The application of $\star$-products to noncommutative geometry and gauge theory

Andreas Sykora

The application of $\star$-products to noncommutative geometry and gauge theory

Andreas Sykora

Dissertation der Fakultät für Physik
der Ludwig Maximilian Universität München[1]

Juni 2004

[1]Erstgutachter: Prof. Dr. Julius Wess
Zweitgutachter: Prof. Dr. Ivo Sachs
Tag der mündlichen Prüfung: 2.11.2004

Abstract

Due to the singularities arising in quantum field theory and the difficulties in quantizing gravity it is often believed that the description of spacetime by a smooth manifold should be given up at small length scales or high energies. In this work we will replace spacetime by noncommutative structures arising within the framework of deformation quantization. The ordinary product between functions will be replaced by a $\star$-product, an associative product for the space of functions on a manifold.

We develop a formalism to realize algebras defined by relations on function spaces. For this porpose we construct the Weyl-ordered $\star$-product and present a method how to calculate $\star$-products with the help of commuting vector fields.

Concepts developed in noncommutative differential geometry will be applied to this type of algebras and we construct actions for noncommutative field theories. In the classical limit these noncommutative theories become field theories on manifolds with nonvanishing curvature. It becomes clear that the application of $\star$-products is very fruitful to the solution of noncommutative problems. In the semiclassical limit every $\star$-product is related to a Poisson structure, every derivation of the algebra to a vector field on the manifold. Since in this limit many problems are reduced to a couple of differential equations the $\star$-product representation makes it possible to construct noncommutative spaces corresponding to interesting Riemannian manifolds.

Derivations of $\star$-products makes it further possible to extend noncommutative gauge theory in the Seiberg-Witten formalism with covariant derivatives. The resulting noncommutative gauge fields may be interpreted as one forms of a generalization of the exterior algebra of a manifold. For the Formality $\star$-product we prove the existence of the abelian Seiberg-Witten map for derivations of these $\star$-products. We calculate the enveloping algebra valued non abelian Seiberg-Witten map pertubatively up to second order for the Weyl-ordered $\star$-product. A general method to construct actions invariant under noncommutative gauge transformations is developed. In the commutative limit these theories are becoming gauge theories on curved backgrounds.

We study observables of noncommutative gauge theories and extend the concept of so called open Wilson lines to general noncommutative gauge theories. With help of this construction we give a formula for the inverse abelian Seiberg-Witten map on noncommutative spaces with nondegenerate $\star$-products.

Contents

Acknowledgements

I am indebted to Julius Wess for letting me join his group in Munich and drawing my attention to noncommutative gauge theory and $\star$-products. I thank Claudia Jambor and Wolfgang Behr for their fruitful collaboration on parts of the material included in this work. I thank Fabian Bachmaier, Christian Blohmann, Marija Dimitrijevic, Larisa Jonke, Branislav Jurco, Florian Koch, John Madore, Frank Meyer, Dzo Mikulovic, Lutz Möller, Harold Steinacker, Alexander Schmidt, Effrosini Tsouchnika, Hartmut Wachter and Michael Wohlgenannt for many useful and inspiring discussions.

Chapter 1

Introduction

All experiments in physics support the assumption that spacetime should be described by a differential manifold and all successful theories may be formulated as field theories on such manifolds. But in quantum field theories there are some intrinsic difficulties at high energy or short distances that can not be resolved. No hints are given by experiment where and how these difficulties should be solved. But there are other fomulations of successful theories like the algebraic approch to quantum mechanics that leave the setting of differential manifolds.

In the early days of quantum field theory it was already suggested by Heisenberg that spacetime might be modified at very short distances by algebraic properties that could lead to uncertainty relations for the space coordinates. The first one to write an entire article about this subject was Snyder [1]. The idea behind spacetime noncommutativity is mainly inspired by quantum mechanics. Quantum phase space is defined by replacing canonical variables q^i, p_j by hermitian operators which obey the Heisenberg commutation relations $[\hat{q}^i, \hat{p}_j] = i\hbar\delta^i_j$. Now the space becomes smeared out and the notion of a point is replaced by a Planck cell. In the limit $\hbar \to 0$ one can recover the ordinary phase space. In its simplest form spacetime noncommutativity can be described in the same way by replacing the commutative coordinate functions x^i by operators $\hat{x}^i$ of a general algebra obeying the relations

$$[\hat{x}^i, \hat{x}^j] = \hat{c}^{ij}.$$

The righthand side of this equations should tend to zero in a certain limit and one recovers in this way the classical commuting space. Although this idea seemed quite promising the progress was very slow due to the success of renormalization theory on the one hand and the mathematical complexity of noncommutative structures on the other hand. It took a long time until noncommutative geometry was mathematically defined and physical models were formulated [2, 3, 4, 5, 6].

Perhaps one reason for the slow progress is that postulating an uncertainty relation between position measurements will lead to a nonlocal theory, with all of the resulting difficulties. A secondary reason is that noncommutativity of the spacetime coordinates generally conflicts with Lorentz invariance. Although it is not implausible that a theory defined using such coordinates could be effectively local on length scales longer than that of the uncertainty, it is harder to believe that the breaking of Lorentz invariance would be unobservable at these scales.

One big hope associated with the application of noncommutative geometry in physics is a better description of quantized gravity. Quantum gravity has an uncertainty principle which prevents one from measuring positions to better accuracies than the Planck length: the momentum and energy required to make such a measurement will itself modify the geometry at these scales [7]. At least it should be possible to construct effective actions where traces of this unknown theory remain. If one believes that quantum gravity is in a sense a quantum field theory, then its observables are operators on a Hilbert space and therefore elements of an algebra. Some properties of this algebra should be reflected in the noncommutative geometry the effective actions are constructed on. As in this case the noncommutativity should be induced by background gravitational fields, the classical limit of the effective actions should reduce to actions on curved spacetimes [8, 9].

A related motivation is that quantum gravity might not be local in the conventional sense. Nonlocality brings with it deep conceptual and practical problems which have not been well understood, and one might want to understand them in the simplest examples first, before proceeding to a more realistic theory of quantum gravity. Further there is an interesting similiarity in the gauge structure of general relativity and noncommutative gauge theory. In the later gauge transformations can be interpreted as a special subgroup of the group of diffeomorphisms. Again with the growing understanding of noncommutative theories one perhaps improves the knowledge about diffeomorphism invariant theories like general relativity.

There are other reasons for introducing noncommutativity into physics. One of the simplest is that it might improve the renormalizability properties of a theory at short distances or even render it finite. However it is known today that certain models develop new devergencies absent in commutative theories [10, 11].

At the moment most of the applications of noncommutativity to physics are done with noncommutative field theory [12, 13]. As one thinks of these models as analogs to classical physics there are also attempts to quantize these theories [14, 15, 16]. A new approach is to treat noncommutative geometries as matrix models and take advantage of the noncommutativity to quantize them [17, 18] since they have finite dimensional representations. Noncommutative field theory is also known to appear naturally in condensed matter physics. One example is the theory of electrons in a magnetic field projected to the lowest Landau level, which is naturally thought of as a noncommutative field theory. Thus these ideas are relevant to the theory of the quantum Hall effect, and indeed, noncommutative geometry has been found very useful in this context [19].

Symmtries have always played a very important role in physical models. But noncommutative spaces mostly are not compatible with the symmetry groups of their commutative counterparts. One way to circumvent these problems are quantum groups. One does not only deform the space but also the symmetry group acting on it. Beginning with the noncommutative plane a large number of deformed spaces with deformed symmetries have been constructed. Among others there are for example the q-deformed Lie algebra of rotations [20] and q-deformed Euclidean space [21], the q-deformed Lorentz algebra [22] and q-deformed Minkowski space [23], the q-deformed Poincare algebra [24], κ-deformed Poincare invariant space [25], to name only a few.

Noncommutative geometry may be useful to describe effective field theories derived from the low energy limit of loop quantum gravity. Since here geometric objects are replaced by operators on a Hilbert space [26], it would not be very unexpected if noncommutative structures appeared in the continuum limit of this theory. However the relation between loop quantum gravity and noncommutative geometry has not been explored very well. Nevertheless there are hints that an effective theory should be a noncomutative one. For example there exists a nonperturbative quantization of gravity with an isolated horizon as inner boundary within the formalism of loop quantum gravity. The quantum geometry of the horizon looks like a noncommutative torus [27].

String theory made its first contact with noncommutative geometry with a conjecture called M-theory. It was proposed that all known string theories are the low energy limit of this theory. Further it was conjectured that this M-theory may be formulated in the framework of matrix quantum mechanics leading to the name M(atrix)-theory [28]. It was found in [29] that noncommutative geometry arises very naturally in M(atrix)-theory.

Noncommutative geometry entered string theory a second time with the descriptions of open strings in a background B-field [30, 31]. The D-brane is then a noncommutative space whose fluctuations are governed by a noncommutative version of Yang-Mills theory [32] and noncommutativity is induced by a so called $\star$-product. On a curved brane the B-field becomes position dependent [33]. In the case of a constant B-field it has been shown quite soon that there is an equivalent description in terms of ordinary gauge theory. The two pictures are releated by a choice of regularization [34]. Therefore there must exist a field redefinition mapping the one picture to the other, the Seiberg-Witten map [32].

Most of the noncommutativity in this work will be formulated with the help of $\star$-products, i. e. with associative products defined on function spaces. Throughout this work we will formulate them with the help of differential operators

$$f \star g = fg + \frac{i\hbar}{2}\theta^{ij}\partial_i f\, \partial_j g + \cdots$$

and assume that they may be expanded in some parameter of noncommutativity. $\star$-products first emerged from quantum mechanics. Due to Weyl's quantization procedure [35] one was able to pull back noncommutativity to the classical phase space and the first $\star$-product was formulated [36], an associative product between functions on phase space. In this formulation the classical limit of quantum mechanics is very intuitive, the $\star$-product depends on $\hbar$ and for this parameter tending to zero it becomes the ordinary product between functions. The Poisson bracket can be obtained by looking at the first order deviation in $\hbar$. With this in mind deformation quantization [37] was formulated. One has to look for defomations of algebras of functions of Poisson manifolds and realize quantum mechanics on this manifolds in this way. A more abstract picture of $\star$-products was developed being now an associative product on the space of functions on a manifold.

The formulation of gauge theories in this work will be done with the mentioned Seiberg-Witten map formalism emerging from string theory. After its discovery the Seiberg-Witten map has been extensivly studied and applied to noncommutative field theory. An interesting approch is set within the Kontsevich $\star$-product formalism [38]. Here the Seiberg-Witten map is found to be a part of the Formality map [39, 40, 41, 42]. In particular these studies show that the Seiberg-Witten map is an integral feature of any noncommutative geometry obtained through deformation quantization of a Poisson manifold. Additionally the Seiberg-Witten map was extended to nonabelian gauge groups. The noncommutative gauge transformations are not longer Lie-algebra valued and have to be defined on the enveloping algebra [43].

On noncommutative $\mathbb{R}^N_\theta$ which is characterized by constant parameters θ^{ij} the Seiberg-Witten map can be constructed using various techniques. The

Seiberg-Witten equations lead to a consistency condition which may be solved order by order [44]. Further it can be solved with a cohomological approach within the BRST formalism [45]. There exist few Seiberg-Witten maps on other noncommutative spaces. On the fuzzy sphere a Seiberg-Witten map was constructed up to second order for a $\star$-product that does not truncate the space of functions and for the finite dimensional representations S^2_N [46]. On κ-Minkowski spacetime it has been calculated in [47, 48]. There are extensions of the constant case Seiberg-Witten map to supersymmetric gauge theories [49, 50]. Another remarkable aspect of Seiberg-Witten gauge theory is that it is sensitive to the representation of the gauge group. Due to this grand unified theories do not have unique noncommutative analogs [51].

The first attempt to quantize noncommutative field theories in the $\star$-product representation was done in [52]. This was done similar to the perturbative way interacting commutative field theories are treated and Seiberg-Witten gauge theories are mostly quantized using this method. However it is not quite clear how the quantization of Seiberg-Witten gauge theory can be done in a consistent way since the solution of the Seiberg-Witten equations is not unique and other solutions are related by nonlocal field redefinitions [53]. Nevertheless this was used in [54] to show that noncommutative abelian gauge theory on the $\mathbb{R}^N_\theta$ in the $\star$-product representation is renormalizable. The same was done for $U(n)$ gauge groups up to one loop level in [55].

After this general introduction we begin to deal with $\star$-products and the representation of algebras by them. We begin with the definition and first properties like the semiclassical limit and the equivalence of $\star$-products with respect to linear transformations on function space. The semiclassical limit will be crucial to all applications throughout this work. In this limit the $\star$-products are in one-to-one correspondence to Poisson structures up to the mentioned linear transformations on function space. After that we start with the representation of algebras defined by relations on function spaces and calculate the Weyl-ordered $\star$-product up to second order. The Weyl-ordered $\star$-product will be very important for us to give explict formulas in noncommutative gauge theory. In the end we give closed formulas for $\star$-products for several algebras, mainly quantum spaces like $M(so_q(3))$, $M(so_q(1,3))$ and $M(so_q(4))$. For this we generalize the Moyal-Weyl product with the help of commuting vector fields and give a method how to calculate this type of $\star$-representation for relation-defined algebras. It will become clear that a big amount of the information we have about the algebra is already contained in the Poisson structure of the $\star$-product.

The purpose of chapter 3 is to relate noncommutative differential geometry and $\star$-product algebras. After a short introduction to aspects of noncommutative differential geometry we need later, we apply the $\star$-product formalism to the commuting frame formalism developed in [6]. We will see that in the semiclassical limit, an algebra with nonconstant commutator and therefore nonconstant Poisson structure will in general yield a curved background. With the application of the commuting frame formalism we are now able to construct noncomutative spaces with interesting classical limit. The general considerations yield a system of partial differential equations, which we can try to solve for certain interesting geometries. In two dimensions we are quite successful and we are able to construct algebras for all spaces of constant curvature. In four dimensions this is not the case, since the mentioned system of partial differential equations is getting more and more overdetermined in higher dimensions. At the end of the chapter we give another very interesting application for $\star$-products in noncommutative geometry. We calculate rotational invariant Poisson structures in four dimensions and quantize them with the help of $\star$-products. On the resulting algebra we construct a first order differential calculus having a frame for the Schwarzschild metric as classical limit.

We will see in chapter 3 that derivations are very useful for formulating noncommutative geometry on quantized Poisson manifolds. In chapter 4 therefore we make general considerations about derivations of $\star$-products. We again come to the conclusion that the important informations are included in the semiclassical limit of the $\star$-product. Vector fields in a sense compatible with the Poisson structure of the $\star$-product and derivations are in one-to-one correspondence. We apply our results to the Formality $\star$-product and the Weyl-ordered $\star$-product from the second chapter. An alternative definition of noncommutative forms will be later useful in combination with Seiberg-Witten gauge theory. To make contact with physical application we introduce traces on $\star$-products at the end of the chapter. With them we start to construct actions on noncommutative spaces having field theories on curved backgrounds as classical limit. As an example we give an noncommutative action being the deformation of ϕ^4-theory on a space of constant curvature.

Chapter 5 is dedicated to the application of $\star$-products to noncommutative gauge theory. We start with a introduction to noncommutative gauge theory and the special case of Seiberg-Witten gauge theory, a fomulation of noncommutative gauge theory only possible in the $\star$-product representation. Only in Seiberg-Witten gauge theory at the moment it is possible to formulate noncommutative analogs to general nonabelian gauge theories. Our main purpose in the following is the extension of the Seiberg-Witten map to derivations of $\star$-products. Then we give a closed formula for the abelian

Seiberg-Witten map for the Fomality $\star$-product. The Seiberg-Witten map for the Weyl-ordered $\star$-product is calculated up to second order. We relate the resulting objects to the noncommutative forms introduced in the chapter 4. Now we are able to construct actions on noncommutative spaces invariant under noncommutative gauge transformations. The actions have as classical limit a gauge theory on a curved background. We give an example of a noncommutative version of electrodynamics on a background with constant curvature. At the end we deal with observables of noncommutative gauge theories. Most useful in this context are the so called open Wilson lines introduced in the case of constant commutator. We will generalize them to general $\star$-product algebras. With these observables we are able to give a formula for the inverse abelian Seiberg-Witten map on symplectic manifolds.

Chapter 2

$\star$-products

The first $\star$-product emerged from Weyl's quantization procedure [35]. Assume that $f(q_i, p_j)$ is a function on a classical phase space and associate the following operator with it

$$\hat{f} = \Omega(f) = \int d^n\xi\, d^n\eta\, \tilde{f}(\xi,\eta)\, e^{\frac{i}{\hbar}(\hat{q}\cdot\xi + \hat{p}\cdot\eta)}. \tag{2.1}$$

Here $\tilde{f}(\xi,\eta)$ is the Fourier transform of f, the operators $\hat{q}_i$ and $\hat{p}_j$ should fulfill the canonical commutation relations $[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$. In this case it is possible to give an inverse operation

$$\Omega^{-1}(\hat{f}) = \int d^n\xi\, d^n\eta\, Tr\left(\hat{f}\, e^{-\frac{i}{\hbar}(\hat{q}\cdot\xi + \hat{p}\cdot\eta)}\right)\, e^{i(q\cdot\xi + p\cdot\eta)}.$$

Here Tr is the trace on the Fock space representation of the operator algebra. Now one can pull back the product between two operators to a product between two functions on phase space

$$f \star_M g = \Omega^{-1}(\Omega(f)\Omega(g)),$$

which yields the Moyal product on classical phase space. If $P^{IJ}\partial_I \wedge \partial_J$ ($\partial_I = (\partial_{q_i}, \partial_{p_j})$) is the Poisson structure of the classical phase space, i. e. $\{f,g\} = P^{IJ}\partial_I f \partial_J g$ and $\{q_i, p_j\} = \delta_{ij}$, it is possible to write down an explicit formula

$$f \star_M g = \sum_{n=0}^{\infty} \frac{(i\hbar)^n}{2^n n!} P^{I_1J_1} \cdots P^{I_nJ_n} \partial_{I_1} \cdots \partial_{I_n} f\, \partial_{J_1} \cdots \partial_{J_n} g \tag{2.2}$$

for the Moyal product. A good introduction to this is [36] and references therein. The generalization of the above construction is called deformation quantization [37, 56], where one tries to quantize phase spaces by finding appropriate $\star$-products for functions on phase space.

2.1 Definition and first properties

The Moyal product (2.2) is a special case of a $\star$-product. To define more general $\star$-products let M be an arbitrary (sufficiently smooth) finite dimensional manifold. A $\star$-product on M is an associative, $\mathbb{C}$-linear product on the space of functions (with values in $\mathbb{C}$) on M given by

$$f \star g = fg + \frac{h}{2} B_1(f,g) + (\frac{h}{2})^2 B_2(f,g) + \cdots$$

where f an g are two such functions and the B_i are bidifferential operators on M. The parameter h is called deformation parameter. There is a natural gauge group acting on $\star$-products consisting of $\mathbb{C}$-linear transformations on the space of functions

$$f \to f + hD_1(f) + h^2 D_2(f) + \cdots$$

where the D_i are differential operators. They may be interpreted as a generalization of coordinate transformations. If D is such a linear transformation it maps a $\star$-product to a new $\star$-product

$$f \star' g = D^{-1}(D(f) \star D(g)). \tag{2.3}$$

If one expands this equation in h one sees that a linear transformation D acting on $\star$ only affects the symmetric part of B_1

$$B_1'(f,g) = B_1(f,g) + D_1(fg) - fD_1(g) - D_1(f)g$$

and one can show that the symmetric part of B_1 may be canceled by a linear transformation. For this we may assume B_1 to be antisymmetric. Since $\star$ is associative, the commutator

$$[f \overset{\star}{,} g] = f \star g - g \star f = hB_1(f,g) + \cdots$$

has to be a derivation

$$[f \star g \overset{\star}{,} h] = f \star [g \overset{\star}{,} h] + [f \overset{\star}{,} h] \star g.$$

Up to first order this means that the antisymmetric part of B_1 is a derivation with respect to both functions f and g. Additionally the Jakobi-identity is fulfilled for the commutator

$$[f \overset{\star}{,} [g \overset{\star}{,} h]] + [h \overset{\star}{,} [f \overset{\star}{,} g]] + [g \overset{\star}{,} [h \overset{\star}{,} f]] = 0.$$

Up to second order this implies that B_1 is a Poisson structure

$$\{f,\{g,h\}\}+\{h,\{f,g\}\}+\{g,\{h,f\}\}=0$$

where $\{f,g\}=B_1(f,g)$. Therefore after a certain linear transformation we can always write on a local patch of the manifold (here locally $\{f,g\}=\Pi^{ij}\partial_i f\partial_j g$)

$$f\star g=fg+\frac{ih}{2}\Pi^{ij}\partial_i f\,\partial_j g+\cdots$$

with

$$\Pi^{il}\partial_l\Pi^{jk}+\Pi^{kl}\partial_l\Pi^{ij}+\Pi^{jl}\partial_l\Pi^{ki}=0 \tag{2.4}$$

We have seen that $\star$-products up to second order are classified by Poisson structures on the manifold. On the other hand, if there is a manifold with a given Poisson structure $\{,\}$ on it, it is possible to construct $\star$-products with

$$f\star g=fg+\frac{ih}{2}\{f,g\}+\cdots.$$

This was first done for symplectic manifolds (manifolds with invertible Π^{ij}) in [57, 58]. In [38] a general construction for arbitrary Poisson manifolds has been given (see also [59]). It makes use of the so called formality map that we will use later for constructing noncommutative gauge theories, too.

2.2 Algebras and $\star$-products

Suppose we are taking $\mathbb{R}^N$ as the manifold and parametrize it by N coordinates x^i. Then $\theta^{ij}=const.\,(i,j=1\cdots N)$ clearly fulfills the Poisson condition (2.4). With a view to the original Moyal product (2.2) we can write down a $\star$-product for this Poisson structure

$$f\star g=\sum_{n=0}^{\infty}\frac{(ih)^n}{2^n n!}\theta^{i_1j_1}\cdots\theta^{i_nj_n}\partial_{i_1}\cdots\partial_{i_n}f\,\partial_{j_1}\cdots\partial_{j_n}g \tag{2.5}$$

where f and g are functions on $\mathbb{R}^N$. We will again call it Moyal-Weyl $\star$-product. A proof that it is really associative will be given in (2.3.1). Since

$$[x^i \stackrel{\star}{,} x^j]=ih\theta^{ij},$$

the space of functions on $\mathbb{R}^N$ together with the $\star$-product forms a realization of the algebra

$$\mathcal{A}=\mathbb{C}<\hat{x}^1,\cdots,\hat{x}^N>/([\hat{x}^i,\hat{x}^j]-ih\theta^{ij}).$$

In opposite to the representation on a Hilbert space we will call it a ⋆-product representation. Now the question arises if we can do the same with other relation-defined algebras. We will see that this is possible if we invent an ordering description. Other possibilities for relations are Lie algebra structures with

$$[\hat{x}^i, \hat{x}^j] = ihC^{ij}{}_k \hat{x}^k, \qquad h, C^{ij}{}_k \in \mathbb{C}$$

and quantum space structures [60, 61, 62, 22] with

$$\hat{x}^i \hat{x}^j = qR^{ij}{}_{kl} \hat{x}^k \hat{x}^l, \qquad q = e^h, R^{ij}{}_{kl} \in \mathbb{C}$$

Instead of considerering these special relations we will in the following discuss a more general case. We assume that the algebra $\mathcal{A}$ is generated by N elements $\hat{x}^i$ and relations

$$[\hat{x}^i, \hat{x}^j] = \hat{c}^{ij}(\hat{x}) = ih\widetilde{\hat{c}}^{ij}(\hat{x})$$

where we assume that the right hand side of this formula is containing a parameter h, and is becoming in a sense small if this parameter approches zero. Mathematically more correct we have to use a h-adic expanded algebra

$$\mathcal{A} = \mathbb{C} < \hat{x}^1, \cdots, \hat{x}^N > [[h]]/([\hat{x}^i, \hat{x}^j] - ih\widetilde{\hat{c}}^{ij}(\hat{x})) \tag{2.6}$$

where it is possible to work with formal power series in h. Note that this kind of algebras all have the Poincare-Birkhoff-Witt property since a reordering of two $\hat{x}^i$ never affects the polynomials of same order in h. An algebra with Poicare-Birkhoff-Witt property possesses a basis of lexicographically ordered monomials. For an algebra generated by two elements $\hat{x}$ and $\hat{y}$ this means that the monomials $\hat{x}^n\hat{y}^m$ constitute a basis. For example the algebra defined by the relations $[\hat{x}, \hat{y}] = \hat{x}^2 + \hat{y}^2$ does not have this property. An example of an algebra that is not included in the above three cases but fulfilling the property (2.6) is given later in (3.3.2).

2.2.1 Algebra generator orderings

Note that Weyl's quantization procedure (2.1) does not make reference to any algebra relation. So let us calculate what Weyl is doing on an algebraic level with this formula. For this let

$$f(p) = \int d^n x\, f(x) e^{ip_i x^i}$$

be the Fourier transform of f. Formally we get for a monomial in $\mathbb{R}^N$

$$\int d^n x\, x^1 \cdots x^m\, e^{ip_i x^i} \;\; = (-i\partial_{p_{i_1}}) \cdots (-i\partial_{p_{i_m}})\delta(p).$$

The Weyl operator associated to a function f is defined by

$$W(f) = \int \frac{d^n p}{(2\pi)^n} f(p) e^{-ip_i \hat{x}^i} \tag{2.7}$$

(see e. g. [63]). For a monomial we get

$$W(x^{i_1} \cdots x^{i_m}) \quad = \quad \frac{1}{m!} \partial_{p_{i_1}} \cdots \partial_{p_{i_m}} (p_i \hat{x}^i)^m$$

and therefore the Weyl operator really maps monomials to the corresponding symmetrical ordered polynomial in the algebra, e. g. for three generators

$$W(x^i x^j x^k) = \frac{1}{3!}(\hat{x}^i\hat{x}^j\hat{x}^k + \hat{x}^i\hat{x}^k\hat{x}^j + \hat{x}^k\hat{x}^i\hat{x}^j + \hat{x}^j\hat{x}^i\hat{x}^k + \hat{x}^j\hat{x}^k\hat{x}^i + \hat{x}^k\hat{x}^j\hat{x}^i).$$

A similar calculation may be done for normal ordering with the result that

$$N(f) = \int \frac{d^n p}{(2\pi)^n} f(p) e^{-ip_1 \hat{x}^1} \cdots e^{-ip_n \hat{x}^n}.$$

In the end we see that for calculating a $\star$-product like Weyl we need a ordering description Ω that maps monomials in the coordinates x^i to polynomials in the algebra generators $\hat{x}^i$. Then the $\star$-product is defined by

$$\Omega(f \star_\Omega g) = \Omega(f)\Omega(g) \tag{2.8}$$

for two functions f and g. If we have used another ordering description Ω', the resulting $\star$-product is gauge equivalent to this $\star$-product by the linear transformation

$$D = \Omega^{-1}\Omega'$$

since

$$f \star_{\Omega'} g = D^{-1}(D(f) \star_\Omega D(g)).$$

The choice of different ordering descriptions is equivalent to taking a different gauge of $\star$-product.

For calculating the $\star$-product in the constant and Lie algebra case with the Weyl-ordering operator see e. g. [63]. There a normal ordered $\star$-product is calculated for the Manin plane, too. For the influence of ordering descriptions in the constant case see e. g. [64]. Many examples of $\star$-products for algebras and corresponding ordering descriptions are given in [65].

2.2.2 Weyl-ordered ⋆-products

In this section we will calculate a ⋆-product generated by symmetric ordering (2.7) of the generators of the algebra (2.6), the Weyl-ordered ⋆-product [66]. The algebra of functions equipped with the Weyl-ordered ⋆-product is isomorphic by construction to the noncommutative algebra it is based on.

