Morava motives of projective quadrics

Andrei Lavrenov

Dissertation
an der Fakultät für Mathematik, Informatik und Statistik
der Ludwig-Maximilians-Universität
München

Vorgelegt am
24. September 2020

Erstgutachter: Prof. Dr. Nikita Semenov
Zweitgutachter: Prof. Dr. Alexander Vishik
Drittgutachter: Prof. Dr. Marc Levine

Tag der mündlichen Prüfung: 18.12.2020

Abstract

The present Ph. D. thesis is devoted to Morava motives of projective quadrics, meaning that we replace the Chow theory by another oriented cohomology theory.

We consider arbitrary oriented cohomology theories as we wish to obtain invariants that are simpler than Chow motives. In fact, there exists a series of theories, more precisely, Morava K-theories $\mathrm{K}(n)^*$, which starts from K^0 and tends to CH.

The most important and interesting results are the following ones:

Theorem (Theorem 1.3.9). *Let Q be a generic quadric of dimension $D > 0$, and $n > 1$; we denote $N = 2^n$ for $D = 2d$ even, or $N = 2^n - 1$ for $D = 2d+1$ odd. Then $\mathrm{K}(n)$-motive of Q has an indecomposable summand of rank $\min(N, 2d+2)$, and $\max(0, 2d+2-N)$ summands isomorphic to Tate motives.*

Theorem (Theorem 2.0.1). *For a group $G_m = \mathrm{Spin}_m$ or $G_m = \mathrm{SO}_m$ with $m \geq 2^{n+1} + 1$, $n > 1$, the canonical map $\mathrm{K}(n)^*(G_m; \mathbb{F}_2) \to \mathrm{K}(n)^*(G_{m+2}; \mathbb{F}_2)$ is an isomorphism.*

We also describe several algorithms useful for computer computations of $\mathrm{K}(n)$-motives of small-dimensional varieties.

Zusammenfassung

In der vorliegenden Doktorarbeit betrachten wir die Motive von projektiven Quadriken in Morava K-Theorie, d.h. wir ersetzen die Chow-Theorie durch eine andere orientierte Kohomologietheorie.

Indem wir unseren Focus auf beliebige orientierte Kohomologietheorie erweiten, hoffen wir Invariantanten zu finden, die einfacher sind als Chow-Motive. Genauer betrachten wir eine Reihe von Theorien, die Morava K-Theorien $\mathrm{K}(n)^*$, welche von K^0 ausgehend gegen CH "konvergieren".

Als Hauptergebnisse erhalten wir:

Theorem (Theorem 1.3.9). *Es sei Q eine generische Quadrik von der Dimension $D > 0$ und es sei $n > 1$. Bezeichne $N = 2^n$ für $D = 2d$ gerade oder $N = 2^n - 1$ für $D = 2d + 1$ ungerade. Dann hat das $\mathrm{K}(n)$-Motiv von Q einen unzerlegbaren Summanden vom Rang $\min(N, 2d + 2)$ und $\max(0, 2d + 2 - N)$ Summanden, die isomorph zu Tate Motiven sind.*

Theorem (Theorem 2.0.1). *Für $G_m = \mathrm{Spin}_m$ oder $G_m = \mathrm{SO}_m$ ist der kanonische Homomorphismus $\mathrm{K}(n)^*(G_m; \mathbb{F}_2) \to \mathrm{K}(n)^*(G_{m+2}; \mathbb{F}_2)$ ein Isomorphismus, für $m \geq 2^{n+1} + 1$, $n > 1$.*

Außerdem präsentieren wir auch verschiedene Algorithmen für die Berechnungen von $\mathrm{K}(n)$-Motiven von Varietaten kleiner Dimensionen.

Contents

Introduction

The present Ph. D. thesis is devoted to generalized motives of projective quadrics, meaning that we replace the Chow group CH^* by another oriented cohomology theory A^* in the sense of Levine–Morel [LM] when defining the category of correspondences.

The study of quadratic forms over arbitrary fields began with the work by Witt, who defined the ring $\mathrm{W}(k)$ arising from isometry classes of quadratic forms over a field k, and described it in terms of generators and relations. The theory was further developed by Kaplansky, Cassels, Pfister, Arason, Elman, Lam, and many others. In particular, Kaplansky introduced the u-invariant of a field k equal to the highest dimension of an anisotropic quadratic form over k, and the s-invariant equal to the least number $s(k)$ such that -1 is a sum of $s(k)$ squares in k (if exists) [Kap]. Pfister investigated the ring structure of $\mathrm{W}(k)$, he computed its Krull dimension, zero-divisors and spectrum. He also considered the filtration of the Witt ring by the powers of fundamental ideal $\mathrm{I}(k)$ of even-dimensional forms, and the generators of $\mathrm{I}(k)^n$ which we call now Pfister forms. In particular, he used them to show that $s(k)$ is a power of 2 (if finite). Later, Merkurjev proved that u-invariant can equal any even number [Me91], leaving open the question about the existence of fields with an odd u-invariant.

The category of Chow motives was defined by Grothendieck, and happened to be a powerful tool in the theory of quadratic forms. The computations of Chow groups of projective quadrics allowed to construct fields with an odd u-invariant [Izh], and the motivic decomposition of a Pfister quadric established by Rost [Ro] plays a crucial role in the proof of the Milnor conjecture on étale cohomolgy [Vo03]. The solution to the Milnor conjecture itself together with [OVV, Mo] has various applications to the purely algebraic structure of $\mathrm{W}(k)$ and its filtration by $\mathrm{I}(k)^n$, see, e.g., [EKM, Chapter VII].

Chow motives of projective quadrics were studied by Merkurjev, Vishik, Karpenko, and many others, and there exist now plenty various applications of Chow groups to quadratic forms, and, more generally, to projective homogeneous varieties, including [Ka04, Ka03, KaMe, Ya08, KaZh, Vi05, Vi98, PSZ]. However, for a general smooth projective quadric there are still many open questions about the behaviour of its Chow motive. In contrast, if we change the Chow group by Grothendieck's K^0 in the definition of motives, the resulting category behaves much more simply. Swan computed the K-theory of the projective quadric [Sw], and the results of Panin imply that the decomposition of K^0-motive of a smooth projective quadric depends only on the discriminant and the Clifford algebra of the corresponding quadratic form [Pa94].

The motives of Grothendieck inspired Quillen's paper [Qu] where the complex cobordism theory is described as the universal contravariant functor on the category of C^∞-manifolds endowed with pushforward maps. Later Levine and Morel brought these Quillen's ideas "back" to the "motivic" world defining the *algebraic* cobordism theory, as they write in [LM]. This allowed to consider algebraic analogues of well studied topological oriented cohomology theories, such as Morava K-theories.

We should remark here that *algebraic* Morava K-theory as conjectured by Voevodsky in [Vo95] or as constructed in [LeTr] is a *bi*-graded "big" theory, and in the present thesis we only consider oriented cohomology theories in the sense of [LM], sometimes called "small". Our (small) Morava K-theory is the $(2*,*)$-diagonal of the "big" theory of [LeTr], as shown in [Le09].

The algebraic cobordism theory of Levine–Morel [LM, LePa, Le07, Vi15], and arbitrary oriented cohomology theories [PaSm, ViYa, NeZa, CPZ, Vi19, Se17, GiVi, PS20] are extensively studied now. In particular, Vishik used the algebraic cobordism theory in his construction of fields with u-invariant 2^n+1 [Vi07]. The algebraic cobordism theory inspired Vishiks' excellent connections [Vi11], despite the fact that the proof of the result can be given using Chow groups only. Similarly, the general context of oriented cohomology theories inspired Petrov–Semenov connections of [PS20], despite the fact that their result can also be proven using Chow groups only [Ka20]. An example of a very different application of algebraic cobordism to the theory of quadratic forms is given in Panin's [Pa09].

One can define the category of motives corresponding to any algebraic oriented cohomology theory A^*, and the results of Levine–Morel [LM] and Vishik–Yagita [ViYa] imply that the decomposition of the Chow motive is the "roughest" in the following sense: starting from any decomposition of the Chow motive $\mathcal{M}_{\mathrm{CH}}(X)$ of a smooth projective variety X as a sum of several motivic summands $\mathcal{N}^i_{\mathrm{CH}}$ one can construct corresponding objects $\mathcal{N}^i_A$ in the category of A-motives for any oriented cohomology theory A in such a way that the A-motive $\mathcal{M}_A(X)$ of X is the sum of $\mathcal{N}^i_A$.

However, for an indecomposable Chow summand $\mathcal{N}_{\mathrm{CH}}$ the corresponding A-motive $\mathcal{N}_A$ can be decomposable. For example, the Chow motive of the generic quadric is indecomposable (see [Vi04, Ka12]), however, a K^0-motive of any quadric Q with $\dim Q > 0$ is decomposable.

We are interested in motives constructed with respect to arbitrary oriented cohomology theories because we wish to obtain invariants which are simpler than Chow motives, but keep more information about quadrics than K^0-motives. And, in fact, we can take a series of theories, more precisely, Morava K-theories $\mathrm{K}(n)^*$, which starts from K^0 and tends to CH^* in a certain sense.

To be more precise here, we should first recall that any oriented cohomology theory is endowed with a formal group law, and any formal group law over any ring comes from a certain oriented cohomology theory. Among various theories corresponding to the same formal group law there exists a universal one called *free*. The class of free theories contains algebraic cobordism, Chow, Grothendieck's K^0, and Morava K-theories we consider in the present thesis. The major feature of these theories for our purposes is the Rost nilpotence principle recently proven in [GiVi].

Vishik gives a geometric description of free theories in [Vi19], which allowed him to construct operations on algebraic cobordism, and later was used by Sechin to construct operations from Morava K-theories [Se17, Se18]. His results imply, in particular, that the category of free theories (and multiplicative operations) is equivalent to the category of (1-dimensional commutative graded) formal group laws.

Working with quadratic forms, it is natural to consider localized at 2 coefficients $\mathbb{Z}_{(2)}$ instead of integral, and, therefore, consider only formal group laws over $\mathbb{Z}_{(2)}$-algebras. Then by the theorem of Cartier we can restrict ourselves to a narrower class of formal group laws, called 2-*typical* ones (any formal group law over a $\mathbb{Z}_{(2)}$-algebra is isomorphic to a 2-typical one), in particular, it is natural to consider the *universal* 2-*typical* formal group law. It admits the standard construction as a formal group law over the polynomial ring with infinite number of variables $\mathbb{Z}_{(2)}[v_1, v_2, \dots]$ defined by the recurrent identities, see, e.g., [Ra, Appendix A2]. Then specifying $v_k = 0$ for $k \neq n$, and inverting v_n, we obtain the formal group law over $\mathbb{Z}_{(2)}[v_n^{\pm 1}]$, and the corresponding free theory is called Morava K-theory $\mathrm{K}(n)^*$. We remark that the same definition is used in [PS14, PS20], however, in [Se17, Se18] the term "Morava K-theory" is understood more generally as a free theory which becomes isomorphic to the described one after passing to $\overline{\mathbb{F}_2}$. Observe also that in [Se17, Se18] oriented cohomology theories are usually *non-graded*, but can be graded artificially, as explained in [Se18], and that the only possible (non-graded) free theories with $\overline{\mathbb{F}_2}$-coefficients are Morava K-theories and the Chow theory, in particular, $\mathrm{K}^0 \otimes \overline{\mathbb{F}_2}$ and $\mathrm{K}(1) \otimes \overline{\mathbb{F}_2}$ are isomorphic (the isomorphism between the corresponding formal group laws is called Artin–

Hasse exponent). In this context the definition of [Se17, Se18] becomes very natural, but the grading and the push-forwards also play an important role. In particular, K^0 and $\mathrm{K}(1)$ have different push-forwards.

However, the major advantage of our choice of one particular Morava K-theory from the class of Sechin's Morava K-theories is the possibility to work with closed formulae. We can find a simple multiplication table for $\mathrm{K}(n)^*(Q)$ for a split quadric Q in a certain base (see Corollary1.1.5), and working modulo 2 we have a closed formula for the projectors in $\mathrm{K}(n)^*(Q \times Q)$, in particular, for the diagonal (see Proposition 1.3.3).

Morava K-theories are related to higher powers of the fundamental ideal in the Witt ring. Sechin–Semenov state in [SeSe] the "Guiding Principle" for their research (which dates back to Voevodsky's program [Vo95]) claiming that for a projective homogeneous variety X vanishing of its cohomological invariants with 2-torsion coefficients in degrees no greater than $n+1$ should correspond to the splitting of the $\mathrm{K}(n)$-motive of X. In particular, they proved in [SeSe] that if a class of a quadratic form q in the Witt ring actually lies in I^{n+2}, then n first categories of $\mathrm{K}(n)$-motives do not distinguish q from the hyperbolic form.

The present thesis, however, deals with the case that is in some sense opposite to the hyperbolic quadric, namely, the generic quadric. As we mentioned, the Chow motive of a generic quadric is indecomposable, the K^0-motive is decomposable. Our results describe the behaviour of the $\mathrm{K}(n)$-motive of the generic quadric. The first chapter of the present thesis is devoted to the proof of the following

Theorem (Theorem 1.3.9)**.** *Let Q be a generic quadric of dimension D, and $n > 1$; we denote $N = 2^n$ for $D = 2d$ even, or $N = 2^n - 1$ for $D = 2d + 1$ odd. Then the $\mathrm{K}(n)$-motive of Q has an indecomposable summand of rank* $\min(N, 2d+2)$, *and* $\max(0, 2d+2-N)$ *summands isomorphic to shifted motives of the point.*

In other words, the $\mathrm{K}(n)$-motive of a generic quadric of dimension $D < 2^n - 1$ is indecomposable, and a generic quadric of a large dimension decomposes as the sum of several Tates and an indecomposable summand of rank 2^n or $2^n - 1$ depending on the parity of D.

The plan of the proof of Theorem 1.3.9 is the following. First, we show that the motivic decompositions of any smooth projective quadrics with respect to theories $\mathrm{K}(n)^*$ and $\mathrm{K}(n)^*(-;\mathbb{F}_2)$ coincide, in particular, a decomposition in a sum of indecomposables is unique. Using the result of Gille–Vishik [GiVi] we can repeat the argument from the paper of Chernousov–Merkurjev [ChMe]. Next, we give an explicit description of the composition of correspondences in $\mathrm{K}(n)^*(\overline{Q} \times \overline{Q}; \mathbb{F}_2)$, and find the required amount of rational idempotents. Finally, we prove that the "large" summand is indecomposable using the technique of Petrov–Semenov developed in [PS20], which relates the category of $\mathrm{K}(n)$-motives of twisted forms of projective homogeneous G-varieties to the category of $\mathrm{K}(n)^*(G)$-comodules.

The second chapter of the thesis is naturally related to the above results. It is devoted to the computation of Morava K-theory $\mathrm{K}(n)^*(G)$ of a split orthogonal or spinor group G. We remark that partial computations can also be found in [Ya05, Zo]. The following theorem is proven by the aspirant jointly with Victor Petrov.

Theorem (Theorem 2.0.1)**.** *For the group $G_m = \mathrm{Spin}_m$ or $G_m = \mathrm{SO}_m$ with $m \geq 2^{n+1}+1$, $n > 1$, the canonical map $\mathrm{K}(n)^*(G_m; \mathbb{F}_2) \to \mathrm{K}(n)^*(G_{m-2}; \mathbb{F}_2)$ is an isomorphism.*

There is the following idea behind the proof of Theorem 2.0.1. Observe that for a variety X the $\mathbb{L}$-algebra $\Omega^*(X)$ admits a natural augmentation $\deg\colon \Omega^*(X) \to \Omega^*\big(\operatorname{Spec} k(X)\big) = \mathbb{L}$ by the pullback to the generic point [LM, Remark 1.2.12]. For an augmented $\mathbb{L}$-algebra A we denote A^+ its augmentation ideal, and we say that the sequence of augmented algebras $(A_i, d_i\colon A_i \to A_{i+1})$ is exact if $\operatorname{Ker} d_i$ coincides with the ideal generated by $\operatorname{Im} d_{i-1} \cap A_i^+$.

For G a split semisimple group with a split maximal torus T and Borel subgroup B containing T, the sequence

$$\Omega^*(\mathrm{B}T) \to \Omega^*(G/B) \to \Omega^*(G) \to \mathbb{L}$$

is a right exact sequence of augmented $\mathbb{L}$-algebras.

Since we have a very explicit description of $\Omega^*(\mathrm{B}T) \cong \mathbb{L}[\![x_1, \ldots, x_l]\!]$ and $\Omega^*(G/B) \cong \mathbb{L}^{|W|}$, where W is the Weyl group of G, we can try to compute $\Omega^*(G)$ with the use of the above sequence. However, the map $\Omega^*(\mathrm{B}T) \to \Omega^*(G/B)$ has quite a complicated form after these identifications. One can write a closed formula [CPZ, Equation (8)] for it in terms of BGG–Demazure divided difference operators Δ_i.

Instead, we found it much easier to reduce the statement of Theorem 2.0.1 as in [PS20] and [PS12] to the following result: for $G = \mathrm{SO}_m$ or $G = \mathrm{Spin}_m$ with $m \geq 2^{n+1}+1$, $n > 1$, the natural pullback map

$$\mathrm{K}(n)^*(Q; \mathbb{F}_2) \to \mathrm{K}(n)^*(G; \mathbb{F}_2),$$

where $Q = G/P_1$ is a split quadric, factors through $\mathrm{K}(n)^*(\mathrm{pt}; \mathbb{F}_2)$. Next, we use the divided difference operators in the proof of this result.

The third chapter was actually the starting point of the present work, and it describes several computer algorithms which were applied by the aspirant to describe Morava K-theories of small-dimensional varieties.

Besides technical remarks concerning the work with power series, we explicitly describe the character from the Morava ring $\mathrm{K}(n)^*(Q)$ to the Chow ring $\mathrm{CH}^*(Q)$ of a split quadric Q. This approach was used in [Rü] for computer computations with small-dimensional hypersurfaces, and now it is extended to the case of quadrics of arbitrary dimensions. Several results of the first chapter can be re-proved using this approach.

We also continued the work [PS14], which suggests a certain approach to the description of the T-equivariant Morava ring $\mathrm{K}(n)_T(Q \times Q)$ for a split quadric Q as a subring of $\bigoplus \mathrm{K}(n)_T(\mathrm{pt})$, where the direct sum is taken over the fixed points of the torus action. Forgetting the action of T we expected to find interesting rational projectors in the *ordinary* $\mathrm{K}(n)^*(Q \times Q)$, but we were able to find only Tate summands. This "misfortune" in fact suggested the statement of Theorem 1.3.9. We hope, however, that the algorithm can be useful in the future for experiments with small-dimensional quadrics, and include its description in the present thesis.

Acknowledgements

I wish to thank several people for their constant help. First of all, I would like to express my sincere gratitude to my advisor Nikita Semenov for his guidance and patience. I am also grateful to Victor Petrov, who explained various things to me, and shared many of his ideas, to Pavel Sechin for our endless discussions on Morava K-theory, and to Maksim Zhykhovich and Alexey Ananyevskiy, who answered dozens of my questions.

Chapter 1

Morava motives of generic quadrics

The present chapter is devoted to the Morava K-theory $\mathrm{K}(n)^*$ of a smooth projective quadric and the corresponding motive $\mathcal{M}_{\mathrm{K}(n)}(Q)$. We determine the multiplicative table for $\mathrm{K}(n)^*(Q)$ for a *split* quadric Q in Theorem 1.1.4, and Corollary 1.1.5. We prove the Krull–Schmidt Theorem for the $\mathrm{K}(n)$-motive of any quadric Q in Theorem 1.2.6. Finally, we establish the motivic decomposition of the $\mathrm{K}(n)$-motive of a *generic* quadric Q in Theorem 1.3.9.

1.1 The Cobordism Ring of a Split Quadric

For any cellular variety X and an oriented cohomology theory A^* it is easy to describe the structure of $A^*(X)$ as an abelian group. For a split quadric Q we will describe the multiplication in terms of the formal group law. For certain theories A^*, e.g., for Morava K-theory $\mathrm{K}(n)^*$ with $\mathbb{F}_2$ coefficients, the multiplication table has a very simple description. We also establish an explicit formula for the pushforward map along the structure morphism from Q to the point.

1.1.1 Oriented Cohomology Theories

We briefly recall the notion of an oriented cohomology theory in the sense of Levine–Morel [LM], and fix the notation.

We work over a fixed field k of characteristic 0, and $\mathcal{S}\mathsf{m}_k$ denotes the category of smooth quasi-projective varieties over k. We usually denote $\operatorname{Spec} k$ by pt. An *oriented cohomology theory* is given by the following data.
(D1) An additive contravariant functor from $\mathcal{S}\mathsf{m}_k$ to the category of commu-

tative $\mathbb{Z}$-graded rings

$$A^*\colon \mathcal{S}\mathsf{m}_k^{\mathrm{op}} \to \mathcal{R}\mathsf{ings}^*.$$

For a morphism of smooth varieties $f\colon X \to Y$ we write f^A for $A^*(f)$, and call this morphism the *pullback* map along f; it defines the structure of an $A^*(Y)$-algebra on $A^*(X)$. In particular, $A^*(X)$ has the canonical structure of an $A^*(\mathrm{pt})$-algebra. We sometimes call $A^*(\mathrm{pt})$ the ring of coefficients of the theory A^*.

(D2) Homomorphisms of graded $A^*(Y)$-modules

$$f_A\colon A^*(X) \to A^{*+d}(Y)$$

for each projective morphism $f\colon X \to Y$ of pure relative codimension d. We call these homomorphisms *pushforward* maps along f; the pushforward along the identity map is the identity map, and the pushforward along any composition is the composition of the pushforwards. The data (D1) and (D2) should satisfy the transversal square axiom, the projective bundle formula, and the (strong) homotopy invariance [LM, Definition 1.1.2].

For any oriented cohomology theory one defines Chern classes

$$c_i^A(E) \in A^i(X),\ 0 \leqslant i \leqslant n$$

of a vector bundle $E \to X$ of rank n, $c_0(E) = 1$, in such a way that $c_i^A(f^*E) = f^A\big(c_i^A(E)\big)$ for any f, and one has the Whitney formula [LM, § 4.1.7], i.e., for any exact sequence

$$0 \to E' \to E \to E'' \to 0$$

one has $c_m(E) = \sum_{i=0}^m c_i(E')c_{m-i}(E'')$. Moreover, the class $c_1^A(L)$ of any line bundle L is nilpotent, and there exists a unique formal group law

$$F_A(x, y) = \sum_{i,j} a_{ij} x^i y^j \in A^*(\mathrm{pt})[[x, y]],$$

for some $a_{ij} \in A^{1-i-j}(\mathrm{pt})$, such that

$$c_1^A(L \otimes M) = F_A\big(c_1^A(L), c_1^A(M)\big)$$

for any two line bundles L and M, see [LM, Lemma 1.1.3]. All formal group laws are assumed to be commutative and one-dimensional.

Levine and Morel constructed the *algebraic cobordism* theory Ω^*, which is the universal oriented cohomology theory in the following sense: for any oriented cohomology theory A^* there exists a unique natural transformation

$$\vartheta\colon \Omega^* \to A^*$$

of functors from $\mathcal{S}\mathsf{m}_k^{\mathrm{op}}$ to $\mathcal{R}\mathsf{ings}^*$, commuting with pushforwards [LM, Theorem 1.2.6]. They also proved that $\Omega^*(\mathrm{pt})$ is isomorphic to the Lazard ring $\mathbb{L}$, and the formal group law F_Ω is the universal formal group law [LM, Theorem 1.2.7].

As a consequence, for any commutative $\mathbb{Z}$-graded ring R, and any formal group law $F(x, y)$ homogeneous of degree 1 as an element of $R[[x, y]]$, there exists an oriented cohomology theory A^*, more precisely,

$$A^* = \Omega^* \otimes_{\mathbb{L}} R$$

such that $A^*(\mathrm{pt}) = R$, and the corresponding formal group law $F_A(x, y)$ is equal to $F(x, y)$ [LM, Remark 2.4.14]. Such a theory A^* is called *free*.

The free theory corresponding to the *additive* formal group law $F_{\mathrm{a}}(x, y) = x + y$ is the Chow theory CH^* [LM, Theorem 1.2.19]. Observe that in this case $\Omega^* \otimes_{\mathbb{L}} \mathbb{Z}$ equals $\Omega^*/\mathbb{L}^{<0}\Omega^*$. Another example of a free theory is the graded version of Grothendieck's K-theory $\mathrm{K}^0 \otimes_{\mathbb{Z}} \mathbb{Z}[\beta^{\pm 1}]$ (here elements of K^0 have degree 0, and the Bott element β is a formal Laurent variable of degree -1); this theory corresponds to the *multiplicative* formal group law $F_{\mathrm{m}}(x, y) = x + y - \beta xy$ [LM, Theorem 1.2.18]. The most important example for the present work is Morava K-theory [PS14, Se17, Se18, SePhD, SeSe], see Section 1.1.4.

Free theories keep many properties of algebraic cobordism, for instance, they are *generically constant* in the sense of [LM, Definition 4.4.1], see [LM, Corollary 1.2.11], and satisfy the localization property [LM, Definition 4.4.6, Theorem 1.2.8]. The latter is often included in the definition of the oriented cohomology theory, e.g., in [Pa02, Vi19]. Any free theory A^* also satisfies the following identity, which we call the *Normalization identity* following [Pa02, Theorem 1.1.8]:

$$\iota_A(1_{A^*(D)}) = c_1^A\big(\mathcal{L}(D)\big) \tag{1.1}$$

for any smooth divisor $\iota\colon D \hookrightarrow X$, and $\mathcal{L}(D)$ the corresponding line bundle as in [Har, Chapter II, Proposition 6.13], see also [LM, Proposition 5.1.11], or [Me02, Proposition 3.2]. We will use it, actually, only for D a hypersurface in $\mathbb{P}^n$ with $\mathcal{L}(D) = \mathcal{O}(d)$, and $A^* = \Omega^*$ the algebraic cobordism theory.

1.1.2 Multiplication in $\Omega^*(Q)$ of a Split Quadric Q

Let us denote $H = H_{\mathbb{P}^n} = c_1^\Omega\big(\mathcal{O}_{\mathbb{P}^n}(1)\big)$, and recall that the powers of H^k for $0 \leqslant k \leqslant n$ form a free base of $\Omega^*\big(\mathbb{P}^n\big)$ by the projective bundle formula. Moreover, H^k coincide with the classes of projective subspaces of smaller dimensions in $\mathbb{P}^n$, i.e., with the pushforwards of the identity elements

$1 \in \Omega^*(\mathbb{P}^{n-k})$ under the natural inclusions. We remark that the latter statement follows from (1.1) by induction with the use of projection formula. In particular, the pushforward map from $\Omega^*\big(\mathbb{P}^{n-k}\big)$ to $\Omega^*\big(\mathbb{P}^n\big)$ is injective, and sends $H^i_{\mathbb{P}^{n-k}}$ to $H^{i+k}_{\mathbb{P}^n}$. The detailed exposition of these computation can be found, e.g., in [Pa02, Lemma 1.9.2].

We follow the notation of [EKM, Chapter XIII, § 68]. Let Q be a smooth projective quadric of dimension D over k defined by the non-degenerate quadratic form φ on the vector space V of dimension $D+2$. We write $D = 2d$ for D even, or $D = 2d+1$ for D odd. We assume that the quadric Q is split, i.e., φ has the maximal possible Witt index $d+1$. Let us denote W a maximal isotropic subspace of V, and $W^\perp$ its orthogonal complement in V (for D even $W^\perp = W$, and for D odd W is a hyperplane in $W^\perp$). We denote $\mathbb{P}(V)$ the projective space of V which contains Q as a hypersurface. Then $\mathbb{P}(W)$ is contained in Q, and $Q \setminus \mathbb{P}(W) \to \mathbb{P}(V/W^\perp)$ is a vector bundle

$$\begin{array}{ccccc} \mathbb{P}(W) & \overset{i}{\hookrightarrow} & Q & \overset{j}{\hookleftarrow} & Q \setminus \mathbb{P}(W) \\ & & & & \big\downarrow p \\ & & & & \mathbb{P}(V/W^\perp). \end{array} \tag{1.2}$$

Theorem 1.1.1. *Let us introduce the following notation:*

$$H_{\mathbb{P}(W)} = c_1^\Omega\big(\mathcal{O}_{\mathbb{P}(W)}(1)\big) \in \Omega^*(\mathbb{P}(W)),$$

$$l_i = l_i^\Omega = i_\Omega(H^{d-i}_{\mathbb{P}(W)}), \quad \textit{and} \quad h = h_\Omega = c_1^\Omega(\mathcal{O}_Q(1)).$$

1. *The elements l_i for $0 \leqslant i \leqslant d$, and the powers h^k of h for $0 \leqslant k \leqslant d$ form a free base of $\Omega^*(Q)$ over $\Omega^*(\mathrm{pt}) = \mathbb{L}$.*

2. *The multiplication table is determined by the identities*

$$h \cdot l_i = \begin{cases} l_{i-1}, & i > 0, \\ 0, & i = 0; \end{cases} \tag{1.3}$$

$$l_i \cdot l_j = \begin{cases} l_0, & i = j = d, \ \textit{and} \ D \equiv 0 \mod 4, \\ 0, & \textit{elsewhere}; \end{cases} \tag{1.4}$$

$$h^{d+1} = \frac{[2]_\Omega(h)}{h^{D-2d}}\, l_d = \sum_{i=1}^{D-d} b_i\, l_{D-d-i}\,, \tag{1.5}$$

where $[2]_\Omega(t) = F_\Omega(t,t) = \sum_{i\geq 1} b_i\, t^i \in \mathbb{L}[\![t]\!]$ is the multiplication by 2 in the sense of the universal formal group law.

