) imply

$$
\begin{aligned}
&\left|\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma^{\mathbf{Q}(m_k(\infty))}\right]-\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma^{\mathbf{Q}(m_k(n))}\right]\right|=\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma^{\mathbf{Q}(m_k(\infty))}\right]-\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma^{\mathbf{Q}(m_k(n))}\right] \\
<&\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma^{\mathbf{Q}(m_k(\infty))}\right]-\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right]+\frac{2\varepsilon}{8} \\
<&\left|\mathbb{E}_{\mathbf{Q}}\left[V_\sigma^{\mathbf{Q}(m_k(\infty))}\right]-\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}^{\mathbf{Q}(m_k(\infty))}\right]\right|+\frac{2\varepsilon}{8}<\frac{4\varepsilon}{8}.
\end{aligned}
$$

Combining these steps we obtain

$$
\left|\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}\right]-\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma\right]\right|<\frac{4\varepsilon}{8}+\left|\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma^{\mathbf{Q}(m_k(\infty))}\right]-\mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma^{\mathbf{Q}(m_k(n))}\right]\right|<\varepsilon.
$$

Since the three cases cover all possible scenarios we have

$$\left|\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma\right]\right| < \varepsilon, \quad \text{for all } n \geq N_0(\varepsilon, k(\varepsilon)).$$

Because $\varepsilon > 0$ was arbitrary, we can conclude that

$$\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}\right] \xrightarrow{n\to\infty} \mathbb{E}_{\mathbf{Q}_0}\left[V_\sigma\right].$$

This implies that $(V_t)_{t\in[0,T]}$ admits a right-continuous modification with respect to $\mathbf{Q}_0$ and hence also with respect to $\mathbf{P}$. □

We now formulate Assumption 2.18 and Theorem 2.19 in the numéraire-free setting.

Assumption 2.20. We assume the existence of $\mathbf{Q}_0 \in \mathcal{Q}$ such that for any decreasing sequence of stopping times $0 \leq (\sigma_n)_{n\in\mathbb{N}} \leq T$ with $\sigma_n \downarrow \sigma$ as n tends to infinity, there exists a sequence

$$(\mathbf{Q}(m_k(n)), \tilde{S}^{\mathbf{Q}}(m_k(n))_{k\in\mathbb{N}} \subset \mathrm{CPS}(0,T), \; n \in \bar{\mathbb{N}},$$

such that

$$\left(\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right]\right)_{k\in\mathbb{N}}$$

converges uniformly over all $n \in \bar{\mathbb{N}}$ to $\mathbb{E}_{\mathbf{Q}_0}[V_{\sigma_n}]^2$, i.e., for all $\varepsilon > 0$ exists $K = K(\varepsilon) \in \mathbb{N}$ such that

$$\left|\mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[V_{\sigma_n}\right]\right| < \varepsilon, \text{ for all } k \geq K, \text{ and for all } n \in \bar{\mathbb{N}}, \tag{2.66}$$

and that for all $k \in \mathbb{N}$, $\mathbf{Q}_0(\bigcup_{N\in\mathbb{N}} A_N^{\varepsilon,k}) = 1$, where

$$A_N^{\varepsilon,k} := \left\{\omega \in \Omega : |V_{\sigma_n}^{\mathbf{Q}(m_k(n_0))} - V_{\sigma}^{\mathbf{Q}(m_k(n_0))}|(\omega) < \varepsilon, \; \forall n \geq N, \; \forall n_0 \in \bar{\mathbb{N}}\right\}. \tag{2.67}$$

Theorem 2.21. *Let Assumption 1.5 and 2.20 hold and $s \in [0,T]$. Let $X_T = (X_T^1, X_T^2) \in L^\infty(\mathcal{F}_T, \boldsymbol{P}; \mathbb{R}^2)$ such that*

$$X_T^1 + (X_T^2)^+(1-\lambda)S_T - (X_T^2)^-(1+\lambda)S_T \geq -M_s^1 - M_s^2 S_T \tag{2.68}$$

for some $(M_s^1, M_s^2) \in L_+^1(\mathcal{F}_s, \mathcal{Q}) \times L_+^\infty(\mathcal{F}_s, \mathcal{Q})$. Then

$$\left(\operatorname*{ess\,sup}_{(\boldsymbol{Q},\tilde{S}^{\boldsymbol{Q}})\in CPS} \mathbb{E}_{\boldsymbol{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\boldsymbol{Q}} \mid \mathcal{F}_t\right]\right)_{t\in[s,T]}$$

admits a right-continuous modification with respect to $\boldsymbol{P}$.

Proof. The proof of Theorem 2.19 carries over using Assumption 1.5 and 2.20. □

Theorem 2.19 (resp. Theorem 2.21) provides sufficient conditions such that the process V admits a right-continuous modification.
Next, we give an example where Assumption 2.20 is fulfilled.

[2] Analogously to (2.41), we write $V_t = \operatorname{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\mathrm{CPS}} \mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_t\right]$, $t \in [0,T]$.

Example 2.22. Suppose Assumption 1.5 holds and let $(X_T^1, X_T^2) = (0,1)$. Then, for each $k \in \mathbb{N}$ such that $\frac{1}{k} \leq \lambda$ there exists $(\mathbf{Q}(k), \tilde{S}^{\mathbf{Q}}(k)) \in \mathrm{CPS}(0,T,\frac{1}{k})$. We get

$$(1-\lambda)S_t \leq \tilde{S}_t^{\mathbf{Q}}(k) \leq \frac{1+\lambda}{1+\frac{1}{k}}\tilde{S}_t^{\mathbf{Q}}(k) \leq (1+\lambda)S_t, \quad t \in [0,T],\ k \in \mathbb{N}. \tag{2.69}$$

In particular, $(\mathbf{Q}(k), \mu_k\tilde{S}^{\mathbf{Q}}(k)) \in \mathrm{CPS}(0,T,\lambda)$ for $\mu_k := \frac{1+\lambda}{1+\frac{1}{k}}$. By the martingale property of consistent price systems in the non-local sense we have

$$\operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\mathrm{CPS}} \mathbb{E}_{\mathbf{Q}}\left[\tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_t\right] = \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\mathrm{CPS}} \tilde{S}_t^{\mathbf{Q}} \leq (1+\lambda)S_t.$$

Furthermore, we obtain by the martingale property of $\tilde{S}^{\mathbf{Q}}(k)$ that

$$\begin{aligned}
&\left|\operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\mathrm{CPS}} \mathbb{E}_{\mathbf{Q}}\left[\tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_t\right] - \mathbb{E}_{\mathbf{Q}(k)}\left[\mu_k\tilde{S}_T^{\mathbf{Q}}(k) \mid \mathcal{F}_t\right]\right| \leq \left|(1+\lambda)S_t - \mathbb{E}_{\mathbf{Q}(k)}\left[\mu_k\tilde{S}_T^{\mathbf{Q}}(k) \mid \mathcal{F}_t\right]\right| \\
=&\left|(1+\lambda)S_t - \mu_k\tilde{S}_t^{\mathbf{Q}}(k)\right| \leq \left|(1+\lambda)S_t - \mu_k\left(1-\frac{1}{k}\right)S_t\right| = \left|(1+\lambda)S_t\left(1-\frac{1-\frac{1}{k}}{1+\frac{1}{k}}\right)\right| \\
=&(1+\lambda)S_t\frac{2}{k+1}.
\end{aligned}$$

Then we get for any $\mathbf{Q}_0 \in \mathcal{Q}$, and $t \in [0,T]$ that

$$\begin{aligned}
&\left|\mathbb{E}_{\mathbf{Q}_0}[V_t] - \mathbb{E}_{\mathbf{Q}_0}\left[\mu_k V_t^{\mathbf{Q}(k)}\right]\right| \leq \mathbb{E}_{\mathbf{Q}_0}\left[\left|(1+\lambda)S_t - \mathbb{E}_{\mathbf{Q}(k)}\left[\mu_k\tilde{S}_T^{\mathbf{Q}}(k) \mid \mathcal{F}_t\right]\right|\right] \\
\leq&\mathbb{E}_{\mathbf{Q}_0}\left[(1+\lambda)S_t\frac{2}{k+1}\right] \leq \mathbb{E}_{\mathbf{Q}_0}\left[\frac{1+\lambda}{1-\lambda}\frac{2}{k+1}\tilde{S}_t^{\mathbf{Q}_0}\right] = \frac{1+\lambda}{1-\lambda}\frac{2}{k+1}\tilde{S}_0^{\mathbf{Q}_0} \\
\leq&\frac{(1+\lambda)^2}{1-\lambda}\frac{2}{k+1}S_0.
\end{aligned} \tag{2.70}$$

Therefore, we can easily see that for for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that (2.66) of Assumption 2.20 is fulfilled by the sequence $(\mathbf{Q}(k), \mu_k\tilde{S}^{\mathbf{Q}}(k))_{k\in\mathbb{N}} \subset \mathrm{CPS}(0,T)$ which is independent of $t \in [0,T]$. Let $(\sigma_n)_{n\in\mathbb{N}}$ be any decreasing sequence of stopping times. Note that in this case we get that A_N^k does not depend on $n_0 \in \mathbb{N}$ anymore, i.e.,

$$A_N^k = \left\{\omega \in \Omega : |V_{\sigma_n}^{\mathbf{Q}(k)} - V_{\sigma}^{\mathbf{Q}(k)}|(\omega) < \varepsilon,\ \forall n \geq N\right\}, \tag{2.71}$$

By Definition 1.1 all consistent price systems are càdlàg which yields that $\mathbf{Q}_0(\bigcup_{N\in\mathbb{N}} A_N^k) = 1$. We conclude that under Assumption 1.5 also Assumption 2.20 is fulfilled for $X_T = (0,1)$.

It is worth noting that the process V does have the exact same meaning as the super-replication price in the frictionless setting, see [37], [71]. More concretely, let $S = (S_t)_{t\in[0,T]}$ be a non-negative, locally bounded, càdlàg semimartingale, such that the set of equivalent local martingale measures, $\mathcal{M}_{\mathrm{loc}}(S)$, is non-empty. Then the process $\tilde{V} = (\tilde{V}_t)_{t\in[0,T]} = (\operatorname{ess\,sup}_{\mathbf{Q}\in\mathcal{M}_{\mathrm{loc}}(S)} \mathbb{E}_{\mathbf{Q}}[Y_T \mid \mathcal{F}_t])_{t\in[0,T]}$ defines the wealth of the minimal hedging strategy for a contingent claim Y_T, see Theorem 3.2 of [71]. Further, $\tilde{V}$ is the capital of a self-financing portfolio if and only if $\tilde{V}$ is a local $\mathbf{Q}$-martingale for all $\mathbf{Q} \in \mathcal{M}_{\mathrm{loc}}(S)$. In particular, $\tilde{V}$ indicates the liquidation value of the portfolio and the required capital to start the portfolio.

Under the presence of transaction costs this symmetry fails. The process V defines the capital that is needed for a self-financing, admissible strategy to super-replicate the contingent claim. The liquidation value is usually lower than the required capital.

Chapter 3

Asset price bubbles under proportional transaction costs

This chapter is based on [18]. We define the fundamental value of the asset and introduce the notion of asset price bubbles in Section 3.1. Using the results from Chapter 2, we obtain a dual representation of the fundamental value from which we derive further properties of asset price bubbles in Section 3.2. In Section 3.3, we provide several examples to illustrate our notion of asset price bubbles and to compare it to the frictionless market model. Finally, in Section 3.4, we elaborate the comparison to the frictionless case and discuss the impact of proportional transaction costs on bubbles' formation.

3.1 Fundamental value and asset price bubbles

The term *asset price bubble* is well-known, however, the opinions differ on the exact definition. Most definitions have two ingredients, namely the market price of an asset and its *fundamental* value, where an asset price bubble is defined by the difference of the two. We assume that the market price is given by the price process $S = (S_t)_{t\in[0,T]}$. Following the approach of [52] in the frictionless case, we define the fundamental value by the super-replication price of the asset. A priori, the meaning of "super-repliaction price of the asset" is not clear, as both components of trading strategies are specified in our setting.
In market models without transaction costs, holding the asset or having the market value of the asset in the bank account is equivalent. In contrast, under proportional transaction costs this does no longer hold. One share of the asset at time $t \in [0,T]$ costs $(1+\lambda)S_t$. For selling one share of the asset at time $t \in [0,T]$ a trader only receives $(1-\lambda)S_t$.

Definition 3.1. The *fundamental value* $F = (F_t)_{t\in[0,T]}$ of an asset S at time $t \in [0,T]$ in a market model with proportional transaction costs $0 < \lambda < 1$ is defined by

$$F_t := \operatorname{ess\,inf}\left\{X_t \in L^1_+(\mathcal{F}_t, \mathcal{Q}_{\mathrm{loc}}) :\ \exists \varphi \in \mathcal{V}^{\mathrm{loc}}_{t,T}(X_t,\lambda) \text{ with } \varphi_t = (X_t,0) \text{ and } \varphi_T = (0,1)\right\}.$$

We say there is an asset price bubble in the market model with transaction costs if $\mathbf{P}(F_\sigma < (1+\lambda)S_\sigma) > 0$ for some stopping time σ with values in $[0,T]$. We define the *asset price bubble* as the process $\beta = (\beta_t)_{0\le t\le T}$ given by

$$\beta_t := (1+\lambda)S_t - F_t, \quad t \in [0,T]. \tag{3.1}$$

Remark 3.2. *In Definition 4.2 of [46], the authors provide a robust definition of an asset price bubble, which can also be interpreted as a bubble under proportional transaction costs. In [46], only one component of the trading strategies is specified. Further, the authors consider a worst case scenario in the sense that the strategy begins in cash, but initial capital is all in stock and the strategy ends in cash, but the trader has to deliver one share of the asset. Transferring this to our setting, this means to super-replicate the position* $((1+\lambda)S_T, 0)$ *with initial endowment* $(0, x)$.
Specifying both components of the trading strategies in our model allows to stress out the differences of possible positions associated to S *to be super-replicated, see below.*

Proposition 3.3. *Under Assumption 1.4, we have that the fundamental value* $F = (F_t)_{t\in[0,T]}$ *is such that*

$$F_t \le (1+\lambda)S_t, \quad t \in [0,T],$$

and $F_t \in L^1(\mathcal{F}_t, \mathcal{Q}_{loc})$, $t \in [0,T]$. *Moreover, the bubble* $\beta = (\beta_t)_{t\in[0,T]}$ *has almost surely non-negative paths.*

Proof. Consider the buy and hold strategy starting at time $t \in [0,T]$. With an initial endowment $\varphi_t = ((1+\lambda)S_t, 0)$ it is possible to buy one share of the asset at time t and keep it until the terminal time T. Thus, the buy and hold strategy defines a possible trading strategy and hence $F_t \le (1+\lambda)S_t$ for all $t \in [0,T]$. Therefore, the bubble has almost surely non-negative paths. The fact that $F_t \in L^1_+(\mathcal{F}_t, \mathcal{Q}_{\text{loc}})$ follows by

$$F_t \le (1+\lambda)S_t \le \frac{1+\lambda}{1-\lambda}\tilde{S}^{\mathbf{Q}}_t, \quad \text{for all } (\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \text{CPS}_{\text{loc}}.$$

□

The definition of the fundamental value for the ask-price, Definition 3.1, requires some more explanation. There are arguably other reasonable positions to super-replicate in the context of asset price bubbles. We consider $\overline{\varphi}_T = ((1+\lambda)S_T, 0)$, $\varphi_T = (0,1)$, and $\underline{\varphi}_T = ((1-\lambda)S_T, 0)$. The first one, $\overline{\varphi}_T = ((1+\lambda)S_T, 0)$ corresponds to Definition 4.2 of [46] and seems to be too high, as it represents a worst case scenario. Super-replicating $\overline{\varphi}_T$ is only reasonable if the trader wants to hold one share of the asset at time T. For this purpose, the strategy to buy the share of the asset at time T might be too expensive. Conversely, $\underline{\varphi}_T$ equals the liquidation value of one share of the asset at time T. Thus, super-replicating $\underline{\varphi}_T$ is for traders who are aiming for cash. As it is not possible to re-buy a share of the asset with the capital $(1-\lambda)S_T$, we do not consider it as a suitable value to define the fundamental value.
Morally, position $\overline{\varphi}_T$ allows to buy on share of the asset and end up with position φ_T. Also liquidating position φ_T leads to $(1-\lambda)S_T$ in the bank account, i.e., we obtain position $\underline{\varphi}_T$. In particular, the super-replication prices of the positions $\overline{\varphi}_T$, φ_T, $\underline{\varphi}_T$ are ordered such that by Proposition 2.8 we have

$$\sup_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\text{CPS}_{\text{loc}}} \mathbb{E}_{\mathbf{Q}}\left[(1+\lambda)S_T\right] \ge \sup_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\text{CPS}_{\text{loc}}} \mathbb{E}_{\mathbf{Q}}\left[\tilde{S}^{\mathbf{Q}}_T\right] \ge \sup_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\text{CPS}_{\text{loc}}} \mathbb{E}_{\mathbf{Q}}\left[(1-\lambda)S_T\right].$$

We interpret the super-replication price of the position $\varphi_T = (0,1)$ as the amount a trader is willing to pay if she had to hold the asset in her portfolio until the terminal time T, see [59]. This definition also allows to model bubble birth, as in [12] and [52], as shown in Example 3.18.

3.2 Properties of the fundamental value and bubbles

In this section we study some basic properties of the fundamental value and of asset price bubbles in our setting. Using the results from Section 2 we obtain a dual representation for the fundamental value F. In Proposition 3.5, we provide sufficient conditions for the absence of asset price bubbles. Further, Proposition 3.6 guarantees that the rise of transaction costs cannot lead to the occurrence of bubbles. The section is concluded by Theorem 3.9 which provides sufficient conditions such that the fundamental value admits a right-continuous modification.

Corollary 3.4. *Suppose Assumption 1.4 holds. Then we have for all $t \in [0,T]$ that*

$$F_t = \operatorname*{ess\,sup}_{(\boldsymbol{Q},\tilde{S}^{\boldsymbol{Q}})\in CPS_{loc}} \mathbb{E}_{\boldsymbol{Q}}\left[\tilde{S}_T^{\boldsymbol{Q}} \mid \mathcal{F}_t\right].$$

Proof. For $X_T = (0,1)$ we have that

$$X_T^1 + (X_T^2)^+(1-\lambda)S_T - (X_T^2)^-(1+\lambda)S_T = (1-\lambda)S_T \geq 0,$$

and because all $\tilde{S}^{\mathbf{Q}}$ such that $(\mathbf{Q},\tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}_{\mathrm{loc}}(0,T)$ are local $\mathbf{Q}$-martingales we get

$$\operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\mathrm{CPS}_{\mathrm{loc}}} \mathbb{E}_{\mathbf{Q}}\left[\tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_t\right] \leq \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\mathrm{CPS}_{\mathrm{loc}}} \tilde{S}_t^{\mathbf{Q}} \leq (1+\lambda)S_t \in L^1(\mathcal{F}_t,\mathcal{Q}_{\mathrm{loc}}).$$

In particular, the conditions of Proposition 2.8 are fulfilled and we get the desired identity. ☐

Proposition 3.5. *Let Assumption 1.5 hold. Then we have for all stopping times $0 \leq \sigma \leq T$ that*

$$\operatorname*{ess\,sup}_{(\boldsymbol{Q},\tilde{S}^{\boldsymbol{Q}})\in CPS} \mathbb{E}_{\boldsymbol{Q}}\left[\tilde{S}_T^{\boldsymbol{Q}} \mid \mathcal{F}_\sigma\right] = (1+\lambda)S_\sigma.$$

In particular, there is no asset price bubble in the market model.

Proof. Let $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} \leq \lambda$. Assumption 1.5 guarantees the existence of $(\mathbf{Q}^n,\tilde{S}^n)_{n\in\mathbb{N}}$ such that $(\mathbf{Q}^n,\tilde{S}^n) \in \mathrm{CPS}(0,T,\frac{1}{n})$ for all $n \geq n_0$. We consider the sequence $(\mathbf{Q}^n,\mu_n\tilde{S}^n)_{n\in\mathbb{N}} \subset \mathrm{CPS}(0,T,\lambda)$, where $\mu_n := \frac{1+\lambda}{1+\frac{1}{n}}$, for $n \geq n_0$. In fact, for $t \in [0,T]$ we have

$$(1-\lambda)S_t \leq \left(1-\frac{1}{n}\right)S_t \leq \tilde{S}_t^n \leq \mu_n\tilde{S}_t^n \leq \mu_n\left(1+\frac{1}{n}\right)S_t = (1+\lambda)S_t.$$

By Proposition 3.3 and by because $\mu_n\tilde{S}^n$ is a $\mathbf{Q}^n$-martingale for all $n \geq n_0$ we have that

$$(1+\lambda)S_\sigma \geq \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\mathrm{CPS}} \mathbb{E}_{\mathbf{Q}}\left[\tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_\sigma\right] \geq \operatorname*{ess\,sup}_{n\geq n_0} \mathbb{E}_{\mathbf{Q}^n}\left[\mu_n\tilde{S}_T^n \mid \mathcal{F}_\sigma\right] = \operatorname*{ess\,sup}_{n\geq n_0} \mu_n\tilde{S}_\sigma^n.$$

Then we get for the essential supremum that

$$\begin{aligned}\left|(1+\lambda)S_\sigma - \operatorname*{ess\,sup}_{n\geq n_0} \mu_n \tilde{S}^n_\sigma\right| &\leq \left|(1+\lambda)S_\sigma - \operatorname*{ess\,sup}_{n\geq n_0} \mu_n \left(1-\frac{1}{n}\right) S_\sigma\right| \\ &= \left|(1+\lambda)S_\sigma \left(1 - \operatorname*{ess\,sup}_{n\geq n_0} \frac{1-\frac{1}{n}}{1+\frac{1}{n}}\right)\right| = 0.\end{aligned}$$

Hence we can conclude that

$$\operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in \mathrm{CPS}} \mathbb{E}_{\mathbf{Q}}\left[\tilde{S}^{\mathbf{Q}}_T \mid \mathcal{F}_\sigma\right] = (1+\lambda)S_\sigma.$$

□

Proposition 3.6. *Let Assumption 1.4 hold. If there exists $\lambda_0 \in (0,1)$ such that there is no bubble in the market model with transaction costs λ_0, then there is no bubble in the market model with transaction costs $\lambda > \lambda_0$.*

Proof. Suppose that for $\lambda_0 \in (0,1)$ there is no bubble in the market model with proportional transaction costs λ_0, i.e.,

$$F^{\lambda_0}_\sigma = \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in \mathrm{CPS}_{\mathrm{loc}}(0,T,\lambda_0)} \mathbb{E}_{\mathbf{Q}}[\tilde{S}^{\mathbf{Q}}_T \mid \mathcal{F}_\sigma] = (1+\lambda_0)S_\sigma,$$

for all $[0,T]$-valued stopping times σ. Fix some $\lambda > \lambda_0$. Clearly, $\mathrm{CPS}_{\mathrm{loc}}(0,T,\lambda_0) \subseteq \mathrm{CPS}_{\mathrm{loc}}(0,T,\lambda)$. Define $c \in \mathbb{R}$ by

$$c := \frac{1+\lambda}{1+\lambda_0}.$$

Then $c > 1$ and $c(1+\lambda_0) = (1+\lambda)$. Let $(\mathbf{Q},\tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}_{\mathrm{loc}}(0,T,\lambda_0)$ be arbitrary. It is easy to see that $(\mathbf{Q},c\tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}_{\mathrm{loc}}(0,T,\lambda)$ as

$$(1-\lambda)S_t \leq (1-\lambda_0)S_t \leq \tilde{S}^{\mathbf{Q}}_t \leq c\tilde{S}^{\mathbf{Q}}_t \leq c(1+\lambda_0)S_t \leq (1+\lambda)S_t, \quad t\in[0,T].$$

This yields

$$\begin{aligned}F^\lambda_\sigma &= \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in \mathrm{CPS}_{\mathrm{loc}}(0,T,\lambda)} \mathbb{E}_{\mathbf{Q}}[\tilde{S}^{\mathbf{Q}}_T \mid \mathcal{F}_\sigma] \geq \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in \mathrm{CPS}_{\mathrm{loc}}(0,T,\lambda_0)} \mathbb{E}_{\mathbf{Q}}[c\tilde{S}^{\mathbf{Q}}_T \mid \mathcal{F}_\sigma] \\ &= c(1+\lambda_0)S_\sigma = (1+\lambda)S_\sigma.\end{aligned}$$

□

Proposition 3.6 guarantees that a rise of transaction costs does not yield bubbles' formation.

