Non-Commutative Analysis on Quantum Spaces

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Zusammenfassung

In der Quantenfeldtheorie treten Singularitäten auf, die im Rahmen des Standardmodells der Teilchenphysik nur unter Verwendung von Renormalisierung behoben werden können. Ebenso findet sich im derzeitigen Standardmodell keine Möglichkeit, Gravitation zu quantisieren und damit die Basis für eine vereinheitlichte Feldtheorie zu schaffen. Die Effekte, die zu diesen Problemen mit dem Standardmodell führen, resultieren aus dem quantenmechanischen Verhalten im Bereich der Planck-Länge. Insbesondere der Einfluss quantenmechanischer Effekte auf die Struktur des Raumes legt nahe, auch den Raum selbst durch nichtkommutative Strukturen zu beschreiben. Diese Strukturen werden explizit dadurch beschrieben, dass die Funktionenalgebra des kommutativen Raumes durch eine Algebra nichtkommutativer Koordinaten ersetzt wird.

Eine besondere Rolle spielen in diesem Zusammenhang Quantenräume, bei denen nicht nur die Raumstruktur deformiert wird, sondern gleichzeitig auch die Symmetriegruppe des Raumes abgeändert wird, sodass die Symmetrien des Raumes nicht gebrochen werden.

Die vorliegende Arbeit beschäftigt sich damit, Elemente der nichtkommutativen Analysis in den kommutativen Raum zu übertragen. Im ersten Kapitel wird eine Möglichkeit präsentiert, wie das Produkt zweier Elemente der nichtkommutativen Algebra auf die kommutative Algebra übertragen werden kann. Unter der Verwendung spezieller Vektorfelder wird ein verallgemeinertes Sternprodukt in Form einer geschlossenen Formel angegeben werden, sodass störungsrechnerische Ansätze verallgemeinert werden können.

Auch die üblichen partiellen Ableitungen werden in der nichtkommutativen Algebra eingebettet. Die daraus resultierende Wirkung wird für den Fall des $q-$deformierten Euklidischen Raumes in n Dimensionen - dargestellt auf dem entsprechenden kommutativen Raum - explizit angegeben, ebenso wie die durch die Nichtkommutativität veränderte Leibnizregel.

Die nichtkommutative Algebra beinhaltet bis zu diesem Punkt noch keinen Operator, der einer Integration entsprechen würde. Um die Algebra jedoch nicht noch mehr erweitern zu müssen, bietet sich in diesem Fall noch eine weitere Möglichkeit an: es wird die durch die partiellen Ableitungen induzierte Gitterstruktur des Raumes ausgenutzt, um das unbestimmte Integral als Summe über die Funktionswerte an allen Gitterpunkten zu beschreiben. Für einige Quantenräume lassen sich damit gute Ergebnisse erzielen. Der anschliessend konstruierte Hilbertraum bietet dafür die nötige mathematische Basis und die Möglichkeit, darüberhinaus das zuvor definierte Integral als Spur eines speziellen Spurklasse-Operators darzustellen.

Contents

List of Figures

Chapter 1

Introduction

One of the main problems in Quantum Field Theory (QFT) is the way how to join QFT and General Relativity in a consistent way. It seems that for very small distances it is impossible to study the geometry of the space. Consider a cube in space with each edge of Planck's length or less. Measuring simultaneously the three coordinates x, y and z of a particle in the cube, the uncertainty relation gives big errors for the momenta and therefore big uncertainty of the energy ΔE. The smaller the cube the bigger is the energy required to measure its dimensions. Beyond certain energies in this way a black hole could be created. Therefore the observation of the geometry of the space gives a different geometry, which makes the observation useless.

A similar problem has already been known in quantum mechanics where one cannot measure some quantities simultaneously. In the language of operators this means that the two corresponding operators do not commute. But in quantum mechanics the operators corresponding to the three space coordinates commute, which leads to the black holes problem. One way to avoid this is to assume that the coordinate operators should not commute. Therefore coordinates cannot be measured simultaneously. This means that the commutative algebra generated by the operators $\hat{x}, \hat{y}$ and $\hat{z}$, which is isomorphic to the algebra of polynomials on $\mathbb{R}^3$, is replaced by a non-commutative algebra on a quantum space [35]. In order to obtain self-adjoint operators this algebra should be a $*$-algebra.

In general deforming just the space leads to a breaking of the space-time symmetry. In order to preserve the notion of a space-time symmetry one has to deform the symmetry group together with the space it acts on.

From the symmetry point of view, Lie groups are of particular interest in physics. Unfortunately they cannot be continuously deformed within their proper category, since they form a countable and hence discrete set. But since they are manifolds they can be naturally embedded in the category of algebras by the Gel'fand-Naimark map [17, 28], so that the additional group structure on the manifold side is translated into a Hopf algebra on the algebra side. Until then hardly any non-trivial example for Hopf algebras was known.

That changed with the discovery of quantum groups [14].

Quantum groups are deformations of usual groups, but they are constructed in the way that they are compatible with the structure of the underlying quantum space. Then it is possible to write down free theories on non-commutative spaces as theories on commutative spaces with deformed interactions, for example a new multiplication called $\star-$product. Some physical relevant examples are deformations of the rotation group [24], the Lorentz group [8, 30, 33, 34] and the Poincaré group [31].

Another property of these deformed theories is that deformations discretise the spectra of space-time observables [16]. Therefore it looks like the deformation puts physics on a space-time lattice, which leads to the hope that field theories might be regularised by themselves.

The aim of this thesis is to work out some tools for non-commutative analysis in quantum spaces. In the first chapter, based upon a close collaboration with A. Sykora [18], we construct a generalised Moyal-Weyl $\star-$product by using Hamiltonian vector fields instead of derivatives. This ansatz leads to a closed formula for the $\star-$product and gives an easy procedure for the construction of a wide class of $\star-$products.
Subsequent to the earlier work on representation of operators on quantum spaces [2] in Chapter 2 we give the deformed action of derivatives on $SO_q(n)$ and additionally their Leibniz rules in a closed formula. The results are represented on the commutative space via the $\star-$product also calculated here.
Following the requirements of physics in the third chapter we construct an integral on quantum spaces. We follow the classical construction of integrals by Riemann and use the lattice structure induced by the non-commutative action of the derivatives. For most of the treated examples this leads to a straightforward summation formula, also easily applicable to computerised simulations.
The question arising from the previous chapter is whether and for which functions the integral converges. In Chapter 4 we construct for this a Hilbert space and show that on this space the integral converges. Furthermore we can express it in terms of a trace via a trace-class operator.

This thesis gives a few tools for the future work with quantum groups on quantum spaces. The $\star-$product constructed in the first chapter might be interesting especially in those cases, where until now just a perturbative $\star-$product was available. This enables deeper analysis of such theories, the group around Julius Wess actually works on.
On one hand the results for the $q-$deformed $n-$dimensional Euclidean space are interesting in mathematics for the analysis of the structure of this space. On the other hand they can be used to obtain results in a $q-$deformed $n-$dimensional Minkowski space by applying a Wick rotation, since we found no direct way to achieve the integral in Chapter 3.
The integrals found for the other $q-$deformed Euclidean spaces make it possible to calculate integrals of functions and therefore also for differential equations explicitly. This

simplifies the analysis of equations and functions on these spaces.
Constructing a Hilbert space for which the integrals converge gives a clear mathematical background for the integration defined in Chapter 3, on which further work concerning the properties of the integral can be done.

Chapter 2

$\star-$products

2.1 Introduction to $\star-$products

In classical mechanics as well as in quantum mechanics the aim of physicists is to study the time evolution of a system. In classical mechanics observables are smooth functions $C^\infty(M)$ on a Poisson manifold M. They form a commutative algebra. In quantum mechanics the set of possible states forms a Hilbert space H with self adjoint operators as physical observables. They form a non-commutative C^*-algebra.
There are various methods to connect a Poisson manifold with a Hilbert space formulation (see [1, 7]). One of them is via the so-called deformation quantisation, introduced in [4, 5]. Instead of constructing a Hilbert space first, one just works on the algebra. Since the product of classical observables is commutative and the one of operators of a quantum system is not, the idea is to deform the commutative product to a non-commutative, associative product. This deformed product has to carry over the necessary relations between classical and quantum system, so one of its properties is that it contains the classical limit for vanishing deformation parameter.

We start with a Poisson algebra $A = C^\infty(M)$ of smooth functions on M. To deform the point wise product on A we define a family of products depending on a deformation parameter h :

$$\begin{aligned} \times_h : A \times A &\rightarrow A \\ (f, g) &\rightarrow f \times_h g \end{aligned}$$

where $\times_0$ would be the undeformed commutative product. Demanding the new product to depend smoothly on the deformation parameter, we express $\times_h$ in terms of formal power series in h. But then $f \times_h g$ is no longer in A but in $A[[h]]$[1]:

$$\times_h : A \times A \rightarrow A[[h]].$$

[1] $A[[h]]$ here means the set of all formal power series in h with coefficients in the algebra A.

By linearity this product can be extended to a product of elements in $A[[h]]$:

$$\star : A[[h]] \times A[[h]] \to A[[h]]$$

Hence we can define $\star$-products:

Definition 1 *A* deformed product *or* $\star$−product *in A is an associative, h−adic continuous, $\mathbb{C}$ bilinear product*

$$\star : A[[h]] \times A[[h]] \to A[[h]]$$

that takes the particular value on A :

$$\begin{aligned} f \star g &= \sum_{n=0}^{\infty} B_n(f,g)h^n \\ f \star g|_{h=0} &= fg \end{aligned}$$

where $B_n : A \times A \to A$ are bi-differential operators.

The condition of associativity on A which extends to $A[[h]]$

$$f \star (g \star h) = (f \star g) \star h$$

then goes to

$$\sum_{m+k=n} B_m(f, B_k(g,h)) = \sum_{m+k=n} B_m(B_k(f,g),h). \tag{2.1}$$

That leads to supplementary conditions for the differential operators B_n. For $n = 1$ and $n = 2$ we obtain

$$\begin{aligned} B_1(fg,h) + B_1(f,g)h &= B_1(f,gh) + f\,B_1(g,h) \qquad (2.2) \\ B_2(fg,h) + B_1(B_1(f,g),h) + B_2(f,g)h &= B_2(f,gh) + B_1(f,B_1(g,h)) + f\,B_2(g,h). \end{aligned}$$

The first of these two equations together with its cyclic permutation of f,g and h leads on one hand to the Leibniz rule for the antisymmetric part $B_1^-(f,g) = \frac{1}{2}(B_1(f,g) - B_1(g,f))$ of B_1

$$B_1^-(f,gh) = B_1^-(f,g)h + gB_1^-(f,h) \tag{2.3}$$

and on the other to the Jacobi identity, so B_1^- can be identified with a Poisson bracket $\{.\,,.\}$ on M.

Definition 2 *A* quantisation *of a Poisson manifold M is a $\star$−product on A in the sense of Def. 1 such that $B_1^- = \{.\,,.\}$.*

This definition is based on an idea of Paul Dirac ([13]): He suggested that, in order to quantise, one should look for an associative, non-commutative product $*$ on A and define the commutator $\{f,g\} := -\frac{\hbar}{2}(f * g - g * f)$.

An example for $\star-$products is the so-called Moyal-Weyl product: let $M = \mathbb{R}^n$ with a Poisson structure $\Pi = \Pi^{ij}\partial_i \wedge \partial_j$, $\Pi^{ij} = -\Pi^{ji} = const. \in \mathbb{R}$ and for the deformation parameter we choose $h = \frac{i\hbar}{2}$ to reproduce quantum mechanics. Then the Moyal-Weyl product is defined as formal power series in Π :

$$\begin{aligned} f \star_M g &:= e^{\frac{i\hbar}{2}\Pi}(f,g) \\ &= \sum_{n=0}^{\infty} \frac{(i\hbar)^n}{2^n n!}(\Pi^{i_1 j_1} \cdots \Pi^{i_n j_n})(\partial_{i_1} \ldots \partial_{i_n} f)(\partial_{j_1} \ldots \partial_{j_n} g) \end{aligned} \tag{2.4}$$

2.2 Gauge group action

As a generalisation of coordinate transformations we now can take $\mathbb{C}[[h]]-$linear maps D, which naturally form a gauge group acting on A[2]:

$$\begin{aligned} D: A &\rightarrow A[[h]] \\ D(f) &= \sum_{n=0}^{\infty} D_n(f)h^n \end{aligned}$$

where D_n are linear differential operators. D is invertible if and only if D_0 is invertible, so we postulate $D_0 = 1$ and obtain its inverse E

$$\begin{aligned} E_0 &= 1 \\ E_n &= -\sum_{m=0}^{n-1} E_m D_{n-m} \quad \text{for}\, n > 0. \end{aligned}$$

If we now take a product $\star$ and a gauge transformation D, we can think of as formal coordinate transformation, we obtain a new product in the new coordinates $f \star' g = D(E(f) \star E(g))$ (see also [23])

$$\begin{array}{ccc} A[[h]] \times A[[h]] & \xrightarrow{\star} & A[[h]] \\ \downarrow{\scriptstyle D\times D} & & \downarrow{\scriptstyle D} \\ A[[h]] \times A[[h]] & \xrightarrow{\star'} & A[[h]] \end{array}$$

As one can immediately see the new product $\star'$ is also associative and a $\star-$product:

$$\begin{aligned} f \star' g &= \sum_{n=0}^{\infty} C_n(f,g)h^n \\ C_n(f,g) &= \sum_{m+k+l+j=n} D_m B_k(E_l f, E_j g) \end{aligned} \tag{2.5}$$

[2]and because of the $\mathbb{C}[[h]]-$linearity also on $A[[h]]$

Now apply the gauge transformation D on the $\star$−product. For (2.5) we obtain at first order ($n = 1$)

$$C_1(f,g) = B_1(f,g) - fD_1(g) + D_1(fg) - D_1(f)g. \tag{2.6}$$

We see that the gauge transformation only affects the symmetric part of B_1, so we always can find a gauge transformation that makes the symmetric part vanish. Hence we can treat (up to gauge equivalence) B_1 to be anti-symmetric and gauge invariant in all calculations. The Leibniz rule for B_1 up to first order in h is then

$$B_1(fg,h) = fB_1(g,h) + gB_1(f,h) \tag{2.7}$$

instead of (2.3). This means that B_1 is a derivation with respect to both functions f and g.

2.3 Further properties

Since the $\star$-product is associative and B_1 anti-symmetric, the commutator

$$[f \stackrel{\star}{,} g] = f \star g - g \star f = 2hB_1(f,g) + \mathcal{O}(h^2)$$

satisfies with (2.7) the Leibniz rule

$$[f \star g \stackrel{\star}{,} h] = f \star [g \stackrel{\star}{,} h] + [f \stackrel{\star}{,} h] \star g \tag{2.8}$$

up to all orders:

$$\begin{aligned} [f \star g \stackrel{\star}{,} h] &= (f \star g) \star h - h \star (f \star g) \\ &= f \star (g \star h) - f \star (h \star g) + (f \star h) \star g - (h \star f) \star g \\ &= f \star [g \stackrel{\star}{,} h] + [f \stackrel{\star}{,} h] \star g \end{aligned}$$

Additionally the Jacobi-identity is fulfilled

$$[f \stackrel{\star}{,} [g \stackrel{\star}{,} h]] + [h \stackrel{\star}{,} [f \stackrel{\star}{,} g]] + [g \stackrel{\star}{,} [h \stackrel{\star}{,} f]] = 0.$$

Up to second order this implies that B_1 is a Poisson structure

$$\{f, \{g,h\}\} + \{h, \{f,g\}\} + \{g, \{h,f\}\} = 0$$

where $\{f,g\} = B_1(f,g)$. So after a certain linear transformation we always can write on a local patch of the manifold

$$f \star g = fg + \frac{ih}{2}\Pi^{ij}\partial_i f\, \partial_j g + \cdots$$

with

$$\Pi^{il}\partial_l\Pi^{jk} + \Pi^{kl}\partial_l\Pi^{ij} + \Pi^{jl}\partial_l\Pi^{ki} = 0. \tag{2.9}$$

We have seen that $\star$-products up to second order are classified by Poisson structures on the manifold. On the other hand, if there is a manifold with a Poisson structure $\{.\,,.\}$, it is possible to construct $\star$-products with

$$f \star g = fg + \frac{i\hbar}{2}\{f,g\} + \cdots.$$

2.4 Algebras and ⋆−products

Suppose we are taking $\mathbb{R}^N$ as the manifold and parametrise it by N coordinates x^i and the antisymmetric matrix $\Pi^{ij} = \theta^{ij} = const.\,(i,j = 1,\ldots,N)$ fulfils the Poisson condition (2.9). For this Poisson structure we can use (2.4) to write down a ⋆-product

$$f \star g = \sum_{n=0}^{\infty} \frac{(i\hbar)^n}{2^n n!}\,\theta^{i_1 j_1}\cdots\theta^{i_n j_n}(\partial_{i_1}\cdots\partial_{i_n} f)\,(\partial_{j_1}\cdots\partial_{j_n} g) \tag{2.10}$$

where f and g are functions on $\mathbb{R}^N$. This special case is called the Moyal-Weyl ⋆-product. Since $[x^i \stackrel{\star}{,} x^j] = i\hbar\theta^{ij}$, the space of functions on $\mathbb{R}^N$ together with the ⋆-product forms a representation of the algebra

$$\mathcal{A} = \frac{\mathbb{C} < \hat{x}^1, \cdots, \hat{x}^N >}{\mathcal{R}},$$

where $\mathcal{R}$ is the ideal formed by the relation $[\hat{x}^i, \hat{x}^j] = i\hbar\theta^{ij}$. In the following we will see that we can do the same with other relation-defined algebras, if we fix an ordering of the coordinates. Possibilities are Lie algebra structures like

$$[\hat{x}^i, \hat{x}^j] = i\hbar C^{ij}{}_k \hat{x}^k, \qquad \hbar, C^{ij}{}_k \in \mathbb{C}$$

and quantum space structures as introduced in [9, 34, 39, 41]

$$\hat{x}^i\hat{x}^j = qR^{ij}{}_{kl}\hat{x}^k\hat{x}^l, \qquad q = e^{\hbar}, R^{ij}{}_{kl} \in \mathbb{C}.$$

Instead of considering these special relations we discuss in the following a more general case. We assume that the algebra $\mathcal{A}$ is generated by N elements $\hat{x}^i$ and relations

$$[\hat{x}^i, \hat{x}^j] = \widetilde{\hat{c}}^{ij}(\hat{x}) = i\hbar\hat{c}^{ij}(\hat{x})$$

where we assume that the rhs. of this formula contains a parameter $\hbar$ and goes to 0, if this parameter vanishes. Mathematically more correct we have to use a $\hbar$-adic expanded algebra

$$\mathcal{A} = \frac{\mathbb{C} < \hat{x}^1, \cdots, \hat{x}^N > [[\hbar]]}{([\hat{x}^i, \hat{x}^j] - i\hbar\hat{c}^{ij}(\hat{x}))} \tag{2.11}$$

where it is possible to work with formal power series in $\hbar$. Note that this kind of algebras all have the Poincaré-Birkhoff-Witt property since a reordering of two $\hat{x}^i$ never affects the polynomials of same order in $\hbar$. This means that the dimension of a subspace spanned by monomials of a fixed degree in $\mathcal{A}$ is the same as the dimension of the subspace spanned by monomials in commutative variables of the same degree. This makes it possible to establish a vector space isomorphism between the non-commutative algebra $\mathcal{A}$ and the associate commutative algebra, if only one chooses a basis in the algebra $\mathcal{A}$, e.g. the lexicographically (normal) ordered monomials.

2.4.1 Algebra generator orderings

The first $\star$−product was a result of Weyl's quantisation procedure (see [42]). Assuming that $f(q_i, p_j)$ are functions of a classical phase space C an operator is introduced by

$$\hat{f} = \Omega(f) := \int \mathrm{d}^n\,\xi\,\mathrm{d}^n\,\eta\,\tilde{f}(\xi,\eta)e^{\frac{i}{\hbar}(\hat{q}\cdot\xi+\hat{p}\cdot\eta)}, \tag{2.12}$$

where $\tilde{f}$ is the inverse Fourier transformed of f and the operators $\hat{q}$, $\hat{p}$ fulfil the canonical commutation relations $[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$. It is possible to give the inverse of the operation (2.12). Since the inverse operation is known one can now pull back the product of two operators to a ordinary product on the phase space C

$$\begin{array}{ccc} C\otimes C & \xrightarrow{\Omega\otimes\Omega} & \mathcal{A}\otimes\mathcal{A} \\ \star\downarrow & & \downarrow\cdot \\ C & \xrightarrow{\Omega} & \mathcal{A} \end{array}$$

and obtain in this way

$$f\star g = \Omega^{-1}(\Omega(f)\cdot\Omega(g)),$$

which is the Moyal product on classical phase space. This quantisation procedure can also be extended to polynomials.