Proof. The fact that $\Omega^*(Q)$ is a free $\mathbb{L}$-module of rank $2d+2$ is well-known, and easily follows, e.g., from [ViYa, Corollary 2.9] or [NeZa, Theorem 6.5]. However, the direct proof is very short.

Combining the localization exact sequence with the homotopy invariance axiom, we obtain a right exact sequence

$$\Omega^*\big(\mathbb{P}(W)\big) \xrightarrow{\quad i_\Omega \quad} \Omega^*(Q) \xrightarrow{(p^\Omega)^{-1} j^\Omega} \Omega^*\big(\mathbb{P}(V/W^\perp)\big)$$

with $\Omega^*(\mathbb{P}(W))$ and $\Omega^*(\mathbb{P}(V/W^\perp))$ free $\mathbb{L}$-modules of rank $d{+}1$ (in particular, the surjection splits). Observe that i_Ω is injective, because the pushforward from $\Omega^*(\mathbb{P}(W))$ to $\Omega^*(\mathbb{P}(V))$ is injective.

Therefore, to prove the first statement of the theorem, we only have to show that $\Omega^*\big(\mathbb{P}(V/W^\perp)\big)$ is freely generated by the images of h^k, $0 \leqslant k \leqslant d$. Since $Q \setminus \mathbb{P}(W) \subseteq \mathbb{P}(V) \setminus \mathbb{P}(W^\perp)$, it remains to be shown that $\Omega^*\big(\mathbb{P}(V/W^\perp)\big)$ is freely generated by the images of $H^k_{\mathbb{P}(V)}$, $0 \leqslant k \leqslant d$, and the latter is clear.

Observe that the base elements l_i and h^k are homogeneous elements of $\Omega^*(Q)$ of degree $D-i$ and k, respectively, and $\mathbb{L}$ is graded by non-positive numbers, therefore in the decomposition

$$h^{d+1} = \sum_i a_i\, l_i + \sum_k c_k\, h^k$$

of h^{d+1} as a aum of base elements we necessarily have $c_k = 0$ for all k for the degree reasons, i.e., h^{d+1} lies in the image of i_Ω. Since $\Omega^*\big(\mathbb{P}(W)\big)$ injects into $\Omega^*\big(\mathbb{P}(V)\big)$ we can pushforward the above equality to $\Omega^*\big(\mathbb{P}(V)\big)$ to determine a_i. Let $I\colon Q \hookrightarrow \mathbb{P}(V)$ denote the inclusion, then

$$I_\Omega\big(I^\Omega(H^{d+1}) \cdot 1\big) = \sum_i a_i\, H^{D+1-i}.$$

The projection formula implies that the left hand side is equal to $H^{d+1} I_\Omega(1)$, and putting $D = Q$ in (1.1) we have

$$I_\Omega(1_{\Omega^*(Q)}) = c_1^\Omega\big(\mathcal{O}(2)\big) = [2]_\Omega\Big(c_1^\Omega\big(\mathcal{O}(1)\big)\Big) = \sum_{i \geq 1} b_i H^i.$$

Then, clearly, $a_{D-d-i} = b_i$. The same argument proves (1.3).

Finally, for the degree reasons we only need to consider (1.4) for D even and $i = j = d$, where $l_d^2 = a l_0$ for some $a \in \mathbb{Z}$. Since $\Omega^D(Q) \cong \mathrm{CH}^D(Q)$ by [LM, Lemma 4.5.10], we can determine a modulo $\mathbb{L}^{<0}$. But for the Chow theory the result is well-known, see, e.g., [EKM, § 68]. $\square$

Our initial computation of the above multiplication table was more awkward, see Proposition 3.1.1, and the above simplification is suggested by Alexey Ananyevskiy.

1.1.3 Pushforwards along Structure Morphisms

For $X \in \mathcal{S}\mathsf{m}_k$ let $\chi\colon X \to \mathrm{pt}$ be the structure morphism, and let us denote $[X] \in \mathbb{L}$ the pushforward of $1 \in \Omega^*(X)$ along χ, i.e., $[X] = \chi_\Omega(1_{\Omega^*(X)})$. We will describe the class $[S]$ of the hypersurface S in $\mathbb{P}^n$ in terms of the classes of projective spaces $[\mathbb{P}^i]$.

Proposition 1.1.2. *Let $\iota\colon S \hookrightarrow \mathbb{P}^n$ be a smooth hypersurface defined by a homogeneous polynomial $f \in k[X_0, \ldots X_n]$ of degree d. Consider the series $[d]_\Omega(t) = \sum_{i\geq 1} a_i\, t^i$. Then $[S] = \sum_{i=1}^n a_i [\mathbb{P}^{n-i}]$.*

Proof. By (1.1) we have $\iota_\Omega(1_{\Omega^*(S)}) = c_1^\Omega\big(\mathcal{O}(d)\big) = [d]_\Omega(H)$. Then

$$[S] = \sum_{i=1}^n a_i [H^i] = \sum_{i=1}^n a_i [\mathbb{P}^{n-i}].$$

□

The same argument can be slightly generalized to obtain the following

Corollary 1.1.3. *In the notation of Theorem 1.1.1 we have the following formulae:*

$$\chi_\Omega(l_i) = [\mathbb{P}^i]; \tag{1.6}$$

$$\chi_\Omega(h^k) = \sum_{j=1}^{D+1-k} b_j\, [\mathbb{P}^{D+1-k-j}]\,. \tag{1.7}$$

Proof. Identity (1.6) is obvious, and (1.7) follows from (1.1) with the use of the projection formula

$$\chi_\Omega(h^k) = \chi_\Omega\big(I_\Omega(I^\Omega H^k)\big) = \chi_\Omega\big(H^k \cdot I_\Omega(1_{\Omega^*(Q)})\big) = \chi_\Omega\left(\sum_{j=1}^{D+1-k} b_j\, H^{k+j}\right).$$

□

Obviously, the results of Theorem 1.1.1 and Corollary 1.1.3 can be applied to any oriented cohomology theory by the universality of algebraic cobordism.

1.1.4 Morava K-theory

For any commutative $\mathbb{Q}$-algebra R and a formal group law $F(u,v) \in R[\![u,v]\!]$ there exists a unique power series $\log_F(t) = t + \ldots \in R[\![t]\!]$ satisfying

$$\log_F\bigl(F(u,v)\bigr) = \log_F(u) + \log_F(v).$$

This power series is called the logarithm of F. Let $\log_\Omega(t)$ denote the logarithm of the universal formal group law over $\mathbb{L} \otimes \mathbb{Q}$. Then the identity

$$\log_\Omega(t) = \sum_{i=1}^{\infty} \frac{[\mathbb{P}^{i-1}]}{i} t^i \tag{1.8}$$

is known as the Mishchenko formula [Sh, Theorem 1], cf. also Subsection 3.1.2. Therefore, Corollary 1.1.3 is especially useful for oriented cohomology theories with the reasonable logarithm of the corresponding formal group law F.

Following [PS14], for a fixed natural $n \geq 2$ consider the series

$$l(t) = \sum_{k \geq 0} 2^{-k}\, v_n^{\frac{2^{nk}-1}{2^n-1}}\, t^{2^{nk}} \in \mathbb{Q}[v_n][[t]] \tag{1.9}$$

where v_n denotes a free polynomial variable of degree $1 - 2^n$, and let $l^{-1}(t)$ be the *composition* inverse of $l(t)$. Then

$$F(x,y) = l^{-1}(l(x) + l(y)) \tag{1.10}$$

is a formal group law over $\mathbb{Z}_{(2)}[v_n]$ (in fact, over $\mathbb{Z}[v_n]$), see [Haz, Chapter I, Section 2]; here $\mathbb{Z}_{(2)}$ denotes the localization of $\mathbb{Z}$ at the ideal $(2) = 2\mathbb{Z}$. Obviously, $l(t)$ is the logarithm of this formal group law over $\mathbb{Q}[v_n]$. We will call the corresponding free theory

$$\mathrm{K}(n)^* = \Omega^* \otimes_{\mathbb{L}} \mathbb{Z}_{(2)}[v_n^{\pm 1}]$$

the (n-th) Morava K-theory. We remark that we consider Morava K-theories only for prime $p = 2$ because we only work with quadrics (cf. Lemma 1.2.5 below).

The term "Morava K-theory" can denote a family of free theories, as in [Se17, Se18, SePhD, SeSe]. In the present work we prefer to use it, in contrast, only for the theory chosen above. As a side remark, we mention that there exists a universal 2-typical formal group law F_{BP} defined over the ring $V \cong \mathbb{Z}_{(2)}[v_1, v_2, \ldots]$, see [Ra, Theorem A2.1.25]. If $\log_{\mathrm{BP}}(t) = \sum_{i \geq 0} l_i t^{p^i}$ is the logarithm of F_{BP} over $V \otimes \mathbb{Q}$, then the mentioned isomorphism $V \cong$

$\mathbb{Z}_{(2)}[v_1, v_2, \ldots]$ can be chosen in such a way that $2l_k = \sum_{i=0}^{k-1} l_i v_{k-i}^{2^i}$, see [Ra, Theorem A2.2.3]. Ravenel in [Ra] calls this choice of v_k *Hazewinkel's generators.* Then sending v_k to 0 for $k \neq n$ we obtain exactly the formal group law with the logarithm (1.9).

For an oriented cohomology theory A^* let us denote $[\mathbb{P}^n]_A = \chi_A(1_{A^*(\mathbb{P}^n)})$. Then by (1.8) we have

$$[\mathbb{P}^i]_{\mathrm{K}(n)} = \begin{cases} 2^{(n-1)k}\, v_n^{\frac{2^{nk}-1}{2^n-1}}, & i = 2^{nk}-1, \\ 0, & i \neq 2^{nk}-1. \end{cases}$$

In particular, our assumption $n \geq 2$ guaranties that $[\mathbb{P}^i]_{\mathrm{K}(n)} \equiv 0 \mod 2$ for $i > 0$. Moreover, it is easy to see that $l(v_n t^{2^n}) = 2(l(t) - t)$, and therefore

$$[2]_F(t) = l^{-1}(2t) +_F (v_n\, t^{2^n}).$$

Unfortunately, we do not have a closed formula for the coefficients of the series $[2]_F(t)$, e.g., for $n = 2$ we have the following first few terms for the series:

$$[2]_F(t) = 2t - 7v_2\, t^4 + 112 v_2^2\, t^7 - 2380 v_2^3\, t^{10} + 58268 v_2^4\, t^{13} - 1566096 v_2^5\, t^{16} + \ldots$$

However, it is not hard to check that $l^{-1}(2t) \in 2\mathbb{Z}[[t]]$, so that

$$[2]_F(t) \equiv v_n t^{2^n} \mod 2$$

(cf. also [Ra, A2.2.4]). Thus, Theorem 1.1.1 and Corollary 1.1.3 imply the following

Theorem 1.1.4. *Consider the free theory* $\mathrm{K}(n)^*(-;\mathbb{F}_2)$ *with the coefficient ring* $\mathbb{F}_2[v_n^{\pm 1}]$*, and the formal group law obtained as a reduction of* (1.10) *modulo 2. Then, for a smooth projective split quadric Q of dimension $D = 2d+1$ or $D = 2d+2$, the ring* $\mathrm{K}(n)^*(Q;\mathbb{F}_2)$ *is a free* $\mathbb{F}_2[v_n^{\pm 1}]$*-module with the base* $l_i^{\mathrm{K}(n)}$*, and* $(h_{\mathrm{K}(n)})^k$*,* $0 \leq i, k \leq d$*, defined in Theorem* 1.1.1*; the multiplication table can be deduced from the identities*

$$h_{\mathrm{K}(n)} \cdot l_i^{\mathrm{K}(n)} = \begin{cases} l_{i-1}^{\mathrm{K}(n)}, & i > 0, \\ 0, & i = 0, \end{cases} \tag{1.11}$$

$$l_i^{\mathrm{K}(n)} \cdot l_j^{\mathrm{K}(n)} = \begin{cases} l_0^{\mathrm{K}(n)}, & i = j = d, \ \textit{and} \ D \equiv 0 \mod 4, \\ 0, & \textit{elsewhere}, \end{cases} \tag{1.12}$$

$$h_{\mathrm{K}(n)}^{d+1} = \begin{cases} v_n\, l_{D-d-2^n}^{\mathrm{K}(n)}, & D \geq 2^{n+1}-1, \\ 0, & D < 2^{n+1}-1, \end{cases} \tag{1.13}$$

and the pushforward along the structure morphism is described by

$$\chi_{\mathrm{K}(n)}(l_0^{\mathrm{K}(n)}) = 1, \tag{1.14}$$

$$\chi_{\mathrm{K}(n)}(l_i^{\mathrm{K}(n)}) = 0, \quad i > 0, \tag{1.15}$$

$$\chi_{\mathrm{K}(n)}(h_{\mathrm{K}(n)}^{k}) = 0, \quad k \neq D + 1 - 2^n, \tag{1.16}$$

$$\chi_{\mathrm{K}(n)}(h_{\mathrm{K}(n)}^{D+1-2^n}) = v_n, \quad \textit{if } D \geq 2^n - 1. \tag{1.17}$$

More generally, take any formal group law F over $\mathbb{Z}_{(2)}$ with $\log_F(t) = \sum_{i\geqslant 1} c_i t^i$ satisfying the property $i \cdot c_i \equiv 0 \mod 2$ for $i \geqslant 2$ (e.g., any 2^n-typical formal group law of height n [SePhD, Proposition 2.9.4]). Then taking the reduction modulo 2 we obtain a theory A satisfying (1.14), (1.15), and

$$\chi_A(h_A^k) \equiv b_{D+1-k} \mod 2,$$

where $[2]_F = F(t,t) = \sum_{i\geq 1} b_i t^i$. We conclude the section with another generalization of Theorem 1.1.4.

1.1.5 Lubin–Tate Formal Group Laws

We will discuss different choices of $\mathbb{Z}_{(2)}$-integral bases for the Morava of a split quadric. Using Quillen's reorientation [PaSm, Pa02], we conclude that any formal group law isomorphic to (1.10) gives rise to a free theory, naturally isomorphic to $\mathrm{K}(n)^*$ as a functor from $\mathcal{S}\mathsf{m}_k^{\mathrm{op}}$ to $\mathcal{R}\mathsf{ings}^*$ (with a different structure of pushforwards), in particular, different laws give us different choices of l_i, and, as a result, different multiplication tables.

Take a ring $\mathbb{Z}_{(2)} \subseteq R \subseteq \mathbb{Z}_2$, and a series $g(t) \in R[[t]]$ such that

$$g(t) \equiv 2t \mod t^2, \tag{1.18}$$

$$g(t) \equiv t^{2^n} \mod 2. \tag{1.19}$$

For any such series there exists a unique formal group law F_g over R satisfying

$$g\big(F_g(x,y)\big) = F(g(x), g(y)), \tag{1.20}$$

and, moreover,

$$[2]_{F_g}(t) = g(t). \tag{1.21}$$

For different $g(t)$ and $h(t)$ satisfying (1.18) and (1.19) the laws F_g and F_h are strictly isomorphic over R, see [Zi, Chapter I, Section 11], and [Haz, Chapter I, Section 8], in particular, [Haz, 8.3.23 (iii)]. We will call all these laws Lubin–Tate formal group laws. E.g., consider a law F obtained from (1.10) evaluating v_n at 1, and $g(t) = [2]_F(t)$. For these F and g hold (1.18), (1.19) and (1.20), therefore the uniqueness condition implies that F is a Lubin–Tate formal group law.

Following [Se18], we can make a formal group law *homogeneous* adding a formal variable ν. We assume that x and y have degree 1, and elements of R have degree zero, and choose a natural m in such a way that any homogeneous form of $F(x, y)$ of degree $\neq 1 + km$, $k \in \mathbb{N}$, equals zero. We can always take $m = 1$, and, e.g., for the law $F(x, y) = x+y-xy$ it is the only possible choice. However, we prefer to take m as large as we can. Now consider the ring $R[\nu^{\pm 1}]$ assuming that $\deg \nu = -m$, and the formal group law $F^{\mathrm{h}}(x, y)$ over it obtained from $F(x, y)$ by means of the multiplication of each homogeneous form of degree $1 + km$ by ν^k. The obtained law is a homogeneous form over $R[\nu^{\pm 1}][\![x, y]\!]$ of degree 1, and therefore we can consider a free theory

$$A^*_F = \Omega^* \otimes_{\mathbb{L}} R[\nu^{\pm 1}]$$

corresponding to this formal group law. Obviously, starting from $F(x, y) = x + y - xy$, and denoting $\beta = \nu$ we obtain exactly the multiplicative formal group law $F^{\mathrm{h}} = F_{\mathrm{m}}$, and $A^*_F = \mathrm{K}^0 \otimes_{\mathbb{Z}} \mathbb{Z}[\beta^{\pm 1}]$. It can be sometimes convenient to consider also the *connective* version of the theory

$$\mathrm{CA}^*_F = \Omega^* \otimes_{\mathbb{L}} R[\nu],$$

which coincides with usual connective K-theory in the case of the multiplicative formal group law.

Returning to the Lubin–Tate formal group laws, we see that there exist free theories A^* for which $h^{d+1}_A \in A^*(Q)$ can be equal to any linear combination of l^A_i compatible with grading and conditions (1.18) and (1.19).

If we assume that for the series g and h satisfying (1.18) and (1.19) we can chose m in such a way that $g(t)/t = g'(t^m)$, and $h(t)/t = h'(t^m)$, then [Haz, Chapter I, Section 8.2] proves that homogeneous versions F^{h}_g and F^{h}_h of the corresponding formal group laws are still isomorphic. By [Pa02, Theorem 2.3.1], cf. also [Vi19, Theorem 6.9], we conclude that $A^*_{F_g}$ and $A^*_{F_h}$ are naturally isomorphic as functors from $\mathcal{S}\mathsf{m}^{\mathrm{op}}_k$ to $\mathcal{R}\mathsf{ings}^*$.

This proves the following

Corollary 1.1.5. *There exist base elements* $\widetilde{l_i} \in \mathrm{K}(n)^*(Q)$ *such that the* $\mathbb{Z}_{(2)}$*-integral multiplication table in* $\mathrm{K}(n)^*(Q)$ *is determined by the identities (1.11), (1.12) with* l_i *changed by* $\widetilde{l_i}$*, and the identity*

$$h^{d+1} = 2\widetilde{l}_{D-d-1} + v_n \widetilde{l}_{D-d-2^n}.$$

Proof. Take F_g^{h} defined by (1.10), and $h(t) = 2t + t^{2^n} \in \mathbb{Z}_{(2)}[[t]]$ in the discussion above. Then the isomorphism between $\mathrm{K}(n)^*$ and $A^*_{F_h}$ gives us the desired base elements. □

Cf. the above statement with [SeSe, Proposition 8.9].

1.2 The Krull–Schmidt Theorem

In the present section we recall several basic facts about the category of motives, and prove the Krull–Schmidt Theorem for the $\mathrm{K}(n)$-motive $\mathcal{M}_{\mathrm{K}(n)}(Q)$ of any smooth projective quadric Q, i.e., we prove that the decomposition of $\mathcal{M}_{\mathrm{K}(n)}(Q)$ into indecomposable summands is unique.

1.2.1 The Category of Motives

For an oriented cohomology theory A^* we consider the category of (effective) A-motives defined as in [Ma].

First, we consider the category of correspondences $\mathcal{C}\mathsf{orr}_A$ whose objects are pairs (X, n) for X a smooth projective variety over k, and n a non-negative integer, and the morphisms are given by

$$\mathcal{C}\mathsf{orr}_A\big((X,n),(Y,m)\big) = \bigoplus A^{d_i+m-n}(X \times Y_i)$$

for the connected components $\coprod Y_i = Y$ of dimensions $d_i = \dim Y_i$. We will write X for $(X, 0)$, and call morphisms from X to Y in this category 0-*correspondences*. We denote (X, n) by $X\{n\}$ and call these objects *twisted* by n. It suffices to define the identity maps and the composition only for connected varieties, and these definitions can be extended linearly to non-connected ones. We recall the constructions to fix notation.

For a connected X the identity morphism of $X\{n\}$ is given by the push-forward

$$\delta_A(1_{A^*(X)}) \in A^{\dim X}(X \times X)$$

along the diagonal map $\delta\colon X \to X \times X$, and we will denote it by $\Delta = \Delta_{X\{n\}}$. The composition of correspondences is given by

$$f \circ g = (\mathrm{pr}_{13})_A\Big((\mathrm{pr}_{23})^A(f) \cdot (\mathrm{pr}_{12})^A(g)\Big),$$

for $\mathrm{pr}_{ij}\colon X \times X \times X \to X \times X$ natural projections. E.g., for a, $b \in A^*(X)$ let us denote

$$a \times b = \mathrm{pr}_1^A(a) \cdot \mathrm{pr}_2^A(b),$$

where pr_i are natural projections from $X \times X$ to X. If $\chi\colon X \to \mathrm{pt}$ denotes the structure morphism, then one has the equality $\big((\mathrm{pr}_{13})_A \circ (\mathrm{pr}_{12} \circ \mathrm{pr}_2)^A\big)(x) = \chi_A(x)$ by the transversal square axiom, and therefore the projection formula implies that

$$a \times b \ \circ \ c \times d \ = \ \chi_A(ad) \cdot c \times b. \tag{1.22}$$

The idempotent completion $\mathcal{M}\mathsf{ot}_A$ of the category $\mathcal{C}\mathsf{orr}_A$ is called the category of A-motives [Ma, §5]. Objects of $\mathcal{M}\mathsf{ot}_A$ are pairs $(X\{n\}, \pi)$ where π is an idempotent in $\mathcal{C}\mathsf{orr}_A(X\{n\}, X\{n\}) = A^{\dim X}(X \times X)$. The pair (X, Δ_X) is called the *motive* of X, and we will denote it $\mathcal{M}(X) = \mathcal{M}_A(X)$.

Morphisms $(X\{n\}, \pi) \to (Y\{m\}, \rho)$ are given by

$$f \in \mathcal{C}\mathsf{orr}_A(X\{n\}, Y\{m\})$$

such that $\rho \circ f \circ \pi = f$ (observe that the definition from [Ma, §5] is obviously equivalent to the above one). In this category each idempotent splits in the sense of [Ba, Chapter I, § 3], and therefore direct sum decompositions of the object $\mathcal{M}(X)$ are in 1-to-1 correspondence with decompositions $\Delta_X = \sum \pi_i$ of Δ_X as a sum of mutually orhogonal idempotents

$$\pi_i \in \mathcal{C}\mathsf{orr}_A(X, X) = A^{\dim X}(X \times X),$$

i.e., such that $\pi_i \pi_j = 0$ for $i \neq j$.

1.2.2 Split Motives

We call twisted motives of the point *Tate motives*, and we say that the motive $\mathcal{M}(X)$ of a smooth projective variety X is *split*, if it is isomorphic to a sum of Tate motives,

$$\mathcal{M}(X) \cong \bigoplus \mathcal{M}(\mathrm{pt})\{n_i\}.$$

If (X, π) is a Tate motive, i.e., $(X, \pi) \cong \mathcal{M}(\mathrm{pt})\{n\}$ for some n, then there exist morphisms

$$\mathcal{M}(\mathrm{pt})\{n\} \xrightarrow{q} \mathcal{M}(X) \xrightarrow{p} \mathcal{M}(\mathrm{pt})\{n\},$$

i.e., $q \in A^{\dim X - n}(X)$, and $p \in A^n(X)$ such that $\chi_A(pq) = 1$. Conversely, having such p and q, we see that $\pi = q \times p$ is an idempotent by (1.22), and (X, π) is a Tate motive.

We call a smooth projective variety X *cellular* if there exists a filtration

$$X = X_0 \supseteq X_1 \supseteq \ldots \supseteq X_{n+1} = \emptyset \tag{1.23}$$

of X by closed subschemes such that $X_i \setminus X_{i+1}$ is a disjoint union of affine spaces for each i. By [ViYa, Corollary 2.9] the cobordism motive $\mathcal{M}_\Omega(X)$ of a cellular variety X is split. Then the universality of algebraic cobordism implies that the A-motive $\mathcal{M}_A(X)$ is split for any theory A^*.

For X cellular, and any smooth projective Z, we can obtain from (1.23) a similar filtration for $X \times Z$, and deduce by the same kind of argument that the motive of $X \times Z$ decomposes to a sum of twisted motives of Z, cf. [NeZa]. This implies, in particular, that the map $A^*(X) \otimes_{A^*(\mathrm{pt})} A^*(Z) \to A^*(X \times Z)$ given by $x \otimes z \mapsto x \times z$ is an isomorphism. We refer to this fact as the *Künneth formula*.

Natural examples of cellular varieties are projective spaces and *split* projective quadrics. In the latter case a cellular filtration can be obtained from (1.2).

1.2.3 The Rost Nilpotence Principle

Below we consider only *free* theories to simplify the notation. Let L/k be a field extension, and for $X \in \mathcal{S}\mathsf{m}_k$ let us denote $X_L = X \times_{\operatorname{Spec} k} \operatorname{Spec} L \in \mathcal{S}\mathsf{m}_L$. We have the natural map

$$\mathrm{res}^{\Omega}_{L/k} \colon \Omega^*(X) \to \Omega^*(X_L)$$

which we call *the extension of scalars*, see [LM, Example 1.2.10], and [GiVi, Example 2.7]. This gives us a map $\mathrm{res}^A_{L/k}$ for any free theory A^*. The case $L = \overline{k}$ an algebraic closure of k is of a special interest. In particular, for $X = Q$ a smooth projective quadric, $Q_{\overline{k}}$ is a split quadric, and the results of Section 1.1 can be applied to it.

We write $\overline{X}$ for $X_{\overline{k}}$ and call the image of $\mathrm{res}_{\overline{k}/k}$ the subring of *rational elements*. The examples of rational elements are, e.g., $1 \in A^0(\overline{X})$ and $\Delta_{\overline{X}} \in A^{\dim X}\big(\overline{X} \times \overline{X}\big)$. For a smooth projective quadric $i \colon Q \hookrightarrow \mathbb{P}^{\dim Q+1}$ the element $i^A\Big(c_1\big(\mathcal{O}_{\mathbb{P}^{\dim Q+1}}(1)\big)\Big) \in A^1(Q)$ is the preimage of $h \in A^1(\overline{Q})$, i.e., h is another example of a rational element.

The map $\mathrm{res}^A_{L/k}$ can be extended to the category of A-motives. In particular, for a motive $\mathcal{M}$ consider the map

$$\mathrm{End}_{\mathcal{M}\mathsf{ot}_A}(\mathcal{M}) \to \mathrm{End}_{\mathcal{M}\mathsf{ot}_A}(\mathcal{M}_L). \tag{1.24}$$

We say that the *Rost nilpotence principle* holds for $\mathcal{M}$ if for any field extension L/k the kernel of the map (1.24) consists of nilpotent elements (with respect to the composition of correspondences as a multiplication).

The Rost nilpotence principle was proven for the Chow motives of projective quadrics by Rost in [Ro]. Alternative proofs of this result are given in [Vi98] and [Br03]. The Rost nilpotence principle for Chow motives of projective homogeneous varieties was proven in [CGM] and [Br05], for generically split Chow motives of smooth projective varieties in [ViZa], and for Chow motives of surfaces in [Gi10, Gi14]. By [ViYa, Corollary 2.8] the Chow theory can be replaced, e.g., by algebraic cobordism or connective K-theory.

The recent theorem of Gille–Vishik [GiVi, Corollary 4.5] asserts that the Rost nilpotence principle holds for the motive $\mathcal{M}(X)$ of any projective homogeneous variety X for a semisimple linear algebraic group, and for any free theory A^* (e.g., for a motive $\mathcal{M}(Q)$ of any smooth projective quadric Q).

The power of the above result can be illustrated by the following purely ring-theoretic proposition [Ba, Chapter III, Proposition 2.10].

Proposition 1.2.1. *Let N be a two-sided ideal in a ring R, and suppose either that N is nil or that R is N-adically complete (i.e., $R \cong \varprojlim R/N^i$). Then finite sets of orthogonal idempotents can be lifted modulo N. I.e., given $\pi_i \in R/N$ such that $\pi_i\pi_j = \delta_{ij}\pi_i$, there exist $\varpi_i \in R$ such that $\pi_i = \varpi_i + N$, and $\varpi_i\varpi_j = \delta_{ij}\varpi_i$.*

If we assume additionally in the statement of the above proposition that the family of idempotents π_i is complete, i.e., $\sum_{i=1}^{m} \pi_i = 1$, then the lifts ϖ_i for $i < m$, and $\varpi_m = 1 - \sum_{i=1}^{m-1} \varpi_i$ is a *complete* family of orthogonal idempotents in R, and $\pi_m = \varpi_m + N$.