With look to (2.8) we start with

$$f \star g = \int \frac{d^n k}{(2\pi)^n} \int \frac{d^n p}{(2\pi)^n} f(k) g(p)\, W^{-1}(e^{-ik_i\hat{x}^i} e^{-ip_i\hat{x}^i})$$

where we have used

$$W(e^{ik_i x^i}) = e^{ik_i\hat{x}^i}.$$

We are therefore able to write down the ⋆-product of the two functions if we know the form of the last expression. For this we expand it in terms of commutators. We use

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A}+\hat{B}} R(\hat{A}, \hat{B})$$

with

$$\begin{aligned} R(\hat{A},\hat{B}) &= 1 + \frac{1}{2}[\hat{A},\hat{B}] \\ &- \frac{1}{6}[\hat{A}+2\hat{B},[\hat{A},\hat{B}]] + \frac{1}{8}[\hat{A},\hat{B}][\hat{A},\hat{B}] + \mathcal{O}(3). \end{aligned}$$

If we set $\hat{A} = -ik_i\hat{x}^i$ and $\hat{B} = -ip_i\hat{x}^i$ the above mentioned expression becomes

$$W^{-1}(e^{-ik_i\hat{x}^i} e^{-ip_i\hat{x}^i}) =$$

$$\begin{aligned} & e^{-i(k_i+p_i)x^i} + \frac{1}{2}(-ik_i)(-ip_j) W^{-1}(e^{-i(k_i+p_i)\hat{x}^i}[\hat{x}^i,\hat{x}^j]) \\ & -\frac{1}{6}(-i)(k_m+2p_m)(-ik_i)(-ip_j) W^{-1}(e^{-i(k_i+p_i)\hat{x}^i}[[\hat{x}^m,[\hat{x}^i,\hat{x}^j]]) \\ & +\frac{1}{8}(-ik_m)(-ip_n)(-ik_i)(-ip_j) W^{-1}(e^{-i(k_i+p_i)\hat{x}^i}[\hat{x}^m,\hat{x}^n][\hat{x}^i,\hat{x}^j]) \\ & +\mathcal{O}(3). \end{aligned}$$

If we assume that the commutators of the generators are written in Weyl ordered form

$$\hat{c}^{ij} = W(c^{ij}),$$

we see that

$$[\hat{x}^m,[\hat{x}^i,\hat{x}^j]] = W(c^{ml}\partial_l c^{ij}) + \mathcal{O}(3),$$

$$[\hat{x}^m, \hat{x}^n][\hat{x}^i, \hat{x}^j] = W(c^{mn}c^{ij}) + \mathcal{O}(3).$$

Further we can derive

$$\begin{aligned} W^{-1}(e^{iq_i\hat{x}^i}W(f)) &= W^{-1}\left(\int \frac{d^n p}{(2\pi)^n} f(p) e^{-i(q_i+p_i)\hat{x}^i} R(-iq_i\hat{x}^i, -ip_i\hat{x}^i)\right) \\ &= e^{-iq_i x^i}\left(f + \frac{1}{2}(-iq_i)c^{ij}\partial_j f\right) + \mathcal{O}(2). \end{aligned}$$

Putting all this together yields

$$\begin{aligned} W^{-1}(e^{-ik_i\hat{x}^i}e^{-ip_i\hat{x}^i}) &= e^{-i(k_i+p_i)x^i}\Big(1 + \frac{1}{2}c^{ij}(-ik_i)(-ip_j) \\ &+ \frac{1}{8}c^{mn}c^{ij}(-ik_m)(-ip_n)(-ik_i)(-ip_j) \\ &+ \frac{1}{12}c^{ml}\partial_l c^{ij}(-i)(k_m - p_m)(-ik_i)(-ip_j)\Big) \\ &+ \mathcal{O}(3), \end{aligned}$$

and we can write down the Weyl ordered $\star$-product up to second order for an arbitrary algebra

$$\begin{aligned} f \star g &= fg + \frac{1}{2}c^{ij}\partial_i f \partial_j g \\ &+ \frac{1}{8}c^{mn}c^{ij}\partial_m\partial_i f \partial_n\partial_j g \\ &+ \frac{1}{12}c^{ml}\partial_l c^{ij}(\partial_m\partial_i f\partial_j g - \partial_i f \partial_m\partial_j g) + \mathcal{O}(3). \end{aligned} \tag{2.9}$$

Let us collect some properties of the just calculated $\star$-product. First

$$[x^i \stackrel{\star}{,} x^j] = c^{ij}$$

is the Weyl ordered commutator of the algebra. Further, if there is a conjugation on the algebra and if we assume that the noncommutative coordinates are real $\overline{\hat{x}^i} = \hat{x}^i$, then the Weyl ordered monomials are real, too. This is also true for the monomials of the commutative coordinate functions. Therefore this $\star$-product respects the ordinary complex conjugation

$$\overline{f \star g} = \overline{g} \star \overline{f}.$$

On the level of the Poisson tensor this means

$$\overline{c^{ij}} = -c^{ij}.$$

It is very instructive to calculate the action of a linear transformation (2.3)

$$\begin{aligned}\Omega &= e^{\Omega^m\partial_n+\Omega^{mn}\partial_m\partial_n+\cdots}\\ &= 1+(\Omega^m+\frac{1}{2}\Omega^n\partial_n\Omega^m)\partial_m+(\Omega^{mn}+\frac{1}{2}\Omega^m\Omega^n)\partial_m\partial_n\\ &\quad +\cdots\\ \Omega^{-1} &= e^{-\Omega^m\partial_n-\Omega^{mn}\partial_m\partial_n-\cdots}\\ &= 1-(\Omega^m-\frac{1}{2}\Omega^n\partial_n\Omega^m)\partial_m-(\Omega^{mn}-\frac{1}{2}\Omega^m\Omega^n)\partial_m\partial_n\\ &\quad +\cdots\end{aligned}$$

on the Weyl ordered $\star$-product. We find

$$\begin{aligned}\Omega^{-1}(\Omega(f)\star\Omega(g)) &= f\star g\\ &\quad +\frac{1}{2}(c^{in}\partial_n\Omega^j-c^{jn}\partial_n\Omega^i-\Omega^n\partial_n c^{ij})\,\partial_i f\,\partial_j g\\ &\quad -2\Omega^{ij}\,\partial_i f\,\partial_j g\\ &\quad +\cdots.\end{aligned}$$

The first deviation is the Lie derivative of the vector field $\Omega^i\partial_i$ for c^{ij}. Later we will compare the Weyl ordered $\star$-product to another one and give in this case an explicit formula for the transformation Ω.

2.2.3 Example: $M(so_a(n))$

Here we will start the example of a quantum space introduced in [25]. Although this quantum space is covariant under the quantum group $SO_a(n)$, we will never use this property. We have taken it because of its simple relations. Further it has a nontrivial center and there exist outer derivations that will below serve as a useful example.

Since we are using the n-dimensional generalisation introduced in [48, 47] we will simply call it $SO_a(n)$ covariant quantum space or abreviated $M(so_a(n))$. The relations of this quantum space are

$$[\widehat{x}^0,\widehat{x}^i]=ia\widehat{x}^i \quad \text{for} \quad i\neq 0,$$

with a a real number. The $\hat{x}^i$ simply commute with each other. In the following of the example Greek indices will run from 0 to $n-1$, whereas Latin indices will run from 1 to $n-1$. It is easy to see that the Poisson tensor corresponding to the algebra is

$$c^{\mu\nu}=iax^i(\delta_0^\mu\delta_i^\nu-\delta_0^\nu\delta_i^\mu).$$

Since we are dealing here with the case of a Lie algebra we surely have $W(c^{\mu\nu}) = [\hat{x}^\mu, \hat{x}^\nu]$. In this case the Weyl ordered ⋆-product takes the following form (compare [48])

$$\begin{aligned} f \star g &= fg + \frac{ia}{2}\left(\partial_o f\, x^i \partial_i g - x^i \partial_i f\, \partial_o g\right) \\ &-\frac{a^2}{8}\left(\partial_0^2 f\, x^i x^j \partial_i \partial_j g - 2\partial_0 x^i \partial_i f\, \partial_0 x^i \partial_i g + x^i x^j \partial_i \partial_j f\, \partial_0^2 g\right) \qquad (2.10) \\ &-\frac{a^2}{12}\left(\partial_0^2 f\, x^i \partial_i g - \partial_0 f \partial_0\, x^i \partial_i g - \partial_0 x^i \partial_i f\, \partial_o g + x^i \partial_i f\, \partial_0^2 g\right) \\ &+\mathcal{O}(a^3). \end{aligned}$$

We will continue the example when we have derivations of the ⋆-product algebra.

2.3 ⋆-products with commuting vector fields

The ⋆-products of the last section are only given up to second order and we have not been able to derive closed formulas. Here we present a closed formula for ⋆-products that generalise the Moyal-Weyl-⋆-product (2.5) in a simple way. [67] We only have to replace the partial derivatives in the formula by commuting vector fields, since they have the same algebraic properties. We will prove the associativity of this ⋆-product and make considerations of how to get desired algebra relations. After that we calculate ⋆-products for some examples like the $so(3)$ Lie algebra and several quantum spaces.

2.3.1 Definition and proof of associativity

Let X be a vector field. Then it is easy to show that

$$X^i(x)\frac{\partial}{\partial x^i}\left(f(x)\, g(x)\right) = \left(X^i(y)\frac{\partial}{\partial y^i} + X^i(z)\frac{\partial}{\partial z^i}\right)\left(f(y)g(z)\right)\Big|_{y\to x, z\to x}.$$

To write the last formula in a more compact way we introduce the following notation

$$X_1 f_1 g_1 = (X_2 + X_3) f_2 g_3|_{2\to 1, 3\to 1}.$$

With this we can derive a kind of Leibniz rule

$$\begin{aligned} X_1^l f_1 g_1 &= (X_2 + X_3)^l f_2 g_3|_{2\to 1, 3\to 1} \\ P(X_1) f_1 g_1 &= P(X_2 + X_3) f_2 g_3|_{2\to 1, 3\to 1} \end{aligned}$$

where P is a polynomial in X. The last equation can also be written in the form

$$P(X_1)\left(f_1 g_1|_{2\to 1,3\to 1}\right) = P(X_2+X_3) f_2 g_3|_{2\to 1,3\to 1}\,. \tag{2.11}$$

Let now be $X_a = X_a^i \partial_i$ n commuting vector fields, i. e. $[X_a, X_b] = 0$. Note that then locally always a coordinate system $y^a(x)$ may be found with $X_a = \partial_{y^a}$. Globally this does not have to be the case. Further let σ^{ab} be a constant matrix. Then we define a $\star$-product via

$$(f\star g)\big|_1 = e^{\sigma^{ab}X_{a2}X_{b3}} f_2 g_3\Big|_{2=3=1}\,. \tag{2.12}$$

This $\star$ product is associative since

$$\begin{aligned}(f\star(g\star h))\big|_1 &= e^{\sigma^{ab}X_{a2}X_{b3}} f_2\left(e^{\sigma^{cd}X_{c4}X_{d5}} g_4 h_5\Big|_{4\to 3,5\to 3}\right)\Bigg|_{2\to 1,3\to 1}\\ &= e^{\sigma^{ab}X_{a2}(X_{b4}+X_{b5})} f_2 e^{\sigma^{cd}X_{c4}X_{d5}} g_4 h_5\Big|_{4\to 3,5\to 3,2\to 1,3\to 1}\\ &= e^{\sigma^{ab}X_{a1}X_{b2}+\sigma^{ab}X_{a1}X_{b3}} e^{\sigma^{cd}X_{c2}X_{d3}} f_1 g_2 h_3\Big|_{2\to 1,3\to 1}\end{aligned}$$

and

$$\begin{aligned}((f\star g)\star h)\big|_1 &= e_2^{\sigma^{ab}X_{a1}X_{b2}}\left(e^{\sigma^{cd}X_{c3}X_{d4}} f_3 g_4\Big|_{3\to 1,4\to 1}\right) h_2\Bigg|_{2\to 1}\\ &= e^{\sigma^{ab}(X_{a3}+X_{a4})X_{b2}} e^{\sigma^{cd}X_{c3}X_{d4}} f_3 g_4 h_2\Big|_{3\to 1,4\to 1,2\to 1}\\ &= e^{\sigma^{ab}X_{a1}X_{b3}+\sigma^{ab}X_{a2}X_{b3}} e^{\sigma^{cd}X_{c1}X_{d2}} f_1 g_2 h_3\Big|_{2\to 1,3\to 1}\end{aligned}$$

where in the second step we used the relation (2.11). The two expressions are equal since the vector fields commute.

For future use we calculate the $\star$-commutator

$$\begin{aligned}[f \stackrel{\star}{,} g] &= \left(e^{\sigma^{ab}X_{a1}X_{b2}} - e^{\sigma^{ab}X_{a2}X_{b1}}\right) f_1 g_2\Big|_{2\to 1}\\ &= 2\sinh(\sigma^{ab}X_{a1}X_{b2}) f_1 g_2\big|_{2\to 1}\,.\end{aligned}$$

The last line is only valid for an antisymmetric matrix σ.

For the case of two vector fields, which we call $X_1 = X$ and $X_2 = Y$, we write down the explicit formula for $\sigma^{12} = h, \sigma^{21} = 0$

$$f\star g = \sum_{n=0}^{\infty}\frac{h^n}{n!}(X^n f)\,(Y^n g) \tag{2.13}$$

the asymmetric ⋆-product and for $\sigma^{12} = \frac{h}{2}, \sigma^{21} = -\frac{h}{2}$

$$f \star g = \sum_{n=0}^{\infty} \frac{h^n}{2^n n!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (X^{n-i} Y^i f)\,(X^i Y^{n-i} g) \tag{2.14}$$

which yields the antisymmetric ⋆-product. Both ⋆-products have the same Poisson tensor $\Pi = X \wedge Y$.

2.3.2 Linear transformations

If we have a ⋆-product, we have seen that we simply can produce a new ⋆-product by a linear transformation on the space of functions (2.3). Suppose that D is such an invertible operator and that its expansion in derivatives starts with 1. Additionally we now assume, that D is of the form

$$D = e^{\tau(X_a)} \quad , \quad D^{-1} = e^{-\tau(X_a)}$$

where τ is a polynomial of the vector fields X_a. Then for the ⋆-product (2.12) we see that

$$\begin{aligned} f \star' g &= D^{-1}(D(f) \star D(g)) \\ &= e^{-\tau(X_{a1})} \left(e^{\sigma^{ab} X_{a2} X_{b3}} e^{\tau(X_{a2})} f_2 e^{\tau(X_{a3})} g_3 \Big|_{2\to 1, 3\to 1} \right) \\ &= e^{-\tau(X_{a2}+X_{a3}) + \sigma^{ab} X_{a2} X_{b3} + \tau(X_{a2}) + \tau(X_{a3})} f_2 g_3 \Big|_{2\to 1, 3\to 1} . \end{aligned}$$

For τ only quadratic in the X_a (note that τ_2^{ab} is symmetric)

$$\tau = \tau_1^a X_a + \frac{1}{2} \tau_2^{ab} X_a X_b$$

we have

$$\tau(X_{a1}) + \tau(X_{a2}) - \tau(X_{a1} + X_{a2}) = -\tau_2^{ab} X_{a1} X_{b2}$$

and the new ⋆-product becomes

$$f \star' g = e^{(\sigma^{ab} - \tau_2^{ab}) X_{a1} X_{b2}} \Big|_{2\to 1}$$

So we see that the antisymmetric ⋆-product (2.14) and the asymmetric ⋆-product (2.13) are related by a linear transformation in function space.

As already mentioned locally commuting vector fields can be represented by a coordinate transformation. It is very important that this need not to be the case globally. This is the reason that the algebras resulting from the ⋆-product are not isomorphic to the constant case algebra. We will see explicit examples for this later.

2.3.3 Differences to other ⋆-products

The Poisson tensor of the above defined ⋆-product (2.12) is $\Pi^{ij} = \sigma^{ab} X_a^i X_b^j$ with σ^{ab} antisymmetric. This we can plug into the formula of the Weyl ordered ⋆-product (2.9) and make a linear transformation. We can compare the result to the ⋆-product (2.12). After some calculations we get

$$e^{-\rho}\left(e^{\rho}(f) \star_{\text{Weyl}} e^{\rho}(g)\right) = f \star_\sigma g$$

with

$$\begin{aligned}\rho &= 1 + \frac{1}{16}\sigma^{ab}\sigma^{cd}(X_a^m\partial_m X_c^i)(X_b^n\partial_n X_d^j)\partial_i\partial_j \\ &+ \frac{1}{24}\sigma^{ab}\sigma^{cd}(X_a^i X_c^j X_b^n \partial_n X_d^k + X_a^k X_c^i X_b^n \partial_n X_d^j + X_a^j X_c^k X_b^n \partial_n X_d^i)\partial_i\partial_j\partial_k \\ &+ \mathcal{O}(\sigma^3).\end{aligned}$$

Therefore this two ⋆-products are equivalent at least up to second order.

Later we will define the Kontsevich ⋆-product. This ⋆-product can be constructed on every Poisson manifolds and proves that one is able to find a ⋆-product for every Poisson structure. We were not able to show, that there is a equivalence between the Kontsevich ⋆-product and the ⋆-product constructed by commuting vector fields. There may be obstructions since the equivalence is dependent of the Poisson cohomology of the Poisson manifold.

2.3.4 Some examples in two dimensions

We calculate some examples in two dimensions with the asymmetric ⋆-product (2.13).

$X = ax\partial_x$, $Y = \partial_y$

We get

$$[x \overset{\star}{,} y] = ax,$$

the algebra of two dimensional a-euclidian space. The algebra relations follow from

$$\begin{aligned} x \star x &= x^2, \\ x \star y &= xy + ax, \\ y \star x &= xy, \\ y \star y &= y^2. \end{aligned}$$

$X = (a + bx)\partial_x$, $Y = (c + dy)\partial_y$

This is the general linear case. We get

$$x \star y = e^{bd}(y + \frac{c}{d}) \star (x + \frac{a}{b}),$$

which follows from

$$\begin{aligned} x \star x &= x^2, \\ x \star y &= e^{bd}(y + \frac{c}{d})(x + \frac{a}{b}), \\ y \star x &= xy, \\ y \star y &= y^2. \end{aligned}$$

$X = \frac{a}{\sqrt{x^2+y^2}}(x\partial_x + y\partial_y)$, $Y = x\partial_y - y\partial_x$

These are the derivatives ∂_r and ∂_θ of the coordinate transformation $x = r\cos\theta$, $y = r\sin\theta$. We get

$$[x \overset{\star}{,} y] = a\sqrt{x \star x + y \star y},$$

which follows from

$$\begin{aligned} x \star x &= x^2 - a\frac{xy}{r}, \\ x \star y &= xy + a\frac{x^2}{r}, \\ y \star x &= xy - a\frac{y^2}{r}, \\ y \star y &= y^2 + a\frac{xy}{r}. \end{aligned}$$

$X = a(x\partial_x + y\partial_y)$, $Y = x\partial_y - y\partial_x$

This is a simplification of the previous case. We get

$$[x \overset{\star}{,} y] = (\tan a)(x \star x + y \star y),$$

which follows from

$$\begin{aligned} x \star x &= \cos a\, x^2 - \sin a\, xy, \\ x \star y &= \cos a\, xy + \sin a\, y^2, \\ y \star x &= \cos a\, xy - \sin a\, y^2, \\ y \star y &= \cos a\, y^2 + \sin a\, xy, \end{aligned}$$

$$\begin{aligned} x \star x + y \star y &= \cos a \, (x^2 + y^2), \\ x \star y - y \star x &= \sin a \, (x^2 + y^2). \end{aligned}$$

It is interesting that this algebra does not have the Poincare-Birkhoff-Witt property for $\tan a = 1$.

We have seen that even in this simple cases very rich structures surface. But it is not quite clear what happens if we try to replace the formal parameter in the $\star$-product expansion by a number. In the last case we see, that in this case higher order relations can arise.

2.3.5 Realization of algebras

If we want to represent an algebra with the help of a $\star$-product, we have seen that this is possible if we use an ordering description. In this section we propose an other method of how to calculate a $\star$-product with the property, that it reproduces the algebra relations of some desired algebra. With this second approach no ordering description is needed. It is even not quite clear in the end, if there would be an ordering description that would yield the same $\star$-product with the first approach.

We know that the $\star$-commutator of a $\star$-product is a Poisson tensor up to first order

$$[f \stackrel{\star}{,} g] = h\{f, g\} + \mathcal{O}(h^2) = h\Pi(f, g) + \mathcal{O}(h^2).$$

where Π is the Possion-bivector of the Poisson structure. If we would have a $\star$-product that reproduces the algebra relations, the right hand side of the previous equation would be a polynomial in the generators of the algebra, i. e.

$$[x^i \stackrel{\star}{,} x^j] = hc_\star^{ij}(x).$$

The $\star$ in the index of $c_\star^{ij}$ indicates that all products between the coordinate functions in it are $\star$-products. To calculate the leading order of $c_\star^{ij}(x)$ it is not necessary to know the explicit form of the $\star$-product, since it always starts with the ordinary product of functions. We can conclude that

$$\{x^i, x^j\} = \Pi^{ij} = c^{ij}(x), \tag{2.15}$$

For the special case for the $\star$-products (2.12) it is

$$\Pi = \sigma^{ab} X_a \wedge X_b.$$

If we are able to write a general Poisson bivector in this form, we can try to reconstruct the algebra relations with the help of the $\star$-products (2.12).

For this let f be a function and $X_f = \{f, \cdot\}$ the Hamiltonian vector field associated to f. Then the commutator of vector fields is

$$[X_f, X_g] = X_{\{f,g\}},$$

due to the Jakobi identity of the Poisson bracket. If we can find functions with

$$\{f_i, g_j\} = \delta_{ij}, \quad \{f_i, f_j\} = 0, \quad \{g_i, g_j\} = 0,$$

this implies that all commutators between the associated Hamiltonian vector fields vanish. Now one can deduce from the splitting theorem for Poisson manifolds [68] that this is possible in a neighborhood of a point if the rank of the Poisson tensor is constant around this point. Since we do not want to find a $\star$-product on $\mathbb{R}^N$, but a $\star$-product with certain commutation relations, we can reduce $\mathbb{R}^N$ by the set of points where the rank of the Poisson tensor jumps and we have a good chance to find functions with the desired properties on the new manifold. In this case we can write the Poisson tensor as

$$\Pi = \sum_i X_{f_i} \wedge X_{g_i}.$$

In the following we will find functions f_i anf g_i for Poisson tensors of several algebras and will use the corresponding Hamiltonian vector fields in the $\star$-products (2.12). We will calculate the resulting algebra relations from the $\star$-product and compare them to the original algebra relations.

2.3.6 The quantum space $M(so_a(n))$

We will start our examples by giving a closed formula for a second $\star$-product for the quantum space introduced in (2.2.3). It is closely related to the $\star$-product for the two dimensional a-euclidean space given above. As manifold we take $\mathbb{R}^N$ with coordinates x^0 and x^i with $i = 1, \ldots, N-1$ and use the asymmetric $\star$-product (2.13) with the two vectorfields

$$X = iax^i\partial_i, \qquad Y = \partial_o.$$

With this we get

$$[x^i \overset{\star}{,} x^0] = iax^i,$$

the algebra of $M(so_a(n))$. The algebra relations follow from

$$\begin{aligned} x^i \star x^j &= x^i x^j, \\ x^i \star x^0 &= x^i x^0 + iax^i, \\ x^0 \star x^i &= x^i x^0, \\ y \star y &= y^2. \end{aligned}$$

To show the usefulness of the approach proposed in the last section we now make a generalization of the above defined algebra. The new relations are

$$[\hat{x}^\alpha, \hat{x}^\beta] = i(a^\alpha \hat{x}^\beta - a^\beta \hat{x}^\alpha)$$

where a^α are now n deformation parameters. For this relations to be consistent the Jacobi identities have to be fulfilled, which easily can be proofed. In this case the commuting vector fields can not so easy guessed like in the special case.

Since the right hand side of the relation is linear and we are therefore dealing with a Lie algebra the Poisson tensor associated with the algebra is simple

$$\{x^\alpha, x^\beta\} = a^\alpha x^\beta - a^\beta x^\alpha.$$

If we want to find commuting vector fields that reproduce this Poisson tensor we now follow the way outlined in the previous section. The rank of this matrix is 2. Therefore we have to find two functions fulfilling $\{f, g\} = 1$. We make a guess and define

$$f = a^\alpha x^\alpha \qquad \tilde{x}^\alpha = x^\alpha - \frac{a^\alpha a^\beta}{a^2} x^\beta$$

with $a^2 = a^\alpha a^\alpha$. These functions have commutation relations very similar to the special case of $M(so_a(n))$.

$$\{f, \tilde{x}^\alpha\} = a^2 \tilde{x}^\alpha \qquad \{\tilde{x}^\alpha, \tilde{x}^\beta\} = 0$$

If we define $g = \frac{1}{a^2} ln \sqrt{\tilde{x}^\alpha \tilde{x}^\alpha}$ we see that

$$\{f, g\} = 1 \tag{2.16}$$

and the desired functions are found. The commuting vector fields are now easy calculated

$$\begin{aligned} X = \{f, \cdot\} &= a^2 x^\beta \partial_\beta - (a^\alpha x^\alpha) a^\beta \partial_\beta \\ Y = \{\cdot, g\} &= -\frac{1}{a^2} a^\beta \partial_\beta \end{aligned}$$

In this case we are lucky since no singularities have shown up and the ⋆-product can be defined on whole $\mathbb{R}^n$. Again we may use the asymmetric ⋆-product (2.13) and see that the algebra relations are reproduced.

2.3.7 q-deformed Heisenberg algebra

If we take the q-deformed Heisenberg algebra [60] in two dimensions

$$\hat{x}\hat{y} = q\hat{y}\hat{x} + \theta$$

we very easily can calculate a $\star$-product in $h = ln\, q$ and θ. The Poisson tensor Π is

$$\Pi = (xy + \frac{\theta}{h})\partial_x \wedge \partial_y.$$

We see that with $f = \ln(xy + \frac{\theta}{h})$ and $g = \ln y$

$$\{f, g\} = 1.$$

The Hamiltonian vector fields are

$$X = X_f = y\partial_y - x\partial_x, \qquad Y = X_g = -(x + \frac{\theta}{hy})\partial_x.$$

To calculate the $\star$-products we note

$$X^n(x) = (-1)^n x, \qquad X^n(y) = y,$$
$$Y^n(x) = (-1)^n(x + \frac{\theta}{hy}) \text{ for } n > 0, \qquad Y^n(y) = \delta^{n,0} y.$$

For the asymmetric $\star$-product (2.13) this yields

$$\begin{aligned} x \star y &= xy, \\ y \star x &= e^{-h}xy + (e^{-h} - 1)\frac{\theta}{h}. \end{aligned}$$

For the antisymmetric $\star$-product (2.14) we get

$$\begin{aligned} x \star y &= e^{\frac{h}{2}}xy + (e^{+\frac{h}{2}} - 1)\frac{\theta}{h}, \\ y \star x &= e^{-\frac{h}{2}}xy + (e^{-\frac{h}{2}} - 1)\frac{\theta}{h}. \end{aligned}$$

Both $\star$-products have therefore the algebra relation

$$x \star y = e^h y \star x + (e^h - 1)\frac{\theta}{h}$$

and by a redefinition of θ the original alegebra relations are reproduced.

2.3.8 The Lie algebra $so(3)$

First try

The algebra relations of the eveloping algebra of $so(3)$ are $[\hat{x}^i, \hat{x}^j] = i\epsilon^{ijk}\hat{x}^k$. The corresponding poisson tensor is

$$\Pi = ix\partial_y \wedge \partial_z + iy\partial_z \wedge \partial_x + iz\partial_x \wedge \partial_y.$$

For $f = -i \arctan \frac{x}{y}$ we have $\{z, f\} = 1$. The Hamiltonian vector fields are

$$\begin{aligned} X = X_f &= -\partial_z + \frac{z}{x^2+y^2}(x\partial_x + y\partial_y), \\ Y = X_z &= i(y\partial_x - x\partial_y). \end{aligned}$$

We calculate ($\rho = \sqrt{x^2+y^2}$)

$$\begin{aligned} Y^n(z) &= \delta^{n0} z, & X^n(z) &= \delta^{n0} z + \delta^{n1}, \\ Y^{2n}(x) &= (-1)^n x, & Y^{2n+1}(x) &= -(-1)^n y, \\ Y^{2n}(y) &= (-1)^n y, & Y^{2n+1}(y) &= (-1)^n x, \\ X(x) &= -x\frac{z}{\rho^2}, & Y(y) &= -y\frac{z}{\rho^2}, \\ X^n(x) &= x(1+\frac{z^2}{\rho^2}) f_n(\rho, z), & X^n(y) &= y(1+\frac{z^2}{\rho^2}) f_n(\rho, z), \\ f_2 &= -\frac{1}{\rho^2}, & f_{n+1} &= \frac{z}{\rho^2} f_n + \partial_z f_n - \frac{z}{\rho}\partial_\rho f_n, \\ f_3 &= \frac{z}{\rho^4}, & f_4 &= \frac{1}{\rho^4}(1+\frac{5z^2}{\rho^2}). \end{aligned}$$

If we transform to the x^+, x^-coordinate system we are now able to calculate the commutator of $x^+$and $x^-$and get

$$\begin{aligned} [x^+ \stackrel{\star}{,} x^-] &= 2\sum_{n=0}^{\infty} \frac{h^{2n+1}}{(2n+1)!} \sum_{i=0}^{\infty} \binom{n}{i} (X^i x^+)(X^{n-i} x^-) \\ &= -2hz + \frac{h^3}{6}(1+\frac{z^2}{\rho^2})(\frac{4z}{\rho^2}) + \mathcal{O}(h^5). \end{aligned}$$

In this case the original algebra relations are not reproduced.

Second try

After a linear basis transformation we now can start with the algebra relations

$$[\hat{z}, \hat{x}^+] = \hat{x}^+, \quad [\hat{z}, \hat{x}^-] = -\hat{x}^-, \quad [\hat{x}^+, \hat{x}^-] = \hat{z}.$$

With $f = \ln x^-$we have $\{f, z\} = 1$. The Hamiltonian vector fields become now

$$X = X_f = \partial_z - \frac{z}{x^-}\partial_+, \qquad Y = X_z = x^+\partial_+ - x^-\partial_{x^-}.$$

Therefore

$$\begin{aligned} X^n(z) &= \delta^{n0} z + \delta^{n1}, & Y^n(z) &= \delta^{n0} z, \\ X^n(x^+) &= -\delta^{n1}\frac{z}{x^-} + \delta^{n0}x^+ - \delta^{n2}\frac{1}{x^-}, & Y^n(x^+) &= x^+, \\ X^n(x^-) &= \delta^{n0}x^-, & Y^n(x^-) &= (-1)^n x^-. \end{aligned}$$

and with the asymmetric ⋆-product (2.13) we get

$$\begin{aligned} z \star x^+ &= zx^+ + hx^+, & x^+ \star z &= x^+ z, \\ z \star x^- &= zx^- - hx^-, & x^- \star z &= x^- z, \\ x^+ \star x^- &= x^+x^- + hz - \frac{h^2}{2}, & x^- \star x^+ &= x^+x^-. \end{aligned}$$

and therefore

$$[z \stackrel{\star}{,} x^+] = hx^+, \quad [z \stackrel{\star}{,} x^-] = -hx^-, \quad [x^+ \stackrel{\star}{,} x^-] = hz - \frac{h^2}{2}.$$

With $\tilde{z} = z - \frac{h}{2}$ now the correct algebra relations are reproduced.

