Invariance feedback entropy of uncertain nonlinear control systems

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Eidesstattliche Versicherung

Hiermit erkläre ich, Mahendra Singh Tomar, an Eides statt, dass die vorliegende Dissertation ohne unerlaubte Hilfe gemäß Promotionsordnung vom 12.07.2011, § 8, Abs. 2 Pkt. 5, angefertigt worden ist.

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Zusammenfassung

In der klassischen Kontrolltheorie geht man üblicherweise davon aus, dass Sensoren und Regler durch Punkt-zu-Punkt-Verkabelung miteinander verbunden sind. In vernetzten Kontrollsystemen (VKS) sind Sensoren und Regler oft räumlich verteilt und Daten werden mittels eines digitalen Kommunikationsnetzwerks übertragen. Im Vergleich zu klassischen Kontrollsystemen bieten VKS viele Vorteile wie z.B. reduzierte Verkabelung, geringe Installations- und Instandhaltungskosten, größere Systemflexibilität und einfache Modifizierbarkeit. VKS haben Anwendungen in vielen Bereichen, z. B. in der Fahrzeugtechnik, intelligenten Gebäuden und Transportnetzwerken. Jedoch macht die Verwendung von Kommunikationsnetzwerken in Regelschleifen die Analyse und den Entwurf von VKS wesentlich komplexer. Die Verwendung digitaler Kanäle in VKS beschränkt aufgrund der endlichen Bandbreite die Datenmenge, die pro Zeiteinheit von Sensoren zu Reglern übertragen werden kann. Dies führt zu Quantisierungsfehlern, welche die Regelungsperformance ungünstig beeinflussen können. Das Problem der Regelung und Zustandsschätzung über einen digitalen Kommunikationskanal mit beschränkter Bitrate hat in den letzten zwei Jahrzehnten viel Aufmerksamkeit erhalten.

Eine scharfe untere Schranke der Datenrate eines digitalen Kanals zwischen dem Kodierer (in Sensornähe) und dem Regler, die zum Erreichen eines Regelungsziels wie z.B. Stabilisierung oder Invarianz benötigt wird, kann durch einen passenden Entropiebegriff als intrinsische Größe des Systems charakterisiert werden, und hängt nicht von der Wahl des Kodierers und Reglers ab.

Im ersten Teil der Arbeit beschreiben wir die Invarianz-Feedback-Entropie (IFE), die den Begriff der Invarianz-Entropie für deterministische nichtlineare Kontrollsysteme auf unsichere Systeme erweitert. Die IFE charakterisiert die Zustandsinformation, die von einem Regler benötigt wird, um eine Teilmenge Q des Zustandsraums invariant zu machen. Wir diskutieren eine Anzahl von elementaren Eigenschaften der IFE, z.B. Bedingungen für ihre Endlichkeit und die im deterministischen Spezialfall vorliegende Äquivalenz zum wohlbekannten Begriff der Invarianz-Entropie (IED). Wir analysieren unsichere lineare Kontrollsysteme und leiten eine universelle Unterschranke der IFE her.

Im zweiten Teil der Arbeit betrachten wir vernetzte Kontrollsysteme und streben eine obere Schranke der IFE eines Netzwerks in Termen der IFE der Teilsysteme an. Außerdem präsentieren wir drei technische Resultate. Zuerst zeigen wir, dass die IFE einer nichtleeren Teilmenge Q des Zustandsraums eines zeitdiskreten unsicheren Kontrollsystems nach oben durch die größte IFE der Mengen in einer beliebigen endlichen Partition von Q beschränkt

ist. Im zweiten Resultat betrachten wir unsichere Kontrollsysteme Σ_1 und Σ_2 mit identischen Zustands- und Eingangsräumen. Die mengenwertigen Übergangsfunktionen F_1 und F_2 der beiden Systeme sind nach Annahme so beschaffen, dass das Bild eines beliebigen Zustands-Eingangs-Paars unter F_1 in dem entsprechenden Bild unter F_2 enthalten ist. Für eine gegebene nichtleere Teilmenge des Zustandsraums zeigen wir, dass die IFE von Σ_2 größer oder gleich derjenigen von Σ_1 ist. Das dritte Resultat zeigt, dass die IFE niemals kleiner wird, wenn man die Menge der Kontrolleingänge verkleinert. Um die Effektivität der Resultate zu illustrieren, berechnen wir eine Ober- und eine Unterschranke der IFE eines Netzwerks von unsicheren, linearen, zeitdiskreten Systemen, welche den zeitlichen Verlauf der Temperaturen in 100 Räumen eines zirkulären Gebäudes beschreiben.

Im letzten Teil der Arbeit präsentieren wir Algorithmen für die numerische Abschätzung der IFE. Dazu betrachten wir zunächst eine Partition einer gegebenen Teilmenge Q des Zustandsraums. Dann wird ein Regler in Form einer Suchtabelle berechnet, die jedem Element der Partition eine Menge von Kontrollwerten zuordnet, welche die Invarianz von Q garantieren. Nach der Reduktion der Suchtabelle von einer mengenwertigen zu einer einwertigen Abbildung, wird ein gewichteter Graph konstruiert. Für deterministische Systeme liefert der Logarithmus des Spektralradius einer Übergangsmatrix, die aus dem Graphen ermittelt wird, eine obere Schranke der Entropie. Für unsichere Systeme stellt das maximale durchschnittliche Zyklusgewicht des Graphen eine Oberschranke der IFE dar. Im deterministischen Fall zeigen wir, dass der Wert der ersten Oberschranke nicht größer als derjenige der zweiten Oberschranke ist. Als nächstes präsentieren wir die Ergebnisse der Algorithmen angewandt auf drei deterministische Beispielsysteme, für welche der exakte Wert der IED bekannt ist oder durch andere Methoden abgeschätzt werden kann. Zusätzlich liefert unser Algorithmus ein statisches Kodierungs- und Regelungsprotokoll, das der Schranke an die Datenrate entspricht. Schließlich präsentieren wir die berechneten Oberschranken der IFE eines unsicheren linearen Kontrollsystems.

Abstract

In classical control theory, the sensors and controllers are usually connected through point-to-point wiring. In networked control systems (NCS), sensors and controllers are often spatially distributed and involve digital communication networks for data transfer. Compared to classical control systems, NCS provide many advantages such as reduced wiring, low installation and maintenance costs, greater system flexibility and ease of modification. NCS find applications in many areas such as automobiles, intelligent buildings, and transportation networks. However, the use of communication networks in feedback control loops makes the analysis and design of NCS much more complex. In NCS, the use of digital channels for data transfer from sensors to controllers limits the amount of data that can be transferred per unit of time, due to the finite bandwidth of the channel. This introduces quantization errors that can adversely affect the control performance. The problem of control and state estimation over a digital communication channel with a limited bit rate has attracted a lot of attention in the past two decades.

A tight lower bound on the data rate of a digital channel between the coder (near the sensor) and the controller, to achieve some control task such as stabilization or invariance, can be characterized in terms of some appropriate notion of entropy which is described as an intrinsic property of the system and is independent of the choice of the coder-controller.

In the first part of this thesis, we describe invariance feedback entropy (IFE) that extends the notion of invariance entropy of deterministic nonlinear control systems to those with uncertainty. The IFE characterizes the necessary state information required by any controller to render a subset Q of the state space invariant. We discuss a number of elementary properties of the IFE, e.g. conditions for its finiteness and its equivalence to the well-known notion of invariance entropy (IED) in the deterministic case. We analyze uncertain linear control systems and derive a universal lower bound of the IFE.

In the second part of this thesis, we consider interconnected control systems and seek to upper bound the IFE of the network using the IFE of the subsystems. In addition, we present three technical results related to the IFE. First, we show that the IFE of a nonempty subset Q of the state space of a discrete-time uncertain control system is upper bounded by the largest possible IFE among the members of any finite partition of Q. Second, we consider two uncertain control systems, Σ_1 and Σ_2, that have identical state spaces and identical control input sets. The set valued transition functions, F_1 and F_2, of the two systems are such that the image of any state-input pair under F_1 is a subset of that under F_2. For a given nonempty subset of the state space, we show that the IFE of Σ_2 is larger than or equal to the IFE of Σ_1. Third, we show that the IFE will never decrease by reducing the set of control inputs. To illustrate the effectiveness of the results, we compute an upper bound and a lower bound of the IFE of a network of uncertain, linear, discrete-time subsystems describing the evolution of temperatures of 100 rooms in a circular building.

In the last part of this thesis, we present algorithms for the numerical estimation of the IFE. In particular, given a subset Q of the state space, we first partition it. Then a controller, in the form of a lookup table that assigns a set of control values to each cell of the partition, is computed to enforce invariance of Q. After reduction of the lookup table to a single-valued map from a set-valued one, a weighted directed graph is constructed. For deterministic systems, the logarithm of the spectral radius of a transition matrix obtained from the graph gives an upper bound of the entropy. For uncertain systems, the maximum mean cycle weight of the graph upper bounds the IFE. For deterministic systems, the value of the first upper bound is shown to be lower than or equal to the value of the second upper bound. Next, we present the results of the algorithms applied to three deterministic examples for which the exact value of the IED is known or can be estimated by other techniques. Additionally, our algorithm provides a static coder-controller scheme corresponding to the obtained data-rate bound. Finally, we present the computed upper bounds of the IFE for an uncertain linear control system.

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Chapter 1

Introduction

In classical control theory, the sensors and controllers are usually connected through point-to-point wiring. In networked control systems (NCS), sensors and controllers are often spatially distributed and involve digital communication networks for data transfer. Compared to classical control systems, NCS provide many advantages such as reduced wiring, low installation and maintenance costs, greater system flexibility and ease of modification. NCS find applications in many areas such as car automation, remote surgery, intelligent buildings, and transportation networks. However, the use of communication networks in feedback control loops makes the analysis and design of NCS more complex. In NCS, the use of digital channels for data transfer from the sensors to controllers, limit the amount of data that can be transferred per unit of time, due to the finite bandwidth of the channel. This introduces quantization errors that can adversely affect the control performance.

Data rate constrained feedback is a maturate research topic and has been extensively studied for linear control systems and asymptotic stabilizability, see e.g. [60] and references therein. For linear control systems, the critical data rate has been characterized in terms of the unstable eigenvalues of the system matrix under various assumptions on the system model, channel model, communication protocol, and stabilization/estimation objectives [35, 58, 73]. In [73], for discrete-time linear control systems a lower bound on the data rate of the digital channel between the coder and controller was presented such that asymptotic observation and stabilization cannot be realized below this value. Comprehensive reviews of results on data-rate-limited control can be found, e.g., in the articles [60, 2, 27] and books [84, 54, 26, 38].

The topological entropy of a discrete-time linear system is also given by the logarithm of the absolute value of the unstable determinant. This relation between the minimal data rate and the topological entropy apparently inspired researchers to study entropy notions for nonlinear dynamics and different control objectives. Topological entropy characterizes the maximal exponential rate at which information about the initial state is generated by a dynamical system with increasing time. It can also be described as a measure of the growth rate of the smallest number of trajectories necessary to approximate the state of a dynamical system with arbitrarily fine but finite precision.

For nonlinear systems, the smallest bit rate of a digital channel between the coder

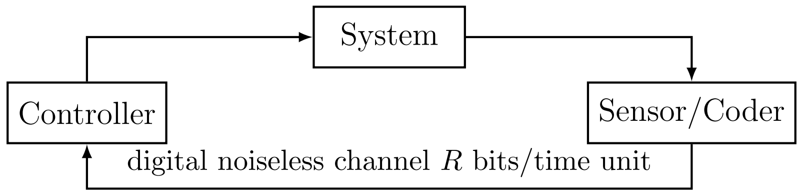


Figure 1.1: Coder-controller feedback loop.

and the controller, to achieve some control task such as stabilization or invariance, can be characterized in terms of certain notions of entropy which are described as intrinsic quantities of the open-loop system and are independent of the choice of the coder-controller. In spirit, they are similar to classical entropy notions used in the theory of dynamical systems to quantify the rate at which a system generates information about the initial state, see e.g. [37].

In this thesis we study the classical feedback control loop, in which a controller that is feedback connected with a given system is used to enforce a prespecified control task in the closed loop. Unlike in the classical setting, we do not assume that the sensor (or coder) is able to transmit an infinite amount of information to the controller, but is restricted to use a digital noiseless channel with a bounded data rate to communicate with the controller. The closed loop of such a feedback is illustrated in Fig. 1.1. In this context, we are interested in characterizing the minimal data rate of the digital channel between coder and controller that enables the controller to achieve the given control task. Or equivalently, we are interested in quantifying the information required by the controller to achieve a given control goal.

We focus on the control task of rendering a given nonempty subset of the state space invariant. Invariance specifications are one of the most fundamental system requirements and are ubiquitous in the analysis and control of dynamical systems [4, 11]. In [59], Nair et. al extended the well-known notion of topological entropy of dynamical systems [1, 12, 24] to discrete-time deterministic control systems by making use of open-loop control functions and open covers. They introduced the notion of topological feedback entropy that quantifies the rate at which a deterministic, discrete-time control system generates information, with states confined in a given compact set. They showed that this notion of entropy is equal to the smallest average data rate at which a subset Q of the state space can be made invariant. In other words, to enforce set-invariance over a noiseless digital channel in the feedback loop, the bit rate of the channel must not be less than the entropy of the plant. Thus for set-invariance, the channel must transfer information at a rate faster than the rate of information generated by the system. The topological feedback entropy is defined based on open covers of Q, where each cover element is associated with an open loop control sequence of some finite length, that ensures that the system with initial state in the cover element evolves inside Q. Then the entropy measures the minimal exponential growth rate, of the smallest cardinality among the subcovers, over increasing time. Later Colonius and Kawan [19] introduced a notion of invariance entropy for continuous-time deterministic control systems. The notion of invariance entropy in [19] is based on the minimum cardinality of all sets of control functions that can make Q invariant. They used the growth rate of the number of open-loop control functions necessary to enforce Q invariant over a growing time horizon to characterize the minimum data rate required to achieve invariance. While the definition in [59] clearly resembles the definition of entropy for dynamical systems in [1] based on open covers, the invariance entropy introduced in [19] is close to the notion of entropy in [12, 24] based on spanning sets. Both notions coincide for discrete-time control systems provided that a strong invariance condition holds [21, 38]. The monograph [38] presents elaborate exposition on invariance entropy and its estimates in terms of dynamical quantities such as Lyapunov exponents.

In this dissertation, we continue this line of research and study a notion of invariance feedback entropy (IFE) [66, 67] for uncertain control systems to quantify the necessary state information required by any controller to render a subset Q of the state space invariant in the closed loop. IFE equivalently also quantifies the smallest asymptotic average bit rate, from the coder to the controller in the feedback loop, above which Q can be made invariant over a digital noiseless channel. Since uncertain systems are considered here, open-loop control functions cannot be used, instead, the IFE is defined using invariant covers of Q. For the case of deterministic control systems, the IFE is shown to be equivalent to the invariance entropy; see Theorem 4.

1.1 Related Literature

Various offshoots of invariance entropy have been proposed to tackle different control problems or other classes of systems, see for instance [15] (exponential stabilization), [43] (invariance in networks of systems), [66] (invariance for uncertain control systems), [17, 78] (measure-theoretic versions of invariance entropy) and [47] (stochastic stabilization).

In [23], for deterministic nonlinear systems, the invariance entropy is shown to vary continuously with respect to system parameters, under some assumptions. This lead to robustness of the critical data-rate with respect to small perturbations. In [20], a version of invariance entropy for partially observed, continuous-time systems with outputs is investigated with the control objective to make a subset of the output space invariant. A lower bound on the invariance entropy for a class of partially hyperbolic sets is discussed in [42]. Two extensions of the topological feedback entropy are studied in [33], one for systems with outputs (partial observation) and one for systems with discontinuous transition function, with the objective to steer the system into a target set. In [43], for networks of discrete-time, deterministic control systems, a notion of subsystem invariance entropy was introduced to characterize the smallest data rate, from a centralized controller to the subsystem, which is required to make a subset Q of the state set invariant.

Stabilization

Minimal bit rates and entropy for exponential stabilization of continuous-time control systems is discussed in [15]. A discussion on a notion of topological entropy and its relation to global exponential stability for switched linear systems is presented in [80]. For stochastic nonlinear systems controlled over a possibly noisy communication channel, the paper [47] analyzes the largest class of channels for which there exist feedback control schemes for stabilization under a given stochastic stability criterion. The authors introduce a notion of entropy to derive lower bounds on the required channel capacity for stabilization. In [22], the authors consider linear deterministic control systems and provide a zero-delay coder-decoder scheme for stabilization that operates at a data rate equal to the topological entropy of the system.

Estimation

The problem of state estimation over digital channels has also been studied extensively by several groups of researchers. As it turns out, the classical notions of entropy used in dynamical systems, namely measure-theoretic and topological entropy (or small variations of them), can be used to describe the smallest data rate or channel capacity above which the state of an autonomous dynamical system can be estimated with arbitrary precision, see [68, 50, 69, 82, 45]. For uncertain dynamical systems, [68] studied state estimation under limited bit rate together with topological entropy (based on spanning sets). The paper [50] introduced the notion of estimation entropy to characterize the critical data rate for exponential state estimation with a given exponent for a continuous-time system on a compact subset of its state-space. The estimation entropy equals the topological entropy in case the value of the exponent is zero. This notion of entropy is defined in terms of the number of system trajectories that approximate all other trajectories up to an exponentially decaying error. Further, they provide an alternative equivalent definition of estimation entropy, which uses approximating functions that can be different from the trajectories of the system. They combined ideas from [68] and [15]. As in [68], the focus is on state estimation rather than control. Similar to [15], they require that state estimates converge at a prescribed exponential rate. The paper [39] provided a lower bound on the estimation entropy in terms of Lyapunov exponents under certain assumptions. An extension of the estimation entropy to a class of stochastic hybrid systems is provided in [6]. In [70], the authors present a notion of topological entropy to lower bound the bit rate needed to estimate the state of a nonlinear dynamical system, with unknown bounded inputs, up to a constant error. For networked systems, relation between observation rate and topological entropy is discussed in [53]. The paper analyzes the rate at which a discrete-time, deterministic, and possibly large network of nonlinear systems generates information, and analyzes the minimal data-rate for observation of the current state of the network.

The study of topological entropy to characterize the minimum data-rate for observation has the drawback that topological entropy can be discontinuous with respect to the dynamical system. This can lead to estimation schemes to suffer from lack of robustness. This lack of robustness and difficulty in implementation of estimation schemes based on topological entropy led to the study of three different types of observability criteria in [51] which later led to the introduction of the notion of restoration entropy for continuous-time systems in [52]. Restoration entropy characterizes the minimal data rate above which the state of a system can be estimated so that the estimation quality is not just preserved but can also be improved. In [40], the authors extended the notion of restoration entropy to discrete-time systems and show that for most dynamical systems it strictly exceeds the topological entropy. This implies that satisfaction of a state estimation objective that is more robust with respect to perturbations requires a higher rate of data transmission than non-robust ones. In [44], the authors describe a new characterization of the restoration entropy that does not need computation of any temporal limit. They show that a proper choice of Riemannian metric can enable the computation of the exact value of the restoration entropy.

The result in [46] analyzes the problem of optimal zero-delay coding and estimation of a stochastic dynamical system over a noisy communication channel under three estimation criteria and derives lower bounds on the smallest channel capacity above which the objective can be achieved with an arbitrarily small error. In [45], the authors investigated the same problem for the case of deterministic systems with discrete noiseless channels.

Algorithms for state estimation over digital channels have been proposed in several works; see [50, 51, 32].

Switched systems

In [69], the notion of estimation entropy was extended to the case of switched nonlinear dynamical systems with unknown switching signals but known dwell time. This entropy lower bounds the data-rate needed to estimate the state with an error that decays exponentially but only after a specified period of time after each switch. In [77], a closed form expression for the estimation entropy [50] is provided for a class of switched linear systems in terms of the system's Lyapunov exponents under mild restrictions on switching signals. Switched Linear Systems (SLSs) are those described by a finite set of linear modes, among which the systems can switch. The paper in [82] introduced a notion of topological entropy for switched systems, defined in terms of the minimal number of initial states needed to approximate all initial states with a finite precision. The notion is studied for different classes of SLSs in [79, 83]. The paper in [80] discussed relation between topological entropy as defined in [82] and global exponential stability. They show that a SLS is globally exponentially stable if its topological entropy remains zero under a destabilizing perturbation. In [82], the topological entropy is analyzed for SLSs with a fixed switching signal. For SLSs with arbitrary switching, [10] introduced the notion of worst-case topological entropy defined as the largest topological entropy over the set of all possible switching signals. It is shown that this quantity is equal to the minimal data rate required for state observation with exponentially decreasing estimation error, and that practical coders–decoders can be designed to operate arbitrarily close to this data rate. A data-rate larger than this entropy will be sufficient to observe the state of the system for every switching signal.

Sufficient data rates for feedback stabilization of SLSs were established in [49, 81, 9, 8].

1.2 Outline of the thesis

In Section 2.2, we motivate the need of the novel notion of invariance feedback entropy. We define the IFE and establish various elementary properties in Section 2.3. In Section 2.4, we establish the data rate theorem. In Section 2.5, we derive a lower bound on the IFE of uncertain linear control systems. The lower bound is invariant under state space transformations and recovers the well-known minimal data rate (sum of the logarithms of the unstable eigenvalues of the system matrix) in the absence of uncertainties. Additionally, we derive a lower bound of the data rate of any static, memoryless coder-controller. We show that the lower bounds are tight for certain classes of systems.

In Section 3.2, we show three additional useful properties of the IFE. In Section 3.3, we show how one can approximate the IFE of a network of uncertain control systems and a set Q using the IFEs of subsystems. In Section 3.4.1, by an example, we demonstrate that this upper bound can be tight. Finally, in Section 3.4.2, we compute an upper bound and a lower bound of the IFE of an uncertain, linear, discrete-time system, that describes the evolution of temperature of 100 rooms in a circular building.

Section 4.2 presents the fundamental definitions for invariance entropy of deterministic systems (IED). In Section 4.3, we describe in detail the implementational steps of our algorithm to compute an upper bound of the IED and illustrate them by a two-dimensional linear example. Section 4.4 presents two upper bounds for the IFE. Section 4.5 describes the relationship between the discussed upper bounds for IED and IFE in the case of deterministic systems. The results of our proposed algorithms for deterministic systems are illustrated on a linear and two nonlinear examples in Section 4.6, in which we also present upper bounds of the IFE computed for a two-dimensional uncertain linear system. Moreover, for the uncertain linear example, we analytically compute a lower bound for comparison. Finally, Section 4.7 contains some comments on the performance of our algorithms.

Chapter 2

Invariance Feedback Entropy

2.1 Introduction

In this chapter we study the notion of invariance feedback entropy (IFE) and establish some of its properties. IFE quantifies the smallest asymptotic average bit rate, from the coder to the controller in the feedback loop, above which a subset Q of the state set can be made invariant over a digital noiseless channel.

2.1.1 Contributions

The contents of this chapter have been published in the journal IEEE Transactions on Automatic Control [75]. It is a joint work with Dr. Matthias Rungger and Prof. Majid Zamani. I established the Theorems 1, 4 and 5, and the Lemma 9. I revised the Example 2, the proof of the Lemmas 4 and 2 and the Remark 2. I also revised the Theorems 7 and 8 to improve the lower bounds through subspace projection and also added the Remark 1. Rest of the work was done by Dr. Matthias Rungger. Prof. Majid Zamani supervised the work.

We establish a number of elementary properties of the IFE, e.g., we provide conditions that ensure that the IFE is finite and show that we recover the well-known notion of entropy for deterministic control systems. When there is a feedback refinement relation [61] from one system to another one, we show that the entropy of the former is not larger than the latter. This result generalizes the fact that the invariance entropy of deterministic control systems cannot increase under semiconjugation [19, Thm 3.5], [38, Prp. 2.13]. We prove the data rate theorem, which shows that the invariance entropy is a tight lower bound of the data rate of any coder-controller that achieves invariance in the closed loop. To this end, we introduce a history-dependent notion of data rate. We discuss possible alternative data rate definitions and motivate our particular choice by two examples. We analyze uncertain linear control systems and derive a universal lower bound of the IFE. The lower bound depends on the absolute value of the determinant of the system matrix and a ratio involving the volume of the invariant set and the set of uncertainties. The lower bound is invariant under state space transformations and recovers the well-known minimal data rate [60] in the absence of uncertainties. Furthermore, we derive a lower bound of the data rate of any static, memoryless coder-controller. Both lower bounds are intimately related and for certain cases it is possible to bound the performance loss due to the restriction to static coder-controllers by 1 bit/time unit. We show that the lower bounds are tight for certain classes of systems.

2.1.2 Notations

We denote by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ the set of natural, integer and real numbers, respectively. We annotate those symbols with subscripts to restrict the sets in the obvious way, e.g. $\mathbb{R}_{>0}$ denotes the positive real numbers. We denote the closed, open and half-open intervals in $\mathbb{R}$ with endpoints a and b by $[a,b]$, $]a,b[$, $[a,b[$, and $]a,b]$, respectively. The corresponding intervals in $\mathbb{Z}$ are denoted by $[a;b]$, $]a;b[$, $[a;b[$, and $]a;b]$, i.e., $[a;b] = [a,b] \cap \mathbb{Z}$ and $[a;a[= \varnothing$.

For a set A, we use $\#A \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ to denote the number of elements of A, i.e., if A is finite we have $\#A \in \mathbb{Z}_{\geq 0}$ and $\#A = \infty$ otherwise. Given two sets A and B, we say that A is smaller (larger) than B if $\#A \leq \#B$ ($\#A \geq \#B$) holds. A set J of subsets of A is said to *cover* B, where $B \subseteq A$, if B is a subset of the union of the elements of J. A *cover* of a set B, is a set of subsets of B that covers B.

We use $\exists_{a \in A} x = a$ to refer to: there exists a in A such that $x = a$. In a similar way, $\forall_{a \in A} x = a$ is used. Given two sets $A, B \subseteq \mathbb{R}^n$, we define the set addition by $A + B := \{x \in \mathbb{R}^n \mid \exists_{a \in A}, \exists_{b \in B}\ x = a + b\}$. For $A = \{a\}$, we slightly abuse notation and use $a + B = \{a\} + B$. The symbols $\operatorname{cl} A$, $\operatorname{int} A$ and $\wp(A)$ denote the closure, the interior and the power set of a set A, respectively. We call a set $A \subseteq \mathbb{R}^n$ measurable if it is Lebesgue measurable and use $\mu(A)$ to denote its measure [72]. We use id to denote an identity map. For a linear space $\mathbb{E}$, we denote it's dimension by $\texttt{dim}(\mathbb{E})$.

We follow [63] and use $f \colon A \rightrightarrows B$ to denote a *set-valued map* from A into B, whereas $f \colon A \to B$ denotes an ordinary map. If f is set-valued, then f is *strict* if for every $a \in A$ we have $f(a) \neq \varnothing$. The restriction of f to a subset $M \subseteq A$ is denoted by $f|_M$. By convention we set $f|_\varnothing := \varnothing$. The composition of $f : A \rightrightarrows B$ and $g : C \rightrightarrows A$, $(f \circ g)(x) = f(g(x))$ is denoted by $f \circ g$. We use B^A to denote the set of all functions $f : A \to B$. For a relation $R \subseteq A \times B$ and $D \subseteq A$, we define $R(D) := \cup_{d \in D} R(d)$.

The concatenation of two functions $x : [0;a[\to X$ and $y : [0;b[\to X$ with $a \in \mathbb{N}$ and $b \in \mathbb{N} \cup \{\infty\}$ is denoted by xy which we define by $xy(t) := x(t)$ for $t \in [0;a[$ and $xy(t) := y(t-a)$ for $t \in [a, a+b[$. We use $\inf \varnothing = \infty$, $\log_2 \infty = \infty$ and $0 \cdot \infty = 0$.

For scalars a, b and sets A, B, by $a \cdot b$ and $A \times B$ we denote the scalar product and the Cartesian product, respectively. For a set A, a *partition* is a collection of disjoint nonempty subsets of A that have A as their union. By $[a_0 a_1 \ldots a_{N-1}]$, $a_i \in \mathbb{N}$, we denote a finite sequence of integers of length N, also called a *word*. An element of the set $\mathbb{N}^{\mathbb{Z}}$ is referred to as a bi-infinite word. We use the notation $|\cdot|$ for the absolute value of a complex number. For an $n \times n$ matrix B, by $\lambda(B)$, $\rho(B)$ and $B_{i,j}$ we denote an eigenvalue of B, the spectral radius of B and the entry in the j-th column of the i-th row of B, respectively.

2.2 Motivation

We study data rate constrained feedback for discrete-time *uncertain control systems* described as

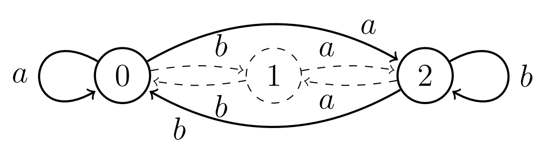
$$\xi(t+1) \in F(\xi(t), \nu(t)) \tag{2.1}$$

where $\xi(t) \in X$ is the *state signal* and $\nu(t) \in U$ is the *input signal*. The sets X and U are referred to as *state alphabet* and *input alphabet*, respectively. The map $F : X \times U \rightrightarrows X$ is called the *transition function*.

We are interested in coder-controllers that force the system (2.1) to evolve inside a nonempty set Q of the state alphabet X, i.e., every state signal ξ of the closed loop illustrated in Fig. 1.1 with $\xi(0) \in Q$ satisfies $\xi(t) \in Q$ for all $t \in \mathbb{Z}_{\geq 0}$. Specifically, we are interested in the average data rate of such coder-controllers.

Notably, our system description is rather general and, depending on the structure of alphabets X and U, we can represent a variety of commonly used system models. If we assume X and U to be discrete, we can use (2.1) to represent discrete event systems[1] [14] and digital/embedded systems [7]. Let us consider the following simple example.

Example 1. Consider a system with state alphabet and input alphabet given by $X := \{0, 1, 2\}$ and $U := \{a, b\}$, respectively. The transition function is illustrated by:



The set of interest is defined to be $Q := \{0, 2\}$. The states that are outside Q, and the transitions that lead to them, are indicated by dashed lines. When the system is in state 0 the only valid input is given by a. Similarly, if the system is in state 2 the only valid input is given by b. If the input a is applied at 0 at time t, the system can either be in 0 or 2 at time $t + 1$. Note that the valid control inputs for the states 0 and 2 differ and the controller is required to have exact state information at every point in time. Due to the nondeterministic transition function, it is not possible to determine the current state of the system based on the knowledge of the past states, the past control inputs and the transition function. Therefore, the controller can obtain the state information only through measurement, which implies that at least one bit needs to be transmitted at every time step. $\square$

Current theories from [59, 19, 38, 16] are unable to explain the minimal data rate of one bit per time step observed in Example 1.

If we allow X and U to be (subsets of) Euclidean spaces, we are able to recover one of the most fundamental system models in control theory, i.e., the class of nonlinear control systems with bounded uncertainties [28, 11]. If the system description is given in continuous-time, we can use (2.1) to represent the sampled-data system [48] with sampling

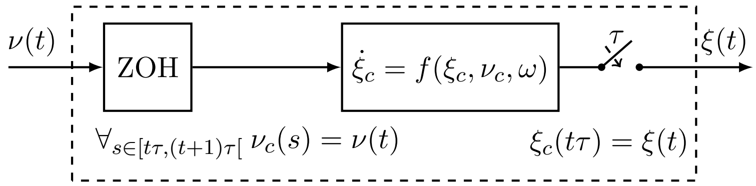


Figure 2.1: Sampled-data discrete-time system.

time $\tau \in \mathbb{R}_{>0}$ as illustrated in Fig. 2.1. The disturbance signal ω is assumed to be bounded $\omega(s) \in W \subseteq \mathbb{R}^p$ for all times $s \in \mathbb{R}_{\geq 0}$. The transition function $F(x, u)$ is defined as the set of states that are reachable by the continuous-time system at time τ from initial state x under a constant input signal $\nu_c(s) = u$ and a bounded disturbance signal ω. If the continuous-time dynamic is linear, the sampled-data system is of the form

$$\xi(t+1) \in A\xi(t) + B\nu(t) + W \tag{2.2}$$

where A and B are matrices of appropriate dimension and W is a nonempty set representing the uncertainties.

Example 2. Consider an instance of (2.2) with $X := \mathbb{R}$, $U := [-1, 1]$ and

$$F(x, u) := \tfrac{1}{2}x + u + [-3, 3]$$

with the set of constraints given by $Q := [-4, 4]$. $\square$

For Example 2, we establish in Section 2.5, that the smallest possible data rate of a coder-controller that enforces Q to be invariant is one bit per time step. This example demonstrates that in contrast to linear systems without disturbances, where the data rate depends only on the unstable eigenvalues, see e.g. [19, Thm. 5.1] or [73], for systems of the form (2.2) the data rate depends among other things also on the *stable* eigenvalues.

2.3 Invariance Feedback Entropy

In this section, we recall the notion of invariance feedback entropy and establish some elementary properties.

2.3.1 The entropy

Formally, we define a *system* as triple

$$\Sigma := (X, U, F) \tag{2.3}$$

where X and U are nonempty sets and $F : X \times U \rightrightarrows X$ is assumed to be strict. A *trajectory* of (2.3) on $[0; \tau[$ with $\tau \in \mathbb{N} \cup \{\infty\}$ is a pair of sequences (ξ, ν), consisting of a state signal

[1]If (2.1) represents a discrete event system, the data rate unit is given in bits/event.

$\xi : [0; \tau + 1[\to X$ and an input signal $\nu : [0; \tau[\to U$, that satisfies (2.1) for all $t \in [0; \tau[$. We denote the set of all trajectories on $[0; \infty[$ by $\mathcal{B}(\Sigma)$.

Throughout this chapter, we call a system (X, U, F) *finite* if X and U are finite.

We follow [59] and [19, Sec. 6] and define the invariance feedback entropy with the help of covers of Q. Consider the system $\Sigma = (X, U, F)$ and a nonempty set $Q \subseteq X$. A cover $\mathcal{A}$ of Q and a function $G : \mathcal{A} \to U$ is called an *invariant cover* $(\mathcal{A}, G)$ of (Σ, Q) if $\mathcal{A}$ is finite and for all $A \in \mathcal{A}$ we have $F(A, G(A)) \subseteq Q$.

