On limiting curvature in mimetic gravity

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München 2021

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Dissertation
an der Fakultät für Physik
der Ludwig–Maximilians–Universität
München

vorgelegt von
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aus Güssing, Österreich

München, den 11. Mai 2021

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Zweitgutachter: Prof. Dr. Paul Steinhardt, Princeton University
Tag der mündlichen Prüfung: 8. Juli 2021

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Zusammenfassung

Das Singularitätenproblem der Allgemeinen Relativitätstheorie (ART) und die Rätsel im Zusammenhang mit dem Endzustand von evaporierenden schwarzen Löchern deuten auf eine fundamentale Unvollständigkeit unseres theoretischen Verständnisses von Gravitation hin. Laut gängiger Volksweisheit muss ART unausweichlich zusammenbrechen, sobald die Planck Skala erreicht ist um dann von einer unbekannten, nicht-perturbativen Theorie der Quantengravitation abgelöst zu werden. Trotz zahlreicher laufender Anstrengungen kann bis heute keiner der Versuche zur Konstruktion einer solchen Theorie eine zufriedenstellende Lösung vorweisen. Als Versuch diesen scheinbaren gordischen Knoten zu zerschlagen verfolgen wir einen alternativen Zugang: Da keine experimentellen Nachweise für die Gültigkeit von ART bis hin zur Planck Skala vorliegen, erlauben wir Abweichungen schon auf sub-Planck Skalen. Wenn solch eine modifizierte Gravitation nun eine eingebaute obere Schranke für Krümmungen besitzen würde, könnte man hoffen Singularitäten innerhalb der klassischen Theorie aufzulösen und womöglich das Planck-Regime gänzlich zu vermeiden.

Ziel dieser Arbeit war es, eine konkrete modifizierte Theorie der Gravitation zu konstruieren in welcher diese Idee realisiert ist. Dies wurde durchgeführt im Kontext von "mimetischer Gravitation", einer kürzlich vorgeschlagenen Modifikation von ART, basierend auf einer Reparametrisierung der Freiheitsgrade der Raumzeit Metrik under Verwendung eines einer Zwangsbedingung unterliegenden skalaren Feldes ϕ. Zu diesem Zweck wurde das minimale mimetische Modell durch Einführung einer Skalenabhängigkeit der Gravitationskonstante G und der kosmologischen Konstante Λ erweitert, eingeschränkt durch die Bedingungen von Verschiebungssymmetrie und Abwesenheit höherer Ableitungen der Metrik in der modifizierten Einstein Gleichung.

Die Auflösung von Singularitäten konnte sowohl in isotropen und anisotropen Kosmologien als auch bei nicht-rotierenden schwarzen Löchern verwirklicht werden. Eine notwendige Zutat ist dabei "asymptotische Freiheit", das Verschwinden der Gravitationskonstante sobald die extrinsische Krümmung ihre obere Schranke κ_0 erreicht. Im Fall von räumlich flachen Hyperebenen $\phi = const.$ kann damit sichergestellt werden, dass Singularitäten durch einen glatten Übergang zu einer asymptotischen de Sitter Raumzeit ersetzt werden. Diese Aussage lässt sich auch auf nicht-rotierende schwarze Löcher ausdehnen, für welche eine exakte Lösung gefunden werden konnte. Im Hinblick auf Hawking-Strahlung deutet diese modifizierte Lösung darauf hin, dass die Evaporation zum Erliegen kommt nachdem die Masse eines schwarzen Loches auf Skalen von $M_{\min} \sim \kappa_0^{-1}$ gesunken ist und danach ein stabiler Überrest zurück bleibt.

Die Einführung des mimetischen Feldes ermöglicht außerdem die kovariante Addition von räumlichen Krümmungsinvarianten höherer Ordnung. Einerseits kann dies einen "Bounce" in räumlich gekrümmten Raumzeiten herbeiführen, andererseits bietet sich eine solche Theorie zur Konstruktion einer kovarianten Version der Hořava Gravitation an.

Abschließend untersuchen wir Stabilitätsfragen in kosmologischer Störungstheorie und berechnen primordiale Spektren. Während eine inflationäre Hintergrundlösung aus der Theorie natürlich folgt, finden wir, dass ein erfolgreiches Inflationsszenario nicht mit dem mimetischen Freiheitsgrad alleine umsetzbar ist.

Abstract

The singularity problem of General Relativity (GR) and the puzzles associated with the final state of an evaporating black hole hint at a fundamental incompleteness in our current theoretical understanding of gravity. According to popular wisdom, GR is bound to fail at the Planck scale, where some unknown, non-perturbative theory of quantum gravity has to take over. Despite numerous ongoing efforts, to date none of the attempts to construct such a theory can claim to present a satisfactory solution. In an attempt to cut what seems to be a Gordian knot, we explore an alternative approach: Since there is no experimental evidence for the validity of GR all the way up until the Planck scale, we allow deviations already at sub-Planckian scales. If such a modified gravity would happen to have a built-in upper limit on curvature, one could hope to resolve singularities on a dominantly classical level and potentially avoid the Planck regime altogether.

The goal of this thesis was to construct a concrete modified theory of gravity where this idea is realized. This was done in the context of “mimetic gravity”, a recently proposed modification of GR based on a reparametrization of the degrees of freedom of the physical spacetime metric using a constrained scalar field ϕ. To this end, the minimal mimetic model was extended by introducing a scale dependence of the gravitational constant G and cosmological constant Λ, restricted by requirements of shift symmetry and absence of higher derivatives of the metric in the modified Einstein equation.

On the level of background solutions, the goal of singularity resolution has been achieved in a variety of different settings, including isotropic and anisotropic cosmologies as well as non-rotating black holes. The study of anisotropic singularities shows that a necessary ingredient is the concept of “asymptotic freedom”. This name refers to any modification where a limiting curvature scale κ_0 is implemented by a vanishing gravitational constant. In the case of spatially flat slices $\phi = const.$, a sufficiently fast vanishing of G is enough to ensure that singularities are replaced by a smooth transition to an asymptotic de Sitter space at limiting curvature. This statement can also be extended to the case of a non-rotating black hole, for which an exact solution has been found. Considering Hawking radiation, this modified solution suggests that evaporation will come to a halt once a black hole's mass has dropped to scales $M_{\min} \sim \kappa_0^{-1}$, leaving behind a minimal remnant.

Moreover, it was noticed that the introduction of the mimetic field also enables the addition of higher order spatial curvature invariants in a generally covariant way. On the one hand, this can induce bounces in spatially non-flat spacetimes, on the other hand such a theory lends itself to construct a version of covariantized Hořava gravity.

Finally, we explore cosmological perturbations, address stability issues and calculate primordial spectra in the simplest single component flat Friedmann models. While an inflationary background solution is a natural outcome, we find that a successful inflationary scenario cannot be realized with only the mimetic degree of freedom.

Acknowledgements

First and foremost, I would like to express my gratitude towards my supervisor Slava Mukhanov for his continued support throughout the years since he first accepted me as a Master student. His amazing physical intuition and his striving for simplicity even in seemingly complicated situations are qualities that I will always aspire to.

Many thanks go to my collaborator Ali Chamseddine for everything I learned from him and for our discussions during the semester he spent at LMU.

I am very grateful for the opportunities to talk about my work in front of other research groups that have been given to me. In particular I want to thank Reinhard Alkofer for inviting me to Graz, and Anna Ijjas and Paul Steinhardt for inviting me to their Simons group meeting. Huge thanks to Paul Steinhardt also for agreeing to referee my thesis.

I would like to extend many thanks to all my current and former colleagues in the Theoretical Astroparticle Physics and Cosmology group and at the Arnold Sommerfeld Center for creating an enjoyable work environment. In particular, I want to thank my former office mates Adiel Meyer and Luca Mattiello, my current office mates Anamaria Hell and Cecilia Giavoni, as well as Katrin Hammer. I also want to thank Herta Wiesbeck-Yonis for all the administrative support she provided.

I cannot conclude this section without expressing my incredible gratitude for the never-ending support that is provided to me by my family and those who are close to me.

Publications

This is a cumulative dissertation including the following articles:

[1] A. H. Chamseddine, V. Mukhanov and T. B. Russ, *Asymptotically Free Mimetic Gravity,* Eur. Phys. J. **C79** (2019) 558, arXiv:1905.01343 [hep-th].
DOI: 10.1140/epjc/s10052-019-7075-y.

[2] A. H. Chamseddine, V. Mukhanov and T. B. Russ, *Mimetic Hořava gravity,* Physics Letters B, **798** (2019) 134939, arXiv:1908.01717 [hep-th].
DOI: 10.1016/j.physletb.2019.134939

[3] A. H. Chamseddine, V. Mukhanov and T. B. Russ, *Black Hole Remnants,* JHEP **1910** (2019) 104, arXiv:1908.03498 [hep-th] .
DOI: 10.1007/JHEP10(2019)104

[4] A. H. Chamseddine, V. Mukhanov and T. B. Russ, *Non-Flat Universes and Black Holes in Asymptotically Free Mimetic Gravity,* Fortsch. Phys.**68** (2020), 1900103, arXiv:1912.03162 [hep-th].
DOI: 10.1002/prop.201900103

[5] T. B. Russ, *On Stability of Asymptotically Free Mimetic Hořava Gravity,* (2021), arXiv:210312442 [gr-qc].

The order of authors in these publications was chosen alphabetically by tradition. TBR was the main author of publications [1], [3], [4] and [5] and one of the main authors of the letter [2].

The results in [1] were obtained by TBR as a continuation of his Master thesis, supervised by V. Mukhanov. V. Mukhanov contributed the section about quantum fluctuations and was involved in the writing of the manuscript. Checking and revision of the manuscript was done by A. Chamseddine. In [2], A. Chamseddine extended the idea to use the mimetic field to covariantly extract spatial curvature invariants (partially originated by TBR) to build a covariantized model of Hořava gravity. Considerations about subtraction of the boundary term as well as checking and revision of the manuscript were done by TBR. V. Mukhanov provided useful commentary on the manuscript. The results published in [3] and [4] were obtained and written down by TBR, supervised by V. Mukhanov and A. Chamseddine. A. Chamseddine checked calculations, V. Mukhanov gave helpful input in many discussions and helped to improve the manuscript, especially for [3]. [5] was a single author publication by TBR.