2.3.9 The quantum space $M(so_q(3))$

Here we give as a second example the ⋆-product for the quantum space $M(so_q(3))$ invariant under the quantum group $SO_q(3)$. [21] The algebra relations in the basis adjusted to the quantum group terminology are

$$\hat{z}\hat{x}^+ = q^2\hat{x}^+\hat{z}, \quad \hat{z}\hat{x}^- = q^{-2}\hat{x}^-\hat{z}, \quad [\hat{x}^-, \hat{x}^+] = (q - q^{-1})\hat{z}^2.$$

For the commutators we get

$$[\hat{z}, \hat{x}^+] = (q^2 - 1)\hat{x}^+\hat{z}, \quad [\hat{z}, \hat{x}^-] = (q^2 - 1)\hat{x}^-\hat{z}, \quad [\hat{x}^-, \hat{x}^+] = (q - \frac{1}{q})\hat{z}^2$$

and therefore the Poisson brackets are

$$\{z, x^+\} = 2zx^+, \quad \{z, x^-\} = -2zx^-, \quad \{x^-, x^+\} = 2z^2.$$

For $f = \ln x^-$and $g = \frac{1}{2}\ln z$ we have $\{f, g\} = 1$ and the Hamiltonian vector fields become

$$X_f = 2z\partial_z + \frac{2z^2}{x^-}\partial_+, \qquad X_g = x^+\partial_+ - x^-\partial_-.$$

For the $\star$-product we try now

$$X = z\partial_z + \frac{\alpha z^2}{x^-}\partial_+, \qquad Y = x^+\partial_+ - x^-\partial_-.$$

We have

$$\begin{aligned} Y^n(x^+) = x^+, \quad Y^n(x^-) = (-1)^n x^-, \quad Y^n(z) = \delta^{n,0} z, \\ X^n(x^-) = \delta^{n,0} x^-, \quad X^n(z) = z, \\ X^n(x^+) = \alpha 2^{n-1}\frac{z^2}{x^-} \qquad \text{for } n > 0. \end{aligned}$$

For the asymmetric $\star$-product (2.13) we calculate

$$\begin{aligned} x^+ \star z &= x^+ z, & z \star x^+ &= e^h x^+ z, \\ x^- \star z &= x^- z, & z \star x^- &= e^{-h} x^- z, \\ x^+ \star x^- &= x^+ x^- + \frac{\alpha}{2}(e^{-2h} - 1)z^2, & x^- \star x^+ &= x^+ x^-, \\ z \star z &= z^2 \end{aligned}$$

and get algebra relations

$$z \star x^+ = e^h x^+ \star z, \quad z \star x^- = e^{-h} x^- \star z, \quad [x^+ \overset{\star}{,} x^-] = \frac{\alpha}{2}(e^{-2h} - 1) z \star z.$$

If we set

$$q = e^{\frac{h}{2}}, \qquad \alpha = \frac{2q^2}{q + \frac{1}{q}}$$

this reproduces the algebra relations.

For the antisymmetric ⋆-product (2.14) we calculate

$$
\begin{aligned}
x^+ \star z &= e^{-\frac{h}{2}} x^+ z, & z \star x^+ &= e^{\frac{h}{2}} x^+ z,\\
x^- \star z &= e^{\frac{h}{2}} x^- z, & z \star x^- &= e^{-\frac{h}{2}} x^- z,\\
x^+ \star x^- &= x^+ x^- + \frac{\alpha}{2}(e^{-h} - 1) z^2, & x^- \star x^+ &= x^+ x^- + \frac{\alpha}{2}(e^{h} - 1) z^2,\\
z \star z &= z^2.
\end{aligned}
$$

The algebra relations now are

$$
z \star x^+ = e^h x^+ \star z, \quad x^- z \star x^- = e^{-h} x^- \star z, \quad [x^+ \stackrel{\star}{,} x^-] = -\frac{\alpha}{2}(e^h - e^{-h}) z \star z.
$$

Here we can reproduce the algebra relations when we set

$$
q = e^{\frac{h}{2}}, \qquad \alpha = -\frac{2}{q + \frac{1}{q}}.
$$

2.3.10 The quantum space $M(so_q(1,3))$

We try to generalize the previous example and start with the more general commuting vector fields

$$
X = z\partial_z + \frac{1}{x^-}(\alpha z^2 + \beta z)\partial_+, \qquad Y = x^+\partial_+ - x^-\partial_-.
$$

The only relation, that is changed is

$$
X^n(x^+) = \frac{1}{x^-}(2^{n-1}\alpha z^2 + \beta z)
$$

and we calculate with the antisymmetric ⋆-product (2.14)

$$
\begin{aligned}
x^+ \star z &= e^{-\frac{h}{2}} x^+ z, & z \star x^+ &= e^{\frac{h}{2}} x^+ z,\\
x^- \star z &= e^{\frac{h}{2}} x^- z, & z \star x^- &= e^{-\frac{h}{2}} x^- z,\\
z \star z &= z^2,
\end{aligned}
$$

$$
x^+ \star x^- = x^+ x^- + \frac{\alpha}{2}(e^{-h} - 1) z^2 + \beta(e^{-\frac{h}{2}} - 1) z,
$$

$$
x^- \star x^+ = x^+ x^- + \frac{\alpha}{2}(e^{h} - 1) z^2 + \beta(e^{\frac{h}{2}} - 1) z.
$$

The relations become

$$
z \star x^+ = e^h x^+ \star z, \qquad z \star x^- = e^{-h} x^- \star z,
$$

$$[x^+ \stackrel{\star}{,} x^-] = -\frac{\alpha}{2}(e^h - e^{-h})z \star z + -\beta(e^{\frac{h}{2}} - e^{-\frac{h}{2}})z.$$

The algebra relation of $M(so_q(1,3))$ are [21, 23]

$$[\hat{t}, \hat{x}^i] = 0,$$

$$[\hat{x}^-, \hat{x}^+] = (q - \frac{1}{q})(\hat{z}^2 - \hat{t}\hat{z}),$$

$$\hat{z}\hat{x}^+ = q^2 \hat{x}^+ \hat{z} + (1 - q^2)\hat{t}\hat{x}^+,$$

$$\hat{z}\hat{x}^- = q^{-2}\hat{x}^- \hat{z} + (1 - q^{-2})\hat{t}\hat{x}^-.$$

We can define $\tilde{\hat{z}} = \hat{z} - \hat{t}$ and get new relations

$$[\hat{t}, \hat{x}^i] = 0,$$

$$[\hat{x}^-, \hat{x}^+] = (q - \frac{1}{q})(\tilde{\hat{z}}^2 + \hat{t}\tilde{\hat{z}}),$$

$$\tilde{\hat{z}}\hat{x}^+ = q^2 \hat{x}^+ \tilde{\hat{z}},$$

$$\tilde{\hat{z}}\hat{x}^- = q^{-2}\hat{x}^- \tilde{\hat{z}}.$$

These relations are reproduced by the ⋆-product if we set

$$q = e^{\frac{h}{2}} \quad \alpha = -\frac{2}{q + \frac{1}{q}}, \quad \beta = -1.$$

2.3.11 The quantum space $M(so_q(4))$

The algebra relations of $M(so_q(4))$ are [20, 69]

$$\begin{aligned} \hat{x}_1\hat{x}_2 &= q\hat{x}_2\hat{x}_1, & \hat{x}_1\hat{x}_3 &= q\hat{x}_3\hat{x}_1, \\ \hat{x}_3\hat{x}_4 &= q\hat{x}_4\hat{x}_3, & \hat{x}_2\hat{x}_4 &= q\hat{x}_4\hat{x}_2, \\ \hat{x}_2\hat{x}_3 &= \hat{x}_3\hat{x}_2, & [\hat{x}_4, \hat{x}_1] &= (q - \frac{1}{q})\hat{x}_2\hat{x}_3. \end{aligned}$$

The Poisson brackets are

$$\begin{aligned} \{x_1, x_2\} &= x_1 x_2, & \{x_2, x_4\} &= x_2 x_4, \\ \{x_1, x_3\} &= x_1 x_3, & \{x_3, x_4\} &= x_3 x_4, \\ \{x_2, x_3\} &= 0, & \{x_4, x_1\} &= 2x_2 x_3. \end{aligned}$$

Since the Poisson tensor has two Casimir functions, two vector fields will suffice. We take

$$\begin{aligned} f = \ln x_2, &\qquad g = \ln x_4, \\ \{f, g\} = 1, & \\ X = X_f = x_4\partial_4 - x_1\partial_1, &\qquad Y = X_g = -(x_2\partial_2 + x_3\partial_3) + 2\frac{x_2x_3}{x_4}\partial_1. \end{aligned}$$

Therefore

$$\begin{aligned} X^n(x_1) = (-1)^n x_1, &\qquad Y^n(x_1) = \delta^{n0}x_1 + \delta^{ni}(-2)^i\frac{x_2x_3}{x_4}, \\ X^n(x_2) = \delta^{n0}x_2, &\qquad Y^n(x_2) = (-1)^n x_2, \\ X^n(x_3) = \delta^{n0}x_3, &\qquad Y^n(x_3) = (-1)^n x_3, \\ X^n(x_4) = x_4, &\qquad Y^n(x_4) = \delta^{n0}x_4. \end{aligned}$$

For the asymmetric ⋆-product (2.13) we get

$$\begin{aligned} x_1 \star x_2 = e^h x_1x_2, &\qquad x_2 \star x_1 = x_1x_2, \\ x_1 \star x_3 = e^h x_1x_3, &\qquad x_3 \star x_1 = x_1x_3, \\ x_1 \star x_4 = x_1x_4, &\qquad x_4 \star x_1 = x_1x_4 + (e^{-2h} - 1)x_2x_3, \\ x_2 \star x_3 = x_2x_3, &\qquad x_3 \star x_2 = x_2x_3, \\ x_2 \star x_4 = x_2x_4, &\qquad x_4 \star x_2 = e^{-h}x_2x_4, \\ x_3 \star x_4 = x_3x_4, &\qquad x_4 \star x_3 = e^{-h}x_3x_4 \end{aligned}$$

which yields the algebra relations

$$\begin{aligned} x_1 \star x_2 = e^h x_2 \star x_1, &\qquad x_1 \star x_3 = e^h x_3 \star x_1, \\ x_3 \star x_4 = e^h x_4 \star x_3, &\qquad x_2 \star x_4 = e^h x_4 \star x_2, \\ x_2 \star x_3 = x_3 \star x_2, &\qquad [x_1 \overset{\star}{,} x_4] = (e^{-2h} - 1)x_2 \star x_3. \end{aligned}$$

For the antisymmetric ⋆-product (2.14) we calculate

$$\begin{aligned} x_1 \star x_2 = e^{\frac{h}{2}} x_1x_2, &\qquad x_2 \star x_1 = e^{-\frac{h}{2}} x_1x_2, \\ x_1 \star x_3 = e^{\frac{h}{2}} x_1x_3, &\qquad x_3 \star x_1 = e^{-\frac{h}{2}} x_1x_3, \\ x_1 \star x_4 = x_1x_4 + (e^{-h} - 1)x_2x_3, &\qquad x_4 \star x_1 = x_1x_4 + (e^h - 1)x_2x_3, \\ x_2 \star x_3 = x_2x_3, &\qquad x_3 \star x_2 = x_2x_3, \\ x_2 \star x_4 = e^{\frac{h}{2}} x_1x_2, &\qquad x_4 \star x_2 = e^{-\frac{h}{2}} x_1x_2, \\ x_3 \star x_4 = e^{\frac{h}{2}} x_1x_3, &\qquad x_4 \star x_3 = e^{-\frac{h}{2}} x_1x_3 \end{aligned}$$

and we get the relations

$$\begin{aligned} x_1 \star x_2 &= e^h x_2 \star x_1, & x_1 \star x_3 &= e^h x_3 \star x_1, \\ x_3 \star x_4 &= e^h x_4 \star x_3, & x_2 \star x_4 &= e^h x_4 \star x_2, \\ x_2 \star x_3 &= x_3 \star x_2, & [x_1 \overset{\star}{,} x_4] &= (e^h - e^{-h}) x_2 \star x_3. \end{aligned}$$

In this case the relations are exactly reproduced.

2.3.12 Fourdimensional q-deformed Fock space

The algebra relations are [70, 71]

$$\begin{aligned} \hat{x}_1 \hat{x}_2 &= \frac{1}{q} \hat{x}_2 \hat{x}_1, & \hat{y}_1 \hat{y}_2 &= q \hat{y}_2 \hat{y}_1, \\ \hat{y}_1 \hat{x}_2 &= q \hat{x}_2 \hat{y}_1, & \hat{y}_2 \hat{x}_1 &= q \hat{x}_1 \hat{y}_2, \\ \hat{y}_1 \hat{x}_1 &= q^2 \hat{x}_1 \hat{y}_1 + \theta, & \hat{y}_2 \hat{x}_2 &= q^2 \hat{x}_2 \hat{y}_2 + (q^2 - 1) \hat{x}_1 \hat{y}_1 + \theta. \end{aligned}$$

The Poisson tensor becomes

$$\begin{aligned} \{x_1, x_2\} &= -x_1 x_2, & \{y_1, y_2\} &= y_1 y_2, \\ \{x_2, y_1\} &= -x_2 y_1, & \{x_1, y_2\} &= -x_1 y_2, \\ \{y_1, x_1\} &= 2 x_1 y_1 + \theta, & \{y_2, x_2\} &= 2(x_1 y_1 + x_2 y_2) + \theta. \end{aligned}$$

After some calculations we find the desired functions

$$\begin{aligned} f_1 &= -\ln x_1, & f_2 &= \frac{1}{2} \ln(2 x_1 y_1 + \theta), \\ g_1 &= f_2 - \ln x_2, & g_2 &= \frac{1}{2} \ln \frac{2(x_1 y_1 + x_2 y_2) + \theta}{2 x_1 y_1 + \theta} \end{aligned}$$

with $\{f_1, f_2\} = 1$, $\{g_1, g_2\} = 1$, the other brackets vanish. The Hamiltonian vector fields are

$$\begin{aligned} X_1 = X_{f_1} &= x_2 \partial_{x_2} + y_2 \partial_{y_2} + \frac{2 x_1 y_1 + \theta}{x_1} \partial_{y_1}, & Y_1 = X_{f_2} &= x_1 \partial_{x_1} - y_1 \partial_{y_1}, \\ X_2 = X_{g_1} &= \frac{2(x_1 y_1 + x_2 y_2) + \theta}{x_2} \partial_{y_2}, & Y_2 + X_{g_2} &= x_2 \partial_{x_2} - y_2 \partial_{y_2}. \end{aligned}$$

We calculate

$$\begin{aligned} X_1^n(x_1) &= \delta^{n0} x_1, & Y_1^n(x_1) &= x_1, \\ X_1^n(x_2) &= x_2, & Y_1^n(x_2) &= \delta^{n0} x_2, \\ X_1^n(y_1) &= \frac{2^n}{x_1}(x_1 y_1 + \frac{\theta}{2}) \ (n > 0), & Y_1^n(y_1) &= (-1)^n y_1, \\ X_1^n(y_2) &= y_2, & Y_1^n(y_2) &= \delta^{n0} y_2, \end{aligned}$$

$$\begin{aligned} X_2^n(x_1) &= \delta^{n0} x_1, & Y_2^n(x_1) &= \delta^{n0} x_1, \\ X_2^n(x_2) &= \delta^{n0} x_2, & Y_2^n(x_2) &= x_2, \\ X_2^n(y_1) &= \delta^{n0} y_1, & Y_2^n(y_1) &= \delta^{n0} y_1, \\ X_2^n(y_2) &= \frac{2^n}{x_2}(x_1 y_1 + x_2 y_2 + \frac{\theta}{2}) \text{ for } n > 0, & Y_2^n(y_2) &= (-1)^n y_2. \end{aligned}$$

Since we now have four vector fields we use a generalization of the asymmetric $\star$-product

$$f \star g = \sum_{n=0,m=0}^{\infty} \frac{h^{n+m}}{n!m!} \left(X_1^n X_2^m f\right) \left(Y_1^n Y_2^m g\right)$$

and get

$$\begin{aligned} x_1 \star x_2 &= x_1 x_2, & x_2 \star x_1 &= e^h x_1 x_2, \\ y_1 \star y_2 &= y_1 y_2, & y_2 \star y_1 &= e^{-h} y_1 y_2, \\ y_1 \star x_2 &= y_1 x_2, & x_2 \star x_1 &= e^{-h} x_2 y_1, \\ y_2 \star x_1 &= e^h x_1 y_2, & x_1 \star y_2 &= x_1 y_2, \\ x_1 \star y_1 &= x_1 y_1, & x_2 \star y_2 &= x_2 y_2, \end{aligned}$$

$$y_1 \star x_1 = e^{2h} x_1 y_1 + \frac{e^{2h} - 1}{2}\theta,$$

$$y_2 \star x_2 = e^{2h} x_2 y_2 + (e^{2h} - 1) x_1 y_1 + \frac{e^{2h} - 1}{2}\theta.$$

The algebra relations for this $\star$-product are

$$\begin{aligned} x_1 \star x_2 &= e^{-h} x_2 \star x_1, & y_1 \star y_2 &= e^h y_2 \star y_1, \\ y_1 \star x_2 &= e^h x_2 \star y_1, & y_2 \star x_1 &= e^h x_1 \star y_2, \end{aligned}$$

$$y_1 \star x_1 = e^{2h} x_1 \star y_1 + \frac{e^{2h} - 1}{2}\theta,$$

$$y_2 \star x_2 = e^{2h} x_2 \star y_2 + (e^{2h} - 1) x_1 \star y_1 + \frac{e^{2h} - 1}{2}\theta.$$

And we get the same relations as in the original algebra if we set

$$q = e^h, \qquad \theta' = \frac{2}{q^2 - 1}\theta.$$

Chapter 3

Geometry

To study physics in the noncommutative realm, one replaces the commutative algebra of functions on a space with a noncommutative algebra. Such a replacement is generally controlled by a parameter so that in some limit we can get back a commutative space. The same we expect from theories built an a noncommutative space: In the commutative limit they should reduce to a meaningful commutative theory. $\star$-products have shown to be very useful tools for constructing such deformations since their classical limit is very easily calculated. In this chapter we apply $\star$-products to the commuting frame formalism developed in [6]. For a noncommutative space where the commutator of the coordinates is constant, the commutative limit of this formalism is the usual flat spacetime. For noncommutative spaces with more complicated, non-constant commutators this limit can be a curved manifold.

After a short introduction to noncommutative differential geometry where we fix our notation, we will calculate the semi-classical limit of the commuting frame formalism. In this limit we will see that the construction of a Poisson tensor for a given frame reduces to solving a couple of differential equations. The deformation quantization of the Poisson tensor gives us a $\star$-product and we have constructed a noncommutative space with desired classical limit. We give some examples and we will see that the formalism works well in two dimensions, but has its restrictions in four dimensions. In the end we will construct Poisson structures having the same symmetries as the Schwarzschild metric. Here we are able to give a first order differential calculus with the desired classical limit.

3.1 Noncommutative differential geometry

Locally every manifold can be described by N coordinates x^i. The set of all derivations acting on functions on the manifold forms a module over the algebra of functions. The partial derivatives ∂_i form a basis for all these derivations. Dual to the space of derivations is the space of one forms. The differentials dx^i form a basis of this space and they are dual to the partial derivatives

$$dx^i(\partial_j) = \delta^i_j.$$

With help of the differentials one is able to introduce the de Rham differential mapping functions to one-forms

$$df = dx^i \partial_i f.$$

If one introduces higher order forms with the rule

$$dx^i dx^j = -dx^j dx^i,$$

one can extend the de Rham differential to a nilpotent graded derivation. The differential d and all higher order forms are the exterior algebra of the manifold. One can show that the whole topology of the manifold is encoded in the properties of d or the exterior algebra respectively.

In noncommutative geometry one replaces the commutative algebra of functions on the manifold by an noncommutative algebra. Here we again restrict ourself to algebras defined by relations

$$\mathcal{A} = \mathbb{C} < \hat{x}^1, \cdots, \hat{x}^N > /\mathcal{R}. \tag{3.1}$$

To find something similar to differential geometry one can go on and construct differential calculi to these type of algebras. Just like in the commutatitve case a differential calculus on $\mathcal{A}$ is a $\mathbb{Z}$-graded algebra

$$\Omega(\mathcal{A}) = \bigoplus_{r \geq 0} \Omega^r(\mathcal{A})$$

where the spaces $\Omega^r(\mathcal{A})$ are $\mathcal{A}$-bimodules with $\Omega^0(\mathcal{A}) = \mathcal{A}$. The elements of $\Omega^r(\mathcal{A})$ are called r-forms. There is a linear map

$$\hat{d} : \Omega^r(\mathcal{A}) \to \Omega^{r+1}(\mathcal{A})$$

with the same properties as the commutative differential. It is nilpotent

$$\hat{d}^2 = 0$$

and graded

$$\hat{d}(\omega_1\omega_2) = (\hat{d}\omega_1)\omega_2 + (-1)^r\omega_1\hat{d}\omega_2 \qquad (3.2)$$

where $\omega_1 \in \Omega^r(\mathcal{A})$ and $\omega_2 \in \Omega(\mathcal{A})$. Additionally we assume that $\hat{d}$ generates the spaces $\Omega^r(\mathcal{A})$ for $r > 0$ in the sense that

$$\Omega^{r+1}(\mathcal{A}) = \mathcal{A} \cdot \hat{d}\Omega^r(\mathcal{A}) \cdot \mathcal{A}.$$

Using the Leibniz rule (3.2), every element of $\Omega^r(\mathcal{A})$ can be written as a linear combination of monomials $f(\hat{x})\hat{d}\hat{x}^{i_1}\hat{d}\hat{x}^{i_2}\cdots\hat{d}\hat{x}^{i_r}$. The action of $\hat{d}$ is determined by

$$\hat{d}(f(\hat{x})\hat{d}\hat{x}^{i_1}\hat{d}\hat{x}^{i_2}\cdots\hat{d}\hat{x}^{i_r}) = \hat{d}(f(\hat{x}))\hat{d}\hat{x}^{i_1}\hat{d}\hat{x}^{i_2}\cdots\hat{d}\hat{x}^{i_r}.$$

To construct a differential calculus on the algebra $\mathcal{A}$ (3.1) one starts with a first oder differential calculus, that means one restricts to the 1-forms and the differential

$$\hat{d} : \mathcal{A} \to \Omega^1(\mathcal{A}).$$

The Leibniz rule (3.2) and the relations $\mathcal{R}$ of the algebra have to be consistent with the bimodule structure of $\Omega^1(\mathcal{A})$. In the following all relations will be given in terms of commutators $[\hat{x}^i, \hat{x}^j] = c^{ij}(\hat{x})$, therefore

$$[\hat{d}\hat{x}^i, \hat{x}^j] + [\hat{x}^i, \hat{d}\hat{x}^j] = \hat{d}c^{ij}(\hat{x}).$$

For the higher order differential calculus one has to go on in the same way. The relations of the bimodule structure again have to be consistent with $\hat{d}^2 = 0$ and the Leibniz rule.

3.2 Commuting frame formalism

Surely in commutative differential geometry one is not forced to use the partial derivatives of the coordinates as basis for the space of derivations. One can also use a comoving frame

$$e_a = e_a{}^i\partial_i, \qquad (3.3)$$

where $e_a{}^i$ is an invertible matrix. Here $a = 1\cdots N$ is an index numbering the derivations of the frame. The dual frame is therefore $(e_\nu{}^a e_a{}^\mu = \delta^\mu_\nu)$

$$\theta^a = e^a{}_\mu(x)dx^\mu.$$

The differential can be written only with this new basis elements

$$df = \theta^a e_a(f).$$

These formulas all have global extensions to the whole manifold. To go on we can restrict ourselves to special differential calculi related to derivations of the algebra. The set of all derivations on the algebra is not any more a module. But we can take a special set of linear independent derivations $\hat{e}_a$ and introduce a first order differential calculus in the following way. The space of one forms should be a bimodule over the algbra generated by $\hat{\theta}^a$ and the differential is defined by

$$\hat{d}\hat{f} = \hat{\theta}^a \hat{e}_a \hat{f}.$$

The components of the frame may be defined by

$$\hat{e}_a \hat{x}^\alpha = \hat{e}_a{}^\alpha.$$

Since the $\hat{e}_a$ are derivations it is consistend to let the $\hat{\theta}^a$ commute with all generators of the algebra

$$\hat{x}^a \hat{\theta}^b = \hat{\theta}^b \hat{x}^a$$

The $\hat{\theta}^a$ form a commuting frame for the algebra. The differential $\hat{d}$ and the forms $\hat{\theta}^a$ constitute a first order differential calculus on the algebra. To construct an analog to the exterior algebra a higher order calculus is necessary. As we have seen relations for the $\hat{\theta}^a$ among themselves and $\hat{d}\hat{\theta}^a$in terms of two forms have to be given in a consistent way.

A very important structure for physical applications is a metric on the manifolds which turns it into a (pseudo-) Riemannian manifold. It can be shown that there always exists a dual frame

$$\theta^a = e^a{}_\mu(x) dx^\mu$$

for which the metric is constant

$$g_{\mu\nu} dx^\mu dx^\nu = \eta_{ab} \theta^a \theta^b = \eta_{ab} e^a{}_\mu e^b{}_\nu dx^\mu dx^\nu.$$

Note that there are many frames resulting in the same metric. If $M^a{}_b(x)$ is a local $SO(n)$ gauge transformation the metric stays the same if we use the transformed frame

$$\theta'^a = M^a{}_b(x) \theta^b. \tag{3.4}$$

With the above construction it is very easy to generalize this to the noncommutative case. We simply assume that the frame is always adapted to the metric

$$\hat{g} = \eta_{ab}\hat{\theta}^a\hat{\theta}^b.$$

If the derivations are all inner derivations

$$\hat{e}_a\hat{f} = [\hat{\lambda}_a, \hat{f}], \tag{3.5}$$

the algebra has to have a trivial center, if the module of one forms should have the same number of generators as in the commutative case. Otherwise one is not able to find enough linear independent derivations. We will call the $\hat{\lambda}_a$ "momentum maps". The components of the frame are now commutators

$$\hat{e}_a\hat{x}^\alpha = [\hat{\lambda}_a, \hat{x}^\alpha] = \hat{e}_a{}^\alpha$$

and the differential may be written as a commutator with a one form

$$d\hat{f} = \hat{\theta}^a\hat{e}_a\hat{f} = [\hat{\theta}^a\hat{\lambda}_a, \hat{f}].$$

We will call $\hat{\theta} = \hat{\theta}^a\hat{\lambda}_a$ the Dirac operator of the differential calculus. For a Dirac operator in the sense of [2] more conditions have to be fulfilled. It is not clear how to generalize the notion of a local frame transformation (3.4), since after that the frame will not commute any more with functions.

3.2.1 Semiclassical limit of $\star$-product representations

Here we assume that the noncommutative frame consists of inner derivations (3.5). If we have represented the algebra with a $\star$-product then to first order the algebra relations define a Poisson structure

$$[x^\alpha \overset{\star}{,} x^\beta] = h\{x^\alpha, x^\beta\} + \cdots = h\Pi^{\alpha\beta}(x) + \cdots.$$

Further there are functions λ_a that correspond to the momentum maps of the algebra. We now have

$$\{\lambda_a, f\} = e_a{}^\mu\partial_\mu f$$

and we can identify the functions $e_a{}^\mu$ with the coframe of the first section. In the semiclassical limit there is direct correspondence between a frame and the momentum maps.

On the other hand one can ask the question if it is possible to construct a Poisson structure and momentum maps that reproduce with the above formalism a given frame. We know that

$$\{\lambda_a, x^\mu\} = \Pi^{\alpha\mu}\partial_\alpha\lambda_a = e_a{}^\mu \tag{3.6}$$

has to be fullfilled. If we introduce the closed symplectic form $\omega = \Pi^{-1}$ we can translate the last equation into

$$\omega_{\alpha\beta} = -(\partial_\alpha \lambda_a) e^a{}_\beta.$$

From this we derive two equations that have to be fullfilled for λ_a and the frame e^a. ω has to be antisymmetric and closed

$$S_{\alpha\beta} = (\partial_\alpha \lambda_a) e^a{}_\beta + (\partial_\beta \lambda_a) e^a{}_\alpha = \omega_{\beta\alpha} + \omega_{\alpha\beta} = 0,$$

$$d\omega = 0.$$

Since the algebra has trivial center it is neccessary that the dimension of our space is even-dimensional $N = 2M$. The equation $S = 0$ has $\frac{1}{2}N(N+1)$ and $d\omega = 0$ has $\binom{N}{3} = \frac{1}{6}N(N-1)(N-2)$ components. Even in two dimensions these are 3 partial differential equations for the 2 functions λ_a . We see that in higher dimensions it will become very difficult to find a frame in which the above system of equations may be solved. Further we are free to make local frame transformations and coordinate transformations on our classical manifold and there are no hints which frame to use for quantizing the geometry.