Proposition 3.7. *If the asset price $S = (S_t)_{t\in[0,T]}$ is a semimartingale and the set $\mathcal{M}_{loc}(S)$ of equivalent local martingale measures for S is not empty, then $(\boldsymbol{Q},\mu S) \in CPS_{loc}(0,T)$ for $\boldsymbol{Q} \in \mathcal{M}_{loc}(S)$ and $\mu \in [1-\lambda, 1+\lambda]$, and*

$$F_\sigma = \operatorname*{ess\,sup}_{(\boldsymbol{Q},\tilde{S}^{\boldsymbol{Q}})\in CPS_{loc}} \mathbb{E}_{\boldsymbol{Q}}\left[\tilde{S}^{\boldsymbol{Q}}_T \mid \mathcal{F}_\sigma\right] \geq \operatorname*{ess\,sup}_{\boldsymbol{Q}\in\mathcal{M}_{loc}(S)} \mathbb{E}_{\boldsymbol{Q}}\left[\mu S_T \mid \mathcal{F}_\sigma\right], \tag{3.2}$$

for all $[0,T]$-valued stopping times σ.

Proof. Equation (3.2) immediately follows by the observation that

$$\{(\mathbf{Q},\mu S) : \mathbf{Q} \in \mathcal{M}_{\text{loc}}(S), \mu \in [1-\lambda, 1+\lambda]\} \subseteq \text{CPS}_{\text{loc}}(0,T). \tag{3.3}$$

□

Definition 3.8. Let $D \subseteq \mathbb{R}$ be an open set in $\mathbb{R}$. A function $f : D \to \mathbb{R}$ is said to be upper *semi-continuous* at $x \in D$ if

$$\limsup_{y\to x} f(y) \le f(x). \tag{3.4}$$

We say that f is upper semi-continuous from the right at $x \in D$, if (3.4) holds for $y \downarrow x$. Further, f is called upper semi-continuous (from the right) if f is upper semi-continuous (from the right) for all $x \in D$.

Note that Theorem 2.19 and 2.21 also provide sufficient conditions such that the super-replication price process admits a right-continuous modification. In Example 2.22, it is shown that Theorem 2.21 can be applied for $X_T = (0,1)$ if Assumption 1.5 is in place. However, if Assumption 1.5 is in place, there is no asset price bubble in the market model by Proposition 3.5. Under Assumption 1.4 it is not clear if Theorem 2.19 is applicable.

Theorem 3.9. *Suppose that Assumption 1.4 holds and assume that the function*

$$\varphi(t) := \sup_{(\boldsymbol{Q},\tilde{S}^{\boldsymbol{Q}})\in CPS_{loc}} \mathbb{E}_{\boldsymbol{P}}\left[\mathbb{E}_{\boldsymbol{Q}}\left[\tilde{S}_T^{\boldsymbol{Q}} \mid \mathcal{F}_t\right]\right], \quad t \in [0,T], \tag{3.5}$$

is upper semi-continuous from the right. Then $F = (F_t)_{t\in[0,T]}$ admits a right-continuous modification with respect to $\boldsymbol{P}$.

Proof. By Theorem 48 in [36], F admits a right-continuous modification with respect to $\mathbf{P}$ if and only if for every decreasing sequence $(\beta_n)_{n\in\mathbb{N}}$ of bounded stopping times $\lim_{n\to\infty} \mathbb{E}_{\mathbf{P}}\left[F_{\beta_n}\right] = \mathbb{E}_{\mathbf{P}}\left[F_{\lim_{n\to\infty}\beta_n}\right]$. By Lemma 2.15, we get that

$$\mathbb{E}_{\mathbf{P}}\left[\operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\text{CPS}_{\text{loc}}} \mathbb{E}_{\mathbf{Q}}\left[\tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_\sigma\right]\right] = \sup_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\text{CPS}} \mathbb{E}_{\mathbf{P}}\left[\mathbb{E}_{\mathbf{Q}}\left[\tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_\sigma\right]\right], \tag{3.6}$$

for all stopping times σ with values in $[0,T]$. Let now $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of stopping times with values in $[0,T]$ such that $\sigma_n \downarrow \sigma$ as n tends to infinity. We now prove that

$$\lim_{n\to\infty} \mathbb{E}_{\mathbf{P}}\left[F_{\sigma_n}\right] = \mathbb{E}_{\mathbf{P}}\left[F_\sigma\right].$$

Since $\left(\mathbb{E}_{\mathbf{Q}}[\tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_t]\right)_{\sigma\le t\le T}$ is right-continuous we get by Fatou's lemma that

$$\begin{aligned}\liminf_{n\to\infty} \sup_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\text{CPS}_{\text{loc}}} \mathbb{E}_{\mathbf{P}}\left[\mathbb{E}_{\mathbf{Q}}\left[\tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_{\sigma_n}\right]\right] &\ge \liminf_{n\to\infty} \mathbb{E}_{\mathbf{P}}\left[\mathbb{E}_{\mathbf{Q}}\left[\tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_{\sigma_n}\right]\right] \\ &\ge \mathbb{E}_{\mathbf{P}}\left[\liminf_{n\to\infty} \mathbb{E}_{\mathbf{Q}}\left[\tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_{\sigma_n}\right]\right] = \mathbb{E}_{\mathbf{P}}\left[\mathbb{E}_{\mathbf{Q}}\left[\tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_\sigma\right]\right],\end{aligned} \tag{3.7}$$

for arbitrary $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}_{\mathrm{loc}}$. Because (3.7) holds for all $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}_{\mathrm{loc}}(0,T)$ we get that

$$\begin{aligned}\liminf_{n\to\infty} \mathbb{E}_{\mathbf{P}}\left[F_{\sigma_n}\right] &= \liminf_{n\to\infty} \sup_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\mathrm{CPS}_{\mathrm{loc}}} \mathbb{E}_{\mathbf{P}}\left[\mathbb{E}_{\mathbf{Q}}\left[\tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_{\sigma_n}\right]\right] \\ &\geq \sup_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\mathrm{CPS}_{\mathrm{loc}}} \mathbb{E}_{\mathbf{P}}\left[\mathbb{E}_{\mathbf{Q}}\left[\tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_{\sigma}\right]\right] = \mathbb{E}_{\mathbf{P}}\left[F_{\sigma}\right],\end{aligned} \tag{3.8}$$

where the last equality follows by (3.6). By the assumption of upper semi-continuity from the right we directly obtain

$$\limsup_{n\to\infty} \sup_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\mathrm{CPS}_{\mathrm{loc}}} \mathbb{E}_{\mathbf{P}}\left[\mathbb{E}_{\mathbf{Q}}\left[\tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_{\sigma_n}\right]\right] \leq \sup_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\mathrm{CPS}_{\mathrm{loc}}} \mathbb{E}_{\mathbf{P}}\left[\mathbb{E}_{\mathbf{Q}}\left[\tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_{\sigma}\right]\right]. \tag{3.9}$$

In particular, (3.9) also implies that the limit is finite, because

$$\sup_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\mathrm{CPS}_{\mathrm{loc}}} \mathbb{E}_{\mathbf{P}}\left[\mathbb{E}_{\mathbf{Q}}\left[\tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_{\sigma}\right]\right] \leq \mathbb{E}_{\mathbf{P}}[(1+\lambda)S_\sigma] < \infty.$$

Combining (3.8) and (3.9) yields by (3.6) that

$$\limsup_{n\to\infty} \mathbb{E}_{\mathbf{P}}[F_{\sigma_n}] = \mathbb{E}_{\mathbf{P}}[F_\sigma] \leq \liminf_{n\to\infty} \mathbb{E}_{\mathbf{P}}[F_{\sigma_n}].$$

□

Corollary 3.10. *Suppose that Assumption 1.4 holds and assume that there exists $\boldsymbol{Q}_0 \in \mathcal{Q}_{loc}$ such that the function*

$$\varphi(t) := \sup_{(\boldsymbol{Q},\tilde{S}^{\boldsymbol{Q}})\in CPS_{loc}} \mathbb{E}_{\boldsymbol{Q}_0}\left[\mathbb{E}_{\boldsymbol{Q}}\left[\tilde{S}_T^{\boldsymbol{Q}} \mid \mathcal{F}_t\right]\right], \quad t \in [0,T], \tag{3.10}$$

is upper semi-continuous from the right. Then $F = (F_t)_{t\in[0,T]}$ admits a right-continuous modification with respect to $\boldsymbol{P}$.

Proof. With the same arguments as in the proof of Theorem 3.9 it follows that F admits a right-contiunous modification with respect to $\mathbf{Q}_0$. Since $\mathbf{P}$ and $\mathbf{Q}_0$ are equivalent we conclude that F also admits a right-continuous modification with respect to $\mathbf{P}$. □

Remark 3.11. *In order to give the assumption of Theorem 3.9 and of Corollary 3.10 some meaning, we consider the frictionless case. In the frictionless case inequality* (3.9) *is automatically fulfilled because the super-replication price process is a supermartingale. More specifically, let S be a semimartingale such that $\mathcal{M}_{loc}(S) \neq \emptyset$, then $\tilde{F} = (\tilde{F}_t)_{t\in[0,T]}$ given by*

$$\tilde{F}_t = \operatorname*{ess\,sup}_{\boldsymbol{Q}\in\mathcal{M}_{loc}(S)} \mathbb{E}_{\boldsymbol{Q}}[S_T \mid \mathcal{F}_t], \quad t \in [0,T]$$

is a $\boldsymbol{Q}$-supermartingale for all $\boldsymbol{Q} \in \mathcal{M}_{loc}(S)$, see Proposition 4.3 of [71]. Then for any $\boldsymbol{Q}_0 \in \mathcal{M}_{loc}(S)$ it holds that

$$\mathbb{E}_{\boldsymbol{Q}_0}[F_{\sigma_n}] \leq \mathbb{E}_{\boldsymbol{Q}_0}[F_\sigma].$$

In the presence of transaction costs the supermartingale property of the fundamental value F may fail. For instance, suppose the price process S is given by the geometric fractional Brownian motion. In particular, S is no semimartingale, see [14]. Then, S has satisfies the conditional full support property and thus fulfills Assumption 1.5, see Theorem 1.2 and Proposition 4.2 of [47]. In particular, $F_t = S_t$ for all $t \in [0,T]$ by Proposition 3.5, which shows that F is no supermartingale.
Thus, we must require additional regularity conditions on the family of consistent price systems to guarantee the existence of a right-continuous modification for F.

3.3 Examples

In this section, we illustrate our setting and the impact of transaction costs on asset bubbles in several examples. Starting with Example 3.12, we present a market model where the asset price, driven by a fractional Brownian motion, has a bubble in the sense of Definition 3.1. It is well-known that the fractional Brownian motion is not a semimartingale and thus no equivalent martingale measure exists. Therefore, in this case the market model admits arbitrage without transaction costs. Thus, it is not reasonable to consider this price process in a frictionless setting. However, in in the presence of arbitrary small proportional transaction costs, it is possible to consider price processes which are driven by the fractional Brownian motion and still obtain an arbitrage-free market models.
Then we start to work out the different behavior of asset price bubbles in our setting with transaction costs and a frictionless market model. For the market model without transaction costs, we consider the setting of [52] and also the definition of asset price bubbles therein. Of particular interest in this respect is the occurrence and prevention of asset price bubbles. In Section 3.4, we then formalize the observations of the examples.
For this purpose, we start in Example 3.16 with a standard market model where the price process is a true martingale such that there is no bubble, neither with nor without transaction costs. This indicates that the introduction of transaction costs cannot generate asset price bubbles. In Example 3.17, there is a bubble in the frictionless market model in the sense of Definition 3.15 (see also Definition 3.1. of [52]) but in the presence of transaction costs the bubble disappears. Thus, it is an explicit example that transaction costs can possible prevent the appearance of bubbles. Example 3.18 shows that, similar to Example 5.4 of [52], bubble's birth is naturally included in our notion of asset price bubbles and that the presence of transaction costs does to guarantee the prevention of bubbles.

Example 3.12. We follow Example 7.1 of [46]. Let W^H be a fractional Brownian motion with Hurst index $0 < H < 1$, i.e., a centered Gaussian process with continuous sample paths an covariance function

$$\Gamma(t,s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

For $H = 1/2$, we recover the standard Brownian motion. We define $X = (X_t)_{t\geq 0}$ by $X_0 = 1$ and

$$X_t := \exp(W_t^H + \mu t), \quad t > 0,$$

for $\mu \geq 0$. Let $\mathbb{F}^X := (\mathcal{F}_t^X)_{t\geq 0}$ be the (completed) natural filtration of the process X. Define the stopping time

$$\tau := \inf\left\{t \in \mathbb{R} : X_t = \frac{1}{2}\right\},$$

and set

$$S_t := X_{\tau\wedge\tan t}, \quad 0 \leq t < \frac{\pi}{2}, \quad S_t = \frac{1}{2}, \quad t \geq \frac{\pi}{2}.$$

Fix $T \geq \pi/2$ and define $\mathcal{G}_t := \mathcal{F}_{\tan t}$, $0 \leq t < \pi/2$, and $\mathcal{G}_{\pi/2} := \mathcal{F}_\infty$. Although, X admits a consistent price system in the non-local sense on the interval $[0,T]$ for all $T > 0$ by Proposition 4.2 of [47], there exists no consistent price system in the non-local sense for S for any $\lambda \in (0,1)$. Indeed, by contradiction assume that there exists a consistent price system $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}})$ for S in the non-local sense for some $\lambda \in (0,1)$. Then we have

$$\frac{1-\lambda}{2} \leq \tilde{S}_t^{\mathbf{Q}} = \mathbb{E}_{\mathbf{Q}}\left[\tilde{S}_T^{\mathbf{Q}} \mid \mathcal{G}_t\right] \leq \frac{1+\lambda}{2} \quad \text{for all } t \in [0,T],$$

and hence also

$$\frac{1-\lambda}{2(1+\lambda)} \leq S_t \leq \frac{1+\lambda}{2(1-\lambda)},$$

which would imply that S_t is bounded for $0 < t < \pi/2$. Thus, we obtain a contradiction and conclude that there is no consistent price system in the non-local sense.

Still Assumption 1.4 is satisfied by S. By Theorem 1.2 and Proposition 4.2 of [47] X admits a continuous consistent price system $(\widehat{\mathbf{Q}}, \widehat{S}^{\widehat{\mathbf{Q}}})$ on $[0,T]$. Let $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}})$ be defined as the time-changed process obtained from $(\widehat{\mathbf{Q}}, \widehat{S}^{\widehat{\mathbf{Q}}})$. By Proposition (1.5) of [81] it is guaranteed that $(\mathbf{Q}, (\tilde{S}^{\mathbf{Q}})^\tau)$ is consistent local price price for S on $[0,T]$.

We now show that there is a bubble in this market model with transaction costs for $\lambda < 1/3$. For any consistent local price system $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}})$ of S we have

$$(1-\lambda) \leq \tilde{S}_0^{\mathbf{Q}} \leq (1+\lambda),$$

and

$$\frac{1-\lambda}{2} \leq \tilde{S}_T^{\mathbf{Q}} \leq \frac{1+\lambda}{2},$$

where we used that $S_0 = 1$ and $S_T = 1/2$. For $\lambda < 1/3$ this implies that

$$\tilde{S}_0^{\mathbf{Q}} \geq 1-\lambda > \frac{1+\lambda}{2} \geq \tilde{S}_T^{\mathbf{Q}}$$

for all consistent local price systems. Thus, we have

$$(1+\lambda)S_0 \geq \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in \mathrm{CPS}_{\mathrm{loc}}} \tilde{S}_0^{\mathbf{Q}} > \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in \mathrm{CPS}_{\mathrm{loc}}} \mathbb{E}_{\mathbf{Q}}[\tilde{S}_T^{\mathbf{Q}}]. \tag{3.11}$$

Equation (3.11) shows that

$$\beta_0 = (1+\lambda)S_0 - F_0 = (1+\lambda)S_0 - \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in \mathrm{CPS}_{\mathrm{loc}}} \mathbb{E}_{\mathbf{Q}}[\tilde{S}_T^{\mathbf{Q}}] > 0$$

which means that there is a bubble at time $t = 0$ under transaction costs $\lambda < 1/3$.

Remark 3.13. *Recall, that due to the well-known arbitrage arguments, see [33], the processes X and S in Example 3.12 cannot be considered to describe asset price dynamics in a market model without transaction costs if $H \neq 1/2$. Hence in the case a comparison with an analogous frictionless market model makes no-sense. The case $H = 1/2$ is presented [46] but with a different definition of asset price bubbles, see Remark 3.2.*

In the following, we give a brief overview of the framework of [52]. For sake of simplicity and consistency with our setting presented in Section 1.2, we may slightly adapt the setting of [52]. In particular, we assume that the asset price S is given by a càdlàg non-negative semimartingale such that $\mathcal{M}_{\mathrm{loc}}(S) \neq \emptyset$. It is well-known that this guarantess the property NFLVR, see [33]. Set $\mathbf{S} := (B, S)$ with risk-less asset $B \equiv 1$. In [52], there may be no risk-less asset and $\mathbf{S}$ is only assumed to be a semimartingale after being discounted with a generalized numéraire. Let σ be a $[0,T]$-valued stopping time and denote by ${}^{\sigma}L(\mathbf{S})$ the set of all $\mathbb{R}^2$-valued processes $\nu = (\nu_t^1, \nu_t^2)_{\sigma \le t \le T}$ which are predictable on $[\![\sigma, T]\!]$ and for which the stochastic integral process $\int_{\sigma}^{t} \nu_s d\mathbf{S}_s$, $\sigma \le t \le T$, is defined in the sense of 2-dimensional stochastic integration, see [80, Section III.6].

Definition 3.14 (Definition 2.5, [52])**.** Fix a stopping time $0 \le \sigma \le T$. The space ${}^{\sigma}L^{\mathrm{sf}}(\mathbf{S})$ of *self-financing strategies* (for $\mathbf{S}$) on $[\![\sigma, T]\!]$ consists of all 2-dimensional processes ν which are predictable on $[\![\sigma, T]\!]$, belong to ${}^{\sigma}L(S)$, and such that the *value process* $V(\nu)(\mathbf{S})$ of ν satisfies the self-financing condition

$$V(\nu)(\mathbf{S}) := \nu \cdot \mathbf{S} = \nu_{\sigma} \cdot \mathbf{S}_{\sigma} + \int_{\sigma} \nu_u d\mathbf{S}_u \quad \text{on } [\![\sigma, T]\!].$$

Definition 3.15 (Definition 3.1, [52])**.** The *fundamental value* of the asset S at time $t \in [0,T]$ is defined by

$$S_t^* := \operatorname*{ess\,inf} \left\{ v \in L_+^1(\mathcal{F}_t, \mathbf{P}) : \exists \nu \in {}^{t}L_+^{\mathrm{sf}}(\mathbf{S}) \text{ with } V_T(\nu)(\mathbf{S}) \ge S_T \text{ and } V_t(\nu)(\mathbf{S}) \le v \right\}. \quad (3.12)$$

We say that the market model has a *strong bubble* if S^* and S are not indistinguishable, i.e., if $\mathbf{P}(S_{\sigma}^* < S_{\sigma}) > 0$ for some stopping $0 \le \sigma \le T$ and define the process $\beta^{\mathrm{NoTC}} = (\beta_t^{\mathrm{NoTC}})_{0 \le t \le T}$ by $\beta_t^{\mathrm{NoTC}} := S_t - S_t^*$, $t \in [0,T]$.

In contrast to Definition 3.1 of [52] we require in Definition 3.15 that $v \in L_+^1(\mathcal{F}_t, \mathbf{P})$ in (3.12) to be consistent with Definition 3.1. In the presented setting, Theorem 3.2 of [71] guarantees that

$$S_{\sigma}^* = \operatorname*{ess\,sup}_{\mathbf{Q} \in \mathcal{M}_{\mathrm{loc}}(S)} \mathbb{E}_{\mathbf{Q}} \left[S_T \mid \mathcal{F}_{\sigma} \right]. \quad (3.13)$$

In the more general framework of [52] it is possible that the numéraire itself is a bubble process making it a bad choice as numéraire. Then (3.13) does not hold. Under the present assumptions, however, this cannot happen. We refer to Remark 3.11 of [52] and the comment before for more information.

Example 3.16. Let S be a semimartingale such that $\mathcal{M}_{\mathrm{loc}}(S) \neq \emptyset$ and assume that there exists $\mathbf{Q}_0 \in \mathcal{M}_{\mathrm{loc}}(S)$ such that S is a true $\mathbf{Q}_0$-martingale. By Proposition 3.7, $(\mathbf{Q}_0, \tilde{S}^{\mathbf{Q}_0})$ is a consistent price system in the non-local sense for S, where $\tilde{S}^{\mathbf{Q}_0} := ((1+\lambda)S_t)_{0 \le t \le T}$. By the martingale property and by Proposition 3.3 we obtain for any $[0,T]$-valued stopping time σ that

$$(1+\lambda)S_\sigma \geq \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^Q)\in \mathrm{CPS_{loc}}} \mathbb{E}_{\mathbf{Q}}\left[\tilde{S}^{\mathbf{Q}}_T \mid \mathcal{F}_\sigma\right] \geq \mathbb{E}_{\mathbf{Q}_0}\left[(1+\lambda)S_T \mid \mathcal{F}_\sigma\right] = (1+\lambda)S_\sigma.$$

Hence there is no bubble in the market model with transaction costs. Alternatively, we can observe that Assumption 1.5 is satisfied because $(\mathbf{Q}_0, \tilde{S}^{\mathbf{Q}_0}) \in \mathrm{CPS}(0,T,\lambda)$ for all $\lambda > 0$ and thus Proposition 3.5 guarantees that there is no bubble in the market model.
In Section 3.4 we proved that the introduction of transaction costs cannot lead to the formation of a bubble. It is easy to see that this is also the case in this example. For any $[0,T]$-valued stopping time σ we have

$$S_\sigma \geq \operatorname*{ess\,sup}_{\mathbf{Q}\in\mathcal{M}_{loc}(S)} \mathbb{E}_{\mathbf{Q}}\left[S_T \mid \mathcal{F}_\sigma\right] \geq \mathbb{E}_{\mathbf{Q}_0}\left[S_T \mid \mathcal{F}_\sigma\right] = S_\sigma,$$

which means that there is no bubble in the market model without transaction costs in the sense of Definition 3.15.

Example 3.17. Let S be given by a three-dimensional inverse Bessel process on a probability space $(\Omega, \mathcal{F}^S_T, \mathbb{F}^S, \mathbf{P})$, i.e.,

$$S_t := \|B_t\|^{-1}, \quad t \in [0,T], \tag{3.14}$$

where $(B_t)_{t\in[0,T]} = (B^1_t, B^2_t, B^3_t)_{t\in[0,T]}$ is a three-dimensional Brownian motion with $B_0 = (1,0,0)$ and $\mathbb{F}^S$ defined by $\mathcal{F}^S_t := \sigma(S_s : s \leq t)$. Then $\mathcal{M}_{\mathrm{loc}}(S) = \{\mathbf{P}\}$ and S is a strict local $\mathbf{P}$-martingale. For the frictionless case, it is shown in Example 5.2 of [52] that there is a bubble in the sense of Definition 3.15.
We now show that the introduction of proportional transaction costs prevents the occurrence of the bubble in this scenario. By Theorem 5.2 of [46] there exists $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}(0,T)$ for all $\lambda > 0$ and hence Assumption 1.5 is fulfilled. By Proposition 3.5 there is no bubble in the market model with proportional transaction costs in the sense of Definition 3.1. In particular, this proves that proportional transaction costs can prevent bubbles' formation.

