Planar Homotopy Algebras and Open-String Field Theory

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Zusammenfassung

Diese Dissertation beschätigt sich mit der Frage nach der Existenz einer offenen Stringfeldtheorie, welche auf der Quantenebene ohne eine Kopplung an den geschlossenen String konsistent ist. Dies soll durch eine Einschränkung auf planare Feynman Graphen geschehen. Ziel ist eine Formulierung einer solchen Theorie in der mathematischen Sprache der Homotopiealgebren zu finden. Anschließend untersuchen wir ob sich eine solche Formulierung auch auf allgemeine Eichfeldtheorien, insbesondere im Limes großer Eichgruppen übertragen lässt. Zum Schluss gehen wir auf Probleme einer solchen Formulierung ein, und wie sich diese durch eine Erweiterung weg von Beschränkung auf rein planare Graphen beheben lassen.

Diese Arbeit soll gleichzeitig als ausgedehnte Einführung in die Theorie des Batalin-Vilkovisky Formalismus dienen. Wir betrachten verschiedene mathematische Aspekte dieses Formalismus und dessen Bezug zu Homotopiealgebren.

Abstract

This thesis is concerned about the existence of a open-string field theory that is consistent at the quantum level without coupling to the closed string. We want to achieve this via a restriction to planar Feynman graphs. Our aim is to formulate this theory in the mathematical language of homotopy algebras. We further ask whether such a formulation is applies also to general gauge theories, in particular in the limit of large gauge groups. Finally, we will discuss the problems of such a formulation, as well as how these can be solved by lifting the restriction to planar diagrams only.

This work should also serve as an extensive introduction to the Batalin-Vilkovisky formalism. We look at different mathematical aspects of this formalism and its relation to homotopy algebras.

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Contents

1. Introduction

1.1. Gauge Theories and the Batalin-Vilkovisky Formalism

During the development of physics in the last century, quantum field theories crystallized as the most effective tool to describe and predict physical processes. In high energy physics, the standard model puts quarks and leptons as the building blocks of matter in our world. It also describes their interaction in terms of the strong and the electroweak forces. This standard model of particles and forces is formulated as a quantum field theory (QFT). But even away from this fundamental level QFT is dominant as a language. When we zoom out of the scale of quarks and gluons, we enter the realm of hadrons. In this picture, the atomic nucleus is made up of baryons (protons and neutrons, but also more exotic objects are possible), which are held together by mesons (most importantly pions). This model is also described by a quantum field theory. At even larger scales, quantum field theories appear in solid state physics. Finally, at astronomical and cosmological scales, we use general relativity, which can be treated as a classical field theory.

Since quantum field theory is ubiquitous in physics, a proper understanding of it is of course necessary. A major complication arises when the theory is a gauge theory. This means that the theory is invariant under certain variations in the field variables. These variations are called *gauge symmetries* of the theory. The physical quantities in a gauge theory are independet of whether we use one set of fields or a gauge equivalent one. "Unfortunately", all the experimentally tested fundamental theories are gauge theories. Needless to say, it is expected that a more fundamental theory is a gauge theory as well, which may possess even more complicated gauge structure. A potential candidate is string (field) theory, which has a gauge symmetry that is infinitely more complex (in at least two ways) than that of the standard model.

A very simple example of a gauge theory is quantum electrodynamics, the theory of electrons (or any electrically charged particle) interacting through electromagnetic forces. One manifestation of this simplicity lies in the fact that photons, the mediators of the electromagnetic force, are electrically neutral (not charged) and therefore do not influence each other directly. They can only do so using electrons and positrons as mediators. Because of that reason the techniques we explore in this work are not needed in quantum electrodynamics. Nevertheless, they can of course also be applied in this case. Things can change when one considers theories with more than one force carrying particle (gauge bosons). Let us add weak interactions to the game. We get three additional gauge bosons, abbreviated by the letters W^+, W^-, Z. In addition to being electrically charged, they also possess a "charge" called the weak isospin. All particles with weak isospin (like quarks and the gauge bosons themselves) interact via the three gauge particles. In this way the gauge particles can scatter even without creating fermions (matter particles) or photons.

The fundamental interaction among gauge bosons leads to inconsistencies when we treat weak interactions the same way as electromagnetic interactions. The reason is that, naively, gauge bosons related by gauge transformations contribute independently in these interactions. The theory, all of the sudden, distinguishes between gauge bosons related by gauge transformations. To obtain a good theory, one should introduce a mechanism that cancels contributions coming from gauge equivalent bosons. A way, which was, according to [36], originally proposed by R. Feynman[1], is the introduction of certain fake particles. These particles, now called ghosts, violate the spin-statistics theorem, since they are fermions of even spin. The work of L.D. Faddeev and V.N. Popov [37] provided an explanation for their appearance in terms of the path integral formulation of quantum field theory. When the theory has a gauge symmetry, the integral has to be restricted to fields not related by gauge transformations. From the experience with ordinary integrals they proposed that such a restriction gives rise to a Jacobian determinant in the path integral. This determinant can then be expressed in terms of the fake particle fields. In honor of their work, these fields are now known as Faddeev-Popov ghosts.

The introduction of the ghost fields led to new insights into Yang-Mills theories. It was already known that the gauge symmetry of quantum electrodynmaics leads to relations among the physical observables of the theory. These relations are called *Ward-Takahashi identities*. With the result of Faddeev and Popov, A.A. Slavnov [83] and J.C. Taylor [89] generalized these to Yang-Mills theories. C. Becchi, A. Rouet, S. Stora [13, 14] and I.V. Tyutin (unpublished) then observed that these new *Slavnov-Taylor identities* can be derived by using a symmetry which involves the Faddeev-Popov ghosts. The symmetry is generated by an odd (fermionic) operator Q, the *BRST charge*, which has the additional property that it squares to zero. The mathematical theory behind such operators is called *homological algebra*, a concept that we will meet repeatedly in this work.

In 1975, J. Zinn-Justin [98] introduced an even more elegant way to express the Slavnov-Taylor identities. He introduced sources for the BRST transformations and defined a bilinear structure $\{\cdot,\cdot\}$ on the space of functionals. The Slavnov-Taylor identities can then be neatly encoded in the identity

$$\{\Gamma,\Gamma\} = 0, \tag{1.1}$$

where Γ is the quantum effective action, i.e. the action where all quantum effects are absorbed into the interactions and kinematics of the theory. I.A. Batalin and G.A. Vilkovisky then further developed his results in a series of papers [7, 8, 9, 10, 11, 12]. They referred to the sources for the BRST transformations as anti-fields and to the bilinear form $\{\cdot,\cdot\}$ as the anti-bracket. The framework they developed is now called the Batalin-Vilkovisky (BV) formalism/quantization. The central equation encoding the consistency of a quantum field theory is the quantum master equation

$$\frac{1}{2}\{S,S\} - i\hbar\Delta S = 0, \tag{1.2}$$

which is very similar to (1.1). BV quantization can be seen as a vast generalization to the procedure developed by Faddeev and Popov. It can be applied to theories where the Faddeev-Popov procedure fails, e.g. supergravity [60]. The BV treatment allows for field dependent algebras of gauge transformations, which may have gauge symmetries themselves.

[1] A transcript of the lecture is available at [39]. It is mainly concerned about a quantum theory of gravity, but he also commented on a similar mechanism in Yang-Mills theory, which is the general framework for theories like that of the weak force.

1.2. BV Formalism in String Field Theory

The BV approach is now an essential ingredient of string field theory, since it gives a geometric explanation for the condition of gauge invariance of the theory. Scattering amplitudes in string theory are computed directly by integrals of functions over the moduli space of two dimensional surfaces. The moduli space parametrizes the shape of these surfaces up to a local rescaling (conformal transformations). String theory is a theory of S-matrices, that is, it directly provides formulas for scattering amplitudes without relying on some underlying field theory. The amplitudes are not derived by Feynman rules. The goal of string field theory is to remedy this, i.e. to give a quantum field theory from which the S-matrices of string theory derive from. This is done in the following way. Inside each moduli space of an n-point scattering amplitude with g loops, one specifies a certain region to be the fundamental n-vertex with g internal loops $\mathcal{V}_{g,n}$ (for closed strings, this means that the vertex comes with a relative factor of $\hbar^{2g}$). The rest of the moduli space should then be covered by lower vertices plus propagators. Geometrically, the propagators connect the vertices by attaching tubes of varying length along some previously defined circles. When we write $\mathcal{V} = \sum_{n,g} \mathcal{V}_{n,g}$, the condition that the space is covered exactly once is governed by the equation

$$\partial \mathcal{V} + \frac{1}{2}\{\mathcal{V}, \mathcal{V}\} + \Delta \mathcal{V} = 0, \tag{1.3}$$

which has the form of the quantum master equation. String field theory associates to each $\mathcal{V}^{g,n}$ a vertex function $S_I^{g,n}$ and to the boundary operator ∂ the BRST differential Q. Under this association, the form (1.3) of the BV equation is conserved. It becomes

$$QS_I + \frac{1}{2}\{S_I, S_I\} + \Delta S_I = 0, \tag{1.4}$$

where $S_I = \sum_{g,n} S_I^{g,n}$.

The earliest complete version of a string field theory is that for the open bosonic string developed by Witten in [93]. One of its remarkable feature is that the theory is consistent at the classical level with only a cubic vertex, i.e. $S_I^{n,g=0} = 0$ for all $n \neq 3$ (since the theory is classical, it only considers vertices without internal loops). Shortly afterwards, he also proposed a version for the open superstring in [92], which also has only a cubic interaction. However, this interaction has singularities when considered in higher amplitudes. One resolution to this problem was proposed by T. Erler, S. Konopka and I. Sachs in [33], see also [32, 34, 35, 61]. However, resolving the singularity resulted in the necessity to introduce vertices of any valence.

A proper formulation and understanding of closed-string field theory only came after Witten's formulation of open-string field theory, since the moduli space of closed surfaces is not so easily partitioned already at tree level. B. Zwiebach gave a prescription in [99] to obtain such a partition, which applies even at the quantum level. Things are even worse here, since already for the bosonic string we have that $\mathcal{V}^{(g,n)} \neq 0$ for all g and n.[2] A complete description of a string field theory of the closed string was given in [100]. A superstring version of closed strings was only developed much later by A. Sen and collaborators, see [27] for a review. Sen also established the use of the quantum effective action in string field theory [81], which can, for example, be used to determine mass renormalizations.

[2] If we want to be precise we should say that we always have $\mathcal{V}^{0,1} = \mathcal{V}^{0,2} = 0$.

1.2.1. Open-Closed String Field Theory

Although open string field theory is much easier to handle as a classical theory, things change if one considers it as a quantum theory. At the quantum level, the open string only exists when coupled to the closed string due to ultraviolet divergences. To explain this point, we should first talk about why UV divergences are absent in closed string theory. Naively, a scattering process is in the UV when some length l on the surface describing the interaction approaches zero. However, scale invariance allows us to rescale by this length. The limit $l \to 0$ then turns into $l' \to \infty$ for some other length l'. This limit describes an infrared process and is therefore harmless. We could say that whenever a process approaches the UV, we turn it into an IR process by rescaling. The absence of UV divergences is simply due to the fact that there is no UV limit.

The story for the open string is basically the same. However, there is one crucial difference. The rescaling can turn an open string process into a closed string process. We therefore interpret potential UV divergences in the open string sometimes (but not always) as an infrared closed string. Historically, this phenomenon was observed first as a gauge anomaly in the open-string theory. A gauge anomaly means that a classical gauge symmetry does not survive at the quantum level. This results in an inconsistent theory. It was then observed by M. Green and J.H. Schwarz [47] that the anomaly can lifted if two things happen (*Green-Schwarz mechanism*). First, the gauge group of the theory should be $SO(2^{d/2})$, where d is the spacetime dimension in which the string propagates (for the critical superstring it is $d = 10$, while the critical bosonic string has $d = 26$). Second, the open string should interact with a closed string. The fact that the gauge group is $SO(2^{d/2})$ implies that the string is necessarily unoriented, i.e. it has no preferred direction. We therefore should allow for unoriented surfaces. Further, quantum effects of the closed string also introduces handles (genus).

This discussion of course implies that a quantum field theory of open strings should also include closed strings. For oriented bosonic strings this was discussed in [101].[3] This was afterwards extended to unoriented strings in [28]. Explicit constructions with HIKKO-type vertices up to quartic order can be found in [63]. A one-loop calculation [5] confirmed the necessity of $SO(2^{d/2})$ for the bosonic string to that order. A theory with open-closed superstrings was described recently in [38].

1.2.2. A Simple String Field Theory?

The BV formalism provides a good conceptual understanding of string field theories. However, we already pointed out that all theories, except the classical bosonic open string, have an infinite number of vertices. For this reason, good computational progress was only made for the classical bosonic open string, where a large variety of solutions to the equations of motion was found. For all other theories, computations are usually done with the tools provided by ordinary string theory.

Needless to say, a huge step would be made if we would be able bring any of the other string field theories to a form which has only a finite number of vertices. For the closed bosonic string, this means that we should find such a theory already at the classical level. Another direction one can consider is to get a simple quantum theory of open strings. Of course, the Green-Schwarz mechanism immediately renders a theory of open-strings incomplete, and we are again facing an infinite number of vertices involving closed strings. However, there is

[3] This theory would have a gauge anomaly.

a potential loophole. At one loop, there are three different types of surfaces, called planar, unoriented and non-planar [48]. The non-planar diagram requires the closed string. On the other hand, the anomaly in the planar and unoriented diagrams cancel exactly when the gauge group is $SO(2^{d/2})$. From this one may hope that a theory, of planar and unorientable diagrams only, survives quantization without the closed string. This brings us to the first question we want to answer in this work.

Question 1: Is it possible to have a quantum open-string field theory of planar diagrams without the closed string?

Unfortunately, the answer to this question is negative for reasons we explain in section 4.

1.3. Homotopy Algebras in Field Theories

Another theme of this work will be about homotopy algebras, in particular the homotopy versions of associative algebras and Lie algebras, as well as their quantum versions. Given a classical field theory in the BV language, it immediately induces a homotopy Lie algebra (L_∞-algebra) for any solution of the equations of motion of the theory. In fact, these two viewpoints are equivalent in many applications. When the space of fields is linear and equipped with a constant anti-bracket, the BV theory induces a (cyclic) L_∞-algebra. On the other hand, any such cyclic L_∞-algebra induces a BV theory with constant anti-bracket (see [56]).

Homotopy algebras arise when one tries to combine an algebra structure on some vector space (e.g. the space of fields) with a (co-)homological structure on that space and then asks for equivalent descriptions. They have their roots in mathematics. Homotopy associative algebras were introduced as a tool in topology by J. Stasheff in his thesis [85, 86]. They gained popularity in physics when their role was discovered in classical open-string field theory [43]. On the other hand, already before that it was observed that L_∞-algebras arises form the vertices in classical closed-string field theory [87, 100]. The mixed version of classical open and closed strings was covered in [59]. Besides string field theory, L_∞-algebras are crucial in the deformation quantization of classical systems, see [62], and also, from a purely mathematical point of view, in general deformation theory/moduli problems (see for example [66]). In quantum physics, the structure to consider is called quantum (or loop) homotopy algebras for obvious reasons. For closed-string field theory, they appear in the form described in [68, 73]. Quantum open and open-closed string field theories are covered for example in [53, 29]. The relevance of homotopy algebras for field theories other than string field theory was emphazied in [54]. They received a broader attention among high energy physicists and mathematical physicists in recent years [15, 4, 23, 22].

1.4. Quantum A_∞-Algebras and Planar Field Theories

Classical open-string field theory can be formulated as an A_∞-algebra. This approach proved to be very useful in the formulation of a field theory of the open superstring [33]. In ordinary field theories, A_∞-algebras arise when one considers color-ordered diagrams (the word color stems from quantum chromodynamics). The corresponding Feynman rules consists of only those diagrams that can be drawn on a disc with external edges ending on the boundary and such that edges do not overlap. For example, the s and t-channels of a four-point scattering process have planar diagrams, while the u-channel is non-planar due to its overlapping legs. The planar subsector encapsulates all the relevant properties of a classical field theory like gauge invariance and unitarity. Moreover, the full theory can be obtained by symmetrization over the incoming and outgoing states. From this point of view, the A_∞-description is simpler than the description by homotopy Lie algebras.

Color-order diagrams exist also at the quantum level. There, also non-planar diagrams arise. One important observation by G. 't Hooft [88] is that planar diagrams dominate over non-planar ones when the dimension N of the fundamental representation of the gauge group grows. The limit $N \to \infty$ is known as the large N limit, and only planar diagrams survive. We want to ask in this work whether this large N-limit exists as some kind of self-contained homotopy algebra, without reference to the non-planar sector. This is the second main question we want to answer.

Question 2: Is there a consistent planar subsector of the quantum version of A_∞-algebras, and is it realized by the large N limit of gauge theories?

We search for an answer to this question in chapter 5. There, we try to construct a quantum A_∞-algebra similar to a one way how quantum L_∞-algebras are constructed. We will see that this will give a theory whose products are most naturally expressed in terms of planar diagrams. However, we will see that such a theory does not look like theory of planar *field theory* diagrams. We will also explain how the large N-limit fails to give a consistent homotopy algebra.

1.5. Overview

Chapter 2 starts with a short introduction on how differential graded algebra structures and their homotopy versions appear in classical field theories. This will lay a foundation when we traverse into the quantum world. We focus on the advantage of homotopy algebras over ordinary algebras from a mathematical point of view, in particular their stability under quasi-isomorphisms. The content of this part is well known to experts in the field. Nevertheless, we always try to develop the theory from scratch and bring in our own thoughts whenever possible. The second part of chapter 2 (starting with section 2.4) then turns to develop the classical Batalin-Vilkovisky formalism. At this point, we will give several examples. These also motivate some discussions about assumptions on the action functional S made in the literature and whether some of these can or even should be dropped. However, these remain conjectural. The two parts of chapter 2, i.e. homotopy algebras and the BV formalism, are finally related in section 2.5.

In the beginning of chapter 3, we give an introduction to the quantum Batalin-Vilkovisky formalism. One emphasis lies on revealing its relation to the path integral formalism. In particular, we discuss the role of ghosts (section 3.2) and how to gauge fix (section 3.4). This may help the reader to relate the BV approach to more conventional treatments of quantum field theory. On the other hand, we assume that this is known to researchers who study the BV approach, especially those who have some mathematical background. In section (section 3.5), we discuss an approach to regularization that naturally arises when quantum field theories are formulated in the BV language. [4] As an application, we compute the

[4]This approach is related to the concept of *stubs* in string field theory.

anomaly of the chiral Schwinger model. We think that the advantage of this approach over conventional methods is that the latter most of the time feel quite arbitrary. An extended version of this is planned to be published. We conclude in section 3.6 by giving a definition of quantum version of homotopy algebras, that is motivated by the quantum BV formalism.

Chapter 4 is concerned with **Question 1** in this thesis. We first discuss the concept of partitions of moduli spaces in string field theory. This part can be read quite independently from the previous two chapters. However, some knowledge about the quantum Batalin-Vilkovisky formalism is helpful, since the quantum master equation enters as a consistency condition on partitions of moduli spaces. In section 4.4, we quickly discuss how this data can then be turned into a quantum field theory. Answering **Question 1** now amounts to showing whether the Green-Schwarz anomaly can be made to vanish. A view on the anomaly in terms of string field theory is given in section 4.6. We show how a theory of planar diagrams avoids the closed string at one-loop level (section 4.6.1), but fails to do so at higher loops (section 4.6.2).

In chapter 5, we develop an approach to answering **Question 2**. We mimic the construction of quantum L_∞-algebras. The reader may want to recall the end of chapter 3 for that. We develop a definition of higher order (co-)derivations in section 5.1 over non-commutative algebras. It is shown that it is equivalent to a definition given previously in [16]. Our definition allows us to prove some theorems for this type of higher order derivations, which we could not find elsewhere. We also point out some crucial differences to the commutative case that will later show up as an obstruction for a positive answer to **Question 2**. In section 5.4, we give a definition of planar homotopy algebras based on higher order coderivations. However, we point out that this definition does not give what we expect for a quantum field theory. In section 5.5, we follow another route and apply the large N limit to a loop homotopy associative algebra, based on the latter one's definition given in section 3.6. We will see that we encounter a similar obstruction to the one we observed in section 5.4. On the other hand, we show in section 5.6 that planar diagrams give gauge invariant S-matrices, although we restrict to theories with cubic interactions. Finally, in section 5.7, we point towards a consistent subsector of loop homotopy associative algebras, that sits between the planar subsector and the full algebra.

The results of chapter 4 and 5 will be the content of another publication.

2. Homotopy Algebras and the Batalin-Vilkovisky Formalism

The first concept we discuss is that of homotopy associative algebras. As explained in the introduction, this is a language useful in the descrption of open-string field theory and color-ordered field theories. Closed-string field theory and general field theories have the structure of a homotopy Lie algebra. For the most part in this chapter, we will be concerned about the mathematical properties of homotopy algebras, but we try to draw the connection to field theories whenever possible. The following should serve as a motivation from the point of view of a physicist.

The very first concept we meet is that of a differential graded vector space (see the appendix for a mathematical introduction to this concept). The vector space we will be concerned about is a space of field $\mathcal{F}$. The fact that it is graded means that we assign an internal quantum number to elements (fields) in $\mathcal{F}$, which we call ghost number. The field space $\mathcal{F}$ contains more than what is usually considered as the space of fields (e.g. gauge potentials A_μ in Yang-Mills) we may call $\mathcal{F}^{(0)}$. In particular, $\mathcal{F}$ contains the gauge parameters. We assign ghost number zero to all fields in the "classical" field space $\mathcal{F}$. The gauge parameters are given ghost number minus one. We call the subspace of gauge parameters $\mathcal{F}^{(-1)}$. Further, the field space is extended to include a space of *anti-fields* $\mathcal{F}^1$ of ghost number one, as well as a space of *anti-ghosts* $\mathcal{F}^2$ of ghost number two. We will come to their role in a moment.

For a moment let us pick Yang-Mills theory as a specific example to illustrate the ideas. When interactions are turned off, the gauge transformation of a field $A_\mu \in \mathcal{F}^{(0)}$ reads

$$\delta A_\mu = \partial_\mu c, \tag{2.1}$$

where the gauge parameter $c \in \mathcal{F}_{(-)1}$ is function taking values in a Lie algebra. We introduce a linear operator $Q^{(-1)}$ that turns a gauge parameter c into a gauge transformation of A_μ, i.e.

$$(Q^{(-1)}c)_\mu := \partial_\mu c. \tag{2.2}$$

The operator $Q^{(-1)}$ is now a map

$$\mathcal{F}^{-1} \overset{Q^{(-1)}}{\longrightarrow} \mathcal{F}^0. \tag{2.3}$$

Since it raises the ghost number by one unit, we say that Q itself has ghost number one. A fancy way of saying that we only want to consider fields up to gauge transformations is to look at the quotient $\mathcal{F}^{(0)}/\mathrm{Im}\, Q$.

In a classical field theory, we further want to restrict to A_μ that satisfy the equations of motion. Without the presence of sources, the equation of motion (we still don't consider interactions) for A_μ reads

$$\partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = 0. \tag{2.4}$$

We again introduce an operator, which we call $Q^{(0)}$, that encodes the equation of motion. We define

$$(Q^{(0)}A)^\nu := \partial_\mu\partial^\mu A^\nu - \partial_\mu\partial^\nu A^\mu. \tag{2.5}$$

We take its image to lie in $\mathcal{F}^1$, i.e. it takes values in anti-fields. In general, anti-fields will be denoted by A^*_μ. Since the equations of motion are gauge invariant, we have that

$$Q^{(0)} \circ Q^{(-1)} = 0. \tag{2.6}$$

This property allows us to form the quotient $H^0(\mathcal{F}) = \frac{\ker Q^{(0)}}{\operatorname{Im} Q^{(-1)}}$, which we can identify as gauge classes of fields satisfying equations of motion. The space $H^0(\mathcal{F})$ is what we actually consider the set of physically measurable fields.

On anti-fields we define a third operator : $\mathcal{F}^{(1)} \overset{Q^{(1)}}{\to} \mathcal{F}^{(2)}$ by

$$Q^{(1)}A^* = \partial^\mu A^*_\mu. \tag{2.7}$$

Note that again

$$Q^{(1)} \circ Q^{(0)} = 0, \tag{2.8}$$

which encodes the Noether identities.

We can arrange this data in a sequence

$$0 \to \mathcal{F}^{(-1)} \overset{Q^{(-1)}}{\to} \mathcal{F}^{(0)} \overset{Q^{(0)}}{\to} \mathcal{F}^{(1)} \overset{Q^{(1)}}{\to} \mathcal{F}^{(2)} \to 0, \tag{2.9}$$

where we adjoined the zero map on both ends. We can also combine Q to a single operator of ghost number one on all of $\mathcal{F}$. Equations (2.6) and (2.8) then combine to the single identity

$$Q^2 = 0. \tag{2.10}$$

The data $(\mathcal{F}, Q)$ is one example of a differential graded vector space. In general, a graded vector space is some vector space V which admits a decomposition

$$V = \bigoplus_{n\in\mathbb{Z}} V^n. \tag{2.11}$$

If $v \in V^n$, we say that v has degree (or ghost number) n. We then say that V is also *differential*, if it comes equipped with a map $\mathrm{d} : V \to V$ raising the degree by one unit and such that $\mathrm{d}^2 = 0$. We define the degree n cohomology to be the vector space $H^n(V) := \frac{\ker \mathrm{d}|_{V^n}}{\operatorname{Im} \mathrm{d}|_{V^{n-1}}}$. In our previous example, the zeroth cohomology contained the physical fields.

Up to now, we did not consider any interactions. We would like to treat interactions along similar lines as the operator $Q : \mathcal{F} \to \mathcal{F}$. For example, we describe a cubic interaction by a product $m_2 : \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$. If there is only a cubic interaction, the equations of motion then read

$$QA + m_2(A, A) = 0. \tag{2.12}$$

Similar to what we did before, we want to include also gauge parameters to this product, so that it also generates gauge transformations. If the gauge transformation is at most quadratic, we write

$$\delta A = Qc + m_2(c, A). \tag{2.13}$$

One can now derive consistency conditions by demanding that the equations of motion are gauge invariant. There are three.

1. $Q^2 = 0$.
2. $Qm_2(a,b) = (-)^s m_2(Qa,b) + (-)^t m_2(a,Qb),\ s,t \in \mathbb{Z}$.
3. m_2 is a Lie bracket.

Here, the signs depend on the grading. If Q and m_2 satisfy the above compatibility conditions, we have the structure of a differential graded Lie algebra. We can conclude the following. A gauge theory with only a cubic interaction has the structure of a differential graded Lie algebra.

If the gauge theory contains also interactions and gauge transformations of higher order, more compatibility conditions arise. In principle, we may have a theory that is non-polynomial, i.e. one with interactions to any order. In this case, there is an infinite number of conditions. In this case, the structure is called homotopy Lie algebra (or L_∞-algebra) by mathematicians. There is also another important type of homotopy algebra, called a homotopy associative (A_∞) algebra. It generalizes ordinary associative algebras in the sense that it allows for more inputs. As explained in the introduction, they are important in open-string field theory and color-ordered gauge theories. Below, we will be mostly concerned with these.

2.1. Homotopy Algebras

Consider the data of a differential graded algebra (A, d, M). This means that A is a cochain complex of vector spaces with differential d, and it comes equipped with a degree preserving associative product M, which is compatible with d. This means that

$$\mathrm{d}M(a,b) = M(\mathrm{d}a, b) + (-)^a M(a, \mathrm{d}b). \tag{2.14}$$

This condition makes sure that M descends to a product on cohomology.

Given a pair of differential graded algebras (A, d_A, M_A) and (B, d_B, M_B), we define a homomorphism to be a linear map $\phi : A \to B$, such that it is a homomorphism of cochain complexes and, at the same time, of algebras. The former condition says that ϕ is of degree 0 and that it commutes with the differentials, i.e. $\mathrm{d}_B\phi = \phi\mathrm{d}_A$. The latter demands $\phi(M_A(a,b)) = M_B(\phi(a),\phi(b))$. These two conditions are such that ϕ gives an algebra homomorphism between cohomologies.

One way to say that two differential graded algebras are the same is to ask for the existence of an isomorphism between them. In most applications however, this is too strong. In the end, what one really wants is that the cohomologies are the same, since they contain the relevant data. This leads to the definition of a quasi-isomorphism. This is a homomorphism $\phi : A \to B$ which becomes invertible when restricted to cohomologies.

We can go one step further and consider any two morphisms to be the same whenever they are equal on cohomology. An a priori stronger notion of this is the following. Given two morphisms $f, g : (A, \mathrm{d}_A, M_A) \rightrightarrows (B, \mathrm{d}_B, M_B)$, we say that they are *homotopic*, if there exists a map $h : A \to B$ such that $f - g = \mathrm{d}_B h + h\mathrm{d}_A$. Some simple algebra shows that homotopic maps are equal on cohomology. Fortunately, when working with complexes over vector spaces, the converse is also true.[1] The notion of homotopy is only superficially stronger, at least for our purposes.

[1] This fact can be found in any standard textbook on homological algebra, e.g. [91].

Suppose now that we have a quasi-isomorphism $i : A \to B$ of differential graded algebras. We can use the notion of homotopy to write this property algebraically. The map i is a quasi-isomorphism if and only if there is another chain map $p : B \to A$, such that $p \circ i$ is homotopic to id_A.[2] We say that p is a *homotopy inverse* of i. The map p is then automatically also a quasi-isomorphism. However, p cannot always be chosen such that it is a map of algebras. Quasi-isomorphisms between algebras are therefore generically not invertible.

A related problem is the following. Suppose that we are given quasi-isomorphic complexes A and B, and suppose B has a product M_B compatible with its differential. Are we able to define a product on A with this data? This is of course true if A and B happen to be isomorphic. Moreover, we are definitely able to define a product on the cohomology $H^\bullet(A)$, since it is isomorphic to $H^\bullet(B)$. To construct it, let i be a quasi-isomorphism from A to B with homotopy inverse p. The product $M_A = p \circ M_B \circ (i \otimes i)$ descends to a product on $H^\bullet(A)$ since p and i are assumed to be chain maps. We can try to give A an algebra structure using M_A. It is definitely compatible with the differential. On the other hand, M_A usually fails to be associative, as we can easily check. Let h be a homotopy from $p \circ i$ to id_A, i.e. $1 - i \circ p = h\mathrm{d}_A + \mathrm{d}_A h$. The failure of M_A being associative is then

$$M_A(M_A(a,b),c) - M_A(a, M_A(b,c)) = \mathrm{d}H(a,b,c) + H(\mathrm{d}a,b,c) + (-)^a H(a,\mathrm{d}b,c) \tag{2.15}$$

$$+ (-)^{a+b} H(a,b,\mathrm{d}c), \tag{2.16}$$

where

$$H(a,b,c) = p \circ M_B(i(a), hM_B(i(b),i(c))) - p \circ M_B(hM_B(i(a),i(b)),i(c)). \tag{2.17}$$

This is a map $H : A^{\otimes 3} \to A$ of degree -1. When we strip off the inputs from the equations, we see that

$$M_A \circ (\mathrm{id}_A \otimes M_A) - M_A \circ (M_A \otimes \mathrm{id}_A) = \mathrm{d}_A H + H\mathrm{d}_{A^{\otimes 3}}. \tag{2.18}$$

This means that $M_A \circ (\mathrm{id}_A \otimes M_A)$ is merely homotopic to $M_A \circ (M_A \otimes \mathrm{id}_A)$. This weaker notion still ensures that M_A becomes associative when restricted to cohomology.

The story does not stop with the introduction of H. Consider products of four elements in an associative algebra. There are two distinct ways to show that $a(b(cd)) = ((ab)c)d$, either by moving the inner parentheses or the outer parentheses first. In the previous example, there are therefore two distinct homotopies between the four element products made out of M_A. It turns out that these are again equal only up to a homotopy H', where $H' : A^{\otimes 4} \to A$ is a map of degree -2. Needless to say, even the reader unfamiliar with this may guess that this continues to happen in every negative degree.

The notion of a homotopy associative algebra, or A_∞-algebra, encapsulates this behavior. It demands that all associativity relations should hold up to homotopy, and that different homotopies between the same objects are themselves related by homotopy. We give a first property a homotopy associative algebra has to satisfy.

Property 2.1.1 (This is the content of **Theorem 13** in [90])**.** Let (V, d) be a differential graded vector space. An A_∞-structure on V has for each $k \geq 2$ a map $M_k : V^{\otimes k} \to V$ of degree $2-k$, such that the following holds. Denote by $P_k(n)$ the linear subspace of $V^{\otimes n} \to V$, which is generated by all compositions of the $\{M_l \mid l \geq 2\}$, so that the result has n inputs and is of degree $-k$. Consider the chain complex

$$\ldots \to P_k(n) \to \ldots \to P_2(n) \to P_1(n) \to P_0(n) \to 0, \tag{2.19}$$

[2]The equivalence of these statements is again guaranteed by the fact that we work over vector spaces.

where the arrows are the induced action of d on $\mathrm{lin}(V^{\otimes k}, V)$. The M_k are such that the homology groups satisfy satisfies $H_k(P_\bullet(n)) = 0$ for $k \geq 1$ and $H_0(P_\bullet(n))$ is at most one-dimensional.

Let us unravel this. The space $P_0(n)$ is spanned by all the ways we can write multiplication of n elements using M_2. For example, $P_0(3)$ contains $M_2 \circ (M_2 \otimes \mathrm{id}_V)$ and $M_2 \circ (\mathrm{id}_V \otimes M_2)$. All of them should be related by the homotopies living in $P_1(n)$. So we demand that the quotient $P_0(n)/\mathrm{d}P_1(n)$ is one-dimensional (it is zero dimensional in the degenerate case $M_2 = 0$). On the other hand, we want different homotopies between the same objects to be themselves related by a homotopy of one level higher. Two level k homotopies produce the same relation at level $k-1$ if their difference is in $\ker(d|_{P_k(n)})$. A level $k+1$ homotopy between them exists exactly when they are in the image of $d|_{P_{k+1}(n)}$. This is the condition $H_{k\geq 1}(P_\bullet(n)) = 0$.

Property 2.1.1 is not sufficient to describe A_∞-algebras completely. It does not serve as a definition. The reason is that, in order to achieve the vanishing of the homology groups, we can in principle rescale the homotopies with arbitrary factors. For example, any rescaling of (2.17) still leads to the vanishing of the $H_1(P_\bullet(n))$. However, this ambiguity will be fixed once we specified what the homomorphisms between A_∞-algebras are. This also takes us back to the original question concerning the invertability of quasi-isomorphisms between differential graded algebras. Let us go back to our example were we failed to transport a strictly associative algebra structure from a differential graded algebra (B, d_B, M_B) to a quasi-isomorphic vector space (A, d_A). We now know that the proper thing to define is a product on A which is associative up to homotopy. We would like to have a notion of homomorphism, which allows us to relate the algebra B to the homotopy algebra A. The way to to this is to allow for non-linear maps between A and B.

Definition 2.1.1 (section 1.6. in [90]**).** Define a homomorphism from an A_∞-algebra $(A, \mathrm{d}_A, \{M_k\})$ to another $(B, \mathrm{d}_B, \{N_k\})$ to be a collection of linear maps $\{f_n : A^{\otimes n} \to B\}_{n\geq 1}$ of degree $1-n$, such that the following holds.

$$\sum_{k=1}^{N} \sum_{\sum_{j=1}^{k} i_j = N} N_k \circ (f_{i_1} \otimes \cdots \otimes f_{i_k}) = \sum_{k=1}^{N} \sum_{i=0}^{N-k} (-)^{M_k f_{N-k+1}} f_{N-k+1}(\mathrm{id}_A^{\otimes i} \otimes M_k \otimes \mathrm{id}_A^{\otimes N-k-i}). \tag{2.20}$$

In the above we set $M_1 = \mathrm{d}_A$ and $N_1 = \mathrm{d}_B$. We call an A_∞-morphism $\{f_n : A^{\otimes n} \to B\}_{n\geq 1}$ a (quasi-)isomorphism, if f_1 is a (quasi-)isomorphism of cochain complexes.

Although the above rule looks messy, it is easy to remember. Loosely speaking, when we commute the collection $F = \{f_n\}_{n\geq 1}$ through the collection $N = \{N_k\}_{k\geq 1}$ we obtain $M = \{M_k\}_{k\geq 1}$. We may write $N \circ F = F \circ M$.[3] We will later see that the following conditions hold.

1. Any isomorphism of A_∞-algebras has an inverse.

2. Any quasi-isomorphism of A_∞-algebras has a homotopy inverse.

The second statement shows the advantage of working with A_∞-algebras instead of associative algebras when the underlying space is a (co-)chain complex.

[3] We will later introduce a formalism where this becomes a algebraically exact statement.

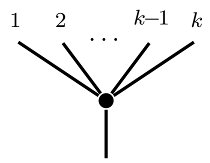
Remark 2.1.1. In this work we will also sometimes meet homotopy Lie (L_∞) algebras. This concept is not very different from its associative counterpart. Aside from skew symmetry, the defining property of a Lie algebra is the Jacobi identity. For L_∞-algebras we require that this identity as well as all higher relations are true only up to homotopy. Given a graded vector space V, for each number of inputs $k \geq 2$, there is a single product

$$M_k : V^{\wedge k} \longrightarrow V. \tag{2.21}$$

of degree $2 - k$. Note that M_k is defined on the exterior power of V, so it is graded anti-symmetric in its entries.

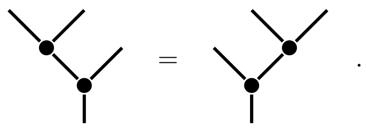
2.2. Diagrammatic Representation of Products

It has proven to be useful to represent the products of an A_∞-algebra by trees. We denote a product $M_k : A^{\otimes k} \to A$ by

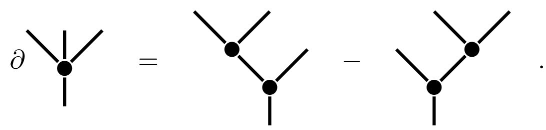
$M_k =$  . (2.22)

We distinguish between its k leaves and its root. The leaves represent the k inputs of M_k. The root represents its output. The numbers are there to remind us to which input a leave corresponds. However, we will most of the time not display them. They are fixed by the convention that the numbers grow from left to right.

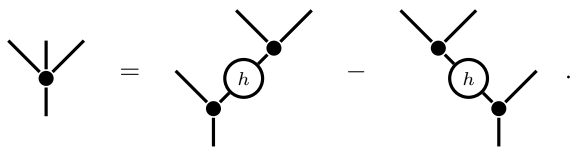
The main advantage of this approach is that allows to write expressions involving multiple products in a simple way. For example, in a (strict) associative algebra, we display associativity as

 (2.23)

Similarly, we draw associativity up to homotopy as

 (2.24)

Sometimes, it may happen that we have to insert linear operators between legs. For example in (2.17), the triple product H depends on the homotopy h. We draw this as

 (2.25)

2.3. Bar Construction of Homotopy Associative and Homotopy Lie Algebras

2.3.1. Tensor (Co-)Algebra

We review the construction and properties of the tensor algebra. Given a finite dimensional graded vector space V over $\mathbb{C}$, we define the free tensor algebra to be the space $TV = \bigoplus_{n\geq 0} V^{\otimes n}$, equipped with an algebra structure given by the tensor product. The free tensor algebra can be desribed by the following universal property. Given a linear map $\phi_1 : V \to A$ from a vector space V into an algebra A. Then there is a unique algebra homomorphism $\phi : TV \to A$, such that

$$\begin{tikzcd} V \arrow[r, hook] \arrow[dr, "\phi_1"'] & TV \arrow[d, "\phi"] \\ & A \end{tikzcd} \tag{2.26}$$

commutes. Therefore, we have a bijection $\mathrm{lin}_{\mathbb{C}}(V, A) \cong \mathrm{hom}(TV, A)$. This also characterizes derivations on TV using the following argument. Take $A = TV$. The universal property tells us that $\mathrm{Lin}_{\mathbb{C}}(V, TV) \cong \mathrm{hom}(TV, TV)$. A derivation δ on TV is an infinitesimal homomorphism $\phi : TV \to TV$, expanded around the identity. We write $\phi = \mathrm{id} + \epsilon\delta$, where $\epsilon^2 = 0$. At the infinitesimal level, the universal property then reads $\mathrm{Lin}_{\mathbb{C}}(V, TV) \cong \mathrm{Der}(TV)$.[4]

Given a basis $\{x^i\}_{i\in I}$ of V, its tensor algebra consists of finite sums of the form

$$\sum_{k=0}^{n} f_{i_1,\dots,i_k} x^{i_1} \otimes \cdots \otimes x^{i_k}. \tag{2.27}$$

Infinite sums are allowed in the completion $\hat{T}(V) = \prod_{k\geq 1} V^{\otimes k}$ of $T(V)$. We can define a topology on $\hat{T}V$. A sequence $(f_n)_{n\in\mathbb{N}}$ converges to zero, if each coefficient of f_n becomes equal to zero when n is large enough. This ensures that

$$\lim_{n\to\infty} \sum_{k=0}^{n} f_{i_1,\dots,i_k} x^{i_1} \otimes \cdots \otimes x^{i_k} = \sum_{k=0}^{\infty} f_{i_1,\dots,i_k} x^{i_1} \otimes \cdots \otimes x^{i_k}, \tag{2.28}$$

where the left hand side is the limit of a sequence in $TV \subseteq \hat{T}V$, while the right hand side is not in TV. On the other hand, the sequence $(\frac{1}{n})_{n\in\mathbb{N}}$ does not converge to 0 in this topology, since the coefficient in zeroth power is never *equal* to zero.

Remark 2.3.1. The difference between a product $\prod_i V_i$ and a sum $\bigoplus_i V_i$ of vector spaces V_i is technical. The former contains elements of the form $(v_1, v_2, ...)$, where in principle infinitely many of the v_i's are non-zero. On the other hand, we may write elements in $\bigoplus_i V_i$ as formal *finite* sums $\sum_{i=1}^{N} v_i$, such that $v_i \in V_i$. The sum is a subspace of the product in the sense that there is an inclusion

$$\sum_{i=1}^{N} v_i \mapsto (v_1, ..., v_N, 0, 0, ...). \tag{2.29}$$

[4] The author learned about this viewpoint in [77]. When one wants do discuss graded derivations, one should consider infinitesimal homomorphisms around the shifted identity (or suspension) map $\mathrm{id} : TV \to TV[k]$.

This suggests that we may also write elements in $\prod_i V_i$ as *infinite* sums when the indexing set happens to be countable. If the indexing set is actually finite, the above inclusion becomes an isomorphism. We also want to emphasize that the product $\prod_i$ is not the tensor product $\bigotimes_i$.

Remark 2.3.2. In TV, the only invertible elements are those living in $\mathbf{k} = V^{\otimes 0}$. On the other hand, an element in $\hat{T}V$ is invertible if and only if it has non-zero constant part, that is, if it has a non-zero projection onto $\mathbf{k}$. This can easily be seen by making use of the geometric series $(1-x)^{-1} = \sum_{k\geq 0} x^k$.

Let V^* be the dual of V, with dual basis $\{x_i^*\}_{i\in I}$. Then, $T(V^*)$ is the linear dual of $\hat{T}(V)$. Define the pairing

$$\langle \cdot,\cdot\rangle : T(V^*) \otimes \hat{T}(V) \to \mathbf{k} \tag{2.30}$$

The tensor product on $\hat{T}(V)$ induces a coproduct $\Delta : T(V^*) \to T(V^*) \otimes T(V^*)$ through $\langle f, a\cdot b\rangle = \langle \Delta f, a \otimes b\rangle$. It acts on basis elements by deconcatenation

$$\Delta(x_1^* \otimes \cdots \otimes x_n^*) = \sum_{k=0}^{n} (x_1^* \otimes \cdots \otimes x_k^*) \otimes (x_{k+1}^* \otimes \cdots \otimes x_n^*). \tag{2.31}$$

The parentheses indicate how we think of it as an element of $T(V^*) \otimes T(V^*)$. When $k = 0$ or n, we define $(\ ) = 1 \in \mathbf{k}$. Associativity of the tensor product implies coassociativity of Δ. This property reads

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta. \tag{2.32}$$

We denote $T(V^*)$ with the coproduct Δ by $T^c(V^*)$. $T^c(V^*)$ has a counit $\varepsilon : T^c(V^*) \to \mathbb{C}$ given by the obvious projection. It satisfies

$$(\mathrm{id}_{T^c(V^*)} \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes \mathrm{id}_{T^c(V^*)}) \circ \Delta = \mathrm{id}_{T^c(V^*)} \tag{2.33}$$

Given any vector space V, we want to give a universal property describing the coalgebra $T^c(V)$. However, there is a small technical difficulty which forces us to restrict the class of coalgebras. We begin with the definition of a *coaugmented* coalgebra.

Definition 2.3.1. A coalgebra (C, Δ, ε) with counit is a vector space C over a field $\mathbf{k}$, such that $\Delta : C \to C \otimes C$ is coassociative as in (2.32), and the counit satisfies $\varepsilon : C \to \mathbf{k}$ satisfies (2.33). We say that C is coaugmented, if it comes equipped with a coaugmentation map $\eta : C \to \mathbf{k}$, such that $\varepsilon \circ \eta = \mathrm{id}_{\mathbf{k}}$. A morphism of coaugmented coalgebras is a linear map

$$\phi : (C, \Delta_C, \varepsilon_C, \eta_C) \to (D, \Delta_D, \varepsilon_D, \eta_D), \tag{2.34}$$

such that $\Delta_D \circ \phi = (\phi \otimes \phi) \circ \Delta_C$, $\varepsilon_D \circ \phi = \varepsilon_C$, and $\phi \circ \eta_C = \eta_D$.

We will occasionally use higher powers $\Delta_n : C \to C^{\otimes n}$ of the coproduct defined by $\Delta_n = (\mathrm{id} \otimes \Delta) \circ \Delta_{n-1}$ and $\Delta_2 = \Delta$.

The tensor coalgebra $T^c(V)$ has a coaugmentation map η given by the inclusion $\mathbf{k} \to T^c(V)$. For any coaugemented coalgebra C, it follows that there is natural isomorphism $C \cong \mathbf{k} \oplus \ker \varepsilon$. This allows us to define the reduced coproduct $\overline{\Delta}$ on $\overline{C} := \ker \varepsilon$ by

$$\overline{\Delta}(x) = \Delta(x) - 1 \otimes x - x \otimes 1. \tag{2.35}$$

Definition 2.3.2. Let (C, Δ, ε) be a coaugmented coalgebra and $\overline{C}$ as above. Define $F^n(\overline{C}) = \ker \overline{\Delta}_n$. C is called conilpotent, if

$$\overline{C} = \bigcup_{n \geq 2} F^n(\overline{C}). \tag{2.36}$$

Elements $p \in \ker \overline{\Delta}$ satisfy $\Delta(p) = 1 \otimes p + p \otimes 1$. In this case, p is called primitive. The condition that C is conilpotent is equivalent to saying that any $x \in C$ splits into primitive elements by applying $\overline{\Delta}$ finitely many times. Clearly, $\ker \overline{\Delta}_{n+1} = \bigoplus_{k=1}^{n} V^{\otimes k}$ in the case of the tensor coalgebra $T^c(V)$. Hence, the tensor coalgebra is conilpotent.

We are finally set up to state the universal property of the tensor coalgebra. Let C be a conilpotent coalgebra and V a vector space. For any linear map $\phi_1 : C \to V$, there is a unique morphism of coaugmented coalgebras $\phi : C \to T^c(V)$, such that

$$\begin{tikzcd} & T^c(V) \arrow[d] \\ C \arrow[ur, "\phi"] \arrow[r, "\phi_1"'] & V \end{tikzcd} \tag{2.37}$$

commutes. This implies in particular that we have a bijection $\mathrm{Hom}(T^cV, T^cV) \cong \mathrm{Lin}(T^cV, V)$. The infinitesimal version of this involves the dual to the notion of a derivation.

Definition 2.3.3. Let (C, Δ) be a coalgebra. A linear map $\delta : C \to C$ is called a coderivation, if it satisfies the co-Leibniz rule

$$\Delta \circ \delta = (\delta \otimes \mathrm{id}_C + \mathrm{id}_C \otimes \delta) \circ \Delta. \tag{2.38}$$

Remark 2.3.3. A graded coalgebra with coderivation δ such that $\delta^2 = 0$ is called a differential graded coalgebra for obvious reasons.

We denote the space of coderivations from C to itself by $\mathrm{Coder}(C)$. We have a linear isomorphism $\mathrm{Coder}(T^cV, T^cV) \cong \mathrm{Lin}(T^cV, V)$. We give an explicit description of the map $D : \mathrm{Lin}(T^cV, V) \to \mathrm{Coder}(T^cV, T^cV)$. Let $m_k \in \mathrm{Lin}(V^{\otimes k}, V) \subseteq \mathrm{Lin}(T^cV, V)$. Then,

$$D(m_k)(a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-k} (-)^{m_k(a_1+\ldots+a_{i-1})} a_1 \otimes \cdots \otimes a_i \otimes m_k(a_{i+1}, ..., a_{i+k}) \otimes a_{i+k+1} \otimes \cdots \otimes a_n. \tag{2.39}$$

As it is clear from the commutative diagram in (2.37) the inverse relation is obtained by projecting the output of a coderivation to $V \subseteq T^c(V)$. If (V, d) is a differential graded vector space, we can lift d to coderivation on $T^c(V)$. This defines a differential graded vector space $(T^c(V), \mathrm{d})$, such that d acts as a coderivation.

2.3.2. Symmetric Tensor (Co-)Algebra

To describe L_∞-algebras, we need the symmetric tensor (co-)algebra. Given a vector space V, we define it to be the quotient of TV modulo the two-sided ideal I generated by elements of the form $a \otimes b - (-)^{ab} b \otimes a$. We write $S(V) = T(V)/I$ and denote its product by $\odot$.

There is also a completed version $\hat{S}(V) = \hat{T}(V)/I$, which has $S(V^*)$ as its linear dual. The product on $\hat{S}(V)$ induces a coproduct $\Delta : S(V^*) \to S(V^*) \otimes S(V^*)$, defined by

$$\Delta(x_1 \odot \cdots \odot x_n) = \sum_{k \geq 0} \sum_{\sigma \in S(n)} (-)^{\epsilon} \frac{1}{k!(n-k)!} (x_{\sigma(1)} \odot \cdots \odot x_{\sigma(k)}) \otimes (x_{\sigma(k+1)} \cdots \odot x_{\sigma(n)}), \quad (2.40)$$

where $(-)^{\epsilon}$ is the sign obtained by permuting the elements with $\sigma \in S(n)$. We write $(S^c(V), \Delta)$ for the symmetric tensor coalgebra of a vector space V. The symmetric tensor coalgebra has the same universal property as the tensor coalgebra, but with respect to coaugmented cocommutative algebras. It follows that both homomorphisms and coderivations are uniquely determined by their restriction to $\mathrm{lin}(S^c V, V)$.

2.3.3. A Definition of Homotopy Associative and Homotopy Lie Algebras

We will use the machinery introduced above to give a compact definition of homotopy associative algebras. The very similar case of homtopy Lie algebras will be discussed further below.

Definition 2.3.4. Let (V, d) be a differential graded vector space. An A_∞-structure on V is a degree one coderivation M on $T^c(V)$, such that M is zero on $\mathbf{k} \oplus V$, and $(\mathrm{d} + M)^2 = 0$. We denote the triple of this data defining an A_∞-algebra by (V, d, M).

The coderivation $\mathrm{d} + M$ gives $T^c(V)$ the structure of a differential graded coalgebra. This suggests an even more efficient definition of A_∞-algebras. We could say that an A_∞-algebra over V is the pair $(T^c(V), M)$ equipped with a degree one coderivation M so that $M|_{V^{\otimes 0}} = 0$ and it makes $T^c(V)$ into a differential graded coalgebra. One can find this definition also in the literature (e.g. [58]). The case $M|_{V^{\otimes 0}} \neq 0$ is also considered. One then refers to them as weak A_∞-algebras. The reason why we prefer definition 2.3.4 is because it makes the structure of V as a differential graded vector space explicit, and the term "homotopy" refers to homotopies with respect to this differential.