The Fourier transform of a function f is

$$f(p) = \int d^n x\, f(x)e^{ip_i x^i}.$$

For a monomial we formally obtain in $\mathbb{R}^N$

$$\int \mathrm{d}^n x\, x^1\cdots x^m\, e^{ip_i x^i} = (-i\partial_{p_{i_1}})\cdots(-i\partial_{p_{i_m}})\delta(p).$$

The Weyl operator associated to the function f is defined by

$$W(f) := \int \frac{d^n p}{(2\pi)^n} f(p)e^{-ip_i\hat{x}^i} \tag{2.13}$$

(see e.g. [26]). Hence we obtain for a monomial

$$W(x^{i_1}\cdots x^{i_m}) = \frac{1}{m!}\partial_{p_{i_1}}\cdots\partial_{p_{i_m}}(p_i\hat{x}^i)^m$$

and therefore the Weyl operator really maps monomials to the corresponding symmetrical ordered polynomial in the algebra, e.g. for three generators

$$W(x^i x^j x^k) = \frac{1}{3!}(\hat{x}^i\hat{x}^j\hat{x}^k + \hat{x}^i\hat{x}^k\hat{x}^j + \hat{x}^k\hat{x}^i\hat{x}^j + \hat{x}^j\hat{x}^i\hat{x}^k + \hat{x}^j\hat{x}^k\hat{x}^i + \hat{x}^k\hat{x}^j\hat{x}^i). \tag{2.14}$$

A similar calculation for normal ordering leads to

$$N(f) = \int \frac{d^n p}{(2\pi)^n} f(p)e^{-ip_1\hat{x}^1}\cdots e^{-ip_n\hat{x}^n}.$$

2.4.2 Equivalence of ⋆−products

There is an isomorphism (modulo $\hbar$) between the polynomial algebra $\mathcal{A} := \mathbb{C}^N$ and the quantum space $\mathcal{A}_\hbar := \frac{\mathbb{C}<\hat{x}^1,\ldots,\hat{x}^N>[[\hbar]]}{\mathcal{R}}$ with $\mathcal{R}$ being the defining relation of the quantum space. The isomorphism μ is defined on the generators via $\mu(x^i) = \hat{x}^i$. Because it is an isomorphism of vector spaces, we can expand it to formal power series yielding a $\mathbb{C}[[\hbar]]-$linear isomorphism of $\hbar-$adic vector spaces $\Omega : \mathcal{A}[[\hbar]] \to \mathcal{A}_\hbar$ which we call an ordering prescription. It is not unique. Two popular ordering prescriptions we have already given above: the normal ordering and the symmetric or Weyl-ordering.
Using the ordering prescriptions, we can transfer the non-commutative multiplication map $m_\hbar$ of $\mathcal{A}_\hbar$ to $\mathcal{A}[[\hbar]]$ by requiring

$$\begin{array}{ccc} \mathcal{A}[[\hbar]]\hat{\otimes}\mathcal{A}[[\hbar]] & \xrightarrow{\Omega\otimes\Omega} & \mathcal{A}_\hbar\hat{\otimes}\mathcal{A}_\hbar \\ \big\downarrow m_\Omega & & \big\downarrow m_\hbar \\ \mathcal{A}[[\hbar]] & \xrightarrow{\Omega} & \mathcal{A}_\hbar \end{array}$$

to be a commutative diagram, where $\hat{\otimes}$ denotes the topological tensor product. The transfered multiplication map

$$m_\Omega := \Omega^{-1} \circ m_\hbar \circ (\Omega \otimes \Omega)$$

is our ⋆−product. By this construction we easily can see that another ordering prescription Ω' yields another multiplication map $m_\Omega \neq m_{\Omega'}$, but the algebras are isomorphic: $(\mathcal{A}[[\hbar]], m_\Omega) \simeq (\mathcal{A}[[\hbar]], m_{\Omega'})$, with $\Omega^{-1} \circ \Omega'$ being an isomorphism.
In less mathematical terms the ⋆-product reads as

$$\Omega(f \star_\Omega g) = \Omega(f)\Omega(g) \tag{2.15}$$

for two functions f and g in $\mathcal{A}[[\hbar]]$. If we had used another ordering description Ω', we would obtain

$$f \star_{\Omega'} g = D^{-1}(D(f) \star_\Omega D(g)) \tag{2.16}$$

with $D = \Omega^{-1}\Omega'$. The choice of different ordering prescriptions is equivalent to taking a different gauge of ⋆-product.

2.5 Formulation of ⋆−products with commuting vector fields

The ⋆−products in [6, 36] are given up to second order, since no closed formula could be found. However, it is possible to generalise the results to a closed formula. For this we replace the partial derivatives in the Moyal-Weyl formula by commuting vector fields, since they have the same algebraic properties. Then the associativity of this ⋆−product is proved and a formalism of how to obtain the desired algebra relations is found. In the next section we show the way our ⋆−product works for some illustrative two-dimensional examples.

2.5.1 Definitions and proof of associativity

Let $X_j = X^i(x_j)\frac{\partial}{\partial(x_j)^i}$ be the components of a vector field X (not necessarily complete) with different coordinates $x_j = ((x_j)^1, \ldots, (x_j)^n)$ and $f_j = f(x_j)$ formal power series in the coordinates. We then introduce the following notation

$$X_j f_j = X^i(x_j)\frac{\partial}{\partial(x_j)^i} f(x_j),$$

where $(x_j)^i$ means the i-th component of the coordinate vector x_j. With this we can write down the Leibniz rule in a intuitive way:

$$\begin{aligned} X_1 f_1 g_1 &= (X_2 + X_3) f_2 g_3|_{2,3\to 1} \\ X_1^l f_1 g_1 &= (X_2 + X_3)^l f_2 g_3\big|_{2,3\to 1} \\ P(X_1) f_1 g_1 &= P(X_2 + X_3) f_2 g_3|_{2,3\to 1} \end{aligned}$$

where P is a polynomial in X. The last equation we also could write in the form

$$P(X_1)\left(f_2 g_3|_{2,3\to 1}\right) = P(X_2 + X_3) f_2 g_3|_{2,3\to 1}\,. \tag{2.17}$$

This we expand for n commuting vector fields $X_a = X_a^i \partial_i$, i.e. $[X_a, X_b] = 0$. Further let σ^{ab} be a constant matrix. Then we can define a $\star$-product via

$$(f \star g)|_1 := e^{\sigma^{ab} X_{a2} X_{b3}} f_2 g_3\Big|_{2,3\to 1}\,. \tag{2.18}$$

This $\star$-product is associative since

$$\begin{aligned} (f \star (g \star h))|_1 &= e^{\sigma^{ab} X_{a2} X_{b3}} f_2 \left(e^{\sigma^{cd} X_{c4} X_{d5}} g_4 h_5 \Big|_{4,5\to 3} \right)\Bigg|_{2,3\to 1} \\ &= e^{\sigma^{ab} X_{a2}(X_{b4}+X_{b5})} f_2 e^{\sigma^{cd} X_{c4} X_{d5}} g_4 h_5 \Big|_{4,5\to 3;2,3\to 1} \\ &= e^{\sigma^{ab} X_{a1} X_{b2} + \sigma^{ab} X_{a1} X_{b3}} e^{\sigma^{cd} X_{c2} X_{d3}} f_1 g_2 h_3 \Big|_{2,3\to 1} \end{aligned}$$

and

$$\begin{aligned} ((f \star g) \star h)|_1 &= e_2^{\sigma^{ab} X_{a1} X_{b2}} \left(e^{\sigma^{cd} X_{c3} X_{d4}} f_3 g_4 \Big|_{3,4\to 1} \right) h_2 \Bigg|_{2\to 1} \\ &= e^{\sigma^{ab}(X_{a3}+X_{a4}) X_{b2}} e^{\sigma^{cd} X_{c3} X_{d4}} f_3 g_4 h_2 \Big|_{3,4,2\to 1} \\ &= e^{\sigma^{ab} X_{a1} X_{b3} + \sigma^{ab} X_{a2} X_{b3}} e^{\sigma^{cd} X_{c1} X_{d2}} f_1 g_2 h_3 \Big|_{2,3\to 1} \end{aligned}$$

where we used the relation (2.17). The two expressions are equal since the vector fields commute.

For antisymmetric σ one obtains for the ⋆-commutator

$$\begin{aligned} [f \overset{\star}{,} g] &= \left(e^{\sigma^{ab} X_{a1} X_{b2}} - e^{-\sigma^{ab} X_{a1} X_{b2}}\right) f_1 g_2\Big|_{2\to 1} \\ &= 2\sinh(\sigma^{ab} X_{a1} X_{b2}) f_1 g_2\big|_{2\to 1}\,. \end{aligned}$$

Further on we take two vector fields $X_1 = X$ and $X_2 = Y$.

With $\sigma^{12} = h, \sigma^{21} = 0$ we get for an **asymmetric** ⋆−product

$$f \star g = \sum_{n=0}^{\infty} \frac{h^n}{n!} (X^n f)\,(Y^n g), \tag{2.19}$$

while for $\sigma^{12} = \frac{h}{2}, \sigma^{21} = -\frac{h}{2}$ we have an **antisymmetric** ⋆−product

$$f \star g = \sum_{n=0}^{\infty} \frac{h^n}{2^n n!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (X^{n-i} Y^i f)\,(X^i Y^{n-i} g). \tag{2.20}$$

2.5.2 Linear transformations

In (2.16) we have seen that we can transfer one ⋆−product into another by a linear transformation on the space of functions for two different orderings in the sense of the previous section.
Let D be such an invertible operator and let its expansion in derivatives start with $\mathcal{O}(0) = 1$. Additionally we assume that D is of the form

$$D = e^{\tau(X_a)}, D^{-1} = e^{-\tau(X_a)} \tag{2.21}$$

where τ is a polynomial in the vector fields X_a. For the ⋆−product (2.18) we then obtain together with (2.21)

$$\begin{aligned} f \star' g &= D^{-1}(D(f) \star D(g)) \\ &= e^{-\tau(X_{a1})} \left(e^{\sigma^{ab} X_{a2} X_{b3}} e^{\tau(X_{a2})} f_2 e^{\tau(X_{a3})} g_3 \Big|_{2,3\to 1} \right) \\ &= e^{-\tau(X_{a2}+X_{a3}) + \sigma^{ab} X_{a2} X_{b3} + \tau(X_{a2}) + \tau(X_{a3})} f_2 g_3 \Big|_{2,3\to 1}\,. \end{aligned}$$

For τ only quadratic in the X_a (τ_2^{ab} is symmetric, since the vector fields commute)

$$\tau = \tau_1^a X_a + \tfrac{1}{2} \tau_2^{ab} X_a X_b$$

we have

$$\tau(X_{a1}) + \tau(X_{a2}) - \tau(X_{a1} + X_{a2}) = -\tau_2^{ab} X_{a1} X_{b2}$$

and the new ⋆-product becomes

$$f \star' g = e^{(\sigma^{ab} - \tau_2^{ab}) X_{a1} X_{b2}} f_1 g_2 \Big|_{2\to 1}\,. \tag{2.22}$$

Therefore the antisymmetric ⋆-product (2.20) and the asymmetric ⋆-product (2.19) are related by a linear transformation in function space:

$$\begin{aligned} f \star' f &= e^{\sigma'^{ab} X_{a1} X_{b2}} f_1 g_2 \Big|_{2\to 1} \\ &= e^{(\sigma^{ab} - \tau_2^{ab}) X_{a1} X_{b2}} f_1 g_2 \Big|_{2\to 1} \end{aligned}$$

$$\Longrightarrow \sigma'^{ab} = \sigma^{ab} - \tau_2^{ab}.$$

With σ' being the antisymmetric matrix from above and σ the asymmetric one we obtain explicitly $\tau_2^{12} = \tau_2^{21} = \frac{h}{2}$, so we can write for the relation between the two ⋆−products

$$f \star' g = e^{-\tau_2^{ab} X_{a1} X_{b2}} f \star g.$$

2.5.3 Reconstruction of algebras

The ⋆−commutator $[. \overset{\star}{,} .]$ of a ⋆−product is a Poisson tensor up to first order (see section 2.3), so we can calculate the Poisson tensor of a given algebra quite easily from the ⋆−commutator relations. The algebra we want to reconstruct reads in terms of generators as follows

$$[\hat{x}^i, \hat{x}^j] = hW(c^{ij}(x))$$

with W mapping the commutative coordinates to the algebra elements (see also equation (2.14)). This directly leads to the Poisson structure of this algebra:

$$\{x^i, x^j\} = c^{ij}(x).$$

On the other hand for the ⋆-commutator of a general ⋆-product it holds that if it is expanded up to first order in the following way

$$\begin{aligned} [f \overset{\star}{,} g] &= h\{f, g\} + \mathcal{O}(h^2) \\ &= h\Pi(f, g) + \mathcal{O}(h^2), \end{aligned}$$

where Π is the Poisson-bivector of the Poisson structure, one directly obtains the Poisson structure of the algebra. For the special case for the ⋆-products (2.18) it is given by

$$\Pi = \sigma^{ab} X_a \wedge X_b.$$

So if we are able to write a general Poisson bivector in this special form, we can reconstruct the algebra relations under use of the ⋆-products (2.18).

Let f now be a function and $X_f = \{f, .\}$ the Hamiltonian vector field associated to f. Then the commutator of two Hamiltonian vector fields is

$$[X_f, X_g] = X_{\{f,g\}}$$

due to the Jacobi identity of the Poisson bracket. If we can find functions, not necessarily unique, with

$$\{f_i, g_j\} = \delta_{ij}, \quad \{f_i, f_j\} = 0, \quad \{g_i, g_j\} = 0, \tag{2.23}$$

then all commutators between the associated Hamiltonian vector fields vanish. The Splitting theorem for Poisson manifolds [38] tells us that this is possible in a neighbourhood of a point if the rank of the Poisson tensor is constant around this point. Since we do not want to find a $\star$-product on $\mathbb{R}^N$, but a $\star$-product with certain commutation relations, we can reduce $\mathbb{R}^N$ by the set of points where the rank of the Poisson tensor jumps and we have a good chance to find functions with the desired properties on the new manifold. In this case we can write the Poisson tensor as

$$\Pi = \sum_i X_{f_i} \wedge X_{g_i}.$$

In the following we give functions f_i and g_i for Poisson tensors of several algebras and use the corresponding Hamiltonian vector fields in the $\star-$products (2.18). We calculate the $\star-$algebra relations coming from the $\star-$product and compare them to the original algebra relations.

2.6 Examples for $\star-$products in two dimensions

For these examples we just use the asymmetric $\star-$product (2.19) and special Hamiltonian vector fields (which we do not justify for the moment). We obtain the $\star-$products for the algebra of the two dimensional Euclidean space and calculate the $\star-$product for commuting general linear Hamiltonian vector fields which includes the Manin plane. Then we treat the case of derivatives in spherical coordinates and the same for the unit circle $x^2+y^2=1$. First we have to find two commuting Hamiltonian vector fields. Then we calculate how powers of these vector fields act on coordinates, so we first do $X^n x$, $X^n y$, $Y^n x$ and $Y^n y$. The results we insert into equation (2.19). With the $\star-$products of the coordinates it is easy to get the according $\star-$commutator.

2.6.1 a-Euclidean space

The vector fields we use are $X = x\partial_x$ and $Y = ia\partial_y$. With $h = 1$ we obtain

$$\begin{aligned} x \star x &= x^2, \\ x \star y &= xy + iax, \\ y \star x &= xy, \\ y \star y &= y^2 \end{aligned}$$

and

$$[x \stackrel{\star}{,} y] = iax,$$

which is the algebra of two dimensional a-Euclidean space [11, 12].

2.6.2 General linear vector fields

Here we use the general linear vector fields $X = (a + bx)\partial_x$ and $Y = (c + dy)\partial_y$. For the coordinates we obtain in this general linear case

$$\begin{aligned} x \star x &= x^2, \\ x \star y &= xy + (e^{bd} - 1)(y + \tfrac{c}{d})(x + \tfrac{a}{b}), \\ y \star x &= xy, \\ y \star y &= y^2. \end{aligned}$$

The commutation relations then read

$$x \star y = (e^{bd} - 1)(y + \tfrac{c}{d}) \star (x + \tfrac{a}{b}), \quad b, d \neq 0.$$

Particular algebras we get, if we take special values for the parameters. For $a, c = 0$ we obtain

$$\begin{aligned} a = c = 0 \ &: \ [x \stackrel{\star}{,} y] = (e^{bd} - 1) y \star x, \\ a = 0 \ &: \ [x \stackrel{\star}{,} y] = (e^{bd} - 1)(y + \tfrac{c}{d}) \star x, \\ c = 0 \ &: \ [x \stackrel{\star}{,} y] = (e^{bd} - 1) y \star (x + \tfrac{a}{b}). \end{aligned}$$

The first relation is exactly that of the two dimensional Heisenberg algebra, the other ones show up a similar structure, but lead to a transformation in ones of the coordinates.
If we take $b, d = 0$ the $\star-$product of x and y has to be rewritten, since $\frac{1}{b}$ and $\frac{1}{d}$ are not defined. We then obtain

$$\begin{aligned} b = d = 0 \ &: \ [x \stackrel{\star}{,} y] = ac = \text{const.}, \\ b = 0 \ &: \ [x \stackrel{\star}{,} y] = c(a + bx), \\ d = 0 \ &: \ [x \stackrel{\star}{,} y] = a(c + dy), \end{aligned}$$

where the first case corresponds to the $\theta = $ const. case treated in various recent publications [10, 22, 40] and the other cases to algebras like the $a-$Euclidean space.

2.6.3 Vector fields in general spherical coordinates

The vector fields $X = \frac{a}{\sqrt{x^2+y^2}}(x\partial_x + y\partial_y)$, $Y = x\partial_y - y\partial_x$ we use in this case are the derivatives ∂_r and ∂_θ in spherical coordinates $x = r\cos\theta$, $y = r\sin\theta$. The $\star-$product provides

$$\begin{aligned} x \star x &= x^2 - a\tfrac{xy}{r}, \\ x \star y &= xy + a\tfrac{x^2}{r}, \\ y \star x &= xy - a\tfrac{y^2}{r}, \\ y \star y &= y^2 + a\tfrac{xy}{r}, \end{aligned}$$

which leads to

$$[x \stackrel{\star}{,} y] = a\sqrt{x \star x + y \star y}.$$

2.6.4 Vector fields in spherical coordinates on the unit circle

Simplifying the previous case by taking the unit circle $x^2 + y^2 = 1$ we have for the vector fields $X = a(x\partial_x + y\partial_y)$, $Y = x\partial_y - y\partial_x$. We find for the coordinates

$$\begin{aligned} x \star x &= x^2 \cos a - xy \sin a, \\ x \star y &= xy \cos a + x^2 \sin a, \\ y \star x &= xy \cos a - y^2 \sin a, \\ y \star y &= y^2 \cos a + xy \sin a, \end{aligned}$$

$$\begin{aligned} x \star x + y \star y &= (x^2 + y^2) \cos a, \\ x \star y - y \star x &= (x^2 + y^2) \sin a. \end{aligned}$$

To compute the last two equations we only used the $\star-$products of the coordinates from above. Altogether we obtain

$$[x \overset{\star}{,} y] = (\tan a)(x \star x + y \star y).$$

This algebra does not have the Poincaré-Birkhoff-Witt property for $\tan a = 1$, so we have to treat a as a formal parameter.

2.7 Examples for $\star-$products constructed from quantum spaces

In the previous section we calculated $\star-$products by taking Hamiltonian vector fields. We gave no justification for how the vector field looked like. In this chapter we want to show the whole way of constructing a $\star-$product only with the help of the algebra relations.

First we obtain the Poisson tensor by using the algebra relations of the quantum space. We know that the functions the needed Hamiltonian vector fields are based on have to satisfy the equation

$$\{f, g\} = 1$$

with the calculated Poisson structure. With this we get the vector fields by

$$X_f = \{f, \,.\,\}.$$

Then we can follow the way we worked out in the previous section.

We show how this construction works for the $q-$deformed Heisenberg algebra. The also treated Lie algebra $so(3)$ is given as well as the $q-$deformed Euclidean spaces in three and four dimensions. The quantum spaces with more physical relevance are the $q-$deformed Minkowski space $M(so_q(1,3))$, the $q-$deformed Fock space in four dimensions and the $a-$deformed $n-$dimensional Euclidean space we want to start with.