In particular, if we have a motivic decomposition $\mathcal{M}(\overline{Q}) = \bigoplus \mathcal{M}_i$, such that in the corresponding decomposition $\Delta_{\overline{Q}} = \sum \pi_i$ all orthogonal projectors π_i are *rational*, then the lifts ϖ_i give us a motivic decomposition of $\mathcal{M}(Q)$.

Remark 1.2.2. We also remark that we can lift Tate summands without the Rost nilpotence principle. More precisely, for Tate summands of $\mathcal{M}_{\overline{k}}$ corresponding to orthogonal idempotents $a_i \times b_i$ with a_i and b_i rational, i.e., coming from some α_i and β_i, we have $\chi_A(\alpha_i\beta_i) = 1$, and $\chi_A(\alpha_i\beta_j) = 0 = \chi_A(\beta_j\alpha_i)$ for $i \neq j$, therefore the lifts $\alpha_i \times \beta_i$ are again orthogonal and correspond to Tate summands.

1.2.4 The Krull–Schmidt Theorem over Fields

We say that the Krull–Schmidt Theorem holds for a motive $\mathcal{M}$ if for any two decompositions of $\mathcal{M}$ as a finite direct sum of *indecomposable* motives

$$\mathcal{M} \cong \bigoplus_{i=1}^{m} \mathcal{N}_i \cong \bigoplus_{j=1}^{m'} \mathcal{N}'_j$$

one has $m = m'$ and $\mathcal{N}'_j \cong \mathcal{N}_{\sigma(i)}$ for some permutation $\sigma \in S_m$.

Recall that a (non-commutative) ring S is *local* if $a + b = 1$ implies that at least one of a, b is invertible for $a, b \in S$. The following proposition is our main tool to prove the Krull–Schmidt Theorem [Ba, Chapter I, Theorem 3.6].

Proposition 1.2.3. *Assume that an additive category $\mathcal{C}$ is idempotent complete. Let A_i, B_j be objects of $\mathcal{C}$ with local endomorphism rings, and $A_1 \oplus \ldots \oplus A_m = B_1 \oplus \ldots \oplus B_{m'}$. Then $m = m'$, and $A_i \cong B_{\sigma(i)}$ for some $\sigma \in S_m$.*

In particular, if any indecomposable direct summand of a motive $\mathcal{M}$ has a local endomorphism ring, we conclude that the Krull–Schmidt Theorem holds for $\mathcal{M}$.

Proposition 1.2.4. *Consider a theory A^* with the property that $A^*(\mathrm{pt})$ is a K-algebra over some field K (of an arbitrary characteristic), and all $A^k(\mathrm{pt})$ are finite dimensional vector spaces over K, and consider a smooth projective variety X over k with $\mathcal{M}(\overline{X})$ split and satisfying the Rost nilpotence principle. Then the Krull–Schmidt Theorem holds for $\mathcal{M}(X)$.*

If A^* is free we can take X to be any projective homogeneous variety for a semisimple group G by [GiVi, Corollary 4.5].

Proof. Since $\mathcal{M}\mathsf{ot}_A\big(\mathcal{M}(\mathrm{pt})\{n\}, \mathcal{M}(\mathrm{pt})\{m\}\big) = A^{m-n}(\mathrm{pt} \times \mathrm{pt})$, the endomorphism ring of $\mathcal{M}(\overline{X})$ can be represented by a matrix with an element from $A^{k_{ij}}(\mathrm{pt})$ in the position (i, j). In particular, it is a finite-dimensional algebra over a field K.

Now take an indecomposable (non-zero) summand $\mathcal{N}$ of $\mathcal{M}(X)$, then the image of $\mathcal{N}$ under the restriction map is the direct summand of $\mathcal{M}(\overline{X})$, and the image S of the endomorphism ring $\mathrm{End}(\mathcal{N})$ is a subring of the described matrix ring, in particular, it is also finite-dimensional.

Since the Rost nilpotence principle holds for $\mathcal{M}(X)$, we conclude that $S \neq 0$, and S is indecomposable by Proposition 1.2.1. This implies that S is local [La01, Corollary 19.19].

Finally, apply the Rost nilpotence principle again to conclude that $\mathrm{End}(\mathcal{N})$ is local itself, and therefore the Krull–Schmidt Theorem holds for $\mathcal{M}(X)$ by Proposition 1.2.3. □

We also remark that we do not avoid decomposable Tate motives in the above proposition, cf. [SeZh, Remark 3.10].

1.2.5 Krull–Schmidt for Quadrics

We have seen in the previous section that the Krull–Schmidt Theorem holds for a motive of a smooth projective quadric Q with respect to Morava K-theory $\mathrm{K}(n)^*(-; \mathbb{F}_2)$. In the present section we prove it for $\mathrm{K}(n)^*$ as well.

We basically repeat the argument from [ChMe]. Observe that for *connective* Morava K-theory the Krull-Schmidt Theorem follows from the case of Chow [ChMe, Corollary 35] by [ViYa, Corollary 2.8].

More generally, we can consider any free theory A^* with $A^*(\mathrm{pt}) = R[\nu^{\pm 1}]$ a Laurent polynomial ring over a d.v.r. R of $\operatorname{char} R \neq 2$ with $\deg \nu = -m$, and $\deg a = 0$ for $a \in R$, cf. Section 1.1.5.

Take an indecomposable summand $\mathcal{N}$ of $\mathcal{M}(Q)$ and let us denote by S the image of $\operatorname{End}_{\mathcal{M}\mathsf{ot}_A}(\mathcal{N})$ under the restriction map $\operatorname{res}_{\overline{k}/k}$. We will prove that S is local.

The ring $\operatorname{End}\big(\mathcal{M}_{\mathrm{K}(n)}(\overline{Q})\big)$ can be described as a matrix ring, and since $A^k(\mathrm{pt})$ is either R or 0, this ring is free as R-module. Its subring S is therefore torsion free and finitely generated over R. Since R is a d.v.r., this implies that S is also free over R [CuRe, Theorem 4.13].

Next, let us denote by K the field of fractions of R and consider the theory $B^* = A^*(-;\, K)$.

Lemma 1.2.5. *For any free theory B^* with $\frac{1}{2} \in B^*(\mathrm{pt})^\times$ the B^*-motive of Q has at least $D + 1 = \dim Q + 1$ Tate summands.*

Proof. By the universality of algebraic cobordism it suffices to consider the case $B^* = \Omega^* \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$ which in turn follows from the case $B^* = \mathrm{CH}^* \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$ by [ViYa, Corollary 2.8]. For such a B^* one can lift the system of rational orthogonal idempotents $\frac{1}{2}\, h^i \times h^{D-i}$, $0 \leq i \leq D$, in $B^D(\overline{Q} \times \overline{Q})$, cf. (1.22) or [EKM, Section 68].

□

We can take any lift $\widehat{h}$ for h and define the lifts $\pi_i = \frac{1}{2}\, \widehat{h}^i \times \widehat{h}^{D-i}$. Let us denote $\varpi = \Delta_Q - \sum_{i=0}^{D} \pi_i$. Observe that ϖ is symmetric with respect to the transpose automorphism of $A^*(Q \times Q)$, induced by the exchange of factors in $Q \times Q$. Since $\mathcal{M}(\overline{Q})$ is a sum of $2d + 2$ Tate motives for $D = 2d$ or $D = 2d + 1$, we see that $\operatorname{res}_{\overline{k}/k}(\varpi) = 0$, for D odd, which implies that $\varpi = 0$ by the Rost nilpotence principle, and that $\operatorname{res}_{\overline{k}/k}(\varpi)$ is a Tate motive $\mathcal{M}(\mathrm{pt})\{d\}$ for D even (we use that the Krull–Schmidt Theorem is already proven over fields, and Tate motives are indecomposable).

Example. As an example, consider $B^* = \mathrm{CH}^* \otimes \mathbb{Q}$, and a zero-dimensional quadric $Q = \operatorname{Spec} k(\sqrt{a})$ corresponding to a 2-dimensional quadratic form $\langle 1,\, -a \rangle$ (assuming that a is not a square in k). Then

$$\pi_0 = \frac{1}{2} \cdot 1 = (1/2,\, 1/2) \in \mathrm{CH}^0\big(\operatorname{Spec} k(\sqrt{a}) \times_{\operatorname{Spec} k} \operatorname{Spec} k(\sqrt{a});\, \mathbb{Q}\big) = \mathbb{Q} \oplus \mathbb{Q}$$

(one can check by definition that $1 \circ 1 = 2$), and B^*-motive of $\operatorname{Spec} k(\sqrt{a})$ is decomposable in contrast to the integral CH^*-motive. However,

$$\mathrm{CH}^*\big(\operatorname{Spec} k(\sqrt{a});\, \mathbb{Q}\big) = \mathbb{Q}$$

is indecomposable, which shows that $\mathcal{M}\mathsf{ot}_B(\varpi,\, \mathbb{Q}) = 0 = \mathcal{M}\mathsf{ot}_B(\mathbb{Q},\, \varpi)$. At the same time, $\mathcal{M}\mathsf{ot}_B\big(\mathrm{res}_{\overline{k}/k}(\varpi),\, \mathbb{Q}\big) = \mathbb{Q} = \mathcal{M}\mathsf{ot}_B\big(\mathbb{Q},\, \mathrm{res}_{\overline{k}/k}(\varpi)\big)$ since $\mathrm{res}_{\overline{k}/k}(\varpi) = \mathbb{Q}$ is a Tate motive.

Returning to our situation, where $\mathcal{N}$ is an indecomposable summand of $\mathcal{M}_A(Q)$, and K the field of fractions of $R = A^*(\mathrm{pt})$, we can pass now to the direct summand $\mathcal{N} \otimes K$ of a B^*-motive of Q, and this direct summand decomposes into a sum of π_{i_k} and possibly ϖ.

If $\mathcal{N} \otimes K$ is a sum of π_{i_k}, then $\mathrm{res}_{\overline{k}/k}\colon \mathrm{End}(\mathcal{N} \otimes K) \to \mathrm{End}(\overline{\mathcal{N}} \otimes K)$ is an isomorphism and $S \otimes_R K = \mathrm{End}(\overline{\mathcal{N}} \otimes K)$ is a product of $m = \deg \nu$ matrix rings of size $n_j \times n_j$ over K, where n_j is a number of motives $\mathcal{M}(\mathrm{pt})\{j + mh\}$, $h \in \mathbb{N}$, among π_{i_k}. We claim that if $\mathcal{N} \otimes K$ is a sum of π_{i_k}-s and ϖ, then $S \otimes_R K$ is a product of matrix rings over K as well.

Observe that for $D = 2d$, and for

$$f \in B^d(Q) = \mathcal{M}\mathsf{ot}_B\big(Q,\, \mathcal{M}(\mathrm{pt})\{d\}\big) = \mathcal{M}\mathsf{ot}_B\big(\mathcal{M}(\mathrm{pt})\{d\},\, Q\big)$$

one has $\varpi \circ f = (\mathrm{pr}_2)_*\big(\varpi \cdot \mathrm{pr}_1^*(f)\big)$, and $f \circ \varpi = (\mathrm{pr}_1)_*\big(\varpi \cdot \mathrm{pr}_2^*(f)\big)$, where pr_i are the natural projections from $Q \times Q$ to Q. Since ϖ is symmetric with respect to the transpose isomorphism, we get $\varpi \circ f = f \circ \varpi$, and therefore

$$\mathcal{M}\mathsf{ot}_B\big(\varpi,\, \mathcal{M}(\mathrm{pt})\{d\}\big) = \mathcal{M}\mathsf{ot}_B\big(\mathcal{M}(\mathrm{pt})\{d\},\, \varpi\big).$$

Now we see that, if $\mathcal{N} \otimes K$ is a sum of π_{i_k}-s and ϖ for D even, then $S \otimes_R K = \mathrm{res}_{\overline{k}/k}\big(\mathrm{End}(\mathcal{N} \otimes K)\big)$ is a product of $m - 1$ matrix rings $n_j \times n_j$ over K, where n_j is a number of motives $\mathcal{M}(\mathrm{pt})\{j + mh\}$ among π_{i_k} for $j \neq d$, and a block with $L = \mathrm{res}_{\overline{k}/k}\big(\mathrm{End}(\varpi)\big)$ in the left upper corner, $M = \mathrm{res}_{\overline{k}/k}\big(\mathcal{M}\mathsf{ot}_B\big(\varpi,\, \mathcal{M}(\mathrm{pt})\{d\}\big)\big)$ in each other cell of the first row and the first column, and K in the other cells. Since L is a K-subalgebra of $\mathrm{End}\big(\mathcal{M}(\mathrm{pt})\{d\}\big) = K$, we conclude that $L = K$. Similarly, M as a K-submodule of K can be either K, or 0. In any case we get that $S \otimes_R K$ is a product of matrix rings over K.

Consequently, we can apply the following particular case of Heller's Theorem [CuRe, Theorem 30.18], cf. also [CuRe, Definitions 30.12, 23.1–4, 7.12, 3.35].

Theorem (Heller)**.** *Let R be a d.v.r. with a field of fractions K, let S be an R-algebra, which is a free R-module of finite rank, and assume that $S \otimes_R K$*

is a product of matrix rings over K. Let us denote by $\widehat{\ }$ the completion with respect to the maximal ideal of R. If S is indecomposable in a sum of left ideals, then $\widehat{S}$ is also indecomposable.

Let us denote the uniformizing element of R by p, then applying the above Theorem to our S we conclude that $S/p = \widehat{S}/p$ is indecomposable by Proposition 1.2.1. Since S/p is finite dimensional over a field R/p, we conclude that it is local [La01, Corollary 19.19].

This implies that S itself is local. Indeed, let $a + b = 1 \in S$, and assume that $\bar{a}$ is invertible modulo p. Then finitely generated R-modules S/aS and S/Sa are equal to 0 by Nakayama's Lemma ($M = pM \Rightarrow M = 0$), i.e., a is right and left invertible.

Finally, by the Rost nilpotence principle we get that the endomorphism ring $\mathrm{End}_{\mathcal{M}\mathsf{ot}_A}(\mathcal{N})$ of any indecomposable summand $\mathcal{N}$ of $\mathcal{M}(Q)$ is local, which implies that the Krull–Schmidt Theorem holds for $\mathcal{M}(Q)$ by Proposition 1.2.3.

Taking $A^* = \mathrm{K}(n)^*$, $R = \mathbb{Z}_{(2)}$, $\nu = v_n$, and $m = 2^n - 1$ in the above consideration, we obtain the following

Theorem 1.2.6. *Let Q be a smooth projective quadric, then the Krull–Schmidt Theorem holds for the* $\mathrm{K}(n)^*$*-motive of* Q.

It is easy to deduce now that motivic decompositions of $\mathcal{M}(Q)$ with respect to A^* and $A^*(-;\, R/p)$ coincide. We remark that for the Chow theory it was shown in [Vi04, Hau].

We remark that for any summand $\mathcal{N}$ of $\mathcal{M}(Q)$, its image under the restriction map is a sum of Tate motives (Tate summands are indecomposable since R is local). Their amount is called the *rank* of $\mathcal{N}$. For $\mathcal{N} \neq 0$ the Rost nilpotence principle implies that $\mathrm{rank}\,\mathcal{N} > 0$.

Then the motive of Q clearly admits a decomposition into a finite direct sum of indecomposable summands, and, moreover, any direct sum decomposition is finite (an induction by the rank). Consider the decomposition of $\mathcal{M}(Q)$ with respect to the theory A^* into a direct sum of indecomposable summands (the Krull–Schmidt Theorem implies that such a decomposition is unique), and consider completions of endomorphism rings E_i of the summands $\mathcal{N}_i$. By Heller's Theorem above (and the Rost nilpotence principle), these completions $\widehat{E_i}$ are indecomposable. Then by Proposition 1.2.1 we obtain that $\widehat{E_i}/p = E_i/p$ are indecomposable. Since E_i/p coincides with the endomorphism ring of $\mathcal{N}_i \otimes R/p$, we see that the motive of Q with respect to $A^*(-;\, R/p)$ is the sum of the indecomposable motives $\mathcal{N}_i \otimes R/p$, and by the Krull–Schmidt Theorem such a decomposition is unique.

Corollary 1.2.7. *The motivic decompositions of a smooth projective quadric Q with respect to Morava K-theories $\mathrm{K}(n)^*$ and $\mathrm{K}(n)^*(-;\mathbb{F}_2)$ are the same. More precisely, for a decomposition of $\mathcal{M}(Q) \cong \bigoplus \mathcal{N}_i$ into indecomposable summands with respect to $\mathrm{K}(n)^*$, the respective summands $\mathcal{N}_i \otimes \mathbb{F}_2$ of $\mathcal{M}(Q) \otimes \mathbb{F}_2$ are again indecomposable.*

As we have seen previously, passing to $\mathbb{F}_2$-coefficients gives us a great technical advantage, which we realize in the next section.

1.3 Morava Motives of Quadrics

In the present section we describe the decomposition of the $\mathrm{K}(n)$-motive $\mathcal{M}_{\mathrm{K}(n)}(Q)$ of a *generic* quadric Q into indecomposable summands.

1.3.1 Height n Theories

It is easy to prove that the $\mathrm{K}(n)$-motive of any quadric Q of large dimension is decomposable.

Proposition 1.3.1. *Let Q be a smooth projective quadric of dimension $D \geq 2^n - 1$, $n \geq 2$. Then its $\mathrm{K}(n)$-motive $\mathcal{M}(Q)$ decomposes as a sum of $D - 2^n + 2$ Tates and a motive $\mathcal{N}$ of rank 2^n for D even, or $2^n - 1$ for D odd.*

Proof. We are free to work with $\mathrm{K}(n)^*(-;\mathbb{F}_2)$ instead of $\mathrm{K}(n)^*$ due to Corollary 1.2.7. Let us denote $D' = D - 2^n + 1$, then applying Theorem 1.1.4 to (1.22) we see that $\pi_i = v_n^{-1}\, h^i \times h^{D'-i}$ for $0 \leq i \leq D'$ is a system of $D' + 1$ rational orthogonal projectors, corresponding to Tate summands. □

Remark. Although we prefer to consider Morava K-theories only for $n \geq 2$, we remark that Lemma 1.2.5 is the parallel result for $\mathrm{K}(0)^* = \mathrm{CH}^*(-;\mathbb{Q})$.

Our objective is to show that the remaining summand $\mathcal{N}$ is generally speaking indecomposable. Before passing to this question we describe possible generalizations of Proposition 1.3.1.

Consider a field K of characteristic 2, and a formal group law F over it. Then the series $[2]_F(t) = F(t,t)$ is either zero or starts with at^{2^n} for some $a \in K^\times$ [Ra, Lemma A2.2.7], and $n \in \mathbb{N} \setminus 0$. In the latter case we say that F has (2-)*height* n, and in the former one F is isomorphic to the additive formal group law (cf., e.g., the remark before [Zi, Theorem 5.33]), and is usually assumed to have height ∞. If $\operatorname{char} K \neq 2$, the series $[2]_F(t)$ starts with $2t$ and we can additionally set the 2-height of F equal to 0. Reductions of Lubin–Tate formal group laws modulo 2 give us examples of formal group laws of all heights, see Section 1.1.5.

Passing to the Laurent polynomial ring $K[\nu^{\pm 1}]$ with $\deg \nu = -m$ we can obtain a homogeneous version F^{h} of F as in Section 1.1.5, and consider the free theory theory A^* with the ring of coefficients $A^*(\mathrm{pt}) = K[\nu^{\pm 1}]$, and the formal group law $F_A = F^{\mathrm{h}}$.

Assume that Q is a smooth projective quadric of dimension $D \geq 2^n - 1$, where n is the height of F. For $D' = D - 2^n + 1$ we see from (1.7) that

$$\chi_A(h_A^{D'}) = a \cdot \nu^{2^n/m},$$

for $a \in K^\times$, and $\chi_A(h_A^k) = 0$ for $k > D'$. Let us denote

$$\pi_i = a^{-1}\nu^{-2^n/m} \cdot h_A^i \times h_A^{D'-i} \in A^D(\overline{Q} \times \overline{Q}), \quad 0 \leqslant i \leqslant D'.$$

It immediately follows from (1.22) that $\pi_i \circ \pi_i = \pi_i$, and $\pi_i \circ \pi_j = 0$ for $i > j$.

Lemma 1.3.2. *Let $\pi'_k, \ldots, \pi'_N$ be orthogonal projectors, and $\pi_0, \ldots, \pi_{k-1}$ be projectors with the property $\pi'_i \circ \pi_j = 0$, and $\pi_i \circ \pi_j = 0$ for $i > j$. Let us denote*

$$\pi'_{k-1} = \pi_{k-1} - \sum_{i=k}^{N} \pi_{k-1} \circ \pi'_i.$$

Then π'_{k-1} is again a projector, it is orhogonal to π'_i for $i > k-1$, and one has $\pi'_{k-1} \circ \pi_j = 0$ for $k-1 > j$. Moreover, if $\pi'_i = a'_i \times b'_i$ for $i \geq k$, and $\pi_j = a_j \times b_j$ for $j \leq k-1$, then it immediately follows from (1.22) that $\pi'_{k-1} = a'_{k-1} \times b'_{k-1}$ for $a'_{k-1} = a_{k-1} - \sum_{i=k}^{N} \chi_A(a_{k-1}b'_i)\, a'_i$, and $b'_{k-1} = b_{k-1}$.

The claim of Lemma 1.3.2 is obvious, and applied to our π_i-s it allows to obtain a system of mutually orthogonal projectors π'_i for $0 \leqslant i \leqslant D'$, corresponding to Tate summands. Then the Rost nilpotence principle allows to decompose the A-motive of Q into $D'+1$ Tates and a remaining summand (cf. also Remark 1.2.2).

1.3.2 Morava with Finite Coefficients

In the present subsection we work with a split quadric Q, and give an explicit decomposition of its $\mathrm{K}(n)^*(-;\mathbb{F}_2)$-motive into Tate summands in terms of orthogonal projectors for $n \geq 2$. We work with the base h^i, l_i, $0 \leq i \leq d$ of the ring $\mathrm{K}(n)^*(Q;\mathbb{F}_2)$, where $D = \dim Q$ is equal to $2d$ or $2d+1$, and sometimes write $h^i = 0 = l_i$ for $i < 0$.

Proposition 1.3.3. *We will use the following notation:*

$$D' = D - 2^n + 1,$$
$$d' = D' - d.$$

Then the following statements hold.

1. *The diagonal* $\Delta \in \mathrm{K}(n)^*(Q \times Q; \mathbb{F}_2)$ *is equal to*

$$\Delta = \sum_{i=0}^{d} \left(h^i \times l_i + l_i \times h^i\right) + v_n \sum_{i=d'}^{d} l_i \times l_{D'-i} +$$
$$+ \delta_{0,\, D \bmod 4} \cdot (h^d + v_n l_{d'}) \times (h^d + v_n l_{d'}),$$

where $\delta_{0,\, D \bmod 4}$ *is* 0 *for* $D \not\equiv 0 \mod 4$*, and* 1 *for* $D \equiv 0 \mod 4$*. Here, in particular,* l_i *is assumed to be* 0 *for* $i < 0$*.*

2. *Projectors* $\pi_i = v_n^{-1} \cdot h^i \times h^{D'-i}$ *for* $0 \leq i \leq D'$*, together with*

$$\varpi_j = (h^j + v_n l_{D'-j}) \times (l_j + v_n^{-1} h^{D'-j})$$

for $d' \leq j \leq d-1$*, and*

$$\varpi_d = \left(h^d + v_n l_{d'}\right) \times \left(l_d + v_n^{-1} h^{d'} + \delta_{0,\, D \bmod 4} \cdot (h^d + v_n l_{d'})\right),$$

define a decomposition of Δ *into a sum of* $2d+2$ *orthogonal Tates. Observe that* $\pi_i = h^i \times l_i$*, and* $\pi_{D'-i} = l_i \times h^i$ *for* $i < d'$*.*

Proof. The proof is straightforward. Recall that $h^{d+1} = v_n\, l_{d'-1}$, and, more generally, $h^m = v_n\, l_{D'-m}$ for $m \geq d+1$, in particular, l_i is rational for $i < d'$. We denote by $\chi \colon Q \to \mathrm{pt}$ the structure morphism to the point, and the pushforward along this morphism is given by the following formulae (see Theorem 1.1.4):

$$\begin{aligned} \chi_{\mathrm{K}(n)}(l_0) &= 1, \\ \chi_{\mathrm{K}(n)}(l_i) &= 0, \quad i > 0, \\ \chi_{\mathrm{K}(n)}(h^{D'}) &= v_n, \\ \chi_{\mathrm{K}(n)}(h^k) &= 0, \quad k \neq D'. \end{aligned}$$

First, assume that $D \not\equiv 0 \mod 4$, then the diagonal in $\mathrm{K}(n)^*(Q \times Q; \mathbb{F}_2)$ is equal to

$$\Delta = \sum_{k=0}^{d} \left(h^k \times l_k + l_k \times h^k\right) + v_n \sum_{i+j=D'} l_i \times l_j.$$

Indeed, it is obvious that $\Delta \circ (a \times l_m) = a \times l_m$; similarly, for $m < d'$ we have $\Delta \circ (a \times h^m) = a \times h^m$. Finally, for $d' \leq m \leq d$ we get three summands in the composition

$$\Delta \circ (a \times h^m) = v_n \cdot a \times l_{D'-m} + a \times h^m + v_n \cdot a \times l_{D'-m},$$

and two of them cancel each other. As we remarked, we mean that $l_i = 0$ for $i < 0$, if D is small. We have just checked that $\Delta \circ (a \times b) = a \times b$ for arbitrary $a, b \in \mathrm{K}(n)^*(Q; \mathbb{F}_2)$, and by symmetry we get $(a \times b) \circ \Delta = a \times b$.

The projectors $\pi_i = v_n^{-1} \cdot h^i \times h^{D'-i}$ are mutually orthogonal for $0 \leq i \leq D'$, see Proposition 1.3.1. Since $\pi_i = h^i \times l_i$, and $\pi_{D'-i} = l_i \times h^i$ for $i < d'$, the projector

$$\Pi = \Delta - \sum_{i=0}^{D'} \pi_i$$

decomposes as follows:

$$\Pi = \sum_{i=d'}^{d} (h^i \times l_i + l_i \times h^i) + v_n \sum_{i=d'}^{d} l_{D'-i} \times l_i + v_n^{-1} \sum_{i=d'}^{d} h^i \times h^{D'-i},$$

and changing the indexing $\sum_{i=d'}^{d} l_i \times h^i = \sum_{i=d'}^{d} l_{D'-i} \times h^{D'-i}$, we conclude that

$$\Pi = \sum_{i=d'}^{d} (h^i + v_n l_{D'-i}) \times (l_i + v_n^{-1} h^{D'-i}).$$

Let us denote

$$\varpi_j = (h^j + v_n l_{D'-j}) \times (l_j + v_n^{-1} h^{D'-j}),$$

and check that ϖ_j, $d' \leq j \leq d$, are orthogonal to π_i and each other. Indeed,

$$\varpi_j \circ \pi_i = v_n^{-1} \cdot \chi_{\mathrm{K}(n)}\big((h^j + v_n l_{D'-j}) h^{D'-i}\big) \cdot h^i \times (l_j + v_n^{-1} h^{D'-j}) = 0$$

for $i \neq j$ since $\chi_{\mathrm{K}(n)}(0) = 0$, and for $i = j$ since $\chi_{\mathrm{K}(n)}(h^{D'}) = \chi_{\mathrm{K}(n)}(v_n l_0)$. Finally,

$$\varpi_j \circ \varpi_i = \chi_{\mathrm{K}(n)}\big((h^j + v_n l_{D'-j})(l_i + v_n^{-1} h^{D'-i})\big) \cdot (h^i + v_n l_{D'-i}) \times (l_j + v_n^{-1} h^{D'-j})$$

is obviously 0 for $i \neq j$, and equals ϖ_j for $i = j$, since two of the three equal summands

$$\chi_{\mathrm{K}(n)}(l_0) + \chi_{\mathrm{K}(n)}(v_n^{-1} h^{D'}) + \chi_{\mathrm{K}(n)}(l_0)$$

cancel each other. By symmetry, we conclude that π_i and ϖ_i give us a decomposition of the diagonal into the sum of orthogonal Tate summands.