At this point, it may not be immediately obvious how definition 2.3.4 relates to our discussion in the beginning of this chapter. To make the connection, recall that the coderivation M equivalent to a map $TV \to V$. This means that we can write

$$M = \sum_{k \geq 2} m_k, \quad (2.41)$$

where m_k is determined by a map $m_k : V^{\otimes k} \to V$. The m_k are the products and homotopies appearing in our original description of A_∞-algebras, up to a difference in degree. Since M is of degree one, so are the m_k. By making the shift $V \to V[-1]$, the map $m_k : V[-1]^{\otimes k} \to V[-1]$ becomes of degree $2-k$, which matches the convention introduced in section 2.1 (see also appendix A).

Property 2.1.1 gave an indirect description of A_∞-algebras, but we refrained from giving an direct formula of the A_∞-relations, i.e. the relations among d and the m_k. They can be read off the condition $(\mathrm{d} + M)^2 = 0$. Fortunately, to find them we don't have to compute the action of $(\mathrm{d} + M)^2$ on all of $T^c(V)$. We can make our lives a bit easier by noticing that, since $\mathrm{d} + M$ is of odd degree, we have $(\mathrm{d} + M)^2 = \frac{1}{2}[\mathrm{d} + M, \mathrm{d} + M]$. Further, the space of coderivations is closed under taking commutators, as it is the case of derivations of an algebra. So $(\mathrm{d}+M)^2$ is also a coderivation, independent of whether it vanishes or not. But this implies that $(\mathrm{d}+M)^2=0$ is equivalent to $\pi_1\circ(\mathrm{d}+M)^2=0$, where $\pi_1 : T^c(V)\to V$ is the canonical projection. Independent of the approach, $(\mathrm{d}+M)^2=0$ produces an infinite set of conditions. We state the first three.

$$d^2(a) = 0, \tag{2.42}$$

$$\mathrm{d}(m_2(a,b)) = -m_2(\mathrm{d}a,b)-(-)^a m_2(a,\mathrm{d}b), \tag{2.43}$$

$$\begin{aligned}\mathrm{d}(m_3(a,b,c)) = &-m_3(\mathrm{d}a,b,c)-(-)^a m_3(a,\mathrm{d}b,c)-(-)^{a+b}m_3(a,b,\mathrm{d}c)\\ &-m_2(m_2(a,b),c)-(-)^a m_2(a,m_2(b,c)).\end{aligned} \tag{2.44}$$

$$\vdots$$

Equation (2.43) and (2.44) are essentially (2.14) and (2.18), up to signs which come from the relative shift in degree.

We now come to the definition of A_∞-morphisms.

Definition 2.3.5. A mapping $\Phi : (V,\mathrm{d}_V,M_V)\to(W,\mathrm{d}_W,M_W)$ is an A_∞-morphism if the following two things hold.

1. Φ is a coalgebra morphism from $T^c(V)$ to $T^c(W)$ such that $\Phi|_{V^{\otimes 0}}=0$.
2. Φ commutes with the A_∞-structure, that is $(\mathrm{d}_W+M_W)\circ\Phi=\Phi\circ(\mathrm{d}_V+M_V)$.

Since Φ is a coalgebra morphism, it is determined by a linear map $F : T^c(V)\to W$ through $F=\pi_1^W\circ\Phi$, where $\pi_1^W : T^c(W)\to W$ is the canonical projection. We write

$$F=\sum_{n\geq 0}F_n, \tag{2.45}$$

where $F_n\in\mathrm{Lin}(V^{\otimes n},W)\subseteq\mathrm{Lin}(T^c(V),W)$. We will henceforth refer to the collection $\{F_n\}_{n\geq 0}$ as the *components* of F or, equivalently, of Φ. The condition $\Phi|_{V^{\otimes 0}}=0$ means that $F_0=0$. Morphisms with $F_0\neq 0$ are also considered and are called weak A_∞-morphisms by some authors.

The approach to A_∞-algebras just described is known as the *bar construction*. It has the obvious advantage that both the A_∞-relations and the composition of A_∞-morphisms can be written in a nice compact form. The drawback of this approach is that it disguises the interplay between algebra and homotopy theory.

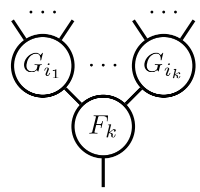
Let us talk a bit more about A_∞-morphisms. The components $F_n : V^{\otimes n}\to W$ are of degree 0. To make the connection to our first definition of A_∞-morphisms, we again shift the degrees of the vector spaces by one unit. Then, $F_n : V[-1]^{\otimes n}\to W[-1]$ is of degree $1-n$. Also, we define (quasi-)isomorphisms as in definition 2.1.1. This means that we call $F : T^c(V)\to W$ a (quasi-)isomorphism, if $F_1 : V\to W$ is a (quasi-)isomorphism of differential graded vector spaces.

Remark 2.3.4. This definitions of quasi-isomorphisms and isomorphisms does not look natural when we think of A_∞-algebras as a certain class of differential graded coalgebras. The common definition is of course that an isomorphism is a morphism which has an inverse. Similarly, a quasi-isomorphism should have an inverse on cohomology. Fortunately, an $F : T^c(V)\to W$ is a (quasi-)isomorphism in this sense, if and only if $F|_V : V\to W$ is a (quasi-)isomorphism.

Lemma 1 (Invertability of isomorphisms between tensor coalgebras)**.** *Let $F : T^c(V) \to W$ be a linear map of degree zero such that $F_0 = 0$. F defines an isomorphism of coalgebras if and only if $F|_V : V \to W$ is invertible as a linear map.*

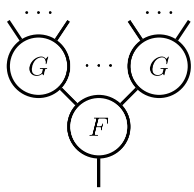
Proof based on [64]. If $F : T^c(V) \to W$ has an inverse $G : T^c(W) \to V$, then G_1 is a linear inverse of F_1. On the other hand, suppose we are given a $F : T^c(V) \to W$ with invertible F_1, and let $G : T^c(W) \to V$ be any linear map. The composition $G \circ F : T^c(V) \to V$ is

$$(G \circ F)_n = \sum_{k \geq 0} \sum_{i_1 + \ldots + i_k = n} \qquad (2.46)$$



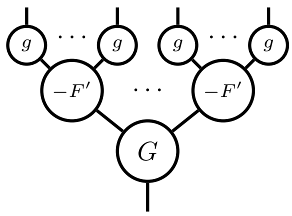
It is convenient to draw this without reference to the components,

$$G \circ F = \qquad (2.47)$$



We want to define G such that it is an inverse of F. Let us denote the linear part of G by g. We obviously need $g = F_1^{-1}$. Let F' be F, except $F_1' = 0$. The inverse of F can now be recursively constructed in the following way.

$$G = \qquad (2.48)$$



with initial condition $G_1 = g$ and $G_{k \neq 1} = 0$. □

Remark 2.3.5. The condition $F_0 = 0$ is necessary since otherwise each G_n would consist of an infinite sum. In principle, it may be possible to obtain an inverse in the more general case $F_0 \neq 0$ if one has a good notion of convergence of these sums.

Remark 2.3.6. Pictorially, G defined in (2.48) consists of all tree level Feynman diagrams with propagator g and vertices $-F_{k \geq 2}$. The propagators are not amputated, so g enters also in the external legs.

We saw that an A_∞-isomorphisms F, in the sense that F_1^{-1} exists, are invertible as coalgebra morphisms. Clearly, the inverse morphism will automatically be a chain map if F is. Hence, A_∞-isomorphisms are isomorphisms of differential graded coalgebras. The same holds true when one considers quasi-isomorphisms in each setting. To see why this is true, we need a little bit more theorems, which we defer to the next section.

We now also discuss the definitions of homotopy Lie algebras and their morphisms. They are obtained from the definition of homotopy associative algebras by using the symmetric coalgebra $S^c(V)$ instead of $T^c(V)$.

Definition 2.3.6. An L_∞-structure on a differential graded vector space (V, d) consists of a degree one coderivation M on $S^c(V)$, such that M is zero on $\mathbf{k} \oplus V$, and $(d + M)^2 = 0$. We denote this data by a triple (V, d, M).

Remark 2.3.7. We again have a shift in degree in this definition with respect to the original description we gave in the introduction to this chapter in remark 2.1.1. To relate to that convention, we apply the shift $V \mapsto V[-1]$. In the bar construction, the coderivation induces/is equivalent as set of graded *symmetric* maps $M_k : V^{\odot k} \to V$. The degree shift turns them into *anti-symmetric* maps $V[-1]^{\wedge k} \to V[-1]$ under the décalage isomorphism, see appendix A.

Definition 2.3.7. A mapping $\Phi : (V, \mathrm{d}_V, M_V) \to (W, \mathrm{d}_W, M_W)$ is an L_∞-morphism if the following hold.

1. Φ is a coalgebra morphism from $S^c(V)$ to $S^c(W)$ such that $\Phi|_{V^{\otimes 0}} = 0$.
2. Φ commutes with the L_∞-structure, that is $(\mathrm{d}_W + M_W) \circ \Phi = \Phi \circ (\mathrm{d}_V + M_V)$.

Lemma 2. *A linear map $F : S^c(V) \to W$ with $F|_{\mathbf{k}} = 0$ defines a an isomorphism between symmetric tensor coalgebras if and only if $F|_V : V \to W$ is invertible.*

Proof. The proof is very similar to the associative case. It can be found in [64]. ☐

2.3.4. The Homological Perturbation Lemma and the Homotopy Transfer Theorem

The *homological perturbation lemma* is a powerful tool to transfer (co-)homological structure from one chain complex to another, so that the two complexes are homotopy equivalent in the end (see [26] for a nice treatment of this lemma). As the name suggests, it does so perturbatively. That is, it assumes that we are already given a homotopy equivalence data. Recall that this means that for a given pair of complexes (A, d_A) and (B, d_B), we have

$$i : (A, \mathrm{d}_A) \rightleftarrows (B, \mathrm{d}_B) : p, \tag{2.49}$$

where p and i are quasi-isomorphisms such that

$$1 - i \circ p = \mathrm{d}_B h + h \mathrm{d}_B, \tag{2.50}$$

with some fixed homotopy h.

Let $\delta : B \to B$ be of degree one and such that $(\mathrm{d}_B + \delta)^2 = 0$. δ can be thought of a perturbation of the cohomological structure on B. We further demand that $(1 - \delta h)^{-1}$ exists. The homological perturbation lemma now tells us that there exists a perturbed homotopy equivalence data

$$i' : (A, \mathrm{d}_A + \delta') \rightleftarrows (B, \mathrm{d}_B + \delta) : p', \tag{2.51}$$

$$1 - i' \circ p' = (\mathrm{d}_B + \delta) \circ h' + h' \circ (\mathrm{d}_B + \delta). \tag{2.52}$$

The perturbed data is

$$i' = i + h(1 - \delta h)^{-1} \delta i, \tag{2.53}$$

$$p' = p + p(1 - \delta h)^{-1} \delta h, \tag{2.54}$$

$$h' = h + h(1 - \delta h)^{-1} \delta h, \tag{2.55}$$

$$\delta' = p(1 - \delta h)^{-1} \delta i. \tag{2.56}$$

Recall that we used a homotopy equivalence data to derive the first homotopy in an A_∞-structure from a differential graded algebra structure, where the homotopy is given by (2.17). As a first application, let us see whether we can use the perturbation lemma to directly compute all the higher homotopies. To apply it, we need to make use of the bar construction of A_∞-algebras. Suppose we are given a differential graded vector space (V, d_V) and an A_∞-algebra (W, d_W, M_W), such that (V, d_V) and (W, d_W) are homotopy equivalent as differential graded vector spaces. As before, let us denote this data by

$$i : (V, \mathrm{d}_V) \rightleftarrows (W, \mathrm{d}_W) : p, \;\; h\mathrm{d}_W + \mathrm{d}_W h = 1 - i \circ p. \tag{2.57}$$

This homotopy equivalence induces a homotopy equivalence on the tensor coalgebras

$$I : (T^c(V), \mathrm{d}_V) \rightleftarrows (T^c(W), \mathrm{d}_W) : P, \tag{2.58}$$

where $I = i \circ \pi_1 : T^c(V) \to W$ and $P = p \circ \pi_1 : T^c(W) \to V$ lift to coalgebra morphisms. There is a homotopy H from 1 to $I \circ P$. It acts on $W^{\otimes n}$ by

$$H = \sum_{k=1}^{n} (i \circ p)^{\otimes k-1} \otimes h \otimes \mathrm{id}_W^{\otimes n-k}. \tag{2.59}$$

It extends diagonally to a degree minus one map $H : T^c(W) \to T^c(W)$.

To apply the perturbation lemma, we view M_W as a perturbation of the coderivation d_W on $T^c(W)$. The lemma then provides us the perturbed data I', P', H', M'_V. One could conclude that we found an A_∞-structure on V. But we should be careful. The lemma only tells us that $\mathrm{d}_V + M_V$ gives $T^c(V)$ the structure of a cochain complex. To conclude that $(T^c(V), \mathrm{d}_V + M_V)$ defines an A_∞-structure, we should check that M_V is a coderivation. It would further be nice to have that I' and P' are A_∞-morphisms, which amounts to showing that they are coalgebra morphisms.

Remark 2.3.8. We should also make sure that $(1 - \delta h)^{-1}$ exists. One way is to ensure its existence is to write $(1 - \delta h)^{-1} = \sum_{k \geq 0} (\delta h)^k$ and demand that a sequence converges if terms with fixed number of inputs become ultimately constant. Note that this demands that δ starts with a term with at least two inputs. This notion of convergence excludes weak A_∞-algebras.

The above checks are not trivial at all. It was pointed out in [50] that a crucial condition for the (co-)algebra structures to be preserved by the perturbation lemma is that the homotopy data is a strong deformation retract. This is the same data as (2.49) and (2.50), plus the additional side conditions

$$pi = \mathrm{id}_A, \quad hi = 0, \quad h^2 = 0. \tag{2.60}$$

In this setup, A can be thought of as a sub-complex of B, which is homotopy equivalent to B. The homotopy h contracts B into A. So whenever we have a strong deformation retract, the homological perturbation lemma tells us that we have a transfer of A_∞-structures and a homotopy equivalence data

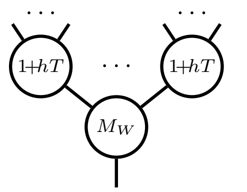
$$I' : (T^c(V), \mathrm{d}_V + N') \rightleftarrows (T^c(W), \mathrm{d}_W + M') : P'. \tag{2.61}$$

Further, one can show that the above data also is a strong deformation retract [26].

The more general case is treated by the *homotopy transfer theorem*, see for example [90] and [64], **section 10.3**.

Theorem 1 (Homotopy Transfer Theorem, HPT)**.** *Let be $i : (V, d_V) \leftrightarrow (W, d_W) : p$ be an homotopy equivalence of differential graded vector spaces, with homotopy h. Given an A_∞-structure on (W, d_W), there is an A_∞-structure on V together with an A_∞-quasi-isomorphism $i' : V \to W$ extending i.*

We state the explicit formula for the transferred structure diagrammatically. Let $M_W : T^c(W) \to W$ be the A_∞-structure on W. The induced structure M_V on V is recursively defined as follows. Let

$T =$ 

, (2.62)

with initial condition $T = 0$. Define $(M_V)_n = p \circ T_n \circ i^{\otimes n}$, which is the induced A_∞-structure on V. The A_∞-quasi-isomorphism $i' : T^c(V) \to W$ is induced from $i'_1 = i_1$ and $i'_n = h \circ T_n \otimes i^{\otimes n}$ when $n \geq 2$.

Unlike the homological perturbation lemma, the homotopy transfer theorem does not provide a homotopy inverse p' of i'. As we will see, a homotopy inverse always exists. However, it is not guaranteed to be a perturbation of p, by which we mean that it reduces to p when $M_W = 0$.

To conclude, let us talk about the ideal theorem one would like to have. This was described for example in [67], where this was called the *ideal perturbation problem.*

Definition 2.3.8 (Ideal Perturbation Problem)**.** Let (A, d_A) and (B, d_B) be (co-)chain complexes together with a two-way homotopy equivalence between them. This means that we are given the following.

$$i : (A, \mathrm{d}_A) \leftrightarrows (B, \mathrm{d}_B) : p, \quad \mathrm{d}_B h + h \mathrm{d}_B = 1 - i \circ p, \quad \mathrm{d}_B l + l \mathrm{d}_B = 1 - p \circ i. \tag{2.63}$$

The ideal perturbation problem then asks: Given a perturbation δ of d_B, does there exist a perturbation δ' of d_A together with perturbed data p', i', h', l' giving us again a two-way homotopy equivalence?

Remark 2.3.9. The difference to the assumption in the homological perturbation lemma is that we also take another homotopy l as the data. A strong deformation retract, with $l = 0$, is a special case of it.

It was proven in [67] that the IPP has in general no solution. Furthermore, an obstruction was given for it to have a solution. The obstruction can be forced to vanish if one changes the initial homotopy

$$L \to L - p(iL - Hi), \tag{2.64}$$

or, equivalently, a similar shift in H. Hence, the IPP has a solution if one allows for a shift in the initial data. When one forgets about the homotopy L afterwards, one gets the statement of the HPL.

Unfortunately, even with the solution to the HPP, (co-)algebra structures are not preserved. A way to circumvent this problem was given in [55] (the authors also derived the shift (2.64) solving the HPP). They used the *mapping cylinder* of i to reduce the problem of perturbing a single homotopy equivalence to the perturbation of two strong deformation retracts. Recall that the mapping cylinder of a map chain map $\alpha : (X, \mathrm{d}_X) \to (Y, \mathrm{d}_Y)$ is a complex Z_α, where

$$Z_\alpha^n = X^n \oplus X^{n+1} \oplus Y^n, \tag{2.65}$$

equipped with the differential

$$\mathrm{d}(x, x', y) = (\mathrm{d}x - x', -\mathrm{d}x', \mathrm{d}y + \alpha(x')). \tag{2.66}$$

The cylinder Z_α admits a strong deformation retract onto Y, and, if α is a homotopy equivalence, a strong deformation retract onto X. Because there are difficulties in giving the cylinder of maps between (co-)algebras itself a (co-)algebra structure, the authors of [55] restricted this construction to free tensor (co-)algebras in the end. This works in the case of A_∞-algebras. A perturbation of differential (co-)algebra structures can then be obtained by doing two perturbations, one into and one from the mapping cylinder. The algebra version of this statement is **theorem** (2.3*) in [55].

Remark 2.3.10. The argument does not apply for L_∞-algebras. Given an (ordinary) retract, which means that we only have $pi = \mathrm{id}_A$, one can always make a shift in the homotopy data to obtain a strong deformation retract (this fact is mentioned for example in [4]). However, the homotopy transfer theorem also applies for L_∞-algebras. The HTT does not demand a retract. From this it looks like that there should be a way to proof the general homotopy transfer theorem from the homological perturbation lemma also in this general case, maybe along the same lines as in [55], i.e. by finding a good cylinder of maps between commutative (co-)algebras.

2.3.5. The Minimal Model

Among all the A_∞-structures one can construct with the HTT, the most important is arguably the structure an A_∞-algebra induces on its cohomology. We recall its construction.

Suppose we have an A_∞-algebra (V, d, M). Let $H(V)$ be the cohomology of V with respect to d. To apply the HTT, we must construct a homotopy equivalence between V and $H(V)$. Since V is a vector space, things are particularly simple. Any cochain complex of vector spaces admits a splitting

$$V^n = B^n(V) \oplus H^n(V) \oplus B^{n+1}(V), \tag{2.67}$$

where B^n are the coboundaries in degree n, i.e. $B^n = \mathrm{Im}(\mathrm{d}|_{V^{n-1}})$. In this picture, the differential acts by inclusion

$$B^{n+1} \hookrightarrow V^{n+1}, \tag{2.68}$$

see [91]. Therefore, we can always find a strong deformation retract,

$$i : (H(V), 0) \leftrightarrows (V, \mathrm{d}) : p, \tag{2.69}$$

where the homotopy $h : V^n \to V^{n-1}$ is given by the inclusion reversing (2.68).

The existence of a strong deformation retract allows us to apply the HPL. As pointed out in the last section, its advantage over the HTT is that it gives us a quasi-isomorphism from V to $H(V)$ between the perturbed structures. We denote this by

$$I_\infty : (H(V), 0, H(M)) \leftrightarrows (V, \mathrm{d}, M) : P_\infty. \tag{2.70}$$

Definition 2.3.9. The A_∞-algebra $(H(V), 0, M_H)$ is called the *minimal model* of (V, d, M).

Theorem 2 (**Theorem 10.4.7** in [64])**.** *Given a quasi-isomorphism*

$$F : (V, d_V, M) \to (W, d_W, N), \tag{2.71}$$

there exists a quasi-isomorphism $G : (W, d_W, N) \to (N, d_N, M)$, which is inverse to F on cohomology.

Proof. Denote by

$$I_\infty : (H(V), 0, H(M)) \leftrightarrows (M, \mathrm{d}, V) : P_\infty \tag{2.72}$$

and

$$J_\infty : (H(W), 0, H(N)) \leftrightarrows (W, \mathrm{d}_W, N) : Q_\infty \tag{2.73}$$

the strong deformation retracts on the respective cohomologies. The composite

$$F' = Q_\infty \circ F \circ I_\infty \tag{2.74}$$

is an A_∞-morphism. Since F is a quasi-isomorphism, its linear part is a quasi-isomorphism of differential graded vector spaces. This implies that the linear part of F' is an isomorphism, hence it admits an inverse G' by lemma 1. The composite $G = I_\infty \circ G' \circ Q_\infty$ is a quasi-isomorphism of the type we were looking for. □

There exists an improvement to the theorem we just stated. With a proper notion of homotopy, we will see that G is in fact a homotopy inverse to F. The notion of homotopy is naturally obtained by thinking of A_∞-algebras as differential graded coalgebras. Applying the HPL to strong deformation retracts automatically gives us a strong homotopy retract in the perturbed data, which, in particular, is a homotopy equivalence.

Corollary 1. *The maps F and G in theorem 2 are homotopy inverse to each other.*

Proof. Recall that $G = Q_\infty \circ G' \circ I_\infty$, where G' is the inverse of $F' = Q_\infty \circ F \circ I_\infty$. In the following, we write $\cong$ whenever equality holds up to homotopy. We know that $I_\infty \circ P_\infty \cong \mathrm{id}_V$ and $J_\infty \circ Q_\infty \cong \mathrm{id}_W$, since we obtained these by application of the HPL. We compute

$$G \circ F = \quad I_\infty \circ G' \circ Q_\infty \circ F \cong \quad I_\infty \circ G' \circ F' \circ P_\infty = \quad I_\infty \circ P_\infty \cong \quad \mathrm{id}_V, \tag{2.75}$$
$$F \circ G = \quad F \circ I_\infty \circ G' \circ Q_\infty \cong \quad J_\infty \circ F' \circ G' \circ Q_\infty = \quad J_\infty \circ Q_\infty \cong \quad \mathrm{id}_W. \tag{2.76}$$

□

Theorem 2 together with corollary 1 imply that any A_∞-quasi-isomorphism admits an inverse up to homotopy. This is not true for quasi-isomorphisms of differential graded algebras. This property is actually a way to define homotopy algebras. It is, up to isomorphisms, the minimal extension of the notion of a differential graded algebra, such that quasi-isomorphisms become invertible. In the mathematical literature, this statement is formulated as follows. One can show that the category of A_∞-algebras with morphisms up to homotopy equivalence is equivalent to the homotopy (or derived) category of differential graded algebras, see **theorem 8** in [90].

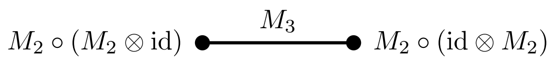


Figure 2.1.: The complex $P_\bullet(3)$.

2.3.6. A_∞-Products as Polytopes

Recall the description of A_∞-algebras given by property 2.1.1. For each number of inputs n, we had a chain complex

$$\to \ldots \to P_2(n) \to P_1(n) \to P_0(n) \to 0. \tag{2.77}$$

For fixed n, this chain complex is isomorphic to the representation of a certain $n-2$ dimensional polytope as a finite cell complex. It has a single top dimensional face associated to the one dimensional space $P_{n-2}(n)$ spanned by the A_∞ product M_n. Each M_n consists of a single closed $n-2$ dimensional cell, that is it is a compact contractible subset of $\mathbb{R}^{n-2}$. Compositions of products are translated to the cartesian product of topological spaces. With this, the A_∞-relations give rise to a unique cellular decomposition of these polytopes.

We will describe this explicitly up to M_5. The product M_2 is a contractible subset of $\mathbb{R}^0$. Hence, it is necessarily a point. It follows that also all compositions of the M_2 are points. The space $P_0(3)$ is spanned by two of these points given by the compositions $M_2 \circ (M_2 \otimes \mathrm{id})$ and $M_2 \circ (\mathrm{id} \otimes M_2)$. The A_∞-relation tells us that they are the boundary of the one-dimensional object M_3. We therefore conclude that $M_3 \in P_1(3)$ is a line. In degree $n \geq 2$, we have that $P_n(3) = 0$. We conclude that, as a cell complex, $P_\bullet(3)$ consists of single 1-cell connecting two 0-cells. This is depicted in figure 2.1.

The next complex on the list is $P_\bullet(4)$. The space $P_0(4)$ is a zero dimensional space and consists of all the cubic powers of M_2. There are five of them. Therefore, $P_0(4)$ consists of five 0-cells. $P_1(4)$ contains all the ways we can combine a single M_2 with a single M_3. Since M_2 is a point and M_3 is a line, their topological representations are again lines. There are again five of them. When we connect these lines along their common 0-cells, we obtain a single closed loop, which forms the boundary of a pentagon. Finally, this loop is the boundary of the 2-cell $M_4 \in P_2(4)$. Therefore, the complex $P_\bullet(4)$ is a pentagon, together with its canonical decomposition into five 0-cells, five 1-cells and a single 2-cell. Pictorially, this is shown in figure 2.2.

At this point, it is obvious how to build up $P_\bullet(k)$ for any k. First, we list all the 0-cells. Then, we list all the 1-cells and glue them along their common 0-cells. We continue this by gluing, for any n, all the n-cells along their common $n-1$-cells. For $P_\bullet(5)$, this means that we have 14 0-cells, 20 1-cells, 9 2-cells and a single 3-cell. Out of all the two cells, three of them are are obtained by composing M_3 with itself. These 2-cells are therefore rectangles. The remaining six consist of compositions of M_2 with M_4, which are topologically the same as M_4 and therefore pentagons. A picture of the resulting polytope can be found in [90], **exercise 16**.

Geometrically, the property that the homology groups of $P_\bullet(n)$ vanish in non-zero degree is easily understood. All the polytopes are contractible to a point. They contract onto an ordinary associative algebra, where, for any n, all the zero cells in $P_0(n)$ are identified.

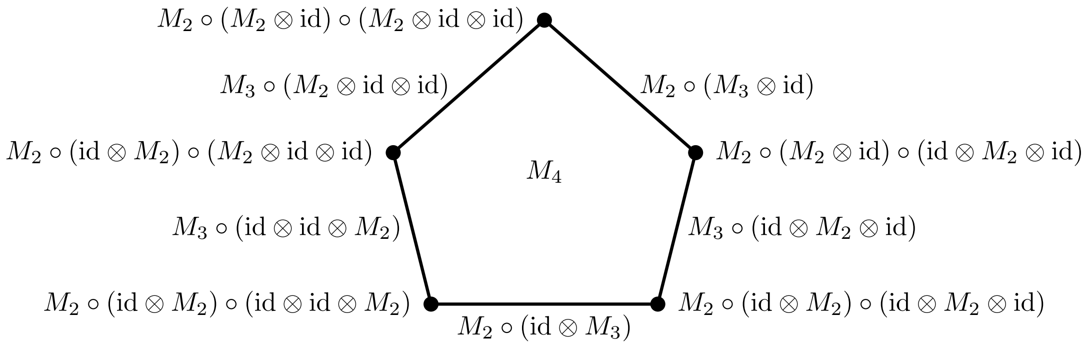


Figure 2.2.: The complex $P_\bullet(4)$.

2.3.7. Cyclic Homotopy Algebras

A physically relevant subclass of A_∞-algebras are *cyclic* A_∞-algebras. We will see that the products $M_k : V^{\otimes k} \to V$ represent the interaction of $k+1$ particles, i.e. M_k represents the sum of all order $(k+1)$-terms in the Lagrangian. The interactions don't distinguish between incoming and outgoing particles. We therefore need a tool to compare the inputs to the output of a product.

Definition 2.3.10. Let (V, d) be a differential graded vector space. An *odd symplectic structure* on V is a degree -1 product $\omega : V \otimes V \to \mathbf{k}$, such that the following holds.

- ω is graded anti-symmetric, that is, $\omega(v, w) = -(-)^{(v-1)(w-1)}\omega(w, v)$.
- ω is non-degenerate.
- ω is compatible with the differential d. This means that $\omega(\mathrm{d}a, b) = -(-)^a \omega(a, \mathrm{d}b)$.

Remark 2.3.11. As it is always the case with compatibility of a differential structure with another structure, compatibility of ω with d implies that ω restricts to a symplectic form on cohomology.

We can use ω to turn an output into an input. We define $\omega M_k : V^{\otimes(k+1)} \to \mathbf{k}$ by the formula

$$(\omega M_k)(a_0, ..., a_n) = \omega(M_k(a_0, ..., a_{n-1}), a_n). \tag{2.78}$$

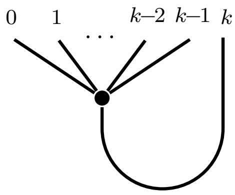
ωM_k takes values in $\mathbf{k}$. This allows us to think of $S = \sum_{k\geq 1} \omega M_k$ as a formal function on the linear space V. We will later interpret S as a (perturbatively expanded) action of a field theory.

Definition 2.3.11. A product $M_k : V^{\otimes k} \to V$ is called cyclic, if ωM_k is invariant under the cyclic permuation of its inputs. An A_∞-algebra or L_∞-algebra (V, d, M_k) is called cyclic, if it comes equipped with a symplectic form ω, such that the ωM_k are cyclic.

Remark 2.3.12. Compatibility of ω with the differential d together with anti-symmetry of ω implies that d is cyclically invariant.

Remark 2.3.13. Products of L_∞-algebras are already symmetric in their inputs. Cyclicity then implies that ωM is symmetric in all its inputs.

We represent the application of ω to an M_k diagrammatically by an arc,

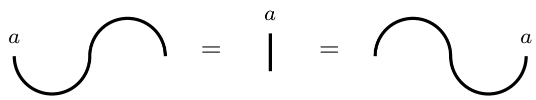
$\omega M_k =$ 

. (2.79)

We also define an inverse operation. Non-degeneracy of ω implies that for a basis $\{e^i\}_{i\in I}$ there is a dual basis vector $\{e_i^*\}_{i\in I}$ such that

$$\omega(e^i, e_j^*) = \delta_j^i. \tag{2.80}$$

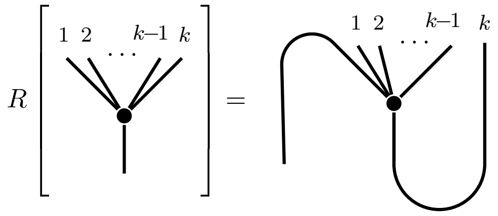
We define $\omega^{-1} = e_i^* \otimes e^i$. We think of the bivector ω^{-1} as an inverse of ω. This is motivated by the identity $\omega(a, e_i^*)e^i = a = e_i^*\omega(e^i, a)$. We draw this as



. (2.81)

The straight line represents the identity. The simplectic form gave us a degree -1 map $\omega : \mathrm{Hom}(T^cV, V) \to \mathrm{Hom}(T^cV, \mathbf{k})$. We can use ω^{-1} to define an inverse of this map. For $F \in \mathrm{Hom}(V^{n+1}, \mathbf{k})$, we define $(\omega^{-1}F)(a_1, a_2, ..., a_n) = F(a_1, ..., a_n, e_i^*)e^i$.

For any product $M_k \in \mathrm{Hom}(V^{\otimes k}, V) \to V$, not necessarily cyclic invariant, we use ω and ω^{-1} to define a rotation of M_k. We denote this by $M_k \mapsto R[M_k]$. Pictorially,



. (2.82)

If we define σ to be the permutation $(\sigma F)(a_0, ..., a_k) = F(a_k, a_0, ..., a_{k-1})$, then $R = \omega^{-1}\sigma\omega$. Cyclic invariance can be restated as $R[M_k] = M_k$.

Arbitrary compositions of products will not give cyclically invariant products. This follows from the identity (c.f. [44])

$$R[M_m \circ (\mathrm{id}^{\otimes k} \otimes M_n \otimes \mathrm{id}^{\otimes (m-k-1)})] = \begin{cases} (-)^{M_m M_n} R[M_n] \circ (\mathrm{id}^{\otimes (n-1)} \otimes R[M_m]) & \text{when } k = 1, \\ R[M_m] \circ (\mathrm{id}^{\otimes k-1} \otimes R[M_n] \otimes \mathrm{id}^{\otimes (m-k)})] & \text{else .} \end{cases}$$

On the other hand, when we think of M_m and M_n as coderivations, their commutator $[M_m, M_n]$ is cyclically invariant. Finally, homotopy transfer restricts to cyclic A_∞-algebras, i.e. cyclic A_∞-algebras give rise to cyclic A_∞-algebras through the application of the homotopy transfer theorem, see [58], **Corollary 6.14**.

2.4. The Batalin-Vilkovisky Formalism of Classical Gauge Theory

In this section, we want to put the previous mathematical definitions and theorems into the context of field theory. We can start by asking how field theories fit into homological algebra. For people who know about quantization of gauge theories, the methods of BRST- and BV-quantization will immediately come to mind. Both concepts are well known to have a homological interpretation. However, let us be more basic for a moment. We will come back to BV-quantization later.

Working with chain complexes is working with redundant information. We do manipulations on the level of cochain complexes. But in the end, we are interested in the cohomology groups. In mathematics, these redundancies are usually introduced to derive simple objects from more complicated ones. For example, to any topological space X one can associate its cohomology groups $\{H^n(X, R)\}_{n\geq 0}$ with coefficients in some abelian group R. The cohomology groups are homotopy invariants, which implies a great loss of information about the space X. On the other hand, topological spaces are often homotopy equivalent to simpler ones, so determining cohomology groups can often be reduced to the study of simpler spaces.

In physics, the story is somewhat turned upside down. One introduces redundant information to make life simpler. In a first course on Maxwell's theory, students are told that, while the primary object one works with is the gauge field A, the physically observable quantities are the electric and magnetic field (equivalently, the field strength $F = \mathrm{d}A$). The redundant information manifests in the gauge invariance $A \to A + \mathrm{d}\lambda$. The advantage in considering A lie in its simple transformation properties.

Aside from gauge invariance, another instance where redundant information occurs is when one treats theories off-shell. This means that one does not demand fields to satisfy equations of motion a priori. The advantage of this approach already shows up in classical mechanics. Before even computing the Euler-Lagrange equations, one picks suitable generalized coordinates to simplify the problem as much as possible. In quantum field theory, allowing the propagation of off-shell degrees of freedom is what made Feynman's formulation to quantum field theory much more accessible than that of Schwinger, see **chapter 4** of [79] for a short historical review.

2.4.1. Quasi-Isomorphisms in Field Theory

The goal of this section is to give a loose motivation why it is natural to consider the notion of quasi-isomorphisms in field theory.

We explained the essential motivation for this already in the previous section. Field theories come with a rudandancies. They arise from gauge invariance and off-shell descriptions. We would like to have a notion of an equivalence of theories, that is insensitive to redundancies. When we store all relevant data in a cohomology of a theories, quasi-isomorphisms between these theories, when they are realized as complexes, are the correct notion of an equivalence of theories.

Without knowledge about the cohomological description of field theories, we can still guess what quasi-isomorphisms should do. Firstly, the restriction of fields to solutions should be a quasi-isomorphism.[5] Secondly, gauge fixing some fields should also be given by a quasi-isomorphism. We will see later that the Batalin-Vilkovisky formalism will do exactly that.

[5] This is true on the classical level. The quantum theoretic equivalent is to integrate out fields.

2.4.2. Homotopy Intersections

We begin by the discussing the homological description of algebraic intersections. We will see that this extends the notion of intersections in classical algebraic geometry.

Spaces in algebraic geometry are equivalently described by the algebra of functions on that space. In the following, we will almost exclusively talk about the algebra of functions rather than the underlying geometric space. The reason is that these algebras of functions are easier to treat in the infinite dimensional setting, at least formally.

The basic building blocks in finite dimensional algebraic geometry are the n-dimensional affine spaces $\mathbb{A}^n$, whose algebra of functions are the polynomials in n variables. We will usually work over the complex numbers. In this case, the algebra is denoted by $\mathbb{C}[x_1, ..., x_n]$, but in general the polynomials can take values in any ring.

Subspaces are zero sets of a collection of functions. Given such a collection $\{f_i\}_{i=1}^k \subseteq \mathbb{C}[x_1, ..., x_n]$, the set $(f_1, ..., f_k) := \{\sum_{i=1}^k f_i g_i \mid g_i \in \mathbb{C}[x_1, ..., x_n]\}$ is called the ideal generated by the f_i. We think of $\mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_k)$ as an algebra of functions describing the set of common zeros $f_1 = ... = f_k = 0$. The word *intersection*, as it appears in the title of this section, is also sometimes used. One can think of $\mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_k)$ as the space obtained by intersecting the graphs of the functions $\{f_k\}$ with the graph of the zero function.

The algebra $\mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_n)$ is generically larger than the set of functions obtained by restriction of functions in $\mathbb{C}[x_1, ..., x_n]$ to the set theoretic subspace $f_1 = ... = f_k = 0$ of $\mathbb{A}^n$. The reason is the following. Let V_0 be the set of common zeros of the f_i. Define by $I(V_0) = \{g \in \mathbb{C}[x_1, ..., x_n] \mid g|_{V_0} = 0\}$ the ideal of functions vanishing on V_0. Two functions in $\mathbb{C}[x_1, ..., x_n]$ restrict to the same function on V_0, if and only if their difference is in $I(V_0)$. We therefore can identify the algebra of functions on V_0 with the quotient $\mathbb{C}[x_1, ..., x_n]/I(V_0)$. It is a trivial check that $(f_1, ..., f_k) \subseteq I(V_0)$. It follows that $\mathbb{C}[x_1, ..., x_n]/I(V_0) \subseteq \mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_n)$. The reverse inclusion can, however, fail. The failure happens whenever the ideal $(f_1, ..., f_k)$ is not *radical*. We say that an ideal I is radical, if and only if $\mathbb{C}[x_1, ..., x_n]/I$ has no nilpotent elements.

Example 2.4.1. In physics, we are primarily interested in the space of critical points of an action S, that is the set of points where the equations of motion $\mathrm{d}S$ are satisfied. Consider for example

$$S(x, y) = \frac{1}{2}x^2 + \frac{1}{3}(x + y)^3. \tag{2.83}$$

The equations of motion are

$$\partial_x S = x + (x + y)^2 = 0, \quad \partial_y S = (x + y)^2 = 0. \tag{2.84}$$

The function $(x+y)$ is not zero in the ring $\mathbb{C}[x, y]/(\partial_x S, \partial_y S)$. However, since $(x+y)^2 = \partial_y S$, it is nilpotent. In this case, the ring $\mathbb{C}[x, y]/(\partial_x S, \partial_y S)$ is strictly larger than the space of functions on the set $\partial_x S = \partial_y S = 0$. Indeed, the only point satisfying the equations of motion is the origin, and functions on a single point are just numbers in $\mathbb{C}$.

One viewpoint about the previous phenomenon is that the algebra $\mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_k)$ contains more information about how the set of common zeros is constructed than the functions on that set. For example, the set $x^k = 0$ for any $k \geq 1$ in $\mathbb{A}^1$ is the origin. On the other hand, the any two members of the family of algebras $\{\mathbb{C}[x]/(x^k)\}_{k\geq 1}$ are never isomorphic.

There is also a homological description of the spaces $\mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_k)$. Since these are quotients, it is natural to describe them by the homology in some complex. This is done as follows. For each f_i, we adjoin an element x_i^* of degree 1. We obtain the graded algebra $\mathbb{C}[x_1, ..., x_n, x_1^*, ..., x_k^*]$. We then define the boundary operator $\delta(x_i^*) = f_i$ and extend it by applying Leibniz' rule.

Definition 2.4.1 ([3], **section 3.2.3).** Define the complex $(K^\bullet(f_1, ..., f_k), \delta)$ with

$$K^\bullet(f_1, ..., f_k) = \mathbb{C}[x_1, ..., x_n, x_1^*, ..., x_k^*] \tag{2.85}$$

as above. We call this complex the derived zero set of the functions $f_1, ..., f_k$.

Clearly, by construction we have that the homology in degree 0, $H_0(K^\bullet(f_1, ..., f_k))$, is equal to $\mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_k)$. The higher homology groups capture the relations among the f_i. By a relation we mean a set of $R^i \in \mathbb{C}[x_1, ..., x_n]$, $i = 1, ..., k$, such that $R^i f_i = 0$. In this case, $\delta(R^i x_i^*) = 0$, so $R^i x_i^*$ defines a cycle in degree one. The degree one boundaries are the trivial relations $f_i f_j - f_j f_i = 0$. These are generated by the $R_t^i = T^{ij} f_j$, with $T^{ij} = -T^{ji}$. Note that the R_t are exactly those relations which vanish when restricted to $\mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_k)$. We conclude that

$$H_1(K^\bullet(f_1, ..., f_k)) = \frac{\text{Relations among the } f_i}{\text{Relations vanishing on } \mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_k)}. \tag{2.86}$$

Since $H_0(K^\bullet(f_1, ..., f_k)) = \mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_k)$, the derived zero sets contain more information than the ordinary algebraic intersection. We previously considered the example $x^k = 0$ on $\mathbb{C}[x]$. As a follow up on this, we could now consider that we impose a set of equations $x^{k_1} = ... = x^{k_n} = 0$, which we order according to $k_1 \leq ... \leq k_n$. The geometric space is still the origin $x = 0$. The algebra of functions is $\mathbb{C}[x]/(x^{k_1}) = H_0(K^\bullet(x^{k_1}, ..., x^{k_n}))$. However, the higher homology groups now also remember in how often we have imposed $x = 0$ (see the introduction to [65]).

What we gave here was a particular construction of the derived zero set. In general, there are many different ways to construct the complex $K^\bullet(f_1, ..., f_k)$. The general approach can be described using the Tor groups. For any ring R, we saw that, given $a \in R$, we can think of $R/(a)$ as the subspace $a = 0$. Consider the space $R/(a)$ as a module over R. The intersection of two subspaces $a = 0$ and $b = 0$ is described by the tensor product

$$R/(a, b) \cong R/(a) \otimes_R R/(b). \tag{2.87}$$

The homological version of this tensor product is known as the derived tensor product. It is denoted by

$$R/(a) \otimes_R^L R/(b). \tag{2.88}$$

To build it, choose a free[6] resolution $P^\bullet$ of $R/(a)$. For example, we could use

$$P^\bullet : 0 \to Rx^* \xrightarrow{x^* \mapsto a} R \to 0. \tag{2.89}$$

We then define the derived tensor product to be the complex

$$R/(a) \otimes_R^L R/(b) = P^\bullet \otimes_R R/(b). \tag{2.90}$$

A standard result in homological algebra is that this is independent of the choice of resolution, up to quasi-isomorphisms, see for example [91]. The homology groups of this complex are denoted by

$$H_i(P^\bullet \otimes_R R/(b)) = \mathrm{Tor}_i^R(R/(a), R/(b)). \tag{2.91}$$

[6]A projective resolution is, in fact, enough.

They are called Tor(-sion) groups.

There are several mathematical reasons why one wants to consider the derived tensor products over the ordinary tensor products. The first obvious reason is that $\mathrm{Tor}_0^R(A,B) = A \otimes_R B$, so the derived tensor product is more general, since it contains the full ordinary tensor product.[7] Secondly, the derived tensor product gives long exact sequences. Given a short exact sequence of R-modules,

$$0 \to B \to C \to D \to 0, \tag{2.92}$$

there is a long exact sequence

$$... \to \mathrm{Tor}_1^R(A,C) \to \mathrm{Tor}_1^R(A,D) \to \mathrm{Tor}_0^R(A,B) \to \mathrm{Tor}_0^R(A,C) \to \mathrm{Tor}_0^R(A,D) \to 0. \tag{2.93}$$

These long exact sequences are useful since once we know the Tor groups of two of the modules in $\{B, C, D\}$, we can determine the Tor group of the third. But most importantly, derived intersections preserve quasi-isomorphisms. By this we mean the following. Suppose B is quasi-isomorphic to C. We write $B \simeq C$. Then,

$$A \otimes_R^L B \simeq A \otimes_R^L C. \tag{2.94}$$

This is not always true if we would use instead the ordinary tensor product. This is discussed in [78]. In conclusion, if we want to consider quasi-isomorphisms as our notion of equivalence, we should use the derived tensor product.

Since we are primarily interested in the critical locus of an action, we repeat the derived construction explicitly again in that case. Suppose we are given an action $S \in \mathbb{C}[x_1, ..., x_n]$. We are interested in the algebra of functions on $\mathrm{d}S = 0$. Therefore, we consider two functions f, g equivalent, if there is a vector field X, such that

$$f - g = \mathrm{d}S(X) = X(S). \tag{2.95}$$

This suggests that we consider vector fields to live in degree 1, and define a differential δ acting on a vector field X by $\delta(X) = -X(S)$. This suggests the following definition.

Definition 2.4.2. We denote by $\Gamma(\mathbb{A}^n, \Lambda^k T\mathbb{A}^n)$ the space of sections of the kth exterior power of the tangent bundle of $\mathbb{A}^n$. We define the derived critical locus of a function S on $\mathbb{A}^n$ to be the complex

$$\mathrm{dcrit}_\bullet(S) : ... \to \Gamma(\mathbb{A}^n, \Lambda^2 T\mathbb{A}^n) \to \Gamma(\mathbb{A}^n, T\mathbb{A}^n) \to \mathbb{C}[x_1, ..., x_n] \to 0 \tag{2.96}$$

with differential $\delta X = -X(S)$ on $X \in \Gamma(\mathbb{A}^n, T\mathbb{A}^n)$, extended by Leibniz' rule.

The complex $\mathrm{dcrit}_\bullet(S)$ has a natural algebra structure, which turns it into a differential graded algebra. It is called the algebra of polyvector fields. It admits a degree -1 bracket, the *Schouten bracket*. We denote it by $\{\,\cdot\,,\,\cdot\,\}$. It is the extension of the Lie bracket of vector fields to polyvector fields. It turns $\mathrm{dcrit}_\bullet(S)$ into a *Gerstenhaber algebra*. As such, it has the following properties.

- Anti-symmetry: $\{A, B\} = -(-)^{(A-1)(B-1)}\{B, A\}$.
- Leibniz rule: $\{A, BC\} = \{A, B\}C + (-)^{(A-1)B} B\{A, C\}$.

[7] We already saw that the derived intersection of surfaces $f_i = 0$ contains the ordinary intersection.

- Jacobi identity:

$$(-)^{(A-1)(C-1)}\{A,\{B,C\}\}+(-)^{(B-1)(C-1)}\{C,\{A,B\}\}+(-)^{(A-1)(B-1)}\{B,\{C,A\}\}=0.$$

A nice feature of the Schouten bracket is that the differential δ becomes the Hamiltonian vector field with respect to S, i.e. $\delta = \{S, \cdot\}$.

Remark 2.4.1. The particular choice of resolution provided us a strict Gerstenhaber bracket. In general, the induced Gerstenhaber structure is only up to homotopy, as it is pointed out in [51].

2.4.3. Homotopy Quotients

Another basic construction on spaces is to take quotients. As for intersections, there exists a homological version of that construction.

Assume that we have a Lie group G acting on a space M. This is given by a group homomorphisms $\rho : G \to \mathrm{Diff}(M)$. Let us provide a description of the quotient M/G. Let us do this not in terms of the geometric space, but with respect to the functions on that space. Let $\mathcal{O}(M)$ denote the algebra of functions on M. Infinitesimally, the Lie algebra $\mathfrak{g}$ acts on $\mathcal{O}(M)$ as derivation. The Lie group action of G induces a Lie algebra representation $\rho : \mathfrak{g} \to \Gamma(M, TM)$. We identify the functions on the quotient M/G with the $\mathfrak{g}$-invariant subspace of $\mathcal{O}(M)$. Our aim is therefore to find a description of that subspace.

First of all, a Lie algebra representation of $\mathfrak{g}$ acting on $\mathcal{O}(M)$ is equivalent to a $U\mathfrak{g}$-module structure on $\mathcal{O}(M)$. Here,

$$U\mathfrak{g} = T\mathfrak{g}/([a,b]-(ab-ba)) \tag{2.97}$$

is called the *universal enveloping algebra* of $\mathfrak{g}$. Its action on $\mathcal{O}(M)$ is induced by the obvious action of $T\mathfrak{g}$ on $\mathcal{O}(M)$. Since ρ is a Lie algebra representation, i.e. $\rho([a,b]) = \rho(a)\rho(b) - \rho(b)\rho(a)$, this action descends to the quotient $U\mathfrak{g}$. Let R denote the trivial $U(\mathfrak{g})$-module. As a vector space it is $\mathbb{C}$, and $U(\mathfrak{g})$ acts as $A(\lambda) = 0$, $A \in U(\mathfrak{g})$, $\lambda \in \mathbb{C}$. Having this set up, we can describe the $\mathfrak{g}$-invariant subspace of $\mathcal{O}(M)$ by

$$\mathrm{Lin}_{U\mathfrak{g}}(R, \mathcal{O}(M)). \tag{2.98}$$

The problem with this object is the same as it was for the ordinary tensor product. In general, it does not preserve quasi-isomorphisms. This means that, given two quasi-isomorphic representations, their $\mathfrak{g}$-invariant subspaces may not be quasi-isomorphic. To cure this, we need a derived version of $\mathfrak{g}$-invariant subspaces (equivalently, derived G-quotients in the geometric language[8]).

The trick to cure this the problem is to build a free resolution of the trivial representation R over $U\mathfrak{g}$. Let $\varepsilon_0 : U\mathfrak{g} \to \mathbb{C}$ by the augmentation defined by projection onto $\mathfrak{g}^{\otimes 0}$ inside $U\mathfrak{g}$. $U\mathfrak{g}$ is clearly free over itself, so we declare $P_0 = U\mathfrak{g}$ to be the degree zero term in our resolution. The degree one space P_1 should be such that $\mathrm{Im}(\delta : P_1 \to P_0) = \ker(\varepsilon_0) = R$. The kernel is obviously the subspace of $U\mathfrak{g}$ with at least one power in $\mathfrak{g}$. We therefore define

$$P_1 = U\mathfrak{g} \otimes_{\mathbb{C}} \mathfrak{g} \xrightarrow{\delta} U\mathfrak{g} = P_0, \tag{2.99}$$

[8]The title of this section is indeed called homotopy *quotients*, although what we determine is actually a subspace (intersection). In the literature, one usually refers to the geometric side, although one often does computations on the algebraic side. Whenever we switch between these two descriptions, quotients become intersections, and vice versa.

where δ multiplies $\mathfrak{g}$ to $U\mathfrak{g}$ from the right. $\delta : P_1 \to P_0$ has a kernel. This is due to the fact that elements in $U\mathfrak{g}$ are subject to the relation $g_1g_2 - g_2g_1 = [g_1, g_2]$. The kernel of δ consists of elements of the form

$$Ag_1 \otimes g_2 - Ag_2 \otimes g_1 - A \otimes [g_1, g_2]. \tag{2.100}$$

We can cancel these by defining $P_2 = U\mathfrak{g} \otimes_{\mathbb{C}} (\mathfrak{g} \wedge \mathfrak{g})$ with the differential

$$\delta(A \otimes (g_1 \wedge g_2)) = Ag_1 \otimes g_2 - Ag_2 \otimes g_1 - A \otimes [g_1, g_2]. \tag{2.101}$$

From here on, the construction continues along the same lines. In degree k, we set $P_k = U\mathfrak{g} \otimes_{\mathbb{C}} \Lambda^k\mathfrak{g}$, and we define a differential similar to (2.101). Each P_k is free as a left $U\mathfrak{g}$ module. The general proof that this construction gives a resolution of R can be found for example in ([91], **Theorem 7.7.2**).

Definition 2.4.3. The derived quotient of M by G is described by the cochain complex

$$CE^\bullet(\mathfrak{g}, \mathcal{O}(M)) := \mathrm{Lin}_{Ug}(P_\bullet, \mathcal{O}(M)). \tag{2.102}$$

It is called the *Chevalley-Eilenberg* cochain complex of the Lie algebra $\mathfrak{g}$ with representation $\mathcal{O}(M)$. The cohomology groups are often called Lie algebra cohomology.