2.7.1 The quantum space $M(so_a(n))$

The quantum space we want to investigate in this section was first introduced in [25]. It is covariant under the quantum group $SO_a(n)$ and has a nontrivial center. The reason for us to choose it as the first example in this section is its weak deformation, which leads to concise formulae for the $\star-$product. This space is closely related to the $\star-$product for the two dimensional a-euclidean space given above. But since we are using the $n-$dimensional generalisation introduced in [11, 12] in the following we just call it $SO_a(n)$ covariant quantum space or abbreviated $M(so_a(n))$. Its algebra relations are

$$[\hat{x}^0, \hat{x}^i] = ia\hat{x}^i \quad \text{for} \quad 0 < i < n-1, \tag{2.24}$$

where $a \in \mathbb{R}$. For all generators it holds that $[\hat{x}^i, \hat{x}^j] = 0$. The Greek indices run from 0 to $n-1$, the Latin ones from 1 to $n-1$.
As manifold we take $\mathbb{R}^n$ with coordinates x^0 and x^i and use the asymmetric $\star-$product (2.19) with the two vector fields

$$X = x^i\partial_i, \quad Y = ia\partial_0, \tag{2.25}$$

where $h = 1$. With this we get for the coordinates

$$\begin{aligned} x^i \star x^j &= x^i x^j, \\ x^i \star x^0 &= x^i x^0 + iax^i, \\ x^0 \star x^i &= x^i x^0, \\ x^0 \star x^0 &= (x^0)^2 \end{aligned}$$

and thus for the algebra relations

$$[x^i \stackrel{\star}{,} x^0] = iax^i,$$

which are the algebra relations (2.24).

2.7.2 q-deformed Heisenberg algebra

We consider the q-deformed Heisenberg algebra [39] in two dimensions

$$\hat{x}\hat{y} = q\hat{y}\hat{x} + \theta \tag{2.26}$$

for which we calculate a $\star$-product in $q = e^h$ and θ. The algebra relations have to be written in Weyl-ordered form to obtain the Poisson structure in the way described in section 2.5.3: first by a general ansatz

$$[\hat{x}, \hat{y}] = a\frac{\hat{x}\hat{y} + \hat{y}\hat{x}}{2} + b$$

for which we find $a = 2\frac{q-1}{q+1}$ and $b = \frac{2\theta}{q+1}$. This we have to be expanded in h. Then the Poisson tensor Π is

$$\Pi = (xy + \frac{\theta}{h})\partial_x \wedge \partial_y.$$

We see that $f = \ln(xy + \frac{\theta}{h})$ and $g = \ln y$ fulfil the requirement

$$\{f, g\} = 1$$

to assure the commutativity of the two vector fields. With this the Hamiltonian vector fields become

$$\begin{aligned} X = X_f &= \{\ln(xy + \tfrac{\theta}{h}), .\} = y\partial_y - x\partial_x, \\ Y = X_g &= \{\ln y, .\} = -(x + \tfrac{\theta}{hy})\partial_x. \end{aligned}$$

With regard to the $\star-$product we calculate the action of powers of the vector fields on the coordinates

$$\begin{aligned} X^n(x) &= (-1)^n x, \\ X^n(y) &= y, \\ Y^n(x) &= \delta^{n0} x + (-1)^n (x + \tfrac{\theta}{hy})\delta^{ni}, \quad i > 1 \\ Y^n(y) &= \delta^{n0} y. \end{aligned}$$

For the **asymmetric** $\star-$product (2.19) this yields

$$\begin{aligned} x \star y &= xy, \\ y \star x &= e^{-h} xy + (e^{-h} - 1)\tfrac{\theta}{h}. \end{aligned}$$

For the **antisymmetric** $\star-$product (2.20) we obtain

$$\begin{aligned} x \star y &= e^{\frac{h}{2}} xy + (e^{+\frac{h}{2}} - 1)\tfrac{\theta}{h}, \\ y \star x &= e^{-\frac{h}{2}} xy + (e^{-\frac{h}{2}} - 1)\tfrac{\theta}{h}. \end{aligned}$$

Both $\star-$products therefore provide the algebra relation

$$x \star y = e^h y \star x + (e^h - 1)\frac{\theta}{h}$$

which is the original algebra relation (2.26) for $q = e^h$ and $\theta' = \frac{e^h - 1}{h}$.

2.7.3 The Lie algebra $so(3)$

We start with the algebra relations in the basis $\hat{x}^+, \hat{z}, \hat{x}^-$:

$$\begin{aligned} [\hat{z}, \hat{x}^+] &= \hat{x}^+, \\ [\hat{z}, \hat{x}^-] &= -\hat{x}^-, \\ [\hat{x}^+, \hat{x}^-] &= \hat{z} \end{aligned} \tag{2.27}$$

for which we find the Poisson tensor

$$\Pi = 2x^+ z \; \partial_z \wedge \partial_+ + 2x^- z \; \partial_z \wedge \partial_- + 2z^2 \; \partial_- \wedge \partial_+.$$

With $f = \ln x^-$, $g = z$ we have $\{f, z\} = 1$. The Hamiltonian vector fields then become

$$\begin{aligned} X &= X_f = \partial_z - \tfrac{z}{x^-}\partial_+, \\ Y &= X_g = x^+\partial_+ - x^-\partial_-. \end{aligned}$$

Hence we obtain

$$\begin{aligned} X^n(\,z\,) &= \delta^{n0} z + \delta^{n1}, \\ X^n(x^+) &= \delta^{n0} x^+ - \delta^{n1}\tfrac{z}{x^-} - \delta^{n2}\tfrac{1}{x^-}, \\ X^n(x^-) &= \delta^{n0} x^-, \\ Y^n(\,z\,) &= \delta^{n0} z, \\ Y^n(x^+) &= x^+, \\ Y^n(x^-) &= (-1)^n x^- \end{aligned}$$

and the **asymmetric** $\star$-product (2.19) gives

$$\begin{aligned} z \star x^+ &= zx^+ + hx^+, & z \star x^- &= zx^- - hx^-, \\ x^+ \star z &= x^+ z, & x^- \star z &= x^- z, \end{aligned}$$

$$\begin{aligned} x^+ \star x^- &= x^+x^- + hz - h^2/2, \\ x^- \star x^+ &= x^+x^-. \end{aligned}$$

and therefore the $\star-$commutator reads as

$$\begin{aligned} [z \stackrel{\star}{,} x^+] &= hx^+, \\ [z \stackrel{\star}{,} x^-] &= -hx^-, \\ [x^+ \stackrel{\star}{,} x^-] &= h(z - h/2). \end{aligned}$$

With $\tilde{z} = z - \frac{h}{2}$ the correct algebra relations (2.27) are reproduced.

2.7.4 The quantum space $M(so_q(3))$

The algebra relations in the basis adjusted to the quantum group terminology [24] which is a generalisation of the basis x^0, $x^\pm$ in the commutative space:

$$\begin{aligned} \hat{z}\hat{x}^+ &= q^2 \hat{x}^+ \hat{z}, \\ \hat{z}\hat{x}^- &= q^{-2} \hat{x}^- \hat{z}, \\ [\hat{x}^-, \hat{x}^+] &= (q - q^{-1})\hat{z}^2. \end{aligned} \tag{2.28}$$

For the Weyl ordered commutators we obtain

$$\begin{aligned} [\hat{z}, \hat{x}^+] &= \tfrac{2(q^2-1)}{q^2+1}\,\tfrac{\hat{z}\hat{x}^+ + \hat{x}^+\hat{z}}{2}, \\ [\hat{z}, \hat{x}^-] &= -\tfrac{2(q^2-1)}{q^2+1}\,\tfrac{\hat{z}\hat{x}^- + \hat{x}^-\hat{z}}{2}, \\ [\hat{x}^-, \hat{x}^+] &= (q - q^{-1})\hat{z}^2 \end{aligned}$$

and therefore the Poisson structure is

$$\Pi = 2zx^+\,\partial_z\wedge\partial_+ - 2zx^-\,\partial_z\wedge\partial_- + 2z^2\,\partial_-\wedge\partial_+.$$

For $f = \frac{1}{2}\ln x^-$ and $g = \ln z$ the necessary $\{f, g\} = 1$ holds and the Hamiltonian vector fields become

$$\begin{aligned} X_f &= z\partial_z + \tfrac{z^2}{x^-}\partial_+, \\ X_g &= 2(x^+\partial_+ - x^-\partial_-). \end{aligned}$$

For the ⋆−product we take the generalisation

$$\begin{aligned} X &= z\partial_z + \tfrac{\alpha z^2}{x^-}\partial_+, \\ Y &= x^+\partial_+ - x^-\partial_-. \end{aligned}$$

The action of potentials of the vector fields read as

$$\begin{aligned} X^n(x^+) &= \delta^{n0}x^+ + \alpha 2^{n-1}\frac{z^2}{x^-}\delta^{ni}, \quad i > 0 \\ X^n(x^-) &= \delta^{n0}x^-, \\ X^n(z) &= z, \\ Y^n(x^+) &= x^+, \\ Y^n(x^-) &= (-1)^n x^-, \\ Y^n(z) &= \delta^{n0}z. \end{aligned}$$

For the **asymmetric** ⋆-product (2.19) we obtain

$$\begin{aligned} z\star x^+ &= e^h x^+ z, & z\star x^- &= e^{-h}x^- z, \\ x^+\star z &= x^+ z, & x^-\star z &= x^- z, \end{aligned}$$

$$\begin{aligned} x^-\star x^+ &= x^+x^-, \\ x^+\star x^- &= x^+x^- + \tfrac{\alpha}{2}(e^{-2h}-1)z^2, \end{aligned}$$

and (with $z\star z = z^2$) the algebra relations become

$$\begin{aligned} z\star x^+ &= e^h x^+\star z, \\ z\star x^- &= e^{-h}x^-\star z, \\ [x^+\stackrel{\star}{,}x^-] &= \tfrac{\alpha}{2}(e^{-2h}-1)z\star z. \end{aligned}$$

If we set

$$q = e^{h/2}, \quad \alpha = -\tfrac{2q^2}{q+q^{-1}}$$

this reproduces exactly the algebra relations (2.28).

For the **antisymmetric** ⋆-product (2.20) we have

$$\begin{aligned} x^+ \star z &= e^{-h/2} x^+ z, & x^- \star z &= e^{h/2} x^- z, \\ z \star x^+ &= e^{h/2} x^+ z, & z \star x^- &= e^{-h/2} x^- z, \end{aligned}$$

$$\begin{aligned} x^+ \star x^- &= x^+ x^- + \tfrac{\alpha}{2}(e^{-h} - 1) z^2, \\ x^- \star x^+ &= x^+ x^- + \tfrac{\alpha}{2}(e^{h} - 1) z^2, \end{aligned}$$

and the algebra relations are with $z \star z = z^2$

$$\begin{aligned} z \star x^+ &= e^h x^+ \star z, \\ z \star x^- &= e^{-h} x^- \star z, \\ [x^+ \overset{\star}{,} x^-] &= -\tfrac{\alpha}{2}(e^h - e^{-h}) z \star z. \end{aligned}$$

Here we can reproduce the algebra relations (2.28), if we set

$$q = e^{h/2}, \quad \alpha = -\tfrac{2}{q+q^{-1}}.$$

2.7.5 The quantum space $M(so_q(1,3))$

The algebra of this quantum space is given in [24]:

$$\begin{aligned} [\hat{x}^0, \hat{x}^A] &= 0 \\ [\hat{x}^-, \hat{x}^+] &= (q - q^{-1})(\hat{x}^3)^2 - (q - q^{-1}) \hat{x}^0 \hat{x}^3 \\ \hat{x}^3 \hat{x}^+ &= q^2 \hat{x}^+ \hat{x}^3 + (1 - q^2) \hat{x}^0 \hat{x}^+ \\ \hat{x}^3 \hat{x}^- &= q^{-2} \hat{x}^- \hat{x}^3 + (1 - q^{-2}) \hat{x}^0 \hat{x}^- \end{aligned} \tag{2.29}$$

The Poisson structure then reads as

$$\Pi = 2x^3(x^3 - x^0)\ \partial_- \wedge \partial_+ + 2x^+(x^3 - x^0)\ \partial_3 \wedge \partial_+ + 2x^-(x^0 - x^3)\ \partial_3 \wedge \partial_-.$$

Our ansatz for the functions the Hamiltonian vector fields are based on is

$$f = \tfrac{1}{2} \ln x^-, \quad g = \ln(x^3 - x^0) \qquad \Rightarrow \{f, g\} = 1$$

with which we obtain

$$\begin{aligned} X_f &= x^3(x^3 - x^0) \tfrac{1}{x^-} \partial_+ + (x^3 - x^0) \partial_3; \\ X_g &= 2(x^+ \partial_+ - x^- \partial_-). \end{aligned}$$

For the calculation we choose vector fields generalising the above ones:

$$\begin{aligned} X &= (\alpha(x^3)^2 - \beta x^3 x^0)\tfrac{1}{x^-}\partial_+ + (x^3 - x^0)\partial_3 \\ Y &= x^+\partial_+ - x^-\partial_-. \end{aligned}$$

Their powers act on the coordinates as follows (with $i > 0$):

$$\begin{aligned} X^n x^0 &= \delta^{n0} x^0, \\ X^n x^3 &= \delta^{n0} x^3 + \delta^{ni}(x^3 - x^0), \\ X^n x^+ &= \delta^{n0} x^+ + \delta^{ni}(x^3 - x^0)[2^{n-1}\alpha(x^3 - x^0) + (2\alpha - \beta)x^0]\tfrac{1}{x^-}, \\ X^n x^- &= \delta^{n0} x^-, \end{aligned}$$

$$\begin{aligned} Y^n x^0 &= \delta^{n0} x^0, \\ Y^n x^3 &= \delta^{n0} x^3, \\ Y^n x^+ &= x^+, \\ Y^n x^- &= (-1)^n x^-. \end{aligned}$$

For the **asymmetric** $\star$−product we find

$$\begin{aligned} x^0 \star x^i &= x^0 x^i, \\ x^i \star x^0 &= x^i x^0, \\ x^3 \star x^+ &= e^h x^3 x^+ - (e^h - 1)x^0 x^+, \\ x^+ \star x^3 &= x^+ x^3, \\ x^3 \star x^- &= e^{-h} x^3 x^- + (1 - e^{-h})x^0 x^-, \\ x^- \star x^3 &= x^- x^3, \\ x^- \star x^+ &= x^- x^+, \\ x^+ \star x^- &= x^+ x^- + \tfrac{1}{2}(e^{-2h} - 1)\alpha(x^3 - x^0)^2 - (e^{-h} - 1)(2\alpha - \beta)(x^3 - x^0)x^0. \end{aligned}$$

The algebra relations we obtain from these $\star$−products are

$$\begin{aligned} [x^0 \overset{\star}{,} x^i] &= 0, \\ [x^- \overset{\star}{,} x^+] &= -\tfrac{1}{2}(e^{-2h} - 1)\alpha(x^3 - x^0)^2 - (e^{-h} - 1)(2\alpha - \beta)(x^3 - x^0)x^0, \\ x^3 \star x^+ &= e^h x^+ \star x^3 + (1 - e^h)x^0 \star x^+, \\ x^3 \star x^- &= e^{-h} x^- \star x^3 + (1 - e^{-h})x^0 \star x^-. \end{aligned}$$

Obviously we have $e^h = q^2$. We can apply the coordinate transformation $x'^3 = x^3 - x^0$ that does not change the other relations. Comparing the new relation for x^- and x^+ to the original one we obtain for the two remaining parameters

$$\alpha = \tfrac{2q^2}{q+q^{-1}}, \qquad \beta = \tfrac{5q^2-1}{q+q^{-1}}$$

to reproduce the original algebra relations (2.29).

The **antisymmetric** ⋆−product provides

$$
\begin{aligned}
x^0 \star x^i &= x^0 x^i,\\
x^i \star x^0 &= x^i x^0,\\
x^3 \star x^+ &= e^{h/2} x^3 x^+ - (e^{h/2}-1) x^0 x^+,\\
x^+ \star x^3 &= e^{-h/2} x^+ x^3 - (e^{-h/2}-1) x^0 x^+,\\
x^3 \star x^- &= e^{-h/2} x^3 x^- - (e^{-h/2}-1) x^0 x^-,\\
x^- \star x^3 &= e^{h/2} x^- x^3 - (e^{h/2}-1) x^0 x^-,\\
x^- \star x^+ &= x^- x^+ + \tfrac{1}{2}(e^h-1)\alpha(x^3-x^0)^2 + (e^{h/2}-1)(2\alpha-\beta)(x^3-x^0)x^0,\\
x^+ \star x^- &= x^+ x^- + \tfrac{1}{2}(e^{-h}-1)\alpha(x^3-x^0)^2 + (e^{-h/2}-1)(2\alpha-\beta)(x^3-x^0)x^0.
\end{aligned}
$$

The resulting algebra ⋆−relations read

$$
\begin{aligned}
[x^0 \stackrel{\star}{,} x^i] &= 0,\\
[x^- \stackrel{\star}{,} x^+] &= \tfrac{1}{2}(e^h - e^{-h})\alpha(x^3-x^0)^2 + (e^{h/2}-e^{-h/2})(2\alpha-\beta)(x^3-x^0)x^0,\\
x^3 \star x^+ &= e^h x^+ \star x^3 + (1-e^h) x^0 \star x^+\\
x^3 \star x^- &= e^{-h} x^- \star x^3 + (1-e^{-h}) x^0 \star x^-.
\end{aligned}
$$

We obtain again $q^2 = e^h$ and with the same coordinate transformation as before the other parameters are

$$\alpha = \tfrac{2}{q+q^{-1}}, \qquad \beta = \tfrac{4+q+q^{-1}}{q+q^{-1}}$$

which exactly reproduce the original relations (2.29).

2.7.6 The quantum space $M(so_q(4))$

The algebra relations we take for $M(so_q(4))$ can be found in [15, 29]:

$$
\begin{aligned}
\hat{x}_1\hat{x}_2 &= q\hat{x}_2\hat{x}_1,\\
\hat{x}_3\hat{x}_4 &= q\hat{x}_4\hat{x}_3,\\
\hat{x}_2\hat{x}_3 &= \hat{x}_3\hat{x}_2,\\
\hat{x}_1\hat{x}_3 &= q\hat{x}_3\hat{x}_1,\\
\hat{x}_2\hat{x}_4 &= q\hat{x}_4\hat{x}_2,\\
[\hat{x}_4, \hat{x}_1] &= (q-q^{-1})\hat{x}_2\hat{x}_3.
\end{aligned}
\tag{2.30}
$$

The Poisson tensor then reads

$$\Pi = x_1 x_2\, \partial_1 \wedge \partial_2 + x_1 x_3\, \partial_1 \wedge \partial_3 + x_2 x_4\, \partial_3 \wedge \partial_4 + 2 x_2 x_3\, \partial_4 \wedge \partial_1.$$

It has two Casimir functions[3], so two vector fields suffice to reproduce the algebra:

$$f = \ln x_2,\; g = \ln x_4 \qquad \Rightarrow \qquad \{f, g\} = 1.$$

The vector fields then become

$$\begin{aligned} X &:= X_f = x_4\partial_4 - x_1\partial_1, \\ Y &:= X_g = -(x_2\partial_2 + x_3\partial_3) + 2\tfrac{x_2x_3}{x_4}\partial_1. \end{aligned}$$

The powers of the two vector fields then act on the coordinates as follows ($i > 0$):

$$\begin{aligned} X^n(x_1) &= (-1)^n x_1, \\ X^n(x_2) &= \delta^{n0} x_2, \\ X^n(x_3) &= \delta^{n0} x_3, \\ X^n(x_4) &= x_4, \\ Y^n(x_1) &= \delta^{n0} x_1 - \delta^{ni}(-2)^n \tfrac{x_2x_3}{x_4}, \\ Y^n(x_2) &= (-1)^n x_2, \\ Y^n(x_3) &= (-1)^n x_3, \\ Y^n(x_4) &= \delta^{n0} x_4. \end{aligned}$$

The **asymmetric** ⋆-product (2.19) therefore gives us

$$\begin{aligned} x_1 \star x_2 &= e^h x_1x_2, & x_2 \star x_3 &= x_2x_3, \\ x_2 \star x_1 &= x_1x_2, & x_3 \star x_2 &= x_2x_3, \\ \\ x_1 \star x_3 &= e^h x_1x_3, & x_2 \star x_4 &= x_2x_4, \\ x_3 \star x_1 &= x_1x_3, & x_4 \star x_2 &= e^{-h} x_2x_4, \\ \\ x_4 \star x_1 &= x_1x_4 - (e^{-2h} - 1)x_2x_3, & x_3 \star x_4 &= x_3x_4, \\ x_1 \star x_4 &= x_1x_4, & x_4 \star x_3 &= e^{-h} x_3x_4 \end{aligned}$$

so that the algebra relations then are

$$\begin{aligned} x_1 \star x_2 &= e^h x_2 \star x_1, \\ x_1 \star x_3 &= e^h x_3 \star x_1, \\ x_3 \star x_4 &= e^h x_4 \star x_3, \\ x_2 \star x_4 &= e^h x_4 \star x_2, \\ x_2 \star x_3 &= x_3 \star x_2, \\ [x_4 \overset{\star}{,} x_1] &= (1 - e^{-2h}) x_2 \star x_3. \end{aligned}$$

[3]This means that for this special Π there are two functions fulfilling equation (2.23).