Now consider the case $D \equiv 0 \mod 4$. Recall that in this case one has $l_d^2 = l_0$. Then the diagonal is given by

$$\Delta = (h^d + v_n l_{d'}) \times (h^d + v_n l_{d'}) + \sum_{k=0}^{d} \big(h^k \times l_k + l_k \times h^k\big) + v_n \sum_{i+j=D'} l_i \times l_j.$$

Indeed, arguing as above we see that $\Delta \circ a \times l_m = a \times l_m$ for $m \neq d$, and $\Delta \circ a \times h^m = a \times h^m$ for $m \neq d'$. Next,

$$\Delta \circ\ a \times l_d = a \times (h^d + v_n l_{d'}) + a \times l_d + a \times h^d + v_n\, a \times l_{d'} = a \times l_d,$$

and

$$\Delta \circ\ a \times h^{d'} = 0 \cdot a \times (h^d + v_n l_{d'}) + v_n\, a \times l_d + a \times h^{d'} + v_n\, a \times l_d = a \times h^{d'}.$$

As above, consider projectors $\pi_i = v_n^{-1}\, h^i \times h^{D'-i}$, and $\Pi = \Delta - \sum_{i=0}^{D'} \pi_i$, then one has

$$\Pi = (h^d + v_n l_{d'}) \times (h^d + v_n l_{d'}) + \sum_{i=d'}^{d} (h^i + v_n l_{D'-i}) \times (l_i + v_n^{-1} h^{D'-i}).$$

Grouping the summand we obtain the following decomposition for Π:

$$\Pi = (h^d + v_n l_{d'}) \times (l_d + v_n^{-1} h^{d'} + h^d + v_n l_{d'}) + \sum_{i=d'}^{d-1} (h^i + v_n l_{D'-i}) \times (l_i + v_n^{-1} h^{D'-i}).$$

Let us denote $\varpi_j = (h^j + v_n l_{D'-j}) \times (l_j + v_n^{-1} h^{D'-j})$ for $d' \leq j \leq d-1$, and

$$\varpi_d = (h^d + v_n l_{d'}) \times (l_d + v_n^{-1} h^{d'} + h^d + v_n l_{d'}),$$

and check that ϖ_j are orthogonal to π_i and each other. Arguing as in the case $D \not\equiv 0 \mod 4$, we conclude that π_i are orhogonal to ϖ_j for $j \neq d$. In fact, the same argument shows that π_i is orhogonal to ϖ_d. Similarly, we conclude as above that $\varpi_i \circ \varpi_j = \delta_{ij} \varpi_j$ for $d \neq i, j$, and that ϖ_j are orthogonal to ϖ_d for $j \neq d, d'$. Moreover $\varpi_d \circ \varpi_{d'} = 0$ and $\varpi_d \circ \varpi_d = \varpi_d$. Therefore, it remains to check that $\varpi_{d'} \circ \varpi_d = 0$. Indeed,

$$\varpi_{d'} \circ \varpi_d = \chi_{\mathrm{K}(n)}\big((h^{d'} + v_n l_d)(l_d + v_n^{-1} h^{d'} + h^d + v_n l_{d'})\big) \cdot (h^d + v_n l_{d'}) \times (l_{d'} + v_n^{-1} h^d),$$

and it is easy to see that this expression is zero (since $l_d^2 = l_0$). □

1.3.3 Co-action and Realization

Let G be a smooth affine algebraic group over the base field k of characteristic 0. We identify the Galois cohomology group $\mathrm{H}^1(k, G) = \mathrm{H}^1\big(\mathrm{Gal}(\overline{k}/k),\ G(\overline{k})\big)$ with the set of isomorphism classes of G-torsors over k. The natural homomorphism $G \to \mathrm{Aut}(G)$ induces the map on the Galois cohomology, therefore each G-torsor defines an element $\xi \in \mathrm{H}^1\big(k, \mathrm{Aut}(G)\big)$, and the corresponding twisted form ${}_{\xi}G$ of G, see [Se65], or [KMRT, Chapter VII].

For a representation $\rho\colon G \hookrightarrow \mathrm{GL}_m$ of G we call $\mathrm{GL}_m \to \mathrm{GL}_m/\rho(G)$ the *standard classifying G-torsor*, and its fiber over the generic point of $X = \mathrm{GL}_m/\rho(G)$ is called the *standard generic G-torsor* over $k(X)$, or just the *generic torsor* see [GMS, Example 3.1].

Consider the (split) group $G = \mathrm{SO}_m$ over k. Let ${}_EG$ be a generic group, i.e., the twisted form of G corresponding to the generic G-torsor E, see [PS17, Section 3]. Strictly speaking, ${}_EG$ is defined over a field extension L of k, but we will denote SO_m over L or $\overline{L}$ by the same letter G. For each parabolic subgroup P of G there exists a unique ${}_EG$-homogeneous variety X which is a twisted form of G/P, see [ChMe, Section 1.1], moreover, $X = E/P$. For a maximal parabolic $P = P_1$ corresponding to the first simple root in the Dynkin diagram (according to the numbering of Bourbaki) we refer to E/P_1 as the *generic quadric* of dimension $D = m-2$. We also remark that E/P_1 is generic with respect to SO_m, and there exist slightly different notions of generic quadric, which are compared in [Ka18]. Untill the end of the chapter, we reserve the notation Q for the split quadric G/P_1.

For any G-torsor, and any free theory A^* Petrov and Semenov define a bi-algebra H^* [PS20, Definition 4.6], and the co-action of H^* on $A^*(G/P)$ for any parabolic P of G [PS20, Definition 4.10]. They also show that for $A^* = \mathrm{CH}(-;\mathbb{F}_p)$ the bi-algebra H^* carries essentially the same information as the *J-invariant* [PSZ, Vi05]. We will be interested only in the case of the generic torsor, and in this case $H^* = A^*(G)$ for any free theory A^* [PS20, Example 4.7], and the co-action .

$$\rho = \rho_A\colon A^*(G/P) \to A^*(G) \otimes_{A^*(\mathrm{pt})} A^*(G/P)$$

is given by the pullback map along the left multiplication by elements of G on G/P composed with the Künneth isomorphism [PS20, Lemma 4.3] (in particular, ρ is a graded $A^*(\mathrm{pt})$-algebra homomorphism).

Consider smooth projective homogeneous varieties G/P, $G/P' \in \mathcal{S}\mathsf{m}_k$, and let X, and X' be the respective E-twisted forms. For an element $\alpha \in A^*(X \times X')$ define the *realization map*

$$\alpha_\star\colon A^*(X) \to A^*(X')$$

by the formula $(\mathrm{pr}_{X'})_A \circ \mathrm{m}_\alpha \circ \mathrm{pr}_X^A$, where m_α stands for the multiplication by α, and pr_X, $\mathrm{pr}_{X'}$ are the natural projections from $X \times X'$ to X or X' respectively. By [PS20, Theorem 4.14],

$$\overline{\alpha}_\star\colon A^*(\overline{X}) \to A^*(\overline{X'})$$

is a homomorphism of $A^*(G)$-comodules. This fact allows to consider an (additive) functor from the full subcategory of $\mathcal{M}\mathsf{ot}_A$ generated by the motives of smooth projective homogeneous ${}_EG$-varieties to the category of graded $A^*(G)$-comodules [PS20, Remark 4.15]. In particular, the motivic decomposition of E/P gives us a decomposition of $A^*(G/P)$ into a sum of $A^*(G)$-comodules (as above, with some abuse of notation, we usually write G/P for $\overline{X}$). In the next subsection we use this fact to show that for the generic quadric E/P_1 the motivic summand $\mathcal{N}$ from Proposition 1.3.1 is indecomposable.

1.3.4 Motives of Generic Quadrics

We proved in Proposition 1.3.1 that the $\mathrm{K}(n)$-motive ($n \geq 2$) of any smooth projective quadric of dimension $D \geq 2^n - 1$ has $D + 2 - 2^n$ Tate summands. In the present subsection we show that the remaining summand of a *generic* quadric is indecomposable. This reproves, in particular, that the Chow motive of a generic quadric is indecomposable, see [Vi04, Ka12].

With this end in view, we describe the co-action of $A^*(G)$ on $A^*(Q)$ in the notation of the previous section. Recall that G denotes the split group SO_m, P_1 is the maximal parabolic corresponding to the first simple root in the Dynkin diagram, and $Q = G/P_1$ is a *split* quadric. Observe that since $A^*(Q)$ is generated as an algebra by the elements h and $l = l_d$ in the notation of Theorem 1.1.1, and since $\rho(h) = 1 \otimes h$ by [PS20, Lemma 4.12], the co-action is determined by $\rho(l)$.

It is convenient here to work with *connective* Morava K-theory $\mathrm{CK}(n)$ as in Section 1.1.5. Sending v_n to 0 we get a surjective map from connective Morava onto Chow theory by [ViYa]. Recall that

$$\mathrm{CH}^*(\mathrm{SO}_m; \mathbb{F}_2) \cong \mathbb{F}_2[e_1, \ldots, e_{\lfloor \frac{m-1}{2} \rfloor}]/(e_i^2 = e_{2i}),$$

see, e.g., [PS20, Lemma 7.1], where $\operatorname{codim} e_i = i$. Assume that

$$r = \left\lfloor \frac{m-1}{2} \right\rfloor < 2^n - 1 = \deg v_n,$$

and take arbitrary homogeneous preimages $\widetilde{e}_i \in \mathrm{CK}(n)^*(\mathrm{SO}_m; \mathbb{F}_2)$ of e_i. Observe that each cyclic submodule generated by $\widetilde{e}_i$ is free for all i, because $\Omega^*(\mathrm{SO}_m)$ can be presented as a graded module over $\mathbb{L}$ by generators in non-negative degrees and relations in positive degrees by [Vi15, Theorem 4.3], and therefore the same is true about $\mathrm{CK}(n)^*(\mathrm{SO}_m; \mathbb{F}_2)$ as a module over $\mathbb{F}_2[v_n]$. Since $\widetilde{e}_i$ are elements of different codimensions $< 2^n - 1$, we conclude that the submodule generated by $\{1, \widetilde{e}_1, \ldots, \widetilde{e}_r\}$ is a $\mathbb{F}_2[v_n]$-free submodule of $\mathrm{CK}(n)^*(\mathrm{SO}_m; \mathbb{F}_2)$.

Then after the localization by the powers of v_n images f_i of $\widetilde{e}_i$ together with 1 generate a $\mathbb{F}_2[v_n^{\pm 1}]$-free submodule of $\mathrm{K}(n)^*(\mathrm{SO}_m;\mathbb{F}_2)$. Since any graded module over $\mathbb{F}_2[v_n^{\pm 1}]$ is free, we can find g_α in such a way that

$$\{1, f_1, \ldots, f_r\} \cup \{g_\alpha\}_{\alpha \in A}$$

form a base of $\mathrm{K}(n)^*(\mathrm{SO}_m;\mathbb{F}_2)$ over $\mathbb{F}_2[v_n^{\pm 1}]$.

Lemma 1.3.4. *Assume that $m \le 2^n$. Then the co-action of $\mathrm{K}(n)^*(\mathrm{SO}_m;\mathbb{F}_2)$ on $\mathrm{K}(n)^*(Q;\mathbb{F}_2)$ is defined by the equation*

$$\rho(l) = \sum_{i=1}^{r} f_i \otimes h^{r-i} + 1 \otimes l + \sum_{k \ge 1} v_n^k \cdot \sum_{\alpha \in A} c_{k,\alpha} \cdot g_\alpha \otimes q_{k,\alpha}$$

for some $c_{k,\alpha} \in \mathbb{F}_2$, and $q_{k,\alpha} \in \mathrm{K}(n)^*(Q)$.

Proof. Observe that $r \le 2^{n-1} - 1 < 2^n - 1$, and $\dim Q \le 2^n - 2$. It follows from [PS20, Lemma 7.2] that

$$\rho_{\mathrm{CK}(n)}(l) = \sum_{i=1}^{r} \widetilde{e}_i \otimes h^{r-i} + 1 \otimes l + v_n \sum g \otimes q$$

for some homogeneous $g \in \mathrm{CK}(n)^*(\mathrm{SO}_m;\mathbb{F}_2)$, and $q \in \mathrm{CK}(n)^*(Q;\mathbb{F}_2)$. Since $\rho_{\mathrm{CK}(n)}(l)$ is homogeneous of codimension r, none of such g can equal $\sum_{j\in J} v_n^{k_j} \widetilde{e}_j$ for any $J \subseteq \{1, \ldots r\}$, and $k_j \ge 0$ by dimensional reasons. Now the result about $\mathrm{K}(n)^*$ follows. □

Lemma 1.3.5. *The $\mathrm{K}(n)$-motive $\mathcal{M}$ of the generic quadric E/P_1 of dimension $0 < D \le 2^n - 2$ is indecomposable.*

Proof. Recall that the motivic decomposition of E/P_1 provides a decomposition of $A^*(G/P_1)$ into a sum of $A^*(G)$-comodules. Assume that $\mathcal{M}$ decomposes as a direct sum $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$. Then by [PS20, Theorem 4.14] and [PS20, Remark 4.15] the realization $D = \mathrm{K}(n)^*(\overline{\mathcal{M}})$ of the motive $\mathcal{M}$ is a direct sum of realizations D_1 and D_2 of the motives $\mathcal{M}_1$ and $\mathcal{M}_2$. Since D_i are graded, each lh^k for $k \ge 0$ lies in one of D_i's, and since D_i are sub-comodules, the description of $\rho(lh^k)$ from Lemma 1.3.4 implies that $h^{r-1} \in D_i$ for the same i. But the sum is direct, so that all lh^k lie in the same D_i, for instance, in D_1. In particular, $l \in D_1$, so that the formula for $\rho(l)$ implies that all h^k for $0 \le k \le r-1$ lie in D_1, and the formula for $\rho(lh)$ implies that $h^r \in D_1$. Therefore, $D_1 = D$, and $D_2 = 0$. The realization of a non-zero motive cannot be 0, therefore $\mathcal{M}$ is indecomposable. □

Lemma 1.3.6. *The co-action of* $\mathrm{K}(n)^*(\mathrm{SO}_{2^n+1};\mathbb{F}_2)$ *on* $\mathrm{K}(n)^*(Q_{2^n-1};\mathbb{F}_2)$ *is defined by*

$$\rho(l)=\sum_{i=1}^{2^{n-1}} f_i\otimes h^{2^{n-1}-i}+1\otimes l+v_n\, f_{2^{n-1}}\otimes lh^{2^{n-1}-1}+\sum_{k\geq 1} v_n^k\cdot\sum_{\alpha\in A} c_{k,\alpha}\cdot g_\alpha\otimes q_{k,\alpha}$$

Proof. As in Lemma 1.3.4, we have that

$$\rho_{\mathrm{CK}(n)}(l)=\sum_{i=1}^{2^{n-1}}\widetilde{e}_i\otimes h^{2^{n-1}-i}+1\otimes l+v_n\sum g\otimes q$$

for some homogeneous $g\in\mathrm{CK}(n)(\mathrm{SO}_m;\mathbb{F}_2)$, and $q\in\mathrm{CK}(n)(Q;\mathbb{F}_2)$. Since $\rho_{\mathrm{CK}(n)}(l)$ is homogeneous of codimension 2^{n-1}, there exists only one possibility for such a g to equal $\sum_{j\in J} v_n^{k_j}\widetilde{e}_j$ by dimensional reasons, more precisely, the case $g=\widetilde{e}_{2^{n-1}}$ cannot be excluded. Moreover, we will show that such a summand indeed appears.

We already see that the $\mathrm{K}(n)$-co-action is determined by the formula

$$\rho(l)=\sum_{i=1}^{2^{n-1}} f_i\otimes h^{2^{n-1}-i}+1\otimes l+c\cdot v_n\, f_{2^{n-1}}\otimes l_0+\sum_{k\geq 1} v_n^k\cdot\sum_{\alpha\in A} c_{k,\alpha}\cdot g_\alpha\otimes q_{k,\alpha}$$

for some c, $c_{k,\alpha}\in\mathbb{F}_2$, and $q_{k,\alpha}\in\mathrm{K}(n)^*(Q)$, and we claim that $c=1$.

Since $\pi=v_n^{-1}\cdot 1\otimes 1$ is a projector in $\mathrm{K}(n)^*(Q_{2^n-1})\otimes\mathrm{K}(n)^*(Q_{2^n-1})$, and the diagonal is equal to

$$\Delta=\sum_{i=0}^{2^{n-1}-1}(h^i\otimes l_i+l_i\otimes h^i)+v_n\cdot l_0\otimes l_0,$$

see Proposition 1.3.3, the projector

$$\Delta-\pi=\sum_{i=1}^{2^{n-1}-1}(h^i\otimes l_i+l_i\otimes h^i)+(1+v_nl_0)\otimes(v_n^{-1}+l_0)$$

defines a summand of $\mathcal{M}_{\mathrm{K}(n)}(E/P_1)$. Then, using [PS20, Theorem 4.14], and [PS20, Remark 4.15], we conclude that the realization functor maps the motivic summands $\mathcal{M}_1=(\mathcal{M}(E/P),\pi)$ and $\mathcal{M}_2=(\mathcal{M}(E/P),\Delta-\pi)$ to the $\mathrm{K}(n)^*(\mathrm{SO}_{2^n+1})$-subcomodules $D_1=\mathrm{K}(n)^*(\overline{\mathcal{M}_1})=\mathbb{F}_2[v_n^{\pm1}]\cdot 1$, and

$$D_2=\mathrm{K}(n)^*(\overline{\mathcal{M}_2})=\mathbb{F}_2[v_n^{\pm1}]\cdot(1+v_nl_0)\oplus\bigoplus_{i=1}^{2^{n-1}-1}(\mathbb{F}_2[v_n^{\pm1}]\cdot h^i\oplus\mathbb{F}_2[v_n^{\pm1}]\cdot l_i)$$

of $\mathrm{K}(n)^*(Q)$, respectively (here $^-$ denotes the restriction map $\mathrm{res}_{\overline{k}/k}$). Since $\rho(l)$ belongs to the submodule $\mathrm{K}(n)^*(\mathrm{SO}_{2^n+1})\otimes D_2$, it remains to collect the terms of $\rho(l)$ lying in $\mathbb{F}_2[v_n^{\pm1}]e_{2^{n-1}}\otimes\mathrm{K}(n)^*(Q)$ to conclude that $c=1$. □

Lemma 1.3.7. *The motivic summand* $\mathcal{M} = (\mathcal{M}(E/P),\, \Delta - v_n^{-1}\cdot 1\otimes 1)$ *from Lemma 1.3.6 is indecomposable.*

Proof. The proof of Lemma 1.3.5 works verbatim. □

Lemma 1.3.8. *A generic odd-dimensional projective quadric* E/P_1 *of any dimension* $\geq 2^n-1$ *has an indecomposable motivic summand of rank* 2^n-1. *A generic even-dimensional projective quadric* E/P *of any dimension* $\geq 2^n-2$ *has an indecomposable motivic summand of rank* 2^n.

Proof. Let $N = 2^n-1$ or $N = 2^n$ depending on the dimension of the quadric, and assume that all indecomposable summands of the $\mathrm{K}(n)$-motive of E/P_1 have rank $< N$. We can assume that E/P_1 with this property has the least possible dimension $\dim(E/P_1) \geq N$. By Lemmas 1.3.5 and 1.3.7, a generic quadric of dimension 2^n-2 or 2^n-1 over any field of characteristic 0 has an indecomposable summand of rank N, in particular, $\dim(E/P_1) > 2^n-1$.

Let C denote the commutator subgroup $[L_1, L_1]$ of the Levi subgroup L_1 of the parabolic $P = P_1$ in G, and F denote a generic C-torsor. In fact, $C = \mathrm{SO}_{m-2}$ and we can assume that F is a C_L-torsor over a field extension L/k, such that F and E_L define the same twisted forms over L.

Then the Chow motive of ${}_F Q_{m-2}$ is isomorphic to a sum of two Tate motives and a (shifted) motive of a *generic* projective quadric of dimension $m-4$,

$$\mathcal{M}_{\mathrm{CH}}({}_F Q_{m-2}) \cong \mathcal{M}_{\mathrm{CH}}(\mathrm{pt}) \oplus \mathcal{M}_{\mathrm{CH}}({}_F(\mathrm{SO}_{m-2}/P_1))\{1\} \oplus \mathcal{M}_{\mathrm{CH}}(\mathrm{pt})\{m-2\},$$

cf. the proof of [PS20, Lemma 7.2]. Therefore, the same is true for the cobordism motive of ${}_F Q_{m-2}$ by [ViYa], and for the $\mathrm{K}(n)$-motive.

However, by our assumption (and Lemmas 1.3.5 and 1.3.7), the motive $\mathcal{M}_{\mathrm{K}(n)}({}_F(\mathrm{SO}_{m-2}/P_1))$ has an indecomposable summand of rank N, therefore $\mathcal{M}_{\mathrm{K}(n)}({}_F Q_{m-2})$ has an indecomposable summand of rank N as well.

□

Now Proposition 1.3.1 and Lemmas 1.3.5 and 1.3.8 give us in a certain sense the upper and lower bounds for the size of the indecomposable summand in the motive of a generic quadric. Combining them with Corollary 1.2.7, we can, moreover, pass to Morava with $\mathbb{Z}_{(2)}$-coefficients.

Theorem 1.3.9. *Let* Q *be a generic* SO_m*-torsor of a split quadric of positive dimension* $D = 2d$ *or* $D = 2d+1$. *For* $n \geq 2$ *consider Morava K-theory* $\mathrm{K}(n)^*$ *(with* $\mathbb{Z}_{(2)}$*-coefficients). If* $D < 2^n-1$*, then the* $\mathrm{K}(n)$*-motive of* Q *is indecomposable, and if* $D \geq 2^n-1$*, then the* $\mathrm{K}(n)$*-motive of* Q *decomposes into a sum of* $D+2-2^n$ *(shifted) Tate summands and an indecomposable summand* $\mathcal{N}$ *of rank* 2^n *for* D *even, or* 2^n-1 *for* D *odd.*

We remark that our assumptions $n \geq 2$ on Morava K-theory $\mathrm{K}(n)^*$ and $D > 0$ on the dimension of the quadric are important here. Indeed, the 0-dimensional quadric Q_0 defined by the quadratic form φ is just a quadratic field extension $\operatorname{Spec} k\sqrt{\operatorname{disc}(\varphi)}$, and for any SO_2-torsor we obtain a quadric with a trivial discriminant. This implies that $\mathrm{K}(n)^*(Q_0)$ is isomorphic to $\mathrm{K}(n)^*(\mathrm{pt} \sqcup \mathrm{pt}) = \mathrm{K}(n)^*(\mathrm{pt}) \oplus \mathrm{K}(n)^*(\mathrm{pt})$, in particular, $l = l_0$ is rational and the motive is split. In terms of the co-action we have $\mathrm{K}(n)^*(\mathrm{SO}_2; \mathbb{F}_2) = \mathrm{K}(n)^*(\mathbb{G}_{\mathrm{m}}; \mathbb{F}_2) = \mathbb{F}_2[v_n^{\pm 1}]$, and $\rho(l) = 1 \times l$. This does not prevent the motive from being decomposable.

In particular, for $\mathrm{K}(1)$ (and similarly for K^0) we do not have a base of induction in the proof of Lemma 1.3.8, since $2^1 - 2 = 0$.

Chapter 2

Stabilization of $\mathbf{K}(\boldsymbol{n})^*(\mathbf{SO}_m)$

As we have seen in the previous section, the behaviour of the $\mathrm{K}(n)$-motive of a quadric is closely related to the structure of the ring $\mathrm{K}(n)^*(\mathrm{SO}_m\,;\mathbb{F}_2)$. In the present section we prove that $\mathrm{K}(n)^*(\mathrm{SO}_m\,;\mathbb{F}_2)$ stabilizes for m large enough. This reflects Theorem 1.3.9 proven in the previous chapter, which is also a kind of stabilization result. The results of the present section are obtained by the aspirant jointly with Victor Petrov.

More precisely, we are interested in the structure of $\mathrm{K}(n)^*(G\,;\mathbb{F}_2)$ for a group variety G equal to SO_m or Spin_m. We will prove the following

Theorem 2.0.1. *For $m \geqslant 2^{n+1}+1$ the pullback maps along the natural closed embeddings induce isomorphisms of rings*

$$\mathrm{K}(n)^*(\mathrm{SO}_m\,;\mathbb{F}_2) \cong \mathrm{K}(n)^*(\mathrm{SO}_{m-2}\,;\mathbb{F}_2), \text{ and}$$
$$\mathrm{K}(n)^*(\mathrm{Spin}_m\,;\mathbb{F}_2) \cong \mathrm{K}(n)^*(\mathrm{Spin}_{m-2}\,;\mathbb{F}_2).$$

We remark that partial computations of algebraic cobordism of SO_m or Spin_m can be found in [Ya05, Zo].

2.1 Schubert Calculus

Most of the preliminary results collected below are not specific for orthogonal groups, and remain true for any split semisimple group. Moreover, they look more natural in this generality.

2.1.1 The Plan of the Proof

As always we work over a fixed field k of characteristic 0, let G denote a split semisimple group of rank l over k, $T \cong (\mathbb{G}_{\mathrm{m}})^{\times l}$ a fixed split maximal torus of G, and B a Borel subgroup containing T. Let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be a set of simple roots of a root system Φ of G. For a subset $\Theta \subseteq \Pi$ consider a corresponding parabolic subgroup P_Θ in G, generated by B and root subgroups $U_{-\alpha}$ for $\alpha \in \Theta$. In particular, $B = P_\emptyset$, and the maximal parabolic subgroups are $P_i = P_{\Pi\setminus\{\alpha_i\}}$.

For a parabolic subgroup P in G we denote its Levi part by L, its unipotent radical by U, and the opposite unipotent radical by U^-. We also denote the commutator subgroup of L by $C = [L, L]$. In the particular case $G = \mathrm{SO}_m$ or $G = \mathrm{Spin}_m$, $P = P_1$, and $L = L_1$ its Levi subgroup, C is isomorphic to SO_{m-2} or Spin_{m-2}.

Let A^* be an oriented cohomolgy theory, F_A the corresponding formal group law, and let M denote the group of characters of T. Then $A^*(\mathrm{B}T)$ denotes the ring $A^*(\mathrm{pt})[\![M]\!]_{F_A}$ as defined in [CPZ, Definition 2.4]. This notation can be justified by the application of Totaro's procedure [To] to the theory A^*, see, e.g., [CZZ, Theorem 3.3]. By [CPZ, Corollary 2.13] we know that $A^*(\mathrm{B}T)$ is isomorphic to the power series ring $A^*(\mathrm{pt})[\![x_1, \dots, x_l]\!]$, and if $M \cong \mathbb{Z}\chi_1 \oplus \ldots \oplus \mathbb{Z}\chi_l$, we can define an element $x_\lambda \in A^*(\mathrm{B}T)$ for $\lambda \in M$ according to the rule $x_{\chi_i} = x_i$, and $x_{\lambda+\mu} = F_A(x_\lambda, x_\mu)$. We return to these definitions in the third chapter, where x_λ are interpreted as T-equivariant Chern classes of one-dimensional representations of weight λ.

Recall that for a variety X the algebraic cobordism ring $\Omega^*(X)$ as an $\mathbb{L}$-algebra admits the natural augmentation $\deg\colon \Omega^*(X) \to \Omega^*\big(\mathrm{Spec}\, k(X)\big) = \mathbb{L}$ by the pullback to the generic point [LM, Remark 1.2.12]. For an augmented $\mathbb{L}$-algebra A we denote A^+ its augmentation ideal, and we say that the sequence of augmented algebras $(A_i, d_i\colon A_i \to A_{i+1})$ is exact if $\mathrm{Ker}\, d_i$ coincides with the ideal generated by $\mathrm{Im}\, d_{i-1} \cap A_i^+$.

Following [CPZ, Definition 10.2] we consider the *characteristic map*

$$\mathfrak{c}\colon A^*(\mathrm{B}T) \to A^*(G/B).$$

By [GiZa, Proposition 5.1, Example 5.6] we know that the sequence

$$\Omega^*(\mathrm{B}T) \to \Omega^*(G/B) \to \Omega^*(G) \to \mathbb{L} \tag{2.1}$$

is a right exact sequence of augmented $\mathbb{L}$-algebras, where the first arrow is the characteristic map, and the second one is the pullback along $G \to G/B$. This, obviously, gives a similar sequence for any free theory A^*, cf. Lemma 2.1.6 below. The basic idea behind our computation of $\mathrm{K}(n)^*(G)$ is to use the exact sequence (2.1) for Morava K-theory. Observe that the above sequence can be continued to the left as in [CZZ, Theorem 10.2].

On the one hand, since $\Omega^*(G/B)$ is a free $\mathbb{L}$-module with a base given by resolutions of singularities ζ_w of Schubert varieties $\overline{BwB/B}$, w in the Weyl group W of G, see [CPZ, Lemma 13.7], we have a very explicit description of $\Omega^*(\mathrm{B}T) \cong \mathbb{L}[\![t_1, \ldots, t_l]\!]$, and $\Omega^*(G/B) \cong \mathbb{L}^{|W|}$. On the other hand, the characteristic map $\mathfrak{c}\colon \Omega^*(\mathrm{B}T) \to \Omega^*(G/B)$ after these identification has a rather complicated form. One can write a closed formula [CPZ, Equation (8)] for it in terms of BGG–Demazure divided difference operators Δ_i, see Section 2.2.1.

Instead of working with $\mathfrak{c}$ directly, we found it much easier to deduce the following Proposition 2.1.1 from [GiZa, Proposition 5.1], and then we prove Proposition 2.1.2 with the use of divided difference operators Δ_i. Theorem 2.0.1 is an obvious corollary of these two propositions.

Proposition 2.1.1. *a) For any free theory A^* and any parabolic subgroup P in a split semisimple group G the pullback map along the closed embedding induces an isomorphism of rings*

$$A^*(P) \cong A^*(G) \otimes_{A^*(G/P)} A^*(\mathrm{pt}).$$

b) For the Levi part L of a parabolic P and $C = [L, L]$ the pullback maps along the natural closed embeddings induce isomorphisms of rings

$$A^*(P) \cong A^*(L) \cong A^*(C).$$

The proof of Proposition 2.1.1 a) follows [PS20, Lemma 6.2], and the proof of b) follows [PS12].