Notation and units. Throughout this thesis (including [1–5]) the $(+,-,-,-)$ sign convention and Planck units $G_0 = G(\Box\phi = 0) = 1$, $\hbar = 1$, $c = 1$, $k_B = 1$ are used.

Greek indices running through $0, 1, 2, 3$ denote spacetime coordinates. Latin indices running through $1, 2, 3$ denote spatial coordinates. Partial derivatives in direction μ are denoted by $\partial_\mu \% \equiv \%_{,\mu}$. The mimetic field is denoted by ϕ.

$g_{\mu\nu}$, $\Gamma^{\alpha}_{\mu\nu}$, $\nabla_\mu \% \equiv \%_{;\mu}$, $R^{\sigma}_{\lambda\mu\nu}$, $R_{\mu\nu}$, $R = g^{\mu\nu} R_{\mu\nu}$ denote the physical spacetime metric and its associated connection coefficients, covariant derivative, Riemann tensor, Ricci tensor and Ricci scalar.

γ_{ij}, λ^{k}_{ij}, $D_i \% \equiv \%_{|i}$, ${}^3R^{n}_{mij}$, ${}^3R_{ij}$, ${}^3R = \gamma^{ij}\,{}^3R_{ij}$, κ_{ij} denote the spatial metric on slices of constant ϕ, its associated connection coefficients, covariant derivative, Riemann tensor, Ricci tensor, Ricci scalar and extrinsic curvature of its embedding into the spacetime.

A tilde $\widetilde{X}_{ij} \equiv X_{ij} - \frac{1}{3}\gamma_{ij} X$ on a spatial $2-$tensor X_{ij} denotes its traceless part.

While an effort was made to keep notation and naming of variables fairly consistent between [1–5], beware that some notations do differ. For example, in [1, 2, 5] t denotes the time coordinate in the slicing given by ϕ, which in [3, 4] is denoted by T, since t is reserved for the Schwarzschild time coordinate there.

Summary

0.1 Introduction

More then a century after its invention [6], General relativity (GR) is commonly agreed to be the gold standard for a successful physical theory, making a multitude of testable predictions based on few conceptually simple assumptions. Then why, one might ask, is there such a flourishing market for modified theories of gravity? One robust theoretical prediction of GR that could be considered to contradict physical intuition is the formation of singularities from physically realistic initial conditions. These are regions of a spacetime beyond which geodesics cannot be extended, e.g. the Big Bang singularity at the beginning of an expanding universe or the singularity inside a Black Hole. As shown by the work of Penrose and Hawking [7–9], the formation of singularities is not an artifact of highly symmetric exact solutions of the Einstein equation, but follows generically from the assumption of validity of GR plus a mild condition on the matter content. Penrose's theorem, as formulated in the modern review [10], states the following: Under the conditions that a spacetime

- (i) has a non-compact, connected Cauchy hypersurface,
- (ii) contains a trapped surface (a compact two dimensional surface such that any future directed congruence of null geodesics starting orthogonal to the surface, both ingoing and outgoing, is converging), and
- (iii) satisfies the null convergence condition (NCC), i.e. $R_{\mu\nu}n^\mu n^\nu \geq 0$ for any null vector n^μ where $R_{\mu\nu}$ is the Ricci tensor,

there exists at least one null geodesic that is future incomplete. In this formulation the theorem does not make any assumption about the validity of GR. With the additional ingredient of assuming that gravity is described by the Einstein equation, the NCC is equivalent to the null energy condition (NEC) $T_{\mu\nu}n^\mu n^\nu \geq 0$, where $T_{\mu\nu}$ is the matter energy momentum tensor.

Any attempt at singularity resolution has to circumvent this theorem one way or another. Many efforts to do so have focused on finding matter models violating the NEC. While this condition is satisfied for most known forms of classical matter, minimally coupled to gravity, it is well known that it can be easily violated even by a simple scalar field condensate. On the other hand, in modified theories of gravity, the implication NEC

$\Rightarrow$ NCC is no longer necessarily true and hence NCC violation is possible without NEC violating matter. There is room for interpretation to write any new terms that modify the Einstein equation on the right hand side and regard them as “matter”. In this case sometimes modified gravities are also described as NEC violating even when they are actually only NCC violating. In the mimetic gravity model discussed in this thesis, however, a splitting into “GR + new terms” seems somewhat artificial. For this reason we will follow the point of view that everything coming from the mimetic gravity action is “gravity” and everything else is “matter”.

Limiting curvature. Although the singularity theorems don’t make any statement about the nature of singularities and their causes, in practically all concrete examples the inextendibility of spacetimes beyond physical singularities (in contrast to coordinate singularities) can be attributed to the blow-up of some invariant measure of curvature. A natural strategy for constructing theories of gravity that avoid spacetime singularities is hence the idea of “limiting curvature”, a built-in upper limit on the magnitude of curvature invariants. A theory with this feature could be viewed as an effective field theory of gravity close to the Planck scale but still in the domain where the classical description of spacetime is valid, such that the singularity problem can be resolved on a dominantly classical level. Moreover, if all curvature invariants would happen to be bounded well below Planckian values, the “Planck regime”, dominated by non-perturbative quantum gravity effects, as conjectured by extrapolation of theories tested only at lower scales, might actually never be entered.

From first considerations by Markov in 1982 about “Limiting density of matter as a universal law of nature”, [11], and first ideas about the existence of a “fundamental length” $l_f > l_{pl}$ exceeding the Planck length $l_{pl} \sim 10^{-33}$ cm [12], this proposal has undergone a number of incarnations during the last decades. Early implementations made use of ad-hoc introduction of energy dependence of fundamental constants [13] or explored the impact of limiting curvature on black hole interiors by ad-hoc introduction of transition layers [14, 15]. First concrete realizations of the limiting curvature idea in $1 + 1$ [16] and $1+3$ dimensions [17–20] made use of actions corrected by higher order curvature invariants and non-dynamical scalars or string theory-motivated effective actions, like dilaton gravity.

In some of these models, including [13–15, 20], one encounters the concept of “asymptotic freedom” in the sense that the coupling between matter and gravity goes to zero in the limiting curvature limit. The conclusions drawn in [13–15] about asymptotic freedom leading to a de Sitter like initial state of the universe or de Sitter like black hole interior, although coming from a different approach to the concept, agree in spirit with the results we will find in this thesis. An important difference are the quantities that the running couplings are taken to depend on: In [13] the scale dependent gravitational constant depends on the energy density of matter, while in the theory presented below it will depend on a particular measure of curvature. At the level of modifications of spatially flat Friedmann universes it could look like, although some modifications arise more naturally than others, both models are equivalent. However, proceeding to singularities in different contexts, like e.g. the Kasner or the Schwarzschild singularity, it is clear that, being vacuum solutions, these singularities could never be removed by a model like [13].

In a generally covariant theory in which the gravitational field is described only by a metric, limiting curvature has to be realized by implementing an upper bound on higher order curvature scalars. In general, any such additional term in the action will include terms quadratic in second derivatives of the metric. As we will see, the novel possibility that mimetic gravity brings to the idea of limiting curvature is the fact that it allows to limit measures of curvature that only include first order derivatives of the metric in a covariant way. For instance, it allows to limit the extrinsic curvature of slices of constant ϕ, which happens to coincide with the covariant quantity $\Box\phi$.

Modified gravity and disformal transformations. By the distinguishing feature of GR as being the unique theory of metric gravity in four spacetime dimensions whose action is local and whose equation of motion is second order (sometimes called the Lovelock theorem, [21, 22]), any modification thereof has to disobey one of these conditions, i.e.

- go to higher dimensions, like in successors of Kaluza-Klein theory [23, 24] or in the DGP model [25],
- allow higher derivatives, like in the Starobinsky model [26],
- allow more geometric structure than only the metric, like in Einstein-Cartan-Kibble-Sciama gravity [27–29],
- allow non-local actions [30], or
- introduce new physical entities separate from the spacetime metric which a priori don't have to be of geometric origin, like in Brans-Dicke theory [31] or more general Scalar-tensor theories, e.g. Horndeski theory [32].

Note that this is not a complete list of modified gravities and some of the given examples could be filed in several of these categories.

Another possibility that lies somewhat outside this list is to modify GR by a reparametrization of degrees of freedom rather than ad hoc introduction of new fields. A pioneering work in the direction of this idea was Bekenstein's 1992 article [33], "The Relation between physical and gravitational geometry", where the notion of a "disformal transformation" or "disformation" was introduced to refer to the reparametrization of the physical metric in terms of an auxiliary metric $\tilde{g}_{\mu\nu}$ and a scalar function φ as

$$g_{\mu\nu} = C(\varphi, w)\tilde{g}_{\mu\nu} + D(\varphi, w)\varphi_{,\mu}\varphi_{,\nu}, \qquad w = \tilde{g}^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta}, \tag{1}$$

where $C(\varphi, w)$, $D(\varphi, w)$ are arbitrary functions and $C \neq 0$. Our intuition might tell us that a simple reparametrization like this can certainly not change anything about the physical content of the theory. For most disformal transformations this intuition is correct: Even when the variation of the Einstein-Hilbert action $S \propto \int \mathrm{d}^4x\sqrt{-g}\, R\,[g_{\mu\nu}]$ is performed with respect to $\tilde{g}_{\mu\nu}$ and φ, the resulting equations of motion in the end are still equivalent to the Einstein equation for $g_{\mu\nu}$.

However, this is not true for all disformations. The subclass of transformations for which $C(\varphi, w)/w + D(\varphi, w) =: E(\varphi)$ is a function of φ only is singular and in this case the resulting equations of motion do not reproduce the Einstein equation. As shown in [34], in the case where the gradient of the scalar field is timelike, all singular disformations can be reduced to "mimetic gravity" in the same way in which non-singular disformations reduce to GR.