Consider an invariant cover $(\mathcal{A}, G)$ of (Σ, Q), fix $\tau \in \mathbb{N}$ and let $\mathcal{S} \subseteq \mathcal{A}^{[0;\tau[}$ be a set of sequences in $\mathcal{A}$. For $\alpha \in \mathcal{S}$ and $t \in [0; \tau - 1[$ we define

$$P_{\mathcal{S}}(\alpha|_{[0;t]}) := \{A \in \mathcal{A} \mid \alpha|_{[0;t]}A = \hat{\alpha}|_{[0;t+1]}, \text{ for some } \hat{\alpha} \in \mathcal{S}\}$$

as the set of immediate successor elements of $\alpha|_{[0;t]}$ in $\mathcal{S}$. The set $P_{\mathcal{S}}(\alpha|_{[0;t]})$ contains the cover elements A so that the sequence $\alpha|_{[0;t]}A$ can be extended to a sequence in $\mathcal{S}$. For $t = \tau - 1$, we have $\alpha|_{[0;\tau-1]} = \alpha$ and we define for notational convenience the set

$$P_{\mathcal{S}}(\alpha|_{[0;t]}) = P_{\mathcal{S}}(\alpha) := \{A \in \mathcal{A} \mid A = \hat{\alpha}(0), \text{ for some } \hat{\alpha} \in \mathcal{S}\} \tag{2.4}$$

which is actually independent of $\alpha \in \mathcal{S}$ and corresponds to the "initial" cover elements A in $\mathcal{S}$, i.e., there exists $\alpha \in \mathcal{S}$ with $A = \alpha(0)$. A set $\mathcal{S} \subseteq \mathcal{A}^{[0;\tau[}$ is called (τ, Q)-*spanning* in $(\mathcal{A}, G)$ if the set $P_{\mathcal{S}}(\alpha)$ with $\alpha \in \mathcal{S}$ covers Q and we have

$$\forall_{\alpha \in \mathcal{S}} \forall_{t \in [0;\tau-1[} \; F(\alpha(t), G(\alpha(t))) \subseteq \bigcup_{A' \in P_{\mathcal{S}}(\alpha|_{[0;t]})} A'. \tag{2.5}$$

We associate with every (τ, Q)-spanning set $\mathcal{S}$ the *expansion number* $\mathcal{N}(\mathcal{S})$, which we define by

$$\mathcal{N}(\mathcal{S}) := \max_{\alpha \in \mathcal{S}} \prod_{t=0}^{\tau-1} \# P_{\mathcal{S}}(\alpha|_{[0;t]}). \tag{2.6}$$

For a given invariant cover $(\mathcal{A}, G)$, we denote by $r_{\text{inv}}(\tau, \mathcal{A}, G, \Sigma)$ the smallest expansion number possible for any (τ, Q)-spanning set in $(\mathcal{A}, G)$, i.e.,

$$r_{\text{inv}}(\tau, \mathcal{A}, G, \Sigma) := \min \{\mathcal{N}(\mathcal{S}) \mid \mathcal{S} \text{ is } (\tau, Q)\text{-spanning in } (\mathcal{A}, G)\}.$$

We define the *entropy* of an invariant cover $(\mathcal{A}, G)$ by

$$h(\mathcal{A}, G) := \lim_{\tau \to \infty} \frac{1}{\tau} \log_2 r_{\text{inv}}(\tau, \mathcal{A}, G, \Sigma). \tag{2.7}$$

As shown in Lemma 1 (stated below), the limit of the sequence in (2.7) exists so that the entropy of an invariant cover $(\mathcal{A}, G)$ is well-defined.

The *invariance feedback entropy* of Σ and Q follows by

$$h_{\text{inv}}(Q, \Sigma) := \inf_{(\mathcal{A}, G)} h(\mathcal{A}, G) \tag{2.8}$$

where we take the infimum over all $(\mathcal{A}, G)$ invariant covers of (Σ, Q). Let us revisit the examples from the previous section to illustrate the various definitions.

Example 1 (Continued)**.** First, we determine an invariant cover $(\mathcal{A}, G)$ of the system in Example 1 and Q. Since the system is finite, we can set $\mathcal{A} := \{\{x\} \mid x \in Q\}$. Recall that $Q = \{0, 2\}$ and a suitable function G is given by $G(\{0\}) := a$ and $G(\{2\}) := b$. Suppose that $\mathcal{S} \subseteq \mathcal{A}^{[0;\tau[}$ is (τ, Q)-spanning with $\tau \in \mathbb{N}$. Let us look at condition (2.5) for $t \in [0; \tau - 1[$ and $\alpha \in \mathcal{S}$. If $\alpha(t) = \{0\}$, we have $P_{\mathcal{S}}(\alpha|_{[0;t]}) = \{\{0\}, \{2\}\}$ since $F(\{0\}, G(\{0\})) = F(0, a) = \{0, 2\}$. If $\alpha(t) = \{2\}$ the same reasoning leads to $P_{\mathcal{S}}(\alpha|_{[0;t]}) = \{\{0\}, \{2\}\}$. Also for $\alpha \in \mathcal{S}$ we have $P_{\mathcal{S}}(\alpha) = \{\{0\}, \{2\}\}$ since $P_{\mathcal{S}}(\alpha)$ is required to be a cover of Q. It follows that $\mathcal{S} = \mathcal{A}^{[0;\tau[}$ and the expansion number $\mathcal{N}(\mathcal{S}) = r_{\text{inv}}(\tau, \mathcal{A}, G, \Sigma) = 2^\tau$ so that the entropy of the $(\mathcal{A}, G)$ follows to $h(\mathcal{A}, G) = 1$. Since $(\mathcal{A}, G)$ is the only invariant cover, we obtain $h_{\text{inv}}(Q, \Sigma) = 1$. $\square$

Example 2 (Continued)**.** Let us recall the linear system in Example 2. An invariant cover $(\mathcal{A}, G)$ is given by $\mathcal{A} := \{a_0, a_1\}$ with $a_0 := [-4, 0]$, $a_1 := [0, 4]$ and $G(a_0) := 1$, $G(a_1) := -1$. Let $\mathcal{S}$ be any (τ, Q)-spanning set in $(\mathcal{A}, G)$. As $P_{\mathcal{S}}(\alpha) \subseteq \mathcal{A}$ is required to cover Q, so $P_{\mathcal{S}}(\alpha) = \mathcal{A}$. For $a_i \in \mathcal{A}$, $i \in \{0, 1\}$, we have $F(a_i, G(a_i)) = [-4; 4]$ which makes $P_{\mathcal{S}}(a_i) = \mathcal{A}$. Thus $\mathcal{S} = \mathcal{A}^{[0;\tau[}$. Since $\#\mathcal{A} = 2$, we obtain that $h(\mathcal{A}, G) = 1$. $\square$

We continue with showing the subadditivity property of $\log_2 r_{\text{inv}}(\cdot, \mathcal{A}, G, \Sigma)$.

Lemma 1. *Consider the system $\Sigma = (X, U, F)$ and a nonempty set $Q \subseteq X$. Let $(\mathcal{A}, G)$ be an invariant cover of (Σ, Q), then the function $\tau \mapsto \log_2 r_{\text{inv}}(\tau, \mathcal{A}, G, \Sigma)$, $\mathbb{N} \to \mathbb{R}_{\geq 0}$, is subadditive, i.e., for all $\tau_1, \tau_2 \in \mathbb{N}$ the inequality*

$$\log_2 r_{\text{inv}}(\tau_1 + \tau_2, \mathcal{A}, G, \Sigma) \leq \log_2 r_{\text{inv}}(\tau_1, \mathcal{A}, G, \Sigma) + \log_2 r_{\text{inv}}(\tau_2, \mathcal{A}, G, \Sigma)$$

holds and we have

$$\lim_{\tau \to \infty} \frac{1}{\tau} \log_2 r_{\text{inv}}(\tau, \mathcal{A}, G, \Sigma) = \inf_{\tau \in \mathbb{N}} \frac{1}{\tau} \log_2 r_{\text{inv}}(\tau, \mathcal{A}, G, \Sigma). \tag{2.9}$$

The following lemma might be of independent interest. It states that the expansion number is not less than the cardinality for any (τ, Q)-spanning set. We use it in the proves of Theorems 4 and 13.

Lemma 2. *Consider an invariant cover $(\mathcal{A}, G)$ of (2.3) and some nonempty set $Q \subseteq X$. Let $\mathcal{S}$ be a (τ, Q)-spanning set, then we have $\#\mathcal{S} \leq \mathcal{N}(\mathcal{S})$.*

The proofs of both lemmas are given in the appendix.

2.3.2 Entropy across related systems

One of the most important properties of entropy of classical dynamical systems is its invariance under any change of coordinates [1, Thm. 1]. In [19] this property has been shown for deterministic control systems in the context of semiconjugation [19, Thm. 3.5]. In the following, we present a result in the context of feedback refinement relations [61], which contains the result on semiconjugation as a special case.

Definition 1. *Let Σ_1 and Σ_2 be two systems of the form*

$$\Sigma_i = (X_i, U_i, F_i) \text{ with } i \in \{1,2\}. \tag{2.10}$$

A strict relation $R \subseteq X_1 \times X_2$ is a feedback refinement relation *from Σ_1 to Σ_2 if there exists a map $r : U_2 \to U_1$ so that the following inclusion holds for all $(x_1, x_2) \in R$ and $u \in U_2$*

$$R(F_1(x_1, r(u))) \subseteq F_2(x_2, u). \tag{2.11}$$

When there is a feedback refinement relation from one system to another one, the following theorem shows that the IFE of the former is not larger than the latter.

Theorem 1. *Consider two systems Σ_i, $i \in \{1,2\}$, of the form* (2.10)*. Let Q_1 and Q_2 be two nonempty subsets of X_1 and X_2, respectively. Suppose that R is a feedback refinement relation from Σ_1 to Σ_2, and $Q_1 = R^{-1}(Q_2)$. Then*

$$h_{1,\mathrm{inv}}(Q_1) \leq h_{2,\mathrm{inv}}(Q_2) \tag{2.12}$$

holds, where $h_{i,\mathrm{inv}}(Q_i)$ is the invariance feedback entropy of Σ_i and Q_i.

Proof. If $h_{2,\mathrm{inv}}(Q_2) = \infty$, the inequality holds and subsequently we consider the case $h_{2,\mathrm{inv}}(Q_2) < \infty$. We will make use of Lemma 9 in the Appendix to show (2.12). Let us pick an invariant cover $(\mathcal{A}_2, G_2)$ of (Σ_2, Q_2) so that $h(\mathcal{A}_2, G_2) < \infty$. Next we define the set $\mathcal{A}_1 := \{A_1 \subseteq Q_1 \mid \exists_{A_2 \in \mathcal{A}_2}\ R^{-1}(A_2) = A_1\}$.

Now let $M = R^{-1}$ and $r : U_2 \to U_1$ in Lemma 9, where R and r are, respectively, the relation and map associated with the feedback refinement relation in Def. 1. We observe that all the conditions 1) - 4) in Lemma 9 hold.

Thus there exists a map $G_1^* : \mathcal{A}_1 \to U_1$ such that $(\mathcal{A}_1, G_1^*)$ is an invariant cover of (Σ_1, Q_1), and

$$h(\mathcal{A}_1, G_1^*) \leq h(\mathcal{A}_2, G_2).$$

Therefore, inequality (2.12) holds. □

2.3.3 Conditions for finiteness

We analyze two particular instances of systems – finite systems and systems with a topological state alphabet – and provide conditions ensuring that the invariance entropy is finite. The results are based on the following lemma.

Lemma 3. *Consider a system $\Sigma = (X, U, F)$ and a nonempty set $Q \subseteq X$. There exists an invariant cover $(\mathcal{A}, G)$ of (Σ, Q) iff $h_{\mathrm{inv}}(Q, \Sigma) < \infty$.*

Proof. It follows immediately from (2.8) that $h_{\mathrm{inv}}(Q) < \infty$ implies the existence of an invariant cover of (Σ, Q). For the reverse direction, we assume that $(\mathcal{A}, G)$ is an invariant cover of (Σ, Q). We fix $\tau \in \mathbb{N}$ and define $\mathcal{S} := \{\alpha \in \mathcal{A}^{[0;\tau[} \mid \forall_{t \in [0;\tau-1[}\ \alpha(t+1) \cap F(\alpha(t), G(\alpha(t))) \neq \varnothing\}$. It is easy to verify that $\mathcal{S}$ is (τ, Q)-spanning and $\mathcal{N}(\mathcal{S}) \leq (\#\mathcal{A})^\tau$. An upper bound on $h_{\mathrm{inv}}(Q, \Sigma)$ follows by $\log_2 \#\mathcal{A}$. □

If Σ is finite, it is rather straightforward to show that the controlled invariance of Q w.r.t. Σ is necessary and sufficient for $h_{\text{inv}}(Q, \Sigma)$ to be finite. Let us recall the notion of controlled invariance [11].

We call $Q \subseteq X$ *controlled invariant* with respect to a system $\Sigma = (X, U, F)$, if for all $x \in Q$ there exists $u \in U$ so that $F(x, u) \subseteq Q$. We refer the interested readers to [64] for a discussion on computation of controlled invariant set for controllable linear discrete-time systems.

Theorem 2. *Consider a finite system $\Sigma = (X, U, F)$ and a nonempty set $Q \subseteq X$. Then $h_{\text{inv}}(Q, \Sigma) < \infty$ if and only if Q is controlled invariant.*

Proof. Let $h_{\text{inv}}(Q, \Sigma)$ be finite. Then there exists an invariant cover $(\mathcal{A}, G)$ so that $h(\mathcal{A}, G) < \infty$. Hence, for every $x \in Q$, we can pick an $A \in \mathcal{A}$ with $x \in A$, so that $F(x, G(A)) \subseteq F(A, G(A)) \subseteq Q$. Hence, Q is controlled invariant w.r.t. Σ.

Assume Q is controlled invariant w.r.t. Σ. For $x \in Q$, let $u_x \in U$ be such that $F(x, u_x) \subseteq Q$. It is easy to check that $(\mathcal{A}, G)$ with $\mathcal{A} := \{\{x\} \mid x \in Q\}$ and $G(\{x\}) := u_x$ is an invariant cover of (Σ, Q), so that the assertion follows from Lemma 3. $\square$

In general controlled invariance of Q is not sufficient to guarantee finiteness of the invariance feedback entropy as shown in the next example.

Example 3. Consider $\Sigma = (\mathbb{R}, [-1, 1], F)$ with the dynamics given by $F(x, u) := x + u + [-1, 1]$. Let $Q := [-1, 1]$, then for every $x \in Q$ we can pick $u = -x$ so that $F(x, u) = [-1, 1] \subseteq Q$, which shows that Q is controlled invariant. Now suppose that $h_{\text{inv}}(Q, \Sigma)$ is finite. Then according to Lemma 3 there exists an invariant cover $(\mathcal{A}, G)$ of (Σ, Q). Since $\mathcal{A}$ is required to be finite, there exists $A \in \mathcal{A}$ with an infinite number of elements and therefore we can pick two different states in A, i.e., $x, x' \in A$ with $x \neq x'$. However, there does not exist a single $u \in U$ so that $F(x, u) \subseteq Q$ and $F(x', u) \subseteq Q$. Hence, $(\mathcal{A}, G)$ cannot be an invariant cover, which implies $h_{\text{inv}}(Q, \Sigma) = \infty$. $\square$

In the subsequent theorem we present some conditions for systems with a topological state alphabet, which imply the finiteness of the invariance entropy. The conditions may be difficult to verify for a particular problem instance. Nevertheless, with these conditions, we follow closely the assumptions based on continuity and strong invariance employed in [60, 21] to ensure finiteness of the invariance entropy for deterministic systems. We use the following notion of continuity of set-valued maps [5] to show the next result.

Let A and B be topological spaces and $f : A \rightrightarrows B$. We say that f is *upper semicontinuous*, if for every $a \in A$ and every open set $V \subseteq B$ containing $f(a)$ there exists an open set $U \subseteq A$ with $a \in U$ so that $f(U) \subseteq V$.

Theorem 3. *Consider a system $\Sigma = (X, U, F)$ and a nonempty compact subset Q of X. Let X be a topological space. If $F(\cdot, u)$ is upper semicontinuous for every $u \in U$ and Q is strongly controlled invariant, i.e., for all $x \in Q$ there exists $u \in U$ so that $F(x, u) \subseteq \operatorname{int} Q$, then $h_{\text{inv}}(Q, \Sigma) < \infty$.*

Proof. For each $x \in Q$, we pick an input $u_x \in U$ so that $F(x, u_x) \subseteq \operatorname{int} Q$. Since $F(\cdot, u_x)$ is upper semicontinuous and $\operatorname{int} Q$ is open, there exists an open subset A_x of X, so that $x \in A_x$ and $F(A_x, u_x) \subseteq \operatorname{int} Q$. Hence, the set $\{A_x \mid x \in Q\}$ of open subsets of X covers Q. Since Q is a compact subset of X, there exists a finite set $\{A_{x_1}, \ldots, A_{x_m}\}$ so that $Q \subseteq \cup_{i\in[1;m]} A_{x_i}$ [31, Ch. 2.6]. Let $\mathcal{A} := \{A_{x_1} \cap Q, \ldots, A_{x_m} \cap Q\}$ and define for every $i \in [1; m]$ the function $G(A_{x_i}) := u_{x_i}$. Then $(\mathcal{A}, G)$ is an invariant cover of (Σ, Q), and the assertion follows from Lemma 3. □

Example 3 (Continued)**.** Let $\varepsilon > 0$, consider Σ from Example 3 with the modified input set $U_\varepsilon := [-1-\varepsilon, 1+\varepsilon]$. Let $Q_\varepsilon := [-1-\varepsilon, 1+\varepsilon]$ then we see that Q_ε is strongly controlled invariant. We construct an invariant cover for (Σ, Q_ε) as follows. We define n as the smallest integer larger than $\frac{1}{2\varepsilon}$ and introduce $\{x_{-n}, \ldots, x_0, \ldots x_n\}$ with $x_i := 2i\varepsilon$ and set $A_i := (x_i + [-\varepsilon, \varepsilon]) \cap Q_\varepsilon$. For each $i \in [-n; n]$, we define $G(A_i) := -x_i$ so that $F(A_i, G(A_i)) \subseteq Q_\varepsilon$. By definition of n we have $x_{-n} \leq -1$ and $1 \leq x_n$ and we see that $(\mathcal{A}, G)$ with $\mathcal{A} := \{A_i \mid i \in [-n; n]\}$ is an invariant cover of (Σ, Q_ε). Hence, it follows from Lemma 3 that $h_{\text{inv}}(Q_\varepsilon, \Sigma)$ is finite. □

2.3.4 Deterministic systems

For deterministic systems we recover the notion of invariance feedback entropy in [59, 21].

Let us consider the map $f : X \times U \to X$ representing a deterministic system

$$\xi(t+1) = f(\xi(t), \nu(t)). \tag{2.13}$$

We can interpret (2.13) as special instance of (2.3), where F is given by $F(x, u) := \{f(x, u)\}$ for all $x \in X$ and $u \in U$ and the notions of a trajectory of (2.3) extend to (2.13) in the obvious way. Given an input $u \in U$, we introduce $f_u : X \to X$ by $f_u(x) := f(x, u)$ and extend this notation to sequences $\nu \in U^{[0;t]}$, $t \in \mathbb{N}$, by

$$f_\nu(x) := f_{\nu(t)} \circ \cdots \circ f_{\nu(0)}(x).$$

We follow [21] to define the entropy of (2.13). Consider a nonempty set $Q \subseteq X$ and fix $\tau \in \mathbb{N}$. A set $\mathcal{S}^{\text{det}} \subseteq U^{[0;\tau[}$ is called (τ, Q)*-spanning* for f and Q, if for every $x \in Q$ there exists $\nu \in \mathcal{S}^{\text{det}}$ so that the associated trajectory (ξ, ν) on $[0; \tau[$ of (2.13) with $\xi(0) = x$ satisfies $\xi([0; \tau]) \subseteq Q$. We use $r^{\text{det}}_{\text{inv}}(\tau, Q)$ to denote the number of elements of the smallest (τ, Q)-spanning set

$$r^{\text{det}}_{\text{inv}}(\tau, Q) := \inf\{\#\mathcal{S}^{\text{det}} \mid \mathcal{S}^{\text{det}} \text{ is } (\tau, Q)\text{-spanning}\}. \tag{2.14}$$

The *(deterministic) invariance entropy* of (X, U, f) and Q is defined by

$$h^{\text{det}}_{\text{inv}}(Q) := \lim_{\tau\to\infty} \frac{1}{\tau} \log_2 r^{\text{det}}_{\text{inv}}(\tau, Q). \tag{2.15}$$

Again the function $\tau \mapsto \log_2 r^{\text{det}}_{\text{inv}}(\tau, Q)$ is subadditive [21, Prop. 2.2] thus by Fekete's Lemma [21, Lem. 2.1] the limit in (2.15) exists.

Now, we have the following theorem.

Theorem 4. *Consider the system $\Sigma = (X, U, F)$ and a nonempty set $Q \subseteq X$. Suppose F satisfy $F(x, u) = \{f(x, u)\}$ for all $x \in X$, $u \in U$, and for some $f : X \times U \to X$. Then the invariance feedback entropy of Σ and Q equals the deterministic invariance entropy of (X, U, f) and Q, i.e.,*

$$h_{\mathrm{inv}}(Q, \Sigma) = h^{\mathrm{det}}_{\mathrm{inv}}(Q). \tag{2.16}$$

Proof. We begin with the inequality $h^{\mathrm{det}}_{\mathrm{inv}}(Q) \geq h_{\mathrm{inv}}(Q, \Sigma)$. If $h^{\mathrm{det}}_{\mathrm{inv}}(Q) = \infty$ the inequality trivially holds and subsequently we assume that $h^{\mathrm{det}}_{\mathrm{inv}}(Q)$ is finite. We fix $\varepsilon > 0$ and pick $\tau \in \mathbb{N}$ so that $\frac{1}{\tau} \log_2 r^{\mathrm{det}}_{\mathrm{inv}}(\tau, Q) \leq h^{\mathrm{det}}_{\mathrm{inv}}(Q) + \varepsilon$. We chose a (τ, Q)-spanning set $\mathcal{S}^{\mathrm{det}}$ for f and Q with $\#\mathcal{S}^{\mathrm{det}} = r^{\mathrm{det}}_{\mathrm{inv}}(\tau, Q)$. For every $\nu \in \mathcal{S}^{\mathrm{det}}$ we define the sets

$$A_0(\nu) := Q \cap \bigcap_{t=0}^{\tau-1} f^{-1}_{\nu|_{[0;t]}}(Q)$$

and for $t \in [0; \tau - 1[$ the sets $A_{t+1}(\nu) := f(A_t(\nu), \nu(t))$. The minimality of $\mathcal{S}^{\mathrm{det}}$ implies that $A_0(\nu) \neq \varnothing$ and $A_0(\nu) \neq A_0(\nu')$ for all $\nu, \nu' \in \mathcal{S}^{\mathrm{det}}$. Let $\mathcal{A}$ be the set of all sets $A_t(\nu)$ with $t \in [0; \tau[$ and $\nu \in \mathcal{S}^{\mathrm{det}}$. With each $A \in \mathcal{A}$ we associate a single pair (ν, t), where $\nu \in \mathcal{S}^{\mathrm{det}}$ and $t \in [0; \tau[$, such that is satisfies $A = A_t(\nu)$ and the following condition: $\nu' \in \mathcal{S}^{\mathrm{det}}$ and $t' \in [0; \tau[$ with $A = A_{t'}(\nu')$ implies $t \leq t'$. Then we define the map $G : \mathcal{A} \to U$ by $G(A) = \nu(t)$ where (ν, t) is associated with A. By the definition of $A_t(\nu)$, it is easy to see that $f(A_t(\nu), G(A_t(\nu))) \subseteq Q$ for all $t \in [0; \tau[$ and $\nu \in \mathcal{S}^{\mathrm{det}}$. Moreover, since $\mathcal{S}^{\mathrm{det}}$ is (τ, Q)-spanning, for every $x \in Q$ there is $\nu \in \mathcal{S}^{\mathrm{det}}$ so that for all $t \in [0; \tau[$ we have $f_{\nu|_{[0;t]}}(x) \in Q$ which implies $x \in A_0(\nu)$ and we see that $\{A_0(\nu) \mid \nu \in \mathcal{S}^{\mathrm{det}}\}$ covers Q. It follows that $(\mathcal{A}, G)$ is an invariant cover of (Σ, Q). Let $\mathcal{S}_{\mathrm{inv}}$ be the set of sequences $\alpha : [0; \tau[\to \mathcal{A}$ defined iteratively as $\alpha(0) \in \{A_0(\nu) \mid \nu \in \mathcal{S}^{\mathrm{det}}\}$ and $\alpha(t+1) = f(\alpha(t), G(\alpha(t)))$. Then $P_{\mathcal{S}_{\mathrm{inv}}}(\alpha)$ covers Q since $\{A_0(\nu) \mid \nu \in \mathcal{S}^{\mathrm{det}}\}$ covers Q as discussed above. For any distinct $\alpha, \alpha' \in \mathcal{S}_{\mathrm{inv}}$ we have $\alpha(0) \neq \alpha'(0)$ so for every $t \in [0; \tau - 1[$ we have $\#P_{\mathcal{S}_{\mathrm{inv}}}(\alpha|_{[0;t]}) = 1$, $f(\alpha(t), G(\alpha(t))) = P_{\mathcal{S}_{\mathrm{inv}}}(\alpha|_{[0;t]})$ and thus $\mathcal{S}_{\mathrm{inv}}$ satisfies (2.5). Therefore, $\mathcal{S}_{\mathrm{inv}}$ is (τ, Q)-spanning in $(\mathcal{A}, G)$. Moreover, as $\nu \neq \nu'$ implies $A_0(\nu) \neq A_0(\nu')$, we have $\#P_{\mathcal{S}_{\mathrm{inv}}}(\alpha) = \#\mathcal{S}^{\mathrm{det}}$, so that $r_{\mathrm{inv}}(\tau, \mathcal{A}, G, \Sigma) \leq \mathcal{N}(\mathcal{S}_{\mathrm{inv}}) = \#\mathcal{S}^{\mathrm{det}} = r^{\mathrm{det}}_{\mathrm{inv}}(\tau, Q)$ follows. Due to Lemma 1, we have $\log_2 r_{\mathrm{inv}}(n\tau, \mathcal{A}, G, \Sigma) \leq n \log_2 r_{\mathrm{inv}}(\tau, \mathcal{A}, G, \Sigma)$ and we see that $\frac{1}{\tau} \log_2 r_{\mathrm{inv}}(\tau, \mathcal{A}, G, \Sigma)$ (and therefore $\frac{1}{\tau} \log_2 r^{\mathrm{det}}_{\mathrm{inv}}(\tau, Q)$) provides an upper bound for $h(\mathcal{A}, G)$ so that we obtain $h_{\mathrm{inv}}(Q, \Sigma) \leq h(\mathcal{A}, G) \leq h^{\mathrm{det}}_{\mathrm{inv}}(Q) + \varepsilon$. Since this holds for any $\varepsilon > 0$ we obtain the desired inequality.

We continue with the inequality $h^{\mathrm{det}}_{\mathrm{inv}}(Q) \leq h_{\mathrm{inv}}(Q, \Sigma)$. If $h_{\mathrm{inv}}(Q, \Sigma) = \infty$ the inequality trivially holds and subsequently we assume $h_{\mathrm{inv}}(Q, \Sigma) < \infty$. We fix $\varepsilon > 0$ and pick an invariant cover $(\mathcal{A}, G)$ of (Σ, Q) so that $h(\mathcal{A}, G) \leq h_{\mathrm{inv}}(Q, \Sigma) + \varepsilon$. We fix $\tau \in \mathbb{N}$ and pick a (τ, Q)-spanning set $\mathcal{S}_{\mathrm{inv}}$ in $(\mathcal{A}, G)$ so that $\mathcal{N}(\mathcal{S}_{\mathrm{inv}}) = r_{\mathrm{inv}}(\tau, \mathcal{A}, G, \Sigma)$. We define for every $\alpha \in \mathcal{S}_{\mathrm{inv}}$ the input sequence $\nu_\alpha : [0; \tau[\to U$ by $\nu_\alpha(t) := G(\alpha(t))$ and introduce the set $\mathcal{S}^{\mathrm{det}} := \{\nu_\alpha \mid \alpha \in \mathcal{S}_{\mathrm{inv}}\}$. For $x \in Q$ we iteratively construct $\alpha \in \mathcal{A}^{[0;\tau[}$ and $\nu \in U^{[0;\tau[}$ as follows: for $t = 0$ we pick $\alpha_0 \in \mathcal{S}_{\mathrm{inv}}$ so that $x \in \alpha_0(0)$ and set $\nu(0) := G(\alpha_0(0))$. For $t \in [0; \tau - 1[$ we pick $\alpha_{t+1} \in \mathcal{S}_{\mathrm{inv}}$ so that $\alpha_{t+1}|_{[0;t]} = \alpha_t$ and $f_{\nu|_{[0;t]}}(x) \in \alpha_{t+1}(t+1)$ and set $\nu(t+1) := G(\alpha_{t+1}(t+1))$. Since $(\mathcal{A}, G)$ is an invariant cover of (Σ, Q), it is easy to show that $f_{\nu|_{[0;t]}}(x) \in Q$ holds for all $t \in [0;\tau[$, which implies that $\mathcal{S}^{\mathrm{det}}$ is (τ, Q)-spanning for f and Q. Thus, we obtain $r_{\mathrm{inv}}^{\mathrm{det}}(\tau, Q) \leq \#\mathcal{S}^{\mathrm{det}} \leq \#\mathcal{S}_{\mathrm{inv}} \leq \mathcal{N}(\mathcal{S}_{\mathrm{inv}}) = r_{\mathrm{inv}}(\tau, \mathcal{A}, G, \Sigma)$, where the inequality $\#\mathcal{S}_{\mathrm{inv}} \leq \mathcal{N}(\mathcal{S}_{\mathrm{inv}})$ follows from Lemma 2. Since this holds for any $\tau \in \mathbb{N}$, we obtain the inequality $\varepsilon + h_{\mathrm{inv}}(Q, \Sigma) \geq h(\mathcal{A}, G) \geq h_{\mathrm{inv}}^{\mathrm{det}}(Q)$ for arbitrary $\varepsilon > 0$ which shows $h_{\mathrm{inv}}(Q, \Sigma) \geq h_{\mathrm{inv}}^{\mathrm{det}}(Q)$. □

2.3.5 Invariant covers with closed elements

We conclude this section with a result on the topological structure of the cover elements for systems with topological state alphabet and lower semicontinuous transition functions and closed sets Q. The result is used in Theorem 7 but might be of interest on its own.

Let A and B be topological spaces and $f : A \rightrightarrows B$. We say that f is *lower semicontinuous* if $f^{-1}(V)$ is open whenever $V \subseteq B$ is open.

Theorem 5. *Consider a system $\Sigma = (X, U, F)$ with topological state alphabet and a nonempty closed set $Q \subseteq X$. Assume that $F(\cdot, u)$ is lower semicontinuous for every $u \in U$. Let $(\mathcal{A}, G)$ be an invariant cover of (Σ, Q) and let $\mathcal{C} := \{\mathrm{cl}\, A \subseteq \mathrm{cl}\, X \mid A \in \mathcal{A}\}$. Then there exists a map $H^* : \mathcal{C} \to U$ such that $(\mathcal{C}, H^*)$ is an invariant cover of (Σ, Q) and*

$$h(\mathcal{C}, H^*) \leq h(\mathcal{A}, G). \tag{2.17}$$

In the proof of the theorem, we use the following lemma, the proof of which follows the standard arguments in [5].

Lemma 4. *Let X be a topological space and $f : X \rightrightarrows X$. If f is lower semicontinuous then $f(\mathrm{cl}\, \Omega) \subseteq \mathrm{cl}\, f(\Omega)$ holds for every nonempty subset $\Omega \subseteq X$.*

Proof. For the sake of contradiction, suppose there exists $x \in \mathrm{cl}\, \Omega$, $y \in f(x)$ and $y \notin \mathrm{cl}\, f(\Omega)$. Then the open set $V := X \setminus \mathrm{cl}\, f(\Omega)$ contains y. Let us define $U := f^{-1}(V) = \{x' \in X \mid f(x') \cap V \neq \varnothing\}$ and since f is lower semicontinuous and V is open so U is open. As $V \cap f(x) \ni y$, thus nonempty, so $x \in U$. By definition, $V \cap f(\Omega) = \varnothing$ so $U \cap \Omega = \varnothing$ and since U is open so $U \cap \mathrm{cl}\, \Omega = \varnothing$ which is in contradiction with $x \in U$ and $x \in \mathrm{cl}\, \Omega$. □

Proof of Theorem 5. In Lemma 9 in the Appendix, let $M = \mathrm{cl}$, $\Sigma_1 = \Sigma_2 = \Sigma$, $Q_2 = Q_1 = Q$, $\mathcal{A}_2 = \mathcal{A}$, $G_2 = G$, $\mathcal{A}_1 = \mathcal{C}$ and $r = \mathrm{id}$, then one can easily verify that conditions 1) - 3) hold, while Lemma 4 implies that 4) is satisfied. Thus there exists a map $H^* : \mathcal{C} \to U$ such that $(\mathcal{C}, H^*)$ is an invariant cover of (Σ, Q), and $h(\mathcal{C}, H^*) \leq h(\mathcal{A}, G)$. □

2.4 Data-Rate-Limited Feedback

We present the data rate theorem associated with the invariance feedback entropy of uncertain control systems. It shows that the invariance feedback entropy is a tight lower bound of the data rate of any coder-controller scheme that renders the set of interest invariant.

We introduce a history-dependent definition of data rates of coder-controllers with which we extend previously used time-invariant [60] and time-varying [59, 38] notions. We interpret the history-dependent definition of data rate as a nonstochastic variant of the notion of data rate used e.g. in [71, Def. 4.1] for noisy linear systems, defined as the average of the expected length of the transmitted symbols in the closed loop. We motivate the particular notion of data rate by two examples; one which illustrates that the time-varying definition [59] results in too large data rates and one which shows that the notion of data rate based on the framework of nonstochastic information theory, used in [56, 57] for estimation [57] and control [56] of linear systems, leads to too small data rates.

2.4.1 The coder-controller

We assume that a coder for the system (2.3) is located at the sensor side (see Fig. 1.1), which at every time step, encodes the current state of the system using the finite *coding alphabet* S. It transmits a symbol $s_t \in S$ via the discrete noiseless channel to the controller. The transmitted symbol $s_t \in S$ might depend on all past states and is determined by the *coder function*

$$\gamma : \bigcup_{t\in\mathbb{Z}_{\geq 0}} X^{[0;t]} \to S.$$

At time $t \in \mathbb{Z}_{\geq 0}$, the controller received $t+1$ symbols $s_0 \dots s_t$, which are used to determine the control input given by the *controller function*

$$\delta : \bigcup_{t\in\mathbb{Z}_{\geq 0}} S^{[0;t]} \to U.$$

A *coder-controller* for (2.3) is a triple $H := (S, \gamma, \delta)$, where S is a coding alphabet and γ and δ are a compatible coder function and controller function, respectively.