Proposition 2.1.2. *For $G = \mathrm{SO}_m$ or $G = \mathrm{Spin}_m$ with $m \geqslant 2^{n+1} + 1$ the natural map*

$$\mathrm{K}(n)^*(G/P_1;\, \mathbb{Z}/2) \to \mathrm{K}(n)^*(G;\, \mathbb{Z}/2)$$

factors through $\mathrm{K}(n)^*(\mathrm{pt};\, \mathbb{Z}/2)$.

The rest of this chapter is organized as follows. We start with the recollection of some known results on Chow. Then we deduce similar results about any free theory essentially by Nakayama's Lemma. Next, we prove Proposition 2.1.1, and the final two Sections are devoted to divided difference operators and the proof of Proposition 2.1.2.

2.1.2 The Results on Chow

In this section we collect several well-known results on Chow, in particular, we describe the isomorphism

$$\mathrm{CH}^*(G/B) \cong \mathrm{CH}^*(G/P) \otimes_{\mathbb{Z}} \mathrm{CH}^*(P/B)$$

(which already appeared, e.g., in [DrTy], [PS20, Lemma 6.2]). In the next section we will replace Chow with an arbitrary theory.

The Weyl group W of G is generated by simple reflections $s_i = s_{\alpha_i}$ corresponding to simple roots $\alpha_i \in \Pi$. We denote $l(v)$ the length of $v \in W$ in simple reflections. The longest word of W is denoted w_0.

For $P = P_\Theta$ consider $W_P = \langle s_i \mid i \in \Theta \rangle$, and let us denote $W^P = \{v \in W \mid l(vs_i) = l(v) + 1 \ \ \forall i \in \Theta\}$. Then the map

$$W^P \times W_P \to W$$

sending a pair (u, v) to the product uv is a bijection, and $l(uv) = l(u) + l(v)$ [BjBr, Proposition 2.2.4]. This immediately implies that W^P is a set of the minimal representatives for the elements of W/W_P.

Let X_w denote the classes $[\overline{BwB/B}]$ of Schubert varieties in $\mathrm{CH}^*(G/B)$. Observe that these varieties are not necessarily smooth. It is well-known that $\{X_w \mid w \in W\}$ is a free base of $\mathrm{CH}^*(G/B)$ [De, Corollaire du Proposition 1]. The pullback map along the natural projection $\pi\colon G/B \to G/P$ defines an isomorphism between $\mathrm{CH}^*(G/P)$ and a free abelian subgroup of $\mathrm{CH}^*(G/B)$ with the base $\{X_{w_0 w} \mid w \in W^P\}$ [Kö, Corollary 1.5, Lemma 1.2], see also [GPS, Section 5.1].

It is convenient to replace B with the corresponding opposite parabolic B^- in the Bruhat decomposition, and introduce notation Z_w for the classes of $\overline{B^- wB/B}$ in $\mathrm{CH}(G/B)$. Observe that G acts on G/B with left translations in such a way that $w_0(BwB/B) = B^- w_0 wB/B$ as subvarieties, and taking closures we get

$$w_0 \left(\overline{BwB/B}\right) = \overline{B^- w_0 wB/B}.$$

By [Gr, Lemma 1] the induced action of G on $\mathrm{CH}(G/B)$ is trivial, and therefore $X_w = Z_{w_0 w}$ (see also [De, Proposition 1]).

Let us denote $B' = B \cap C$ the Borel subgroup of C. Since P/B and C/B' coincide as varieties, (cf. [PS12, Lemma 2.4]), we conclude that the ring $\mathrm{CH}^*(P/B)$ has a free base consisting of the classes Z'_w of closures of Bruhat cells $(B')^- wB'/B'$ in C/B', or, equivalently, of closures of $(B')^- wB/B$ in P/B, $w \in W_P$. We introduce the following notation.

Notation (Subring R and subgroup V)**.** Let us denote by R a free abelian subgroup of $\mathrm{CH}^*(G/B)$ with a base Z_w, $w \in W^P$, and by V a free abelian subgroup with a base Z_w, $w \in W_P$. We have the following isomorphisms of abelian groups:

$$R \cong \mathrm{CH}^*(G/P), \qquad V \cong \mathrm{CH}^*(P/B),$$

moreover, the pullback map $\pi^{\mathrm{CH}}\colon \mathrm{CH}^*(G/P) \to \mathrm{CH}^*(G/B)$ is injective, and $R = \mathrm{Im}(\pi^{\mathrm{CH}})$, i.e., R is a subring of $\mathrm{CH}^*(G/B)$, isomorphic to $\mathrm{CH}^*(G/P)$.

The next lemma is just [EG94, Proposition 1] applied to our situation. We remark that $U_P^- \cdot P/B$ is an open subvariety of G/B, and the multiplication $U_P^- \times (P/B) \to (U_P^- \cdot P)/B$ defines an isomorphism of schemes.

Lemma 2.1.3. *a) The map $R \otimes_{\mathbb{Z}} V \to \mathrm{CH}^*(G/B)$ sending $p \otimes q$ to pq is a bijection.*

b) An isomorphism of abelian groups $\mathrm{CH}^*(P/B) \cong V$, identifying Z'_w with Z_w, $w \in W_P$, is a section for the pullback map along the closed embedding $\iota\colon P/B \to G/B$.

Proof. For a), we take the open subscheme U equal to $(U_P^- \cdot P)/P$ (the fibration $\pi\colon G/B \to G/P$ trivializes over such an U), and $x_0 = 1 \in U$ in the proof of [EG94, Proposition 1]. Then the fiber F_{x_0} is equal to P/B, and we can take $Z'_w \in \mathrm{CH}^*(P/B)$, $w \in W_P$ as its base. It follows from [EG94, Proposition 1] that the closures of $U_P^- \times \overline{(B')^- wB/B}$ in G/B form a base of $\mathrm{CH}^*(G/B)$ over its subring R.

Recall that Schubert cells are reduced, and the closure of a reduced scheme is just the closure of the underlying topological space endowed with the reduced scheme structure on it. Thus, we can conclude that the closure of $U_P^- \times \overline{(B')^- wB/B}$ coincides with the closure of $U_P^- \times (B')^- wB/B = B^- wB/B$, i.e., with Z_w, $w \in W_P$.

For b), we decompose ι as a composition of a closed embedding $i\colon P/B \to (U_P^- \cdot P)/B$, followed by an open embedding $j\colon (U_P^- \cdot P)/B \hookrightarrow G/B$.

We will show that $\iota^*(Z_w) = Z'_w$ for $w \in W_P$. A pullback $j^*(Z_w)$ is given by the intersection of Z_w with an open subvariety $(U_P^- \cdot P)/B$, and therefore is reduced. We claim that it coincides with $U_P^- \times \overline{(B')^- wB/B}$ as a subvariety, and since both schemes are reduced, it is enough to check that underlying topological spaces coincide. The latter is clear.

Next, i is a zero section of the trivial fibration $U_P^- \cdot P/B \to P/B$, and therefore sends $U_P^- \times \overline{(B')^- wB/B}$ to Z'_w. □

Remark. See a different approach to the above lemma for $\mathrm{CH}^* \otimes \mathbb{C}$ in [DrTy] (the proofs work for CH with coefficients in any field).

2.1.3 Nakayama's Lemma

In this section we prove an analogue of Lemma 2.1.3 for any free theory. As a matter of fact, we deduce it from the case of Chow with the use of graded Nakayama's Lemma (below). We will apply it for $N = \Omega^*(X)$, where $X \in \mathcal{S}\mathsf{m}_k$. In this case $N/\mathbb{L}^{<0}N = \mathrm{CH}^*(X)$ [LM, Theorem 1.2.19].

Lemma 2.1.4 (Graded Nakayama's Lemma). *Let M and N be graded $\mathbb{L}$-modules, N be finitely generated, and $f\colon M \to N$ be a homomorphism of $\mathbb{L}$-modules, which preserves grading. Then*
a) $\mathbb{L}^{<0}N = N \Rightarrow N = 0$;
b) if $f \otimes \mathbb{Z}\colon M/\mathbb{L}^{<0}M \to N/\mathbb{L}^{<0}N$ is surjective, then f is surjective as well.

Proof. a) follows from the fact that $\mathbb{L}$ is non-positively graded, and therefore any finitely generated graded module has en element of the maximal degree, see a detailed exposition, e.g., in [La06, Chapter II, Propositions 4.3 and 4.4]. For b) take $N = N/f(M)$ in a). □

The next theorem is proven in [CPZ, Theorem 13.13].

Theorem (Calmès–Petrov–Zainoulline). *For any free theory A^* there exists a free $A^*(\mathrm{pt})$-base of $A^*(G/B)$, consisting of homogeneous elements $\zeta_w = \zeta_w^A$, $w \in W$. Moreover, ζ_w^A coincide with the images of ζ_w^Ω under the canonical map $\Omega^*(G/B) \to A^*(G/B)$, and ζ_w^{CH} coincide with X_w defined above.*

Notation (Subring R^A and free submodule V^A). For an oriented cohomology theory A^* we denote by R^A the image of $A^*(G/P)$ in $A^*(G/B)$ under the pullback map, and by V^A the (free graded) $A^*(\mathrm{pt})$-submodule of $A^*(G/B)$, generated by $\zeta_{w_0w}^A$, $w \in W_P$.

Remark. We do not claim that R^A coincides with the submodule generated by $\zeta_{w_0w}^A$, $w \in W^P$, cf. [LZZ, Remark 3.13].

Lemma 2.1.5. *a) The map $f\colon R^A \otimes_{A^*(\mathrm{pt})} V^A \to A^*(G/B)$ which sends $p \otimes q$ to the product pq is surjective.*
b) The pullback map along the closed embedding $\iota^A\colon A^*(G/B) \to A^*(P/B)$ restricted to V^A defines an isomorphism $V^A \cong A^*(P/B)$.

Proof. Since V^A and $A^*(P/B)$ are free $A^*(\mathrm{pt})$-modules of the same rank, it is enough to prove that $\iota^A|_{V^A}$ is surjective to get b). Therefore we can assume that $A^* = \Omega^*$. Since the $\mathbb{L}$-module $M = R^\Omega \otimes_{\mathbb{L}} V^\Omega$ is graded, f preserves grading, and $f \otimes \mathbb{Z}$ is surjective by Lemma 2.1.3, we can apply Lemma 2.1.4 to get a). Similary, we can apply Lemma 2.1.4 to $M = V^A$, $f = \iota^A|_{V^A}$ to get b). □

2.1.4 The Characteristic Map

Following [LM, Remark 1.2.12], for any free theory A^* and any smooth (irreducible) variety X with a function field K we consider a map $\deg_A$, defined as the restriction to the generic K-point of X_K

$$\deg_A\colon A^*(X) \to A^*(K) \cong A^*(\mathrm{pt}).$$

We remark that the map $\deg_A$ canonically splits by

$$A^*(\mathrm{pt}) \to A^*(X), \quad a \mapsto a \cdot 1_{A^*(X)}.$$

Notation (Ideal $\mathcal{I}_A$)**.** For any free theory A^* consider the characteristic map

$$\mathfrak{c}_A \colon A^*(\mathrm{B}T) \to A^*(G/B),$$

and let $\mathcal{I}_A = \mathcal{I}_A(G/B)$ denote the ideal of $A^*(G/B)$ generated by the set $\mathrm{Im}(\mathfrak{c}_A) \cap \mathrm{Ker}(\deg_A)$.

Lemma 2.1.6. *For a split reductive group G (not necessarily semisimple) the pullback map along the natural projection $p^A \colon A^*(G/B) \to A^*(G)$ is surjective, and its kernel coincides with $\mathcal{I}_A$.*

Proof. First, for the case $A^* = \Omega^*$, the statement follows from [GiZa, Proposition 5.1, Example 5.6].

Next, consider an arbitrary free theory A^*. Then the following sequence is exact:

$$\mathcal{I}_\Omega \otimes_{\mathbb{L}} A^*(\mathrm{pt}) \longrightarrow A^*(G/B) \xrightarrow{p^A} A^*(G) \longrightarrow 0,$$

i.e., any element x from the kernel of p^A has a form

$$x = \sum_k \Big(\sum_j \mathfrak{c}_\Omega(b_{kj}) g_j \Big) \otimes_{\mathbb{L}} c_k = \sum_{k,j} \big(\mathfrak{c}_\Omega(b_{kj}) \otimes_{\mathbb{L}} 1\big) \cdot \big(g_j \otimes_{\mathbb{L}} c_k\big),$$

for some $b_{kj} \in \Omega^*(\mathrm{B}T)$, $\deg_\Omega\big(\mathfrak{c}_\Omega(b_{kj})\big) = 0$, $g_j \in \Omega^*(G/B)$, $c_k \in A^*(\mathrm{pt})$. Now, we can rewrite $\mathfrak{c}_\Omega(b_{kj}) \otimes_{\mathbb{L}} 1 = \mathfrak{c}_A(b_{kj} \otimes_{\mathbb{L}} 1)$, and observe that

$$\deg_A\big(\mathfrak{c}_A(b_{kj} \otimes_{\mathbb{L}} 1)\big) = (\deg_\Omega \otimes_{\mathbb{L}} 1)\big(\mathfrak{c}_\Omega(b_{kj}) \otimes_{\mathbb{L}} 1\big) = 0$$

to obtain the claim. □

In particular, the argument of [PS12, Lemma 2.2] can be used for any free A^* to prove Proposition 2.1.1 b), i.e., to show that the natural pullback maps define isomorphisms $A^*(P) \cong A^*(L) \cong A^*(C)$.

Proof of Proposition 2.1.1 b). Since $P \to L$ is an affine fibration, and the inclusion $L \hookrightarrow P$ is its zero section, we have $A^*(P) \cong A^*(L)$. Next, for $T' = T \cap C$, and $B' = B \cap C$ consider the diagram

$$\begin{array}{ccccc} A^*(\mathrm{B}T) & \xrightarrow{\mathfrak{c}} & A^*(L/B) & \twoheadrightarrow & A^*(L) \\ \downarrow & & \| & & \downarrow \\ A^*\big(\mathrm{B}T'\big) & \xrightarrow{\mathfrak{c}} & A^*(C/B') & \twoheadrightarrow & A^*(C), \end{array}$$

and apply Lemma 2.1.6. □

Now we are ready to prove part a) of Proposition 2.1.1 as well, i.e., that we have an isomorphism of rings

$$A^*(P) \cong A^*(G) \otimes_{A^*(G/P)} A^*(\mathrm{pt}).$$

The argument is essentially the same as in [PS20, Lemma 6.2].

Proof of Proposition 2.1.1 a). First, we consider the surjective map from Lemma 2.1.5 a), followed by the pullback from Lemma 2.1.5 b):

$$R^A \otimes_{A^*(\mathrm{pt})} V^A \to A^*(G/B) \to A^*(P/B),$$

and tensor this sequence with $A^*(\mathrm{pt})$ over $A^*(G/P)$. We obviously have

$$A^*(\mathrm{pt}) \otimes_{A^*(G/P)} R^A \otimes_{A^*(\mathrm{pt})} V^A \cong V^A,$$

and since $P/B \to G/B \to G/P$ factors through a point, $A^*(\mathrm{pt}) \otimes_{A^*(G/P)} A^*(P/B) \cong A^*(P/B)$. Therefore, we get a sequence

$$V^A \to A^*(\mathrm{pt}) \otimes_{A^*(G/P)} A^*(G/B) \to A^*(P/B),$$

and the composite map is an isomorphism by Lemma 2.1.5 b).

Now, we use $A^*(P/B) \cong A^*(L/B)$, and tensor the above sequence (of isomorphisms) with $A^*(\mathrm{pt})$ over $A^*(\mathrm{B}T)$. By Lemma 2.1.6 we get

$$A^*(G/B) \otimes_{A^*(\mathrm{B}T)} A^*(\mathrm{pt}) = A^*(G),$$

and similarly $A^*(L/B) \otimes_{A^*(\mathrm{B}T)} A^*(\mathrm{pt}) = A^*(L)$. We finish the proof with the use of Theorem 2.1.1 b). Since the characteristic map commutes with pullbacks, we can conclude that the isomorphism $A^*(P) \cong A^*(G) \otimes_{A^*(G/P)} A^*(\mathrm{pt})$ is actually induced by the pullback along the inclusion $P \to G$. □

2.2 BGG–Demazure Operators

In this section we finish the proof of Theorem 2.0.1 with the use of BGG–Demazure operators.

2.2.1 The Case of Chow

Divided difference operators Δ_i were defined independently by Bernstein–Gelfand–Gelfand [BGG] and Demazure [De] to describe the characteristic map $\mathfrak{c}_{\mathrm{CH}}$. These operators were generalized to an arbitrary oriented cohomology theory in [CPZ].

Recall that $A^*(\mathrm{B}T)$ is isomorphic to $A^*(\mathrm{pt})[\![x_1,\ldots,x_l]\!]$. If M is a group of characters of T we can identify $A^*(\mathrm{B}T)$ with $A^*(\mathrm{pt})[\![M]\!]_{F_A}$ as defined in [CPZ, Definition 2.4], see [CZZ, Theorem 3.3] (F_A stands for a formal group law of A^*). We fix a base $\chi_1,\ldots,\chi_l$ of M and write x_{χ_i} for $x_i\in A^*(\mathrm{B}T)$. For an arbitrary $\lambda\in M$ we have an element $x_\lambda\in A^*(\mathrm{B}T)$ defined according to the rule $x_{\lambda_i}=x_i$, and $x_{\lambda+\mu}=F_A(x_\lambda,\,x_\mu)$.

Then the action of the Weyl group W on M induces the action of W on $A^*(\mathrm{B}T)$ according to the rule $s_\alpha(x_\lambda)=x_{\lambda-\alpha^\vee(\lambda)\alpha}$. We now define divided difference operators for Ω^* by the formula

$$\Delta_i^\Omega(u)=\frac{u-s_i(u)}{x_{\alpha_i}},$$

and for any other theory by change of coefficients [CPZ, Definition 3.5].

Consider the case $A^*=\mathrm{CH}^*$. Then for any two reduced decompositions $w=s_{i_1}\ldots s_{i_k}=s_{j_1}\ldots s_{j_k}$ the operators $\Delta_{i_1}\circ\ldots\circ\Delta_{i_k}$ and $\Delta_{j_1}\circ\ldots\circ\Delta_{j_k}$ coincide, and we denote such a composition simply Δ_w. Moreover, if the decomposition $s_{i_1}\ldots s_{i_k}$ is not reduced, then $\Delta_{i_1}\circ\ldots\circ\Delta_{i_k}=0$, see [BGG, Theorem 3.4].

We have the following description of $\mathfrak{c}_{\mathrm{CH}}$ in terms of divided difference operators. For a homogeneous $u\in\mathrm{CH}^*(\mathrm{B}T)$ of degree s by [De, Theorem 1] one has

$$\mathfrak{c}_{\mathrm{CH}}(u)=(-1)^{l(w_0)-s}\sum_{l(w)=s}\Delta_w(u)Z_w.$$

Further, for a minimal parabolic $P_{\{\alpha_i\}}$ consider the natural projection

$$\pi_i\colon G/B\to G/P_{\{\alpha_i\}},$$

and define the operator $\widetilde{\Delta}_i(z)=-(\pi_i^{\mathrm{CH}}\circ(\pi_i)_{\mathrm{CH}})(z)$ on $\mathrm{CH}^*(G/B)$. Then for the characteristic map $\mathfrak{c}_{\mathrm{CH}}$ one has $\widetilde{\Delta}_i\circ\mathfrak{c}_{\mathrm{CH}}=\mathfrak{c}_{\mathrm{CH}}\circ\Delta_i$ [CPZ, Theorem 13.13]. We will also write Δ_i for $\widetilde{\Delta}_i$ if it does not lead to confusion.

We also need to describe the action of $\widetilde{\Delta}_i$ on the Schubert base. Take $w\in W$, and assume that $l(ws_i)=l(w)+1$ for some i. Then by [CPZ, Lemma 13.3] and [CPZ, Lemma 12.4] we have $\Delta_i(\mathfrak{t}\cdot X_w)=-\mathfrak{t}\cdot X_{ws_i}$ for some $\mathfrak{t}\in\mathbb{Z}\setminus 0$. Since $\mathrm{CH}^*(G/B)$ is a free $\mathbb{Z}$-module, and Δ_i is $\mathbb{Z}$-linear, this obviously implies that $\Delta_i(X_w)=-X_{ws_i}$.

This means in particular that any X_w can be obtained from $\pm\mathrm{pt}=\pm X_1$ by a sequence of Δ_i, and, moreover, since w_0 is greater than w in weak Bruhat order (see [BjBr, Definition 3.1.1 (i)]) for any $w\in W$ [BjBr, Proposition 3.1.2 (iii)] there always exists a sequence $(i_1,\ldots,i_k)$ such that

$$\Delta_{i_1}\circ\ldots\circ\Delta_{i_k}(X_w)=\pm X_{w_0}=\pm 1\in\mathrm{CH}^*(G/B)$$

(here $s_{i_1}\dots s_{i_k}$ is a reduced expression for $w^{-1}w_0$).

Moreover, by [CPZ, Lemma 13.3] we see that the map $\mathfrak{c}$ becomes surjective after inverting some $\mathfrak{t}\in\mathbb{Z}\setminus 0$, in particular, we can deduce that the sequence $\widetilde{\Delta}_{i_1}\circ\dots\circ\widetilde{\Delta}_{i_k}$ does not depend on reduced decomposition of $w=s_{i_1}\dots s_{i_k}$, and equals 0 if it is not reduced.

Therefore, if we apply the same operator $\Delta_{w^{-1}w_0}$ to another

$$w'=s_{j_1}\dots s_{j_s}\neq w$$

with $s=l(w')=l(w)$, we get

$$\Delta_{w^{-1}w_0}(X_{w'})=\Delta_{i_1}\circ\dots\circ\Delta_{i_k}\circ\Delta_{j_1}\circ\dots\circ\Delta_{j_s}(\mathrm{pt})=0$$

since $s_{i_1}\dots s_{i_k}\cdot s_{j_1}\dots s_{j_s}$ is not a reduced expression. In this sense Δ_w and X_w are dual to each other.

Remark. The proof of [CPZ, Theorem 13.13] contains a misprint, more precisely, $A_{I_w}(z_0)$ should be changed to $A_{I_w^{\mathrm{rev}}}(z_0)$.

2.2.2 The General Case

Similar results are obtained in [CPZ] for any theory A^*, however, the formulae become a bit trickier. We introduce the notation

$$F_A(x,\,y)=x+y+xy\cdot G(x,y),$$

and for any simple root $\alpha_i\in\Pi\subset M$ we denote

$$\kappa_i=\kappa_i^A=G\big(\mathfrak{c}_A(x_{\alpha_i}),\,\mathfrak{c}_A(x_{\alpha_{-i}})\big)$$

(observe that $\kappa_i^{\mathrm{CH}}=0$). As above, for a minimal parabolic $P_{\{\alpha_i\}}$ consider the natural projection $\pi_i\colon G/B\to G/P_{\{\alpha_i\}}$, and define Demazure operator $\Delta_i=\Delta_i^A$ on $A^*(G/B)$ by

$$\Delta_i(z)=\kappa_i\,z-(\pi_i^A\circ(\pi_i)_A)(z),$$

and the action of the simple reflections on $A^*(G/B)$ by

$$s_i(z)=z-\mathfrak{c}_A(x_{\alpha_i})\cdot\Delta_i(z).$$

By [PS20, Lemma 3.8] for Demazure operators satisfy the following Leibniz rule:

$$\Delta_i(uv)=\Delta_i(u)v+s_i(u)\Delta_i(v),$$

and s_i are $A^*(\mathrm{pt})$-algebra homomorphisms defining the action of Weyl group W on $A^*(G/B)$. Moreover, $\mathfrak{c}_A$ respects the actions of Weyl group (see the proof of [PS20, Lemma 3.8]).

2.2.3 Application to Morava of a Quadric

Starting from now n denotes a fixed natural number, $G = \mathrm{SO}_m$ or $G = \mathrm{Spin}_m$ with $m \geqslant 2^{n+1} + 1$, and $P = P_1$.

Recall that for any oriented cohomology theory A^* the ring $A^*(Q)$ of a smooth projective split quadric $Q = G/P$ is a free $A^*(\mathrm{pt})$ module of rank $2d+2$, where $m = 2d+2$ or $m = 2d+3$. $A^*(Q)$ is generated as an $A^*(\mathrm{pt})$-algebra by two elements: h of codimension 1, and l of codimension $m-d-2$ connected with the following equation:

$$h^{d+1} = \frac{F_A(h,h)}{h^{m-2d-2}}\, l.$$

For $A^* = \mathrm{CH}^*$ the pullback map along $\pi\colon G/B \to G/P \cong Q$ can be described explicitly, in particular,

$$\pi^{\mathrm{CH}}(h^d) = \begin{cases} Z_{s_d \dots s_1}, & m = 2d+3, \\ Z_{s_d \dots s_1} + Z_{s_{d+1} s_{d-1} \dots s_1}, & m = 2d+2; \end{cases}$$

and $\pi^{\mathrm{CH}}(h^k) = Z_{s_k \dots s_1}$ for $0 \le k \le d-1$.

Using the observations from Subsection 2.2.1 we can conclude that for $k \le d$ there exists a sequence $I_0 = (i_1, \dots, i_k)$ such that

$$\Delta^{\mathrm{CH}}_{I_0}\big(\pi^{\mathrm{CH}}(h^k)\big) = (-1)^k X_{w_0} = (-1)^k \in \mathrm{CH}^*(G/B).$$

Recall that for $A^* = \mathrm{K}(n)^*(-;\mathbb{Z}/2)$ one has the equation

$$h^{d+1} = v_n\, h^{k_0} l \tag{2.2}$$

for $k_0 = 2^n$ for m even, and $k_0 = 2^n - 1$ for m odd by Theorem 1.1.4. In our assumption $m \geqslant 2^{n+1} + 1$, we have $d \geqslant 2^n$ for m even, and $d \geqslant 2^n - 1$ for m odd, which means that $k_0 \le d$. Our objective is to compute Δ_{I_0} chosen for $k = k_0$ from the right hand and left hand sides of the above equality.

Further we prefer to work with $A^* = \Omega^*$, and only at the end pass to Morava.

Notation. Denote $J = \mathrm{Ker}(\deg_\Omega)$ an ideal in $\Omega^*(G/B)$ generated by homogeneous elements of positive codimension, cf. [LM, Theorem 1.2.14], and J_s is the ideal generated by homogeneous elements of codimension at least s. Then $J_s \cdot J_r \subseteq J_{s+r}$, in particular, J is nilpotent.

Lemma 2.2.1. *For any set of indices I and a homogeneous element x of codimension $|x| \ge |I|$, we claim that $\Delta^{\Omega}_I(x)$ is equal to a homogeneous element y of codimension $|x| - |I|$ modulo $J_{|x|-|I|+1}$.*

Proof. We proceed by induction on $|I|$. For a homogeneous y of codimension $|y|$ we see that $\Delta_i^\Omega(y)$ is a sum of the homogeneous element $-\big(\pi_i^\Omega \circ (\pi_i)_\Omega\big)(y)$ of codimension $|y|-1$ and $\kappa_i y \in J_{|y|}$. Then by Leibniz rule for any a we have that $\Delta_i(ay)$ lies in $J_{|y|-1}$. If $\Delta_I^\Omega(x) = y + \sum a_i y_i$ by induction hypothesis, with $|y| = |x| - |I|$ and $|y_i| \geq |x| - |I| + 1$, then $\Delta_i^\Omega\big(\Delta_I^\Omega(x)\big)$ is a sum of $y' = -\big(\pi_i^\Omega \circ (\pi_i)_\Omega\big)(y)$ and an element of $J_{|x|-|I|}$. □

We apply this observation to $x = \pi^\Omega(h^{k_0})$ and $I = I_0$ as above, and get some homogeneous element y of codimension 0 such that $z = \Delta_I^\Omega(x) - y$ lies in J. Then the image z^{CH} of z in $\mathrm{CH}^*(G/B)$ actually lies in $\mathrm{CH}^{>0}(G/B)$, and the image y^{CH} of y lies in $\mathrm{CH}^0(G/B)$. However, we know that $y^{\mathrm{CH}} + z^{\mathrm{CH}}$ is equal to $(-1)^k$, which implies that $z \in \mathbb{L}^{<0}\,\Omega^*(G/B)$, and $y \in (-1)^k + \mathbb{L}^{<0}\cdot J$. Since J is nilpotent, and $\Delta_I^\Omega(x) \in (-1)^k + J$, we conclude that it is invertible. This obviously implies the following

Lemma 2.2.2. *For any theory A^* let us denote $H_A = \pi^A(h^{k_0})$. Then the element $\Delta_{I_0}^A(H_A)$ is invertible in $A^*(G/B)$.*

Let E be a generic torsor of G, see Section 1.3.3. For each parabolic subgroup P' of G there exists a unique ${}_EG$-homogeneous variety $X = E/P'$ which is a twisted form of G/P', see [ChMe, Section 1.1].

Let us denote by $\overline{A^*}(G/B)$ the image of the restriction map

$$A^*(E/B) \to A^*(G/B).$$

We will call elements of this set *rational* ones. Since Δ_i can be defined on $A^*(E/B)$ by the same formula, we conclude that Δ_i of a rational element is again rational.

For any free theory A^* we have the following statement.