Remark 2.4.2. The description of $CE^\bullet(\mathfrak{g}, \mathcal{O}(M))$ can be simplified slightly. Given a $\mathbf{k}$-algebra A, a $\mathbf{k}$-vector space V and a left A-module M, there is an isomorphisms

$$\mathrm{Lin}_A(A \otimes_{\mathbf{k}} V, M) \cong \mathrm{Lin}_{\mathbf{k}}(V, M). \tag{2.103}$$

In our case,

$$\mathrm{Lin}_{U\mathfrak{g}}(P_\bullet, \mathcal{O}(M)) = \mathrm{Lin}_{U\mathfrak{g}}(U\mathfrak{g} \otimes_{\mathbb{C}} \Lambda^\bullet\mathfrak{g}, \mathcal{O}(M)) \cong \mathrm{Lin}_{\mathbb{C}}(\Lambda^\bullet\mathfrak{g}, \mathcal{O}(M)) \cong \mathcal{O}(M) \otimes_{\mathbb{C}} \Lambda^\bullet\mathfrak{g}^*.$$

This allows us to forget about the role of the universal enveloping algebra, which was used to construct $CE^\bullet(\mathfrak{g}, \mathcal{O}(M))$.

Example 2.4.2. Let $\mathfrak{g} = \Gamma(M, TM)$ be the Lie algebra of vector fields on a manifold M. The complex $CE^\bullet(\Gamma(M, TM), C^\infty(M))$ is the de Rham complex. It is useful to keep this example in mind when one wants to memorize the explicit formula for the differential. One can simply use the coordinate-free formula of the de Rham differential.[9]

In degree zero, the cohomology consists of elements $f \in \mathcal{O}(M)$ satisfying $\rho(a)f = 0$ for all $a \in \mathfrak{g}$. Hence, it is the subspace of $\mathfrak{g}$-invariant functions, i.e. the functions on the ordinary quotient M/G. The derived quotient fully encodes the ordinary quotient.

2.4.4. The Batalin-Vilkovisky Formalism

We argued that there are two types of redundancies in field theories, gauge redundancies and off-shell degrees of freedom. The Batalin-Vilkovisky (BV) construction can be divided into two steps. The restriction to on-shell degrees of freedom by formation of a derived critical locus, and the identification of gauge equivalences by forming a derived quotient. To

[9] The formula can be found in [91], **Corollary 7.7.3**.

the knowledge of the author, the interpretation of BV theory in context of derived geometry has been advocated primarily by K. Costello and O. Gwilliam, see [22, 51, 23].

Observables of a classical field theory are functions on a field space $\mathcal{F}_0$. To avoid unnecessary notational complications, we will denote a field by a single field variable $\phi(x)$, although a general field theory usually depends of course on multiple fields. Let us call the space of observables $\mathcal{O}(\mathcal{F}_0)$. We usually think of the observables as polynomials in the field variables $\phi(x)$. In this case, an observable $F \in \mathcal{O}(M)$ of homogeneous degree n is of the form

$$F[\phi] = \int \mathrm{d}x_1 \cdots \mathrm{d}x_n \, F(x_1, ..., x_n)\phi(x_1)\cdots\phi(x_n). \tag{2.104}$$

A functional derivative on $F[\phi]$ is defined by the rule $\frac{\delta\phi(x)}{\delta\phi(y)} = \delta(x-y)$. This allows us to define also vector fields and forms on $\mathcal{F}_0$. Of course, in any concrete setting, one should care about existence of all the integrals involved. Nevertheless, we will only be interested in the algebraic nature of these operations and think of all the operations on the formal level. A good setup for the discussion of analytic problems can be found in the appendix of [42]. With potential analytic issues aside, the space $\mathcal{O}(\mathcal{F}_0)$ behaves much like polynomial algebras in finite dimensions.

The laws of physics in a field theory are governed by an action $S_0 \in \mathcal{O}(\mathcal{F}_0)$. The equations of motion are the critical points of S_0, $\mathrm{d}S_0 = 0$. We want to form the derived critical locus of S_0. Recall that this is, by definition, the complex

$$\mathrm{dcrit}_\bullet(S_0) = ... \to \Gamma(\mathcal{F}_0, \Lambda^n T(\mathcal{F}_0)) \to ... \to \Gamma(\mathcal{F}_0, T(\mathcal{F}_0)) \to \mathcal{O}(\mathcal{F}_0) \to 0. \tag{2.105}$$

An element in $\Gamma(\mathcal{F}_0, \Lambda^n T(\mathcal{F}_0))$ is of the form

$$\int \mathrm{d}x_1 \cdots \mathrm{d}x_n F(\phi, x_1, ..., x_n)\frac{\delta}{\delta\phi(x_1)}\cdots\frac{\delta}{\delta\phi(x_n)} \tag{2.106}$$

with anti-commuting functional derivatives. Physicists usually write $\frac{\delta}{\delta\phi(x)} = \phi^*(x)$ and call ϕ^* the anti-field (of ϕ). Recall that the complex $\mathrm{dcrit}_\bullet(S_0)$ has an odd Poisson bracket, the Schouten bracket. In terms of fields and anti-fields, it acts as

$$\{\phi^*(x), \phi(y)\} = \frac{\delta\phi(y)}{\delta\phi(x)} = \delta(x-y). \tag{2.107}$$

We could have equivalently used this as a definition for the Schouten bracket. Its extension to general elements in $\mathrm{dcrit}_\bullet(S_0)$ is given by the formula

$$\{F, G\} = \int \mathrm{d}x \, \frac{\delta_r F}{\delta\phi^*(x)}\frac{\delta G}{\delta\phi(x)} - \frac{\delta_r F}{\delta\phi(x)}\frac{\delta G}{\delta\phi^*(x)}. \tag{2.108}$$

The expression $\frac{\delta_r F}{\delta\phi(x)} := (-)^{(F+1)\phi}\frac{\delta F}{\delta\phi(x)}$ denotes the derivative from the right. The sign is such that it satisfies graded Leibniz rule when we think of it as acting on functionals from the right.

The complex $\mathrm{dcrit}_\bullet(S_0)$, as we defined it, is homologous rather than cohomologous, i.e. it has a differential of degree -1. We will now turn it into a cochain complex by defining $\mathrm{dcrit}^\bullet(S_0) = \mathrm{dcrit}_{-\bullet}(S_0)$. It becomes a graded algebra concentrated in non-positive degree. This is necessary, since we want to reserve positive degrees for the Chevalley-Eilenberg complex. The bracket $\{\,\cdot\,,\,\cdot\,\}$ becomes of cohomological degree 1.

We recall the interpretation of the first two cohomology groups of $\mathrm{dcrit}^\bullet(S_0)$. Two functionals F and G describe the same object in $H^0(\mathrm{dcrit}^\bullet(S_0))$, if their difference can be written as

$$G - F = -X(S_0) = -\int \mathrm{d}x X(\phi, x)\frac{\delta S_0}{\delta\phi(x)} = \{S_0, X\} \tag{2.109}$$

for some vector field

$$X = \int \mathrm{d}x X(\phi, x)\frac{\delta}{\delta\phi(x)}. \tag{2.110}$$

We define $H^0(\mathrm{dcrit}^\bullet(S_0)) = \mathcal{O}(\mathrm{crit}(S_0))$, the algebra of functions on the critical locus of S_0.

The cohomology in degree -1, $H^{-1}(\mathrm{dcrit}^\bullet(S_0))$, encodes the symmetries of S_0. A vector field Y is closed under the differential of $\mathrm{dcrit}^\bullet(S_0)$, if $Y(S_0) = 0$. On the other hand, it is exact if it is of the form

$$Y = \int \mathrm{d}x\mathrm{d}y Y(x, y)\frac{\delta S_0}{\delta\phi(x)}\frac{\delta}{\delta\phi(y)}, \quad Y(x, y) = -Y(y, x). \tag{2.111}$$

These vanish on the solution space of S_0. Symmetries of this kind are usually called *trivial*. We quotient them out to obtain $H^{-1}(\mathrm{dcrit}^\bullet(S_0))$. In doing so, $H^{-1}(\mathrm{dcrit}^\bullet(S_0))$ becomes the space of symmetries of S_0 *on* $\mathrm{crit}(S_0)$. Indeed, a pair of symmetries

$$X = \int \mathrm{d}x X(\phi, x)\frac{\delta}{\delta\phi(x)}, \quad Y = \int \mathrm{d}x Y(\phi, x)\frac{\delta}{\delta\phi(x)}, \tag{2.112}$$

is equal on $\mathrm{crit}(S_0)$, if their "components" $X(\phi, x), Y(\phi, y)$ differ by a functional proportional to the equations of motion. We want to stress that, in this way, the vector fields are restricted to $\mathrm{crit}(S_0)$ in the algebraic sense. They are taken modulo the ideal generated by the equations of motions, rather than modulo the vanishing ideal of the set $\mathrm{d}S = 0$.

The fact that $H^{-1}(\mathrm{dcrit}^\bullet(S_0))$ encodes the symmetries of S shows that this space knows about possible gauge symmetries of S_0. We will give a construction to get rid of all gauge symmetries in $H^{-1}(\mathrm{dcrit}^\bullet(S_0))$. The reason is to do this is the following. The main motiviation for the construction presented here was to identify equivalent theories using the notion of a quasi-isomorphism. One equivalence should be related to gauge fixing certain fields. By passing from an action S_0 to a gauge fixed action S_0^{gf}, the action will lose all the gauge symmetries we fix. Therefore, there will be no trace of these symmetries left in $H^{-1}(\mathrm{dcrit}^\bullet(S_0^{gf}))$. If we want $\mathrm{dcrit}^\bullet(S_0^{gf})$ to be quasi-isomorphic to $\mathrm{dcrit}^\bullet(S_0)$, we should get rid of these symmetries in $H^{-1}(\mathrm{dcrit}^\bullet(S_0))$ as well.

Suppose that we have a group of gauge transformations $\mathcal{G}$[10] acting on field space $\mathcal{F}_0$ through a Lie algebra homomorphism $\rho : \mathfrak{G} \to \Gamma(\mathcal{F}_0, T\mathcal{F}_0)$ from the Lie algebra $\mathfrak{G}$ of $\mathcal{G}$ into the space of vector fields on $\mathcal{F}_0$. We want to demote the vector fields in the image of ρ to coboundaries in $\mathrm{dcrit}^\bullet(S_0)$. Therefore, we put the elements of $\mathfrak{G}$ in degree -2 and extend the differential $\delta_{crit} \mapsto \delta_{crit} + \delta_g$ by $\delta_g(a) = \rho(a)$ for all $a \in \mathfrak{G}$. This gets rid of all the gauge symmetries in $\mathrm{dcrit}^{-1}(S_0)$.

Let us summarize what we have done so far. We build the graded vector space $\mathrm{Sym}(\mathfrak{G}^\bullet) \oplus \Gamma(\mathcal{F}_0, \mathrm{Sym}(T[-1]\mathcal{F}_0))$. The differential splits into two parts, $\delta = \delta_{crit} + \delta_g$. We defined it on

[10] Let us emphasize that, in a gauge theory, $\mathcal{G}$ is not the gauge group G of that theory! Rather, it denotes the set of all gauge transformations. This means that its elements are G-valued functions $\mathbb{R}^{1,3} \to G$; $x \mapsto g(x)$, or more generally sections of a bundle with fibers isomorphic to G. In order to not forget about this distinction, we denote the group of gauge transformations by $\mathcal{G}$, and reserve the letter G for gauge groups of gauge theories.

vector fields by $\delta_{crit} X = -X(S)$ and on gauge transformations $a \in \mathfrak{G}$ by $\delta_g(a) = \rho(a)$. In degree zero, the cohomology of this complex describes functions on the critical locus $\mathrm{d}S = 0$. In degree -1, the cohomology contains all symmetries that are neither trivial nor gauge.

We successfully described the space of on-shell fields in terms of the derived critical locus. Let us now turn our attention towards the description of gauge invariant functionals. We already discussed that the proper cohomological description is realized by the Chevalley-Eilenberg complex $CE^{\bullet}(\mathfrak{G}, \mathcal{O}(\mathcal{F}_0))$. Its zeroth cohomology consists of the gauge invariant functionals.

Recall that $\mathrm{Sym}(\mathfrak{G}[1]^*) \otimes \mathcal{O}(\mathcal{F}_0)$ is the underlying space of the Chevalley-Eilenberg complex. The elements of $\mathrm{Sym}(\mathfrak{G}[1]^*)$ are functions on $\mathfrak{G}[1]$. Coordinates on $\mathfrak{G}[1]$ are called c-ghosts. We denote them by $\mathcal{C}^i(x)$. In this notation, a general element in $\mathrm{Sym}(\mathfrak{G}[1]^*) \otimes \mathcal{O}(\mathcal{F}_0)$ of degree n looks like

$$\int \mathrm{d}x_1 \cdots \mathrm{d}x_n \, F_{i_1 \dots i_n}(\phi, x_1, ..., x_n) \mathcal{C}^{i_1}(x_n) \cdots \mathcal{C}^{i_n}(x_n). \tag{2.113}$$

In the BV formalism, one expresses the Chevalley-Eilenberg differential as a Hamiltonian vector field, similar to what we have seen in case of differential of the derived intersection. Let us define a Poisson bracket for ghosts. The natural pairing on $\mathfrak{G}[2] \oplus \mathfrak{G}[1]^*$ gives rise to a Poisson bracket on $\mathrm{Sym}(\mathfrak{G}[2] \oplus \mathfrak{G}[1]^*)$. The variables dual to the c-ghosts $\mathcal{C}^i(x)$ are denoted by $\mathcal{C}^*_j(y)$. They are called anti-ghosts. We have already encountered them in our discussion following the derived critical locus, where we used them to get rid of gauge symmetries in the degree -1 cohomology $H^{-1}(\mathrm{dcrit}(S))$ of the derived critical locus. In terms of ghost and anti-ghost variables, the Poisson bracket on $\mathrm{Sym}(\mathfrak{G}[2] \oplus \mathfrak{G}[1]^*)$ reads

$$\{F, G\}_{gh} = \int \mathrm{d}x \frac{\delta_r F}{\delta \mathcal{C}^*_i(x)} \frac{\delta G}{\delta \mathcal{C}^i(x)} - \frac{\delta_r F}{\delta \mathcal{C}^i(x)} \frac{\delta G}{\delta \mathcal{C}^*_i(x)}. \tag{2.114}$$

The Chevalley-Eilenberg differential δ_{CE} consists of two parts. One is related to the Lie bracket and the other one is related to the representation. Let us first look at the part of δ_{CE} involving the representation. δ_{CE} differential on a functional $F \in \mathcal{O}(\mathcal{F}_0)$ is

$$\delta_{CE} F = \int \mathrm{d}x \mathrm{d}y \, \frac{\delta F}{\delta \phi(x)} R_j(\phi, x, y) \mathcal{C}^j(y). \tag{2.115}$$

To write this as a Hamiltonian vector field, we need the Poisson bracket coming from the Schouten bracket. We find that

$$\delta_{CE} F = \{S_1, F\}_S, \quad S_1 = -\rho.^{11} \tag{2.116}$$

The representation $S_1 = -\rho$ also serves another purpose. When we use the ghost/anti-ghost Poisson bracket, it generates the differential δ_g on anti-ghosts. Explicitly,

$$\delta_g \mathcal{C}^*(x)_i = \{S_1, \mathcal{C}^*(x)\}_g = \int \mathrm{d}y R_i(\phi, x, y) \phi^*(y). \tag{2.117}$$

$\int \mathrm{d}y R_j(\phi, x, y) \phi^*(y)$ is the vector field representing the element $\mathcal{C}^*(x)_i \in \mathfrak{G}^*$.

[11] The reader may worry in which sense the action $S_0 + S_1$ is a function. Since the representation is a map $\rho : \mathfrak{G} \to \Gamma(\mathcal{F}_0, T\mathcal{F}_0)$, we can view S_1 as a function on $\mathfrak{G} \otimes T^*[-1]\mathcal{F}_0$. This will become clear further below when we define the space of fields.

On ghosts $\mathcal{C}^i(x) \in \mathfrak{G}^*$, δ_{CE} acts as

$$\delta_{CE}\mathcal{C}^i(x) = \int \mathrm{d}y\mathrm{d}z\, T^i_{jk}(\phi, x, y, z)\mathcal{C}^j(y)\mathcal{C}^k(z). \tag{2.118}$$

In terms of variables, the Lie bracket can be expressed as

$$[\,\cdot\,,\,\cdot\,] = \int \mathrm{d}x\mathrm{d}y\mathrm{d}z\, \mathcal{C}^*_i(x)T^i_{jk}(\phi, x, y, z)\mathcal{C}^j(y)\mathcal{C}^k(z) =: S_2. \tag{2.119}$$

We observe that

$$\delta_{CE}\mathcal{C}^i(x) = \{S_2, \mathcal{C}^i\}_{gh}. \tag{2.120}$$

Hence, the Lie bracket generates δ_{CE} on ghosts via the Poisson bracket $\{\,\cdot\,,\,\cdot\,\}_{gh}$.

Remark 2.4.3. We may call the $T^i_{jk}(\phi, x, y, z)$ the structure constants of the algebra of gauge transformations $\mathfrak{G}$. In a gauge theory with gauge algebra $\mathfrak{g}$, they read

$$T^i_{jk}(\phi, x, y, z) = T^i_{jk}\delta(x-y)\delta(x-z), \tag{2.121}$$

where the T^i_{jk} are the structure constants of the algebra $\mathfrak{g}$.

We finally combine the Chevalley-Eilenberg differential with differential of the derived critical locus. We define the total field space

$$\mathcal{F} = T^*[-1]\mathcal{F}_0 \oplus \mathfrak{G}[1] \oplus \mathfrak{G}[2]^*. \tag{2.122}$$

Dually, its space of functions is

$$\mathcal{O}(\mathcal{F}) = \Gamma(\mathcal{F}_0, \mathrm{Sym}(T[1]\mathcal{F}_0)) \oplus \mathrm{Sym}(\mathfrak{G}[1]^* \oplus \mathfrak{G}[2]). \tag{2.123}$$

This is the graded vector space underlying the Chevalley-Eilenberg complex and the derived critical locus. Its grading is called the ghost number. There is also another useful grading, the anti-field number. By definition it is minus the ghost number, if the ghost number is negative and 0 otherwise. This means that ϕ^* has anti-field number 1 and $\mathcal{C}^*_i$ has anti-field number two.

We equip $\mathcal{O}(\mathcal{F})$ with a Poisson bracket

$$\{\,\cdot\,,\,\cdot\,\} = \{\,\cdot\,,\,\cdot\,\}_S + \{\,\cdot\,,\,\cdot\,\}_{gh}. \tag{2.124}$$

We define the action $S = S_0 + S_1 + S_2$, where $S_1 = -\rho$ and $S_2 = [\,\cdot\,,\,\cdot\,]$. This action generates the vector field

$$Q = \{S, \,\cdot\,\}, \tag{2.125}$$

which, as we have seen, contains the Chevalley-Eilenberg differential δ_{CE}, the differential of the derived critical locus δ_{crit}, as well as its extension δ_g. In fact, it contains even more. Due to the presence of S_2, the action of δ_g on anti-ghosts $\mathcal{C}^*(x)$ is extended. Further, S_1 extends the action of δ_{crit} on vector fields ϕ^*.

In case of Yang-Mills theory, the differential Q squares to zero. We define the complex

$$(\mathcal{O}(\mathcal{F}), Q). \tag{2.126}$$

It is the complex describing Yang-Mills in the BV formalism. However, the Batalin-Vilkovisky formalism was invented to describe theories with gauge symmetries more complex than those of Yang-Mills (for example supergravity). In these cases, the construction we presented is not sufficient. In particular, the differential Q does not necessarily square to zero. This is due to the fact that in some cases, the representation ρ is a Lie algebra representation only up to equations of motion,

$$\rho([a,b]) = [\rho(a), \rho(b)] + \delta_{crit}(F), \tag{2.127}$$

for some functional F. Gauge algebras in theories where this happens are called *open*. Another generalization allows for *reducible* gauge symmetries. These are symmetries where the representation ρ has a kernel, or, in the case of open algebras, has elements in its domain whose image is proportional to the equations of motion. In this case, one introduces ghosts for ghosts, which deal with redundancies among gauge transformations.

The general story of BV quantization goes as follows. Start with a classical action S_0. Introduce anti-fields to describe its derived critical locus. Then account for all gauge symmetries similar as we did above. Finally, if the differential $Q = \{S, \cdot\}$ involving the extended action does not square to zero, add further terms in higher anti-field number so that it squares to zero. Observe that $Q^2 = 0$ is equivalent to $\{S, S\} = 0$ by the Jacobi identity. This motivates the following definition.

Definition 2.4.4. Given an algebra of functionals $\mathcal{O}(\mathcal{F})$ equipped with an odd Poisson bracket $\{\ ,\ \}$. We say that a functional $F \in \mathcal{O}(\mathcal{F})$ satisfies the *classical master equation*, if

$$\{F, F\} = 0. \tag{2.128}$$

In the light of this definition, properly extending S so that $Q^2 = 0$ amounts to solving the classical master equation. An action satisfying the classical master equation is then the action of a BV theory.

Remark 2.4.4. It would be interesting to see whether the general BV construction has a similar interpretation in terms of the derived critical locus and (some generalization of) the Chevalley-Eilenberg complex. The author thinks that such a generalization can be achieved by extending the Chevalley-Eilenberg complex from Lie algebras to L_∞-algebras. It is easy to see that such an extension exists whenever the Chevalley-Eilenberg complex is constructed with respect to the trivial representation. In that case, the complex is just the dual of the bar construction of an L_∞-algebra. With a good notion of a representation of an L_∞-algebra, it may be possible to generalize this construction to non-trivial representations. A definition of L_∞-algebras acting on graded manifolds was given in [71]. This is not completely sufficient, since this will not reproduce open algebras properly. We should rather ask for representations into differential graded manifolds, so that the representation knows about trivial transformations.

2.4.5. Solving the Classical Master Equation

A sufficient condition for solving the classical master equation is given in [41], which we recall here. The condition is that the presence of anti-ghosts turns the complex

$$(\Gamma(\mathcal{F}_0, \mathrm{Sym}(T[1]\mathcal{F}_0)) \oplus \mathrm{Sym}(\mathfrak{G}[2]), \delta_g + \delta_{gh}) \tag{2.129}$$

into a resolution of its zeroth cohomology. By definition, this means that all non-zero cohomologies are trivial.

In [41] it was shown that, when the complex $(Q_{BV}, \mathcal{F}^-)$ is a resolution, a solution to the master equation can be constructed perturbatively in anti-field number. On the other hand, even when the $(Q_{BV}, \mathcal{F}^-)$ is not a resolution, a solution to the BV master equation may still be obtainable. In the literature, it is often considered to be a necessary to construct a resolution of the critical locus. However, in the following examples we want to argue that one may not always want to remove all the lower cohomology groups, since some arise because we have physical degrees of freedom instead of gauge degrees of freedom.

Example 2.4.3. Let us discuss what happens when we resolve the critical locus of S_0. We begin with a finite dimensional action S_0, i.e. some function on a finite dimensional manifold. We automatically have a resolution when their are no non-trivial symmetries. Let us take for example

$$S(x, y) = x^n + y^n, \quad n \geq 2, \tag{2.130}$$

as a function on the two dimensional plane. The derived intersection is described by the complex

$$\mathrm{dcrit}^\bullet(S) : 0 \to \mathbb{R}[x, y](\partial_x \wedge \partial_y) \to \mathbb{R}[x, y]\partial_x \oplus \mathbb{R}[x, y]\partial_y \to \mathbb{R}[x, y] \to 0 \tag{2.131}$$

and differential $Q_{BV}(\partial_i) = -\partial_i S$. The equations of motion are $x^{n-1} = 0$ and $y^{n-1} = 0$. Algebraically, this means that we take

$$H^0(\mathrm{dcrit}^\bullet(S)) = \mathcal{O}(\mathrm{crit}(S)) = \mathbb{R}[x, y]/(x^{n-1}, y^{n-1}) \tag{2.132}$$

as our space of on-shell functions. Let $X = f(x, y)\partial_x + g(x, y)\partial_y$ be any vector field. It is a symmetry of S if

$$fx^{n-1} + gy^{n-1} = 0, \tag{2.133}$$

i.e. $f = -g\frac{y^{n-1}}{x^{n-1}}$. In order for X not to be singular, g should be proportional to x^{n-1}. Hence, the most general symmetries of S are generated by vector fields of the form

$$X = f(x, y)(y^{n-1}\partial_x - x^{n-1}\partial_y). \tag{2.134}$$

But these vanish on $\mathcal{O}(\mathrm{crit}(S))$. This suggests that they are trivial, i.e. that they will not contribute to $H^1(\mathrm{dcrit}^\bullet(S))$. Indeed, they are the image of the bi-vector fields of the form $f(x, y)\partial_x \wedge \partial_y$ under the differential Q_{BV}. This tells us that $H^{-1}(\mathrm{dcrit}^\bullet(S)) = 0$. It is an easy check that also $H^{-2}(\mathrm{dcrit}^\bullet(S)) = 0$. We find that the complex $\mathrm{dcrit}^\bullet(S)$ resolves $\mathcal{O}(\mathrm{crit}(S))$. This is expected, since all equations of motion are independent and the solution manifold is the single point $x = y = 0$.

Example 2.4.4. Consider the action

$$S_1(x, y) = (x - y)^n, \quad n \geq 2. \tag{2.135}$$

It has an obvious shift symmetry $(x, y) \mapsto (x + a, y + a)$. Hence, we expect that the derived critical locus will not be a resolution. The derived critical locus is again of the form

$$\mathrm{dcrit}^\bullet(S_1) : 0 \to \mathbb{R}[x, y](\partial_x \wedge \partial_y) \to \mathbb{R}[x, y]\partial_x \oplus \mathbb{R}[x, y]\partial_y \to \mathbb{R}[x, y] \to 0, \tag{2.136}$$

with differential $Q_{BV}(\partial_i) = -\partial_i S_1$. The space of on-shell functionals is

$$H^0(\mathrm{dcrit}^\bullet(S_1)) = \mathcal{O}(\mathrm{crit}(S_1)) = \mathbb{R}[x, y]/(x^{n-1} - y^{n-1}). \tag{2.137}$$

This time, the underlying geometric space is the diagonal $\{(x,y) \in \mathbb{R}^2 \,|\, x = y\}$. In particular, it is of dimension one. This is obviously due to the fact that the equations of motion are not independent. We have $\partial_x S_1 = -\partial_y S_1$. It follows that all vector fields of the form $X = f(x,y)(\partial_x + \partial_y)$ are in the kernel of Q_{BV} in degree -1. On the other hand, the trivial vector fields are of the form

$$Q_{BV}(f(x,y)(\partial_x \wedge \partial_y)) = f(x,y)(x^{n-1} - y^{n-1})(\partial_x + \partial_y). \tag{2.138}$$

The cohomology in degree -1 is $H^{-1}(\mathrm{dcrit}^\bullet(S)) = \mathcal{O}(\mathrm{crit}(S_1))(\partial_x + \partial_y)$. These are the vector fields tangent to $\mathrm{crit}(S_1)$. On the other hand, since (2.138) is zero only if $f = 0$, we find that $H^{-2}(\mathrm{dcrit}^\bullet(S)) = 0$.

To get a resolution of $\mathcal{O}(\mathrm{crit}(S_1))$, we should declare the symmetry $(x,y) \mapsto (x+a, y+a)$ to be gauge. Recall that we achieve this by introducing an anti-ghost c^* of degree -2 and declare $Q_{BV}(c^*) = \partial_x + \partial_y$. This kills cohomology in degree -1. Since the anti-ghost c^* is even, the complex becomes unbounded in negative degree. We therefore may worry about new cohomologies in lower degrees. Indeed, there are new cocycles of the form

$$\int_0^{c^*} \mathrm{d}z\, g(x,y,z)(x^{n-1} - y^{n-1}) + g(x,y,c^*)\partial_x \wedge \partial_y. \tag{2.139}$$

However, this is also a coboundary. It is the image of

$$\int_0^{c^*} \mathrm{d}z\, g(x,y,z)\partial_y. \tag{2.140}$$

This shows that we have a resolution of $\mathcal{O}(\mathrm{crit}(S_1))$.

The BV formalism now tells us to also introduce ghosts and to add the Chevalley-Eilenberg differential. This introduces a non-trivial condition on functions in degree zero. It is

$$Q_{CE}(f(x,y)) = (\partial_x + \partial_y)f(x,y) = 0. \tag{2.141}$$

This means of course that the gauge invariant functions depend only on the difference $x - y$. The combined cohomology of Q_{CE} and Q_{BV} is isomorphic to $\mathbb{R}[x]/(x^{n-1})$. The underlying geometry is again zero dimensional. This is of course expected, since the whole one dimensional line is identified by the gauge symmetry.

From this example we see that, in finite dimensions, when we resolve the critical locus of a function by introducing anti-ghosts, the space of physical fields (i.e. on-shell and modulo gauge transformations) is always zero dimensional. Higher dimensional spaces only appear when the number of independent equations is lower than the dimension of the ambient space. But, as we have seen, building up a resolution will always lead to an identification of all points on (a connected component in) the critical locus of S by gauge symmetries.

Example 2.4.5. Consider a free massless scalar field theory,

$$S[\phi] = \frac{1}{2}\int \mathrm{d}^4x\, \phi(x)\Box\phi(x). \tag{2.142}$$

We assume for simplicity that all four spacetime dimensions are compactified to a circle, so we don't have to worry about convergence of the integral. The action has a symmetry under the shift $\phi \to \phi + \phi_0$, whenever ϕ_0 satisfies the equations of motion $\Box\phi_0(x) = 0$. This symmetry is generated by the vector field

$$X = \int \mathrm{d}^4x\, \phi_0(x) \frac{\delta}{\delta\phi(x)}. \tag{2.143}$$

In this case, the derived critical locus of S_0 has again cohomology in degree -1. The vector field (2.143) is independent of ϕ, so it does in particular not vanish when ϕ is on-shell. Therefore, it is not trivial. The derived critical locus is not a resolution of the space of on-shell functionals. Also, we usually do not want to fix this through anti-ghosts, since we consider fields satisfying the equations of motions to be physical degrees of freedom rather than gauge degrees of freedom. Nevertheless, a solution to the master equation trivially exists, since S already satisfies the master equation (as does any action without ghosts and anti-fields).

Example 2.4.6. General relativity may be a physically observable example for a theory where not all symmetries of the action are considered gauge symmetries. The *pp-wave spacetimes* are exact solutions to the field equations in vacuum. Their metric is can be written as

$$\mathrm{d}s^2 = H(u, x, y)\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}v + \mathrm{d}x^2 + \mathrm{d}y^2, \tag{2.144}$$

where H is an arbitrary function in u, x, y. The particular choice of coordinate system is called *Brinkmann coordinates.* The metric solves the Einstein equations in vacuum if $(\partial_x^2 + \partial_y^2)H(u, x, y) = 0$. It has non-zero curvature if the Hessian of H with respect to x and y is non-zero.[12] When $H = 0$, we obtain Minkowski spacetime. The function H, with $(\partial_x^2 + \partial_y^2)H(u, x, y) = 0$, parametrizes solutions close to flat spacetime. If the solution has non-zero curvature, it cannot be gauge equivalent (diffeomorphic) to flat spacetime.

Let us define the vector field

$$X = \int \sqrt{g}\mathrm{d}^4x\, h_{\mu\nu}(x) \frac{\delta}{\delta g_{\mu\nu}(x)}, \tag{2.145}$$

where $h_{uu} = H_0$ and $h_{\mu\nu} = 0$ otherwise. It generates a shift of the uu-component of the spacetime metric by H_0. Let $g(H)$ denote the metric (2.144). The vector field X defines a symmetry of the Einstein-Hilbert action $S[g] = \int \sqrt{g}\mathrm{d}^4x\, R$ at all points of the form $g(H)$, as long as $(\partial_x^2 + \partial_y^2)H_0(x, y) = 0$. We would like adapt X, so that it defines an element in $H^{-1}(\mathrm{dcrit}^\bullet(S))$, i.e. to extend it such that it defines a symmetry everywhere. If this is possible, general relativity would be an example of a theory where not all symmetries of the action correspond to gauge degrees of freedom, i.e. where one does not want to resolve the critical locus of S.

Let us emphasize again that the space $\mathcal{O}(\mathrm{crit}(S_0)) = H^0(\mathrm{dcrit}^\bullet(S))$, which appears as the degree zero cohomology of the derived critical locus, is the space of functions $\mathcal{O}(\mathcal{F}_0)$ modulo the ideal generated by the equations of motion. As we pointed out before, this space is sometimes bigger than the space of functionals obtained by restricting $\mathcal{O}(\mathcal{F}_0)$ to $\mathrm{d}S_0 = 0$. In the literature, existence of a resolution of the space of functionals $\mathcal{O}(\mathrm{crit}(S_0))$ usually comes with the assumption that S is regular enough so that $\mathcal{O}(\mathrm{crit}(S_0))$ is *equal to* the space of functionals obtained by restriction to $\mathrm{d}S_0 = 0$, see for example [52] and [46]. In that case, the vanishing ideal J is equal to the ideal I generated by the equations of motion. However,

[12] All these facts can be found for example in [82].

the regularity assumption is no longer necessary if we are willing to work with $\mathcal{O}(\mathcal{F}_0)/I$ rather than $\mathcal{O}(\mathcal{F}_0)/J$.

The regularity assumption on S_0 is satisfied whenever the Hessian of S_0 is locally constant on-shell. Assuming that all symmetries of S_0 are gauge (meaning that we have a resolution of $\mathcal{O}(\mathrm{crit}(S_0))$), the propagator becomes invertible on non-gauge degrees of freedom , i.e. after we impose a gauge. In physical theories however, the propagator is usually not completely invertible, since, even after gauge fixing, we cannot invert it on fields satisfying the free equations of motion. These fields serve as external states in S-matrices. Therefore, assuming that S_0 is regular would mean that this theory has no non-trivial S-matrices.

Let us illustrate this with a finite dimensional example. We already considered the action

$$S_0(x,y) = \frac{1}{2}x^2 + \frac{\lambda}{3!}(x+y)^3. \tag{2.146}$$

The following two things are important.

- $S_0(x,y)$ has no gauge symmetries. Therefore, the derived critical locus $\mathrm{dcrit}^\bullet(S_0)$ is already a resolution.
- S_0 is not regular at the origin.

Since S_0 is not regular, the hessian $\partial_i\partial_j S_0|_{x=y=0}$ has a kernel, which are the vectors along the y-axis. Thinking of S_0 as a field theory, we would say that the fields y are the physical fields entering the S-matrix. Indeed, this example has non-trivial S-matrices. They are represented by trivalent graphs with vertex factors λ and propagators equal to 1.

From this example it is clear that, in order for a theory to have non-trivial S-matrices, it should not be regular. At the formal level, this is quite obvious. Assume that we can split fields $\phi = \phi_{\text{off-shell}} + \phi_{\text{on-shell}}$, the on-shell fields will always enter in cubic order or higher.

To really see what is happening in field theories, it may be worth to study regularity of S_0 with well founded analytic assumptions on the fields in the theory. After all, the question whether or not a map is invertible depends on the domain it is defined on. With the tools we have developed here, we really can only compare to the finite dimensional case, which of course may hide important properties of more physical infinite dimensional field theories.

Let us repeat what we learned in this section. For finite dimensional theories, the critical locus is resolved by $\delta_{crit} + \delta_g$, if the space of solutions to the equations of motion modulo gauge transformations is zero dimensional. Moreover, if the action functional is regular on-shell, the space of functionals $\mathcal{O}(\mathrm{crit}(S_0))$ is reduced, i.e. it is obtained by restricting functionals to $\mathrm{d}S_0 = 0$. For non-regular action functionals, the space $\mathrm{d}S_0 = 0$ is still zero dimensional. However, we say that it has infinitesimal directions. By this we mean the set of tangent vectors, which we define to be the kernel of the hessian at a solution modulo gauge transformations, may have non-zero dimensions. This is the space of free on-shell fields, which enter in the S-matrix. Finally, if $\mathcal{O}(\mathrm{crit}(S))$ is not resolved, the underlying geometric space has dimension bigger than 0. In this case we say that it admits finite physical directions. The space of infinitesimal directions (tangents) can still be bigger, but some of these directions may integrate finitely. We will later state a necessary and sufficient condition for this to happen. One of the examples showed that gravity is a theory where actual finite physical directions (pp-waves) exist. Since gravity is infinite dimensional, we unfortunately cannot say whether this implies that a BV formulation of gravity should not resolve the critical locus. Maiking a precise statement would require a more thorough discussion about domains of definition in gravity.

2.4.6. Symplectic Geometry in BV

The fact that the BV formalism has a Poisson bracket and the cohomological vector field $Q = \{S, \cdot\}$ is Hamiltonian suggests that there exists a formulation in terms of symplectic geometry. The difference to the symplectic geometry appearing in Hamiltonian mechanics is now that the variables are graded and the symplectic form has a degree shift.[13] The symplectic viewpoint will be useful once we discuss gauge fixing.

Definition 2.4.5. Let M be a $\mathbb{Z}$-graded manifold. We call a two-form ω an odd symplectic structure on M if ω is closed, non-degenerate and of degree -1.

Just as in ordinary symplectic geometry, ω can be written locally in terms of Darboux coordinates (c.f. [80]). This means that there is a there are local coordinates (x^i, x_i^*), such that

$$\omega = \sum_i (-)^{x_i^*} \mathrm{d}x_i^* \wedge \mathrm{d}x^i. \tag{2.147}$$

This of course implies that $|x^i| + |x_i^*| = -1$. Given a function F, we define its Hamiltonian vector field X_F via

$$(-)^F \mathrm{d}F = i_{X_F}\omega. \tag{2.148}$$

The sign conventions are such that $i_{\partial_j}\mathrm{d}x^k = \delta_j^k$. In that case, we have

$$X_F = \frac{\partial_r F}{\partial x_i^*}\frac{\partial}{\partial x^i} - \frac{\partial_r F}{\partial x^i}\frac{\partial}{\partial x_i^*}. \tag{2.149}$$

Therefore,

$$X_F(G) = \{F, G\}. \tag{2.150}$$

Remark 2.4.5. Most field theory examples are constructed on a linear space of fields with constant symplectic structure, i.e. they are formulated in global Darboux coordinates from the start. One instance where this is not the case is background independent open-string field theory [95], which is now more commonly known as boundary string field theory. The field theory manifold is parametrized by perturbations of the worldsheet (a disc) action by operators inserted at the boundary. The symplectic structure is then just given by the expectation value of a porduct of two boundary operators with respect to that worldsheet action. This theory was studied by Ivo Sachs and the author in [20], where it was shown that it directly reproduces the minimal model (in the A_∞-sense) of open-string field theory.

The symplectic structure allows us to talk about canonical transformations. These are conveniently stated in terms of generating functions. They provide a coordinate transformation $(x^i, x_i^*) \mapsto (X^i(x, x^*), X_i^*(x, x^*))$. We will only need a type 3 (in the sense of [45]) generating function of the form $F_3(x^*, X) = -x_i^* X^i - \Psi(X)$, where Ψ is some function of degree -1. The coordinate transformations are

$$x^i = -\frac{\partial F_3}{\partial x_i^*} = X^i, \quad X_i^* = -\frac{\partial F_3}{\partial X^i} = x_i^* + \left.\frac{\partial \Psi}{\partial X^i}\right|_{X^i = x^i}. \tag{2.151}$$

For this particular generating function it is obvious that the symplectic form is preserved. The function Ψ is called the gauge fixing fermion.

[13] The Hamiltonian version of BV is called BFV (Batalin-Fradkin-Vilkovisky), which also has ghosts. Its symplectic form extends the usual one defined on phase space. In this formulation, the symplectic form has degree 0.

2.4.7. More Field Theory Examples of BV Theories

We will here discuss the classical BV formulation of the two most standard gauge theories in physics, namely Chern-Simons theory and Yang-Mills theory. These examples and their BV formulation can also be found in [22].

Chern-Simons Theory: Let M be a three dimensional manifold and G a Lie group with Lie algebra $\mathfrak{g}$. Before applying the BV procedure, the field space consists of connections A of the bundle $M \times G$[14]. The classical field space is therefore $\mathcal{F}_0 = C^\infty(M) \otimes \mathfrak{g}$. As usual, we denote the gauge fields by A. Let us fix a $\mathfrak{g}$-invariant pairing $(\cdot,\cdot)_\mathfrak{g} : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$. We define a pairing $(\cdot,\cdot)$ on $\Omega^\bullet(M) \otimes \mathfrak{g}$ by

$$(\alpha \otimes v, \beta \otimes w) = \int_M \alpha \wedge \beta (v,w)_\mathfrak{g}. \tag{2.152}$$

The Chern-Simons action can then be written as

$$S_0 = \int_M \frac{1}{2}(A, \mathrm{d}A) + \frac{1}{6}(A, [A,A]). \tag{2.153}$$

The gauge group consists of smooth functions from M into G. Therefore, $\mathcal{G} = C^\infty(M) \otimes G$ and $\mathfrak{G} = C^\infty(M) \otimes \mathfrak{g}$. Given an $\lambda \in \mathfrak{G}$, its action on a gauge connection A is

$$\delta A = \mathrm{d}\lambda + [\lambda, A]. \tag{2.154}$$

The ghost fields therefore live in $\Omega^0(M) \otimes \mathfrak{g}[1]$.

A nice feature of the BV formulation of Chern-Simons theory is that it can be obtained from ordinary Chern Simons theory with essentially no extra effort. We simply have to extend the field space $\mathcal{F}_0 = \Omega^1(M) \otimes \mathfrak{g}[1]$ to $\mathcal{F} = \Omega^\bullet(M) \otimes \mathfrak{g}[1]$. We have the following classes of fields.

1. Ghosts of degree -1 in $\Omega^0(M) \otimes \mathfrak{g}[1]$.
2. Fields of degree 0 in $\Omega^1(M) \otimes \mathfrak{g}[1]$.
3. Anti-fields of degree 1 in $\Omega^2(M) \otimes \mathfrak{g}[1]$.
4. Anti-ghosts of degree 2 in $\Omega^3(M) \otimes \mathfrak{g}[1]$.

Note that the grading is opposite to that of field variables. The anti-bracket is induced from the degree -1 pairing $(\cdot,\cdot) : \mathcal{F} \otimes \mathcal{F} \to \mathbb{R}$[15], which defines an odd symplectic form on $\mathcal{F}$. Let us denote a general element in $\mathcal{F}$ by $\mathcal{A}$. The BV extended action is

$$S = \int_M \frac{1}{2}(\mathcal{A}, \mathrm{d}\mathcal{A}) + \frac{1}{6}(\mathcal{A}, [\mathcal{A},\mathcal{A}]). \tag{2.155}$$

We expand $\mathcal{A} = C + A + A^* + C^*$. Here, the form degrees increase from zero (zero-form C) to three (three-form C^*) from left to right. Beside the action S_0, the BV extended action contains the following two terms,

$$S_1 = (A^*, \mathrm{d}C + [C,A]), \quad S_2 = \frac{1}{2}(C^*, [C,C]). \tag{2.156}$$

[14] For simplicity we work with a trivial G-bundle.

[15] As it stands, the inner product $(\cdot,\cdot)$ is symmetric in its entries. We therefore have to change some signs to obtain a anti-symmetric form.

These are exactly the terms we encountered in our introduction to the classical BV formalism. $S_1 = -\rho$ generates the action of the Lie algebra $\mathfrak{G}$ on fields and $S_2 = [\cdot, \cdot]$ is the Lie bracket.

Yang-Mills Theory: Let M be a four dimensional manifold. The fields in Yang-Mills are the same as in Chern-Simons, namely connections on the bundle $M \times G$, where G is a Lie group. Therefore, $\mathcal{F}_0 = C^\infty(M) \otimes \mathfrak{g}$. Let $A \in \mathcal{F}_0$ a connection. Its field strength (curvature) is given by

$$F = \mathrm{d}A + \frac{1}{2}[A, A]. \tag{2.157}$$

Yang-Mills theory requires a metric as an additional data over Chern-Simons theory. Given a metric g we can the define the Hodge star $*$. Denote by $(\cdot, \cdot)$ the pairing defined in equation (2.152). The Yang-Mills action takes the form

$$S_0 = \frac{1}{2}(F, *F). \tag{2.158}$$

Since the gauge symmetry of Yang-Mills is the same as for Chern-Simons, the BV extended action is constructed essentially along the same lines as for Chern-Simons. The only difference lies in the form degree of the additional fields, which changes since we now consider a four dimensional manifold. We would still like to have $(\cdot, \cdot)$ as the symplectic form inducing the anti-bracket. Therefore, anti-fields will necessarily be three-forms and, since ghosts are $\mathfrak{g}$-valued functions, anti-ghosts are four-forms. As for Chern-Simons, we write (A^*, C, C^*) for the additional fields. The extended action consists again of the terms

$$S_1 = (A^*, \mathrm{d}C + [C, A]), \quad S_2 = \frac{1}{2}(C^*, [C, C]). \tag{2.159}$$

2.5. From Batalin-Vilkovisky to Homotopy Algebras

We recall the basic setup in BV. Let S be a BV action on a field space $\mathcal{F}$. The space of functionals $\mathcal{O}(\mathcal{F})$ comes equipped with a degree 1 Poisson bracket $\{\cdot, \cdot\}$. The action S defines a cohomological vector field $Q = \{S, \cdot\}$. The zeroth of the vector field Q in $\mathcal{F}$ are the fields satisfying the equations of motion.

Let us analyze the local structure around a given solution ϕ_0. Consider the tangent space $T_{\phi_0}\mathcal{F}$. Since ϕ_0 is a zero of Q, Q induces a linear map

$$Q_1 : T_{\phi_0}\mathcal{F} \to T_{\phi_0}\mathcal{F}. \tag{2.160}$$

This is obtained by taking the Lie bracket with an arbitrary extension of a vector $v \in T_{\phi_0}\mathcal{F}$. Because Q is cohomological, we have $Q_1^2 = 0$. We therefore have a cochain complex $(T_{\phi_0}\mathcal{F}, Q_1)$. Its cohomology can be identified as the space of free on-shell fields in a given background ϕ_0. We call the complex $(T_{\phi_0}\mathcal{F}, Q_1)$ the tangent complex at ϕ_0 inside the differential graded manifold $(\mathcal{F}, Q)$. By passing to cohomology, we obtain the tangent space $H(T_{\phi_0}\mathcal{F})$ to $H(\mathcal{F})$ at ϕ_0.

We argued before that, due to singularties, the vectors $v \in H(T_{\phi_0}\mathcal{F})$ may or may not extend finitely to curves inside $H(\mathcal{F})$. To analyze this problem, we need to consider Q beyond its linear part Q_1. To this end, we expand Q as a Taylor series at ϕ_0,

$$Q = Q_1 + Q_2 + \dots . \tag{2.161}$$

The kth term defines a multilinear map $Q_k : (T_{\phi_0}\mathcal{F})^{\otimes k} \to T_{\phi_0}\mathcal{F}$. It follows from $Q^2 = 0$ that the Q_k give $T_{\phi_0}\mathcal{F}$ the structure of an L_∞-algebra, see for example [18], **appendix C**. Locally, a field theory can then be described equivalently as an L_∞-algebra. Given an L_∞-structure $(\mathcal{F}, Q, \{m_k\}_{k\geq 2})$, the equations of motion for a degree zero field a become the Maurer-Cartan equation

$$Qa + \sum_{k\geq 2} \frac{1}{k!} m_k(a, ..., a) = 0. \tag{2.162}$$

On the other hand, gauge transformations connected infinitesimally by

$$\delta a = Qb + \sum_{k\geq 2} \frac{1}{(k-1)!} m_k(b, a, ..., a), \tag{2.163}$$

where b is an element of degree -1. In fact, gauge transformations can be described as solution to another Maurer-Cartan equation. We can combine our original L_∞-algebra with the differentical graded structure of the de Rham complex $(\Omega^\bullet([0,1]), \mathrm{d}_{dR})$. A general degree 0 element $\alpha \in \Omega^\bullet([0,1]) \otimes \mathcal{F}$ can be written as

$$\alpha = a(t) + \mathrm{d}t\, b(t). \tag{2.164}$$

The Maurer-Cartan equation on α then induces two separate equations,

1. $$Q\alpha(t) + \sum_{k\geq 2} \frac{1}{k!} m_k(a(t), ..., a(t)) = 0 \tag{2.165}$$

2. $$\mathrm{d}t\, \partial_t \alpha(t) - \mathrm{d}t\, Qb(t) - \mathrm{d}t \sum_{k\geq 2} \frac{1}{(k-1)!} m_k(b(t), a(t), ..., a(t)) = 0. \tag{2.166}$$

The first one tells us that $a(t)$ satisfies the Maurer-Cartan equation for all t, while the second one provides the infinitesimal gauge transformation (2.163). In this way, the family $a(t)$ is a homotopy between the solutions $a(0)$ and $a(1)$.

The consideration above reflects something we already know from the BV perspective. The degree -1 ghost b[16] generates gauge transformations for the physical field a. So what about ghosts for ghosts? From our experience it is now easy to guess how they can be implemented. We should allow for differential two forms. Arguably the simplest two dimensional space with boundaries is the 2-simplex $\Delta^2 := \{(x, y, z) \in [0,1]^3 | x+y+z = 1\}$. This is a straightforward generalization of the unit interval $[0,1]$, which is the 1-simplex. We can now ask for solutions to the master equation on $\mathcal{F} \otimes \Omega^\bullet(\Delta^2)$. Higher gauge transformations are obtained similarly by tensoring the L_∞-algebra with the complex $\Omega^\bullet(\Delta^n)$.

Remark 2.5.1. The convention to use the de Rham complex of simplices can be found for example in [23]. Another convenient choice are of course (hyper-)cubes $[0,1]^n$ like they are used in reference [57].

Remark 2.5.2. Gauge transformations explain the role of ghosts (fields in negative degree) in L_∞-algebras. On the other hand, anti-fields (fields in positive degree) do not appear

[16] b has degree -1 since it is a field and not a field variable.

in this way. They appear when one tensors the L_∞-algebra with complexes that are non-trivial in negative degree. In the spirit of derived moduli problems ([65, 23]) one uses *local differential graded artinian algebras* that are concentrated in non-positive degree. The local artinian rings are used to measure local properties up to some finite order. For example,

$$R = \mathbb{C}[\varepsilon]/(\varepsilon^n) \tag{2.167}$$

is local artinian. The solutions to the Maurer-Cartan equations of an L_∞-algebra tensored with R are the solutions to order ε^{n-1}. Things are similar when R comes in addition equipped with a differential.

At the level of cohomology, we have a minimal L_∞-algebra on $H(T_{\phi_0}\mathcal{F})$. We denote the minimal L_∞-algebra on $H(T_{\phi_0}\mathcal{F})$ as

$$Q^{\min} = Q_2^{\min} + Q_3^{\min} + ..., \ . \tag{2.168}$$

These maps have a physical interpretation. $Q_k^{\min}$ describes the tree level $k+1$-point (on-shell, amputated) scattering amplitudes, cf. [73], **section III.C** for this statement in the (more general) quantum case. From the viewpoint of deformation theory, the S-matrices measure the failure of a tangent $v \in H(T_{\phi_0}\mathcal{F})$ to integrate to a finite solution of the equations of motion. Therefore, whenever a background ϕ_0 has non-trivial S-matrices, the space $H(\mathcal{F})$ has a singularity at that point. The tangent space (the space of infinitesimal directions) is bigger than the dimension of the space $H(\mathcal{F})$.