For the **antisymmetric** ⋆-product (2.20) we do the same calculations

$$\begin{aligned} x_1 \star x_2 &= e^{h/2} x_1 x_2, & x_2 \star x_3 &= x_2 x_3, \\ x_2 \star x_1 &= e^{-h/2} x_1 x_2, & x_3 \star x_2 &= x_2 x_3, \end{aligned}$$

$$\begin{aligned} x_1 \star x_3 &= e^{h/2} x_1 x_3, & x_2 \star x_4 &= e^{h/2} x_4 x_2, \\ x_3 \star x_1 &= e^{-h/2} x_1 x_3, & x_4 \star x_2 &= e^{-h/2} x_4 x_2, \end{aligned}$$

$$\begin{aligned} x_4 \star x_1 &= x_1 x_4 + (1 - e^{-h}) x_2 x_3, & x_3 \star x_4 &= e^{h/2} x_4 x_3, \\ x_1 \star x_4 &= x_1 x_4 + (1 - e^{h}) x_2 x_3, & x_4 \star x_3 &= e^{-h/2} x_4 x_3 \end{aligned}$$

which leads to the following algebra relations

$$\begin{aligned} x_1 \star x_2 &= e^h x_2 \star x_1, \\ x_1 \star x_3 &= e^h x_3 \star x_1, \\ x_3 \star x_4 &= e^h x_4 \star x_3, \\ x_2 \star x_4 &= e^h x_4 \star x_2, \\ x_2 \star x_3 &= x_3 \star x_2, \\ [x_4 \overset{\star}{,} x_1] &= (e^h - e^{-h}) x_2 \star x_3. \end{aligned}$$

This reproduces exactly the original algebra relations (2.30) with $q = e^h$.

2.7.7 4-dimensional q-deformed Fock space

One finds the algebra relations for this example in [19, 20]

$$\begin{aligned} \hat{x}_1 \hat{x}_2 &= q^{-1} \hat{x}_2 \hat{x}_1, \\ \hat{y}_1 \hat{x}_1 &= q^2 \hat{x}_1 \hat{y}_1 + \theta, \\ \hat{y}_1 \hat{x}_2 &= q \hat{x}_2 \hat{y}_1, \\ \hat{y}_2 \hat{x}_1 &= q \hat{x}_1 \hat{y}_2, \\ \hat{y}_2 \hat{x}_2 &= q^2 \hat{x}_2 \hat{y}_2 + (q^2 - 1) \hat{x}_1 \hat{y}_1 + \theta, \\ \hat{y}_1 \hat{y}_2 &= q \hat{y}_2 \hat{y}_1. \end{aligned} \tag{2.31}$$

Thus, the Poisson tensor is

$$\begin{aligned} \Pi = & -x_1 x_2\, \partial_{x_1} \wedge \partial_{x_2} + x_2 y_1\, \partial_{y_1} \wedge \partial_{x_2} + (2 x_1 y_1 + \theta)\, \partial_{y_1} \wedge \partial_{x_1} \\ & + y_1 y_2\, \partial_{y_1} \wedge \partial_{y_2} + x_1 y_2\, \partial_{y_2} \wedge \partial_{x_1} + [2(x_1 y_1 + x_2 y_2) + \theta]\, \partial_{y_2} \wedge \partial_{x_2}. \end{aligned}$$

Here there are four Casimir functions of the Poisson tensor, so we have to find four functions fulfilling (2.23) which are

$$\begin{aligned} f_1 &= -\ln x_1, \\ f_2 &= \tfrac{1}{2}\ln(2x_1y_1+\theta), \\ g_1 &= f_2 - \ln x_2, \\ g_2 &= \tfrac{1}{2}\ln\tfrac{2(x_1y_1+x_2y_2)+\theta}{2x_1y_1+\theta}. \end{aligned}$$

In this case the Hamiltonian vector fields are

$$\begin{aligned} X_1 := X_{f_1} &= x_2\partial_{x_2} + y_2\partial_{y_2} + (2x_1y_1+\theta)\tfrac{1}{x_1}\partial_{y_1}, \\ Y_1 := X_{f_2} &= x_1\partial_{x_1} - y_1\partial_{y_1}, \\ \\ X_2 := X_{g_1} &= [2(x_1y_1+x_2y_2)+\theta]\tfrac{1}{x_2}\partial_{y_2}, \\ Y_2 := X_{g_2} &= x_2\partial_{x_2} - y_2\partial_{y_2}. \end{aligned}$$

The action of its powers on the coordinates is then given by $(i > 0)$

$$\begin{aligned} Y_1^n(x_1) &= x_1, & X_1^n(x_1) &= \delta^{n0}x_1, \\ Y_1^n(x_2) &= \delta^{n0}x_2, & X_1^n(x_2) &= x_2, \\ Y_1^n(y_1) &= (-1)^n y_1, & X_1^n(y_1) &= \delta^{n0}y_1 + \delta^{ni}\,2^{n-1}\,\tfrac{2x_1y_1+\theta}{x_1}, \\ Y_1^n(y_2) &= \delta^{n0}y_2, & X_1^n(y_2) &= y_2, \\ \\ Y_2^n(x_1) &= \delta^{n0}x_1, & X_2^n(x_1) &= \delta^{n0}x_1, \\ Y_2^n(x_2) &= x_2, & X_2^n(x_2) &= \delta^{n0}x_2, \\ Y_2^n(y_1) &= \delta^{n0}y_1, & X_2^n(y_1) &= \delta^{n0}y_1, \\ Y_2^n(y_2) &= (-1)^n y_2 & X_2^n(y_2) &= \delta^{n0}y_2 + \delta^{ni}\,2^{n-1}\,\tfrac{2(x_1y_1+x_2y_2)+\theta}{x_2}. \end{aligned}$$

Our ⋆−products (2.19) and (2.20) are just defined for two vector fields, but now we have to handle four of them. This we do by generalising for example the asymmetric ⋆−product to the following form:

$$f\star g = \sum_{n,m=0}^{\infty} \frac{h^{n+m}}{n!\,m!}\,(X_1^n X_2^m f)\,(Y_1^n Y_2^m g)\,.$$

This fulfils the same conditions as the ones with just two Hamiltonian vector fields, since the vector fields are Hamiltonian and the ⋆−product fits the definition (2.18) as can be seen by

$$f\star g = \sum_{n,m=0}^{\infty} \tfrac{h^{n+m}}{n!\,m!}(X_1^n X_2^m f)(Y_1^n Y_2^m g)$$

$$
\begin{aligned}
&\overset{1}{=} \sum_{n,m=0}^{\infty} \tfrac{h^{n+m}}{n!\,m!}(X_1^n X_2^m \otimes Y_1^n Y_2^m)(f,g) \\
&\overset{2}{=} \sum_{n,m=0}^{\infty} \tfrac{h^{n+m}}{n!\,m!}(X_1^n \otimes Y_1^n)(X_2^m \otimes Y_2^m)(f,g) \\
&= \sum_{n=0}^{\infty} \tfrac{h^n}{n!}(X_1^n \otimes Y_1^n) \sum_{m=0}^{\infty} \tfrac{h^m}{m!}(X_2^m \otimes Y_2^m)(f,g) \\
&= \sum_{n=0}^{\infty} \tfrac{h^n}{n!}(X_1 \otimes Y_1)^n \sum_{m=0}^{\infty} \tfrac{h^m}{m!}(X_2 \otimes Y_2)^m (f,g) \\
&= e^{X_1\otimes Y_1 + X_2 \otimes Y_2}(f,g).
\end{aligned}
$$

In the $\star$−product $y_2 \star x_2$ we need to know the action of X_1^n on some functions of coordinates, so first we have to calculate $(i > 0)$

$$
\begin{aligned}
X_1^n(x_1y_1) &= \delta^{n0}x_1y_1 + \delta^{ni}\, 2^{n-1}(2x_1y_1 + \theta), \\
X_1^n f(x_2) &= f(x_2), \\
X_1^n f(y_2) &= f(y_2), \\
&\Downarrow \\
X_1^n \, \tfrac{2(x_1y_1 + x_2y_2)+\theta}{x_2} &= \tfrac{1}{x_2} X_1^n (2(x_1y_1 + x_2y_2) + \theta) \\
&= \tfrac{1}{x_2}\left[2^n(2x_1y_1 + \theta) + 2x_2y_2\right],
\end{aligned}
$$

and then compute

$$
\begin{aligned}
x_1 \star x_2 &= x_1x_2, & y_2 \star x_1 &= e^h y_2 x_1, \\
x_2 \star x_1 &= e^h x_2 x_1, & x_1 \star y_2 &= x_1 y_2, \\
\\
y_1 \star x_2 &= y_1 x_2, & y_1 \star x_1 &= e^{2h} x_1 y_1 + \tfrac{1}{2}(e^{2h}-1)\theta, \\
x_2 \star y_1 &= e^{-h} x_2 y_1, & x_1 \star y_1 &= x_1 y_1, \\
\\
y_1 \star y_2 &= y_1 y_2, & y_2 \star x_2 &= e^{2h} x_2 y_2 + (e^{2h}-1)x_1y_1 + \tfrac{1}{2}(e^{2h}-1)\theta, \\
y_2 \star y_1 &= e^{-h} y_2 y_1, & x_2 \star y_2 &= x_2 y_2.
\end{aligned}
$$

The $\star$−relations of this algebra then are

$$
\begin{aligned}
x_1 \star x_2 &= e^{-h} x_2 \star x_1, \\
y_1 \star x_2 &= e^{h} x_2 \star y_1,
\end{aligned}
$$

[1] with $(a \otimes b)(f,g) := (af)\cdot(bg)$

[2] we used the definition of the product on a tensor product algebra

$$
\begin{aligned}
y_1 \star y_2 &= e^h y_2 \star y_1, \\
y_2 \star x_1 &= e^h x_1 \star y_2, \\
y_1 \star x_1 &= e^{2h} x_1 \star y_1 + \tfrac{1}{2}(e^{2h} - 1)\theta, \\
y_2 \star x_2 &= e^{2h} x_2 \star y_2 + (e^{2h} - 1) x_1 \star y_1 + \tfrac{1}{2}(e^{2h} - 1)\theta.
\end{aligned}
$$

To reproduce the original algebra relations (2.31) we just have to set

$$q = e^{2h}, \quad \theta' = \tfrac{1}{2}(e^{2h} - 1)\theta.$$

Chapter 3

Leibniz rule on the n-dimensional q-deformed Euclidean space

In classical theories it is clear how to apply mathematical operations like derivation and integration of functions. Generalising the classical theory in the sense of deformation quantisation all these prescriptions do chance.

Coming from usual quantum mechanics generalised states are power series in the commuting coordinates of the system. This means that the action of operators (usually depending on the coordinates themselves and partial derivatives) on these functions can be calculated and interpreted quite easily.

In non-commutative geometry we therefore use formal power series in the algebra generators $\hat{x}_i$. As explicitly explained in [21] one has first to define a first order differential calculus (FODC) Γ by a linear mapping d from the non-commutative algebra χ to the FODC $\mathrm{d} : \chi \to \Gamma$. Then there are unique elements $\partial_i(x) \in \chi$ which are called the partial derivatives such that $\mathrm{d}x = \sum_i \mathrm{d}x_i \cdot \partial_i(x)$. With these elements one can construct an expanded algebra $\mathcal{A}_q(n)$ with the $2N$ generators $\hat{x}_1, \ldots, \hat{x}_n, \hat{\partial}_1, \ldots, \hat{\partial}_n$, where for calculations the algebra has to be specified by the commutation relations of its generators.

The quantum space we want to treat here is the $q-$deformed $n-$dimensional Euclidean space, which is based on the so-called Quantum Weyl algebra [21]:

$$\begin{aligned}
\hat{x}_i \hat{x}_j &= q \hat{x}_j \hat{x}_i; \quad i < j \\
\hat{\partial}_i \hat{\partial}_j &= q^{-1} \hat{\partial}_j \hat{\partial}_i; \quad i < j \\
\hat{\partial}_i \hat{x}_j &= q \hat{x}_j \hat{\partial}_i; \quad i \neq j \\
\hat{\partial}_i \hat{x}_i - q^2 \hat{x}_i \hat{\partial}_i &= 1 + (q^2 - 1) \sum_{j>i} \hat{x}_j \hat{\partial}_j.
\end{aligned} \tag{3.1}$$

Our aim is to develop the tools to construct a non-commutative field theory: the action of the derivatives on formal power series. One can do it the easy way by shifting coordinates

as done in [21], but that leads only to a closed formula for the action of the $\hat{\partial}_i$. But since we do not want to lose information and need the whole Leibniz rule to enable the derivative of products of functions, we have to do it the way we developed in [2, 3].
For the quantum Weyl algebra there is no $\star-$product given by other authors in the way we need it here, so we first have to give it by ourselves. It enables us to compare our results to the ones in classical theories, especially we have to prove the right classical limit for the action of the derivatives and the Leibniz rule.

3.1 $\star-$product

The coordinate algebra (3.1) satisfies the Poincaré-Birkhoff-Witt property meaning that the subspace of monomials of a fixed degree has the same dimension as its analogue in commutative space. Therefore we can find an algebra isomorphism between the commutative algebra $\mathcal{A}$ and the non-commutative algebra by choosing a fixed ordering of the coordinates in the deformed algebra:

$$W: \quad \mathcal{A} \rightarrow \mathcal{A}_q$$

$$W(x_1^{m_1} \cdots x_n^{m_n}) = \hat{x}_1^{m_1} \cdots \hat{x}_n^{m_n}.$$

In order to treat formal power series $f(\hat{x}_1 \ldots \hat{x}_n)$ this isomorphism can be extended to an isomorphism of algebras introducing a new product called $\star$-product which is defined by

$$W(f \star g) = W(f)W(g).$$

With this basic introduction we now can start to calculate the $\star$-product for the algebra (3.1) along [37].

First we calculate the commutation relations fulfilled by the powers of the coordinates:

$$(\hat{x}_i)^{m_i}(\hat{x}_j)^{m_j} = q^{m_i m_j}(\hat{x}_j)^{m_j}(\hat{x}_i)^{m_i}\,, i < j$$

Then we choose a fixed ordering for this quantum space, for its simplicity the normal ordering

$$(\hat{x}_1)^{m_1}(\hat{x}_2)^{m_2} \cdots (\hat{x}_n)^{m_n}.$$

Since we can expand the isomorphism W to arbitrary power series in the coordinates, it suffices to calculate the product of two monomials in the algebra:

$$\begin{aligned}
(\hat{x}_1)^{m_1}(\hat{x}_2)^{m_2} \cdots (\hat{x}_n)^{m_n} \cdot (\hat{x}_1)^{k_1}(\hat{x}_2)^{k_2} \cdots (\hat{x}_n)^{k_n} = \\
&= q^{-k_1 \sum\limits_{i=2}^{n} m_i - k_2 \sum\limits_{i=3}^{n} m_i - \ldots - k_{n-1} m_n} (\hat{x}_1)^{m_1+k_1}(\hat{x}_2)^{m_2+k_2} \cdots (\hat{x}_n)^{m_n+k_n} \\
&= q^{-\sum\limits_{i=1}^{n-1} \sum\limits_{j=i+1}^{n} k_i m_j} (\hat{x}_1)^{m_1+k_1}(\hat{x}_2)^{m_2+k_2} \cdots (\hat{x}_n)^{m_n+k_n}
\end{aligned}$$

This enables us to write down a $\star-$product for the monomials:

$$\begin{aligned} W(x_1^{m_1}\cdots x_n^{m_n})\cdot W(x_1^{k_1}\cdots x_n^{k_n}) &= \hat{x}_1^{m_1}\cdots\hat{x}_n^{m_n}\,\cdot\,\hat{x}_1^{k_1}\cdots\hat{x}_n^{k_n} \\ &= q^{-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} k_i m_j}\hat{x}_1^{m_1+k_1}\cdots\hat{x}_n^{m_n+k_n} = W(q^{-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} k_i m_j}x_1^{m_1+k_1}\cdots x_n^{m_n+k_n}) \\ &= W((x_1^{m_1}\cdots x_n^{m_n})\star(x_1^{k_1}\cdots x_n^{k_n})) \end{aligned}$$

For the generalisation of this formula we need to replace the specific powers of the coordinates by an operator which produces these powers as it acts on the monomials or functions. The simplest operators that fulfil the specifications are

$$\begin{aligned} \sigma_i &:= x_i\partial_i = m_i \\ \sigma_i' &:= x_i'\partial_i' = k_i \end{aligned}$$

where ∂_i are the ordinary partial derivatives and ∂' are derivatives only acting on coordinates x'. With this we obtain the following formula generalised to arbitrary power series f and g :

$$f\star g = q^{-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sigma_i'\sigma_j} f(x)g(x')\Big|_{x'=x}.$$

3.2 Leibniz rule

We start the calculations with the commutation relations of the derivatives $\hat{\partial}_i$ with the powers of single coordinates $(\hat{x}_i)^{n_i}$ using (3.1). The following relations can be proved by complete induction concerning the power of the coordinate:

$$\begin{aligned} \hat{\partial}_i(\hat{x}_j)^{m_j} &= q^{m_j}(\hat{x}_j)^{m_j}\hat{\partial}_i \\ \hat{\partial}_i(\hat{x}_i)^{m_i} &= [[m_i]]_{q^2}(\hat{x}_i)^{m_i-1} + q^{2m_i}(\hat{x}_i)^{m_i}\hat{\partial}_i + (q^2-1)[[m_i]]_{q^2}(\hat{x}_i)^{m_i-1}\sum_{j>i}\hat{x}_j\hat{\partial}_j \end{aligned}$$

Application of these formulae to a monomial leads to

$$\begin{aligned} &\hat{\partial}_i\,(\hat{x}_1)^{m_1}\cdots(\hat{x}_i)^{m_i}\cdots(\hat{x}_n)^{m_n} = \\ =\;& q^{\sum_{k=1}^{i-1} m_k}[[m_i]]_{q^2}\,(\hat{x}_1)^{m_1}\cdots(\hat{x}_i)^{m_i-1}\cdots(\hat{x}_n)^{m_n} + q^{m_i+\sum_{k=1}^{n} m_k}(\hat{x}_1)^{m_1}\cdots(\hat{x}_n)^{m_n}\hat{\partial}_i \\ +\;& q^{\sum_{k=1}^{i-1} m_k}(q^2-1)[[m_i]]_{q^2}\,(\hat{x}_1)^{m_1}\cdots(\hat{x}_i)^{m_i-1}\sum_{j_1>i}\underline{\hat{x}_{j_1}\hat{\partial}_{j_1}(\hat{x}_{i+1})^{m_{i+1}}\cdots(\hat{x}_n)^{m_n}}. \end{aligned}$$

At first sight this does not seem to be a closed formula, since the underlined part leads to a recursive relation that might not be solvable. The first intuitive try is to calculate straight along the obvious way: we try to solve an arbitrary relation of the form $\sum_{j_m>i}\hat{x}_{j_m}\partial_{j_m}(\hat{x}_{i+1})^{m_i+1}\cdots(\hat{x}_n)^{m_n}$. But this just leads us to more recursive relations of the same form, which obviously is no progress. If we instead reverse the summation, the calculations become much easier to handle:

$$\sum_{j_m>i} \hat{x}_{j_m}\partial_{j_m}(\hat{x}_{i+1})^{m_i+1}\cdots(\hat{x}_n)^{m_n} = \sum_{j_m=0}^{n-i-1} \hat{x}_{n-j_m}\partial_{n-j_m}(\hat{x}_{i+1})^{m_i+1}\cdots(\hat{x}_n)^{m_n}.$$

For the new term under the summation we can find a closed formula, proved by complete induction:

$$\begin{aligned}
&\hat{x}_{n-j}\hat{\partial}_{n-j}(\hat{x}_{i+1})^{m_{i+1}}\cdots(\hat{x}_n)^{m_n} \\
&= \; q^{2\sum\limits_{k=0}^{j-1} m_{n-k}}[[m_{n-j}]]_{q^2}(\hat{x}_{i+1})^{m_{i+1}}\cdots(\hat{x}_n)^{m_n} \\
&\quad + q^{\sum\limits_{k=0}^{j-1} m_{n-k}}(\hat{x}_{i+1})^{m_{i+1}}\cdots(\hat{x}_{n-j})^{m_{n-j}+1}\cdots(\hat{x}_n)^{m_n}\hat{\partial}_{n-j} \\
&\quad + q^{\sum\limits_{k=0}^{j-1} m_{n-k}}(q^2-1)[[m_{n-j}]]_{q^2}\sum_{l=0}^{j} q^{\sum\limits_{s=0}^{j-l-1} m_{n-l-s}}(\hat{x}_{i+1})^{m_{i+1}}\cdots(\hat{x}_{n-l})^{m_{n-l}+1}\cdots(\hat{x}_n)^{m_n}\hat{\partial}_{n-l}.
\end{aligned}$$

With the abbreviations

$$\begin{aligned}
\hat{P}_{\pm}(i) := \hat{P}(\pm i) &\equiv (\hat{x}_1)^{m_1}\cdots(\hat{x}_i)^{m_i\pm 1}\cdots(\hat{x}_n)^{m_n} \\
\hat{P}(0) &:= (\hat{x}_1)^{m_1}\cdots(\hat{x}_n)^{m_n}
\end{aligned}$$

we obtain for the Leibniz rule for monomials on the non-commutative algebra

$$\begin{aligned}
\hat{\partial}_i\hat{P}(0) &= q^{\sum\limits_{k=1}^{i-1} m_k + \sum\limits_{k=i+1}^{n} 2m_k}[[m_i]]_{q^2}\hat{P}(-i) + q^{m_i+\sum\limits_{k=1}^{i-1} m_k}\hat{P}(0)\hat{\partial}_i \\
&\quad + q^{\sum\limits_{k=1}^{i-1} m_k}(q^2-1)[[m_i]]_{q^2}\sum_{j=0}^{n-i-1} q^{\sum\limits_{k=0}^{j-1} m_{n-k}} \cdot \Big[\hat{P}(+(n-j),-i)\hat{\partial}_{n-j} + \\
&\qquad + (q^2-1)[[m_{n-j}]]_{q^2}\sum_{l=0}^{j} q^{\sum\limits_{s=0}^{j-l-1} m_{n-l-s}}\hat{P}(+(n-l),-i)\,\hat{\partial}_{n-l}\Big].
\end{aligned} \tag{3.2}$$

Since we found an appropriate $\star-$product we now can work with the commutative algebra and introduce the Jackson derivatives

$$D^i_{q^a}f := \frac{f-f(q^a x_i)}{(1-q^a)x_i},$$

where f is an arbitrary power series in the commutative space according to [3].