3.2.2 The flat metric

First suppose we want to apply the formalism to the frame $\theta^a = \delta^a{}_\alpha dx^\alpha$. Then for $S = 0$ we get

$$\partial_\alpha \lambda_\beta + \partial_\beta \lambda_\alpha = 0.$$

After some calculations one finds that

$$\lambda_\alpha = c_{\alpha\beta} x^\beta + \delta_\alpha$$

is the most general solution. c is a constant antisymmetric matrix and δ_α are some constants. For the inverse of Π this yields

$$\omega_{\alpha\beta} = c_{\alpha\beta},$$

which is clearly a closed form. We have reproduced the formalism with constant invertible Poisson tensor.

Secondly we want to investigate the case of a holonomic frame $\theta^a = \partial_\alpha f^a dx^\alpha$. After a coordinate transformation one sees immediatly that now

$$\lambda_a = c_{ab} f^a + \delta_a$$

with c and δ again constant. Therefore

$$\omega_{\alpha\beta} = \partial_\alpha f^a \partial_\beta f^b c_{ab},$$

which is again closed. The first order formalism is invariant under coordinate transformations, which we could have seen from the definitions of S and ω, too.

3.2.3 Two dimensional examples

In the following we will apply the formalism to some two dimensional examples. We will see that it works quite well in this case since the equation $d\omega = 0$ is fulfilled for all two forms in two dimensions.

Sphere

The metric of the sphere in polar coordinates is

$$ds^2 = d\vartheta^2 + \sin^2\vartheta d\varphi^2.$$

The most obvious frame is

$$\theta^1 = d\vartheta, \qquad \theta^2 = \sin\vartheta d\varphi.$$

$S = 0$ yields

$$\begin{aligned} \partial_\vartheta\lambda_1 + \partial_\vartheta\lambda_1 &= 0, \\ \partial_\vartheta\lambda_2 \sin\vartheta + \partial_\varphi\lambda_1 &= 0, \\ \partial_\varphi\lambda_2 + \partial_\varphi\lambda_2 &= 0. \end{aligned}$$

Therefore

$$\lambda_1 = -\frac{1}{h}\varphi + \delta_1, \qquad \partial_\vartheta\lambda_2(\vartheta) = \frac{1}{h\sin\vartheta}$$

and ω may be calculated

$$\omega_{\vartheta\varphi} = \frac{1}{h}.$$

In two dimensions every two form is closed and therefore

$$\{\vartheta, \varphi\} = h$$

fulfills the Jacobi-identities. An algebra having this Poisson structure is the Heisenberg algebra $[\hat{\vartheta}, \hat{\varphi}] = h$ in two dimensions. Since the second momentum map λ_2 is not a polynomial in the algebra generators a quantization of this momentum map seems to be very unnatural.

Constant curvature

It is known that all two dimensional spaces with constant curvature can be written in the following form (see e. g. [6])

$$ds^2 = f^2(u,v)(du^2 + dv^2)$$

with

$$\begin{aligned} f &= \frac{1}{1+u^2+v^2} && \text{sphere,} \\ f &= \frac{1}{1-u^2-v^2} && \text{Poincare disk,} \\ f &= \frac{1}{v} && \text{Lobachewski plane.} \end{aligned}$$

For the sphere this are stereographic coordinates. We use the frame

$$\theta^1 = f du, \qquad \theta^2 = f dv.$$

$S = 0$ yields

$$\lambda_1 = -\frac{1}{h}v + \delta_1, \qquad \lambda_2 = \frac{1}{h}u + \delta_2$$

and we can calculate

$$\{u, v\} = \frac{h}{f}.$$

This Poisson bracket easily may be generalized to algebra relations. All the momentum maps are linear in the coordinates. Therefore they correspond to the algebra generators, no ordering ambiguity is present. For the sphere in stereographic coordinates we get

$$[\hat{u}, \hat{v}] = h(1 + \hat{u}^2 + \hat{v}^2).$$

Since we have started from the stereographic projection of the sphere the resulting algebra for the sphere makes no reference to the different topology. The resulting algebra is a noncommutative sphere with a hole at the south pole and in this sense a noncommutative plane with a non constant metric. Similar we get for the Poincare disk

$$\{\hat{u}, \hat{v}\} = h(1 - \hat{u}^2 - \hat{v}^2)$$

and for the Lobachewski plane

$$\{\hat{u}, \hat{v}\} = h\hat{v}.$$

The resulting noncommutative Lobacheweski plane is known to the literature [72].

Metric with one translational symmetry

We start with the rather general ansatz

$$ds^2 = \pm e^{2\psi(r)}dt^2 + e^{2\phi(r)}dr^2,$$

which is invariant under translations $t \to t + c$ in the t direction. We use the frame

$$\theta^t = e^{\psi}dt, \qquad \theta^r = e^{\phi}dr.$$

$S = 0$ yields

$$\lambda_r = \frac{1}{h}t, \qquad \partial_r \lambda_t(r) = -\frac{1}{h}e^{\phi-\psi}$$

and the Poisson structure becomes

$$\{t, r\} = he^{-\phi}.$$

Two dimensional Schwarzschild

We specialise now to the case of the t-r-slice through the Schwarzschild metric. Here

$$e^{2\psi} = 1 - \frac{r_0}{r}, \qquad e^{2\phi} = \frac{1}{1 - \frac{r_0}{r}}$$

and we get

$$\{t, r\} = h\sqrt{1 - \frac{r_0}{r}}. \tag{3.7}$$

This is well defined if we restrict the manifold to $t \in \mathbb{R}$ and $r \geq r_0$. In the limit $r \to \infty$ this Poisson structure tends to the constant one. The momentum maps are

$$\lambda_r = \frac{1}{h}t,$$

$$\lambda_t = -\frac{1}{h}(1 + r_0 \ln(r - r_0)).$$

We can write down an algebra which has this Poisson bracket as semiclassical limit

$$[\hat{t}, \hat{r}] = h\sqrt{1 - \frac{r_o}{\hat{r}}}, \tag{3.8}$$

where the square root is considered as a Taylor series in $\hat{r}$. This algebra may be represented with a $\star$-product constructed out of the Poisson bracket (3.7). We will use the resulting algebra in (3.3.3) to construct a noncommutative frame for the four dimensional Schwarzschild metric.

Higher order differential calculus

We want to construct a higher order differential calculus for the algebra (3.8) for

$$\hat{d}\hat{f} = \hat{\theta}^r[\hat{\lambda}_r, \hat{f}] + \hat{\theta}^t[\lambda_t, \hat{f}]$$

with

$$\hat{\lambda}_r = \frac{1}{h}\hat{t}, \qquad \hat{\lambda}_t = -\frac{1}{h}(1 + r_0 \ln(\hat{r} - r_0)).$$

First we calculate

$$\hat{d}\hat{t} = \hat{\theta}^t \frac{1}{\sqrt{1 - \frac{r_0}{\hat{r}}}}, \qquad \hat{d}\hat{r} = \hat{\theta}^r \sqrt{1 - \frac{r_0}{\hat{r}}}.$$

With this we get ($\rho(\hat{r}) = \frac{r_0}{2\hat{r}^2}\frac{1}{\sqrt{1-\frac{r_0}{\hat{r}}}}$)

$$[\hat{t}, \hat{d}\hat{t}] = h\,\hat{d}\hat{t}\,\rho(\hat{r}), \qquad [\hat{r}, \hat{d}\hat{t}] = 0,$$
$$[\hat{t}, \hat{d}\hat{r}] = h\,\hat{d}\hat{r}\,\rho(\hat{r}), \qquad [\hat{r}, \hat{d}\hat{r}] = 0.$$

From this we derive

$$2\hat{d}\hat{t}\,\hat{d}\hat{t} = 4\hat{d}\hat{t}\,\hat{d}\hat{r}\,\rho'(\hat{r}),$$
$$\hat{d}\hat{r}\,\hat{d}\hat{t} + \hat{d}\hat{t}\,\hat{d}\hat{r} = 0,$$
$$\hat{d}\hat{r}\,\hat{d}\hat{t} + \hat{d}\hat{t}\,\hat{d}\hat{r} = \hat{d}\hat{r}\,\hat{d}\hat{r}\,\rho(\hat{r}),$$
$$\hat{d}\hat{r}\,\hat{d}\hat{r} = 0.$$

The first condition implies

$$0 = [\hat{t}, \hat{d}\hat{t}\,\hat{d}\hat{t}] = 4\hat{d}\hat{t}\,\hat{d}\hat{r}\,[\hat{t}, \rho'(\hat{r})].$$

Therefore

$$\hat{d}\hat{t}\,\hat{d}\hat{r} = 0,$$

which has not the desired classical limit.

3.2.4 Metrics in four dimensions

Although the formalism works well in two dimensions we will see that this is not the case in four dimensions. We tried to solve the system of partial differential equations for diverse physically interesting frames but we never were really successful.

Schwarzschild metric

The best known form of the Schwarzshild metric is

$$ds^2 = -(1 - \frac{r_0}{r})dt^2 + \frac{1}{1 - \frac{r_0}{r}}dr^2 + r^2 d\Omega^2.$$

Here the most obvious frame is

$$\theta^t = e^{\psi} dt, \qquad \theta^r = e^{-\psi} dt,$$
$$\theta^{\vartheta} = d\vartheta, \qquad \theta^{\varphi} = \sin\vartheta d\varphi,$$

with $e^{\psi} = \sqrt{1 - \frac{r_o}{r}}$. Here the $S = 0$ equations have no solution for arbitrary ψ except $\psi = 0$. In another coordinate sytem the Schwarzschild metric becomes [73]

$$ds^2 = -dt^2 + (dx^i - x^i \sqrt{\frac{2m}{r^3}} dt)^2.$$

A more general frame ist

$$\theta^0 = dt, \qquad \theta^i = dx^i - x^i f(r) dt,$$

with $f = \sqrt{\frac{2m}{r^3}}$. For general f again the $S = 0$ equations imply $f = 0$.

Kasner metric

One form of the Kasner metric is [73]

$$ds^2 = -dt^2 + (dx^i - p^i_j \frac{x^j}{t} dt)^2.$$

To be more flexible we start with the following frame

$$\theta^0 = dt, \qquad \theta^i = dx^i + P^i{}_j(t) x^j dt.$$

The $S = 0$ equations become

$$\begin{aligned} \partial_0 \lambda_0 + \partial_0 \lambda_i P^i{}_j x^j &= 0, \\ \partial_0 \lambda_i + \partial_i \lambda_l P^l{}_m x^m + \partial_i \lambda_0 &= 0, \\ \partial_i \lambda_j + \partial_j \lambda_i &= 0. \end{aligned}$$

From the last equation we deduce that

$$\lambda_i = c_{jk}(t) x^k + \delta_i(t).$$

We make the ansatz

$$\lambda_x = -\alpha(t)y \quad \lambda_y = \alpha(t)x \quad \lambda_z = \beta(t),$$
$$P(t) = \begin{pmatrix} p(t) & 0 & 0 \\ & p(t) & 0 \\ & & q(t) \end{pmatrix}$$

and the $S = 0$ equations reduce to

$$\dot{\alpha} = p\alpha, \qquad \ddot{\beta} = -q\dot{\beta},$$

with

$$\lambda_0 = -\dot{\beta}z + \gamma.$$

The two form ω becomes

$$\begin{aligned} \omega &= -\dot{\alpha}(ydx - xdy) \wedge dt - \alpha dx \wedge dy + \dot{\beta}dz \wedge dt \\ &= (xdy - ydx) \wedge d\alpha - \alpha dx \wedge dy + dz \wedge d\beta. \end{aligned}$$

It is not closed. To cure this problem we make a slight modification

$$\omega = (axdy - (1-a)ydx) \wedge \alpha - \alpha dx \wedge dy + dz \wedge d\beta$$

with a some constant. We get now

$$\{z, x\} = a\frac{\dot{\alpha}}{\alpha\dot{\beta}}x, \qquad \{z, t\} = -\frac{1}{\dot{\beta}},$$
$$\{z, y\} = a\frac{\dot{\alpha}}{\alpha\dot{\beta}}y, \qquad \{x, y\} = \frac{1}{\alpha}.$$

If we use the definitions for the λ_a from above, we can calculate the coframe

$$\{\lambda_a, x^\alpha\} = \begin{pmatrix} 1 & -a\frac{\dot{\alpha}}{\alpha}x & -(1-a)\frac{\dot{\alpha}}{\alpha}y & -\frac{\ddot{\beta}}{\dot{\beta}}z \\ 0 & 1 & 0 & -a\frac{\dot{\alpha}}{\dot{\beta}}y \\ 0 & 0 & 1 & (1-a)\frac{\dot{\alpha}}{\dot{\beta}}x \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The frame becomes

$$\begin{aligned} \theta^0 &= dt, \\ \theta^x &= dx + a\frac{\dot{\alpha}}{\alpha}x\,dt, \\ \theta^y &= dy + (1-a)\frac{\dot{\alpha}}{\alpha}y\,dt, \\ \theta^z &= dz + (\frac{\ddot{\beta}}{\dot{\beta}} - (a^2 - (1-a)^2)\frac{\dot{\alpha}^2}{\alpha\dot{\beta}}xy)dt \\ &\quad + \frac{\dot{\alpha}}{\dot{\beta}}(aydx - (1-a)xdy. \end{aligned}$$

We see that with the commuting frame formalism it is impossible to construct a frame, becoming in the classical limit the above given frame of the Kasner metric.

In [74] a noncommutative version of the Kasner algebra was constructed using the commuting frame formalism we started from. To relate our example to the one there we now further assume

$$a\frac{\dot\alpha}{\alpha} = \frac{p_1}{t}, \qquad (1-a)\frac{\dot\alpha}{\alpha} = \frac{p_2}{t}.$$

Therefore

$$\frac{p_1}{a} = \frac{p_2}{1-a} \qquad \alpha = t^{\frac{p_1}{a}}.$$

Further we set

$$\frac{\ddot\beta}{\dot\beta} = \frac{p_3}{t},$$

so

$$\dot\beta = t^{p_3}.$$

Now

$$\frac{\dot\alpha}{\dot\beta} = \frac{p_1}{a} t^{\frac{p_1}{a}-p_3-1}$$

and the frame becomes

$$\begin{aligned}
\theta^0 &= dt, \\
\theta^x &= dx + p_1\frac{x}{t}dt, \\
\theta^y &= dy + p_1\frac{1-a}{a}\frac{y}{t}dt, \\
\theta^z &= dz + p_3\frac{z}{t}dt + (a^2-(1-a)^2)\frac{p_1^2}{a^2}t^{\frac{p_1}{a}-p_3}xydt \\
&\quad + \frac{p_1}{a}t^{\frac{p_1}{a}-p_3-1}(aydx-(1-a)xdy).
\end{aligned}$$

The last term of θ^z has the same order of magnitude as the ordinary deviation from the flat metric in θ^x and θ^y.

With this solution the commutation relations become

$$\{z,x\} = p_1 t^{-p_3-1}x, \qquad \{z,t\} = t^{-p_3},$$
$$\{z,y\} = \frac{1-a}{a}p_1 t^{-p_3-1}y, \qquad \{x,y\} = t^{-\frac{p_1}{a}}.$$

If one sets the parameters p_1 and p_3 to zero the above relations become constant commutator relations between the coordinates. If we let t go to infinity we can fit the parameters p_1,p_3 and a in such a way that all relations vanish.

3.3 $SO(3) \times T$ invariant Poisson structures and algebras

In this section we try to construct algebras having the same symmetries as the Schwarzschild metric. Meaning invariance under rotations and time translations. For this we first construct non-degenerate Poisson structures with these properties. Since we know that every Poisson structure may be quantized by a $\star$-product we are able to write down all possible algebras with trivial center. We will see that these are quite unique. With the help of one of these algebras, we propose a non-commuting frame, that becomes in the classical limit a frame for the Schwarzschild metric.

3.3.1 The Poisson structures

We start with following Ansatz

$$\begin{aligned} \{x^i, x^j\} &= \beta(r,t)\epsilon^{ij}{}_k x^k, \\ \{t, x^i\} &= \alpha(r,t)x^i. \end{aligned}$$

where $i = x, y, z$ and $r = \sqrt{x^2+y^2+z^2}$. These equations are obviously covariant under rotations. It would be invariant under translations in the t direction if α and β do not depend on t, but we keep them in this form to be more general. The bracket with a function f is

$$\begin{aligned} \{x^i, f\} &= -\alpha x^i \partial_t f + \beta \epsilon^{ij}{}_k \partial_j f, \\ \{t, f\} &= \alpha x^i \partial_i f. \end{aligned}$$

With this the Jacobi identities are

$$\begin{aligned} \{x^i, \{x^j, x^k\}\} + \text{cyc.} &= -\alpha \partial_t \beta \epsilon^{ijk} r^2, \\ \{t, \{x^j, x^k\}\} + \text{cyc.} &= \alpha(r\partial_r \beta - \beta)\epsilon^{ij}{}_k x^k. \end{aligned}$$

The brackets become a Poisson structure if the right hand side of the above equations vanishes. This is the case if either

$$\alpha = 0, \qquad \beta = \beta(r,t)$$

or

$$\alpha = \alpha(r,t), \qquad \beta = br,$$

where b is a constant. In the first case t commutes with all functions. Only in the second case, there is the possibility for all derivations to be inner derivations. Therefore we will later restrict us to this case.

3.3.2 Algebras and isomorphisms

After quantization the Poisson structures become algebras. Note that in both cases

$$\{r, x^i\} = 0$$

and there is no ordering problem on the right hand side of the algebra relations if α and β do not depend on t. We will assume from now on that β does not depend on $\hat{t}$. The first case is now

$$[\hat{x}^i, \hat{x}^j] = \beta(\hat{r})\epsilon^{ij}{}_k\hat{x}^k, \qquad [\hat{t}, \hat{x}^i] = 0$$

with $\beta \neq b\hat{r}$. If we define $\hat{s}^i = \beta^{-1}(\hat{r})\hat{x}^i$ these relations become

$$[\hat{s}^i, \hat{s}^j] = \epsilon^{ij}{}_k\hat{s}^k, \qquad [x^0, \hat{s}^i] = 0.$$

Therefore this algebra is isomorphic to $U(su(2)) \times \mathbb{C}$.

The second case is

$$[\hat{x}^i, \hat{x}^j] = b\hat{r}\epsilon^{ij}{}_k\hat{x}^k, \qquad [\hat{t}, \hat{x}^i] = \alpha(\hat{r}, \hat{t}).$$

If we again define $\hat{s}^i = \hat{r}^{-1}\hat{x}^i$ we get the constraint $\hat{s}^i\hat{s}_i = 1$. The relations become

$$\begin{aligned} [\hat{s}^i, \hat{s}^j] &= b\epsilon^{ij}{}_k\hat{s}^k, \qquad [\hat{s}^i, \hat{r}] = 0, \\ [\hat{t}, \hat{s}^i] &= 0, \qquad [\hat{t}, \hat{r}] = \alpha(\hat{r}, \hat{t})\hat{r}. \end{aligned}$$

This algebra is isomorphic to $S_b \times \mathbb{C}^2$ for $\alpha = 0$ or $S_b \times A$ otherwise. For $b = \sqrt{\frac{4}{N^2-1}}$ and N an integer S_b is a fuzzy sphere with deformation parameter b and has finite dimensional representations. A can be any two dimensional algebra.

3.3.3 Rotational invariant metrics with a minimal non-commuting frame

In the last section we have seen that the second algebra decomposes into a three dimensional rotational covariant algebra and a two dimensional algebra for which we can now use the algebra from Section 3.2.3. The relations become now ($\hat{x}^i = \hat{r}\hat{s}^i$, $\delta_{ij}\hat{s}^i\hat{s}^j = 1$)

$$\begin{aligned} [\hat{s}^i, \hat{s}^j] &= b\epsilon^{ij}{}_k\hat{s}^k, \qquad [\hat{s}^i, \hat{r}] = 0, \\ [\hat{t}, \hat{s}^i] &= 0, \qquad [\hat{t}, \hat{r}] = he^{-\phi(\hat{r})}. \end{aligned}$$

We now use five inner derivations to define a frame

$$\begin{aligned}\hat{e}^0_i &= [\frac{\hat{s}^i}{b}, \cdot] \to -\epsilon_{ij}{}^k x^j \partial_k,\\ \hat{e}_r &= [\hat{\lambda}_r, \cdot] = [\frac{\hat{t}}{h}, \cdot] \to e^{-\phi}\partial_r,\\ \hat{e}_t &= [\hat{\lambda}_t, \cdot] \to e^{-\psi}\partial_t.\end{aligned}$$

The arrows indicate the classical limit. $\hat{\lambda}_t$ is defined in the classical limit via

$$\partial_r \lambda_t(r) = -\frac{1}{h} e^{\phi-\psi}.$$

In the classical limit $x^i e_i = 0$ and the derivations are linearly dependent. The dual one forms to the $\hat{e}_i$ are well known, they form the differential calculus on the fuzzy sphere. The dual forms of $\hat{e}_r$ and $\hat{e}_t$ are easily constructed. The classical limit of these forms is

$$\begin{aligned}\hat{\theta}^i_0 &\to -\frac{1}{r^2}\epsilon^i{}_{jk} x^j dx^k,\\ \hat{\theta}^r &\to e^{\phi} dr,\\ \hat{\theta}^t &\to e^{\psi} dt.\end{aligned}$$

We know

$$\delta_{ij}\hat{\theta}^i_0\hat{\theta}^j_0 \to d\Omega^2.$$

We define now some forms that do not commute with functions

$$\hat{\theta}^i = \hat{r}\hat{\theta}^i_0,$$

$$\hat{f}\hat{\theta}^i = \hat{\theta}^i \hat{r}^{-1}\hat{f}\hat{r}.$$

Only $\hat{t}$ does not commute with $\hat{\theta}^i$. With these forms we can now construct a noncommutative version of the four dimensional metric

$$ds^2 = -(\hat{\theta}^t)^2 + (\hat{\theta}^r)^2 + \delta_{ij}\hat{\theta}^i\hat{\theta}^j \to -e^{2\psi}dt^2 + e^{2\phi}dr^2 + r^2 d\Omega^2.$$

Note that dual to the $\hat{\theta}^i$ are the following deformed derivations

$$\hat{e}_i = \hat{r}^{-1}[\frac{\hat{s}^i}{b}, \cdot] \to \frac{1}{r}\epsilon_{ij}{}^k x^j \partial_k$$

with

$$\hat{e}_i(\hat{f}\cdot\hat{g}) = \hat{e}_i\hat{f}\cdot\hat{g} + \hat{r}^{-1}\hat{f}\hat{r}\cdot\hat{e}_i\hat{g}.$$

The inner isomorphism defined by $\hat{r}$ is

$$\hat{r}^{-1}\hat{t}\hat{r} = \hat{t} + h\frac{e^{-\phi(\hat{r})}}{\hat{r}}.$$

In the case of the Schwarzschild metric this becomes

$$\hat{r}^{-1}\hat{t}\hat{r} = \hat{t} + \frac{h}{\hat{r}}\sqrt{1 - \frac{r_o}{\hat{r}}}.$$

Again this only makes sense if the spectrum of $\hat{r}$ has no values smaller than r_o. In the limit $r \to \infty$ this tends to the identity.

If we define the one form

$$\hat{\theta} = \hat{x}^i\hat{\theta}^i + \hat{\lambda}_r\hat{\theta}^r + \hat{\lambda}_t\hat{\theta}^t$$

it follows that $(a = r, t)$

$$[\hat{\theta}, \hat{f}] = \hat{\theta}^i\hat{e}_i\hat{f} + \hat{\theta}^a\hat{e}_a\hat{f} = \hat{d}\hat{f}$$

$\hat{\theta}$ is the Dirac operator of the differential calculus.

It would be nice to have a higher order differential calculus for the first order one. To construct this we note that

$$\begin{aligned}
[\hat{e}_i, \hat{e}_j] &= \frac{1}{\hat{r}^2}\epsilon_{ij}{}^k\hat{e}_k, \\
[\hat{e}_t, \hat{e}_i] &= 0, \\
[\hat{e}_r, \hat{e}_i] &= (\hat{e}_r\frac{1}{\hat{r}})\hat{r}\hat{e}_i = -\frac{1}{\hat{r}}\hat{e}_i, \\
[\hat{e}_r, \hat{e}_t] &= -\frac{1}{h}[e^{-\psi(\hat{r})}, \cdot].
\end{aligned}$$

It is consistent to define

$$\begin{aligned}
\hat{\theta}^i\hat{\theta}^j &= -\hat{\theta}^j\hat{\theta}^i, \\
\hat{\theta}^a\hat{\theta}^i &= -\hat{\theta}^i\hat{\theta}^a
\end{aligned}$$

and

$$\begin{aligned}
\hat{d}\hat{\theta}^k &= \frac{1}{2\hat{r}^2}\epsilon_{ij}{}^k\hat{\theta}^i\hat{\theta}^j + \frac{1}{\hat{r}}\hat{\theta}^r\hat{\theta}^k, \\
\hat{d}\hat{\theta}^r &= 0.
\end{aligned}$$

The claim $\hat{d}^2 = 0$ reduces to

$$d\hat{\theta}^t\hat{e}_t\hat{f} - \hat{\theta}^a\hat{\theta}^b\hat{e}_b\hat{e}_a\hat{f} = 0.$$

In (1.5.4) we have shown that this implies $(\hat{\theta}^r)^2 = (\hat{\theta}^t)^2 = \hat{\theta}^r\hat{\theta}^t = \hat{\theta}^t\hat{\theta}^r = \hat{d}\hat{\theta}^t = 0$. Again we are not able to extend the first order differential calculus to a differnential calculus of higher order.

Chapter 4

Derivations of $\star$-products

We have seen that if we want to make noncommutative geometry in the $\star$-product formulation we have been very successful interpreting derivations as a noncommutative analog of frames. In the last chapter we used invertible Poisson structures, the resulting algebras therefore had trivial center and all derivations have been inner derivations. To be more general we now relax our restrictions and take degenerated Poisson structures in consideration. Consequently the algebras will have central elements. This is due to the fact that in the classical limit all vector fields formed from the inner derivations will commute with the functions associated to these central elements. We are forced to deal with outer derivations and in this chapter we first will examine derivations of $\star$-products without referring to any abstract algebraic construction.

In the beginning we will introduce Kontsevich's Formality map [38] to make some statements about derivations on quantized Poisson manifolds. Then we will calculate the general form of derivations on the Weyl ordered $\star$-product. Knowing the restrictions and the form of the derivations we are able to construct an interesting differential calculus on the $\star$-product algebra. This will be used later to make contact with Seiberg-Witten gauge theory. After some considerations how to construct traces for $\star$-product algebras we are able to write down consistent actions for noncommutative theories becoming non abelian gauge theories on curved manifolds in the classical limit.

4.1 The Formality Map

Kontsevich's Formality map [38] is a very useful tool for studying the relations between Poisson tensors and $\star$-products. To make use of the Formality map we first want to recall some definitions. A polyvector field is a skew-symmetric tensor in the sense of differential geometry. Every n-polyvector field α may locally be written as

$$\alpha = \alpha^{i_1 \dots i_n} \, \partial_{i_1} \wedge \dots \wedge \partial_{i_n}.$$

We see that the space of polyvector fields can be endowed with a grading n. For polyvector fields there is a grading respecting bracket that in a natural way generalizes the Lie bracket $[\cdot,\cdot]_L$ of two vector fields, the Schouten-Nijenhuis bracket (see A.1). If π is a Poisson tensor, the Hamiltonian vector field H_f for a function f is

$$H_f = [\pi, f]_S = -\pi^{ij} \partial_i f \partial_j.$$

Note that $[\pi,\pi]_S = 0$ is the Jakobi identity of a Poisson tensor.

On the other hand a n-polydifferential operator is a multilinear map that maps n functions to a function. For example, we may write a 1-polydifferential operator D as

$$D(f) = D_0 f + D_1^i \partial_i f + D_2^{ij} \partial_i \partial_j f + \dots .$$

The ordinary multiplication $\cdot$ is a 2-polydifferential operator. It maps two functions to one function. Again the number n is a grading on the space of polydifferential operators. Now the Gerstenhaber bracket (see A.2) is natural and respects the grading.