Mimetic dark matter and "vanilla" mimetic gravity. "Mimetic gravity" corresponds to the singular disformation $C(\varphi, w) = w$, $D(\varphi, w) = 0$, i.e. to the reparametrization

$$g_{\mu\nu} = \tilde{g}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}\,\tilde{g}_{\mu\nu}, \tag{2}$$

where ϕ is now called the "mimetic field". This particular disformal transformation is special for two reasons: (i) it is singular, explaining how a simple reparametrization can alter the degrees of freedom of the theory and (ii) the physical metric is invariant under Weyl transformations $\tilde{g}_{\mu\nu} \mapsto \Omega^2\tilde{g}_{\mu\nu}$ of the auxiliary metric.

This construction was first considered in [35] with the intention to isolate what was called the "conformal degree of freedom of gravity" in a covariant way and make it dynamical. The initial motivation to do this was to show that dark matter does not necessarily have to consist of new particles but could also be an effect of geometry. Indeed, using (2), when performing the variation of the Einstein-Hilbert action $S \propto \int \mathrm{d}^4x\sqrt{-g}\,R\,[g_{\mu\nu}]$ with respect to the auxiliary metric $\tilde{g}_{\mu\nu}$ and mimetic field ϕ one finds a new term that modifies the Einstein equation. Since this new contribution mimics the behaviour of a dust-like component, but its origin and interactions are purely geometrical, the authors have called this contribution "mimetic dark matter". Subsequent extensions of this minimal modification have started to go under name of "mimetic gravity" as an umbrella term. This is somewhat of a misnomer since the theory does not "mimic" gravity but is rather a modification of GR which, among other things, contains a component that mimics dark matter.

A direct consequence of (2) is the mimetic constraint

$$g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} = 1. \tag{3}$$

The assumption of a global solution to this equation presents a restriction on the causal structure of admissible spacetimes. For example, the existence of a global function with timelike gradient is equivalent to stable causality, a causality condition that ensures, roughly speaking, that there are no closed causal curves even when the metric is slightly perturbed, see [36]. As shown in [37], an equivalent way to introduce the mimetic field is to impose the constraint (3) with a Lagrange multiplier field. This formulation has the advantage that instead of the auxiliary metric and the mimetic field one is always dealing with the physical metric and the mimetic field.

In the following this minimal mimetic model with action

$$\mathcal{S} = -\frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}\left\{R\left[g_{\mu\nu}\right] + \lambda\left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - 1\right)\right\}, \tag{4}$$

still given by the Einstein-Hilbert action modified only by the constraint, will be referred to as “vanilla” mimetic gravity. As shown in [38], vanilla mimetic gravity is free of ghosts, provided that the energy density of mimetic dark matter is positive.

Note that the mimetic construction viewed as a Scalar-Tensor theory goes beyond Horndeski theory, as was shown in [39]. This should not be confused with an extension of this minimal model that was introduced under the name of “mimetic Horndeski theory” [40], where the mimetic constraint was added to a general Horndeski action.

Extensions and earlier limiting curvature models. Many extensions of the vanilla mimetic model have been considered since its introduction in 2013. See [41] for a review of the developments in mimetic gravity until 2016.

Limiting curvature in the context of mimetic gravity has first been considered in the earlier mimetic model [42, 43]. In cosmological applications of this model, singularities are resolved by a regular bounce at finite scale factor which takes place during a timespan given by the inverse limiting curvature scale. This bouncing solution has been compared to loop quantum gravity bounces and lead to considerations of limiting curvature mimetic gravity as an “effective” loop quantum cosmology theory, [44–46]. However, not only do these bouncing solutions require a change of branch in the modified Friedmann equation, as was realized in [47], they also exhibit a ghost instability in the region around the bounce which is unavoidable, [48]. In [44] this ghost instability in the bouncing region together with the wrong sign of the gradient term was wrongly interpreted as stability in this region, followed by a gradient instability in the region after the bounce.

The application of the first limiting curvature mimetic model to non-rotating black holes in [43] resulted in the picture of a Russian-nesting-doll-like black hole, where the internal structure consists of a sequence of black holes of ever smaller gravitational radius. After unpacking a finite number of these smaller black holes, one reaches a region in which the approximation used to derive the solution breaks down. In this model, the conclusions that after passing this complicated internal structure a spacetime at limiting curvature is reached and that the near horizon geometry of a minimal size black hole is changed in such a way that the final product of evaporation is a stable remnant remain at the level of conjectures.

Covariantized Hořava gravity. Apart from the singularity problem, another big incentive that has driven the quest for quantum gravity has been the perturbative non-renormalizability of Einstein gravity, the main problem being the negative mass dimension of the Newton constant $[G_N] = -2$, [49].

Relativistic higher derivative corrections can modify the graviton propagator to cure UV divergences, but will at the same time introduce a second pole in the propagator, corresponding to ghost excitations. A way to benefit from the former without having to pay the price of the latter was suggested by Hořava in [50], drawing analogies from condensed matter systems. Hořava gravity breaks general covariance to introduce an explicit "asymmetry" between space and time. The scaling of the modified graviton propagator with the four-momentum $k_\mu = (\omega, \mathbf{k})$ can then be written schematically as

$$\frac{1}{\omega^2 - c_T^2 \mathbf{k}^2 - \sigma(\mathbf{k}^2)^z}. \tag{5}$$

The minimal value of the "dynamical critical exponent" z for a power counting renormalizable theory is $z = 3$.

An example for a concrete gravity action where this critical value of z is attained is given by

$$\mathcal{S} = \int \mathrm{d}t\mathrm{d}^3\mathbf{x} N^2 \left\{ \frac{1}{16\pi G_N} \left(K^{ij}K_{ij} - \lambda K^2 \right) - 16\pi G_N \sigma^4 \left({}^3C^{ij}\, {}^3C_{ij} \right) \right\}, \tag{6}$$

where the spacetime $\mathcal{M} = \mathbb{R} \times \Sigma$ has been sliced into hypersurfaces Σ with extrinsic curvature K_{ij} and intrinsic Cotton tensor ${}^3C_{ij}$, and σ is a dimensionless coupling. The case where the lapse $N = N(t)$ does not depend on spatial coordinates of the slicing is called *projectable* Hořava gravity. The full renormalization analysis of projectable Hořava gravity has been performed in [51, 52].

The problems associated with an extra scalar mode that arises from the explicit breaking of general covariance (see [53]) have been addressed by a number of covariantized versions of Hořava gravity, like [54], which typically possess also a "scalar graviton" or other propagating degrees of freedom. Since the mimetic field provides a global time function whose gradient is everywhere timelike, in [2] it was shown that it can be used to isolate any scalar quantity that is invariant under spatial diffeomorphisms in constant ϕ slices in a fully covariant way. In this way it is easy to write down a covariantized Hořava-gravity-like theory with only higher spatial derivatives but no higher time derivatives or mixed derivatives. Interestingly, several connections between mimetic gravity and Hořava gravity have been drawn: In [55] it was explored how a dust-like component emerges as a constant of integration in Hořava-Lifshitz gravity. In [56] an equivalence between the IR limit of projectable Hořava gravity and a mimetic matter scenario has been found. Another Hořava-like mimetic model has been presented in [57].

0.2 The theory

The theory considered in this thesis was built up in steps over the course of [1–5]. Starting with a general mimetic gravity action where the mimetic field is introduced through a Lagrange multiplier constraint [37],

$$\mathcal{S} = -\frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}\left\{\mathcal{L} + \lambda\left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - 1\right)\right\}, \tag{7}$$

the requirements of shift symmetry and absence of higher derivatives of the metric in the modified Einstein equation restrict the possibilities for such a theory: For instance, if $\mathcal{L}$ contains a running gravitational constant G, i.e. a function $f \equiv 1/G$ multiplying the Ricci scalar, it can only depend on the trace of extrinsic curvature of slices of constant ϕ, a quantity that happens to coincide with the covariant expression $\Box\phi$.

All applications considered in [1–5] are encompassed by the general Lagrangian

$$\begin{aligned}\mathcal{L} = \mathcal{L}_{\mathsf{nhd}} + \mathcal{L}_{\mathsf{hd}} = f(\Box\phi)R + 2\Lambda(\Box\phi) + (f(\Box\phi) - 1)\widetilde{R} \\ + \alpha(\widetilde{R}) + \beta(\widetilde{R})\widetilde{R}^{\mu\nu}\widetilde{R}_{\mu\nu} + \sigma_T^4\,\widetilde{C}^{\mu}_{\nu}\widetilde{C}^{\nu}_{\mu} - \frac{\sigma_S^4}{8}P^{\mu}_{\nu}\nabla_{\mu}\widetilde{R}\nabla^{\nu}\widetilde{R},\end{aligned} \tag{8}$$

which is symmetric under shifts $\phi \mapsto \phi + c$. The meaning and purpose of all individual terms will be elaborated on in the following sections. Reviewing the development over the course of [1–5], it is worth to point out that every step represents a true generalization. In particular, the exact solutions obtained in [1, 3] stand unchanged in the more general theory considered in [5].

A theory without higher derivatives. Starting with the Lagrangian first introduced in [3],

$$\mathcal{L}_{\mathsf{nhd}} = f(\Box\phi)R + 2\Lambda(\Box\phi) + (f(\Box\phi) - 1)\widetilde{R}, \tag{9}$$

where

$$\widetilde{R} = 2\phi^{,\mu}\phi^{,\nu}G_{\mu\nu} - (\Box\phi)^2 + \nabla^{\mu}\nabla^{\nu}\phi\nabla_{\mu}\nabla_{\nu}\phi, \tag{10}$$

we encounter a theory whose modified Einstein equation is free of any higher derivatives of the metric. As argued already in [1], the only quantity that f can depend on without introducing higher time[*] derivatives of the metric in the modified Einstein equation is $\Box\phi$. However, in the theory considered in [1], it still contained higher mixed and spatial derivatives because it was missing the last term of (9). In fact, these higher mixed derivatives of the theory $\tilde{\mathcal{L}} = f(\Box\phi)R + 2\Lambda(\Box\phi)$ have been used in [58] to adjust the otherwise wrong sign of the gradient term of the scalar degree of freedom of mimetic gravity in perturbations around a flat Friedmann universe. However, later in [59] this theory was shown to possess an additional second scalar degree of freedom which is hidden when considering only homogeneous backgrounds but which can lead to instabilities already when perturbing

[*]In the following the distinction between time and space is referring to the slicing into spatial hypersurfaces of constant mimetic field ϕ.

around Minkowski spacetime with a non-homogeneous field profile, [60]. In the process of searching for black hole solutions, leading up to [3], it was realized that there is a unique, covariant way to remove the term responsible for higher mixed and spatial derivatives, resulting in the Lagrangian (9). In fact, this theory turned out to be significantly simpler than $\tilde{\mathcal{L}}$, mirrored by the fact that it allowed to find an exact black hole solution. Since in the modified Einstein equation of (9) no higher derivatives of the metric are present, one could hope that no second scalar degree of freedom is hiding in this theory. Of course this argument is purely heuristic, as a full Hamiltonian analysis has not yet been performed.