2.5.1. Example: Deformations in ϕ^3-theory

Consider a scalar ϕ^3-theory

$$S[\phi] = \frac{1}{\mathrm{vol}(M)} \int_M \frac{1}{2}\phi\Box\phi + \frac{\lambda}{3!}\phi^3. \tag{2.169}$$

The factor $\frac{1}{\mathrm{vol}(M)}$ is put in for convenience. To keep things simple, we put the field ϕ on a two dimensional torus $M = S^1 \times S^1$. This allows us to expand the field in Fourier modes,

$$\phi(t,x) = \sum_{n,m\in\mathbb{Z}} \phi_{n,m} e^{-int+imx}. \tag{2.170}$$

The action becomes

$$S(\{\phi_{nm}\}) = \frac{1}{2}\sum_{n,m} \phi_{-n,-m}(n^2 - m^2)\phi_{n,m} + \frac{\lambda}{3!}\sum_{n,m,k,l} \phi_{n,m}\phi_{k,l}\phi_{-n-k,-m-l}. \tag{2.171}$$

The equations of motion are

$$\frac{\partial S}{\partial \phi_{-n,-m}} = (n^2 - m^2)\phi_{n,m} + \frac{\lambda}{2}\sum_{k,l}\phi_{k,l}\phi_{n-k,m-l} = 0. \tag{2.172}$$

The homological vector field is therefore

$$Q = \{S, \cdot\} = -\sum_{n,m}(n^2 - m^2)\phi_{n,m}\frac{\delta}{\delta\phi^*_{-n,-m}} - \frac{\lambda}{2}\sum_{k,l,n,m}\phi_{k,l}\phi_{-n-k,-m-l}\frac{\delta}{\delta\phi^*_{-n,-m}}. \tag{2.173}$$

Let us look at the local structure of Q around $\phi = 0$. A general vector at that point can be written as

$$X = v_{n,m}\frac{\delta}{\delta\phi_{n,m}} + v^*_{n,m}\frac{\delta}{\delta\phi^*_{n,m}}. \tag{2.174}$$

In these coordinates, the Taylor coefficients of Q at 0 are

$$Q_1 = -\sum_{m,n}(n^2 - m^2)v_{n,m}\frac{\delta}{\delta\phi^*_{-n,-m}}, \quad Q_2 = -\frac{\lambda}{2}\sum_{m,n,k,l} v_{k,l}v_{-n-k,-m-l}\frac{\delta}{\delta\phi^*_{-n,-m}}, \tag{2.175}$$

and $Q_{k\geq 3} = 0$. The check that these give rise to an L_∞-algebra is trivial. Any combination quadratic in the Q_i is zero, since the tangent complex only has two non-zero degrees,

$$0 \longrightarrow (T_0\mathcal{F})^0 \xrightarrow{Q_1} (T_0\mathcal{F})^1 \longrightarrow 0. \tag{2.176}$$

We compute its cohomology. In degree zero, a vector $v_{n,m}\frac{\delta}{\delta\phi_{n,m}}$ is in $\ker Q_1$ if and only if $n = \pm m$. On the other hand, the image of Q_1 in degree zero consists of vectors of the form

$$\sum_{m,n}(n^2 - m^2)v_{n,m}\frac{\delta}{\delta\phi^*_{-n,-m}}. \tag{2.177}$$

This tells us that, whenever $n \neq \pm m$, a vector $v^*_{n,m}\frac{\delta}{\delta\phi^*_{n,m}} \in (T_0\mathcal{F})^1$ is exact. We therefore find that both $H^0(T_0\mathcal{F}^\bullet)$ and $H^1(T_0\mathcal{F}^\bullet)$ con be identified with fields satisfying the linear equations of motion $\Box\phi = 0$.

We now want to find conditions for a vector in $H^0(T_0\mathcal{F}^\bullet)$ to integrate to a finite solution. We make a perturbative ansatz

$$\phi_{n,m} = \sum_{k\geq 1}\epsilon^k\phi^{(k)}_{n,m}. \tag{2.178}$$

To linear order in ϵ, the coefficient $\phi^{(1)}_{n,m}$ describes tangents. Form our study of the tangent complex, we already know that these have to satisfy the linear equations of motion. To quadratic order in ϵ, we obtain

$$(n^2 - m^2)\phi^{(2)}_{n,m} = -\frac{\lambda}{2}\sum_{k,l}\phi^{(1)}_{k,l}\phi^{(1)}_{n-k,m-l} =: m_{2;m,n}(\{\phi^{(1)}_{k,l}\},\{\phi^{(1)}_{k,l}\}), \tag{2.179}$$

or equivalently

$$\phi^{(2)}_{n,m} = -\frac{1}{(n^2 - m^2)}m_{2;m,n}(\{\phi^{(1)}_{k,l}\},\{\phi^{(1)}_{k,l}\}), \quad n \neq |m|. \tag{2.180}$$

A careful reader should be hesitant at this point, since something can go wrong here. Equation (2.179) can only be satisfied if $m_{2;m,m}$ and $m_{2;m,-m}$, when restricted to free on-shell fields, are zero for all m. This gives an obstruction for a vector $\phi^{(1)}_{n,m}$ to integrate to a second order solution. The subset of $m_{2;m,n}$ with $n = \pm m$ is consists of exactly these maps which restrict to $H^\bullet(T_0\mathcal{F}^\bullet)$. They therefore define a map

$$m_{2;m,\pm m} : H^\bullet(T_0\mathcal{F}^\bullet) \otimes H^\bullet(T_0\mathcal{F}^\bullet) \to H^\bullet(T_0\mathcal{F}^\bullet). \tag{2.181}$$

Of degree 1[17]. They are the components of the bilinear product of the minimal L_∞-algebra and obviously describe the cubic S-matrices. We can conclude the following. A first order deformation at $\phi = 0$ integrates to a second order deformation, if and only if the cubic S-matrix of the first order deformation vanishes.

Let us do one more order. In order ϵ^3, the equation of motion is

$$(n^2 - m^2)\phi^{(3)}_{n,m} = -\lambda \sum_{k,l} \phi^{(2)}_{k,l}\phi^{(1)}_{n-k,m-l} = 2m_{2;n,m}(\{\phi^{(2)}_{k,l}\}, \{\phi^{(1)}_{k,l}\}). \tag{2.182}$$

Again, for this to be consistent we need that the right hand side is zero whenever $m = \pm n$. From (2.180), we know the dependence of $\phi^{(2)}_{k,l}$ in terms of $\phi^{(1)}_{m,n}$ for $m \neq \pm n$. The values of $\phi_{n,\pm n}$ can in principle be arbitrary. However, they since these components describe free on-shell fields, these deformations are not very interesting. They are already covered by the first order deformations. Because of this, we set $\phi^{(2)}_{n,\pm n} = 0$ to zero. Expressing everything in terms of tangents $\phi^{(1)}_{n,m}$ we obtain as a equation for the third order deformation

$$\phi^{(3)}_{n,m} = -\frac{2}{n^2 - m^2} m_{2;n,m}(\{-\frac{1}{k^2 + l^2} m_{2;k,l}(\{\phi^{(1)}_{s,t}\}, \{\phi^{(1)}_{s,t}\})\}_{k \neq \pm l}, \{\phi^{(1)}_{k,l}\}) \tag{2.183}$$

$$=: m_{3;n,m}(\{\phi^{(1)}_{k,l}\}, \{\phi^{(1)}_{k,l}\}, \{\phi^{(1)}_{k,l}\}), \quad m \neq \pm n, \tag{2.184}$$

together with the condition

$$m_{3;m,\pm m}(\{\phi^{(1)}_{k,l}\}, \{\phi^{(1)}_{k,l}\}, \{\phi^{(1)}_{k,l}\}) = 0. \tag{2.185}$$

The restriction of $m_{3;n,m}$ to $m = \pm n$ gives rise to a trilinear map

$$m_{3;m,\pm m} : H^\bullet(T_0\mathcal{F}^\bullet) \otimes H^\bullet(T_0\mathcal{F}^\bullet) \otimes H^\bullet(T_0\mathcal{F}^\bullet) \to H^\bullet(T_0\mathcal{F}^\bullet), \tag{2.186}$$

of degree 1, which we identify as the quartic S-matrix. It is the cubic product of the minimal L_∞-algebra. We can conclude that a first order deformation lifts to a third order deformation, if both the cubic and the quartic S-matrix on the first order deformation vanish.

This generalizes to arbitrary order. We will see that the following is true in any BV theory. An order k deformation exists, if all S-matrices to order $k+1$ vanish. This again shows the relation between S-matrices and singular actions discussed in section 2.4.5.

2.5.2. General Deformations

We will now describe the general procedure to show the relation between S-matrices and deformations.[18] Let $(\mathcal{O}(\mathcal{F}), Q = \{S, \cdot\})$ be the complex of a BV theory. Let ϕ_0 be an element of degree zero and a solution to the equations of motion. This implies that $Q|_{\phi_0} = 0$. We look for perturbed solutions of the form

$$\phi = \sum_{k \geq 0} \epsilon^k \phi_k. \tag{2.187}$$

The linear term ϕ_1 describes tangents inside $\mathcal{O}(\mathcal{F})$ at ϕ_0. As noted earlier, the assumption that ϕ_0 is a zero of Q implies that Q induces a differential $Q_1 : T_{\phi_0}\mathcal{F} \to T_{\phi_0}\mathcal{F}$, which

[17] The fact that we want to think of them as maps in degree 1 comes from the fact that the equations of motion are the zeros of the cohomological vector field Q.

[18] We follow the procedure described in [58] for A_∞-algebras, which applies similarly to L_∞-algebras.

turns $(Q_1, T_{\phi_0}\mathcal{F})$ into a cochain complex. Equivalently, Q_1 is the linear part in the Taylor expansion

$$Q = \sum_{k\geq 1} Q_k \tag{2.188}$$

of Q around ϕ_0.

Since $T_{\phi_0}\mathcal{F}$ is a vector space, there is a strong deformation retract $i : H^\bullet(T_{\phi_0}\mathcal{F}) \leftrightarrows T_{\phi_0}\mathcal{F} : p$. Recall that this means that we have a homotopy $h : T_{\phi_0}\mathcal{F} \to T_{\phi_0}\mathcal{F}$, such that

$$hQ_1 + hQ_1 = 1 - ip, \quad pi = 1, \quad hi = 0, \quad h^2 = 0. \tag{2.189}$$

We will write $P = ip$. This map projects onto $H^\bullet(T_{\phi_0}\mathcal{F})$ as a subspace of $T_{\phi_0}\mathcal{F}$. By the properties of a strong deformation retract, there are further projections $P_t = Q_1 h$ and $P_u = hQ_1$. Since

$$P + P_t + P_u = 1, \tag{2.190}$$

this gives a decomposition of $T_{\phi_0}\mathcal{F}$ into three independent subspaces. Explicitly,

$$(T_{\phi_0}\mathcal{F})^n = H^n(T_{\phi_0}\mathcal{F}) \oplus B^n(T_{\phi_0}\mathcal{F}) \oplus B^{n+1}(T_{\phi_0}\mathcal{F})[1], \tag{2.191}$$

where $\operatorname{Im} P_u|_{(T_{\phi_0}\mathcal{F})^n}$ is identified with $B^{n+1}(T_{\phi_0}\mathcal{F})[1]$ under Q_1. Non-zero tangents in $\operatorname{Im} P_u$ are therefore elements which are not annihilated by Q_1. We call these *unphysical*, while we refer to tangents in $\operatorname{Im} P_t = B^\bullet(T_{\phi_0}\mathcal{F})$ as *trivial* and tangents in $H^\bullet(T_{\phi_0}\mathcal{F})$ as *physical*. We write

$$\phi = \phi_p + \phi_t + \phi_u. \tag{2.192}$$

In the following, we write $Q := Q_1$ and $m_k := Q_k$ as we usually do for A_∞-algebras. We wish to solve the Maurer-Cartan equation

$$Q\phi + \sum_{k\geq 2} m_k(\phi, ..., \phi) = 0. \tag{2.193}$$

Before doing so, we impose the linear gauge $h\phi = 0$. This is equivalent to setting $\phi_t = 0$, since

$$h\phi = h\phi_t \quad \text{and} \quad Qh\phi = \phi_t. \tag{2.194}$$

Let us make a small comment about this choice. It is definitely a good gauge choice in a theory with interactions turned off, since it picks a unique representative out of each gauge orbit $\{\phi + Q\alpha\}_\alpha$.[19] We then assume that it is also a good choice also for the interacting theory. As a side result below we will observe that this is actually not completely true and that there are in general residual gauge symmetries on the physical fields ϕ_p.

With the choice of gauge, we now write $\phi_{gf} = \phi_u + \phi_p$. Plugging this into (2.193) then gives

$$Q\phi_u + \sum_{k\geq 2} m_k(\phi_{gf}, ..., \phi_{gf}) = 0. \tag{2.195}$$

Since $\phi_u = P_u\phi_u = hQ\phi_u$, a solution to this equation satisfies

$$\phi_u = -h \sum_{k\geq 2} m_k(\phi_u + \phi_p, ..., \phi_u + \phi_p). \tag{2.196}$$

[19]One can even turn this around and say that a choice of homotopy h is a choice of gauge.

We can solve this perturbatively. We take our ansatz (2.187) and write

$$\phi_{gf} = \sum_{k\geq 1} \epsilon^k \phi_{gf,k}. \tag{2.197}$$

Since $\phi_{gf,1}$ satisfies the linear equations of motion, we have $\phi_{gf,1} = \phi_p \in H^\bullet(T_{\phi_0}\mathcal{F})$. Therefore, ϕ_u starts at order ϵ^2. We can use this to repeatedly plug ϕ_u into the right hand side of (2.196). This eliminates ϕ_u to arbitrarily high orders. We can therefore write

$$\phi_u = -h \sum_{k\geq 2} \tilde{m}_k(\phi_p, ..., \phi_p). \tag{2.198}$$

It is straightforward to see that the $\tilde{m}_k$ are the order k tree-level Feynman amplitudes with vertices m_k and propagator $-h$. We observe that they determine ϕ_u in terms of the tangents ϕ_p.

We now check under what conditions is our ϕ_u a consistent solution. Plugging (2.196) back into (2.195) implies that

$$0 = (-Qh+1)\sum_{k\geq 2} m_k(\phi_{gf}, ..., \phi_{gf}) = (P+hQ)\sum_{k\geq 2} m_k(\phi_{gf}, ..., \phi_{gf}) = P\sum_{k\geq 2} m_k(\phi_{gf}, ..., \phi_{gf}),$$

where we used that

$$Q\sum_{k\geq 2} m_k(\phi_{gf}, ..., \phi_{gf}) = -Q^2\phi_{gf} = 0, \tag{2.199}$$

since we assume that ϕ_{gf} is a solution. Eliminating ϕ_u from (2.199) in favor of ϕ_p gives

$$0 = \sum_{k\geq 2} P\tilde{m}_k(\phi_p, ..., \phi_p) = \sum_{k\geq 2} m_k^{\min}(\phi_p, ..., \phi_p). \tag{2.200}$$

The m_k^{min} define the minimal A_∞-structure on $H^\bullet(T_{\phi_0}\mathcal{F})$. It is now straightforward to deduce from (2.200) that a tangent ϕ_p integrates to a solution of order ϵ^k, if all tree-level S-matrices vanish to that order. Note that the minimal model also indicates that there may be residual gauge freedoms on the level of cohomology not fixed by the condition $h\phi = 0$.

Equation (2.200) is the Maurer-Cartan equation on the cohomology. We just saw that solving the original Maurer-Cartan equation is equivalent to solving the minimal one. From the homological point of view, this should not come as a surprise. Our philosophy is that field theories (independent of whether we describe them in the BV language or as A_∞/L_∞-algebras) are equivalent, if and only if they are quasi-isomorphic.

2.5.3. Chern-Simons and Yang-Mills Theory as L_∞-Algebras

We saw that an L_∞-algebra arises as the local description of BV theories around critical points. This construction becomes very simple when the field space is already linear. In this case, we can equivalently talk about the theories in the BV language or in the L_∞ language.

Assume that we have a linear field space $(\mathcal{F}, \omega)$, where ω is the symplectic form inducing the anti-bracket. Given an L_∞-algebra with differential d, products $\{m_k\}_{k\geq 2}$, and cyclic with respect to ω. Then the action

$$S[\phi] = \frac{1}{2}\omega(\phi, \mathrm{d}\phi) + \sum_{k\geq 2}\frac{1}{(k+1)!}\omega(\phi, m_k(\phi, ..., \phi)) \tag{2.201}$$

defines a classical BV action, i.e. solves the master equation with respect to the bracket induced from ω.

Both Chern-Simons theory and Yang-Mills theory, as discussed in section 2.4.7, are already formulated in this way. Let us first discuss the more simple Chern-Simons theory. Recall that the field space was $\mathcal{F} = \Omega^\bullet(M) \otimes \mathfrak{g}[1]$, where $\mathfrak{g}$ is some Lie algebra. The linear differential is the de-Rham differential

$$\mathrm{d} : \Omega^\bullet(M) \otimes \mathfrak{g}[1] \to \Omega^{\bullet+1}(M) \otimes \mathfrak{g}[1] \tag{2.202}$$

and is obviously of degree 1. The product is induced by the Lie bracket on $\mathfrak{g}$,

$$[\cdot,\cdot] : (\Omega^k(M) \otimes \mathfrak{g}[1]) \otimes (\Omega^l(M) \otimes \mathfrak{g}[1]) \to \Omega^{k+l}(M) \otimes \mathfrak{g}[1]. \tag{2.203}$$

Without the shift $\mathfrak{g} \mapsto \mathfrak{g}[1]$, the bracket would be of degree 0. The shift, however, turns it into a map of degree 1. All higher products are zero. The compatibility conditions therefore reduce to those of a differential graded Lie algebra, which are obviously satisfied in this case. Moreover, the products should be cyclic with respect to ω. This is guaranteed by the fact that we defined $(\cdot,\cdot)$ as a integral (cyclicity with respect to d) and by a choice of $\mathfrak{g}$ invariant product (cyclicity with respect to $[\cdot,\cdot]$).

Yang-Mills theory is a little bit harder to discuss since the form of the vertices are not independent of ghost number the way they are in Chern-Simons. Let us first focus on the classical part S_0. We have

$$S_0 = \frac{1}{2}(F,*F) = \frac{1}{2}(\mathrm{d}A,*F) + \frac{1}{4}([A,A],*F) = \frac{1}{2}(A,\mathrm{d}*F) + \frac{1}{4}(A,[A,*F]) \tag{2.204}$$

$$= \frac{1}{2}(A,\mathrm{d}*\mathrm{d}A) + \frac{1}{4}(A,\mathrm{d}*[A,A]) + \frac{1}{4}(A,[A,*\mathrm{d}A]) + \frac{1}{8}(A,[A,*[A,A]]). \tag{2.205}$$

From this we can read of the L_∞-products on the gauge fields.

$$m_2(A,B) = \mathrm{d}*[A,B] + [A,*\mathrm{d}B] + [B,*\mathrm{d}A] \tag{2.206}$$

$$m_3(A,B,C) = [A,*[B,C]] + [B,*[C,A]] + [C,*[A,B]], \tag{2.207}$$

while the differential is

$$Q(A) = \mathrm{d}*\mathrm{d}A. \tag{2.208}$$

For all other combinations, we have that $Q = \mathrm{d}$, $m_2 = [\cdot,\cdot]$ and $m_3 = 0$. The linear differential Q gives rise to the complex

$$0 \longrightarrow \Omega^0(M)\otimes\mathfrak{g} \xrightarrow{\mathrm{d}} \Omega^1(M)\otimes\mathfrak{g} \xrightarrow{\mathrm{d}*\mathrm{d}} \Omega^3(M)\otimes\mathfrak{g} \xrightarrow{\mathrm{d}} \Omega^4(M)\otimes\mathfrak{g} \longrightarrow 0. \tag{2.209}$$

Unlike Chern-Simons theory, the L_∞ products of Yang-Mills are not universally defined independent of the degree. We therefore need to check the L_∞-relations explicitly for each combination of inputs. This also allows us to determine relative signs.[20] We begin with the derivation property. Let $C, D \in \Omega^0(M)\otimes\mathfrak{g}$. Then $m_2(C,D)$ is again a zero form. Therefore Q acts as the de-Rham differential and the derivation property follows immediately from the derivation property of d on forms.

[20] In Chern-Simons theory, the signs can be fixed by using the decalange isomorphism (see appendix A) on d and $[\cdot,\cdot]$. The maps then automatically satisfy the L_∞ relation when d and $[\cdot,\cdot]$ define a differential graded Lie algebra. Since the maps in Yang-Mills are more involved, this argument cannot be applied here.

The next on the list is m_2 on a field A and a ghost C. We find

$$\begin{aligned} Qm_2(A,C) &= \mathrm{d}*\mathrm{d}[A,C] = \mathrm{d}*[\mathrm{d}A,C] - \mathrm{d}*[A,\mathrm{d}C] && (2.210)\\ &= [\mathrm{d}*\mathrm{d}A,C] + [*\mathrm{d}A,\mathrm{d}C] - \mathrm{d}*[A,\mathrm{d}C] && (2.211)\\ &= -m_2(QA,C) - d*[A,\mathrm{d}C] - [\mathrm{d}C,*\mathrm{d}A] - [A,*\mathrm{dd}C] && (2.212)\\ &= -m_2(QA,C) - m_2(A,QC). && (2.213) \end{aligned}$$

To get the correct signs, we defined $m_2(QA,C) = -[QA,C]$. Since QA and C are both odd, this also tells us that $m_2(C,QA) = [C,QA] = [QA,C] = -m_2(QA,C)$.

Consider now an anti-field (three-form) A^* and a ghost C. Then,

$$\begin{aligned} Qm_2(A^*,C) &= -\mathrm{d}[A^*,C] = -[\mathrm{d}A^*,C] + [A^*,\mathrm{d}C] && (2.214)\\ &= -m_2(QA^*,C) - (-)^{A^*} m_2(A^*,QC). && (2.215) \end{aligned}$$

On the right hand side of the equation, we see that all the m_2 act as $[\cdot,\cdot]$, i.e. there is no extra minus sign.

The next natural candidates to put into Qm_2 would be a anti-ghost and a ghost. However, all terms in the derivation property are actually zero since we already exceed the maximal form degree by one. We therefore now check Qm_2 on two fields A and B.

$$\begin{aligned} Qm_2(A,B) &= \mathrm{d}(\mathrm{d}*[A,B] + [A,*\mathrm{d}B] + [B,*\mathrm{d}A]) && (2.216)\\ &= [\mathrm{d}A,*\mathrm{d}B] - [A,\mathrm{d}*\mathrm{d}B] + [\mathrm{d}B,*\mathrm{d}A] - [B,\mathrm{d}*\mathrm{d}A] && (2.217)\\ &= -m_2(QA,B) - m_2(A,QB). && (2.218) \end{aligned}$$

Here we used that $[\mathrm{d}B,*\mathrm{d}A] = -[*\mathrm{d}A,\mathrm{d}B] = -[\mathrm{d}A,*\mathrm{d}B]$.

Just like the ghost/anti-ghost case, all other possible combinations vanish trivally due to the fact that the form degree exceeds four. The next thing to check is therefore that m_2 satisfies the Jacobi identity, up to the homotopy m_3. The homotopy m_3 is non-zero only if all inputs are fields. Therefore, let $A,B,C\in\Omega^1(M)\otimes\mathfrak{g}$. We compute

$$\begin{aligned} Qm_3(A,B,C) &= \mathrm{d}[A,*[B,C]] + \text{cyclic}\\ &= [\mathrm{d}A,*[B,C]] - [A,\mathrm{d}*[B,C]] + \text{cyclic}\\ &= [\mathrm{d}A,*[B,C]] - [A,m_2(B,C)] + [A,[C,*\mathrm{d}B]] + [A,[B,*\mathrm{d}C]] + \text{cyclic}\\ &= -[A,m_2(B,C)] + [\mathrm{d}A,*[B,C]] - [*\mathrm{d}A,[B,C]] + \text{cyclic}\\ &= -[A,m_2(B,C)] + \text{cyclic} = -m_2(A,m_2(B,C)) + \text{cyclic}. && (2.219) \end{aligned}$$

Again, we used that $[\mathrm{d}A,*[B,C]] = [*\mathrm{d}A,[B,C]]$. Further, we combined $[A,[C,*\mathrm{d}B]] + [A,[B,*\mathrm{d}C]]$ with terms hidden in "cyclic" by using the Jacobi identity. This proves Jacobi up to homotopy for these particular inputs. Note that we do not have any terms of the form $m_3(QA,B,C)$ etc., since QA is an anti-field, and m_3 is non-zero only if all inputs are fields. On the other hand, this shows that we also have to check the identity when C is a ghost field, since QC is then a field. We have

$$m_3(A,B,QC) = [A,*[B,\mathrm{d}C]] + \text{cyclic}. \tag{2.220}$$

This computation is by far the longest, so let us split it up a little bit.

$$[A, *[B, \mathrm{d}C]] = -[A, *\mathrm{d}[B, C]] + [A, *[\mathrm{d}B, C]] = -[A, *\mathrm{d}[B, C]] + [A, [*\mathrm{d}B, C]]. \tag{2.221}$$

$$= -[A, *\mathrm{d}[B, C]] + [[A, *\mathrm{d}B], C] + [[C, A], *\mathrm{d}B] \tag{2.222}$$

$$[\mathrm{d}C, *[A, B]] = \mathrm{d}[C, *[A, B]] - [C, \mathrm{d} * [A, B]] = \mathrm{d} * [C, [A, B]] - [C, \mathrm{d} * [A, B]]. \tag{2.223}$$

$$= \mathrm{d} * [[C, A], B] + \mathrm{d} * [[C, B], A] - [C, \mathrm{d} * [A, B]] \tag{2.224}$$

$$[B, *[\mathrm{d}C, A]] = [B, *\mathrm{d}[C, A]] - [B, *[C, \mathrm{d}A]] = [B, *\mathrm{d}[C, A]] - [B, [C, *\mathrm{d}A]] \tag{2.225}$$

$$= [B, *\mathrm{d}[C, A]] - [[B, C], *\mathrm{d}A] - [C, [B, *\mathrm{d}A]]. \tag{2.226}$$

We wrote each product as a sum of three terms, therefore in total we have nine terms. Each m_2 on a field and a ghost is simply given by the commutator. On the other hand, m_2 on two fields is a sum of three terms. Hence,

$$m_2(A, m_2(B, C)) + \text{cyclic} \tag{2.227}$$

also has a total number of nine terms. These are exactly equal to minus those we found before. We found the correct L_∞ relation also in this case.

For all other cases, we have that $m_3 = 0$, so m_2 should be satisfies the Jacobi identity strictly. In most cases, all the m_2 are given by the usual commutator bracket, which of course satisfy Jacobi identity. There are two remaining cases with two fields and either an anti-field or an anti-ghost. Because of the two fields, m_2 on them is not given by the Jacobi identity. However, for these combinations the Jacobi identity is trivially satisfied because again the form degree is too high.

This concludes the check on the L_∞ relations for Yang-Mills. As expected, all the relations are satisfied. However, the explicit computations where quite tedious. On the other hand, the check that the Yang-Mills action satisfies the classical master equation is arguably shorter.

2.5.4. From L_∞-Algebras to A_∞-Algebras in Field Theory

The scattering of four particles in Yang-Mills at tree level has the form

$$A = \mathrm{tr}([T_1, T_2][T_3, T_4])M_s + \mathrm{tr}([T_1, T_4][T_2, T_3])M_t + \mathrm{tr}([T_1, T_3][T_4, T_2])M_u. \tag{2.228}$$

In the above expression, M_s stands for the kinematic factor describing an s-channel process $12 \to 34$. Likewise, M_t and M_u describe the t-channel and u-channel. This expression is symmetric under the simultaneous exchange of matrices $T_i \leftrightarrow T_j$ and momenta $p_i \leftrightarrow p_j$, so it has an S_4 symmetry. For this to see one has to know that each kinetic factor $M_{s,t,u}$ has the same symmetry properties as the trace factor multiplying it. For example, in the case of the s-channel,

$$\begin{aligned} M_s(p_1, p_2, p_3, p_4) &= -M_s(p_2, p_1, p_3, p_4) = -M_s(p_1, p_2, p_4, p_3) \\ &= M_s(p_2, p_1, p_4, p_3) = M_s(p_3, p_4, p_1, p_2). \end{aligned} \tag{2.229}$$

Further, different channels are related through crossing symmetries,

$$(2 \leftrightarrow 4): \qquad M_t(p_1, p_2, p_3, p_4) = -M_s(p_1, p_4, p_2, p_3), \tag{2.230}$$

$$(2 \leftrightarrow 3): \qquad M_u(p_1, p_2, p_3, p_4) = -M_s(p_1, p_3, p_2, p_4). \tag{2.231}$$

Let us store the symmetry data inside the letters s, t, u. For this we write $s = \{[1,2],[3,4]\}$. Here, for any two strings of numbers, a and b, $[a,b] = ab - ba$ and $\{a,b\} = ab + ba$. The product is given by string concatenation. Explicitly,

$$s = 1234 - 2134 - 1243 + 2143 + 3412 - 4312 - 3421 + 4321. \tag{2.232}$$

Likewise, $t = \{[1,4],[2,3]\}$, and $u = \{[1,3],[4,2]\}$. S_4 acts on s, t, u in the obvious way. This allows us to write (2.228) in a very compact form,

$$A = \frac{1}{2} \sum_{\sigma \in S_4} \mathrm{tr}(T_{\sigma(1)} T_{\sigma(2)} T_{\sigma(3)} T_{\sigma(4)}) M_{\sigma(s)}, \tag{2.233}$$

where $M_{-i} = -M_i$ for $i = s, t, u$.

The cyclic symmetry of the trace in (2.233) allows us two simplify further. In order to do so, we split the sum by writing $\sigma_4 = \sigma_3 \cdot (1234)^n$, where σ_4 is a general element in S_4 and σ_3 a permutes (234). This decomposition generates each permutation exactly once. The cycle (1234) leaves the trace invariant, but interchanges $s \leftrightarrow -t$. Therefore

$$A = \sum_{\sigma \in S_3} \mathrm{tr}(T_1 T_{\sigma(2)} T_{\sigma(3)} T_{\sigma(4)})(M_{\sigma(s)} - M_{\sigma(t)}). \tag{2.234}$$

The combination $M = M_s - M_t$ is cyclically symmetric (it is invariant under (1234)). This amplitude is called primitive or color-stripped in QCD.

Primitive amplitudes likewise exist for processes involving an arbitrary number of fields a_i in field space $\mathcal{F}$. At tree level, they are maps

$$M : \mathcal{F}^{\otimes n+1} \to \mathbb{C}, \tag{2.235}$$

graded symmetric under cyclic permutation of its entries. We can reconstruct the full amplitude by defining

$$A(a_0 \otimes T_0, ..., a_n \otimes T_n) = \sum_{\sigma \in S_n} \pm \mathrm{tr}(T_0 T_{\sigma(1)} \cdots T_{\sigma(n)}) M(a_0, a_{\sigma(1)}, ..., a_{\sigma(n)}), \tag{2.236}$$

where the sign is determined by the permutation of the $a_i \otimes T_i$ according to their degree.

From the viewpoint of homological algebra, the primitive amplitudes induce a minimal cyclic A_∞-structure. This structure can be deduced from a non-minimal A_∞-structure given by the vertices, from which the amplitudes are constructed via planar[21] tree-level diagrams only. For example, the four point scattering then consists of the s-channel and the t-channel, as we have seen above.

2.5.5. Chern-Simons and Yang-Mills Theory as A_∞-Algebras

Given a cyclic A_∞-algebra on a symplectic vector space (V, ω) with differential Q and products $\{m_k\}_{k \geq 2}$, the action is defined to be

$$S(a) = \frac{1}{2}\omega(a, Qa) + \sum_{k \geq 2} \frac{1}{k+1} \omega(a, m_2(a,a)). \tag{2.237}$$

[21] A graph with external edges is planar, if it can be put on a disc without overlapping lines, and such that the external edges end on the boundary of the disc.

Note that the symmetry factor is here merely $\frac{1}{k+1}$, in contrast to (2.201) before. It accounts for the cyclic symmetry of the vertices.

As we have seen above, A_∞-algebras can be obtained from field theories by considering only color-ordered/primitive diagrams. The amplitudes in this case are merely cyclic symmetric instead in their entries instead of being fully symmetric.

As always, let us start simple and consider Chern-Simons theory. Recall that its action is given by

$$S = \frac{1}{2}(\mathcal{A}, \mathrm{d}\mathcal{A}) + \frac{1}{6}(\mathcal{A}, [\mathcal{A}, \mathcal{A}]). \tag{2.238}$$

We remember also that $\mathcal{A}$ contains all the fields of the BV extended formalism. It has only a single L_∞ product given by the Lie bracket

$$a \otimes b \mapsto (-)^a [a, b]. \tag{2.239}$$

To obtain an A_∞-algebra that reproduces Chern-Simons theory, we should find products, so that, after symmetrization, the lowest product becomes the Lie bracket, while all higher products become zero. Let us represent the underlying Lie algebra of the theory by a matrix algebra. In this sense we can define products of Lie algebra elements. There are continuously many choices for the product. We consider two extreme cases,

$$m_2(a, b) = (-)^a a \wedge b, \quad \tilde{m}_2(a, b) = \frac{1}{2}(-)^a [a, b], \tag{2.240}$$

but in principle any properly normalized linear combination of these two would also work. The product $\tilde{m}_2$ is not associative. So if we would like to use it, we should search for higher products so that we have associativity at least up to homotopy. On the other hand, m_2 clearly is associative, since

$$\begin{aligned} m_2(m_2(a, b), c) &= (-)^{a+b+1} m_2(a, b) \wedge c = (-)^{b+1} a \wedge b \wedge c = -a \wedge m_2(b, c) && (2.241) \\ &= -(-)^a m_2(a, m_2(b, c)), && (2.242) \end{aligned}$$

which is exactly associativity as we demand it for a degree one product. It is therefore advantageous to take m_2 as a product.

In terms of m_2, the Chern-Simons action reads

$$S = \frac{1}{2}\omega(\mathcal{A}, Q\mathcal{A}) + \frac{1}{3}\omega(\mathcal{A}, m_2(\mathcal{A}, \mathcal{A})), \tag{2.243}$$

which is obviously of the form (2.237). To make sense of this, we should of course demand that ω is defined on the matrix algebra, which is the case for example when it is given by the trace of the matrix representation. In this case, m_2 is cyclic with respect to ω.

Remark 2.5.3. Open bosonic string field theory as defined by Witten in [93] was motivated precisely by the associative algebra version of Chern-Simons. Cubic open-string field theory has therefore an underlying differential graded associative algebra. On the other hand, by taking the commutator with respect to the product, one can also give cubic open-string field theory the structure of a differential graded Lie algebra.

For Yang-Mills theory, we take $m_2(a, b) = \pm a \wedge b$ as long as not both a and b are fields/one-form. The sign, which is dependent on the inputs, will be fixed by the A_∞-relations. Also,

$m_3(a,b,c) = 0$ if at least one entry is not a field/one-form. On the other hand, if all entries are fields, we define

$$m_2(a,b) = \mathrm{d} * (a \wedge b) + a \wedge *\mathrm{d}b - (*\mathrm{d}a) \wedge b, \tag{2.244}$$

$$m_3(a,b,c) = a \wedge *(b \wedge c) - (*(a \wedge b)) \wedge c. \tag{2.245}$$

We check the A_∞ relations explicitly. Let c, d be ghosts. Then,

$$Qm_2(c,d) = -\mathrm{d}(c \wedge d) = -(\mathrm{d}c) \wedge d + c \wedge \mathrm{d}d = -m_2(Qc,d) - (-)^c m_2(c,Qd). \tag{2.246}$$

Here we chose $m_2(c,d) = -c \wedge d$. In turn, the sign rules demand that $m_2(c,a) = -c \wedge a$ and $m_2(a,c) = a \wedge c$ for a field a and a ghost c. For this configuration, we have that

$$Qm_2(a,c) = \mathrm{d} * \mathrm{d}(a \wedge c) = (\mathrm{d} * \mathrm{d}a) \wedge c + (*\mathrm{d}a) \wedge \mathrm{d}c - \mathrm{d} * (a \wedge \mathrm{d}c) - a \wedge (*\mathrm{d}\mathrm{d}c) \tag{2.247}$$

$$= -m_2(Qa,c) - m_2(a,Qc). \tag{2.248}$$

Here it was necessary to define $m_2(a^*,c) = -a^* \wedge c$ for some anti-field a^* and ghost c. Note that this is consistent with what we found in the L_∞-algebra treatment, where we had $m_2(a^*,c) = -[a^*,c]$. We continue with this combination,

$$Qm_2(a^*,c) = -\mathrm{d}(a^* \wedge c) = -\mathrm{d}a^* \wedge c + a^* \wedge \mathrm{d}c = -m_2(Qa^*,c) - (-)^{a^*} m_2(a^*,Qc). \tag{2.249}$$

We find that we have to choose the following signs. $m_2(c^*,c) = c^* \wedge c$ and $m_2(a^*,a) = a^* \wedge a$, for some anti-ghost c^*, ghost c, and field a. Finally, the last non-trivial combination is the product of two fields a and b. We have

$$Qm_2(a,b) = \mathrm{d}(\mathrm{d} * (a \wedge b) + a \wedge *\mathrm{d}b - (*\mathrm{d}a) \wedge b) \tag{2.250}$$

$$= \mathrm{d}a \wedge *\mathrm{d}b - a \wedge \mathrm{d} * \mathrm{d}b - (\mathrm{d} * \mathrm{d}a) \wedge b - \mathrm{d}a \wedge *\mathrm{d}b \tag{2.251}$$

$$= -m_2(Qa,b) - m_2(a,Qb). \tag{2.252}$$

All other derivation properties are trivially satisfied due to form degree exceeding four.

We also have to check associativity up to homotopy. The non-trivial cases are when $m_3 \neq 0$. Let a, b, c be fields. We have

$$Qm_3(a,b,c) = \mathrm{d}(a * (bc) - *(ab)c) = \mathrm{d}a * (bc) - a\mathrm{d} * (bc) - (\mathrm{d} * (ab))c - *(ab)\mathrm{d}c \tag{2.253}$$

$$= (*\mathrm{d}a)(bc) - a\mathrm{d} * (bc) - (\mathrm{d} * (ab))c - (ab) * \mathrm{d}c - a(*\mathrm{d}b)c + a(*\mathrm{d}b)c \tag{2.254}$$

$$= -m_2(a,b)c - am_2(b,c) = -m_2(m_2(a,b),c) - m_2(a,m_2(b,c)). \tag{2.255}$$

Next take c not a field but a ghost. Since m_3 is now non-commutative (in contrast to the L_∞-products), we should check the identity for Qc at any position in m_3. In order to not get bored, let us focus only on one. We take Qc to be in last entry. Then

$$m_3(a,b,Qc) = a * (b\mathrm{d}c) - *(ab)\mathrm{d}c \tag{2.256}$$

$$= -a * \mathrm{d}(bc) + a(*\mathrm{d}b)c - \mathrm{d}(*(ab)c) + (\mathrm{d} * (ab))c + (*\mathrm{d}a)bc - (*\mathrm{d}a)bc \tag{2.257}$$

$$= -a * \mathrm{d}m_2(b,c) + m_2(a,b)c - \mathrm{d} * (am_2(b,c)) + (*\mathrm{d}a)m_2(b,c) \tag{2.258}$$

$$= -m_2(m_2(a,b),c) - m_2(a,m_2(b,c)), \tag{2.259}$$

which is the correct relation.

3. Quantum Homotopy Algebras and the Batalin-Vilkovisky Formalism

Measurable effects in quantum physics are computed via expectation values of observables. This is the quantum analog of evaluation of functionals on the equations of motion. Given any functional f on field space $\mathcal{F}_0$, its expectation value is defined formally by the path integral

$$\langle f\rangle = \int_{\mathcal{F}_0} \prod_x \mathrm{d}\phi(x) e^{\frac{i}{\hbar}S_0[\phi]} f[\phi]. \tag{3.1}$$

Of course, this integral over the infinite dimensional space $\mathcal{F}_0$ is not defined. Therefore, we proceed as usual in this situation and pretend that we work in the finite dimensional setup and develop the quantum BV formalism there. The result will again lead to a cohomological description of the integral (3.1), which makes sense also over field spaces.

3.1. Path Integral without Gauge Symmetries - The Twisted de Rham Complex

The starting point of this section is de Rhams theorem. Let M be an n dimensional smooth manifold without boundary. An integral of a k-form $\alpha \in \Omega^k(M)$ is defined over any singular chain in $C_k(M, \mathbb{R})$. Recall that a degree k singular chain C is a superposition

$$C = \sum_{i=1}^{N} \lambda_i f_i \tag{3.2}$$

of continuous maps $f_i : \Delta^k \to M$ from the k-simplex Δ^k into M. The boundary map

$$\partial : C_k(M, \mathbb{R}) \to C_{k-1}(M, \mathbb{R}) \tag{3.3}$$

is defined by the restriction of each f_i to the boundary of Δ_k. The integral of α over the chain C in (3.2) is defined by

$$\int_C \alpha = \sum_{i=1}^{N} \lambda_i \int_{\Delta_k} f_i^* \alpha, \tag{3.4}$$

where $f_i^*\alpha$ is the pullback of the form α to Δ_k by f_i. It follows that any k-form α defines a linear map on $C_k(M, \mathbb{R})$. Stokes theorem tells us that this is in fact a chain map $\phi : \Omega^\bullet(M) \to C^\bullet(M, \mathbb{R})$ into the space of singular cochains, the objects dual to singular chains. We have the following theorem by de Rham.

Theorem 3 (de Rham)**.** *The map ϕ is a quasi-isomorphism.*

We assume that M is compact. In this case, the (co-)homologies are finite dimensional. Our aim is to show that the computation of the de-Rham cohomology is equivalent to computing integrals over closed forms. Let C_i be a basis of $H_k(C_\bullet(M,\mathbb{R}))$ and α^i a basis for $H^k(\Omega^\bullet(M))$. Since the two spaces are dual to each other, we can normalize the two bases such that

$$\int_{C_i} \alpha^j = \delta_i^j. \tag{3.5}$$

By assumption, any $\alpha \in H^k(\Omega^\bullet(M))$ can be written as

$$\alpha = \sum_i \lambda_i \alpha^i. \tag{3.6}$$

By our choice of normalization, the coefficients are

$$\lambda_i = \int_{C_i} \alpha. \tag{3.7}$$

We observe that once we know the cohomology class of a closed form α, we in principle also know the values of all its integrals over k-dimensional subspaces of M without boundary. Computing integrals is therefore equivalent to computing cohomologies.

As a special case, we look at the integration of top dimensional forms. It is well known that $H^n(\Omega^\bullet(M)) \cong \mathbb{R}$. Let ω be volume form with non-zero integral,

$$\int_M \omega \neq 0. \tag{3.8}$$

ω generates $H^n(\Omega^\bullet(M))$. Given any other top form ω', we can compute its integral relative to ω by writing

$$\omega' = \lambda\omega + \mathrm{d}j, \quad \lambda \in \mathbb{R}. \tag{3.9}$$

Clearly, $\lambda = \frac{\int_M \omega'}{\int_M \omega}$. We can also use ω to define a volume on M. We are then able to integrate functions over M. Similarly to what we did in (3.9), we can write

$$f\omega = \lambda\omega + \mathrm{d}j, \tag{3.10}$$

where λ is the normalized integral

$$\frac{\int_M f\omega}{\int_M \omega}. \tag{3.11}$$

Most of the time, we will work with a twisted de Rham differential. This is motivated by quantum field theory, where integrals are weighted by the phase $e^{\frac{i}{\hbar}S_0(x)}$. On the level of the de Rham complex, this induces the twist

$$\mathrm{d} \mapsto \mathrm{d}_t = e^{-\frac{i}{\hbar}S_0}\mathrm{d}e^{\frac{i}{\hbar}S_0} = \mathrm{d} + \frac{i}{\hbar}\mathrm{d}S \wedge\ . \tag{3.12}$$

An integral with respect to the weighted measure $e^{\frac{i}{\hbar}S_0}\omega$ is computed by replacing d with d_t in equation (3.11).

We successfully encoded the weighted integral into the complex by introducing a twisted differential. Our next goal is to also make the volume form ω part of the differential. This procedure is described in [94]. To achieve this, we normalize our forms with respect to ω.

On an n-dimensional manifold, any k-form can be obtained by contracting ω with $n-k$ vector fields. A more refined way is to say that we contract ω with a single polyvector field. In this way, we obtain any k-form in a unique way. Another way of saying this is that ω induces an isomorphism of $C^\infty(M)$-modules[1]

$$\Omega^\bullet(M)[n] \cong \Gamma(M, \mathrm{Sym}^{n-\bullet}(T[1]M)). \tag{3.13}$$

This isomorphism allows us to turn $\Gamma(M, \mathrm{Sym}^{n-\bullet}(T[1]M))$ into a cochain complex. The constant function $1 \in \Gamma(M, \mathrm{Sym}^0(T[1]M))$ corresponds to our volume form ω. Therefore, 1 represents the cohomology class of ω. This is what we consider to be normalized with respect to ω.

Remark 3.1.1. Another way to think about this was proposed in [94]. We can consider $1 \in \Omega^0(M)$ as the ground state in the fermionic space of forms. A coordinate one form $\mathrm{d}x^i$ serves as a creation operator. The adjoint annihilation operator is given by i_{∂_i}, the inner derivation with respect to the partial derivative $\frac{\partial}{\partial x^i}$. They satisfy the usual anti-commutation relations

$$\{i_{\partial_i}, \mathrm{d}x^j\} = \delta_i^j, \ \{\mathrm{d}x^i, \mathrm{d}x^j\} = \{i_{\partial_i}, i_{\partial_j}\} = 0. \tag{3.14}$$

What we have is a lowest weight representation of a space of fermions. But we could equally likely consider a highest weight representation. In this case, we would consider 1 as a state of highest energy and multiplication by partial derivatives $\frac{\partial}{\partial x^i}$ as an operation lowering its energy. The result is the space of polyvector fields $\Gamma(M, \mathrm{Sym}^\bullet(T[1]M))$. For a finite number of fermions, their exists a state of highest energy in the lowest weight representation. This is our volume form ω. It provides an isomorphism between the representations. An important observation is that this is no longer true for an infinite number of fermions. Highest and lowest weight representations are not equivalent. In infinite dimensions, an isomorphism like (3.13) does not exist. We conclude that their are in general two notions of integration, which are only equivalent in finite dimensions.

We want to see how the twisted differential d_t translates through (3.13). As we did in classical field theory, we call the odd directions of $\Gamma(M, \mathrm{Sym}^\bullet(T[1]M))$ "anti-fields", denoted by x_i^*. The isomorphism (3.13) ϕ can be described by an odd Fourier transform

$$\alpha \mapsto \phi(\alpha) = \int \mathrm{d}(\mathrm{d}x^n)...\mathrm{d}(\mathrm{d}x^1)\omega^{-1}(x)e^{-x_i^*\mathrm{d}x^i}\alpha(x^i, \mathrm{d}x^j). \tag{3.15}$$

The integral over the odd $\mathrm{d}x^i$ obeys the usual rules of Grassmann integration. The transformation is written in local coordinates. $\omega(x)$ is the component of the $\omega = \omega(x)\mathrm{d}x^1 \wedge ... \wedge \mathrm{d}x^n$. The factor $\mathrm{d}(\mathrm{d}x^n)...\mathrm{d}(\mathrm{d}x^1)\omega^{-1}(x)e^{x_i^*\mathrm{d}x^i}$ is coordinate independent.[2] This makes sure that α has the correct transformation properties. We also give the inverse transformation

$$f \mapsto \phi^{-1}(f) = \int \mathrm{d}x_1^*...\mathrm{d}x_n^*\omega(x)e^{x_i^*\mathrm{d}x^i}f(x^i, x_j^*). \tag{3.16}$$

As a consistency check we note that $\phi^{-1}(1) = \omega$.

[1] It is not an isomorphism of commutative algebras.

[2] The author is aware of the fact that the Fourier transformation in reference [80] uses ω instead of ω^{-1}. However, with the former choice the transform $\phi(\alpha)$ would depend on local coordinates, as one can easily check in the case when α is a top form. Also, our convention coincides with reference [51].

The odd Fourier transform has a property very similar to the even Fourier transform. Let L_a denote the left-multiplication by an element a. We observe that $\phi \circ L_{\mathrm{d}x^i} \circ \phi^{-1} = (-)^{n+1}\frac{\partial}{\partial x_i^*}$ and $\phi^{-1} \circ L_{x_i^*} \circ \phi = (-)^n \frac{\partial}{\partial(\mathrm{d}x^i)}$. In other words, differentiation and multiplication gets interchanged under a Fourier transform. With this property, we can determine how the de Rham differential d acts on polyvector fields. We have

$$\phi \circ \mathrm{d} \circ \phi^{-1} = \phi \circ L_{\mathrm{d}x^i} \circ \frac{\partial}{\partial x^i} \circ \phi^{-1} = (-)^{n+1} L_{\omega^{-1}(x)} \frac{\partial^2}{\partial x^i \partial x_i^*} L_{\omega(x)}. \tag{3.17}$$

We actually can get rid of the n dependence by recalling that ϕ is an isomorphism between polyvector fields $\Gamma(M, \mathrm{Sym}^\bullet TM)$ and the *shifted* de Rham complex $\Omega^\bullet(M)[n]$, where natural choice of differential on the latter is $(-)^n\mathrm{d}$. We define

$$\Delta(f)(x^i, x_j^*) = -\omega^{-1}(x) \frac{\partial^2}{\partial x^i \partial x_i^*}(\omega(x) f(x^i, x_j^*)). \tag{3.18}$$

We call Δ the *BV-Laplacian.* On a vector field $X = X^i x_i^*$, it acts as

$$\Delta(X) = -\omega^{-1}\partial_i(\omega X^i) = -\mathrm{div}\, X. \tag{3.19}$$

The Fourier transform ϕ is a an isomorphism of modules. For this reason, we can compute the effect of the shift coming from $e^{\frac{i}{\hbar}S_0}$ directly on the level of polyvector fields. We have

$$\phi \circ \mathrm{d}_t \circ \phi^{-1} = \phi \circ e^{-\frac{i}{\hbar}S_0} \mathrm{d} e^{\frac{i}{\hbar}S_0} \circ \phi^{-1} = -e^{-\frac{i}{\hbar}S_0} \Delta e^{\frac{i}{\hbar}S_0} = -\Delta - \frac{i}{\hbar}\frac{\partial S_0}{\partial x^i}\frac{\partial}{\partial x_i^*}. \tag{3.20}$$

There is something familiar from classical BV. The twist produces $\frac{i}{\hbar}\{S_0, \cdot\}$, which is, up to a prefactor, the differential computing the derived critical locus. In fact, in BV one normalizes the differential by $i\hbar$ to obtain

$$i\hbar\Delta + \{S_0, \cdot\}. \tag{3.21}$$

Remark 3.1.2. It is not surprising that the critical locus appears as part of this differential. First of all, we expect to obtain classical physics by setting $\hbar = 0$. Furthermore, for non-zero $\hbar$ the main contribution in a path integral still comes from the classical trajectory.

Remark 3.1.3. An important property is that Δ fully determines the anti-bracket. We have the relation

$$\Delta(FG) = \Delta(F)G + (-)^F\{F, G\} + (-)^F F\Delta G. \tag{3.22}$$

We can think of $\{\cdot, \cdot\}$ as a measure for the failure of Δ to be of first order. The general properties of Δ are captured in the following two definitions.

Definition 3.1.1. Let A be a graded commutative algebra. A linear map $\Delta : V \to V$ is called of second order, if

$$\{a, b\} := (-)^{a\Delta}\Delta(ab) - (-)^{a\Delta}\Delta(a)b - a\Delta b \tag{3.23}$$

is of first order differential operator in both variables. This means that $\{a, \cdot\}$ and $\{\cdot, b\}$ act as ordinary derivations.

Definition 3.1.2. A *Batalin-Vilkovisky algebra* is a graded commutative algebra A together with a degree one second order differential operator Δ, such that $\Delta^2 = 0$.

A simple exercise is to show the following.

Corollary 2. *If a second order operator Δ satisfies $\Delta^2 = 0$, then the bracket defined in (3.23) has the following properties.*

- $\{\,\cdot\,,\,\cdot\,\}$ *satisfies the graded Jacobi identity.*
- Δ *is a derivation with respect to* $\{\,\cdot\,,\,\cdot\,\}$,

$$\Delta\{a, b\} = \{\Delta(a), b\} + (-)^{(a+1)}\{a, \Delta(b)\}. \tag{3.24}$$

Corollary 2 tells us that all necessary information is encoded in the Laplacian. The antibracket together with its properties derive from it.

We stress again that in finite dimensions, the complex $(\Gamma(M, \mathrm{Sym}^{\bullet} T[1]M), -i\hbar\Delta + \{S_0, \cdot\,\})$ is still that of de Rham, with a measure weighted by the action S_0. In degree $-k$, it computes integrals over homology classes of subspaces with codimension k. From the path integral point of view, it is therefore necessary to restrict to forms α, so that

$$\mathrm{d}(e^{\frac{i}{\hbar}S_0}\alpha) = 0. \tag{3.25}$$

In the Fourier transformed language, this means that $\phi(\alpha)$ is closed under the differential $-i\hbar\Delta + \{S_0, \cdot\,\}$. Closed elements in the algebra $(\Gamma(M, \mathrm{Sym}^{\bullet} T[1]M)$ are usually called gauge invariant functions, although we did not yet include gauge symmetries. To this point it just means that their integral is invariant under continuous deformations of the integration cycle.