Given a coder-controller (S, γ, δ) for (2.3) and $\xi \in X^{[0;t]}$ with $t \in \mathbb{Z}_{\geq 0}$, let us use the mapping

$$\Gamma_t : X^{[0;t]} \to S^{[0;t]}$$

to denote the sequence $\zeta = \Gamma_t(\xi)$ of coder symbols generated by ξ, i.e., $\zeta(t') = \gamma(\xi|_{[0;t']})$ holds for all $t' \in [0;t]$. Subsequently, for $\zeta \in S^{[0;t[}$ with $t \in \mathbb{N}$, we use

$$Z(\zeta) := \{s \in S \mid \exists_{(\xi,\nu)\in\mathcal{B}(\Sigma)}\ \zeta s = \Gamma(\xi|_{[0;t]}) \wedge \forall_{t'\in[0;t[}\nu(t') = \delta(\zeta|_{[0;t']})\} \tag{2.18}$$

to denote the possible successor coder symbols s of the symbol sequence ζ in the closed loop illustrated in Fig. 1.1. For notational convenience, let us use the convention $Z(\varnothing) := S$, so that $Z(\zeta|_{[0;0[}) = S$ for any sequence ζ in S. For $\tau \in \mathbb{N} \cup \{\infty\}$, we introduce the set

$$\mathcal{Z}_\tau := \{\zeta \in S^{[0;\tau[} \mid \zeta(0) \in \gamma(X) \wedge \forall_{t\in]0;\tau[}\ \zeta(t) \in Z(\zeta|_{[0;t[})\}$$

and define the *transmission data rate* of a coder-controller H by

$$R(H) := \limsup_{\tau\to\infty} \max_{\zeta\in\mathcal{Z}_\tau} \frac{1}{\tau} \sum_{t=0}^{\tau-1} \log_2 \#Z(\zeta|_{[0;t[}) \tag{2.19}$$

as the asymptotic average numbers of symbols in $Z(\zeta)$ considering the worst-case of possible symbol sequences $\zeta \in \mathcal{Z}_\tau$.

A coder-controller $H = (S, \gamma, \delta)$ for (2.3) is called *Q-admissible* where Q is a nonempty subset of X, if for every trajectory (ξ, ν) on $[0; \infty[$ of (2.3) that satisfies

$$\xi(0) \in Q \text{ and } \forall_{t \in \mathbb{Z}_{\geq 0}}\ \nu(t) = \delta(\Gamma_t(\xi|_{[0;t]})), \tag{2.20}$$

we have $\xi(\mathbb{Z}_{\geq 0}) \subseteq Q$. Let us use $\mathcal{B}_Q(H)$ to denote the set of all trajectories (ξ, ν) on $[0; \infty[$ of (2.3) that satisfy (2.20).

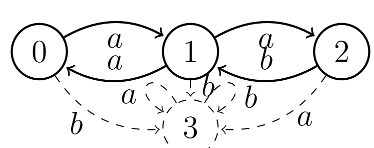
Data rate definition with time-varying coding alphabet

We follow [59] and introduce a notion of data rate, based on time-varying coding alphabet, for a coder-controller $H = (S, \gamma, \delta)$ for (2.3). Let $(S_t)_{t \geq 0}$ be the sequence in the power set of S that for each $t \in Z_{\geq 0}$ contains the smallest number of symbols so that $\gamma(\xi) \in S_t$ holds for all $\xi \in X^{[0;t]}$. Then the data rate of H follows by

$$R_{\mathrm{tv}}(H) := \liminf_{\tau \to \infty} \frac{1}{\tau} \sum_{t=0}^{\tau-1} \log_2 \#S_t.$$

In the following we use an example to show that there exists a Q-admissible coder-controller H, which satisfies $R(H) < R_{\mathrm{tv}}(\bar{H})$ for any Q-admissible coder-controller $\bar{H}$. Note that this inequality is purely a nondeterministic phenomenon: if the control system is deterministic, it follows from the deterministic and the nondeterministic data rate theorem ([59, Thm. 1] and Theorem 6 below) and the equivalence $h^{\mathrm{det}}_{\mathrm{inv}}(Q) = h_{\mathrm{inv}}(Q, \Sigma)$ (Theorem 4) that the different notions of data rates coincide in the sense that $\inf_H R(H) = \inf_H R_{\mathrm{tv}}(H)$ (at least if the strong invariance condition in [59, Thm. 1] holds).

Example 4. Consider an instance of (2.3) with $U := \{a, b\}$, $X := \{0, 1, 2, 3\}$ and F is illustrated by



Let $Q := \{0, 1, 2\}$. The transitions that lead outside Q and the states that are outside Q are marked by dashed lines. Consider the coder-controller $H = (S, \gamma, \delta)$ with $S := X$ and γ and δ are given for $\xi \in X^{[0;t]}$, $t \in \mathbb{Z}_{\geq 0}$, by $\gamma(\xi) := \xi(t)$ and $\delta(\xi) := a$ if $\xi(t) \in \{0, 1, 3\}$ and $\delta(\xi) := b$ if $\xi(t) = 2$. We compute the number of possible successor symbols $Z(\xi)$ for $\xi \in X^{[0;t]}$, $t \in \mathbb{Z}_{\geq 0}$, by $\#Z(\xi) = 1$ if $\xi(t) \in \{0, 2, 3\}$ and $\#Z(\xi) = 2$ if $\xi(t) = 1$. It is easy to verify that H is Q-admissible. Since the state $\xi(t) = 1$ occurs only every other time step for any element (ξ, ν) of the closed loop, we compute the data rate to $R(H) = 1/2$. Consider a time-varying Q-admissible coder-controller $\bar{H} = (\bar{S}, \bar{\gamma}, \bar{\delta})$. Initially, the states $\{0, 1\}$ and $\{2\}$ need to be distinguishable at the controller side in order to confine the system to Q

so that $\#\bar{S}_0 \geq 2$ follows. At time $t = 1$, the system is possibly again in any of the states $\{0, 1, 2\}$ (depending on the initial condition) and we have $\#\bar{S}_1 \geq 2$. By continuing this argument, we see that $\#\bar{S}_t \geq 2$ for all $t \in \mathbb{Z}_{\geq 0}$ and $R_{\mathrm{tv}}(\bar{H}) \geq 1$ follows. $\square$

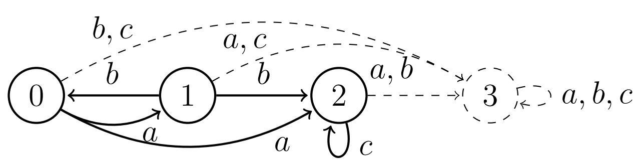
Zero-error capacity of uncertain channels

Alternatively to the definition of the data rate of a coder-controller in (2.19) we could follow [56, 57] and define the data rate of a coder-controller as the zero-error capacity C_0 of an *ideal stationary memoryless uncertain channel* (SMUC) in the nonstochastic information theory framework presented in [57, Def. 4.1]. Although zero-error capacity is a characteristic of the channel and is independent of the chosen coder-controller and the plant, we slightly abuse notation and use C_0 to refer to the special case of the channel input function space restricted to the set of all possible symbol sequences generated in the closed loop. The input alphabet of the SMUC equals the output alphabet and is given by S. The channel is ideal and does not introduce any error in the transmission. Hence, the transition function is the identity, i.e., $T(s) = s$ holds for all $s \in S$. The input function space $\mathcal{Z}_\infty \subseteq S^{[0;\infty[}$ is the set of all possible symbol sequences that are generated by the closed loop, which represents the total amount of information that needs to be transmitted by the channel. For the ideal SMUC, the zero-error capacity [57, Eq. (25)], for a coder-controller H results in

$$C_0(H) := \lim_{\tau \to \infty} \frac{1}{\tau} \log_2 \#\mathcal{Z}_\tau.$$

We use the following example to demonstrate that the zero-error capacity is too low, i.e., $C_0(H) = 0$ while $R(H) \geq 1$.

Example 5. Consider an instance of (2.3) with $U := \{a, b, c\}$, $X := \{0, 1, 2, 3\}$ and F is illustrated by



The transitions and states that lead, respectively, are outside the set of interest $Q := \{0, 1, 2\}$ are dashed. Consider the Q-admissible coder-controller $H = (S, \gamma, \delta)$ with $S := X$ and γ and δ are given for $\xi \in X^{[0;t]}$, $t \in \mathbb{Z}_{\geq 0}$ by $\gamma(\xi) := \xi(t)$ and

$$\delta(\xi) := \begin{cases} a & \text{if } \xi(t) \in \{0, 3\} \\ b & \text{if } \xi(t) = 1 \\ c & \text{if } \xi(t) = 2. \end{cases}$$

We pick the trajectory $(\xi, \nu) \in \mathcal{B}_Q(H)$ given for $t \in \mathbb{Z}_{\geq 0}$ by $\xi(2t) = 0$ and $\xi(2t + 1) = 1$. We obtain $Z(\xi|_{[0;t]}) = \{1, 2\}$ if $\xi(t) = 0$ and $Z(\xi|_{[0;t]}) = \{0, 2\}$ if $\xi(t) = 1$. Since $\#F(x, u) \leq$

2 for all $x \in X$ and $u \in U$, it is straightforward to see that $\sum_{t=0}^{\tau-1} \log_2 \#Z(\xi|_{[0;t[}) = \max_{\zeta \in \mathcal{Z}_\tau} \sum_{t=0}^{\tau-1} \log_2 \#Z(\zeta|_{[0;t[})$ holds for all $\tau \in \mathbb{N}$. Hence, we obtain $R(H) = 1$.
We are going to derive $C_0(H)$. Consider the set $\mathcal{Z}_\tau \subseteq X^{[0;\tau[}$ and the hypothesis for $\tau \in \mathbb{N}$: there exists at most one $\xi \in \mathcal{Z}_\tau$ with $\xi(\tau-1) = 1$ and there exists at most one $\xi \in \mathcal{Z}_\tau$ with $\xi(\tau-1) = 0$. For $\tau = 1$ we have $\mathcal{Z}_1 = X$ and the hypothesis holds. Suppose the hypothesis holds for $\tau \in \mathbb{N}$ and let $\xi \in \mathcal{Z}_\tau$. We have $Z(\xi) = \{0, 2\}$ if $\xi(t) = 1$, $Z(\xi) = \{1, 2\}$ if $\xi(t) = 0$, $Z(\xi) = \{2\}$ if $\xi(t) = 2$ and $Z(\xi) = \{3\}$ if $\xi(t) = 3$, so that the hypothesis holds for $\tau + 1$, which shows that the hypothesis holds for every $\tau \in \mathbb{N}$. Therefore, we obtain a bound of the number of elements in $\mathcal{Z}_\tau$ by $4 + 2(\tau - 1)$ and the zero-error capacity of H follows by $C_0(H) = 0$. $\square$

Example 5 shows that even though, the asymptotic average of the total amount of information that needs to be transmitted (= symbol sequences generated by the closed loop) via the channel is zero, the necessary (and sufficient) data rate to confine the system Σ within Q is one. The discrepancy results from the causality constraints that are imposed on the coder-controller structure by the invariance condition, i.e., at each instant in time the controller needs to be able to produce a control input so that all successor states are inside Q, see e.g. [71]. Contrary to this observation, the zero-error capacity is an adequate measure for data rate constraints for the invariance for deterministic linear systems or for uniform boundedness for linear systems with disturbances [56, 57].

Periodic coder-controllers

Now, we introduce periodic coder-controllers that will be utilized to establish the data rate theorem in the next subsection. Given $\tau \in \mathbb{N}$ and a coder-controller $H = (S, \gamma, \delta)$, we say that H is *τ-periodic* if for all $t \in \mathbb{Z}_{\geq 0}$, $\zeta \in S^{[0;t]}$ and $\xi \in X^{[0;t]}$ we have

$$\begin{aligned} \gamma(\xi) &= \gamma(\xi|_{[\tau\lfloor t/\tau \rfloor;t]}), \\ \delta(\zeta) &= \delta(\zeta|_{[\tau\lfloor t/\tau \rfloor;t]}). \end{aligned} \tag{2.21}$$

For such periodic coder-controllers, the transmission data rate is equal to the smallest average number of bits, sufficient enough for every possible τ length symbol sequence in the closed loop, needed to encode the sets of possible successor coder symbols where the average is taken over the length of a symbol sequence. The following lemma formalizes this statement.

Lemma 5. *The transmission data rate of a τ-periodic coder-controller $H = (S, \gamma, \delta)$ for* (2.3) *is given by*

$$R(H) = \max_{\zeta \in \mathcal{Z}_\tau} \frac{1}{\tau} \sum_{t=0}^{\tau-1} \log_2 \#Z(\zeta|_{[0;t]}). \tag{2.22}$$

Proof. Let L denote the right-hand-side of (2.22). Consider $T \in \mathbb{N}$, $\zeta \in \mathcal{Z}_T$ and set $a := \lfloor T/\tau \rfloor$ and $\bar{\tau} := T - \tau a$. We define $\zeta_i := \zeta|_{[i\tau;(i+1)\tau[}$ for $i \in [0; a[$ and $\zeta_a := \zeta|_{[a\tau;T[}$. Since γ is τ-periodic, we see that each ζ_i with $i \in [0; a[$ is an element of $\mathcal{Z}_\tau$, and we obtain for $N_i := \sum_{t=0}^{\tau-1} \log_2 \#Z(\zeta_i|_{[0;t]})$ the bound $N_i \leq L\tau$ for all $i \in [0;a[$. We define $N_a := \sum_{t=0}^{\bar{\tau}-1} \log_2 \#Z(\zeta_a|_{[0;t]})$ which is bounded by $N_a \leq \tau \log_2 \#S$. Note that $a\tau + \bar{\tau} = T$, so that for $C := \tau \log_2 \#S$ we have

$$\tfrac{1}{T} \sum_{t=0}^{T-1} \log_2 \#Z(\zeta|_{[0;t]}) = \tfrac{1}{T}(\sum_{i=0}^{a-1} N_i + N_a) \leq \tfrac{1}{T}(aL\tau + L\bar{\tau} + C) = L + \tfrac{C}{T}.$$

Since C is independent of T, the assertion follows. □

The following lemma states that there always exists a τ-periodic coder-controller with a data rate not larger than that of a given coder-controller.

Lemma 6. *For every coder-controller $H = (S, \delta, \gamma)$ for (2.3) and $\varepsilon > 0$, there exists a τ-periodic coder-controller $\hat{H} = (S, \hat{\delta}, \hat{\gamma})$ that satisfies*

$$R(\hat{H}) \leq R(H) + \varepsilon.$$

Proof. For $\varepsilon > 0$, we pick $\tau \in \mathbb{N}$ so that $\log_2 \#\mathcal{Z}_0/\tau \leq \varepsilon/2$ and $\max_{\zeta \in \mathcal{Z}_\tau} \frac{1}{\tau} \sum_{t=0}^{\tau-2} \log_2 \#Z(\zeta|_{[0;t]}) \leq R(H) + \varepsilon/2$. We define $\hat{\gamma}$ and $\hat{\delta}$ for all $\xi \in X^{[0;t]}$, $\zeta \in S^{[0;t]}$ with $t \in \mathbb{Z}_{\geq 0}$ by

$$\hat{\gamma}(\xi) := \gamma(\xi|_{[\tau\lfloor t/\tau \rfloor;t]}) \quad \text{and} \quad \hat{\delta}(\zeta) := \delta(\zeta|_{[\tau\lfloor t/\tau \rfloor;t]}).$$

Let $\hat{Z}$ be defined in (2.18) w.r.t. $\hat{\gamma}$. Then we have for all $\zeta \in S^{[0;t]}$ with $t \in [0; \tau - 1[$ the equality $Z(\zeta) = \hat{Z}(\zeta)$ and for every $\zeta \in S^{[0;\tau[}$ we have $\hat{Z}(\zeta) = \mathcal{Z}_0$ which follows from the fact that $\hat{\gamma}$ is τ-periodic. The transmission data rate of $\hat{H}$ follows by (2.22) which is bounded by

$$\max_{\zeta \in \hat{\mathcal{Z}}_\tau} \frac{1}{\tau} \Big(\sum_{t=0}^{\tau-2} \log_2 \#\hat{Z}(\zeta|_{[0;t]}) + \log_2 \#\mathcal{Z}_0 \Big) \leq R(H) + \varepsilon. \qquad \square$$

The theorem in the next subsection shows that the data rate of a coder-controller able to make a subset of the state-space invariant cannot be less than the IFE of the subset.

2.4.2 The data rate theorem

The next result establishes the data rate theorem.

Theorem 6. *Consider a system $\Sigma = (X, U, F)$ and a nonempty set $Q \subseteq X$. The invariance feedback entropy of Σ and Q satisfies*

$$h_{\text{inv}}(Q, \Sigma) = \inf_{H \in \mathcal{H}} R(H) \tag{2.23}$$

where $\mathcal{H}$ is the set of all Q-admissible coder-controllers for Σ.

We use the following two technical lemmas to show Theorem 6.

Lemma 7. *Let $H = (S, \gamma, \delta)$ be a Q-admissible τ-periodic coder-controller for $\Sigma = (X, U, F)$. Then there exists an invariant cover $(\mathcal{A}, G)$ of (Σ, Q) and a (τ, Q)-spanning set $\mathcal{S}$ in $(\mathcal{A}, G)$ so that*

$$\frac{1}{\tau} \log_2 \mathcal{N}(\mathcal{S}) \leq R(H).$$

Proof. For every $t \in [0;\tau[$ and every $\zeta \in \mathcal{Z}_{t+1}$ we define $A(\zeta) := \{x \in Q \mid \exists_{(\xi,\nu)\in\mathcal{B}_Q(H)}\ \zeta = \Gamma_t(\xi|_{[0;t]}) \wedge \xi(t) = x\}$, $G(A(\zeta)) := \delta(\zeta)$ and $\mathcal{A} := \{A(\zeta) \mid \zeta \in \mathcal{Z}_{t+1} \wedge t \in [0;\tau[\}$. We show that $(\mathcal{A}, G)$ is an invariant cover of (Σ, Q). Clearly, $\mathcal{A}$ is finite and every element of $\mathcal{A}$ is a subset of Q. Since H is Q-admissible, for every $x \in Q$ there exists $(\xi, \nu) \in \mathcal{B}_Q(H)$ so that $\xi(0) = x$. Hence, $\{A(s) \mid s \in \mathcal{Z}_1\}$ covers Q and we see that $\mathcal{A}$ covers Q. Let $A \in \mathcal{A}$ and suppose that there exists $x \in A$ so that $F(x, G(A)) \nsubseteq Q$. Since $A \in \mathcal{A}$, there exists $t \in [0;\tau[$, $\zeta \in \mathcal{Z}_{t+1}$ and $(\xi, \nu) \in \mathcal{B}_Q(H)$ so that $A = A(\zeta)$, $\zeta = \Gamma_t(\xi|_{[0;t]})$ and $x = \xi(t)$. Note that ν satisfies (2.20) so that $\nu(t) = G(A(\zeta))$ holds. We fix $x' \in F(x, G(A))\backslash Q$ and pick a trajectory (ξ', ν') of Σ on $[0;\infty[$ such that $\xi'(0) = x'$ and $\nu'(t') = \delta(\Gamma_t((\xi|_{[0;t]}\xi')|_{[t;t+t'+1]}))$ holds for all $t' \in \mathbb{Z}_{\geq 0}$. We define $(\bar{\xi}, \bar{\nu})$ by $\bar{\xi} := \xi|_{[0;t]}\xi'$ and $\bar{\nu} := \nu|_{[0;t]}\nu'$, which by construction is a trajectory of Σ on $[0;\infty[$ which satisfies (2.20) but $\bar{\xi}([0;\infty[) \nsubseteq Q$. This contradicts the Q-admissibility of H and we can deduce that $F(A, G(A)) \subseteq Q$ for all $A \in \mathcal{A}$, which shows that $(\mathcal{A}, G)$ is an invariant cover of (Σ, Q).
We are going to construct a (τ, Q)-spanning set $\mathcal{S} \subseteq \mathcal{A}^{[0;\tau[}$ with the help of $\mathcal{Z}_\tau$. For each $\zeta \in \mathcal{Z}_\tau$ we define a sequence $\alpha_\zeta : [0;\tau[\to \mathcal{A}$ by $\alpha_\zeta(t) := A(\zeta|_{[0;t]})$ for all $t \in [0;\tau[$ and use $\mathcal{S}$ to denote the set of all such sequences $\{\alpha_\zeta \mid \zeta \in \mathcal{Z}_\tau\}$. Note that $P_\mathcal{S}(\alpha_\zeta) = \{A(s) \mid s \in \mathcal{Z}_1\}$ holds for all $\alpha_\zeta \in \mathcal{S}$, and we see that $P_\mathcal{S}(\alpha_\zeta)$ covers Q. Let us show (2.5). Let $\alpha_\zeta \in \mathcal{S}$, $t \in [0;\tau-1[$ so that $\alpha_\zeta(t) = A(\zeta|_{[0;t]})$. We define $\zeta_t := \zeta|_{[0;t]}$ and fix $x_0 \in A(\zeta_t)$ and $x_1 \in F(x_0, G(A(\zeta_t)))$. Since $x_0 \in A(\zeta_t)$ there exists $(\xi, \nu) \in \mathcal{B}_Q(H)$ so that $\zeta_t = \Gamma_t(\xi|_{[0;t]})$ with $\xi(t) = x_0$ and we use (2.20) to see that $G(A(\zeta_t)) = \delta(\zeta_t) = \nu(t)$. Therefore, $(\xi, \nu)|_{[0;t]}$ can be extended to a trajectory in $(\bar{\xi}, \bar{\nu}) \in \mathcal{B}_Q(H)$ with $\bar{\xi}(t+1) = x_1$. Let $s = \gamma(\bar{\xi}|_{[0;t+1]})$, then we have $s \in Z(\zeta_t)$ and $\zeta_{t+1} := \zeta_t s \in \mathcal{Z}_{t+2}$ holds. Moreover, $\zeta_{t+1} = \Gamma_{t+1}(\bar{\xi}|_{[0;t+1]})$ and we conclude that $x_1 \in A(\zeta_{t+1})$. We repeat this process for $x_i \in F(A(\zeta_{t+i}), G(A(\zeta_{t+i}))$, $i \in [0;k]$ until $t + k = \tau - 1$ at which point we arrive at $\zeta_{t+k} \in \mathcal{Z}_\tau$ and we see that the associated sequence $\alpha_{\zeta_{t+k}}$ is an element of $\mathcal{S}$ that satisfies $x_1 \in \alpha_{\zeta_{t+k}}(t+1)$ and $\alpha_{\zeta_{t+k}}|_{[0;t]} = \alpha_\zeta|_{[0;t]}$. Since such a sequence can be constructed for every $x_1 \in F(x_0, G(A(\zeta_t)))$ and $x_0 \in A(\zeta_t)$, we see that (2.5) holds and it follows that $\mathcal{S}$ is (τ, Q)-spanning in $(\mathcal{A}, G)$.
We claim that $\#P_\mathcal{S}(\alpha_\zeta|_{[0;t]}) \leq \#Z(\zeta|_{[0;t]})$ for every $\alpha_\zeta \in \mathcal{S}$ and $t \in [0;\tau-1[$. Let $A \in P_\mathcal{S}(\alpha_\zeta|_{[0;t]})$, then there exists $\alpha_{\zeta'} \in \mathcal{S}$ such that $A = \alpha_{\zeta'}(t+1)$ and $\zeta'|_{[0;t]} = \zeta|_{[0;t]}$. Hence $\zeta'(t+1) \in Z(\zeta|_{[0;t]})$. Moreover, for $A, \bar{A} \in P_\mathcal{S}(\alpha_\zeta|_{[0;t]})$ with $A \neq \bar{A}$ there exists $\alpha_{\zeta'}, \alpha_{\bar{\zeta}'} \in \mathcal{S}$ such that $A = A(\zeta'|_{[0;t+1]})$ and $\bar{A} = A(\bar{\zeta}'|_{[0;t+1]})$, which shows that $\zeta'(t+1) \neq \bar{\zeta}'(t+1)$ and $\zeta'(t+1), \bar{\zeta}'(t+1) \in Z(\zeta|_{[0;t]})$ and we obtain $\#P_\mathcal{S}(\alpha_\zeta|_{[0;t]}) \leq \#Z(\zeta|_{[0;t]})$ for all $t \in [0;\tau-1[$ and $\zeta \in \mathcal{Z}_\tau$. For $t = \tau - 1$ we have $P_\mathcal{S}(\alpha_\zeta) = \{A(s) \mid s \in \mathcal{Z}_1\}$. For $Z(\zeta)$ we have $Z(\zeta) = \gamma(X)$, since H is τ-periodic and we obtain $\#P_\mathcal{S}(\alpha_\zeta) \leq \#Z(\zeta)$ for every $\zeta \in \mathcal{Z}_\tau$. Hence, $\mathcal{N}(\mathcal{S}) \leq \max_{\zeta\in\mathcal{Z}_\tau} \prod_{t=0}^{\tau-1} \#Z(\zeta|_{[0;t]})$ follows and we obtain $\frac{1}{\tau}\log_2 \mathcal{N}(\mathcal{S}) \leq R(H)$. □

In the proof of the following lemma, we use an *enumeration* of a finite set A, which is a function $e : A \to [1;\#A]$ such that $e(A) = [1;\#A]$.

Lemma 8. *Consider an invariant cover $(\mathcal{A}, G)$ of $\Sigma = (X, U, F)$ and some nonempty set $Q \subseteq X$. Let $\mathcal{S}$ be a (τ, Q)-spanning set in $(\mathcal{A}, G)$. Then there exists a Q-admissible τ-periodic coder-controller $H = (S, \gamma, \delta)$ for Σ so that*

$$\frac{1}{\tau}\log_2 \mathcal{N}(\mathcal{S}) \geq R(H).$$

Proof. We define $\mathcal{S}_t := \{\alpha \in \mathcal{A}^{[0;t]} \mid \exists_{\hat{\alpha}\in\mathcal{S}}\ \hat{\alpha}|_{[0;t]} = \alpha\}$ for $t \in [0;\tau[$ and observe that $\mathcal{S}_{\tau-1} = \mathcal{S}$ and for every $\alpha \in \mathcal{S}$ we have $P_{\mathcal{S}}(\alpha) = \mathcal{S}_0$. For $\alpha \in \mathcal{S}_t$ with $t \in [0;\tau-1[$ let $e(\alpha)$ be an enumeration of $P_{\mathcal{S}}(\alpha)$. We slightly abuse the notation, and use $e(\varnothing)$ to denote an enumeration of $\mathcal{S}_0$ so that $e(\alpha|_{[0;0[}) = e(\varnothing)$ for all $\alpha \in \mathcal{S}$. Let $m \in \mathbb{N}$ be the smallest number so that every co-domain of $e(\alpha)$ is a subset of $[1;m]$. We use this interval to define the set of symbols $S := [1;m]$. We are going to define $\gamma(\xi)$ and $\delta(\zeta)$ for all sequences $\xi \in X^{[0;t]}$, respectively, $\zeta \in S^{[0;t]}$ with $t \in [0;\tau[$, which determines γ and δ for all elements in their domain, since γ and δ are τ-periodic. We begin with γ, which we define iteratively. For $t = 0$ and $x \in X$ we set $\gamma(x) := e(\varnothing)(A)$ if there exists $A \in \mathcal{S}_0$ with $x \in A$. If there are several $A \in \mathcal{S}_0$ that contain x we simply pick one. If there does not exist any $A \in \mathcal{S}_0$ with $x \in A$ we set $\gamma(x) := 1$. For $t \in]0;\tau[$ and $\xi \in X^{[0;t]}$ we define $\gamma(\xi) := e(\alpha|_{[0;t[})(\alpha(t))$ for $\alpha \in \mathcal{S}_t$ that satisfies i) $\xi(t) \in \alpha(t)$ and ii) $\gamma(\xi|_{[0;t']}) = e(\alpha|_{[0;t'[})(\alpha(t'))$ holds for all $t' \in [0;t[$. Again, if there are several such $\alpha \in \mathcal{S}_t$ we simply pick one. If there does not exist any α in $\mathcal{S}_t$ that satisfies i) and ii), we set $\gamma(\xi) := 1$. We define δ for $t \in [0;\tau[$ and $\zeta \in S^{[0;t]}$ as follows: if there exists $\alpha \in \mathcal{S}_t$ that satisfies $e(\alpha|_{[0;t'[})(\alpha(t')) = \zeta(t')$ for all $t' \in [0;t]$, we set $\delta(\zeta) := G(\alpha(t))$, otherwise we set $\delta(\zeta) := u$ for some $u \in U$. Let us show that the coder-controller is Q-admissible. We fix $(\xi,\nu) \in \mathcal{B}_Q(H)$ and proceed by induction with the hypothesis parameterized by $t \in [0;\tau[$: there exists $\alpha \in \mathcal{S}_t$ so that $\xi(t) \in \alpha(t)$, $\gamma(\xi|_{[0;t']}) = e(\alpha|_{[0;t'[})(\alpha(t'))$ and $\nu(t') = G(\alpha(t'))$ hold for all $t' \in [0;t]$. For $t = 0$, we know that $\mathcal{S}_0$ covers Q so that for $\xi(0) \in Q$ there exists $A \in \mathcal{S}_0$ with $x \in A$ and it follows from the definition of γ and δ that $\gamma(\xi(0)) = e(\varnothing)(\bar{A})$ for some $\bar{A} \in \mathcal{S}_0$ with $\xi(0) \in \bar{A}$ and $\nu(0) = \delta(\gamma(\bar{A})) = G(\bar{A})$. Now suppose that the induction hypothesis holds for $t \in]0;\tau-1[$. Since $\xi(t) \in \alpha(t)$ and $\nu(t) = G(\alpha(t))$ for some $\alpha \in \mathcal{S}_t$, we use (2.5) to see that there exists $\bar{\alpha} \in \mathcal{S}$ so that $\bar{\alpha}|_{[0;t]} = \alpha$ and $\xi(t+1) \in \bar{\alpha}(t+1)$, so that $\bar{\alpha}$ satisfies i) and ii) in the definition of γ and we have $\gamma(\xi|_{[0;t+1]}) = e(\alpha)(\hat{\alpha}(t+1))$ for some $\hat{\alpha} \in \mathcal{S}_{t+1}$ with $\xi(t+1) \in \hat{\alpha}(t+1)$ and $\hat{\alpha}|_{[0;t]} = \alpha$. Since $\hat{\alpha}$ is uniquely determined by the symbol sequence $\zeta \in S^{[0;t+1]}$ given by $\zeta(t') = e(\hat{\alpha}|_{[0;t'[})(\hat{\alpha}(t'))$ for all $t' \in [0;t+1]$, we have $\nu(t+1) = \delta(\zeta) = G(\hat{\alpha}(t+1))$, which completes the induction. Note that the induction hypothesis implies that $F(\xi(t),\nu(t)) \subseteq Q$ for all $t \in [0;\tau[$, since $\xi(t) \in \alpha(t)$ and $\nu(t) = G(\alpha(t))$. We obtain $\xi([0;\infty[) \subseteq Q$ from the τ-periodicity of H and the Q-admissibility follows.
We derive a bound for $R(H)$. Since H is τ-periodic, we have for any $\zeta \in \mathcal{Z}_\tau$ the equality $Z(\zeta) = e(\varnothing)(\mathcal{S}_0)$ and we see that $\#Z(\zeta) = \#e(\varnothing)(\mathcal{S}_0) = \#P_{\mathcal{S}}(\alpha)$ for any $\alpha \in \mathcal{S}$. We fix $\zeta \in \mathcal{Z}_\tau$ and pick $\alpha \in \mathcal{S}$ so that $\alpha(t) = e^{-1}(\alpha|_{[0;t[})(\zeta(t))$ holds for all $t \in [0;\tau[$. By definition, the set $Z(\zeta|_{[0;t]})$ is the co-domain of an enumeration of $P_{\mathcal{S}}(\alpha|_{[0;t]})$, which shows $\#Z(\zeta|_{[0;t]}) = \#P_{\mathcal{S}}(\alpha|_{[0;t]})$. Therefore, we have $\max_{\zeta\in\mathcal{Z}_\tau}\prod_{t=0}^{\tau-1}\#Z(\zeta|_{[0;t]}) \leq \max_{\alpha\in\mathcal{S}}\prod_{t=0}^{\tau-1}\#P_{\mathcal{S}}(\alpha|_{[0;t]})$ and the assertion follows by (2.22). □

Now, we continue with the proof of Theorem 6.

Proof of Theorem 6. Let us first prove the inequality $h_{\mathrm{inv}}(Q,\Sigma) \leq \inf_{H\in\mathcal{H}} R(H)$. If the right-hand-side of (2.23) equals infinity the inequality trivially holds and subsequently we assume the right-hand-side of (2.23) is finite. We fix $\varepsilon > 0$ and pick a coder-controller $\bar{H} = (S,\bar{\gamma},\bar{\delta})$ so that $R(\bar{H}) \leq \inf_{H\in\mathcal{H}} R(H) + \varepsilon$. According to Lemma 6 there exists a τ-

periodic coder-controller $H = (S, \gamma, \delta)$ so that $R(H) \leq R(\bar{H}) + \varepsilon$. It is straightforward to see that for every $(\xi, \nu) \in \mathcal{B}_Q(H)$ and $\xi_i := \xi|_{[i\tau;(i+1)\tau[}$, $i \in \mathbb{Z}_{\geq 0}$, there exists $(\bar{\xi}, \bar{\nu}) \in \mathcal{B}_Q(\bar{H})$, so that $\xi_i = \bar{\xi}|_{[0;\tau[}$, which shows that H is Q-admissible. From Lemma 7 it follows that there exists an $(\mathcal{A}, G)$ of Σ and Q and a (τ, Q)-spanning set in $(\mathcal{A}, G)$ so that $\frac{1}{\tau}\log_2 \mathcal{N}(\mathcal{S}) \leq R(H)$. We use Lemma 1 to see that $r_{\text{inv}}(n\tau, \mathcal{A}, G, \Sigma) \leq n r_{\text{inv}}(\tau, \mathcal{A}, G, \Sigma)$ so that $h(\mathcal{A}, G) = \lim_{n\to\infty} \frac{1}{n\tau}\log_2 r_{\text{inv}}(n\tau, \mathcal{A}, G, \Sigma) \leq \frac{1}{\tau}\log_2 r_{\text{inv}}(\tau, \mathcal{A}, G, \Sigma) \leq \frac{1}{\tau}\log_2 \mathcal{N}(\mathcal{S})$. By the choice of H we obtain $2\varepsilon + \inf_{H\in\mathcal{H}} R(H) \geq R(H) \geq h_{\text{inv}}(Q, \Sigma)$. Since this holds for arbitrary $\varepsilon > 0$ we arrive at the desired inequality.