For illustrative purposes, in the following the equations of motion of (9) will be reviewed first in covariant form, then in the slicing given by the mimetic field. For the sake of simplicity, it is convenient to express $\Lambda(\Box\phi)$ without loss of generality in terms of $\overline{\Lambda}(\Box\phi)$ as

$$\Lambda(\Box\phi) = \tfrac{2}{3}\,\Box\phi\left[\Box\phi f(\Box\phi) - F(\Box\phi) - \tfrac{3}{2}H(\Box\phi)\right], \tag{11}$$

where $\mathrm{d}F/\mathrm{d}\Box\phi = f(\Box\phi)$ and $(\Box\phi)^2\,\mathrm{d}H/\mathrm{d}\Box\phi = f(\Box\phi)\overline{\Lambda}(\Box\phi)$.

Variation with respect to the Lagrange multiplier field λ yields the mimetic constraint

$$g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} = 1. \tag{12}$$

The equation of motion one finds by variation with respect to ϕ reads

$$\nabla_\nu\left[(\lambda + (f-1)R)\phi^{,\nu} - (f^{,\nu}\,\phi^{,\mu})_{;\mu} - Z^{,\nu} - (f-1)R^{\mu\nu}\phi_{,\mu}\right] = 0. \tag{13}$$

It is worth mentioning that this equation is nothing like a Klein-Gordon type equation for a dynamical scalar field. Since the mimetic field ϕ is already completely fixed by the constraint (12), (13) has to be used to determine the Lagrange multiplier field λ. Finally, variation with respect to the metric results in the equation of motion

$$\begin{aligned} R_{\mu\nu} - \left(\tfrac{1}{2}\mathcal{L}_{\mathsf{nhd}} + \nabla_\alpha(Z\phi^{,\alpha})\right)g_{\mu\nu} + \nabla_\alpha\left(f^{,\alpha}\phi_{,\mu}\phi_{,\nu} - (f-1)\phi^{,\alpha}\nabla_\mu\nabla_\nu\phi\right) \\ + 2(f-1)\phi^{,\alpha}\phi_{(,\mu}R_{\nu)\alpha} + 2\phi_{(,\mu}Z_{,\nu)} - (\lambda + (f-1)R)\phi_{,\mu}\phi_{,\nu} = 8\pi T_{\mu\nu}, \end{aligned} \tag{14}$$

where $T^{\mu\nu}$ is the matter energy momentum tensor and

$$Z := \tfrac{1}{2}f'\left((\Box\phi)^2 + \phi^{;\mu\nu}\phi_{;\mu\nu}\right) - \Lambda'. \tag{15}$$

By the constraint (12), ϕ is a global time function whose gradient has constant unit norm. It may hence be used as the time coordinate $t = \phi$ of a synchronous coordinate system

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{16}$$

in which the modified Einstein equation will take its simplest form. In these coordinates $\nabla_\mu\nabla_0\phi = 0$ and

$$-\nabla_i\nabla_j\phi = \kappa_{ij} = \tfrac{1}{2}\partial_0\gamma_{ij} \tag{17}$$

is equal to the extrinsic curvature of slices of constant ϕ. The trace of extrinsic curvature of these slices can then by extracted in a covariant form as

$$\Box\phi = \kappa = \gamma^{ij}\kappa_{ij} = \frac{1}{\sqrt{\gamma}}\partial_0\sqrt{\gamma}, \tag{18}$$

where $\gamma = \det\gamma_{ij}$ is the spatial metric determinant. Moreover,

$$\widetilde{R} = {}^3R \tag{19}$$

coincides with the spatial curvature of slices of constant ϕ. In this slicing, (13) can be integrated and solved for the Lagrange multiplier field λ and the components of the modified Einstein equation can be simplified to the system

$$f\left(\frac{1}{3}\kappa^2 - \overline{\Lambda}\right) - \frac{1}{2}(f+\kappa f')\tilde{\kappa}^i_j\tilde{\kappa}^j_i + \frac{1}{2}{}^3R - \Xi = 8\pi T_{00} \tag{20}$$

$$fR_{0i} + Z_{,i} + \kappa^j_i f_{,j} = 8\pi T_{0i} \tag{21}$$

$$-\frac{1}{\sqrt{\gamma}}\,\partial_0\left(\sqrt{\gamma}\,\left(f\kappa^i_j + Z\delta^i_j\right)\right) - \tfrac{1}{2}\mathcal{L}_{\mathsf{nhd}}\,\delta^i_j - {}^3R^i_j = 8\pi T^i_j, \tag{22}$$

where

$$\Xi = \tfrac{1}{\sqrt{\gamma}}\int \mathrm{d}t\sqrt{\gamma}\,D_i\left(8\pi T^i_0 - R^i_0\right). \tag{23}$$

Another important equation is obtained by subtracting the trace from (22),

$$-\frac{1}{\sqrt{\gamma}}\,\partial_0\left(\sqrt{\gamma}\,f\,\tilde{\kappa}^i_j\right) - {}^3\widetilde{R}^i_j = 8\pi\widetilde{T}^i_j, \tag{24}$$

where the tracefree part of spatial 2-tensors is denoted by $\tilde{\kappa}^i_j \equiv \kappa^i_j - \frac{1}{3}\kappa\delta^i_j$, etc. For details, see [4]. The time reversal invariance of the Einstein equation is maintained, provided that f and $\overline{\Lambda}$ depend symmetrically on $\Box\phi \equiv \kappa$. In this case the contracting counterpart $\kappa < 0$ to any expanding solution $\kappa > 0$ can be found simply by the transformation $t \mapsto -t$.

The modified Einstein equations (20-24) do not contain any higher derivatives of the metric γ_{ij}. However, note that (20) together with (23) constitutes an integro-differential equation. Thus the dynamical equations of motion in the synchronous frame consist in general not only of the spatial components (22), but also of the second order differential equation

$$\frac{1}{\sqrt{\gamma}}\partial_0\left\{\sqrt{\gamma}\left[f\left(\frac{1}{3}\kappa^2 - \overline{\Lambda}\right) - \frac{1}{2}(f+\kappa f')\tilde{\kappa}^i_j\tilde{\kappa}^j_i + \frac{1}{2}{}^3R\right]\right\} + D_iR^i_0 = -8\pi\kappa_{ij}T^{ij}, \tag{25}$$

that one obtains by taking $\frac{1}{\sqrt{\gamma}}\partial_0(\sqrt{\gamma}\%)$ of (20). Here the continuity equation

$$0 = \nabla_\mu T^{0\mu} = \frac{1}{\sqrt{\gamma}}\partial_0\left(\sqrt{\gamma}T_{00}\right) + D_iT^i_0 + \kappa_{ij}T^{ij} \tag{26}$$

was used, which, like in GR, still follows from the equation of motion (14). The additional dynamical equation of motion (25) is a manifestation of the fact that mimetic gravity possesses a scalar degree of freedom. It is easy to see what the nature of this degree of freedom is in a homogeneous spacetime: In this case (23) is given only by a constant of integration,

$$\Xi = \frac{c_{\mathsf{MDM}}(x^i)}{\sqrt{\gamma}}, \tag{27}$$

and (25) has a simple first integral: the temporal component (20) of the modified Einstein equation which is now only a constraint, containing a contribution of mimetic dark matter (MDM).

Subtraction of boundary terms and higher spatial derivatives. Expanding the Ricci scalar R in the synchronous coordinates (16) and then using (17) and (19) to rewrite it in covariant form, we find the expansion

$$-R = 2\nabla_\mu \left(\Box\phi \, \phi^{;\mu}\right) - \left(\Box\phi\right)^2 + \phi^{;\mu\nu}\phi_{;\mu\nu} + \widetilde{R}. \tag{28}$$

Noting that

$$f(\Box\phi)\nabla_\mu \left(\Box\phi \, \phi^{;\mu}\right) = \nabla_\mu \left(F(\Box\phi)\phi^{;\mu}\right) + \Box\phi \left[f(\Box\phi)\Box\phi - F(\Box\phi)\right],$$

where $\mathrm{d}F/\mathrm{d}\Box\phi = f(\Box\phi)$, the Lagrangian (9) can be rewritten as

$$\mathcal{L}_{\mathsf{nhd}} = -2\nabla_\mu \left(F(\Box\phi)\phi^{;\mu}\right) + \tfrac{4}{3}\ell(\Box\phi) - f(\Box\phi)\left(\nabla^\mu\nabla^\nu\phi\nabla_\mu\nabla_\nu\phi - \tfrac{1}{3}(\Box\phi)^2\right) - \widetilde{R}, \tag{29}$$

where

$$\ell(\Box\phi) = \tfrac{3}{2}\Box\phi F - (\Box\phi)^2 f + \tfrac{3}{2}\Lambda \tag{30}$$

collects all terms depending only on $\Box\phi$. The first term in (29) is a total derivative and thus subtracting it changes the action only by a boundary term. For boundaries that lie along hypersurfaces of constant ϕ, this is analogous to the addition of the Gibbons-Hawking boundary term in GR. A note of caution is due here: The usage of (28), which was derived assuming validity of the mimetic constraint, in the action (7) before variation is in general not a valid procedure. However, one can explicitly verify that the equations of motion of (9) and (29) are identical.