Example 3.18. This example is based on Example 5.4 of [52]. It illustrates that bubble birth (see [79], [12]) is naturally included in our model.
Let $W = (W_t)_{t\in[0,1]}$ be a Brownian motion and denote by $\mathbb{F}^W$ the natural filtration generated by W. We introduce a random variable γ with values in $(0,1]$ independent of W and assume that γ satisfies

$$0 < \mathbf{P}(\gamma = 1) < 1, \quad \text{and} \quad \mathbf{P}(0 < t_0 \leq \gamma) = 1,$$

for some $t_0 \in (0,1)$. By $\mathbb{F}^\gamma$ we denote the filtration generated by $H_t = \mathbb{1}_{\{\gamma\leq t\}}$, $t \in [0,1]$. This makes γ a $\mathbb{F}^\gamma$ stopping time which will be the time of the bubble's birth. Further, define the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,1]}$ by $\mathcal{F}_t := \mathcal{F}^W_t \vee \mathcal{F}^\gamma_t \vee \mathcal{N}$, $t \in [0,1]$, where $\mathcal{N}$ denotes the $\mathbf{P}$-nullsets of $\mathcal{F}^W_1 \vee \mathcal{F}^\gamma_1$. Then $\mathbb{F}$ is complete and γ is also a $\mathbb{F}$ stopping time. Let $S = (S_t)_{0\leq t\leq 1}$ be the unique strong solution to the SDE

$$dS_t = S_t\left(\mu \mathrm{d}t + v(t,\gamma)\mathrm{d}W_t\right), \quad S_0 = 1, \tag{3.15}$$

with $\mu \in \mathbb{R}$ and $v : [0,1]^2 \to [v_0, \infty)$ given by

$$v(t,u) = v_0 \left(1 + \frac{1}{1-t} \mathbb{1}_{\{u \le t < 1\}}\right), \tag{3.16}$$

for $v_0 > 0$. Up to time γ, S is a geometric Brownian motion. When γ occurs, the term $1/(1-t)$ is activated and the volatility process starts to blow up and finally explodes at time 1. Thus, S converges to 0 as t tends to 1, i.e.,

$$S_t \mathbb{1}_{\{\gamma < 1\}} \xrightarrow{\mathbf{P}\text{-a.s.}} 0, \quad t \to 1. \tag{3.17}$$

In particular, $S_1(\omega_0) = 0$ for $\omega_0 \in \{\omega \in \Omega : \gamma(\omega) < 1\}$. We now show that the fundamental value F_σ of S for an arbitrary $[0,T]$-valued stopping time σ is given by

$$F_\sigma = (1+\lambda) S_\sigma \mathbb{1}_{\{\gamma > \sigma\}}. \tag{3.18}$$

Consider the strategy with initial capital $(1+\lambda) S_\sigma \mathbb{1}_{\{\gamma > \sigma\}}$ such that we buy and hold the asset S at time σ if γ has not occurred yet or we wait until time 1 when $S_1 = 0$ if γ has already occurred. At time σ, we decide whether we buy the strategy or wait until time 1 according to whether γ has happened before time σ or not. If γ happens strictly after σ we cannot be sure if the volatility blows up. Mathematically speaking, define $\varphi = (\varphi_t^1, \varphi_t^2)_{t \in [\![\sigma, T]\!]}$ on $[\![\sigma, T]\!]$ by

$$(\varphi_t^1, \varphi_t^2) = \begin{cases} \left((1+\lambda) S_\sigma \mathbb{1}_{\{\gamma > \sigma\}}, 0\right), & \text{for } t = \sigma, \\ \left(0, \mathbb{1}_{\{\gamma > \sigma\}}\right), & \text{for } \sigma < t < 1, \\ (0,1), & \text{for } t = 1. \end{cases}$$

Note that φ is admissible in the numéraire-based sense because $(1+\lambda) S_\sigma \mathbb{1}_{\{\gamma > \sigma\}} \in L_+^1(\mathcal{F}_\sigma, \mathcal{Q}_{\text{loc}})$. As the strategy φ super-replicates the position $X_T = (0,1)$, we conclude by (3.17) that $F_\sigma \le (1+\lambda) S_\sigma \mathbb{1}_{\{\gamma > \sigma\}}$.
For the reverse direction, "≥" we apply Proposition 2.8 and Proposition 3.7. In Example 5.4 of [52] it is proved that

$$\operatorname*{ess\,sup}_{\mathbf{Q} \in \mathcal{M}_{\text{loc}}(S)} \mathbb{E}_{\mathbf{Q}} \left[S_1 \mid \mathcal{F}_\sigma\right] = S_\sigma \mathbb{1}_{\{\gamma > \sigma\}}.$$

Thus, by Proposition 3.7 we get that

$$F_\sigma = \operatorname*{ess\,sup}_{(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \text{CPS}_{\text{loc}}} \mathbb{E}_{\mathbf{Q}} \left[\tilde{S}_1^{\mathbf{Q}} \mid \mathcal{F}_\sigma\right] \ge \operatorname*{ess\,sup}_{\mathbf{Q} \in \mathcal{M}_{\text{loc}}(S)} \mathbb{E}_{\mathbf{Q}} \left[(1+\lambda) S_1 \mid \mathcal{F}_\sigma\right] = (1+\lambda) S_\sigma \mathbb{1}_{\{\gamma > \sigma\}}.$$

Hence, we conclude

$$F_\sigma = (1+\lambda) S_\sigma \mathbb{1}_{\{\gamma > \sigma\}}.$$

In particular, $F_\sigma = (1+\lambda) S_\sigma$ on $\{\sigma < \gamma\}$ and $F_\sigma = 0 < (1+\lambda) S_\sigma$ on $\{\sigma \ge \gamma\}$. Thus, if $\sigma < \gamma$ there is no asset price bubble in the market but if $\sigma \ge \gamma$ there is bubble in the market. So, γ is the time when the bubble is born.

3.4 Impact of transaction costs on bubbles' formation

In this section we study the impact of transaction costs on asset price bubbles. The focus is on the question of whether the occurrence of bubbles can be prevented by the introduction transaction costs. In addition, we study the impact of transaction costs on bubbles' size. These problems have been discussed in detail in the economic literature. Recall the discussion on the economic literature in the introduction. Here, we address the issue of whether and which impact transaction costs may of on asset price bubbles from a mathematical point of view in our setting.
For the frictionless market model, we consider the setting of [52] and also the definition of asset price bubbles therein, see Section 3.3. Recall that we assume that $\mathcal{M}_{\text{loc}}(S) \neq \emptyset$. In particular, we can apply Proposition 3.7 and get

$$(1+\lambda)S_t \geq F_t = \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^Q)\in \mathrm{CPS}_{\text{loc}}} \mathbb{E}_{\mathbf{Q}}\left[\tilde{S}_T^Q \mid \mathcal{F}_t\right] \geq \operatorname*{ess\,sup}_{\mathbf{Q}\in\mathcal{M}_{loc}(S)} \mathbb{E}_{\mathbf{Q}}[(1+\lambda)S_T \mid \mathcal{F}_t] = (1+\lambda)S_t^*,$$

where S^* denotes the fundamental value for the frictionless case in the sense of Definition 3.15, for all $t \in [0,T]$. It follows that

$$\beta_t = (1+\lambda)S_t - F_t \leq (1+\lambda)\left(S_t - \operatorname*{ess\,sup}_{\mathbf{Q}\in\mathcal{M}_{loc}(S)} \mathbb{E}_{\mathbf{Q}}[S_T \mid \mathcal{F}_t]\right) = (1+\lambda)\beta_t^{\text{NoTC}}, \quad t \in [0,T]. \tag{3.19}$$

By (3.19) it is guaranteed that the introduction of proportional transaction costs cannot lead to bubbles' formation. In fact, if $\beta_\sigma^{\text{NoTC}} = 0$ for a $[0,T]$-valued stopping time, then also $\beta_\sigma = 0$. For the impact of transaction costs on the size of the bubble, we can follow from (3.19) that

$$\frac{\beta_\sigma}{\beta_\sigma^{\text{NoTC}}} \mathbb{1}_{\{\beta_\sigma^{\text{NoTC}}>0\}} \leq 1+\lambda, \tag{3.20}$$

which means that the quotient of the bubbles is bounded by the factor $(1+\lambda)$. In particular, the bubble under transaction costs is smaller or equal than the size of the bubble without transaction costs multiplied by the factor $(1+\lambda)$. Furthermore, we can derive from (3.19) that

$$-\lambda\beta^{\text{NoTC}} \leq \beta_t^{\text{NoTC}} - \beta_t \leq \beta_t^{\text{NoTC}}. \tag{3.21}$$

Both bound in (3.21) are sharpe in the sense that they can be obtained. For the left hand side of (3.21), we consider Example 3.18. In this examples we have for a $[0,1]$-valued stopping time σ that

$$\beta_\sigma - \beta_\sigma^{\text{NoTC}} = (1+\lambda)S_\sigma \mathbb{1}_{\{\gamma\leq\sigma\}} - S_\sigma \mathbb{1}_{\{\gamma\leq\sigma\}} = \lambda S_t \mathbb{1}_{\{\gamma\leq\sigma\}} = \lambda\beta_\sigma^{\text{NoTC}}, \tag{3.22}$$

where γ is also a stopping time satisfying $\mathbf{P}(\gamma = 1) < 1$ representing the birth of the bubble. For the right hand side of (3.21), we consider Example 3.17. Here, $\beta_t \equiv 0$ and hence

$$\beta_t^{\text{NoTC}} - \beta_t = \beta_t^{\text{NoTC}}.$$

In particular, Example 3.17 proves that transaction costs can prevent bubbles' formation as $\beta_0^{\text{NoTC}} > 0$. Multiplying the bubble without transaction by the factor $(1+\lambda)$ we obtain

$$\begin{aligned}
&(1+\lambda)\beta_t^{\text{NoTC}} - \beta_t \\
=&(1+\lambda)\left(S_t - \operatorname*{ess\,sup}_{\mathbf{Q}\in\mathcal{M}_{loc}(S)} \mathbb{E}_{\mathbf{Q}}[S_T \mid \mathcal{F}_t]\right) - (1+\lambda)S_t + \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^Q)\in\text{CPS}_{\text{loc}}} \mathbb{E}_{\mathbf{Q}}\left[\tilde{S}_T^Q \mid \mathcal{F}_t\right] \\
=& \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^Q)\in\text{CPS}_{\text{loc}}} \mathbb{E}_{\mathbf{Q}}\left[\tilde{S}_T^Q \mid \mathcal{F}_t\right] - (1+\lambda) \operatorname*{ess\,sup}_{\mathbf{Q}\in\mathcal{M}_{loc}(S)} \mathbb{E}_{\mathbf{Q}}[S_T \mid \mathcal{F}_t] =: \Delta_{t,T}(\lambda).
\end{aligned} \tag{3.23}$$

By rearranging equation (3.23) we then obtain

$$\beta_t = (1+\lambda)\beta_t^{\text{NoTC}} - \Delta_{t,T}(\lambda). \tag{3.24}$$

Clearly, it holds $\Delta_{t,T}(\lambda) \in [\![0,(1+\lambda)\beta_t^{\text{NoTC}}]\!]$. For Example 3.17 we can determine $\Delta_{0,T}$ explicitly. Using (7) from [39], Proposition 3.5 and $S_0 = 1$ we obtain that

$$\begin{aligned}
\Delta_{0,T}(\lambda) &= \sup_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\text{CPS}_{\text{loc}}} \mathbb{E}_{\mathbf{Q}}\left[\tilde{S}_T^{\mathbf{Q}}\right] - (1+\lambda) \sup_{\mathbf{Q}\in\mathcal{M}_{\text{loc}}(S)} \mathbb{E}_{\mathbf{Q}}\left[S_T\right] \\
&= (1+\lambda)S_0 - (1+\lambda)\mathbb{E}_{\mathbf{P}}\left[S_T\right] \\
&= (1+\lambda) - (1+\lambda)\left(2\Phi\left(\frac{1}{\sqrt{T}}\right) - 1\right) \\
&= 2(1+\lambda)\left(1 - \Phi\left(\frac{1}{\sqrt{T}}\right)\right),
\end{aligned}$$

where Φ denotes the cumulative distribution function of the standard normal distribution. With the same calculation we also get that

$$\beta_0^{\text{NoTC}} = 2\left(1 - \Phi\left(\frac{1}{\sqrt{T}}\right)\right)$$

In particular, $\Delta_{0,T}(\lambda) = (1+\lambda)\beta_0^{\text{NoTC}}$. For T tending to infinity we obtain

$$\lim_{T\to\infty} \beta_0^{\text{NoTC}}(T) = \lim_{T\to\infty} 2\left(1 - \Phi\left(\frac{1}{\sqrt{T}}\right)\right) = 1,$$

and

$$\lim_{T\to\infty} \Delta_{0,T} = \lim_{T\to\infty} 2(1+\lambda)\left(1 - \Phi\left(\frac{1}{\sqrt{T}}\right)\right) = \lim_{T\to\infty} (1+\lambda)\beta_0^{\text{NoTC}}(T) = (1+\lambda).$$

Remark 3.19. *Equation* (3.19) *guarantees that the introduction of transaction costs cannot generate asset price bubbles. In particular, if there is a bubble in a market model with transaction costs, the corresponding frictionless market model has an asset price bubble as well. The size of the bubble under transaction costs cannot be bigger than the size of the bubble in the frictionless market model times the factor* $(1+\lambda)$.
As noted above, the introduction of transaction costs cannot cause bubbles' formation. Example 3.17 of Section 3.3 shows that transaction costs can possibly prevent bubbles' formation as there is a bubble in the sense of Definition 3.15 but no bubble in the presence

of transaction costs in the sense of Definition 3.1.
However, it is not guaranteed that the introduction of transaction costs prevent the occurrence bubbles. For instance, in Example 3.18 the bubble occurs in both market models, with and without transaction costs.
The impact of transaction costs on the occurrence of bubbles coincides with the expectations based on the economic literature which we discussed in the introduction.

Part II

Neural network-based approximation for the superhedging price

Chapter 4

Quantile hedging and the process of consumption

This chapter is based on Section 2, 3.1, and 4.1 of [13]. In this section, we build the theoretical basis for the neural network-based approximation of the superhedging process. After a short motivation of Part II, we introduce the setting of the second part of the thesis. Further, we present the notion of quantile hedging and prove that the α-quantile hedging price converges to the superhedging price as α tends to 1, see Theorem 4.9. In Corollary 4.15 we prove the analogous result for success ratios. Finally, for $t > 0$ we represent the process of consumption by essential supremums, see Proposition 4.16, and use the uniform Doob decomposition to rewrite the superhedging process.

4.1 Motivation Part II

In incomplete markets perfect replication of a contingent claim may not be possible. Superhedging offers an alternative method but it presents two main disadvantages. On the one hand, the superhedging strategy not only reduces the risk but also the possibility to profit. On the other hand, the superhedging price may be considered to be too high from a theoretical perspective as it does not define an arbitrage-free price. For instance, consider the discrete time Black-Scholes model in one period, given by $(S_t^0, S_t^1)_{t=0,1}$. In [24], it was proved that the market model is incomplete and the price of a call option with strike price $K > 0$ is given by

$$\sup_{\mathbf{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbf{Q}}\left[(S_1^1 - K)^+\right] = S_0^1, \tag{4.1}$$

i.e., the cheapest strategy to superhedge the call option is to buy and hold the underlying asset. Thus buying the asset always offers a better payout than the call option at the same price. Especially, in a Black-Scholes model, where, due to the normal distribution, extreme events are very rare, it seems not reasonable to superhedge the call option as above.

Quantile hedging, introduced in [38], offers a solution to this issue. There is the approach of quantile hedging with budget constraint, where a trader determines a fixed budget that she is willing to pay to secure a given contingent claim as good as possible in the sense

that she maximizes the probability of the set where the claim is dominated by the portfolio. For the second approach of quantile hedging, a trader chooses the probability of superhedging and minimizes the required capital for this goal. We refer to this as the α-quantile hedging approach, where $\alpha \in (0,1)$ denotes the probability of the success set, see (4.4). It may be difficult to calculate the α-quantile hedging price or the superhedging price. Recently, machine learning methods were successfully applied to similar calculations. In the concrete case of hedging, there is the approach of [21], where the authors approximate the superhedging price and the corresponding strategy by neural networks and not only present a numerical solution but also prove in a mathematical framework that the approximation and implementation is feasible. Latter approach was proved with the help of the universal approximation theorem, see [55].
In the discrete time market model of [40], we prove that the α-quantile hedging price converges to the superhedging price, see Theorem 4.9. For $t > 0$ we use the uniform Doob decomposition to obtain the superhedging price process. More precisely, assuming that the superhedging price and the corresponding superhedging strategy is known, it is sufficient to determine the process of consumption. In Proposition 4.16, we represent the process of consumption by essential supremums. Then, in Theorem 5.5, we show that the α-quantile hedging price, and thus also the superhedging price can be approximated by neural networks. We define the approximated process of consumption by essential supremums of neural networks and show in Proposition 5.6 and Theorem 5.7 that the process of consumption can be approximated by neural networks. Therefore, we obtain an approximation of the complete superhedging price process for all $t \geq 0$. Finally, we present numerical results. Since superhedging depends only depends on null sets our numerical results have to be considered with care when the market models is given by a price process with unbounded support. In this case, our results indicate that error caused by the discretization of the probability space is significant. On the other side, in finite models we obtain good results with our methodology in the sense that we get a high superhedging probability with a reasonable superhedging price which are also consistent on trainings and test set. In particular, finite models include sample-based frameworks or possibly prediction sets, see [6], [7] [56]. Prediction sets allow to include believes on future price developments or restrict the model to a certain set of relevant paths.

4.2 Setting

We now present the discrete time model of [40]. Let $T > 0$ denote a finite time horizon and consider a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ endowed with a filtration $\mathbb{F} := (\mathcal{F}_t)_{t=0,1,\dots,T}$. We assume $\mathcal{F}_t = \sigma(Y_0, \dots, Y_t)$ for $t = 0, \dots, T$ and for some $\mathbb{R}^m$-valued process $Y = (Y_t)_{t=0,\dots,T}$ for some $m \in \mathbb{N}$, and write $\mathcal{Y}_t = (Y_0, \dots, Y_t)$ for $t \geq 0$. Further, we suppose that $\mathcal{F} = \mathcal{F}_T$ and that Y_0 is constant $\mathbf{P}$-almost surely which implies that $\mathcal{F}_0 = \{\emptyset, \Omega\}$.
The asset prices are modeled by a non-negative, adapted, stochastic process

$$\bar{S} = (S^0, S) = (S_t^0, S_t^1, \dots, S_t^d)_{t=0,1,\dots,T},$$

with $d \geq 1$, $d \in \mathbb{N}$. In particular, $m \geq d$. Further, we assume that

$$S_t^0 > 0 \quad \text{for all } t = 0, 1, \dots, T,$$

and define $S^0 = (S^0_t)_{t=0,1,\dots,T}$ to be the numéraire. The discounted price process $\bar{X} = (X^0, X) = (X^0_t, X^1_t, \dots, X^d_t)_{t=0,1,\dots,T}$ is given by

$$X^i_t := \frac{S^i_t}{S^0_t}, \quad t = 0, 1, \dots, T, \ i = 0, \dots, d.$$

In order to be consistent with [40] and to avoid confusion with Part I, we use a different notation for equivalent martingale measures. By $\mathcal{P}$ we denote the set of equivalent martingale measures, i.e., X is a $\mathbf{P}^*$-martingale for all $\mathbf{P}^* \in \mathcal{P}$. To guarantee that the market model is arbitrage-free, we assume $\mathcal{P} \neq \emptyset$, see Theorem 5.16 of [40].

Definition 4.1. A trading strategy is a predictable $\mathbb{R}^{d+1}$-valued process

$$\bar{\xi} = (\xi^0, \xi) = (\xi^0_t, \xi^1_t, \dots, \xi^d_t)_{t=1,\dots,T}.$$

The (discounted) value process $V = (V_t)_{t=0,\dots,T}$ associated with a trading strategy $\bar{\xi}$ is given by

$$V_0 := \bar{\xi}_1 \cdot \bar{X}_0 \quad \text{and} \quad V_t := \bar{\xi}_t \cdot \bar{X}_t \quad \text{for } t = 1, \dots, T.$$

A trading strategy $\bar{\xi}$ is called self-financing if

$$\bar{\xi}_t \cdot \bar{S}_t = \bar{\xi}_{t+1} \cdot \bar{S}_t \quad \text{for } t = 1, \dots, T-1.$$

A self-financing trading strategy is called an admissible strategy if its value process satisfies $V_T \geq 0$.
By $\mathcal{A}$ we denote the set of all admissible strategies $\bar{\xi}$ and by $\mathcal{V}$ the associated value processes, i.e.,

$$\mathcal{V} := \left\{ V = (V_t)_{t=0,1,\dots,T} : V_t = \bar{\xi}_t \cdot \bar{X}_t \text{ for } t = 0, \dots, T, \text{ and } \bar{\xi} \in \mathcal{A} \right\}$$

By Proposition 5.7 of [40], a trading strategy $\bar{\xi}$ is self-financing if and only if

$$V_t = V_0 + \sum_{k=1}^{t} \xi_k \cdot (X_k - X_{k-1}) \quad \text{for all } t = 0, \dots, T,$$

with $V_0 := \bar{\xi}_1 \cdot \bar{X}_0$. In particular, given an $\mathbb{R}^d$-valued predictable process ξ and $V_0 \in \mathbb{R}$ the pair (V_0, ξ) uniquely defines a self-financing strategy.

Remark 4.2. *For a self-financing strategy with value process V, Theorem 5.14 of [40] guarantees that V is a $\boldsymbol{P}^*$-martingale for all $\boldsymbol{P}^* \in \mathcal{P}$ if $V_T \geq 0$. In particular $V_T \geq 0$ implies that $V_t \geq 0$ for all $t = 0, \dots, T$.*

Definition 4.3. A non-negative, random variable C on $(\Omega, \mathcal{F}_T, \mathbf{P})$ is called a European contingent claim.

Let H be a discounted European contingent claim such that

$$\sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H] < \infty.$$

Definition 4.4. Let H be a discounted European contingent claim. A self-financing trading strategy $\bar{\xi}$ whose value process V satisfies

$$V_T \geq H$$

is called a superhedging strategy for H. In particular, any superhedging strategy is admissible since $H \geq 0$ by definition.

Set

$$\mathcal{U}_t := \left\{ \tilde{U}_t \in L^0(\mathcal{F}_t, \mathbf{P}) : \exists \tilde{\xi} \text{ pred. s.t. } \tilde{U}_t + \sum_{k=t+1}^{T} \tilde{\xi}_k \cdot (X_k - X_{k-1}) \geq H \right\}. \tag{4.2}$$

Then, $\mathcal{U}_t$ describes the set of initial capital required at time $t = 0, 1, \ldots, T$ to superhedge the discounted European claim H and the superhedging price of H is defined by $\inf \mathcal{U}_0$. See Appendix A for further details.
We say that V is a $\mathcal{P}$-(super/sub)-martingale if V is a (super/sub)-martingale for all $\mathbf{P}^* \in \mathcal{P}$.

4.3 Quantile hedging

In incomplete market models it may only be possible to superhedge a given contingent claim but not to hedge it perfectly. As mentioned in Section 4.1, the superhedging price may be too high in some sitautions from a practical as well as from a theoretical perspective. In [38] quantile hedging was introduced to address this problem. We distinguish two different cases, namely, with budget constraint and with given probability of success. For the approximation of the superhedging price we only need quantile hedging with a given probability of success which we simply refer to as (α-)quantile hedging.

4.3.1 Budget constraint

For completeness, we give a brief overview to quantile hedging with budget constraint. For this purpose, let us put our self in the position of the seller of an option. Assume that we either cannot or are not willing to spend the superhedging price $\inf \mathcal{U}_0$ to super-replicate the option we have sold. Fix $v < \inf \mathcal{U}_0$, where v represents the budget that we can spend for the super-replication. The aim is now to find a strategy ξ^* with value process $V^* = (V_t^*)_{t=0,1,\ldots,T}$ which maximizes the probability of the set on which we superhedge the claim H, i.e.,

$$\mathbf{P}(V_T^* \geq H) = \sup_{\xi \text{ adm.}} \mathbf{P}(V_T^\xi \geq H)$$

under the constraint

$$V_0^\xi \leq v,$$

where $V^\xi = (V_t^\xi)_{t=0,1,\ldots,T}$ denotes the value process of a strategy ξ. Such a strategy with value process V^* does not necessarily exists. As we will see below, the analogous α-quantile hedging may not allow an explicit solution either. For this reason, the problems are extended by using the Neyman-Pearson lemma. In Section 4.3.3, we present some more details on this extension.

4.3.2 Success sets

Now, we consider a given probability of the set on which we superhedge the claim H and we want to minimize the required capital. Given a probability of success $\alpha \in (0,1)$ we consider the minimization problem

$$\inf \mathcal{U}_0^\alpha := \inf\{u \in \mathbb{R} : \exists \xi = (\xi_t)_{t=1,\dots,T} \text{ predictable process with values in } \mathbb{R}^d \text{ such that } (u,\xi) \text{ is admissible and } \mathbf{P}\left(u + \sum_{k=1}^T \xi_k \cdot (X_k - X_{k-1}) \geq H\right) \geq \alpha\}. \tag{4.3}$$

Here $1-\alpha$ is called the shortfall probability. Quantile hedging may be considered as a dynamic version of the value at risk concept, see [38].
For an admissible strategy (u,ξ) with associated value process V, we call

$$\{V_T \geq H\} \tag{4.4}$$

the *success set.*

Remark 4.5. *In contrast to the classical superhedging price* $\inf \mathcal{U}_0$ *defined in* (4.2)*, we required in* (4.3) *that* (u,ξ) *is admissible since this is not automatically implied by the definition of quantile hedging. In* (4.2) *it is sufficient that* ξ *is predictable because admissibility is always fulfilled by the definition of superhedging strategies in Definition 4.4.*

For the problem of quantile hedging of (4.3) there exists an equivalent formulation which is convenient for calculations, see Proposition 4.6 below. Note that this formulation is also used in [38] without proof.