3.2. Gauge Symmetries

Whenever gauge symmetries appear, we want to restrict the integral over a submanifold which is transverse to the gauge orbits. Recall form the classical treatment that we describe a symmetry by a representation $\rho : \mathfrak{g} \to \Gamma(M, TM)$ of a Lie algebra $\mathfrak{g}$. The image of ρ are the vector fields that are tangent to the gauge orbit. We can produce a differential form on a gauge slice by contracting a volume form ω with all the vectors in the image of ρ.

In dual picture of polyvector fields, a contraction with X translates to a left multiplication by X. A volume form is represented by 1. Let $a_1, ..., a_k$ be a basis of $\mathfrak{g}[1]$. A form on a gauge slice is obtained by multiplication of 1 with all the tangents to the orbit $\rho(a_i)$, i.e.

$$\rho(a_1) \cdots \rho(a_k). \tag{3.26}$$

This contraction can actually obtained by doing an integral over $\mathfrak{g}[1]$. The kth power ρ^k of our representation ρ defines a function on $\mathfrak{g}[1]$ taking values in the space of polyvector fields. A Grassmannian integration over $\mathfrak{g}[1]$ precisely produces (3.26),

$$\int [\mathrm{d}c^1, ..., \mathrm{d}c^k]\rho^k = \rho(a_1) \cdots \rho(a_k).^3 \tag{3.27}$$

The Grassmannian integration automatically projects onto the kth power of c. We can therefore also write

$$\int [\mathrm{d}c^1, ..., \mathrm{d}c^k]\rho^k = (i\hbar)^k \int [\mathrm{d}c^1 \cdots \mathrm{d}c^k] e^{-\frac{i}{\hbar}\rho}. \tag{3.28}$$

[3] We normalize the coordinates with respect to the basis $\{a_i\}$, i.e. $c^i(a_j) = \delta^i_j$.

This motivates the shift $S_0 \mapsto S_0 - \rho =: S_0 + S_1$ at the quantum level. Also, writing $e^{-\frac{i}{\hbar}\rho}$ instead of ρ^k allows for a transition to infinite dimensional integrals Lie algebras (like the algebra $\mathfrak{G}$ of gauge transformations), where we do not have a highest power k.

The upshot is that the proper integral in a gauge theory is

$$\int [\mathrm{d}c^1, ..., \mathrm{d}c^k]\, \omega\, \mathrm{d}^n x\, e^{\frac{i}{\hbar}(S_0+S_1)} \tag{3.29}$$

We would like to have a cohomological theory computing these integrals. It should extend the complex $(\Gamma(M, \mathrm{Sym}^\bullet T[1]M), -i\hbar\Delta + \{S_0, \cdot\})$ developed for integrals without gauge symmetries. We first want to put the ghost variables c on the same footing as the ordinary variables x. We do a Fourier transformation with respect to the even variables $\mathrm{d}c^k$.

To obtain the correct result, we need to mention that integrals in odd variables are not taken over differential forms. The reason is that the $\mathrm{d}c^i$ are even variables, and therefore there is no top form to integrate. This is related to what we have seen for bosonic variables. In finite dimensions, highest and lowest weight representations over anticommuting creation and annihilation operators are isomorphic. We mentioned that this is no longer true in infinite dimensions, and it is more natural to consider highest weight representations in order to have a top form. Over commuting variables, like the $\mathrm{d}c^i$, there is no top form in the lowest weight representation, even in finite dimensions. It is therefore necessary to use the highest weight representation. Forms of this type are called *integral forms* in the literature, in order to distinguish them from differential forms.

A top form ω should be annihilated by all the $\mathrm{d}c^i$. This motivates the notation

$$[\mathrm{d}c^1, ..., \mathrm{d}c^k] = \delta(\mathrm{d}c^1) \cdots \delta(\mathrm{d}c^k). \tag{3.30}$$

The $\delta(\mathrm{d}c^i)$ are anticommuting, unlike the $\mathrm{d}c^i$. Descendants are obtained by differentiation with respect to the $\frac{\partial}{\partial \mathrm{d}c^i}$. As usual, derivatives of delta functions are defined by the rule

$$f(c^i, \mathrm{d}c^i)\partial^n_{\mathrm{d}c^i}\delta(\mathrm{d}c^i) = (-)^n(\partial^n_{\mathrm{d}c^i} f(c^i, 0))\delta(\mathrm{d}c^i). \tag{3.31}$$

The exterior derivative $\mathrm{d} = \mathrm{d}c^i \frac{\partial}{\partial c^i}$ acts on integral forms in the usual way. For example,

$$\mathrm{d}\delta(\mathrm{d}c^i) = 0, \quad \mathrm{d}(c^i\delta(\mathrm{d}c^i)) = \mathrm{d}c^i\delta(\mathrm{d}c^i) = 0, \quad \mathrm{d}(c^i\delta'(\mathrm{d}c^i)) = -\delta(\mathrm{d}c^i). \tag{3.32}$$

We have Stokes' theorem

$$\int (\mathrm{d}\alpha)(c^i, \mathrm{d}c^i) = 0, \tag{3.33}$$

for any integral form $\alpha(c^i, \mathrm{d}c^i)$.

Remark 3.2.1. For a given set of fermionic variables θ^i, one could also consider forms which are differential in some $\mathrm{d}\theta^i$, while integral in the other. This is related to picture number in string theory, where the picture number p of a form is defined to be minus the number of $\delta(\mathrm{d}\theta^i)$ (including derivatives), see for example [96]. De Rham cohomology in different picture numbers has been studied in [17][4].

We are now set up to do the Fourier transformation with respect to the $\mathrm{d}c^i$. It is *defined* by

$$\phi : \prod_i \partial^{n_i}_{\mathrm{d}c^i}\delta(\mathrm{d}c^i) \mapsto \prod_i \frac{1}{n_i!}(c^*_i)^{n_i}, \tag{3.34}$$

[4] Their picture number is minus the picture number considered in the physics literature.

where, as we will see, the c_i^* are the anti-ghosts appearing also in the classical description. The effect of (left-)multiplication $L_{\mathrm{d}c^i}$ in the transformed space is

$$\phi \circ L_{\mathrm{d}c^i} \circ \phi^{-1}(\prod_j (c_j^*)^{n_j}) = -n_i (c_i)^{n_i - 1} \prod_{j \neq i} (c_j^*)^{n_j}. \tag{3.35}$$

$L_{\mathrm{d}c^i}$ is transformed to the derivative $-\frac{\partial}{\partial c_i^*}$, as we would expect. It is easy to see that the same rule applies for higher powers in $L_{\mathrm{d}c^i}$. This implies that the exterior derivative d on ghosts gets mapped to the second order differential operator

$$\Delta_{gh} = -\sum_i \frac{\partial^2}{\partial c^i \partial c_i^*}. \tag{3.36}$$

This operator again determines the anti-bracket via (3.23), however this time with an additional minus sign. If we want (3.23) to hold also in this case, we should replace $\Delta_{gh} \mapsto -\Delta_{gh}$.

3.3. General Quantum BV Formalism

At the classical level, we saw that $Q = \{S_0 + S_1, \cdot\}$ may no longer square to zero. To obtain a cohomological theory at the classical level, we add also terms in higher anti-ghost number, so that S satisfies the classical master equation $\{S, S\} = 0$. Quantum physics now also lead to introduction of a new operator, the second order Laplacian Δ. Its canonical representation is

$$\Delta = \sum_i (-)^{\phi_i + 1} \frac{\partial^2}{\partial \phi^i \partial \phi_i^*}, \tag{3.37}$$

where ϕ^i runs over the whole set of fields plus ghosts. The operator Δ represents the de Rham differential in the Fourier transformed space. Since the integral is weighted by the action $S_0 + S_1$, we obtain a twist

$$\Delta \mapsto (-i\hbar) e^{-\frac{i}{\hbar}(S_0 + S_1)} \Delta e^{\frac{i}{\hbar}(S_0 + S_1)}. \tag{3.38}$$

For non-zero S_1, this operator may no longer square to zero. This is related to the fact that S_1 is constructed such that it kills gauge directions of the integral form of physical fields. The form is therefore no longer a top form and hence not necessarily closed. To check closedness, we need to act with Δ on the twisted measure $e^{\frac{i}{\hbar}(S_0 + S_1)}$,

$$\Delta(e^{\frac{i}{\hbar}(S_0 + S_1)}) = e^{\frac{i}{\hbar}(S_0 + S_1)} (\frac{i}{\hbar} \Delta(S_0 + S_1) - \frac{1}{2\hbar^2} \{S_0 + S_1, S_0 + S_1\}). \tag{3.39}$$

This motivates the following definition.

Definition 3.3.1. Given a BV algebra $(A, \cdot, \Delta)$, we say that $S(\hbar) \in A \otimes \mathbb{C}[[\hbar]]$ satisfies the *quantum master equation*, if

$$\frac{1}{2}\{S, S\} - i\hbar \Delta S = 0. \tag{3.40}$$

Corollary 3. *If S satisfies the quantum master equation, the operator*

$$\Delta_S := (-i\hbar) e^{-\frac{i}{\hbar} S} \Delta e^{\frac{i}{\hbar} S} \tag{3.41}$$

squares to zero. Furthermore,

$$\Delta_S = -i\hbar \Delta + \{S, \cdot\}. \tag{3.42}$$

Definition 3.3.1 and corollary 3 are the quantum analogs of the classical master equation $\{S,S\}=0$ and nilpotency of $\{S,\cdot\}$. A solution $S(\hbar)$ to (3.40) is called quantum action. It is a power series in $\hbar$. When $S(\hbar)$ solves the quantum master equation, then $S(\hbar=0)$ automatically solves the classical one. To find a solution to the quantum master equation, it is therefore a good idea to first find a solution $S(\hbar=0)$ of the classical master equation, and then to determine $S(\hbar)$ perturbatively in $\hbar$.

Quantum anomalies can be identified as the obstructions to solve the quantum master equation. In the path integral formalism, anomalies arise because the path integral measure is not gauge invariant. Recall that $S_1=-\rho$, where $\rho:\mathfrak{g}[-1]\to\Gamma(M,T[1]M)$. In terms of ghosts we can write $\rho=-X_ic^i$, where X_i is a vector field generating a gauge transformation. By acting on $-\rho$ with Δ, we obtain

$$\Delta(-\rho)=-\Delta(X_i)c^i=\mathrm{div}(X_i)c^i, \tag{3.43}$$

where we used the fact that Δ computes minus the divergence when acting on vector fields. The divergence of a vector field X_i is defined to be the change of the volume form under X_i,

$$L_{X_i}\omega:=\mathrm{div}(X_i)\omega, \tag{3.44}$$

where L_{X_i} denotes the Lie derivative. We therefore can indeed identify $\Delta(S_1)$ with the change of the measure under gauge transformations. Adding $\hbar$ dependent terms to the action so that the quantum master equation is satisfied then means that we change the measure and/or the gauge transformations so that the former is again invariant.

3.4. Integration and Gauge Fixing

Let $\mathcal{F}$ be the field space of a BV theory with fields x^i and anti-fields x^*_i. We have a canonical symplectic form

$$\omega=\sum_i(-)^{x^*_i}\,\mathrm{d}x^*_i\wedge\mathrm{d}x^i. \tag{3.45}$$

Integration over a function $f\in\mathcal{O}(\mathcal{F})$ is defined to be

$$\int_{\mathcal{L}}\mu f, \tag{3.46}$$

where $\mathcal{L}$ is a Lagrangian submanifold with respect to ω and μ is some measure[5]. We still pretend that we work in finite dimensions. In that case, a submanifold $\mathcal{L}\subseteq\mathcal{F}$ is called Lagrangian, if $\omega|_{\mathcal{L}}=0$ and $\dim\mathcal{L}=\frac{1}{2}\dim\mathcal{F}$. We recall a standard fact about the local properties of a Lagrangian submanifold. Locally, we can describe $\mathcal{L}$ as the graph of a function $p_j=\Psi_j(q)$, where $P=\{p_1,...,p_n\}$ is a subset of the all coordinates $\{x^1,...,x^n,x^*_1,...,x^*_n\}$ and $Q=\{q_1,...,q_n\}$ its complement. The fact that $\mathcal{L}$ is Lagrangian implies that whenever $x^i\in Q$, we have $x^*_i\notin Q$, and vice versa. Up to possible redefinition of coordinates by signs, it follows that

$$\omega=\sum_i\mathrm{d}p_i\wedge\mathrm{d}q^i. \tag{3.47}$$

[5]The local description of μ will be given below. A global approach can be described by considering half-densities on $\mathcal{F}$, which restrict to densities on $\mathcal{L}$.

In this form, it is an easy check that the 1-form $\psi_i \mathrm{d}q^i$ is necessarily closed. We can therefore locally write $\psi_i = \frac{\partial \psi}{\partial q^i} = p_i$. We can conclude that, locally, any Lagrangian submanifold can be described by a single function ψ. It is called the gauge fixing fermion and necessarily of degree -1. The integration measure is $\mu = \mathrm{d}q^1 \wedge ... \wedge \mathrm{d}q^n$.

To see whether we obtain something familiar, let us work out integration over a two dimensional Lagrangian submanifold in a four dimensional superspace $\mathbb{R}^{2|2}$ concentrated in degree 0 and -1. In that case, we should obtain ordinary integrals in the bosonic plane $\mathbb{R}^2$.

We have coordinates x, y, x^*, y^*. The symplectic form is

$$\omega = -\mathrm{d}x^* \wedge \mathrm{d}x - \mathrm{d}y^* \wedge \mathrm{d}y. \tag{3.48}$$

We want to look for a gauge fixing fermion ψ. The fact that it is necessarily of degree -1 limits the possible choices. Also, it should depend only on two of the four coordinates. Depending on that choice, we obtain three classes of possible ψ.

1. $\psi_1 = \psi(x, y)$. Since x and y are of degree zero, the only possible choice is $\psi_1 = 0$.
2. $\psi_2 = \psi(x, y^*)$. The most general choice is $\psi_2 = y^* l(x)$. We could also equivalently consider $\psi_2 = \psi(x^*, y)$.
3. $\psi_3 = \psi(x^*, y^*)$. The only possible gauge fixing fermions are of the form $\psi_3 = ax^* + by^*$, $a, b \in \mathbb{R}$.

We wish to integrate functions $F = f + gx^* + hy^* + kx^*y^*$, such that $\Delta f = 0$. A gauge fixing fermion of the first type, $\psi(x, y) = 0$, considers $(x, y) = (q_1, q_2)$ as position variables and $(x^*, y^*) = (-p_1, -p_2)$ as conjugate momenta. Since $p_i = \frac{\partial \psi}{\partial q^i} = 0$, the function F gets projected onto $F = f(x, y)$, integrated over the (x, y)-plane. We obtain

$$\int_{p_i = \frac{\partial \psi_1}{\partial q^i}} F = \int \mathrm{d}x\mathrm{d}y f(x, y). \tag{3.49}$$

When we think of the quantum BV complex as a Fourier transformed de Rham complex, this result makes sense. Under Fourier transformation, f becomes the two form $f \mathrm{d}x \wedge \mathrm{d}y$.

Consider now a gauge fixing fermion of the form $\psi_2(x, y^*) = y^* l(x)$. The position variables are $(q_1, q_2) = (x, y^*)$ and their conjugate momenta are $(p_1, p_2) = (-x^*, -y)$. The gauge fixing fermion sets $x^* = -\frac{\partial \psi}{\partial x} = -y^* l'(x)$ and $y = -\frac{\partial \psi}{\partial y^*} = -l(x)$. This already suggests that we will effectively integrate over the one dimensional space $y = -l(x)$. The function F becomes

$$F|_{p_i = \frac{\partial \psi}{\partial q^i}} = f(x, -l(x)) + y^*[h(x, -l(x)) - l'(x) g(x, -l(x))]. \tag{3.50}$$

Integration with respect to the odd variable y^* projects F to the term proportional to y^*. We find

$$\int_{p_i = \frac{\partial \psi_2}{\partial q^i}} F = \int \mathrm{d}x[h(x, -l(x)) - l'(x) g(x, -l(x))] \tag{3.51}$$

This is an ordinary integral of the one form $\mathrm{d}xh - \mathrm{d}yg$ over $y = -l(x)$. This is exactly the form obtained from Fourier transforming the degree -1 part of F.

Gauge fixing with respect to $\psi_3 = ax^* + by^*$ forces $x = -a$ and $y = -b$. Integration amounts to evaluation of k at that point.

The above examples demonstrate how integration in the BV formalism reduces to ordinary integration in the most simple cases. Arguably the most interesting case came from the gauge fixing fermion ψ_2, where we were able to obtain integration over the graph $y = -f(x)$. Integrals along more general curves can be obtained by splitting the curves into regions where they can be described by a graph and then integrate their using the appropriate gauge fixing fermion. On the other hand, any integral over a Lagrangian submanifold described locally by a gauge fixing fermion of type ψ_2 represents a one dimensional curve in this way.

Remark 3.4.1. In the physics literature, the fields are usually taken as position variables and the anti-fields as their conjugate momenta. Since the usual BV construction introduces fields only in nonnegative degrees, one has to add new fields in negative degree to be able to write down non-trivial gauge fixing fermions. These fields are called trivial pairs[6] because their action is completely decoupled from the rest of the fields. They become related to the other fields only after gauge fixing.

Let us now discuss why Lagrangian submanifolds are the spaces we want to integrate. This is the essence of the following theorem, whose statement and proof can found in [80].

Theorem 4. *Consider a field space $\mathcal{F}$, together with a symplectic form ω and associated BV Laplacian Δ. We then have the following.*

1. *Given a fixed Lagrangian submanifold $\mathcal{L}$ of $\mathcal{F}$, we have*

$$\int_{\mathcal{L}} \Delta(f) = 0, \tag{3.52}$$

for any function $f \in \mathcal{O}(\mathcal{F})$.

2. *Given a smooth family of Lagrangian submanifolds $\mathcal{L}_t$, $t \in [0,1]$ and a function $f \in \mathcal{O}(\mathcal{F})$ such that $\Delta f = 0$. Then,*

$$\int_{\mathcal{L}_0} f = \int_{\mathcal{L}_1} f. \tag{3.53}$$

3.4.1. Perturbative Evaluation of Integrals

In quantum field theories, it is often only possible to evaluate path integrals perturbatively. The perturbative expansion is expressed in terms of Feynman diagrams. The cohomological nature of BV theory will allow us to rederive Feynman diagrams using the homological perturbation lemma.

Let S be a BV action on a linear field space $\mathcal{S}$. It satisfies the quantum master equation. We split $S = S_F + S_I$, where S_F is the free and classical part of S. Equivalently, S_F is determined by the fact that it is quadratic in the fields and independent of $\hbar$. The interacting part S_I should be such that its classical part $(S_I|_{\hbar=0})$ starts at cubic order. We therefore assume that the origin of our field space $\mathcal{F}$ is a solution to the classical equations of motion.

The integral we would like to evaluate is of the form

$$\int_{\mathcal{L}} e^{\frac{i}{\hbar}(S_F+S_I)} f, \tag{3.54}$$

where $\mathcal{L}$ is some subspace of field space $\mathcal{F}$ and $f \in \mathcal{O}(\mathcal{F})$. By a choice of basis Φ^i on field space, we can write $S_F = \int \Phi^i(x) K_{ij}(x,y) \Phi^j(y)$. The subspace $\mathcal{L}$ should be chosen such

[6] The word *pair* stems from the fact that one always has to introduce two fields and two anti-fields at a time in order to be able to write down a non-zero action.

that K_{ij} becomes invertible on it. So what are the zero eigenvectors of K_{ij}? Recall that S_F generates the linear differential $Q_F = \{S_F, \cdot\}$. Q_F therefore restricts to an operator on $\mathcal{F}$. If we identify $\mathcal{F}$ with its tangent space at the origin, we can write $S_F[\phi] = \omega(\phi, Q_F\phi)$. By comparison we see that $K_{ij}(x, y)$ are the components of the bilinear form $\omega(\phi, Q_F\phi)$. By non-degeneracy of ω, the zero eigenspace of K is the kernel of Q_F. We should therefore look for an $\mathcal{L}$ transversal to $\ker Q_F$. Since $\ker Q_F$ is linear, no harm is done when we choose $\mathcal{L}$ to be linear as well. Transversality then is equivalent to

$$\mathcal{F} = \mathcal{L} \oplus \ker Q_F. \tag{3.55}$$

An analogous way to describe $\mathcal{L}$ is the following.

Lemma 3. *$\mathcal{L}$ is transversal to $\ker Q_F$ if and only if $Q_F : \mathcal{L} \to \operatorname{Im} Q_F[1]$ is an isomorphism.*

Proof. This follows from the fact that

$$0 \longrightarrow \ker Q_F \longrightarrow \mathcal{F} \xrightarrow{Q_F} \operatorname{Im} Q_F[1] \longrightarrow 0 \tag{3.56}$$

is exact, and therefore $\mathcal{F} = \ker Q_F \oplus \operatorname{Im} Q_F[1]$ since we work with vector spaces. □

From what we know from the last section, we would like to choose $\mathcal{L}$ to be Lagrangian with respect to ω. As it turns out, this is often impossible. The reason is the following. Since $\mathcal{F} = \mathcal{L} \oplus \ker Q_F$ and $\ker Q_F{}^{\perp} = \operatorname{Im} Q_F$, it follows that

$$\mathcal{F} = \mathcal{L}^{\perp} \oplus \operatorname{Im} Q_F. \tag{3.57}$$

Therefore, any $x \in \ker Q_F$ has a unique decomposition $x = x' + \mathrm{d}y$, where $x' \in \mathcal{L}^{\perp} \cap \ker Q_F$. This implies that we can identify $H^{\bullet}(\mathcal{F}, Q_F)$ with $\ker Q_F \cap \mathcal{L}^{\perp}$. If we assume that $\mathcal{L}$ is Lagrangian ($\mathcal{L} = \mathcal{L}^{\perp}$), we could on the other hand conclude that $\ker Q_F \cap \mathcal{L}^{\perp} = 0$ since

$$\mathcal{F} = \mathcal{L} \oplus \ker Q_F = \mathcal{L}^{\perp} \oplus \ker Q_F. \tag{3.58}$$

Hence, the obstruction for $\mathcal{L}$ to be Lagrangian is given by cohomology of $Q_{\mathcal{F}}$.

Although we cannot choose $\mathcal{L}$ to be Lagrangian, it will be useful to put some restrictions on it (aside from its transversality to $\ker Q_F$). To have better control over the spaces involved, we would like to introduce projectors. The splitting $\mathcal{F} = \mathcal{L} \oplus \ker Q_F$ provides orthogonal projections $P_{\mathcal{L}}, P_{\ker Q}$ onto $\mathcal{L}$ and $\ker Q_F$. We would further like to split $\ker Q_F$ into cohomology and $\operatorname{Im} Q_F$. A projection P_t onto the latter can be obtained from the decomposition $\mathcal{F} = \mathcal{L}^{\perp} \oplus \operatorname{Im} Q_F$. Unfortunately, it is not generically true that P_t is orthogonal to $P_{\mathcal{L}}$. The necessary and sufficient condition of orthogonality is provided by the following lemma.

Lemma 4. *The projectors $P_t, P_{\mathcal{L}}$ are orthogonal, i.e. $P_t \circ P_{\mathcal{L}} = P_{\mathcal{L}} \circ P_t = 0$, if and only if $\mathcal{L} \subseteq \mathcal{L}^{\perp}$.*

Proof. First of all, since $\operatorname{im} Q_F \subseteq \ker Q_F$, we always have

$$P_{\mathcal{L}} \circ P_{\mathcal{L}^{\perp}} = (1 - P_{\ker Q_F})(1 - P_t) = 1 - P_{\ker Q_F} = P_{\mathcal{L}} \tag{3.59}$$

It follows that $P_{\mathcal{L}} \circ P_t = P_{\mathcal{L}} \circ P_{\mathcal{L}^{\perp}} \circ P_t = 0$ without further assumptions on $\mathcal{L}$. The lemma now follows from the fact that $\mathcal{L} \subseteq \mathcal{L}^{\perp}$ is equivalent to $P_{\mathcal{L}}^{\perp} \circ P_{\mathcal{L}} = P_{\mathcal{L}}$, which is in turn equivalent to $P_t \circ P_{\mathcal{L}} = 0$. □

The condition $L \subseteq L^{\perp}$ can be equivalently written as $\omega|_{\mathcal{L}} = 0$. In symplectic geometry, spaces having this property are called *isotropic*. This is the condition we want to put on $\mathcal{L}$. It is the closest we can get to $\mathcal{L}$ being Lagrangian. Another consequence of L being isotropic is that $\mathcal{L}^{\perp} = \mathcal{L} \oplus (\ker Q_F \cap \mathcal{L}^{\perp})$, which can be achieved by splitting elements in $\mathcal{L}^{\perp}$ according to $\mathcal{F} = \mathcal{L} \oplus \ker Q_F$. It follows that

$$\mathcal{F} = \mathcal{L} \oplus (\mathcal{L}^{\perp} \cap \ker Q_F) \oplus \operatorname{Im} Q_F. \tag{3.60}$$

The space $\mathcal{L}^{\perp} \cap \ker Q_F$ can be reached by first projecting onto $\mathcal{L}^{\perp}$ with $1 - P_t$ and then onto $\ker Q$ with $1 - P_{\mathcal{L}}$. We therefore define a third projector $P_p = (1 - P_{\mathcal{L}})(1 - P_t)$. By orthogonality of $P_{\mathcal{L}}$ and P_t it follows that

$$1 = P_p + P_{\mathcal{L}} + P_t, \tag{3.61}$$

which gives us the split (3.60). Recall that $\mathcal{L}^{\perp} \cap \ker Q_F$ is a representation of the cohomology of Q_F. For this reason we say that P_p projects onto *physical* fields. Likewise, we call the image of P_t the *trivial* fields.

We can use the machinery developed above to obtain a direct sum decomposition of ω on $\mathcal{F}$. We have

$$\omega = \omega_p + \omega'. \tag{3.62}$$

where ω_p is the restriction to $\mathcal{L}^{\perp} \cap \ker Q_F$ and ω' is the restriction to $\mathcal{L} \oplus \operatorname{Im} Q_F$. In fact, this turns out to be true without assuming that $\mathcal{L}$ is isotropic, since $(\mathcal{L} \oplus \operatorname{Im} Q_F)^{\perp} = L^{\perp} \cap \ker Q_F$.[7] The advantage of considering isotropic $\mathcal{L}$ lies in the observation that $\mathcal{L}$ is in fact Lagrangian with respect to ω', which defines a non-degenerate form on $\mathcal{L} \oplus \operatorname{Im} Q_F$. We can therefore think of integration over $\mathcal{L}$ as an ordinary BV integration in the space $\mathcal{L} \oplus \operatorname{Im} Q_F$. Physical field in this case act as background fields. In this sense, the integration defines a map

$$a \mapsto e^{\frac{i}{\hbar} S_p(a)} := \int_{e \in \mathcal{L}} e^{\frac{i}{\hbar}(\omega(e, Qe) + S_I(e+a))}, \tag{3.63}$$

see for example [22], **chapter 5, lemma 2.7.1**. This is in fact a special case of a more general concept called the *BV pushforward* in [19].

Theorem 5 (Linear version of **Theorem 2.9** in [19])**.** *Given a field space $\mathcal{F}$ with symplectic form ω, which is a sum of two symplectic vector spaces, i.e. $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''$ and $\omega = \omega' + \omega''$. Given $\mathcal{L}$, a Lagrangian subspace in $\mathcal{F}''$, then $\mathcal{L}$ defines a map $\int_{\mathcal{L}} : \mathcal{O}(\mathcal{F}) \to \mathcal{O}(\mathcal{F}'')$, such that*

$$\int_{\mathcal{L}} \Delta f = \Delta'' \int_{\mathcal{L}} f, \tag{3.64}$$

where $f \in \mathcal{O}(\mathcal{F})$. Further, given a family of Lagrangian subspaces $\mathcal{L}_t, t \in [0,1]$ and a function f such that $\Delta f = 0$, then

$$\int_{\mathcal{L}_1} f - \int_{\mathcal{L}_0} f = \Delta'' g, \tag{3.65}$$

for some function g on $\mathcal{F}''$.

[7] Any orthogonality operation $\perp$ with respect to a given bilinear form satisfies $\operatorname{span}(A, B)^{\perp} = A^{\perp} \cap B^{\perp}$, where A, B are some linear subspaces.

For the case at hand, the theorem tells us that the integration defines a chain map between the original theory with action W on $\mathcal{F}$, and an action S_p on the space $\mathcal{L}^\perp \cap \ker Q_F$ of on-shell fields. We will see below that this is in fact a quasi-isomorphism. The second statement says that homotopic modifications of the integration cycle $\mathcal{L}$ lead to Δ_p exact deformations of the exponentiated action $e^{i\frac{S_p}{\hbar}}$, where Δ_p is the Laplacian induced by ω_p.

We will now give a homological interpretation of theorem 5. Recall that one condition on $\mathcal{L}$ was that $Q_F|_\mathcal{L}$ is an isomorphism onto $\operatorname{Im} Q_F$. We therefore have an inverse $H : \operatorname{Im} Q_F \to \mathcal{L}$. The map H is necessarily of degree -1. Using the decomposition $\mathcal{F} = \mathcal{L} \oplus \mathcal{L}^\perp \cap \ker Q_F \oplus \operatorname{Im} Q_F$, we can extend H to a map on all of $\mathcal{F}$. By construction, we have

$$Q_F H = P_t, \quad H Q_F = P_\mathcal{L}, \quad H^2 = 0. \tag{3.66}$$

It then follows that

$$\{Q_F, H\} = 1 - P_p, \tag{3.67}$$

i.e. H is a homotopy between P_p and the identity. If we denote by $i : \mathcal{L}^\perp \cap \ker Q_F \to \mathcal{F}$ the inclusion and $p : \mathcal{F} \to \mathcal{L}^\perp \cap \ker Q_F$ the projection, we obtain a homotopy equivalence data,

$$i : (\mathcal{L}^\perp \cap \ker Q_F, 0) \leftrightarrows (\mathcal{F}, Q_F) : p. \tag{3.68}$$

One can check that $p \circ i = 1$, $H \circ i = 0$ and $p \circ H = 0$, so H actually defines a strong deformation retract. Using the homotopy equivalence, we can apply the homological perturbation lemma, where the perturbation is $-i\hbar\Delta + \{S_I, \cdot\}$. In [30], the following was shown.

Theorem 6 (**Theorem 4** in [30])**.** *The perturbed differential δ' is of the form*

$$\delta' = i\hbar\Delta_p - \{S_p, \cdot\}. \tag{3.69}$$

Remark 3.4.2. A priori it is not clear that the homological perturbation lemma produces a differential of the form (3.69), meaning that it is a second order differential operator and that the only second order contribution is given by Δ_p.

The theorem tells us that the theory obtained from the homological perturbation lemma is the same as the theory we got from integrating over $\mathcal{L}$. An equivalent viewpoint is therefore the following. Instead of actual integration, we can replace the notion of perturbative path integral with an application of the perturbation lemma. The perurbative integral is then given by the projection operator $p' : \mathcal{O}(\mathcal{F}) \to \mathcal{O}(\mathcal{L}^\perp \cap \ker Q_F)$ obtained from $p : \mathcal{F} \to \mathcal{L} \cap \ker Q_F$ through the perturbation lemma. This also verifies a statement from earlier. The path integral is indeed a quasi-isomorphism.

In the classical case, we saw that computing the minimal model of a theory amounts to summing over tree level Feynman diagrams. The story here is similar. The vertices of S_p consists of full Feynman amplitudes, including loops. Considering the relation of the homological perturbation to path integrals, this is of course not surprising. Integrating over $\mathcal{L}$ amounts to integrating out *all* off-shell degrees of freedom. The resulting effective action should still reproduce the usual S-matrix elements. But since it has no propagator, the only way to do so is when the vertices itself contain the full S-matrix.

3.4.2. Example: Gauge Fixing Electromagnetism without Trivial Pairs

Let us illustrate the gauge fixing procedure in the case of electromagnetism. The BV extended action has the form

$$S = \int -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A^*_\mu \partial^\mu C =: S_0 + S_1. \tag{3.70}$$

Our goal is to gauge fix this theory and compare it to the action obtained by applying the Faddeev-Popov procedure. The gauge condition in that case is given by an equation $F(A^\mu, x) = 0$. Locally, we can solve this for one of the components of A^μ. Let us assume that that component is A^0. Hence, there is a function $G^0(A^i, x)$, such that

$$F(G^0(A^i, x), A^i(x), x) = 0. \tag{3.71}$$

We can impose this constraint with the gauge fixing fermion

$$\Psi(A_0^*, A^i) = -\int \mathrm{d}^4 y A_0^*(x)\, G^0(A^i, x). \tag{3.72}$$

It imposes

$$A^0(x) = -\frac{\delta\Psi}{\delta A_0^*} = G^0(A^i, x), \quad A_j^*(x) = -\int \mathrm{d}^4 y A_0^*(y) \frac{\delta G^0(A^i, y)}{\delta A^j(x)}. \tag{3.73}$$

We find that

$$S_1 = -\int \mathrm{d}^4 x A_0^*(x) \partial^0 C(x) + \int \mathrm{d}^4 y\, \mathrm{d}^4 x A_0^*(y) \frac{\delta G^0(A^i, y)}{\delta A^j(x)} \partial^j C(x). \tag{3.74}$$

We want to rewrite this in terms of F. We can differentiate (3.71) with respect to A^i. We find

$$0 = \int \mathrm{d}^4 y \frac{\delta F(G^0, A^i, x)}{\delta A^0(y)} \frac{\delta G^0(A^i, y)}{\delta A^i(z)} + \frac{\delta F(G^0, A^i, x)}{\delta A^i(z)}. \tag{3.75}$$

We define a new field variable $B(x)$ of ghost number -1 through $A_0^*(x) = \int \mathrm{d}^4 y B(y) \frac{\delta F(G^0, A^i, y)}{\delta A^0(x)}$. This allows us to use (3.75) in S_1. We find

$$S_1|_{gf} = -\int \mathrm{d}^4 y \mathrm{d}^4 x B(y) \frac{\delta F(G^0, A^i, y)}{\delta A^0(x)} \partial^0 C(y) - \int \mathrm{d}^4 y \mathrm{d}^4 x B(y) \frac{\delta F(G^0, A^i, y)}{\delta A^j(x)} \partial^j C(x). \tag{3.76}$$

This has the form exactly as one finds in the Faddeev Popov approach. A more familiar form may be

$$S_1|_{gf} = -\int \mathrm{d}^4 y B(y) Q(F)(y), \tag{3.77}$$

where Q is the BRST operator.

3.5. An Anomaly Computation

We conclude this chapter by giving an example for an anomaly computation. This will also serve as some sort of introduction to string field theory, since we use a regularization methods which draws heavily from the picture of geometric vertices in string field theory.

We recall the notion of an anomaly in field theory. We say that a theory has an anomaly, if the theory has a symmetry at the classical level that is broken at the quantum level. In the path integral formalism, the source of anomalies is a nonsymmetric measure. Such a measure is usually defined through a certain regularization scheme. If the regularization scheme preserves gauge invariance, anomalies cannot arise. For this reason, the vector symmetry of fermions coupled to a gauge field can always be defined without having an anomaly, since we can apply either dimensional or Pauli-Villars regularization, both preserving the vector symmetry. On the other hand, a chiral symmetry potentially has a anomaly. Neither of the two schemes preserves this symmetry. Beside the chiral anomalies there can also be a gravitational anomaly, since no known regularization scheme preserves diffeomorphism invariance. The conformal anomaly in $D \neq 26$ bosonic string theory is an example of this.

We need to distinguish between anomalies of global symmetries and gauge symmetries. The former are harmless. They result in physical processes at the quantum level excluded at the classical level. For example, the neutral pion decays through a process forbidden at the classical level. On the other hand, also gauge symmetries can suffer from anomalies, which makes the theory inconsistent. This happens when only left-handed (equivalently only right-handed) fermions couple to the gauge fields. In this case, one explicitly needs to check that there is no anomaly is order to have a consistent theory, which fortunately is the case in the theory of weak interaction.

A very simple theory with a gauge anomaly is the chiral Schwinger model. This is chiral quantum electrodynamics in two dimensions. It is given by the action

$$S_0 = \int \sqrt{2}\bar{\psi}(i\partial_- + A_-)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{3.78}$$

The field $\bar{\psi}$ denotes the complex conjugate of ψ. Also, we use lightcone coordinates $x^\pm = \frac{1}{\sqrt{2}}(t \pm x)$. In the BV formalism, we account for the gauge symmetry by adding

$$S_1 = \int ic\bar{\psi}^*\bar{\psi} - ic\psi^*\psi - A^*_\mu\partial^\mu c. \tag{3.79}$$

As it is now, the action of the BV Laplacian is not defined on S_1. We can fix this by considering a new action S', which is equivalent to $S_0 + S_1$ on the classical level. It can be constructed with the help of a homotopy with respect to the linear differential generated by $S_0 + S_1$. The linear part reads

$$Q_0 = \{S^{(2)}, \cdot\} = \int_x \sqrt{2}i\partial_-\psi(x)\frac{\delta}{\delta\bar{\psi}^*(x)} + \sqrt{2}i\partial_-\bar{\psi}(x)\frac{\delta}{\delta\psi^*(x)} + \ldots, \tag{3.80}$$

where we omitted everything that only involves the gauge field and the ghost, since they will not be important for us. The reason is that the anomaly is completely due to the action of the Laplacian on fermions. One way to regularize is through the introduction of a heat kernel. This effectively means that we do a replacement

$$\psi \mapsto K\psi := e^{-\tau_0\Box}\psi, \tag{3.81}$$

and similarly for the complex conjugate and all the anti-fields. This can be justified by noting that $e^{-\tau_0\Box}$ is homotopic to the identity operator. The corresponding homotopy is

$$H = -i\sqrt{2}(\int_x\int_0^{\tau_0} \bar{\psi}^*(x)e^{-\tau\Box}\partial_+\frac{\delta}{\delta\psi(x)} + \psi^*(x)e^{-\tau\Box}\partial_+\frac{\delta}{\delta\bar{\psi}(x)}). \tag{3.82}$$

One then computes

$$\{Q_0, H\} = \int_x \psi(x)\frac{\delta}{\delta\psi(x)} - \int_x \psi(x)e^{-\tau_0\Box}\frac{\delta}{\delta\psi(x)} + \ldots \tag{3.83}$$

where the dots contain three more contributions which contain are obtained by replacing $\psi \mapsto (\psi^*, \bar{\psi}, \bar{\psi}^*)$. The operator $\int_x \psi(x)\frac{\delta}{\delta\psi(x)}$ is the identity on field space, while $\int_x \psi(x)e^{-\tau_0\Box}\frac{\delta}{\delta\psi(x)}$ is of course the heat kernel. An equivalent relation is the following. Write

$$P(x,y) = -i\sqrt{2}\int_0^{\tau_0} e^{-\tau\Box}\partial_+\delta(x-y) \tag{3.84}$$

and

$$K(x,y) = e^{-\tau_0\Box}\delta(x-y). \tag{3.85}$$

Then,

$$\sqrt{2}i\partial_{x^-}P(x,y) = \delta(x-y) - K(x,y). \tag{3.86}$$

Denote by V the field space of the spinor fields. We denote the homotopy equivalence data by

$$i : (V, Q_0) \leftrightarrows (V, Q_0) : p. \tag{3.87}$$

There is some freedom in choosing p and i. All we need is that $i \circ p = K$. We make a symmetric choice. We define

$$K_{1/2}\psi := e^{-\frac{\tau_0}{2}\Box}\psi \tag{3.88}$$

and pick $p = i = K_{1/2}$. Similarly, we denote by $K_{1/2}(x,y)$ the integral kernel of $K_{1/2}$.

The cohomological vector field $\{S_0+S_1, \cdot\}$ defines an L_∞-algebra. The homotopy transfer theorem provides a new L_∞-structure related to the original one by the homotopy H. It is obtained by constructing Feynman diagrams with propagator H and external legs $K_{1/2}$. To cubic order, the new action S' is given by

$$S'_{\text{cubic}} = \sqrt{2}\bar{\psi}(z)K_{1/2}(z,y)A_-(y)K_{1/2}(y,x)\psi(x) \tag{3.89}$$

$$-i\psi^*(z)K_{1/2}(z,y)c(y)K_{1/2}(y,x)\psi(x) \tag{3.90}$$

$$-i\bar{\psi}(z)K_{1/2}(z,y)c(y)K_{1/2}(y,x)\bar{\psi}^*(x). \tag{3.91}$$

The above expression uses Einstein - de Witt integration convention, which we will use from now on. If we had a homotopy also affecting the ghost and gauge field, then there would be a similar external leg contribution for these fields. Unless $K_{1/2} = 1$, the above expression gives a non-vanishing contribution to the master equation,

$$\{S'_{\text{cubic}}, S'_{\text{cubic}}\} = -\sqrt{8}i\bar{\psi}(u)K_{1/2}(u,w)c(w)K(w,y)A_-(y)K_{1/2}(y,x)\psi(x) \tag{3.92}$$

$$+\sqrt{8}i\bar{\psi}(u)K_{1/2}(u,w)A_-(w)K(w,y)c(y)K_{1/2}(y,x)\psi(x) \tag{3.93}$$

$$+2\psi^*(u)K_{1/2}(u,w)c(w)K(w,y)c(y)K_{1/2}(y,x)\psi(x) \tag{3.94}$$

$$+2\bar{\psi}^*(u)K_{1/2}(u,w)c(w)K(w,y)c(y)K_{1/2}(y,x)\bar{\psi}(x). \tag{3.95}$$

The induced quartic interaction exactly cancels this. It is

$$S'_{\text{quartic}} = -(\sqrt{2}\bar{\psi}(u)K_{1/2}(u,w)A_-(w) + i\psi^*(u)K_{1/2}(u,w)c(w)) \tag{3.96}$$

$$P(w,y)(\sqrt{2}A_-(y)K_{1/2}(y,x)\psi(x) + ic(y)K_{1/2}(y,x)\bar{\psi}^*(x)). \tag{3.97}$$

This can be checked by noting the following. First of all, we have that

$$Q_0(\sqrt{2}\bar{\psi}(u)K_{1/2}(u,w)A_-(w) + i\psi^*(u)K_{1/2}(u,w)c(w)) \tag{3.98}$$

$$= -\sqrt{2}\partial_{w^-}(\bar{\psi}(u)K_{1/2}(u,w)c(w)), \tag{3.99}$$

$$Q_0(\sqrt{2}A_-(y)K_{1/2}(y,x)\psi(x) + ic(y)K_{1/2}(y,x)\bar{\psi}^*(x)) \tag{3.100}$$
$$= \sqrt{2}\partial_{y^-}(c(y)K_{1/2}(y,x)\psi(x)). \tag{3.101}$$

It follows that

$$Q_0 S'_{\text{quartic}} = -\sqrt{2}\bar{\psi}(u)K_{1/2}(u,w)c(w)\partial_{w^-}P(w,y)(\sqrt{2}A_-(y)K_{1/2}(y,x)\psi(x) \tag{3.102}$$
$$+ic(y)K_{1/2}(y,x)\bar{\psi}^*(x)) - \sqrt{2}(\sqrt{2}\bar{\psi}(u)K_{1/2}(u,w)A_-(w) \tag{3.103}$$
$$+i\psi^*(u)K_{1/2}(u,w)c(w))\partial_{y^-}P(w,y)c(y)K_{1/2}(y,x)\psi(x) \tag{3.104}$$

We can now use (3.86) and obtain

$$Q_0 S'_{\text{quartic}} = -i\bar{\psi}(u)K_{1/2}(u,w)c(w)K(w,y)(\sqrt{2}A_-(y)K_{1/2}(y,x)\psi(x)+ \tag{3.105}$$
$$ic(y)K_{1/2}(y,x)\bar{\psi}^*(x)) + i(\sqrt{2}\bar{\psi}(u)K_{1/2}(u,w)A_-(w) \tag{3.106}$$
$$+i\psi^*(u)K_{1/2}(u,w)c(w))K(w,y)c(y)K_{1/2}(y,x)\psi(x). \tag{3.107}$$

One can check that

$$2Q_0 S'_{\text{quartic}} + \{S'_{\text{cubic}}, S'_{\text{cubic}}\} = 0. \tag{3.108}$$

Hence, the quartic vertex exactly cancels the cubic vertex, as it should according to the homotopy transfer theorem.

We can now move on and compute $\Delta S'$ in cubic and quartic order. We find

$$\Delta S'_{\text{cubic}} = -iK_{1/2}(x,y)c(y)K_{1/2}(y,x) + iK_{1/2}(x,y)c(y)K_{1/2}(y,x) = 0 \tag{3.109}$$

and

$$\Delta S'_{\text{quartic}} = -i\sqrt{2}K_{1/2}(x,w)A_-(w)P(w,y)c(y)K_{1/2}(y,x) \tag{3.110}$$
$$+i\sqrt{2}K_{1/2}(x,w)c(w)P(w,y)A_-(y)K_{1/2}(y,x) = i\sqrt{8}K(y,w)c(w)P(w,y)A_-(y). \tag{3.111}$$

We used that $K_{1/2}(x,y)K_{1/2}(x,z) = K(x,z)$ and $P(x,y) = -P(y,x)$. This is of course a τ_0 dependent expression. Let us see whether there is a universal part. We consider the limit $\tau_0 \to 0$. We have that

$$P(x,y) = \sqrt{2}i\tau_0\partial_{x^+}\delta(x-y) + \mathcal{O}(\tau_0^2). \tag{3.112}$$

Also,

$$K(x,y) = \frac{1}{4\pi i\tau_0}e^{-\frac{(x-y)^2}{4\tau_0}}. \tag{3.113}$$

It follows that

$$\Delta S'_{\text{quartic}} = \frac{i}{\pi}\int_x c(x)\partial_+ A_-(x). \tag{3.114}$$

3.5.1. Canceling the Anomaly by Introducing New Particles

The fact that $\Delta S \neq 0$ does not necessarily imply that the theory has an anomaly. In principle, it may happen that the we can introduce additional $\hbar$-dependent terms to S that cancel ΔS through the classical part $\{S,S\}$ of the master equation. If we would allow for non-local term, we could even cancel the chiral anomaly. Consider

$$S_{nl} = \frac{\hbar}{2\pi}\int_x A_-(x)\frac{\partial_+}{\partial_-}A_-(x). \tag{3.115}$$

Then, the gauge transformation of S_{nl} exactly cancels the anomaly in the master equation.

Non-local terms in quantum field theories can be the result of integrating out fields. So we may hope that integrating in new fields may cure the non-locality. Indeed, as it is pointed out in [6], a Wess-Zumino term can cancel this. Let us introduce a scalar particle θ with the usual kinetic term

$$S_{\theta,kin} = \int_x \frac{1}{2}(\partial_\mu \theta(x))^2. \tag{3.116}$$

We couple it linearly to A_- via

$$S_{\theta,int} = \sqrt{\frac{2\hbar}{\pi}} \int_x \theta(x)\partial_+ A_-(x). \tag{3.117}$$

Integrating out θ yields (3.115).

Introducing the new scalar particle has destroyed gauge invariance already at the classical level. We can cure this by demanding that θ transforms as $\delta\theta = \sqrt{\frac{\hbar}{2\pi}}c$. With this choice, the full quantum master equation is satisfied. The transformation of θ also cancels the anomaly. This example is a baby version of the Green-Schwarz mechanism in string theory. In this picture, the gauge field A_- is the open string and the scalar field θ is the closed string. What we saw above already reveals many features of the Green-Schwarz mechanism. For example, the one-loop anomaly is canceled by introducing an interaction at tree level. Also, the appearance of the factor $\sqrt{\hbar}$ is very similar to string theory. In the latter case, the interaction is proportional to $\hbar$.[8]

Remark 3.5.1. That the interaction between A and θ is linear is of course due to the fact that the original anomaly ΔS only contains a single power of the gauge field. For every extra two dimensions, the anomaly gets another power of A. In that process, the interaction between A and θ becomes more and more non-linear. In the case of the superstring, which lives in ten dimensional spacetime, the closed string interacts with both two and with four open strings [48].

3.6. From Quantum BV to Quantum Homotopy Algebras

In section 2.5 we explored the relation between classical BV theory and homotopy algebras. This correspondence carries over at the quantum level. The BV-Laplacian is essentially the only difference between the classical and the quantum theory. We need a similar operation for homotopy algebras. In terms of diagrams, this operator should create loops and in some sense be determined by the sypmlectic structure ω.

3.6.1. Higher Order Coderivations over Commutative Coalgebras

A quantum structure on L_∞-algebras can be introduced by extension of the bar construction. This is described in reference [68]. Recall that a (classical) L_∞-structure is defined as a degree one coderivation on the symmetric tensor coalgebra $S^c(V)$ of a vector space V. We will henceforth call ordinary coderivations to be of order one to make room for higher order

[8] In string theory, there are additional factors of $\hbar$. For example, a closed string loop contributes two powers of $\hbar$, while the open string has only one power. On the other hand, open strings already interact with a power of $\hbar$ at tree level. What we observe in our point particle example may be thought of as the square root of the stringy case.

coderivations. But before that, let us first consider the more familiar case of higher order derivations. For commutative algebras, one approach is due to Koszul.

Definition 3.6.1. Given a **k**-linear endomorphism $\delta : A \to A$ on some graded commutative **k**-algebra A, we can recursively define multilinear maps $K_i : A^{\otimes i} \to A$ through

$$\begin{aligned} K_1(\delta)(f) &= \delta(f), \\ K_{i+1}(\delta)(f_0, ...f_{i-1}, f_i) &= K_i(\delta)(f_0, ..., f_{i-2}, f_{i-1}f_i) - K_i(\delta)(f_0, ..., f_{i-2}, f_{i-1})f_i \\ &\quad - (-)^{f_{i-1}\delta} f_{i-1} K_i(\delta)(f_0, ..., f_{i-2}, f_i). \end{aligned} \tag{3.118}$$

We call the multilinear maps K_i Koszul braces. One says that δ is a derivation of order i if $K_{i+1}(\delta) = 0$. We denote the space of order k derivations over A by $\mathrm{Der}_k(A)$. We also define $\mathrm{Der}_l(A) = 0$ if $l < 0$.

Remark 3.6.1. The map K_{i+1} measures the failure of K_i to act as a derivation in any of its entries. When A has a unit, derivations in this sense necessarily satisfy $\delta(1) = 0$.

Proposition 1. *The following is true on any commutative algebra A.*

1. *$Der_k(A) \subseteq Der_{k+1}(A)$ for all $k \geq 0$.*
2. *$Der_k(A) \circ Der_l(A) \subseteq Der_{k+l}(A)$.*
3. *$[Der_k(A), Der_l(A)] \subseteq Der_{k+l-1}(A)$.*

Proof. See [1]. □

The notion of order k coderivation is obtained by dualizing order k derivations. Our main interest lies in coderivations on the symmetric tensor algebra $S^c(V)$. In that case, an order n coderivation is determined by its image in $\sum_{1 \leq k \leq n} V^{\odot k}$.

Proposition 2. *Given a linear map $f : S^c(V) \to \sum_{1 \leq k \leq n} V^{\odot k}$, denote by $\tilde{f} : S^c(V) \to S^c(V)$ its extension by the inclusion $\sum_{1 \leq k \leq n} V^{\odot k} \hookrightarrow S^c(V)$. Define*

$$D(f) = \nabla_2 \circ (\tilde{f} \otimes \mathbb{1}) \circ \Delta_2. \tag{3.119}$$

Here, ∇_2 is the product and Δ_2 is the coproduct on the symmetric tensor algebra. Then, $D(f)$ defines an order k coderivation.

Proof. **Proposition 3.4** in [68]. The way we write the formula for $D(f)$ is taken from **Appendix A.2** in [73]. □

Remark 3.6.2. The above proposition is a generalization of the fact that order one coderivations are determined by its image in $V \subseteq S^c(V)$.