Similarly to $\kappa = \Box\phi$, $\kappa^{ij}\kappa_{ij} = \nabla^\mu\nabla^\nu\phi\nabla_\mu\nabla_\nu\phi$, and ${}^3R = \widetilde{R}$, any quantity that is invariant under spatial diffeomorphisms on slices of constant ϕ may be rewritten in a fully covariant form using the mimetic field and the projector $P^\nu_\mu = \delta^\nu_\mu - \phi_{,\mu}\phi^{,\nu}$ to project out the time direction, as shown in [2]. Thus, this theory lends itself to be used for a covariantized version of Hořava gravity. For example, the spatial Cotton tensor squared term from (6) can be covariantized as

$${}^3C^i_j \, {}^3C^j_i = \widetilde{C}^\mu_\nu \widetilde{C}^\nu_\mu, \tag{31}$$

where

$$\widetilde{C}^\mu_\nu = -\frac{\epsilon^{\mu\rho\kappa\lambda}}{\sqrt{-g}} \nabla_\lambda\phi \, \nabla_\rho \left(\widetilde{R}_{\nu\kappa} - \frac{1}{4} g_{\nu\kappa}\widetilde{R}\right), \tag{32}$$

with the Levi-Civita symbol $\epsilon^{0123} = 1$ and the covariantized spatial Ricci tensor and scalar

$$\widetilde{R}_{\mu\nu} = P_\mu^\alpha P_\nu^\beta R_{\alpha\beta} + \nabla_\alpha \left(\phi^{,\alpha} \nabla_\mu \nabla_\nu \phi\right), \qquad \widetilde{R} = -g^{\mu\nu} \widetilde{R}_{\mu\nu}. \tag{33}$$

More generally, in (8) we included the higher spatial derivative terms

$$\mathcal{L}_{\text{hd}} = \alpha(\widetilde{R}) + \beta(\widetilde{R}) \widetilde{R}^{\mu\nu} \widetilde{R}_{\mu\nu} + \sigma_T^4 \, \widetilde{C}^\mu_\nu \widetilde{C}^\nu_\mu - \frac{\sigma_S^4}{8} P^\mu_\nu \nabla_\mu \widetilde{R} \nabla^\nu \widetilde{R}. \tag{34}$$

The other sixth order derivative term σ_S will turn out to be necessary to have a chance of averting the gradient instability of the scalar degree of freedom of mimetic gravity.

The higher spatial derivative terms α and β do not play any role in a spatially flat spacetime, also in linear perturbations around a spatially flat background, provided that $\alpha = \mathcal{O}(\widetilde{R}^{\geq 3})$, $\beta = \mathcal{O}(\widetilde{R}^{\geq 1})$. As it was found in [4], in the presence of spatial curvature, limiting extrinsic curvature is not sufficient for singularity resolution. In [4] it was shown that in a non-flat universe, the isotropic spatial curvature dependent potential

$$V(\widetilde{R}) := \alpha(\widetilde{R}) + \tfrac{1}{3} \beta(\widetilde{R}) \widetilde{R}^2 \tag{35}$$

can be used to resolve the Big Bang singularity by inducing a bounce driven by higher order spatial curvature terms. This has also been extended to Bianchi type V and some special cases of Bianchi types II, VI_0, VII_0, VIII, IX. The remaining term

$$\beta(\widetilde{R}) \left(\widetilde{R}^{\mu\nu} \widetilde{R}_{\mu\nu} - \tfrac{1}{3} \widetilde{R}^2 \right), \tag{36}$$

corresponding to anisotropic spatial curvature, then plays a role in cosmological perturbations around these bouncing background solutions. As shown in [5], these spatially non-flat, bouncing solutions in fact always suffer from some form of instability. In the case of positive curvature, a ghost instability is avoided, but a gradient instability is unavoidable for any universe that undergoes any significant amount of expansion. Interestingly, the instability is most severe in the GR limit and the bounce itself actually is the “least unstable” region.

0.3 Results & Conclusions

In this section the main results of this thesis, as published in [1–5], are presented and discussed. First, in subsection 0.3.1 we review the results from [1] for the background dynamics of a flat Friedmann universe, in the light of the condition to be free of ghosts, as found in [5]. Next, in subsection 0.3.2 the argument why "asymptotic freedom" becomes a necessary condition to resolve anisotropic singularities is presented in application to a Bianchi type I universe. The next subsection 0.3.3 reviews the non-singular modified black hole solution found in [3] and extended in [4]. Finally, in subsection 0.3.4 we review the main results of the linear stability analysis performed in [5] and argue for the necessity of higher order spatial curvature terms (34), first introduced in [2], to avoid the gradient instability of the scalar degree of freedom of mimetic gravity.

0.3.1 Big Bang replaced by initial dS

To see the theory introduced in the last section at work, the first application we consider is to a flat Friedmann universe. Compared to shift-symmetry breaking mimetic models, where a freely adjustable, time dependent background enables any conceivable background solution to be put in more or less by hand [61], we find that in the shift symmetric theory (8) there is essentially only one possibility for the qualitative behaviour of a non-singular, flat Friedmann universe: to replace the Big Bang singularity by a smooth transition to an asymptotic de Sitter spacetime at limiting curvature.

In a homogeneous and isotropic spacetime the solution of (12) consistent with these symmetries is, up to shifts, $t = \phi$. That is, the synchronous time given by the mimetic field agrees with the cosmological time of isotropic observers. The trace of the extrinsic curvature of slices of constant ϕ hence coincides with a multiple of the Hubble parameter,

$$\Box\phi \equiv \kappa = 3H \equiv 3\tfrac{\dot{a}}{a}, \tag{37}$$

where a dot denotes t-derivatives. Applied to a flat Friedmann metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{38}$$

the modified Einstein equations (20-24) result in the modified Friedmann equation

$$H^2 - \frac{1}{3}\overline{\Lambda}(\kappa) = \frac{1}{3}G(\kappa)\left(\frac{c_{\mathsf{MDM}}}{a^3} + 8\pi\varepsilon^m\right) = \frac{8\pi}{3}G(\kappa)\,\varepsilon. \tag{39}$$

where the suggestive notation $G = 1/f$ has been used and the total energy density ε consists of the matter energy density ε^m and the constant of integration corresponding to mimetic dark matter. The modified analogue of acceleration equation reads

$$\frac{\dot{H}}{H}\frac{\mathrm{d}}{\mathrm{d}\kappa}\left[f\left(\kappa^2 - 3\overline{\Lambda}\right)\right] = -8\pi\left(\varepsilon + p\right). \tag{40}$$

Note that the modified Friedmann equation (39) is still formulated in terms of the same quantities H and ε as the usual Friedmann equation. Only the relation between curvature and energy density is altered at high curvatures.

In contrast, in a shift-symmetry breaking theory where a potential $U(\phi)$ is added to the Lagrangian, we would have found the time dependent background $U(t)$ appearing on the right hand site of (40). Moreover, in this case the contribution of mimetic matter

$$c_{\mathsf{MDM}}(t) \propto \int \mathrm{d}(a^3) U(t) \tag{41}$$

would not be constant in time. It is clear that in this case the time dependent background can be chosen in such a way to obtain any solution one can think of. Examples for both bouncing as well as inflationary solutions obtained in this way have been provided in [61]. In this work we refrain from introducing such a time dependent background and try to realize limiting curvature in shift-symmetric mimetic gravity.

Assuming that the total energy density ε is a monotonic function of the scale factor a, the modified Friedmann equation (39) can be viewed as an integral curve in the phase space spanned by a and $H = \dot{a}/a$. This allows to understand the qualitative behaviour of any such relation without obtaining explicit solutions, cf. [4]. Plotting H^2 vs. ε for different possible modified Friedmann equations (39) (see figure 1) it is easy to proof that the only thing that can replace the Big Bang singularity is a smooth transition to an initial de Sitter stage at limiting curvature, requiring that the following three conditions are satisfied:

(i) Limiting curvature:

$$\kappa^2 < \kappa_0^2 \tag{42}$$

Combined with $|\dot{\kappa}| < \mathcal{O}(\kappa_0^2)$, which follows together with condition (iii), this is enough to ensure that all curvature invariants are bounded in a flat Friedmann universe.

(ii) GR-limit:

$$G(\kappa) = 1 + \mathcal{O}\left((\kappa/\kappa_0)^2\right), \qquad \overline{\Lambda}(\kappa) = \Lambda_0 + \mathcal{O}\left((\kappa/\kappa_0)^2\right)\kappa^2 \tag{43}$$

This condition ensures that in the low curvature regime we recover the usual Friedmann equation and modifications restrict to the limiting curvature regime. Moreover, the requirement that corrections to $\overline{\Lambda}$ appear only at order κ^4 means that in the low curvature limit perturbations in the scalar degree of freedom of mimetic gravity remain non-propagating, like in vanilla mimetic gravity.

(iii) No ghost instability of the scalar degree of freedom of mimetic gravity:

$$0 < \frac{\partial H^2}{\partial \varepsilon} < \frac{8\pi}{3} G(\kappa) \tag{44}$$

For a derivation of this condition, see [5].