The Formality map is a collection of skew-symmetric multilinear maps U_n, $n = 0, 1, \dots$, that maps n polyvector fields to a m-differential operator. To be more specific let $\alpha_1, \dots, \alpha_n$ be polyvector fields of grade $k_1, \dots, k_n$. Then $U_n(\alpha_1, \dots, \alpha_n)$ is a polydifferential operator of grade

$$m = 2 - 2n + \sum_i k_i.$$

In particular the map U_1 is a map from a k-vectorfield to a k-differential operator. It is defined by

$$U_1(\alpha^{i_1 \dots i_n} \partial_{i_1} \wedge \dots \wedge \partial_{i_n})(f_1, \dots, f_n) = \alpha^{i_1 \dots i_n} \partial_{i_1} f_1 \cdot \dots \cdot \partial_{i_n} f_n.$$

The formality maps U_n fulfill the formality condition [38, 75]

$$Q_1' U_n(\alpha_1, \dots, \alpha_n) + \frac{1}{2} \sum_{\substack{I \sqcup J = \{1,\dots,n\} \\ I,J \neq \emptyset}} \epsilon(I,J) Q_2'(U_{|I|}(\alpha_I), U_{|J|}(\alpha_J)) \qquad (4.1)$$

$$= \frac{1}{2}\sum_{i\neq j}\epsilon(i,j,\ldots,\hat{i},\ldots,\hat{j},\ldots,n)U_{n-1}(Q_2(\alpha_i,\alpha_j),\alpha_1,\ldots,\widehat{\alpha}_i,\ldots,\widehat{\alpha}_j,\ldots,\alpha_n).$$

The hats stand for omitted symbols, $Q'_1(\Upsilon) = [\Upsilon,\mu]$ with μ being ordinary multiplication and $Q'_2(\Upsilon_1,\Upsilon_2) = (-1)^{(|\Upsilon_1|-1)|\Upsilon_2|}[\Upsilon_1,\Upsilon_2]_G$ with $|\Upsilon_s|$ being the degree of the polydifferential operator Υ_s, i.e. the number of functions it is acting on. For polyvectorfields $\alpha_s^{i_1\ldots i_{k_s}}\partial_{i_1}\wedge\ldots\wedge\partial_{i_{k_s}}$ of degree k_s we have $Q_2(\alpha_1,\alpha_2) = -(-1)^{(k_1-1)k_2}[\alpha_2,\alpha_1]_S$.

For a bivectorfield π we can now define a bidifferential operator

$$\star = \sum_{n=0}^{\infty}\frac{1}{n!}U_n(\pi,\ \ldots,\pi)$$

i.e.

$$f\star g = \sum_{n=0}^{\infty}\frac{1}{n!}U_n(\pi,\ \ldots,\pi)(f,g).$$

We further define the special polydifferential operators

$$\begin{aligned}\Phi(\alpha) &= \sum_{n=1}^{\infty}\frac{1}{(n-1)!}U_n(\alpha,\pi,\ \ldots,\pi),\\ \Psi(\alpha_1,\alpha_2) &= \sum_{n=2}^{\infty}\frac{1}{(n-2)!}U_n(\alpha_1,\alpha_2,\pi,\ldots,\pi).\end{aligned}$$

For g a function, X and Y vectorfields and π a bivectorfield we see that

$$\delta_X = \Phi(X)$$

is a 1-differential operator and that both $\phi(g)$ and $\Psi(X,Y)$ are functions.

We now use the formality condition (4.1) to calculate

$$[\star,\star]_G = \Phi([\pi,\pi]_S), \tag{4.2}$$

$$[\Phi(f),\star]_G = -\Phi([f,\pi]_S), \tag{4.3}$$

$$[\delta_X,\star]_G = \Phi([X,\pi]_S), \tag{4.4}$$

$$\begin{aligned}&[\delta_X,\delta_Y]_G + [\Psi(X,Y),\star]_G\\ &\quad = \delta_{[X,Y]_S} + \Psi([\theta,Y]_S,X) - \Psi([\theta,X]_S,Y),\end{aligned} \tag{4.5}$$

$$\begin{aligned}&[\Phi(\pi),\Phi(g)]_G + [\Psi(\pi,g),\star]_G\\ &\quad = -\delta_{[\pi,g]_S} - \Psi([\theta,g]_S,\pi) - \Psi([\theta,\pi]_S,g),\end{aligned}$$

$$\begin{aligned}&[\delta_X, \Phi(g)]_G \\ &= \phi([X,g]_S) - \Psi([\theta,g]_S, X) - \Psi([\theta, X]_S, g).\end{aligned} \qquad (4.6)$$

If π is Poisson, i. e. $[\pi,\pi]_S = 0$ and if X and Y are Poisson vector fields, i. e. $[X,\pi]_S = [Y,\pi]_S = 0$, the relations (4.2) to (4.5) become

$$\begin{aligned} f\star(g\star h) &= (f\star g)\star h, \\ \delta_{H_f}(g) &= -[\Phi(f) \stackrel{\star}{,} g], \\ \delta_X(f\star g) &= \delta_X(f)\star g + f\star\delta_X(g), \\ ([\delta_X,\delta_Y] - \delta_{[X,Y]_L})(g) &= [\Psi(X,Y) \stackrel{\star}{,} g]. \end{aligned} \qquad (4.7)$$

when evaluated on functions. $[\cdot,\cdot]$ are now ordinary brackets. $\star$ defines an associative product, the Hamiltonian vector fields are mapped to inner derivations and Poisson vector fields are mapped to outer derivations of the $\star$-product. Additionally the map δ preserves the bracket up to an inner derivation. The last equation can be cast into a form which we will use in the definition of deformed forms in (4.3)

$$[\delta_X, \delta_Y] = \delta_{[X,Y]_\star}$$

with

$$[X,Y]_\star = [X,Y]_L + H_{\Phi^{-1}\Psi(X,Y)}.$$

For every Poisson manifold there not only exists a quantization with the Formality $\star$-product, but additionally there is a deformation of the Lie bracket going with the derivations of this $\star$-product.

4.2 Weyl-ordered ⋆-products

The formality $\star$-product is the obvious choice if we start from a Poisson manifold and therefore if we only need a $\star$-product that reproduces the Poisson structure to first order. But starting from an algebra, we need a $\star$-product that reproduces the whole algebra, not just the Poisson structure. If we extract a Poisson structure from an algebra generated by noncommutative coordinates fulfilling certain commutation relations, there's no way of knowing if the formality $\star$-product built from this Poisson structure will reproduce the commutation relations or not. For this purpose the Weyl ordered $\star$-product (2.2.2) is more suitable. In the following we will calculate the derivations for this type of $\star$-product.

We have shown that for the Formality $\star$-product there exists a map δ from the derivations of the Poisson manifold $T_\pi M = \{X \in TM | [X,\pi]_S = 0\}$

to the derivations of the $\star$-product $T_\star M = \{\delta \in T_{poly} | [\delta, \star]_G = 0\}$. Since an arbitrary $\star$-product is equivalent to the Formality $\star$-product, we can expect that such a map exists for every $\star$-product. Here we state some facts, that we can say about such a map in general. For this we expand it on a local patch in terms of partial derivatives

$$\delta_X = \delta_X^i \partial_i + \delta_X^{ij} \partial_i \partial_j + \cdots .$$

Due to its property to be a derivation, it is now easy to see that δ_X is completely determined by the first term $\delta_X^i \partial_i$. This means that if the first term is zero, the other terms have to vanish, too. If further e is an arbitrary derivation of the $\star$-product there must exist a vector field X_e such that

$$\delta_{X_e} = e.$$

If $X, Y \in T_\pi M$, then $[\delta_X, \delta_Y]$ is again a derivation of the $\star$-product and we can conclude that

$$[\delta_X, \delta_Y] = \delta_{[X,Y]_\star}, \tag{4.8}$$

where $[X, Y]_\star$ is a deformation of the ordinary Lie bracket of vector fields. Obviously it is linear, antisymmetric and fulfills the Jacobi identity.

We will now calculate δ and $[\cdot, \cdot]_\star$ up to second order for the Weyl ordered $\star$-product. We assume that δ_X can be expanded in the following way

$$\delta_X = X^i \partial_i + \delta_X^{ij} \partial_i \partial_j + \delta_X^{ijk} \partial_i \partial_j \partial_k + \cdots .$$

Expanding the equation

$$\delta_X (f \star g) = \delta_X (f) \star g + f \star \delta_X (g)$$

order by order and using $[X, c]_S = 0$ we find that

$$\begin{aligned} \delta_X \;=\; & X^i \partial_i - \frac{1}{12} c^{lk} \partial_k c^{im} \partial_l \partial_m X^j \partial_i \partial_j \\ & + \frac{1}{24} c^{lk} c^{im} \partial_l \partial_i X^j \partial_k \partial_m \partial_j + \mathcal{O}(3). \end{aligned} \tag{4.9}$$

For $[\cdot, \cdot]_\star$ we simply calculate $[\delta_X, \delta_Y]$ and get

$$\begin{aligned} [X, Y]_\star \;=\; & [X, Y]_L \\ & - \frac{1}{12} (c^{lk} \partial_k c^{im} \partial_l \partial_m X^j \partial_i \partial_j Y^n - c^{lk} \partial_k c^{im} \partial_l \partial_m Y^j \partial_i \partial_j X^n) \partial_n \\ & + \frac{1}{24} (c^{lk} c^{im} \partial_l \partial_i X^j \partial_k \partial_m \partial_j Y^n - c^{lk} c^{im} \partial_l \partial_i Y^j \partial_k \partial_m \partial_j X^n) \partial_n \\ & + \mathcal{O}(3). \end{aligned}$$

4.3 Forms

Now we want to introduce noncommutative forms, which will later be used in Seiberg-Witten gauge theory (5.3). If we have a map δ from the Poisson vector fields to the derivations of the $\star$-product, we have seen that there is a natural Lie-algebra structure $[\cdot,\cdot]_\star$ (4.8) over the space of Poisson vector fields, the space of derivations of the Poisson structure. On the space of derivations of the $\star$-product we can easily construct the Chevalley cohomology. Further, again with the map δ, we can lift derivations of the Poisson structure to derivations of the $\star$-product. Therefore it should be possible to pull back the Chevalley cohomology from the space of derivations to vector fields. This will be done in the following.

A deformed k-form is defined to map k Poisson vector fields to a function and has to be skew-symmetric and linear over $\mathbb{C}$. This is a generalization of the undeformed case, where a form has to be linear over the algebra of functions. Functions are defined to be 0-forms. The space of forms $\Omega_\star M$ is now a $\star$-bimodule via

$$(f \star \omega \star g)(X_1, \ldots, X_k) = f \star \omega(X_1, \ldots, X_k) \star g. \tag{4.10}$$

As expected, the exterior differential is defined with the help of the map δ.

$$\begin{aligned}\delta\omega(X_0, \ldots, X_k) &= \sum_{i=0}^{k} (-1)^i\, \delta_{X_i}\omega(X_0, \ldots, \hat{X}_i, \ldots, X_k) \\ &+ \sum_{0\le i<j\le k} (-1)^{i+j}\omega([X_i, X_j]_\star, X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k).\end{aligned} \tag{4.11}$$

With the properties of δ and $[\cdot,\cdot]_\star$ it follows that

$$\delta^2\omega = 0.$$

To be more explicit we give formulas for a function f, a one form A and a two form F

$$\begin{aligned}\delta f(X) &= \delta_X f, \\ \delta A(X, Y) &= \delta_X A_Y - \delta_Y A_X - A_{[X,Y]_\star}, \\ \delta F(X, Y, Z) &= \delta_X F_{Y,Z} - \delta_Y F_{X,Z} + \delta_Z F_{X,Y}, \\ &\quad - F_{[X,Y]_\star,Z} + F_{[X,Z]_\star,Y} - F_{[Y,Z]_\star,X}.\end{aligned}$$

A wedge product may be defined

$$\omega_1 \wedge \omega_2(X_1, \ldots, X_{p+q}) = \frac{1}{p!q!} \sum_{I,J} \varepsilon(I, J)\, \omega_1(X_{i_1}, \ldots, X_{i_p}) \star \omega_2(X_{j_1}, \ldots, X_{j_q})$$

where (I, J) is a partition of $(1, \ldots, p+q)$ and $\varepsilon(I, J)$ is the sign of the corresponding permutation. The wedge product is linear and associative and generalizes the bimodule structure (4.10). We note that it is no more graded commutative. We again give some formulas.

$$\begin{aligned}(f \wedge a)_X &= f \star a_X, \\ (a \wedge f)_X &= a_X \star f, \\ (a \wedge b)_{X,Y} &= a_X \star b_Y - a_Y \star b_X.\end{aligned}$$

The differential (4.11) fulfills the graded Leibniz rule

$$\delta(\omega_1 \wedge \omega_2) = \delta\omega_1 \wedge \omega_2 + (-1)^{k_2}\, \omega_1 \wedge \delta\omega_2.$$

4.4 Construction of actions

All field theories in physics may be formulated by an action principle. For field theories on a curved manifold the action is of the form

$$\mathcal{S} = \int d^n x\, \sqrt{g}\, \mathcal{L}(g^{ij}, \phi, \partial_i \phi)$$

with g the determinant of the metric. $\mathcal{L}$ is a local Lagrange density depending on the fields and its partial derivatives. A simple examples is a single scalar field ϕ. In this case we have $\mathcal{L} = g^{ij}\partial_i\phi\partial_j\phi + p(\phi)$ where p is a polynomial in the field ϕ. We can formulate the action in the language of frames and in this case

$$\mathcal{S} = \int d^n x\, e\, \mathcal{L}(\phi, e_a\phi)$$

where e_a are the vectorfields of the frame as defined in (3.3). Now e is the determinant of the frame and in the example of the single scalar field the Lagrange density becomes $\mathcal{L} = \eta^{ab} e_a\phi e_b\phi + p(\phi)$.

If we do not want to give up the action principle we have to generalize the notion of an action functional to noncommutative geometry. The generalization of the Lagrange density is quite clear. We only have to replace the fields by algebra elements and if we believe that the commuting frame formalism is the right one we can use derivations of the algebra as a substitute for the frame of derivations .

What to use for the integral and the measure function e is not so clear in the beginning. As an action maps fields to numbers from an algebraic point of view, the most obvious candidate is a trace on a representation of the algebra. A trace is cyclic with respect to the product of algebra elements

$$Tr\, \hat{f}\hat{g} = Tr\, \hat{g}\hat{f}.$$

Therefore it is possible to do partial integration with inner derivations

$$Tr\,[\hat{\lambda}, \hat{f}]\hat{g} = -Tr\,\hat{f}[\hat{\lambda}, \hat{g}].$$

Another argument for using a trace comes from noncommutative gauge theory (5.2): Suppose we have a field invariant under the following transformation

$$\delta_{\hat{\alpha}}\hat{\phi} = i[\hat{\alpha}, \hat{\phi}],$$

then the trace of a polynomial in $\hat{\phi}$ is invariant under this type of gauge transformations. We will see in the following that the use of a trace in the $\star$-product representation will solve the problem with the measure function in a quite remarkable way.

4.4.1 Traces for $\star$-products

If the algebra is in the $\star$-product representation, the algebra elements are functions on some manifold and we are able to integrate them. But the pure integral is not cyclic with respect to the $\star$-product. To cure this we introduce a measure function Ω and make the following ansatz for the trace of the $\star$-product

$$tr\,f = \int d^n x\,\Omega(x)\,f(x).$$

If we expand the equation of cyclicity

$$\int d^n x\,\Omega(x)\,[f(x) \stackrel{\star}{,} g(x)] = 0$$

up to first order we see that Ω has to fulfill

$$\partial_i(\Omega\Pi^{ij}) = 0, \tag{4.12}$$

where Π^{ij} is the Poisson structure corresponding to the $\star$-product $[f \stackrel{\star}{,} g] = h\Pi^{ij}\partial_i f\,\partial_j g + \cdots$. It is known [76] that there is always a gauge equivalent $\star$-product in the sense of (2.3) for which cyclicity is guaranteed to all orders.

If the Poisson structure Π^{ij} is invertible then a solution to the equation (4.12) can be given. In this case the inverse of the Pfaffian

$$\frac{1}{\Omega} = Pf(\Pi) = \sqrt{det(\Pi)} = \frac{1}{2^n n!}\epsilon_{i_1 i_2 \cdots i_{2n}}\Pi^{i_1 i_2}\cdots\Pi^{i_{2n-1}i_{2n}}$$

is the measure function.

4.4.2 Commuting frames from inner derivations

Form (3.6) we know that in the commuting frame formalism with inner derivations ${e_a}^i = \Pi^{li}\partial_l\lambda_a$. In two dimensions we have

$$
{e_a}^i = \Pi^{12}\begin{pmatrix} -\partial_2\lambda_1 & -\partial_1\lambda_1 \\ \partial_2\lambda_2 & \partial_1\lambda_2 \end{pmatrix},
$$

$$
e^{-1} = \det({e_a}^i) = (\Pi^{12})^2(\partial_1\lambda_1\,\partial_2\lambda_2 - \partial_2\lambda_1\,\partial_1\lambda_2).
$$

e^{-1} is the inverse of the measure function induced by the metric. On the other hand the inverse of the measure function induced by the trace is

$$
\Omega^{-1} = \frac{1}{2}\epsilon_{ij}\Pi^{ij} = \Pi^{12}.
$$

If we want these two measure functions to be equal,

$$
\partial_1\lambda_1\,\partial_2\lambda_2 - \partial_2\lambda_1\,\partial_1\lambda_2 = \frac{1}{\Pi^{12}} \tag{4.13}
$$

has to be fulfilled. This is not the case in any of the examples from (3.2.3).

In four dimensons the measure function induced by the trace is

$$
\Omega^{-1} = \frac{1}{8}\epsilon_{ijkl}\Pi^{ij}\Pi^{kl} = \Pi^{12}\Pi^{34} - \Pi^{13}\Pi^{24} + \Pi^{14}\Pi^{23},
$$

which is quadratic in the elements of Π. Due to (3.6) the measure function induced by the metric contains monomials of order four. Again there are constraints of the form (4.13) if want the two functions to be equal. This is also the case in higher dimensions.

4.4.3 Commuting frames and quantum spaces

We will now propose another method how to find Poisson structures with compatible frames. On several quantum spaces deformed derivations have been constructed [62, 21, 77]. In many cases the deformed Leibniz rule may be written in the following form

$$
\hat\partial_\mu(\hat f\hat g) = \hat\partial_\mu\hat f\hat g + {\hat T_\mu}{}^\nu(\hat f)\hat\partial_\nu\hat g,
$$

where $\hat T$ is an algebra morphism from the quantum space to its matrix ring

$$
{\hat T_\mu}{}^\nu(\hat f\hat g) = {\hat T_\mu}{}^\alpha(\hat f){\hat T_\alpha}{}^\nu(\hat g).
$$

Again in some cases it is possible to implement this morphism with some kind of inner morphism

$$\hat{T}_\mu{}^\nu(\hat{f}) = \hat{e}_\mu{}^a \hat{f} \hat{e}_a{}^\nu,$$

where $\hat{e}_a{}^\mu$ is an invertible matrix with entries from the quantum space. If we define

$$\hat{e}_a = \hat{e}_a{}^\mu \hat{\partial}_\mu,$$

the $\hat{e}_a$ are derivations

$$\hat{e}_a(\hat{f}\hat{g}) = \hat{e}_a(\hat{f})\hat{g} + \hat{f}\hat{e}_a(\hat{g}).$$

The dual formulation of this with covariant differential calculi on quantum spaces is the formalism with commuting frames investigated for example in [78, 5, 79, 6].

We can now represent the quantum space with the help of a ⋆-product. For example, we can use the Weyl-ordered ⋆-product we have constructed in section 2.2.2. Further we can calculate the action of the operators $\hat{e}_a$ on functions. Since these are now derivations of a ⋆-product, their classical limits are necessarily a Poisson vector fields e_a for the Poisson structure of the ⋆-product and with (4.9) the derivations are represented by

$$\hat{e}_a = \delta_{e_a}.$$

4.4.4 Example: $M(so_a(n))$

Now we continue the example (2.2.3). As a special frame we take the deformed commuting derivations acting on the coordinates like

$$\begin{aligned}
\hat{\partial}_o \hat{x}^0 &= 1 + \hat{x}^0 \hat{\partial}_o, \\
\hat{\partial}_o \hat{x}^i &= \hat{x}^i \hat{\partial}_o, \\
\hat{\partial}_i \hat{x}^j &= \delta_i^j + \hat{x}^j \hat{\partial}_i, \\
\hat{\partial}_i \hat{x}^0 &= (\hat{x}^0 + ia)\hat{\partial}_i.
\end{aligned}$$

Note that the $\hat{\partial}_i$ are not derivations on the quantum space. But we can apply the procedure described previously. If we define

$$\hat{\rho} = \sqrt{\sum_i (\hat{x}^i)^2}$$

and assume that it is invertible, we can write the above formulas for $\hat{\partial}_i$ in another way

$$\hat{\partial}_i \hat{f}\hat{g} = \hat{\partial}_i \hat{f} \cdot \hat{g} + \hat{\rho}^{-1}\hat{f}\hat{\rho} \cdot \hat{\partial}_i \hat{g},$$

since

$$\hat{\rho}^{-1}\hat{x}^0\hat{\rho} = \quad \hat{x}^0 + ia, \quad \hat{\rho}^{-1}\hat{x}^i\hat{\rho} = \hat{x}^i.$$

Therefore as we have seen

$$\hat{e}_o = \hat{\partial}_0, \qquad \hat{e}_i = \hat{\rho}\hat{\partial}_i$$

is a frame on the quantum space. The classical limit of this is obviously

$$e_o = \partial_0, \qquad e_i = \rho\partial_i.$$

The derivations (4.9) going with the Weyl ordered $\star$-product are identical up to third order

$$\begin{aligned} \delta_0 &= \partial_o + \mathcal{O}(a^3), \\ \delta_i &= \rho\partial_i + \mathcal{O}(a^3). \end{aligned}$$

In the classical limit we have n linear independent derivations and we can apply the commutating frame formalism. The forms dual to the derivations are

$$\theta^0 = dt, \qquad \theta^i = \frac{1}{\rho}dx^i,$$

and the classical metric (with $\eta_{ab} = diag(1,-1,-1,\cdots)$) becomes

$$g = \eta_{ab}\theta^a\theta^b = (dx^0)^2 - \rho^{-2}((dx^1)^2 + \cdots + (dx^{n-1})^2).$$

We know that we can write

$$(dx^1)^2 + \cdots + (dx^{n-1})^2 = d\rho^2 + \rho^2 d\Omega^2_{n-2},$$

where $d\Omega^2_{n-2}$ is the metric of the $n-2$ dimensional sphere. Therefore in this new coordinate system

$$g = (dx^0)^2 - (d\ln\rho)^2 + d\Omega^2_{n-2}$$

and we see that the classical space is a cross product of two dimensional Euclidean space and a $(n-2)$-sphere. Therefore it is a space of constant non vanishing curvature. Further we calculate that

$$\sqrt{\det g} = \rho^{-(n-1)}$$

fulfills the equation for the measure function (4.12) of the $\star$-product trace. Here we are lucky and are able to write down actions for field theories on this special quantum space with the correct classical limit. For example

$$\begin{aligned} S &= Tr\,(\eta^{\alpha\beta}\hat{e}_\alpha\hat{\phi}\,\hat{e}_b\hat{\phi} + m^2\hat{\phi}^2 + a\hat{\phi}^4) \\ &= \int \frac{d^n x}{\rho^{n-1}}\,\delta_o\phi\star\delta_o\phi - \delta_i\phi\star\delta_i\phi + m^2\phi\star\phi + a\phi\star\phi\star\phi\star\phi \end{aligned}$$

is a well defined action with the $\star$-product (2.10) and reduces in the classical limit to ϕ^4-theory on above described manifold. We will continue this example at (5.3.7), where we will have explicit formulas for the Seiberg-Witten-maps and we are able to do gauge theory.

Chapter 5

Gauge theory

In this chapter we will investigate noncommutative gauge theory formulated in the $\star$-product formalism, where it is possible to formulate general non-abelian gauge theories on noncommutative spacetime. Nonexpanded theories can in general only deal with $U(n)$-gauge groups, but using Seiberg-Witten-maps relating noncommutative quantities with their commutative counterparts makes it possible to consider arbitrary nonabelian gauge groups [32, 63, 44].

The case of an algebra with constant commutator has been extensively studied. This theory reduces in the classical limit to a theory on a flat spacetime. Therefore it is necessary to develop concepts working with more general algebras, since one would expect that curved backgrounds are related to algebras with nonconstant commutation relations. We present here a method using derivations of $\star$-products to build covariant derivatives for Seiberg-Witten gauge theory. Further we are able to write down a noncommutative action by linking the derivations to a frame field induced by a nonconstant metric as explaned in the last chapter. An example is given where the action reduces in the classical limit to scalar electrodynamics on a curved background.

5.1 Classical gauge theory

First let us recall some properties of a general classical gauge theory. A non-abelian gauge theory is based on a Lie group with Lie algebra

$$[T^a, T^b] = i\, f^{ab}{}_c T^c.$$

Matter fields transform under a Lie algebra valued infinitesimal parameter

$$\delta_\alpha \psi = i\alpha\psi, \qquad \alpha = \alpha_a T^a \tag{5.1}$$

in the fundamental representation. It follows that

$$(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\psi = \delta_{-i[\alpha,\beta]}\psi. \tag{5.2}$$

The commutator of two consecutive infinitesimal gauge transformation closes into an infinitesimal gauge transformation. Further a Lie algebra valued gauge potential is introduced with the transformation property

$$\begin{aligned} a_i &= a_{ia}T^a, \\ \delta_\alpha a_i &= \partial_i\alpha + i[\alpha, a_i]. \end{aligned} \tag{5.3}$$

With this the covariant derivative of a field is

$$D_i\psi = \partial_i\psi - ia_i\psi.$$

The field strength of the gauge potential is defined to be the commutator of two covariant derivatives

$$iF_{ij} = [D_i, D_j] = \partial_i a_j - \partial_j a_i - i[a_i, a_j].$$

The last equations can all be stated in the language of forms. For this a connection one form is introduced

$$a = a_{ia}T^a dx^i.$$

The covariant derivative now acts as

$$D\psi = d\psi - ia\psi.$$

The field strength becomes a two form

$$F = da - ia \wedge a.$$

We note that all this may be formulated with finite gauge transformations g. They are related to the infinitesimal gauge parameter by

$$g = e^{i\alpha} = e^{i\alpha_a(x)t^a}.$$

With this finite gauge transformations for the covariant derivative and a field in the fundamental representation are

$$\begin{aligned} T_g D_\mu &= gD_\mu g^{-1}, \\ T_g a_\mu &= gag^{-1} + ig\partial_\mu g^{-1}, \\ T_g\psi &= g\psi. \end{aligned}$$

5.2 Noncommutative gauge theory

In a gauge theory on a noncommutative space, fields should again transform like (5.1)

$$\hat{\delta}_{\hat{\Lambda}}\hat{\Psi} = i\hat{\Lambda}\hat{\Psi}. \tag{5.4}$$

It follows again that

$$(\hat{\delta}_{\hat{\Lambda}}\hat{\delta}_{\hat{\Gamma}} - \hat{\delta}_{\hat{\Gamma}}\hat{\delta}_{\hat{\Lambda}})\hat{\Psi} = \hat{\delta}_{-i[\hat{\Lambda},\hat{\Gamma}]}\hat{\Psi}. \tag{5.5}$$

Since multiplication of a function with a field is not again a covariant operation we are forced to introduce a covariantizer with the transformation property

$$\hat{\delta}_{\hat{\Lambda}}D(\hat{f}) = i[\hat{\Lambda}, D(\hat{f})]. \tag{5.6}$$

From this it follows that

$$\hat{\delta}_{\hat{\Lambda}}(D(\hat{f})\hat{\Psi}) = i\hat{\Lambda}D(\hat{f})\hat{\Psi}. \tag{5.7}$$

If we covariantize the coordinate functions $\hat{x}^i$ we get covariant coordinates

$$\hat{X}^i = D(\hat{x}^i) = \hat{x}^i + \hat{A},^i \tag{5.8}$$

where the gauge field now transforms according to

$$\hat{\delta}_{\hat{\Lambda}}\hat{A}^i = -i[\hat{x}^i, \hat{\Lambda}] + i[\hat{\Lambda}, \hat{A}^i]. \tag{5.9}$$

Unluckily, this does not have a meaninful commutative limit, a problem that can only be fixed for the canonical case (i.e. $[\widehat{x}^i, \widehat{x}^j] = i\theta^{ij}$ with θ a constant) and invertible θ.

For noncommutative algebras where we already have derivatives with a commutative limit, it therefore seems natural to gauge these. But due to their nontrivial coproduct the resulting gauge field would have to be derivative-valued to match the rather awkward behaviour under gauge transformations. The physical reason for this might be the following: The noncommutative derivatives are in general built to reduce to derivatives on flat spacetime, which might not be the correct commutative limit.

We therefore advocate a solution using derivations that will later on (see section 5.3.6) be linked to derivatives on curved spacetime:

If we have a derivation $\hat{\partial}$, i. e. a map with the property

$$\hat{\partial}(\hat{f}\hat{g}) = (\hat{\partial}\hat{f})\hat{g} + \hat{f}(\hat{\partial}\hat{g})$$

for arbitrary elements $\hat{f}$ and $\hat{g}$ of the algebra, we can introduce a noncommutative gauge parameter $\hat{A}_{\hat{\partial}}$ and demand that the covariant derivative (or covariant derivation) of a field

$$\hat{D}\hat{\Psi} = (\hat{\partial} - i\hat{A}_{\hat{\partial}})\hat{\Psi}$$

again transforms like a field

$$\hat{\delta}_{\hat{\Lambda}}\hat{D}\hat{\Psi} = i\hat{\Lambda}\hat{D}\hat{\Psi}.$$

From this it follows that $\hat{A}_{\hat{\partial}}$ has to transform like

$$\hat{\delta}_{\hat{\Lambda}}\hat{A}_{\hat{\partial}} = \hat{\partial}\hat{A}_{\hat{\partial}} + i[\hat{\Lambda}, \hat{A}_{\hat{\partial}}]. \tag{5.10}$$

This is the transformation property we would expect a noncommutative gauge potential to have, and in the next section we will show that for this object we can indeed construct a Seiberg-Witten map in a natural way. If we have an involution on the algebra, we can demand that the gauge potential is hermitian $\hat{A}_{\hat{\partial}} = \overline{\hat{A}_{\hat{\partial}}}$. Additionally the field $\overline{\hat{\Psi}}$ transforms on the right hand side. In this case expressions of the form

$$\overline{\hat{\Psi}}\hat{\Psi} \quad \text{and} \quad \overline{\hat{D}\hat{\Psi}}\hat{D}\hat{\Psi}$$

become gauge invariant quantities.