We continue with the inequality $h_{\text{inv}}(Q, \Sigma) \geq \inf_{H\in\mathcal{H}} R(H)$. If $h_{\text{inv}}(Q, \Sigma) = \infty$ the inequality trivially holds and subsequently we consider $h_{\text{inv}}(Q, \Sigma) < \infty$. We fix $\varepsilon > 0$ and pick an invariant cover $(\mathcal{A}, G)$ of (Σ, Q) so that $h(\mathcal{A}, G) < h_{\text{inv}}(Q, \Sigma) + \varepsilon$. We pick $\tau \in \mathbb{N}$ so that $\frac{1}{\tau}\log_2 r_{\text{inv}}(\tau, \mathcal{A}, G, \Sigma) < h(\mathcal{A}, G) + \varepsilon$. Let $\mathcal{S}$ be (τ, Q)-spanning set that satisfies $r_{\text{inv}}(\tau, \mathcal{A}, G, \Sigma) = \mathcal{N}(\mathcal{S})$. It follows from Lemma 8 that there exists a Q-admissible coder-controller H so that $\frac{1}{\tau}\log_2 \mathcal{N}(\mathcal{S}) \geq R(H)$ holds, and hence, we obtain $2\varepsilon + h_{\text{inv}}(Q, \Sigma) \geq R(H)$. This inequality holds for any $\varepsilon > 0$, which implies that $h_{\text{inv}}(Q, \Sigma) \geq \inf_{H\in\mathcal{H}} R(H)$. □

2.5 Uncertain Linear Control Systems

We derive a lower bound of the invariance feedback entropy of uncertain linear control systems (2.2) and compact sets Q. In this setting, we also derive a lower bound of the data rate of any *static* or *memoryless* coder-controller. Similar to [60, Section II], we employ the Brunn-Minkowsky inequality to obtain a lower bound on the growth of the size of the uncertainty set of the state at the controller side in one time step. For the general case, we use this inequality to derive a lower bound on the expansion number, which in turn leads to the entropy. For static coder-controllers the derivation of the lower bound is substantially simpler, see the proof of [60, Thm 1] and the proof of Theorem 8.

2.5.1 Universal lower bound

Theorem 7. *Consider the matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and two nonempty measurable sets $W, Q \subseteq \mathbb{R}^n$ with $\mu(W) < \mu(Q)$ and suppose that Q is compact. Let Σ be given by $X = \mathbb{R}^n$, $U \subseteq \mathbb{R}^m$ with $U \neq \varnothing$ and F according to*

$$\forall_{x\in X}\forall_{u\in U} \qquad F(x, u) = Ax + Bu + W. \tag{2.24}$$

Let $\mathbb{R}^n = \mathbb{E}_1 \oplus \mathbb{E}_2$, where $\mathbb{E}_1$ is an A invariant subspace of $\mathbb{R}^n$ with $\mathbb{E}_1 \neq \{0\}$, and $\oplus$ stands for the direct sum. Let $\pi_1 : \mathbb{R}^n \to \mathbb{E}_1$ be the projection onto $\mathbb{E}_1$ along $\mathbb{E}_2$, and[2] *$\mu_1(\pi_1 W) < \mu_1(\pi_1 Q)$, also let $n_1 = \textbf{dim}(\mathbb{E}_1)$ and μ_1 denote the n_1-dimensional Lebesgue measure. Then, the invariance feedback entropy of Σ and Q satisfies*

$$\log_2\left(|\det A|_{\mathbb{E}_1}| \frac{\mu_1(\pi_1 Q)}{(\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1})^{n_1}}\right) \leq h_{\text{inv}}(Q, \Sigma). \tag{2.25}$$

[2]Since map π_1 is linear, we use notation $\pi_1 A$ instead of $\pi_1(A)$, $\forall A \subseteq \mathbb{R}^n$, for the sake of simpler presentation.

Proof. Let us first point out that every compact set has finite Lebesgue measure.

If $|\det A|_{\mathbb{E}_1}| = 0$ the left-hand-side is $-\infty$ and (2.25) holds. In the remainder we consider the case $|\det A|_{\mathbb{E}_1}| > 0$. If $h_{\text{inv}}(Q,\Sigma) = \infty$ the inequality (2.25) holds independent of the left-hand-side and subsequently we assume that $h_{\text{inv}}(Q,\Sigma) < \infty$. We pick $\varepsilon \in \mathbb{R}_{>0}$ and an invariant cover $(\mathcal{C}, H)$ of (Σ, Q), so that $h(\mathcal{C}, H) \leq h_{\text{inv}}(Q,\Sigma) + \varepsilon$. Given Theorem 5, we can assume that the cover elements of $\mathcal{C}$ are closed, which yields by the compactness of Q that the cover elements are compact and therefore Lebesgue measurable.

We fix $\tau \in \mathbb{N}$ and pick a (τ, Q)-spanning set $\mathcal{S}$ so that $r_{\text{inv}}(\tau, \mathcal{C}, H, \Sigma) = \mathcal{N}(\mathcal{S})$, which exists, since for fixed τ, the number of (τ, Q)-spanning set is finite.

We are going to show that there exists $\alpha \in \mathcal{S}$ that satisfies

$$\left(|\det A|_{\mathbb{E}_1}| \frac{\mu_1(\pi_1 Q)}{(\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1})^{n_1}}\right)^{\tau} \leq \prod_{t=0}^{\tau-1} \# P_{\mathcal{S}}(\alpha|_{[0;t]}). \tag{2.26}$$

We construct $\alpha \in \mathcal{S}$ iteratively over $t \in [0; \tau[$. For $t = 0$ we introduce $S_0 := \{\alpha(0) \mid \alpha \in \mathcal{S}\}$ and define

$$m_0 := \max\{\mu_1(\pi_1 \alpha(0))^{1/n_1} \mid \alpha \in \mathcal{S}\}.$$

We pick $\Omega_0 \in S_0$ so that $m_0 = \mu_1(\pi_1 \Omega_0)^{1/n_1}$. For $t \in [1; \tau - 1[$ we set $\alpha_{t'} := \Omega_0 \cdots \Omega_{t'}$ for $t' \in [0; t]$ and assume that $\Omega_{t'} \in P_{\mathcal{S}}(\alpha|_{[0;t'[})$ and $\mu_1(\pi_1 \Omega_{t'})^{1/n_1} = m_{t'}$ holds for all $t' \in [1; t]$ where

$$m_{t'} := \max\{\mu_1(\pi_1 \Omega)^{1/n_1} \mid \Omega \in P_{\mathcal{S}}(\alpha|_{[0;t'[})\}.$$

Then we set $m_{t+1} := \max\{\mu_1(\pi_1 \Omega)^{1/n_1} \mid \Omega \in P_{\mathcal{S}}(\alpha|_{[0;t+1[})\}$ and pick $\Omega_{t+1} \in P_{\mathcal{S}}(\alpha|_{[0;t+1[})$ so that $m_{t+1} = \mu_1(\pi_1 \Omega_{t+1})^{1/n_1}$. For $t = \tau - 1$ we obtain a sequence $\alpha := \Omega_0 \cdots \Omega_{\tau-1}$ that is an element of $\mathcal{S}$. Hence, it follows from (2.5) that α satisfies for all $t \in [0; \tau[$ the inclusion

$$\pi_1\big(A\alpha(t) + BH(\alpha(t)) + W\big) \subseteq \pi_1\left(\bigcup_{\Omega \in P_{\mathcal{S}}(\alpha|_{[0;t]})} \Omega\right). \tag{2.27}$$

For $t \in [0; \tau - 1[$, we use the Brunn-Minkowsky inequality for compact, measurable sets [34]

$$\mu_1(\pi_1 A\alpha(t))^{1/n_1} + \mu_1(\pi_1 W)^{1/n_1} \leq \mu_1(\pi_1 A\alpha(t) + \pi_1 BH(\alpha(t)) + \pi_1 W)^{1/n_1}$$

and the equality [72]

$$\mu(A\alpha(t))^{1/n} = |\det A|^{1/n} \mu(\alpha(t))^{1/n}$$

together with $\mu_1(\pi_1 \alpha(t))^{1/n_1} = m_t$ and (2.27), to derive

$$|\det A|_{\mathbb{E}_1}|^{1/n_1} m_t + \mu_1(\pi_1 W)^{1/n_1} \leq m_{t+1} (\# P_{\mathcal{S}}(\alpha|_{[0;t+1[}))^{1/n_1} \tag{2.28}$$

for all $t \in [0; \tau - 1[$. Note that we also used the fact that $\mathbb{E}_1$ is A invariant to show inequality (2.28). Also, for every $t \in [0; \tau[$ we have

$$|\det A|_{\mathbb{E}_1}|^{1/n_1} m_t + \mu_1(\pi_1 W)^{1/n_1} \leq \mu_1(\pi_1 Q)^{1/n_1} \tag{2.29}$$

since $A\alpha(t) + BH(\alpha(t)) + W \subseteq Q$ which follows from the fact that $\alpha(t) \in \mathcal{C}$ and $(\mathcal{C}, H)$ is an invariant cover. To ease the notation, let us introduce $N_0 := (\#P_{\mathcal{S}}(\alpha))^{1/n_1}$ and $N_t := (\#P_{\mathcal{S}}(\alpha|_{[0;t[}))^{1/n_1}$ for $t \in [1;\tau[$. We use induction over $\tau' \in [0;\tau[$ to show

$$\left(|\det A|_{\mathbb{E}_1}|^{1/n_1} \frac{\mu_1(\pi_1 Q)^{1/n_1}}{\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1}}\right)^{\tau'+1} \leq \prod_{t=0}^{\tau'} N_t. \tag{2.30}$$

Let us show (2.30) for $\tau' = 0$. Since $P_{\mathcal{S}}(\alpha)$ is a cover of Q and $\#P_{\mathcal{S}}(\alpha)^{1/n_1} = N_0$ we obtain

$$\mu_1(\pi_1 Q)^{1/n_1} \leq m_0 N_0. \tag{2.31}$$

From (2.29) we obtain $m_0 \leq (\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1})/|\det A|_{\mathbb{E}_1}|^{1/n_1}$ and (2.30) follows for $\tau' = 1$.

If $\tau = 1$ we have shown (2.30) and subsequently we consider $\tau > 1$. We fix $\tau'' \in [1;\tau[$ and assume that (2.30) holds for all $\tau' \in [0;\tau''[$. We use (2.28) recursively to derive

$$m_0 \leq \frac{m_{\tau''}}{|\det A|_{\mathbb{E}_1}|^{\tau''/n_1}} \left(\prod_{t=1}^{\tau''} N_t\right) - \sum_{t=1}^{\tau''} \frac{\mu_1(\pi_1 W)^{1/n_1}}{|\det A|_{\mathbb{E}_1}|^{t/n_1}} \prod_{t'=1}^{t-1} N_{t'} \tag{2.32}$$

with the convention that $\prod_{t=a}^{b} x_t = 1$ for $b < a$. Using (2.31) and rearranging the terms in (2.32) we obtain

$$\mu_1(\pi_1 Q)^{1/n_1} + \sum_{t=1}^{\tau''} \frac{\mu_1(\pi_1 W)^{1/n_1}}{|\det A|_{\mathbb{E}_1}|^{t/n_1}} \prod_{t'=0}^{t-1} N_{t'} \leq \frac{m_{\tau''}}{|\det A|_{\mathbb{E}_1}|^{\tau''/n_1}} \prod_{t=0}^{\tau''} N_t. \tag{2.33}$$

We invoke the induction hypothesis and use the inequality
$\prod_{t'=0}^{t-1} N_{t'} \geq ((|\det A|_{\mathbb{E}_1}|\mu_1(\pi_1 Q))^{1/n_1}/(\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1}))^t$ to derive

$$\mu_1(\pi_1 Q)^{1/n_1} + \sum_{t=1}^{\tau''} \frac{\mu_1(\pi_1 W)^{1/n_1} \mu_1(\pi_1 Q)^{t/n_1}}{(\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1})^t} \leq \frac{m_{\tau''}}{|\det A|_{\mathbb{E}_1}|^{\tau''/n_1}} \prod_{t=0}^{\tau''} N_t. \tag{2.34}$$

From Lemma 10 (given in the Appendix) it follows that the left-hand-side of (2.34) evaluates to

$$\mu_1(\pi_1 Q)^{1/n_1} + \sum_{t=1}^{\tau''} \frac{\mu_1(\pi_1 W)^{1/n_1} \mu_1(\pi_1 Q)^{t/n_1}}{(\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1})^t} = \frac{\mu_1(\pi_1 Q)^{(\tau''+1)/n_1}}{(\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1})^{\tau''}}. \tag{2.35}$$

We combine $m_{\tau''} \leq (\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1})/|\det A|_{\mathbb{E}_1}|^{1/n_1}$ (that follows from (2.29)) with (2.34) and (2.35) to get

$$\frac{\mu_1(\pi_1 Q)^{(\tau''+1)/n_1}}{(\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1})^{\tau''}} \leq \frac{\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1}}{|\det A|_{\mathbb{E}_1}|^{(\tau''+1)/n_1}} \prod_{t=0}^{\tau''} N_t \tag{2.36}$$

which shows that (2.30) holds for $\tau' = \tau''$. Hence, (2.30) holds for all $\tau' \in [0;\tau[$. In particular, for $\tau' = \tau - 1$ and we conclude that (2.26) holds.

Inequality (2.26) together with the definition of $\mathcal{N}(\mathcal{S})$ yields

$$\left(|\det A|_{\mathbb{E}_1}|\frac{\mu_1(\pi_1 Q)}{(\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1})^{n_1}}\right)^{\tau} \leq \mathcal{N}(\mathcal{S}) = r_{\mathrm{inv}}(\tau, \mathcal{C}, H, \Sigma)$$

where the equality follows by our choice of $\mathcal{S}$. From (2.7) we get

$$\log_2\left(|\det A|_{\mathbb{E}_1}|\frac{\mu_1(\pi_1 Q)}{(\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1})^{n_1}}\right) \leq h(\mathcal{C}, H) \leq h_{\mathrm{inv}}(Q, \Sigma) + \varepsilon \tag{2.37}$$

which implies (2.25) since (2.37) holds for every $\varepsilon > 0$. $\square$

Remark 1. *Let* $\texttt{spec}(A)$ *denote the spectrum of* A, $\mathbb{E}^{\lambda}$ *denote the eigenspace of* A *associated with* $\lambda \in \texttt{spec}(A)$ *and* $B \subseteq \texttt{spec}(A)$. *In Theorem 7 if* $\mathbb{E}_1 = \bigoplus_{\lambda \in B} \mathbb{E}^{\lambda}$, *then a good choice of* $\mathbb{E}_1$ *will be the one that gives the largest lower bound in* (2.25).

Remark 2. *Note that the lower bound, i.e., the left-hand-side of inequality* (2.25), *is invariant under coordinate transformation. Let* $z = Tx$ *for some invertible matrix* $T \in \mathbb{R}^{n \times n}$ *so that the transition function* $\bar{F}$ *of the system in the new coordinates is*

$$\bar{F}(z, u) = TAT^{-1}z + TBu + TW \tag{2.38}$$

and $\bar{Q} = TQ$. *Let* $\bar{\mathbb{E}}_i = T\mathbb{E}_i$, $i \in \{1, 2\}$, $\bar{\pi}_1 : \mathbb{R}^n \to \bar{\mathbb{E}}_1$ *be the projection on* $\bar{\mathbb{E}}_1$ *along* $\bar{\mathbb{E}}_2$. *Then we obtain*

$$\begin{aligned}
&|\det(TAT^{-1})|_{\bar{\mathbb{E}}_1}|\frac{\mu_1(\bar{\pi}_1 TQ)}{(\mu_1(\bar{\pi}_1 TQ)^{1/n_1} - \mu_1(\bar{\pi}_1 TW)^{1/n_1})^{n_1}} = \\
&|\det A|_{\mathbb{E}_1}|\frac{\mu_1(T\pi_1 Q)}{(\mu_1(T\pi_1 Q)^{1/n_1} - \mu_1(T\pi_1 W)^{1/n_1})^{n_1}} = \\
&|\det A|_{\mathbb{E}_1}|\frac{\mu_1(\pi_1 Q)}{(\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1})^{n_1}}.
\end{aligned}$$

When W is a singleton set, by taking $\mathbb{E}_1$ as the unstable subspace, we get the largest lower bound in (2.25) which recovers the well-known value of the invariance entropy [38, Th. 3.1] for deterministic linear control systems, i.e., the invariance entropy equals $\log_2 |\det A|_{\mathbb{E}_1}|$. This matches also other results known from stabilization with rate limited feedback [73].

2.5.2 Static coder-controllers

We restrict our attention to static coder-controllers and derive a lower bound of the data rate of such coder-controllers.

Let $(\mathcal{C}, H)$ be an invariant cover of (2.3) and a nonempty set $Q \subseteq X$. We define the *data rate* of $(\mathcal{C}, H)$ by

$$R(\mathcal{C}, H) := \log_2 \#\mathcal{C}. \tag{2.39}$$

The definition is motivated by the fact that any invariant cover $(\mathcal{C}, H)$ immediately provides a *static* or *memoryless* coder-controller scheme: given $x \in Q$ at the coder side, it is sufficient that the coder transmits one of the cover elements $C \in \mathcal{C}$ that contains the current state $x \in C$, to ensure that the controller is able to confine the successor states of x to Q, i.e.,

$$Ax + BH(C) + W \subseteq Q. \tag{2.40}$$

The number of different cover elements that need to be transmitted via the digital, noiseless channel at any time $t > 0$ is bounded by $\#\mathcal{C}$. Neither the coder nor the controller requires any past information for a correct functioning. Hence, we speak of $(\mathcal{C}, H)$ as static or memoryless coder-controller for (X, U, F).

The next result provides a lower bound on the data rate of any static coder-controller.

Theorem 8. *Consider the matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and two nonempty measurable sets $W, Q \subseteq \mathbb{R}^n$ with $\mu(W) < \mu(Q)$ and suppose that Q is compact. Let Σ in* (2.3) *be given by $X = \mathbb{R}^n$, $U \subseteq \mathbb{R}^m$ with $U \neq \varnothing$, F according to* (2.24), *$\mathbb{E}_1$, $\mathbb{E}_2$, μ_1, n_1 and π_1 as in Theorem 7 and $\mu_1(\pi_1 W) < \mu_1(\pi_1 Q)$. Then, we have*

$$\log_2 \left\lceil |\det A|_{\mathbb{E}_1}| \frac{\mu_1(\pi_1 Q)}{(\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1})^{n_1}} \right\rceil \leq \inf_{(\mathcal{C},H)} R(\mathcal{C}, H) \tag{2.41}$$

where we take the infimum over all invariant covers $(\mathcal{C}, H)$ of (Σ, Q).

Proof. If $|\det A|_{\mathbb{E}_1}| = 0$ the left-hand-side of (2.41) evaluates to $-\infty$ so that (2.41) holds. Let us consider $|\det A|_{\mathbb{E}_1}| > 0$. If the right-hand-side of (2.41) evaluates to ∞ nothing needs to be shown and we consider $\inf_{(\mathcal{C},H)} R(\mathcal{C}, H) < \infty$. Since $\inf_{(\mathcal{C},H)} R(\mathcal{C}, H)$ is finite, there exists an invariant cover $(\mathcal{D}, G)$ of (Σ, Q). Let $(\mathcal{C}, H)$ be the invariant cover with closed cover elements as constructed from $(\mathcal{D}, G)$ in Theorem 5. Then $(\mathcal{C}, H)$ is an invariant cover of (Σ, Q) and we have $R(\mathcal{C}, H) \leq R(\mathcal{D}, G)$.

As $(\mathcal{C}, H)$ is an invariant cover of (Σ, Q), we have for every $\Omega \in \mathcal{C}$ the inclusion

$$\pi_1(A\Omega + BH(\Omega) + W) \subseteq \pi_1 Q. \tag{2.42}$$

We use the Brunn-Minkowsky inequality for compact, measurable sets (see proof of Theorem 7) together with the identity [72] $\mu(A\Omega)^{1/n} = |\det A|^{1/n} \mu(\Omega)^{1/n}$ to derive $|\det A|_{\mathbb{E}_1}|^{1/n_1} \mu_1(\pi_1 \Omega)^{1/n_1} + \mu_1(\pi_1 W)^{1/n_1} \leq \mu_1(\pi_1 Q)^{1/n_1}$ which yields the bound

$$\mu_1(\pi_1 \Omega)^{1/n_1} \leq \frac{\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1}}{|\det A|_{\mathbb{E}_1}|^{1/n_1}}. \tag{2.43}$$

As $\#\mathcal{C}$ is an upper bound on the number of cover elements needed to cover $F(\Omega, H(\Omega))$, we have

$$\mu_1(\pi_1 Q)^{1/n_1} \leq (\#\mathcal{C})^{1/n_1} \max\{\mu_1(\pi_1 \Omega)^{1/n_1} \mid \Omega \in \mathcal{C}\}. \tag{2.44}$$

We use (2.43) (which holds for every $\Omega \in \mathcal{C}$) in (2.44) and rearrange the result to obtain

$$|\det A|_{\mathbb{E}_1}|^{1/n_1} \frac{\mu_1(\pi_1 Q)^{1/n_1}}{\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1}} \leq (\#\mathcal{C})^{1/n_1}.$$

Since this inequality holds for every invariant cover $(\mathcal{C}, H)$, we obtain (2.41). $\square$

It is easy to bound the difference between the universal lower bound in (2.25) and the lower bound of data rates for static coder-controllers in (2.41) so that we arrive at the following corollary, which allows us to quantify the performance loss due to the restriction to static coder-controllers.

Corollary 1. *In the context and under the assumptions of Theorem 8, let $a \in \mathbb{R}_{\geq 0}$ be given by*

$$a := |\det A|_{\mathbb{E}_1}| \frac{\mu_1(\pi_1 Q)}{(\mu_1(\pi_1 Q)^{1/n_1} - \mu_1(\pi_1 W)^{1/n_1})^{n_1}}.$$

Suppose that $a < \infty$ and there exists an invariant cover $(\mathcal{C}, H)$ of (Σ, Q) with $R(\mathcal{C}, H) = \log_2 \lceil a \rceil$. Then, the data rate R of $(\mathcal{C}, H)$ satisfies

$$R \leq h_{\mathrm{inv}}(Q, \Sigma) + 1. \tag{2.45}$$

Proof. Let $b \in [0, 1[$ be so that $a + b = \lceil a \rceil$. We use $a \leq 2^{h_{\mathrm{inv}}(Q,\Sigma)}$ and $0 \leq h_{\mathrm{inv}}(Q, \Sigma)$ to derive

$$\begin{aligned} R =& \log_2(a + b) \leq \log_2(2^{h_{\mathrm{inv}}(Q,\Sigma)} + b) \\ \leq& h_{\mathrm{inv}}(Q, \Sigma) + \log_2(1 + 2^{-h_{\mathrm{inv}}(Q,\Sigma)}) \\ \leq& h_{\mathrm{inv}}(Q, \Sigma) + 1. \end{aligned}$$

□

2.5.3 Tightness of the lower bounds

We show for a particular class of scalar linear difference inclusions of the form

$$\xi(t + 1) \in a\xi(t) + \nu(t) + [w_1, w_2] \tag{2.46}$$

with $a \in \mathbb{R}_{\neq 0}$, $w_1, w_2 \in \mathbb{R}$ and $w_1 \leq w_2$ that the lower bounds established in the previous subsections are tight.

Subsequently, we assume that Q is given as an interval containing $[w_1, w_2]$

$$Q := [q_1, q_2], \qquad q_1, q_2 \in \mathbb{R}, q_1 < w_1, w_2 < q_2.$$

We are going to construct a static coder-controller $(\mathcal{C}, H)$ and show that its data rate equals the lower bound in Theorem 8. To this end, we introduce

$$\begin{aligned} \Delta q &:= q_2 - q_1, & \Delta w &:= w_2 - w_1, \\ q_c &:= (q_2 + q_1)/2 \quad \text{and} & w_c &:= (w_2 + w_1)/2 \end{aligned} \tag{2.47a}$$

and consider

$$m := \left\lceil |a| \frac{\Delta q}{\Delta q - \Delta w} \right\rceil \text{ and } d := \frac{\Delta q}{m}. \tag{2.47b}$$

Given q_c and d, we introduce the intervals $\Lambda_i \subseteq \mathbb{R}$, $i \in \mathbb{Z}$

$$\Lambda_i := \begin{cases} q_c + [id, (i + 1)d] & \text{if } m \text{ is even} \\ q_c + \left[(i - \frac{1}{2})d, (i + \frac{1}{2})d\right] & \text{if } m \text{ is odd} \end{cases} \tag{2.47c}$$

which we use to define

$$\mathcal{C} := \{\Lambda_i \cap Q \mid \Lambda_i \cap (\mathrm{int} Q) \neq \varnothing\}. \tag{2.47d}$$

The control function follows for every $C_i \in \mathcal{C}$ by

$$H(C_i) := q_c - aq_c - w_c - \begin{cases} ad(i + \frac{1}{2}) & \text{if } m \text{ is even} \\ adi & \text{if } m \text{ is odd.} \end{cases} \tag{2.47e}$$

For this construction of $(\mathcal{C}, H)$, we have the following result.

Theorem 9. *Consider the scalars $a \in \mathbb{R}_{\neq 0}$, $w_1, q_1, w_2, q_2 \in \mathbb{R}$ with $q_1 < w_1 \leq w_2 < q_2$. Let Σ in* (2.3) *be given by $X = U = \mathbb{R}$ and F by $F(x, u) = ax + u + [w_1, w_2]$. Then, $(\mathcal{C}, H)$ defined in* (2.47) *is an invariant cover of $(\Sigma, [q_1, q_2])$ and we have*

$$\log_2 \left\lceil |a| \frac{\Delta q}{\Delta q - \Delta w} \right\rceil = R(\mathcal{C}, H). \tag{2.48}$$

Proof. We show the theorem for odd m. The case for even m, follows along the same arguments. It is rather straightforward to show that $\mathcal{C}$ is a cover of Q and subsequently we show that $\#\mathcal{C} = m$. Note that $i > m/2 - 1/2$ implies that the left limit of Λ_i satisfies $q_c + (i - \frac{1}{2})d \geq q_c + m/2d = q_2$, which shows that $i > m/2 - 1/2$ implies $\Lambda_i \cap (\mathrm{int} Q) = \varnothing$. Similarly, $i < -m/2 + 1/2$ implies $\Lambda_i \cap (\mathrm{int} Q) = \varnothing$, and we see that $\Lambda_i \cap (\mathrm{int} Q) \neq \varnothing$ implies $-m/2 + 1/2 \leq i \leq m/2 - 1/2$ so that $\#\mathcal{C} \leq m$ holds.

We continue to show that $F(C_i, H(C_i)) \subseteq [q_1, q_2]$ holds for every $C_i \in \mathcal{C}$. Given (2.47e) we obtain for $F(C_i, H(C_i))$ the interval

$$a((q_c + d\big[i - \tfrac{1}{2}, i + \tfrac{1}{2}\big]) \cap Q) + q_c - aq_c - w_c - adi + [w_1, w_2]$$

which is a subset of $I := q_c + |a|\frac{d}{2}[-1, 1] + \frac{\Delta w}{2}[-1, 1]$. Let us show that $I \subseteq Q$. Since I is centered at q_c, it is sufficient to show $|a|d/2 + \Delta w/2 \leq \Delta q/2$. Note that $m \geq |a|\Delta q/(\Delta q - \Delta w)$ so that $d \leq (\Delta q - \Delta w)/|a|$ follows and we obtain the desired inequality $|a|d/2 + \Delta w/2 \leq \Delta q/2$ which shows $F(C_i, H(C_i)) \subseteq [q_1, q_2]$. Hence $(\mathcal{C}, H)$ is an invariant cover with $R(\mathcal{C}, H) \leq \log_2 m$, which together with the inequality in Theorem 8 shows the assertion. □

Example 2 (Continued)**.** Let us recall the linear system in Example 2 with $a = 1/2$, $W = [-3, 3]$ and $Q = [-4, 4]$. For this case, $m = 2$ and $d = 4$. The cover elements of $\mathcal{C}$ are given according to (2.47c) by

$$C_{-1} = [-4, 0] \text{ and } C_0 = [0, 4].$$

The inputs follow according to (2.47e) by

$$H(C_{-1}) = 1 \text{ and } H(C_0) = -1.$$

The data rate of $(\mathcal{C}, H)$ is given by $\log_2 2 = 1$ bits per time unit.

We can use Corollary 1 to conclude that the performance loss due to the restriction to static coder-controllers in Example 2 is no larger than 1 bit/time unit. However, for this example, and in general for scalar systems of the form (2.46) for which $|a|\Delta q/(\Delta q - \Delta w)$ is in $\mathbb{N}$, we see that the data rate of the proposed static coder-controller matches the best possible data rate $h_{\mathrm{inv}}(Q, \Sigma)$ since in this case $R(\mathcal{C}, H)$ equals the lower bound in Theorem 7.

Chapter 3

Compositional quantification of IFE

3.1 Introduction

In systems theory, a large system is often represented as an interconnected network of smaller subsystems. Such a representation is then utilized to establish properties on the network through the study of the subsystems.

In this chapter, we once again consider discrete-time uncertain control systems described by difference inclusions. The main theorem of the chapter establishes an upper bound of the IFE of a network of interconnected control subsystems in terms of the IFEs of smaller subsystems.

To the best of our knowledge, this is the first work on compositionally quantifying a notion of entropy for interconnected control systems. Although, the results in [43] also talk about networks of systems, in [43], subsystems are fully isolated and they are not interconnected physically to each other. Further, the results in [43] only deal with deterministic systems, whereas we deal with general nondeterministic systems.

In addition, there is no work so far on the design of coder-controllers with the data rate close to the IFE, though only for scalar linear systems (see Subsection 2.5.3). One can leverage results of this chapter to provide coder-controllers for multi-dimensional linear control systems by looking at them as interconnections of scalar subsystems (if possible); see the second case study in Subsection 3.4.2.

3.1.1 Contributions

The contents of this chapter have been published in the journal IEEE Control Systems Letters [76]. It is a joint work with Prof. Majid Zamani. The results were established and written by myself. Prof. Majid Zamani supervised the work.

In this chapter we deal with the IFE of networks composed of smaller subsystems and seek to quantify the IFE of the network using those of smaller subsystems. For a network Σ composed of subsystems $\Sigma^{(i)}$, $1 \leq i \leq n$, and a nonempty set Q, which is a Cartesian product of $Q^{(i)}$ for subsystems $\Sigma^{(i)}$, we provide an upper bound of the IFE in terms of the IFEs of systems $\bar{\Sigma}^{(i)}$ which have a much lower dimensional state space and, thus, easier to deal with. The system $\bar{\Sigma}^{(i)}$ is derived from the subsystem $\Sigma^{(i)}$ by considering the states of the other subsystems $\Sigma^{(j)}$, $j \neq i$, as disturbances. We also present three technical results related to IFE. First, given a nonempty set Q and a finite partition of it, we show that the IFE of Σ and Q is upper bounded by the largest IFE of Σ and any member in the partition. The second result relates the IFE of two uncertain systems Σ_1 and Σ_2 which are identical except for their transition functions. The set valued transition functions F_1 and F_2 of the two systems are such that the image of any state-input pair under F_1 is a subset of that under F_2. Clearly, the dynamics of Σ_2 involves larger uncertainty. For a given Q, we show that the IFE of Σ_1 cannot be greater than that of Σ_2. The third result states that the IFE of any new system created by reducing the set of control inputs cannot be smaller than that of the original system. Further, via an example, we show that the upper bound, computed compositionally, is tight for some systems. Finally, to illustrate the effectiveness of the results, we compute an upper bound and a lower bound of the IFE of a network of uncertain, linear, discrete-time subsystems describing the evolution of temperatures of 100 rooms in a circular building.

3.2 Some more properties of the IFE

3.2.1 Partition of Q

The following proposition states that the IFE of a system Σ and a nonempty set Q cannot be greater than the largest IFE of Σ and any member of a finite partition of Q.

Proposition 1. *Consider the system in* (2.3) *and a nonempty set $Q \subseteq X$ and assume that $\{Q_1, \dots, Q_n\}$ is a partition of Q for some $n \in \mathbb{N}$ (i.e., $Q = \cup_{i=1}^{n} Q_i$). Then:*

$$h_{\text{inv}}(Q, \Sigma) \leq \max_{i \in [1;n]} h_{\text{inv}}(Q_i, \Sigma). \tag{3.1}$$

Proof. If for some $i \in [1;n]$, $h_{\text{inv}}(Q_i, \Sigma) = \infty$ then (3.1) holds trivially, hence we assume $h_{\text{inv}}(Q_i, \Sigma) < \infty$ for all $i \in [1;n]$. From Lemma 3, for $i \in [1;n]$, when $h_{\text{inv}}(Q_i, \Sigma)$ is finite we have the existence of an invariant cover $(\mathcal{A}_i, G_i)$ of (Σ, Q_i). Consider $\mathcal{A} := \cup_{i \in [1;n]} \mathcal{A}_i$, then for $A \in \mathcal{A}$, $\#\{\mathcal{A}_i \mid \mathcal{A}_i \ni A, i \in [1;n]\} = 1$ as $\{Q_1, \dots, Q_n\}$ is a partition of Q. We define $G : \mathcal{A} \to U$ by $G(A) = G_i(A)$ if $A \in \mathcal{A}_i$. Since $(\mathcal{A}_i, G_i)$ is an invariant cover, for any $A \in \mathcal{A}_i$ we have $F(A, G(A)) = F(A, G_i(A)) \subseteq Q_i \subseteq Q$. Thus $(\mathcal{A}, G)$ is an invariant cover of (Σ, Q). For $\tau \in \mathbb{N}$, let $\mathcal{S}_i \subseteq \mathcal{A}_i^{[0;\tau[}$ be a (τ, Q_i)-spanning set in $(\mathcal{A}_i, G_i)$ such that $\mathcal{N}(\mathcal{S}_i) = r_{\text{inv}}(\tau, \mathcal{A}_i, G_i, \Sigma)$. The set $\mathcal{S} := \cup_{i \in [1;n]} \mathcal{S}_i$ is (τ, Q)-spanning in $(\mathcal{A}, G)$ as $\{Q_1, \dots, Q_n\}$ covers Q and $\mathcal{S}_i$ is (τ, Q_i)-spanning in $(\mathcal{A}_i, G_i)$. Then, the expansion number for the set $\mathcal{S}$ is $\mathcal{N}(\mathcal{S}) = \max_{\alpha \in \mathcal{S}} \prod_{t=0}^{\tau-1} \# P_{\mathcal{S}}(\alpha|_{[0;t]}) = \max_{i \in [1;n]} \mathcal{N}(\mathcal{S}_i)$ $= \max_{i \in [1;n]} r_{\text{inv}}(\tau, \mathcal{A}_i, G_i, \Sigma)$. Thus, $r_{\text{inv}}(\tau, \mathcal{A}, G, \Sigma) \leq \max_{i \in [1;n]} r_{\text{inv}}(\tau, \mathcal{A}_i, G_i, \Sigma)$ which concludes (3.1). $\square$

3.2.2 Systems with higher uncertainty

Consider two systems that are identical except for the transition function. The second system is such that, for every state action pair, the value of its transition function is a superset of that of the first system. In other words, the two systems are identical except that the second system has higher uncertainty. The next result shows that as the uncertainty in the system increases, the IFE also increases.