Proposition 4.6. *Fix* $\alpha \in (0,1)$*. Then*

$$\inf \mathcal{U}_0^\alpha = \inf\left\{\sup_{\boldsymbol{P}^* \in \mathcal{P}} \mathbb{E}^*[H\mathbb{1}_A] : \ A \in \mathcal{F}_T, \ \boldsymbol{P}(A) \geq \alpha\right\}.$$

Proof. " $\leq$ ": Let $A \in \mathcal{F}_T$ such that $\mathbf{P}(A) \geq \alpha$. We prove that

$$\sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H\mathbb{1}_A] \in \left\{u \in \mathbb{R} : \exists \xi \text{ adm. s.t. } \mathbf{P}\left(u + \sum_{k=1}^T \xi_k \cdot (X_k - X_{k-1}) \geq H\right) \geq \alpha\right\}. \tag{4.5}$$

Applying the well-known superhedging duality, see Corollary A.4, on the modified claim $\tilde{H} := H\mathbb{1}_A$ we get that

$$\sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[\tilde{H}] = \inf\left\{u \in \mathbb{R} : \exists \xi \text{ pred. s.t. } u + \sum_{k=1}^T \xi_k \cdot (X_k - X_{k-1}) \geq \tilde{H}\right\}.$$

Further, by Corollary A.5 there exists a superhedging strategy $\hat{\xi}$ with value process $\hat{V} = (\hat{V}_t)_{t=0,1,\dots,}$ such that $\hat{V}_0 = \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[\tilde{H}]$, i.e.,

$$\hat{V}_T = \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[\tilde{H}] + \sum_{k=1}^T \hat{\xi}_k \cdot (X_k - X_{k-1}) \geq \tilde{H} \geq 0. \tag{4.6}$$

Hence, (4.6) implies for $\hat{\xi}$ that

$$\mathbf{P}\left(\sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H\mathbb{1}_A] + \sum_{k=1}^{T} \hat{\xi}_k \cdot (X_k - X_{k-1}) \geq H\right) \geq \mathbf{P}(A) \geq \alpha.$$

This implies (4.5) and it follows that

$$\inf \mathcal{U}_0^\alpha \leq \inf\left\{\sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H\mathbb{1}_A] : \ A \in \mathcal{F}_T, \ \mathbf{P}(A) \geq \alpha\right\}.$$

" $\geq$ ": Let $\tilde{u} \in \mathcal{U}_0^\alpha$ and denote by $\tilde{\xi} = (\tilde{\xi}_k)_{k=1}^T$ the corresponding admissible strategy such that

$$\mathbf{P}\left(\tilde{u} + \sum_{k=1}^{T} \tilde{\xi}_k \cdot (X_k - X_{k-1}) \geq H\right) \geq \alpha.$$

Define the set $\tilde{A}$ by

$$\tilde{A} := \left\{\omega \in \Omega : \tilde{u} + \sum_{k=1}^{T} \tilde{\xi}_k(\omega) \cdot (X_k(\omega) - X_{k-1}(\omega)) \geq H(\omega)\right\}.$$

Then $\tilde{A} \in \mathcal{F}_T$ such that $\mathbf{P}(\tilde{A}) \geq \alpha$ and $\tilde{u} \in \mathcal{U}_0(H\mathbb{1}_{\tilde{A}})$ because we have by construction that

$$\left(\tilde{u} + \sum_{k=1}^{T} \tilde{\xi}_k \cdot (X_k - X_{k-1})\right)\mathbb{1}_{\tilde{A}} \geq H\mathbb{1}_{\tilde{A}},$$

and as $\tilde{\xi}$ is assumed to be admissible we also have

$$\left(\tilde{u} + \sum_{k=1}^{T} \tilde{\xi}_k \cdot (X_k - X_{k-1})\right)\mathbb{1}_{\tilde{A}^c} \geq 0.$$

By Corollary A.4, this implies

$$\tilde{u} \geq \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H\mathbb{1}_{\tilde{A}}] \in \left\{\sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H\mathbb{1}_A] : \ A \in \mathcal{F}_T, \ \mathbf{P}(A) \geq \alpha\right\}. \tag{4.7}$$

In particular, we have constructed a set $\tilde{A}$ for an arbitrary $\tilde{u} \in \mathcal{U}_0^\alpha$ such that (4.7) holds. Thus, we conclude that

$$\inf \mathcal{U}_0^\alpha \geq \inf\left\{\sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H\mathbb{1}_A] : \ A \in \mathcal{F}_T, \ \mathbf{P}(A) \geq \alpha\right\}.$$

□

Corollary A.5 guarantees that there exists a superhedging strategy with initial value $\inf \mathcal{U}_0$. If an explicit solution to the optimization problem (4.3) exists, in the sense that there is an admissible strategy ξ^* with initial capital $V_0^* = \inf \mathcal{U}^\alpha$ and such that $\mathbf{P}(V_T^* \geq H) \geq \alpha$, then Proposition 4.6 shows that this solution is given by the classical superhedging strategy for the knockout option $H\mathbb{1}_A$ for some suitable $A \in \mathcal{F}_T$. It is not guaranteed that such a set $A \in \mathcal{F}_T$ exists. In general, there may be no explicit solution to the optimization problem (4.3). A possible extension of the problem, which is also presented in [38], is based on the Neyman-Pearson lemma and considers so-called success ratios, randomized

tests, respectively, instead of success sets. There are two advantages of success ratios. On the one hand, there exists an explicit solution to the analogous problem, see Proposition 4.14. On the other hand, success ratios take into account the loss outside the success set. In Section 4.3.3 below, we provide a brief overview of success ratios.
We now show that α-quantile hedging price $\inf \mathcal{U}_0^\alpha$ converges to the superhedging price $\inf \mathcal{U}_0$, as α tends to 1. An essential ingredient of the proof of Theorem 4.9 is Lemma 1.70 of [40]. For random variables $(\xi_n)_{n\in\mathbb{N}} \subset L^0(\mathcal{F}_T, \mathbf{P})$ we denote by $\operatorname{conv}\{\xi_1, \xi_2, \dots\}$ the convex hull of $\xi_1, \xi_2, \dots$ which is defined ω-wise.

Lemma 4.7 (Lemma 1.70, [40])**.** *Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence in $L^0(\mathcal{F}, \boldsymbol{P}; \mathbb{R}^d)$ such that $\sup_{n\in\mathbb{N}} |\xi_n| < \infty$. Then there exists a sequence of convex combinations*

$$\eta_n \in conv\{\xi_n, \xi_{n+1}, \dots\}, \quad n \in \mathbb{N},$$

which converges $\boldsymbol{P}$-almost surely to some $\eta \in L^0(\mathcal{F}, \boldsymbol{P}; \mathbb{R}^d)$.

Definition 4.8. For $\alpha \in (0,1)$ we define

$$\mathcal{F}^\alpha := \{A \in \mathcal{F}_T : \mathbf{P}(A) \geq \alpha\}.$$

Theorem 4.9. *The α-quantile hedging price converges to the superhedging price as α tends to 1, i.e.,*

$$\inf \mathcal{U}_0^\alpha \xrightarrow{\alpha\uparrow 1} \inf \mathcal{U}_0.$$

Proof. By Proposition 4.6 it is sufficient to prove that

$$\inf_{A\in\mathcal{F}^\alpha} \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H\mathbb{1}_A] \xrightarrow{\alpha\uparrow 1} \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H].$$

Let $(\alpha_n)_{n\in\mathbb{N}} \subset (0,1)$ be an increasing sequence such that α_n converges to 1 as n tends to infinity. Because $\mathcal{F}^{\alpha_{n+1}} \subset \mathcal{F}^{\alpha_n}$ it holds that

$$\inf_{A\in\mathcal{F}^{\alpha_n}} \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H\mathbb{1}_A] \leq \inf_{A\in\mathcal{F}^{\alpha_{n+1}}} \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H\mathbb{1}_A] \leq \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H]. \tag{4.8}$$

This implies that $(\inf_{A\in\mathcal{F}^{\alpha_n}} \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H\mathbb{1}_A])_{n\in\mathbb{N}}$ is a monotone and bounded sequence and hence the limit exists and is finite. Given $n \in \mathbb{N}$, for $l \in \mathbb{N}$ there exists $A(n,l) \in \mathcal{F}^{\alpha_n}$ such that

$$\left| \inf_{A\in\mathcal{F}^{\alpha_n}} \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H\mathbb{1}_A] - \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*\left[H\mathbb{1}_{A(n,l)}\right] \right| < \frac{1}{l}.$$

For $n \in \mathbb{N}$ we define $A_n := A(n,n)$. Thus, we obtain for each $n \in \mathbb{N}$ that

$$\inf_{A\in\mathcal{F}^{\alpha_n}} \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H\mathbb{1}_A] \leq \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H\mathbb{1}_{A_n}] < \inf_{A\in\mathcal{F}^{\alpha_n}} \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H\mathbb{1}_A] + \frac{1}{n}. \tag{4.9}$$

By (4.8) and (4.9) the limit of $(\sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H\mathbb{1}_{A_n}])_{n\in\mathbb{N}}$ exists and we get that

$$\lim_{n\to\infty} \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H\mathbb{1}_{A_n}] = \lim_{n\to\infty} \inf_{A\in\mathcal{F}^{\alpha_n}} \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H\mathbb{1}_A] \leq \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H]. \tag{4.10}$$

In order to prove that the right hand of inequality in (4.10) actually equality holds, we use Lemma 4.7 (see also Lemma 1.70 of [40]). There exists a sequence $\psi_n \in \operatorname{conv}\{\mathbb{1}_{A_n}, \mathbb{1}_{A_{n+1}}, \dots\}$, $n \in \mathbb{N}$, which converges $\mathbf{P}$-almost surely to some $\psi \in L^\infty(\mathcal{F}_T, \mathbf{P}; [0,1])$. Yet, it is not clear if ψ is also an indicator function of some $\mathcal{F}_T$ measurable set as $(A_n)_{n\in\mathbb{N}}$ could have non-empty intersections. We now show that $\psi = 1$. By Lemma 4.7 (see also Lemma 1.70 of [40]), ψ_n, $n \in \mathbb{N}$, is a convex combination of $\{\mathbb{1}_{A_n}, \mathbb{1}_{A_{n+1}}, \dots\}$, i.e., ψ_n is of the form

$$\psi_n = \sum_{k=n}^{\infty} \lambda_k^n \mathbb{1}_{A_k}, \tag{4.11}$$

for some $(\lambda_k^n)_{k=n}^\infty \geq 0$ with $\sum_{k=n}^\infty \lambda_k^n = 1$.
Note that $0 \leq (\psi_n)_{n\in\mathbb{N}} \leq 1$ and thus obtain by dominated convergence and (4.11) that

$$\mathbb{E}_{\mathbf{P}}[\psi] = \lim_{n\to\infty} \mathbb{E}_{\mathbf{P}}[\psi_n] = \lim_{n\to\infty} \mathbb{E}_{\mathbf{P}}\left[\sum_{k=n}^{\infty} \lambda_k^n \mathbb{1}_{A_k}\right] = \lim_{n\to\infty}\left(\sum_{k=n}^{\infty} \lambda_k^n \mathbb{E}_{\mathbf{P}}\left[\mathbb{1}_{A_k}\right]\right). \tag{4.12}$$

Using the definition of the limes inferior we get by (4.12) that

$$\begin{aligned}\mathbb{E}_{\mathbf{P}}[\psi] &\geq \lim_{n\to\infty}\left(\sum_{k=n}^{\infty} \lambda_k^n \inf_{l\geq n} \mathbb{E}_{\mathbf{P}}\left[\mathbb{1}_{A_l}\right]\right) = \lim_{n\to\infty}\left(\inf_{l\geq n} \mathbb{E}_{\mathbf{P}}\left[\mathbb{1}_{A_l}\right]\right)\\ &= \liminf_{n\to\infty} \mathbb{E}_{\mathbf{P}}[\mathbb{1}_{A_n}] = \liminf_{n\to\infty} \mathbf{P}(A_n) \geq \liminf_{n\to\infty} \alpha_n = 1,\end{aligned} \tag{4.13}$$

where we also used that $\sum_{k=n}^\infty \lambda_k^n = 1$. Now, we can conclude that $\psi = 1$ as $\mathbb{E}_{\mathbf{P}}[\psi] = 1$ and $0 \leq \psi \leq 1$. We now use similar arguments as in (4.12) and (4.13) for the supremum instead of the infimum in (4.9). Then, by dominated convergence we obtain for any $\mathbf{P}^* \in \mathcal{P}$ that

$$\begin{aligned}\limsup_{n\to\infty}\left(\inf_{A\in\mathcal{F}^{\alpha_n}} \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H\mathbb{1}_A] + \frac{1}{n}\right) &\geq \limsup_{n\to\infty} \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H\mathbb{1}_{A_n}]\\ &\geq \limsup_{n\to\infty} \mathbb{E}^*\left[H\mathbb{1}_{A_n}\right]\\ &= \lim_{n\to\infty}\left(\sup_{l\geq n} \mathbb{E}^*\left[H\mathbb{1}_{A_l}\right]\right)\\ &= \lim_{n\to\infty}\left(\sum_{k=n}^{\infty} \lambda_k^n \sup_{l\geq n} \mathbb{E}^*\left[H\mathbb{1}_{A_l}\right]\right)\\ &\geq \lim_{n\to\infty}\left(\sum_{k=n}^{\infty} \lambda_k^n \mathbb{E}^*\left[H\mathbb{1}_{A_k}\right]\right)\\ &= \lim_{n\to\infty} \mathbb{E}^*[H\psi_n] = \mathbb{E}^*[H\psi] = \mathbb{E}^*[H].\end{aligned} \tag{4.14}$$

By (4.10) the limit on the left hand side of (4.14) exists and hence

$$\lim_{n\to\infty}\left(\inf_{A\in\mathcal{F}^{\alpha_n}} \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H\mathbb{1}_A] + \frac{1}{n}\right) = \limsup_{n\to\infty}\left(\inf_{A\in\mathcal{F}^{\alpha_n}} \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H\mathbb{1}_A] + \frac{1}{n}\right) \geq \mathbb{E}^*\left[H\right]. \tag{4.15}$$

Since (4.15) holds for all $\mathbf{P}^* \in \mathcal{P}$, we get

$$\lim_{n\to\infty}\left(\inf_{A\in\mathcal{F}^{\alpha_n}} \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H\mathbb{1}_A] + \frac{1}{n}\right) \geq \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H], \tag{4.16}$$

and thus we obtain together with (4.8) that

$$\lim_{n\to\infty}\left(\inf_{A\in\mathcal{F}^{\alpha_n}}\sup_{\mathbf{P}^*\in\mathcal{P}}\mathbb{E}^*[H\mathbb{1}_A]\right)\le\sup_{\mathbf{P}^*\in\mathcal{P}}\mathbb{E}^*[H]\le\lim_{n\to\infty}\left(\inf_{A\in\mathcal{F}^{\alpha_n}}\sup_{\mathbf{P}^*\in\mathcal{P}}\mathbb{E}^*[H\mathbb{1}_A]+\frac{1}{n}\right).$$

We conclude that

$$\lim_{n\to\infty}\left(\inf_{A\in\mathcal{F}^{\alpha_n}}\sup_{\mathbf{P}^*\in\mathcal{P}}\mathbb{E}^*[H\mathbb{1}_A]\right)=\sup_{\mathbf{P}^*\in\mathcal{P}}\mathbb{E}^*[H].$$

□

Theorem 4.9 guarantees that the superhedging price at $t=0$ can be approximated by the α-quantile hedging price for α sufficient large. We now present a brief discursion to success ratios.

Proposition 4.10. *If $(\Omega,\mathcal{F},\mathbb{F},\boldsymbol{P})$ is a finite probability space, then there exists $\alpha_0\in(0,1)$ such that*

$$\inf\mathcal{U}_0=\inf\mathcal{U}_0^{\alpha_0}.$$

Proof. Since Ω is finite, we can define

$$\varepsilon:=\min_{\omega\in\Omega}\mathbf{P}\left(\{\omega\}\right)\in(0,1).$$

Choose $\alpha_0\in(1-\varepsilon,1)$ arbitrary.Then, any set $A\in\mathcal{F}_T$ such that $\mathbf{P}(A)\ge\alpha_0$ satisfies that $\mathbf{P}(A)=1$. Thus, we conclude that

$$\inf\mathcal{U}_0=\inf\mathcal{U}_0^{\alpha_0}.$$

□

4.3.3 Success ratios

As mentioned above, one disadvantage of quantile hedging is the possible lack of an explicit solution to (4.3). The Neyman-Pearson lemma suggests to use randomized tests to guarantee the existence of a solution. In particular, in the context of quantile hedging, we consider a special family of randomized tests called success ratios.

Let $\mathcal{R}:=L^\infty(\mathcal{F}_T,\mathbf{P};[0,1])$ be the set of randomized tests. For $\alpha\in(0,1)$ we denote by $\mathcal{R}^\alpha$ the set

$$\mathcal{R}^\alpha:=\{\varphi\in\mathcal{R}:\ \mathbb{E}_{\mathbf{P}}[\varphi]\ge\alpha\}.$$

The analogous problem of quantile hedging in terms of randomized tests is given by

$$\inf\left\{\sup_{\mathbf{P}^*\in\mathcal{P}}\mathbb{E}^*\left[H\varphi\right]:\varphi\in\mathcal{R}^\alpha\right\}. \tag{4.17}$$

In a first step, we prove that this problem admits an explicit solution. In a second step, we show that the solution is given by the so-called success ratio, see Definition 4.12 below. In particular, (4.17) can be formulated in terms of success ratios, see also [38]. In Proposition 4.11 and 4.14 we provide proofs for results that are mentioned in [38] without proof. See also Section 8.1 of [40].

Proposition 4.11. *There exists a randomized test $\tilde{\varphi} \in \mathcal{R}$ such that*

$$\mathbb{E}_{\boldsymbol{P}}[\tilde{\varphi}] = \alpha,$$

and

$$\inf_{\varphi \in \mathcal{R}^{\alpha}} \sup_{\boldsymbol{P}^* \in \mathcal{P}} \mathbb{E}^*[H\varphi] = \sup_{\boldsymbol{P}^* \in \mathcal{P}} \mathbb{E}^*[H\tilde{\varphi}]. \tag{4.18}$$

Proof. Let $(\varphi_n)_{n \in \mathbb{N}} \subset \mathcal{R}^{\alpha}$ be a sequence such that

$$\lim_{n \to \infty} \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H\varphi_n] = \inf_{\varphi \in \mathcal{R}^{\alpha}} \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H\varphi]. \tag{4.19}$$

Using Lemma 4.7 (see also Lemma 1.70 of [40]), we obtain a sequence of convex combinations $\tilde{\varphi}_n \in \operatorname{conv}\{\varphi_n, \varphi_{n+1}, \dots\}$ converging $\mathbf{P}$-almost surely to a function $\tilde{\varphi} \in \mathcal{R}$. Note that $\tilde{\varphi}_n$ is of the form

$$\tilde{\varphi}_n = \sum_{k=n}^{\infty} \lambda_k^n \varphi_n, \tag{4.20}$$

for some $0 \leq (\lambda_k^n)_{k \geq n} \leq 1$ with $\sum_{k=n}^{\infty} \lambda_k^n = 1$. In particular, by dominated convergence we get that

$$\mathbb{E}_{\mathbf{P}}\left[\tilde{\varphi}_n\right] = \mathbb{E}_{\mathbf{P}}\left[\sum_{k=n}^{\infty} \lambda_k^n \varphi_n\right] = \sum_{k=n}^{\infty} \lambda_k^n \mathbb{E}_{\mathbf{P}}\left[\varphi_n\right] \geq \sum_{k=n}^{\infty} \lambda_k^n \alpha = \alpha,$$

and thus $\tilde{\varphi}_n \in \mathcal{R}^{\alpha}$ for all $n \in \mathbb{N}$. Hence, dominated convergence yields that

$$\mathbb{E}_{\mathbf{P}}[\tilde{\varphi}] = \lim_{n \to \infty} \mathbb{E}_{\mathbf{P}}[\tilde{\varphi}_n] \geq \alpha, \tag{4.21}$$

and it follows that $\tilde{\varphi} \in \mathcal{R}^{\alpha}$. Following similar arguments as in the proof of Theorem 4.9, we obtain by (4.20) and dominated convergence for any $\mathbf{P}^* \in \mathcal{P}$ that

$$\begin{aligned}\limsup_{n \to \infty} \mathbb{E}^*\left[H\varphi_n\right] &= \lim_{n \to \infty}\left(\sup_{k \geq n} \mathbb{E}^*\left[H\varphi_k\right]\right) \geq \lim_{n \to \infty}\left(\sum_{k=n}^{\infty} \lambda_k^n \mathbb{E}^*\left[H\varphi_k\right]\right) \\ &= \lim_{n \to \infty} \mathbb{E}^*\left[H\tilde{\varphi}_n\right] = \mathbb{E}^*\left[H\tilde{\varphi}\right]. \end{aligned} \tag{4.22}$$

Moreover, (4.19), (4.22) and dominated convergence yield

$$\inf_{\varphi \in \mathcal{R}^{\alpha}} \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H\varphi] = \limsup_{n \to \infty} \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H\varphi_n] \geq \limsup_{n \to \infty} \mathbb{E}^*[H\varphi_n] \geq \mathbb{E}^*[H\tilde{\varphi}]. \tag{4.23}$$

Since (4.23) holds for all $\mathbf{P}^* \in \mathcal{P}$ we obtain

$$\inf_{\varphi \in \mathcal{R}^{\alpha}} \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H\varphi] \geq \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H\tilde{\varphi}].$$

Note that $\tilde{\varphi} \in \mathcal{R}^{\alpha}$ by (4.21) and hence

$$\inf_{\varphi \in \mathcal{R}^{\alpha}} \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H\varphi] = \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H\tilde{\varphi}],$$

i.e., $\tilde{\varphi}$ is a minimizer.
It is left to show that $\mathbb{E}_{\mathbf{P}}[\tilde{\varphi}] = \alpha$ holds. Assume $\mathbb{E}_{\mathbf{P}}[\tilde{\varphi}] > \alpha$. Then there exists $\varepsilon > 0$ such that $\varphi_{\varepsilon} := (1-\varepsilon)\tilde{\varphi} \in \mathcal{R}^{\alpha}$, and

$$\sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H\varphi_{\varepsilon}] = (1-\varepsilon) \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H\tilde{\varphi}] < \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H\tilde{\varphi}], \tag{4.24}$$

which contradicts the minimality property of $\tilde{\varphi}$. Therefore, we conclude that

$$\mathbb{E}_{\mathbf{P}}[\tilde{\varphi}] = \alpha.$$

$\square$

Proposition 4.11 shows that there exists an explicit solution of the analogous optimization problem in (4.17). But the concept of randomized tests is abstract without clear economic meaning. We now introduce the family of success ratios, which is a subset of randomized tests, allowing an economic interpretation.