Equation. (3.2) now reads as

$$\begin{aligned}\partial_i P(0) &= q^{\sum\limits_{k=1}^{i-1} m_k + \sum\limits_{k=i+1}^{n} 2m_k} D^i_{q^2} P(0) + q^{m_i + \sum\limits_{k=1}^{i-1} m_k} P(0)\partial_i \\ &+ q^{\sum\limits_{k=1}^{i-1} m_k} (q^2-1) D^i_{q^2} \sum_{j=0}^{n-i-1} q^{\sum\limits_{k=0}^{j-1} m_{n-k}} x_{n-j} \cdot \\ &\cdot \Big[P(0)\partial_{n-j} + (q^2-1) \sum_{l=0}^{j} q^{\sum\limits_{s=0}^{j-l-1} m_{n-l-s}} x_{n-l} D^{n-j}_{q^2} P(0) \partial_{n-l} \Big].\end{aligned}$$

To simplify the notation we introduce scaling operators

$$L^i_a \hat{x}_i^{m_i} := (q^a \hat{x}_i)^{m_i}$$

with the composition

$$L^{i,j}_a := L^i_a \cdots L^{i+j}_a.$$

With this the Leibniz rule of the n-dimensional q-deformed Euclidean space represented on the commuting n-dimensional Euclidean space reads as

$$\begin{aligned}\partial_i f = D^i_{q^2} L^{1,i-1}_1 L^{i+1,n}_2 f + L^{1,n}_1 L^i_1 f \partial_i + (q^2-1) D^i_{q^2} L^{1,i-1}_1 \sum_{j=0}^{n-i-1} x_{n-j} \cdot \\ \cdot \Big[L^{n-j+1,n}_1 f \partial_{n-j} + (q^2-1) D^{n-j}_{q^2} \sum_{l=0}^{j} x_{n-l} L^{n-j+1,n}_1 L^{n-j+1,n-l} f \partial_{n-l} \Big].\end{aligned}$$

Note that we find no higher orders in the derivatives D^i. The main reason is that from the beginning we postulated a classical limit for the whole deformation and derivatives would be contradictory to this. That also holds for every single step of the calculation, so also in (3.2) we only insert terms with classical limit. Altogether we find terms quadratic in derivatives D^k at the most.

For the action of the derivatives on functions we find (analogue to [2])

$$\partial_i \triangleright f(x_1, \ldots, x_n) = D^i_{q^2} L^{1,i-1}_1 L^{i+1,n}_2 f(x_1, \ldots, x_n),$$

which is the same result as in [21] up to isomorphism.

Chapter 4

Integration on $q-$deformed Quantum spaces

4.1 Ideas and interpretation

The geometrical interpretation of integration is marked by the approach of Riemann. His idea was to calculate the area under a function by summing up the rectangles determined by the difference of two points with the according function value. To obtain the exact value of the needed area he took the limit of the difference of the two points going to zero which lead to the transition between the discrete sum and a continuous integration by the postulation of continuous points. Since we want to follow this very basic construction we start with the basics of Riemann summation. These basics we want to use in a generalised form.

Riemann sums are defined on a closed interval $I = [a, b]$ with a partition $a = x_0 < x_1 < \ldots < x_n = b$ where an arbitrary $\xi_k \in [x_{k-1}, x_k]$. Then the Riemann sum of a function $f : [a, b] \to \mathbb{R}$ is

$$\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}).$$

For $(x_k - x_{k-1}) \to 0$ the Riemann sum converges to the Riemann integral $\int_I f(x)\,\mathrm{d}x$.

The proper way to extend the Riemann integral to infinite integration limits is to write it as an infinite sum of proper integrals:

$$\lim_{a \to -\infty} \int_a^b f(x)\,\mathrm{d}x = \int_{-\infty}^{b} f(x)\,\mathrm{d}x = \int_{a_0}^{b} f(x)\,\mathrm{d}x + \int_{a_{-1}}^{a_0} f(x)\,\mathrm{d}x + ...$$

where $b > a_0 > a_{-1} > \ldots$. This integral only converges, if f decreases fast enough in infinity like every physical relevant function. We now express the single integral in terms of Riemann sums:

$$\int_{a_{i-1}}^{a_i} f(x)\,\mathrm{d}x = \lim_{\Delta \to 0} \sum_{k_i=1}^{n} f(\xi_{k_i})(x_{k_i} - x_{k_i-1})$$

with $\Delta := (x_{k_i} - x_{k_i-1})$, $a_{i-1} < x_0 < \ldots < a_i$ and $\xi_{k_i} \in [x_{k_i-1}, x_{k_i}]$ arbitrary, so we can always choose the left limit of the interval $[x_{k_i-1}, x_{k_i}]$, and still the formula holds:

$$\int_{a_{i-1}}^{a_i} f(x)\,\mathrm{d}x = \lim_{\Delta\to 0} \sum_{k_i=1}^{n} f(x_{k_i})(x_{k_i} - x_{k_i-1}).$$

With this the improper integral reads as

$$\int_{-\infty}^{b} f(x)\,\mathrm{d}x = \sum_{i=-\infty}^{0} \int_{a_{i-1}}^{a_i} f(x)\,\mathrm{d}x = \lim_{\Delta\to 0} \sum_{i=-\infty}^{0} \sum_{k_i=1}^{n} f(x_{k_i})(x_{k_i} - x_{k_i-1}).$$

This double summation we replace by summation over all single partition points renamed:

$$\sum_{i=-\infty}^{0} \sum_{k_i=1}^{n} f(x_{k_i})(x_{k_i} - x_{k_i-1}) = \sum_{i=-\infty}^{0} f(x_i)(x_i - x_{i-1}).$$

With this we now have for the integral with $\Delta_i := |x_i - x_{i-1}|$

$$\int_{-\infty}^{b} f(x)\,\mathrm{d}x = \lim_{\Delta_i\to 0} \sum_{i=-\infty}^{0} f(x_i)\Delta_i.$$

In an analogous way we can extend this integral also to an infinite upper limit:

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{\Delta_i\to 0} \sum_{i=-\infty}^{\infty} f(x_i)\Delta_i.$$

For a quantised space with an induced lattice there is usually a way to express all x_i by a starting point and the lattice. Therefore we also find such an expression for Δ_i, which does not vanish for a lattice, so we can drop out the limit. This simplifies the calculations a lot.

In our case of quantum spaces there is indeed a lattice structure induced by the action of the derivatives (see [2]). This structure suggests that the integral can be written as a kind of Riemann sum as in the classical case [27].
The ansatz for our integral therefore is close to the Riemann sum. In one dimension it reads as

$$J(f) = \sum_{n=-\infty}^{+\infty} f(x_n)\rho_n$$

where ρ_n is the weight of the integral.

For obtaining the initially unknown parameters x_n and ρ_n in this equation we do not need many conditions: it suffices to postulate translation invariance for the integral and ask for the Riemann integral being the classical limit of our integral. The translation invariance of the integral $J(\partial \triangleright f) = 0$ gives us the arguments x_n in detail and also the weight ρ_n up to a constant c. We denote by $\partial\triangleright$ the already deformed action on the commutative space, which includes the $\star-$structure according to [2, 3]. This works best for lattice structures of the form $f(x_n) - f(x_{n+1})$, which we get for all $q-$deformed Euclidean spaces. The remaining constant c is determined by the condition of the right classical limit for $q \to 1$.

4.2 1-dimensional quantum space with an explicit example

To clarify our approach we take a look at a simple algebra for which we assume to know a $\star$-product. The commutation relation for the partial derivative $\hat{\partial}$ and the coordinate $\hat{x}$ here reads as

$$\hat{\partial}\hat{x} = 1 + q^2\hat{x}\hat{\partial}$$

which directly leads to $\partial \triangleright x = 1$. But we want to know the action of $\hat{\partial}$ on formal power series of $\hat{x}$, so we need the Leibniz rule for arbitrary powers of the coordinate

$$\hat{\partial}\hat{x}^n = [[n]]_{q^2}\hat{x}^{n-1} + q^{2n}\hat{x}^n\hat{\partial}.$$

Using this we get for the action in commutative space

$$\partial \triangleright f = [[n]]_{q^2}x^{-1}f = \frac{(1-L_2)f}{(1-q^2)x}, \tag{4.1}$$

where L_a is a scaling operator: $L_a f(x) := f(q^a x)$ and f is a formal power series as usual. This scaling is characteristic for the lattice Γ_n

$$\Gamma_n = \left\{x_n | x_n = (L_2)^n x_0 = q^{2n}x_0;\ n \in \mathbb{Z}\right\}.$$

It consists of points x_n, defined by an initial point x_0 and the scaling operator L_2 :

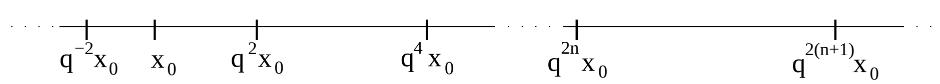

Figure 4.1: Lattice induced by (4.1)

According to the previous statements we define the integral as sum over the functions values of all points with an unknown weight ρ_n:

$$J(f) = \sum_{n=-\infty}^{+\infty} f((L_2)^n x_0)\rho_n.$$

Using the translation invariance we obtain for the integral of the derivative

$$J(\partial \triangleright f) = \sum_{n=-\infty}^{+\infty} \frac{f(q^{2n}x_0) - f(q^{2n+2}x_0)}{(1-q^2)q^{2n}x_0}\rho_n \overset{!}{=} 0 \qquad \Rightarrow \qquad \rho_n = q^{2n} \cdot c \tag{4.2}$$

with $c =$ const.
The last constraint is the postulation of the classical limit of this integral being the Riemann integral. For this we calculate the difference between to lattice points:

$$\Delta x_n = |x_{n+1} - x_n| = (1 - q^2) q^{2n} x_0, \quad 1 \geq |q| \in \mathbb{C}.$$

We can see immediately that our integral formula is a modified Riemann sum: every summand is the product of the length of the interval with the function's value at the left point of the interval (see figure 4.2), but this time with a specific weight modifying the usual Riemann sum. For obtaining an ordinary Riemann integral in the classical limit we

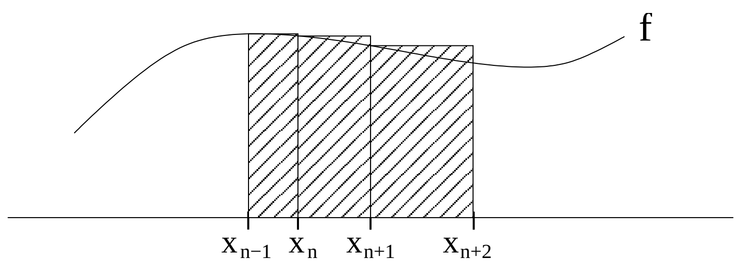

Figure 4.2: Geometrical interpretation of the integral

have to show that for our integral $\Delta x_n \xrightarrow{q \to 1} 0$ for any n, which is obviously fulfilled in the one-dimensional example. Δx_n provides the missing summation independent constant factor we need for the classical limit. For this example we find for $c = (1 - q^2) x_0$ which we insert in equation (4.2).
The new integral in the first example then reads as

$$J(f) = (1 - q^2) x_0 \sum_{n=-\infty}^{+\infty} q^{2n} f(q^{2n} x_0).$$

But if we take a closer look at the above formula for the integral, we see that for $q \to 1$ we only cover the positive half line (or the negative half line, depending on the sign of x_0). Therefore we have to add the same term also for negative (positive) x_0

$$J(f) = (1 - q^2) x_0 \sum_{n=-\infty}^{+\infty} q^{2n} \left(f(q^{2n} x_0) + f(-q^{2n} x_0) \right). \tag{4.3}$$

An explicit example for the new integral is the calculation of the integral of the Gaussian distribution over the whole 1-dimensional space. We take the power series expansion of the function:

$$f(x) = \tfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} = \tfrac{1}{\sqrt{2\pi}} \sum_{i=0}^{\infty} \frac{(-1)^i 2^{-i}}{i!} x^{2i}$$

with which the integral reads as

$$J(f) = (1-q^2)x_0 \sum_{n=-\infty}^{+\infty} q^{2n} \tfrac{1}{\sqrt{2\pi}} \sum_{i=0}^{\infty} \frac{(-1)^i 2^{-i}}{i!} \left(q^{4ni} x_0^{2i} + q^{4ni}(-x_0)^{2i} \right).$$

After some calculation we obtain

$$J(f) = \tfrac{2}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \underbrace{(1-q^2)x_0 q^{2n}}_{\Delta x_n} \underbrace{e^{-\frac{1}{2}(q^{2n}x_0)^2}}_{f(x_n)}.$$

Now we want to compare this result to the result in commutative space by taking the classical limit of Δx_n, which vanishes for $q \to 1$, so we here have the transition to the Riemann integral:

$$\stackrel{q\to 1}{\longrightarrow} \tfrac{2}{\sqrt{2\pi}} \int_0^{\infty} f(x)\,\mathrm{d}x = 1$$

as expected from the classical limit, if we choose $x_0 = 0$.

In an analogous way the quantum spaces discussed in [2, 3] provide a lattice structure in almost the same way. This suggests that a similar ansatz for the integral might also work in the same way.

4.3 3-dimensional Euclidean space

The ansatz we work with is analogue to that in the previous one-dimensional case, but takes into account that there is a 3-dimensional lattice and all 3 dimensions do influence the weight:

$$J(f) = \sum_{i,j,k=-\infty}^{+\infty} f(q^{\alpha i}x_0^+, q^{\beta j}x_0^3, q^{\gamma k}x_0^-)\rho_{ijk}.$$

First we plug in the translation invariance according to the action of the derivatives calculated in [2]

$$\begin{aligned} \partial^- \triangleright f &= -q^{-1} D^+_{q^4} f\,, \\ \partial^3 \triangleright f &= D^3_{q^2} f(q^2 x^+)\,, \\ \partial^+ \triangleright f &= -q D^-_{q^4} f(q^2 x^3) - q\lambda x^+ (D^3_{q^2})^2 f \end{aligned}$$

with $\lambda = q - q^{-1}$. We use the notations explicitly set in appendix A, namely we list only the scaled arguments of the function and the derivatives $D^A_{q^i}$ are the Jackson derivatives.

The first result for the weight and the powers of the function's arguments we obtain from the action of ∂^3, since this is the simplest one

$$\begin{aligned} J(\partial^3 \triangleright f) &= \sum_{i,j,k=-\infty}^{+\infty} \frac{q^{-\beta j}\rho_{ijk}}{(1-q^2)x_0^3} \left(f(q^{\alpha i+2}x_0^+, q^{\beta j}x_0^3, q^{\gamma k}x_0^-) - f(q^{\alpha i+2}x_0^+, q^{\beta j+2}x_0^3, q^{\gamma k}x_0^-)\right) \\ &\stackrel{!}{=} 0. \end{aligned}$$

Performing the summation over j first we see that the integral only vanishes, if we have

$$\rho_{ijk} = q^{\beta j}\rho_{ik} \qquad \text{and} \qquad \beta = 2\,.$$

Inserting the action of ∂^- we obtain by first summing over i

$$\rho_{ik} = q^{4i}\rho_k.$$

Calculation with respect to the derivative ∂^+ gives an anomaly: the sum splits up into two terms with incompatible powers of the coordinates

$$\begin{aligned} &J(\partial^+ \triangleright f) = \\ &= -\tfrac{q}{(1-q^4)x_0^-} \sum_{i,j,k=-\infty}^{+\infty} q^{-\gamma k}\left(f(q^{4i}x_0^+, q^{2j+2}x_0^3, q^{\gamma k}x_0^-) - f(q^{4i}x_0^+, q^{2j+2}x_0^3, q^{\gamma k+4}x_0^-)\right)\rho_{ijk} \\ &+ \tfrac{x_0^+}{(1-q^2)(x_0^3)^2} \sum_{i,j,k=-\infty}^{+\infty} q^{2(4i-j)}\rho_k \Big(f(q^{4i}x_0^+, q^{2j}x_0^3, q^{\gamma k}x_0^-) - f(q^{4i}x_0^+, q^{2j+2}x_0^3, q^{\gamma k}x_0^-) \\ &\qquad -q^{-2}f(q^{4i}x_0^+, q^{2j+2}x_0^3, q^{\gamma k}x_0^-) + q^{-2}f(q^{4i}x_0^+, q^{2j+4}x_0^3, q^{\gamma k}x_0^-)\Big)\,. \end{aligned}$$

These two terms can not cancel each other, so they have to vanish separately to ensure the translation invariance. Fortunately the second of these terms vanishes by itself, because we have an infinite sum and the terms cancel recursively, so we obtain one condition for the weight as usual, leading to (with $c = const.$)

$$\rho_k = q^{4k} \cdot c.$$

For the right classical limit we are doing the obvious calculations of the difference of two arbitrary lattice points in all directions, which leads to

$$\begin{aligned} \Delta x_i^+ &= (1-q^4)q^{4i}x_0^+, \\ \Delta x_j^3 &= (1-q^2)q^{2j}x_0^3, \\ \Delta x_k^- &= (1-q^4)q^{4k}x_0^-. \end{aligned} \tag{4.4}$$

The classical limit then is

$$\lim_{q\to 1}(\Delta x_i^+ \cdot \Delta x_j^3 \cdot \Delta x_k^-) = \lim_{q\to 1}\left(q^{2(i+j+2k)}(1-q^2)(1-q^4)^2\; x_0^+ x_0^3 x_0^-\right) = 0 \tag{4.5}$$

for all $i, j, k \in \mathbb{N}$ which is expected to get the Riemann integral in the classical case. The summand independent constant in this case is $C = (1-q^2)(1-q^4)^2 x_0^+ x_0^3 x_0^-$, so that for the whole integral we obtain

$$J(f) = (1-q^2)(1-q^4)^2 x_0^+ x_0^3 x_0^- \sum_{i,j,k=-\infty}^{+\infty} q^{2(2i+j+2k)} \tilde{f}(q^{4i} x_0^+, q^{2j} x_0^3, q^{4k} x_0^-),$$

where we denote by $\tilde{f}$ the sum

$$\tilde{f}(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) + f(-x_1, x_2, \ldots, x_n) + \ldots + f(-x_1, \ldots, -x_n) \qquad (4.6)$$

including all possible configurations of signs of the arguments.