5.3 Seiberg-Witten gauge theory

In [44] a method how to construct noncommutative non abelian gauge theories using Seiberg-Witten-maps was presented. In the case of constant Poisson structure treated there, it is possible to introduce the momenta via covariant coordinates: $\partial_i = [\theta^{-1}_{ij} x^j, \cdot]$. In general this approach does not yield the desired classical limit. The momentum operators have to be introduced in another way. We will approach the problem by considering derivations of $\star$-products.

5.3.1 Gauge transformations and derivations

If we have a look at (5.5), we see that the commutator of two gauge transformations only closes into the Lie algebra in the fundamental representation of $U(n)$. For non abelian gauge groups, we are forced to go to the enveloping algebra, giving us infinitely many degrees of freedom. But this problem can be solved using Seiberg-Witten maps [32, 44].

The noncommutative gauge parameter and the noncommutative gauge potential will be enveloping algebra valued, but they will only depend on their commutative counterparts, therefore preserving the right number of degrees of freedom. These Seiberg-Witten maps Λ, Ψ and D are functionals of their classical counterparts and additionally of the gauge potential a_i. Their transformation properties (5.6) and (5.7) should be induced by the classical ones (5.1) and (5.3) like

$$\begin{aligned} \widehat{\Lambda}_\beta[a] + \widehat{\delta}_\alpha \widehat{\Lambda}_\beta[a] &= \widehat{\Lambda}_\beta[a + \delta_\alpha a], \\ \widehat{\Psi}_\psi[a] + \widehat{\delta}_\alpha \widehat{\Psi}_\psi[a] &= \widehat{\Psi}_{\psi+\delta_\alpha\psi}[a + \delta_\alpha a], \\ \widehat{A}[a] + \widehat{\delta}_\alpha \widehat{A}[a] &= \widehat{A}[a + \delta_\alpha a]. \end{aligned}$$

The Seiberg-Witten maps can be found order by order using a $\star$-product to represent the algebra on a space of functions. Translated into this language we get for the fields [44]

$$\delta_\alpha \Psi_\psi[a] = i\Lambda_\alpha[a] \star \Psi_\psi[a]. \tag{5.11}$$

From (5.11) we can derive a consistency condition for the noncommuative gauge parameter [43]. Insertion into (5.5) and the use of (5.2) yields

$$i\delta_\alpha \Lambda_\beta - i\delta_\beta \Lambda_\alpha + [\Lambda_\alpha \stackrel{\star}{,} \Lambda_\beta] = i\Lambda_{-i[\alpha,\beta]}. \tag{5.12}$$

The transformation law for the covariantizer is now

$$\delta_\alpha(D[a](f)) = i[\Lambda_\alpha[a] \stackrel{\star}{,} D[a](f)]. \tag{5.13}$$

We now want to extend the Seiberg-Witten-map to derivations of the $\star$-product. In the next section we will see that we are able to identify derivations of a $\star$-products with Poisson vector fields of the Poisson structure associated with the $\star$-product. To be more explicit, let us assume that X is a Poisson vector field

$$X^i \partial_i \{f, g\} = \{X^i \partial_i f, g\} + \{f, X^i \partial_i g\},$$

then we know that there exists a polydifferential operator δ_X with the following property (see chapter 4.1 esp. (4.8) and 4.2)

$$\delta_X(f \star g) = \delta_X f \star g + f \star \delta_X g.$$

It is easy to see that all derivations of this kind exhaust the space of derivations of the $\star$-product. Since the commutator of two derivations is again a derivation we have concluded that there has to be a deformed Lie bracket $[\cdot,\cdot]_\star$ with the following property

$$[\delta_X, \delta_Y] = \delta_{[X,Y]_\star}.$$

With help of the operator δ_X we can now introduce the covariant derivative of a field and the gauge potential in the following way

$$D_X[a]\Psi_\psi[a] = \delta_X \Psi_\psi[a] - iA_X[a] \star \Psi_\psi[a]. \tag{5.14}$$

It follows that the gauge potential has to transform like

$$\delta_\alpha A_X[a] = \delta_X \Lambda_\alpha[a] + i[\Lambda_\alpha[a] \stackrel{\star}{,} A_X[a]]. \tag{5.15}$$

A field strength may be defined

$$iF_{X,Y}[a] = [D_X[a] \stackrel{\star}{,} D_Y[a]] - D_{[X,Y]_\star}[a]. \tag{5.16}$$

The properties of $\delta.$ and $[\cdot,\cdot]_\star$ ensure that this is really a function and not a polydifferential operator

$$F_{X,Y}[a] = \delta_X A_Y[a] - \delta_Y A_X[a] - i[A_X[a] \stackrel{\star}{,} A_Y[a]] - A_{[X,Y]_\star}[a].$$

We are able to translate Seiberg-Witten gauge theory into the language of the forms introduced in (4.3). A_X is the connection one form A evaluated on the vector field X. It transforms like

$$\delta_\alpha A = \delta\Lambda_\alpha + i\Lambda_\alpha \wedge A - A \wedge \Lambda_\alpha.$$

The covariant derivative of a field Ψ is now

$$D\Psi = \delta\Psi - iA \wedge \Psi,$$

and the field strength becomes

$$F = DF = \delta A - iA \wedge A.$$

One easily can show that the field strength is covariant constant

$$DF = \delta F - iA \wedge F = 0.$$

5.3.2 Finite Seiberg-Witten gauge transformations

Ii is interesting that noncommutative infinitesimal and finite gauge transformations may be related like in the classical case. To show this let us first define noncommutative finite gauge transformations similar to the classical case [42]

$$T_g \Psi_\psi[a] = \Psi_{T_g\psi}[T_g a] = G_g[a] \star \Psi_\psi[a],$$
$$T_g D_X[a] = D_X[T_g a] = G_g[a] \star D_X[a] \star G_g[a]^{-1}.$$

If we apply two consecutive gauge transformations on a field

$$T_{g_2} T_{g_1} \Psi_\psi[a] = G_{g_1}[T_{g2} a] \star (G_{g_2}[a] \star \Psi_\psi[a]),$$

we can derive a consistency condition for finite Seiberg-Witten gauge transfomations

$$G_{g_1 g_2}[a] = G_{g_1}[T_{g_2} a] \star G_{g_2}[a].$$

Now we are able to relate finite and infinitesimal gauge transformations. In the classical case

$$T_g \psi = e^{i\alpha} \psi = e^{\delta_\alpha} \psi$$

where δ_α is the action of the infinitesimal gauge transformation on the field ψ. We can use the same fomula to define the noncommutative gauge transformations

$$G_{e^{i\alpha}}[a] \star \Psi_\psi[a] = e_\star^{\delta_\alpha} \star \Psi_\psi[a].$$

To get an explicit formula we note that

$$(\delta_\alpha - i\Lambda_\alpha[a]) \star \Psi_\alpha[a] = 0$$

and calculate

$$\begin{aligned} e_\star^{\delta_\alpha} \star \Psi_\psi[a] &= e_\star^{\delta_\alpha} \star e_\star^{-\delta_\alpha + i\Lambda_\alpha[a]} \star \Psi_\psi[a] \\ &= e_\star^{i\Lambda_\alpha[a] + \frac{i}{2}\delta_\alpha \Lambda_\alpha[a] + \cdots} \star \Psi_\psi[a] \end{aligned}$$

where we have used the Baker-Campbell-Hausdorff formula.

5.3.3 Enveloping algebra valued gauge transformations

Gauge theories on noncommutative spaces cannot be formulated with Lie algebra valued infinitesimal transformations and therefore not with Lie algebra valued gauge fields. To see this we assume that the noncommutative gauge parameter is Lie algebra valued

$$\hat\alpha = \hat\alpha_a T^a,$$

where the $\hat{\alpha}_a$ are elements of the algebra describing the noncommutative space and the T^a are generators of a Lie algebra with $[T^a, T^b] = if^{ab}{}_c T^c$. We have seen (5.5) that due to consistency the commutator of two gauge parameters again has to be a gauge parameter, but now

$$[\hat{\alpha}, \hat{\beta}] = \frac{i}{2}\{\hat{\alpha}_a \hat{\beta}_b\} f^{ab}{}_c T^a + \frac{1}{2}[\hat{\alpha}_a, \hat{\beta}_b]\{T^a, T^b\}$$

where $\{,\}$ denotes the anticommutator. All higher powers of the generators T^a of the gauge group may be created in this way. Thus the enveloping algebra of the Lie algebra seems to be a proper setting for nonabelian noncommutative gauge theory. In general this is not very attractive because the enveloping algebra is infinite dimensional and consequently requires an infinite number of gauge parameters and gauge potentials.

In Seiberg-Witten gauge theory, however, it is possible to restrict the number of infinitesimal enveloping algebra valued gauge parameters to the usual ones [43]. In this case the gauge parameter depends on the Lie algebra valued parameters and its derivatives of the corresponding commutative gauge theory. The construction of this kind of enveloping valued gauge parameter is based on the Seiberg-Witten map. We have seen that in Seiberg-Witten gauge theory the gauge parameter $\Lambda_\alpha[a]$ is a functional of the classical one $\alpha = \alpha_a T^a$ and the classical potential $a_i = a_{ia} T^a$. We expand it order by order in the expansion parameter

$$\Lambda_\alpha[a] = \Lambda_\alpha^0[a] + \Lambda_\alpha^1[a] + \cdots. \tag{5.17}$$

Further it has to fulfill the consistency condition (5.12). If we plug (5.17) into this equation we get to zeroth and first order

$$i\delta_\alpha \Lambda_\beta^0 - i\delta_\beta \Lambda_\alpha^0 + [\Lambda_\alpha^0, \Lambda_\beta^0] = i\Lambda^0_{-i[\alpha,\beta]},$$

$$i\delta_\alpha \Lambda_\beta^1 - i\delta_\beta \Lambda_\alpha^1 + [\Lambda_\alpha^0, \Lambda_\beta^1] - [\Lambda_\beta^0, \Lambda_\alpha^1] - i\Lambda^1_{-i[\alpha,\beta]} = -\frac{1}{2}c^{ij}[\partial_i \Lambda_\alpha^0, \partial_j \Lambda_\beta^0]$$

where we have assumed that we use as usual $f \star g = fg + \frac{1}{2}c^{ij}\partial_i f \partial_j g + \cdots$. The first equation is fulfilled by the commutative gauge parameter. Since this yields additionally the correct classical limit we set

$$\Lambda_\alpha^0 = \alpha.$$

With this, a solution to the first order equation is

$$\Lambda_\alpha^1 = -\frac{i}{4}c^{ij}\{\partial_i \alpha, a_j\} = -\frac{i}{4}c^{ij}\partial_i \alpha_a \, a_{jb}\{T^a, T^b\}.$$

This is now obviously enveloping algebra valued. The solution is not unique, since a solution to the homogeneous part of the first order equation may be added.

We have made the above considerations only for the noncommutative gauge parameter. It should be clear that the method can be extended to the gauge potential with help of (5.15), to fields with (5.11) and to the covariantizer with (5.13). This will be done in (5.3.4) for the special case of the Weyl-ordered $\star$-product. Again all these solutions are not unique, due to cohomologies induced by the homogeneous parts of the equations. Other methods have to be used to restrict the possible solutions. In [54] this is done for the constant case by demanding that the resulting action is renormalizable up to all orders.

5.3.4 Seiberg-Witten map for Weyl-ordered $\star$-product

With the methods developed in the last section we will now present a consistent solution for the Seiberg-Witten maps up to second order for the Weyl ordered $\star$-product and non-abelian classical gauge transformations. The solutions have been chosen in such a way that they reproduce the ones obtained in [44] for the constant case. In the following we will use the Weyl-ordered $\star$-product expanded order by order

$$f \star g = fg + f \star_1 g + f \star_2 g + \cdots$$

with e. g.

$$f \star_1 g = \frac{1}{2} c^{ij} \partial_i f \, \partial_j g.$$

Noncommutative gauge parameter

As we have seen we have to expand Λ in terms of the deformation parameter

$$\Lambda_\alpha[a] = \alpha + \Lambda^1_\alpha[a] + \Lambda^2_\alpha[a] + \cdots .$$

To zeroth order the consistency condition (5.12) is equal to the commutative one (5.2). Therefore, we already have set $\Lambda^0_\alpha = \alpha$.

To first order we obtain

$$i\delta_\alpha \Lambda^1_\beta - i\delta_\beta \Lambda^1_\alpha + [\alpha, \Lambda^1_\beta] - [\beta, \Lambda^1_\alpha] - i\Lambda^1_{-i[\alpha,\beta]} = -[\alpha \overset{\star_1}{,} \beta]$$

$$= -\frac{1}{2} c^{ij} [\partial_i \alpha, \partial_j \beta]$$

and to second order

$$i\delta_\alpha \Lambda^2_\beta - i\delta_\beta \Lambda^2_\alpha + [\alpha\,\Lambda^2_\beta] - [\beta, \Lambda^2_\alpha] - i\Lambda^2_{-i[\alpha,\beta]}$$

$$= -[\alpha \overset{\star_1}{,} \Lambda^1_\beta] - [\beta \overset{\star_1}{,} \Lambda^1_\alpha] - [\Lambda^1_\alpha, \Lambda^1_\beta] - [\alpha \overset{\star_2}{,} \beta]$$

$$= -\frac{1}{2}c^{ij}[\partial_i\alpha, \partial_j\Lambda^1_\beta] - \frac{1}{2}c^{ij}[\partial_i\beta, \partial_j\Lambda^1_\alpha] - [\Lambda^1_\alpha, \Lambda^1_\beta]$$

$$-\frac{1}{8}c^{mn}c^{ij}[\partial_m\partial_i\alpha, \partial_n\partial_j\beta] - \frac{1}{12}c^{ml}\partial_l c^{ij}([\partial_m\partial_i\alpha, \partial_j\beta] - [\partial_i\alpha, \partial_m\partial_j\beta]).$$

A solution to this is

$$\begin{aligned}
\Lambda_\alpha[a] \quad = \quad & \alpha - \frac{i}{4}c^{ij}\{\partial_i\alpha, a_j\} \\
& + \frac{1}{32}c^{ij}c^{kl}\Big(4\{\partial_i\alpha, \{a_k, \partial_l a_j\}\} - 2i[\partial_i\partial_k\alpha, \partial_j a_l] \\
& \quad +2[\partial_j a_l, [\partial_i\alpha, a_k]] - 2i[[a_j, a_l], [\partial_i\alpha, a_k]] \\
& \quad +i\{\partial_i\alpha, \{a_k, [a_j, a_l]\}\} + \{a_j, \{a_l, [\partial_i\alpha, a_k]\}\}\Big) \\
& + \frac{1}{24}c^{kl}\partial_l c^{ij}\Big(\{\partial_i\alpha, \{a_k, a_j\}\} - 2i[\partial_i\partial_k\alpha, a_j]\Big) + \mathcal{O}(3).
\end{aligned}$$

Noncommutative matter field

Now we derive formulas for fields that transforms according to (5.11). We again expand the noncommuting field in terms of the deformation parameter

$$\Psi_\psi[a] = \psi + \Psi^1_\psi[a] + \Psi^2_\psi[a] + \cdots.$$

To first order (5.11) reduces again to the classical transformation law (5.1). To first order we obtain

$$\delta_\alpha \Psi^1_\psi - i\alpha\Psi^1_\psi = i\left(\alpha \star_1 \psi + \Lambda^1_\alpha\psi\right) = i\left(\frac{1}{2}c^{ij}\partial_i\alpha\partial_j\psi + \Lambda^1_\alpha\psi\right)$$

and to second order

$$\delta_\alpha \Psi^2_\psi - i\alpha\Psi^2_\psi = i\left(\alpha \star_2 \psi + \alpha \star_1 \Psi^1_\psi + \Lambda^1_\alpha \star_1 \psi + \Lambda^1_\alpha\Psi^1_\psi + \Lambda^2_\alpha\psi\right)$$

$$= i\Big(\frac{1}{8}c^{mn}c^{ij}\partial_m\partial_i\alpha, \partial_n\partial_j\psi + \frac{1}{12}c^{ml}\partial_l c^{ij}(\partial_m\partial_i\alpha\partial_j\psi - \partial_i\alpha\partial_m\partial_j\psi)$$

$$+\frac{1}{2}c^{ij}\partial_i\alpha\partial_j\Psi^1_\psi + \frac{1}{2}c^{ij}\partial_i\Lambda^1_\alpha\partial_j\psi + \Lambda^1_\alpha\Psi^1_\psi + \Lambda^2_\alpha\psi\Big).$$

A solution for the noncommutative field is

$$\begin{aligned}
\Psi_\psi[a] &= \psi + \frac{1}{4}c^{ij}\Big(2ia_i\partial_j\psi + a_ia_j\psi\Big) \\
&+\frac{1}{32}c^{ij}c^{kl}\Big(4i\partial_ia_k\partial_j\partial_l\psi - 4a_ia_k\partial_j\partial_l\psi - 8a_i\partial_ja_k\partial_l\psi \\
&\quad +4a_i\partial_ka_j\partial_l\psi + 4ia_ia_ja_k\partial_l\psi - 4ia_ka_ja_i\partial_l\psi \\
&\quad +4ia_ja_ka_i\partial_l\psi - 4\partial_ja_ka_i\partial_l\psi + 2\partial_ia_k\partial_ja_l\psi \\
&\quad -4ia_ia_l\partial_ka_j\psi - 4ia_i\partial_ka_ja_l\psi + 4ia_i\partial_ja_ka_l\psi \\
&\quad -3a_ia_ja_la_k\psi - 4a_ia_ka_ja_l\psi - 2a_ia_la_ka_j\psi\Big) \\
&+\frac{1}{24}c^{kl}\partial_lc^{ij}\Big(2ia_j\partial_k\partial_i\psi + 2i\partial_ka_i\partial_j\psi + 2\partial_ka_ia_j\psi \\
&\quad -a_ka_i\partial_j\psi - 3a_ia_k\partial_j\psi - 2ia_ja_ka_i\psi\Big) + \mathcal{O}(3).
\end{aligned}$$

Covariantizer

As in the preceeding cases we again expand the covariantizer in terms of the deformation parameter

$$D(f) = f + D^1(f) + D^2(f) + \cdots.$$

Since the transformation law (5.13) to zeroth order is trivial we can assume that D starts with the identity. To first order we get

$$\delta_\alpha D^1(f) = i[\alpha \overset{\star_1}{,} f] + i[\alpha, D^1(f)] = \frac{i}{2}c^{ij}[\partial_i\alpha, \partial_jf] + i[\alpha, D^1(f)]$$

and to second order

$$\begin{aligned}
\delta_\alpha D^2(f) &= i[\alpha \overset{\star_2}{,} f] + i[\alpha \overset{\star_1}{,} D^1(f)] + i[\Lambda^1_\alpha \overset{\star_1}{,} f] + i[\alpha, D^2(f)] + i[\Lambda^1_\alpha, D^2(f)] \\
&= \frac{i}{8}c^{mn}c^{ij}[\partial_m\partial_i\alpha, \partial_n\partial_jf] + \frac{i}{12}c^{ml}\partial_lc^{ij}([\partial_m\partial_i\alpha, \partial_jf] - [\partial_i\alpha, \partial_m\partial_jf]) \\
&+\frac{i}{2}c^{ij}[\partial_i\alpha, \partial_jD^1(f)] + \frac{i}{2}c^{ij}[\partial_i\Lambda^1_\alpha, \partial_jf] + i[\alpha, D^2(f)] + i[\Lambda^1_\alpha, D^2(f)].
\end{aligned}$$

A solution to this is

$$\begin{aligned}
D[a](f) &= f + ic^{ij}a_i\partial_jf \\
&+\tfrac{1}{4}c^{ij}c^{kl}\Big(-2\{a_i, \partial_ja_k\}\partial_lf + \{a_i, \partial_ka_j\}\partial_lf \\
&\quad +i\{a_i, [a_j, a_k]\}\partial_lf - \{a_i, a_k\}\partial_j\partial_lf\Big) \\
&+\tfrac{1}{4}c^{il}\partial_lc^{jk}\{a_i, a_k\}\partial_jf + \mathcal{O}(3).
\end{aligned}$$

Noncommutative gauge potential

Again we expand the noncommutative gauge potential, starting with the usual one

$$A_X = X^n a_n + A^1_X + A^2_X + \cdots .$$

Since it is a noncommutative form in the sense of (4.3), we have to evaluate it on a Poisson vector field X. We again expand the equation (5.15) and obtain to first order

$$\delta_\alpha A^1_X = X^i \partial_i \Lambda^1_\alpha + \delta^1_X \alpha + i[\alpha \overset{\star_1}{,} X^n a_n] + i[\alpha, A^1_X] + i[\Lambda^1_\alpha, X^n a_n]$$

$$= X^i \partial_i \Lambda^1_\alpha + \frac{i}{2} c^{ij} [\partial_i \alpha, \partial_j (X^n a_n)] + i[\alpha, A^1_X] + i[\Lambda^1_\alpha, X^n a_n].$$

For the second order we get

$$\delta_\alpha A^2_X = i[\alpha \overset{\star_2}{,} X^n a_n] + i[\alpha \overset{\star_1}{,} A^1_X] + i[\Lambda^1_\alpha \overset{\star_1}{,} X^n a_n]$$

$$+ X^i \partial_i \Lambda^2_\alpha + \delta^2_X \alpha + \delta^1_X \Lambda^1_\alpha + i[\alpha, A^2_X] + i[\Lambda^1_\alpha, A^1_X] + i[\Lambda^2_\alpha, X^n a_n]$$

$$= \frac{i}{8} c^{mn} c^{ij} [\partial_m \partial_i \alpha, \partial_n \partial_j (X^n a_n)]$$

$$+ \frac{i}{12} c^{ml} \partial_l c^{ij} ([\partial_m \partial_i \alpha, \partial_j (X^n a_n)] - [\partial_i \alpha, \partial_m \partial_j (X^n a_n)])$$

$$+ \frac{i}{2} c^{ij} [\partial_i \alpha, \partial_j A^1_X] + \frac{i}{2} c^{ij} [\partial_i \Lambda^1_\alpha, \partial_j (X^n a_n)]$$

$$- \frac{1}{12} c^{lk} \partial_k c^{im} \partial_l \partial_m X^j \partial_i \partial_j \alpha + \frac{1}{24} c^{lk} c^{im} \partial_l \partial_i X^j \partial_k \partial_m \partial_j \alpha$$

$$+ X^i \partial_i \Lambda^2_\alpha + \delta^1_X \Lambda^1_\alpha + i[\alpha, A^2_X] + i[\Lambda^1_\alpha, A^1_X] + i[\Lambda^2_\alpha, X^n a_n].$$

We found the following solution to the noncommutative gauge potential

$$\begin{aligned} A_X \; = \; & X^n a_n + \frac{i}{4} c^{kl} X^n \{a_k, \partial_l a_n + f_{ln}\} + \frac{i}{4} c^{kl} \partial_l X^n \{a_k, a_n\} \\ & + \frac{1}{32} c^{kl} c^{ij} X^n \Big(-4i[\partial_k \partial_i a_n, \partial_l a_j] \\ & \quad + 2i[\partial_k \partial_n a_i, \partial_l a_j] - 4\{a_k, \{a_i, \partial_j f_{ln}\}\} \\ & \quad - 2[[\partial_k a_i, a_n], \partial_l a_j] + 4\{\partial_l a_n, \{\partial_i a_k, a_j\}\} \\ & \quad - 4\{a_k, \{f_{li}, f_{jn}\}\} + i\{\partial_n a_j, \{a_l, [a_i, a_k]\}\} \\ & \quad + i\{a_i, \{a_k, [\partial_n a_j, a_l]\}\} - 4i[[a_i, a_l], [a_k, \partial_j a_n]] \\ & \quad + 2i[[a_i, a_l], [a_k, \partial_n a_j]] + \{a_i, \{a_k, [a_l, [a_j, a_n]]\}\} \\ & \quad - \{a_k, \{[a_l, a_i], [a_j, a_n]\}\} - [[a_i, a_l], [a_k, [a_j, a_n]]] \Big) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{32}c^{kl}c^{ij}\partial_j X^n\Big(2i[\partial_k a_i,\partial_l a_n]+2i[\partial_i a_k,\partial_l a_n]\\
&\qquad +2i[\partial_i a_k,\partial_l a_n-\partial_n a_l]+4\{a_n,\{a_l,\partial_k a_i\}\}\\
&\qquad +4\{a_k,\{a_i,\partial_n a_l-\partial_l a_n\}\}-2i\{a_k,\{a_i,[a_n,a_l]\}\}\\
&\qquad +i\{a_i,\{a_l,[a_n,a_k]\}\}+i\{a_n,\{a_l,[a_i,a_k]\}\}\Big)\\
&+\frac{1}{24}c^{kl}c^{ij}\partial_l\partial_j X^n\Big(\partial_i\partial_k a_n-2i[a_i,\partial_k a_n]-\{a_n,\{a_k,a_i\}\}\Big)\\
&+\frac{1}{24}c^{kl}\partial_l c^{ij}X^n\Big(2i[a_j,\partial_k\partial_i a_n]+2i[\partial_k a_i,f_{jn}]\\
&\qquad -\{\partial_j a_n,\{a_k,a_i\}\}+2\{a_i,\{a_k,f_{nj}\}\}\Big)\\
&+\frac{1}{24}c^{kl}\partial_l c^{ij}\partial_j X^n\Big(-4i[a_i,\partial_k a_n]+2i[a_k,\partial_i a_n]-\{a_n,\{a_k,a_i\}\}\Big)\\
&-\frac{1}{12}c^{kl}\partial_l c^{ij}\partial_j\partial_k X^n\partial_i a_n+\mathcal{O}(3).
\end{aligned}
$$

5.3.5 Seiberg-Witten map for Formality $\star$-products

We will now apply the formalism of Seiberg-Witten gauge theory to Kontsevich's formality $\star$-product. Here we are able to calculate the abelian Seiberg-Witten map up to all orders. We have seen that derivations for this $\star$-product are easily obtained from Poisson vector fields. With them we have all the key ingredients to do noncommutative gauge theory on any Poisson manifold. To relate the noncommutative theory to commutative gauge theory, we need the Seiberg-Witten maps for the formality $\star$-product. In [40] and [41] the Seiberg-Witten maps for the noncommutative gauge parameter and the covariantizer were already constructed to all orders in θ for abelian gauge theory. We will extend the method developed there to the Seiberg-Witten map for covariant derivations.

Semi-classical construction

We will first do the construction in the semi-classical limit, where the star commutator is replaced by the Poisson bracket. As in [40] and [41], we define, with the help of the Poisson tensor $\theta = \frac{1}{2}\theta^{kl}\partial_k \wedge \partial_l$

$$d_\theta = -[\cdot, \theta]$$

and (locally)

$$a_\theta = \theta^{ij} a_j \partial_i .$$

Note that the bracket used in the definition of d_θ is not the Schouten-Nijenhuis bracket (A.1). For polyvectorfields π_1 and π_2 it is

$$[\pi_1, \pi_2] = -[\pi_2, \pi_1]_S,$$

giving an extra minus sign for π_1 and π_2 both even (see A.5.2). Especially, we get for d_θ acting on a function g

$$d_\theta g = -[g, \theta] = [g, \theta]_S = \theta^{kl} \partial_l g \partial_k.$$

Now a parameter t and t-dependent $\theta_t = \frac{1}{2}\theta_t^{kl}\partial_k \wedge \partial_l$ and $X_t = X_t^k \partial_k$ are introduced, fulfilling

$$\partial_t \theta_t = f_\theta = -\theta_t f \theta_t \quad \text{and} \quad \partial_t X_t = -X_t f \theta_t,$$

where the multiplication is ordinary matrix multiplication. Given the Poisson tensor θ and the Poisson vectorfield X, the formal solutions are

$$\theta_t = \theta \sum_{n=0}^{\infty} (-t\ f\theta)^n = \frac{1}{2}(\theta^{kl} - t\theta^{ki} f_{ij} \theta^{jl} + \ldots)\partial_k \wedge \partial_l$$

and

$$X_t = X \sum_{n=0}^{\infty} (-t\ f\theta)^n = X^k \partial_k - tX^i f_{ij} \theta^{jk} \partial_k + \ldots .$$

θ_t is still a Poisson tensor and X_t is still a Poisson vectorfield, i.e.

$$[\theta_t, \theta_t] = 0 \qquad \text{and} \qquad [X_t, \theta_t] = 0.$$

For the proof see A.3.

With this we calculate

$$f_\theta = \partial_t \theta_t = -\theta_t f \theta_t = -[a_\theta, \theta] = d_\theta a_\theta. \tag{5.18}$$

We now get the following commutation relations

$$[a_{\theta_t} + \partial_t, d_{\theta_t}(g)] = d_{\theta_t}((a_{\theta_t} + \partial_t)(g)), \tag{5.19}$$
$$[a_{\theta_t} + \partial_t, X_t] = -d_{\theta_t}(X_t^k a_k), \tag{5.20}$$

where g is some function which might also depend on t (see A.5.1).