Proposition 2. *Consider two systems Σ_1 and Σ_2 of the form* (2.3) *with $X_1 = X_2 = X$, $U_1 = U_2 = U$ and $F_1(x, u) \subseteq F_2(x, u)$ for all $x \in X$, $u \in U$. For a nonempty set $Q \subseteq X$, the invariance feedback entropies of the two systems are related as*

$$h_{\text{inv}}(Q, \Sigma_1) \le h_{\text{inv}}(Q, \Sigma_2). \tag{3.2}$$

Proof. Without loss of generality, we assume that $h_{\text{inv}}(Q, \Sigma_2)$ is finite. From Lemma 3, we know that there exists an invariant cover $(\mathcal{A}, G)$ of (Σ_2, Q). For $\tau \in \mathbb{N}$, let $\mathcal{S} \subseteq \mathcal{A}^{[0;\tau[}$ be a (τ, Q)-spanning set in $(\mathcal{A}, G)$ for Σ_2 such that $\mathcal{N}(\mathcal{S}) = r_{\text{inv}}(\tau, \mathcal{A}, G, \Sigma_2)$. Since for every $x \in X$, $u \in U$ we have $F_1(x, u) \subseteq F_2(x, u)$, $\mathcal{S}$ is also (τ, Q)-spanning in $(\mathcal{A}, G)$ for Σ_1. Thus, if $r_{\text{inv}}(\tau, \mathcal{A}, G, \Sigma_1)$ is the smallest expansion number for any (τ, Q)-spanning set in $(\mathcal{A}, G)$ for Σ_1, then $r_{\text{inv}}(\tau, \mathcal{A}, G, \Sigma_1) \le \mathcal{N}(\mathcal{S}) = r_{\text{inv}}(\tau, \mathcal{A}, G, \Sigma_2)$. Hence, we obtain (3.2). □

3.2.3 Smaller set of control inputs

If the set of control inputs is reduced, then the entropy of the new system cannot be less than that of the original. The next proposition formalizes this statement.

Proposition 3. *Consider two systems Σ_1 and Σ_2 of the form* (2.3) *with $X_1 = X_2 = X$, $U_1 \supseteq U_2$ and $F_1 = F_2 = F$. For a nonempty set $Q \subseteq X$, the following holds*

$$h_{\text{inv}}(Q, \Sigma_1) \le h_{\text{inv}}(Q, \Sigma_2).$$

Proof. Let $h_{\text{inv}}(Q, \Sigma_2) < \infty$. From Lemma 3, we have the existence of an invariant cover $(\mathcal{A}, G)$ of (Σ_2, Q). For $\tau \in \mathbb{N}$, let $\mathcal{S} \subseteq \mathcal{A}^{[0;\tau[}$ be a (τ, Q)-spanning set in $(\mathcal{A}, G)$ for Σ_2 such that it has the smallest expansion number, i.e., $\mathcal{N}(\mathcal{S}) = r_{\text{inv}}(\tau, \mathcal{A}, G, \Sigma_2)$. Clearly $(\mathcal{A}, G)$ is also an invariant cover of (Σ_1, Q) and $\mathcal{S}$ is also a (τ, Q)-spanning set for Σ_1. This leads to $r_{\text{inv}}(\tau, \mathcal{A}, G, \Sigma_1) = \mathcal{N}(\mathcal{S})$ and therefore the entropy of the invariant cover $(\mathcal{A}, G)$ is equal for both the systems, i.e., $h_{\Sigma_2}(\mathcal{A}, G) = h_{\Sigma_1}(\mathcal{A}, G)$. For $i \in 1, 2$, let $\mathcal{C}_i$ denote the set of all invariant covers of (Σ_i, Q). Because $U_2 \subseteq U_1$, clearly we have $\mathcal{C}_2 \subseteq \mathcal{C}_1$. Thus from (2.8) we obtain $h_{\text{inv}}(Q, \Sigma_1) \le h_{\text{inv}}(Q, \Sigma_2)$. □

In the following section, we consider a network of uncertain control subsystems and a subset Q of its state set, and provide an upper bound of its IFE in terms of the IFEs of smaller subsystems.

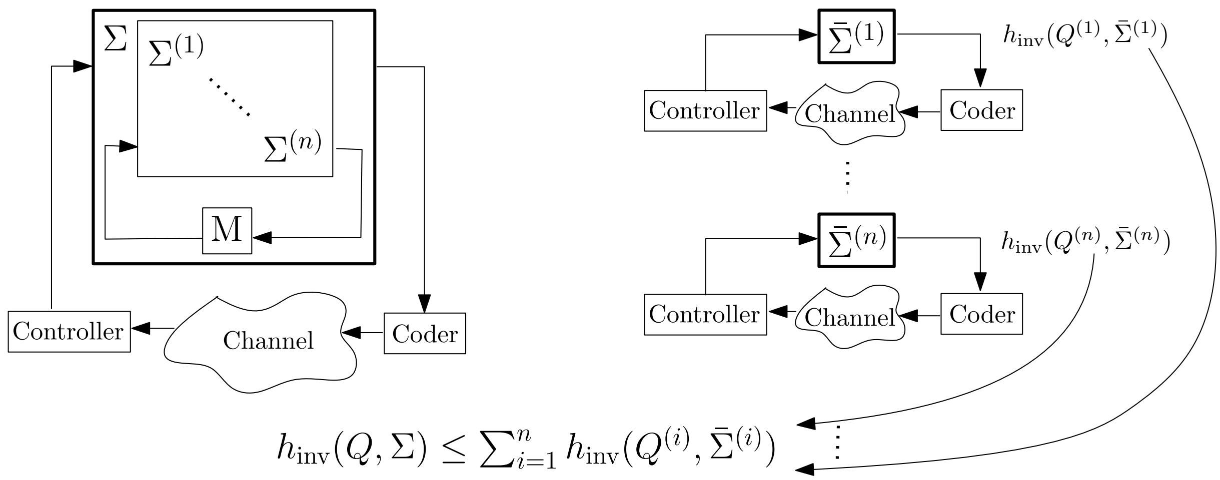


Figure 3.1: An interconnected control system Σ is composed of n subsystems $\Sigma^{(i)}$ with M as the interconnection map. The smaller systems $\bar{\Sigma}^{(i)}$ are obtained from subsystems $\Sigma^{(i)}$ as per (3.4). State of Σ is desired to be kept invariant within the set $Q = Q^{(1)} \times \cdots \times Q^{(n)}$. The channel between the coder and controller is assumed to be digital and noiseless with finite bit-rate.

3.3 Networks of uncertain control systems

Consider a discrete-time control system Σ composed of n subsystems $\Sigma^{(1)}, \ldots, \Sigma^{(n)}$:

$$\begin{aligned} \Sigma &: x_{k+1} \in F(x_k, u_k), \\ \Sigma^{(i)} &: x_{k+1}^{(i)} \in F^{(i)}(x_k, u_k^{(i)}), \end{aligned} \tag{3.3}$$

where $x_k = (x_k^{(1)}, \ldots, x_k^{(n)}) \in X$, $X = X^{(1)} \times \cdots \times X^{(n)}$, $U = U^{(1)} \times \cdots \times U^{(n)}$, $F^{(i)} \colon X \times U^{(i)} \rightrightarrows X^{(i)}$, $F \colon X \times U \rightrightarrows X$, and $F(x_k, u_k) = F^{(1)}(x_k, u_k^{(1)}) \times \cdots \times F^{(n)}(x_k, u_k^{(n)})$.

Let $Q^{(i)}$ be a nonempty subset of $X^{(i)}$ and $Q = Q^{(1)} \times \cdots \times Q^{(n)}$.

Given subsystems $\Sigma^{(i)}$, we define new subsystems $\bar{\Sigma}^{(i)}$ by considering states $x^{(j)}$, $j \neq i$, as bounded disturbances lying in the sets $Q^{(j)}$:

$$\bar{\Sigma}^{(i)} : x_{k+1}^{(i)} \in \bar{F}^{(i)}\left(x_k^{(i)}, u_k^{(i)}\right), \tag{3.4}$$

where $\bar{F}^{(i)} \colon X^{(i)} \times U^{(i)} \rightrightarrows X^{(i)}$,
$\bar{F}^{(i)}\big(x_k^{(i)}, u_k^{(i)}\big) := F^{(i)}\big(Q^{(1)} \times \cdots \times Q^{(i-1)} \times \{x_k^{(i)}\} \times Q^{(i+1)} \times \cdots \times Q^{(n)}, u_k^{(i)}\big)$.

For such a network Σ and a set $Q = Q^{(1)} \times \cdots \times Q^{(n)}$, the following theorem presents an upper bound of the IFE of Σ and Q in terms of the IFEs of the smaller systems $\bar{\Sigma}^{(i)}$ and $Q^{(i)}$.

Theorem 10. *For Σ as in (3.3), $\bar{\Sigma}^{(i)}$ as in (3.4) and the set $Q \subseteq X$ as $Q = Q^{(1)} \times \cdots \times Q^{(n)}$, $Q^{(i)} \neq \varnothing$, the following holds:*

$$h_{inv}(Q, \Sigma) \leq \sum_{i=1}^{n} h_{inv}(Q^{(i)}, \bar{\Sigma}^{(i)}). \tag{3.5}$$

Proof. If for any $i \in [1; n]$, $h_{\text{inv}}(Q^{(i)}, \bar{\Sigma}^{(i)}) = \infty$, then the inequality (3.5) holds. Hence, we assume that $h_{\text{inv}}(Q^{(i)}, \bar{\Sigma}^{(i)}) < \infty$, for each $i \in [1; n]$. Then from Lemma 3, we conclude the existence of an invariant cover for $(\bar{\Sigma}^{(i)}, Q^{(i)})$. For $\varepsilon > 0$, let $(\mathcal{A}^{(i)}, G^{(i)})$ be an invariant cover of $(\bar{\Sigma}^{(i)}, Q^{(i)})$ such that

$$h(\mathcal{A}^{(i)}, G^{(i)}) \leq h_{\text{inv}}(Q^{(i)}, \bar{\Sigma}^{(i)}) + \varepsilon/n.$$

In the following, we denote by $\pi_i^X : X \to X^{(i)}$ and $\pi_i^U : U \to U^{(i)}$ the projection to the i-th component of the state set and the input set, respectively.

Let us define $\mathcal{A} := \{A^{(1)} \times \cdots \times A^{(n)} \mid A^{(i)} \in \mathcal{A}^{(i)}, i = 1, \ldots, n\}$ and for $A \in \mathcal{A}$ define $G(A) := (G^{(1)}(\pi_1^X A), \ldots, G^{(n)}(\pi_n^X A))$. For $i \in [1; n]$, from the definition of $\bar{F}^{(i)}$, we have $F^{(i)}(A, \pi_i^U G(A)) \subseteq \bar{F}^{(i)}(\pi_i^X A, \pi_i^U G(A)) = \bar{F}^{(i)}(\pi_i^X A, G^{(i)}(\pi_i^X A))$. Since $(\mathcal{A}^{(i)}, G^{(i)})$ is an invariant cover of $(\bar{\Sigma}^{(i)}, Q^{(i)})$, we have $\bar{F}^{(i)}(\pi_i^X A, G^{(i)}(\pi_i^X A)) \subseteq Q^{(i)}$ and therefore,

$$F(A, G(A)) = F^{(1)}(A, \pi_1^U G(A)) \times \cdots \times F^{(n)}(A, \pi_n^U G(A)) \subseteq Q,$$

Thus, $(\mathcal{A}, G)$ is an invariant cover of (Σ, Q).

Let $\mathcal{S}^{(i)} \subseteq \mathcal{A}^{(i)[0;\tau[}$ be a $(\tau, Q^{(i)})$-spanning set in $(\mathcal{A}^{(i)}, G^{(i)})$ with minimal expansion number, i.e., $\mathcal{N}(\mathcal{S}^{(i)}) = r_{\text{inv}}(\tau, \mathcal{A}^{(i)}, G^{(i)}, \bar{\Sigma}^{(i)})$. Then, $\{\alpha^{(i)}(0) \mid \alpha^{(i)} \in \mathcal{S}^{(i)}\}$ covers $Q^{(i)}$ and for all $\alpha^{(i)} \in \mathcal{S}^{(i)}$, $t \in [0; \tau - 1[$, we have

$$\bar{F}^{(i)}\big(\alpha^{(i)}(t), G^{(i)}(\alpha^{(i)}(t))\big) \subseteq \bigcup_{A \in P_{\mathcal{S}^{(i)}}(\alpha^{(i)}|_{[0;t]})} A. \tag{3.6}$$

Consider $\mathcal{S} := \big\{\big(\alpha^{(1)}(t) \times \cdots \times \alpha^{(n)}(t)\big)_{t=0}^{\tau-1} \mid \big(\alpha^{(i)}(t)\big)_{t=0}^{\tau-1} \in \mathcal{S}^{(i)}, i = 1, \ldots, n\big\}$. Let $\alpha \in \mathcal{S}$ such that $\alpha(t) = \alpha^{(1)}(t) \times \cdots \times \alpha^{(n)}(t)$ for $0 \leq t \leq \tau - 1$ where $\alpha^{(i)} \in \mathcal{S}^{(i)}$. Then, the set of successor elements of the sequence $\alpha|_{[0;t]}$ with respect to the set $\mathcal{S}$ is

$$P_{\mathcal{S}}(\alpha|_{[0;t]}) = \big\{A_1 \times \cdots \times A_n \mid A_i \in P_{\mathcal{S}^{(i)}}(\alpha^{(i)}|_{[0;t]}), i \in [1; n]\big\}.$$

We observe that

$$\begin{aligned}
&\Bigg(\bigcup_{A_1 \in P_{\mathcal{S}^{(1)}}(\alpha^{(1)}|_{[0;t]})} A_1\Bigg) \times \Bigg(\bigcup_{A_2 \in P_{\mathcal{S}^{(2)}}(\alpha^{(2)}|_{[0;t]})} A_2\Bigg) \times \cdots \times \Bigg(\bigcup_{A_n \in P_{\mathcal{S}^{(n)}}(\alpha^{(n)}|_{[0;t]})} A_n\Bigg) \\
&= \bigcup_{A_1 \in P_{\mathcal{S}^{(1)}}(\alpha^{(1)}|_{[0;t]})} \Bigg(A_1 \times \Bigg(\bigcup_{A_2 \in P_{\mathcal{S}^{(2)}}(\alpha^{(2)}|_{[0;t]})} A_2\Bigg) \times \cdots \times \Bigg(\bigcup_{A_n \in P_{\mathcal{S}^{(n)}}(\alpha^{(n)}|_{[0;t]})} A_n\Bigg)\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \bigcup_{\substack{A_1 \in P_{\mathcal{S}^{(1)}}(\alpha^{(1)}|_{[0;t]}) \\ A_2 \in P_{\mathcal{S}^{(2)}}(\alpha^{(2)}|_{[0;t]})}} \left(A_1 \times A_2 \times \left(\bigcup_{A_3 \in P_{\mathcal{S}^{(3)}}(\alpha^{(3)}|_{[0;t]})} A_3 \right) \times \cdots \times \left(\bigcup_{A_n \in P_{\mathcal{S}^{(n)}}(\alpha^{(n)}|_{[0;t]})} A_n \right) \right) \\
&= \bigcup_{\substack{A_1 \in P_{\mathcal{S}^{(1)}}(\alpha^{(1)}|_{[0;t]}) \\ \vdots \\ A_n \in P_{\mathcal{S}^{(n)}}(\alpha^{(n)}|_{[0;t]})}} (A_1 \times \cdots \times A_n) \\
&= \bigcup_{A \in P_{\mathcal{S}}(\alpha|_{[0;t]})} A.
\end{aligned} \tag{3.7}
$$

From the definition of $\bar{F}^{(i)}$ and (3.6), we have

$$
\begin{aligned}
F^{(i)}(\alpha(t), \pi_i^U G(\alpha(t))) &\subseteq \bar{F}^{(i)}(\alpha^{(i)}(t), G^{(i)}(\alpha^{(i)}(t))) \\
&\subseteq \bigcup_{A \in P_{\mathcal{S}^{(i)}}(\alpha^{(i)}|_{[0;t]})} A.
\end{aligned} \tag{3.8}
$$

From the definition of F, we have

$$
F(\alpha(t), G(\alpha(t))) = F^{(1)}(\alpha(t), \pi_1^U G(\alpha(t))) \times \cdots \times F^{(n)}(\alpha(t), \pi_n^U G(\alpha(t))).
$$

The equation above together with (3.8) and (3.7) give

$$
F(\alpha(t), G(\alpha(t))) \subseteq \bigcup_{A \in P_{\mathcal{S}}(\alpha|_{[0;t]})} A.
$$

Thus, $\mathcal{S}$ is (τ, Q)-spanning for Σ in $(\mathcal{A}, G)$.

For $t \in [0; \tau - 1]$, we observe that

$$
\begin{aligned}
\#P_{\mathcal{S}}(\alpha|_{[0;t]}) &= \#P_{\mathcal{S}^{(1)}}(\alpha^{(1)}|_{[0;t]}) \cdot \ldots \cdot \#P_{\mathcal{S}^{(n)}}(\alpha^{(n)}|_{[0;t]}), \\
\prod_{t=0}^{\tau-1} \#P_{\mathcal{S}}(\alpha|_{[0;t]}) &= \prod_{t=0}^{\tau-1} \#P_{\mathcal{S}^{(1)}}(\alpha^{(1)}|_{[0;t]}) \cdot \ldots \cdot \prod_{t=0}^{\tau-1} \#P_{\mathcal{S}^{(n)}}(\alpha^{(n)}|_{[0;t]}), \\
\max_{\alpha \in \mathcal{S}} \prod_{t=0}^{\tau-1} \#P_{\mathcal{S}}(\alpha|_{[0;t]}) &= \max_{\alpha^{(1)} \in \mathcal{S}^{(1)}} \prod_{t=0}^{\tau-1} \#P_{\mathcal{S}^{(1)}}(\alpha^{(1)}|_{[0;t]}) \cdot \ldots \cdot \max_{\alpha^{(n)} \in \mathcal{S}^{(n)}} \prod_{t=0}^{\tau-1} \#P_{\mathcal{S}^{(n)}}(\alpha^{(n)}|_{[0;t]}).
\end{aligned}
$$

Thus, the expansion number of the set $\mathcal{S}$ is

$$
\mathcal{N}(\mathcal{S}) = \mathcal{N}(\mathcal{S}^{(1)}) \cdot \ldots \cdot \mathcal{N}(\mathcal{S}^{(n)}).
$$

If $r_{\text{inv}}(\tau, \mathcal{A}, G, \Sigma)$ is the smallest possible expansion number for any (τ, Q)-spanning set in $(\mathcal{A}, G)$, then $r_{\text{inv}}(\tau, \mathcal{A}, G, \Sigma) \leq \mathcal{N}(\mathcal{S})$ and since $\mathcal{N}(\mathcal{S}^{(i)}) = r_{\text{inv}}(\tau, \mathcal{A}^{(i)}, G^{(i)}, \bar{\Sigma}^{(i)})$ by assumption, we obtain

$$
\begin{aligned}
r_{\mathrm{inv}}(\tau, \mathcal{A}, G, \Sigma) &\leq \prod_{i=1}^{n} r_{\mathrm{inv}}(\tau, \mathcal{A}^{(i)}, G^{(i)}, \bar{\Sigma}^{(i)}), \\
\frac{1}{\tau} \log r_{\mathrm{inv}}(\tau, \mathcal{A}, G, \Sigma) &\leq \frac{1}{\tau} \sum_{i=1}^{n} \log r_{\mathrm{inv}}(\tau, \mathcal{A}^{(i)}, G^{(i)}, \bar{\Sigma}^{(i)}), \\
h_{\mathrm{inv}}(Q, \Sigma) &\leq \sum_{i=1}^{n} h_{\mathrm{inv}}(Q^{(i)}, \bar{\Sigma}^{(i)}) + \varepsilon.
\end{aligned}
$$

Since ε is arbitrary, we get

$$
h_{\mathrm{inv}}(Q, \Sigma) \leq \sum_{i=1}^{n} h_{\mathrm{inv}}(Q^{(i)}, \bar{\Sigma}^{(i)}),
$$

as desired. □

In the next section, we demonstrate by an example that the bound in (3.5) is tight.

3.4 Examples

In this section, we present two case studies, the first of which demonstrates that the bound in (3.5) is tight, and the other one describes the compositional computation of an upper bound and a lower bound of the IFE for a linear, discrete-time, uncertain model describing the evolution of room temperatures in a circular building with 100 rooms.

3.4.1 Tightness

The following example shows that the bound in (3.5) is tight.

Consider system Σ as

$$
\Sigma : x_{k+1} \in A x_k + u_k + W, \quad A = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.75 & 0 \\ 0 & 0 & 2 \end{bmatrix},
$$

with $i \in [1;3]$, $Q^{(i)} = [-4, 4]$, $W^{(i)} = [-3, 3]$, $U^{(i)} = [-7, 7]$, $U = U^{(1)} \times U^{(2)} \times U^{(3)}$, $Q = Q^{(1)} \times Q^{(2)} \times Q^{(3)}$, and the disturbance set $W = W^{(1)} \times W^{(2)} \times W^{(3)}$. The dynamics of each state can be considered as a scalar subsystem

$$
\begin{aligned}
\bar{\Sigma}^{(1)} &: x_{k+1}^{(1)} \in 0.5 x_k^{(1)} + u_k^{(1)} + W^{(1)}, \\
\bar{\Sigma}^{(2)} &: x_{k+1}^{(2)} \in 0.75 x_k^{(2)} + u_k^{(2)} + W^{(2)}, \\
\bar{\Sigma}^{(3)} &: x_{k+1}^{(3)} \in 2 x_k^{(3)} + u_k^{(3)} + W^{(3)},
\end{aligned}
$$

where $x_k^{(i)}$ and $u_k^{(i)}$ denote the i-th component of x_k and u_k, respectively. For $\bar{\Sigma}^{(1)}$ and $Q^{(1)}$, using Theorem 7, we get $h_{\text{inv}}(Q^{(1)}, \bar{\Sigma}^{(1)}) \geq 1$ and, using Theorems 6 and 9, we get $h_{\text{inv}}(Q^{(1)}, \bar{\Sigma}^{(1)}) \leq 1$. Thus $h_{\text{inv}}(Q^{(1)}, \bar{\Sigma}^{(1)}) = 1$. Similarly, we obtain $h_{\text{inv}}(Q^{(2)}, \bar{\Sigma}^{(2)}) = 1.585$ and $h_{\text{inv}}(Q^{(3)}, \bar{\Sigma}^{(3)}) = 3$.

On the other hand, for Σ and Q and using Theorem 7, we get $h_{\text{inv}}(Q, \Sigma) \geq 5.585$ and using (3.5), we get $h_{\text{inv}}(Q, \Sigma) \leq 5.585$. Thus, $h_{\text{inv}}(Q, \Sigma) = 5.585$ which shows that $h_{\text{inv}}(Q, \Sigma)$ attains the upper bound in (3.5).

3.4.2 Computation of an upper and a lower bound for a network of uncertain control subsystems

Consider the problem of temperature regulation in a circular building with $n = 100$ rooms, each equipped with a heater. The temperatures T of the rooms is described by the discrete-time model adapted from [55] with some modifications:

$$\Sigma : T(k+1) = AT(k) + \beta T^e \mathbf{1} + \gamma T^h \nu(k) + w(k), \tag{3.9}$$

where $A \in \mathbb{R}^{n \times n}$ is a matrix with elements $\{A\}_{i,i} = (1 - 2\alpha - \beta)$, $\{A\}_{i,i+1} = \{A\}_{i+1,i} = \{A\}_{1,n} = \{A\}_{n,1} = \alpha$, $\forall i \in [1; n-1]$, and all other elements are zero, $\nu(k) \in \mathcal{V}^n$, $\mathcal{V} = [-1.35, 0.372]$, is the input, $\mathbf{1}$ is the n-dimensional vector with all elements being one, $T^e = -1^o C$ is the outside temperature, $T^h = 50^o C$ is the heater temperature, $w(k) \in W^n$, $W = [5.7, 7]$, is the disturbance, and α, β and γ are the conduction factors given by $\alpha = 0.1$, $\beta = 0.045$, $\gamma = 0.09$. The temperature is desired to be kept invariant in the set Q^n where $Q = [19, 21]$.

Let us denote by Σ_i the dynamics of the temperature of the i-th room,

$$\Sigma_i : T_i(k+1) = (1 - 2\alpha - \beta)T_i(k) + \alpha(T_{i-1}(k) + T_{i+1}(k)) + \beta T^e + \gamma T^h \nu_i(k) + w_i(k),$$

where the subscript i in T_i, ν_i and w_i denotes the i-th-component. Now, we define systems $\bar{\Sigma}_i$ by considering the states T_j, $j \neq i$, as disturbances lying within the set Q,

$$\bar{\Sigma}_i : T_i(k+1) \in (1 - 2\alpha - \beta)T_i(k) + \alpha 2Q + \beta T^e + u_i(k) + W, \tag{3.10}$$

where $u_i(k) \in U := \gamma T^h \mathcal{V}$. With $\bar{W} := 2\alpha Q + \beta T^e + W$, we can rewrite (3.10) as

$$\bar{\Sigma}_i : T_i(k+1) \in (1 - 2\alpha - \beta)T_i(k) + u_i(k) + \bar{W}.$$

From Theorems 7, 6, and 9, we obtain $2.3315 \leq h_{\text{inv}}(Q, \bar{\Sigma}_i) \leq 2.5850$. From Theorems 10 and 7, we get $108.3 \leq h_{\text{inv}}(Q^n, \Sigma) \leq 258.5$. Figure 3.2 shows the maximum and minimum temperature for a closed loop trajectory of Σ under a static memoryless coder-controller with data rate 258.49.

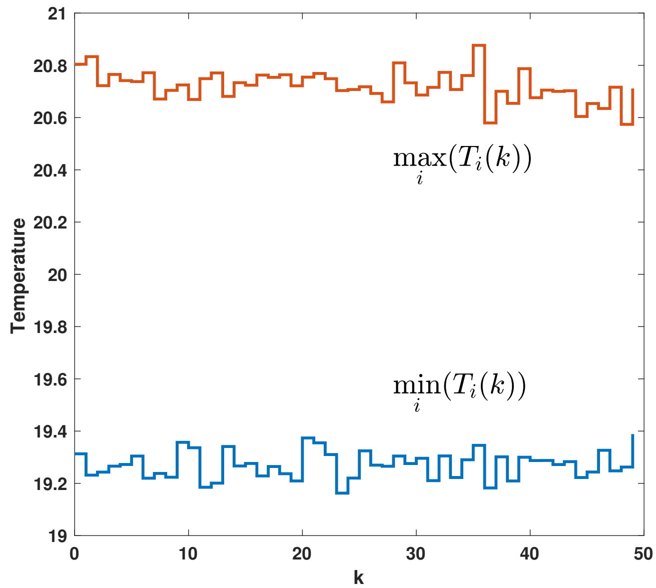


Figure 3.2: Maximum and minimum temperature for a trajectory of Σ in (3.9), with an initial temperature $T(0) \in Q^n$, where $Q = [19; 21]$, under a static memoryless coder-controller, with a channel data rate 258.49, keeping temperature T invariant inside Q^n.

3.5 Discussion

In Section 3.3, for a large network of interconnected uncertain control subsystems, to compute an upper bound of the IFE of the network, we defined new subsystems $\bar{\Sigma}^{(i)}$ such that the sum of their IFEs gives us a desired upper bound. Now it is natural to ask, weather a lower bound of the IFE of the network can also be computed based on the subsystem IFEs. The example below shows that the IFE of any subsystem $\bar{\Sigma}^{(i)}$ may not necessarily lower bound the IFE of the network.

Consider a system Σ as

$$\Sigma : x_{k+1} = Ax_k + u_k, \quad A = \begin{bmatrix} 2 & 0.9 \\ 0.9 & 3 \end{bmatrix},$$

with $i \in [1; 2]$, $Q^{(i)} = [1, 4]$, $U^{(i)} = [-15, 15]$, $u_k \in U = U^{(1)} \times U^{(2)}$, and $Q = Q^{(1)} \times Q^{(2)}$.

The dynamics of each state can be considered as a scalar subsystem

$$\bar{\Sigma}^{(1)} : x_{k+1}^{(1)} \in 2x_k^{(1)} + u_k^{(1)} + W,$$
$$\bar{\Sigma}^{(2)} : x_{k+1}^{(2)} \in 3x_k^{(2)} + u_k^{(2)} + W,$$

where $x_k^{(i)}$ and $u_k^{(i)}$ denote the i-th component of x_k and u_k, respectively, and $W = [0.9, 3.6]$.

Using Theorems 7 and 9 we get $h_{\mathrm{inv}}(Q^{(1)}, \bar{\Sigma}^{(1)}) = 4.3219$, $h_{\mathrm{inv}}(Q^{(2)}, \bar{\Sigma}^{(2)}) = 4.9069$ and $h_{\mathrm{inv}}(Q, \Sigma) = 2.3757$. Thus, the statement that the smallest IFE among the subsystems lower bounds the IFE of the network may not always hold.

Chapter 4

Numerical Overapproximation

4.1 Introduction

The invariance entropy for deterministic (IED) and uncertain systems (IFE) are equivalent in the deterministic case, see Theorem 4. In this chapter, we present algorithms for the numerical computation of these two quantities.

In the first two sections of this chapter we consider deterministic control systems and focus on a notion of *invariance entropy* (IED) which was introduced in [19] as a measure for the smallest average data rate above which a given compact and controlled invariant subset Q of the state space can be made invariant. We present the first attempt to numerically compute upper bounds of invariance entropy. Our approach combines different algorithms. First, we compute a symbolic abstraction [65] of the given control system over the set Q and the corresponding invariant controller. Particularly, we subdivide Q into small boxes and assign control inputs (from a grid on the input set) to these boxes that guarantee invariance in one time step. This results in a typically huge look-up table whose entries are the pairs (x, u) of states and control inputs which are admissible for maintaining invariance of Q. In the second step, the look-up table is significantly reduced by building a binary decision tree via a decision tree learning algorithm. This tree, in turn, leads to a typically much smaller partition of Q with one control input assigned to each partition element that will guarantee invariance of Q in one time step. This data defines a map $T : Q \to Q$ to which, in the third step, we apply an algorithm that approximates the exponential growth rate of the total number of length-n T-orbits [30] which are distinguishable via the given partition. The output of this algorithm then serves as an upper bound for the invariance entropy.

For the implementation of the first step –the construction of an invariant controller– we use `SCOTS`, a software tool written in C++ designed for exactly this purpose [65]. `SCOTS` relies on a rectangular grid, and assigns to each grid box in Q a set of permissible control inputs. For the second step, we use the software tool `dtControl` [3], which builds the decision tree and determinizes the invariant controller by choosing from the set of permissible control inputs exactly one for each box. `dtControl` also groups together all the boxes which are assigned the same control input. For such a grouping, classification techniques such as logistic regression and linear support vector machines are employed. Finally, the third step is accomplished via an algorithm proposed in [30], originally designed for the estimation of topological entropy. This algorithm is based on the theory of symbolic dynamical systems and breaks up into standard graph-theoretic constructions.

In addition, we also focus on uncertain control systems and the IFE. For a discrete-time, uncertain control system Σ, given a nonempty set Q, if the IFE of Q is finite, then an upper bound can be computed by solving a mean-payoff-game which is constructed using a finite abstraction of Σ [66, Sec. 6]. However, the number of vertices in the mean-payoff-game is of the order of 2^{2^n}, where n is the number of states in the finite abstraction; in other words, the size of the mean-payoff-game increases doubly exponentially with n. In this chapter, we present two upper bounds for the IFE that can be computed from a weighted directed graph which is constructed from an invariant partition $(\bar{\mathcal{A}}, G)$ of Q. Here, $\bar{\mathcal{A}}$ is a finite partition of the set Q and $G : \bar{\mathcal{A}} \to U$ is a map into the space of control inputs such that image of $A \in \bar{\mathcal{A}}$, with $G(A)$ as the control input, under the system dynamics is contained in Q. Both upper bounds can be computed in linear time given the number of nodes and edges in the graph. First result (cf. Theorem 11) characterizes the entropy of the invariant partition $(\bar{\mathcal{A}}, G)$ in terms of the weights of the graph and also presents a simple upper bound for it. Second result (cf. Theorem 12) establishes that the entropy of $(\bar{\mathcal{A}}, G)$ is the same as the maximum mean weight over all cycles in the graph. Finally, for deterministic systems, the relationship between those upper bounds will be explicitly explained (cf. Theorem 13).

4.1.1 Contributions

The contents of this chapter are based on [74]. It is a joint work with Dr. Christoph Kawan and Prof. Majid Zamani. Most of the work is done by myself. Dr. Christoph Kawan contributed in the initial discussions and in the writing of the introduction section. Prof. Majid Zamani supervised the work.

We present algorithms for the numerical computation of an upper bound of the IED and two upper bounds of the IFE. The algorithms also provide static coder-controller schemes corresponding to the obtained upper bounds. For the deterministic case, we establish the relation between the upper bounds of the IED and the IFE. We also present the results of the algorithms applied to four examples.

4.2 Upper bound for invariance entropy of deterministic systems

In this section, we focus on deterministic control systems. We recall the definition of the IED from Section 2.3.4 and present our algorithm to compute an upper bound for it.

Consider a discrete-time control system

$$\Sigma : \quad \xi(t+1) = f(\xi(t), \nu(t)), \tag{4.1}$$

where $f : X \times U \to X$, $X \subseteq \mathbb{R}^n$, $U \subseteq \mathbb{R}^m$, is continuous. Same as in Section 2.3.4, for $\nu \in U^{[0;t]}, t \in \mathbb{Z}_{\geq 0}$, we define a map $f_\nu(x) : X \to X$ by $f_\nu(x) := f_{\nu(t)} \circ \cdots \circ f_{\nu(0)}(x)$ where $f_{\nu(t)}(x) := f(x, \nu(t))$.

We call a triple $(\mathcal{A}, \tau, G)$ an *invariant partition* of Q, where $Q \subseteq X$, if $\mathcal{A}$ is a partition of Q, $\tau \in \mathbb{N}$, and $G : \mathcal{A} \to U^\tau$ is a map such that, for every $A \in \mathcal{A}$, $t \in [0; \tau - 1]$ and $\nu = G(A)$, we have $f_{\nu|_{[0;t]}}(A) \subseteq Q$. For a given $\mathcal{C} = (\mathcal{A}, \tau, G)$, we define a map $T_\mathcal{C} : Q \to Q$ as

$$T_\mathcal{C}(x) := f_{G(A_x)}(x),$$

where $A_x \in \mathcal{A}$ is such that $x \in A_x$.