Definition 4.12. For an admissible strategy with value process $V \in \mathcal{V}$ we define its success ratio by

$$\varphi_V := \mathbb{1}_{\{V_T \geq H\}} + \frac{V_T}{H} \mathbb{1}_{\{V_T < H\}}. \tag{4.25}$$

For $\alpha \in (0,1)$ we denote by $\mathcal{V}^\alpha$ the set

$$\mathcal{V}^\alpha := \{V \in \mathcal{V} : \ \mathbb{E}_{\mathbf{P}}[\varphi_V] \geq \alpha\}.$$

Remark 4.13. *Note that for $V \in \mathcal{V}$ we have that $V_T \geq 0$. In particular, $\boldsymbol{P}(\{H = 0\} \cap \{V_T < H\}) = 0$ and hence* (4.25) *is well-defined.*

The first part of the definition of a success ratio in (4.25), coincides with the success set, see (4.4), and the second part of (4.25) penalizes where superhedging fails.
First, we provide the analogous optimization problem of (4.3) in terms of success ratios. In Proposition 4.14 below, we then prove that there exists an explicit solution to (4.26) and that the solution coincides with the solution of optimization problem in terms of randomized tests, see (4.17).
The optimization problem in terms of success ratios is given by

$$\inf \left\{ \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[\varphi_V] : \ V \in \mathcal{V}^\alpha \right\}. \tag{4.26}$$

Proposition 4.14. *There exists an admissible strategy with value process $\tilde{V}$ such that*

$$\mathbb{E}_{\boldsymbol{P}}\left[\varphi_{\tilde{V}}\right] = \alpha,$$

and

$$\inf_{V \in \mathcal{V}^\alpha} \sup_{\boldsymbol{P}^* \in \mathcal{P}} \mathbb{E}^*[H\varphi_V] = \sup_{\boldsymbol{P}^* \in \mathcal{P}} \mathbb{E}^*\left[H\varphi_{\tilde{V}}\right], \tag{4.27}$$

where φ_V denotes the success ratio associated to a portfolio $V \in \mathcal{V}$ as in (4.25)*. Moreover, $\varphi_{\tilde{V}}$ coincides with the solution $\tilde{\varphi}$ from Proposition 4.11.*

Proof. We observe that

$$\inf_{\varphi \in \mathcal{R}^\alpha} \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H\varphi] \leq \inf_{V \in \mathcal{V}^\alpha} \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H\varphi_V], \tag{4.28}$$

because

$$\{\varphi_V \in \mathcal{R} : \ V \in \mathcal{V}^\alpha\} \subseteq \mathcal{R}^\alpha.$$

On the left hand side of (4.28) we apply Proposition 4.11 to obtain a solution $\tilde{\varphi} \in \mathcal{R}$ in the sense that

$$\inf_{\varphi \in \mathcal{R}^\alpha} \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H\varphi] = \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*\left[H\tilde{\varphi}\right].$$

We prove that there exists $\tilde{V} \in \mathcal{V}^\alpha$ such that

$$\tilde{\varphi} = \varphi_{\tilde{V}},$$

which suffices to conclude the proof by (4.28). Define the the modified claim

$$\tilde{H} := H\tilde{\varphi}.$$

By Corollary A.5 it is guaranteed that there is a superhedging strategy $\tilde{\xi}$ with value process $\tilde{V}$ for $\tilde{H}$ such that

$$\tilde{V}_0 = \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*\left[\tilde{H}\right].$$

Recall that $(V_0, \tilde{\xi})$ defines an admissible strategy by Remark 4.5 and hence $\tilde{V} \in \mathcal{V}$. Further, $\tilde{V}$ also satisfies $\mathbb{E}_{\mathbf{P}}[\varphi_{\tilde{V}}] \geq \alpha$, i.e., $\tilde{V} \in \mathcal{V}^\alpha$, since

$$\varphi_{\tilde{V}} = \mathbb{1}_{\{\tilde{V}_T \geq H\}} + \frac{\tilde{V}_T}{H} \mathbb{1}_{\{\tilde{V}_T < H\}} \geq \tilde{\varphi} \mathbb{1}_{\{\tilde{V}_T \geq H\}} + \frac{H\tilde{\varphi}}{H} \mathbb{1}_{\{\tilde{V}_T < H\}} = \tilde{\varphi}, \tag{4.29}$$

where we used that $\tilde{V}_T$ dominates $\tilde{H} = H\tilde{\varphi}$ by definition and $0 \leq \tilde{\varphi} \leq 1$. In particular, (4.29) implies that

$$\mathbb{E}_{\mathbf{P}}[\varphi_{\tilde{V}}] \geq \mathbb{E}_{\mathbf{P}}[\tilde{\varphi}] \geq \alpha,$$

such that $\tilde{V} \in \mathcal{V}^\alpha$ and $\varphi_{\tilde{V}} \in \mathcal{R}^\alpha$. To this end, we show that $\tilde{\varphi} = \varphi_{\tilde{V}}$. The first direction, $\varphi_{\tilde{V}} \geq \tilde{\varphi}$ follows by (4.29). For the other direction, we start to show that $\varphi_{\tilde{V}}$ is also a minimizer of the problem (4.18), which will imply

$$\sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H\varphi_{\tilde{V}}] \leq \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H\tilde{\varphi}].$$

Indeed, by Theorem 5.14 of [40], $\tilde{V}$ is a $\mathcal{P}$-martingale, i.e., a $\mathbf{P}^*$-martingale for all $\mathbf{P}^* \in \mathcal{P}$, because $\tilde{V}_T \geq \tilde{H} \geq 0$ and hence

$$\begin{aligned}\sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H\varphi_{\tilde{V}}] &= \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*\left[H\left(\mathbb{1}_{\{\tilde{V}_T \geq H\}} + \frac{\tilde{V}_T}{H}\mathbb{1}_{\{\tilde{V}_T < H\}}\right)\right] \\ &\leq \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[\tilde{V}_T] = \tilde{V}_0 = \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H\tilde{\varphi}],\end{aligned} \tag{4.30}$$

where we used in the last equality that $\tilde{V}_0$ is the superhedging price of $\tilde{H} = H\tilde{\varphi}$. Therefore, we conclude that $\varphi_{\tilde{V}}$ is a solution to the minimization problem of (4.27). With the same arguments as in (4.24) in the proof of Proposition 4.11 it follows that

$$\mathbb{E}_{\mathbf{P}}[\varphi_{\tilde{V}}] = \alpha. \tag{4.31}$$

Therefore, by (4.24) and (4.31)

$$\mathbb{E}_{\mathbf{P}}[\varphi_{\tilde{V}}] = \alpha = \mathbb{E}_{\mathbf{P}}[\tilde{\varphi}],$$

and hence

$$\mathbb{E}_{\mathbf{P}}\left[\varphi_{\tilde{V}} - \tilde{\varphi}\right] = 0.$$

Because $\varphi_{\tilde{V}} \geq \tilde{\varphi}$ by (4.29) we can now follow that $\varphi_{\tilde{V}} = \tilde{\varphi}$. In particular, the quantile hedging formulations of (4.17) and (4.26) are equivalent. $\square$

Analogously to Theorem 4.9, we can approximate the superhedging price by the α-quantile hedging price given in terms of success ratios, see (4.26).

Corollary 4.15. *The following convergence holds:*

$$\inf_{V \in \mathcal{V}^{\alpha}} \sup_{\boldsymbol{P}^* \in \mathcal{P}} \mathbb{E}^*[H\varphi_V] \xrightarrow{\alpha \uparrow 1} \sup_{\boldsymbol{P}^* \in \mathcal{P}} \mathbb{E}^*[H],$$

where φ_V denotes the success ratio associated to a portfolio $V \in \mathcal{V}$ as in (4.25).

Proof. Let $(\alpha_n)_{n\in\mathbb{N}} \subset (0,1)$ be an increasing sequence such that $\alpha_n \uparrow 1$ as n tends to infinity. By Proposition 4.14, for all $n \in \mathbb{N}$ there exists $\varphi_{\tilde{V}^n}$ such that

$$\inf_{V \in \mathcal{V}^{\alpha_n}} \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H\varphi] = \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H\varphi_{\tilde{V}^n}]. \tag{4.32}$$

It is easy to see that

$$\sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H\varphi_{\tilde{V}^n}] \leq \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H\varphi_{\tilde{V}^{n+1}}] \leq \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H] \tag{4.33}$$

because $0 \leq (\varphi_{\tilde{V}^n})_{n\in\mathbb{N}} \leq 1$ and

$$\left\{\varphi_V \in \mathcal{R} : \; V \in \mathcal{V}^{\alpha^{n+1}}\right\} \subseteq \left\{\varphi_V \in \mathcal{R} : \; V \in \mathcal{V}^{\alpha^{n}}\right\}.$$

So, we obtain a sequence $(\varphi_{\tilde{V}^n})_{n\in\mathbb{N}} \subset \mathcal{R}$ of randomized tests such that $\tilde{V}^n \in \mathcal{V}^{\alpha_n}$ by (4.33), for all $n \in \mathbb{N}$. Further, the sequence $(\sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H\varphi_{\tilde{V}^n}])_{n\in\mathbb{N}}$ is monotone and bounded and thus convergent. By Lemma 4.7 (see also Lemma 1.70 of [40]), there exists a sequence

$$\psi_n \in \operatorname{conv}\{\varphi_{\tilde{V}^n}, \varphi_{\tilde{V}^{n+1}}, \dots\}$$

such that ψ_n converges $\mathbf{P}$-a.s. to some $\psi \in \mathcal{R}$. With the same arguments as in (4.12) and (4.13) of the proof of Theorem 4.9 we get by dominated convergence that

$$\begin{aligned}
\liminf_{n\to\infty} \mathbb{E}_{\mathbf{P}}\left[\varphi_{\tilde{V}^n}\right] &= \lim_{n\to\infty} \left(\inf_{k\geq n} \mathbb{E}_{\mathbf{P}}\left[\varphi_{\tilde{V}^k}\right]\right) = \lim_{n\to\infty} \left(\sum_{k=n}^{\infty} \lambda_k^n \inf_{k\geq n} \mathbb{E}_{\mathbf{P}}\left[\varphi_{\tilde{V}^k}\right]\right) \\
&\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \lambda_k^n \mathbb{E}_{\mathbf{P}}\left[\varphi_{\tilde{V}^k}\right] = \lim_{n\to\infty} \mathbb{E}_{\mathbf{P}}\left[\psi_n\right] = \mathbb{E}_{\mathbf{P}}\left[\psi\right],
\end{aligned}$$

for some $0 \leq (\lambda_k^n)_{k=n}^{\infty} \leq 1$ with $\sum_{k\geq n} \lambda_k^n = 1$ satisfying

$$\sum_{k=n}^{\infty} \lambda_k^n \varphi_{\tilde{V}^k} = \psi_n.$$

In particular, we have

$$\mathbb{E}_{\mathbf{P}}[\psi] = \lim_{n\to\infty} \mathbb{E}_{\mathbf{P}}[\psi_n] \geq \liminf_{n\to\infty} \mathbb{E}_{\mathbf{P}}[\varphi_{\tilde{V}^n}] \geq \liminf_{n\to\infty} \alpha_n = 1,$$

and thus $\psi = 1$. By (4.33) we get

$$\lim_{n\to\infty} \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H\varphi_{\tilde{V}^n}] \le \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H], \tag{4.34}$$

and conversely by dominated convergence we obtain with similar arguments as in (4.14) of the proof of Theorem 4.9 for any $\mathbf{P}^* \in \mathcal{P}$ that

$$\limsup_{n\to\infty} \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H\varphi_{\tilde{V}^n}] \ge \limsup_{n\to\infty} \mathbb{E}^*[H\varphi_{\tilde{V}^n}] \ge \lim_{n\to\infty} \mathbb{E}^*[H\psi_n] = \mathbb{E}^*[H\psi] = \mathbb{E}^*[H]. \tag{4.35}$$

Because the limit on the left side of (4.35) exists and (4.35) holds for all $\mathbf{P}^* \in \mathcal{P}$ we obtain that

$$\lim_{n\to\infty} \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H\varphi_{\tilde{V}^n}] = \limsup_{n\to\infty} \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H\varphi_{\tilde{V}^n}] \ge \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H], \tag{4.36}$$

Putting (4.32), (4.34) and (4.36) together yields

$$\lim_{n\to\infty} \inf_{\varphi\in\mathcal{R}^{\alpha_n}} \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H\tilde{\varphi}] = \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H].$$

□

4.4 Process of consumption

The aim of this section is a characterization of the superhedging price process

$$\left(\operatorname*{ess\,sup}_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H \mid \mathcal{F}_t]\right)_{t=0,1,\dots,T}.$$

Recall that by the uniform Doob decomposition and Corollary A.5 that we have

$$\operatorname*{ess\,sup}_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*\left[H \mid \mathcal{F}_t\right] = \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*\left[H\right] + \sum_{k=1}^{t} \xi_k \cdot (X_k - X_{k-1}) - B_t, \quad \text{for all } t = 0,\dots,T, \tag{4.37}$$

where $B = (B_t)_{t=0,1,\dots,T}$ with $B_0 = 0$ is a non-negative, increasing process. We refer to B as the process of consumption. Using (4.37), and assuming that we know the superhedging strategy given by $(\sup_{\mathbf{P}\in\mathcal{P}} \mathbb{E}^*[H], \xi)$ it is sufficient to calculate the process B. Exploiting that $(\operatorname{ess\,sup}_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*\left[H \mid \mathcal{F}_t\right])_{t=0,1,\dots,T}$ is the smallest $\mathcal{P}$-supermartingale whose terminal value dominates H by Corollary A.5, we define $\tilde{B} = (\tilde{B}_t)_{t=0,\dots,T}$ by $\tilde{B}_0 := 0$ and for $t = 1,\dots,T$,

$$\tilde{B}_t := \operatorname{ess\,sup} \mathcal{B}_t, \tag{4.38}$$

where

$$\mathcal{B}_t := \left\{ D_t \in L^0(\mathcal{F}_t, \mathbf{P}) : \tilde{B}_{t-1} \le D_t \le \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H] + \sum_{k=1}^{T} \xi_k \cdot (X_k - X_{k-1}) - H \right\} \tag{4.39}$$

Proposition 4.16. *We have that*

$$B_t = \tilde{B}_t \quad \textit{for all } t = 0,\dots,T,$$

where B is defined in (4.37) *and $\tilde{B}$ in* (4.38)*, respectively.*

Proof. We prove the assertion by induction. For $t = 0$ we have $B_0 = 0 = \tilde{B}_0$ by definition. For the induction step assume that

$$B_{t-1} = \tilde{B}_{t-1} \tag{4.40}$$

for some $1 \le t \le T$. As B is increasing and by (4.40), we get that $B_t \ge \tilde{B}_{t-1}$ and by (A.5) that

$$\sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H] + \sum_{k=1}^{T} \xi_k \cdot (X_k - X_{k-1}) - H \ge B_t. \tag{4.41}$$

Therefore, $B_t \in \mathcal{B}_t$ and thus $B_t \le \tilde{B}_t = \operatorname{ess\,sup} \mathcal{B}_t$. For the converse direction we assume that $\mathbf{P}(B_t < \tilde{B}_t) > 0$ and lead this to a contradiction. To this end, let ξ denote the minimal superhedging strategy with initial capital $\sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H]$ and define $\tilde{V} = (\tilde{V}_s)_{s=0,\dots,T}$ by

$$\tilde{V}_s := \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H] + \sum_{k=1}^{s} \xi_k \cdot (X_k - X_{k-1}) - \tilde{B}_s. \tag{4.42}$$

Since $(\sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H], \xi)$ was assumed to be a superhedging strategy, we have that

$$\sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H] + \sum_{k=1}^{T} \xi_k \cdot (X_k - X_{k-1}) \ge H \ge 0. \tag{4.43}$$

Further, by the definition of $\tilde{B}$ in (4.38) and (4.39) we have

$$0 \le \tilde{B}_s \le \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H] + \sum_{k=1}^{T} \xi_k \cdot (X_k - X_{k-1}) - H \quad \text{for all } s = 0, \dots, T. \tag{4.44}$$

By (4.43) and Theorem 5.14 of [40] we get that

$$\left(\sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H] + \sum_{k=1}^{s} \xi_k \cdot (X_k - X_{k-1}) \right)_{s=0,\dots,T}$$

is $\mathcal{P}$-martingale for all $\mathbf{P}^* \in \mathcal{P}$ and hence

$$\sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H] + \sum_{k=1}^{T} \xi_k \cdot (X_k - X_{k-1}) - H \in L^1(\mathcal{F}, \mathbf{P}^*) \quad \text{for all } \mathbf{P}^* \in \mathcal{P},$$

where we used that $\sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H] < \infty$. Thus (4.44) implies that $\tilde{V}_s \in L^1(\mathcal{F}_s, \mathbf{P}^*)$ for all $\mathbf{P}^* \in \mathcal{P}$ and all $s = 0, \dots, T$. Further, by (4.42) $\tilde{V}$ can be decomposed in a martingale part M and an increasing non negative process such that

$$\tilde{V}_s = M_s - \tilde{B}_s \quad s = 0, 1, \dots, T,$$

which implies that $\tilde{V}$ is a $\mathcal{P}$-supermartingale. Note that $\tilde{V}$ is non-negative because its terminal value dominates H by construction. We have already proved that $B_s \le \tilde{B}_s$ for all $s = 0, \dots, T$ and thus by (4.42) we have

$$\tilde{V}_s \le \operatorname*{ess\,sup}_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H \mid \mathcal{F}_s] = \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H] + \sum_{k=1}^{t} \xi_k \cdot (X_k - X_{k-1}) - B_s \quad \text{for all } s = 0, 1, \dots, T.$$

Then we obtain

$$\mathbf{P}(\tilde{V}_t < \operatorname*{ess\,sup}_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H \mid \mathcal{F}_t]) = \mathbf{P}(B_t < \tilde{B}_t) > 0,$$

which contradicts the fact that $(\operatorname{ess\,sup}_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H \mid \mathcal{F}_s])_{s=0,\dots,T}$ is the smallest $\mathcal{P}$ - supermartingale whose terminal value dominates H. Thus $\mathbf{P}(B_t < \tilde{B}_t) = 0$ and hence $B_t = \tilde{B}_t$. This concludes the proof. $\square$

Remark 4.17. *In the definition of* (4.38) *we can equivalently consider* $\operatorname{ess\,sup} \widehat{\mathcal{B}}_t$, *where*

$$\widehat{\mathcal{B}}_t := \left\{ D_t \in L^0(\mathcal{F}_t, \boldsymbol{P}) : 0 \le D_t \le \sup_{\boldsymbol{P}^* \in \mathcal{P}} \mathbb{E}^*[H] + \sum_{k=1}^{T} \xi_k \cdot (X_k - X_{k-1}) - H \right\},$$

for $t = 1, \dots, T$. *This is due to the fact that, on the one hand* $\mathcal{B}_t \subset \widehat{\mathcal{B}}_t$ *for all* $t = 1, \dots, T$. *On the other hand, for* $D_t \in \widehat{\mathcal{B}}_t$ *we define* $\tilde{D}_t := D_t \vee B_{t-1}$. *Then* $\tilde{D}_t \in \mathcal{B}_t$ *and* $D_t \le \tilde{D}_t$. *Therefore,* $\operatorname{ess\,sup} \widehat{\mathcal{B}}_t = \operatorname{ess\,sup} \mathcal{B}_t = B_t$ *for all* $t = 1, \dots, T$.

Chapter 5

Neural network based approximation

This chapter is based on Section 3.2 and 4.2 of [13]. We present the neural network based approximation of the superhedging price process using the universal approximation theorem, Theorem 1 of [55]. First, we provide the mathematical definition of a neural network and prove Theorem 5.2 which is an implication of the universal approximation theorem, see [55, Theorem 1 and Section 3]. For $t = 0$ we use the results of Section 4.3 to approximate the superhedging price by neural networks via the α-quantile hedging price, see Theorem 5.5 . For $t > 0$ we express the approximated process of consumption also by essential supremums of neural networks in order to apply the methodology explained in Section 4.4. In Theorem 5.7, we show that the approximated process is arbitrary close to the process of consumption using the representation of Proposition 4.16.

5.1 Neural networks

We recall the following definition of neural networks, see e.g. [21].

Definition 5.1. Consider $L, N_0, N_1, \dots, N_L \in \mathbb{N}$ with $L \geq 2$, $\sigma\colon (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ measurable and for any $\ell = 1, \dots, L$, let $W_\ell\colon \mathbb{R}^{N_{\ell-1}} \to \mathbb{R}^{N_\ell}$ be an affine function. A function $F\colon \mathbb{R}^{N_0} \to \mathbb{R}^{N_L}$ defined as

$$F(x) = W_L \circ F_{L-1} \circ \dots \circ F_1 \text{ with } F_\ell = \sigma \circ W_\ell \text{ for } \ell = 1, \dots, L-1$$

is called a *(feed forward) neural network*. Here the *activation function* σ is applied componentwise. L denotes the number of layers, $N_1, \dots, N_{L-1}$ denote the dimensions of the hidden layers and N_0, N_L the dimension of the input and output layers, respectively. For any $\ell = 1, \dots, L$ the affine function W_ℓ is given as $W_\ell(x) = A^\ell x + b^\ell$ for some $A^\ell \in \mathbb{R}^{N_\ell \times N_{\ell-1}}$ and $b^\ell \in \mathbb{R}^{N_\ell}$. For any $i = 1, \dots N_\ell, j = 1, \dots, N_{\ell-1}$ the number A^ℓ_{ij} is interpreted as the weight of the edge connecting the node i of layer $\ell - 1$ to node j of layer ℓ. The number of non-zero weights of a network is $\sum_{\ell=1}^L \|A^\ell\|_0 + \|b^\ell\|_0$, i.e. the sum of the number of non-zero entries of the matrices A^ℓ, $\ell = 1, \dots, L$, and vectors b^ℓ, $\ell = 1, \dots, L$. By $\mathcal{NN}^\sigma_{N_0,N_1}$ we denote the set of neural networks $F : \mathbb{R}^{N_0} \to \mathbb{R}^{N_1}$ with activation function σ.

Common choices for the activation function σ are the tangens hyperbolicus, relu, or sigmoid function, i.e., $\sigma_1(x) = \tanh(x)$, $\sigma_2(x) = \max(x,0)$ or $\sigma_3(x) = \frac{1}{1+e^{-x}}$. For each $k = 1, \dots, T+1$ we denote the set of all possible neural network parameters corresponding to neural networks mapping $\mathbb{R}^{mk} \to \mathbb{R}^d$ by

$$\Theta_k = \cup_{L\geq 2} \cup_{(N_0,\dots,N_L)\in\{mk\}\times\mathbb{N}^{L-1}\times\{d\}} \left(\times_{\ell=1}^{L} \mathbb{R}^{N_\ell \times N_{\ell-1}} \times \mathbb{R}^{N_\ell}\right).$$

We identify a neural network $F^{\theta_k} : \mathbb{R}^{mk} \to \mathbb{R}^d$ by its parameters specified by $\theta_k \in \Theta_k$, see Definition 5.1. Recall that $m \in \mathbb{N}$ denotes the dimension of the stochastic process $Y = (Y_t)_{t=0,1,\dots,T}$ and that $\mathcal{F}_t = \sigma(Y_0, \dots, Y_t) = \sigma(\mathcal{Y}_t)$. In particular, any $\mathcal{F}_t$-measurable random variable Z can be represented by $Z = f_t(\mathcal{Y}_t)$ for some measurable function f_t. To approximate f_t by a neural network we use the universal approximation theorem from [55] and Theorem 5.5, below.

Theorem 5.2 (Theorem 1, [55])**.** *Suppose σ is bounded and non-constant. Then, for any finite measure μ on $(\mathbb{R}^{N_0}, \mathcal{B}(\mathbb{R}^{N_0}))$ and $1 \leq p < \infty$ the set $\mathcal{NN}^{\sigma}_{N_0,1}$ is dense in $L^p(\mathcal{B}(\mathbb{R}^{N_0}), \mu)$.*

The following result essentially follows from [55, Theorem 1], but it is only mentioned in Section 3 of [55] without proof. Thus, we include a proof here. The idea is simply to approximate a measurable function by an L^1-function which then can be approximated by a neural network.

Theorem 5.3. *Assume σ is bounded and non-constant. Let $f\colon (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \to (\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m))$ be a measurable function and μ be a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Then for any $\varepsilon, \tilde{\varepsilon} > 0$ there exists a neural network g such that*

$$\mu\left(\left\{x \in \mathbb{R}^d \colon \|f(x) - g(x)\| > \tilde{\varepsilon}\right\}\right) < \varepsilon.$$

Proof. Let $\varepsilon, \tilde{\varepsilon} > 0$ be arbitrary. Choose $C > 0$ such that

$$\mu\left(\left\{x \in \mathbb{R}^d \colon \|f(x)\| > C\right\}\right) < \frac{\varepsilon}{2}. \tag{5.1}$$

Set $\tilde{f} = \mathbb{1}_{\{x\in\mathbb{R}^d : \|f(x)\|\leq C\}} f$. Then $\tilde{f} \in L^1(\mathcal{B}(\mathbb{R}^d), \mu)$ and hence by Theorem 5.2 (see also Theorem 1 of [55]) there exists a neural network g with

$$\int_{\mathbb{R}^d} \left\|\tilde{f}(x) - g(x)\right\| \mu(dx) < \frac{\varepsilon\tilde{\varepsilon}}{4}.$$

By Markov's inequality we obtain that

$$\mu\left(\left\{x \in \mathbb{R}^d \colon \|\tilde{f}(x) - g(x)\| > \frac{\tilde{\varepsilon}}{2}\right\}\right) \leq \frac{2}{\tilde{\varepsilon}} \int_{\mathbb{R}^d} \left\|\tilde{f}(x) - g(x)\right\| \mu(dx) < \frac{\varepsilon}{2}. \tag{5.2}$$

Using (5.1) and (5.2) we get

$$\begin{aligned}
&\mu\left(\left\{x \in \mathbb{R}^d \colon \|f(x) - g(x)\| > \tilde{\varepsilon}\right\}\right) \\
\leq&\mu\left(\left\{x \in \mathbb{R}^d \colon \|f(x) - \tilde{f}(x)\| > \frac{\tilde{\varepsilon}}{2}\right\} \cup \left\{x \in \mathbb{R}^d \colon \|\tilde{f}(x) - g(x)\| > \frac{\tilde{\varepsilon}}{2}\right\}\right) \\
<&\mu\left(\left\{x \in \mathbb{R}^d \colon \|f(x)\| > C\right\}\right) + \frac{\varepsilon}{2} < \varepsilon,
\end{aligned}$$

where we used that $f - \tilde{f} = f\mathbb{1}_{\{x\in\mathbb{R}^d : \|f(x)\|>C\}}$. ∎

5.2 Neural network based approximation of the superhedging price

In this section we prove that the superhedging price at $t = 0$ can be approximated by neural networks, see Theorem 5.5. For this purpose, we define the (truncated) approximate superhedging price and prove then, using the truncated α-quantile hedging price, that the approximation is arbitrary close to the superhedging price at $t = 0$ under Assumption 5.4. With the notation introduced in Section 5.1, we define the approximate superhedging price at $t = 0$ by

$$\inf \mathcal{U}_0^{\Theta} = \inf \left\{ u \in \mathbb{R} : \exists\, \theta_{k,\xi} \in \Theta_k, k = 1, \dots, T, \text{ s.t. } u + \sum_{k=1}^{T} F^{\theta_{k,\xi}}(\mathcal{Y}_{k-1}) \cdot (X_k - X_{k-1}) \geq H \right\}, \quad (5.3)$$

where $F^{\theta_{k,\xi}}$ denotes the neural network specified by the parameters $\theta_{k,\xi} \in \Theta_k$ representing the strategy ξ at time k, i.e., $F^{\theta_{k,\xi}}(\mathcal{Y}_{k-1})$ approximates ξ_k. For $\alpha \in (0,1)$ the approximate α-quantile hedging price is defined by

$$\inf \mathcal{U}_0^{\Theta,\alpha} = \inf \left\{ u \in \mathbb{R} : \exists\, \theta_{k,\xi} \in \Theta_k, k = 1, \dots, T \text{ s.t. } \mathbf{P}\left(u + \sum_{k=1}^{T} F^{\theta_{k,\xi}}(\mathcal{Y}_{k-1}) \cdot (X_k - X_{k-1}) \geq H \right) \geq \alpha \right\}. \quad (5.4)$$

Let $C > 0$. Then, we define the truncated approximate superhedging price $\inf \mathcal{U}_0^{\Theta,C}$ and the truncated approximate α-quantile hedging price $\inf \mathcal{U}_0^{\Theta,C,\alpha}$ by

$$\mathcal{U}_0^{\Theta,C} := \left\{ u \in \mathbb{R} : \exists \theta_{k,\xi} \in \Theta_k, k = 1, \dots, T \text{ s.t. } u + \textstyle\sum_{k=1}^{T} \left(\left(F^{\theta_{k,\xi}} \wedge C \right) \vee (-C) \right) (\mathcal{Y}_{t-1}) \cdot (X_k - X_{k-1}) \geq H \right\} \quad (5.5)$$

and

$$\mathcal{U}_0^{\Theta,C,\alpha} := \left\{ u \in \mathbb{R} : \exists \theta_{k,\xi} \in \Theta_k, k = 1, \dots, T \text{ s.t. } \mathbf{P}\left(u + \textstyle\sum_{k=1}^{T} \left(\left(F^{\theta_{k,\xi}} \wedge C \right) \vee (-C) \right) (\mathcal{Y}_{t-1}) \cdot (X_k - X_{k-1}) \geq H \right) \geq \alpha \right\}, \quad (5.6)$$

where the maximum and minimum are taken componentwise.