To construct the Seiberg-Witten map for the gauge potential A_X, we first define

$$K_t = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (a_{\theta_t} + \partial_t)^n.$$

With this, the semi-classical gauge parameter reads [40, 41]

$$\Lambda_\lambda[a] = K_t(\lambda)\Big|_{t=0}.$$

To see that this has indeed the right transformation properties under gauge transformations, we first note that the transformation properties of a_{θ_t} and $X_t^k a_k$ are

$$\delta_\lambda a_{\theta_t} = \theta_t^{kl} \partial_l \lambda \partial_k = d_{\theta_t} \lambda \tag{5.21}$$

and

$$\delta_\lambda (X_t^k a_k) = X_t^k \partial_k \lambda = [X_t, \lambda]. \tag{5.22}$$

Using (5.21), (5.22) and the commutation relations (5.19), (5.20), a rather tedious calculation (see A.4) shows that

$$\delta_\lambda K_t(X_t^k a_k) = X_t^k \partial_k K_t(\lambda) + d_{\theta_t}(K_t(\lambda)) K_t(X_t^k a_k).$$

Therefore, the semi-classical gauge potential is

$$A_X[a] = K_t(X_t^k a_k)\Big|_{t=0}.$$

Quantum construction

We can now use the Kontsevich formality map to quantise the semi-classical construction. All the semi-classical expressions can be mapped to their counterparts in the $\star$-product formalism without loosing the properties necessary for the construction. One higher order term will appear, fixing the transformation properties for the quantum objects.

The star-product we will use is

$$\star = \sum_{n=0}^{\infty} \frac{1}{n!} U_n(\theta_t, \ldots, \theta_t).$$

We define

$$d_\star = -[\cdot, \star]_G \ ,$$

which for functions f and g reads

$$d_\star(g)\, f = [f \stackrel{\star}{,} g].$$

The bracket used in the definition of $d_\star$ is the Gerstenhaber bracket (A.2). We now calculate the commutators (5.19) and (5.20) in the new setting (see A.5.2). We get

$$\begin{aligned}
[\Phi(a_{\theta_t}) + \partial_t, d_\star(\Phi(g))] &= d_\star((\Phi(a_{\theta_t}) + \partial_t)\Phi(f)), \\
[\Phi(a_{\theta_t}) + \partial_t, \Phi(X_t)] &= -d_\star(\Phi(X_t^k a_k) - \Psi(a_{\theta_t}, X_t)).
\end{aligned}$$

The higher order term $\Psi(a_{\theta_t}, X_t)$ has appeared, but looking at the gauge transformation properties of the quantum objects we see that it is actually necessary. We get

$$\delta_\lambda \Phi(a_{\theta_t}) = \Phi(d_{\theta_t}\lambda) = d_\star \Phi(\lambda)$$

with (4.8) and (5.21) and

$$\begin{aligned}
\delta_\lambda(\Phi(X_t^k a_k) - \Psi(a_\theta, X_t)) &= \Phi([X_t, \lambda]) - \Psi(d_\theta\lambda, X_t) \\
&= [\Phi(X_t), \Phi(\lambda)] - \Psi([\theta_t, \lambda], X_t) \\
&\quad + \Psi([\theta_t, X_t], \lambda) - \Psi(d_\theta\lambda, X_t) \\
&= [\Phi(X_t), \Phi(\lambda)] \\
&= \delta_{X_t}\Phi(\lambda),
\end{aligned}$$

where the addition of the new term preserves the correct transformation property. With

$$K_t^\star = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\Phi(a_{\theta_t}) + \partial_t)^n,$$

a calculation analogous to the semi-classical case gives

$$\begin{aligned}
\delta_\lambda(K_t^\star(\Phi(X_t^k a_k) - \Psi(a_{\theta_t}, X_t))) &= \delta_{X_t} K_t^\star(\Phi(\lambda)) \\
&\quad + d_\star(K_t^\star(\Phi(\lambda))) K_t^\star(\Phi(X_t^k a_k) - \Psi(a_{\theta_t}, X_t)).
\end{aligned}$$

As in [40, 41], the noncommutative gauge parameter is

$$\Lambda_\lambda[a] = K_t^\star(\Phi(\lambda))\Big|_{t=0},$$

and we therefore get for the noncommutative gauge potential

$$A_X[a] = K_t^\star(\Phi(X_t^k a_k) - \Psi(a_{\theta_t}, X_t))\Big|_{t=0},$$

transforming with

$$\delta_\lambda A_X = \delta_X \Lambda_\lambda - [\Lambda_\lambda \stackrel{\star}{,} A_X].$$

5.3.6 Construction of gauge invariant actions

We have seen that in the noncommutative realm the integral may be replaced by a trace on a representation of the algebra describing the noncommutative space. In the $\star$-product formalism this has been the ordinary integral together with a measure function (4.12). With the covariantizer $D[a](f)$ (5.13) for functions at hand it is now easy to construct actions invariant under noncommutative gauge transformations that reduce in the classical limit to gauge theory on a flat space. For example the measure function can be compensated by $D[a](\Omega^{-1}(x))$. But from the point of view of noncommutative gauge theory this looks quite unnatural. To make contact with the commuting frame formalism we will have to go another way.

First we want to translate classical gauge theory (5.1) into the language of frames. Since forms are dual to vector fields, they may be evaluated on a frame. In the special case of the connection one form this yields

$$a_a = a(e_a) = a_i dx^i(e_a) = a_i e^i{}_a.$$

The same we can do with the covariant derivate

$$(D\psi)(e_a) = e_a\psi - ia_a\psi.$$

The field strength becomes

$$f(e_a, e_b) = f_{ab} = e_a a_b - e_b a_a - a([e_a, e_b]) - i[a_a, a_b].$$

Since in scalar electrodynamics we do not need a spin connection, it is simple to write down its action on an curved manifold with the frame formalism

$$\mathcal{S} = \int d^n x\, e\, (-\frac{1}{4}\eta^{ab}\eta^{cd} f_{ac} f_{bd} + \eta^{ab} D_a\overline{\phi} D_b\phi + m^2\overline{\phi}\phi).$$

Here again

$$e = (\det\, e_a{}^{\mu})^{-1} = \sqrt{\det\,(g_{\mu\nu})}$$

is the measure function for the curved manifold.

The considerations above can be generalized to a curved noncommutative space, i.e. a noncommutative space with a Poisson structure that is compatible with a frame e_a. For a curved noncommutative space we are now able to mimic the previous classical constructions and evaluate the noncommutative covariant derivative (5.14) and field strength (5.16) on it

$$D_a\Phi = \delta_{e_a}\Phi - iA_{e_a} \star \Phi,$$

$$F_{ab} = F(e_a, e_b).$$

Using the measure function and our noncommutative versions of field strength and covariant derivative we end up with the following action

$$\mathcal{S} = tr \int d^n x \, \Omega \, (-\frac{1}{4}\eta^{ab}\eta^{cd} F_{ac} \star F_{bd} + \eta^{ab} D_a \overline{\Phi} \star D_b \Phi - m^2 \overline{\Phi} \star \Phi). \quad (5.23)$$

tr is the trace of the Lie algebra representation. By construction this action is invariant under noncommutative gauge transformations

$$\delta_\alpha S = 0.$$

To lowest order we obtain

$$\mathcal{S}_0 = tr \int d^n x \, \Omega \, (-\frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} f_{\alpha\gamma} f_{\beta\delta} + g^{\alpha\beta} D_\alpha \bar{\phi} D_\beta \phi - m^2 \bar{\phi}\phi),$$

with $g_{\alpha\beta}$ the metric induced by the frame. If $g = \Omega$, the commuting frame formalism yields the desired classical limit.

5.3.7 Example: $M(so_a(n))$

We have seen that the components of the frame ($e_\alpha = X_\alpha{}^\mu \partial_u$) are

$$\begin{aligned} X_0^\mu &= \delta_0^\mu, \\ X_i^\mu &= \rho \delta_i^\mu. \end{aligned}$$

These we can plug into our solution of the Seiberg-Witten map and the derivation corresponding to the Weyl-ordered $\star$-product and get

$$\begin{aligned} \Lambda_\lambda[a] &= \lambda + \frac{a}{4} x^i \{\partial_0 \lambda, a_i\} - \frac{a}{4} x^i \{\partial_i \lambda, a_0\} + \mathcal{O}(a^2), \\ \Phi_\phi[a] &= \phi - \frac{a}{2} x^i a_0 \partial_i \phi + \frac{a}{2} x^i a_i \partial_0 \phi + \frac{ia}{4} x^i [a_0, a_i] \phi + \mathcal{O}(a^2), \\ A_{X_0} &= a_0 - \frac{a}{4} x^i \{a_0, \partial_i a_0 + f_{i0}\} + \frac{a}{4} x^i \{a_i, \partial_0 a_0\} + \mathcal{O}(a^2), \\ A_{X_j} &= \rho a_j - \frac{a}{4} \rho \{a_j, a_0\} - \frac{a}{4} \rho x^i \{a_0, \partial_i a_j + f_{ij}\} \\ &\quad + \frac{a}{4} \rho x^i \{a_i, \partial_0 a_j + f_{0j}\} + \mathcal{O}(a^2), \\ \delta_{X_\mu} &= X_\mu^\nu \partial_\nu + \mathcal{O}(a^2). \end{aligned}$$

The action (5.23) becomes up to first order

$$S = \int d^n x \Big(\quad -\frac{1}{2} \rho^{3-n} \eta^{00} \eta^{ij} Tr(f_{0i} f_{0j}) - \frac{1}{4} \rho^{5-n} \eta^{kl} \eta^{ij} Tr(f_{ki} f_{lj})$$

$$
\begin{aligned}
&+\rho^{1-n}\eta^{00}\overline{D_0\phi}D_0\phi+\rho^{3-n}\eta^{kl}\overline{D_k\phi}D_l\phi \\
&-\frac{a}{2}\rho^{3-n}\eta^{00}\eta^{ij}x^pTr(f_{0p}f_{0i}f_{0j}) \\
&+\frac{a}{4}\rho^{5-n}\eta^{kl}\eta^{ij}x^pTr(f_{0p}f_{ki}f_{lj}) \\
&-\frac{a}{2}\rho^{5-n}\eta^{kl}\eta^{ij}x^pTr(f_{jp}\{f_{ki},f_{l0}\}) \\
&-\frac{a}{2}\rho^{3-n}\eta^{kl}x^i\overline{D_k\phi}f_{l0}D_i\phi+\frac{a}{2}\rho^{3-n}\eta^{kl}x^i\overline{D_k\phi}f_{li}D_0\phi \\
&-\frac{a}{2}\rho^{3-n}\eta^{kl}x^i\overline{D_i\phi}f_{l0}D_k\phi+\frac{a}{2}\rho^{3-n}\eta^{kl}x^i\overline{D_0\phi}f_{li}D_k\phi \\
&-a\rho^{3-n}\eta^{kl}x^i\overline{D_k\phi}f_{0i}D_l\phi\Big)+\mathcal{O}(a^2).
\end{aligned}
$$

We know that in the classical limit $a\to 0$ the action reduces to scalar electrodynamics on a curved background or its nonabelian generalization, respectively.

5.4 Observables

In the previous sections we have seen that gauge theory on noncommutative spaces is a very interesting and fruitful subject. Nevertheless we need a method to extract physical predictions from the theory. Since a gauge transformation should not affect the predictions we make, we have to find gauge invariant objects. Such observables are not easy to find if we want them to have a sensible classical limit.

A second reason for studying observables is the similarity between noncommutative gauge theory and gravity in view of the gauge structure. The equations of general relativity transform covariantly under coordinate transformations. Therefore the group of local diffemorphisms is part of the gauge group. Now take a gauge transformation in the $\star$-product representation of a noncommutative $U(1)$-gauge theory. Then we have seen that the coordiantes are not invariant under gauge transformations

$$x^i\to e_\star^{i\alpha(x)}\star x^i\star e_\star^{-i\alpha(x)}=x^i+\theta^{ij}\partial_j\alpha+\cdots \tag{5.24}$$

In the whole section we will assume that the $\star$-product looks up to first order like

$$f\star g=fg+\frac{i}{2}\theta^{ij}\partial_i f\partial_j g+\cdots, \tag{5.25}$$

where θ^{ij} is antisymmetric and fulfills the Poisson equation. In the semi-classical limit the transformations (5.24) become the Hamiltonian flows of the Poisson manifold. In a sense the gauge group of noncommutative gauge theories contains a large class of diffeomorphisms. Since it is not easy to find a full set of meaningful observables in general relativity (see e. g. [80] and for a more general review [81]), the study of nonncommutative gauge theory will perhaps give new insights into this subject.

In the case of constant commutator so called open Wilson lines [82] have been introduced as observables of noncommutative gauge theory. We will use covariant coordinates (5.8) to generalize this construction to general $\star$-products. In [83] they were used to give an exact formula for the inverse Seiberg-Witten map. We will generalize this construction for $\star$-products with invertible Poisson structure θ^{ij}.

5.4.1 Classical Wilson lines

Let us first recall some aspects of the commutative gauge theory. For this let a_μ be a gauge field. Then an infinitesimal parallel transporter (infinitesimal wilson line) may be defined via

$$\begin{aligned} U(x, x+l) &= 1 + i l^\mu a_\mu(x) \\ &= e^{i l^\mu a_\mu(x)} + \mathcal{O}(l^2), \end{aligned}$$

where l^μ is an infinitesimal constant vector. The infinitesimal Wilson line transforms like

$$T_g U(x, x+l) = g(x) U(x, x+l) g^{-1}(x+l) + \mathcal{O}(l^2).$$

Now let Γ_{yz} a Path connecting the points a and b. And let $\{x_i\}_{i\in 0\ldots N}$ be a partition of this Path. Then we define

$$\begin{aligned} U_N[\Gamma_{yz}] &= \prod_{i=1}^{N} U(x_{i-1}, x_i) \\ &= \prod_{i=1}^{N} \left(1 + i(x_i^\mu - x_{i-1}^\mu) a_\mu(x_i)\right) \\ &= \prod_{i=1}^{N} \left(1 + i l_i^\mu a_\mu(y + \sum_{j=1}^{i} l_i)\right). \end{aligned}$$

U_N transforms in the following way

$$T_g U_N[\Gamma_{xy}] = g(x) U_N[\Gamma_{xy}] g^{-1}(y) + \mathcal{O}(l_i^2).$$

Further the Wilson line of the Path Γ_{xy} is the continuum limit of the U_N

$$\begin{aligned} U[\Gamma_{xy}] &= \lim_{N\to\infty} U_N[\Gamma_{xy}] \\ &= P \exp(i \int dx^\mu a_\mu), \end{aligned}$$

where P denotes path ordering of the exponential. If one acts with a gauge transformation on the Wilson line

$$T_g U[\Gamma_{xy}] = g(x)U[\Gamma_{xy}]g^{-1}(y),$$

one sees that it transforms only at its endpoints.

5.4.2 Noncommutative Wilson lines

In the case $\theta^{ij} = const.$ the basic observation was that translations in space are gauge transformations [82]. They are realized by

$$T_l x^j = x^j + l_i\theta^{ij} = e^{il_i x^i} \star f \star e^{-il_i x^i}.$$

Now one can pose the question what happens if one uses covariant coordinates [84]. In this case the inner automorphism

$$f \to e^{il_i X^i} \star f \star e^{-il_i X^i}$$

should consist of a translation and a gauge transformation dependent of the translation. If we subtract the translation again only the gauge transformation remains and the resulting object

$$W_l = e^{il_i X^i} \star e^{-il_i x^i}$$

has a very interesting transformation behavior under a gauge transformation

$$W_l'(x) = g(x) \star W_l(x) \star g^{-1}(x + l_i\theta^{ij}).$$

It transforms like a Wilson line starting at x and ending at $x + l\theta$.

As in the constant case we can start with

$$W_l = e_\star^{il_i X^i} \star e_\star^{-il_i x^i},$$

where now $e_\star$ is the $\star$-exponential. Every multiplication in its Taylor series is replaced by the $\star$-product. In contrast to the constant case, $e_\star^{l_i x^i} = e^{l_i x^i}$ isn't true any more. The transformation property of W_l is now

$$W_l'(x) = g(x) \star W_l(x) \star g^{-1}(T_l x),$$

where

$$T_l x^j = e^{il_i x^i} \star x^j \star e^{-il_i x^i}$$

is an inner automorphism of the algebra, which can be interpreted as a quantized coordinate transformation. If we replace commutators by Poisson brackets, the classical limit of this coordinate transformation may be calculated

$$T_l x^k = e^{il_i[x^i \stackrel{\star}{,} \cdot]} x^k \approx e^{-l_i\{x^i,\cdot\}} x^j = e^{-l_i \theta^{ij}\partial_j} x^k,$$

the formula becoming exact for θ^{ij} constant or linear in x. We see that the classical coordinate transformation is the flow induced by the Hamiltonian vector field $-l_i\theta^{ij}\partial_j$. At the end we may expand W_l in terms of θ and get

$$W_l = e^{il_i\theta^{ij}a_j} + \mathcal{O}(\theta^2),$$

where we have replaced A^i by its Seiberg-Witten expansion. We see that for l small this really is a Wilson line starting at x and ending at $x + l\theta$.

5.4.3 Observables

Now we are able to write down a large class of observables for the above defined noncommutative gauge theory, namely

$$U_l = \int d^n x \, \Omega(x) \, W_l(x) \star e_\star^{il_i x^i} = \int d^n x \, \Omega(x) \, e_\star^{il_i X^i(x)}$$

or more general

$$f_l = \int d^n x \, \Omega(x) \, f(X^i) \star e_\star^{il_i X^i(x)}$$

with f an arbitrary function of the covariant coordinates. Obviously they are invariant under gauge transformations.

5.4.4 Inverse Seiberg-Witten-map

As an application of the above constructed observables we generalize [83] to arbitrary $\star$-products, i. e. we give a formula for the inverse Seiberg-Witten map for $\star$-products with invertible Poisson structure. In order to map noncommutative gauge theory to its commutative counterpart we need a functional $f_{ij}[X]$ fulfilling

$$f_{ij}[g \star X \star g^{-1}] = f_{ij}[X],$$

$$df = 0$$

and

$$f_{ij} = \partial_i a_j - \partial_j a_i + \mathcal{O}(\theta^2).$$

f is a classical field strength and reduces in the limit $\theta \to 0$ to the correct expression.

To proove the first and the second property we will only use the algebra properties of the $\star$-product and the cyclicity of the trace. All quantities with a hat will be elements of an algebra. With this let $\hat{X}^i$ be covariant coordinates in an algebra, transforming under gauge transformations like

$$\hat{X}^{i\prime} = \hat{g}\hat{X}^i\hat{g}^{-1}$$

with $\hat{g}$ an invertible element of the algebra. Now define

$$\hat{F}^{ij} = -i[\hat{X}^i, \hat{X}^j]$$

and

$$(\hat{F}^{n-1})_{ij} = \frac{1}{2^{n-1}(n-1)!}\epsilon_{iji_1i_2\cdots i_{2n-2}}\hat{F}^{i_1i_2}\cdots\hat{F}^{i_{2n-3}i_{2n-2}}.$$

Since an antisymmetric matrix in odd dimensions is never invertible we have assumed that the space is $2n$ dimensional. The expression

$$\mathcal{F}_{ij}(k) = str_{\hat{F},\hat{X}}\left((\hat{F}^{n-1})_{ij}e^{ik_j\hat{X}^j}\right) \tag{5.26}$$

clearly fulfills the first property due to the properties of the trace. str is the symmetrized trace, every monomial in $\hat{F}^{ij}$and $\hat{X}^k$ should be symmetrized. For an exact definition see [83]. Note that symmetrization is only necessary for spaces with dimension higher than 4 due to the cyclicity of the trace. In dimensions 2 and 4 we may replace str by the ordinary trace tr. $\mathcal{F}_{ij}(k)$ is the Fourier transform of a closed form if

$$k_{[i}\mathcal{F}_{jk]} = 0$$

or if the current

$$J^{i_1\cdots i_{2n-2}} = str_{\hat{F},X}\left(\hat{F}^{[i_1i_2}\cdots\hat{F}^{i_{2n-3}i_{2n-2}]}e^{ik_j\hat{F}^j}\right)$$

is conserved, respectively

$$k_iJ^{i\cdots} = 0.$$

This is easy to show, if one uses

$$str_{\hat{F},\hat{X}}\left([k\hat{X}, \hat{X}^l]e^{ik_j\hat{X}^j}\cdots\right) = str_{\hat{F},\hat{X}}\left([\hat{X}^l, e^{ik_j\hat{X}^j}]\cdots\right) = str_{\hat{F},\hat{X}}\left(e^{ik_j\hat{X}^j}[\hat{X}^l,\cdots]\right),$$

which can be calculated by simple algebra.

To show the last property we have to switch to the $\star$-product formalism and expand the formula in θ^{ij}. The expression (5.26) now becomes

$$\mathcal{F}[X]_{ij}(k) = \int \frac{d^{2n}x}{Pf(\theta)} \left((F_\star^{n-1})_{ij} \star e_\star^{ik_j X^j} \right)_{sym\, F,X}.$$

The expression in brackets has to be symmetrized in F^{ij} and X^i for $n > 2$. Up to third order in θ^{ij}, the commutator F^{ij} of two covariant coordinates is

$$F^{ij} = -i[X^i \stackrel{\star}{,} X^j] = \theta^{ij} - \theta^{ik} f_{kl} \theta^{lj} - \theta^{kl} \partial_l \theta^{ij} a_k + \mathcal{O}(3)$$

with $f_{ij} = \partial_i a_j - \partial_j a_i$ the ordinary field strength. Furthermore we have

$$e_\star^{ik_i X^i} = e^{ik_i x^i}(1 + ik_i \theta^{ij} a_j) + \mathcal{O}(2).$$

If we choose the antisymmetric $\star$-product (5.25), the symmetrization will annihilate all the first order terms of the $\star$-products between the F^{ij} and X^i, and therefore we get

$$-\mathcal{F}[X]_{ij}(k) =$$

$$= -2n \int \frac{d^{2n}x}{\epsilon\theta^n} \left(\epsilon_{ij}\theta^{n-1} - (n-1)\epsilon_{ij}\theta^{n-2}\theta f \theta - \theta^{kl}\partial_l(\epsilon_{ij}\theta^{n-1})a_k \right) e^{ik_i x^i} + \mathcal{O}(1)$$

$$= -2n \int \frac{d^{2n}x}{\epsilon\theta^n} \left(\epsilon_{ij}\theta^{n-1} - (n-1)\epsilon_{ij}\theta^{n-2}\theta f \theta - \frac{1}{2}\epsilon_{ij}\theta^{n-1} f_{kl}\theta^{kl} \right) e^{ik_i x^i} + \mathcal{O}(1)$$

$$= d^{2n}x \left(\theta_{ij}^{-1} + 2n(n-1)\frac{\epsilon_{ij}\theta^{n-2}\theta f \theta}{\epsilon\theta^n} - \frac{1}{2}\theta_{ij}^{-1} f_{kl}\theta^{kl} \right) e^{ik_i x^i} + \mathcal{O}(1),$$

using partial integration and $\partial_i(\epsilon\theta^n\theta^{ij}) = 0$. To simplify notation we introduced

$$\epsilon_{ij}\theta^{n-1} = \epsilon_{iji_1j_1\cdots i_{n-1}j_{n-1}}\theta^{i_1j_1}\cdots\theta^{i_{n-1}j_{n-1}}$$

etc. In the last line we have used

$$\theta_{ij}^{-1} = -\frac{(\theta^{n-1})_{ij}}{Pf(\theta)} = -2n\frac{\epsilon_{ij}\theta^{n-1}}{\epsilon\theta^n}.$$

We will now have a closer look at the second term, noting that

$$\theta^{ij}\frac{\epsilon_{ij}\theta^{n-2}\theta f \theta}{\epsilon\theta^n} = -\frac{1}{2n}\theta_{kl}^{-1}\theta^{kr} f_{rs}\theta^{sl} = -\frac{1}{2n} f_{rs}\theta^{rs}$$

and therefore

$$\frac{\epsilon_{ij}\theta^{n-2}\theta f \theta}{\epsilon\theta^n} = a\frac{\epsilon_{ij}\theta^{n-1}}{\epsilon\theta^n} f_{rs}\theta^{rs} + b f_{ij} \tag{5.27}$$

with $a+b=-\frac{1}{2n}$. Taking e. g. $i=1, j=2$ we see that

$$\epsilon_{12\cdots kl}\theta^{n-2}\theta^{kr}f_{rs}\theta^{sl} = \epsilon_{12\cdots kl}\theta^{n-2}(\theta^{k1}\theta^{2l}-\theta^{k2}\theta^{1l})f_{12} + \text{terms without } f_{12}.$$

Especially there are no terms involving $f_{12}\theta^{12}$ and we get for the two terms on the right hand side of (5.27)

$$2a\epsilon_{12}\theta^{n-1}f_{12}\theta^{12} = -2nb\epsilon_{12}\theta^{12}\theta^{n-1}f_{12}$$

and therefore $b=-\frac{a}{n}$. This has the solution

$$a = -\frac{1}{2(n-1)} \quad \text{and} \quad b = \frac{1}{2n(n-1)}.$$

With the resulting

$$2n(n-1)\frac{\epsilon_{ij}\theta^{n-2}\theta f\theta}{\epsilon\theta^{n}} = \frac{1}{2}\theta^{-1}_{ij}f_{kl}\theta^{kl} + f_{ij}$$

we finally get

$$-\mathcal{F}[X]_{ij}(k) = \int d^{2n}x \, \left(\theta^{-1}_{ij} + f_{ij}\right) e^{ik_i x^i} + \mathcal{O}(1).$$

Therefore

$$f[X]_{ij} = \mathcal{F}[X]_{ij}(k) - \mathcal{F}[x]_{ij}(k)$$

is a closed form that reduces in the classical limit to the classical Abelian field strength. We have found an expression for the inverse Abelian Seiberg-Witten map.

Appendix A

Definitions and calculations

A.1 The Schouten-Nijenhuis bracket

The Schouten-Nijenhuis bracket for multivector fields $\pi_s^{i_1\dots i_{k_s}}\partial_{i_1}\wedge\dots\wedge\partial_{i_{k_s}}$ can be written as ([75], IV.2.1):

$$[\pi_1,\pi_2]_S=(-1)^{k_1-1}\pi_1\bullet\pi_2-(-1)^{k_1(k_2-1)}\pi_2\bullet\pi_1,$$

$$\pi_1\bullet\pi_2=\sum_{l=1}^{k_1}(-1)^{l-1}\pi_1^{i_1\dots i_{k_1}}\partial_l\pi_2^{j_1\dots j_{k_2}}\partial_{i_1}\wedge\dots\wedge\widehat{\partial_{i_l}}\wedge\dots\wedge\partial_{i_{k_1}}\wedge\partial_{j_1}\wedge\dots\wedge\partial_{j_{k_2}},$$

where the hat marks an omitted derivative.

For a function g, vectorfields $X=X^k\partial_k$ and $Y=Y^k\partial_k$ and a bivectorfield $\pi=\frac{1}{2}\pi^{kl}\partial_k\wedge\partial_l$ we get:

$$\begin{aligned}
[X,g]_S &= X^k\partial_k g,\\
[\pi,g]_S &= -\pi^{kl}\partial_k g\partial_l,\\
[X,\pi]_S &= \frac{1}{2}(X^k\partial_k\pi^{ij}-\pi^{ik}\partial_kX^j+\pi^{jk}\partial_kX^i)\partial_i\wedge\partial_j,\\
[\pi,\pi]_S &= \frac{1}{3}(\pi^{kl}\partial_l\pi^{ij}+\pi^{il}\partial_l\pi^{jk}+\pi^{jl}\partial_l\pi^{ki})\partial_k\wedge\partial_i\wedge\partial_j.
\end{aligned}$$

A.2 The Gerstenhaber bracket

The Gerstenhaber bracket for polydifferential operators A_s can be written as ([75], IV.3):

$$[A_1, A_2]_G = A_1 \circ A_2 - (-1)^{(|A_1|-1)(|A_2|-1)} A_2 \circ A_1,$$
$$(A_1 \circ A_2)(f_1, \dots f_{m_1+m_2-1}) =$$
$$\sum_{j=1}^{m_1} (-1)^{(m_2-1)(j-1)} A_1(f_1, \dots f_{j-1}, A_2(f_j, \dots, f_{j+m_2-1}), f_{j+m_2}, \dots, f_{m_1+m_2-1}),$$

where $|A_s|$ is the degree of the polydifferential operator A_s, i.e. the number of functions it is acting on.