IED: Let Q be compact and controlled invariant and $\tau \in \mathbb{N}$. We define a set $\mathcal{S}^{\text{det}} \subseteq U^{[0;\tau[}$ to be (τ, Q)-spanning if for every $x \in Q$ there exists $\nu \in \mathcal{S}^{\text{det}}$ so that the associated trajectory (ξ, ν) on $[0; \tau[$ of Σ with $\xi(0) = x$ satisfies $\xi([0; \tau]) \subseteq Q$. The smallest possible cardinality for any (τ, Q)-spanning set is denoted by $r_{\text{inv}}^{\text{det}}(\tau, Q)$. Then the IED of Q is

$$h_{\text{inv}}^{\text{det}}(Q) := \lim_{\tau \to \infty} \frac{1}{\tau} \log_2 r_{\text{inv}}^{\text{det}}(\tau, Q),$$

if $r_{\text{inv}}^{\text{det}}(\tau, Q)$ is finite for all $\tau > 0$. Otherwise $h_{\text{inv}}^{\text{det}}(Q) := \infty$. Next we define the counting entropy of the map $T_\mathcal{C}$ for a given $\mathcal{C} = (\mathcal{A}, \tau, G)$. The ratio of the counting entropy to τ upper bounds the IED.

Counting entropy: Consider a set Q, a map $T : Q \to Q$ and a finite partition $\mathcal{A} = \{A_1, \dots, A_q\}$ of Q. For $N \in \mathbb{N}$, consider the set $\mathcal{W}_N(T, \mathcal{A}) := \{[a_0 a_1 \dots a_{N-1}] : \exists x \in Q \text{ with } T^i(x) \in A_{a_i}, 0 \leq i < N\}$.

The counting entropy of T with respect to the partition $\mathcal{A}$ is defined as

$$h^*(T, \mathcal{A}) := \lim_{N \to \infty} \frac{1}{N} \log_2 \#\mathcal{W}_N(T, \mathcal{A}),$$

where the existence of the limit follows from the subadditivity of the sequence $(\log_2 \#\mathcal{W}_N(T, \mathcal{A}))_{N \in \mathbb{N}}$. Then the IED of Q satisfies [38, Thm. 2.3]

$$h_{\text{inv}}^{\text{det}}(Q) = \inf_{\mathcal{C} = (\mathcal{A}, \tau, G)} \frac{1}{\tau} h^*(T_\mathcal{C}, \mathcal{A}),$$

where the infimum is taken over all invariant partitions $\mathcal{C} = (\mathcal{A}, \tau, G)$ of Q.

To find an upper bound of $h^*(T_\mathcal{C}, \mathcal{A})$, we select a refinement $\mathcal{B} = \{B_1, \dots, B_{\bar{n}}\}$ of $\mathcal{A}$, i.e., $\mathcal{B}$ is a partition of Q such that each element of $\mathcal{A}$ is the union of some elements of $\mathcal{B}$. Let us define an $\bar{n} \times \bar{n}$ *transition matrix* Γ by

$$\Gamma_{i,j} := \begin{cases} 1 & \text{if } T_\mathcal{C}(B_i) \cap B_j \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

A sequence $[b_0 \dots b_{N-1}]$ is called a $\mathcal{B}$-word if $\Gamma_{b_i,b_{i+1}} = 1$ for every $i \in [0; N-2]$. Next, we define the set

$$\mathcal{W}_N(\mathcal{B},\mathcal{A}) := \big\{[a_0 \dots a_{N-1}] \mid \exists \text{ a } \mathcal{B}\text{-word } [b_0 \dots b_{N-1}] \text{ s.t. } B_{b_i} \subseteq A_{a_i}, i \in [0; N-1]\big\}. \tag{4.3}$$

From [30, Sec. 2.2], we have

$$h(\mathcal{B},\mathcal{A}) := \lim_{N\to\infty} \frac{\log_2 \#\mathcal{W}_N(\mathcal{B},\mathcal{A})}{N} \geq h^*(T_{\mathcal{C}},\mathcal{A}). \tag{4.4}$$

Moreover, under certain assumptions it can be shown that $h(\mathcal{B},\mathcal{A})$ converges to $h^*(T_{\mathcal{C}},\mathcal{A})$ as the maximal diameter of the elements of $\mathcal{B}$ tends to zero, see [30, Thm. 4].

To compute $h(\mathcal{B},\mathcal{A})$, we first construct a directed graph $\mathcal{G}$ from the transition matrix Γ. Let us define a map $L : \mathcal{B} \to \{1,\dots,\#\mathcal{A}\}$ by

$$L(B_i) := j, \quad \text{where } j \text{ satisfies } B_i \subset A_j, \tag{4.5}$$

and call $L(B_i)$ the *label* of B_i. The graph $\mathcal{G}$ has $\mathcal{B}$ as its set of nodes. If $\Gamma_{i,j} = 1$, $i, j \in [1; \bar{n}]$, then there is a directed edge from the node B_i to B_j with the edge label $L(B_i)$. Elements of $\mathcal{W}_N(\mathcal{B},\mathcal{A})$ are generated by concatenating labels along walks of length N on the graph $\mathcal{G}$. Next, we construct a second graph $\mathcal{G}_R$ which is deterministic (i.e., no two outgoing edges have the same label) and is such that the set of all bi-infinite words that are generated by walks on $\mathcal{G}_R$ is the same as the set generated by walks on $\mathcal{G}$. For details on the construction of $\mathcal{G}_R$ from $\mathcal{G}$, see [30, Sec. 2.4]. Each node in the deterministic graph $\mathcal{G}_R$ denotes a subset of $\mathcal{B}$ and has at most one outgoing edge for any given label. We use the graph $\mathcal{G}_R$ to define an *adjacency matrix* R by $R_{i,j} := l$, where $i, j \in [1; \tilde{n}]$, and l is the number of edges from the node i to the node j of $\mathcal{G}_R$ and $\tilde{n}$ is the number of nodes in $\mathcal{G}_R$. If $\mathcal{G}$ is strongly connected, then from [30, Prop. 7], we have $h(\mathcal{B},\mathcal{A}) = \log_2 \rho(R)$. Thus, we have an upper bound for the IED of Q:

$$h_{\text{inv}}^{\text{det}}(Q) \leq \frac{1}{\tau} \log_2 \rho(R).$$

If $\mathcal{G}$ is not strongly connected, we need to determine its strongly connected components and apply the algorithm separately to each component (cf. [30, Rem. 9]). Then the maximum of the spectral radii of the obtained adjacency matrices will serve as an upper bound for $h_{\text{inv}}^{\text{det}}(Q)$. In the rest of this chapter, wherever $\tau = 1$, we write $(\mathcal{A}, G)$ instead of $(\mathcal{A}, 1, G)$.

4.3 Implementation of the algorithm for IED

In this section, we present our algorithm for the computation of upper bounds for the IED. We illustrate the steps involved in the algorithm with the help of the following example.

Example 6. *Consider the linear control system*

$$x_{k+1} = Ax_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_k, \quad A = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix},$$

with $x_k \in \mathbb{R}^2$ *and* $u_k \in U = [-1, 1]$. *For the compact controlled invariant set* $\bar{Q} = [-1, 1] \times [-2, 2]$, *see [18, Ex. 21], we intend to compute an upper bound of the IED.* □

Given a discrete-time system Σ as in (4.1) and a set $\bar{Q} \subseteq X$, we proceed according to the following steps:

1. Compute a symbolic invariant controller for the set $\bar{Q}$. Consider the hyperrectangle $\bar{Q}_X$ of smallest volume that encloses $\bar{Q}$. We use `SCOTS` to compute an invariant controller for Σ with $\bar{Q}_X$ as the state set and η_s and η_i as the grid parameters for the state and input sets, respectively. The use of small η_s results in a finer grid on the state set, i.e., a grid with boxes of smaller volumes, which generally results in a better upper bound. We denote the set of boxes in the domain of the computed controller by $\mathcal{B}$. Let $\mathcal{B} = \{B_1, \ldots, B_{\bar{n}}\}$ and $Q := \cup_{B_i \in \mathcal{B}} B_i \subseteq \bar{Q}$.

   Example 6 (Continued). *We used* ***`SCOTS`*** *with* $\bar{Q}_X = \bar{Q}$ *as the state set and the state and input set grid parameters* $\eta_s = [0.57142 \quad 0.57142]^T$ *and* $\eta_i = 0.005$, *respectively. This results in a state set grid with* 21 *boxes,* $\mathcal{B} = \{B_1, \ldots, B_{21}\}$, *and* $Q := \cup_{B_i \in \mathcal{B}} B_i$ *(see Fig. 4.1).* □

2. The controllers obtained in the previous step are in general non-deterministic, thus in this step, we determinize the obtained controller. We denote the closed-loop system (Σ with the determinized controller C) by Σ_C. To determinize the controller efficiently, one can use the state-of-the-art toolbox `dtControl` [3], which utilizes the *decision tree learning algorithm* to provide different determinized controllers with various choices of the input arguments 'Classifier' and 'Determinizer'. The tool not only determinizes the controller but also provides the required coarse partition $\mathcal{A}$ (of which $\mathcal{B}$ is a refinement). We refer the interested reader to [3] for a detailed discussion about `dtControl`.

   Example 6 (Continued). *For the example, we used* ***`dtControl`*** *with parameters Classifier = 'cart' and Determinizer = 'maxfreq'. This results in an invariant partition* $(\mathcal{A}, G)$ *for the set* $Q := \cup_{B \in \mathcal{B}} B$, *where* $\mathcal{A}$ *is a partition of* Q *such that every* $A \in \mathcal{A}$ *is a union of some elements in* $\mathcal{B}$ *and* $G(A) \in U$ *is the control input assigned to the set* A *given by* ***`dtControl`***. *Figure 4.1 shows the obtained partitions* $\mathcal{A}$ *and* $\mathcal{B}$. □

3. For the dynamical system Σ_C, we obtain the transition matrix Γ (defined in (4.2)) for the boxes in Q.

   Example 6 (Continued). Γ *comes out to be a* 21×21 *matrix, where each entry takes value* 0 *or* 1 *according to* (4.2). □

4. We obtain the map L as given in (4.5) that assigns a label to every member of the partition $\mathcal{B}$.

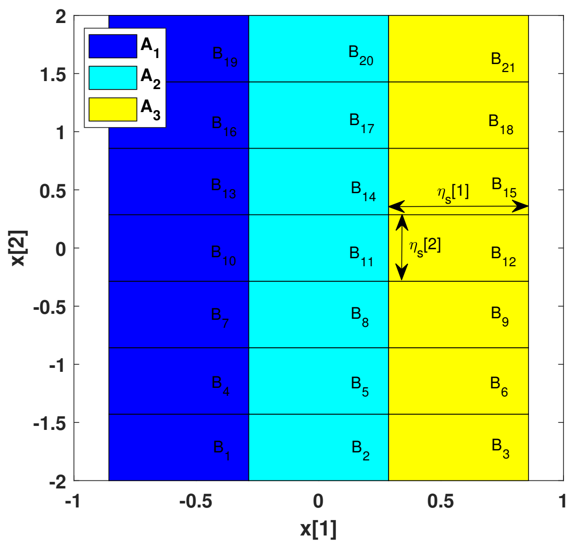


Figure 4.1: The partitions $\mathcal{A} = \{A_1, A_2, A_3\}$ and $\mathcal{B} = \{B_1, \ldots, B_{21}\}$ for Example 6.

Example 6 (Continued)**.** *For $B_i \in \mathcal{B}$*

$$L(B_i) = \begin{cases} 1 & \text{if } i = 1 + 3t, 0 \leq t \leq 6, \\ 2 & \text{if } i = 2 + 3t, 0 \leq t \leq 6, \\ 3 & \text{if } i = 3 + 3t, 0 \leq t \leq 6. \end{cases}$$

□

5. We construct a directed graph $\mathcal{G}$ with $\mathcal{B}$ as the set of nodes. If $\Gamma_{i,j} = 1$, then there is a directed edge from the node B_i to B_j with label $L(B_i)$.

6. We obtain the set $\mathcal{G}_{\mathrm{SCC}} := \{\mathcal{G}_{\mathrm{SCC},1}, \ldots, \mathcal{G}_{\mathrm{SCC},p}\}$, where $\mathcal{G}_{\mathrm{SCC},i}$, $1 \leq i \leq p$, are the strongly connected components of the graph $\mathcal{G}$. A directed graph is called strongly connected if for every pair of nodes u and v there exists a directed path from u to v and vice versa.

Example 6 (Continued)**.** *$\mathcal{G}$ is strongly connected. Thus, $\mathcal{G}_{SCC} = \{\mathcal{G}\}$.* □

7. For every $\bar{\mathcal{G}} \in \mathcal{G}_{\mathrm{SCC}}$, we find an associated deterministic graph $\bar{\mathcal{G}}_R$. The directed graph $\bar{\mathcal{G}}_R$ is deterministic in the sense that for every node no two outgoing edges have the same label.

Example 6 (Continued)**.** *Figure 4.2 shows the directed graph $\bar{\mathcal{G}}_R$. Each node in Fig. 4.2 refers to a subset of $\mathcal{B}$: $R_1 = \{B_i \mid 13 \leq i \leq 18\}$, $R_2 = \{B_i \mid 7 \leq i \leq 12\}$, $R_3 = \{B_i \mid 4 \leq i \leq 9\}$, $R_4 = \{B_i \mid 16 \leq i \leq 21\}$, $R_5 = \{B_i \mid 10 \leq i \leq 15\}$, and $R_6 = \{B_i \mid 1 \leq i \leq 6\}$.* □

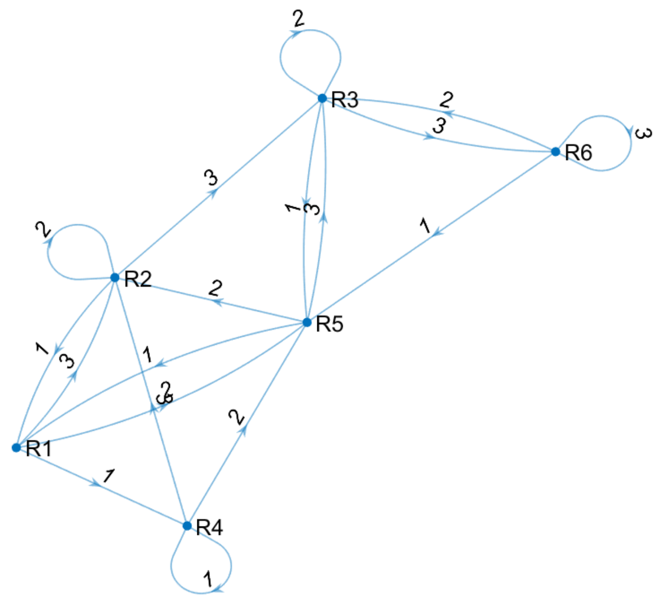


Figure 4.2: The deterministic directed graph $\bar{\mathcal{G}}_R$ for Example 6.

8. Using $\bar{\mathcal{G}}_R$, we construct an adjacency matrix $R^{\bar{\mathcal{G}}}$ with

$$R^{\bar{\mathcal{G}}}_{i,j} = l,$$

where l is the number of edges from node i to node j in $\bar{\mathcal{G}}_R$. Then, we obtain

$$h(\mathcal{B}, \mathcal{A}) = \max_{\bar{\mathcal{G}} \in \mathcal{G}_{\mathrm{SCC}}} \log_2 \rho(R^{\bar{\mathcal{G}}}).$$

Example 6 (Continued)**.** *From $\bar{\mathcal{G}}_R$, we get*

$$R^{\bar{\mathcal{G}}} = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix},$$

$\rho(R^{\bar{\mathcal{G}}}) = 3$ *and* $h^{\mathrm{det}}_{\mathrm{inv}}(Q) \leq h(\mathcal{B}, \mathcal{A}) \approx 1.5850$*. For discrete-time, deterministic linear systems, we know that the IED is independent of the set Q and is given by the logarithm of the unstable determinant,* $h^{\mathrm{det}}_{\mathrm{inv}}(Q) = 1$*.* □

4.4 Upper bounds of the invariance feedback entropy for uncertain systems

In this section, we focus on uncertain control systems. We describe the construction of a weighted directed graph which serves as the basis for computing two upper bounds for the IFE presented later in Theorem 11. Theorem 12 presents the proposed upper bound in Theorem 11 in a much simplified form as the maximum mean weight for any cycle in the graph.

Consider a discrete-time uncertain control system Σ as defined in (2.3) and a nonempty set $Q \subseteq X$. By an *invariant partition*, we refer to an invariant cover $(\bar{\mathcal{A}}, G)$ of (Σ, Q) for which $\bar{\mathcal{A}}$ is a partition of Q (which is consistent with the terminology used for deterministic systems in Section 4.2).

Given an invariant partition $(\bar{\mathcal{A}}, G)$, we define a set-valued map $T : Q \rightrightarrows Q$, $T(x) := F(x, G(A_x))$, where $x \in A_x \in \bar{\mathcal{A}}$.

We construct $\boldsymbol{\mathcal{G}}$, a directed weighted graph with $\bar{\mathcal{A}}$ as the set of nodes. For $A_1, A_2 \in \bar{\mathcal{A}}$, there is an edge in $\boldsymbol{\mathcal{G}}$ from A_1 to A_2 if $T(A_1) \cap A_2 \neq \varnothing$. Let $e_{A_1 A_2}$ refer to the edge from A_1 to A_2. We define maps $D : \bar{\mathcal{A}} \rightrightarrows \bar{\mathcal{A}}$ and $w : \bar{\mathcal{A}} \to \mathbb{R}_{\geq 0}$ as

$$D(A_1) := \left\{ A \in \bar{\mathcal{A}} \mid T(A_1) \cap A \neq \varnothing \right\}, \tag{4.6}$$

$$w(A_1) := \log_2 \# D(A_1). \tag{4.7}$$

The weight of edge $e_{A_1 A_2}$ is defined to be $w(A_1)$. We observe that

$$T(A) \subseteq \bigcup_{\hat{A} \in D(A)} \hat{A}. \tag{4.8}$$

Given the graph $\boldsymbol{\mathcal{G}}$ and $\tau \in \mathbb{N}$, we define sets

$$W_\tau(\boldsymbol{\mathcal{G}}) := \left\{ (A_i)_{i=0}^{\tau-1} \mid A_i \in \bar{\mathcal{A}}, (A_i)_{i=0}^{\tau-1} \text{ is a path in } \boldsymbol{\mathcal{G}} \right\}, \tag{4.9}$$

$$W_\infty(\boldsymbol{\mathcal{G}}) := \left\{ (A_i)_{i=0}^{\infty} \mid A_i \in \bar{\mathcal{A}}, (A_i)_{i=0}^{\infty} \text{ is a path in } \boldsymbol{\mathcal{G}} \right\}. \tag{4.10}$$

For every $(x, u) \in X \times U$, by assumption, we have $F(x, u) \neq \varnothing$, thus every node in $\boldsymbol{\mathcal{G}}$ has an outgoing edge. Therefore, for every $\tau \in \mathbb{N}$, we have

$$W_\tau(\boldsymbol{\mathcal{G}}) = \left\{ (A_i)_{i=0}^{\tau-1} \mid (A_i)_{i=0}^{\infty} \in W_\infty(\boldsymbol{\mathcal{G}}) \right\}.$$

Consider a cycle $c = (e_{A_i A_{i+1}})_{i=1}^{k}$, $A_{k+1} = A_1$ in $\boldsymbol{\mathcal{G}}$. The *mean cycle weight* for c is defined to be the ratio of the sum of the weights and the number of edges in the cycle, i.e.,

$$w_{\mathrm{m}}(c) := \frac{1}{k} \sum_{i=1}^{k} w(A_i).$$

The *maximum mean cycle weight*, $w_{\mathrm{m}}^*(\boldsymbol{\mathcal{G}})$, is then defined as $w_{\mathrm{m}}^*(\boldsymbol{\mathcal{G}}) := \max_c w_{\mathrm{m}}(c)$, where the maximum is taken over all cycles in the graph $\boldsymbol{\mathcal{G}}$. The following theorem presents two numerical upper bounds for the IFE.

Theorem 11. *For an uncertain control system* Σ *as in* (2.3)*, a nonempty set* $Q \subseteq X$*, and an invariant partition* $(\bar{\mathcal{A}}, G)$*, the IFE satisfies*

$$h_{\text{inv}}(Q, \Sigma) \leq h(\bar{\mathcal{A}}, G) = \lim_{\tau\to\infty} \frac{1}{\tau} \max_{\alpha \in W_\infty(\boldsymbol{\mathcal{G}})} \sum_{t=0}^{\tau-2} w(\alpha(t)). \tag{4.11}$$

A rough upper bound for the IFE of (Σ, Q) *is*

$$h_{\text{inv}}(Q, \Sigma) \leq h(\bar{\mathcal{A}}, G) \leq \max_{A \in \bar{\mathcal{A}}} w(A).$$

The entropy of $(\bar{\mathcal{A}}, G)$ turns out to be equal to the maximum mean cycle weight for the graph $\boldsymbol{\mathcal{G}}$, as described in the next theorem.

Theorem 12. *In Theorem 11, let* $\boldsymbol{\mathcal{G}}$ *be the directed weighted graph as defined above. Then*

$$h(\bar{\mathcal{A}}, G) = w^*_{\text{m}}(\boldsymbol{\mathcal{G}}).$$

There exist algorithms to compute the maximum mean cycle weight of a directed weighted graph, see e.g. [36].

The rest of this section is devoted to the proofs of the above two theorems. First, we present three propositions that establish some properties of the set $W_\tau(\boldsymbol{\mathcal{G}})$. Then the proof of Theorem 11 follows. Finally, we present the proof of Theorem 12.

Proposition 4. $W_\tau(\boldsymbol{\mathcal{G}})$ *is a* (τ, Q)*-spanning set in* $(\bar{\mathcal{A}}, G)$*.*

Proof. By assumption we have $F(x,u) \neq \varnothing$ for all $(x,u) \in X \times U$ which results in $T(A) \neq \varnothing$ for all $A \in \bar{\mathcal{A}}$. Since $(\bar{\mathcal{A}}, G)$ is an invariant cover, for every $A \in \bar{\mathcal{A}}$, we have $D(A) \neq \varnothing$. Thus, for every $A \in \bar{\mathcal{A}}$, there is $\hat{A} \in \bar{\mathcal{A}}$ such that $T(A) \cap \hat{A} \neq \varnothing$. This ensures that for every node in $\boldsymbol{\mathcal{G}}$, there exists an outgoing edge. Hence, for all $\tau \in \mathbb{N}$, $A \in \bar{\mathcal{A}}$ we have paths of length τ starting from A. Thus,

$$\{\alpha(0) \mid \alpha \in W_\tau(\boldsymbol{\mathcal{G}})\} = \bar{\mathcal{A}}.$$

Consider any $\alpha \in W_\tau(\boldsymbol{\mathcal{G}})$ and $t \in [0; \tau-1]$. From the definition of $\boldsymbol{\mathcal{G}}$, we have an edge from $\alpha(t)$ to every $A \in D(\alpha(t))$. Thus, for every $t \in [0; \tau-2]$ we have

$$P_{W_\tau(\boldsymbol{\mathcal{G}})}(\alpha|_{[0;t]}) = D(\alpha(t)). \tag{4.12}$$

Using (4.8) and (4.12), we conclude that $W_\tau(\boldsymbol{\mathcal{G}})$ satisfies the condition in (2.5) to be a (τ, Q)-spanning set in $(\bar{\mathcal{A}}, G)$. □

Proposition 5. *For every* (τ, Q)*-spanning set* $\mathcal{S}$ *in* $(\bar{\mathcal{A}}, G)$*, we have*

$$W_\tau(\boldsymbol{\mathcal{G}}) \subseteq \mathcal{S}.$$

Proof. Let $\mathcal{S}$ be a (τ, Q)-spanning set in $(\bar{\mathcal{A}}, G)$. Then by definition, $P_{\mathcal{S}}(\alpha) = \{\alpha(0) \mid \alpha \in \mathcal{S}\} \subseteq \bar{\mathcal{A}}$ covers Q. Since $\bar{\mathcal{A}}$ is a partition of Q, $P_{\mathcal{S}}(\alpha) = \bar{\mathcal{A}}$. If $\alpha \in \mathcal{S}$ and $t \in [0; \tau - 1]$, then again from the definition of a (τ, Q)-spanning set in (2.5) it follows that $P_{\mathcal{S}}(\alpha|_{[0;t]})$ covers $F(\alpha(t), G(\alpha(t))) = T(\alpha(t))$. As $\bar{\mathcal{A}}$ is a partition, $D(\alpha(t))$, which is defined in (4.6), must be contained in every subset of $\bar{\mathcal{A}}$ that covers $T(\alpha(t))$, thus $P_{\mathcal{S}}(\alpha|_{[0;t]}) \supseteq D(\alpha(t))$. Let $\beta \in W_\tau(\boldsymbol{\mathcal{G}})$. Then $\beta(0) \in \bar{\mathcal{A}} = \{\alpha(0) \mid \alpha \in \mathcal{S}\}$ which gives the existence of $\alpha \in \mathcal{S}$ with $\alpha(0) = \beta(0)$. From (4.12), we have $P_{W_\tau(\boldsymbol{\mathcal{G}})}(\beta(0)) = D(\beta(0))$. Similarly to the arguments above, as $\bar{\mathcal{A}}$ is a partition of Q, $D(\beta(0))$ is contained in every subset of $\bar{\mathcal{A}}$ which covers $T(\beta(0))$. As $\mathcal{S}$ is (τ, Q)-spanning, from (2.5) we know that $T(\alpha(0))$ is covered by $P_{\mathcal{S}}(\alpha(0))$ which implies $P_{\mathcal{S}}(\alpha(0)) \supseteq D(\beta(0))$. From the definition of the graph $\boldsymbol{\mathcal{G}}$, we obtain $\beta(1) \in D(\beta(0))$ leading to $\beta(1) \in P_{\mathcal{S}}(\alpha(0))$. Thus, there exists an $\alpha \in \mathcal{S}$ with $\alpha|_{[0;1]} = \beta|_{[0;1]}$. Inductively, we obtain the existence of $\alpha \in \mathcal{S}$ with $\alpha = \beta$, which concludes the proof. $\square$

From (2.6) and Proposition 5, we conclude that for every (τ, Q)-spanning set $\mathcal{S}$ in $(\bar{\mathcal{A}}, G)$, we have

$$\mathcal{N}(W_\tau(\boldsymbol{\mathcal{G}})) \leq \mathcal{N}(\mathcal{S}).$$

Let $r_{\text{inv}}(\tau, \bar{\mathcal{A}}, G, \Sigma)$ be the minimum of $\mathcal{N}(\mathcal{S})$, where $\mathcal{S}$ is a (τ, Q)-spanning set in $(\bar{\mathcal{A}}, G)$. We observe that

$$r_{\text{inv}}(\tau, \bar{\mathcal{A}}, G, \Sigma) = \mathcal{N}(W_\tau(\boldsymbol{\mathcal{G}})) \quad \text{for all } \tau \in \mathbb{N}. \tag{4.13}$$

Proposition 6. *The expansion number of the (τ, Q)-spanning set $W_\tau(\boldsymbol{\mathcal{G}})$ satisfies*

$$\log_2 \mathcal{N}(W_\tau(\boldsymbol{\mathcal{G}})) = \max_{\alpha \in W_\tau(\boldsymbol{\mathcal{G}})} \sum_{t=0}^{\tau-2} w(\alpha(t)) + \log_2 \#\bar{\mathcal{A}}.$$

Proof. By taking logarithms on both sides of (2.6), we obtain

$$\log_2 \mathcal{N}(W_\tau(\boldsymbol{\mathcal{G}})) = \max_{\alpha \in W_\tau(\boldsymbol{\mathcal{G}})} \sum_{t=0}^{\tau-1} \log_2 \# P_{W_\tau(\boldsymbol{\mathcal{G}})}(\alpha|_{[0;t]}).$$

From (2.4), (4.12), and $\bar{\mathcal{A}}$ being a partition of Q, we have

$$\log_2 \mathcal{N}(W_\tau(\boldsymbol{\mathcal{G}})) = \max_{\alpha \in W_\tau(\boldsymbol{\mathcal{G}})} \sum_{t=0}^{\tau-2} \log_2 \# D(\alpha(t)) + \log_2 \#\bar{\mathcal{A}}.$$

This together with (4.7) concludes the proof. $\square$

Now, we have all the ingredients to prove Theorems 11 and 12.

Proof of Theorem 11. From (4.13) and Proposition 6, we have

$$\begin{aligned}\log_2 r_{\text{inv}}(\tau, \bar{\mathcal{A}}, G, \Sigma) &= \log_2 \mathcal{N}(W_\tau(\boldsymbol{\mathcal{G}})) \\ &= \max_{\alpha \in W_\tau(\boldsymbol{\mathcal{G}})} \sum_{t=0}^{\tau-2} w(\alpha(t)) + \log_2 \#\bar{\mathcal{A}}.\end{aligned}$$

Therefore, the entropy of invariant partition $(\bar{\mathcal{A}}, G)$ is

$$\begin{aligned} h(\bar{\mathcal{A}}, G) &= \lim_{\tau\to\infty} \frac{1}{\tau} \log_2 r_{\text{inv}}(\tau, \bar{\mathcal{A}}, G, \Sigma) \\ &= \lim_{\tau\to\infty} \frac{1}{\tau} \max_{\alpha\in W_\tau(\boldsymbol{\mathcal{G}})} \sum_{t=0}^{\tau-2} w(\alpha(t)). \end{aligned}$$

Since the set $W_\tau(\boldsymbol{\mathcal{G}})$ is recovered by restricting the elements of $W_\infty(\boldsymbol{\mathcal{G}})$ to $[0; \tau-1]$, therefore

$$h(\bar{\mathcal{A}}, G) = \lim_{\tau\to\infty} \frac{1}{\tau} \max_{\alpha\in W_\infty(\boldsymbol{\mathcal{G}})} \sum_{t=0}^{\tau-2} w(\alpha(t)).$$

This proves the first claim in Theorem 11.

For any $\tau \in \mathbb{N}$, consider

$$\begin{aligned} \frac{1}{\tau} \max_{\alpha\in W_\infty(\boldsymbol{\mathcal{G}})} \sum_{t=0}^{\tau-2} w(\alpha(t)) &\leq \frac{1}{\tau} \max_{\alpha\in W_\infty(\boldsymbol{\mathcal{G}})} (\tau-1) \max_{A\in\bar{\mathcal{A}}} w(A) \\ &= (1 - \frac{1}{\tau}) \max_{A\in\bar{\mathcal{A}}} w(A). \end{aligned}$$

Taking limit on both sides, we obtain $h(\bar{\mathcal{A}}, G) \leq \max_{A\in\bar{\mathcal{A}}} w(A)$. This completes the proof. $\square$

Proof of Theorem 12. First we construct a mean-payoff-game (MPG) for which the maximum of the value function over a given set equals the entropy of the invariant partition $(\bar{\mathcal{A}}, G)$.

Consider the system in (2.3), a nonempty set $Q \subseteq X$, an invariant partition $(\bar{\mathcal{A}}, G)$, the maps $T : Q \rightrightarrows Q$ and $D : \bar{\mathcal{A}} \rightrightarrows \bar{\mathcal{A}}$ as defined in Section 4.4. We consider the definition of the MPG (V, E, ω) as described in Appendix A.

We construct an MPG (V, E, ω), where $V := V_1 \cup V_2$, $V_1 := \bar{\mathcal{A}}$, $V_2 := \{D(A) \mid A \in \bar{\mathcal{A}}\}$ and $V_1 \cap V_2 = \varnothing$. The set of edges $E := E_1 \cup E_2$ of the MPG is defined by

$$\begin{aligned} E_1 :=& \{(v_1, v_2) \in V_1 \times V_2 \mid v_2 = D(v_1)\}, \\ E_2 :=& \{(v_2, v_1) \in V_2 \times V_1 \mid v_1 \in v_2\}. \end{aligned}$$

The weights for $(v_1, v_2) \in E_1$ and $(v_2, \bar{v}_1) \in E_2$ are given by $\omega(v_1, v_2) := \log_2 \#v_2 = \log_2 \#D(v_1)$ and $\omega(v_2, \bar{v}_1) := \log_2 \#v_2$.

Consider a play $e_0e_1e_2\ldots$ which is an infinitely long sequence of edges. Player 1 wants to minimize the payoff

$$\nu_{\min}(e_0e_1e_2\ldots) := \limsup_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} \omega(e_j),$$

while player 2 wants to maximize the payoff

$$\nu_{\max}(e_0e_1e_2\ldots) := \liminf_{k\to\infty} \frac{1}{k}\sum_{j=0}^{k-1}\omega(e_j).$$

We use (σ_i^*) to denote the optimal positional strategy for the player i. By $\mathcal{P}(v,\sigma_i) \subseteq E^{[0;\infty[}$, we denote the set of all plays that start from the position v and wherein the player i follows the positional strategy σ_i. From (A.2) and (A.3) we have the existence of constants c_1 and c_2, so that for every $\tau \in \mathbb{N}$, $v \in V$, $e \in \mathcal{P}(v,\sigma_1^*)$ and $\hat{e} \in \mathcal{P}(v,\sigma_2^*)$ we have

$$\frac{1}{\tau}\sum_{j=0}^{\tau-1}\omega(e_j) \le \nu(v) + \frac{c_1}{\tau},$$

$$\frac{1}{\tau}\sum_{j=0}^{\tau-1}\omega(\hat{e}_j) \ge \nu(v) + \frac{c_2}{\tau},$$

or

$$\max_{e\in\mathcal{P}(v,\sigma_1^*),v\in V_1} \frac{1}{\tau}\sum_{j=0}^{\tau-1}\omega(e_j) \le \max_{v\in V_1}\nu(v) + \frac{c_1}{\tau}, \tag{4.14}$$

$$\max_{\hat{e}\in\mathcal{P}(v,\sigma_2^*),v\in V_1} \frac{1}{\tau}\sum_{j=0}^{\tau-1}\omega(\hat{e}_j) \ge \max_{v\in V_1}\nu(v) + \frac{c_2}{\tau}. \tag{4.15}$$

In the preceding inequalities, we consider the maximum over the set $V_1 = \bar{\mathcal{A}}$ only, because, in the later parts of the proof, we will relate the graph of the MPG with the graph $\boldsymbol{\mathcal{G}}$ that involves only the elements of V_1 as its nodes. Note that, in our construction of the MPG, player 1 always plays with a fixed strategy, σ_1^*, i.e., for every $v \in V_1$, the next position selected by player 1 is always $\sigma_1^*(v) = D(v)$. Thus, the course of any play is dictated by only player 2, and if the player 2 uses a positional strategy then there will be only one play for any given starting position $v_0 \in V$. This gives $|\mathcal{P}(v,\sigma_2^*)| = 1$ and $\mathcal{P}(v,\sigma_2^*) \subset \mathcal{P}(v,\sigma_1^*)$.