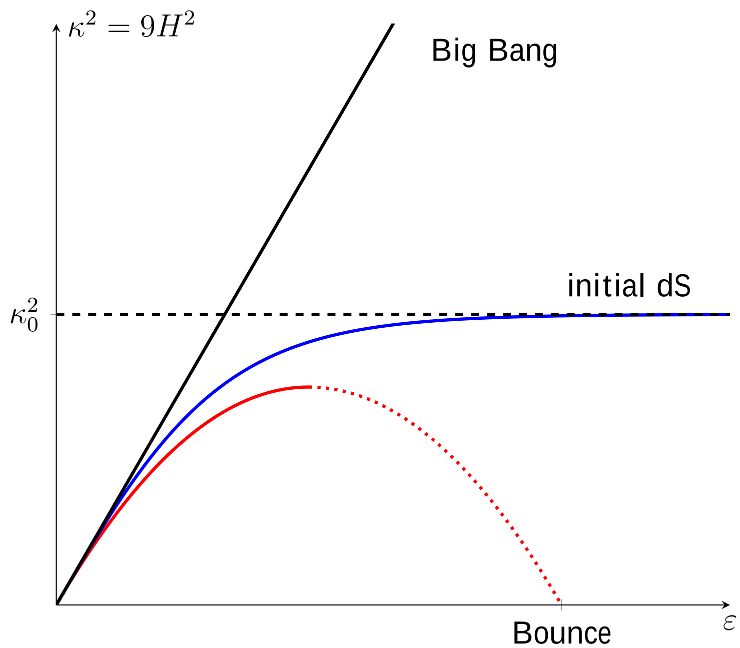


Figure 1: Modified Friedmann equation (39) as a relation between extrinsic curvature $\kappa^2 = 9H^2$ and energy density ε. At low curvatures, any modified relation has to reproduce the Friedmann equation, which is a straight line in this diagram. Following this straight line to arbitrarily high energy densities, one inevitably runs into a Big Bang singularity. Thus, this relation has to be modified at high curvatures to achieve a non-singular solution. Two options for this are: (a) the blue curve: the extrinsic curvature asymptotically approaches the constant limiting curvature as $\varepsilon \to 0$. Since asymptotically constant extrinsic curvature $\kappa \to \kappa_0$ corresponds to initially exponential expansion $a^3 \propto e^{\kappa_0 t}$, in this case the Big Bang is replaced by an initial asymptotic de Sitter spacetime at limiting curvature. (b) the red curve: the extrinsic curvature reaches a maximum after which it starts to decrease as ε continues to increase (dashed part) until it reaches a first order zero at finite ε, corresponding to a bounce. Note that the dashed part of the red curve violates condition (44), meaning that in this region the scalar degree of freedom is a ghost.

Note that condition (iii) could be replaced with the weaker condition of bijectivity of the relation $a(H^2)$, i.e. the first inequality $0 < \frac{\partial H^2}{\partial \varepsilon}$ in (44) is enough to arrive at the above conclusion. The condition of bijectivity could be violated by the modified Friedmann equation (39) only by allowing G or $\overline{\Lambda}$ to be a multivalued function, cf. [47]. Expressed as a condition on G and $\overline{\Lambda}$, the upper bound of the inequality (44) implies that

$$\frac{\mathrm{d}\ln G}{\mathrm{d}\kappa^2} < \frac{1}{(\frac{1}{3}\kappa^2 - \overline{\Lambda})} \frac{\mathrm{d}\overline{\Lambda}}{\mathrm{d}\kappa^2}. \tag{45}$$

In the case where $\overline{\Lambda}$ is constant, one can infer that the running gravitational constant G can only decrease with increasing curvature. In fact, in this case a limiting curvature modification, as realized by blue curve in figure 1, can only be achieved if the product $G\,\varepsilon$ remains bounded as $\varepsilon \to \infty$. This is only possible if $G(\kappa \to \kappa_0) \to 0$, a feature that we will refer to as “asymptotic freedom”.

For concreteness, we will present a concrete solution for the simple choice

$$G(\kappa) = 1 - \left(\frac{\kappa}{\kappa_0}\right)^2, \qquad \overline{\Lambda} = 0, \tag{46}$$

for which the modified Friedmann equation (47) becomes

$$\frac{H^2}{1 - H^2/H_l^2} = \frac{8\pi}{3}\varepsilon, \tag{47}$$

where $H_l = \kappa_0/3$ is the asymptotically constant value of the Hubble parameter during the initial inflationary stage. This is the same modification considered in [5], where linear perturbations around the background solutions of (47) have been analyzed. In the case where a single component with equation of state $p/\varepsilon = w$ is dominating the energy density $\varepsilon \propto a^{-3(1+w)}$, taking the time derivative of the logarithm of (47), one finds a separable differential equation for $H(t)$ with the implicit solution

$$\tfrac{3(1+w)}{2} H_l\, t = \frac{H_l}{H} - \mathrm{atanh}\frac{H}{H_l}. \tag{48}$$

The origin $t = 0$ of time was fixed to agree with the late time Friedmann asymptotic

$$H(t) \approx \frac{2}{3(1+w)\,t}, \qquad \text{at} \qquad t \gg t_f \tag{49}$$

Instead of reaching a singularity at $t = 0$, in the case of $H_l < \infty$ the solution can be extended all the way into the past until $t \to -\infty$ where the asymptotic (49) is now smoothly connected to the early time asymptotic

$$H(t) \approx H_l\left(1 - 2\exp\left[3(1+w)H_l t - 2\right]\right) \qquad \text{at} \quad t \ll t_f \tag{50}$$

Note that this early time solution is independent of the matter content, as long as $w > -1/3$. The initial stage of accelerated expansion ends at $t_f \sim \mathcal{O}(1)/H_l$, where $H = H_f = \sqrt{(1+3w)/(3(1+w))}H_l$. The exact solutions for $a(t)$ and $H(t)$ are plotted in figure 2.

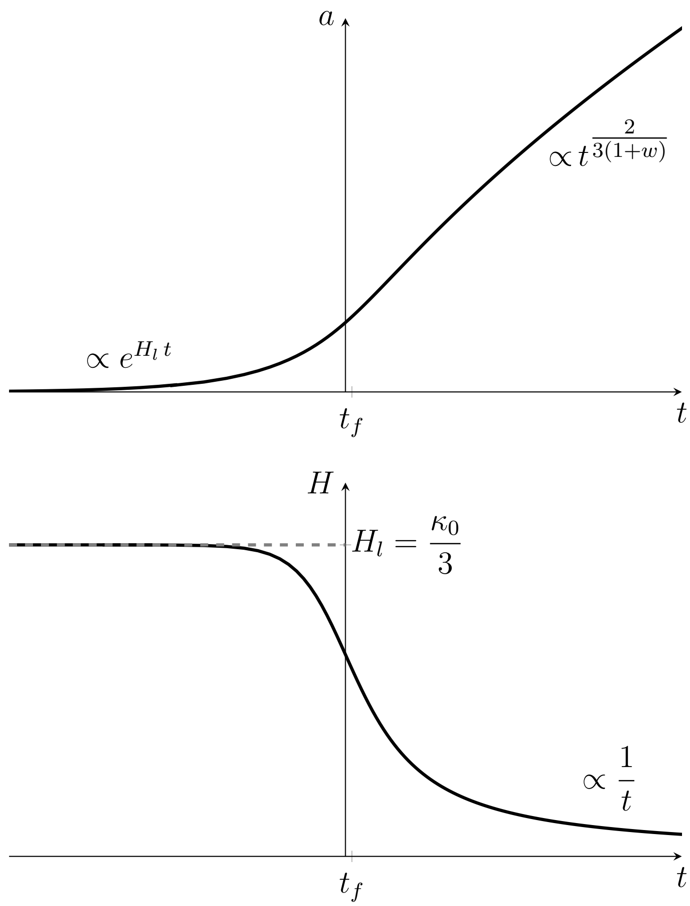


Figure 2: Plot of the expanding branch of the implicit solution (48) for $H(t)$ and the corresponding solution for $a(t)$ found from (47). The end of accelerated expansion happens at $t = t_f \sim \mathcal{O}(1)/H_l$.

Note that the initial inflationary part of this solution does not fit into a slow-roll description since even though

$$-\frac{\dot{H}}{H^2} = \frac{3(1+w)}{2} G(\kappa) \ll 1,$$

before the end of inflation, the second "slow-roll" parameter

$$\frac{\ddot{H}}{2H\dot{H}} = \frac{3(1+w)}{2}$$

is constant and of order of unity at all times.

0.3.2 Anisotropic singularity resolution requires asymptotic freedom

While in the last section "asymptotic freedom" as a means of realizing limiting curvature in a flat, isotropic universe was still only a choice, in this section we find that it becomes a necessity to resolve the anisotropic Kasner singularity. The reason for this can be traced back to the crucial appearance of f in the trace-free spatial modified Einstein equations (24). In GR, approaching an anisotropic singularity where the metric determinant $\gamma \to 0$, the contribution of anisotropies to the Einstein equation scales as $\tilde{\kappa}^i_j \tilde{\kappa}^j_i \sim 1/\gamma = 1/a^6$, where $a = (a_1 a_2 a_3)^{1/3}$ is the averaged scale factor. This result will be modified by a running $G(\kappa)$: Contracting (24) with $\tilde{\kappa}^j_i$ one finds that that

$$\frac{1}{2f\gamma}\partial_0\left[\gamma f^2 \tilde{\kappa}^i_j \tilde{\kappa}^j_i\right] = \tilde{\kappa}^i_j\,{}^3R^j_i + 8\pi \tilde{\kappa}^i_j T^j_i. \tag{51}$$

Assuming that in approach to a singularity spatial curvature and matter contributions are subdominant, the dominant contribution of anisotropy to the temporal modified Einstein equation (20) is hence $\tilde{\kappa}^i_j \tilde{\kappa}^j_i \sim G^2(\kappa)/\gamma = G^2(\kappa)/a^6$, where now G makes an appearance.

As an example to study the contraction of an initially anisotropic spacetime, consider the Bianchi type I metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \sum_{i=1}^{3} a_i^2(t)\mathrm{d}x_i^2, \tag{52}$$

again with the time coordinate $t = \phi$ coinciding with the splitting into homogeneous slices. The Kasner metric is the non-trivial vacuum solution of the Einstein equation given by $a_i = |t|^{p_i}$, where the Kasner exponents satisfy $p_1 + p_2 + p_3 = 1$, $p_1^2 + p_2^2 + p_3^2 = 1$. The contracting part of the solution is covered by $-\infty < t < 0$. Without modification, a curvature singularity occurs at $t = 0$, where the Kretschmann scalar $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = -16p_1p_2p_3/t^4$ diverges.

For the spatially flat metric ansatz (52) and assuming the case of vacuum (or, more generally, isotropic matter) the spatial modified Einstein equations (24) have the first integral (no sum over i)

$$\tilde{\kappa}^i_i = \frac{\dot{a}_i}{a_i} - \frac{\dot{a}}{a} = \left(\tfrac{1}{3} - p_i\right)\frac{G(\kappa)}{\sqrt{\gamma}}, \tag{53}$$

with the metric determinant $\sqrt{\gamma} = a_1 a_2 a_3 = a^3$. Constants of integration have been fixed to agree with the Kasner solution in the early time limit $t \to -\infty$.