Lemma 2.2.3. *Ideal $\mathcal{I}_A$ from Subsection 2.1.4 coincides with the ideal generated by the set $\overline{A^*}(G/B) \cap J$.*

Proof. It is enough to check that the kernel of the map $p^A\colon A^*(G/B) \to A^*(G)$ coincides with the ideal generated by the above set. This is proven in [PS20, Lemma 5.3] (cf. also [PS20, Example 4.7]). □

Now we can prove the following

Lemma 2.2.4. *For any set of indices I and $s > |I|$ we get that an element $\Delta_I^\Omega\big(\pi^\Omega(h^s)\big)$ lies in $\mathcal{I}_\Omega$.*

Proof. Since $h = c_1^\Omega\big(\mathcal{O}_Q(1)\big)$ we conclude that $\pi^\Omega(h^s)$ is rational, and, therefore, $\Delta_I^\Omega\big(\pi^\Omega(h^s)\big)$ is rational as well. On the other hand, a homogeneous element of positive degree by definition lies in J so that $\Delta_I^\Omega\big(\pi^\Omega(h^s)\big)$ lies in J by Lemma 2.2.1. □

Now, let us denote $L = \pi^{\Omega}(l)$, and $H = \pi^{\Omega}(h^{k_0})$. We prove the final

Lemma 2.2.5. *One obtains* $\Delta^{\Omega}_{I_0}(HL) \in \Delta^{\Omega}_{I_0}(H)L + \mathcal{I}_{\Omega}$.

Proof. We show by induction on $|I| \leq k_0$ that $\Delta^{\Omega}_{I}(HL)$ is a sum of elements of the following types:

- the only summand $\Delta_I(H)L$;
- a multiple of $\mathfrak{c}_{\Omega}(x_\lambda)\Delta_I(H)$ for some $\lambda \in M$;
- a multiple of $\Delta_{I'}(H)$ for $|I'| < |I|$.

As a step of induction we apply Δ_i to each of these summands. First,

$$\Delta_i\big(\Delta_I(H)L\big) = (\Delta_i \circ \Delta_I)(H)\, L + s_i\big(\Delta_I(H)\big)\Delta_i(L),$$

where

$$s_i\big(\Delta_I(H)\big) = \Delta_I(H) - \mathfrak{c}_{\Omega}(x_{\alpha_i})\big((\Delta_i \circ \Delta_I)(H)\big).$$

Therefore we get one summand of each type. Next,

$$\Delta_i\big(a\, \mathfrak{c}_{\Omega}(x_\lambda)\Delta_I(H)\big) = \Delta_i\big(a\, \mathfrak{c}_{\Omega}(x_\lambda)\big)\Delta_I(H) + s_i\big(a\, \mathfrak{c}_{\Omega}(x_\lambda)\big)(\Delta_i \circ \Delta_I)(H),$$

where $s_i\big(a\, \mathfrak{c}_{\Omega}(x_\lambda)\big) = s_i(a)\, \mathfrak{c}_{\Omega}(x_{s_i(\lambda)})$, and we get summands of the third and second type. Finally,

$$\Delta_i\big(a\, \Delta_{I'}(H)\big) = \Delta_i(a)\Delta_{I'}(H) + s_i(a)\, (\Delta_i \circ \Delta_{I'})(H),$$

i.e., we get two summands of the third type.

Since $\mathfrak{c}_{\Omega}(x_\lambda) \in \mathcal{I}_{\Omega}$, with the use of Lemma 2.2.4 we get the claim. □

Now we can finish the proof of Proposition 2.1.2, i.e., we can show that the map

$$\mathrm{K}(n)^*(Q;\, \mathbb{Z}/2) \to \mathrm{K}(n)^*(G;\, \mathbb{Z}/2)$$

factors through $\mathrm{K}(n)^*(\mathrm{pt};\, \mathbb{Z}/2)$.

Proof of Proposition 2.1.2. We pullback the equality (2.2)

$$h^{d+1} = v_n\, h^{k_0} l$$

to $\mathrm{K}(n)^*(G/B)$ and chose I_0 as above. On the one hand, $\Delta_{I_0}\big(\pi^{\mathrm{K}(n)}(h^{d+1})\big)$ lies in $\mathcal{I}_{\mathrm{K}(n)}$ by Lemma 2.2.4. On the other hand, $\Delta_{I_0}\big(\pi^{\mathrm{K}(n)}(v_n h^{k_0} l)\big) \in a\, \pi^{\mathrm{K}(n)}(l) + \mathcal{I}_{\mathrm{K}(n)}$ for some invertible $a \in \mathrm{K}(n)^*(G/B)$ by Lemmas 2.2.5 and 2.2.2. This implies that $\pi^{\mathrm{K}(n)}(l)$ lies in $\mathcal{I}_{\mathrm{K}(n)}$. Then l goes to 0 in $\mathrm{K}(n)^*(G)$ (obviously, h goes to 0 as well).

□

Chapter 3

Computational algorithms

The present chapter is actually written earlier than the previous ones, and describes several algorithms necessary for computer computations with varieties of small dimensions. These computations in fact suggested us the statements of Theorems 1.1.4 and 1.3.9.

3.1 A Character from Morava to Chow

In the present section we describe a character from the Morava ring $\mathrm{K}(n)^*(Q)$ of a split quadric Q to the Chow ring $\mathrm{CH}^*(Q)$. This character should play the same role for $\mathrm{K}(n)^*$ which the Chern character plays for K^0. The aspirant used this description for computer experiments with small-dimensional examples, but it can also be applied to a general quadric, e.g., it allows to give an alternative proof to Theorem 1.1.4 from the first chapter. We remark that the same approach was used in [Rü] for computer computations with small-dimensional hypersurfaces.

3.1.1 Multiplicative Operations between Free Theories

For free theories A^* and B^* we denote the corresponding formal group laws F_A and F_B, respectively. By the projective bundle formula we have

$$A^*(\mathrm{pt})[t]/t^{n+1} \cong A^*(\mathbb{P}^n)$$

induced by $t \mapsto c_1^A\big(\mathcal{O}(1)\big)$. Define $A^*(\mathbb{P}^\infty)$ as the inverse limit $A^*\big(\mathbb{P}(V)\big)$ (in the category of non-graded rings), where the projective system is induced by embeddings of finite-dimensional subspaces V in k^∞ (where k is the base field). Then $A^*(\mathbb{P}^\infty) \cong A^*(\mathrm{pt})[[t]]$.

Let $\Phi\colon A^* \to B^*$ be a multiplicative operation, and

$$\Phi_\infty = \Phi(\mathbb{P}^\infty)\colon A^*(\mathbb{P}^\infty) \to B^*(\mathbb{P}^\infty).$$

We consider Φ_∞ as a homomorphism from $A^*(\mathrm{pt})[[t_A]]$ to $B^*(\mathrm{pt})[[t_B]]$, then $\Phi_\infty(t_A)$ is a formal series $\varphi(t_B) \in B^*(\mathrm{pt})[[t_B]]$. This series φ is a homomorphism of formal group laws,

$$\varphi\colon F_B \to F_A \otimes_{A^*(\mathrm{pt})} B^*(\mathrm{pt}),$$

see [Pa02, Section 2.7.5], i.e., one has the identity

$$\varphi\big(F_B(x,y)\big) = \big(F_A \otimes B^*(\mathrm{pt})\big)\big((\varphi(x), \varphi(y)\big) \in B^*(\mathrm{pt})[[x,y]].$$

Therefore, sending a free theory A^* to the corresponding formal group law $(A^*(\mathrm{pt}),\ F_A)$, and a multiplicative operation Φ to the constructed above series φ, we obtain a functor FGL from the category of free theories and multiplicative operations to the category of formal group laws, see [Pa02, Section 2.7.5]. The following statement is proven in [Vi19, Theorem 6.9].

Theorem (Vishik)**.** *In the above notation, the functor* FGL *is, actually, an equivalence between the categories of free theories and formal group laws.*

3.1.2 The Riemann–Roch Theorem

Multiplicative operations do not commute with pushforward maps, however, they preserve them in a certain sense.

Let B^* denote an arbitrary oriented cohomology theory, and $R = B^*(\mathrm{pt})$. For any series $\varphi(t) \in R[[t]]$ and a line bundle L over any $X \in \mathcal{S}\mathsf{m}_k$ we write $\varphi(L)$ for $\varphi\big(c_1^B(L)\big) \in B^*(X)$. For a direct sum of line bundles $E = \oplus_{i=1}^n L_i$ over X we denote

$$\varphi(E) = \prod_{i=1}^n \varphi(L_i).$$

The splitting principle [LM, Remark 4.1.2], [Pa02, Lemma 1.6.1] and the nilpotence of Chern classes then allow to define $\varphi(E)$ for any vector bundle E over X. The obvious properties of this definition are the following: one has the identity $f^B(\varphi(E)) = \varphi(f^*(E))$ for any $f\colon Y \to X$ as it is true for Chern classes; for any other series $\psi(t) \in B^*(\mathrm{pt})[[t]]$ one has the identity $(\varphi \cdot \psi)(E) = \varphi(E)\psi(E)$; for an exact sequence of vector bundles $E_1 \rightarrowtail E \twoheadrightarrow E_2$ one has the identity $\varphi(E) = \varphi(E_1)\varphi(E_2)$. The details can be found, e.g., in [Pa02, Proposition 2.2.3], or in [Fu, Appendix B] (we remark that below we can restrict ourselves to the theories $B^* = \mathrm{CH}^* \otimes R$).

Let $\Phi\colon A^* \to B^*$ be a multiplicative operation between free theories, and $\varphi(t) \in B^*(\mathrm{pt})[[t]]$ a corresponding morphism of formal group laws. Assume additionally that the series $\varphi(t)/t$ is invertible, and consider the *Todd series* $\mathrm{td}_\Phi(t) = t/\varphi(t)$, corresponding to the operation Φ. In this situation one has the following

Theorem (Panin–Smirnov)**.** *For any morphism of smooth projective varieties $f\colon X \to Y$ consider the (non-commutative) diagram*

$$\begin{array}{ccc} A^*(X) & \xrightarrow{f_A} & A^*(Y) \\ \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\Phi} \\ B^*(X) & \xrightarrow{f_B} & B^*(Y). \end{array}$$

Then for any element $\alpha \in A^*(X)$ one has the identity

$$f_B\big(\Phi(\alpha)\,\mathrm{td}_\Phi(\mathcal{T}_X)\big) = \Phi(f_A(\alpha)) \cdot \mathrm{td}_\Phi(\mathcal{T}_Y),$$

where $\mathcal{T}_X$ and $\mathcal{T}_Y$ let us denote the tangent bundles to X and Y respectively.

We will refer to the above result as the Riemann–Roch theorem. Its proof can be found in [PaSm, Sm], and [Pa02, Theorem 2.5.4].

3.1.3 A Remark on the Mishchenko Formula

We can illustrate the power of the Riemann–Roch theorem by the following simple remark: it immediately implies that the Mishchenko formula reduces to the Lagrange inversion formula.

We recall that any formal group law F over a $\mathbb{Q}$-algebra R is strictly isomorphic to the additive formal group law $F_{\mathrm{a}}(x, y) = x + y$, and we call the unique strict isomorphism from F_{a} to F *the exponent* of the law F, and let us denote it by $\exp_F(t) \in R[\![t]\!]$; is satisfies the identity

$$\exp_F(x + y) = \exp_F(x) +_F \exp_F(y) \in R[[x, y]]. \tag{3.1}$$

The term *strict isomorphism* means a series starting from the polynomial variable $t + \ldots$.

Quillens' reorientation [Pa02, Theorem 2.3.1] (generalized in [Vi19, Theorem 6.9]) implies that for a theory $A^* = \Omega^* \otimes_F R$ there exists a unique natural multiplicative operation (in fact, a natural isomorphism)

$$\mathfrak{c} = \mathfrak{c}_A\colon A^* \to \mathrm{CH}^* \otimes R,$$

corresponding to the series $\exp_F$.

Consider a theory $A^* = \Omega^* \otimes_{\mathbb{Z}} \mathbb{Q}$, i.e., $R = \mathbb{L} \otimes_{\mathbb{Z}} \mathbb{Q}$, and apply the Riemann–Roch theorem to the structure morphism $\chi\colon \mathbb{P}^r \to \mathrm{pt}$,

$$\begin{array}{ccc} A^*(\mathbb{P}^r) & \xrightarrow{\quad \chi_A \quad} & R \\ \Big\downarrow{\scriptstyle \mathfrak{c}} & & \Big\downarrow{\scriptstyle \cong} \\ \mathrm{CH}^*(\mathbb{P}^r) \otimes R & \xrightarrow{\chi_{\mathrm{CH}}} & \mathbb{Z} \otimes R, \end{array}$$

then one has the equaltiy $\chi_A(1_{\mathbb{P}^r}) = \chi_{\mathrm{CH}}\big(\mathrm{td}_{\mathfrak{c}}(\mathcal{T}_{\mathbb{P}^r})\big)$. Next, let us denote $H_{\mathrm{CH}} = c_1^{\mathrm{CH}}\big(\mathcal{O}_{\mathbb{P}^r}(1)\big)$, and consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^r} \longrightarrow \mathcal{O}_{\mathbb{P}^r}(1)^{\oplus(r+1)} \longrightarrow \mathcal{T}_{\mathbb{P}^r} \longrightarrow 0. \tag{3.2}$$

Then $\mathrm{td}_{\mathfrak{c}}(\mathcal{T}_{\mathbb{P}^r}) = \mathrm{td}_{\mathfrak{c}}(\mathcal{O}_{\mathbb{P}^r}(1))^{r+1} = \left(\dfrac{H_{\mathrm{CH}}}{\exp_F(H_{\mathrm{CH}})}\right)^{r+1}$. In other words,

$$\chi_A(1_{\mathbb{P}^r}) = \chi_{\mathrm{CH}}\left(\left(\frac{H_{\mathrm{CH}}}{\exp_F(H_{\mathrm{CH}})}\right)^{r+1}\right).$$

Since $\chi_{\mathrm{CH}}(H_{\mathrm{CH}}^r) = 1$ and $\chi_{\mathrm{CH}}(H_{\mathrm{CH}}^k) = 0$ for $k \neq r$, we see that $\chi_A(1)$ coincides with the coefficient of the series $\Big(\dfrac{t}{\exp_F(t)}\Big)^{r+1}$ at t^r. If we denote the *composition* inverse of $\exp_F$ by $\log_F(t) = \sum_{i\geq 1} c_i t^i$, then its coefficients can be computed with the use of the Lagrange inversion formula [Cha, Theorem 11.11]

$$\frac{\mathrm{d}^i}{\mathrm{d}t^i}\Big(\frac{t}{\exp_F(t)}\Big)^{i+1}\Big|_{t=0} = \frac{\mathrm{d}^{i+1}}{\mathrm{d}t^{i+1}}\log_F(t)\Big|_{t=0} \tag{3.3}$$

In other words, $\chi_A(1_{\mathbb{P}^r})$ is equal to $(r+1)c_{r+1}$. Observe that as a consequence one has the same identity $\chi_\Omega(1_{\mathbb{P}^r}) = (r+1)c_{r+1}$ for algebraic cobordism itself, since $\mathbb{L}$ is torsion-free. Therefore, we conclude that the logarithm of the universal formal group law $F = F_\Omega$ over $\mathbb{L} \otimes \mathbb{Q}$ is given by

$$\log_{F_\Omega}(t) = \sum_{i=1}^{\infty} \frac{\chi_\Omega(1_{\mathbb{P}^{i-1}})}{i} t^i.$$

We used the above identity, called the Mishchenko formula, in the first chapter, see (1.8). The above discussion shows the connection of this formula with the classical combinatorial identity (3.3).

3.1.4 Description of the Character

We can use the same argument as in the previous section to construct a character from Morava K-theory to the rational Chow theory. More precisely, for $A^* = \mathrm{K}(n)^* \otimes_{\mathbb{Z}} \mathbb{Q}$ we denote $\exp_n$ the unique strict isomorphism from F_{a} to $F_{\mathrm{K}(n)}$, i.e., *the exponent* of the law $F_{\mathrm{K}(n)}$ defined in Section 1.1.4. The series $\exp_n(t)$ is the composition inverse to (1.9). Using the theorem of Vishik (see Section 3.1.1), we get the corresponding multiplicative operation $\mathfrak{c}(n)_{\mathbb{Q}}$ from A^* to $\mathrm{CH}^* \otimes \mathbb{Q}[v_n^{\pm 1}]$ satisfying the identity

$$\mathfrak{c}(n)_{\mathbb{Q}}\big(c_1^A(\mathcal{O}_{\mathbb{P}^\infty}(1))\big) = \exp_n\big(c_1^{\mathrm{CH}}(\mathcal{O}_{\mathbb{P}^\infty}(1))\big) \in \mathrm{CH}^*(\mathbb{P}^\infty) \otimes \mathbb{Q}[v_n^{\pm 1}].$$

The composition of $\mathfrak{c}(n)_{\mathbb{Q}}$ with the localization map $\mathrm{K}(n)^* \to \mathrm{K}(n)^* \otimes \mathbb{Q}$ will be denoted by $\mathfrak{c}(n)$,

$$\mathfrak{c}(n)\colon \mathrm{K}(n)^* \longrightarrow A^* \xrightarrow{\mathfrak{c}(n)_{\mathbb{Q}}} \mathrm{CH}^* \otimes \mathbb{Q}[v_n^{\pm 1}].$$

This character should play the same role for $\mathrm{K}(n)^*$ which the Chern character plays for K^0.

From the computational point of view the character $\mathfrak{c}(n)$ is especially useful for cellular varieties, or, more generally, for a smooth projective variety X with a split $\mathrm{K}(n)$-motive as in Section 1.2.2. The $\mathrm{K}(n)^*(X)$ for such an X is isomorphic to a direct sum of $\mathbb{Z}_{(2)}[v_n^{\pm 1}]$, in particular, it is torsion-free and injects into $\mathrm{K}(n)^*(X;\, \mathbb{Q})$ via the localization map. Since the operation $\mathfrak{c}(n)_{\mathbb{Q}}$ is a natural isomorphism, we obtain the inclusion

$$\mathfrak{c}(n)\colon \mathrm{K}(n)^*(X) \hookrightarrow \mathrm{CH}^*(X;\, \mathbb{Q}[v_n^{\pm 1}]).$$

An explicit description of the above inclusion can be used as a method to determine the multiplication in the ring $\mathrm{K}(n)^*(X)$. In particular, for $X = Q$ a split projective quadric this method gives an alternative proof of Theorem 1.1.4.

We remark that for a quadric the geometric approach we used in the Section 1.1.2 is simpler than the argument below. However, the method itself seems interesting, and we want to illustrate it using the quadric as an example.

Let Q be a smooth projective split quadric, and take the free base $l_i^{\mathrm{K}(n)}$ for $0 \leqslant i \leqslant d$, and $h_{\mathrm{K}(n)}^k$ for $0 \leqslant k \leqslant d$ from Theorem 1.1.1. We will describe the images of the base elements under the map $\mathfrak{c}(n)$.

Since $h = c_1^{\Omega}(\mathcal{O}_Q(1))$, we obviously have $\mathfrak{c}(n)(h_{\mathrm{K}(n)}^k) = \exp_n(h_{\mathrm{CH}})^k$. We use the Riemann–Roch theorem to compute the images of $l_i^{\mathrm{K}(n)}$. We denote $i\colon \mathbb{P}^d \hookrightarrow Q$ the inclusion of the maximal isotropic subspace, and recall that

$l_i^{\mathrm{K}(n)}$ is equal by definition to $i_{\mathrm{K}(n)}(H^{d-i}_{\mathrm{K}(n)})$, $H_{\mathrm{K}(n)} \in \mathrm{K}(n)^*(\mathbb{P}^d)$. Then the Riemann–Roch theorem applied to the square

$$\begin{array}{ccc} \mathrm{K}(n)^*(\mathbb{P}^d) & \xrightarrow{i_{\mathrm{K}(n)}} & \mathrm{K}(n)^*(Q) \\ \downarrow{\scriptstyle \mathfrak{c}(n)} & & \downarrow{\scriptstyle \mathfrak{c}(n)} \\ \mathrm{CH}^*(\mathbb{P}^d;\, \mathbb{Q}) & \xrightarrow{i_{\mathrm{CH}}} & \mathrm{CH}^*(Q;\, \mathbb{Q}), \end{array}$$

and the element $\alpha = H^{d-i}_{\mathrm{K}(n)} = c_1^{\mathrm{K}(n)}\big(\mathcal{O}_{\mathbb{P}^d}(1)\big)^{d-i}$ gives us the identity

$$i_{\mathrm{CH}}\Big(\mathfrak{c}(n)\big(H^{d-i}_{\mathrm{K}(n)}\big)\,\mathrm{td}_{\,\mathfrak{c}(n)}(\mathcal{T}_{\mathbb{P}^d})\Big) = \mathfrak{c}(n)\big(i_{\mathrm{K}(n)}(H^{d-i}_{\mathrm{K}(n)})\big)\cdot \mathrm{td}_{\,\mathfrak{c}(n)}(\mathcal{T}_Q).$$

We have already computed $\mathrm{td}_{\,\mathfrak{c}(n)}(\mathcal{T}_{\mathbb{P}^d}) = \big(H_{\mathrm{CH}}/\exp_n(H_{\mathrm{CH}})\big)^{d+1}$ using (3.2). Similarly, we can compute $\mathrm{td}_{\,\mathfrak{c}(n)}(\mathcal{T}_Q)$ with the use of the sequence

$$0 \longrightarrow \mathcal{T}_Q \longrightarrow I^*\mathcal{T}_{\mathbb{P}^{D+1}} \longrightarrow \mathcal{O}_Q(2) \longrightarrow 0,$$

where $D = \dim Q$, and $I\colon Q \hookrightarrow \mathbb{P}^{D+1}$ denotes the closed embedding of the quadric into the projective space.

Let h_{CH} denote $c_1(\mathcal{O}_Q(1)) \in \mathrm{CH}^*(Q) \otimes \mathbb{Q}[v_n^{\pm 1}]$, then

$$c_1^{\mathrm{CH}}(\mathcal{O}_Q(2)) = 2\,c_1^{\mathrm{CH}}(\mathcal{O}_Q(1)) = 2h_{\mathrm{CH}},$$

therefore $\mathrm{td}_{\mathfrak{c}(n)}(\mathcal{O}_Q(2)) = 2h_{\mathrm{CH}}/\exp_n(2h_{\mathrm{CH}})$. Since Chern classes respect pullbacks, we get $\mathrm{td}_{\mathfrak{c}(n)}(i^*\mathcal{T}_{\mathbb{P}^{D+1}}) = \big(h_{\mathrm{CH}}/\exp_n(h_{\mathrm{CH}})\big)^{D+2}$, and finally

$$\mathrm{td}_{\mathfrak{c}(n)}(\mathcal{T}_Q) = \left(\frac{h_{\mathrm{CH}}}{\exp_n(h_{\mathrm{CH}})}\right)^{D+2} \cdot \frac{\exp_n(2h_{\mathrm{CH}})}{2\,h_{\mathrm{CH}}}.$$

Recall that $l_i^{\mathrm{K}(n)} = i_{\mathrm{K}(n)}(H^{d-i}_{\mathrm{K}(n)})$, and $\mathfrak{c}(n)(H_{\mathrm{K}(n)}) = \exp_n(H_{\mathrm{CH}})$, therefore the Riemann–Roch theorem gives us the identity

$$\mathfrak{c}(n)(l_i^{\mathrm{K}(n)}) =$$
$$= i_{\mathrm{CH}}\left(\exp_n(H_{\mathrm{CH}})^{d-i}\left(\frac{H_{\mathrm{CH}}}{\exp_n(H_{\mathrm{CH}})}\right)^{d+1}\right)\cdot\left(\frac{\exp_n(h_{\mathrm{CH}})}{h_{\mathrm{CH}}}\right)^{D+2}\frac{2\,h_{\mathrm{CH}}}{\exp_n(2h_{\mathrm{CH}})}.$$

Recall that for $i < D-d$ the pushforward i_{CH} of the inclusion of the maximal isotropic subspace $\mathbb{P}^d \hookrightarrow Q$ sends H^{d-i}_{CH} to $l_i^{\mathrm{CH}} = \frac{1}{2}\,h^{D-i}_{\mathrm{CH}}$. Then for

$\alpha = \sum_{i=0}^{D-d-1} a_i H_{\mathrm{CH}}^{d-i}$ one has $i_{\mathrm{CH}}(\alpha) = \frac{1}{2}\, h_{\mathrm{CH}}^{D-d}\big(\sum_{i=0}^{D-d-1} a_i\, h_{\mathrm{CH}}^{d-i}\big)$. In particular, for $i < D-d$ we get

$$\mathfrak{c}(n)(l_i^{\mathrm{K}(n)}) =$$
$$= \frac{1}{2}\, h_{\mathrm{CH}}^{D-d} \exp_n(h_{\mathrm{CH}})^{d-i} \left(\frac{h_{\mathrm{CH}}}{\exp_n(h_{\mathrm{CH}})}\right)^{d+1} \cdot \frac{\exp_n(h_{\mathrm{CH}})^{D+2} \cdot 2h_{\mathrm{CH}}}{h_{\mathrm{CH}}^{D+2} \cdot \exp_n(2h_{\mathrm{CH}})} =$$
$$= \frac{\exp_n(h_{\mathrm{CH}})^{D+1-i}}{\exp_n(2h_{\mathrm{CH}})}\,.$$

In the only remaining case $i = d$ and D even, the description of $\mathfrak{c}(n)(l_d^{\mathrm{K}(n)})$ is not much harder. The series $\big(H_{\mathrm{CH}}/\exp_n(H_{\mathrm{CH}})\big)^{d+1}$ starts from 1, which maps to $l_d^{\mathrm{CH}} \neq \frac{1}{2}\, h_{\mathrm{CH}}^d$ under the pushforward map, i.e.,

$$i_{\mathrm{CH}}\left(\left(\frac{H_{\mathrm{CH}}}{\exp_n(H_{\mathrm{CH}})}\right)^{d+1}\right) = \frac{1}{2}\, h_{\mathrm{CH}}^{D-d}\left(\frac{h_{\mathrm{CH}}}{\exp_A(h_{\mathrm{CH}})}\right)^{d+1} + \left(l_d^{\mathrm{CH}} - \frac{1}{2}\, h_{\mathrm{CH}}^d\right).$$

Since multiplication by h_{CH} maps $\big(l_d^{\mathrm{CH}} - \frac{1}{2}\, h_{\mathrm{CH}}^d\big)$ to 0, we obtain

$$\mathfrak{c}(n)(l_d^{\mathrm{K}(n)}) = \frac{\exp_n(h_{\mathrm{CH}})^{D+1-d}}{\exp_n(2h_{\mathrm{CH}})} + \left(l_d^{\mathrm{CH}} - \frac{1}{2}\, h_{\mathrm{CH}}^d\right).$$

As a result, we get the following

Proposition 3.1.1. *In the above notation, the character*

$$\mathfrak{c}(n)\colon \mathrm{K}(n)^*(Q) \hookrightarrow \mathrm{CH}^*(Q;\, \mathbb{Q}[v_n^{\pm 1}])$$

on a smooth projective split quadric Q of dimension $D = 2d$ or $D = 2d+1$ is determined by the identities

$$\mathfrak{c}(n)(h_{\mathrm{K}(n)}^k) = \exp_n(h_{\mathrm{CH}})^k,$$
$$\mathfrak{c}(n)(l_i^{\mathrm{K}(n)}) = \frac{\exp_n(h_{\mathrm{CH}})^{D+1-i}}{\exp_n(2h_{\mathrm{CH}})} \quad \textit{for } i < D-d,$$
$$\mathfrak{c}(n)(l_d^{\mathrm{K}(n)}) = \frac{\exp_n(h_{\mathrm{CH}})^{D+1-d}}{\exp_n(2h_{\mathrm{CH}})} + \left(l_d^{\mathrm{CH}} - \frac{1}{2}\, h_{\mathrm{CH}}^d\right) \quad \textit{for } D \textit{ even.}$$

As a corollary, we can reprove Theorem 1.1.4. Indeed, if we try to decompose $\mathfrak{c}(n)(h_{\mathrm{K}(n)}^{d+1}) = \exp_n(h_{\mathrm{CH}})^{d+1}$ as a linear combination of $\mathfrak{c}(l_i^{\mathrm{K}(n)})$,

$$\mathfrak{c}(n)(h_{\mathrm{K}(n)}^{d+1}) = \sum c_i \cdot \mathfrak{c}(n)(l_i^{\mathrm{K}(n)}) = \frac{\exp_n(h_{\mathrm{CH}})^{d+1}}{\exp_n(2h_{\mathrm{CH}})} \sum c_i \cdot \exp_n(h_{\mathrm{CH}})^{D-d-i},$$

we see that the task is equivalent to the decomposition of $\exp_n(2h_{\mathrm{CH}})$ as a linear combination of the powers of $\exp_n(h_{\mathrm{CH}})$. Such a decomposition is exactly the series $[2]_{F_{\mathrm{K}(n)}}(t) = F_{\mathrm{K}(n)}(t,t)$, cf. (3.1).

3.1.5 Computation of the Morava Exponent

We close the present Section with the following remark concerning the computer computations with the series $\exp_n$, see (3.1). The formulae below should be well-known, and we include them only because we do not have a good reference. We would also like to mention that a similar approach is used in [BaJi] to obtain closed formulae for the coefficients of formal group laws. In our opinion, however, it is faster to compute the exponent, and then deduce the coefficients of a formal group law using it.