4.4 4-dimensional Euclidean space

In analogy to the previous quantum space we take as the ansatz for the integral

$$J(f) = \sum_{i,j,k,l=-\infty}^{+\infty} f(q^{\alpha i} x_0^1, q^{\beta j} x_0^2, q^{\gamma k} x_0^3, q^{\delta l} x_0^4)\, \rho_{ijkl}. \qquad (4.7)$$

The actions of the derivatives we need for assuring the translation invariance are [2]:

$$\begin{aligned}
\partial^1 \triangleright f &= q^{-1} D_{q^2}^4 f(qx^2, qx^3)\,, \\
\partial^2 \triangleright f &= D_{q^2}^3 f(qx^1, q^2x^4) - q\lambda x^2 D_{q^2}^1 D_{q^2}^4 f(q^{-1}x^1, qx^2, qx^3)\,, \\
\partial^3 \triangleright f &= D_{q^2}^2 f(qx^1, q^2x^4) - q\lambda x^3 D_{q^2}^1 D_{q^2}^4 f(q^{-1}x^1, qx^2, qx^3)\,, \\
\partial^4 \triangleright f &= qD_{q^2}^1[f + q\lambda(x^2 D_{q^2}^2 + x^3 D_{q^2}^3) f(q^2x^4) + \lambda^2 x^4 D_{q^2}^4 f] \\
&\quad -q^{-1}\lambda x^4 D_{q^{-2}}^2 D_{q^{-2}}^3 f(q^2x^1, qx^2, qx^3, q^2x^4) \\
&\quad -\lambda^2 x^2 x^3 ((D_{q^2}^1)^2 D_{q^2}^4 f)(q^{-2}x^1, qx^2, qx^3)\,.
\end{aligned}$$

We start the determination of the weight with the simplest action $\partial^1 \triangleright f$

$$J(\partial^1 \triangleright f) \overset{!}{=} 0 \quad \Rightarrow \quad \delta = 2,\ \rho_{ijkl} = q^{2l} \rho_{ijk}.$$

Then we take the action of ∂^2 which gives us a similar problem as we had it in the 3-dimensional case: the sum splits up into two incompatible sums which have to vanish separately. This time none of these two terms vanishes by itself so we obtain two conditions for the weight out of one condition of translation invariance:

$$J(\partial^2 \triangleright f) \overset{!}{=} 0 \quad \Rightarrow \quad \left\{ \begin{array}{l} \gamma = 2,\ \rho_{ijk} = q^{2k} \rho_{ij} \\ \alpha = 2,\ \rho_{ijk} = q^{2i} \rho_{jk} \end{array} \right\} \quad \Rightarrow \quad \rho_{ijk} = q^{2(i+k)} \rho_j.$$

With the action of ∂^3 we obtain one last condition for the weight:

$$J(\partial^3 \triangleright f) \overset{!}{=} 0 \quad \Rightarrow \quad \beta = 2,\ \rho_j = q^{2j} \cdot c.$$

Since we did not use the action of ∂^4 for determining the weight, we can now use it to check our results: inserting $\partial^4 \triangleright f$ into (4.7) gives that all terms cancel recursively as expected.

Calculating the difference of two arbitrary lattice points we obtain

$$\begin{aligned} \Delta x_i^1 &= (1-q^2)q^{2i}x_0^1 \\ \Delta x_j^2 &= (1-q^2)q^{2j}x_0^2 \\ \Delta x_k^3 &= (1-q^2)q^{2k}x_0^3 \\ \Delta x_l^4 &= (1-q^4)q^{4l}x_0^4. \end{aligned}$$

The classical limit for the volume element is then

$$\lim_{q\to 1}\left(q^{2(i+j+k+2l)}(1-q^2)^3(1-q^4)x_0^1x_0^2x_0^3x_0^4\right) = 0 \quad \forall i,j,k,l \in \mathbb{N}$$

as wanted for the Riemann integral being the classical limit of our integral. The whole integral then with (4.6) reads as

$$J(f) = (1-q^2)^4 x_0^1x_0^2x_0^3x_0^4 \sum_{i,j,k,l=-\infty}^{+\infty} q^{2(i+j+k+l)}\tilde{f}(q^{2i}x_0^1, q^{2j}x_0^2, q^{2k}x_0^3, q^{2l}x_0^4)$$

with $\tilde{f}$ again the sum over all possible combinations of signs in the arguments of f.

4.5 $q-$deformed Minkowski space

The actions on the Minkowski space are more complicated than on the Euclidean spaces [2, 3]. Therefore we need a lot of preparing calculations in order to understand which scaling operators characterise the lattice of this quantum space.
As proved in [3] one can switch between the two differential calculi of a quantum space by using another ordering of the coordinates and therefore another $\star$-product and some linear transformations. This means that we can choose the differential calculus with easier expressions for our calculations, because the changing between the different orderings concerns all coordinate dependent terms in the same way and does not change the lattice structure itself. We start by writing down explicitly all the basic derivative operators, the so-called New Jackson derivatives (see appendix A and [2]) to find the lattice structure they impose

$$\begin{aligned} (D^3_{1,q})^{k,l}f \;:=\; & \sum_{m=0}^{k}(-1)^{k-m}\tbinom{k+l-m-1}{l-1}\left((1-q^2)x^3\right)^{m-k-l}\tfrac{1}{m!}\tfrac{\partial^m}{\partial(x^3)^m}f \\ & -(-1)^k\sum_{m=0}^{l-1}\tbinom{k+l-m-1}{k}\left((1-q^2)x^3\right)^{m-k-l}q^{-2m}\tfrac{1}{m!}\tfrac{\partial^m}{\partial(x^3)^m}f(q^2x^3), \end{aligned}$$

$$
\begin{aligned}
(D^3_{2,q})^{k,l} f \;:=\; & -(\tfrac{\lambda}{\lambda_+}\tilde{x}^3)^{-k} \sum_{m=0}^{l-1} (x^3 - q^2 y_-)^{m-l} q^{-2m} \tfrac{1}{m!} \tfrac{\partial^m}{\partial y_-^m} f(q^2 y_-) \\
& - \sum_{m=0}^{k-1} (\tfrac{\lambda}{\lambda_+}\tilde{x}^3)^{m-k} \Bigg[\sum_{i=0}^{m} (-1)^{m-i} \tbinom{m+l-i-1}{l-1} ((1-q^2)y_-)^{i-m-l} \tfrac{1}{i!} \tfrac{\partial^i}{\partial y_-^i} f(y_-) \\
& \qquad - (-1)^m \sum_{i=0}^{l-1} \tbinom{m+l-i-1}{m} ((1-q^2)y_-)^{i-m-l} q^{-2i} \tfrac{1}{i!} \tfrac{\partial^i}{\partial y_-^i} f(q^2 y_-) \Bigg].
\end{aligned}
$$

To get the explicit form of $(D^3_{3,q})^{k,l}_{i,j} f$ we have to calculate first

$$
\begin{aligned}
D^{(k,m)}_{q^{-2},1} f \;=\; & \sum_{s=0}^{m} (-1)^{m-s} \tbinom{m+k-s-1}{k-1} (x^3 - q^{-2}x^3)^{s-m-k} \tfrac{1}{s!} \tfrac{\partial^s}{\partial (x^3)^s} f \\
& - (-1)^m \sum_{s=0}^{k-1} \tbinom{m+k-s-1}{m} (x^3 - q^{-2}x^3)^{s-m-k} q^{2s} \tfrac{1}{s!} \tfrac{\partial^s}{\partial (x^3)^s} f(q^{-2}x^3), \\
D^{(k,m)}_{y_+/y_-,1} f \;=\; & \sum_{s=0}^{m} (-1)^{m-s} \tbinom{m+k-s-1}{k-1} (x^3 - \tfrac{y_+}{y_-}x^3)^{s-m-k} \tfrac{1}{s!} \tfrac{\partial^s}{\partial (x^3)^s} f \\
& - (-1)^m \sum_{s=0}^{k-1} \tbinom{m+k-s-1}{m} (x^3 - \tfrac{y_+}{y_-}x^3)^{s-m-k} \tfrac{1}{s!} \tfrac{\partial^s}{\partial (\frac{y_+}{y_-}\, x^3)^s} f(\tfrac{y_+}{y_-}\, x^3).
\end{aligned}
$$

For the last of the New Jackson derivatives we obtain explicitly

$$
\begin{aligned}
(D^3_{3,q})^{k,l}_{i,j} f := & -(x^3 - q^2 y_+)^{-l} (\tfrac{\lambda}{\lambda_+}\tilde{x}^3)^{-i} (x^3 - q^2 y_-)^{-j} \sum_{m=0}^{k-1} (-\tfrac{\lambda}{\lambda_+}\tilde{x}^3)^{m-k} \tfrac{1}{m!} \tfrac{\partial}{\partial y_+^m} f(y_+) \\
- \;& (\tfrac{\lambda}{\lambda_+}\tilde{x}^3)^{-i} (x^3 q^2 y_-)^{-j} \sum_{m=0}^{l-1} (x^3 - q^2 y_+)^{m-l} \cdot \\
& \cdot \Bigg[\sum_{s=0}^{m} (-1)^{m-s} \tbinom{m+k-s-1}{k-1} ((q^2-1)y_+)^{s-m-k} q^{-2s} \tfrac{1}{s!} \tfrac{\partial^s}{\partial y_+^s} f(q^2 y_+) \\
& \qquad - (-1)^m \sum_{s=0}^{k-1} \tbinom{m+k-s-1}{m} ((q^2-1)y_+)^{s-m-k} \tfrac{1}{s!} \tfrac{\partial^s}{\partial y_+^s} f(y_+) \Bigg] \\
- \;& (x^3 - q^2 y_-)^{-j} \sum_{m=0}^{i-1} (\tfrac{\lambda}{\lambda_+}\tilde{x}^3)^{m-i} \cdot \\
& \cdot \Bigg[\sum_{s=0}^{m} (-1)^{m-s} \tbinom{m+l-s-1}{l-1} (y_- - q^2 y_+)^{s-m-l} (-2\tfrac{\lambda}{\lambda_+}\tilde{x}^3)^{-s-k} \cdot \\
& \cdot \Bigg(\sum_{t=0}^{s} (-1)^{s-t} \tbinom{s+k-t-1}{k-1} (-2\tfrac{\lambda}{\lambda_+}\tilde{x}^3)^t \tfrac{1}{t!} \tfrac{\partial^t}{\partial y_-^t} f(y_-) - (-1)^s \sum_{t=0}^{k-1} \tbinom{s+k-t-1}{s} (-2\tfrac{\lambda}{\lambda_+}\tilde{x}^3)^t \tfrac{1}{t!} \tfrac{\partial^t}{\partial y_+^t} f(y_+) \Bigg)
\end{aligned}
$$

$$-(-1)^m \sum_{s=0}^{l-1} \binom{m+l-s-1}{m} (y_- - q^2 y_+)^{s-m-l} ((q^2-1)y_+)^{-s-k} (-1)^s \cdot$$

$$\cdot \left(\sum_{t=0}^{s} (-1)^t \binom{s+k-t-1}{k-1} ((q^2-1)y_+)^t q^{-2t} \frac{1}{t!} \frac{\partial^t}{\partial y_+^t} f(q^2 y_+) - \sum_{t=0}^{k-1} \binom{s+k-t-1}{s} ((q^2-1)y_+)^t \frac{1}{t!} \frac{\partial^t}{\partial y_+^t} f(y_+) \right) \Bigg]$$

$$- \sum_{m=0}^{j-1} (x^3 - q^2 y_-)^{m-j} \cdot$$

$$\cdot \Bigg[\sum_{s=0}^{m} (-1)^{m-s} \binom{m+i-s-1}{i-1} ((q^2-1)y_-)^{s-m-i} \sum_{t=0}^{s} (-1)^{s-t} \binom{s+l-t-1}{l-1} (-2q^2 \tfrac{\lambda}{\lambda_+} \tilde{x}^3)^{t-s-l} \cdot$$

$$\cdot \left(\sum_{u=0}^{t} (-1)^{t-u} \binom{t+k-u-1}{k-1} (q^2 y_- - y_+)^{u-t-k} q^{-2u} \frac{1}{u!} \frac{\partial^u}{\partial y_-^u} f(q^2 y_-) \right.$$

$$\left. -(-1)^t \sum_{u=0}^{k-1} \binom{t+k-u-1}{t} (q^2 y_- - y_+)^{u-t-k} \frac{1}{u!} \frac{\partial^u}{\partial y_+^u} f(y_+) \right)$$

$$- \sum_{s=0}^{m} (-1)^{m-s} \binom{m+i-s-1}{i-1} ((q^2-1)y_-)^{s-m-i} \sum_{t=0}^{l-1} \binom{s+l-t-1}{s} (-2q^2 \tfrac{\lambda}{\lambda_+} \tilde{x}^3)^{t-s-l} \cdot$$

$$\cdot \left(\sum_{u=0}^{t} (-1)^{t-u} \binom{t+k-u-1}{k-1} ((q^2-1)y_+)^{u-t-k} q^{-2u} \frac{1}{u!} \frac{\partial^u}{\partial y_+^u} f(q^2 y_+) \right.$$

$$\left. -(-1)^t \sum_{u=0}^{k-1} \binom{t+k-u-1}{t} ((q^2-1)y_+)^{u-t-k} \frac{1}{u!} \frac{\partial^u}{\partial y_+^u} f(y_+) \right)$$

$$-(-1)^m \sum_{s=0}^{i-1} \binom{m+i-s-1}{m} ((q^2-1)y_-)^{s-m-i} \sum_{t=0}^{s} (-1)^{s-t} \binom{s+l-t-1}{l-1} (y_- - q^2 y_+)^{t-s-l} \cdot$$

$$\cdot \left(\sum_{u=0}^{t} (-1)^{t-u} \binom{t+k-u-1}{k-1} (-2 \tfrac{\lambda}{\lambda_+} \tilde{x}^3)^{u-t-k} \frac{1}{u!} \frac{\partial^u}{\partial y_-^u} f(y_-) \right.$$

$$\left. -(-1)^t \sum_{u=0}^{k-1} \binom{t+k-u-1}{t} (-2 \tfrac{\lambda}{\lambda_+} \tilde{x}^3)^{u-t-k} \frac{1}{u!} \frac{\partial^u}{\partial y_+^u} f(y_+) \right)$$

$$- \sum_{s=0}^{i-1} \binom{m+i-s-1}{m} ((q^2-1)y_-)^{s-m-i} \sum_{t=0}^{l-1} \binom{s+l-t-1}{s} (y_- - q^2 y_+)^{t-s-l} \cdot$$

$$\left(\sum_{u=0}^{t} (-1)^{t-u} \binom{t+k-u-1}{k-1} ((q^2-1)y_+)^{u-t-k} q^{-2u} \frac{1}{u!} \frac{\partial^u}{\partial y_+^u} f(q^2 y_+) \right.$$

$$\left. -(-1)^t \sum_{u=0}^{k-1} \binom{t+k-u-1}{t} ((q^2-1)y_+)^{u-t-k} \frac{1}{u!} \frac{\partial^u}{\partial y_+^u} f(y_+) \right) \Bigg] .$$

The abbreviations and notations can be found in appendix A, some of them also in [2, 3]. Now we can write down the actions of the derivatives explicitly and therefore get the lattice structure of the $q-$deformed Minkowski space. In this section we just write down the two simple cases of $\hat{\tilde{\partial}}^3 \triangleright f$ and $\hat{\partial}^- \triangleright f$ to show the problems appearing here and giving just the interpretation of the results of the two other derivatives $\hat{\partial}^+ \triangleright f$ and $\hat{\partial}^0 \triangleright f$. The explicit actions for these can be found in appendix B.

$$\begin{aligned}
\hat{\tilde{\partial}}^3 \triangleright f \;=\; & \sum_{k=0}^{\infty} \alpha_+^k \sum_{0 \le i+j \le k} (M^-)^k_{i,j}(\vec{x}) \cdot \qquad (4.8)\\
& \cdot \Bigg[-(\tfrac{\lambda}{\lambda_+} q^{2j} \tilde{x}^3)^{-i} \sum_{m=0}^{i} (\tilde{x}^3_j - q^2 \tilde{y}^j_-)^{m-i-1} q^{-2m} \tfrac{1}{m!} \tfrac{\partial^m}{\partial (\tilde{y}^j_-)^m} f(q^2 x^+, \tilde{y}^j_-) \\
& \quad - \sum_{m=0}^{i-1} (\tfrac{\lambda}{\lambda_+} q^{2j} \tilde{x}^3)^{m-i} ((1-q^2)\tilde{y}^j_-)^{-m-i-1} \cdot \\
& \qquad \cdot \Bigg(\sum_{s=0}^{m} (-1)^{m-s} \tbinom{m+i-s}{i} ((1-q^2)\tilde{y}^j_-)^s \tfrac{1}{s!} \tfrac{\partial^s}{\partial (\tilde{y}^j_-)^s} f(q^2 x^+, \tilde{y}^j_-) \\
& \qquad - (-1)^m \sum_{s=0}^{i} \tbinom{m+i-s}{m} ((1-q^2)\tilde{y}^j_-)^s q^{-2s} \tfrac{1}{s!} \tfrac{\partial^s}{\partial (\tilde{y}^j_-)^s} f(q^2 x^+, q^2 \tilde{y}^j_-) \Bigg) \Bigg],
\end{aligned}$$

$$\begin{aligned}
\hat{\partial}^- \triangleright f = & -\tfrac{q^{-1}}{1-q^2} (x^+)^{-1} (f - f(q^2 x^+)) \qquad (4.9)\\
& -\tfrac{\lambda}{\lambda_+} \sum_{k=0}^{\infty} \alpha_+^k \sum_{0 \le i+j \le k} (M^+)^k_{i,j}(\vec{x}) x^- \, (\tilde{x}^3_{j+1} - y_-)^{i+1} \cdot \\
& \cdot \sum_{m=0}^{i} \Bigg[(\tilde{x}^3_{j+1} - q^2 \tilde{y}^{j+1}_-)^{m-i-1} q^{-2m} \tfrac{1}{m!} \tfrac{\partial^m}{\partial (\tilde{y}^{j+1}_-)^m} f(q^2 x^+, q^{2(j+1)} \tilde{x}^3, q^2 \tilde{y}^{j+1}_-) \\
& \quad + (\tfrac{\lambda}{\lambda_+} q^{2(j+1)} \tilde{x}^3)^m ((1-q^2)\tilde{y}^{j+1}_-)^{-m-i-1} \cdot \\
& \qquad \cdot \Bigg(\sum_{s=0}^{m} (-1)^{m-s} \tbinom{m+i-s}{i} ((1-q^2)\tilde{y}^{j+1}_-)^s \tfrac{1}{s!} \tfrac{\partial^s}{\partial (\tilde{y}^{j+1}_-)^s} f(q^2 x^+, q^{2(j+1)} \tilde{x}^3, \tilde{y}^{j+1}_-) \\
& \qquad - (-1)^m \sum_{s=0}^{i} \tbinom{m+i-s}{m} ((1-q^2)\tilde{y}^{j+1}_-)^s q^{-2s} \tfrac{1}{s!} \tfrac{\partial^s}{\partial (\tilde{y}^{j+1}_-)^s} f(q^2 x^+, q^{2(j+1)} \tilde{x}^3, q^2 \tilde{y}^{j+1}_-) \Bigg) \Bigg] \\
& + q^{-2} \tfrac{1}{\lambda_+} \sum_{k=0}^{\infty} \alpha_+^k \sum_{0 \le i+j \le k} (M^-)^k_{i,j}(\vec{x}) \tfrac{q^{2j} \tilde{x}^3}{q^2 x^+} (\tfrac{\lambda}{\lambda_+} q^{2j} \tilde{x}^3)^{-i} \cdot \\
& \cdot \Bigg[\sum_{m=0}^{i} (\tilde{x}^3_j - q^2 \tilde{y}^j_-)^{m-i-1} q^{-2m} \tfrac{1}{m!} \tfrac{\partial^m}{\partial (\tilde{y}^j_-)^m} (f(q^2 x^+, q^{2j} \tilde{x}^3, q^2 \tilde{y}^j_-) - f(q^4 x^+, q^{2j} \tilde{x}^3, q^2 \tilde{y}^j_-)) \\
& \quad + \sum_{m=0}^{i-1} (-1)^m (\tfrac{\lambda}{\lambda_+} q^{2j} \tilde{x}^3)^{m-i-1} ((1-q^2)\tilde{y}^j_-)^{-m-i-1} \cdot \\
& \qquad \cdot \Bigg(\sum_{s=0}^{m} (-1)^s \tbinom{m+i-s}{i} ((1-q^2)\tilde{y}^j_-)^s \tfrac{1}{s!} \tfrac{\partial^s}{\partial (\tilde{y}^j_-)^s} (f(q^2 x^+, q^{2j} \tilde{x}^3, q^2 \tilde{y}^j_-) - f(q^4 x^+, q^{2j} \tilde{x}^3, q^2 \tilde{y}^j_-))
\end{aligned}$$

$$-\sum_{s=0}^{i}\binom{m+i-1}{m}((1-q^2)\tilde{y}_-^j)^s q^{-2s}\tfrac{1}{s!}\tfrac{\partial^s}{\partial(\tilde{y}_-^j)^s}(f(q^2x^+,q^{2j}\tilde{x}^3,q^2\tilde{y}_-^j)-f(q^4x^+,q^{2j}\tilde{x}^3,q^2\tilde{y}_-^j))\Bigg)\Bigg].$$

Looking for the usual lattice structure necessary for our ansatz for the integral we are almost lost in this space: equation (4.8) gives us no structure of the form $f(x)-f(q^\alpha x)$ and all the other actions give back this structure only in a few terms.
These regular terms give for (4.9) the scalar operator L_2^+, in equation (B.1) we find $L_2^{\tilde{3}}$ and L_2^- and in equation (B.2) the scaling operators are L_2^+, $L_2^{\tilde{3}}$ and L_2^-. For these terms it does not matter that the coordinate x^3 is replaced by $\tilde{y}_+^k$ in some terms, but in others by $\tilde{y}_-^k$, since they still cancel each other. The summation causing the cancelling does affect other coordinates while the substitution remains the same in the affected term. Considering only these terms we would get the postulated translation invariance of the integral.
The irregular terms do not show any structure that would be compatible with the postulation of translation invariance, because they do not cancel each other, no matter which summation we try first. Although we also tried some variations we could not find one for which our ansatz for the integral works.