The temporal modified Einstein equation (20) becomes

$$f\left(\frac{1}{3}\kappa^2 - \overline{\Lambda}\right) = \frac{1}{2}\left(f + \kappa f'\right)\tilde{\kappa}^i_j \tilde{\kappa}^j_i. \tag{54}$$

Using (53), it is again an integral curve in the (still only 2-dimensional) phase space spanned by $a = (a_1 a_2 a_3)^{1/3}$ and $\kappa = 3\frac{\dot{a}}{a}$. The same considerations from the last section apply (replacing $\varepsilon \mapsto 1/\gamma$ in figure 1), and in order for κ^2 to be bounded, it must hold that

$\kappa^2 \to \kappa_0^2$ as $\gamma \to 0$. In a flat, isotropic universe all there is to curvature is extrinsic curvature κ and this was enough to ensure a non-singular solution. However, now we must also require that

$$\tilde{\kappa}^i_j \tilde{\kappa}^j_i = \left(\frac{2G(\kappa)}{3\sqrt{\gamma}}\right)^2 < \kappa_0^2. \tag{55}$$

This is only possible if G decreases at least like $G \sim \sqrt{\gamma}$ as $\sqrt{\gamma} \to 0$ and $\kappa^2 \to \kappa_0^2$. Moreover, if G decreases faster than this, the resulting solution will eventually become isotropic during contraction at limiting curvature due to asymptotic freedom. In this way the anisotropic singularity at $t = 0$ is resolved similar to the isotropic singularity in the last section. The solution can be extended to $t \to \infty$ where all curvature invariants remain bounded and we find again an asymptotic approach to a piece of de Sitter spacetime. In [1] it was shown how to obtain explicit solutions of the modified Einstein equations where this is realized. For the particular choice

$$G(\kappa) = 1 - \left(\frac{\kappa}{\kappa_0}\right)^2, \qquad \overline{\Lambda} = -\frac{1}{3}\kappa^2 \left(\frac{\kappa}{\kappa_0}\right)^2 \tag{56}$$

one can find a simple exact solution of the modified Einstein equation: The contracting branch of the implicit solution for $a(t) = (a_1(t)a_2(t)a_3(t))^{1/3}$ is given by

$$-\kappa_0\, t = \sqrt{1 + \kappa_0^2\, a^6(t)} - \text{acoth}\sqrt{1 + \kappa_0^2\, a^6(t)} \tag{57}$$

and the time dependence of the individual scale factors $a_i(t)$ for every direction can be expressed through $a(t)$ as

$$a_i(t) = a(t) \left(\frac{2}{a^3(t) + \sqrt{1 + \kappa_0^2\, a^6(t)}}\right)^{\frac{1}{3} - p_i}. \tag{58}$$

In the limit $t \to \infty$ the second factor in (58) quickly becomes constant and all three a_i become proportional to $a(t) \approx e^{-H_l t}$. Figure 3 shows a plot of this exact solution for the particular Kasner exponents $p_1 = (1 + \sqrt{3})/3$, $p_2 = 1/3$, $p_3 = (1 - \sqrt{3})/3$.

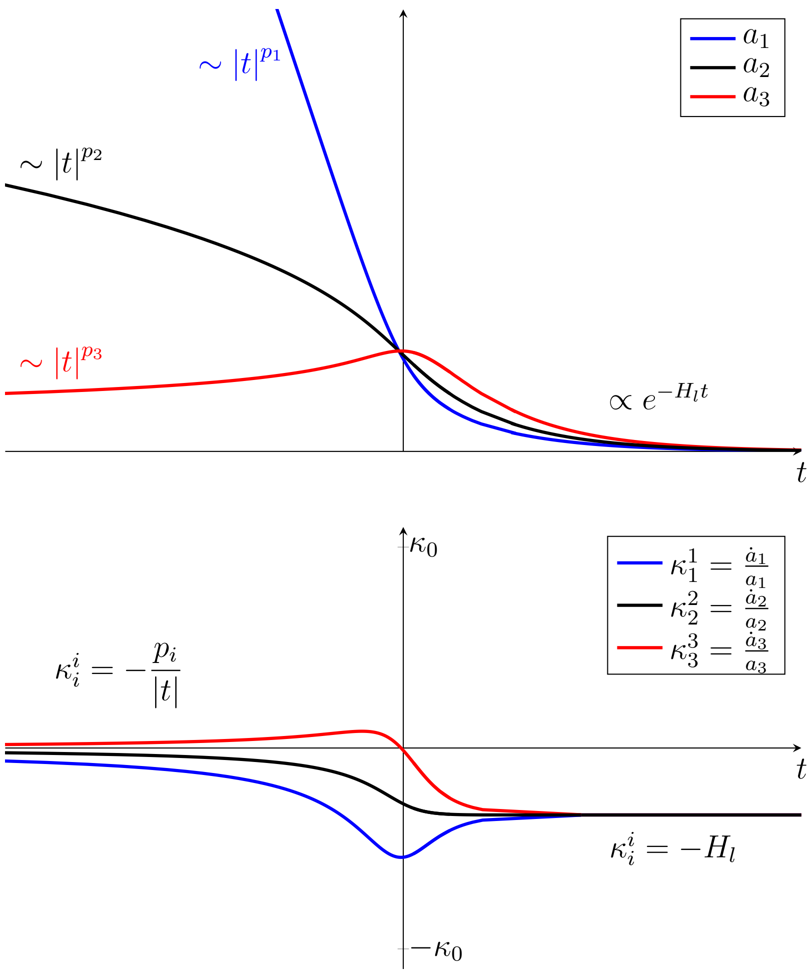


Figure 3: Exemplary modified contracting Kasner solution with $p_1 = (1+\sqrt{3})/3$, $p_2 = 1/3$, $p_3 = (1-\sqrt{3})/3$ for the modification (56). (The index in κ_i^i is not summed.)

0.3.3 Modified black hole with stable remnant

This section is dedicated to the application of the theory (8) to black hole spacetimes. First we examine the qualitative behaviour of non-rotating black holes in the spatial flatness approximation and find that it is analogous to the results of the last section. As a proof of principle that it is possible to obtain a concrete solution in which the spatial flatness approximation is actually exact, we then proceed to review the exact solution first published in [3]. Staying true to the storyline from the last sections, this regular black hole replaces the singularity by a transition to a static de Sitter patch at limiting curvature. The internal structure of this solution is significantly simpler than the earlier limiting curvature mimetic model [43] and thus allows to actually show statements that had to remain conjectures in [43]. For example, the exact solution permits a simple analysis of modified black hole thermodynamics. Extensions of this solution to the case of small charges and slow rotation have been presented in [4].

For an overview of the many existing non-singular black hole modifications see e.g. [62, 63] and references therein. Regular, non-rotating black holes with de Sitter interiors have been studied for a long time, e.g. in [15, 64–66], also in the context of asymptotically free gravity [14]. In contrast to most of these non-singular black holes where the transition to a non-singular interior occurs through a singular hypersurface, subject to the Israel conditions for surface layers, in our model the transition happens gradually and smoothly. Moreover, our modified black hole is based on a concrete action rather than mere assumption of the existence of limiting curvature. While some of our conclusions do agree with renormalization group approaches to asymptotically free gravity, like [67, 68], it is important to point out the distinction that in these works the "renormalization group improved" running gravitational constant is directly inserted into the otherwise unchanged Schwarzschild solution as $G(r)$, without presenting a concrete theory that would admit such a solution.

To find how our modified gravity will affect non-rotating black holes, we look for the high curvature modification of the Schwarzschild metric, expressed in Schwarzschild coordinates t, r, ϑ, φ as

$$\mathrm{d}s^2 = \left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2, \tag{59}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2$, the physical singularity occurs at $r = 0$ and the coordinates are singular at the horizon $r = 2M$. The same metric, expressed in Lemaître coordinates T, R, ϑ, φ with range $0 < x = R - T < \infty$ reads

$$\mathrm{d}s^2 = \mathrm{d}T^2 - \left(\frac{x}{x_+}\right)^{-2/3}\mathrm{d}R^2 - \left(\frac{x}{x_+}\right)^{4/3}(2M)^2\mathrm{d}\Omega^2. \tag{60}$$

The Lemaître coordinates are regular at the horizon which happens at $x = x_+ = 4M/3$ and the physical singularity occurs at $x = 0$. In the Schwarzschild spacetime the only global solution of the mimetic constraint (12) which respects the same symmetries is, up to shifts, $\phi = T$. This is the reason why Lemaître coordinates are best suited for the following investigation. The relation between Lemaître coordinates and Schwarzschild coordinates can be found in [3, 4].

To generalize (60), consider the spherically symmetric ansatz with the time coordinate $T = \phi$,

$$\mathrm{d}s^2 = \mathrm{d}T^2 - a^2(x)\,\mathrm{d}R^2 - b^2(x)\,\mathrm{d}\Omega^2, \tag{61}$$

where the functions a and b still depend only on $x = R - T$. Like the Schwarzschild metric, (61) possesses the additional Killing vector field

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial R} + \frac{\partial}{\partial T}. \tag{62}$$

The norm of the Killing vector field (62) changes sign wherever $a^2(x) = 1$, marking a Killing horizon with surface gravity $g_s = -a'(x) \equiv -\mathrm{d}a/\mathrm{d}x$.

The condition for spatial flatness of the constant T slices of (61) amounts to the single equation $b'(x) \equiv \mathrm{d}b/\mathrm{d}x = a(x)$. Note that that the Schwarzschild metric (60) is spatially flat in Lemaître coordinates. Suppose this property will continue to hold true, at least approximately, for some time after the high curvature modification has taken over. In this case, the modified Einstein equations (20-24) for the functions $a(x)$ and $b(x)$ become formally identical to the ones for $a_1(t)$ and $a_2(t) = a_3(t)$ in a contracting Kasner universe with exponents $p_1 = -1/3$, $p_2 = p_3 = 2/3$, if one interchanges the Kasner time coordinate with $t \leftrightarrow x$. Hence, the same asymptotically free models that lead to asymptotic isotropy during contraction at limiting curvature in a Bianchi type I universe, will cause the metric functions a and b to become alike, i.e. $a(x) \propto b(x)$, after the high curvature modification has kicked in. The solution can then be extended until $x \to -\infty$ and smoothly connects the asymptotics

$$a(x) \approx \begin{cases} \left(\frac{x}{x_+}\right)^{-1/3} & x \gg x_* \\ a_0 \exp\left(\frac{\kappa_0}{3} x\right) & x \ll x_* \end{cases}, \qquad b(x) \approx \begin{cases} \left(\frac{x}{x_+}\right)^{2/3} 2M & x \gg x_* \\ b_0 \exp\left(\frac{\kappa_0}{3} x\right) & x \ll x_* \end{cases} \tag{63}$$

where $x_* \sim \kappa_0^{-1}$. Note that since the Schwarzschild radial coordinate is related to x by $r = b(x)$, the extension of the solution to $x \to -\infty$ corresponds to $r \to 0$.