In contrast to Section 1.1.4, we remark that Morava K-theory $\mathrm{K}(n)^*$ makes sense for any prime p. On the other hand, we will omit v_n in the identities below to make them easier to read.

We can define a free theory corresponding to the formal group law F over $\mathbb{Z}_{(p)}$ determined by its logarithm

$$l(t) = \sum_{k\geq 0} p^{-k} t^{p^{nk}},$$

cf. [PS14]. For any series $l(t) = \sum_{k\geq 0} c_k t^{k+1}$ with $c_0 = 1$ one can find coefficients of its composition inverse $e(t) = \sum_{k\geq 0} d_k t^{k+1}$ by the formula

$$d_m = \sum_{\substack{k_1,\ldots,k_m\geq 0\\ k_1+2k_2+\ldots+mk_m=m}} (-1)^{k_1+\ldots+k_m} \frac{(m+k_1+\ldots+k_m)!}{(m+1)!k_1!\ldots k_m!} c_1^{k_1}\ldots c_m^{k_m},$$

see [Cha, p. 437]. For our logarithm series $l(t)$ this simplifies to

$$d_m = \sum (-1)^{(\sum k_i)} \frac{(m+\sum k_i)!}{(m+1)!\prod(k_i!)}\, p^{-\sum i\cdot k_i}$$

where the sum is taken over all sequences of non-negative integers $(k_1, k_2, \ldots)$, $k_i \geq 0$ subject to an equation

$$\sum (p^{ni}-1)k_i = m.$$

For computer computations the above expression is totally acceptable. For example, we can consider the case $p = 2$ and $n = 2$, and compute a few first coefficients of $\exp_2(t)$ using the above formula:

$$\exp_2(t) = t - \frac{t^4}{2} + t^7 - \frac{11}{4}t^{10} + \frac{35}{4}t^{13} - \frac{977}{32}t^{16} + \frac{1811}{16}t^{19} - \frac{14007}{32}t^{22} + \frac{111735}{64}t^{25}\ldots$$

3.2 Equivariant Computations

The present section does not contain any theorems or propositions. Here we describe the computer algorithms used by the aspirant to look for rational elements on $A^*(Q)$ and rational projectors on $A^*(Q \times Q)$ for Q a split quadric. The main ideas of the algorithm are due to Victor Petrov and Nikita Semenov, see [PS14], and the aspirant continued their work. The obtained results for small-dimensional quadrics helped the aspirant to state the theorems of the first chapter. Here we give a version adopted to the Spin-torsors, and we hope that it can still be helpful in future.

3.2.1 Equivariant Oriented Cohomology Theories

We give an axiomatic definition of *equivariant* oriented cohomology theories following [CZZ]. This paper is a part of the project [CZZ1, CZZ2, CZZ]. Classical examples of these theories are the equivariant Chow groups CH_G of Totaro [To] and Eddidin–Graham [EG98], and equivariant K-theory K_G of Thomason [Th], see also [Pa94, Me05] (we remark that in this chapter it is more convenient to work with the non-graded version of K^0-theory).

For G a smooth linear algebraic group we denote by $\mathcal{S}\mathsf{m}_k^G$ the category of quasi-projective varieties endowed with action of G (and G-equivariant maps). An *equivariant oriented cohomology theory* is given by the following data.

(D1) An additive functor $A_G\colon \left(\mathcal{S}\mathsf{m}_k^G\right)^{\mathrm{op}} \to \mathcal{R}\mathsf{ings}$ for any smooth linear algebraic group G. As usual, we denote $A_G(f)$ by f^A and call it the pullback map along f.

(D2) A morphism $f_A\colon A_G(X) \to A_G(Y)$ of $A_G(Y)$-modules called the pushforward along f for any projective equivariant morphism $f\colon X \to Y$. Pushforwards should map identity morphisms to identity morphisms, and preserve compositions.

(D3) A natural transformation of functors $c^G\colon \mathrm{K}_G \to \widetilde{A}_G$, where $\widetilde{A}_G(X)$ denotes the multiplicative group of the polynomial ring $A_G(X)[t]$. The coefficient at t^i is denoted by c_i^G and called an i-th equivariant Chern class.

(D4) A natural transformation of functors $\mathrm{res}_\phi\colon A_G \to A_H \circ \mathrm{Res}_\phi$ called the *restriction map* for any morphism of algebraic groups $\phi\colon H \to G$, and Res_ϕ stands for the restriction of the action of G to the action of H. Restriction maps should respect composition of morphisms of groups, and commute with pushforwards.

These data should satisfy the following axioms.

(A1) For the transversal square defined as in [LM, Definition 1.1.1], the pullbacks should commute with pushforwards exactly as in [LM, Defini-

tion 1.1.2 (A2)].
(A2) For $p\colon \mathbb{A}^n \times X \to X$ a G-equivariant projection, with G acting linearly on $\mathbb{A}^n$, the map $p^A\colon A_G(X) \to A_G(\mathbb{A}^n \times X)$ is an isomorphism.
(A3) For any inclusion of a smooth divisor $i\colon D \hookrightarrow X$ in $\mathcal{S}\mathsf{m}_k^G$, and $\mathcal{L}(D)$ the corresponding line bundle as in [Har, Chapter II, Proposition 6.13] one has the normalization identity $c_1^G\big(\mathcal{L}(D)\big) = i_A(1_{A_G(D)})$.
(A4) Let $p\colon X \to Y$ be in $\mathcal{S}\mathsf{m}_k^G$, and H a closed normal subgroup of G acting trivially on Y such that $p\colon X \to Y$ is an H-torsor. Consider the quotient map $\pi\colon G \to G/H$. Then the composite $p^A \circ \operatorname{res}_\pi\colon A_{G/H}(Y) \to A_G(X)$ is an isomorphism.
(A5) For $G = 1$ the theory $A = A_1$ should satisfy the axioms of Levine–Morel, in particular, we can associate a formal group law $F = F_A$ with it.
(A6) Let $i\colon Y \hookrightarrow X$ be a regular embedding of codimension d in $\mathcal{S}\mathsf{m}_k^G$. Then the normal bundle $\mathcal{N}_{Y/X}$ to Y in X is naturally G-equivariant, and one has the equality $i^A \circ i_A(1_{A_G(Y)}) = c_d^A\big(\mathcal{N}_{Y/X}\big)$.

In the next axiom we prefer to restrict ourselves only to the cases where G is a *split* reductive group or a parabolic subgroup in it.
(A7) For any closed subvariety $i\colon Z \hookrightarrow X$ in $\mathcal{S}\mathsf{m}_k^G$ with an open complement $j\colon U \hookrightarrow X$, the sequence

$$A_G(Z) \xrightarrow{i_A} A_G(X) \xrightarrow{j^A} A_G(U) \longrightarrow 0$$

is exact.

For $X \in \mathcal{S}\mathsf{m}_k^G$, we consider following [CZZ] the γ-filtration on $A_G(X)$, where $\gamma^i A_G(X)$ is defined as an ideal of $A_G(X)$ generated by products of (equivariant) Chern classes of total degree at least i. Then an equivariant oriented cohomology theory is called *Chern complete* for G, if the ring $A_G(\mathrm{pt})$ is separated and complete with respect to the γ-filtration. We remark that if the ring $A_G(\mathrm{pt})$ is separated for all G, and $\widehat{A}_G(\mathrm{pt})$ denotes its completion with respect to the γ-filtration, then tensoring $- \otimes_{A_G(\mathrm{pt})} \widehat{A}_G(\mathrm{pt})$ defines a Chern complete theory, and this procedure does not affect non-equivarinat groups $A = A_1$, cf. [CZZ, Remark 2.2]. Below we always assume that our theories are Chern complete.

Consider now the case $G = T \cong \mathbb{G}_{\mathrm{m}}^{\times l}$ a split torus, and let

$$M \cong \mathbb{Z}\chi_1 \oplus \ldots \oplus \mathbb{Z}\chi_l$$

be its group of characters. Recall the definition of $A^*(\mathrm{B}T) \cong A^*(\mathrm{pt})[\![M]\!]_{F_A}$ from Section 2.1.1, cf. also [CPZ, Definition 2.4, Corollary 2.13]. We know that $A^*(\mathrm{pt})[\![M]\!]_{F_A}$ is isomorphic to the power series ring $A^*(\mathrm{pt})[\![x_1, \ldots, x_l]\!]$, and we defined the elements $x_\lambda \in A^*(\mathrm{B}T)$ for $\lambda \in M$ according to the rule $x_{\chi_i} = x_i$, and $x_{\lambda+\mu} = F_A(x_\lambda, x_\mu)$.

Assume now that we have a Chern complete equivariant cohomology theory A_G with $A = A_1$. Then by [CZZ, Theorem 3.3] we have an isomorphism

$$A_T(\mathrm{pt}) \cong A^*(\mathrm{pt})[\![M]\!]_{F_A}.$$

More precisely, let L^λ denote the T equivariant line bundle over pt of weight λ, i.e., just a one-dimensional vector space V over k with the action of T, such that $t \cdot v = \lambda(t)v$ for any $t \in T$ and $v \in V$. Then the above isomorphism is given by the identification of $c_1^A(L^\lambda)$ with x_λ, see [CZZ, Theorem 3.3].

Let now G be a split reductive group, T its split maximal torus, and Φ a root system of G. Following [CZZ2, Definition 4.4] we call $A^*(\mathrm{pt})[\![M]\!]_{F_A}$ *regular* (with respect to Φ) if x_α is not a zero divisor for any $\alpha \in \Phi$. Working with a projective homogeneous variety for a split reductive group G we always assume below that the above regularity assumption holds.

3.2.2 Chern Classes and Weights

Let G be a *simply-connected* split semisimple group, P its parabolic subgroup, and let E/P denote an inner twisted form of a flag variety G/P, see Section 1.3.3. Then (non-graded) $\mathrm{K}^0(E/P)$ is isomorphic to $\mathrm{R}(P) \otimes_{\mathrm{R}(G)} \mathbb{Z}$, see [Pa94, Theorem 2.2], where $\mathrm{R}(H)$ stands for the $\mathrm{K}^0\mathrm{Rep}\, H$ of the category of representations of H.

We do not have analogous results for an arbitrary oriented cohomology theory A^* (e.g., for $A^* = \mathrm{CH}^*$), however, we can try to calculate the Chern classes c_i^A of elements of $\mathrm{K}^0(E/P)$, and describe the subring in $A^*(E/P)$ they generate.

Instead of $A^*(E/P)$ itself we describe as usual its image in $A^*(G/P)$ under the extension of scalars map, i.e., the subring of rational elements corresponding to E. The theorem of Panin [Pa94, Theorem 2.2] shows in fact that the subring of $A^*(G/P)$ generated by the images of the Chern classes $c_i \colon \mathrm{K}^0(G/P) \to A^*(G/P)$ is always rational. The description of this subring gives, therefore, a certain information about $A^*(E/P)$.

Any representation of P can be identified with a G-equivariant bundle on G/P as in [Pa94, Lemma 1.3], see also [Me05, Corollary 2.6],[Ana], and under this identification the map $\mathrm{R}(P) \twoheadrightarrow \mathrm{K}^0(G/P)$ of [Pa94, Theorem 2.2] coincides with the map $\mathrm{res}_1^G \colon \mathrm{K}_G^0(G/P) \twoheadrightarrow \mathrm{K}^0(G/P)$ which forgets the action of G. It is convenient for us to decompose this map as a composition of two forgetting maps res_T^G and res_1^T, first, restricting the action of G to the action of its split maximal torus T, and then forgetting the action of T.

The advantage of this approach is explained by the following result. If we identify T-fixed points of G/P with $W^P = W/W_P$ (where W and W_P are the

Weyl groups of G and P), and let us denote the corresponding embeddings $\iota_w\colon \mathrm{pt} \hookrightarrow G/P$, see [Br05, (6.2)], then the map

$$\oplus\iota_w^A\colon A_T(G/P) \to \bigoplus_{w\in W^P} A_T(\mathrm{pt})$$

is injective, see [CZZ, Theorem 8.11]. We can identify, therefore, $A_T(G/P)$ with a subring of $\bigoplus_{w\in W^P} A_T(\mathrm{pt})$ (with the component-wise multiplication).

Using the above identification, we can compute, in fact, the *equivariant* Chern classes c_i^T of the elements of $\mathrm{R}(P) \cong \mathrm{K}_G^0(G/P)$ in $A_T(G/P)$. In other words, given a representation $\rho\colon H \to \mathrm{GL}(V)$ of a parabolic subgroup P we can identify it with a G-equivariant bundle on G/P and describe explicitly its Chern classes as elements of $\bigoplus A_T(\mathrm{pt})$.

Consider the following diagram

$$\begin{array}{ccccccccc}
\mathrm{K}^0\mathrm{Rep}\,P & \xrightarrow{\cong} & \mathrm{K}_G^0(G/P) & \xrightarrow{\mathrm{res}} & \mathrm{K}_P^0(G/P) & \xrightarrow{\mathrm{res}} & \mathrm{K}_T^0(G/P) & \xrightarrow{c_i^T} & A_T(G/P) \\
 & & & & \downarrow{\scriptstyle \iota^{\mathrm{K}}} & & \downarrow{\scriptstyle \iota^{\mathrm{K}}} & & \downarrow{\scriptstyle \iota^{A}} \\
 & & & & \mathrm{K}_P^0(\mathrm{pt}) & \xrightarrow{\mathrm{res}} & \mathrm{K}_T^0(\mathrm{pt}) & \xrightarrow{c_i^T} & A_T(\mathrm{pt})
\end{array}$$

for $\iota\colon \mathrm{pt} = P/P \hookrightarrow G/P$ the embedding of the distinguished point (in particular, ι coincides with the above ι_1, $1 \in W^P$).

The correspondence between representations of P and G-equivariant bundles on G/P is given by $V \mapsto P\backslash\big(G \times \mathbb{A}(V)\big)$, see [Me05, Corollary 2.6], therefore after the restriction to the action of P and the pullback to pt, we get the map $\mathrm{R}(P) \to \mathrm{K}_P^0(\mathrm{pt})$ sending $V \mapsto \mathbb{A}(V)$. Considering $\mathbb{A}(V)$ as a T-equivariant bundle, we can decompose it as a sum of line bundles $\mathbb{A}(V) = \bigoplus_\lambda L^\lambda$ corresponding to the weights λ of V with multiplicities. After the identification $A_T(\mathrm{pt}) \cong A^*(\mathrm{pt})[\![x_{\varpi_1}, \ldots, x_{\varpi_l}]\!]_{F_A}$ we have $c_1^T(L^\lambda) = x_\lambda$, see [CZZ, Theorem 3.3]. Therefore, $c_i^T(V)$ is an i-th symmetric polynomial in x_λ.

Similarly, for any fixed point $\iota_w\colon \mathrm{pt} \hookrightarrow G/P$ we can describe the pullback of $V \mapsto P\backslash\big(G \times \mathbb{A}(V)\big)$ along ι_w. For a $v \in L^\lambda$ we have

$$(t\cdot w,\, v) = (w, w^{-1}tw \cdot v) = (w,\, \lambda(t^w)v),$$

i.e., the weights at a fixed point $w \in W^P$ are equal to $w(\lambda)$ for λ the weights of V, and the Chern classes are the symmetric polynomials in $x_{w(\lambda)}$, cf. the proof of [CZZ, Lemma 6.1].

It is well-known that the representation ring of G is generated by (the classes of) its fundamental representations V_{ϖ_i} (and, in fact, $R(G)$ a polynomial ring on V_{ϖ_i}). For the representation V_{ϖ_i} of the highest weight ϖ_i one can describe all weights of V_{ϖ_i}, see, e.g., the tables in [PSV]. We usually can similarly describe $R(L)$ as well, where L denotes the Levi subgroup of P, $R(P) = R(L)$, and the above procedure gives us a recipe to compute a certain subring of rational cycles in $A_T(G/P)$.

The next section explains how to describe the obtained rational element in more geometric terms, e.g., as pushforwards of certain subvarieties.

3.2.3 Description of Pushforwards

Let G be a split reductive group or even more generally, a parabolic subgroup in such a group, and let P be a parabolic subgroup in G. We will need a closed formula for pushforwards of elements in $A_T(G/P)$ considered as a subring in $\bigoplus A_T(\mathrm{pt})$. We include this formula here, since it is hard to find a reference.

Consider the T-equivariant map $f\colon G'/P' \to G/P$ for G' and G with the same torus T, and parabolic subgroups P' and P. Then we have a diagram

$$\begin{array}{ccc} A_T(G'/P') & \xrightarrow{f_A} & A_T(G/P) \\ \Big\downarrow {\scriptstyle \oplus\iota_w^A} & & \Big\downarrow {\scriptstyle \oplus\iota_v^A} \\ \bigoplus\limits_{w\in W^{P'}} A_T(\mathrm{pt}) & & \bigoplus\limits_{v\in W^{P}} A_T(\mathrm{pt}). \end{array}$$

By [CZZ, Corollary 8.12], the injective map

$$\oplus\iota_w^A\colon A_T(G/P) \to \bigoplus_{w\in W^P} A_T(\mathrm{pt})$$

becomes an isomorphism after the localization at the multiplicative subset x_α where α is a root. Therefore, the pushforward f_A can be computed with the use of the self-intersection formula. Recall that we always assume that x_α are not zero divisors.

More precisely, for the embedding of the closed point $i_w\colon \mathrm{pt} \to G/P$, $w \in W^P$, we have

$$\big(i_w\big)^A \circ \big(i_w\big)_A (1) = c^A_{\mathrm{top}}\big(\mathcal{T}_{G/P,w}\big),$$

where the weights of the tangent bundle $\mathcal{T}_{G/P,w}$ to G/P at the fixed point $w \in W^P$ are equal to $w(\alpha)$ for α the roots of U_P^-, and c^A_{top} can be computed as the top symmetric polynomial of weights, i.e., their product, see [CZZ, Lemmas 6.1 and 6.2].

By [CZZ, Proposition 6.2] we know that $(i_{w'})^A\big(a \cdot (i_w)_A\,(1)\big) = 0$ for $w \neq w' \in W^{P'}$. On the other hand, $(i_w)^A\big(a \cdot (i_w)_A\,(1)\big) = a \cdot c^A_{\mathrm{top}}\big(\mathcal{T}_{G'/P',\,w}\big)$. Therefore, after inverting x_α for α a root we can take $a_w = \Big(c^A_{\mathrm{top}}\big(\mathcal{T}_{G'/P',\,w}\big)\Big)^{-1} \in A_T(\mathrm{pt})$ and obtain an element $e_w = a_w \cdot (i_w)_A\,(1)$ with

$$(i_{w'})^A(e_w) = \delta_{w,w'}.$$

By $A_T(\mathrm{pt})$-linearity of pushforwards, we only have to describe $f_A(e_w)$. First, observe that $f \circ i_w$ for $w \in W^{P'}$ is obviously an inclusion of a fixed point, i.e., coincides with i_v for some $v \in W^P$. Then taking a pullback to this point we get

$$i_v^A\Big(f_A\big((i_w)_A\,(1)\big)\Big) = i_v^A \circ \big(i_v\big)_A\,(1) = c^A_{\mathrm{top}}\big(\mathcal{T}_{G/P,\,v}\big).$$

Since the pullback and the pushforward are $A_T(\mathrm{pt})$-linear, we see that for $a \in A_T(\mathrm{pt})$ one has

$$i_v^A\Big(f_A\big(a \cdot (i_w)_A\,(1)\big)\Big) = a \cdot c^A_{\mathrm{top}}\big(\mathcal{T}_{G/P,\,v}\big).$$

Then for the pushforward of e_w one has the identity

$$(i_v)^A\big(f_A(e_w)\big) = \frac{c^A_{\mathrm{top}}\big(\mathcal{T}_{G/P,\,v}\big)}{c^A_{\mathrm{top}}\big(\mathcal{T}_{G'/P',\,w}\big)}.$$

Now a pushforward of an $A_T(\mathrm{pt})$-linear combination $x = \sum e_w \cdot c_w$ of e_w can be computed by the formula

$$(i_v)^A\big(f_A(x)\big) = \sum_{\substack{w \in W^P \\ f(w)=v}} c_w \cdot \frac{c^A_{\mathrm{top}}\big(\mathcal{T}_{G/P,\,v}\big)}{c^A_{\mathrm{top}}\big(\mathcal{T}_{G'/P',\,w}\big)}. \tag{3.4}$$

Obviously, since x_α are not zero divisors for α a root, and for an element x with the values $c_w = i_w^A(x)$, we can compute $f_A(x)$ by the same formula without inverting x_α's. We remark, however, that the individual summands may have x_α in denominators.

3.2.4 An Example

Consider the example $G = \mathrm{Spin}_5$ with a split maximal torus $T = \mathbb{G}_{\mathrm{m}}^{\times 2}$, and let us denote its characters ϖ_1 and ϖ_2 according to the numbering of Bourbaki. Then for $G' = P' = P = P_1$ we consider $G'/P' = \mathrm{pt}$, and G/P a 3-dimensional quadric. The natural inclusion $P \hookrightarrow G$ induces

$$f\colon \mathrm{pt} \hookrightarrow G/P,$$

and we can compute $f_A(1)$ according to the above method, more precisely, we can show that

$$f_A(1) = (4x_{\varpi_1}x_{\varpi_2}^2 - 4x_{\varpi_1}^2 x_{\varpi_2} + \dots,\, 0,\, 0,\, 0) \in \oplus_{w\in W^P} A_T(\mathrm{pt}), \qquad (3.5)$$

where "…" stands for the terms of higher total degree in x_{ϖ_i}. Moreover, we can further rewrite it in terms of the Chern classes of certain representations of P as in Subsection 3.2.2. The Levi subgroup of our P is isomorphic to GL_2, and denoting by V_{det} and V_{nat} its determinant and natural representation, we will show that

$$(f)_A\,(1_{\mathrm{pt}}) \equiv -\,c_1^A(V_{\mathrm{det}}) \cdot c_2^A(V_{\mathrm{nat}}) \mod (x_{\varpi_1},\, x_{\varpi_2}).$$

We start with the equality (3.5). We have $W = W(\mathrm{B}_2) = \langle s_1,\, s_2\rangle$, and $W_P = \langle s_2\rangle$, so that $W^P = \{1,\, s_1,\, s_2s_1,\, s_1s_2s_1\}$. The only fixed point that has a pre-image is $1 \in W^P$, therefore $f_A(1) = (c,\, 0,\, 0,\, 0) \in \oplus_{w\in W^P} A_T(\mathrm{pt})$, where c can be computed by the formula

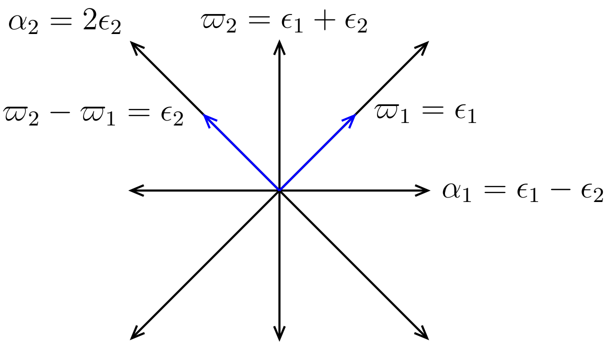
$$c = 1 \cdot \frac{c_{\mathrm{top}}^A\big(\mathcal{T}_{G/P,\,w}\big)}{1}.$$

The roots of U_1^- are $-\alpha_1 = 2(\varpi_2 - \varpi_1)$, $-\alpha_1 - \alpha_2 = -\varpi_1$, and $-\alpha_1 - 2\alpha_2 = -2\varpi_2$. Observe that we cannot take, e.g., $A = \mathrm{CH}_T(-;\mathbb{F}_2)$ here since the roots should not divide zero. We will write x_i for x_{ϖ_i}, then $c = 4x_1x_2^2 - 4x_1^2x_2 + \dots$ where "…" stands for the terms of higher total degree in x_i.

We know that any Sp_{2l} torsor is trivial, in particular, any $\mathrm{Spin}_5 = \mathrm{Sp}_4$ torsor is trivial. As a consequence, for any oriented cohomology theory A^* and any Spin_5 torsor E, the pushforward of a point in $l_0 \in A^*(G/P_1)$ is rational. But we can prove by our method that the pushforward of a point is in fact a product of Chern classes of certain concrete bundles.

The easiest way to prove our claim is to use the mentioned isomorphism of Spin_5 with Sp_4. Then for a split torus $T = \mathbb{G}_{\mathrm{m}}^{\times 2}$ of $G = \mathrm{Sp}_4$ its weights ϖ_1 and ϖ_2 will interchange. We now prefer to write ϖ_1 for our old ϖ_2 and vice versa to have the same numbering as Bourbaki for the system C_l. With this renumbering we now have $P = P_2$, and the corresponding Levi subgroup L_2 coincides with the subgroup of Sp_4 consisting of block-diagonal matrices $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ for $A, B \in \mathrm{M}_{2\times 2}$. This subgroup of Sp_4 is isomorphic to GL_2, i.e., A can be an arbitrary matrix from GL_2, and B is determined by A.

The weights of the natural representation V_{nat} of GL_2 are ϵ_1 and ϵ_2, where $\alpha_1 = \epsilon_1 - \epsilon_2$, and $\alpha_2 = 2\epsilon_2$, i.e., $\epsilon_1 = \varpi_1$, and $\epsilon_2 = \varpi_2 - \varpi_1$.



We have $W = W(\mathrm{C}_2) = \langle s_1, s_2\rangle$, and $W_P = \langle s_1\rangle$, so that $W^P = \{1, s_2, s_1s_2, s_2s_1s_2\}$. Further, $s_2(\epsilon_1) = \epsilon_1$, $s_1s_2(\epsilon_1) = \epsilon_2 = \varpi_2 - \varpi_1$, and $s_2s_1s_2(\epsilon_1) = -\epsilon_2 = \varpi_1 - \varpi_2$. Similarly, $s_2(\epsilon_2) = -\epsilon_2 = \varpi_1 - \varpi_2$, $s_1s_2(\epsilon_2) = -\epsilon_1 = -\varpi_1$, and $s_2s_1s_2(\epsilon_2) = -\epsilon_1 = -\varpi_1$.

We will write x_i for our new x_{ϖ_i}, then the second Chern class of the natural representation of GL_2 is an element of the form

$$c_2^A(V_{\mathrm{nat}}) = \big(x_1(x_2 - x_1) + O(3),\ \ x_1(x_1 - x_2) + O(3),\\ (x_2 - x_1)(-x_1) + O(3),\ \ (x_1 - x_2)(-x_1) + O(3)\big) \in \bigoplus_{w\in W^P} A_T(\mathrm{pt})$$

where "$O(3)$" stands for the terms of the total degree higher than 2. The above element is equivalent to

$$\big(2x_1(x_2 - x_1) + O(3),\ O(3),\ O(3),\ 2x_1(x_2 - x_1) + O(3)\big)$$

modulo the ideal (x_1, x_2).

The determinant representation V_{det} of GL_2 has the weight $\epsilon_1 + \epsilon_2 = \varpi_2$. We have $s_2(\varpi_2) = \alpha_1 = 2\varpi_1 - \varpi_2$, $s_1s_2(\varpi_2) = -\alpha_1 = \varpi_2 - 2\varpi_1$ and $s_2s_1s_2(\varpi_2) = -\varpi_2$. Therefore, the first Chern class of the determinant representation gives us an element

$$c_1^A(V_{\mathrm{det}}) = \big(x_2 + O(2),\ 2x_1 - x_2 + O(2),\ x_2 - 2x_1 + O(2),\ -x_2 + O(2)\big)$$

which is equivalent to

$$\big(2x_2 + O(2),\ 2x_1 + O(2),\ 2x_2 - 2x_1 + O(2),\ O(2)\big)$$

modulo the ideal (x_1, x_2). Then the product of the obtained elements is equal to

$$\big(4x_1x_2(x_2 - x_1) + O(4),\ O(4),\ O(4),\ O(4)\big).$$

This completes our description of the class of the point $f\colon \mathrm{pt} \hookrightarrow G/P$ in terms of the Chern classes,

$$(f)_A\,(1_{\mathrm{pt}}) \equiv -\,c_1^A(V_{\mathrm{det}}) \cdot c_2^A(V_{\mathrm{nat}}) \mod (x_{\varpi_1}, x_{\varpi_2}).$$

Remark. Assume that A is a graded equivariant theory, and, moreover, that $A^*(\mathrm{pt})$ is concentrated in non-positive degrees, then $A^*(G/P)$ does not have elements of degree greater than 3, and since the map $A_T(G/P) \to A^*(G/P)$ forgetting the action of T preserves grading, it sends $O(4)$ to zero (and obviously it sends x_{ϖ_1} and x_{ϖ_2} to zero).

Then the above equality shows that the element $c_1^A(V_{\mathrm{det}}) \cdot c_2^A(V_{\mathrm{nat}})$ goes exactly to the pushforward of the point up to a sign after forgetting the action of T.

3.2.5 Filtration on a Product of Quadrics

Victor Petrov and Nikita Semenov in [PS14] propose an algorithm based on the above observations, which allows to find idempotents in $A_T(Q \times Q)$ with respect to a composition of correspondences. The aspirant used this algorithm to look for projectors in small-dimensional quadrics. This approach helped the aspirant to state the conjecture about the motive of a generic quadric proven in Chapter I. We will not find any new examples of projectors with this algorithm in the present thesis (as compared to Chapter I), however, the algorithm still looks promising for the study of generic Spin_m-torsors of quadrics.