There exists also a generalization, which is due to Grothendieck (c.f. [2]). Given a commutative algebra A, we define (left-)multiplication L_v by elements $v \in A$ to be of order zero. We then say that a linear map $\delta : A \to A$ is of order k, if $[\delta, v_k]$ is of order $k - 1$. One drawback is that order one derivations no longer satisfy Leibniz rule. Instead they satisfy the more general identity

$$\delta(ab) = \delta(a)b + (-)^a a\delta(b) - (-)^a a\delta(1)b. \tag{3.120}$$

Note that this is necessary since a order one derivation no longer needs tp satisfy $\delta(1) = 0$, so consistency demands the extra term in (3.120). Given a derivation δ of any order in the Grothendieck sense, we obtain a derivation of the same order in the Koszul sense by redefining $\delta \mapsto \delta - L_{\delta(1)}$.

3.6.2. Quantum Homotopy Lie Algebras

We now come to the definition of quantum homotopy Lie algebras in its most general form. They are obtained by allowing higher order coderivations. We want to introduce $\hbar$ to measure the quantumness of the coderivations. The rule is that, the higher the order of a coderivation, the more it raises the order of $\hbar$. We first introduce $\hbar$ to the tensor coalgebra by defining $S^c(V)[[\hbar]] := S^c(V) \otimes \mathbf{k}[[\hbar]]$. We then write

$$\mathrm{Coder}_\hbar(S^c(V)[[\hbar]]) := \prod_{n\geq 1} \hbar^{n+1}\,\mathrm{Coder}_n(S^c(V)). \tag{3.121}$$

Definition 3.6.2. A quantum homotopy Lie algebra on a vector space V is defined by a degree one element

$$D \in \mathrm{Coder}_\hbar(S^c(V)[[\hbar]]), \tag{3.122}$$

such that $D^2 = 0$ and $D(\hbar = 0)|_{V^{\otimes 0}} = 0$. Algebras of this type are also called IBL_∞ algebras, see [72].

Definition 3.6.3. An IBL_∞ morphism F is determined by some linear function $f \in \mathrm{Lin}(S^c(V), \prod_{n\geq 1}\hbar^{n-1}V^{\odot n})$ via

$$F = \sum_{n\geq 1} \frac{1}{n!}\nabla_n \circ f^{\otimes n} \circ \Delta_n. \tag{3.123}$$

Here, ∇_n and Δ_n denote the n-ary multiplication and comultiplication.

One may wonder why we should even bother introducing $\hbar$ as a parameter. There are several reasons. First of all, the $\hbar = 0$ part of the algebra defines an ordinary L_∞-algebra. Hence, by keeping track of $\hbar$, we can always reduce to the classical part of the quantum theory. One may conjecture that we could simply take the order one part of D to obtain the classical part. However, quantum effects generally introduce order one coderivations, that are nevertheless proportional to powers of $\hbar$. This is not in contradiction with (3.121), since any order one coderivation is also a coderivation of any higher order. Another reason to keep $\hbar$ is that it introduces some additional structure. The problem is that almost any linear map on $S^c(V)$ can be expanded in terms of an infinite sum of coderivations of any order. To see why, let $f : S^c(V) \to S^c(V)$ be any linear map such that $\mathrm{Im}(f) \cap V^0 = 0$. Take the part of f whose image lies in V and lift it to an order one derivation $D_1(f)$. We then take $f - D_1(f)$, identify its part lying in $V^{\odot 2}$ and lift it to an order two coderivation. Continuing this indefinitely allows us to write

$$f = \sum_{k\geq 1} D_k(f). \tag{3.124}$$

The limit is taken pointwise. Therefore, in the definition of quantum homotopy algebras we could just take a degree one linear map D such that $D^2 = 0$, without even bothering about the notion of coderivations. A third reason to keep $\hbar$ is to have a better control over convergence of infinite sums. We say that a sequence in $S^c(V)[[\hbar]]$ converges, if the sequence converges in each fixed order $\hbar$. This notion permits tadpoles as long as they are at least proportional to $\hbar$. This is the origin of the condition $D(\hbar = 0)|_{V^{\otimes 0}} = 0$.

The BV Laplacian is introduced to homotopy algebras as a second order coderivation. Recall that classical BV theory induces an L_∞ structure on the tangent space $T_{\phi_0}\mathcal{F}$ at a classical solution to the equations of motion. Quantum effects can add $\hbar$ dependent terms to it (this may destroy the classical L_∞-relations). Further, the symplectic structure on $\mathcal{F}$ induces a linear symplectic structure ω on $T_{\phi_0}\mathcal{F}$. Let $\omega^{-1} = e_i^* \odot e^i$ be its inverse (in the sense of section 2.3.7). This in turn induces a map

$$\omega^{-1} : S^c(T_{\phi_0}\mathcal{F}) \longrightarrow T_{\phi_0}\mathcal{F} \odot T_{\phi_0}\mathcal{F}, \tag{3.125}$$

mapping numbers $\lambda \in \mathbb{C}$ to $\lambda e_i^* \odot e^i$, and all other elements to zero. ω^{-1} now lifts to a second order coderivation

$$\theta := \hbar D(\omega^{-1}). \tag{3.126}$$

The action of θ is actually rather simple to describe. It acts via left-multiplication by the element $\hbar e_i^* \odot e^i$. Since we work with a graded commutative algebra and $e_i^* \odot e^i$ is of degree one, it immediately follows that $\theta^2 = 0$. We combine this with the linear differential Q and the degree one coderivation D coming from the vertices to $D_q = Q + D(\hbar) - i\hbar\theta$. It defines a quantum L_∞-structure if D_q squares to zero. Since D_q is a coderivation of order two, $2D_q^2 = [D_q, D_q]$ is of order three, and is therefore determined by its image in

$$T_{\phi_0}\mathcal{F} \oplus (T_{\phi_0}\mathcal{F})^{\odot 2} \oplus (T_{\phi_0}\mathcal{F})^{\odot 3}. \tag{3.127}$$

The following can be shown [68].

Proposition 3. *Let $D_q = Q + D(\hbar) - i\hbar\theta$ as above and such that $D_q^2 = 0$. We have the following.*

1. *D_q^2 is always zero in $(T_{\phi_0}\mathcal{F})^{\odot 3}$.*
2. *The vanishing of D_q^2 in $(T_{\phi_0}\mathcal{F})^{\odot 2}$ is equivalent to cyclicity of $d + D_q$.*

By the above proposition, arguably the most non-trivial part to show is that $D_q^2 = 0$ in $T_{\phi_0}\mathcal{F}$. This puts a condition on $D(\hbar)$ extending the classical L_∞-relations $D(\hbar = 0)^2 = 0$. The condition on $D(\hbar)$ is the equivalent of the quantum master equation. This suggests another, more restrictive definition of quantum homotopy Lie algebras, called *loop homotopy Lie algebras* in [68].[9]

Definition 3.6.4. A loop homotopy Lie algebra on a odd symplectic differential graded vector space $(V[[\hbar]], \mathrm{d}, \omega)$ is a cyclic degree one and order one coderivation $D(\hbar)$ such that

$$(\mathrm{d} + D(\hbar) - i\hbar\theta)^2 = 0 \tag{3.128}$$

and $D(\hbar = 0)|_{V^{\otimes 0}} = 0$.

It can be shown that any loop homotopy Lie algebra induces a loop homotopy Lie algebra on cohomology via an IBL_∞-morphism, see [73]. The algebra maps are the full quantum S-matrices. The IBL_∞-morphism is the equivalent of the perturbative path integral (3.63).

[9]The author does not know whether this distinction is standard in the literature.

3.6.3. Quantum Homotopy Associative Algebras

We will now review the quantum version of homotopy associative algebras. Later in this work, we will try to construct a planar version of it.

In terms of operads, loop homotopy associative algebras were defined in [29]. By reformulating it as a BV theory on a linear field space, the relations among the products can be stated in the form of a quantum master equation. We recall the basic construction. Let $\sigma_n = (12...n) \in S_n$ denote the cyclic permutation of n elements. We define $C^n(V) = V^{\otimes n}/(x \sim \sigma_n x)$ and $C(V) = \bigoplus_{n\geq 0} C^n(V)$. Vertices are degree zero functions

$$f \in \prod_{k\geq 1} \hbar^{k-1}\mathrm{lin}(C(V)^{\odot k}, \mathbf{k}). \tag{3.129}$$

To define the quantum A_∞-relations, we need a symplectic structure ω. Denote by $\omega^{-1} = e_i^* \otimes e^i$ its inverse in the sense described in section 2.3.7. We use it to define two types of operations. Given two functions $f_1 \in \mathrm{lin}(C^k(V), \mathbf{k})$, $f_2 \in \mathrm{lin}(C^l(V), \mathbf{k})$, we define $\{f_1, f_2\} \in \mathrm{lin}(C^{k+l-1}(V), \mathbf{k})$ by

$$\{f_1, f_2\}(a_1, ..., a_{k+l-1}) = \sum_\sigma \pm f_1(a_{\sigma(1)}, ..., a_{\sigma(l-1)}, e_i^*) f_2(e^i, a_{\sigma(l)}, ..., a_{\sigma(k+l-1)}), \tag{3.130}$$

where the sum runs over all cyclic permutations σ of length $k+l-1$. The sign is the usual sign determined by the permutation of graded objects. The second operation is

$$\tilde{\Delta} : \mathrm{lin}(C(V), \mathbf{k}) \to \mathrm{lin}(C(V), \mathbf{k}) \odot \mathrm{lin}(C(V), \mathbf{k}), \tag{3.131}$$

where

$$\tilde{\Delta}(f)(a_1, ..., a_k)(b_1, ..., b_l) \tag{3.132}$$

$$= \sum_{\sigma_k, \sigma_l} -(-)^{\epsilon + e^i(1+a_1+...+a_l)} f(e_i^* a_{\sigma_k(1)}, ..., a_{\sigma_k(k)}, e^i, b_{\sigma_l(1)}, ..., b_{\sigma_l(l)}). \tag{3.133}$$

σ_k, σ_l denote the cyclic permutations as before, and $(-)^\epsilon$ is the sign produced by the permutation. By cyclic symmetry of f, this product is invariant under $(a_1, ..., a_k) \leftrightarrow (b_1, ..., b_l)$, as it should be.

We extend the bracket $\{\cdot, \cdot\}$ to general elements by demanding that it satisfies Leibniz rule with respect to the symmetric product. On the other hand, we inductively define

$$\Delta(fg) = \Delta(fg) + (-)^f f\Delta(g) + (-)^f \{f, g\}, \quad \Delta(f) = \tilde{\Delta}(f) \text{ when } f \in \mathrm{lin}(C(V), \mathbf{k}). \tag{3.134}$$

This turns $\mathrm{lin}(\bigoplus_{n\geq 0} C(V)^{\odot n}, k)$ into a BV algebra. The loop A_∞-relations on a differential graded vector space (V, d) are then encoded in the quantum master equation

$$\mathrm{d}f + \frac{1}{2}\{f, f\} - i\hbar\Delta(f) = 0. \tag{3.135}$$

It would be nice to have a construction of quantum A_∞-algebras that is similar to the bar construction of the quantum L_∞-algebras. This would require a good notion of higher order coderivations on non-commutative algebras. In section 5.4 we will give one definition and show that it naturally leads to an algebra of planar graphs. For the full non-planar case this does not suffice. On the other hand, it is quite obvious how to define the non-commutative equivalent of θ. Recall that θ acted on $S^c(V)$ by left multiplication. We can do something similar on $T^c(V)$. However, the correct object is not the left multiplication of the free tensor algebra, but rather a multiplication by shuffling in $e^i \otimes e_i^*$ into elements of the form $a_1 \otimes \cdots \otimes a_n$. The shuffle product is actually commutative. It then immediately follows that $\theta^2 = 0$, since θ is odd. We would then define a loop A_∞-algebra to satisfy

$$(\mathrm{d} + D(\hbar) - i\hbar\theta)^2 = 0, \tag{3.136}$$

where $\mathrm{d} + D(\hbar)$ is an ordinary coderivation of order one with respect to the coproduct on the free tensor algebra $T^c(V)$.

The shuffle product is in fact special with respect to the free tensor coalgebra $T^c(V)$. The shuffle product and the free coproduct combine to form a bialgebra. This statement can be found for example in [49], **Example 1.3.11** (this example states the dual version). This property may be a hint that the definition of θ is indeed correct. This would also be in agreement to what we find for quantum L_∞-algebras. θ was defined by symmetric multiplication, and the latter combines with symmetric comultiplication into a bialgebra.

4. Geometric Vertices and String Field Theory

In this section we want to answer the first question posed in the introduction. Is there a quantum theory of open strings without the closed string? We begin by introducing Riemann surfaces and moduli spaces, and how moduli spaces can be partitioned in terms of vertices and propagators. The consistency of the partition is then stated in terms of a quantum BV master equation. We then review how to obtain a quantum theory of strings from this data. Then, in order to answer our original question, we ask whether there are partitions of the moduli spaces of open-string diagrams without the necessity to rely on closed strings.

4.1. Unoriented Riemann Surfaces with Boundary

Let us first recall the definition of a Riemann surface. It is a two dimensional compact manifold Σ, with a choice of local coordinates $z_i : U_i \longrightarrow \mathbb{C}$, such that the transition functions

$$z_j \circ z_i^{-1} : z_i(U_i \cap U_j) \longrightarrow \mathbb{C} \tag{4.1}$$

are holomorphic. A function $f : \Sigma_1 \to \Sigma_2$ is holomorphic, if it is so in every chart. If f is invertible, Σ_1 and Σ_2 isomorphic as Riemann surfaces. We will also consider surfaces with boundary, which can be obtained by removing open discs from a given surface. On a boundary, the coordinates are adapted so that they map into the upper half-plane, with boundary points mapped to the real line.

Holomorphic transition functions automatically lead to oriented surfaces. To include non-orientable surfaces, we should also allow for purely anti-holomorphic transition functions. A simple example for such a surface is the Klein bottle, which can be obtained by considering the quotient of the complex plane modulo the relations

$$z \sim z + i, \quad z \sim \bar{z} + 1. \tag{4.2}$$

The second identification forces some of the transition functions to be anti-holomorphic. An automorphism is then a homeomorphism f, which is either holomorphic or anti-holomorphic when expressed in local coordinates.

The next step is to allow punctures/marked points on a Riemann surface. This is an object

$$(\Sigma, f, \{s_1, ..., s_b\}), \tag{4.3}$$

where Σ is a Riemann surface. Define $[n] := \{1, ..., n\}$. f is an injective map $f : [n] \to \Sigma - \partial\Sigma$ labeling n distinct points in the interior of Σ. Likewise, each s_i is an injective map $s_i : [m_i] \to \partial\Sigma$, such that each s_i maps into a different boundary component. Furthermore, we demand that the labels are cyclically ordered with respect to an arbitrarily chosen orientation on the boundary (see figure 4.4).[1] Given an isomorphism $\phi : \Sigma \to \phi(\Sigma)$ of Riemann surfaces, we consider $(\Sigma, f, \{s_1, ..., s_b\})$ and $(\phi(\Sigma), \phi \circ f, \{\phi \circ s_1, ..., \phi \circ s_b\})$ to be isomorphic.

To decide whether two Riemann surfaces are isomorphic, one can first check whether their topologies agree. A well known result in the theory of topological manifolds is that any compact topological surface can be build by taking a sphere and attaching g handles and c crosscaps to it. Concretely, attaching handles can be done by cutting two holes into a surface and join them with a cylinder. Crosscaps are created by cutting a single hole and attaching a Möbius strip to it. Further, boundaries are created by cutting b additional holes. Hence, a surface Σ can be characterized by the three numbers (g, c, b). These numbers are, however, not unique. When $g, c \geq 1$, there is an isomorphism $(g, c, b) \sim (g-1, c+2, b)$. Two obtain a unique classification, one can for example restrict to $c \leq 2$. We can further add the marked points to the topological data. A surface is then uniquely characterized by $(g, c, b, n, \{m_1, ..., m_b\})$.

Another advantage of the topological classification is that it is related to the dimension of the moduli space $\mathcal{M}_{g,b,c}^{n,\{m_i\}}$, the space of complex structures (marked Riemann surfaces up to isomorphisms). Its real dimension is

$$6g - 6 + 3b + 3c + 2n + m, \tag{4.4}$$

whenever this number is bigger than zero. In the remaining exceptional cases, this number merely states the dimension (moduli) minus the number of independent conformal Killing vector fields (CKV). For completeness, let us state what happens in these cases (see [101, 28]).

- The sphere $(g = 0, b = 0, c = 0)$ with $n \leq 3$ closed string punctures has $6 - 2n$ CKVs.
- The disc $(g = 0, b = 1, c = 0)$ and the projective plane $(g = 0, b = 0, c = 1)$ with $m \leq 3$ open string punctures have $3 - m$ CKVs.
- The disc and the projective plane with one closed string puncture and $m \leq 1$ open string punctures have $1 - m$ CKVs.
- The torus $(g = 1, b = 0, c = 0)$ with no punctures has 2 CKVs and 2 moduli.
- The annulus $(g = 0, b = 2, c = 0)$, the Möbius strip $(g = 0, b = 1, c = 1)$ and the Klein bottle $(g = 0, b = 0, c = 2)$ with no punctures have 1 CKV and 1 modulus.

4.2. Local Parametrizations

In this section, we introduce local parametrizations around punctures. Local parametrizations will allow us to sew Riemann surfaces. Furthermore, these will tell us how conformally dependent operators enter into correlators in string field theory.

Define

$$D = \{z \in \mathbb{C} \,|\, |z| \leq 1\}, \quad D_H = D \cap \{z \in \mathbb{C} \,|\, \mathrm{Im}\, z \geq 0\}, \tag{4.5}$$

[1] If we would restrict to orientable surfaces only, we could demand that the cylic order is with respect to the induced orientation on the boundary. Of course, for unorientable surfaces, we don't have such an induced orientation.

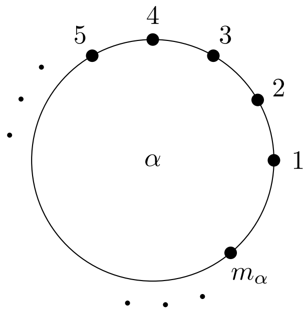


Figure 4.1.: Cyclic ordering of open string punctures on a boundary labeled by α.

the closed unit disc and the closed unit half-disc respectively. Fix a punctured Riemann surface $(\Sigma, f, \{s_1, ..., s_b\})$. Local parametrizations are basically an extension of the maps f and s_i specifying the punctures. By that we mean that we have analytic embeddings

$$F : D \times [n] \longrightarrow \Sigma, \quad S_i : D_H \times [m_i] \longrightarrow \Sigma, \tag{4.6}$$

such that $F|_{0\times[n]} = f$ and $S_i|_{0\times[m_i]} = s_i$. The S_i should map the real lines to the boundary of Σ, and in such a way that the boundary orientation goes in the positive direction. Further, we restrict to those $F, S_1, ..., S_b$, which have pairwise disjoint image. This is necessary if we want to obtain a well defined self-sewing operation.

As in the case of punctured surfaces, we denote a Riemann surface with local parametrizations by $(\Sigma, F, \{S_i\})$. Given a holomorphic automorphism $\phi : \Sigma \to \phi(\Sigma)$, we consider $(\phi(\Sigma), \phi \circ F, \{\phi \circ S_i\})$ to be equivalent to $(\Sigma, F, \{S_i\})$. We denote the space of punctured surfaces in the moduli space $\mathcal{M}_{g,b,c}^{n,\{m_i\}}$ with local parametrizations by $\mathcal{P}_{g,b,c}^{n,\{m_i\}}$. There is a natural projection $\mathcal{P}_{g,b,c}^{n,\{m_i\}} \to \mathcal{M}_{g,b,c}^{n,\{m_i\}}$.

4.3. Geometric Vertices and Sewing Operations

Amplitudes in string theory are defined using integral forms over $\mathcal{M}_{g,b,c}^{n,\{m_i\}}$. In string field theory, these get lifted to forms on $\mathcal{P}_{g,b,c}^{n,\{m_i\}}$. Geometric vertices specify the integration region over which these forms are integrated. We follow [25] and begin by considering singular chains in $\mathcal{P}_{g,b,c}^{n,\{m_i\}}$ with real coefficients. These are formal superpositions

$$\mathcal{V}_{g,b,c}^{n,\{m_i\}} = \sum_{k=1}^{n} \hbar^{2g+b+c} a_k f_k \tag{4.7}$$

of continuous maps $f_k : \Delta^{n_k} \to \mathcal{P}_{g,b,c}^{n,\{m_i\}}$ over $a_k \in \mathbb{R}$. The functions take values in the n-dimensional simplices,

$$\Delta^n = \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i = 1, x_i \geq 0 \, \forall i\}. \tag{4.8}$$

Definition 4.3.1. A geometric vertex $\mathcal{V}$ is a finite superposition of chains of the form (4.7) over real coefficients.

Oriented sewing	Action on topology
sew_c, same surface	$(g,b,c) \mapsto (g+1,b,c)$
sew_c, different surfaces	$(g_1,b_1,c_1) \times (g_2,b_2,c_1) \mapsto$ $(g_1+g_2,b_1+b_2,c_1+c_2)$
sew_o, same surface, same boundary	$(g,b,c) \mapsto (g,b+1,c)$
sew_o, same surface, different boundary	$(g,b,c) \mapsto (g+1,b-1,c)$
sew_o, different surfaces	$(g_1,b_1,c) \times (g_2,b_2,c) \mapsto$ $(g_1+g_2,b_1+b_2-1,c_1+c_2)$
Unoriented sewing	
$\overline{\mathrm{sew}}_c$, same surface	$(g,b,c) \mapsto (g,b,c+2)$
$\overline{\mathrm{sew}}_c$, different surfaces	$(g_1,b_1,c_1) \times (g_2,b_2,c_1)$ $\mapsto (g_1+g_2,b_1+b_2,c_1+c_2)$
$\overline{\mathrm{sew}}_o$, same surface, same boundary	$(g,b,c) \mapsto (g,b,c+1)$
$\overline{\mathrm{sew}}_o$, same surface, different boundary	$(g,b,c) \mapsto (g,b-1,c+2)$
$\overline{\mathrm{sew}}_o$, different surfaces	$(g_1,b_1,c) \times (g_2,b_2,c) \mapsto$ $(g_1+g_2,b_1+b_2-1,c_1+c_2)$

Table 4.1.: Effect of sewing on topology (see [28], **Table 1**)

Remark 4.3.1. The above definition means that we allow for the images of the chains to not be fixed. In other words, the images may consist of several different $\mathcal{P}^{n,\{m_i\}}_{g,b,c}$.

The presence of local coordinates allows us to define a sewing operation. We can either sew along bulk (interior) or boundary coordinates. Suppose we have two boundary coordinates z and w. These can either be part of two different surfaces or a single surface. Since we allow for unorientable surfaces, there are two kinds of sewing operations [28]

$$\mathrm{sew}_o : zw = -1, \quad \overline{\mathrm{sew}}_o : z\overline{w} = 1. \tag{4.9}$$

Similarly, if z and w are bulk coordinates, we set

$$\mathrm{sew}_c(\theta) : zw = e^{i\theta}, \quad \overline{\mathrm{sew}}_c(\theta) : z\overline{w} = e^{i\theta}. \tag{4.10}$$

Note that the parameter of the open string sewing operation is fixed by the condition that the identification should happen on the upper half-plane. On the other hand, for closed string punctures, the relative angle θ is not fixed. The effect of sewing on topology is summarized in table 4.1.

The sewing operations are used to define a BV structure on the space of geometric vertices. Before we can do that, we need a grading on the space of geometric vertices. Suppose a singular chain $\mathcal{V}$ is defined only by maps from k-simplices. In this case, we define $\dim \mathcal{V} := k$. The degree of a singular chain $\mathcal{V}$ into $\mathcal{P}^{n,\{m_i\}}_{g,b,c}$ of homogeneous dimension is then defined to be [101, 28]

$$\deg \mathcal{V} = \dim \mathcal{M}^{n,\{m_i\}}_{g,b,c} - \dim \mathcal{V}. \tag{4.11}$$

In particular, this places those vertices in degree 0, which have the same dimension as the moduli space they project to.

We want to consider a subclass of vertices which are graded symmetric under certain relabelings of the local coordinates. First of all, we want that all vertices are invariant under arbitrary relabeling of the bulk punctures. Further, let $\mathcal{V}$ be a vertex with a boundary having m local coordinates. Denote by $C(\mathcal{V})$ the vertex where the m local coordinates are relabeled cyclically, i.e. $1 \to 2, 2 \to 3, ..., m \to 1$. We demand that

$$C(\mathcal{V}) = (-)^{m-1}\mathcal{V}. \tag{4.12}$$

Finally, let i, j be the labels of two different boundaries and denote by $P_{i,j}$ the operation that switches these labels. We demand

$$P_{i,j}(\mathcal{V}) = (-)^{(m_i+1)(m_j+1)}\mathcal{V}, \tag{4.13}$$

where m_i denotes the number of punctures on the ith boundary.

Definition 4.3.2. We denote the graded vector space of geometric vertices by K. We call vertices with the symmetry properties described above admissible. There is a projector $S : K \to K_s$ onto admissible vertices.

Remark 4.3.2. The signs from the symmetry operations can be memorized by thinking of boundary coordinates and boundaries themselves to be odd and bulk coordinates to be even. This is consistent with string theory, since bosonic open strings always come accompanied with the insertion of one odd ghost and bosonic closed strings with the insertion of two odd ghosts. Boundaries can also described by a state which comes with three ghosts.

Given two vertices $\mathcal{V}, \mathcal{W}$, we define a sewing operation of boundary punctures

$$\mathcal{V} \circ \mathcal{W} \tag{4.14}$$

in the following way. Let z be the last boundary coordinate on the last boundary of $\mathcal{V}$ and w be the first boundary coordinate on the first boundary of $\mathcal{W}$ and identify via

$$\mathrm{SEW}_o = \frac{1}{2}(\mathrm{sew}_o + \overline{\mathrm{sew}}_o). \tag{4.15}$$

We define the boundary bracket as the projection of this product onto admissible vertices,

$$\{\mathcal{V}, \mathcal{W}\}_o = S(\mathcal{V} \circ \mathcal{W}). \tag{4.16}$$

Similarly, we define a self-sewing operation. We apply SEW_o by pairwise picking boundary punctures of the surface. We denote this operation by Δ_o. Both $\{\cdot, \cdot\}_o$ and Δ_o have degree one by formula (4.4).

For bulk punctures, we basically do the same. However, to obtain a degree one operation we should include the whole family of angles in (4.19) to the new chain. The two operations are denoted by $\{\cdot, \cdot\}_c$ and Δ_c. Finally, we combine all the operations into $\{\cdot, \cdot\} = \{\cdot, \cdot\}_c + \{\cdot, \cdot\}_o$ and $\Delta = \Delta_c + \Delta_o$.

We are now ready to state the geometric master equation. The final ingredient is the boundary operator ∂ on chains By (4.11) it is also of degree 1.

Definition 4.3.3. Given a degree 0 chain $\mathcal{V}$, we say that $\mathcal{V}$ satisfies the geometric master equation if

$$\partial\mathcal{V} + \frac{1}{2}\{\mathcal{V}, \mathcal{V}\} + \hbar\Delta\mathcal{V} = 0. \tag{4.17}$$

Admittably, we introduced the geometric master equation in a rather hurried fashion. For more details, the reader can consult [101, 28].

Remark 4.3.3. With our definitions, we do not have a BV algebra, since we are lacking a product. The latter could be introduced by defining it via the disjoint union of Riemann surfaces. In this case, we should also allow for moduli spaces of surfaces with more than one connected component.

Let us explain the meaning of the geometric master equation (4.17). We first introduce a variant of the sewing operations (4.9) and (4.19) by introducing an additional real parameter q. We define

$$\mathrm{sew}_o(q) : zw = -q^{-1}, \quad \overline{\mathrm{sew}}_o(q) : z\overline{w} = q^{-1} \tag{4.18}$$

for the open-string sewing, and

$$\mathrm{sew}_c(q,\theta) : zw = q^{-1}e^{i\theta}, \quad \overline{\mathrm{sew}}_c(q,\theta) : z\overline{w} = q^{-1}e^{i\theta} \tag{4.19}$$

for the closed version. Note that, in order for this to be well defined, we need that $q \geq 1$, since our coordinates are restricted to the unit disc. When $q = 1$, we recover our original definition. In that case, the sewing amounted to cutting out coordinate discs and sew them along the generated boundaries. When q is greater than zero, we insert a strip/tube of length $\tau = \ln q$ in between. We interpret them as open and closed strings propagating with proper time τ between the surfaces. In this way, the propagator is used to cover the regions of the moduli space associated to infrared processes. In particular, the limit $\tau \to \infty$ is the part where the propagating string goes on-shell. On the other hand, there may be regions beyond the point $q = 1$ where the propagator has length zero. We need geometric vertices to account for these regions. The condition that the geometric vertex starts where the propagator ends is precisely the geometric master equation. The boundary of a vertex $\partial\mathcal{V}$ should be where we transition to a region covered by propagators. On the other hand, the boundary of propagator regions is where a propagator approaches length zero. These boundaries are the surfaces obtained by the BV operations $\{\mathcal{W}_1, \mathcal{W}_2\}$ and $\Delta\mathcal{W}$. The geometric master equation says that the boundary of the fundamental vertex $\mathcal{V}$ agrees with the boundaries of the regions where a propagator connects $\mathcal{W}_1$ to $\mathcal{W}_2$, or a propagator connects $\mathcal{W}$ to itself.

4.4. A (Very Brief) Introduction to String Field Theory

A two dimensional conformal field theory associates amplitudes to the geometric vertices by computing correlators on Riemann surfaces. A conformal field theory defines vertex operators A_i and associates an amplitude

$$\langle A_1 \cdots A_k \rangle_M \tag{4.20}$$

for each fixed Riemann surface with parametrized punctures $M \in \mathcal{P}^{n,\{m_i\}}_{g,b,c}$. The vertex operators are inserted at the location of the punctures, so we should have $k = n + \sum_i m_i$. If an A_i has non-zero conformal dimension, the amplitude depends on the choice of local parametrizations.

Given a geometric vertex $\mathcal{V}^{n,\{m_i\}}_{g,b,c}$ with values in $\mathcal{P}^{n,\{m_i\}}_{g,b,c}$, we define the string field theory vertex to be

$$\mathcal{V}^{n,\{m_i\}}_{g,b,c}(A_1, ..., A_k) = \int_{\mathcal{V}^{n,\{m_i\}}_{g,b,c}} \mathrm{d}M \, \langle A_1 \cdots A_k \rangle_M \,, \tag{4.21}$$

where $\mathrm{d}M$ is some measure on $\mathcal{P}^{n,\{m_i\}}_{g,b,c}$. They define the interaction part of the string field theory action

$$S_{\mathcal{V}}(A) = \sum_{n,\{m_i\},g,b,c} \mathcal{V}^{n,\{m_i\}}_{g,b,c}(A,...,A), \tag{4.22}$$

where

$$\mathcal{V} = \sum_{n,\{m_i\},g,b,c} \mathcal{V}^{n,\{m_i\}}_{g,b,c}. \tag{4.23}$$

An important property is the following. Let Q be the BRST operator of the conformal field theory that is obtained from Faddeev-Popov gauge fixing of the Diff $\times$ Weyl symmetry. It gives the following ward identity among amplitudes

$$\mathcal{V}^{n,\{m_i\}}_{g,b,c}(Q(A_1),...,A_k) \pm ... \pm \mathcal{V}^{n,\{m_i\}}_{g,b,c}(A_1,...,Q(A_k)) = \partial\mathcal{V}^{n,\{m_i\}}_{g,b,c}(A_1,...,A_k). \tag{4.24}$$

This follows from the fact that the integral form

$$\mathrm{d}M\, \langle [Q, A_1 \cdots A_k] \rangle \tag{4.25}$$

is exact. The BRST operator enters as the free part in the action,

$$S_F(A) = \frac{1}{2}\omega(A, Q(A)). \tag{4.26}$$

The odd symplectic structure ω is naturally defined in terms of the conformal field theory, where it is called the BPZ (Belavin, Polyakov, Zamolodchikov) inner product. It defines a BV algebra structure, i.e. a Laplacian Δ_{SFT} and an anti-bracket $\{\cdot,\cdot\}_{SFT}$. We can restate (4.24) as

$$\{S_F, S_{\mathcal{V}}\}_{SFT} = S_{\partial\mathcal{V}} \tag{4.27}$$

Given two vertices $\mathcal{V}$ and $\mathcal{W}$, we have the following relation

$$\{S_{\mathcal{V}}, S_{\mathcal{W}}\}_{SFT} = S_{\{\mathcal{V},\mathcal{W}\}}, \quad \Delta_{SFT} S_{\mathcal{V}} = S_{\Delta\mathcal{V}}. \tag{4.28}$$

The following identity follows immediately

$$S_{\partial\mathcal{V}+\frac{1}{2}\{\mathcal{V},\mathcal{V}\}+\hbar\Delta\mathcal{V}} = \frac{1}{2}\{S_F + S_{\mathcal{V}}, S_F + S_{\mathcal{V}}\}_{SFT} + \hbar\Delta_{SFT}S_{\mathcal{V}}. \tag{4.29}$$

We conclude the following. If $\mathcal{V}$ satisfies the geometric master equation, then $S_F + S_{\mathcal{V}}$ satisfies the (euclidean) quantum master equation. One can also say that a 2d conformal field theory defines a morphism between solutions to the quantum master equation, where the morphism is given by

$$\mathcal{V} \mapsto S_F + S_{\mathcal{V}}. \tag{4.30}$$

4.5. Stubs in String Field Theory and Regularization

A nice concept arising from the geometric picture of the vertices is that of stubs. These can be generated by a redefinition of the local parametrizations. We want to illustrate this with the Witten vertex from bosonic open-string field theory [93].

The Witten vertex is an element of $\mathcal{P}^{0,\{3\}}_{0,1,0}$, i.e. a disc with three parametrized boundary punctures. It describes the interaction of three open strings. As a geometric vertex, it is a

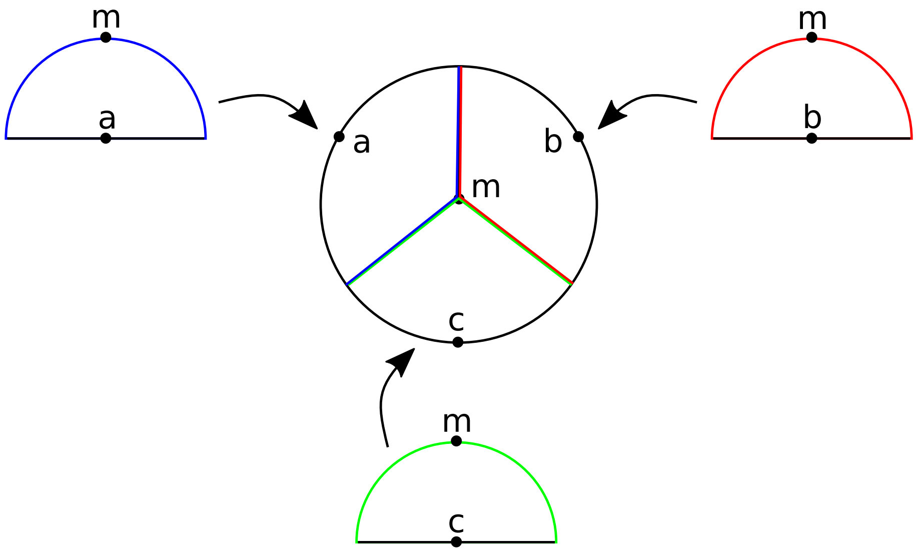


Figure 4.2.: The Witten vertex embeds three coordinate half-discs, labelled by (a,b,c) into the Riemann surface (in this case a disc). The letter m marks the midpoint of the semi-circle of each coordinate disc. All of them are mapped to the same point on the disc. The colors indicate the images of the semi-circles. Here, they are taken such that they overlap exactly.

chain from the zero dimensional simplex into $\mathcal{P}^{0,\{3\}}_{0,1,0}$, which is the same as a point in $\mathcal{P}^{0,\{3\}}_{0,1,0}$. It is constructed such that the boundaries of the coordinate discs exactly meet, see figure 4.2.

Pictorially, we think of the actual interaction as that part of the surface, that is obtained by removing the interiors of the coordinate discs. To see why this makes sense, we first recall how the Riemann surfaces describe interactions of strings. The vertex operators, which are inserted on the origin of the coordinate half-discs, create strings on the arc of the semi-discs. These strings then interact by propagation on the Riemann surface. The region of propagation is therefore exactly that part of the surface that is not covered by the coordinate discs. In this picture, the Witten vertex describes an extreme case. The strings are created exactly on top of each other. They interact without any intermediate propagation.

We can introduce a region of propagation by shrinking the images of the coordinate discs inside the Riemann surface. Let $f : D_H \to D,\ z \mapsto f(z)$ be one of the local coordinates of the Witten vertex. Given a real parameter $q > 1$, we define a new coordinate $\tilde{f}(z) = f(z/q)$. When we do this for all three coordinates, we end up with a picture like the one in figure 4.3. The string states are no longer created on top of each other. Before they interact, they have to propagate over a certain part of the disc.

The process of rescaling the local coordinates is commonly called *adding stubs*. This is because we can think of the process as gluing a small rectangles (tubes in the case of closed strings) along each line that represents an incoming string. Each of these rectangles represents a worldsheet that the incoming string has to transverse.

The effect on the spacetime theory is the following. Rescaling the local coordinates by q

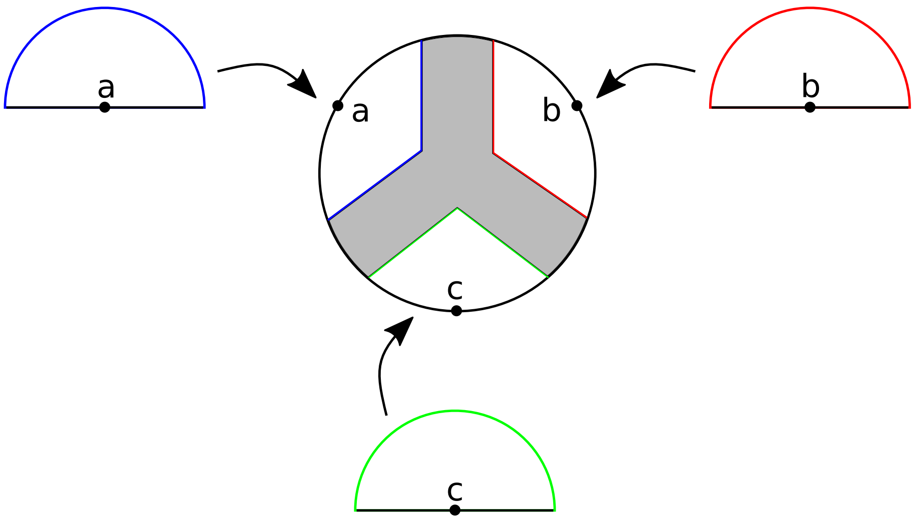


Figure 4.3.: The Witten vertex (figure 4.2) with stubs. The images of the arcs do no longer overlap on the disc. The shaded part shows the region of propagation.

amounts to the insertion of the operator

$$q^{-L_0} = e^{-\tau(\Box + m^2)}. \tag{4.31}$$

Here, $q = \ln \tau$ and we indicated that the operator L_0 acts as $\Box + m^2$ in terms of spacetime fields. The net effect is that the fields propagate an additional imaginary time $t = i\tau$. Recall that we used a similar operation when regularizing the chiral Schwinger model in section 3.5. There, we called the operator K and it was part of a homotopy equivalence data. The same is true is string field theory. We have a homotopy

$$H = \int_0^\tau \mathrm{d}\tau' b_0 e^{-\tau' L_0}, \tag{4.32}$$

where b_0 is the zero mode of the Faddeev-Popov b-ghost on the worldsheet and it satisfies $\{Q, b_0\} = L_0$. It follows that the addition of stubs can be described as a homological perturbation. In particular, if the original theory satisfies either the classical or the quantum master equation, then so does the theory with stubs.

Stubs in string field theory are a natural tool to regularize the Witten vertex in string field theory. Unfortunately, they destroy the cubic nature of the theory. On the other hand, stubs will make the appearance of the closed string explicit at the quantum level. This is again similar to the anomaly computation in section 3.5.

We conclude by describing how stubs affect the covering of the moduli space in the example of the five point interaction in bosonic open-string theory. The BV theory of open string theory has an underlying A_∞-algebra. From that structure, we can deduce that the fundamental five-point vertex covers a pentagon-shaped region (recall the content of section 2.3.6). Adding stubs extends the region covered by the fundamental vertex, see figure 4.4.

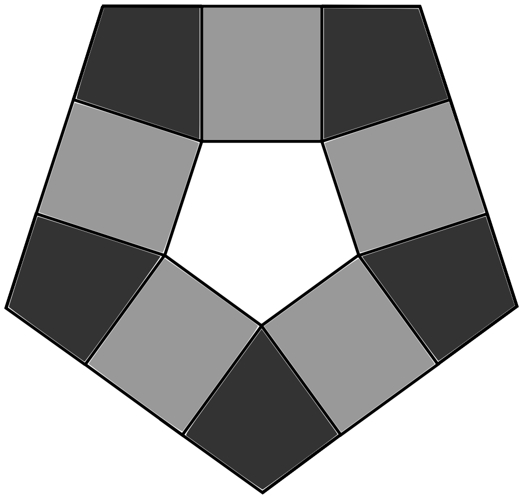


Figure 4.4.: A pentagon shaped region covering the moduli space of the disc with five boundary punctures. The white region displays the covering of a the fundamental vertex before the addition of stubs. After applying a homological perturbation/adding stubs, the region gets extended. The light grey areas are the parts covered by a cubic and a quartic vertex connected with a single homotopy. The dark grey areas are covered by three cubic vertices connected with two homotopies (recall the diagrammatic representation of the transferred structure below the homotopy transfer theorem in section 2.3.4). A similar picture in a related context can be found in [97].

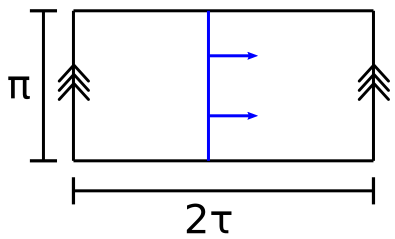


Figure 4.5.: A representation of $\Delta\mathcal{V}^{0,\{3\}}_{s;0,1,0}$. The arrows on the first and the last edge indicate that they are identified. The diameter of the open string is normalized to π. The motion of the open string is displayed in blue. It travels a length of 2τ on the worldsheet before it closes the loop. The boundary puncture is not displayed explicitly. It can be placed anywhere on the boundary by shift symmetry.

4.6. The Green-Schwarz Anomaly in String Field Theory

4.6.1. Cancellation at the One-Loop Level

For unoriented open-strings, there are three types of diagrams at the one-loop level. At the level of Riemann surfaces, two of them come from the annulus. The annulus has two boundaries and we distinguish between those surfaces where all punctures are on a single boundary (planar), and those where this is not the case (non-planar). Since the annulus is orientable, these diagrams are present also in a theory of oriented strings only. Exclusive to unoriented strings is the Möbius strip. Since it has only one boundary, no further distinction can be made that depends on the locations of the punctures.

Different geometric vertices are related by the fact that we want a solution to the geometric master equation (4.17). Quantum anomalies arise whenever we are not able to cancel terms of the form $\Delta\mathcal{V}$. Let us discuss this in the example of cubic bosonic open-string field theory. We mentioned already before that it has a single cubic vertex, which we denote by $\mathcal{V}^{0,\{3\}}_{0,1,0}$, since its underlying topology is the three-punctured disc. At the classical level, the vertex $\mathcal{V}^{0,\{3\}}_{0,1,0}$ is enough to cover the whole moduli space. We would like to compute $\Delta\mathcal{V}^{0,\{3\}}_{0,1,0}$. The sewing process is a superposition of sew_o and $\overline{\text{sew}}_o$. The problem here is that both processes result in a singular surface. Let us first consider sew_o. It produces an annulus with zero circumference. A regular surface can be obtained by introducing stubs. In the notation of (4.31), we rescale the local coordinates by a parameter q. We call the new vertex $\mathcal{V}^{0,\{3\}}_{s;0,1,0}$. The oriented part of $\Delta\mathcal{V}^{0,\{3\}}_{s;0,1,0}$ results in annulus of length 2τ, where $\tau = \ln q$, see figure 4.5. Similarly, the unoriented part is a Möbius strip of length 2τ, see figure 4.6.

Since $\Delta\mathcal{V}^{0,\{3\}}_{s;0,1,0}$ results in two non-vanishing surfaces, their contribution should be canceled by another surface in the geometric master equation. From surfaces with boundary punctures only, we cannot get any contribution from the anti-bracket $\{\cdot,\cdot\}$. On the other hand, we can introduce two fundamental tadpole vertices $\mathcal{V}^{0,\{1,0\}}_{s;0,2,0}$ and $\mathcal{V}^{0,\{1\}}_{s;0,1,1}$ that cover the annulus and Möbius strip with length parameter from 0 to 2τ. In that case, we would write

$$\partial(\mathcal{V}^{0,\{1,0\}}_{s;0,2,0} + \mathcal{V}^{0,\{1\}}_{s;0,1,1}) \stackrel{?}{=} -\hbar\Delta\mathcal{V}^{0,\{3\}}_{s;0,1,0}. \tag{4.33}$$

This is, however, not completely satisfactory. We should be careful when writing $\partial(\mathcal{V}^{0,\{1,0\}}_{s;0,2,0} +$

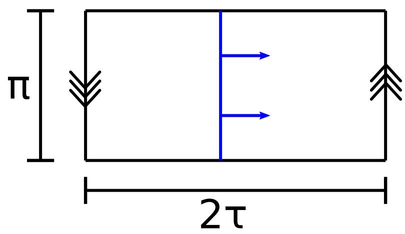


Figure 4.6.: The Möbius strip analog of figure 4.5.

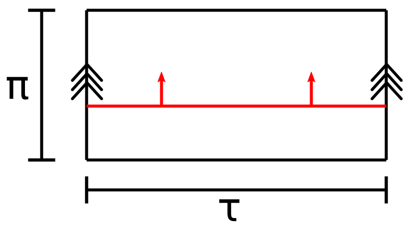


Figure 4.7.: The annulus diagram as seen from the closed string perspective (red).

$\mathcal{V}^{0,\{1\}}_{s;0,1,1}$), since now the tadpole vertices contain the singular surfaces at one end of the boundary. The problem has merely moved to another place. Here is where the closed string comes to the rescue. What we do is the following. We first exclude all surfaces with length parameter $l \leq \tau$ from $\mathcal{V}^{0,\{1,0\}}_{s;0,2,0}$ and $\mathcal{V}^{0,\{1\}}_{s;0,1,1}$. This creates a new boundary at $l = \tau$. From there on, we describe the processes with the closed string. The picture is the following. Consider figure 4.5 and 4.6 with a string not moving from left to right, but from the bottom to the top, see figures 4.7 and 4.8. There, the string does not end on a boundary like an open string, but rather forms a closed loop. From this point of view, we are looking at a closed string. We can interpret them the following way. On the annulus (figure 4.7), a closed string of length τ is created at one boundary, travels a distance[2] of π, and is again annihilated at the other boundary. On the other hand, when considering the Möbius strip (figure 4.8), a closed string of length 2τ is created, travels for $\frac{\pi}{2}$, and then annihilates itself.

We obtain an unambiguous notion of what we call closed string once we fix its length. The standard choice of 2π (twice the length of the open string). Scale invariance of string theory allows us to rescale figures 4.7 and 4.8 so that the length of the closed string is indeed 2π. In case of the annulus, we need to divide all length scales by $\frac{\tau}{2\pi}$. On the other hand, since the closed string on the Möbius strip is twice as long, we merely divide by $\frac{\tau}{\pi}$. In the new scale, the closed string now travels a distance of $\frac{2\pi^2}{\tau}$ on the annulus and $\frac{\pi^2}{2\tau}$ on the Möbius strip. An equivalent representation of the Möbius strip that makes the closed string propagation more visible is shown in figure 4.9.

So how does the closed string help us? The advantage is that the distance the closed string transverses now grows when τ approaches zero. This means that this region can

[2]In the picture of Riemann surfaces, it is most natural to use the word *distance*. From the field theory point of view, we should replace the word "distance" with "proper time".

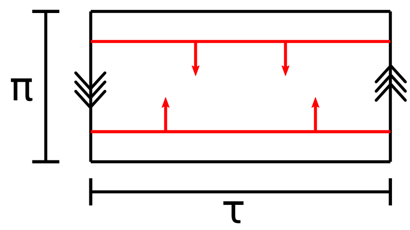


Figure 4.8.: The Möbius strip from the point of view of the closed string (red). Note that, in order to have a closed red line, we need to wind around the sheet twice.

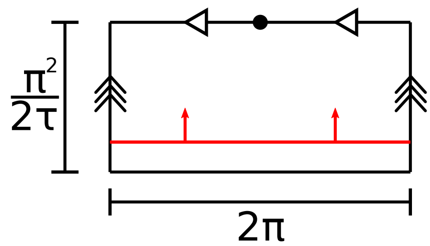


Figure 4.9.: Another representation of the Möbius strip. In this case, the closed string (red) is created at the lower boundary and annihilated at the upper crosscap. The arrows on the upper horizontal line indicate how the two halves are identified to form the crosscap. We already rescaled the picture so that the closed string has length 2π.

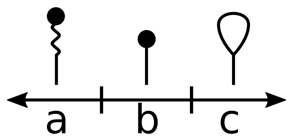


Figure 4.10.: The partition of the moduli space of the one-punctured annulus in terms of open-closed vertices. Straight lines stand for open strings, wiggly lines represent closed strings. Black bullets denote a tadpole vertex. Region (a) is the closed string tadpole $\mathcal{V}^{0,\{0\}}_{s;0,1,0}$ connected by a closed string propagator to the open-closed interaction vertex $\mathcal{V}^{1,\{1\}}_{s;0,1,0}$. Region (b) is the fundamental one-loop open string tadpole $\mathcal{V}^{0,\{1,0\}}_{s;0,2,0}$. Region (c) is covered by the cubic interaction $\mathcal{V}^{0,\{3\}}_{s;0,1,0}$ with a open-string self-loop. Since region (a) and (c) are represented by a propagator, the moduli space continues to infinity in both directions.

be described by a propagator. The closed string allows us to introduce three new vertices $\mathcal{V}^{1,\{1\}}_{s;0,1,0}, \mathcal{V}^{0,\{0\}}_{s;0,1,0}, \mathcal{V}^{0,\{\}}_{s;0,0,1}$. The first one, $\mathcal{V}^{1,\{1\}}_{s;0,1,0}$, is a disc with one bulk and one boundary puncture. It describes a quadratic open-closed interaction vertex (this is similar to the Wess-Zumino term in the chiral Schwinger model we encountered in 3.5). On the other hand, $\mathcal{V}^{0,\{0\}}_{s;0,1,0}$ is a disc with one bulk puncture, while $\mathcal{V}^{0,\{\}}_{s;0,0,1}$ is a crosscap (also called the real projective plane $\mathbb{RP}^2$) with one bulk puncture. They both constitute a closed sting tadpole. We can now cover the region $\tau \to 0$ with the vertex $\mathcal{V}^{1,\{1\}}_{s;0,1,0}$ connected to $\mathcal{V}^{0,\{\}}_{s;0,0,1}$ and $\mathcal{V}^{0,\{0\}}_{s;0,1,0}$ by a closed string propagator. In terms of the geometric master equation, this is

$$\partial(\mathcal{V}^{0,\{1,0\}}_{s;0,2,0} + \mathcal{V}^{0,\{1\}}_{s;0,1,1}) + \{\mathcal{V}^{0,\{0\}}_{s;0,1,0} + \mathcal{V}^{0,\{\}}_{s;0,0,1}, \mathcal{V}^{1,\{1\}}_{s;0,1,0}\} + \hbar\Delta\mathcal{V}^{0,\{3\}}_{s;0,1,0} = 0. \tag{4.34}$$

All operations in the above are now well defined, since none of them produces a singular surface. Figure 4.10 shows how the moduli space of the one-punctured annulus is covered by the vertices we introduced. The same picture applies also to the one-punctured Möbius strip.

So since we were able to solve the master equation, where is the anomaly? The short answer is that there is none due to the closed string. However, we should talk about the fact that our theory has tadpoles. In a field theory, the presence of tadpoles means that the we cannot do perturbation theory, since the theory is not in a vacuum when all fields are zero. The solution to this problem is that we first should shift the fields $\phi \mapsto \phi + \phi_0$, so that all terms linear in the fields (i.e. tadpoles) disappear. The field ϕ_0 then defines a background on which we can safely apply perturbation theory.