4.6 $\kappa-$deformed Minkowski space

The $\kappa-$deformed Minkowski space shows a less complicated deformation structure that its $q-$deformed analogue. Since a $\star-$product for this space is already known the only relations for the construction of our integral are [12]:

$$\begin{aligned}
[\hat{\partial}_n,\hat{x}^i] &= 0\\
[\hat{\partial}_n,\hat{x}^n] &= \eta_n^n\\
[\hat{\partial}_i,\hat{x}^j] &= \eta_i^j\\
[\hat{\partial}_i,\hat{x}^n] &= -ia\eta^{nn}\hat{\partial}_i.
\end{aligned}\tag{4.10}$$

Here are $i,j=0,\ldots,n-1$ and $\mu,\nu=0,\ldots,n$. The metric of this space is $\eta^{\mu\nu}=diag(1,-1,\ldots,-1)$. We do not insert the components of the metric, because then one could easily switch to the Euclidean space.

Because we want to treat this space in the same way as the other ones we need to calculate the action of the derivatives explicitly in any argument. We have

$$\begin{aligned}
\hat{\partial}_n(\hat{x}^i)^m &= (\hat{x}^i)\hat{\partial}_n\\
\hat{\partial}_n(\hat{x}^n)^m &= m\eta_m^n(\hat{x}^n)^{m-1}+(\hat{x}^n)^m\hat{\partial}_n\\
\hat{\partial}_i(\hat{x}^j)^m &= m\eta_i^j(\hat{x}^j)^{m-1}+(\hat{x}^j)^m\hat{\partial}_i\\
\hat{\partial}_i(\hat{x}^n)^m &= (\hat{x}^n-ia\eta^{nn})^m\hat{\partial}_i \stackrel{!}{=} (\tilde{\hat{x}}^n)^m\hat{\partial}_i.
\end{aligned}$$

For the whole monomial it is

$$\begin{aligned}
\hat{\partial}_i(\hat{x}^0)^{k_0}\cdots(\hat{x}^n)^{k_n} &= k_i\eta_i^i(\hat{x}^0)^{k_0}\cdots(\hat{x}^i)^{k_i-1}\cdots(\hat{x}^n)^{k_n}+(\hat{x}^0)^{k_0}\cdots(\tilde{\hat{x}}^n)^{k_n}\hat{\partial}_i\\
\hat{\partial}_n(\hat{x}^0)^{k_0}\cdots(\hat{x}^n)^{k_n} &= k_n\eta_n^n(\hat{x}^0)^{k_0}\cdots(\hat{x}^n)^{k_n-1}+(\hat{x}^0)^{k_0}\cdots(\hat{x}^n)^{k_n}\hat{\partial}_n.
\end{aligned}$$

Then we obtain for the action in the commutative space

$$\begin{aligned} \partial_i \triangleright f &= \eta_i^i \partial_i f \\ \partial_n \triangleright f &= \eta_n^n \partial_n f \end{aligned} \tag{4.11}$$

which are just the ordinary derivatives on commutative space. This is surprising, because this might lead to an ordinary integration for an element of a general function space because of the weak deformation of this space.

Nevertheless there is a deformation which shows up, if we apply the derivative on a product of functions. For this we first introduce a coordinate transformation we will need in the following:

$$\hat{x}^n \rightarrow \tilde{\hat{x}}^n = \hat{x}^n - ia\eta^{nn}.$$

This can be interpreted as first order of the more general transformation

$$\hat{x}^n \rightarrow \tilde{\hat{x}}^n = e^{-ia\eta^{nn}\hat{\partial}_n}\hat{x}^n.$$

Applied to an arbitrary power series in all coordinates we obtain

$$f \rightarrow f(\tilde{\hat{x}}^n) = e^{-ia\eta^{nn}\hat{\partial}_n} f(\hat{x}^n) \stackrel{!}{=} \tilde{f}(\hat{x}).$$

With these transformations we find for the action of the derivatives on a product of functions[1] or Leibniz rule

$$\begin{aligned} \partial_n \triangleright (f \cdot g) &= (\eta_n^n \partial_n f) \cdot g + f \cdot (\eta_n^n \partial_n g) \\ \partial_i \triangleright (f \cdot g) &= (\eta_i^i \partial_i f) \cdot g + \tilde{f}(x) \cdot (\eta_i^i \partial_i g) \end{aligned} \tag{4.12}$$

as we expect it from the results of [12]. This means that we just obtain a lattice structure, if we let the derivative act on a product of functions, but not on a single function of the function space.

Therefore one takes a closer look on the product of two functions the action of which provides a lattice structure. Then there would be two possibilities for an ansatz of the integral: the scalar operator should act only on the first function as induced by the Hopf structure or it acts on the product of functions.

In the first case the ansatz would be

$$J(f \cdot g) = \sum_{k=-\infty}^{\infty} \rho_k \, T^k(f) \cdot g$$

[1] We commute the coordinates step by step with the derivatives as done for one function.

where $T: \quad f \to \tilde{f}, \quad T = e^{-ia\eta^{nn}\partial_n}$. Now inserting the action $\partial_i \triangleright (f \cdot g)$ we get incompatible (in the sense of not-cancelling) terms, since in the first term we find a partial derivative of f and in the second term a derivative of g which cannot cancel in general:

$$J(\partial_i \triangleright (f \cdot g)) = \eta_i^i \sum_{k=-\infty}^{\infty} \rho_k [(e^{-iak\eta^{nn}\partial_n} \partial_i f) \cdot g + (e^{-ia(k+1)\eta^{nn}\partial_n} f) \cdot (\partial_i g)].$$

The second ansatz is

$$J(f \cdot g) = \sum_{k=-\infty}^{\infty} \rho_k \, T^k (f \cdot g).$$

To calculate this we first need to know how T^k acts on a product of functions:

$$e^{-iak\eta^{nn}\partial_n}(f \cdot g) = \sum_{l=0}^{\infty} \tfrac{1}{l!}(-iak\eta^{nn})^l \sum_{j=0}^{l} \tbinom{l}{j} (\partial_n^{l-j} f)(\partial_n^j g).$$

With this we have

$$J(\partial_i \triangleright (f \cdot g)) = \sum_{l=0}^{\infty} \tfrac{(-ia\eta^{nn})^l}{l!} \sum_{j=0}^{l} \tbinom{l}{j} \sum_{k=-\infty}^{\infty} k^l \rho_k [(\partial_n^{l-j} \partial_i f)(\partial_n^j g) + (\partial_n^{l-j} e^{-ia\eta^{nn}\partial_n} f)(\partial_n^j \partial_i g)]$$

and again there is no way to make the two terms with the derivatives cancel in general. This means that also this weaker deformed Minkowski space can not be treated by the ansatz we make for the integral.

4.7 $n-$dimensional Euclidean space

The action of derivatives we have to deal with in this most general Euclidean case we calculated in chapter 3. For completeness we give it here again:

$$\begin{aligned} \partial_i \triangleright f &= D^i_{q^2} L_1^{1,i-1} L_2^{i+1,n} f \\ &= \tfrac{1}{1-q^2}(x_i)^{-1} \Big[f(qx_1, .., qx_{i-1}, \mathbf{x}_i, q^2 x_{i+1}, .., q^2 x_n) - f(qx_1, .., qx_{i-1}, \mathbf{q}^2\mathbf{x}_i, q^2 x_{i+1}, .., q^2 x_n) \Big]. \end{aligned}$$

We see at once that this action leads to a q^2-scaling in all coordinates, so our ansatz for the integral is quite simple this time:

$$J(f) = \sum_{i=1}^{n} \sum_{k_i=-\infty}^{+\infty} f(q^{2k_1} x_1^0, \ldots, q^{2k_n} x_n^0) \rho_{k_1 \ldots k_n},$$

where the x_i^0 are fixed values of the according coordinates.

To check our integral formula we insert $\partial_j \triangleright f$:

$$\begin{aligned} J(\partial_j \triangleright f) &= \sum_{i=1}^{n} \sum_{k_i=-\infty}^{+\infty} \rho_{k_1..k_n} \tfrac{1}{1-q^2} (q^{2k_j} x_j^0)^{-1} \cdot \\ &\qquad \cdot \Big[f(q^{2k_1+1}x_1^0, ., q^{2k_{j-1}+1}x_{j-1}^0, \mathbf{q^{2k_j} x_j}, q^{2k_{j+1}+2}x_{j+1}^0, ., q^{2k_n+2}x_n^0) - \\ &\qquad - f(q^{2k_1+1}x_1^0, ., q^{2k_{j-1}+1}x_{j-1}^0, \mathbf{q^{2k_j+2} x_j^0}, q^{2k_{j+1}+2}x_{j+1}^0, ., q^{2k_n+2}x_n^0) \Big] \\ &\stackrel{!}{=} 0. \end{aligned}$$

First we take the sum along k_j to get the property of $\rho_{k_1 \ldots k_n} = \rho_{k_1} \cdot \ldots \cdot \rho_{k_n} \cdot c$. We find $\rho_j = q^{2k_j}$ to fulfil the translation invariance and since this formula is not restricted concerning j it also works for any other j, so altogether we find

$$\rho_{k_1 \ldots k_n} = q^{2(k_1 + \ldots + k_n)} \cdot c.$$

To determine c we come back to the classical limit of the integral and calculate the difference between two lattice points in x_j direction to be

$$\Delta(x_j)_{k_j} = |(x_j)_{k_j+1} - (x_j)_{k_j}| = (1-q^2) q^{k_j} x_j^0.$$

Since every direction enters the formula for c to assure the classical limit, we get this factor for every direction. The obtained volume element obviously vanishes for $q \to 1$, therefore we obtain $c = (1-q^2)^n q^{2(k_1+\ldots+k_n)} x_1^0 \cdot \ldots \cdot x_n^0$ and thus

$$\rho_{k_1 \ldots k_n} = (1-q^2)^n q^{2(k_1+\ldots+k_n)} x_1^0 \cdot \ldots \cdot x_n^0.$$

Together with the summation over all possible signs in the arguments the integral for the $n-$dimensional $q-$deformed Euclidean space reads

$$J(f) = (1-q^2)^n x_1^0 \cdot \ldots \cdot x_n^0 \sum_{i=1}^{n} \sum_{k_i=-\infty}^{+\infty} q^{2(k_1+\ldots+k_n)} \tilde{f}(q^{2k_1} x_1^0, \ldots q^{2k_n} x_n^0).$$

Chapter 5

Construction of a Hilbert space

In this section we construct a Hilbert space for the integral we defined in the previous chapter. This enables us to express the integral in terms of a trace of a trace-class operator. For this we take the one-dimensional integral (4.3)

$$J(f) = (1-q^2)x_0 \sum_{n=-\infty}^{+\infty} q^{2n}(f(q^{2n}x_0) + f(-q^{2n}x_0)), \quad q \leq 1, \, x_0 \geq 0$$

but the procedure also works for higher dimensional spaces in the according generalisation.

A Hilbert space is a Banach space with associated norm defined by an inner (scalar) product [32]. Therefore the way to construct a Hilbert space is to build a normed space. A Banach space is a complete metric space and if the norm is defined via an inner (scalar) product, we obtain a Hilbert space.

First of all we have to find a space of functions, for which the integral converges. This space will be infinite dimensional. For f to be an element of this space we demand

$$\Lambda^2 := \{f; \, J(|f|^2) < \infty\}.$$

The scalar product of $f, g \in \Lambda^2$ can be defined as follows

$$\langle f, g \rangle_J := J(\bar{f}g). \tag{5.1}$$

Accordingly the norm is

$$\|f\|_J := \sqrt{\langle f, f \rangle_J} = \sqrt{J(|f|^2)}. \tag{5.2}$$

To show that this is a proper definition of a norm, we have to prove that

$$\|\lambda f\|_J = |\lambda| \cdot \|f\|_J, \quad \lambda \in \mathbb{C}, f \in \Lambda^2 \tag{5.3}$$

$$\|f+g\|_J \leq \|f\|_J + \|g\|_J, \quad f, g \in \Lambda^2 \quad \text{(Minkowski inequality)} \tag{5.4}$$

$$\|f\|_J = 0 \Rightarrow f = 0. \tag{5.5}$$

We start with (5.2). Using (5.3) we obtain

$$\begin{aligned}
\|\lambda f\|_J &= \sqrt{J(|\lambda f|^2)} = \sqrt{J(|\lambda|^2 \cdot |f|^2)} \\
&= \sqrt{(1-q^2)x_0 \sum_{n=-\infty}^{+\infty} q^{2n}|\lambda|^2(|f(q^{2n}x_0)|^2 + |f(-q^{2n}x_0)|^2)} \\
&= |\lambda|\sqrt{(1-q^2)x_0 \sum_{n=-\infty}^{+\infty} q^{2n}(|f(q^{2n}x_0)|^2 + |f(-q^{2n}x_0)|^2)} \\
&= |\lambda| \cdot \sqrt{J(|f|^2)} = |\lambda| \cdot \|f\|_J.
\end{aligned}$$

Now we show that the scalar product $\langle f, g\rangle_J$ converges absolutely and additionally gives the Hölder inequality. We find the following estimate for the absolute value of the partial sum of the corresponding series

$$\begin{aligned}
&|1-q^2|x_0\Big[\sum_{k=-n}^{n} |q^k \bar{f}(q^{2k}x_0)| \cdot |q^k g(q^{2k}x_0)| + \sum_{k=-n}^{n} |q^k \bar{f}(-q^{2k}x_0)| \cdot |q^k g(-q^{2k}x_0)|\Big] \\
&\leq |1-q^2|x_0\sqrt{\sum_{k=-n}^{n} q^{2k}(|f(q^{2k}x_0)|^2 + |f(-q^{2k}x_0)|^2)} \cdot \sqrt{\sum_{k=-n}^{n} q^{2k}(|g(q^{2k}x_0)|^2 + |g(-q^{2k}x_0)|^2)} \\
&\leq |1-q^2|x_0\sqrt{\sum_{k=-\infty}^{\infty} q^{2k}(|f(q^{2k}x_0)|^2 + |f(-q^{2k}x_0)|^2)} \cdot \sqrt{\sum_{k=-\infty}^{\infty} q^{2k}(|g(q^{2k}x_0)|^2 + |g(-q^{2k}x_0)|^2)} \\
&= \sqrt{J(|f|^2)} \cdot \sqrt{J(|g|^2)} < \infty,
\end{aligned} \tag{5.6}$$

where for the first inequality we used the usual Cauchy-Schwarz inequality in the form

$$\sum_{k=-n}^{n} a_k b_k + \sum_{k=-n}^{n} a'_k b'_k \leq \sqrt{\sum_{k=-n}^{n}(a_k^2 + a_k'^2)}\sqrt{\sum_{k=-n}^{n}(b_k^2 + b_k'^2)}.$$

The inequality (5.6) holds for any $n \in \mathbb{Z}$, because the last two lines are independent of n and therefore do not change, even if we take the limit $n \to \infty$ for the whole estimate. Since the lhs. is equal to $J(\bar{f}g)$ in this limit, it is absolutely convergent and we obtain the Hölder inequality

$$J(\bar{f}g) \leq \sqrt{J(|f|^2)} \cdot \sqrt{J(|g|^2)}. \tag{5.7}$$

We use this result to prove the Minkowski inequality. Defining $h := |f+g|$ leads to $\|h\|_J = \|f+g\|_J$. With this we find

$$J(|f+g| \cdot h) \leq J(|fh|+|gh|) = J(|fh|)+J(|gh|) \overset{(5.7)}{\leq} \sqrt{J(|f|^2)}\sqrt{J(|h|^2)}+\sqrt{J(|g|^2)}\sqrt{J(|h|^2)}.$$

The lhs. of this equation we can rewrite with the definition of h

$$J(|f+g|\cdot h) = J(|f+g|^2) = \|f+g\|_J^2,$$

while the rhs. is

$$(\|f\|_J + \|g\|_J)\cdot\|h\|_J = (\|f\|_J + \|g\|_J)\cdot\|f+g\|_J.$$

Altogether this leads to the Minkowski inequality

$$\|f+g\|_J \leq \|f\|_J + \|g\|_J.$$

The last equation to be proved is (5.5). Since for $\|f\|_J = \sqrt{J(|f|^2)} = 0$ we know that

$$0 = J(|f|^2) = (1-q^2)x_0 \sum_{n=-\infty}^{\infty} q^{2n}[|f(q^{2n}x_0)|^2 + |f(-q^{2n}x_0)|^2].$$

All terms summed up here are positive. Now take $f \neq 0$. Then we have $J(|f|^2) > 0$ which contradicts the assumption $\|f\|_J = 0$. Therefore equation (5.5) holds.

On a normed space constructed this way a metric is induced naturally:

$$\mathrm{d}_J(f,g) := \|f-g\|_J. \tag{5.8}$$

The axioms to be shown are

$$\begin{aligned}
\mathrm{d}_J(f,g) &\geq 0, \quad f,g,h \in \Lambda^2 && (5.9)\\
\mathrm{d}_J(f,g) &= \mathrm{d}_J(g,f), && (5.10)\\
\mathrm{d}_J(f,h) &\leq \mathrm{d}_J(f,g) + \mathrm{d}_J(g,h), && (5.11)\\
\mathrm{d}_J(f,g) = 0 &\Leftrightarrow f = g. && (5.12)
\end{aligned}$$

The first axiom follows from the fact that in $J(|f-g|^2)$ all summands are positive and therefore also $\|f-g\|_J = \mathrm{d}_J(f,g)$.
Also (5.10) is easy to see by

$$\mathrm{d}_J(f,g) = \|f-g\|_J = \sqrt{J(|f-g|^2)} = \sqrt{J(|g-f|^2)} = \|g-f\|_J = \mathrm{d}_J(g,f).$$

For the proof of (5.11) we calculate

$$\mathrm{d}_J(f,h) = \|f-h\|_J = \|(f-g)+(g-h)\|_J \leq \|f-g\|_J + \|g-h\|_J = \mathrm{d}_J(f,g) + \mathrm{d}_J(g,h),$$

where we used (5.4).
The last axiom (5.12) we prove by

$$\begin{aligned}
\mathrm{d}_J(f,g) = \|f-g\|_J = 0 \quad &\overset{(5.5)}{\Rightarrow} \quad f-g=0,\\
f-g=0 \Rightarrow \|f-g\|_J = \|0\|_J = 0 \quad &\Rightarrow \quad \mathrm{d}_J(f,g) = 0.
\end{aligned}$$

Up to now we showed that Λ^2 is a normed and metric space for the definitions (5.2) and (5.8). We show in the following that every Cauchy sequence converges on this space. With this it becomes a Hilbert space, since we constructed the norm (5.2) via the inner product (5.1).

For this we recall the definition of a Cauchy sequence:

Definition: *A sequence $(x_n)_{n\in\mathbb{N}}$ on a metric space is a* Cauchy sequence*, if*

$$\forall \epsilon > 0 \ \exists N(\epsilon) \in \mathbb{N} \quad \forall n, m \geq N(\epsilon): \quad \|x_n - x_m\| \leq \epsilon.$$

In the following we take the completeness of $\mathbb{C}$ for granted, hence it holds for any Cauchy sequence $(f_n(x))_{n\in\mathbb{N}}$ that $f_n(x) \overset{n\to\infty}{\longrightarrow} f(x)$.
For a Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ in Λ^2 we know from the definition of Cauchy sequences that

$$\forall \epsilon > 0 \ \exists N(\epsilon) \in \mathbb{N}: \quad \|f_n - f_m\|_J \leq \epsilon \quad \forall n, m \geq N(\epsilon).$$

Especially for the partial sums it holds $\forall n, m \geq N(\epsilon)$, $\forall M \in \mathbb{N}$

$$\Big((1-q^2)x_0 \sum_{k=-M}^{M} q^{2k} \left[|f_n(x_k) - f_m(x_k)|^2 + |f_n(-x_k) - f_m(-x_k)|^2\right]\Big)^{1/2} \leq \|f_n - f_m\|_J \leq \epsilon.$$

We now take the limit $m \to \infty$ and obtain $\forall n \geq N(\epsilon), \forall M \in \mathbb{N}$

$$\Big((1-q^2)x_0 \sum_{k=-M}^{M} q^{2k} \left[|f_n(x_k) - f(x_k)|^2 + |f_n(-x_k) - f(-x_k)|^2\right]\Big)^{1/2} \leq \epsilon,$$

since we defined $(f_n(x))_{n\in\mathbb{N}}$ to be a Cauchy sequence in $\mathbb{C}$. Since this inequality holds for any M, we have

$$\Big((1-q^2)x_0 \sum_{k=-\infty}^{\infty} q^{2k} \left[|f_n(x_k) - f(x_k)|^2 + |f_n(-x_k) - f(-x_k)|^2\right]\Big)^{1/2} \leq \epsilon \quad \forall n \geq N(\epsilon).$$

Rewriting this in terms of the norm we obtain

$$\|f_n - f\|_J \leq \epsilon, \quad \forall n \geq N(\epsilon),$$

which says that the Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ converges with the limit f.