The starting point of the algorithm is the following filtration on the product of quadrics, see [PS14, Section 5]. Let Q be a smooth projective quadric over a field k, and $\varphi\colon V \to k$ be a respective quadratic form, i.e., $Q = \{\langle u \rangle \mid \varphi(u) = 0\}$, where angle brackets denote the class of $u \in V$ in the projective space $\mathbb{P}(V)$. Let us denote

$$X = \Big\{ \big(\langle u \rangle,\, \langle v \rangle\big) \in Q \times Q \;\Big|\; \mathrm{b}_\varphi(u,\, v) = 0 \Big\},$$

where b_φ denotes the bilinear form corresponding to φ. Then we have a filtration

$$Q \overset{\delta}{\hookrightarrow} X \hookrightarrow Q \times Q,$$

for δ the diagonal embedding, where the first projection map

$$\mathrm{pr}_1\colon (Q \times Q) \setminus X \to Q$$

is an $\mathbb{A}^D$-fibration for $D = \dim Q$, and

$$X \setminus Q \to \mathrm{OGr}(1, 2;\, Q), \quad \big(\langle u \rangle,\, \langle v \rangle\big) \mapsto \big(\langle u \rangle \leq \langle u,\, v \rangle\big)$$

is an $\mathbb{A}^1$-fibration; here $\mathrm{OGr}(1, 2;\, Q)$ denotes the Grassmannian of isotropic flags of dimensions 1 and 2.

However, X is not smooth, and a possible choice for the resolution of singularities for X is the projectivised tautological bundle τ_2 of rank 2 over the $\mathrm{OGr}(1,2;\,Q)$. More precisely, τ_2 is the bundle which associates with the point $\big(\langle u\rangle \le \langle u,\, v\rangle\big)$ of the Grassmannian the space spanned on u and v, and $\mathbb{P}(\tau_2)$ consists of pairs

$$\big(\langle u\rangle \le \langle u,\, v\rangle,\, \langle w\rangle \le \langle u,\, v\rangle\big).$$

Then the map $f\colon \mathbb{P}(\tau_2) \to Q \times Q$ sending the above pair to $\big(\langle u\rangle,\, \langle w\rangle\big)$ is a resolution of singularities for X.

Now consider the map

$$F\colon A^*(Q) \oplus A^{*-1}\big(\mathrm{OGr}(1,2;\,Q)\big) \oplus A^{*-D}(Q) \to A^*(Q \times Q) \tag{3.6}$$

sending $a \oplus b \oplus c$ to $\mathrm{pr}_1^A(a) + f_A \circ \pi^A(b) + \delta_A(c)$, where π denotes the natural projection from $\mathbb{P}(\tau_2)$ to $\mathrm{OGr}(1,2;\,Q)$. Then the map F is surjective for any free theory by Nakayama's Lemma and the case of Chow. As usual, it is more convenient to assume that Q is *split*, and work with *rational* elements of $A^*(Q)$ and $A^*\big(\mathrm{OGr}(1,2;\,Q)\big)$. Then F is an isomorphism since its domain and codomain are free over $A^*(\mathrm{pt})$ of the same rank, cf. also [NeZa, Theorem 4.4]. In particular, starting from rational cycles a, c on Q and b on $\mathrm{OGr}(1,2;\,Q)$, we can obtain the rational cycle $F(a \oplus b \oplus c)$ on $Q \times Q$.

Petrov and Semenov remark in [PS14] that [CZZ, Theorem 8.11] can be extended to the cases $X = \mathbb{P}(\tau_2)$ and $X = Q \times Q$, i.e., the map

$$A_T(X) \to \bigoplus_{x \in X^T} A_T(x)$$

induced by the embedding of the T-fixed points $X^T \subset X$ is injective for these X, and the description of pushforwards given in Section 3.2.3 remains valid for them as well, cf. [PS14, Section 6]. On the other hand, the representations of the Levi subgroups L_1 and $L_{1,2}$ of the Spin_{D+2} give us explicit elements in $A_T(Q)$ and $A_T\big(\mathrm{OGr}(1,2;\,Q)\big)$, respectively, which are rational with respect to every inner form of Spin_{D+2}.

Since the composition of correspondences can be computed in terms of pullbacks and pushforwards, we obtain an algorithm which allows to find projectors in $A^*(Q \times Q)$. We reproduce the description of this algorithm from [PS14] for the even-dimensional case, and add the odd-dimensional case for the sake of completeness.

3.2.6 Description of the Algorithm

Here we will describe the map F given by (3.6) in terms of the inclusions $A_T(X) \hookrightarrow \bigoplus_{x \in X^T} A_T(x)$. For an element $a \in A_T(X)$ and a fixed point

$x\colon \mathrm{pt} \hookrightarrow X$ we denote $x^A(a)$ by a_x and sometimes call a_x the coordinate of a at x. We identify a with the tuple of its coordinates $(a_x \mid x \in X^T)$.

Proposition 3.2.1. *Consider F from (3.6), and let a, c be elements of $A_T(Q)$, and b be an element of $A_T\big(\mathrm{OGr}(1,2;\,Q)\big)$. Then we can number the fixed points of Q by $i \in \{-l,\dots,-1,1,\dots,l\}$, the fixed points of $\mathrm{OGr}(1,2;\,Q)$ by the pairs (i,j) with $i \neq \pm j$ from this set, and the fixed points of $Q \times Q$ by the arbitrary pairs (i,j) in the natural way we describe below. For a, b, c, $F(a \oplus b \oplus c)$ we denote their pullbacks to the fixed points by a_i, $b_{i,j}$, c_i, and $F(a \oplus b \oplus c)_{i,j}$ respectively. In these terms*

$$F(a \oplus b \oplus c)_{i,j} = a_i + x_{-\epsilon_i-\epsilon_j} \cdot b_{i,j}$$

for $i \neq \pm j$, $F(a \oplus b \oplus c)_{i,-i} = a_i$, and

$$F(a \oplus b \oplus c)_{i,i} = a_i + \sum_{j \neq \pm i} b_{i,j} \cdot \frac{x_{\epsilon_i-\epsilon_j}\, x_{-\epsilon_i-\epsilon_j}\, c^A_{\mathrm{top}}(\mathcal{T}_{Q,i})}{x_{\epsilon_j-\epsilon_i}\, c^A_{\mathrm{top}}(\mathcal{T}_{Q,j})} + c_i \cdot c^A_{\mathrm{top}}(\mathcal{T}_{Q,i}).$$

Here, in the even-dimensional case one has

$$c^A_{\mathrm{top}}\big(\mathcal{T}_{Q,\,k}\big) = \prod_{\substack{-l \le i \le l \\ i \neq \pm k}} x_{\epsilon_i-\epsilon_k},$$

and ϵ_i are the characters $\epsilon_1 = \varpi_1$, $\epsilon_i = \varpi_i - \varpi_{i-1}$ for $1 < i < l-1$ or $i = l$, and $\epsilon_{l-1} = \varpi_l + \varpi_{l-1} - \varpi_{l-2}$. In the odd-dimensional case one has

$$c^A_{\mathrm{top}}\big(\mathcal{T}_{Q,\,k}\big) = x_{-\epsilon_k} \cdot \prod_{\substack{-l \le i \le l \\ i \neq \pm k}} x_{\epsilon_i-\epsilon_k},$$

and ϵ_i are the characters $\epsilon_1 = \varpi_1$, $\epsilon_i = \varpi_i - \varpi_{i-1}$ for $1 < i < l$, and $\epsilon_l = 2\varpi_l - \varpi_{l-1}$.

Proof. We number the fixed points of the split quadric $Q = \mathrm{Spin}_m/P_1$ by the elements of $W^{P^1} = W/W_{P_1}$, where 1 corresponds to P_1/P_1. Then the fixed points of $Q \times Q$ are exactly the pairs of the fixed points of Q, and we identify them with the set $W^{P_1} \times W^{P_1}$.

For the projection $\mathrm{pr}_1\colon Q \times Q \to Q$ on the first coordinate we have therefore a formula

$$\big(\mathrm{pr}_1^A(a)\big)_{(u,v)} = a_u$$

for u, $v \in W^{P_1}$, $a \in A_T(Q) \hookrightarrow \bigoplus_{w \in W^{P_1}} A_T(\mathrm{pt})$, since pullbacks comute with pullbacks.

Further, we can describe the pushforward along the diagonal embedding $\delta\colon Q\hookrightarrow Q\times Q$ as in Section 3.2.3. As there, the weights of the tangent bundle $\mathcal{T}_{Q,w}$ to Q at the fixed point $w\in W^{P_1}$ are equal to $w(\alpha)$ for α the roots of U_1^-, and $c^A_{\mathrm{top}}(\mathcal{T}_{Q,w})$ is the product of the weights, cf. [CZZ, Lemmas 6.1 and 6.2]. Since $\mathcal{T}_{Q\times Q,(u,v)}=\mathcal{T}_{Q,u}\oplus\mathcal{T}_{Q,v}$, and one has the Whitney formula for the Chern classes, we obtain

$$c^A_{\mathrm{top}}(\mathcal{T}_{Q\times Q,(u,v)})=c^A_{\mathrm{top}}(\mathcal{T}_{Q,u})\cdot c^A_{\mathrm{top}}(\mathcal{T}_{Q,v}).$$

Further, we obtain

$$\big(\delta_A(a)\big)_{(u,v)}=\sum_{\substack{w\in W^P\\ \delta(w)=(u,v)}} a_w\cdot\frac{c^A_{\mathrm{top}}\big(\mathcal{T}_{Q\times Q,\,(u,v)}\big)}{c^A_{\mathrm{top}}\big(\mathcal{T}_{Q,\,w}\big)}$$

as in (3.4), i.e.,

$$\big(\delta_A(a)\big)_{(u,v)}=\begin{cases}a_u\cdot c^A_{\mathrm{top}}\big(\mathcal{T}_{Q,\,u}\big), & \text{for } u=v,\\ 0, & \text{elsewhere.}\end{cases}$$

For computer computations it is preferable to avoid any packages working with Weyl groups, and here it is easy to give an explicit formula for $c^A_{\mathrm{top}}\big(\mathcal{T}_{Q,\,u}\big)$. We can realize root system B_l as in Bourbaki tables, more precisely, take a base ϵ_i of a Euclidean space $\mathbb{R}^l$, and identify B_l with the set $\{\pm\epsilon_i \mid 1\le i\le l\}\cup\{\pm\epsilon_i\pm\epsilon_j \mid 1\le i<j\le l\}$. Then the simple roots are $\alpha_i=\epsilon_i-\epsilon_{i+1}$ for $i<l$, and $\alpha_l=\epsilon_l$. The Weyl group $W(\mathrm{B}_l)$ can be identified with a subgroup of $S_{\{\pm\epsilon_1,\ldots,\pm\epsilon_l\}}$ consisting of such σ that $\sigma(-\epsilon_k)=-\sigma(\epsilon_k)$. The fundamental transpositions s_i corresponding to simple roots coincide after this identification with $(\epsilon_i,\epsilon_{i+1})(-\epsilon_{i+1},-\epsilon_i)$ for $i<l$, and $(\epsilon_l,-\epsilon_l)$ for $i=l$. Now the action of the element $w\in W(\mathrm{B}_l)^{P_1}$ on the roots of U_1^- can be easily computed. Indeed, the roots of U_1^- are the negative roots which cannot be obtained as a sum of $\alpha_2,\ldots,\alpha_l$. This is exactly the set $\{-\epsilon_1\}\cup\{-\epsilon_1\pm\epsilon_i \mid 1<i\le l\}$. Then the set $\{w(\alpha)\mid\alpha\in U_1^-\}$ coincides with

$$\{-w(\epsilon_1)\}\cup\{-w(\epsilon_1)\pm\epsilon_i \mid 1<i\le l\}.$$

Obviously, the classes $w\in W^{P_1}=W(\mathrm{B}_l)/\langle s_2,\ldots s_l\rangle$ are determined by $w(\epsilon_1)=\pm\epsilon_k$ or just by $\pm k$, therefore W^{P_1} can be identified in this way with the set $\{-l,\ldots,-1,1,\ldots,l\}$. More precisely, if we look at W^{P_1} as at the set of shortest representatives, then $s_i\ldots s_1$ corresponds to $i+1$ for $i<l$, and $s_i\ldots s_{l-1}s_ls_{l-1}\ldots s_1$ corresponds to $-i$. Introduce the notation $\epsilon_k=-\epsilon_{-k}$

for k negative. Now for $w = w_k$ corresponding to $k \in \{-l, \dots, -1, 1, \dots, l\}$, i.e., with $w(\epsilon_1) = \epsilon_k$, we have

$$c^A_{\text{top}}(\mathcal{T}_{Q,w_k}) = x_{-\epsilon_k} \cdot \prod_{\substack{-l \le i \le l \\ i \ne \pm k}} x_{\epsilon_i - \epsilon_k}.$$

The decomposition of ϵ_i in terms of fundamental weights ϖ_i is given by $\epsilon_1 = \varpi_1$, $\epsilon_i = \varpi_i - \varpi_{i-1}$ for $1 < i < l$, and $\epsilon_l = 2\varpi_l - \varpi_{l-1}$.

Similarly, we can identify D_l with the set of long roots in B_l, i.e., with the set $\{\pm\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le l\}$. Then the simple roots are $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i < l$, and $\alpha_l = \epsilon_{l-1} + \epsilon_l$. Next, the Weyl group $W(\mathrm{D}_l)$ is also a subgroup of $S_{\{\pm\epsilon_1,\dots,\pm\epsilon_l\}}$, and the fundamental transpositions s_i are equal to $(\epsilon_i, \epsilon_{i+1})(-\epsilon_{i+1}, -\epsilon_i)$ for $i < l$, and $s_l = (\epsilon_{l-1}, -\epsilon_l)(\epsilon_l, -\epsilon_{l-1})$. Similarly, the elements of W^{P_1} are determined by $w(\epsilon_1) = \epsilon_k$ or just by k (possibly negative). This allows to identify W^{P_1} with the set $\{-l, \dots, -1, 1, \dots, l\}$, and we denote $w = w_k$ if $w(\epsilon_1) = \epsilon_k$ (here we actually should assume $l \ge 3$). If we look on the shortest representatives of W^{P_1}, then $s_i \dots s_1$ corresponds to $i+1$ for $i < l$, $s_l s_{l-2} \dots s_1$ corresponds to $-l$, and $s_i \dots s_{l-2} s_l s_{l-1} s_{l-2} \dots s_1$ corresponds to $-i$ for $i < l$. The root of U_1^- are $-\epsilon_1 \pm \epsilon_i$ for $i \ne \pm 1$, and after the action of $w_k \in W^{P_1}$ we obtain the set

$$\{-\epsilon_k + \epsilon_i \mid -l \le i \le l,\ i \ne \pm k\}.$$

Therefore for the top Chern class we obtain the formula

$$c^A_{\text{top}}(\mathcal{T}_{Q,w_k}) = \prod_{\substack{-l \le i \le l \\ i \ne \pm k}} x_{\epsilon_i - \epsilon_k}.$$

The decomposition of ϵ_i in terms of fundamental weights is given by $\epsilon_1 = \varpi_1$, $\epsilon_i = \varpi_i - \varpi_{i-1}$ for $1 < i < l-1$ or $i = l$, and $\epsilon_{l-1} = \varpi_l + \varpi_{l-1} - \varpi_{l-2}$.

For the fixed points of $\mathrm{OGr}(1, 2;\, Q)$ we can obtain a similar description of the fixed points. Indeed, the class w in $W/\langle s_2, \dots, s_l\rangle$ is determined by the images of ϵ_1 and ϵ_2. We cannot have $w(\epsilon_1,\, \epsilon_2) = (\epsilon_i,\, -\epsilon_i)$ because $w^{-1}(-i) \ne -w^{-1}(i)$ in this case. But any pair $(\epsilon_i,\, \epsilon_j)$ with $j \ne \pm i$ can be an image of $(\epsilon_1,\, \epsilon_2)$ for both odd- and even-dimensional cases (for D_l we should assume $l \ge 3$). In this way we identify the fixed points of $\mathrm{OGr}(1, 2;\, Q)$ with the pairs $(i,\, j) \in \{-l, \dots, -1, 1, \dots, l\}^{\times 2}$, $i \ne \pm j$. The roots of $U_{1,2}^-$ are $\epsilon_i - \epsilon_1$ for $i \ne \pm 1$, and $\epsilon_j - \epsilon_2$ for $j \ne \pm 1, \pm 2$ in the even-dimensional case (here $i,\, j$ can be negative), and the same set in union with $\{-\epsilon_1,\, -\epsilon_2\}$ in the odd-dimensional case.

We remark that the above description of the fixed points can be obtained in more geometric terms if we identify $\mathrm{Spin_m}/P$ with SO_m/P. The split quadratic form can be identified with the form q on the space V with the base numbered $e_{-l}, \dots, e_{-1}, e_1, \dots, e_l$ in the even-dimensional case, and $e_{-l}, \dots, e_{-1}, e_0, e_1, \dots, e_l$ in the odd-dimensional one, given by $q(e_i) = 0$ for $i \neq 0$, and $q(e_i + e_j) = \delta_{i,-j}$ for $(i, j) \neq (0, 0)$, and $q(e_0) = 1$ for m odd. Then for $\mathrm{SO}_m = \mathrm{SO}(q)$ and T the split maximal torus in it, the subspaces $\langle e_i \rangle$ of V are the weight subspaces for the action of T, and their weights can be identified with the ϵ_i above for $i \neq 0$, and $\langle e_0 \rangle$ has a zero weight. Then these $\langle e_i \rangle$ as points on the quadric $Q = \mathrm{SO}(q)/P_1$ corresponding to q are exactly the fixed points. We identify $\langle e_i \rangle$ with $i \in \{-l, \dots, -1, 1, \dots, l\}$. Similarly, the fixed points on the orthogonal Grassmannian $\mathrm{OGr}(1, 2;\, Q) = \mathrm{SO}(q)/P_{1,2}$ are exactly $\langle e_i \rangle \leq \langle e_i, e_j \rangle$ which we identify with the pairs (i, j). This point of view is also helpful in the description of the fixed points on $\mathbb{P}(\tau_2)$. They should be the pairs

$$\big(\langle u \rangle \leq \langle u,\, v \rangle,\, \langle w \rangle \leq \langle u,\, v \rangle\big),$$

where $\langle u \rangle \leq \langle u,\, v \rangle$ is a fixed point of the Grassmannian, i.e., coincides with $\langle e_i \rangle \leq \langle e_i, e_j \rangle$, and $\langle w \rangle$ should also be a fixed point of $\langle e_i, e_j \rangle$, i.e., it can be either e_i, or e_j. In the first case we identify the fixed point of $\mathbb{P}(\tau_2)$ with the triple $(i, j, +)$, and in the second one with $(i, j, -)$, $i \neq \pm j$. We have the exact sequence

$$0 \longrightarrow \mathcal{T}_{\mathbb{P}^1 \cong \langle e_i, e_j \rangle, \langle e_i \rangle} \longrightarrow \mathcal{T}_{\mathbb{P}(\tau_2), (i,j,+)} \longrightarrow \mathcal{T}_{\mathrm{OGr}(1,2;\, Q), (i,j)} \longrightarrow 0$$

which shows that the top Chern class of $\mathcal{T}_{\mathbb{P}(\tau_2)}$ at the point $(i, j, +)$ is the product of the top Chern class of $\mathcal{T}_{\mathrm{OGr}(1,2;\, Q)}$ at the point (i, j) on the (c_1^A of the) weight of the tangent bundle to $\mathrm{SL}_2/B \cong \mathbb{P}^1 \cong \langle e_i, e_j \rangle$ at $\langle e_i \rangle$ which equals x_α for α the only root of U_B^- equal to $\epsilon_j - \epsilon_i$. Similarly, the top Chern class of $\mathcal{T}_{\mathbb{P}(\tau_2)}$ at the point $(i, j, -)$ is the product of the top Chern class of $\mathcal{T}_{\mathrm{OGr}(1,2;\, Q)}$ at the point (i, j) on $x_{\epsilon_i - \epsilon_j}$.

Now it is easy to describe π^A for π the natural projection from $\mathbb{P}(\tau_2)$ to $\mathrm{OGr}(1, 2;\, Q)$, and f_A for $f \colon \mathbb{P}(\tau_2) \to Q \times Q$ from Section 3.2.5. For $a \in A_T\big(\mathrm{OGr}(1, 2;\, Q)\big) \subseteq \bigoplus_{w \in W^{P_{1,2}}} A_T(\mathrm{pt})$, we denote the coordinates of a by a_w or $a_{i,j}$ if w corresponds to the pair (i, j). Similarly, for $b \in A_T\big(\mathbb{P}(\tau_2)\big)$ we denote its coordinates by $b_{i,j,\pm}$. Then, obviously, $\pi^A(a)_{i,j,\pm} = a_{i,j}$.

Finally, we describe the pushforward map along $f \colon \mathbb{P}(\tau_2) \to Q \times Q$ sending $\big(\langle u \rangle \leq \langle u,\, v \rangle,\, \langle w \rangle \leq \langle u,\, v \rangle\big)$ to $\big(\langle u \rangle,\, \langle w \rangle\big)$. We identify the fixed points of $Q \times Q$ with the pairs $(i,\, j) \in \{l, \dots, -1, 1, \dots, l\}^{\times 2}$, but now $(i,\, j)$ is the pair $\big(\langle e_i \rangle,\, \langle e_j \rangle\big)$, and not the flag $\langle e_i \rangle \leq \langle e_i,\, e_j \rangle$, in particular, i can be equal to $\pm j$. The fixed point $(i,\, j)$ on $Q \times Q$ for $i \neq \pm j$ has exactly one preimage

$(i, j, -)$ with respect to f, and therefore for $b \in A_T\big(\mathbb{P}(\tau_2)\big)$ we get

$$f_A(b)_{i,j} = b_{i,j,-} \cdot \frac{c^A_{\text{top}}(\mathcal{T}_{Q\times Q,(i,j)})}{c^A_{\text{top}}(\mathcal{T}_{\mathbb{P}(\tau_2),(i,j,-)})}.$$

Here

$$c^A_{\text{top}}(\mathcal{T}_{\mathbb{P}(\tau_2),(i,j,-)}) = x_{\epsilon_i - \epsilon_j} \cdot c^A_{\text{top}}(\mathcal{T}_{\mathrm{OGr}(1,2;\,Q),(i,j)}),$$

and

$$c^A_{\text{top}}(\mathcal{T}_{\mathrm{OGr}(1,2;\,Q),(i,j)}) = \prod_{k\neq\pm i} x_{\epsilon_k-\epsilon_i} \cdot \prod_{k\neq\pm i,\pm j} x_{\epsilon_k-\epsilon_j}$$

in the even-dimensional case, and

$$c^A_{\text{top}}(\mathcal{T}_{\mathrm{OGr}(1,2;\,Q),(i,j)}) = x_{-\epsilon_i}\, x_{-\epsilon_j} \prod_{k\neq\pm i} x_{\epsilon_k-\epsilon_i} \cdot \prod_{k\neq\pm i,\pm j} x_{\epsilon_k-\epsilon_j}$$

in the odd-dimensional one. On the other hand,

$$c^A_{\text{top}}(\mathcal{T}_{Q\times Q,(i,j)}) = c^A_{\text{top}}(\mathcal{T}_{Q,i})\, c^A_{\text{top}}(\mathcal{T}_{Q,j}) = \prod_{k\neq\pm i} x_{\epsilon_k-\epsilon_i} \cdot \prod_{k\neq\pm j} x_{\epsilon_k-\epsilon_j}$$

in the even-dimensional case, and

$$c^A_{\text{top}}(\mathcal{T}_{Q\times Q,(i,j)}) = \left(x_{-\epsilon_i} \prod_{k\neq\pm i} x_{\epsilon_k-\epsilon_i}\right) \cdot \left(x_{-\epsilon_j} \prod_{k\neq\pm j} x_{\epsilon_k-\epsilon_j}\right)$$

in the odd-dimensional one. In any case we obtain

$$f_A(b)_{i,j} = b_{i,j,-} \cdot x_{-\epsilon_i-\epsilon_j}$$

for $i \neq \pm j$. Next, a fixed point $(i, -i)$ does not have any pre-images with respect to f, and therefore

$$f_A(b)_{i,-i} = 0.$$

Finally, a fixed point (i, i) on $Q \times Q$ has $2l - 2$ pre-images

$$\big(\langle e_i\rangle \leq \langle e_i,\, e_j\rangle,\, \langle e_i\rangle \leq \langle e_i,\, e_j\rangle\big)$$

with respect to f, and

$$f_A(b)_{i,i} = \sum_{j\neq\pm i} b_{i,j,+} \cdot \frac{c^A_{\text{top}}(\mathcal{T}_{Q\times Q,(i,i)})}{c^A_{\text{top}}(\mathcal{T}_{\mathbb{P}(\tau_2),(i,j,+)})}.$$

Here

$$c^A_{\text{top}}(\mathcal{T}_{\mathbb{P}(\tau_2),(i,j,+)}) = x_{\epsilon_j-\epsilon_i}\, c^A_{\text{top}}(\mathcal{T}_{\mathrm{OGr}(1,2;\,Q),(i,j)}) = x_{\epsilon_j-\epsilon_i} \prod_{k\neq\pm i} x_{\epsilon_k-\epsilon_i} \prod_{k\neq\pm i,\pm j} x_{\epsilon_k-\epsilon_j}$$

for D_l, and

$$c^A_{\text{top}}(\mathcal{T}_{\mathbb{P}(\tau_2),(i,j,+)}) = x_{\epsilon_j-\epsilon_i}\, x_{-\epsilon_i}\, x_{-\epsilon_j} \prod_{k\neq\pm i} x_{\epsilon_k-\epsilon_i} \prod_{k\neq\pm i,\pm j} x_{\epsilon_k-\epsilon_j}$$

for B_l, and $c^A_{\text{top}}(\mathcal{T}_{Q\times Q,(i,i)}) = c^A_{\text{top}}(\mathcal{T}_{Q,i})^2$. Then

$$f_A(b)_{i,i} = \sum_{j\neq\pm i} b_{i,j,+} \cdot \frac{x_{\epsilon_i-\epsilon_j}\, x_{-\epsilon_i-\epsilon_j}\, c^A_{\text{top}}(\mathcal{T}_{Q,i})}{x_{\epsilon_j-\epsilon_i}\, c^A_{\text{top}}(\mathcal{T}_{Q,j})}.$$

□

We close the section with a remark on composition of correspondences. We number the fixed points on $Q\times Q\times Q$ by the triples

$$(i,j,k) \in \{-l,\dots,-1,1,\dots,l\}^{\times 3},$$

and we denote $\mathrm{pr}_{i,j}\colon Q\times Q\times Q \to Q\times Q$ the natural projections. Then for a, $b \in A_T(Q\times Q)$ one has

$$b\circ a = \big(\mathrm{pr}_{13}\big)_A \big(\mathrm{pr}^A_{12}(a)\cdot \mathrm{pr}^A_{23}(b)\big).$$

Obviously, $\mathrm{pr}^A_{12}(a)_{ijk} = a_{ij}$, and $\mathrm{pr}^A_{23}(b)_{ijk} = b_{jk}$, therefore we have $\big(\mathrm{pr}^A_{12}(a)\cdot \mathrm{pr}^A_{23}(b)\big)_{ijk} = a_{ij}b_{jk}$, and

$$(b\circ a)_{i,k} = \sum_j a_{ij}b_{jk}\cdot \frac{c^A_{\text{top}}(\mathcal{T}_{Q\times Q,\,(i,k)})}{c^A_{\text{top}}(\mathcal{T}_{Q\times Q\times Q,\,(i,j,k)})} = \sum_j \frac{a_{ij}b_{jk}}{c^A_{\text{top}}(\mathcal{T}_{Q,\,j})}.$$

In particular, we can check whether the element $F(a\oplus b\oplus c)$ computed above is a projector. For example, if we take $a=b=0$, $c=1$, then

$$F(0\oplus 0\oplus 1)_{i,j} = \begin{cases} c^A_{\text{top}}(\mathcal{T}_{Q,\,i}), & \text{for } i=j,\\ 0, & \text{elsewhere.}\end{cases}$$

Then, obviously, $(F(0\oplus 0\oplus 1)^{\circ 2})_{i,j}$ can only be non-zero for $i=j$, and in this case in the sum we have only one non-zero summand

$$(F(0\oplus 0\oplus 1)^{\circ 2})_{i,i} = \frac{c^A_{\text{top}}(\mathcal{T}_{Q,\,i})\cdot c^A_{\text{top}}(\mathcal{T}_{Q,\,i})}{c^A_{\text{top}}(\mathcal{T}_{Q,\,i})} = c^A_{\text{top}}(\mathcal{T}_{Q,\,i}),$$

i.e., $F(0\oplus 0\oplus 1)$ is a projector. It obviously coincides with $\delta_A(1)=\Delta$ the identity element with respect to the composition of correspondences.

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Eidesstattliche Versicherung

(Siehe Promotionsordnung vom 12.07.11, § 8, Abs. 2 Pkt. .5.)

Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

Lavrenov Andrei

Name, Vorname

13.09.2020. St. Petesburg

Ort, Datum Unterschrift Doktorand/in