For $v \in V_1$, $e \in \mathcal{P}(v,\sigma_1^*)$ we denote the edge e_i by the tuple of vertices (v_i, v_{i+1}). Consider the set $W_\infty(\boldsymbol{\mathcal{G}})$ as defined in (4.10). Now we show the existence of an $\alpha \in W_\infty(\boldsymbol{\mathcal{G}})$ such that $\alpha(t) = v_{2t}$ for all $t \in [0;\infty[$. Since $v_0 \in V_1 = \bar{\mathcal{A}}$, there exists an $\alpha \in W_\infty(\boldsymbol{\mathcal{G}})$ such that $\alpha(0) = v_0$. By definition of E_2, we have $v_2 \in v_1 = D(v_0)$ and from that of $W_\infty(\boldsymbol{\mathcal{G}})$ we have $\alpha(1) \in D(\alpha(0))$. Thus, there exists an $\alpha \in W_\infty(\boldsymbol{\mathcal{G}})$ such that $\alpha(t) = v_{2t}$ for all $t \in [0;1]$. Iteratively, we have the existence of $\alpha_v \in W_\infty(\boldsymbol{\mathcal{G}})$ such that $\alpha_v(t) = v_{2t}$ for all $t \in [0;\infty[$. With similar reasoning, we also obtain the existence of a play $e \in \mathcal{P}(v,\sigma_1^*)$ for every $\alpha_v \in W_\infty(\boldsymbol{\mathcal{G}})$, $\alpha_v(0) = v$ such that $\alpha_v(t) = v_{2t}$ for all $t \in [0;\infty[$. Thus, every element of $W_\infty(\boldsymbol{\mathcal{G}})$ corresponds to some element of $\cup_{v\in V_1}\mathcal{P}(v,\sigma_1^*)$ and vice versa.

Now consider a play $e = e_0e_1e_2\ldots \in \mathcal{P}(v,\sigma_1^*)$, $v \in V_1$. By definition of the map ω, when $e_0 \in E_1$ we have $\omega(e_{2j}) = \omega(e_{2j+1})$ for all $j \in [0;\infty[$, leading to

$$\frac{1}{2k}\sum_{j=0}^{2k-1}\omega(e_j) = \frac{1}{2k}\sum_{j=0}^{k-1}2\omega(e_{2j}) = \frac{1}{k}\sum_{j=0}^{k-1}\omega(e_{2j}).$$

Let $\alpha_v \in W_\infty(\boldsymbol{\mathcal{G}})$ be such that $\alpha_v(t) = v_{2t}$ for all $t \in [0;\infty[$. Then

$$\begin{aligned}
\frac{1}{k}\sum_{j=0}^{k-1}\omega(e_{2j}) =&\frac{1}{k}\sum_{j=0}^{k-1}\log_2 \#D(v_{2j})\\
=&\frac{1}{k}\sum_{j=0}^{k-1}\log_2 \#D(\alpha_v(j))\\
=&\frac{1}{k}\sum_{j=0}^{k-1}w(\alpha_v(j)),
\end{aligned}$$

where the map $w : \bar{\mathcal{A}} \to \mathbb{R}_{\geq 0}$ is defined in (4.7).

Next, consider the set $\hat{W}_\infty(\boldsymbol{\mathcal{G}})$ which is constituted by all such paths in the graph $\boldsymbol{\mathcal{G}}$ that correspond to some play $\hat{e} \in \mathcal{P}(v, \sigma_2^*)$, $v \in V_1$, and is defined as

$$\hat{W}_\infty(\boldsymbol{\mathcal{G}}) := \{\hat{\alpha} \in W_\infty(\boldsymbol{\mathcal{G}}) \mid \exists \hat{e} \in \cup_{v\in V_1}\mathcal{P}(v,\sigma_2^*) \text{ so that } \hat{\alpha}(t) = \hat{v}_{2t}\ \forall t \in [0;\infty[\}.$$

The inequalities (4.14) and (4.15) can now be rewritten as

$$\begin{aligned}
\max_{\alpha\in W_\infty(\boldsymbol{\mathcal{G}})}\frac{1}{\tau}\sum_{j=0}^{\tau-1}w(\alpha(j)) \leq&\max_{v\in V_1}\nu(v) + \frac{c_1}{2\tau},\\
\max_{\hat{\alpha}\in \hat{W}_\infty(\boldsymbol{\mathcal{G}})}\frac{1}{\tau}\sum_{j=0}^{\tau-1}w(\hat{\alpha}(j)) \geq&\max_{v\in V_1}\nu(v) + \frac{c_2}{2\tau},
\end{aligned}$$

or

$$\begin{aligned}
\max_{\alpha\in W_\infty(\boldsymbol{\mathcal{G}})}\frac{1}{\tau}\sum_{j=0}^{\tau-2}w(\alpha(j)) \leq&\max_{v\in V_1}\nu(v) + \frac{c_1}{2\tau},\\
\max_{\hat{\alpha}\in \hat{W}_\infty(\boldsymbol{\mathcal{G}})}\frac{1}{\tau}\sum_{j=0}^{\tau-2}w(\hat{\alpha}(j)) \geq&\max_{v\in V_1}\nu(v) + \frac{\bar{c}_2}{2\tau},
\end{aligned}$$

where $\bar{c}_2 = c_2 - 2\max_{A\in\bar{\mathcal{A}}} w(A)$. Since $\hat{W}_\infty(\boldsymbol{\mathcal{G}}) \subseteq W_\infty(\boldsymbol{\mathcal{G}})$, $\hat{W}_\infty(\boldsymbol{\mathcal{G}})$ can be replaced by $W_\infty(\boldsymbol{\mathcal{G}})$, therefore the above two equations lead to

$$\limsup_{\tau\to\infty}\max_{\alpha\in W_\infty(\boldsymbol{\mathcal{G}})}\frac{1}{\tau}\sum_{j=0}^{\tau-2}w(\alpha(j)) = \max_{v\in V_1}\nu(v).$$

Thus,

$$h(\bar{\mathcal{A}}, G) = \max_{v\in V_1}\nu(v).$$

Let $\boldsymbol{\mathcal{G}}_M$ refer to the graph of the MPG. By definition of $\boldsymbol{\mathcal{G}}_M$ and $\boldsymbol{\mathcal{G}}$, for every cycle c_M in $\boldsymbol{\mathcal{G}}_M$ there exists a corresponding cycle c in $\boldsymbol{\mathcal{G}}$ such that, although the length of c_M is twice that of c, the mean weight is the same for both cycles. In an MPG, if one of the player follows a fixed positional strategy, then $\nu(v)$ is the maximum mean weight of a cycle in $\boldsymbol{\mathcal{G}}_M$ reachable from $v \in V$, see [85, Sec. 4]. Thus, $\max_{v\in V_1}\nu(v) = w_{\mathrm{m}}^*(\boldsymbol{\mathcal{G}})$. $\square$

In the next section, for deterministic systems, we establish the relationship between the discussed upper bounds of IED and IFE.

4.5 Relationship between the upper bounds for IED and IFE

Consider a control system as defined in (4.1), a set $Q \subseteq X$ and an invariant partition $(\mathcal{A}, G)$ of Q. Let $\mathcal{B} = \{B_1, \ldots, B_{n_b}\}$ be a refinement of $\mathcal{A} = \{A_1, \ldots, A_{n_a}\}$. We define a map $T : Q \to Q$, $T(x) := f(x, G(A))$ where $x \in A \in \mathcal{A}$. Now we construct a directed graph $\mathcal{G}_{\mathcal{B}}$ with $\mathcal{B}$ as the set of nodes. Let $L : \mathcal{B} \to \{1, \ldots, n_a\}$ be a map as defined in (4.5). For $B_i, B_j \in \mathcal{B}$, there is an edge in $\mathcal{G}_{\mathcal{B}}$ from B_i to B_j if and only if $T(B_i) \cap B_j \neq \varnothing$, with $L(B_i)$ as the edge label. Using the graph $\mathcal{G}_{\mathcal{B}}$, we construct a matrix $R_{\mathcal{G}_{\mathcal{B}}}$ exactly the same way that the matrix R is constructed from the graph $\mathcal{G}$ in Section 4.2.

Let $\mathcal{W}_N(\mathcal{B}, \mathcal{A})$ and $h(\mathcal{B}, \mathcal{A})$ be as defined in (4.3) and (4.4), respectively. Assuming that $\mathcal{G}_{\mathcal{B}}$ is strongly connected, we have

$$h(\mathcal{B}, \mathcal{A}) = \lim_{N \to \infty} \frac{\log_2 \#\mathcal{W}_N(\mathcal{B}, \mathcal{A})}{N} = \log_2 \rho(R_{\mathcal{G}_{\mathcal{B}}}).$$

The next theorem shows that the upper bound of the IED is no larger than that of the IFE.

Theorem 13. *For Σ as in* (4.1)*, a nonempty set $Q \subseteq X$ and a given invariant partition $(\mathcal{A}, G)$ of Q, the entropy of $(\mathcal{A}, G)$ satisfies*

$$h(\mathcal{A}, G) \geq h(\mathcal{B}, \mathcal{A}) = \log_2 \rho(R_{\mathcal{G}_{\mathcal{B}}}).$$

Proof. Similarly to a $\mathcal{B}$-word as defined in Section 4.2, we define $\mathcal{A}$-words. A sequence $[a_0 \ldots a_{N-1}]$ is called an $\mathcal{A}$-word if $T(A_{a_i}) \cap A_{a_{i+1}} \neq \varnothing$ for every $i \in [0; N-2]$. Then similarly to (4.3), we define

$$\mathcal{W}_N(\mathcal{A}) := \big\{\{A_{a_j}\}_{j=0}^{N-1} \mid [a_0 \ldots a_{N-1}] \text{ is an } \mathcal{A}\text{-word}\big\}.$$

Similarly to the graph $\mathcal{G}_{\mathcal{B}}$, we construct a graph $\mathcal{G}_{\mathcal{A}}$. Since $\mathcal{A}$ is a partition, $\mathcal{W}_N(\mathcal{A})$ is the set of all N-length sequences in $\mathcal{A}$ generated by traversing the paths in the graph $\mathcal{G}_{\mathcal{A}}$. Thus,

$$\mathcal{W}_N(\mathcal{A}) = W_N(\mathcal{G}_{\mathcal{A}}),$$

where $W_N(\mathcal{G}_{\mathcal{A}})$ is defined in (4.9). From Lemma 2 and (4.13), we obtain

$$\#\mathcal{W}_N(\mathcal{A}) = \#W_N(\mathcal{G}_{\mathcal{A}}) \leq \mathcal{N}(W_N(\mathcal{G}_{\mathcal{A}})) = r_{\text{inv}}(N, \mathcal{A}, G, \Sigma).$$

Similarly to (4.4), we define

$$h(\mathcal{A}) := \lim_{N \to \infty} \frac{\log_2 \#\mathcal{W}_N(\mathcal{A})}{N}.$$

This together with (2.7) yields

$$h(\mathcal{A}) \leq h(\mathcal{A}, G).$$

From [30, Sec. 2.5], we have $h(\mathcal{B}, \mathcal{A}) \leq h(\mathcal{A})$. This gives a lower bound for the entropy of the invariant partition $(\mathcal{A}, G)$:

$$\log_2 \rho(R_{\mathcal{G}_{\mathcal{B}}}) \leq h(\mathcal{A}, G).$$

□

4.6 Examples

In the first two examples, we use known formulas for the IED, which have been proved for versions of invariance entropy that slightly differ from the one we introduced in Section 4.2. However, from a numerical point of view, this should not make a considerable difference. In any case, the claimed values for $h_{\mathrm{inv}}^{\mathrm{det}}(\bar{Q})$ in both cases are theoretical lower bounds, while our algorithm provides upper bounds.

The description of our implementation for the upper bounds of the IFE is presented in Example 9.

4.6.1 A linear discrete-time system

Example 6 (Continued)**.** *Again consider the linear control system and the set $\bar{Q}$ as in Section 4.3. The IED of $\bar{Q}$ is given by*

$$h_{\mathrm{inv}}^{\mathrm{det}}(\bar{Q}) = \sum_{|\lambda(A)| \geq 1} \log_2 |\lambda(A)| = 1.$$

Table 4.1 lists the obtained upper bounds $h(\mathcal{B}, \mathcal{A})$ of $h_{\mathrm{inv}}^{\mathrm{det}}(\bar{Q})$ with `SCOTS` *parameters $\eta_s = [0.01 \quad 0.01]^T$ and $\eta_i = 0.5$, for different choices of options in* `dtControl`*. For the same values of η_s and η_i and with 'cart' and 'maxfreq', Table 4.2 presents the variation of the upper bound $h(\mathcal{B}, \mathcal{A})/\tau$ with increasing length τ of the control sequences. By 'control sequence length' we refer to the length of the sequence assigned to any element of the cover $\mathcal{A}$ by the map G as described in Section 4.2. With the same values of η_s and η_i, the two other bounds are $w_m^*(\boldsymbol{\mathcal{G}}) = \max_{A \in \mathcal{B}} w(A) = 2.5850$ and their computation times are 0.593 sec and 0.304 sec, respectively. For the implementational details of $w_m^*(\boldsymbol{\mathcal{G}})$, see Example 9.* □

4.6.2 A scalar continuous-time nonlinear control system

Example 7. *Consider the following scalar continuous-time control system discussed in [38, Ex. 7.2]:*

$$\Sigma : \quad \dot{x} = (-2b \sin x \cos x - \sin^2 x + \cos^2 x) + u \cos^2 x,$$

Table 4.1: Upper bound $h(\mathcal{B}, \mathcal{A})$ for Example 6 with different choices of the determinization options in `dtControl`. Here, we have $h_{\rm inv}^{\rm det}(\bar{Q}) = 1$.

Classifier	Determinizer	$\#\mathcal{A}$	$h(\mathcal{B}, \mathcal{A})$	Computation time (sec)
cart	maxfreq	4	1.0149	26
logreg	maxfreq	4	1.0149	28
cart	minnorm	5	1.0517	28
logreg	minnorm	5	1.0517	33

Table 4.2: Upper bound $h(\mathcal{B}, \mathcal{A})/\tau$ for Example 6 with control sequences of length τ, Classifier = 'cart', and Determinizer = 'maxfreq' in `dtControl`. Here, we have $h_{\rm inv}^{\rm det}(\bar{Q}) = 1$.

τ	$h(\mathcal{B}, \mathcal{A})/\tau$	Computation time
1	1.0149	26 sec
2	1.0092	55 sec
3	1.0053	4 min
4	1.0029	14 min

where $u \in [-\rho, \rho]$, $b > 0$ and $0 < \rho < b^2 + 1$. The equation describes the projectivized linearization of a controlled damped mathematical pendulum at the unstable position, where the control acts as a reset force.

The following set is controlled invariant:

$$\bar{Q} = \left[\arctan(-b - \sqrt{b^2 + 1 + \rho}), \arctan(-b - \sqrt{b^2 + 1 - \rho})\right].$$

In fact, $\bar{Q}$ is the closure of a control set, i.e., a maximal set of complete approximate controllability. With $\mathcal{T}_s \in \mathbb{R}_{>0}$ as the sampling time, we first obtain a discrete-time system as in (4.1)*. Theory suggests that the following formula holds for the IED of Σ, see*[1] *[38, Ex. 7.2]:*

$$h_{\rm inv}^{\rm det}(\bar{Q}) = \frac{2}{\ln 2}\sqrt{b^2 + 1 - \rho}.$$

Discretizing the continuous-time system with sampling time $\mathcal{T}_s$ results in a discrete-time system $\Sigma^{\mathcal{T}_s}$ that satisfies

$$h_{\rm inv}^{\rm det}(\bar{Q}; \Sigma^{\mathcal{T}_s}) \geq \mathcal{T}_s \cdot h_{\rm inv}^{\rm det}(\bar{Q}) = \frac{2\mathcal{T}_s}{\ln 2}\sqrt{b^2 + 1 - \rho}.$$

The inequality is due to the fact that continuous-time open-loop control functions are lost due to the sampling (since only the piecewise constant control functions, constant on each interval of the form $[k\mathcal{T}_s, (k+1)\mathcal{T}_s)$, $k \in \mathbb{Z}_{\geq 0}$, are preserved under sampling). Table 4.3 and

[1] The factor ln(2) appears due to the choice of the base-2 logarithm instead of the natural logarithm, which is typically used for continuous-time systems.

Table 4.3: Values of $h(\mathcal{B},\mathcal{A})$ and the IED $h^{\mathrm{det}}_{\mathrm{inv}}(\bar{Q})$ for Example 7 with $\rho = 1$, $b = 1$ and different choices of the sampling time $\mathcal{T}_s$. Here, we have $h^{\mathrm{det}}_{\mathrm{inv}}(\bar{Q}) = 2.8854$.

$\mathcal{T}_s$	$\#\mathcal{A}$	$h(\mathcal{B},\mathcal{A})/\mathcal{T}_s$	Computation time
0.8	11	4.0207	21.23 hr
0.5	6	4.0847	2.98 hr
0.1	2	4.744	3.33 min
0.01	2	5.1994	55 sec
0.001	2	24.7	60 sec

Table 4.4: Values of $h(\mathcal{B},\mathcal{A})$ and the IED $h^{\mathrm{det}}_{\mathrm{inv}}(\bar{Q})$ for Example 7 with $\rho = 50$, $b = 10$ and different choices of the sampling time $\mathcal{T}_s$. Here, we have $h^{\mathrm{det}}_{\mathrm{inv}}(\bar{Q}) = 20.6058$.

$\mathcal{T}_s$	$\#\mathcal{A}$	$h(\mathcal{B},\mathcal{A})/\mathcal{T}_s$	Computation time
0.11	15	28.5012	1.9 hr
0.1	11	29.1723	1.35 hr
0.01	2	34.4707	13 sec
0.001	2	55.5067	12 sec
0.0001	2	1.5635e+03	31 sec

4.4 list the values of $h(\mathcal{B},\mathcal{A})$ for different choices of the sampling time with the parameters ($\rho = 1$, $b = 1$, $\eta_s = 10^{-6}$, $\eta_i = 0.2\rho$) and ($\rho = 50$, $b = 10$, $\eta_s = 10^{-6}$, $\eta_i = 0.2\rho$), respectively. For both of the tables, the `dtControl` parameters are Classifier = 'cart' and Determinizer = 'maxfreq'. Table 4.5 shows the values of $h(\mathcal{B},\mathcal{A})$ for different selections of the coarse partition $\mathcal{A}$ with the parameters $\mathcal{T}_s = 0.01$, $\eta_s = 10^{-6}$, $\eta_i = 0.2\rho$, $\rho = 1$, and $b = 1$. For the same selection of parameters as in Table 4.3 with $\mathcal{T}_s = 0.01$, Table 4.6 presents the variation of the upper bound $h(\mathcal{B},\mathcal{A})/(\tau\mathcal{T}_s)$ with increasing length τ of the control sequences. □

Table 4.5: Values of $h(\mathcal{B},\mathcal{A})$ for Example 7 with different choices of `dtControl` parameters. Here, we have $h^{\mathrm{det}}_{\mathrm{inv}}(\bar{Q}) = 2.8854$.

Classifier	Determinizer	$\#\mathcal{A}$	$h(\mathcal{B},\mathcal{A})/\mathcal{T}_s$	Computation time (sec)
cart	maxfreq	2	5.1994	55
logreg	maxfreq	2	5.1994	65
linsvm	maxfreq	2	5.1994	61
cart	minnorm	11	6.4475	57
logreg	minnorm	11	6.4475	74

Table 4.6: Upper bound $h(\mathcal{B},\mathcal{A})/(\tau\mathcal{T}_s)$ for Example 7 with control sequences of length τ, Classifier = 'cart', and Determinizer = 'maxfreq' in `dtControl`. Here, we have $h_{\mathrm{inv}}^{\mathrm{det}}(\bar{Q}) = 2.8854$.

τ	$h(\mathcal{B},\mathcal{A})/(\tau\mathcal{T}_s)$	Computation time
1	5.1994	57 sec
2	5.0036	7.5 min
3	4.9547	1.91 hr
4	4.9266	27.27 hr

4.6.3 A 2d uniformly hyperbolic set

Example 8. *Consider the map*

$$f(x,y) := (5 - 0.3y - x^2, x), \quad f : \mathbb{R}^2 \to \mathbb{R}^2,$$

which is a member of the Hénon family, one of the most-studied classes of dynamical systems that exhibit chaotic behavior. We extend f to a control system with additive control:

$$\Sigma : \quad \begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} 5 - 0.3y_k - x_k^2 + u_k \\ x_k + v_k \end{bmatrix},$$

where $\max\{|u_k|, |v_k|\} \leq \varepsilon$*. It is known that f has a non-attracting uniformly hyperbolic set Λ, which is a topological horseshoe (called the* Hénon horseshoe*). This set is contained in the square centered at the origin with side length [62, Thm. 4.2]*

$$r := 1.3 + \sqrt{(1.3)^2 + 20} \approx 5.9573.$$

If the size ε of the control range of Σ is chosen small enough, the set Λ is blown up to a compact controlled invariant set Q^ε with nonempty interior which is not much larger than Λ; see [41]. Moreover, the theory suggests that as $\varepsilon \to 0$, $h_{\mathrm{inv}}^{\mathrm{det}}(Q^\varepsilon)$ converges to the negative topological pressure of $f_{|\Lambda}$ with respect to the negative unstable log-determinant on Λ; see [13] for definitions. A numerical estimate for this quantity, obtained in [29, Table 2] via Ulam's method, is 0.696*.*

Consider the set $\tilde{Q} = [-r/2, r/2] \times [-r/2, r/2]$. For $\varepsilon = 0.08$, using ***SCOTS*** *with parameter values $\eta_s = [0.009 \quad 0.009]^T$ and $\eta_i = [0.01 \quad 0.01]^T$, through iteration, we obtain an all-time controlled invariant set $Q \subseteq \tilde{Q}$. For the iteration, we begin with the set $\tilde{Q}$ and as the first step we compute an invariant controller for the system Σ. Let Q_1 be the domain of the obtained controller. Given Σ, consider its time-reversed system*

$$\Sigma^- : \quad \begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} y_k - v_k \\ \frac{1}{0.3}(5 - (y_k - v_k)^2 + u_k - x_k) \end{bmatrix}.$$

In the second step we compute an invariant controller for the system Σ^- in the set Q_1, and denote the controller domain by Q_2. In the third step we compute an invariant controller

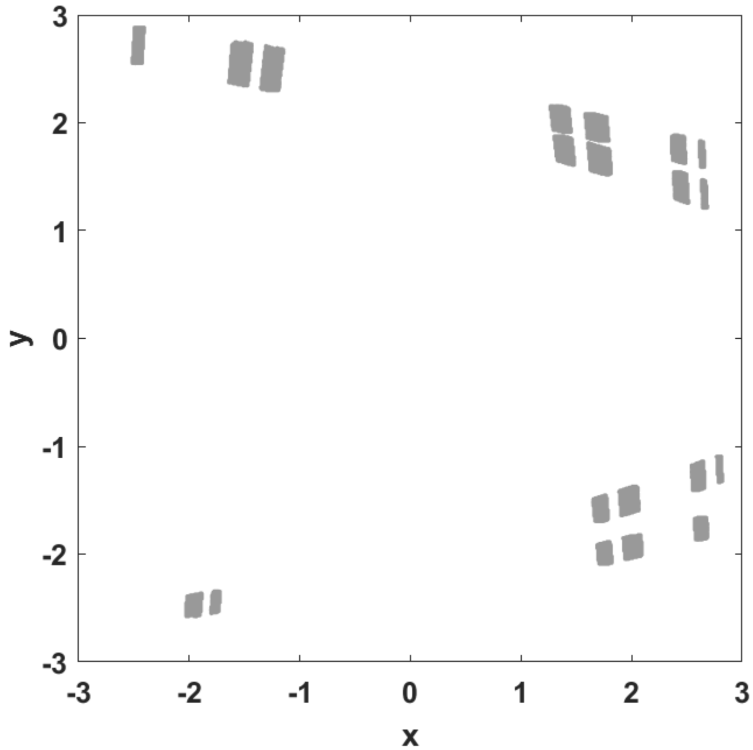


Figure 4.3: The set Q for Example 8.

for Σ but in the set Q_2 and denote the controller domain by Q_3. The steps are repeated until $Q_i = Q_{i+1} =: Q$. The set Q likely approximates the (all-time controlled invariant) set Q^ϵ. Figure 4.3 shows the set Q. For the parameter values $\varepsilon = 0.08$, $\eta_s = [0.009 \quad 0.009]^T$, $\eta_i = [0.01 \quad 0.01]^T$, Table 4.7 lists the values of $h(\mathcal{B}, \mathcal{A})$ for different selections of the coarse partition $\mathcal{A}$. For the same values of ε, η_s and η_i, the obtained values for the other two bounds are $w_m^*(\boldsymbol{\mathcal{G}}) = 3.5646$ and $\max_{A \in \mathcal{B}} w(A) = 3.8074$ with computation times 2.27 sec and 1.99 sec, respectively.

4.6.4 An uncertain linear control system

Example 9. *We consider an uncertain linear control system*

$$\Sigma : \quad x_{k+1} \in F(x_k, u_k) = Ax_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_k + W,$$
$$A = \begin{bmatrix} 2 & 1 \\ -0.4 & 0.5 \end{bmatrix},$$

where the state $x_k \in \mathbb{R}^2$, the control input $u_k \in U = [-1, 1]$ and the disturbance set $W = [-0.1, 0.1]^2$. For a set $Q \subseteq [-1, 1] \times [-2, 2]$, we compute an upper and a lower bound

Table 4.7: Values of $h(\mathcal{B},\mathcal{A})$ for Example 8 with different selections of `dtControl` options. Here, we have $h^{\text{det}}_{\text{inv}}(Q) \approx 0.696$.

Classifier	Determinizer	$\#\mathcal{A}$	$h(\mathcal{B},\mathcal{A})$	Computation time
cart	maxfreq	573	2.3884	0.95 min
linsvm	maxfreq	567	2.3956	1.82 min
logreg	maxfreq	454	2.3994	1.4 min
cart	minnorm	1921	2.9342	1 min
logreg	minnorm	1533	2.9215	2 min
linsvm	minnorm	1923	2.9376	2.15 min

of the IFE of (Σ, Q). *We used* ***SCOTS*** *to obtain an invariant controller for the state-space subset* $\bar{Q} = [-1,1]\times[-2,2]$ *with* $[0.2 \quad 0.2]^T$ *and* 0.05 *as the state and input grid parameters, respectively. The set* Q *is taken to be the domain of the obtained controller that consists of* 109 *state-grid cells each of size* 0.2×0.2. *Figure 4.4 shows the subset* Q.

Computation of the lower bound

We utilize Theorem 7 to compute a lower bound. From Remark 2, we know that the lower bound in Theorem 7 is invariant under coordinate transformations. After a similarity transformation $x = Vz$ *with* $V = \begin{bmatrix} 0.9448 & -0.6552 \\ -0.3277 & 0.7555 \end{bmatrix}$, *we have*

$$z_{k+1} = \tilde{A} z_k + V^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_k + V^{-1} W,$$
$$\tilde{A} = \begin{bmatrix} 1.6531 & 0 \\ 0 & 0.8469 \end{bmatrix}.$$

For $i \in \{1,2\}$, *let* $\pi_i Q$ *denote the projection of the set* Q *to the* i*-th coordinate. Then* $\pi_1 V^{-1} Q = [-2.1207, 2.1207]$, $\pi_2 V^{-1} Q = [-3.4, 3.4]$, $\pi_1 V^{-1} W = [-0.2827, 0.2827]$ *and* $\pi_2 V^{-1} W = [-0.2550, 0.2550]$. *Thus, we have* $0.9316 \le h_{\text{inv}}(Q,\Sigma)$.

Computation of the upper bound

We construct an invariant partition $(\bar{\mathcal{A}}, G)$ *of* (Σ, Q) *by selecting the set of grid cells in the domain of the controller obtained from* ***SCOTS*** *as the cover* $\bar{\mathcal{A}}$. *Let* $C : \bar{\mathcal{A}} \rightrightarrows U$ *denote the controller from* ***SCOTS***. *For* $A \in \bar{\mathcal{A}}$, $C(A)$ *is the list of control inputs in the controller assigned to cell* A *such that each of the control inputs in the list ensures invariance of the states in* A *with respect to the set* Q. *For each* $A \in \bar{\mathcal{A}}$, *we define* $G(A) := u \in C(A)$, *where* u *is such that* $F(A,u)$ *has non-empty intersection with a minimum number of elements of* $\bar{\mathcal{A}}$. *If there are multiple such control values, then one of them is selected randomly. Using* $(\bar{\mathcal{A}}, G)$ *and the transition function* F, *we construct a weighted directed graph* $\boldsymbol{\mathcal{G}}$ *as described in Section 4.4. We used the Boost Graph Library*[2] *to compute the maximum mean cycle*

[2] `https://www.boost.org/doc/libs/1_74_0/libs/graph/doc/index.html`

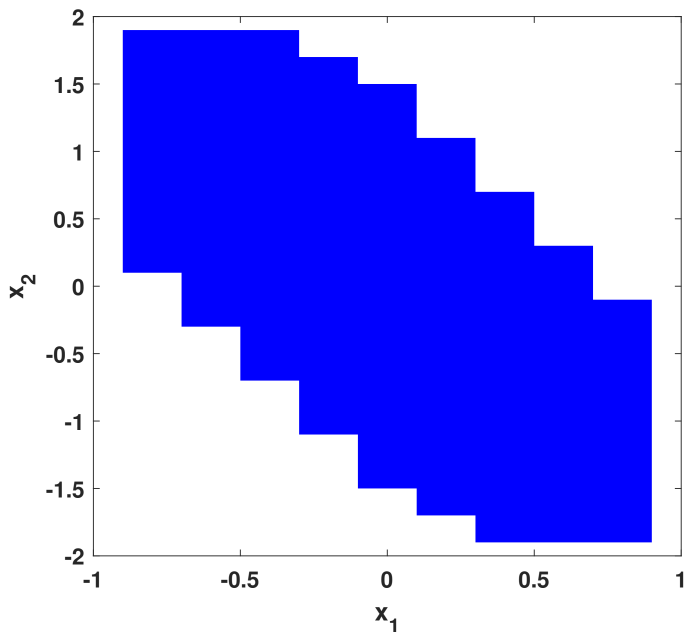


Figure 4.4: The set Q in Example 9 which is defined to be the domain of the invariant controller computed from `SCOTS`.

weight for the graph $\boldsymbol{\mathcal{G}}$ *and obtained* $w_{\mathrm{m}}^*(\boldsymbol{\mathcal{G}}) = \max_{A \in \bar{\mathcal{A}}} w(A) = 3.3219$ *with computation times* 0.0164 *sec and* 0.0133 *sec, respectively. Thus,* $h_{\mathrm{inv}}(Q, \Sigma) \leq 3.3219$. $\square$

4.7 Discussion

Tables 4.2 and 4.6 clearly show that the upper bound of the invariance entropy gets better and better with increasing length τ of the control sequences in the co-domain of the map $G : \mathcal{A} \to U^\tau$. The `dtControl` parameter Determinizer = 'maxfreq' gives the best upper bounds in all three deterministic examples. Let $C : X \rightrightarrows U$ denote the controller that is fed to `dtControl` and S_n denote the subset of the state space corresponding to the node n of the decision tree. When 'maxfreq' is selected, then, during the construction of the decision tree, for every node n, the corresponding part of the controller $(C|_{S_n})$ is determinized through the selection of the control values that have the maximum frequency of appearance in the set $\cup_{x \in S_n} C(x)$. In contrast, with 'minnorm' the controller is determinized by the selection of the control values with the smallest norm.

All the computations in this work were performed on an Intel Core i5-8250U processor with 8 GB RAM. The code is publicly accessible at `https://github.com/mahendrasinghtomar/Invariance_Entropy_upper_bounds`. The computation time and memory requirement of `SCOTS` increase with the reduction of the grid parameter values and the increase in volume of the state and input sets. For `dtControl`, the time increases with the size of the controller file obtained from `SCOTS`. The part of the implementation which computes the deterministic graph $\bar{\mathcal{G}}_R$ from the directed one $\bar{\mathcal{G}}$ is written as a `MATLAB` `mex` function. The computation time of the `MATLAB` code increases with the increase in the number of nodes and the number of edges in the graph $\bar{\mathcal{G}}$.

For the maximum mean cycle weight computation in the Boost Graph Library, the emperical time complexity[3] is $O(\#E)$, where E is the set of edges in the graph. Thus, both upper bounds of the IFE can be computed in linear time. It is a major improvement over the time complexity of the method proposed in [66], see [66, Rem. 1]. As Example 6 and 8 show, the upper bounds of the IFE are close to that of the IED. Thus, Theorem 11 gives upper bounds for the IED that are easier to compute, takes less time and are close to the values obtained by using the algorithm of Section 4.2.

[3] `https://www.boost.org/doc/libs/1_64_0/libs/graph/doc/howard_cycle_ratio.html`

Chapter 5

Conclusions and Future Directions

5.1 Conclusion

In this work we studied invariance feedback entropy for uncertain control systems that characterizes the critical data rate to achieve invariance. We established a number of elementary properties including the relation between the invariance feedback entropies of two systems which are related under a feedback refinement relation. We also studied conditions for finiteness of the entropy. For the deterministic case, the invariance feedback entropy and the invariance entropy are shown to be equivalent. We also described the existence of an invariant cover with closed cover elements, such that its entropy is not more than the entropy of the initial invariant cover. For uncertain linear control systems, we derived lower bounds for the invariance feedback entropy and the data rate of any static, memoryless coder-controller. We showed that for certain linear control systems the lower bounds are tight.

For a large network of uncertain control subsystems, the time and memory requirements can be very high for the computation of an upper bound of IFE using the available method which utilizes a mean payoff game. The resource requirement increases with the state dimension. In Chapter 3, we advocated a different approach for the computation of an upper bound of IFE. In particular, we provided an upper bound which can be computed compositionally by working with much smaller subsystems. With an example, we demonstrated that the bound is tight. Additionally, we presented a relation between the IFEs of a system and a set Q, and the partition elements of Q. Further we showed that as the uncertainty in the system increases the IFE also increases. In Section 4.4, we provided an algorithmic technique to over-approximate the IFE by constructing finite abstractions of original systems. Unfortunately, the complexity of the proposed approach grows exponentially with respect to the state dimension of the overall interconnected system due to the discretization of the state set for constructing the overall finite abstractions. By combining the ideas of Sections 3.3 and 4.4, the complexity will grow linearly in the number of subsystems because one can construct finite abstractions of subsystems independently and then apply the proposed algorithmic technique in Section 4.4 to each subsystem separately.