For functions g and f, differential operators $D_1$and D_2 of degree one and P of degree two we get

$$\begin{aligned} [D, g]_G &= D(g), \\ [P, g]_G(f) &= P(g, f) - P(f, g), \\ [D_1, D_2]_G(g) &= D_1(D_2(g)) - D_2(D_1(g)), \\ [P, D]_G(f, g) &= P(D(f), g) + P(f, D(g)) - D(P(f, g)). \end{aligned} \tag{A.1}$$

A.3 Calculation of $[\theta_t, \theta_t]$ and $[\theta_t, X_t]$

We want to show that θ_t is still a Poisson tensor and that X_t still commutes with θ_t. For this we first define $\theta(n)_l^k = (\theta f)^n = \theta^{ki} f_{ij} \dots \theta^{rs} f_{sl} = f_{li} \theta^{ij} \dots f_{rs} \theta^{sk} = (f\theta)^n$ and $\theta(n)^{kl} = \theta(f\theta)^n = \theta^{ki} f_{ij} \dots f_{rs} \theta^{sl}$. In the calculations to follow we will sometimes drop the derivatives of the polyvectorfields and associate $\pi^{k_1 \dots k_n}$ with $\pi^{k_1 \dots k_n} \frac{1}{n} \partial_{k_1} \wedge \dots \wedge \partial_{k_n}$for simplicity. All the calculations are done locally.

We evaluate

$$\begin{aligned} [\theta_t, \theta_t]_S &= \theta_t^{kl} \partial_l \theta_t^{ij} + \text{c.p. in } (kij) \\ &= \sum_{n,m=0}^{\infty} \sum_{o=0}^{m} (-t)^{n+m} \theta(n)_r^k \theta(o)_s^i \theta(m-o)_p^j \theta^{rl} \partial_l \theta^{sp} + \text{c.p. in } (kij) \\ &\quad + \sum_{n,m=0}^{\infty} \sum_{o=0}^{m} (-t)^{n+m+1} \theta(n)^{kl} \theta(o)^{is} \theta(m-o)^{pj} \partial_l f_{sp} + \text{c.p. in } (kij) \\ &= \sum_{n,m,o=0}^{\infty} (-t)^{n+m+o} \theta(n)_r^k \theta(o)_s^i \theta(m)_p^j \theta^{rl} \partial_l \theta^{sp} + \text{c.p. in } (kij) \\ &\quad - \sum_{n,m,o=0}^{\infty} (-t)^{n+m+o+1} \theta(n)^{kl} \theta(o)^{is} \theta(m)^{jp} \partial_l f_{sp} + \text{c.p. in } (kij). \end{aligned}$$

The first part vanishes because θ_t is a Poisson tensor, i.e.

$$[\theta, \theta]_S = \theta^{kl}\partial_l\theta^{ij} + \text{c.p. in } (kij) = 0, \tag{A.2}$$

the second part because of

$$\partial_k f_{ij} + \text{c.p. in } (kij) = 0. \tag{A.3}$$

To prove that X_t still commutes with θ_t, we first note that

$$X_t = X\sum_{n=0}^{\infty}(-tf\theta) = X(1 - tf\theta_t).$$

With this we can write

$$\begin{aligned}
[X_t, \theta_t] &= [X, \theta_t] - t[Xf\theta_t, \theta_t] \qquad\qquad \text{(A.4)}\\
&= X^n\partial_n\theta_t^{kl} - \theta_t^{kn}\partial_n X^l + \theta_t^{ln}\partial_n X^k \\
&\quad -tX^m f_{mi}\theta_t^{in}\partial_n\theta_t^{kl} + t\theta_t^{kn}\partial_n(X^m f_{mi}\theta_t^{il}) - t\theta_t^{ln}\partial_n(X^m f_{mi}\theta_t^{ik}) \\
&= X^n\partial_n\theta_t^{kl} - \theta_t^{kn}\partial_n X^l + \theta_t^{ln}\partial_n X^k \\
&\quad +t\theta_t^{kn}\partial_n X^m f_{mi}\theta_t^{il} - t\theta_t^{ln}\partial_n X^m f_{mi}\theta_t^{ik} \\
&\quad +t\theta_t^{kn}X^m\partial_n f_{mi}\theta_t^{il} - t\theta_t^{ln}X^m\partial_n f_{mi}\theta_t^{ik}.
\end{aligned}$$

In the last step we used (A.2). To go on we note that

$$t\theta_t^{kn}X^m\partial_n f_{mi}\theta_t^{il} - t\theta_t^{ln}X^m\partial_n f_{mi}\theta_t^{ik} = tX^n\theta_t^{km}\partial_n f_{mi}\theta_t^{il},$$

where we used (A.3). Making use of the power series expansion and the fact that X commutes with θ, i.e.

$$[X, \theta] = X^n\partial_n\theta^{kl} - \theta^{kn}\partial_n X^l + \theta^{ln}\partial_n X^k = 0,$$

we further get

$$\begin{aligned}
X^n\partial_n\theta_t^{kl} + tX^n\theta_t^{km}\partial_n f_{mi}\theta_t^{il} &= \sum_{r,s=0}^{\infty}(-t)^{r+s}\theta(r)^k_i X^n\partial_n\theta^{ij}\theta(s)^l_j \\
&= \sum_{r,s=0}^{\infty}(-t)^{r+s}\theta(r)^k_i\theta^{in}\partial_n X^j\theta(s)^l_j \\
&\quad - \sum_{r,s=0}^{\infty}(-t)^{r+s}\theta(r)^k_i\theta^{jn}\partial_n X^i\theta(s)^l_j.
\end{aligned}$$

Therefore (A.4) reads

$$\begin{aligned}
[X_t, \theta_t] &= \sum_{r,s=0}^{\infty}(-t)^{r+s}\theta(r)^k_i\theta(s)^l_j\theta^{in}\partial_n X^j - \sum_{r,s=0}^{\infty}(-t)^{r+s}\theta(r)^k_i\theta(s)^l_j\theta^{jn}\partial_n X^i \\
&\quad -\theta_t^{kn}\partial_n X^l + \theta_t^{ln}\partial_n X^k + t\theta_t^{kn}\partial_n X^m f_{mi}\theta_t^{il} - t\theta_t^{ln}\partial_n X^m f_{mi}\theta_t^{ik} \\
&= 0.
\end{aligned}$$

A.4 The transformation properties of K_t

To calculate the transformation properties of $K_t(X_t^k a_k)$, we first evaluate

$$\begin{aligned}
\delta_\lambda((a_\theta+\partial_t)^n)X^k a_k &= \sum_{i=0}^{n-1}(a_\theta+\partial_t)^i d_\theta(\lambda)(a_\theta+\partial_t)^{n-1-i}X^k a_k \\
&= \sum_{i=0}^{n-1}\sum_{l=0}^{i}\binom{i}{l} d_\theta((a_\theta+\partial_t)^l(\lambda))(a_\theta+\partial_t)^{n-1-l}X^k a_k
\end{aligned}$$

and

$$\begin{aligned}
&(a_\theta+\partial_t)^n\delta_\lambda(X^k a_k) \\
&= (a_\theta+\partial_t)^n X^k\partial_k\lambda \\
&= X^k\partial_k(a_\theta+\partial_t)^n - \sum_{i=0}^{n-1}(a_\theta+\partial_t)^i d_\theta(X^k a_k)(a_\theta+\partial_t)^{n-1-i}\lambda \\
&= X^k\partial_k(a_\theta+\partial_t)^n \\
&\quad -\sum_{i=0}^{n-1}\sum_{j=0}^{n-1-i}\binom{n-1-i}{j}(-1)^{n-1-i-j} \\
&\qquad (a_\theta+\partial_t)^{i+j}d_\theta((a_\theta+\partial_t)^{n-1-i-j}(X^k a_k))(\lambda) \\
&= X^k\partial_k(a_\theta+\partial_t)^n \\
&\quad +\sum_{i=0}^{n-1}\sum_{j=0}^{n-1-i}\binom{n-1-i}{j}(-1)^{n-1-i-j} \\
&\qquad (a_\theta+\partial_t)^{i+j}d_\theta(\lambda)((a_\theta+\partial_t)^{n-1-i-j}(X^k a_k)) \\
&= X^k\partial_k(a_\theta+\partial_t)^n
\end{aligned}$$

$$+\sum_{i=0}^{n-1}\sum_{j=0}^{n-1-i}\sum_{l=0}^{i+j}\binom{n-1-i}{j}\binom{i+j}{l}(-1)^{n-1-i-j}$$
$$d_\theta((a_\theta+\partial_t)^l(\lambda))((a_\theta+\partial_t)^{n-1-l}(X^k a_k)).$$

We go on by simplifying these expressions. Using

$$\binom{i}{l}=\binom{i-1}{l}+\binom{i-1}{l-1} \quad \text{for} \quad i>l, \tag{A.5}$$

we get

$$\sum_{m=l}^{n-1}\sum_{i=0}^{m}\binom{n-1-i}{m-i}\binom{m}{l}(-1)^{n-1-m}=\sum_{m=l}^{n-1}\binom{n}{m}\binom{m}{l}(-1)^{n-1-m}.$$

Using (A.5) again two times and then using induction we go on to

$$\sum_{m=l}^{n-1}\binom{n}{m}\binom{m}{l}(-1)^{n-1-m}=\sum_{i=0}^{l}\binom{n-1-i}{n-1-l},$$

giving, after using (A.5) again

$$\sum_{i=0}^{l}\binom{n-1-i}{n-1-l}=\binom{n}{l}.$$

Together with

$$\sum_{i=l}^{n-1}\binom{i}{l}=\binom{n}{l+1}$$

these formulas add up to give

$$\sum_{m=l}^{n-1}\sum_{i=0}^{m}\binom{n-1-i}{m-i}\binom{m}{l}(-1)^{n-1-m}+\sum_{i=l}^{n-1}\binom{i}{l}=\binom{n+1}{l+1}$$

and therefore

$$\delta_\lambda(K_t(X^k a_k))=X^k\partial_k(K_t(\lambda))+d_\theta(K_t(\lambda))K_t(X^k a_k).$$

A.5 Calculation of the commutators

A.5.1 Semi-classical construction

We calculate the commutator (5.19) (see also [41]), dropping the t-subscripts on θ_t for simplicity and using local expressions.

$$\begin{aligned}
[a_\theta, d_\theta(g)] &= -\theta^{ij} a_j \partial_i \theta^{kl} \partial_k g \partial_l - \theta^{ij} a_j \theta^{kl} \partial_i \partial_k g \partial_l \\
&\quad +\theta^{kl} \partial_k g \partial_l \theta^{ij} a_j \partial_i + \theta^{kl} \partial_k g \theta^{ij} \partial_l a_j \partial_i \\
&= -\theta^{kl} \partial_k \theta^{ij} a_j \partial_i g \partial_l - \theta^{kl} \theta^{ij} a_j \partial_k \partial_i g \partial_l - \theta^{kl} \theta^{ij} \partial_j a_k \partial_i g \partial_l \\
&= +\theta^{ij} f_{jk} \theta^{kl} \partial_i g \partial_l - \theta^{kl} \partial_k (\theta^{ij} a_j \partial_i g) \partial_l \\
&= -d_{\theta f \theta} g + d_\theta(a_\theta(g)) \\
&= -\partial_t(d_\theta) g + d_\theta(a_\theta(g)).
\end{aligned}$$

For (5.20) we get

$$\begin{aligned}
[a_\theta, X_t] &= \theta^{ij} a_j \partial_i X^k \partial_k - X^k \partial_k \theta^{ij} a_j \partial_i - X^k \theta^{ij} \partial_k a_j \partial_i \\
&= -\theta^{ij} X^k \partial_k a_j \partial_i - \theta^{ik} \partial_k X^j a_j \partial_i \\
&= X^k f_{ki} \theta^{ij} \partial_j + \theta^{ij} \partial_i (X^k a_k) \partial_j \\
&= -\partial_t X - d_\theta(X^k a_k).
\end{aligned}$$

A.5.2 Quantum construction

In [85], (4.3,4.4,4.6) have already been calculated, unluckily (and implicitly) using a different sign convention for the brackets of polyvectorfields. In [41], again a different sign convention is used, coinciding with the one in [85] in the relevant cases. In order to keep our formulas consistent with the ones used in [85, 41], we define our bracket on polyvectorfields π_1 and π_2 as in [85] to be

$$[\pi_1, \pi_2] = -[\pi_2, \pi_1]_S,$$

giving an extra minus sign for π_1 and π_2 both even. The bracket on polydifferential operators is always the Gerstenhaber bracket.

With these conventions and

$$d_\star = -[\cdot, \star],$$

we rewrite the formulas (4.6,4.5,4.3,4.4) so we can use them in the following

$$\begin{aligned}
[\Phi(X), \Phi(g)]_G &= \Phi([X, g]) + \Psi([\theta, g], X) - \Psi([\theta, X], g), && \text{(A.6)} \\
[\Phi(X), \Phi(Y)]_G &= d_\star \Psi(X, Y) && \text{(A.7)} \\
&\quad +\Phi([X, Y]) + \Psi([\theta, Y], X) - \Psi([\theta, X], Y), \\
d_\star \Phi(g) &= \Phi(d_\theta(g)), && \text{(A.8)} \\
d_\star \Phi(X) &= \Phi(d_\theta(X)). && \text{(A.9)}
\end{aligned}$$

For the calculation of the commutators of the quantum objects we first define

$$a_\star = \Phi(a_{\theta_t})$$

and

$$f_\star = \Phi(f_{\theta_t}).$$

With (A.9) we get the quantum version of (5.18)

$$f_\star = d_\star a_\star.$$

For functions f and g we get

$$\partial_t(f \star g) = \sum_{n=0}^{\infty} \frac{1}{n!} \partial_t U_n(\theta_t, \ldots, \theta_t)(f, g)$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} U_n(f_\theta, \ldots, \theta_t)(f, g) = f_\star(f, g).$$

With these two formulas we can now calculate the quantum version of (5.19) as in [41]. On two functions f and g we have

$$\begin{aligned} \partial_t(f \star g) &= f_\star(f, g) \\ &= d_\star a_\star(f, g) \\ &= -[a_\star, \star](f, g) \\ &= -a_\star(f \star g) + a_\star(f) \star g + f \star a_\star(g), \end{aligned}$$

where we used (A.1) in the last step. Therefore

$$\begin{aligned} [a_\star, d_\star(g)](f) &= a_\star(d_\star(g)(f)) - d_\star(g)(a_\star(f)) \\ &= a_\star([f \stackrel{\star}{,} g]) - [a_\star(f) \stackrel{\star}{,} g] \\ &= -\partial_t[f \stackrel{\star}{,} g] - [a_\star(g) \stackrel{\star}{,} f] \\ &= -\partial_t d_\star(g)(f) + d_\star(a_\star(g))(f). \end{aligned}$$

For a function g which might also depend on t the quantum version of (5.19) now reads

$$[a_\star + \partial_t, d_\star(g)] = d_\star(a_\star(g)).$$

We go on to calculate the quantum version of (5.20). We first note that

$$\partial_t \Phi(X_t) = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \partial_t U_n(X_t, \theta_t, \ldots, \theta_t) = \Phi(\partial_t X_t) + \Psi(f_\theta, X_t).$$

With this we get

$$\begin{aligned}
[\Phi(a_\theta), \Phi(X_t)] &= d_\star \Psi(a_\theta, X_t) + \Phi([a_\theta, X_t]) - \Psi([\theta_t\, a_\theta]) + \Psi([\theta_t, X_t], a_\theta) \\
&= d_\star \Psi(a_\theta, X_t) + \Phi(-d_\theta(X_t^k a_k)) + \Phi(-\partial_t X_t) - \Psi(f_\theta, X_t) \\
&= -d_\star(\Phi(X_t^k a_k) - \Psi(a_\theta, X_t)) - \partial_t \Phi(X_t),
\end{aligned}$$

where we have used (A.7).

Bibliography

[1] H. S. Snyder, *Quantized space-time*, Phys. Rev. **71**, 38 (1947).

[2] A. Connes, *Noncommutative geometry* (Academic Press, Inc., San Diego, New York, Boston, 1994).

[3] G. Landi, *An introduction to noncommutative spaces and their geometry*, (1997), hep-th/9701078.

[4] J. M. Gracia-Bondia, J. C. Varilly, and H. Figueroa, *Elements of noncommutative geometry*, Boston, USA: Birkhaeuser (2001) 685 p.

[5] J. Madore, *Noncommutative geometry for pedestrians*, (1999), gr-qc/9906059.

[6] J. Madore, *An introduction to noncommutative differential geometry and physical applications*, (2000), Cambridge, UK: Univ. Pr. (2000) 371p.

[7] B. DeWitt, *in Gravitation, edited by L. Witten*, p. 266 (1962).

[8] J. Madore, *Gravity on fuzzy space-time*, (1997), gr-qc/9709002.

[9] A. Connes, *Gravity coupled with matter and the foundation of noncommutative geometry*, Commun. Math. Phys. **182**, 155 (1996), hep-th/9603053.

[10] M. Chaichian, A. Demichev, and P. Presnajder, *Quantum field theory on noncommutative space-times and the persistence of ultraviolet divergences*, Nucl. Phys. **B567**, 360 (2000), hep-th/9812180.

[11] A. Matusis, L. Susskind, and N. Toumbas, *The IR/UV connection in the non-commutative gauge theories*, JHEP **12**, 002 (2000), hep-th/0002075.

[12] M. R. Douglas and N. A. Nekrasov, *Noncommutative field theory*, Rev. Mod. Phys. **73**, 977 (2001), hep-th/0106048.

[13] H. Grosse, J. Madore, and H. Steinacker, *Field theory on the q-deformed fuzzy sphere. I*, J. Geom. Phys. **38**, 308 (2001), hep-th/0005273.

[14] R. J. Szabo, *Quantum field theory on noncommutative spaces*, Phys. Rept. **378**, 207 (2003), hep-th/0109162.

[15] M. Chaichian, A. Demichev, and P. Presnajder, *Quantum field theory on the noncommutative plane with E(q)(2) symmetry*, J. Math. Phys. **41**, 1647 (2000), hep-th/9904132.

[16] H. Grosse, J. Madore, and H. Steinacker, *Field theory on the q-deformed fuzzy sphere. II: Quantization*, J. Geom. Phys. **43**, 205 (2002), hep-th/0103164.

[17] H. Steinacker, *Quantized gauge theory on the fuzzy sphere as random matrix model*, Nucl. Phys. **B679**, 66 (2004), hep-th/0307075.

[18] G. Landi, F. Lizzi, and R. J. Szabo, *A new matrix model for noncommutative field theory*, Phys. Lett. **B578**, 449 (2004), hep-th/0309031.

[19] J. Bellissard, A. van Elst, and H. Schulz-Baldes, *The Non-Commutative Geometry of the Quantum Hall Effect*, J. Math. Phys. **35**, 5373 (1994), hep-th/0002075.

[20] L. D. Faddeev, N. Y. Reshetikhin, and L. A. Takhtajan, *Quantization of Lie Groups and Lie Algebras*, Leningrad Math. J. **1**, 193 (1990).

[21] A. Lorek, W. Weich, and J. Wess, *Non-commutative Euclidean and Minkowski structures*, Z. Phys. **C76**, 375 (1997).

[22] W. B. Schmidke, J. Wess, and B. Zumino, *A q deformed Lorentz algebra*, Z. Phys. **C52**, 471 (1991).

[23] B. L. Cerchiai and J. Wess, *q-Deformed Minkowski Space based on a q-Lorentz Algebra*, Eur. Phys. J. **C5**, 553 (1998), math.qa/9801104.

[24] O. Ogievetsky, W. B. Schmidke, J. Wess, and B. Zumino, *q deformed Poincare algebra*, Commun. Math. Phys. **150**, 495 (1992).

[25] S. Majid and H. Ruegg, *Bicrossproduct structure of kappa Poincare group and noncommutative geometry*, Phys. Lett. **B334**, 348 (1994), hep-th/9405107.

[26] A. Ashtekar, A. Corichi, and J. A. Zapata, *Quantum theory of geometry. III: Non-commutativity of Riemannian structures*, Class. Quant. Grav. **15**, 2955 (1998), gr-qc/9806041.

[27] A. Ashtekar, J. C. Baez, and K. Krasnov, *Quantum geometry of isolated horizons and black hole entropy*, Adv. Theor. Math. Phys. **4**, 1 (2000), gr-qc/0005126.

[28] A. Konechny and A. Schwarz, *Introduction to M(atrix) theory and noncommutative geometry*, Phys. Rept. **360**, 353 (2002), hep-th/0012145.

[29] A. Connes, M. R. Douglas, and A. Schwarz, *Noncommutative geometry and matrix theory: Compactification on tori*, JHEP **02**, 003 (1998), hep-th/9711162.

[30] C.-S. Chu and P.-M. Ho, *Noncommutative open string and D-brane*, Nucl. Phys. **B550**, 151 (1999), hep-th/9812219.

[31] V. Schomerus, *D-branes and deformation quantization*, JHEP **06**, 030 (1999), hep-th/9903205.

[32] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, JHEP **09**, 032 (1999), hep-th/9908142.

[33] L. Cornalba and R. Schiappa, *Nonassociative star product deformations for D-brane worldvolumes in curved backgrounds*, Commun. Math. Phys. **225**, 33 (2002), hep-th/0101219.

[34] O. Andreev and H. Dorn, *On open string sigma-model and noncommutative gauge fields*, Phys. Lett. **B476**, 402 (2000), hep-th/9912070.

[35] H. Weyl, *Quantum mechanics and group theory*, Z. Phys. **46**, 1 (1927).

[36] D. Sternheimer, *Deformation quantization: Twenty years after*, AIP Conf. Proc. **453**, 107 (1998), math.qa/9809056.

[37] F. Bayen, M. Flato, C. Frønsdal, A. Lichnerowicz, and D. Sternheimer, *Deformation Theory and Quantization*, Ann. Phys. **111**, 61 (1978).

[38] M. Kontsevich, *Deformation quantization of Poisson manifolds, I*, (1997), q-alg/9709040.

[39] B. Jurco and P. Schupp, *Noncommutative Yang-Mills from equivalence of star products*, Eur. Phys. J. **C14**, 367 (2000), hep-th/0001032.

[40] B. Jurco, P. Schupp, and J. Wess, *Noncommutative gauge theory for Poisson manifolds*, Nucl. Phys. **B584**, 784 (2000), hep-th/0005005.

[41] B. Jurco, P. Schupp, and J. Wess, *Nonabelian noncommutative gauge theory via noncommutative extra dimensions*, Nucl. Phys. **B604**, 148 (2001), hep-th/0102129.

[42] B. Jurco, P. Schupp, and J. Wess, *Noncommutative line bundle and Morita equivalence*, Lett. Math. Phys. **61**, 171 (2002), hep-th/0106110.

[43] B. Jurco, S. Schraml, P. Schupp, and J. Wess, *Enveloping algebra valued gauge transformations for non- Abelian gauge groups on non-commutative spaces*, Eur. Phys. J. **C17**, 521 (2000), hep-th/0006246.

[44] B. Jurco, L. Moller, S. Schraml, P. Schupp, and J. Wess, *Construction of non-Abelian gauge theories on noncommutative spaces*, Eur. Phys. J. **C21**, 383 (2001), hep-th/0104153.

[45] B. L. Cerchiai, A. F. Pasqua, and B. Zumino, *The Seiberg-Witten map for noncommutative gauge theories*, (2002), hep-th/0206231.

[46] J. M. Grimstrup, T. Jonsson, and L. Thorlacius, *The Seiberg-Witten map on the fuzzy sphere*, JHEP **12**, 001 (2003), hep-th/0310179.

[47] M. Dimitrijevic *et al.*, *Deformed field theory on kappa-spacetime*, Eur. Phys. J. **C31**, 129 (2003), hep-th/0307149.

[48] M. Dimitrijevic, F. Meyer, L. Moller, and J. Wess, *Gauge theories on the kappa-Minkowski spacetime*, (2003), hep-th/0310116.

[49] D. Mikulovic, *Seiberg-Witten map for superfields on canonically deformed $N = 1$, $d = 4$ superspace*, (2003), hep-th/0310065.

[50] V. Putz and R. Wulkenhaar, *Seiberg-Witten map for noncommutative super Yang-Mills theory*, Int. J. Mod. Phys. **A18**, 3325 (2003), hep-th/0205094.

[51] P. Aschieri, B. Jurco, P. Schupp, and J. Wess, *Non-commutative GUTs, standard model and C, P, T*, Nucl. Phys. **B651**, 45 (2003), hep-th/0205214.

[52] T. Filk, *Divergencies in a field theory on quantum space*, Phys. Lett. **B376**, 53 (1996).

[53] T. Asakawa and I. Kishimoto, *Comments on gauge equivalence in noncommutative geometry*, JHEP **11**, 024 (1999), hep-th/9909139.

[54] A. Bichl *et al.*, *Renormalization of the noncommutative photon self-energy to all orders via Seiberg-Witten map*, JHEP **06**, 013 (2001), hep-th/0104097.

[55] I. L. Buchbinder and V. A. Krykhtin, *One-loop renormalization of general noncommutative Yang- Mills field model coupled to scalar and spinor fields*, Int. J. Mod. Phys. **A18**, 3057 (2003), hep-th/0207086.

[56] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, *Deformation Theory and Quantization. 2. Physical Applications*, Ann. Phys. **111**, 111 (1978).

[57] M. DeWilde and P. B. A. Lecomte, *Existence of Star-Products and of Formal Deformations of the Poisson Lie Algebra of Arbitrary Symplectic Manifolds*, Lett. Math. Phys. **7**, 487 (1983).

[58] B. V. Fedosov, *A Simple Geometrical Construction of Deformation Quantization*, J. Diff. Geom. **40**, 213 (1994).

[59] A. S. Cattaneo and G. Felder, *A path integral approach to the Kontsevich quantization formula*, Commun. Math. Phys. **212**, 591 (2000), math.qa/9902090.

[60] J. Wess, *q-deformed Heisenberg Algebras*, (1999), math.ph/9910013.

[61] B. L. Cerchiai, R. Hinterding, J. Madore, and J. Wess, *A Calculus Based on a q-deformed Heisenberg Algebra*, Eur. Phys. J. **C8**, 547 (1999), math.qa/9809160.

[62] J. Wess and B. Zumino, *Covariant Differential Calculus on the Quantum Hyperplane*, Nucl. Phys. Proc. Suppl. **18B**, 302 (1991).

[63] J. Madore, S. Schraml, P. Schupp, and J. Wess, *Gauge theory on noncommutative spaces*, Eur. Phys. J. **C16**, 161 (2000), hep-th/0001203.

[64] F. Meyer, *Models of gauge field theory on noncommutative spaces*, (2003), hep-th/0308186.

[65] S. Schraml, *Untersuchung nichtkommutativer Räume als Grundlage für physikalische Probleme*, (2000).

[66] W. Behr and A. Sykora, *Construction of gauge theories on curved noncommutative spacetime*, (2003), hep-th/0309145.

[67] C. Jambor and A. Sykora, *Realization of star-products with the help of commuting vector fields*, (2004), hep-th/0405268.

[68] A. Weinstein, *The local structure of Possion manifolds*, J. Diff. Geom. **18**, 523 (1983).

[69] H. Ocampo, *$SO_q(4)$ quantum mechanics*, Z. Phys. C **70**, 525 (1996).

[70] A. Kempf, *Quantum group symmetric Bargmann-Fock construction*, (1992), hep-th/9211022.

[71] A. Kempf, *Quantum group symmetric Fock spaces with Bargmann-Fock representation*, Lett. Math. Phys. **26**, 1 (1992).

[72] A. Aghamohammadi, *The two-parametric extension of h deformation of $GL(2)$ and the differential calculus on its quantum plane*, Mod. Phys. Lett. A **8**, 2607 (1993).

[73] P. Nurowski, E. Schücking, and A. Trautman, *Relativistic gravitational fields with close Newtonian analogs*, Published as Chapter 23 in *On Einstein's Path*, A. Harvey, editor, Springer, New York 1999.

[74] M. Maceda, J. Madore, P. Manousselis, and G. Zoupanos, *Can noncommutativity resolve the big-bang singularity?*, (2003), hep-th/0306136.

[75] D. Arnal, D. Manchon, and M. Masmoudi, *Choix des signes pour la formalite de M. Kontsevich*, (2000), math.qa/0003003, math.qa/0003003.

[76] G. Felder and B. Shoikhet, *Deformation quantization with traces*, (2000), math.qa/0002057.

[77] B. L. Cerchiai, R. Hinterding, J. Madore, and J. Wess, *The Geometry of a q-Deformed Phase Space*, Eur. J. Phys. **C8**, 533 (1999), math.qa/9807123.

[78] A. Dimakis and J. Madore, *Differential calculi an linear connections*, J. Math. Phys. **37**, 4647 (1996).

[79] B. L. Cerchiai, G. Fiore, and J. Madore, *Frame formalism for the N-dimensional quantum Euclidean spaces*, (2000), math.qa/0007044.

[80] A. Perez and C. Rovelli, *Observables in quantum gravity*, (2001), gr-qc/0104034.

[81] C. Rovelli, *The century of the incomplete revolution: Searching for general relativistic quantum field theory*, J. Math. Phys. **41**, 3776 (2000), hep-th/9910131.

[82] N. Ishibashi, S. Iso, H. Kawai, and Y. Kitazawa, *Wilson loops in noncommutative Yang-Mills*, Nucl. Phys. **B573**, 573 (2000), hep-th/9910004.

[83] Y. Okawa and H. Ooguri, *An exact solution to Seiberg-Witten equation of noncommutative gauge theory*, Phys. Rev. **D64**, 046009 (2001), hep-th/0104036.

[84] W. Behr and A. Sykora, *NC Wilson lines and the inverse Seiberg-Witten map for nondegenerate star products*, (2003), hep-th/0312138.

[85] D. Manchon, *Poisson bracket, deformed bracket and gauge group actions in Kontsevich deformation quantization*, (2000), math.qa/0003004, math.qa/0003004.