Assumption 5.4. Suppose that

$$\inf \mathcal{U}_0 = \inf \mathcal{U}_0^{\text{bdd}} := \inf \left\{ u \in \mathbb{R} : \exists \xi \text{ pred. s.t. } \xi_k \in L^\infty \ \forall k \in \{1, \dots, T\}, \ u + \textstyle\sum_{k=1}^{T} \xi_k \cdot (X_k - X_{k-1}) \geq H \right\}.$$

For instance, Assumption 5.4 is satisfied if Ω is finite. But also in the discrete Black-Scholes model, where $H = (S_T - K)^+$ for some $K > 0$, Assumption 5.4 is satisfied because the superhedging strategy is here given by the buy and hold strategy, see [24]. A wide class of discrete time, incomplete market models are presented in [24] such that Assumption 5.4 is satisfied.

In Theorem 5.5, we now prove that the superhedging price can be approximated arbitrary well by $\inf \mathcal{U}_0^{\Theta,C,\alpha}$ for suitable $C > 0$ and $\alpha \in (0,1)$.

Theorem 5.5. *Assume σ is bounded and non-constant. Further, suppose Assumption 5.4 is fulfilled. Then for any $\varepsilon > 0$ there exists $\alpha = \alpha(\varepsilon) \in (0,1)$ and $C = C(\varepsilon) \in (0,\infty)$ such that*

$$\inf \mathcal{U}_0 + \varepsilon \geq \inf \mathcal{U}_0^{\Theta,C,\alpha} \geq \inf \mathcal{U}_0 - \varepsilon. \quad (5.7)$$

Proof. We start with the right inequality of (5.7). By Assumption 5.4 we can consider $\inf \mathcal{U}_0^{\mathrm{bdd}}$ instead of $\inf \mathcal{U}_0$. Set $\tilde{u}_0 = \inf \mathcal{U}_0^{\mathrm{bdd}}$ and fix $\varepsilon > 0$. There exists an admissible strategy given by $(\tilde{u}_0 + \frac{\varepsilon}{2}, \tilde{\xi})$ such that $(\tilde{u}_0 + \frac{\varepsilon}{2}) \in \mathcal{U}_0^{\mathrm{bdd}}$. In particular, $\sup_{1\le k\le T} \|\tilde{\xi}_k\|_\infty < \infty$ and

$$\tilde{u}_0 + \frac{\varepsilon}{2} + \sum_{k=1}^{T} \tilde{\xi}_k \cdot (X_k - X_{k-1}) \ge H.$$

Define $C = C(\varepsilon)$ by

$$C := \sup_{1\le k\le T} \|\tilde{\xi}_k\|_\infty + 1. \tag{5.8}$$

Note that C only depends on ε and will be used for the second part for the proof as well. Analogously to (5.6), we define for $\alpha \in (0,1]$ the truncated α-quantile hedging price $\inf \mathcal{U}_0^{C,\alpha}$ by

$$\mathcal{U}_0^{C,\alpha} := \left\{ u \in \mathbb{R} : \exists \xi \text{ pred. s.t. } \sup_{1\le k\le T} \|\xi_k\|_\infty \le C,\ \mathbf{P}\left(u + \sum_{k=1}^{T} \xi_k \cdot (X_k - X_{k-1}) \ge H \right) \ge \alpha \right\}.$$

The truncated superhedging price at $t = 0$ is defined by $\inf \mathcal{U}_0^{C} := \inf \mathcal{U}_0^{C,1}$. In the first step, we prove that the limit of $\inf \mathcal{U}_0^{C,\alpha}$ for α tending to 1 exists and that

$$\inf \mathcal{U}_0^{\mathrm{bdd}} \le \lim_{\alpha\to 1} \inf \mathcal{U}_0^{C,\alpha} \le \inf \mathcal{U}_0^{\mathrm{bdd}} + \varepsilon. \tag{5.9}$$

Let $(\alpha_n)_{n\in\mathbb{N}} \subset (0,1)$ be an increasing sequence such that $\alpha_n \uparrow 1$ as n tends to infinity. By the definition of $(\mathcal{U}_0^{C,\alpha_n})_{n\in\mathbb{N}}$ we have

$$\mathcal{U}_0^{C,\alpha_n} \supseteq \mathcal{U}_0^{C,\alpha_{n+1}},$$

and thus $\inf \mathcal{U}_0^{C,\alpha_n} \le \inf \mathcal{U}_0^{C,\alpha_{n+1}} \le \inf \mathcal{U}_0^{C}$. For $n \in \mathbb{N}$, set $u_n := \inf \mathcal{U}_0^{C,\alpha_n}$. Since $(u_n)_{n\in\mathbb{N}}$ is monotone and bounded, the limit $u^C := \lim_{n\to\infty} u_n$ is well-defined and $u^C \le \inf \mathcal{U}_0^{C}$. Let $\delta > 0$ be arbitrary. For all $n \in \mathbb{N}$ there exists $\xi^{(n)} = (\xi_t^{(n)})_{t=1,\dots,T}$ predictable with $\sup_{1\le k\le T} \|\xi_k^{(n)}\|_\infty \le C$ such that $(u_n, \xi^{(n)})$ defines an admissible strategy and

$$\mathbf{P}\left(u_n + \delta + \sum_{k=1}^{T} \xi_k^{(n)} \cdot (X_k - X_{k-1}) \ge H \right) \ge \alpha_n. \tag{5.10}$$

Note that such $(u_n, \xi^{(n)})_{n\in\mathbb{N}}$ exists because $(\tilde{u}_0, \tilde{\xi})$ already fulfills (5.9). For $n \in \mathbb{N}$, define the set of success $A_n \in \mathcal{F}_T$ by

$$A_n := \left\{ u_n + \delta + \sum_{k=1}^{T} \xi_k^{(n)} \cdot (X_k - X_{k-1}) \ge H \right\}.$$

By definition, $\mathbf{P}(A_n) \ge \alpha_n$ for all $n \in \mathbb{N}$ and because $\alpha_n \uparrow 1$ as n tends to infinity we also get $\mathbf{P}(A_n) \uparrow 1$ as n tends to infinity. Since $\sup_{1\le k\le T} \|\xi_k^{(n)}\|_\infty \le C$ for all $n \in \mathbb{N}$, Theorem 5.14 of [40] guarantees that the associated value process of $(u_n, \xi^{(n)})$ is a $\mathcal{P}$-martingale and thus we get for all $\mathbf{P}^* \in \mathcal{P}$ that

$$\begin{aligned} u_n + \delta &= \mathbb{E}^*\left[u_n + \delta + \sum_{k=1}^T \xi_k^{(n)} \cdot (X_k - X_{k-1})\right] \qquad (5.11) \\ &\geq \mathbb{E}^*\left[H\mathbb{1}_{A_n}\right] + \mathbb{E}^*\left[\left(u_n + \delta + \sum_{k=1}^T \xi_k^{(n)} \cdot (X_k - X_{k-1})\right)\mathbb{1}_{A_n^c}\right] \\ &\geq \mathbb{E}^*[H\mathbb{1}_{A_n}] + \mathbb{E}^*\left[\left(u_n + \delta - \sum_{k=1}^T \sum_{i=1}^d |\xi_k^{i,(n)}|\, |X_k^i - X_{k-1}^i|\right)\mathbb{1}_{A_n^c}\right] \\ &\geq \mathbb{E}^*[H\mathbb{1}_{A_n}] + \mathbb{E}^*\left[\left(u_n + \delta - C\sum_{k=1}^T \sum_{i=1}^d |X_k^i - X_{k-1}^i|\right)\mathbb{1}_{A_n^c}\right]. \qquad (5.12) \end{aligned}$$

We will now prove that the right hand side of (5.12) converges to $\mathbb{E}^*[H]$. First, $\mathbb{1}_{A_n}$ converges in probability to 1 as n tends to infinity, as for any $\gamma \in (0,1)$ we have

$$\mathbf{P}\left(|\mathbb{1}_{A_n} - 1| > \gamma\right) = \mathbf{P}\left(\mathbb{1}_{A_n^c} > \gamma\right) = \mathbf{P}(A_n^c) \xrightarrow{n\to\infty} 0,$$

because of (5.10). It is in general not true that $\mathbb{1}_{A_n} \xrightarrow{\mathbf{P}\text{-a.s.}} 1$ since A_n may be not contained in A_{n+1}. Therefore, we obtain convergence in probability for

$$H\mathbb{1}_{A_n} \xrightarrow{\mathbf{P}} H, \quad \text{as } n \to \infty. \qquad (5.13)$$

Using that $X = (X^1, \dots, X^d)$ is a d-dimensional $\mathcal{P}$-martingale and $u_n \leq u^C$, for all $n \in \mathbb{N}$ we get that

$$\left|u_n + \delta - C\sum_{k=1}^T \sum_{i=1}^d |X_k^i - X_{k-1}^i|\right| \leq \left(\left|u^C + \delta\right| + |C|\sum_{k=1}^T \sum_{i=1}^d |X_k^i - X_{k-1}^i|\right) \in L^1(\mathcal{F}_T, \mathbf{P}^*).$$

and hence

$$\left(u_n + \delta - C\sum_{k=1}^T \sum_{i=1}^d |X_k^i - X_{k-1}^i|\right)\mathbb{1}_{A_n^c} \xrightarrow{\mathbf{P}} 0 \quad \text{as } n \to \infty. \qquad (5.14)$$

By (5.13) and (5.14) Dominated convergence yields

$$\lim_{n\to\infty} \mathbb{E}^*[H\mathbb{1}_{A_n}] = \mathbb{E}^*[H],$$

and

$$\lim_{n\to\infty} \mathbb{E}^*\left[\left(u_n + \delta - C\sum_{k=1}^T \sum_{i=1}^d |X_k^i - X_{k-1}^i|\right)\mathbb{1}_{A_n^c}\right] = 0,$$

where we used that for dominated convergence it is sufficient that only convergence in probability holds in (5.13) and (5.14). By letting n go to infinity in (5.11) and (5.12) we obtain

$$\lim_{n\to\infty} u_n + \delta = u^C + \delta \geq \lim_{n\to\infty}\left(\mathbb{E}^*[H\mathbb{1}_{A_n}] + \mathbb{E}^*\left[\left(u_n + \delta - C\sum_{k=1}^T \sum_{i=1}^d |X_k^i - X_{k-1}^i|\right)\mathbb{1}_{A_n^c}\right]\right) = \mathbb{E}^*[H]. \qquad (5.15)$$

As (5.15) holds for all $\mathbf{P}^* \in \mathcal{P}$ we can take the supremum on the right hand side and get

$$u^C + \delta \geq \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H].$$

By the superhedging duality, see Corollary A.4, we then obtain

$$\lim_{n\to\infty} \inf \mathcal{U}_0^{C,\alpha_n} + \delta = u^C + \delta \geq \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H] = \inf \mathcal{U}_0 = \inf \mathcal{U}_0^{\mathrm{bdd}}.$$

Since $\delta > 0$ was arbitrary, we get

$$\lim_{\alpha\to 1} \inf \mathcal{U}_0^{C,\alpha} \geq \inf \mathcal{U}_0 = \inf \mathcal{U}_0^{\mathrm{bdd}},$$

which proves the left hand side of (5.9). For the right hand side of (5.9) recall that $(\tilde{u}_0 + \frac{\varepsilon}{2}) \in \mathcal{U}_0^C$ by definition and $\mathcal{U}_0^C \subseteq \mathcal{U}_0^{\mathrm{bdd}}$. On the one hand, this implies that $\inf \mathcal{U}_0^{\mathrm{bdd}} \leq \inf \mathcal{U}_0^C$ and on the other hand,

$$\lim_{\alpha\to 1} \inf \mathcal{U}_0^{C,\alpha} \leq \inf \mathcal{U}_0^C \leq \tilde{u}_0 + \frac{\varepsilon}{2} \leq \inf \mathcal{U}_0^{\mathrm{bdd}} + \varepsilon.$$

Thus, we obtain (5.9). By (5.9) there exists $\alpha = \alpha(\varepsilon) \in (0,1)$ such that

$$\inf \mathcal{U}_0 - \varepsilon = \inf \mathcal{U}_0^{\mathrm{bdd}} - \varepsilon \leq \inf \mathcal{U}_0^{C,\alpha}. \tag{5.16}$$

To conclude the right hand side of (5.7), note that $\mathcal{U}_0^{\Theta,C,\alpha} \subseteq \mathcal{U}_0^{C,\alpha}$ and thus $\inf \mathcal{U}_0^{\Theta,C,\alpha} \geq \inf \mathcal{U}_0^{C,\alpha}$. By (5.16) this yields (5.7).
Let $\alpha \in (0,1)$ be given. We now prove the first part of (5.7). For this purpose, define for $n \in \mathbb{N}$ the set

$$M_n := \left\{ \tilde{u}_0 + \frac{\varepsilon}{2} + \sum_{k=1}^{T} \tilde{\xi}_k \cdot (X_k - X_{k-1}) \geq H \right\} \cap \{\|X_k - X_{k-1}\| \leq n \text{ for } k = 1,\dots,T\}.$$

Then $M_n \subset M_{n+1}$ and thus by the definition of $\tilde{u}_0$ we get

$$1 = \mathbf{P}\left(\tilde{u}_0 + \frac{\varepsilon}{2} + \sum_{k=1}^{T} \tilde{\xi}_k \cdot (X_k - X_{k-1}) \geq H \right) = \mathbf{P}\left(\bigcup_{n\in\mathbb{N}} M_n \right) = \lim_{n\to\infty} \mathbf{P}(M_n).$$

In particular, there exists $n \in \mathbb{N}$ such that $\mathbf{P}(M_n) \geq \frac{\alpha+1}{2}$. Now, we want to approximate the trading strategy $\tilde{\xi}$ by neural networks. For this propose, we note that for each $k = 1,\dots,T$ there exists a measurable function $f_k\colon (\mathbb{R}^{mk}, \mathcal{B}(\mathbb{R}^{mk})) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that $\tilde{\xi}_k = f_k(\mathcal{Y}_{k-1})$, where we used that $\tilde{\xi}$ is predictable. Then, the universal approximation theorem [55, Theorem 1 and Section 3], see also Theorem 5.3 guarantees that for each $k = 1,\dots,T$ there exists $\theta_{k,\tilde{\xi}} \in \Theta$ specifying the neural network $F^{\theta_{k,\xi}}$ such that

$$\mathbf{P}(D_k) < \frac{1-\alpha}{2T}, \quad \text{where } D_k = \left\{ \omega \in \Omega\colon \|f_k(\mathcal{Y}_{k-1}(\omega)) - F^{\theta_{k,\tilde{\xi}}}(\mathcal{Y}_{k-1}(\omega))\| > \left(\frac{\varepsilon}{2nT} \wedge \frac{1}{2} \right) \right\}, \tag{5.17}$$

where $\|\cdot\|$ denotes the Euclidean norm. Define

$$\tilde{F}^{\theta_{k,\tilde{\xi}}} := \left(F^{\theta_{k,\tilde{\xi}}} \wedge C \right) \vee (-C), \quad k = 1,\dots,T,$$

and

$$\tilde{D}_k = \left\{ \omega \in \Omega\colon \|f_k(\mathcal{Y}_{k-1}(\omega)) - \tilde{F}^{\theta_{k,\tilde{\xi}}}(\mathcal{Y}_{k-1}(\omega))\| > \left(\frac{\varepsilon}{2nT} \wedge \frac{1}{2} \right) \right\}.$$

Note that (5.8) implies

$$\|\tilde{\xi}_k\|_\infty + \left(\frac{\varepsilon}{2nT} \wedge \frac{1}{2}\right) < C \quad \text{for all } k = 1, \dots, T.$$

We show that $D_k = \tilde{D}_k$. For $\omega \in D_k^c$ we get for $i \in \{1, \dots, d\}$ that

$$\left|F_i^{\theta_k,\tilde{\xi}}(\mathcal{Y}_{k-1})(\omega)\right| \le \left\|F^{\theta_k,\tilde{\xi}}(\mathcal{Y}_{k-1})(\omega)\right\| \le \left\|\tilde{\xi}_k\right\|_\infty + \left(\frac{\varepsilon}{2nT} \wedge \frac{1}{2}\right) < C,$$

and hence $\tilde{F}_i^{\theta_k,\tilde{\xi}}(\mathcal{Y}_{k-1}) = F_i^{\theta_k,\tilde{\xi}}(\mathcal{Y}_{k-1})$ on D_k^c. Similarly, for $\omega \in \tilde{D}_k^c$ such that we get for $i \in \{1, \dots, d\}$ that

$$\left|\tilde{F}_i^{\theta_k,\tilde{\xi}}(\mathcal{Y}_{k-1}(\omega))\right| \le \left\|\tilde{F}^{\theta_k,\tilde{\xi}}(\mathcal{Y}_{k-1}(\omega))\right\| \le \left\|\tilde{\xi}_k\right\|_\infty + \left(\frac{\varepsilon}{2nT} \wedge \frac{1}{2}\right) < C,$$

and hence $\tilde{F}_i^{\theta_k,\tilde{\xi}}(\mathcal{Y}_{k-1}(\omega)) = F_i^{\theta_k,\tilde{\xi}}(\mathcal{Y}_{k-1}(\omega))$. Therefore

$$D_k^c = \tilde{D}_k^c \quad \text{for all } k = 1, \dots, T,$$

and thus also $D_k = \tilde{D}_k$. In particular,

$$\mathbf{P}(\tilde{D}_k) < \frac{1-\alpha}{2T}, \quad k = 1, \dots, T.$$

To this end, we show that

$$M_n \cap \tilde{D}_1^c \cap \ldots \cap \tilde{D}_T^c \subset \left\{\tilde{u}_0 + \varepsilon + \sum_{k=1}^T \tilde{F}^{\theta_k,\tilde{\xi}}(\mathcal{Y}_{k-1}) \cdot (X_k - X_{k-1}) \ge H\right\}. \tag{5.18}$$

Indeed, on $M_n \cap \tilde{D}_1^c \cap \ldots \cap \tilde{D}_T^c$ it holds that

$$\begin{aligned}
\sum_{k=1}^T \tilde{\xi}_k \cdot (X_k - X_{k-1}) &= \sum_{k=1}^T (\tilde{\xi}_k - \tilde{F}^{\theta_k,\tilde{\xi}}(\mathcal{Y}_{k-1})) \cdot (X_k - X_{k-1}) + \sum_{k=1}^T \tilde{F}^{\theta_k,\tilde{\xi}}(\mathcal{Y}_{k-1}) \cdot (X_k - X_{k-1}) \\
&\le \sum_{k=1}^T \|f_k(\mathcal{Y}_{k-1}) - \tilde{F}^{\theta_k,\tilde{\xi}}(\mathcal{Y}_{k-1})\| \|X_k - X_{k-1}\| \\
&+ \sum_{k=1}^T \tilde{F}^{\theta_k,\tilde{\xi}}(\mathcal{Y}_{k-1}) \cdot (X_k - X_{k-1}) \\
&\le \sum_{k=1}^T \left(\frac{\varepsilon}{2nT} \wedge \frac{1}{2}\right) n + \sum_{k=1}^T \tilde{F}^{\theta_k,\tilde{\xi}}(\mathcal{Y}_{k-1}) \cdot (X_k - X_{k-1}) \\
&\le \frac{\varepsilon}{2} + \sum_{k=1}^T \tilde{F}^{\theta_k,\tilde{\xi}}(\mathcal{Y}_{k-1}) \cdot (X_k - X_{k-1}).
\end{aligned}$$

Thus, we conclude the left inequality of (5.7) by (5.18) and the Fréchet inequalities[1], which yield

$$\begin{aligned}
\mathbf{P}\left(\tilde{u}_0 + \varepsilon + \sum_{k=1}^T \tilde{F}^{\theta_k,\tilde{\xi}}(\mathcal{Y}_{k-1}) \cdot (X_k - X_{k-1}) \ge H\right) &\ge \mathbf{P}(M_n \cap \tilde{D}_1^c \cap \ldots \cap \tilde{D}_T^c) \\
&\ge \mathbf{P}(M_n) + \mathbf{P}(\tilde{D}_1^c) + \cdots + \mathbf{P}(\tilde{D}_T^c) - T \\
&\ge \frac{\alpha+1}{2} + T\left(1 - \frac{1-\alpha}{2T}\right) - T \\
&= \alpha.
\end{aligned}$$

[1]For $C_1, \dots, C_l \in \mathcal{F}$ it holds that $P(C_1 \cap \cdots \cap C_l) \ge \max\{0, \mathbf{P}(C_1) + \cdots + \mathbf{P}(C_l) - (l-1)\}$.

This concludes the proof. □

5.3 Neural network based approximation of the superhedging process

Finally, we prove that the process of consumption can be approximated by neural networks and also introduce an $\tilde{\varepsilon}$-approximative process of consumption, see (5.26), which indicates how the method can be implemented. For this purpose, we show in Proposition 5.6 that for the process B there exist neural networks that are arbitrary close in probability. Then, in Theorem 5.7, we show that for the $\tilde{\varepsilon}$-approximative process is also $\tilde{\varepsilon}$ close to B.
We recall the notation of neural networks from Section 5.1. By Θ we denote the set of parameters corresponding to all neural networks. For each $k = 1, \dots, T+1$ we denote the set of all possible neural network parameters corresponding to neural networks mapping $\mathbb{R}^{mk} \to \mathbb{R}^d$ by

$$\Theta_k = \cup_{L\geq 2} \cup_{(N_0,\dots,N_L)\in\{mk\}\times\mathbb{N}^{L-1}\times\{d\}} \left(\times_{\ell=1}^{L} \mathbb{R}^{N_\ell\times N_{\ell-1}} \times \mathbb{R}^{N_\ell}\right).$$

For $\varepsilon, \tilde{\varepsilon} \in (0,1)$ we define the set

$$\mathcal{B}_t^{\theta_t^*,\varepsilon,\tilde{\varepsilon}} := \left\{F^{\theta_t}(\mathcal{Y}_t) : \theta_t \in \Theta_{t+1} \text{ and } \mathbf{P}\left(B_{t-1} - \tilde{\varepsilon} \leq F^{\theta_t}(\mathcal{Y}_t) \leq \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H] + \textstyle\sum_{k=1}^{T} \xi_k \cdot (X_k - X_{k-1}) - H + \tilde{\varepsilon}\right) \geq 1-\varepsilon\right\}. \tag{5.19}$$

We now construct an approximation of B by neural networks.