In the last main chapter, we presented algorithms for the numerical computation of the IED and the IFE. In particular, given a subset Q of the state set, we first partition it. Then a controller, in the form of a lookup table that assigns a set of control values to each cell of the partition, is computed to enforce invariance of Q. After determinizing the controller, a weighted directed graph is constructed. For deterministic systems, the logarithm of the spectral radius of a transition matrix obtained from the graph gives an upper bound of the entropy. For uncertain systems, the maximum mean cycle weight of the graph upper bounds the entropy. With three deterministic examples, for which the exact value of the invariance entropy is known or can be estimated by other means, we demonstrated that the upper bound obtained by our algorithm is of the same order of magnitude as the actual value. Additionally, our algorithm provides a static coder-controller scheme corresponding to the obtained data-rate bound. Finally, we presented the computed upper bounds of invariance entropy for an uncertain linear control system as well.

5.2 Future Directions

- As noted in Section 2.5.2, any invariant cover $(\mathcal{A}, G)$ immediately provides a static or memoryless coder-controller scheme. Therefore, if $Q \subseteq X$ is controlled invariant, then it is sufficient to use a static coder-controller to achieve invariance in the closed loop. The lower bound in Theorem 7 holds for general (possibly dynamic) coder-controllers, while the lower bound in Theorem 8 holds for only static coder-controllers. A static coder-controller scheme that achieves the lower bound in Theorem 8 for scalar linear uncertain control systems is described in Section 2.5.3. The similarity of the lower bounds in Theorems 7 and 8 allows us to easily make a comparison. The data rate of the static coder-controller for the considered scalar linear uncertain control systems in Theorem 8 is no worse than 1 bit/unit time than the best possibly achievable data rate. The development of constructive coder-controller schemes, with data rate close to the invariance feedback entropy for uncertain nonlinear control systems and equal to the invariance feedback entropy for uncertain multidimensional linear control systems, needs further investigation. Since the implementation effort of dynamic coder-controller schemes is often too large to be useful in practice, it would be interesting to also develop static schemes and then analyze the performance gap between the two.

- In Theorem 7, a lower bound of the IFE, for uncertain linear control systems and compact subset Q of the state space, is provided in terms of the determinant of the system matrix and the Lebesgue measures of the set Q and the disturbance set. The paper [19] presents an upper and a lower bound of the IED, for compact sets and deterministic control systems with Euclidean state spaces, in terms of the derivative of the system dynamics with respect to the system state. A much more detailed study of upper and lower bounds in a more general setting is described in [38]. For uncertain nonlinear control systems, the problem of lower and upper bounds of the

IFE in closed-form expression, is still open. For nonlinear control systems, in contrast to the exact entropy, the upper and lower bounds are often computable. They will help to evaluate how close the data rate of a given coder-controller scheme is to the optimal one.

- For any invariant cover, we can always obtain an invariant partition. For example, given a subset Q of the state space and an invariant cover $(\mathcal{A}, G)$, one possbile way to obtain an invariant partition is through sequential set difference. If $\mathcal{A} = \{A_1, \ldots, A_n\}$, then define $\hat{A}_1 := A_1$, $\hat{A}_2 := A_2 \backslash A_1$, $\hat{A}_3 := A_3 \backslash \cup_{1 \leq i \leq 2} A_i$ and so on. The set $\hat{\mathcal{A}} := \{\hat{A}_1 \ldots \hat{A}_n\}$ will be a partition of the set Q, and the pair $(\hat{\mathcal{A}}, G)$ will be an invariant partition. Given an invariant partition of a set Q and a system Σ, we have a quantization of the set Q such that for every state in Q we have a control input available that will keep all possible successor states of Σ inside the set Q. Intuitively, it seems that invariant partitions should be sufficient to define the IFE, rather than the general invariant covers. If true, then it will simplify the definition of the IFE by simplifying its interpretation. This can be inferred from the discussion in Section 4.4. From an invariant partition, we can construct a weighted directed graph for the closed loop system, where the map G will serve as an invariant controller. The entropy of the invariant partition $h(\hat{A}, G)$ will then be given by the maximum mean cycle weight of the graph, that in fact measures the worst case average number of bits needed to encode the number of outgoing edges for each node in a cycle in the graph.

- Given any invariant cover $(\mathcal{A}, G)$ and a (τ, Q)-spanning set, one can construct a set of control sequences of length τ by taking the image of each cover element under the map G. An alternative definition of the IFE in terms of closed loop control sequences, if possible, may help to extend some of the properties of the IED as studied in [38] to the case of uncertain control systems. Such a definition may also help to describe IFE using only invariant partitions. This will need an appropriate new definition of the spanning sets.

- Given the notion of the IFE based on the set of finite length sequences in elements of some cover, it seems possible to analyze the minimal data rates for other specifications too, in particular, reachability. In contrast to invariance, reachability is concerned with the finite behavior of the closed loop. For two subsets of the state set T and Q with $T \subseteq Q$, a reachability specification may require search for a controller that all closed loop trajectories that start from any state in Q shall reach the target set T in some finite time, while never leaving the set Q. For this, the definition of (τ, Q)-spanning set may have to be appropriately modified. Similar to an invariant cover, we can consider a pair $(\mathcal{A}, G)$ where $\mathcal{A}$ is a finite cover of Q and $G : \mathcal{A} \to U$ is a map that associates with every cover element an input. For finite N, let $\mathcal{R} = \{\alpha_1, \ldots, \alpha_N\}$ be a set of sequences in the cover elements where $\alpha_i \in \mathcal{A}^{[0;n_i[}$, $n_i \in \mathbb{N}$, $1 \leq i \leq N$. Each element α_i in the set $\mathcal{R}$ has it's own length n_i. Now a spanning set can be defined to be a set $\mathcal{R}$ that satisfies the following:

1. the set of first elements of the sequences α_i cover Q, i.e., $Q \subseteq \cup_{i\in[1;N]}\alpha_i(0)$,
2. the last element of every sequence is a subset of T, i.e., $\alpha_i(n_i - 1) \subseteq T$,
3. for every sequence α_i and $t \in [0; n_i - 2]$, the image of the cover element $\alpha_i(t)$, with $G(\alpha_i(t))$ as the input, under the map F is covered by the set $\{\alpha_j(t + 1) \mid j \in J\}$ where J is the set of indices of the sequences α_j whose length n_j is larger than $t + 1$ and which share the same prefix with α_i, i.e., $\alpha_i|_{[0;t]} = \alpha_j|_{[0;t]}$ for all $j \in J$.

With such a spanning set, we can construct a coder-controller scheme so that the closed loop reaches the target set T from every initial state in Q in finite time, while never leaving Q. For example, at $t = 0$, for the initial state $x_0 \in Q$, the coder can transmit the index i of a sequence α_i whose initial element contains the initial state, i.e., $x_0 \in \alpha_i(0)$. Then the controller applies the input $u_0 = G(\alpha_i(0))$ to the system. For $t = 1$, from the definition of the spanning set, we have that for all successor states $x_1 \in F(x_0, u_0)$ there exists a sequence $\alpha_j \in \mathcal{R}$ with $\alpha_i(0) = \alpha_j(0)$ such that $x_1 \in \alpha_j(1)$. The coder, which has access to x_1, transmits the index j to the controller, which in turn applies $u_1 = G(\alpha_j(1))$ to the system. The scheme repeats until the end of a sequence is reached, at which point the system has reached the target set T, i.e., $x_{n_i-1} \in T$. The bit rate of the channel needs to be sufficiently large to support the error free transmission of the indices. For example, at time $t = 0$, the channel needs to support $\log_2 \#\{\alpha(0) \mid \alpha \in \mathcal{R}\}$ bits/unit time. At subsequent times $t > 0$, the channel needs to support $\log_2 \#J$ bits/unit time, where the index set J depends on the history, i.e., the sequence $\alpha_i|_{[0;t]}$ transmitted so far. Following this reasoning, we can associate with every spanning set the minimal bit rate that allows the successful transmission of the necessary indices. By taking the infimum over all spanning sets and reachability covers we obtain a state information measure. Now it remains to show that this information measure provides a tight lower bound on the data rate of any coder-controller scheme that solves the reachability problem. Once it is established that we have a reasonable information measure for reachability properties, then further analysis need to be pursued to derive upper and lower bounds.

Appendix A

Mean-Payoff Games

A *mean-payoff game* (MPG) [25] is played by two players, player 1 and player 2, on a finite, directed, edge-weighted graph $G = (V, E, w)$, where $V := V_1 \cup V_2$, $V_1 \cap V_2 = \varnothing$ with V_i, $i \in \{1, 2\}$ being two nonempty sets, $E \subseteq V \times V$, $w : E \to \mathbb{R}_{>0}$ and for every $v \in V$ there exists $v' \in V$ so that $(v, v') \in E$. The vertices V are also referred to as *positions* of the game. Starting from an initial position $v_0 \in V$, player 1 and player 2 take turns in picking the next position depending on the current position of the game: given $v_0 \in V_i$ for $i \in \{1, 2\}$ player i picks the successor vertex $v_1 \in V$ so that $(v_0, v_1) \in E$ and the play continues with v_1. The infinite sequence of edges $e = (e_k)_{k\in[0;\infty[}$ with $e_k = (v_k, v_{k+1}) \in E$ is called a play. Player 1 wants to minimize the payoff

$$\nu_{\min}(e_0 e_1 e_2 \ldots) := \limsup_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} w(e_j)$$

while player 2 wants to maximize the payoff

$$\nu_{\max}(e_0 e_1 e_2 \ldots) := \liminf_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} w(e_j).$$

A *positional strategy* for player i is a function $\sigma_i : V_i \to V$ so that $(v, \sigma_i(v)) \in E$ holds for all $v \in V_i$. By $\mathcal{P}_i(v, \sigma_i) \subseteq E^{[0;\infty[}$ we denote the set of all plays that start from the position v and wherein the player i follows the positional strategy σ_i.

As it turns out, there exist *optimal positional strategies* σ_i^* for each player i and a function $\nu : V \to \mathbb{R}$ so that player 1 is able to secure a payoff of $\nu(v)$ against any other strategy of player 2 and vice versa, i.e., for all sequences $\check{e} \in \mathcal{P}_1(v, \sigma_1^*)$ and $\hat{e} \in \mathcal{P}_2(v, \sigma_2^*)$ we have

$$\nu_{\min}(\check{e}) \leq \nu(v) \leq \nu_{\max}(\hat{e}). \tag{A.1}$$

We call ν the *value function* of the MPG (V, E, w), see e.g. [25] for details. Note that σ_1^* is optimal in the sense that any deviation of player 1 from σ_1^* can only lead to a larger or equal payoff than $\nu(v)$ considering the worst case with respect to possible strategies of player 2. Similarly, a deviation of player 2 from σ_2^* may only lead to suboptimal payoff.

We exploit the following fact, which follows from the proof of [25, Lemma. 1]: there exist constants c_1 and c_2, so that for every $\tau \in \mathbb{N}$, $v \in V$, $\check{e} \in \mathcal{P}_1(v, \sigma_1^*)$ and $\hat{e} \in \mathcal{P}_2(v, \sigma_2^*)$ we have

$$\frac{1}{\tau} \sum_{j=0}^{\tau-1} w(\check{e}_j) \leq \nu(v) + \frac{c_1}{\tau}, \tag{A.2}$$

$$\frac{1}{\tau} \sum_{j=0}^{\tau-1} w(\hat{e}_j) \geq \nu(v) + \frac{c_2}{\tau}. \tag{A.3}$$

Appendix B

Lemmas and Proofs

B.0.1

Proof of Lemma 1. We fix $\tau_1, \tau_2 \in \mathbb{N}$ and choose two minimal (τ_i, Q)-spanning sets $\mathcal{S}_i$, $i \in \{1,2\}$ in $(\mathcal{A}, G)$ so that $r_{\text{inv}}(\tau_i, \mathcal{A}, G, \Sigma) = \mathcal{N}(\mathcal{S}_i)$. Let $\mathcal{S}$ be the set of sequences $\alpha : [0; \tau_1 + \tau_2[\to \mathcal{A}$ given by $\alpha(t) := \alpha_1(t)$ for $t \in [0; \tau_1[$ and $\alpha(t) := \alpha_2(t - \tau_1)$ for $t \in [\tau_1; \tau_1 + \tau_2[$, where $\alpha_i \in \mathcal{S}_i$ for $i \in \{1,2\}$. We claim that $\mathcal{S}$ is $(\tau_1 + \tau_2, Q)$-spanning in $(\mathcal{A}, G)$. It is easy to see that $\{A \in \mathcal{A} \mid \exists_{\alpha \in \mathcal{S}} A = \alpha(0)\}$ covers Q, since $\{A \in \mathcal{A} \mid \exists_{\alpha \in \mathcal{S}_1} A = \alpha(0)\}$ covers Q. Let $t \in [0; \tau_1 + \tau_2[$ and $\alpha \in \mathcal{S}$. If $t \in [0; \tau_1 - 1[$, we immediately see that $F(\alpha(t), G(\alpha(t))) \subseteq \cup_{A' \in P_{\mathcal{S}}(\alpha|_{[0;t]})} A'$ since $\alpha_1 := \alpha|_{[0;\tau_1[} \in \mathcal{S}_1$ and $\mathcal{S}_1$ satisfies (2.5). Similarly, if $t \in [\tau_1; \tau_1 + \tau_2 - 1[$, we have $F(\alpha(t), G(\alpha(t))) \subseteq \cup_{A' \in P_{\mathcal{S}}(\alpha|_{[0;t]})} A'$ since $\alpha_2 := \alpha|_{[\tau_1;\tau_1+\tau_2[} \in \mathcal{S}_2$ and $\mathcal{S}_2$ satisfies (2.5). For $t = \tau_1 - 1$, we know that $P_{\mathcal{S}}(\alpha|_{[0;\tau_1[})$ equals $\{A \mid \exists_{\alpha_2 \in \mathcal{S}_2}\ \alpha_2(0) = A\}$ which covers Q and the inclusion $F(\alpha(t), G(\alpha(t))) \subseteq \cup_{A' \in P_{\mathcal{S}}(\alpha|_{[0;t]})} A'$ follows. Hence, $\mathcal{S}$ satisfies (2.5) and we see that $\mathcal{S}$ is (τ, Q)-spanning. Subsequently, for $i \in \{1,2\}$ and $\alpha \in \mathcal{S}_i$, $t \in [0; \tau_i - 1[$, let us use $P_{\mathcal{S}_i}(\alpha|_{[0;t]}) := \{A \in \mathcal{A} \mid \exists_{\hat{\alpha} \in \mathcal{S}_i}\ \hat{\alpha}|_{[0;t]} = \alpha|_{[0;t]} \wedge A = \hat{\alpha}(t+1)\}$. Then we have $P_{\mathcal{S}}(\alpha|_{[0;t]}) = P_{\mathcal{S}_1}(\alpha_1|_{[0;t]})$ with $\alpha_1 := \alpha|_{[0;\tau_1[}$ if $t \in [0; \tau_1 - 1[$ and $P_{\mathcal{S}}(\alpha|_{[0;t]}) = P_{\mathcal{S}_2}(\alpha_2|_{[0;t-\tau_1]})$ with $\alpha_2 := \alpha|_{[\tau_1;\tau_1+\tau_2[}$ if $t \in [\tau_1; \tau_1 + \tau_2 - 1[$, while for $t = \tau_1 - 1$ we have $P_{\mathcal{S}}(\alpha|_{[0;t]}) = P_{\mathcal{S}_2}(\alpha_2)$ with $\alpha_2 := \alpha|_{[\tau_1;\tau_1+\tau_2[}$ and $P_{\mathcal{S}}(\alpha) := P_{\mathcal{S}_1}(\alpha_1)$ with $\alpha_1 := \alpha|_{[0;\tau_1[}$. Therefore, $\mathcal{N}(\mathcal{S})$ is bounded by $\mathcal{N}(\mathcal{S}_1) \cdot \mathcal{N}(\mathcal{S}_2)$ and we have $r_{\text{inv}}(\tau_1 + \tau_2, \mathcal{A}, G, \Sigma) \leq r_{\text{inv}}(\tau_1, \mathcal{A}, G, \Sigma) \cdot r_{\text{inv}}(\tau_2, \mathcal{A}, G, \Sigma)$. Hence, $\tau \mapsto \log_2 r_{\text{inv}}(\tau, \mathcal{A}, G, \Sigma)$, $\mathbb{N} \to \mathbb{R}_{\geq 0}$ is a subadditive sequence of real numbers and (2.9) follows by Fekete's Lemma [21, Lem. 2.1]. $\square$

B.0.2

Proof of Lemma 2. For every $t \in [0; \tau[$, we define the set $\mathcal{S}_t := \{\alpha \in \mathcal{A}^{[0;t]} \mid \exists_{\alpha' \in \mathcal{S}}\ \alpha'|_{[0;t]} = \alpha\}$. By definition of $P_{\mathcal{S}}$, we have for all $\alpha \in \mathcal{S}$ the equality $P_{\mathcal{S}}(\alpha) = \mathcal{S}_0$, which shows the assertion for $\tau = 1$ since in this case we have $\mathcal{S}_0 = \mathcal{S}$. Subsequently, we assume $\tau > 1$. For $t \in [0; \tau[$ and $a_0 \ldots a_t \in \mathcal{S}_t$, we use $Y(a_0 \ldots a_t) := \{\alpha \in \mathcal{S} \mid a_0 \ldots a_t = \alpha|_{[0;t]}\}$ to denote the sequences in $\mathcal{S}$ whose initial part is restricted to $a_0 \ldots a_t$. For $t \in [0; \tau - 1[$ and

$a_0 \dots a_t \in \mathcal{S}_t$, we have

$$\begin{aligned}\#Y(a_0 \dots a_t) &= \sum_{a_{t+1} \in P_{\mathcal{S}}(a_0 \dots a_t)} \#Y(a_0 \dots a_{t+1}) \\ &\leq \#P_{\mathcal{S}}(a_0 \dots a_t) \max_{a_{t+1} \in P_{\mathcal{S}}(a_0 \dots a_t)} \#Y(a_0 \dots a_{t+1}).\end{aligned}$$

For every $a_0 \dots a_{\tau-2} \in \mathcal{S}_{\tau-2}$ we have $\#Y(a_0 \dots a_{\tau-2}) = \#P_{\mathcal{S}}(a_0 \dots a_{\tau-2})$ and we obtain a bound for $\#Y(a_0)$ by

$$\#P_{\mathcal{S}}(a_0) \max_{a_1 \in P_{\mathcal{S}}(a_0)} \#P_{\mathcal{S}}(a_0 a_1) \cdots \max_{a_{\tau-2} \in P_{\mathcal{S}}(a_0 \dots a_{\tau-3})} \#P_{\mathcal{S}}(a_0 \dots a_{\tau-2})$$

so that $\#Y(a_0) \leq \max_{\alpha \in \mathcal{S}} \prod_{t=0}^{\tau-2} \#P_{\mathcal{S}}(\alpha|_{[0;t]})$ holds for any $a_0 \in \mathcal{S}_0$. As $\cup_{a_0 \in \mathcal{S}_0} Y(a_0) = \mathcal{S}$ we observe $\#\mathcal{S} = \sum_{a_0 \in \mathcal{S}_0} \#Y(a_0) \leq \#\mathcal{S}_0 \max_{\alpha \in \mathcal{S}} \prod_{t=0}^{\tau-2} \#P_{\mathcal{S}}(\alpha|_{[0;t]})$. Since $\mathcal{S}_0 = P_{\mathcal{S}}(\alpha) = P_{\mathcal{S}}(\alpha|_{[0;\tau-1]})$, we obtain the desired inequality $\#\mathcal{S} \leq \max_{\alpha \in \mathcal{S}} \prod_{t=0}^{\tau-1} \#P_{\mathcal{S}}(\alpha|_{[0;t]})$. $\square$

B.0.3

Lemma 9. *Consider two systems $\Sigma_i = (X_i, U_i, F_i)$, $i \in \{1, 2\}$, a map $r : U_2 \to U_1$ and let Q_i be nonempty subsets of X_i. Suppose that $M : \wp(X_2) \to \wp(X_1)$ maps subsets of X_2 to subsets of X_1 and satisfies for every $u \in U_2$ and $A_2, A_2' \subseteq Q_2$ the following conditions*

1. *$M(Q_2) = Q_1$,*
2. *$A_2 \subseteq A_2' \implies M(A_2) \subseteq M(A_2')$,*
3. *$M(A_2 \cup A_2') = M(A_2) \cup M(A_2')$ and*
4. *$F_1(M(A_2), r(u)) \subseteq M(F_2(A_2, u))$.*

Let $(\mathcal{A}_2, G_2)$ be an invariant cover of Σ_2 and Q_2 and let

$$\mathcal{A}_1 := \{M(A) \mid A \in \mathcal{A}_2\}.$$

Then there exists a map $G_1^* : \mathcal{A}_1 \to U_1$ such that $(\mathcal{A}_1, G_1^*)$ is an invariant cover of Σ_1 and Q_1, and

$$h(\mathcal{A}_1, G_1^*) \leq h(\mathcal{A}_2, G_2). \tag{B.1}$$

Proof. Let us first point out that $\mathcal{A}_1$ is a cover of Q_1. We use 1) and 3) to derive

$$Q_1 = M(Q_2) = M(\cup_{A_2 \in \mathcal{A}_2} A_2) = \cup_{A_2 \in \mathcal{A}_2} M(A_2)$$

and we see that $\mathcal{A}_1$ is a cover of Q_1.

Consider the map $G_1 : \mathcal{A}_1 \rightrightarrows U_1$ defined by

$$G_1(A_1) := \{r(G_2(A_2)) \mid A_2 \in \mathcal{A}_2, M(A_2) = A_1\}$$

and let

$$\mathcal{V}(A_1) := \big\{(V,u) \mid V \subseteq \mathcal{A}_1, u \in G_1(A_1), F_1(A_1,u) \subseteq \cup_{A\in V} A\big\}.$$

We show that $\mathcal{V}(A_1)$ is nonempty for every $A_1 \in \mathcal{A}_1$. Let $A_1 \in \mathcal{A}_1$ and $u \in G_1(A_1)$. Then there exists $A_2 \in \mathcal{A}_2$ so that $A_1 = M(A_2)$ and $u = r(G_2(A_2))$. We use 4) to see that $F_1(A_1,u) \subseteq M(F_2(A_2, G_2(A_2)))$. Since $(\mathcal{A}_2, G_2)$ is an invariant cover we have $F_2(A_2, G_2(A_2)) \subseteq Q_2$ and it follows from 2) that $F_1(A_1,u) \subseteq M(Q_2)$. Since $\mathcal{A}_1$ covers $M(Q_2) = Q_1$, we see that $F_1(A_1,u) \subseteq \cup_{A\in\mathcal{A}_1} A$, which ensures that $\mathcal{V}(A_1) \neq \varnothing$.

Given Σ_1 and $(\mathcal{A}_1, G_1)$ we construct an MPG (V,E,w). Let $V_1 := \mathcal{A}_1$ and $V_2 := \cup_{A\in V_1}\mathcal{V}(A)$ then the *positions* of the MPG follow by $V = V_1 \cup V_2$. We introduce the *edges* $E := E_1 \cup E_2$ of the MPG by

$$\begin{aligned} E_1 &:= \{(v_1,v_2) \in V_1 \times V_2 \mid v_2 \in \mathcal{V}(v_1)\} \\ E_2 &:= \{(v_2,v_1) \in V_2 \times V_1 \mid v_1 \in V', v_2 = (V',u)\}. \end{aligned}$$

For $v \in V_2$ with $v = (V',u)$ by $\#v$ we refer to $\#V'$. The weights for $(v_1,v_2) \in E_1$ and $(v_2,v_1) \in E_2$ are given by $w(v_1,v_2) := \log_2 \#v_2$ and $w(v_2,v_1) := \log_2 \#v_2$. We refer to (V,E,w) as the MPG associated with Σ_1 and $(\mathcal{A}_1, G_1)$. Subsequently, we use σ_i^*, $i \in \{1,2\}$ to denote the optimal positional strategy for player i.

Fix $\tau \in \mathbb{N}$ and let $r_{\mathrm{inv}}(\tau, \mathcal{A}_2, G_2, \Sigma_2)$ denote the smallest possible expansion number associated with the invariant cover $(\mathcal{A}_2, G_2)$ at time τ. Let $\mathcal{S}_2$ be a (τ,Q)-spanning set in $(\mathcal{A}_2, G_2)$ such that $N(\mathcal{S}_2) = r_{\mathrm{inv}}(\tau, \mathcal{A}_2, G_2, \Sigma_2)$. We observe that $Q_1 = M(Q_2) = M(\cup_{\alpha\in\mathcal{S}_2}\alpha(0)) \subseteq \cup_{\alpha\in\mathcal{S}_2} M(\alpha(0))$. Thus $V_0 := \{M(\alpha(0)) \mid \alpha \in \mathcal{S}_2\}$ covers Q_1. We pick $\bar{v} \in V_0$ so that $\nu(\bar{v}) = \max_{v\in V_0} \nu(v)$. We show by induction over $t \in [0;\tau-1[$ the existence of an $\alpha \in \mathcal{S}_2$ and an $(v_k, v_{k+1})_{k\in[0;\infty[} \in \mathcal{P}_2(\bar{v}, \sigma_2^*)$ such that

$$v_{2k} = M(\alpha(k)) \text{ and } v_{2k+1} = (\{M(A) \mid A \in P_{\mathcal{S}_2}(\alpha|_{[0;k]})\}, u_k) \tag{B.2}$$

with $u_k = r(G_2(\alpha(k)))$ holds for all $k \in [0;t]$. Let $t = 0$, then there exists $\alpha \in \mathcal{S}_2$ with $M(\alpha(0)) = \bar{v}$. As $\mathcal{S}_2$ is (τ,Q)-spanning we have $F_2(\alpha(0), G_2(\alpha(0))) \subseteq \cup_{A\in P_{\mathcal{S}_2}(\alpha(0))} A$. For $u = G_2(\alpha(0))$ and $V' = \{M(A) \mid A \in P_{\mathcal{S}_2}(\alpha(0))\}$ we use 4), 2) and 3) to derive

$$F_1(\bar{v}, r(u)) \subseteq M(F_2(\alpha(0),u)) \subseteq M(\cup_{A\in P_{\mathcal{S}_2}(\alpha(0))} A) \subseteq \cup_{A\in V'} A. \tag{B.3}$$

Hence, for $v_1 := (V', r(u))$ we have $v_1 \in \mathcal{V}(\bar{v})$ and $(\bar{v}, v_1) \in E_1$ thus $e_0 = (\bar{v}, v_1)$ for some $e \in \mathcal{P}_2(\bar{v}, \sigma_2^*)$. Now suppose that the induction hypothesis (B.2) holds for $t \in [0;\tau-2[$, $\alpha \in \mathcal{S}_2$ and $(v_k, v_{k+1})_{k\in[0;\infty[} \in \mathcal{P}_2(\bar{v}, \sigma_2^*)$. Let $v_{2t+1} = (V',u)$. From the definition of E_2 we have $v_{2t+2} = \sigma_2^*(v_{2t+1}) \in V'$. Hence, together with (B.2) we see that there exists $A \in P_{\mathcal{S}_2}(\alpha|_{[0;t]})$ with $M(A) = v_{2t+2}$. Then we can pick $\hat{\alpha} \in \mathcal{S}_2$ such that $\hat{\alpha}|_{[0;t]} = \alpha|_{[0;t]}$ and $\hat{\alpha}(t+1) = A$. Further let $\hat{v}_{2t+3} = (V', r(u))$ with $u = G_2(\hat{\alpha}(t+1))$ and $V' = \{M(A) \mid A \in P_{\mathcal{S}_2}(\hat{\alpha}|_{[0;t+1]})\}$. Then, by using the same arguments used to derive (B.3) with v_{2t+2} and $P_{\mathcal{S}_2}(\hat{\alpha}|_{[0;t+1]})$ in place of $\bar{v}$ and $P_{\mathcal{S}_2}(\alpha(0))$ we obtain $F_1(v_{2t+2}, r(u)) \subseteq \cup_{A\in V'} A$. Thus $(v_{2t+2}, \hat{v}_{2t+3}) \in E_1$ and there exists $e \in \mathcal{P}_2(\bar{v}, \sigma_2^*)$ such that $e_k = (v_k, v_{k+1})$ for all $k \in [0;2t+1]$ and $e_{2t+2} = (v_{2t+2}, \hat{v}_{2t+3})$ which completes the induction. Let α and $e := (v_k, v_{k+1})_{k\in[0;\infty[}$ satisfy (B.2) for all $k \in [0;\tau-1[$, which implies $\#v_{2t+1} \leq \#P_{\mathcal{S}_2}(\alpha|_{[0;t]})$ for every $t \in [0;\tau-1[$. As $e \in \mathcal{P}_2(\bar{v},\sigma_2^*)$ from (A.3) we have

$$\begin{aligned}\nu(\bar{v}) + \frac{c_2}{2\tau} &\leq \frac{1}{2\tau}\sum_{j=0}^{2\tau-1} w(e_j)\\ &\leq \frac{1}{\tau}\sum_{t=0}^{\tau-1}\log_2 \#P_{\mathcal{S}_2}(\alpha|_{[0;t]}) + \frac{1}{\tau}\log_2 \#v_{2\tau-1} - \frac{1}{\tau}\log_2 \#V_0\\ &\leq \frac{1}{\tau}\log_2 r_{\mathrm{inv}}(\tau,\mathcal{A}_2,G_2,\Sigma_2) + \frac{\bar{c}_2}{\tau} \end{aligned} \tag{B.4}$$

where $\bar{c}_2 = \log_2 \max_{v\in V_2} \#v$.

We define $G_1^* : \mathcal{A}_1 \to U_1$ based on the value of $\sigma_1^*(A)$, i.e., $G_1^*(A) := u$ where $\sigma_1^*(A) = (V',u)$. For any $A_1 \in \mathcal{A}_1$ and $u = G_1^*(A_1)$ there exists $A_2 \in \mathcal{A}_2$ such that $A_1 = M(A_2)$ and $u = r(G_2(A_2))$. Hence, we use 4), 2) and 1) to derive $F_1(A_1, G_1^*(A_1)) \subseteq M(F_2(A_2, G_2(A_2))) \subseteq M(Q_2) = Q_1$. Thus $(\mathcal{A}_1, G_1^*)$ is an invariant cover of Σ_1 and Q_1.

Now consider the set $\mathcal{S}_1 \subseteq \mathcal{A}_1^{[0;\tau[}$ implicitly defined by $\alpha \in \mathcal{S}_1$ if and only if there exists $(v_k, v_{k+1})_{k\in[0;\infty[} \in \mathcal{P}_1(v_0,\sigma_1^*)$ with $v_0 \in V_0$ so that $\alpha(t) = v_{2t}$ holds for all $t \in [0;\tau[$. The set $\{\alpha(0) \mid \alpha \in \mathcal{S}_1\}$ equals V_0 therefore it covers Q_1. Consider any $\alpha \in \mathcal{S}_1$ and a play $(v_k, v_{k+1})_{k\in[0;\infty[} \in \mathcal{P}_1(v_0,\sigma_1^*)$ such that $\alpha(t) = v_{2t}$ holds for all $t \in [0;\tau[$. For $k \in [0;\tau-1[$ if $v_{2k+1} = (V',u)$ then from the definition of $\mathcal{S}_1$ we have $P_{S_1}(\alpha|_{[0;k]}) = V'$ and from the definition of the MPG we have that V' covers $F_1(v_{2k},u)$. Thus

$$\forall_{\alpha\in\mathcal{S}_1}\forall_{t\in[0;\tau-1[} F(\alpha(t), G_1^*(\alpha(t))) \subseteq \cup_{A'\in P_{S_1}(\alpha|_{[0;t]})} A'.$$

Therefore $\mathcal{S}_1$ is a (τ, Q)-spanning set in $(\mathcal{A}_1, G_1^*)$. Let $\alpha \in \mathcal{S}_1$ such that $\prod_{t=0}^{\tau-1} \#P_{S_1}(\alpha|_{[0;t]}) = N(\mathcal{S}_1)$. Pick an $e \in \mathcal{P}_1(\alpha(0),\sigma_1^*)$ such that $\alpha(t) = v_{2t}$ holds for all $t \in [0;\tau[$. Then from (A.2) we have

$$\begin{aligned}\nu(v_0) + \frac{c_1}{2\tau} &\geq \frac{1}{2\tau}\sum_{j=0}^{2\tau-1} w(e_j)\\ &= \frac{1}{\tau}\sum_{t=0}^{\tau-1}\log_2 \#P_{S_1}(\alpha|_{[0;t]}) + \frac{1}{\tau}\log_2 \#v_{2\tau-1} - \frac{1}{\tau}\log_2 \#V_0\\ &\geq \frac{1}{\tau}\log_2 r_{\mathrm{inv}}(\tau,\mathcal{A}_1,G_1^*,\Sigma_1) + \frac{\bar{c}_1}{\tau}\end{aligned} \tag{B.5}$$

where $\bar{c}_1 = -\log_2 \#V_1$.

From (B.4) and (B.5) we get

$$\begin{aligned}&\frac{1}{\tau}\log_2 r_{\mathrm{inv}}(\tau,\mathcal{A}_1,G_1^*,\Sigma_1) + \frac{\bar{c}_1}{\tau} \leq \nu(v_0) + \frac{c_1}{2\tau} \leq \nu(\bar{v}) + \frac{c_1}{2\tau}\\ &\leq \frac{1}{\tau}\log_2 r_{\mathrm{inv}}(\tau,\mathcal{A}_2,G_2,\Sigma_2) + \frac{c_1 + 2\bar{c}_2 - c_2}{2\tau}.\end{aligned}$$

Since this inequality holds for every $\tau \in \mathbb{N}$, we get

$$h(\mathcal{A}_1, G_1^*) \leq h(\mathcal{A}_2, G_2).$$

□

B.0.4

Lemma 10. *For $a, b \in \mathbb{R}$ and $T \in \mathbb{N}$, it holds*

$$a + \sum_{t=1}^{T} \frac{ba^t}{(a-b)^t} = \frac{a^{T+1}}{(a-b)^T}. \tag{B.6}$$

Proof. We show the identity by induction over T. For $T = 1$, equation (B.6) is easy to verify and subsequently, we assume that the equality holds for $T-1$ with $T \in \mathbb{N}_{\geq 2}$. Now we obtain

$$\begin{aligned} a + \sum_{t=1}^{T} \frac{ba^t}{(a-b)^t} &= \frac{ba^T}{(a-b)^T} + a + \sum_{t=1}^{T-1} \frac{ba^t}{(a-b)^t} \\ = \frac{ba^T}{(a-b)^T} + \frac{a^T}{(a-b)^{T-1}} &= \frac{ba^T + a^T(a-b)}{(a-b)^T} = \frac{a^{T+1}}{(a-b)^T} \end{aligned}$$

which completes the proof. $\square$

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