We now come to the construction of a trace formula for our integral. The integral consists of two parts: the one with positive arguments of the function and the the other one with negative components. Therefore we can split up Λ^2 into the part Λ^2_+ with only functions with positive arguments $f(q^\alpha x_0)$ and Λ^2_- consisting just of functions $f(-q^\alpha x_0)$. Then the integral J can be written as

$$J(f) = J_+(f) + J_-(f)$$

with

$$J_+(f) = (1-q^2)x_0 \sum_{n=-\infty}^{\infty} q^{2n} f(q^{2n}x_0)$$

and

$$J_-(f) = (1-q^2)x_0 \sum_{n=-\infty}^{\infty} q^{2n} f(-q^{2n}x_0).$$

In the following we treat the case of Λ^2_+, since the arguments and calculation for Λ^2_- are completely analogue and we can can reconstruct Λ^2 out of it.

We take a function $f \in \Lambda^2_+$ and an orthonormal system g_k of Λ^2_+ with

$$g_k(x_l) = \frac{1}{\sqrt{(1-q^2)x_0}\, q^l}\delta_{kl}$$

where x_l are coordinates of the Euclidean space. For the following operator we want to show that it is trace-class[1]. We assume

$$\begin{aligned} T_f : \Lambda^2_+ &\rightarrow \Lambda^2_+ \\ g &\mapsto f\varphi \cdot g \end{aligned}$$

with a weight $\varphi(x_k) = (1-q^2)x_0\, q^{2k}$. Then we insert this operator into the usual definition of a trace and obtain

$$\begin{aligned} \operatorname{tr} T_f &= \sum_{k=-\infty}^{+\infty} \langle g_k, T_f g_k \rangle = \sum_{k=-\infty}^{+\infty} \langle g_k, f\varphi \cdot g_k \rangle = {}^2 \\ &= \sum_{k=-\infty}^{+\infty} \langle g_k, f(x_k)\varphi(x_k) g_k \rangle = \sum_{k=-\infty}^{+\infty} f(x_k)\varphi(x_k)\langle g_k, g_k \rangle = \sum_{k=-\infty}^{+\infty} f(x_k)\varphi(x_k) = \\ &= \sum_{k=-\infty}^{+\infty} (1-q^2)x_0\, q^{2k} f(q^{2k}x_0) = J_+(f) < \infty. \end{aligned}$$

Since $\operatorname{tr} T_f < \infty$ it holds that T_f is a trace-class operator, so we get that $J(f)$ can be expressed in terms of the trace.

[1] A trace-class operator is defined to be a positive operator u on a Hilbert space $\mathcal{H}$ for which $\|u\|_1 := \sum_{e_j \in \mathcal{H}} \langle e_j, u e_j \rangle < +\infty$ with e_j the elements of the orthonormal system of $\mathcal{H}$. This ensures that the trace we want to define converges. See also [28].

[2] Here only the term where we insert x_k contributes, since $g_k(x_l) = 0$ for $k \neq l$.

Appendix A

Notations

(1) We use the $q-$numbers taken from [21]:

$$[[n]]_{q^A} = \frac{1-q^{nA}}{1-q^A}.$$

Acting on the coordinate $\hat{x}$ we also can write

$$[[n]]_{q^A}(\hat{x})^n = \frac{1-(L_A)^n}{1-q^A}(\hat{x})^n = \frac{(\hat{x})^n-(q^A\hat{x})^n}{1-q^A}$$

with L_A the scaling operator defined in chapter 3.

(2) Since our formulae would become very complex we only write down explicitly the arguments of the functions that are scaled or transformed:

$$f(q^A x^-) \equiv f(x^+, \tilde{x}^3, x^3, q^A x^-).$$

(3) The ordinary Jackson derivatives are [21]:

$$D^A_{q^i} := \frac{f(x^A) - f(q^i x^A)}{(1-q^i)x^A}.$$

(4) For some repetitive expressions in the coordinates we introduced the following short cuts:

$$\begin{aligned}
\tilde{x}^3_j &:= x^0 + q^{2j}\tilde{x}^3 \\
y_\pm &:= x^0 + \tfrac{2q^{\pm1}}{\lambda_+}\tilde{x}^3 \\
\tilde{y}^j_\pm &:= y_\pm|_{\tilde{x}^3 \to q^{2j}\tilde{x}^3} = x^0 + \tfrac{2q^{\pm1}}{\lambda_+}q^{2j}\tilde{x}^3
\end{aligned}$$

(5) For the $q-$deformed Minkowski space we use the following polynomials

$$\begin{aligned}(M^{\pm})^{k}_{i,j}(\vec{x}) &= \tbinom{k}{i}\lambda_+^j(a_\pm(q^{2j}\tilde{x}^3))^i(x^+x^-)^jS_{k-i,j}(x^0,\tilde{x}^3)\\ (M^{+-})^{k,l}_{i,j,u}(\vec{x}) &= \tbinom{k}{i}\tbinom{l}{j}\lambda_+^u(a_+(q^{2u}\tilde{x}^3))^{k-i}(a_-(q^{2u}\tilde{x}^3))^{l-j}(x^+x^-)^uS_{i+j,u}(x^0,\tilde{x}^3)\end{aligned}$$

where we used $a_\pm(\tilde{x}^3)=q^{\pm1}\tilde{x}^3(q^{\pm1}\tilde{x}^3+\lambda_+x^0)$ and the sum

$$S_{i,j}(x^0,\tilde{x}^3)=\begin{cases}1, & i=j\\ \sum_{p_1=0}^{j}\sum_{p_2=0}^{p_1}\cdots\sum_{p_{i-j}=0}^{p_{i-j-1}}\prod_{l=1}^{i-j}a_+(x^0,q^{2p_l}\tilde{x}^3)\end{cases}$$

taken from [37].

(6) To cover all quadrants with the integral we need to sum over all possible combinations of signs in the argument of the function f, so we define

$$\begin{aligned}\tilde{f}(x_1,x_2,\ldots,x_n) &:= f(x_1,\ldots,x_n)+\\ \text{(1 “–”-sign)}\quad &+f(-x_1,x_2,\ldots,x_n)+\ldots+f(x_1,x_2,\ldots,-x_n)+\\ \text{(2 “–”-signs)}\quad &+f(-x_1,-x_2,\ldots,x_n)+\ldots+f(x_1,\ldots,-x_{n-1},-x_n)+\\ &+\ldots+\\ \text{(1 “+”-sign)}\quad &+f(x_1,-x_2,\ldots,-x_n)+\ldots+f(-x_1,\ldots,-x_{n-1},x_n)+\\ \text{(0 “+”-signs)}\quad &+f(-x_1,-x_2,\ldots,-x_n).\end{aligned}$$

(7) The New Jackson derivatives [2, 3] read

$$D^{(k_1,\ldots,k_l)}_{a_1,\ldots,a_l}f(x)=-\sum_{i=1}^{l}(\prod_{j=i+1}^{l}(x-a_jx)^{-k_j})\sum_{m=0}^{k_i-1}(x-a_ix)^{m-k_i}\left(D^{(k_1,\ldots,k_{i-1},m)}_{a_1/a_i,\ldots,a_{i-1}/a_i,1}f\right)(a_ix)$$

where $a_i\neq1$ and

$$\begin{aligned}D^{(k_1,\ldots,k_l)}_{a_1,\ldots,a_{l-1},1}f(x) &= \sum_{m=0}^{k_l}(-1)^{k_l-m}\tbinom{k_l+k_{l-1}-m-1}{k_{l-1}-1}(x-a_{l-1}x)^{m-k_l-k_{l-1}}D^{(k_1,\ldots,k_{l-2},m)}_{a_1,\ldots,a_{l-2},1}f(x)\\ &-(-1)^{k_l}\sum_{m=0}^{k_{l-1}-1}\tbinom{k_l+k_{l-1}-m-1}{k_l}(x-a_{l-1}x)^{m-k_l-k_{l-1}}\left(D^{(k_1,\ldots,k_{l-2},m)}_{a_1/a_{l-1},\ldots,a_{l-2}/a_{l-1},1}f\right)(a_{l-1}x).\end{aligned}$$

With this formula one reduces the order of the derivative order by order up to

$$D^{(k)}_1f(x)=\tfrac{1}{k!}\tfrac{\partial^k}{\partial x^k}f(x).$$

Appendix B

Action of $\hat{\partial}^+$ and $\hat{\partial}^0$ in the Minkowski space

The action we want to consider is

$$
\begin{aligned}
\hat{\partial}^+ \triangleright f = -q \sum_{k=0}^{\infty} \alpha_+^k \sum_{0\leq i+j\leq k} \Big\{ & \hat{g}_1 \left(f(q^{2j}\tilde{x}^3, q^2\tilde{y}_-^j) - f(q^{2j}\tilde{x}^3, q^2\tilde{y}_-^j, q^2x^-)\right) \\
& +\hat{g}_2 \left(f(q^{2j}\tilde{x}^3, \tilde{y}_-^j) - f(q^{2j}\tilde{x}^3, \tilde{y}_-^j, q^2x^-)\right) \\
& +\hat{g}_3 \left(f(q^{2j}\tilde{x}^3, \tilde{y}_+^j) - f(q^{2j+2}\tilde{x}^3, \tilde{y}_+^j)\right) \\
& +\hat{g}_4 \left(f(q^{2j}\tilde{x}^3, q^2\tilde{y}_+^j) - f(q^{2j+2}\tilde{x}^3, q^2\tilde{y}_+^j)\right)\Big\} \\
-q\tfrac{\lambda}{\lambda_+} \sum_{0\leq k+l\leq\infty} \alpha_+^{k+l} \sum_{i=0}^{k}\sum_{j=0}^{l} \sum_{0\leq u\leq i+j} & (M^{+-})_{i,j,u}^{k,l}(\vec{x}) \cdot \\
\cdot\Big\{ \hat{h}_1 f(q^{2u}\tilde{x}^3, \tilde{y}_+^u) + \hat{h}_2 f(q^{2u}\tilde{x}^3, q^2\tilde{y}_+^u) & + \hat{h}_3 f(q^{2u}\tilde{x}^3, \tilde{y}_-^u) + \hat{h}_4 f(q^{2u}\tilde{x}^3, q^2\tilde{y}_-^u)\Big\} \\
-\tfrac{\lambda}{\lambda_+} \sum_{0\leq k+l\leq\infty} \alpha_+^{k+l+1} \sum_{i=0}^{k}\sum_{j=0}^{l+1} \sum_{0\leq u\leq i+j} & (M^{+-})_{i,j,u}^{k,l+1}(\vec{x}) x^+ \cdot \\
\cdot\Big\{ \hat{h}_5 f(q^{2u}\tilde{x}^3, \tilde{y}_+^u) + \hat{h}_6 f(q^{2u}\tilde{x}^3, q^2\tilde{y}_+^u) & + \hat{h}_7 f(q^{2u}\tilde{x}^3, \tilde{y}_-^u) + \hat{h}_8 f(q^{2u}\tilde{x}^3, q^2\tilde{y}_-^u)\Big\}
\end{aligned} \tag{B.1}
$$

where the $\hat{h}_A$ and $\hat{g}_B$ denote operators consisting of sums of functions of the coordinates multiplied with derivatives. Those terms do not affect the lattice structure, so we can abbreviate them. The $\hat{g}_B$ are used to shorten the terms of usual form $f - f(q^i x^k)$ whereas the $\hat{h}_A$ are used for terms without that recursive structure. The other abbreviations we used can be found in appendix A.

For the action of $\hat{\partial}^0$ we find with the same convention for the abbreviations $\hat{g}$ and $\hat{h}$:

$$\begin{aligned}
\hat{\partial}^0 \triangleright f = \sum_{k=0}^{\infty} \alpha_+^k \sum_{0\leq i+j\leq k} (M^+)^k_{i,j}(\vec{x}) \cdot \Big[& \hat{g}_1 (f(q^{2j}\tilde{x}^3, \tilde{y}_+^j) - f(q^{2j+2}\tilde{x}^3, \tilde{y}_+^j)) \\
& + \hat{g}_2 (f(q^{2j}\tilde{x}^3, q^2\tilde{y}_+^j) - f(q^{2j+2}\tilde{x}^3, q^2\tilde{y}_+^j)) \\
& + \hat{g}_3 (f(q^{2j}\tilde{x}^3, \tilde{y}_+^j) - f(q^{2j+2}\tilde{x}^3, \tilde{y}_+^j) - f(q^2x^+, q^{2j}\tilde{x}^3, \tilde{y}_+^j) + f(q^2x^+, q^{2j+2}\tilde{x}^3, \tilde{y}_+^j)) \\
& + \hat{g}_4 (f(q^{2j}\tilde{x}^3, q^2\tilde{y}_+^j) - f(q^{2j+2}\tilde{x}^3, q^2\tilde{y}_+^j) - f(q^2x^+, q^{2j}\tilde{x}^3, q^2\tilde{y}_+^j) + f(q^2x^+, q^{2j+2}\tilde{x}^3, q^2\tilde{y}_+^j)) \\
& + \hat{g}_5 (f(q^{2j}\tilde{x}^3, q^2\tilde{y}_+^j) - f(q^{2j}\tilde{x}^3, q^2\tilde{y}_+^j, q^2x^-)) \\
& + \hat{g}_6 (f(q^{2j}\tilde{x}^3, \tilde{y}_+^j) - f(q^{2j}\tilde{x}^3, \tilde{y}_+^j, q^2x^-)) \\
- \; q^2 \tfrac{\lambda}{\lambda_+} \sum_{0\leq k+l<\infty} \alpha_+^{k+l} \sum_{i=0}^{k}\sum_{j=0}^{l} \sum_{0\leq u\leq i+j} & (M^{+-})^{k,l}_{i,j,u}(\vec{x}) \cdot \\
\cdot \Big[& \hat{g}_7 (f(q^{2u}\tilde{x}^3, \tilde{y}_+^u) - f(q^2x^+, q^{2u}\tilde{x}^3, \tilde{y}_+^u)) + \hat{g}_8 (f(q^{2u}\tilde{x}^3, q^2\tilde{y}_+^u) - f(q^2x^+, q^{2u}\tilde{x}^3, q^2\tilde{y}_+^u))\Big] \\
+ \; \tfrac{1}{\lambda_+} \sum_{0\leq k+l<\infty} \alpha_+^{k+l+1} \sum_{i=0}^{k}\sum_{j=0}^{l+1} \sum_{0\leq u\leq i+j} & (M^{+-})^{k,l+1}_{i,j,u}(\vec{x}) \cdot \\
\cdot \Big[& \hat{g}_9 (f(q^{2u}\tilde{x}^3, \tilde{y}_+^u) - f(q^2x^+, q^{2u}\tilde{x}^3, \tilde{y}_+^u)) + \hat{g}_{10} (f(q^{2u}\tilde{x}^3, q^2\tilde{y}_+^u) - f(q^2x^+, q^{2u}\tilde{x}^3, q^2\tilde{y}_+^u))\Big] \\
- \; q \tfrac{\lambda}{\lambda_+} \sum_{k=0}^{\infty} \alpha_+^k \sum_{0\leq i+j\leq k} (M^-)^k_{i,j}(\vec{x}) \cdot \Big[& \hat{g}_{11} (f(q^{2j}\tilde{x}^3, q^2\tilde{y}_-^j) - f(q^2x^+, q^{2j}\tilde{x}^3, q^2\tilde{y}_-^j)) \\
& + \hat{g}_{12} (f(q^{2j}\tilde{x}^3, \tilde{y}_-^j) - f(q^2x^+, q^{2j}\tilde{x}^3, \tilde{y}_-^j)) \\
& + \hat{g}_{13} (f(q^{2j}\tilde{x}^3, q^2\tilde{y}_-^j) - f(q^2x^+, q^{2j}\tilde{x}^3, q^2\tilde{y}_-^j) - f(q^{2j}\tilde{x}^3, q^2\tilde{y}_-^j, q^2x^-) + f(q^2x^+, q^{2j}\tilde{x}^3, q^2\tilde{y}_-^j, q^2x^-)) \\
& + \hat{g}_{14} (f(q^{2j}\tilde{x}^3, \tilde{y}_-^j) - f(q^2x^+, q^{2j}\tilde{x}^3, \tilde{y}_-^j) - f(q^{2j}\tilde{x}^3, \tilde{y}_-^j, q^2x^-) + f(q^2x^+, q^{2j}\tilde{x}^3, \tilde{y}_-^j, q^2x^-))\Big] \\
- \; q^2 \tfrac{\lambda}{\lambda_+} \sum_{0\leq k+l<\infty} \alpha_+^{k+l} \sum_{i=0}^{k}\sum_{j=0}^{l} \sum_{0\leq u\leq i+j} & (M^{+-})^{k,l}_{i,j,u}(\vec{x}) \cdot \\
\cdot \Big[& \hat{g}_{15} (f(q^{2u}\tilde{x}^3, \tilde{y}_-^u) - f(q^2x^+, q^{2u}\tilde{x}^3, \tilde{y}_-^u)) + \hat{g}_{16} (f(q^{2u}\tilde{x}^3, q^2\tilde{y}_-^u) - f(q^2x^+, q^{2u}\tilde{x}^3, q^2\tilde{y}_-^u))\Big] \\
+ \; \tfrac{1}{\lambda_+} \sum_{0\leq k+l<\infty} \alpha_+^{k+l+1} \sum_{i=0}^{k}\sum_{j=0}^{l+1} \sum_{0\leq u\leq i+j} & (M^{+-})^{k,l+1}_{i,j,u}(\vec{x}) \cdot \\
\cdot \Big[& \hat{g}_{17} (f(q^{2u}\tilde{x}^3, \tilde{y}_-^u) - f(q^2x^+, q^{2u}\tilde{x}^3, \tilde{y}_-^u)) + \hat{g}_{18} (f(q^{2u}\tilde{x}^3, q^2\tilde{y}_-^u) - f(q^2x^+, q^{2u}\tilde{x}^3, q^2\tilde{y}_-^u))\Big] \\
- \; q^2 \tfrac{\lambda}{\lambda_+} \sum_{0\leq k+l<\infty} \alpha_+^{k+l} \sum_{i=0}^{k}\sum_{j=0}^{l} \sum_{0\leq u\leq i+j} & (M^{+-})^{k,l}_{i,j,u}(\vec{x}) \cdot \\
\cdot \Big[& \hat{h}_1 f(q^{2u}\tilde{x}^3, \tilde{y}_+^u) + \hat{h}_2 f(q^{2u}\tilde{x}^3, q^2\tilde{y}_+^u) + \hat{h}_3 f(q^{2u}\tilde{x}^3, \tilde{y}_-^u) + \hat{h}_4 f(q^{2u}\tilde{x}^3, q^2\tilde{y}_-^u)\Big] \\
+ \; \tfrac{\beta}{\lambda_+} \sum_{0\leq k+l<\infty} \alpha_+^{k+l+1} \sum_{i=0}^{k+1}\sum_{j=0}^{l} \sum_{0\leq u\leq i+j} & (M^{+-})^{k+1,l}_{i,j,u}(\vec{x}) \cdot
\end{aligned} \tag{B.2}$$

$$\cdot \Big[\hat{h}_5 f(q^{2u}\tilde{x}^3, \tilde{y}_+^u) + \hat{h}_6 f(q^{2u}\tilde{x}^3, q^2\tilde{y}_+^u) + \hat{h}_7 f(q^{2u}\tilde{x}^3, \tilde{y}_-^u) + \hat{h}_8 f(q^{2u}\tilde{x}^3, q^2\tilde{y}_-^u) \Big]$$

$$+ \quad \tfrac{1}{\lambda_+} \sum_{0 \leq k+l < \infty} \alpha_+^{k+l+1} \sum_{i=0}^{k} \sum_{j=0}^{l+1} \sum_{0 \leq u \leq i+j} (M^{+-})_{i,j,u}^{k,l+1}(\vec{x}) \cdot$$

$$\cdot \Big[\hat{h}_9 f(q^{2u}\tilde{x}^3, \tilde{y}_+^u) + \hat{h}_{10} f(q^{2u}\tilde{x}^3, q^2\tilde{y}_+^u) + \hat{h}_{11} f(q^{2u}\tilde{x}^3, \tilde{y}_-^u) + \hat{h}_{12} f(q^{2u}\tilde{x}^3, q^2\tilde{y}_-^u) \Big].$$

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