In the case of open-closed string field theory, there are is both an open-string and a closed-string tadpole. Let us first talk about the open-string tadpole. This one does not look problematic for the following reason. Its geometric vertex connects the intermediate region between the open-string and the closed-string regime (region (b) in figure 4.10). But from our analysis it is clear that this region can be shrunk to zero. To see this, recall that in our parametrization, the vertex covered the interval $[\tau, 2\tau]$. We can shrink this region on both ends, by forcing either of the two regions (a) and (c) to cover a larger region. For example, we could use the parameter $\sqrt{q}$ instead of q to give stubs to the Witten vertex.

The vertex would still be regularized. The closed-string area could also be increased by shortening of the closed-string stubs.[3] In this way, the open string tadpole disappears. On the field theory side, this would correspond to a particular background shift that removes the tadpole without affecting the spectrum of the theory.

Things are different in case of the closed string tadpole. The moduli space of both the one-punctured disc and the projective plane is zero dimensional, so there is no possibility in enlarging any regions in order to remove them. We need to get rid of them on the level of the field theory. One approach would be to choose a conformal field theory that maps the geometric tadpoles to zero. In other words, the CFT should be such that all one-point functions on Riemann surfaces vanish. However, it is not clear whether this is possible for all choices of geometric vertices, since one can always end up with a different string field theory by changing local coordinates without affecting the CFT. In this way one may be able to generate tadpoles simply by a redefinition of the vertices.

One way to get our hands on what we could call the "parametrization independent" tadpole is to consider the limit where we make region (c) in figure 4.10 infinitely large. This would correspond to the limit $q \to 1$ (or $\tau \to 0$) in the regularization we applied to the cubic vertex, hence it approaches again the original Witten vertex. The stubs then shrink to zero length. On the other hand, we should adapt the closed string tadpoles, so that regions (a) and (c) do not overlap (we assume that we already got rid of the open string tadpole, i.e. there is no region (b)). This is achieved by shrinking the local coordinates around the closed string puncture on both tadpoles to zero. In this process, closed-string vertex operators inserted at the puncture are suppressed if they are not on-shell, i.e. if they do not satisfy $L_0 = 0$. Hence, we can restrict ourselves to on-shell operators when we ask whether one-point functions in our CFT vanish. However, in this case it is already known that, when the CFT corresponds to a string theory on flat spacetime, the tadpoles cancel if and only if the gauge group is $SO(2^{d/2})$, where d is the spacetime dimension of the string theory. An argument based on closed string partition functions for the bosonic case can be found in [74]. The superstring analog is found in [76].

Remark 4.6.1. The restriction to $SO(2^{d/2})$ gets lifted if we do allow for Lie supergroups as gauge groups. In the case of the Klein bottle, there is an infinite family $SO(2^{d/2}+N, N)$ of possibilities for the unoriented string. For the oriented string, one could also consider $GL(N, N)$. Open string tadpoles vanish in that case, since whenever there is a free boundary, the amplitude gets a factor of $\mathrm{tr}(1)$, which is zero in the case of $GL(n, n)$. This was used for example in [24]. There, they considered the limit $N \to \infty$ and referred to it as "large N". However, this is not a planar limit in the sense of 't Hooft.

For general gauge groups, a physical interpretation is that the brane, on which the open strings end, has non-zero energy density. We can interpret this as a non-zero cosmological constant, which in turn leads to a non-trivial closed-string background, i.e. spacetime becomes curved. Therefore, the closed string in that case becomes necessary in open-string theory in order to generate the curved background. Linearized gravity in the background of the brane was computed in [75, 31].

Up to this point, the closed string does not seem to be necessary at the quantum level if we restrict ourselves to $SO(2^{d/2})$. However, since we only considered one-punctured surfaces,

[3] Without an explicit description of the vertices, it is ambiguous what we mean by a shortening of stubs, since in that case, the we would have to increase the area covered by the local coordinates. With an explicit metric, we could for example increase the area geodesically. On the other hand, local parametrizations always provide a preferred choice of shrinking the domain of definition by simple rescaling.

the non-planar diagram did not show up. To observe it, we should not forget that the introduction of stubs induces higher order vertices, in particular a quartic vertex $\mathcal{V}^{0,\{4\}}_{s;0,1,0}$. Let us draw once more from our experience with the chiral Schwinger model. There, the regularization also amounted to introducing a quartic vertex in order to satisfy the classical master equation. Consistency requires that the quartic vertex $S'_{quartic}$ vanishes when the stub parameter goes to zero. On the other hand, we found that $\Delta S'_{quartic}$ remains finite. We expect a similar phenomenon for the open string. Hence, for the moment, we work with a finite stub parameter. In this case, we get several contributions from $\Delta\mathcal{V}^{0,\{4\}}_{s;0,1,0}$. First of all, the twist sewing $\overline{\mathrm{sew}}_o$ always produces a two-punctured Möbius strip. The oriented sewing instead gives two topologically distinct surfaces, depending on whether or not the identified punctures are next to each other relative to the cyclic order. If we sew neighboring punctures, one of the two boundaries will be empty and the resulting surface is planar. The canceling of this surface in the geometric master equation involves the closed string tadpole. The analysis goes along similar lines as before. On the other hand, sewing non-neighboring punctures results in an annulus with a single puncture on each boundary. Similar to what we did before, we could introduce a fundamental vertex $\mathcal{V}^{0,\{1,1\}}_{s;0,2,0}$ to govern the region where the modulus of the annulus is small. However, we face a similar problem to what we had before. The singular annulus is part of the vertex. So we interpret again the short open string propagation as a long closed string propagation. The master equation in that case reads

$$\partial\mathcal{V}^{0,\{1,1\}}_{s;0,2,0} + \frac{1}{2}\{\mathcal{V}^{1,\{1\}}_{s;0,1,0}, \mathcal{V}^{1,\{1\}}_{s;0,1,0}\} + \hbar\Delta\mathcal{V}^{0,\{4\}}_{s;0,1,0} = 0. \tag{4.35}$$

Notice that this equation involves the open-closed vertex we already introduced before.

In general, fundamental closed strings appear only in non-planar diagrams (apart from tadpoles). Non-planar diagrams are those surfaces having more than one boundary that has punctures. In that case, there are regions in moduli space where the corresponding surface connects the two boundaries by a long tube. This corresponds to a process where several incoming open strings turn into a closed string, and then become open strings again.

This suggests that a theory with planar diagrams, i.e. diagrams where only one boundary has punctures, may avoid the closed string, as long as we assume in addition that the gauge group is $SO(2^{d/2})$. From a topological perspective, we would restrict ourselves to surfaces of the form

$$\mathcal{V}^{0,\{n,0,\ldots,0\}}_{0,b,c}, \quad b \geq 1, c \geq 0, \tag{4.36}$$

i.e. no interior punctures (closed string states) and no genus (closed string loops). On the other hand, we have to allow for unorientable surfaces ($c \neq 0$), because we need $SO(2^{d/2})$ as our gauge group.

It is actually easy restrict the self-sewing operation Δ to produce planar diagrams. Since boundary punctures are cyclically ordered, we have a notion on whether punctures are next to each other. If we restrict Δ only to sew two neighboring punctures, all additional boundaries created in that way will have no punctures.

However, we can already see that such a theory creates closed strings at higher loops. The reason is that any surface with genus g and at least three crosscaps c is topologically equivalent to a surface with genus $g+1$ and number of crosscaps $c-2$. So at the level of three open string loops ($b=1, c=3$) and higher, the closed string pops up again, unless we restrict to surfaces with $c \leq 2$[4]. However, if we would do that, the tadpole anomaly

[4] We will see below that, in fact, we would need to restrict to surfaces with $c \leq 1$

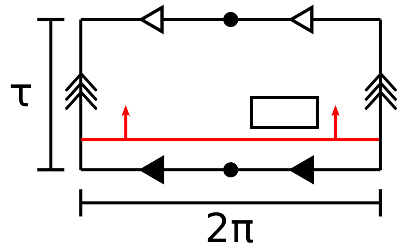


Figure 4.11.: The diagram where a closed string (red) is created at one crosscap, turns into an open string, turns into a closed string again, which ultimately gets annihilated at another crosscap. It is normalized so that the string has length 2π and travels for proper time τ. The rectangle in the middle is the boundary on which open string punctures sit. This diagram is needed in order to cancel the anomaly from the surface where the closed string ends in a boundary.

appears again at three loops. For example, we need the surface with $b = 1, c = 3$ to cancel the tadpole anomaly in $b = 2, c = 2$. We are now able to draw the main conclusion we want to give in this section.

Conclusion 1: There is no consistent string theory with planar and unorientable diagrams only. In particular, the closed string is necessary to obtain a consistent theory of open strings at the quantum level.

4.6.2. The Closed String Reappearing at Two Loops

Although we already saw that it is impossible to avoid the closed string, let us nevertheless work out a specific example. We look at a surface with $b = 1$ and $c = 2$. This surface it automatically planar in the sense that there is only one occupied boundary. Topologically, a surface with two crosscaps is the Klein bottle. We cut a hole in it to have a boundary where open strings can attach.

In the moduli space of the bordered Klein bottle, we consider two regions where the closed string goes on-shell. We begin by considering the case where the closed string is created in a crosscap and then annihilated in another one, see figure 4.11. This is the harmless case. In fact, we actually need it in order to cancel the tadpole anomaly which sits in the region where $\tau \to \infty$.

Figure 4.11 has a dual representation. This is similar to what we saw at one loop, where the open string diagram could be interpreted as a diagram for a closed string. We could call this phenomenon an open-closed duality. In this case, there is a closed-closed duality[5]. In figure 4.12, we look at the picture from right to left. In this picture, the closed string makes a loop, although there is a small difference to the case when the surface has a genus. Here, assume that we observe the closed string at a particular moment. Then, after it made one loop, we do not observe the original closed string, but a mirror image of it. Only after a second loop, we observe the string in its original form. This type of surface is obtained by

[5]Of course, we also have an open-closed duality here. The surface could also be interpreted as a process involving open strings. After all, this was our original motivation to consider these types of diagrams

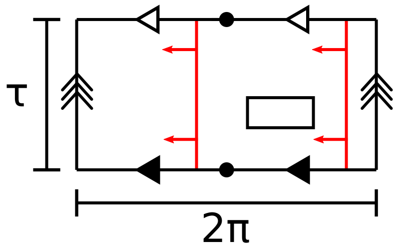


Figure 4.12.: Another interpretation of figure 4.11. In this case, the closed string (red) travels from right to left. It length 2τ and travels for a loop with distance π. Note that the string gets mirrored each time it makes a full loop.

sewing $\mathcal{V}^{2,\{n\}}_{s;0,1,0}$ with the operation $\overline{sew}_c$. A propagator with both ends attached to $\mathcal{V}^{2,\{n\}}_{s;0,1,0}$ accounts for the region where $\tau \to 0$. We find what we already concluded in the last section. The closed string cannot be avoided, even when we restrict to planar diagrams.

4.7. Does the Loop A_∞-algebra know about the L_∞-algebra?

We will now turn to a related problem concerning the quantum version of A_∞-algebras. First of all, let us state that the classical part of the geometric master equation when restricted to open strings ($n = 0, g = 0, b = 1, c = 0$) can be equivalently stated in terms of A_∞-relations. Similarly, open string field theory can be (and often is) described in terms of an A_∞-algebra. We saw that, at the quantum level, the open string needs the closed string. So we may want to ask the question whether the quantum A_∞-algebra needs an L_∞ algebra to be consistent.

Since quantum A_∞-relations are known to exist, the answer is of course negative. However, we will see (at least for some particular cases) that there is a property, which we call non-compactness, of the quantum A_∞-relations that seems to be lifted when an L_∞-algebra ("closed strings") is included. Recall that the classical A_∞-relations have a representation in terms of polytopes. To each product m_k, we associate a polytope of dimension $k-2$. The A_∞-relations describe the boundary of that polytope.

The property we want to point out is that all the polytopes are topologically compact spaces. This follows simply from the fact that all of them admit a decomposition as a finite CW-complex. The map m_2 is a point, while m_3 is a line with boundary the two points $m_2 \circ (m_2 \otimes \mathrm{id})$ and $m_2 \circ (\mathrm{id} \otimes m_2)$. At higher levels, compactness can be deduced from the fact that we always need to introduce only a single product m_k to kill all non-trivial homologies at one level lower. This means that the $k-2$ dimensional polytope consists of a single $k-2$ face representing m_k, and the boundary $[\mathrm{d}, m_k]$, which is, by induction, also a finite CW complex.

At the quantum level, this fails already at lowest order. Let us extend the notation to $m_{k,l}$, where k is the number of inputs and l is an internal number which we can interpret as the number of boundaries plus one. There is only a single point $\Delta m_{2,0}$. We want, however, that it has trivial homology. This means that we need to introduce a vertex $m_{0,1}$. The vertex should be such that $[\mathrm{d}, m_{0,1}] = \Delta m_{2,0}$. If we want to represent it as a topological space, $m_{0,1}$ would be a semi-infinite line with boundary point Δm_2. In particular, it is not compact. However, we can make it compact again by adding a point at infinity. In the language of string field theory, the closed string provides this point. Let us write $m^n_{k,l}$ if the vertex has n closed-string insertions.[6] We introduce a closed string tadpole $m^1_{0,0}$ and an open-closed vertex $m^1_{1,0}$. We can then define $m^1_{1,0} \circ m^1_{0,0}$ to be the point at infinity. Then,

$$[\mathrm{d}, m_{0,1}] + m^1_{1,0} \circ m^1_{0,0} + \Delta m_{2,0} = 0 \tag{4.37}$$

shows that the boundary of $m_{0,1}$ has two points and is therefore a finite line.

The introduction of an L_∞-algebra as part of our quantum A_∞-algebra made the vertices again compact. We now argue that we can expect that this phenomenon continuous to exist at higher orders from what we know from string theory. There, the points at infinity parametrize singular surfaces which would signalize that the open-string vertices suffer from UV divergences. The closed string then removes the singular surfaces (points at infinity) from the vertex regions. The regions covered by the fundamental vertices become compact. Of course, the singular surfaces did not disappear. They are now covered by diagrams involving closed string propagators. However, they are no longer part of any fundamental vertex.

This undoubtedly has an important physical significance. After all, it is what makes string theory free from UV divergences. On the other hand, it is not known to the author whether this has any mathematical significance. It may be that finite CW complexes as models are better behaved from the perspective of homotopy theory.

[6] We want to emphasize that the analysis here is independent of string field theory. We will nevertheless use the string theory language for convenience, since we are already familiar with it.

5. A planar quantum A_∞-algebra

In this chapter we explore the possibility of a quantum theory of planar graphs. We did not succeed in the case of open-string field theory because of the presence of unoriented strings. For gauge field theories it is known however that in the limit where the gauge group becomes large, the planar Feynman diagrams survive. This called the *large* N-limit. Again, in string field theory we cannot take this limit because we are forced to work with $SO(2^{d/2})$. Despite that, we may hope that we obtain a planar theory in the large N-limit of generic quantum field theories.

5.1. (Co-)derivations in Higher Orders for Noncommutative Algebras

We want to mimic the bar construction of quantum L_∞-algebras as it is described in section 3.6. In order to do this, we need a good notion of higher order (co-)derivations over non-commutative algebras, in particular in the case of the free tensor algebra $\hat{T}(V)$ and coalgebra $T^c(V)$. The approach we take below is motivated by the following observation in the case of the commutative tensor algebra $\hat{S}(V)$. If we define derivations and coderivations in the Koszul sense (definition 3.6.1), we have the following symmetry (see for example [21]). A coderivation of arbitrary order with k inputs is an order k derivation. Likewise, a derivation of arbitrary order with l outputs defines a coderivation of order l. For the simplest case, this also holds automatically for derivations on the tensor algebra $\hat{T}V$. The coderivation lift of a linear map $\delta : V \to V$ is indeed a derivation with respect to the tensor product. We would like to have a definition where this holds true also for higher order (co-)derivations over $\hat{S}V$.

We give a first definition. At the moment we still work in the realm of general non-commutative algebras. We will later restrict our attention to the free tensor algebra.

Definition 5.1.1. Given an algebra A over $\mathbb{C}$. Let $\delta \in \mathrm{Lin}_{\mathbb{C}}(A, A)$. We associate to δ a collection of multilinear maps $B_k(\delta) : A^{\otimes k} \to A$, where $k \geq 1$, inductively defined as follows. We set $B_1(\delta)(f) = \delta(f)$, while we demand the higher maps to satisfy

$$B_1(\delta)(f_1 \cdots f_n) = \sum_{l=1}^{n} \sum_{i=1}^{n-l} (-)^{\delta(f_1+\ldots+f_{i-1})} f_1 \cdots f_{i-1} B_l(\delta)(f_i, \ldots, f_{i+l}) f_{i+l+1} \cdots f_n, \quad (5.1)$$

for any $f_1, \ldots, f_n \in A$. We call δ a derivation of order k if $B_{k+1}(\delta) = 0$ on all *non-invertible* elements.

We will see below why we test the B_k only on non-invertible elements to find a proper notion of order. The rule (5.1) is in a sense a generalization of the Leibniz rule. We view $\delta : A \to A$ as acting via the Leibniz rule in a first iteration. The higher products $B_k(\delta)$ provide corrections to this rule. They also act via some kind of Leibniz rule, but instead they take multiple arguments at a time, instead of only one. This is reminiscent of the way a coderivation with k inputs acts.

Another definition for higher order derivations in the purely associative case has already been given before in [16], see also [69, 70]. In [16], each $B_k(\delta)$ was purely defined in terms of δ.

$$B_1(\delta) = \delta, \tag{5.2}$$

$$B_2(\delta)(f_1, f_2) = \delta(f_1 f_2) - \delta(f_1) f_2 - (-)^{f_1 \delta} f_1 \delta(f_2), \tag{5.3}$$

For $k \geq 3$ the formula is

$$\begin{aligned} B_k(\delta)(f_1, ..., f_k) = &\delta(f_1 \cdots f_k) - \delta(f_1 \cdots f_{k-1}) f_k - (-)^{f_1 \delta} f_1 \delta(f_2 \cdots f_k) \\ &+ (-)^{f_1 \delta} f_1 \delta(f_2 \cdots f_{k-1}) f_k. \end{aligned} \tag{5.4}$$

In [69], these maps were called *Börjeson braces*, after the author of [16]. The definition just given has the advantage that it is more robust, since all braces are defined with respect to δ. In contrast, from (5.1) it is not obvious whether or not it constrains some of the products to be zero due to their interdependence. However, the two definitions are actually equivalent. We will show that Börjeson's definition is equivalent to (5.1).

Proposition 4. *The Börjeson braces, as defined in equations* (5.2 − 5.4)*, satisfy (5.1). The other way around, any set of maps satisfying (5.1) are necessarily of the form (5.2 - 5.4).*

Proof. We show this by induction. By definition, the Börjeson braces satisfy (5.1) when $n \in \{1, 2\}$. Suppose that all the Börjeson braces $B_k(\delta)(f_1, ..., f_k)$ satisfy (5.1) for $k \leq n$. The induction step simply amounts to showing that certain sums partially cancel. We have

$$B_1(\delta)(f_1 \cdots f_{n+1}) = B_{n+1}(\delta)(f_1, ..., f_{n+1}) + \delta(f_1 \cdots f_n) f_{n+1} + (-)^{f_1 \delta} f_1 \delta(f_2 \cdots f_{n+1}) \tag{5.5}$$

$$- (-)^{f_1 \delta} f_1 \delta(f_2 \cdots f_n) f_{n+1} \tag{5.6}$$

$$= B_{n+1}(\delta)(f_1, ..., f_{n+1}) \tag{5.7}$$

$$+ \sum_{l=1}^{n} \sum_{i=1}^{n-l} (-)^{m(f_1 + ... + f_{i-1})} f_1 \cdots f_{i-1} B_l(\delta)(f_i, ..., f_{i+l}) f_{i+l+1} \cdots f_{n+1} \tag{5.8}$$

$$+ \sum_{l=1}^{n} \sum_{i=2}^{n+1-l} (-)^{m(f_1 + ... + f_{i-1})} f_1 \cdots f_{i-1} B_l(\delta)(f_i, ..., f_{i+l}) f_{i+l+1} \cdots f_{n+1} \tag{5.9}$$

$$- \sum_{l=1}^{n} \sum_{i=2}^{n-l} (-)^{m(f_1 + ... + f_{i-1})} f_1 \cdots f_{i-1} B_l(\delta)(f_i, ..., f_{i+l}) f_{i+l+1} \cdots f_{n+1} \tag{5.10}$$

$$= \sum_{l=1}^{n+1} \sum_{i=1}^{n+1-l} (-)^{m(f_1 + ... + f_{i-1})} f_1 \cdots f_{i-1} B_l(\delta)(f_i, ..., f_{i+l}) f_{i+l+1} \cdots f_{n+1}. \tag{5.11}$$

Hence also $B_1(\delta)$ satisfies (5.1).

On the other hand, reading the above computation backwards shows the other direction.

□

In [16] it was proven that the $B_k(\delta)$, defined as maps $A^{\otimes k} \to A$ over some algebra A, form an A_∞-algebra over A, if they are generated by a δ such that $\delta^2 = 0$. The fact that this is true may be reason to consider them as a good definition. Their commutative counterpart, the Koszul braces, can be shown to form an L_∞-algebra (c.f. [69]).

Other useful identities arise when one tries to express $B_k(\delta)$ purely in terms of $B_{k-1}(\delta)$.

Lemma 5. *The following identities hold.*

$$\begin{aligned}
B_2(\delta)(f,g) &= B_1(\delta)(fg) - B_1(\delta)(f)g - (-)^{f\delta} f B_1(g) \\
B_3(\delta)(f,g,h) &= B_2(\delta)(f,gh) - B_2(\delta)(f,g)h = B_2(\delta)(fg,h) - (-)^{f\delta} f B_2(\delta)(g,h) \\
B_{k+2}(\delta)(f_0,f_1,...,f_k,f_{k+1}) &= B_3(\delta)(f_0, f_1 \cdots f_k, f_{k+1}), \\
&= B_{k+1}(\delta)(f_0,...,f_k \cdot f_{k+1}) - B_{k+1}(\delta)(f_0,...,f_k)f_{k+1}, \\
&= B_{k+1}(\delta)(f_0 \cdot f_1,...,f_{k+1}) - (-)^{f_0\delta} f_0 B_{k+1}(\delta)(f_1,...,f_{k+1}),
\end{aligned}$$

where $k \geq 1$.

We don't give a proof for these statements. The above relations explain why we want to restrict to test only on non-invertible elements. Otherswise a derivation is not of second order, i.e. $B_3 \neq 0$, the higher braces $B_k(\delta)$ will also be non-zero when applied to units, since

$$B_{k+2}(f,g,1,...,1,h) = B_3(f,g,h) \text{ for any } k \geq 1. \tag{5.12}$$

There exists a quite brutal way to include invertible elements. This is done by considering *augmented* algebras. If A is an algebra over a field $\mathbf{k}$, we call a homomorphism $\varepsilon : A \to \mathbf{k}$ an augmentation if it satisfies $\varepsilon \circ \eta = \mathrm{id}_{\mathbf{k}}$, where $\eta : \mathbf{k} \to A$ is the homomorphism representing the unit in A. The augmentation allows us to project out the invertible part of an element through the projector $P = \mathrm{id}_A - \eta \circ \varepsilon$. We then could consider the braces $B_k(\delta) \circ P^{\otimes k}$. The drawback of this is that it yields more complicated relations than those given in lemma 5. This is also the reason why we don't want to consider this approach.

Remark 5.1.1. The condition for zero order derivations, $B_1(\delta)(f) = \delta(f) = 0$, naively does not seem to allow non-trivial order zero derivations. However, since we only test on non-invertible elements, a map which is non-zero only on invertible elements is technically of order 0. At first glance, this seems to be highly artificial. But, as we will see below, these fit nicely into the classification of derivations on tensor algebras.

Remark 5.1.2. As we mentioned in the end of section 3.6.1, in the commutative case, zero order derivations act by multiplication. In our case, multiplication from the left or from the right is of order two.

Next we show that, like in the commutative case, derivations are closed under taking commutators.

Proposition 5. *Given an algebra, A, the spaces* $Der_k(A)$ *satisfy*

$$[Der_k(A), Der_l(A)] \subseteq Der_{k+l}(A), \tag{5.13}$$

where the commutator is graded, i.e. $[\delta_1, \delta_2] = \delta_1 \circ \delta_2 - (-)^{\delta_1\delta_2} \delta_2 \circ \delta_1$.

Proof. Let δ_1 and δ_2 be a pair of linear maps on A. Using (5.1), a straightforward computation shows that

$$B_n([\delta_1,\delta_2]) = \sum_{\{(s,t)|s+t=n+1\}} \sum_{i=0}^{s+1} B_s(\delta_1)(\mathrm{id}^{\otimes i} \otimes B_t(\delta_2) \otimes \mathrm{id}^{\otimes(s-i-1)}) - (-)^{\delta_1\delta_2}(\delta_1 \leftrightarrow \delta_2). \tag{5.14}$$

In the following, we ignore the part of the commutator where the roles of δ_1 and δ_2 are interchanged, since the argument for this part works along the same lines as below. Assume $n \geq k+l$. We necessarily need $s > k$ or $t > l$ to satisfy the constraint in the above sum. Hence, if $\delta_1 \in \mathrm{Der}_k(A)$ and $\delta_k \in \mathrm{Der}_l(A)$, the terms in the sum are zero when applied to non-invertible elements, except for one case we now describe. Take $s = k+1$ and $t = l$. The value of $B_l(\delta_2)$ may be invertible. Hence, we cannot guarantee that

$$B_{k+1}(\delta_1)(\mathrm{id}^{\otimes i} \otimes B_l(\delta_2) \otimes \mathrm{id}^{\otimes(s-i-1)}) \tag{5.15}$$

is always zero. On the other hand, the case $s = k+2, t = l$ will always give zero, since $B_{k+2}(\delta_1) = 0$ when applied to a set of elements with at most one of them non-invertible. This can be seen by using the formulas given in lemma 5 together with the fact that the product of an invertible element with a non-invertible element is always non-invertible. We can therefore conclude that $B_n([\delta_1,\delta_2]) = 0$ whenever $n \geq k+l+1$. By definition, this means that $[\delta_1,\delta_2]$ is of order $k+l$. □

Remark 5.1.3. The anomalous behavior of the braces with respect to invertible elements leads to a mismatch between the commutative and non-commutative versions of the above proposition. Its commutative version reads

$$[\mathrm{Der}_k(A), \mathrm{Der}_l(A)] \subseteq \mathrm{Der}_{k+l-1}(A). \tag{5.16}$$

In our case, this rule is only satisfied generally when $k,l \in 1,2$. For $k = l = 1$, this is not surprising, since (order one) derivations form a Lie algebra, even over non-commutative algebras.

We now turn to the case when $A = \hat{T}V$. Using the relations we derived above, we can give a concrete description of derivations of some order k. We know that $\mathrm{Der}_k(\hat{T}V) \subseteq \mathrm{Der}_{k+1}(\hat{T}V)$, so whatever we call derivation of order k is also a derivation of order $l > k$. To really distinguish derivations of different orders, we can look at the quotients $\mathrm{sDer}_k(\hat{T}V) = \mathrm{Der}_k(\hat{T}V)/\mathrm{Der}_{k-1}(\hat{T}V)$, which we call the space of *strict* derivations of order k. We will show the following.

Proposition 6. *The quotient $sDer_k(\hat{T}V)$ is isomorphic to the space $\{\delta \in Der_k(\hat{T}V) \mid \delta_k(f) = 0$ for all $f \in T^{<k}V\}$. We further have $sDer_k(\hat{T}V) \cong \hom(V^{\otimes k}, \hat{T}V)$.*

Proof. The statement of the proposition is trivially true for $k = 0$. Suppose δ is of order $k \geq 1$. We can define a derivation δ_k of order k by the formula

$$\delta_k(a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-k} (-)^{l(a_1+\ldots+a_i)} a_1 \otimes \cdots \otimes a_i \otimes B_k(\delta)(a_{i+1}, ..., a_{i+k}) \otimes a_{i+k+1} \otimes \cdots \otimes a_n. \tag{5.17}$$

We also define it to be zero on less than k generators, so it has the property given in the lemma. To proof that δ_k is of order k it suffices to show that $B_l(\delta_k)$ is zero on generators for all $l > k$, since we can always write $B_l(\delta)(f_1, ..., f_{k+1})$ as superposition over TV of higher braces acting on generators. We are left to compute

$$B_l(\delta)(a_1, ..., a_l) = \delta_k(a_1 \cdots a_n) - \delta_k(a_1 \otimes \cdots \otimes a_{n-1})a_n - (-)^{\delta_k a_1} a_1 \delta_k(a_2 \otimes \cdots \otimes a_n) \tag{5.18}$$

$$+ (-)^{\delta_k a_1} a_1 \delta_k(a_2 \otimes \cdots \otimes a_{n-1})a_n, \tag{5.19}$$

which is straightforward to check that this vanishes whenever $n > k$. Note that $\delta_k(a_1 \cdots a_k) = B_k(\delta_k)(a_1, ..., a_k)$, so it is not of order lower than k unless δ was of order lower than k. The next step is to prove that $\delta - \delta_k$ is of order $k-1$. This follows immediately, since $B_l(\delta - \delta_k) = B_l(\delta) - B_l(\delta_k)$, and this difference is zero on generators for all $l \geq k$. Therefore it is zero on all inputs by the same argument as before.

The above proves that we can write any order $k \geq 1$ derivation δ as $\delta = \delta_k + \delta'$, where $\delta' \in \mathrm{Der}_{k-1}(TV)$. We are left to show that all derivations in $\{\delta \in \mathrm{Der}_k(\hat{T}V) \mid \delta_k(f) = 0 \text{ for all } f \in T^{<k}V\}$ act like (5.17). Assume we have such a δ. Substract δ_k, which we define using (5.17). Again, write $\delta - \delta_k = B_1(\delta) - B_1(\delta_k)$ as a superposition of $B_{l \geq 1}$ on generators. But these vanish, since for $l < k$, they are zero by definition of δ and δ_k, while for $l \geq k$, they are zero since $\delta - \delta_k$ is of order $k-1$. This proves that

$$\mathrm{sDer}_k(TV) \cong \{\delta \in \mathrm{Der}_k(TV) \mid \delta_k(f) = 0 \text{ for all } f \in T^{<k}V\}. \tag{5.20}$$

The isomorphism $\mathrm{sDer}_k(\hat{T}V) \cong \hom(V^{\otimes k}, \hat{T}V)$ is an immediate consequence. □

Proposition 6 provides a complete characterization of the space $\mathrm{Der}_n(\hat{T}V)$. We have $\mathrm{Der}_n(\hat{T}V) = \bigoplus_{k=0}^{n} \mathrm{sDer}_k(\hat{T}V)$. A strict derivation of order $k \geq 1$ is a map $\delta_k : V^{\otimes k} \to \hat{T}V$, acting on $\hat{T}V$ via formula (5.17). We also give another formula. Given a map $f : V^{\otimes k} \to \hat{T}V$, we can think of it as a map $f : \hat{T}V \to \hat{T}V$ by extending it to zero on all powers different from k. We define $D(f) \in \mathrm{Der}_k(\hat{T}V)$ by

$$D(f) = \nabla_3 \circ (\mathrm{id}_{\hat{T}V} \otimes f \otimes \mathrm{id}_{\hat{T}V}) \circ \Delta_3. \tag{5.21}$$

It is straightforward to check that this reproduces (5.17). A remarkable feature of this formula is that it is obviously invariant under taking its dual. This already hints to the fact that there will be a duality between higher order derivations of the tensor product and higher order coderivations of the tensor coproduct.

Remark 5.1.4. The commutative counterpart to this is a map $f : V^{\odot k} \to SV$, acting on SV as

$$f = f_{i_1 \ldots i_k}(a) \frac{\partial}{\partial a_{i_1}} \cdots \frac{\partial}{\partial a_{i_k}}. \tag{5.22}$$

Here, a strict order k derivation is generated by kth powers of ordinary derivations. This does not hold in our case. The action of the kth power of first order derivations over $\hat{T}V$ will generically not look like (5.17).

We should also discuss order zero derivations on $\hat{T}V$. As pointed out, these are maps which are non-zero only on invertible elements. The invertible elements of $\hat{T}V$ are exactly those $f \in \hat{T}V$ who satisfy $\pi_0(f) \neq 0$, where $\pi_0 : \hat{T}V \to \mathbb{C}$ is the canonical projection. In other words, they are the elements with non-zero constant part. Observe that to actually have non-trivial order zero derivations, it is important to work on the completed $\hat{T}V$ instead of TV. In TV, the only invertible elements are the constants. But these can always be written as a sum of non-constant elements, e.g. $1 = (1 + x) - x$, for some non-zero $x \in V$. Using linearity, we could then conclude that order zero derivations are zero also on constants.

A contrary approach to order zero derivations would be to use (5.21) to define order zero derivations as lifts of maps $\mathbb{C} \to \hat{T}V$. The statement of proposition 6 would then still be true. It is, however, not possible to give a linear brace $B_1(\delta)$ so that a lift δ_0 under (5.21) ever satisfies $B_1(\delta) = 0$. The most general expression for B_1 we can write is

$$B_1(\delta)(a) = X\delta(a) + Ya\delta(1) + Z\delta(1)a, \quad X, Y, Z \in \mathbb{C}, \tag{5.23}$$

as long as we want to preserve linearity in both arguments. But this does never vanish on δ_0, unless $B_1 = 0$ or $\delta_0 = 0$ of course. On the other hand, δ_0 satisfies $B_2(\delta_0 - \delta_0(1)) = 0$, so we can implement them by broaden our notion of order one derivations. Since this is also considered for derivations on commutative algebras (see the end of section 3.6.1) this appears to be a natural thing to do.

Fortunately, this does not destroy the hierarchy among derivations. Using the broader notion of first order derivations, we still have $\mathrm{Der}_1(A) \subseteq \mathrm{Der}_2(A)$. In fact, applying the shift $\delta \mapsto \delta - \delta(1)$ to the higher braces doesn't do anything.

As it turns out, any linear map $\delta : \hat{T}V \to \hat{T}V$ can be approximated by a sum of strict derivations.

Corollary 4. *Let $\delta \in Lin(T\hat{V}, T\hat{V})$. Then we can write*

$$\delta = \sum_{k=0}^{\infty} \delta_k, \tag{5.24}$$

with $\delta_k \in sDer_k(T\hat{V})$. The limit is taken pointwise.

Proof. Let δ_0 be the map acting by multiplication of $\delta_0(1)$ on $\mathbb{C} \subseteq \hat{T}V$. Further, define $\delta_{k\geq 1}$ via formula (5.17). Consider the difference

$$s_n = \delta - \sum_{k=0}^{n} \delta_k. \tag{5.25}$$

Using (5.1) it follows immediately that $s_n(a_1 \cdots a_k) = 0$ for all $k \leq n$ and $a_i \in V$. Observe that for any set of elements $a_1, ..., a_k \in V$, $s_n(a_1 \otimes \cdots \otimes a_k) = 0$ for $n > k$. This means that $\lim_{n\to\infty} s_n(f) = 0$ for any $f \in \hat{T}V$ in the topology of $\hat{T}V$. Therefore, $\sum_{k=0}^{n} \delta_k$ converges pointwise to δ. ☐

We are finally able to give a definition of higher order coderivations over $T^c(V)$, which was the original goal of this section. We dualize Börjesons definition (5.4) instead of (5.1), since the former leads to shorter expressions.

Definition 5.1.2. Given a linear map $\delta : T^c(V) \to T^c(V)$, we define a collection of linear maps $B^k(\delta) : T^c(V) \to T^c(V)^{\otimes k}, k \geq 2$ by the formula

$$B^k(\delta) = \Delta_k \circ \delta - (\mathrm{id} \otimes \Delta_{k-1} \circ \delta) \circ \Delta_2 - (\Delta_{k-1} \circ \delta \otimes \mathrm{id}) \circ \Delta_2 + (\mathrm{id} \otimes \Delta_{k-2} \circ \delta \otimes \mathrm{id}) \circ \Delta_3. \tag{5.26}$$

We call δ an order k coderivation if $B_{k+1}(\delta)$ is zero when projected onto $T_0^c(V) \otimes \cdots \otimes T_0^c(V)$.[1] In the case of $k = 1$, we obtain the usual notion of a coderivation. The space of order k coderivations on $T^c(V)$ is denoted by $\mathrm{Coder}_k(TV)$.

[1] This is dual to testing the Börjeson only on non-invertible elements.

The properties of higher order coderivations in thise sense follows from dualizing the statements about derivations. In particular, we have the following.

Corollary 5. *There is an isomorphism* $\hom(TV, V^{\otimes k}) \cong sCoder(TV)$. *The isomorphism is realized by*

$$f \longmapsto D(f) = \nabla_3 \circ (id_{TV} \otimes f \otimes id_{TV}) \circ \Delta_3. \tag{5.27}$$

As noted before, there is an obvious duality between derivations on coderivations.

Corollary 6. *Given a map* $f : V^{\otimes k} \to V^{\otimes l}$, $k, l \geq 1$. f *defines both a strict derivation of order* k *and a strict coderivation of order* l *via the lift (5.27).*

5.2. Some Examples of Derivations in Higher Orders

We want to discuss some examples of higher order derivations on $\hat{T}V$. Of course, the reader can construct an infinite number of examples using (5.27). We will therefore consider examples which are not of this form in an obvious manner.

The first type of map we consider is left/right multiplication by an element in $X \in \hat{T}V$. Let $L_X : \hat{T}V \to \hat{T}V$ be X acting by left multiplication. It is immediately obvious that L_X is neither of order zero nor of order one. Let us check whether it is of order 2. We compute

$$B_3(L_X)(f, g, h) = Xfgh - Xfgh - (-)^{Xf} fXgh - (-)^{-Xf} fXgh = 0. \tag{5.28}$$

We see that L_X is in fact of order 2. The same is true for right multiplications. We find this example particularly interesting since (left) multiplication is what is usually considered to be of order 0 when working with commutative algebras.

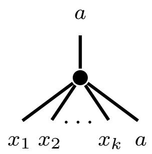
According to our analysis, we should be able to write L_X as a sum of strict derivations. In order zero, it obviously acts as

$$1 \mapsto X, \tag{5.29}$$

and zero on $V^{\otimes k \geq 1}$. In order one, it acts on $a \in V$ as

$$a \mapsto X \otimes a, \tag{5.30}$$

and extended as a derivation on all of $\hat{T}V$. If $X = x_1 \otimes \cdots \otimes x_k$, we may draw this diagrammatically as

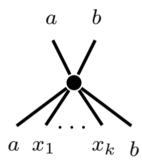


(5.31)

In order two, its action is defined using the bilinear brace B_2. It is

$$B_2(L_X)(a_1, a_2) = Xa_1a_2 - Xa_1a_2 - (-)^{a_1 X} a_1 X a_2 = -(-)^{a_1 X} a_1 X a_2. \tag{5.32}$$

Pictorially,

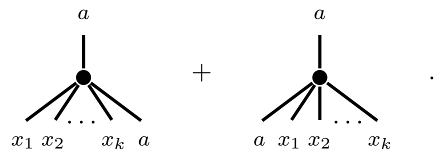
$-$  .

(5.33)

Another example we want to have a look at is the following. Let again $X \in \hat{T}V$. When we think of it as a map $\mathbb{C} \to \hat{T}V$, we can lift it using (5.27). We denote the lift by $D(X)$. It acts on generators as

$$D(X)(a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n+1} (-)^{X(a_1+\ldots+a_{i-1})} a_1 \otimes \cdots \otimes a_{i-1} \otimes X \otimes a_i \otimes \cdots \otimes a_n. \quad (5.34)$$

We have $B_{k\geq 3}(D(X)) = 0$, which is straightforward to check on generators using (5.34). Therefore, $D(X)$ is of degree 2. The splitting into strict derivations turns out to give almost the same as the splitting of L_X. In fact, $B_2(L_X) = B_2(D(X))$. They only differ in order 1, where $D(X)$ acts as



(5.35)

This shows where equation (5.27) does not meet the expectation that a function $V^{\otimes k} \to \hat{T}V$ lifts to a derivation of order k also in the case $k = 0$.

5.3. The Anomalous Behavior of the Braces with Respect to Invertible Elements

We saw in 5.1 that, in order to obtain a good notion of higher order derivations, we need to restrict the vanishing condition of the braces to non-invertible elements. This was due to the fact that $B_k(g, 1, ..., 1, h, l) = B_3(g, k, l)$. This also lead to the observation that

$$[\mathrm{Der}_k(A), \mathrm{Der}_l(A)] \subseteq \mathrm{Der}_{k+l}(A), \quad (5.36)$$

while we would expect $\mathrm{Der}_{k+l-1}(A)$ on the right hand side in analogy to what happens for derivations over commutative algebras.

We would avoid all this trouble if we would restrict to algebras without units. We would never need to mention the phrase “on non-invertible elements”. On top of that, we would indeed have

$$[\mathrm{Der}_k(A), \mathrm{Der}_l(A)] \subseteq \mathrm{Der}_{k+l-1}(A). \quad (5.37)$$

Also, for $A = \hat{T}_0 V := \prod_{k\geq 1} V^{\otimes k}$, formula (5.27) would apply without the exception of lifts of maps $\mathbb{C} \to \hat{T}V$.

There is, however, a reason why we will work on $\hat{T}V$ instead of $\hat{T}_0 V$ (dually, we work on $T^c(V)$ instead of $T^c_0(V)$). We obviously would exclude derivations with non-zero image in $\mathbb{C} \subseteq \hat{T}V$. Even when we are willing to pay that price, there is also an issue on the coalgebra side. The dual notion of a derivation with image in $\mathbb{C}$ is the lift of an element $X \in T^c(V)$ under (5.27). This would actually restrict to a map $T^c_0(V) \to T^c_0(V)$, so it would not be excluded from the start. However, it will not be a coderivation of any definite order. On the other hand, it would have a finite order on $T^c(V)$ according to corollary 6. The reason for this is that when switching from $T^c(V)$ to $T^c_0(V)$, one needs to replace the coproduct Δ in the definition of the co-Braces B^k with the reduced coproduct $\overline{\Delta}$. This already happens when one only considers order 1 coderivations defined by elements $m_0 \in V$, as one would for weak A_∞-algebras.

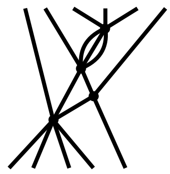


Figure 5.1.: A planar graph. In this way we may display a connected contribution obtained by repeated application of four coderivations.

5.4. Definition of Planar Quantum Homotopy Algebras

We now come to the definition of what we would call a planar quantum homotopy algebra.

Definition 5.4.1. A planar quantum homotopy algebra on a graded vector space V consists of a degree one linear map D on $T^c(V)$ of the form

$$D \in \prod_{k=0}^{\infty} \hbar^k \operatorname{Coder}^{k+1}(TV) \tag{5.38}$$

such that $D(\hbar = 0)|_{V^0} = 0$ and $D^2 = 0$.

The coderivation lift from corollary 5 makes it apparent why it is natural to interpret coderivations as planar graphs. Given an element $v_1 \otimes \cdots \otimes v_n$ and a map $f : V^{\otimes k} \to V^{\otimes l}$, the coderivation lift of f acts as

$$v_1 \otimes \cdots \otimes v_n \mapsto \sum_{s=0}^{n-k+1} (-)^{(v_1 + \ldots v_{s-1})f} v_1 \otimes \cdots \otimes v_{s-1} \otimes f(v_s, \ldots, v_{s+k-1}) \otimes v_{s+k} \otimes \cdots \otimes v_n. \tag{5.39}$$

In figure 5.1 shows how parts of the composition of four coderivations are depicted. We now would like to introduce an operation that is similar to the second order coderivation entering loop L_∞-algebras. The natural candidate is $\omega^{-1} = e_i^* \otimes e^i$, or rather its lift $\theta = D(\omega^{-1})$ to a second order coderivation. An argument on why θ is a likely candidate comes from what we know from string field theory. Recall that empty boundaries are created from Δ whenever it connects neighboring punctures on a single boundary. But this is precisely what θ does. With this settled, let us mimic definition 3.6.4. Given a differential graded vector space (V, d) and a first order coderivation $M(\hbar)$ on $T^c(V)$, we would like to impose

$$(D(\mathrm{d}) + M(\hbar) - i\hbar\theta)^2 = 0. \tag{5.40}$$

Now since $D := D(\mathrm{d}) + M(\hbar) - i\hbar\theta$ is of order two, its square $D^2 = \frac{1}{2}[D, D]$ is at most of order four by the dual of proposition 5. Here, we cannot avoid the anomalous behavior of our definition of higher order coderivations. The problem is that θ is a lift of an element of $T^c(V)$, i.e. it is a map with no inputs. But the fact that D^2 is generally of order four reveals another problem. The only order four contribution comes from θ^2. Hence, a necessary condition for (5.40) to hold is that θ to square to zero. But it is easy to see that this is not the case. In fact

$$\theta^2 = D((-)^{e_i^*} e_i^* \otimes e_j^* \otimes e^j \otimes e^i). \tag{5.41}$$

We find that it is not possible to obtain a planar quantum version of A_∞-by adding an operation that creates loops by connecting neighboring inputs. On the other hand, we may still consider 5.4.1 as a good definition. However, it is most likely that this is too general in order to have a good application in any form of quantum field theory.

5.5. The Large N Limit

Let us now take another route and explore the large N limit of a Yang-Mills in the color-ordered language. We may be lucky and obtain a good notion of a planar quantum algebra by considering the $N \to \infty$ limit of the full non-planar algebra. Let us first recall why only planar Feynman diagrams survive in that limit. In the A_∞ formulation of Yang-Mills, vertices come accompanied by traces

$$\mathrm{tr}_f(T_A T_B T_C), \quad \mathrm{tr}_f(T_A T_B T_C T_D), \tag{5.42}$$

where the T_A are a basis for the Lie algebra normalized such that

$$\mathrm{tr}_f(T_A T_B) = \frac{1}{2}\delta_{AB}. \tag{5.43}$$

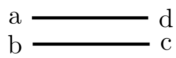
By tr_f, we mean that the trace is taken in the fundamental representation. This is just a choice of inner product on the Lie algebra. Whenever there is a propagator connecting two legs, the Lie algebra matrices are identified an summed over. For example, connecting two cubic vertices leads to expressions like

$$\mathrm{tr}(T_A T_B T_C)\,\mathrm{tr}(T_C T_D T_E). \tag{5.44}$$

Suppose now for a moment that we work with $U(N)$. In evaluating the traces above, we use the Fierz identity

$$T_A{}^{ab} T_A{}^{cd} = \frac{1}{2}\delta^{ad}\delta^{bc}. \tag{5.45}$$

We now use the double line notation of 't Hooft to display this identity graphically.

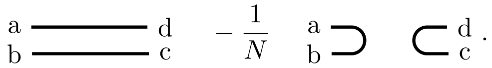


(5.46)

The lines represent the delta functions. Let us do a similar thing for other gauge groups. If we consider $SU(N)$ instead, the Fierz identity is

$$T_A{}^{ab} T_A{}^{cd} = \frac{1}{2}\delta^{ad}\delta^{bc} - \frac{1}{2N}\delta^{ab}\delta^{cd}. \tag{5.47}$$

In this case, we would draw

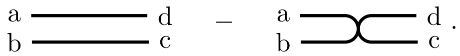


(5.48)

Another interesting case is $SO(N)$. We have

$$T_A{}^{ab} T_A{}^{cd} = \frac{1}{2}\delta^{ad}\delta^{bc} - \frac{1}{2}\delta^{ac}\delta^{bd}. \tag{5.49}$$

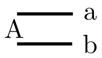
In the double line notation, this is



(5.50)

At this point, we can draw a connection to open-string theory. We can think of the two lines as actual boundaries of a strip. The propagator in the $U(N)$ case is just the strip. On the other hand, if we work with $SO(N)$, we add a twisted propagator. This is the equivalent to the twist sewing producing the unoriented surfaces in open-string theory.

We represent an external state with matrix ${T_A}^{ab}$ by

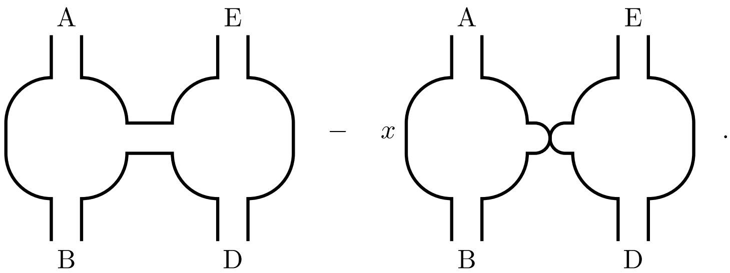


(5.51)

In this picture, let us evaluate the four-point scattering with cubic vertices at tree level. We write (5.44) in components,

$$
\begin{aligned}
{T_A}^{ab}{T_B}^{bc}{T_C}^{ca}{T_C}^{de}{T_D}^{ef}{T_E}^{fd} &= \frac{1}{2}{T_A}^{ab}{T_B}^{bc}{T_D}^{cf}{T_E}^{fa} - \frac{x}{2}{T_A}^{ab}{T_B}^{bc}{T_E^T}^{cf}{T_D^T}^{fa} && (5.52)\\
&= \frac{1}{2}\,\mathrm{tr}_f(T_AT_BT_DT_E) - \frac{x}{2}\,\mathrm{tr}_f(T_AT_BT_E^TT_D^T). && (5.53)
\end{aligned}
$$

Here, $x = 0$ for $U(N)$ and $x = 1$ for $SO(N)$. We denoted by A^T the transposed of a matrix A. Diagrammatically, this is



(5.54)

Whenever fundamental indices (small letters) are contracted, we connected them by a single line. The pictures in the above equation are again very reminiscent of what we saw in string theory. Both pictures look topologically like a disc with four open strings attached to its boundary. Also, we notice that the states D and E enter with inverse orientation in the second diagram relatively to how the enter in the first. This is of course just a way to say that the matrices T_D and T_E get transposed. On the other hand, since we obtain this contribution only in the $SO(n)$ case, we have that $T_D^T = -T_D$ and $T_E^T = -T_E$. We could say that the states in $SO(n)$ are twist invariant (up to a sign). This is also the case in open string theory. In unoriented string theory, we only allow for twist invariant states (c.f. [74]).

In the large N limit, all of the diagrams discussed above enter equally, since at no point did we pick up a factor $\mathrm{tr}_f(1)$. In the double line picture, there was no free boundary, i.e. no closed single-line loop. The pictures in (5.54) are straightforwardly generalized to tree level amplitudes with any number of external states. It is easily seen that no free boundaries appear along that way. Things change when we include loops. We consider the group theory factor in the gluon self-energy. It is

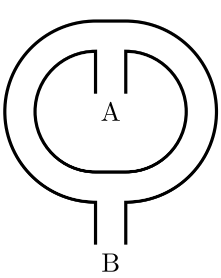
$$
\mathrm{tr}_f(T_CT_AT_D)\,\mathrm{tr}_f(T_CT_BT_D) + \mathrm{tr}_f(T_CT_DT_A)\,\mathrm{tr}_f(T_CT_BT_D) =: A_1 + A_2. \qquad (5.55)
$$

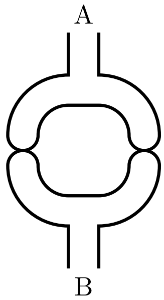
We evaluate A_1 and A_2 with the help of the Fierz identities. We find

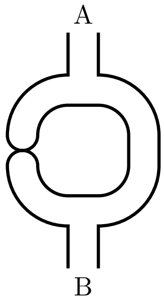
$$A_1 = \frac{1}{4}\operatorname{tr}_f T_A \operatorname{tr}_f T_B - \frac{x}{2}\operatorname{tr}_f(T_A T_B^T) + \frac{x^2}{4}\operatorname{tr}_f(1)\operatorname{tr}_f(T_A T_B), \tag{5.56}$$

$$A_2 = \frac{1}{4}\operatorname{tr}_f(1)\operatorname{tr}_f(T_A T_B) - \frac{x}{2}\operatorname{tr}_f(T_A T_B^T) + \frac{x^2}{4}\operatorname{tr}_f(T_A)\operatorname{tr}_f(T_B). \tag{5.57}$$

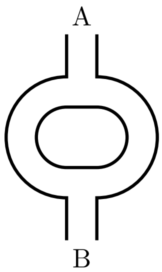
Like before, $x = 0$ is $U(N)$ and $x = 1$ is $SO(N)$. In both cases, we have a contribution that scales with N. Their origin become clear in the double line notation.