Proposition 5.6. *Assume σ is bounded and non-constant. Then for any $\varepsilon, \tilde{\varepsilon} > 0$ there exist neural networks $(F^{\theta_0,\varepsilon,\tilde{\varepsilon}}, \dots, F^{\theta_T,\varepsilon,\tilde{\varepsilon}})$ such that $F^{\theta_t,\varepsilon,\tilde{\varepsilon}}(\mathcal{Y}_t) \in \mathcal{B}_t^{\theta_t^*,\varepsilon,\tilde{\varepsilon}}$ for all $t = 0, \dots, T$ and*

$$\boldsymbol{P}\left(\left|F^{\theta_t,\varepsilon,\tilde{\varepsilon}}(\mathcal{Y}_t) - B_t\right| > \tilde{\varepsilon}\right) < \varepsilon, \quad \text{for all } t = 0, \dots, T.$$

In particular, there exists a sequence of neural networks $\left(F^{\theta_0^n}, \dots, F^{\theta_T^n}\right)_{n\in\mathbb{N}}$ with $F^{\theta_t^n}(\mathcal{Y}_t) \in \mathcal{B}_t^{\theta_t^*,\frac{1}{n},\frac{1}{n}}$ for all $n \in \mathbb{N}$ and for all $t = 0, \dots, T$ such that

$$\left(F^{\theta_0^n}(\mathcal{Y}_0), \dots, F^{\theta_T^n}(\mathcal{Y}_T)\right) \xrightarrow{\boldsymbol{P}\text{-a.s.}} (B_0, \dots, B_T) \quad \text{for } n \to \infty. \tag{5.20}$$

Proof. Fix $\varepsilon, \tilde{\varepsilon} > 0$ and $t \in \{1, \dots, T\}$. Note that $B_0 = 0$ by definition. We use the representation of B given in (4.39), see Proposition 4.16. The set $\mathcal{B}_t$ defined in (4.39) is directed upwards, (see Definition 2.2) since for $B_t^1, B_t^2 \in \mathcal{B}_t$ also $\tilde{B}_t \in \mathcal{B}_t$, where

$$\tilde{B}_t := B_t^1 \vee B_t^2.$$

Thus, by Theorem A.33 of [40], there exists an increasing sequence

$$(B_t^k)_{k\in\mathbb{N}} \subset \mathcal{B}_t,$$

such that

$$B_t^k \xrightarrow{\mathbf{P}\text{-a.s.}} \tilde{B}_t = B_t.$$

Because almost sure convergence implies convergence in probability, there exists $K = K(\varepsilon, \tilde{\varepsilon}) \in \mathbb{N}$ such that

$$\mathbf{P}\left(\left|B_t^k - B_t\right| > \frac{\tilde{\varepsilon}}{2}\right) < \frac{\varepsilon}{2}, \quad \text{for all } k \geq K. \tag{5.21}$$

Fix $k \geq K$. Then there exists a measurable function $f_t^k : (\mathbb{R}^{m(t+1)}, \mathcal{B}(\mathbb{R}^{m(t+1)}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $B_t^k = f_t^k(\mathcal{Y}_t)$. Using the universal approximation theorem [55, Theorem 1 and Section 3] (see also Theorem 5.3) we obtain $\theta_t = \theta_t^k \in \Theta_{t+1}$ and $F^{\theta_t} = F^{\theta_t^k, \varepsilon, \tilde{\varepsilon}}$ satisfying

$$\mathbf{P}\left(\left|f_t^k(\mathcal{Y}_t) - F^{\theta_t}(\mathcal{Y}_t)\right| > \frac{\tilde{\varepsilon}}{2}\right) < \frac{\varepsilon}{2}. \tag{5.22}$$

By the triangle inequality and by De Morgan's law we obtain that

$$\begin{aligned}
&\left\{\omega \in \Omega : \left|B_t(\omega) - F^{\theta_t}(\mathcal{Y}_t(\omega))\right| > \tilde{\varepsilon}\right\} \\
&\subseteq \left\{\omega \in \Omega : \left|B_t(\omega) - B_t^k(\omega)\right| + \left|B_t^k - F^{\theta_t}(\mathcal{Y}_t(\omega))\right| > \tilde{\varepsilon}\right\} \\
&= \left(\left\{\omega \in \Omega : \left|B_t(\omega) - B_t^k(\omega)\right| + \left|B_t^k - F^{\theta_t}(\mathcal{Y}_t(\omega))\right| \leq \tilde{\varepsilon}\right\}\right)^c \\
&\subseteq \left(\left\{\omega \in \Omega : \left|B_t(\omega) - B_t^k(\omega)\right| \leq \frac{\tilde{\varepsilon}}{2}\right\} \cap \left\{\omega \in \Omega : \left|B_t^k - F^{\theta_t}(\mathcal{Y}_t(\omega))\right| \leq \frac{\tilde{\varepsilon}}{2}\right\}\right)^c \\
&= \left\{\omega \in \Omega : \left|B_t(\omega) - B_t^k(\omega)\right| > \frac{\tilde{\varepsilon}}{2}\right\} \cup \left\{\omega \in \Omega : \left|B_t^k(\omega) - F^{\theta_t}(\mathcal{Y}_t(\omega))\right| > \frac{\tilde{\varepsilon}}{2}\right\}.
\end{aligned} \tag{5.23}$$

By (5.21), (5.22), (5.23) and sub-addidivity get that

$$\mathbf{P}\left(\left|B_t - F^{\theta_t}(\mathcal{Y}_t)\right| > \tilde{\varepsilon}\right) \leq \mathbf{P}\left(\left|B_t - B_t^k\right| > \frac{\tilde{\varepsilon}}{2}\right) + \mathbf{P}\left(\left|B_t^k - F^{\theta_t}(\mathcal{Y}_t)\right| > \frac{\tilde{\varepsilon}}{2}\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \tag{5.24}$$

which proves the first part of Proposition 5.6. We note that

$$B_{t-1} \leq B_t \leq \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H] + \sum_{k=1}^{T} \xi_k \cdot (X_k - X_{k-1}) - H,$$

and thus by (5.24)

$$\mathbf{P}\left(B_{t-1} - \tilde{\varepsilon} \leq F^{\theta_t}(\mathcal{Y}_t) \leq \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H] + \textstyle\sum_{k=1}^{T} \xi_k \cdot (X_k - X_{k-1}) - H + \tilde{\varepsilon}\right) \geq \mathbf{P}\left(\left|B_t - F^{\theta_t}(\mathcal{Y}_t)\right| \leq \tilde{\varepsilon}\right) \geq 1 - \varepsilon,$$

which implies that $F^{\theta_t}(\mathcal{Y}_t) = F^{\theta_t^k, \varepsilon, \tilde{\varepsilon}}(\mathcal{Y}_t) \in \mathcal{B}_t^{\theta_t^*, \varepsilon, \tilde{\varepsilon}}$. Now, it is straightforward to construct a sequence of neural networks satisfying (5.20). For $n \in \mathbb{N}$, we set $\varepsilon = \frac{1}{n} = \tilde{\varepsilon}$ and consider the neural network

$$F^{\theta_t^n} := F^{\theta_t^{K(n)}, \frac{1}{n}, \frac{1}{n}}, \quad t \in \{1, \dots, T\}, \ n \in \mathbb{N},$$

where $K(n) = K(\frac{1}{n}, \frac{1}{n})$ is given by (5.21). Then, $F^{\theta_t^n} \in \mathcal{B}_t^{\theta_t^*, \frac{1}{n}, \frac{1}{n}}$ for all $n \in \mathbb{N}$ and for all $t = 0, \dots, T$. By (5.24) we have

$$\mathbf{P}\left(\left|F^{\theta_t^n}(\mathcal{Y}_t) - B_t\right| > \frac{1}{n}\right) < \frac{1}{n} \quad \text{for all } t = 1, \dots, T,$$

which implies convergence in probability, i.e.,

$$F^{\theta_t^n}(\mathcal{Y}_t) \xrightarrow{\mathbf{P}} B_t \quad \text{for } n \to \infty, \text{ for all } t = 0, \dots, T.$$

By passing to a suitable subsequence, convergence also holds $\mathbf{P}$-a.s. simultaneously for all $t = 0, \dots, T$. □

Although, Proposition 5.6 guarantees that B can be approximated by neural networks, it does not give any help how this method could be implemented in practice. For this reason, we introduce $\mathcal{B}_t^{\theta_t^*,\tilde{\varepsilon}}$ in (5.25) below.
Let $\tilde{\varepsilon} > 0$. Recursively, we define the set

$$\begin{aligned}\tilde{\mathcal{B}}_t^{\theta_t^*,\tilde{\varepsilon}} := \{&F^{\theta_t}(\mathcal{Y}_t)\mathbb{1}_A + B_{t-1}^{\theta_{t-1}^*,\tilde{\varepsilon}}\mathbb{1}_{A^c} : \theta_t \in \Theta_{t+1}, A \in \mathcal{F}_t,\\ &B_{t-1}^{\theta_{t-1}^*,\tilde{\varepsilon}} \leq F^{\theta_t}(\mathcal{Y}_t)\mathbb{1}_A + B_{t-1}^{\theta_{t-1}^*,\tilde{\varepsilon}}\mathbb{1}_{A^c} \leq \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H] + \sum_{k=1}^T \xi_k \cdot (X_k - X_{k-1}) - H + \tilde{\varepsilon}\ \}\end{aligned} \tag{5.25}$$

for $t = 1, \dots, T$, and the approximate process of consumption by $B_0^{\theta_0^*,\tilde{\varepsilon}} = 0$ and

$$B_t^{\theta_t^*,\tilde{\varepsilon}} := \operatorname{ess\,sup} \tilde{\mathcal{B}}_t^{\theta_t^*,\tilde{\varepsilon}} \quad \text{for } t = 1, \dots, T. \tag{5.26}$$

Theorem 5.7. *Assume σ is bounded and non-constant. Then*

$$\left|B_t^{\theta_t^*,\tilde{\varepsilon}} - B_t\right| \leq \tilde{\varepsilon} \quad \textit{for all } t = 0, \dots, T.$$

Proof. The proof follows by induction. For $t = 0$ we have by definition $B_0^{\theta_0^*,\tilde{\varepsilon}} = B_0 = 0$. Assume now that

$$\left|B_{t-1}^{\theta_{t-1}^*,\tilde{\varepsilon}} - B_{t-1}\right| \leq \tilde{\varepsilon} \tag{5.27}$$

for some $t \in \{1, \dots, T\}$. First, we prove that

$$B_t + \tilde{\varepsilon} = \operatorname{ess\,sup}\left\{D_t \in L^0(\mathcal{F}_t, \mathbf{P}) : -\tilde{\varepsilon} \leq D_t - \tilde{\varepsilon} \leq \sup\nolimits_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H] + \textstyle\sum_{k=1}^T \xi_k \cdot (X_k - X_{k-1}) - H\ \right\}. \tag{5.28}$$

On the one hand we have

$$\begin{aligned}&\operatorname{ess\,sup}\left\{\tilde{D}_t \in L^0(\mathcal{F}_t, \mathbf{P}) : 0 \leq \tilde{D}_t \leq \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H] + \sum_{k=1}^T \xi_k \cdot (X_k - X_{k-1}) - H\ \right\} + \tilde{\varepsilon}\\ &= \operatorname{ess\,sup}\left\{D_t \in L^0(\mathcal{F}_t, \mathbf{P}) : 0 \leq D_t - \tilde{\varepsilon} \leq \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H] + \sum_{k=1}^T \xi_k \cdot (X_k - X_{k-1}) - H\ \right\}\\ &\leq \operatorname{ess\,sup}\left\{D_t \in L^0(\mathcal{F}_t, \mathbf{P}) : -\tilde{\varepsilon} \leq D_t - \tilde{\varepsilon} \leq \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H] + \sum_{k=1}^T \xi_k \cdot (X_k - X_{k-1}) - H\ \right\}.\end{aligned} \tag{5.29}$$

By (5.29) and Remark 4.17 we obtain that

$$B_t + \tilde{\varepsilon} \leq \operatorname{ess\,sup}\left\{D_t \in L^0(\mathcal{F}_t, \mathbf{P}) : -\tilde{\varepsilon} \leq D_t - \tilde{\varepsilon} \leq \sup\nolimits_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H] + \textstyle\sum_{k=1}^T \xi_k \cdot (X_k - X_{k-1}) - H\ \right\}.$$

On the other hand, define $\tilde{D}_t := D_t \vee \tilde{\varepsilon}$ for

$$D_t \in \left\{\bar{D}_t \in L^0(\mathcal{F}_t, \mathbf{P}) : -\tilde{\varepsilon} \leq \bar{D}_t - \tilde{\varepsilon} \leq \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H] + \sum_{k=1}^T \xi_k \cdot (X_k - X_{k-1}) - H\ \right\}.$$

Then $D_t \leq \tilde{D}_t$ and

$$\tilde{D}_t \in \left\{\bar{D}_t \in L^0(\mathcal{F}_t, \mathbf{P}) : 0 \leq \bar{D}_t - \tilde{\varepsilon} \leq \sup_{\mathbf{P}^*\in\mathcal{P}} \mathbb{E}^*[H] + \sum_{k=1}^T \xi_k \cdot (X_k - X_{k-1}) - H\ \right\}.$$

Taking the essential supremum and by Remark 4.17 we get

$$B_t + \tilde{\varepsilon} \geq \operatorname{ess\,sup}\left\{ D_t \in L^0(\mathcal{F}_t, \mathbf{P}) : -\tilde{\varepsilon} \leq D_t - \tilde{\varepsilon} \leq \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H] + \sum_{k=1}^{T} \xi_k \cdot (X_k - X_{k-1}) - H \right\},$$

and thus (5.28) follows as

$$\begin{aligned}
&\left\{ D_t \in L^0(\mathcal{F}_t, \mathbf{P}) : 0 \leq D_t \leq \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H] + \sum_{k=1}^{T} \xi_k \cdot (X_k - X_{k-1}) - H + \tilde{\varepsilon} \right\} \\
&= \left\{ D_t \in L^0(\mathcal{F}_t, \mathbf{P}) : -\tilde{\varepsilon} \leq D_t - \tilde{\varepsilon} \leq \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H] + \sum_{k=1}^{T} \xi_k \cdot (X_k - X_{k-1}) - H \right\}.
\end{aligned} \tag{5.30}$$

Let $F^{\theta_t}(\mathcal{Y}_t)\mathbb{1}_A + B_{t-1}^{\theta_{t-1}^*, \tilde{\varepsilon}} \mathbb{1}_{A^c} \in \tilde{\mathcal{B}}_t^{\theta_t^*, \tilde{\varepsilon}}$ be arbitrary, i.e., $\theta_t \in \Theta_{t+1}$ and $A \in \mathcal{F}_t$ such that

$$0 \leq B_{t-1}^{\theta_{t-1}^*, \tilde{\varepsilon}} \leq F^{\theta_t}(\mathcal{Y}_t)\mathbb{1}_A + B_{t-1}^{\theta_{t-1}^*, \tilde{\varepsilon}} \mathbb{1}_{A^c} \leq \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H] + \sum_{k=1}^{T} \xi_k \cdot (X_k - X_{k-1}) - H + \tilde{\varepsilon}.$$

By (5.25) and (5.26) $B_s^{\theta_s^*, \tilde{\varepsilon}} \leq B_{s+1}^{\theta_{s+1}^*, \tilde{\varepsilon}}$ for all $s \in \{0, T-1\}$ and $B_0^{\theta_0^*, \tilde{\varepsilon}} = 0$. In particular, $B_s^{\theta_s^*, \tilde{\varepsilon}} \geq 0$ for all $s \geq 0$ and thus

$$F^{\theta_t}(\mathcal{Y}_t)\mathbb{1}_A + B_{t-1}^{\theta_{t-1}^*, \tilde{\varepsilon}} \mathbb{1}_{A^c} \in \left\{ D_t \in L^0(\mathcal{F}_t, \mathbf{P}) : 0 \leq D_t \leq \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H] + \sum_{k=1}^{T} \xi_k \cdot (X_k - X_{k-1}) - H + \tilde{\varepsilon} \right\}.$$

Therefore, (5.28) and (5.30) imply

$$F^{\theta_t}(\mathcal{Y}_t)\mathbb{1}_A + B_{t-1}^{\theta_{t-1}^*, \tilde{\varepsilon}} \mathbb{1}_{A^c} \leq B_t + \tilde{\varepsilon},$$

and by taking the essential supremum on the left hand side also that

$$B_t^{\theta_t^*, \tilde{\varepsilon}} \leq B_t + \tilde{\varepsilon}. \tag{5.31}$$

For the converse direction let $\varepsilon \in (0,1)$. By Proposition 5.6 there exists a neural network $F^{\tilde{\theta}_t} = F^{\tilde{\theta}_t, \varepsilon, \tilde{\varepsilon}}$ such that

$$\mathbf{P}\left(\left| F^{\tilde{\theta}_t}(\mathcal{Y}_t) - B_t \right| > \tilde{\varepsilon} \right) < \varepsilon.$$

We define the sets $A_1, A_2 \in \mathcal{F}_t$ by

$$A_1 := \left\{ \omega \in \Omega : B_t(\omega) - \tilde{\varepsilon} \leq F^{\tilde{\theta}_t}(\mathcal{Y}_t(\omega)) \leq B_t(\omega) + \tilde{\varepsilon} \right\},$$

and

$$A_2 := \left\{ \omega \in \Omega : B_{t-1}^{\theta_{t-1}^*, \tilde{\varepsilon}}(\omega) \leq F^{\tilde{\theta}_t}(\mathcal{Y}_t(\omega)) \right\}.$$

Then, $\mathbf{P}(A_1) > 1 - \varepsilon$. Recall that by monotonicity of B and by (5.27)

$$B_{t-1}^{\theta_{t-1}^*, \tilde{\varepsilon}} \leq B_{t-1} + \tilde{\varepsilon} \leq B_t + \tilde{\varepsilon}. \tag{5.32}$$

Because

$$B_t \leq \sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H] + \sum_{k=1}^{T} \xi_k \cdot (X_k - X_{k-1}) - H,$$

we get by construction for $A := A_1 \cap A_2$ that

$$F^{\tilde{\theta}_t}(\mathcal{Y}_t)\mathbb{1}_A + B_{t-1}^{\theta^*_{t-1},\tilde{\varepsilon}}\mathbb{1}_{A^c} = F^{\tilde{\theta}_t}(\mathcal{Y}_t)\mathbb{1}_{A_1\cap A_2} + B_{t-1}^{\theta^*_{t-1},\tilde{\varepsilon}}\mathbb{1}_{A_1^c\cup A_2^c} \in \tilde{\mathcal{B}}_t^{\theta^*_t,\tilde{\varepsilon}}.$$

We now prove that

$$\mathbf{P}\left(\left|\left(F^{\tilde{\theta}_t}(\mathcal{Y}_t)\mathbb{1}_A + B_{t-1}^{\theta^*_{t-1},\tilde{\varepsilon}}\mathbb{1}_{A^c}\right) - B_t\right| > \tilde{\varepsilon}\right) < \varepsilon.$$

For $\omega \in A_1 \cap A_2^c$ we get that

$$F^{\tilde{\theta}_t}(\mathcal{Y}_t(\omega))\mathbb{1}_{A_1\cap A_2}(\omega) + B_{t-1}^{\theta^*_{t-1},\tilde{\varepsilon}}(\omega)\mathbb{1}_{A_1^c\cup A_2^c}(\omega) = B_{t-1}^{\theta^*_{t-1},\tilde{\varepsilon}}(\omega)$$

and by definition of $A_1 \cap A_2^c$ and (5.32) that

$$B_t(\omega) - \tilde{\varepsilon} \le F^{\tilde{\theta}_t}(\mathcal{Y}_t(\omega)) < B_{t-1}^{\theta^*_{t-1},\tilde{\varepsilon}}(\omega) \le B_t(\omega) + \tilde{\varepsilon}.$$

For $\omega \in A_1 \cap A_2$ we have

$$F^{\tilde{\theta}_t}(\mathcal{Y}_t(\omega))\mathbb{1}_{A_1\cap A_2}(\omega) + B_{t-1}^{\theta^*_{t-1},\tilde{\varepsilon}}(\omega)\mathbb{1}_{A_1^c\cup A_2^c}(\omega) = F^{\tilde{\theta}_t}(\mathcal{Y}_t(\omega))$$

and by definition of A_1 that

$$\left|F^{\tilde{\theta}_t}(\mathcal{Y}_t(\omega)) - B_t(\omega)\right| \le \tilde{\varepsilon}.$$

Thus, we conclude that

$$\mathbf{P}\left(\left|\left(F^{\tilde{\theta}_t}(\mathcal{Y}_t)\mathbb{1}_A + B_{t-1}^{\theta^*_{t-1},\tilde{\varepsilon}}\mathbb{1}_{A^c}\right) - B_t\right| > \tilde{\varepsilon}\right) \le \mathbf{P}(A_1^c) < \varepsilon, \tag{5.33}$$

where we used that $A_1 = (A_1 \cap A_2) \cup (A_1 \cap A_2^c)$ and $\mathbf{P}(A_1) > 1 - \varepsilon$. Because

$$F^{\tilde{\theta}_t}(\mathcal{Y}_t)\mathbb{1}_A + B_{t-1}^{\theta^*_{t-1},\tilde{\varepsilon}}\mathbb{1}_{A^c} \in \tilde{\mathcal{B}}_t^{\theta^*_t,\tilde{\varepsilon}}$$

(5.33) implies that

$$\mathbf{P}\left(B_t^{\theta^*_t,\tilde{\varepsilon}} < B_t - \tilde{\varepsilon}\right) \le \mathbf{P}\left(F^{\tilde{\theta}_t}(\mathcal{Y}_t)\mathbb{1}_A + B_{t-1}^{\theta^*_{t-1},\tilde{\varepsilon}}\mathbb{1}_{A^c} < B_t - \tilde{\varepsilon}\right) < \varepsilon. \tag{5.34}$$

Since $\varepsilon \in (0,1)$ was arbitrary, we conclude that $B_t \le B_t^{\theta^*_t,\tilde{\varepsilon}} + \tilde{\varepsilon}$ by (5.34). Putting (5.31) and (5.34) together, we obtain that $|B_t^{\theta^*_t,\tilde{\varepsilon}} - B_t| \le \tilde{\varepsilon}$ for all $t = 0, \ldots, T$. □

Chapter 6

Numerical results

This chapter is based on Section 5 of [13]. We apply the method introduced in Chapter 4 and 5 on simulated data. Using Theorems 4.9 and 5.5 we can approximate the superhedging price at $t = 0$ in two steps. More precisely, we first approximate the α-quantile hedging for some $\alpha \in (0,1)$. In the second step we increase α via a parameter in the loss function, see (6.1), in order to approximate the superhedging price at $t = 0$. In particular, we obtain an approximate superhedging strategy for the complete interval which will be used in the next step for $t > 0$. Then, by Section 4.4 it is sufficient to know the process of consumption to obtain the superhedging price process for $t > 0$. By Theorem 5.7 we obtain an approximation of the process of consumption which is used to simulate the superhedging price process for $t > 0$.
The details of the algorithm and the implementation are presented in Section 6.1.1. In Section 6.1.2, we apply the method for $t = 0$ in a discrete time, finite trinomial model and a European Call option. In this case, we illustrate the impact of α on the approximated price, see Figure 6.1. Then, we consider a European Barrier Up and Out option and a European Call option in a discrete time Black-Scholes model in Section 6.1.3. For the European Call option in the discrete Black-Scholes model we also approximate the superhedging price process for $t > 0$ and compare it to the discretized δ-hedging strategy, see Section 6.2. Finally, we discuss our numerical results in Section 6.3.

6.1 Case $t = 0$

6.1.1 Algorithm and implementation

Let $N \in \mathbb{N}$ denote a fixed batch size. The learning process of the neural networks proceeds iteratively. At each step i of the iteration we generate i.i.d. samples $Y(\omega_0^{(i)}), \dots, Y(\omega_N^{(i)})$ of Y and consider the empirical loss function

$$
\begin{aligned}
L_\lambda^{(i)}(\theta) = \left|F^{\theta_u}\left(\mathcal{Y}_0\left(\omega_0^{(i)}\right)\right)\right|^2 + \frac{\lambda}{N}\sum_{j=1}^{N} l\Bigg(H\left(\omega_j^{(i)}\right) \\
- \left[F^{\theta_u}\left(\mathcal{Y}_0\left(\omega_j^{(i)}\right)\right) + \sum_{k=1}^{T} F^{\theta_{k,\xi}}\left(\mathcal{Y}_{k-1}\left(\omega_j^{(i)}\right)\right)\cdot\left(X_k\left(\omega_j^{(i)}\right) - X_{k-1}\left(\omega_j^{(i)}\right)\right)\right]\Bigg),
\end{aligned}
$$

with $\theta = (\theta_u, \theta_{1,\xi}, \dots, \theta_{T,\xi})$ and $l : \mathbb{R} \to [0, \infty)$ denoting the squared *rectifier* function, i.e.,

$$l(x) = (\max\{x, 0\})^2 .$$

Using the *Adam* optimizer, see [69], we calculate the gradient of $L_\lambda^{(i)}(\theta)$ and update the weights from $\theta^{(i)}$ to $\theta^{(i+1)}$ to find a local minimum. The Adam optimizer is an extension of stochastic gradient descent that is computationally efficient. After sufficiently many iterations i, the parameter $\theta^{(i)}$ should be sufficiently close to a local minimum of the loss function

$$L_\lambda(\theta) = \left|F^{\theta_u}(\mathcal{Y}_0)\right|^2 + \lambda \mathbb{E}\left[l\left(H - \left(F^{\theta_u}(\mathcal{Y}_0) + \sum_{k=1}^{T} F^{\theta_{k,\xi}}(\mathcal{Y}_{k-1}) \cdot (X_k - X_{k-1})\right)\right)\right]. \tag{6.1}$$

The first term of L_λ represents the approximated superhedging price. Since $\mathcal{Y}_0$ is constant also $F^{\theta_u}(\mathcal{Y}_0)$ is constant and only depends on the information available at $t = 0$. In particular, the approximated price is small if the first term of L_λ is small. The second term in (6.1) is equal 0 when the portfolio dominates the claim H $\mathbf{P}$ almost surely. In this sense, minimizing the second summand of (6.1) corresponds to maximizing the superhedging probability. If $F^{\theta_u}(\mathcal{Y}_0)$ grows, we can usually observe that the second term of (6.1) decreases because higher initial capital facilitates to dominate the claim H. For this reason, the weight λ offers the opportunity to balance between a small initial price of the portfolio and a high superhedging probability. We illustrate the impact of λ in Section 6.1.2 and particularly in Figure 6.1. At the minimum of the loss function L_λ, $F^{\theta_u}(\mathcal{Y}_0)$ is close to the minimal price required to superhedge the claim H with a certain probability, i.e., to the quantile hedging price for a certain $\alpha = \alpha(\lambda)$. Increasing the weight λ leads to a higher superhedging probability $\alpha(\lambda)$ and based on Theorem 5.5 we expect $F^{\theta_u}(\mathcal{Y}_0) \approx \inf \mathcal{U}_0$ for suitable λ.

The loss function in (6.1) can also be modified by other choices of l. For instance, we considered a scaled *sigmoid* function for l in (6.1) such that l can be seen as an approximation of the indicator function. However, we did not obtain stable results with this choice of l. If l was the indicator function, the second term of (6.1) would be equal to the probability of superhedging.

The algorithm is implemented in Python, using the Keras library with backend TensorFlow to build and train the neural networks. More specifically, we create a *Sequential* object and build a Long-Short-Term-Memory network (LSTM), see [54], with the following architecture: the network has two LSTM layers of size 30, which return sequences and one dense layer of size 1. Between the layers we use the *swish* activation function. The activation functions within the LSTM layers are set to default, i.e., the activation function between cells is *tanh* and the recurrent activation function is set to the *sigmoid* function. Further, the kernel and bias of the first LSTM layer are initialized according to the *truncated normal* distribution, i.e., the initial weights are drawn from a standard normal distribution but we discard and re-draw values, which are more than two standard deviations from the mean. With this architecture the neural network has 11191 trainable parameters. The model is then compiled with a customized loss function which is given by (6.1). We generate 1024000 samples, which we split in 70% for the training set and 30% for the test set. The batch size is set to 1024. For the training of the neural network, i.e., for minimizing the loss function, the Adam optimizer is used with a learning rate of 0.001 or 0.0001. We apply the procedure described above in two examples, which we present in the following.

Remark 6.1. *Note that in Keras a loss function has two input arguments, which are typically referred as y_{true} and y_{pred}. The first argument, y_{true} represents the true outcomes and y_{pred} are the outcomes predicted by the neural network. Commonly, the training of a neural network works as follows: the loss function is given by a distance of the true outcomes and the predicted outcomes. Then the loss function is minimized by some optimizer. However, in our case the true outcomes are not known.*

6.1.2 Trinomial model

We consider a discrete time financial market model given by an arbitrage-free trinomial model with $X_0 = 100$ and

$$X_t = X_0 \prod_{k=1}^{t} (1 + R_t), \quad t \in \{0, \dots, T\},$$

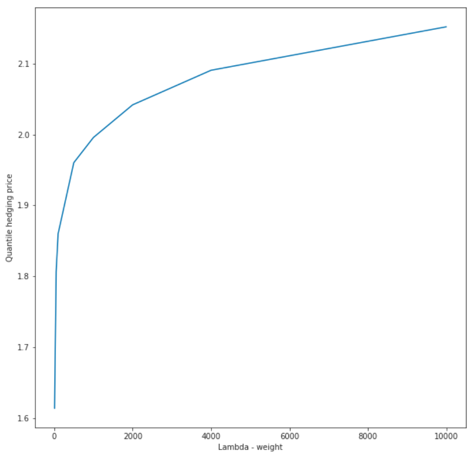
where R_t is $\mathcal{F}_t$-measurable for $t \in \{1, \dots, T\}$, and takes values in $\{d, m, u\}$ with equal probability, where $-1 < d < m < u$. Here, we set $d = -0.01$, $m = 0$, and $u = 0.01$ and $T = 29$. By simple combinatorial arguments this setting admits 3^{29} possible paths. The aim in this model is superhedging a European Call option $H = (X_T - K)^+$ with strike price $K = 100$. The theoretical price of H can be calculated with the results of [26] and with the given parameters the theoretical price is 2.17.
To illustrate the impact of λ in (6.1) in this, we train and evaluate the neural network for $\lambda \in \{10, 50, 100, 500, 1000, 2000, 4000, 10000\}$. Of particular interest are the superhedging probability $\alpha(\lambda)$ and the corresponding $\alpha(\lambda)$-quantile hedging price. For each λ the network is trained over 40 epochs.
We observe that $\alpha(\lambda)$ as well as the $\alpha(\lambda)$-quantile hedging price increases in λ, see Figure 6.1(a),(b). The $\alpha(\lambda)$-quantile hedging price increases also in $\alpha(\lambda)$, see Figure 6.1(c). This observation is consistent with Theorem 4.9. In Figure 6.1(d) we show the superhedging performance on the test set for all λ's, i.e., samples of

$$F^{\theta_u(\lambda)}(\mathcal{Y}_0) + \sum_{k=0}^{T} F^{\theta_{k,\xi}(\lambda)}(\mathcal{Y}_{k-1}) \cdot (X_k - X_{k-1}) - H, \tag{6.2}$$

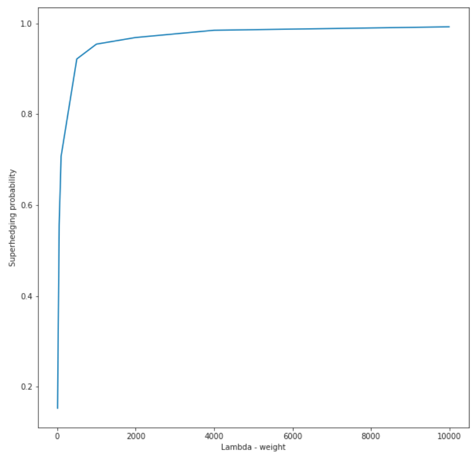
for each λ. Table 6.1 summarizes the values for λ, $\alpha(\lambda)$ and the $\alpha(\lambda)$-quantile hedging price. We particularly note that for $\lambda = 10000$ we obtain a numerical price of 2.15 and $\alpha(\lambda) = 99.24\%$.

6.1.3 Discretized Black Scholes model

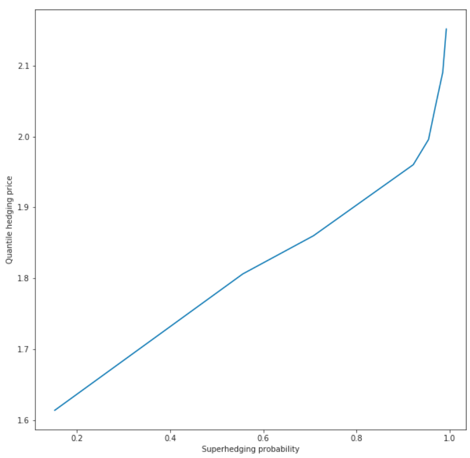
Here we consider a discrete time financial market, where the discounted asset price process X is given by a discretized Black-Scholes model. We set $X_0 = 100$, $\sigma = 0.3$ and $\mu = 0$. Let $\tilde{H}$ be a Barrier Up and Out Call option, i.e., $\tilde{H} = \prod_{t=0}^{T} \mathbb{1}_{\{X_t < U\}} (X_T - K)^+$ with strike $K = 100$ and upper bound $U = 105$ such that $K < U$ and $X_0 < U$. We assume that one



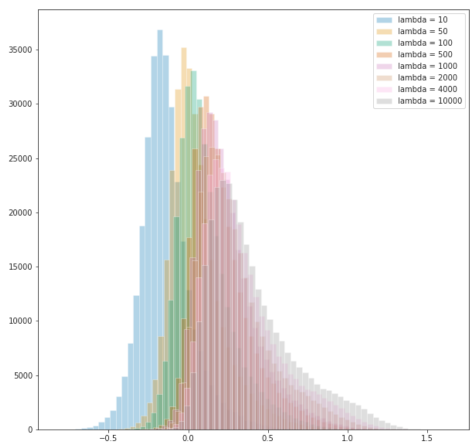
(a) $\alpha(\lambda)$-quantile hedging price depending on λ



(b) $\alpha(\lambda)$ depending on λ



(c) $\alpha(\lambda)$-quantile hedging price depending on $\alpha(\lambda)$



(d) Superhedging performance

Figure 6.1: Impact of λ on the quantile hedging price and on the superhedging probability.

λ	$\alpha(\lambda)$	$\alpha(\lambda)$-quantile hedging price
10	15.23%	1.61
50	55.61%	1.81
100	70.75%	1.86
500	92.16%	1.96
1000	95.42%	2.00
2000	96.88%	2.04
4000	98.48%	2.09
10000	99.24%	2.15

Table 6.1: Impact of λ on $\alpha(\lambda)$ and on the $\alpha(\lambda)$-quantile hedging price.

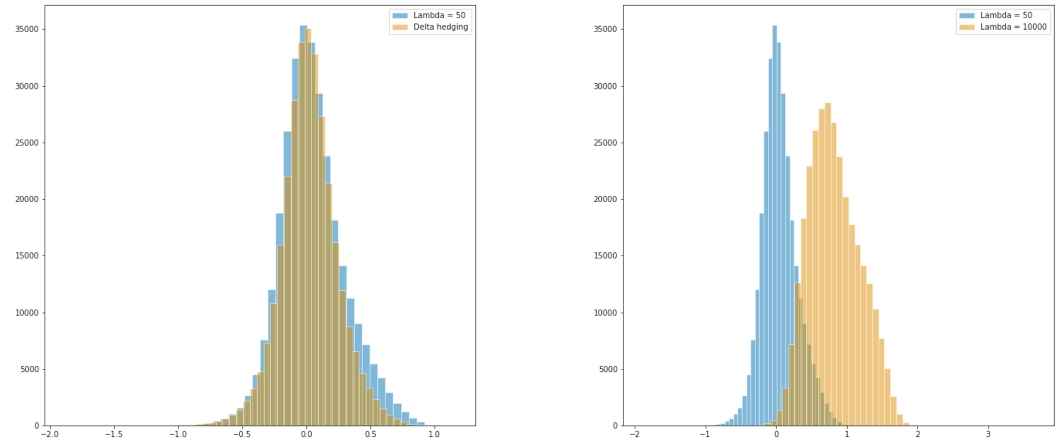
year consists of 250 trading days with daily balancing and consider a time horizon T of 30 trading days. Hence, the maturity of $\tilde{H}$ is $\tau = 30/250$.
In order to achieve a high probability of superhedging we set $\lambda = 10000000$. In fact, evaluating the portfolio predicted by the neural network, we obtain a superhedging probability of 100% on the test set with an approximate price of 3.73. By [24], the theoretical superhedging price $\pi^{\tilde{H}}$ is given by

$$\pi^{\tilde{H}} = X_0 \left(1 - \frac{K}{U}\right) \approx 4.76.$$

In the Black-Scholes model the asset price process at time $t > 0$ has unbounded support and thus the additional error, which arises from the discretization of the probability space, is non-negligible. Although the Barrier option artificially bounds the support of the model, the numerical price still significantly deviates from the theoretical price.
We now consider a European call option $H = (X_T - K)^+$ with strike $K = 100$. Here, we set $X_0 = 100$, $\sigma = 0.1$ and $\mu = 0$. By [24] the theoretical price of H for the discrete time version of the Black-Scholes model is equal to 100, i.e., in order to superhedge H an agent must buy one share of the underlying asset X at $t = 0$ and hold it until T. In contrast, in a standard Black-Scholes model in continuous time with parameters as above the theoretical price of H is 1.38. In the continuous time the δ-hedging strategy provides a perfect hedge of a European Call option. By following the discretized δ-hedging strategy we superhedge H with a probability of 53.69%.
In order to compare our method to the discretized δ-hedging strategy of the Black-Scholes model, we first consider $\lambda = 50$. For $\lambda = 50$ we obtain a superhedging probability of 54.43% and an approximate price of 1.41. In Figure 6.2(a) we compare the δ-hedging strategy with the approximated superhedging strategy obtained for $\lambda = 50$. Further, we set $\lambda = 10000$, which yields a superhedging probability of 99.79% with an approximated price of 2.18. Finally, we compare the results for $\lambda = 50$ and $\lambda = 10000$, respectively, in Figure 6.2(b).

6.2 Case $t > 0$

Following Sections 4.4 and 5.3 we approximate the process of consumption to obtain an approximation of the superhedging price process for $t > 0$. For this purpose, we implement



(a) δ-hedging strategy compared to approximate strategy for $\lambda = 50$

(b) Approximate strategy for $\lambda = 50$ and $\lambda = 10000$

Figure 6.2: Hedging losses for $\lambda = 50$, $\lambda = 10000$ and for the δ-hedging strategy.

the same iterative procedure as introduced in Section 6.1.1. We define $G^{(i)}$ as the difference of the approximated superhedging strategy obtained from Section 6.1 and some European option H, i.e.,

$$G_j^{(i)}(\theta^*) := \left[F^{\theta_u^*}\left(\mathcal{Y}_0\left(\omega_j^{(i)}\right)\right) + \sum_{k=1}^T F^{\theta_{k,\xi}^*}\left(\mathcal{Y}_{k-1}\left(\omega_j^{(i)}\right)\right)\cdot\left(X_k\left(\omega_j^{(i)}\right) - X_{k-1}\left(\omega_j^{(i)}\right)\right) - H\left(\omega_j^{(i)}\right)\right].$$

Then, the empirical loss function is given by

$$\tilde{L}_{t,\beta}^{(i)}(\theta_t) = \frac{1}{N}\sum_{j=1}^{N} -\left|B_t^{\theta_t}\left(\omega_j^{(i)}\right)\right|^2 + \beta \max\left\{\left(B_t^{\theta_t}\left(\omega_j^{(i)}\right) - G_j^{(i)}(\theta^*)\right), 0\right\},$$

where $B_t^{\theta_t}$ is given by

$$B_t^{\theta_t}\left(\omega_j^{(i)}\right) := \max\left\{F^{\theta_t}\left(\mathcal{Y}_t\left(\omega_j^{(i)}\right)\right), B_{t-1}^{\theta_{t-1}}\left(\omega_j^{(i)}\right)\right\}.$$

The two terms of $\tilde{L}_{t,\beta}$ guarantee that F^{θ_t} is as big as possible but less or equal than . By the weight β in $\tilde{L}_{t,\beta}$ it is possible to balance if it is more important that F^{θ_t} is big or if F^{θ_t} does not exceed $G(\theta^*)$.
For each $t > 0$ the algorithm and implementation is similar to Section 6.1.1 using the loss function $\tilde{L}_{t,\beta}$ and the following architecture: the neural network consists of two LSTM layers of size 30 and 20 respectively, which return sequences, one LSTM layer of size 20 providing one single value and one dense layer of size 1. The remaining parameters are chosen as in Section 6.1.1.
For our approach we consider a discrete time financial market given by a discretized Black-Scholes model for the asset price X as in Section 6.1.3. But we only consider a time horizon of 10 trading days and set $X_0 = 100$, $\sigma = 0.1$ and $\mu = 0$.

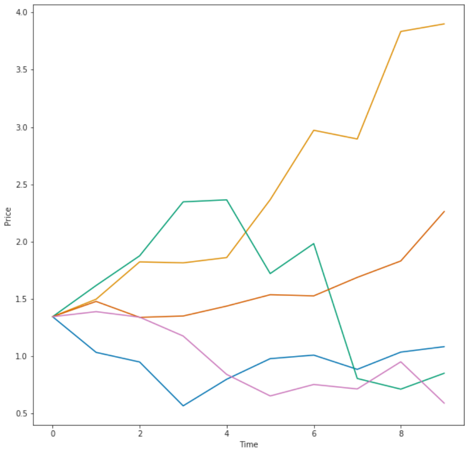
In the first step, we compute an approximate superhedging price and strategy as in Section 6.1.3. For $\lambda = 1024$ we obtain a an approximated price pf 1.35 and a superhedging probability of 98.87% for $t = 0$. For $t \geq 1$ we set $\beta = 500$. For each $t \geq 1$ we use (4.37) to obtain an approximated superhedging price at $t \geq 1$ and the corresponding strategy. In this setting we obtain a superhedging probability of 98.78%. Figure 6.3(a) shows trajectories of the approximated superhedging price process generated by this method. We generate a price process by using the discretized δ-hedging strategy of the Black-Scholes model and plot the corresponding trajectories in Figure 6.3(b). We plot the difference of the approximated superhedging price processes and the corresponding price process obtained by the δ-hedging strategy in Figure 6.3(c).

6.3 Discussion

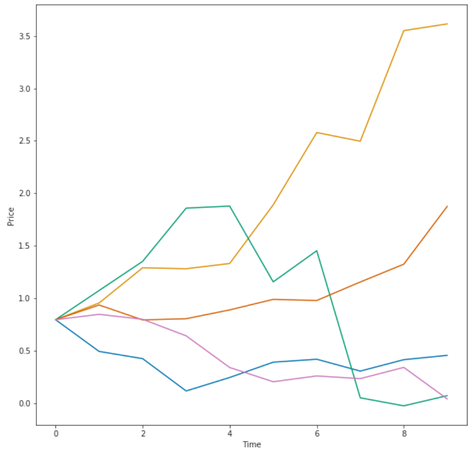
In finite market models as in Section 6.1.2, we obtain an approximation of α-quantile hedging and approximated superhedging prices with small approximation error by our methodology. In this case, the probability space is well represented by the generated data. It is also worth noting, that we get consistent results on the training and test set, respectively, in the sense that the predicted superhedging price as well as the superhedging probability are coincide.

In contrast, our numerical results in Section 6.1.3 indicate that the additional error coming from the discretization of the probability space is non-negligible in models in which the asset price process has unbounded support. Still, our results of the α-quantile hedging price for the generated data are consistent on the training and test set. In particular, for sufficiently large λ we obtain a superhedging probability of 100% for the Barrier Up and Out Call option in Section 6.1.3 on the test set, i.e., on data which is new to the neural network.

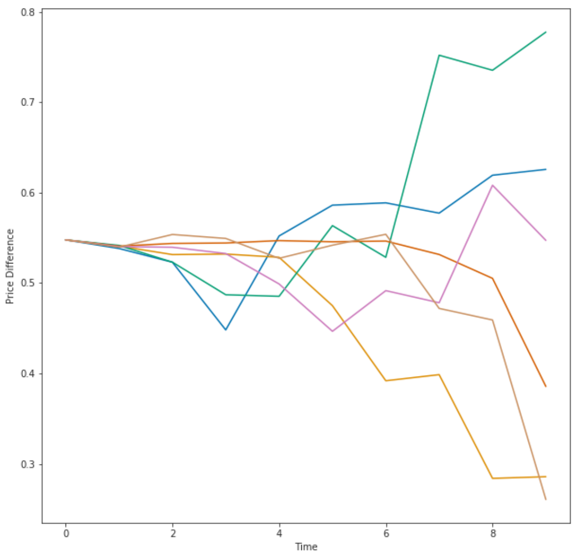
If the data contained in the training set is representative for the complete relevant data or possible price paths, we obtain consistent results with our methodology. In particular, a further possible application of our methodology is given by superhedging in a (model-free) setting on prediction sets, see [6], [7], [56]. Prediction sets offer the opportunity to include beliefs on future price developments or to choose relevant price paths.



(a) Superhedging price process



(b) δ-hedging price process



(c) Difference of the price processes

Figure 6.3: Superhedging price process compared to the δ-hedging price process.

Appendix A

Superhedging

We provide some important results on superhedging from Chapter 7 of [40] and summarize the essential implications in Corollary A.5.
The upper Snell envelope for a discounted European claim H is defined by

$$U_t^{\uparrow}(H) = U_t^{\uparrow} := \operatorname*{ess\,sup}_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H \mid \mathcal{F}_t], \quad \text{for } t = 0, 1, \ldots, T. \tag{A.1}$$

Corollary A.1 (Corollary 7.3, [40])**.** *Let H be a discounted European claim such that*

$$\sup_{\boldsymbol{P}^* \in \mathcal{P}} \mathbb{E}^+ [H] < \infty.$$

Then $(U_t^{\uparrow})_{t=0,1,\ldots,T}$ defined in (A.1) *is the smallest $\mathcal{P}$-supermartingale whose terminal value dominates H.*

Theorem A.2 (Theorem 7.5, [40])**.** *For an adapted, non-negative process U, the following two statements are equivalent:*

i) U is a $\mathcal{P}$-supermartingale.

ii) There exists an adapted increasing process B with $B_0 = 0$ and a d-dimensional predictable process ξ such that

$$U_t = U_0 + \sum_{k=1}^{t} \xi_k \cdot (X_k - X_{k-1}) - B_t \quad \text{for all } t = 0, 1, \ldots, T.$$

Corollary A.3 (Corollary 7.15, [40])**.** *For any discounted European claim H such that*

$$\sup_{\boldsymbol{P}^* \in \mathcal{P}} \mathbb{E}^* [H] < \infty,$$

there exists a d-dimensional predictable process ξ such that

$$\operatorname*{ess\,sup}_{\boldsymbol{P}^* \in \mathcal{P}} \mathbb{E}^* [H \mid \mathcal{F}_t] + \sum_{k=t+1}^{T} \xi_k \cdot (X_k - X_{k-1}) \geq H.$$

Set

$$\mathcal{U}_t := \left\{ \tilde{U}_t \in L^0(\mathcal{F}_t, \mathbf{P}) : \exists \tilde{\xi} \text{ pred. s.t. } \tilde{U}_t + \sum_{k=t+1}^{T} \tilde{\xi}_k \cdot (X_k - X_{k-1}) \geq H \right\}. \tag{A.2}$$

Then, $\mathcal{U}_t$ describes the set of initial capital required at time $t = 0, 1, \ldots, T$ to superhedge the discounted European claim H.

Corollary A.4 (Corollary 7.18, [40]). *Suppose H is a discounted European claim with*

$$\sup_{\boldsymbol{P}^* \in \mathcal{P}} \mathbb{E}^*[H] < \infty.$$

Then

$$\operatorname*{ess\,sup}_{\boldsymbol{P}^* \in \mathcal{P}} \mathbb{E}^*[H \mid \mathcal{F}_t] = \operatorname{ess\,inf} \mathcal{U}_t(H).$$

For the convenience of the reader we summarize these results here.

Corollary A.5. *Suppose H is a discounted European claim with*

$$\sup_{\boldsymbol{P}^* \in \mathcal{P}} \mathbb{E}^*[H] < \infty.$$

The process $(U_t^{\uparrow})_{t=0,1,\dots,T}$ defined in (A.1) is the smallest $\mathcal{P}$-supermartingale whose terminal value dominates H. Furthermore, there exists an adapted increasing process $B = (B_t)_{t=0,\dots,T}$ with $B_0 = 0$ and a d-dimensional predictable process $\xi = (\xi_t)_{t=1,\dots,T}$ such that

$$\operatorname*{ess\,sup}_{\boldsymbol{P}^* \in \mathcal{P}} \mathbb{E}^*[H \mid \mathcal{F}_t] = \sup_{\boldsymbol{P}^* \in \mathcal{P}} \mathbb{E}^*[H] + \sum_{k=1}^{t} \xi_k \cdot (X_k - X_{k-1}) - B_t \quad \textit{for all } t = 0, \dots, T. \tag{A.3}$$

Moreover, $\operatorname{ess\,sup}_{\boldsymbol{P}^* \in \mathcal{P}} \mathbb{E}^*[H \mid \mathcal{F}_t] \in \mathcal{U}_t$, $\operatorname{ess\,sup}_{\boldsymbol{P}^* \in \mathcal{P}} \mathbb{E}^*[H \mid \mathcal{F}_t] = \operatorname{ess\,inf} \mathcal{U}_t$ *and*

$$\operatorname*{ess\,sup}_{\boldsymbol{P}^* \in \mathcal{P}} \mathbb{E}^*[H \mid \mathcal{F}_t] + \sum_{k=t+1}^{T} \xi_k \cdot (X_k - X_{k-1}) \geq H, \quad \textit{for all } t = 0, \dots, T. \tag{A.4}$$

Proof. The process $U^{\uparrow}$ is the smallest $\mathcal{P}$-supermartingale dominating the terminal value of H by Corollary A.1. The decomposition in (A.3) follows by Theorem A.2. Then, by Corollary A.3 and by the definition of $\mathcal{U}_t$ in (A.2), we get that

$$\operatorname*{ess\,sup}_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H \mid \mathcal{F}_t] \in \mathcal{U}_t.$$

Further, Corollary A.4 guarantees that $\operatorname{ess\,sup}_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H \mid \mathcal{F}_t] = \operatorname{ess\,inf} \mathcal{U}_t$ and equation (A.4) follows by Corollary A.3. □

We may call the process B (A.3) process of consumption, see also [71]. Equations (A.3) and (A.4) yield

$$\sup_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H] + \sum_{k=1}^{T} \xi_k \cdot (X_k - X_{k-1}) - H \geq B_t \geq B_{t-1} \geq 0 \quad \text{for all } t = 1, \dots, T. \tag{A.5}$$

Corollary A.5 guarantees that $U_t^{\uparrow}$ is the minimal amount needed at time t to start a superhedging strategy and thus there exists a predictable process ξ such that

$$\operatorname*{ess\,sup}_{\mathbf{P}^* \in \mathcal{P}} \mathbb{E}^*[H \mid \mathcal{F}_t] + \sum_{k=t+1}^{T} \xi_k \cdot (X_k - X_{k-1}) \geq H.$$

Further, $U_0^{\uparrow}$ is called the superhedging price at time $t = 0$ of H and coincides with the upper bound of the set of arbitrage-free prices.

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