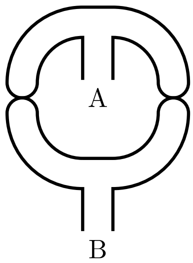
$\operatorname{tr}_f T_A \operatorname{tr}_f T_B =$ 

$=$ 

, $\operatorname{tr}_f(T_A T_B^T) =$ 

, (5.58)

$\operatorname{tr}_f(1)\operatorname{tr}_f(T_A T_B) =$ 

$=$ 

. (5.59)

We observe the following. In $tr_f(A)tr_f(B)$ we have two disjoint boundaries. One is ending in the external state A, the other one in B. These give the traces over A and B. Further, we see that $tr_f(AB^T)$ has only a single boundary with both states A and B attached. At last, there is also $\operatorname{tr}_f(1)\operatorname{tr}_f(AB)$. It has two boundaries, and one of them has no external state attached. Therefore, this last one is the only one that scales with N. The structure is clearly the same as in open-string theory at one-loop. There is a planar, an unorientable and a non-planar diagram.

We now combine A_1 and A_2 and find

$$A_1 + A_2 = \frac{N}{4}(1+x^2)\operatorname{tr}_f(AB) - x\operatorname{tr}_f(AB^T) + \frac{1}{4}(1+x^2)\operatorname{tr}_f(A)\operatorname{tr}_f(B). \tag{5.60}$$

Note that, apart from a factor of two, the only difference between $U(N)$ and $SO(N)$ is the unorientable diagram. This one gets suppressed along with the non-planar diagram when we send N to infinity. So for large N, the theories with gauge groups $U(N)$ and $SO(N)$ essentially become the same (apart from the fact that $U(N)$ has more gauge bosons than $SO(N)$ at fixed N). This also holds at higher loop level. Unorientable diagrams will always come with fewer traces and will therefore always be suppressed. The dominating contribution will always come from a diagram with maximal number of traces (boundaries) and such that only one boundary is occupied with external states. Both $SO(N)$ and $U(N)$ have these. Therefore, in the following we can pretend to work with $U(N)$ without loosing anything interesting.

Let us now consider gauge theory as a full quantum A_∞-algebra and see what happens to the quantum A_∞ relations when we take the limit $N \to \infty$. We use the definition we gave in section 3.6.3. A general quantum A_∞-algebra has products with internal loops, i.e. products which inherently involve multiple traces, including "empty" traces $\mathrm{tr}_f(1)$. Out of these, only the planar products will survive the large N limit. For example, let us consider quartic one-loop vertices with group theory factors

$$f_{2,2} = \mathrm{tr}_f(AB)\,\mathrm{tr}_f(CD), \quad \text{and} \quad f_{4,0} = \mathrm{tr}_f(1)\,\mathrm{tr}_f(ABCD). \tag{5.61}$$

When $N \to \infty$, $f_{2,2}$ becomes negligible. In a large N limit of quantum A_∞, we therefore simply discard $f_{2,2}$. We also assume that the theory satisfies the quantum master equation for all N. It follows that

$$Qf + \frac{1}{2}\{f, f\} - i\hbar\Delta(f) = 0. \tag{5.62}$$

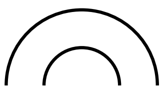
is also true when $N \to \infty$. There is, however, one crucial difference between taking the large N on the products and the large N on the relations. The relations involve Δ. Let us take a look again at $f_{2,2}$, the product we discarded previously. Before taking the planar limit, it enters into the master equation. The action of Δ identifies inputs pairwise. We obtain

$$\Delta(f_{2,2})(AB) = 2\,\mathrm{tr}_f(AT_C)\,\mathrm{tr}_f(BT_C) + 2\,\mathrm{tr}_f(T_CT_C)\,\mathrm{tr}_f(AB) = \mathrm{tr}_f(AB) + N^2\,\mathrm{tr}_f(AB). \tag{5.63}$$

Both contributions only have a single occupied boundary. However, the first one was obtained by identifying two seperate boundaries. From the perspective of string field theory, we would say that this vertex has a genus. So there is only a single boundary which is occupied by two fields, and no empty boundary that would contribute a factor of N. On the other hand, identifying two legs on the same boundary led to a quadratic sacling in N. We created a new boundary by identifying two adjacent fields on a single boundary. Moreover, by removing all fields from one of the original boundaries, we created another free boundary. This is problematic. In the large N limit of the master equation, we would no longer discard the factor $N^2\,\mathrm{tr}_f(AB)$. Also, in general it is needed to cancel contributions coming from both $\{f, f\}$ and $\mathrm{d}f$. Consider $f_{1,2}(A, B, C) = \mathrm{tr}_f(A)\,\mathrm{tr}_f(BC)$ and $f_{1,0}(A) = \mathrm{tr}_f(1)\,\mathrm{tr}_f(A)$. $\{f_{1,0}, f_{1,2}\}$ then would have a contribution of the form $N^2\,\mathrm{tr}_f(AB)$. At the same time, a fundamental vertex $f_{2,0,0} = N^2\,\mathrm{tr}_f(AB)$ would also contribute via $\mathrm{d}f_{2,0,0}$. If we would only have a contribution from $\{f_{1,0}, f_{1,2}\}$, the large N limit would not be problematic, since we would also discard $f_{1,2}$ from the start. But a non-zero $f_{2,0,0}$ would survive $N \to \infty$.

We find that, in order to have a planar solution to the quantum master equation, we should at least have $\mathrm{d}g = 0$ for all planar vertices g with internal loops. However, such a solution would in most cases be useless from the perspective of homotopy theory. We may want to apply a homotopy transfer in order to integrate out some fields. But also, as we have seen, we can use the homotopy transfer to regularize the theory. But we cannot expect that the condition $\mathrm{d}g = 0$ is preserved under homotopies.

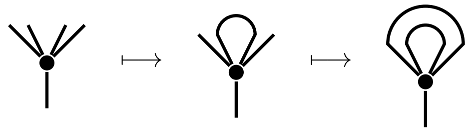
The difficulties we observed here are actually related to the problem that $\theta^2 \neq 0$ we found in section 5.4. Graphically, we would draw this as a double arc,

$\theta^2 =$  . (5.64)

Its action on a vertex $M : V^{\otimes n} \to V$ can be expressed in terms of commutators,

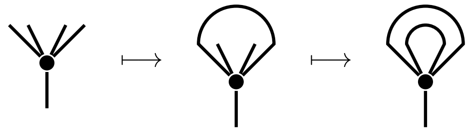
$$[\theta, [\theta, M]] = \frac{1}{2}[[\theta, \theta], M] = [\theta^2, M]. \tag{5.65}$$

Let us display this as a sequence of applications on a five-point vertex.



. (5.66)

There are of course also contributions where θ connects other inputs, but these add up to zero after θ acts the second time. In the non-planar sector, we obtain another contribution in the following way.



. (5.67)

By degree reasons, this exactly cancels (5.66). Notice that the product appearing in the intermediate step of (5.67) is non-planar. It has two boundaries, one has two inputs, while the other one has only one. The final step then restored planarity of the product by connecting the legs on the two-input boundary. This process created two free boundaries at a time in the same way we obtained two powers of N by a single application of Δ. The absence of this process in a planar theory leads to a violation of both the quantum master equation (5.62) and to the condition we proposed previously in (5.40).

5.6. Gauge Invariance of Planar Amplitudes

We gave several arguments why a consistent planar subsector of loop A_∞-algebras does not exist. So does this mean that one should abandon the planar limit all together? In the literature, the planar limit is usually taken at the level of amplitudes. One consistency condition for the amplitudes is gauge invariance. Recall that amplitudes can be obtained as the minimal model of homotopy algebras. In the particular case of loop L_∞-algebras, the chain map between a theory and its minimal model is the perturbative path integral, as we saw in section 3.4.1. However, this form of the amplitudes does not allow for a check of gauge invariance, since the fields are already restricted to cohomology of the linear differential Q. We therefore need amplitudes that are defined on Q-exact states (at least). A straightforward way to do this is to remove the projector p and the inclusion i from external legs. As a reminder, we recall that the maps p and i entering the definition of the minimal model are parts of a homotopy equivalence

$$i : (H(V), 0) \leftrightarrows (V, Q) : p. \tag{5.68}$$

between a differential graded vector space (V, Q) and its cohomology $H(V)$. In this way, amplitudes are even defined on unphysical states (states that are not Q-closed). In this setup, gauge invariance of amplitudes amounts to showing that amplitudes are invariant under Q-exact shifts

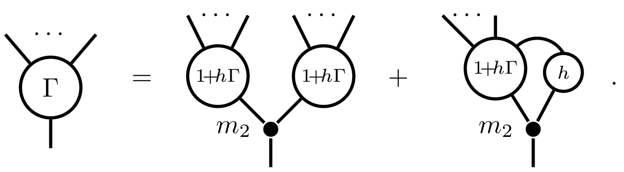
$$\phi \mapsto \phi + Q\lambda. \tag{5.69}$$

To lowest order, this is equivalent to saying that amplitudes, when expressed as maps $\Gamma : T^c(V) \to V$, commute with Q. Hence, gauge invariance is proven by showing that $[Q, \Gamma] = 0$.

Without any further assumptions, $[Q, M] = 0$ is generally not true even when a theory is gauge invariant by conventional means. The crucial assumption we need is that amplitudes vanish when one propagator (homotopy h) is replaced by the projector $i \circ p$. For example, this is generally is believed to be true in string theory, where amplitudes are assumed to vanish in the infrared limit of the moduli space. We want to stress that certain graphs do not need to vanish individually under $h \mapsto i \circ p$, but only their sum has to.

We now want to show that planar graphs are indeed gauge invariant. However, we make two rather strong simplifications. First of all, we consider a theory with cubic vertices only. This implies that we never have to consider the effect of $[Q, \cdot]$ on vertices. All non-zero contributions come from $[Q, \cdot]$ acting on a homotopy h. The second simplification deals with the action on the homotopy. We assume that graphs vanish individually under the replacement $h \mapsto i \circ p$. The net effect is that whenever we have a $[Q, h] = 1 - i \circ p$, we can drop the $i \circ p$.

Planar graphs are defined according to the following recursive definition



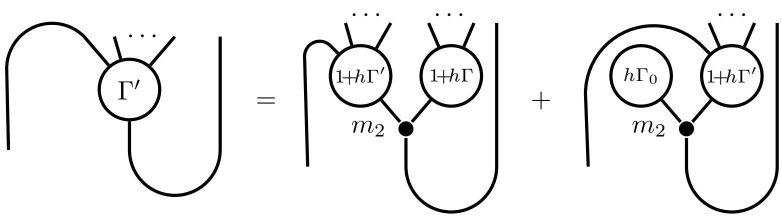
(5.70)

The initial condition is $\Gamma = 0$. From that on, we grow the graph by repeatedly applying the rule (5.70).

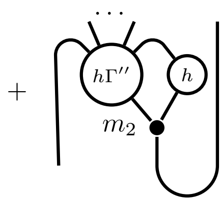
The first thing we need to proof is cyclicity.

Proposition 7. *Assuming that the cubic vertex m_2 is cyclic, the graphs defined via (5.70) are also cyclic.*

Proof. We prove this by induction over the number of vertices n. For $n = 1$, the only graphs are the vertex itself, which we assume to be cyclic, and the tadpole, which has no leaves and hence is trivially cyclic. Now assume that we have proven cyclicity up to $n = k$ and consider a graph Γ with $k + 1$ vertices. We rotate it using ω. To make use of the induction hypothesis, we want to write Γ using the rule (5.70), so that it is expressed in terms of graphs with at most k vertices. We have to be careful with tadpoles when we do this, since ω^{-1} cannot attach to a tadpole. Write Γ_i for the part of Γ with i inputs. Further, define $\Gamma' = \Gamma - \Gamma_0$, $\Gamma'' = \Gamma' - \Gamma_1$ and $\Gamma''' = \Gamma - \Gamma_2$. Note that, in order to proof cyclicity of Γ it suffices to do so for Γ', since any tadpole diagram is trivially cyclic. We have

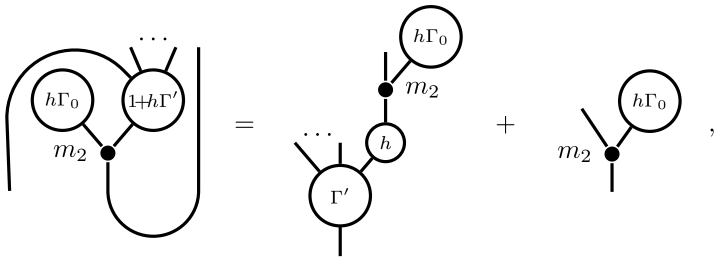


(5.71)

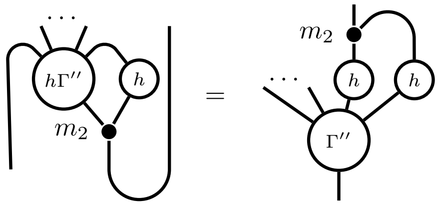


(5.72)

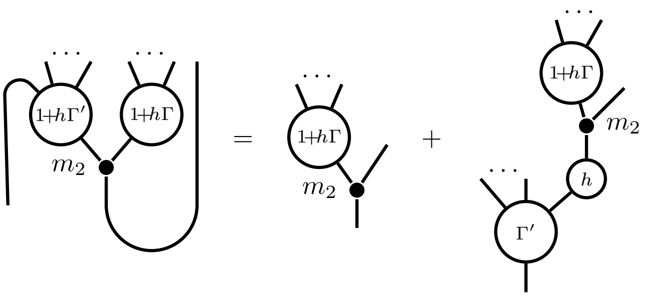
We carelessly used the letter Γ on both sides of the equation, however it should be clear how this is to be read. Both sides should contain only these graphs so the result is a graph with $k+1$ vertices. For example, the number of vertices of the two $1+h\Gamma$ of the tree contribution on the right hand side should add up to k. By the induction hypothesis, all the objects appearing on the right hand side are cyclic. We can use this to kill the external arcs.



(5.73)

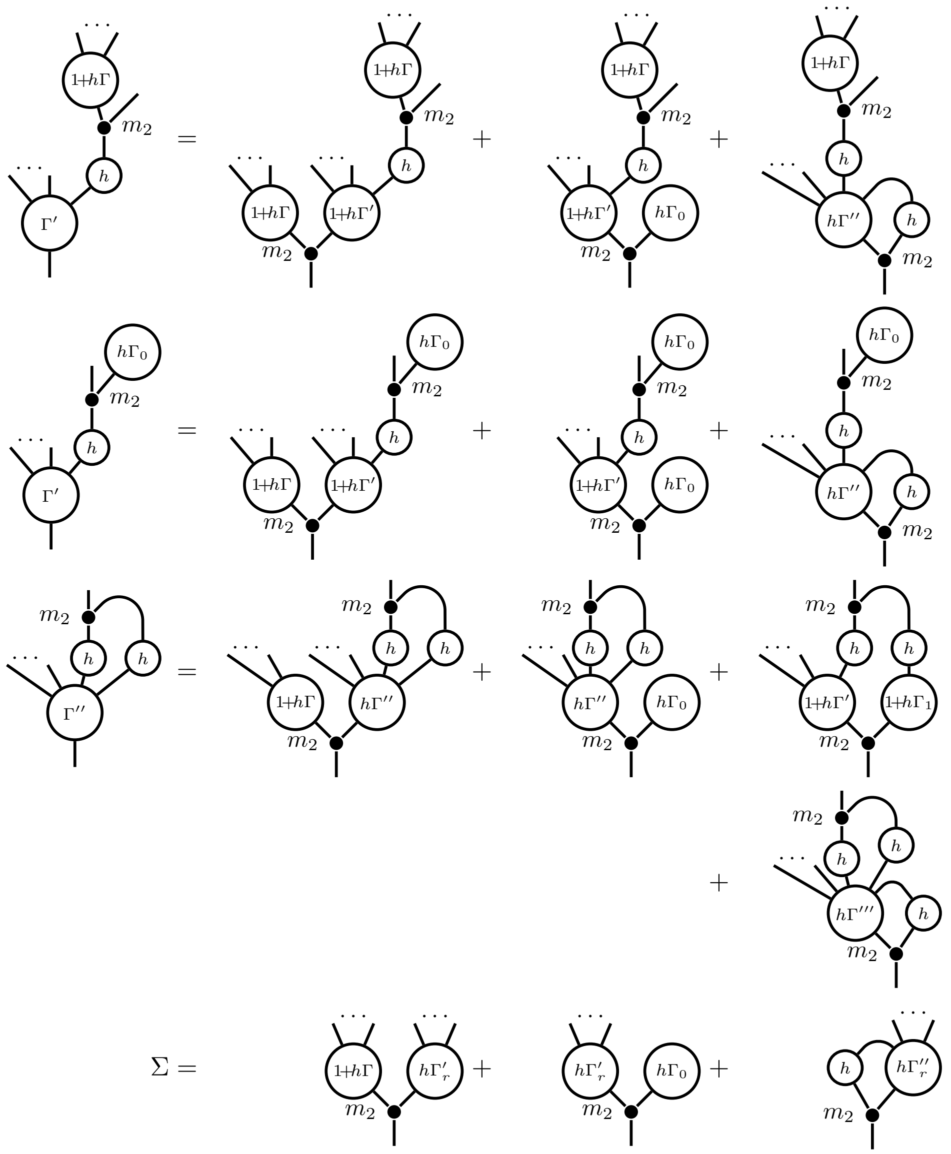


(5.74)



(5.75)

The strategy now is to expand the graphs one more time and identify cyclically rotated subgraphs. These will have at most k vertices, and therefore will be cyclically invariant by assumption.



(5.76)

(5.77)

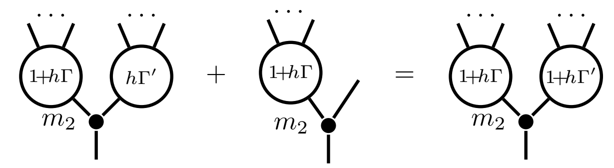
(5.78)

(5.79)

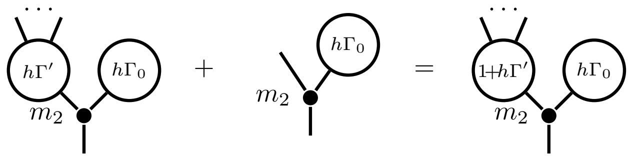
(5.80)

The above is supposed to be read as follows. The letter Σ denotes the sum of all the three graphs above of it. It itself is stated as a sum of graphs, each of which is the sum of the graphs appearing straight above of it. The subscript r says that the graph is cyclically rotated. By the induction hypothesis, they are cyclically invariant and so the subscript can be dropped.

We are almost there. We did not include two very simple graphs in the above. With the graphs in Σ, they combine to

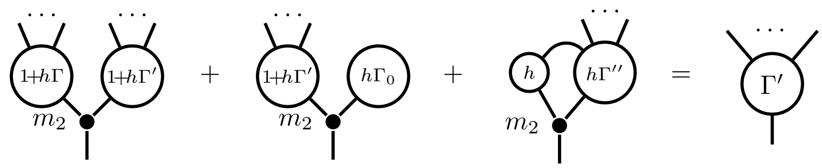


(5.81)



(5.82)

We obtain Γ' by adding up all the remaining graphs.

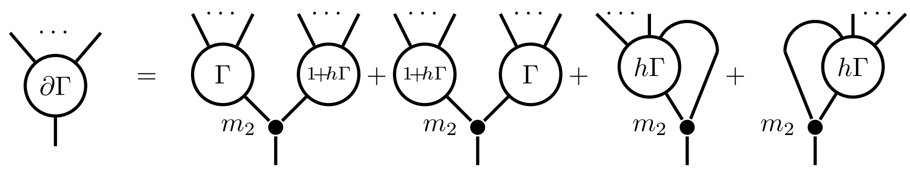


(5.83)

□

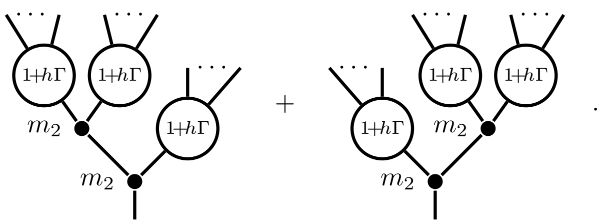
Theorem 7. *Assuming that $\partial h = [Q, h] = 1$, the graphs defined by (5.70) are gauge invariant.*

Proof. We assume we have already been able to show gauge invariance for all graphs up to a fixed number of vertices n. We then take a graph with $n+1$ vertices, and expand it using the recursive definition. On the right hand side, we then only have graphs with up to n vertices. By the induction hypothesis, the action of $\partial = [Q, \cdot]$ is then zero on these. We therefore only get contributions only when ∂ acts on propagators h.



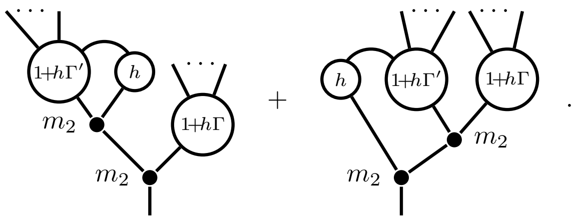
(5.84)

We already made use of one of the A_∞-relations, which is $\partial m_2 = 0$. The next step is to use the recursive definition of Γ in (5.84) again, so that we can apply $(m_2)^2 = 0$. There will be a total of eight graphs. We can distinguish some of them through their number of explicit loops (loops that are actually drawn, we don't count loops hidden in the various Γ). There are two grahps with zero loops, four with one loop, and again two with two loops. We begin with zero loops. They add up to



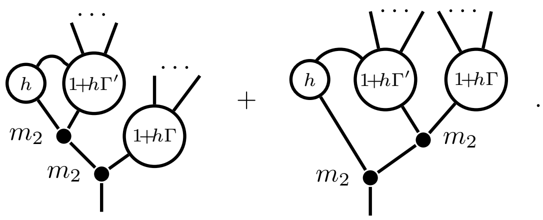
(5.85)

Since $(m_2)^2 = 0$, this vanishes. Two of the one-loop graphs are



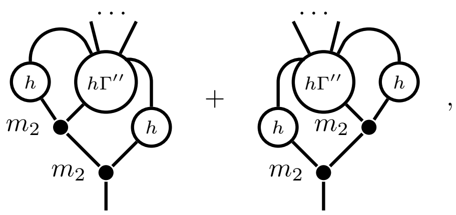
(5.86)

We can rearrange the vertices of the left graph using the cyclicity of Γ,



(5.87)

These cancel again by $(m_2)^2 = 0$. The argument for the other two one-loop graphs works the same (they are the mirror images of the graphs above). We also omitted two graphs with explicit tadpoles. They can be shown to vanish along similar lines. Finally, we have in two loops



(5.88)

which again adds to zero. □

5.7. A Larger Subsector of Loop Homotopy Associative Algebras

A consistent planar subsector of loop homotopy associative algebras does not exist. We saw that the main reason for this was that $\theta^2 \neq 0$ in (5.40). Equivalently, we could say that restricting Δ in (5.62) to connect only neighboring punctures also produces an operation that does not square to zero. The reason is that we miss the operation pictured in equation (5.67). If we want some consistent restriction of Δ, we should make sure that an operation like (5.67) is included. This can in fact be achieved. Consider an operation $\tilde{\Delta}$ that only connects legs that sit on the same boundary. It is straightforward to see that this operation squares to zero.

To get a clearer picture about what $\tilde{\Delta}$ does, let us look at what happens to geometric vertices in terms of their topology. According to table 4.1, connecting punctures on different boundaries always reduces the total number of boundaries by one unit. At the same time, it increases the genus. Dropping these operations allow us to stay at surfaces of genus zero, but admits surfaces with any number of boundaries. Punctures are distributed among them without any restrictions.

Of course to obtain a consistent string theory, we should include the closed string to cancel anomalies. Also, we saw that unoriented strings give rise to non-zero genus at higher loops. We therefore should restrict to oriented strings only. But this means that we necessarily work on a curved background, since our gauge group cannot be $SO(2^{d/2})$. This would be a theory of oriented quantum open strings coupled to classical closed strings. At the massless level, this is a quantum theory of gauge fields coupled to classical gravity.

Remark 5.7.1. During the final stages of this work, the author found out that, in the context of oriented open-closed string theory, this subsector was already mentioned in [101], in particular **equation (3.2.4)**.

The subsector just described may also be studied outside of string field theory, for example in the context of non-chiral gauge theories, since these never have an anomaly. On these, $\tilde{\Delta}$ always increases the number of traces. It would be interesting to see whether this does only allow for "unoriented" gauge groups like $SO(N)$.

6. Conclusion and Outlook

Let us recall the two original questions that motivated this work. The first one was concerned about planar string field theory.

Question 1: Is it possible to have a quantum open-string field theory of planar diagrams without the closed string?

The following answer was given in section 4.

Conclusion 1: There is no consistent string theory with planar and unorientable diagrams only. In particular, the closed string is necessary to obtain a consistent theory of open strings at the quantum level.

We can summarize the essential argument in two sentences. The tadpole anomaly can only be canceled when the unoriented string is included. However, the unoriented string creates closed strings appearing in loops.

The non-polynomial nature of closed-string field theory makes it difficult to include the closed string to the quantum open string beyond the conceptual level. Of course, a mayor step would be made if someone is able to find a polynomial formulation. However, it is shown in [84] that a cubic theory cannot be obtained in terms of the standard method of decomposing of the moduli space with geometric vertex.

The second main question was concerned about the existence of a planar subsector of A_∞-algebras at the quantum level.

Question 2: Is there a consistent planar subsector of the quantum version of A_∞-algebras, and is it realized by the large N limit of gauge theories?

We did not give a conclusion to this question, yet, but we will do so now. First of all, we saw that the large N limit of planar field theories cannot exist. The reason is that there are in general non-planar vertices that nevertheless give rise to planar Feynman graphs. Another way of saying this is the following. Given a vertex M and a the BV Laplacian Δ, it is generally not true that

$$\lim_{N\to\infty} \Delta M = \Delta \lim_{N\to\infty} M, \tag{6.1}$$

since ΔM may be planar while M is not. In this case, the right hand side of (6.1) may be zero, while the left hand side is not. It then can happen that the quantum master equation is not zero when restricted to planar vertices only. The latter statement is of course independent of the Large N limit and can be taken as an answer to the first part of question 2. A consistent planar subsector of A_∞-algebras does not exist.

However, not all results where negative. By mimicking the construction of quantum L_∞-algebras in terms of higher order coderivations, we were able to give a reasonable definition of what we could call a planar homotopy algebra. There is still some work to do. First of all, one should provide a good definition of morphisms of these algebras. These morphisms should naturally arise under the application of the homological perturbation lemma when transferring the algebra structure to the cohomology of the underlying differential graded vector space. Apart from a more complete description, it would also be nice to have an application of this type of algebra.

We also commented on the existence of a larger subsector of loop A_∞-algebras. Further, from the string field theory point of view it looks like that this algebra can be coupled to a classical L_∞-algebra. It the future, it may be interesting to study these further.

A. Graded Vector Spaces and the Décalage Isomorphism

A $\mathbb{Z}$-graded vector space over a field $\mathbf{k}$ is a collection of $\mathbf{k}$-vector spaces, $V = \{V(n) \,|\, n \in \mathbb{Z}\}$. It is convenient to combine them into a single vector space $V = \bigoplus_{n\in\mathbb{Z}} V(n)$. If $v \in V(n)$, we define its degree by $\deg(v) = n$. If there is no point of confusion, we usually will write simply v instead of $\deg(v)$. Homomorphisms between graded vector spaces V and W are collections of linear maps $f(n) : V(n) \to W(n)$. They themselves combine into a linear space denoted by $\hom(V, W)$. It has the structure of a (degree zero) vector space. It can be embedded into the graded space $\mathrm{lin}(V, W)$, which is defined by $\mathrm{lin}(V, W)(n) = \bigoplus_{k\in\mathbb{Z}} \hom(V(k), W(k+n))$, where in this case, $\hom(V(k), W(k+n))$ is the space of linear maps from $V(k)$ to $W(k+n)$ in the sense of ordinary vector spaces. Note that $\hom(V, W) = \mathrm{lin}(V, W)(0)$.

Given two graded vector spaces V an W, we define their tensor product

$$(V \otimes W)(n) = \bigoplus_{k+l=n} V(k) \otimes W(l). \tag{A.1}$$

Graded vector spaces differ from their ungraded counterpart in how the natural isomorphism $V \otimes W \cong W \otimes V$ is implemented. It is defined by

$$\begin{aligned} \sigma_{V,W} : V \otimes W &\longrightarrow W \otimes V, \\ v \otimes w &\longmapsto (-)^{vw} w \otimes v. \end{aligned} \tag{A.2}$$

This is usually called the Koszul sign rule. The ground field $\mathbf{k}$ defines the unit of the tensor product, when we interpret it as a graded vector space in degree zero, i.e. $\mathbf{k}(0) = \mathbf{k}$ and $\mathbf{k}(n) = 0$ otherwise. Given a graded vector space V, we define its dual $V^* = \mathrm{lin}(V, \mathbf{k})$. Note that $V^*(n) \cong \mathrm{lin}(V(-n), \mathbf{k})$.

When evaluating tensor powers of functions, we have to be careful with signs. To get consistent signs, we use the following convention. Consider the evaluation map

$$\mathrm{ev} : \mathrm{lin}(V, W) \otimes V \longrightarrow W, \tag{A.3}$$

$$f \otimes v \longmapsto f(v), \tag{A.4}$$

which we defined without any signs. Note that ev lives in degree zero in the graded vector space $\mathrm{lin}(\mathrm{lin}(V, W) \otimes V, W)$. Now, let us use this map as a basis to fix signs. For example, we want to know how the isomorphism $W \otimes V^* \cong \mathrm{lin}(V, W)$ is implemented. We demand that

$$\begin{tikzcd} \mathrm{lin}(V,W) \otimes V \arrow[r, "\cong"] \arrow[dr, "\mathrm{ev}"'] & W \otimes V^* \otimes V \arrow[d, "\mathrm{id}\otimes\mathrm{ev}"] \\ & W \end{tikzcd} \tag{A.5}$$

commutes. From this one can see that the element

$$w \otimes \alpha \in W \otimes V^* \tag{A.6}$$

corresponds to the map

$$v \mapsto w\,\alpha(v) \tag{A.7}$$

in $\mathrm{lin}(V, W)$. the other hand, the isomorphism $\mathrm{lin}(V, W) \cong V^* \otimes W$ has a sign. $\alpha \otimes w$ defines the linear map

$$x \mapsto (-)^{\alpha w} w\,\alpha(x) = (-)^{\alpha w} \alpha(x)\,w = (-)^{xw} \alpha(x)\,w. \tag{A.8}$$

The sign naturally arises since we have to move x past α.

Moving on, consider now

$$\begin{tikzcd}
(V \otimes W)^* \otimes (V \otimes W) \arrow[r, "\cong"] \arrow[dr, "\mathrm{ev}"'] & (W^* \otimes V^* \otimes V) \otimes W \arrow[d, "\mathrm{ev}\circ(\mathrm{ev}\otimes\mathrm{id})"] \\
& {[0].}
\end{tikzcd} \tag{A.9}$$

Following this diagram, one finds that $\alpha \otimes \beta \in W^* \otimes V^*$ is identified with $v \otimes w \mapsto \beta(v)\alpha(w)$. Again, the isomorphism $(V \otimes W)^* \cong V^* \otimes W^*$ involves a sign coming from the natural isomorphism $V^* \otimes W^* \cong W^* \otimes V^*$. Finally, we use these two observations to determine the isomorphism $\mathrm{lin}(A \otimes B, V \otimes W) \cong \mathrm{lin}(A, V) \otimes \mathrm{lin}(B, W)$. For this, consider the following chain of identifications

$$\mathrm{lin}(A \otimes B, V \otimes W) \cong (V \otimes W) \otimes (A \otimes B)^* \cong V \otimes W \otimes B^* \otimes A^* \tag{A.10}$$

$$\overset{\sigma_{W\otimes B^*, A^*}}{\cong} (V \otimes A^*) \otimes (W \otimes B^*) \cong \mathrm{lin}(A, V) \otimes \mathrm{lin}(B, W). \tag{A.11}$$

When we use the isomorphism $\sigma_{W\otimes B^*, A^*}$ in the intermediate step produces signs. For a fixed pair $f \in \mathrm{lin}(A, V)$, $g \in \mathrm{lin}(B, W)$, their tensor product acts as

$$(f \otimes g)(v \otimes w) = (-)^{gv} f(v)g(w). \tag{A.12}$$

An easy way to memorize this is that, in going from left to right, we have to move v past g. In general, when evaluating expressions, we rearrange the objects in a way so that functions are to the left of the element they act on, using the isomorphism $\sigma_{-,-}$. We then apply the evaluation map to each pair. The sign is then solely produced by $\sigma_{-,-}$.

We will now introduce the décalage isomorphism, where we follow [40]. An important operation one meets when working with graded vector spaces is the degree shift. This process is called suspension. Given a graded vector V, we define another graded vector space $V[n]$ via

$$V[n](k) = V(n + k). \tag{A.13}$$

We naturally identify

$$\mathbf{k}[1] \otimes V \cong V[1]. \tag{A.14}$$

Note that this implies that the identification $V \otimes \mathbf{k}[1] \cong V[1]$ is given by $v \mapsto (-)^v v$. Further, the natural isomorphism

$$(V[1])^{\otimes k} \cong V^{\otimes k}[k] \tag{A.15}$$

is given by

$$v_1 \otimes \cdots \otimes v_n \mapsto (-)^{\sum_{i=1}^n (n-i)v_i} v_1 \otimes \cdots \otimes v_n. \tag{A.16}$$

There is a canonical action $S_n \times V^{\otimes n} \to V^{\otimes n}$ induced by the natural transformation $\sigma_{V,W} : V \otimes W \to W \otimes V$. We can define the symmetric tensor power $V^{\odot n}$ to be the invariant of $V^{\otimes n}$ under the action S_n. Similarly the exterior tensor power $V^{\wedge n}$ is the invariant subspace of $V^{\otimes n}$ under the action of S_n twisted by the sign $(-)^{\sigma}$ of permutations $\sigma \in S_n$.

We can arrange the even and odd action of S_n together with the isomorphism (A.16) into the following commutative diagram

$$\begin{tikzcd}
(V[1])^{\otimes n} \arrow[r, "\cong"] \arrow[d, "\sigma"] & (V)^{\otimes n}[n] \arrow[d, "(-)^{\sigma}\sigma"] \\
(V[1])^{\otimes n} \arrow[r, "\cong"] & (V)^{\otimes n}[n]
\end{tikzcd}. \tag{A.17}$$

It follows that (A.16) induces an isomorphism

$$\mathrm{dec} : (V[1])^{\odot n} \cong V^{\wedge n}[n]. \tag{A.18}$$

This is the first version of the décalage isomorphism. We can further use

$$\mathrm{lin}(V, W[k]) \cong \mathrm{lin}(V, W)[k] \tag{A.19}$$

to obtain an identification

$$\mathrm{dec} : \mathrm{lin}(V^{\otimes n}, W) \cong \mathrm{lin}((V[1])^{\otimes n}, W[l])[k-l], \tag{A.20}$$

where

$$\mathrm{dec}(f)(v_1, ..., v_n) = (-)^{nf+\sum_{i=1}^{n}(n-i)v_i} f(v_1, ..., v_n). \tag{A.21}$$

This in particular induces

$$\mathrm{dec} : \mathrm{Hom}(V^{\otimes n}, V) \cong \mathrm{Hom}((V[1])^{\otimes n}, V[1])[n-1], \tag{A.22}$$

and

$$\mathrm{dec} : \mathrm{Hom}(V^{\wedge n}, V) \cong \mathrm{Hom}((V[1])^{\odot n}, V[1])[n-1]. \tag{A.23}$$

We will meet these identifications when discussing homotopy algebras. For example, mathematicians often prefer to work over anti-symmetric maps when discussing L_{∞}-algebras. This is because its a direct generalization of how ordinary Lie algebras are defined. The Lie bracket is a degree zero anti-symmetric map. On the other hand, we could as well define it as a graded symmetric map of degree one via the décalage isomorphism. This is convention naturally arises from the bar construction as well as the BV formalism.

B. Chain complexes over Vector Spaces

Chain complexes exist in general for any abelian group. In this work we will, however, only meet chain complexes over vector spaces, so we restrict to this case.

Definition B.0.1. A *chain complex* is a collection of vector spaces $V(i)$ together with maps $\partial_i : V(i) \to V(i-1)$, such that $\partial_{i-1} \circ \partial_i = 0$. Equivalently, a chain complex can be defined as a graded vector space V and a degree -1 map $\partial : V \to V$ such that $\partial^2 = 0$. A morphism is a degree zero map $f : (V, \partial_V) \to (W, \partial_W)$ such that $f \circ \partial_V = \partial_W \circ f$. In that case, f is called a chain map. A *cochain complex* is dual to a chain complex, that is ∂ is of degree one.

We will refer to chain complexes over vector spaces as *differential graded* vector spaces. An important notion is that of (co-)homology.

Definition B.0.2. Given a differential graded vector space (V, ∂), we define its homology $H(V) := \frac{\ker \partial}{\operatorname{Im} \partial}$. It has the structure of a graded vector space. Elements in $\ker \partial$ ($\operatorname{Im} \partial$) are called ∂-closed (∂-exact).

An important property of homology is that it is functorial. This means that a chain map $f : (V, \partial_V) \to (W, \partial_W)$ induces a linear map $H(f) : H(V) \to H(W)$ on homology.

Bibliography

[1] Fusun Akman. On some generalizations of Batalin-Vilkovsky algebras. 6 1995.

[2] Fusun Akman and Lucian M. Ionescu. Higher Derived Brackets and Deformation Theory I. *arXiv Mathematics e-prints*, page math/0504541, April 2005.

[3] Mathieu Anel. The Geometry of Ambiguity. 2016.

[4] Alex S. Arvanitakis, Olaf Hohm, Chris Hull, and Victor Lekeu. Homotopy Transfer and Effective Field Theory I: Tree-level. 7 2020.

[5] Tsuguhiko Asakawa, Taichiro Kugo, and Tomohiko Takahashi. One loop tachyon amplitude in unoriented open - closed string field theory. *Prog. Theor. Phys.*, 102:427–466, 1999.

[6] J. Barcelos-Neto. Axial and gauge anomalies in the BV formalism. *Int. J. Mod. Phys. A*, 12:5053–5066, 1997.

[7] I.A. Batalin and G.A. Vilkovisky. Relativistic S Matrix of Dynamical Systems with Boson and Fermion Constraints. *Phys. Lett. B*, 69:309–312, 1977.

[8] I.A. Batalin and G.A. Vilkovisky. Gauge Algebra and Quantization. *Phys. Lett. B*, 102:27–31, 1981.

[9] I.A. Batalin and G.a. Vilkovisky. FEYNMAN RULES FOR REDUCIBLE GAUGE THEORIES. *Phys. Lett. B*, 120:166–170, 1983.

[10] I.A. Batalin and G.A. Vilkovisky. Quantization of Gauge Theories with Linearly Dependent Generators. *Phys. Rev. D*, 28:2567–2582, 1983. [Erratum: Phys.Rev.D 30, 508 (1984)].

[11] I.A. Batalin and G.A. Vilkovisky. Closure of the Gauge Algebra, Generalized Lie Equations and Feynman Rules. *Nucl. Phys. B*, 234:106–124, 1984.

[12] I.A. Batalin and G.A. Vilkovisky. Existence Theorem for Gauge Algebra. *J. Math. Phys.*, 26:172–184, 1985.

[13] C. Becchi, A. Rouet, and R. Stora. The Abelian Higgs-Kibble Model. Unitarity of the S Operator. *Phys. Lett. B*, 52:344–346, 1974.

[14] C. Becchi, A. Rouet, and R. Stora. Renormalization of Gauge Theories. *Annals Phys.*, 98:287–321, 1976.

[15] Roberto Bonezzi and Olaf Hohm. Leibniz Gauge Theories and Infinity Structures. *Commun. Math. Phys.*, 377(3):2027–2077, 2020.

[16] Kaj Börjeson. A-infinity Algebras Derived from Associative Algebras with a Non-Derivation Differential. *arXiv e-prints*, page arXiv:1304.6231, Apr 2013.

[17] R. Catenacci, M. Debernardi, P.A. Grassi, and D. Matessi. Cech and de Rham Cohomology of Integral Forms. *J. Geom. Phys.*, 62:890–902, 2012.

[18] Aberto S. Cattaneo, Pavel Mnev, and Nicolai Reshetikhin. Classical BV theories on manifolds with boundary. *Commun. Math. Phys.*, 332:535–603, 2014.

[19] Alberto S. Cattaneo, Pavel Mnev, and Nicolai Reshetikhin. Perturbative quantum gauge theories on manifolds with boundary. *Commun. Math. Phys.*, 357(2):631–730, 2018.

[20] Christoph Chiaffrino and Ivo Sachs. Classical open string amplitudes from boundary string field theory. *JHEP*, 06:086, 2019.

[21] Kai Cieliebak, Kenji Fukaya, and Janko Latschev. Homological algebra related to surfaces with boundary, 2015.

[22] K. Costello. *Renormalization and Effective Field Theory.* Mathematical surveys and monographs.

[23] Kevin Costello and Owen Gwilliam. Factorization algebras in quantum field theory (vol. 2).

[24] Kevin Costello and Si Li. Quantization of open-closed BCOV theory, I. 2015.

[25] Kevin Costello and Barton Zwiebach. Hyperbolic String Vertices. 8 2019.

[26] M. Crainic. On the perturbation lemma, and deformations. *arXiv Mathematics e-prints*, page math/0403266, March 2004.

[27] Corinne de Lacroix, Harold Erbin, Sitender Pratap Kashyap, Ashoke Sen, and Mritunjay Verma. Closed Superstring Field Theory and its Applications. *Int. J. Mod. Phys. A*, 32(28n29):1730021, 2017.

[28] Oliver DeWolfe. Minimal area nonorientable string diagrams. 8 1997.

[29] Martin Doubek, Branislav Jurčo, and Korbinian Münster. Modular operads and the quantum open-closed homotopy algebra. *Journal of High Energy Physics*, 2015(12):1–55, 2015.

[30] Martin Doubek, Branislav Jurčo, and Ján Pulmann. Quantum L_∞ Algebras and the Homological Perturbation Lemma. *Commun. Math. Phys.*, 367(1):215–240, 2019.

[31] Ian Ellwood, Jessie Shelton, and Washington Taylor. Tadpoles and closed string backgrounds in open string field theory. *JHEP*, 07:059, 2003.

[32] Theodore Erler, Sebastian Konopka, and Ivo Sachs. NS-NS Sector of Closed Superstring Field Theory. *JHEP*, 08:158, 2014.

[33] Theodore Erler, Sebastian Konopka, and Ivo Sachs. Resolving Witten's superstring field theory. *JHEP*, 04:150, 2014.

[34] Theodore Erler, Sebastian Konopka, and Ivo Sachs. Ramond Equations of Motion in Superstring Field Theory. *JHEP*, 11:199, 2015.

[35] Theodore Erler, Yuji Okawa, and Tomoyuki Takezaki. Complete Action for Open Superstring Field Theory with Cyclic A_∞ Structure. *JHEP*, 08:012, 2016.

[36] L. Dmitrievich Faddeev. Faddeev-Popov ghosts. *Scholarpedia*, 4(4):7389, 2009. revision #137028.

[37] L.D. Faddeev and V.N. Popov. Feynman Diagrams for the Yang-Mills Field. *Phys. Lett. B*, 25:29–30, 1967.

[38] Seyed Faroogh Moosavian, Ashoke Sen, and Mritunjay Verma. Superstring Field Theory with Open and Closed Strings. *JHEP*, 01:183, 2020.

[39] Richard Feynman. Quantum Theory of Gravitation. 1963.

[40] Domenico Fiorenza and Marco Manetti. L-infinity structures on mapping cones, 2006.

[41] Jean Fisch and Marc Henneaux. Homological perturbation theory and the algebraic structure of the antifield-antibracket formalism for gauge theories. *Comm. Math Phys.*, 128, 03 1990.

[42] Klaus Fredenhagen and Katarzyna Rejzner. Batalin-Vilkovisky formalism in the functional approach to classical field theory. *Commun. Math. Phys.*, 314:93–127, 2012.

[43] Matthias R. Gaberdiel and Barton Zwiebach. Tensor constructions of open string theories. 1: Foundations. *Nucl. Phys. B*, 505:569–624, 1997.

[44] E. Getzler and M. M. Kapranov. Modular operads. *eprint arXiv:dg-ga/940800*, pages dg–ga/9408003, August 1994.

[45] Herbert Goldstein, Charles Poole, and John Safko. Classical mechanics, 2002.

[46] Joaquim Gomis, Jordi Paris, and Stuart Samuel. Antibracket, antifields and gauge theory quantization. *Phys. Rept.*, 259:1–145, 1995.

[47] Michael B. Green and John H. Schwarz. Anomaly Cancellation in Supersymmetric D=10 Gauge Theory and Superstring Theory. *Phys. Lett. B*, 149:117–122, 1984.

[48] Michael B. Green, John H. Schwarz, and Edward Witten. *Superstring Theory Vol. 2: 25th Anniversary Edition*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 11 2012.

[49] Darij Grinberg and Victor Reiner. Hopf algebras in combinatorics, volume 2. 2020.

[50] V. K. A. M. Gugenheim, L. A. Lambe, and J. D. Stasheff. Perturbation theory in differential homological algebra ii. *Illinois J. Math.*, 35(3):357–373, 09 1991.

[51] Owen Gwilliam. Factorization Algebras and Free Field Theories.

[52] Marc Henneaux. Lectures on the Antifield-BRST Formalism for Gauge Theories. *Nucl. Phys. B Proc. Suppl.*, 18A:47–106, 1990.

[53] Manfred Herbst. Quantum A-infinity structures for open-closed topological strings. 2 2006.

[54] Olaf Hohm and Barton Zwiebach. L_∞ Algebras and Field Theory. *Fortsch. Phys.*, 65(3-4):1700014, 2017.

[55] Johannes Huebschmann and Tornike Kadeishvili. Small models for chain algebras. *Mathematische Zeitschrift*, 207:245–280, 01 1991.

[56] Branislav Jurčo, Tommaso Macrelli, Lorenzo Raspollini, Christian Saemann, and Martin Wolf. L_∞-Algebras, the BV Formalism, and Classical Fields. *Fortsch. Phys.*, 67(8-9):1910025, 2019.

[57] Branislav Jurčo, Lorenzo Raspollini, Christian Saemann, and Martin Wolf. L_∞-Algebras of Classical Field Theories and the Batalin-Vilkovisky Formalism. *Fortsch. Phys.*, 67(7):1900025, 2019.

[58] Hiroshige Kajiura. Noncommutative homotopy algebras associated with open strings. *Rev. Math. Phys.*, 19:1–99, 2007.

[59] Hiroshige Kajiura and Jim Stasheff. Homotopy algebras inspired by classical open-closed string field theory. *Commun. Math. Phys.*, 263:553–581, 2006.

[60] R.E. Kallosh. Modified Feynman Rules in Supergravity. *Nucl. Phys. B*, 141:141–152, 1978.

[61] Sebastian Konopka and Ivo Sachs. Open Superstring Field Theory on the Restricted Hilbert Space. *JHEP*, 04:164, 2016.

[62] Maxim Kontsevich. Deformation quantization of Poisson manifolds. 1. *Lett. Math. Phys.*, 66:157–216, 2003.

[63] Taichiro Kugo and Tomohiko Takahashi. Unoriented open - closed string field theory. *Prog. Theor. Phys.*, 99:649–690, 1998.

[64] Jean-Louis Loday and Bruno Vallette. *Algebraic Operads*, volume 346. 01 2012.

[65] Jacob Lurie. Derived Algebraic Geometry.

[66] Jacob Lurie. Derived algebraic geometry x: Formal moduli problems, 2011. *Preprint available at http://www. math. harvard. edu/˜ lurie/papers/DAG-X. pdf.*

[67] Martin Markl. Ideal perturbation lemma. *Communications in Algebra*, 29(11):5209–5232, 2001.

[68] Martin Markl. Loop homotopy algebras in closed string field theory. *Commun. Math. Phys.*, 221:367–384, 2001.

[69] Martin Markl. On the origin of higher braces and higher-order derivations. 2013.

[70] Martin Markl. Higher braces via formal (non)commutative geometry. *arXiv e-prints*, page arXiv:1411.6964, Nov 2014.

[71] Rajan Mehta and Marco Zambon. L-∞ algebra actions. *Differ. Geom. Appl.*, 30:576–587, 2012.

[72] Korbinian Münster and Ivo Sachs. Quantum Open-Closed Homotopy Algebra and String Field Theory. *Commun. Math. Phys.*, 321:769–801, 2013.

[73] Korbinian Münster and Ivo Sachs. Homotopy Classification of Bosonic String Field Theory. *Commun. Math. Phys.*, 330:1227–1262, 2014.

[74] J. Polchinski. *String theory. Vol. 1: An introduction to the bosonic string.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 12 2007.

[75] Joseph Polchinski. Factorization of Bosonic String Amplitudes. *Nucl. Phys. B*, 307:61–92, 1988.

[76] Joseph Polchinski and Yunhai Cai. Consistency of Open Superstring Theories. *Nucl. Phys. B*, 296:91–128, 1988.

[77] Michael Reiterer. A homotopy BV algebra for Yang-Mills and color-kinematics. 2019.

[78] Emily Riehl. *Categorical Homotopy Theory.* New Mathematical Monographs. Cambridge University Press, 2014.

[79] Matthew D. Schwartz. *Quantum Field Theory and the Standard Model.* Cambridge University Press, 3 2014.

[80] Albert S. Schwarz. Geometry of Batalin-Vilkovisky quantization. *Commun. Math. Phys.*, 155:249–260, 1993.

[81] Ashoke Sen. Gauge Invariant 1PI Effective Action for Superstring Field Theory. *JHEP*, 06:022, 2015.

[82] Absos Ali Shaikh, Tran Quoc Binh, and Haradhan Kundu. Curvature properties of generalized pp-wave metric. *arXiv e-prints*, page arXiv:1702.07497, February 2017.

[83] A.A. Slavnov. Ward Identities in Gauge Theories. *Theor. Math. Phys.*, 10:99–107, 1972.

[84] Hidenori Sonoda and Barton Zwiebach. COVARIANT CLOSED STRING THEORY CANNOT BE CUBIC. *Nucl. Phys. B*, 336:185–221, 1990.

[85] James Dillon Stasheff. Homotopy associativity of h-spaces. i. *Transactions of the American Mathematical Society*, 108(2):275–292, 1963.

[86] James Dillon Stasheff. Homotopy associativity of h-spaces. ii. *Transactions of the American Mathematical Society*, 108(2):293–312, 1963.

[87] Jim Stasheff. Towards a closed string field theory: topology and convolution algebra. In *Topology Proceedings*, volume 17, pages 371–386, 1992.

[88] Gerard 't Hooft. A Planar Diagram Theory for Strong Interactions. *Nucl. Phys.*, B72:461, 1974. [,337(1973)].

[89] J.C. Taylor. Ward Identities and Charge Renormalization of the Yang-Mills Field. *Nucl. Phys. B*, 33:436–444, 1971.

[90] Bruno Vallette. Algebra+homotopy=operad. *Math. Sci. Res. Inst. Publ.*, 62, 02 2012.

[91] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994.

[92] Edward Witten. Interacting Field Theory of Open Superstrings. *Nucl. Phys. B*, 276:291–324, 1986.

[93] Edward Witten. Noncommutative Geometry and String Field Theory. *Nucl. Phys. B*, 268:253–294, 1986.

[94] Edward Witten. A Note on the Antibracket Formalism. *Mod. Phys. Lett. A*, 5:487, 1990.

[95] Edward Witten. On background independent open string field theory. *Phys. Rev. D*, 46:5467–5473, 1992.

[96] Edward Witten. Notes On Supermanifolds and Integration. 9 2012.

[97] Edward Witten. The Feynman $i\epsilon$ in String Theory. *JHEP*, 04:055, 2015.

[98] Jean Zinn-Justin. Renormalization of Gauge Theories. *Lect. Notes Phys.*, 37:1–39, 1975.

[99] Barton Zwiebach. Quantum closed strings from minimal area. *Mod. Phys. Lett. A*, 5:2753–2762, 1990.

[100] Barton Zwiebach. Closed string field theory: Quantum action and the B-V master equation. *Nucl. Phys. B*, 390:33–152, 1993.

[101] Barton Zwiebach. Oriented open - closed string theory revisited. *Annals Phys.*, 267:193–248, 1998.