If a modification satisfies $a_0/b_0 = \kappa_0/3$, then the solution is spatially flat also in the $x \ll x_*$ asymptotic and the spatial flatness approximation is valid everywhere. Moreover, in this case the solution in the far interior is given by a static patch of de Sitter spacetime for which all curvature invariants are bounded by the limiting curvature scale.

On the other hand, in general if $a_0/b_0 \neq \kappa_0/3$, then in the $x \ll x_*$ asymptotic the spatial curvature components scale as $\exp(-\frac{2}{3}\kappa_0 x)$. One can explicitly verify that even in this case spatial curvature is still a subdominant contribution to the modified Einstein equation. Hence, the spatial flatness approximation and the following conclusions about horizon geometry still remain valid to leading order. However, the curvature singularity would not be resolved in this case, as ${}^3R$ diverges.

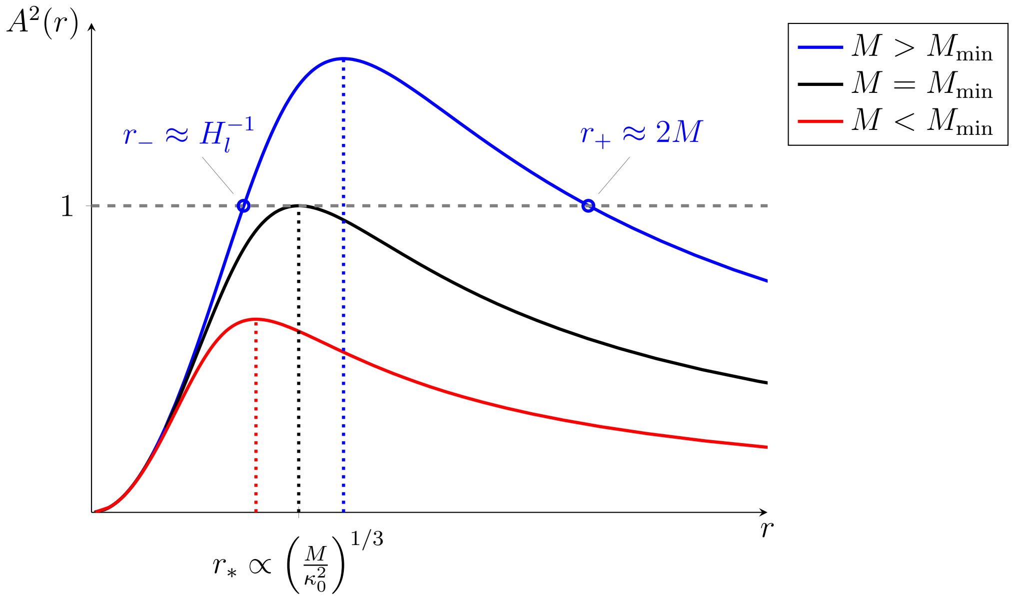


Figure 4: Plot of the exact solution from [3] expressed through $A(r) = a(x(r))$ in Schwarzschild coordinates $r = b(x)$. A Killing horizon occurs when $A^2(r) = 1$ and for this spatially flat solution it has the surface gravity $g_S = -\frac{1}{2}\frac{\mathrm{d}A^2}{\mathrm{d}r}$.

Transitioning between the asymptotics $a \propto b^{-1/2} = r^{-1/2}$ at $x \gg x_*$ and $a \propto b = r$ at $x \ll x_*$ means that at some point in the intermediate region $x \sim x_*$, where the modification takes over, the metric function a has to assume a maximum. The scale of this maximum is set by the mass of the black hole. Thus, a minimal mass $M_{\rm min} \sim \kappa_0^{-1}$ arises which divides the causal structure of solutions into three distinct classes:

$M > M_{\rm min}$ **:** There are two horizons, one on each side of the maximum, where $a(x_\pm) = 1$.

$M = M_{\rm min}$ **:** There is one degenerate horizon at $a(x_*) = 1$ with vanishing surface gravity $g_s = -a'(x_*) = 0$.

$M < M_{\rm min}$ **:** There is no horizon, the Killing vector field (62) is everywhere timelike and hence this solution does not describe a black hole.

Of course it would be troubling if during the process of Hawking evaporation horizons would disappear and the spacetime would switch between these different causal structures. Since the Hawking temperature $T_H = g_s/2\pi$ is given by the surface gravity of the exterior horizon which vanishes for the minimal black hole we can already guess that this will not happen.

Exact solution. The above conclusions have been derived in the spatial flatness approximation. While the statements about the modified horizon geometry are generic for asymptotically free models with a fast enough vanishing of G, the singularity resolution did rely on the special property $a_0/b_0 = \kappa_0/3$. As shown in [3], not only can we find a modification such that this condition is satisfied, it is even possible to obtain a solution which satisfies $b'(x) = a(x)$ everywhere. In this case spatial flatness is not an approximation and the solution is exact. The concrete modified theory for which this was done is given by

$$G(\kappa) = \frac{1+(\kappa/\kappa_0)^2}{1+3\,(\kappa/\kappa_0)^2}\left(1-(\kappa/\kappa_0)^2\right)^2, \tag{64}$$

$$\overline{\Lambda}(\kappa) = \tfrac{4}{3}\kappa^2(\kappa/\kappa_0)^4\frac{9+14(\kappa/\kappa_0)^2+9(\kappa/\kappa_0)^4}{(1+(\kappa/\kappa_0)^2)\,(1+3(\kappa/\kappa_0)^2)^3}. \tag{65}$$

See [4] for details. In Schwarzschild coordinates the spatially flat exact solution takes the form

$$\mathrm{d}s^2 = \left(1-A^2\left(r\right)\right)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1-A^2\left(r\right)} - r^2\mathrm{d}\Omega^2, \tag{66}$$

where $A(r) := a(x(r))$ smoothly interpolates between the asymptotics

$$A^2(r) = \begin{cases} \dfrac{2M}{r}\left[1-\frac{5}{16}(r_*/r)^3+\mathcal{O}\left((r_*/r)^6\right)\right], & r \gg r_* \\ (H_l\,r)^2\left[1-\frac{4}{5}(r/r_*)^3+\mathcal{O}\left((r/r_*)^6\right)\right], & r \ll r_* \end{cases} \tag{67}$$

Here $H_l = \kappa_0/3$ is the Hubble rate of the asymptotic de Sitter patch at limiting curvature, and the location of the maximum of $A(r)$ in Schwarzschild coordinates is

$$r_* = 2\left(\frac{18M}{5\kappa_0^2}\right)^{1/3}. \tag{68}$$

The minimal black hole mass of this solution is given by

$$M_{\min} = \frac{5^{5/2}}{18\,\kappa_0}. \tag{69}$$

The full solution for $A^2(r)$ is plotted in figure 4. The conformal diagrams for the three di erent causal structures can be found in [4] in figure 4.2 and the maximally extend conformal diagram for the eternal black hole solution with $M > M_{\min}$ is presented in figure 4.3. While superficially looking similar to the conformal diagram of a Reissner-Nordström or Kerr metric, the most important di erences are the facts that there is no singularity at $r = 0$ and that the interior region (IIa) is similar to a static de Sitter patch such that the inner horizon r_- is not a Cauchy horizon but rather like a de Sitter horizon.

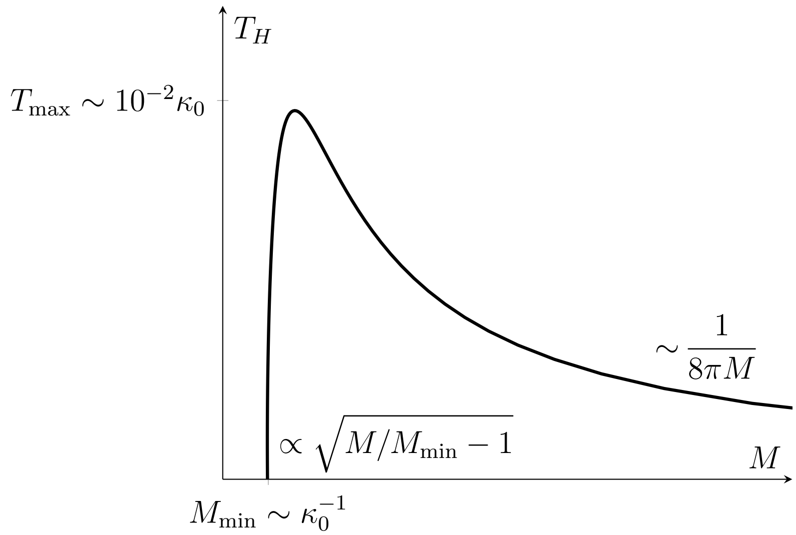


Figure 5: Modified Hawking formula (3.26, 3.27) for the spatially flat exact solution from section 3.5.

The modified Hawking formula following from this solution is described by the implicit curve (3.26, 3.27), plotted in figure 5. It has the asymptotics

$$T_H \approx \begin{cases} \dfrac{1}{8\pi M}\left[1 - \frac{3^6}{5^5}\left(M_{\min}/M\right)^2\right], & M \gg M_{\min} \\ \dfrac{\kappa_0}{2\sqrt{21}\pi}\sqrt{M/M_{\min} - 1}, & M \gtrsim M_{\min} \end{cases} \tag{70}$$

The exterior of a black hole with large initial mass $M_0 \gg M_{\min}$ is still described by the Schwarzschild metric and we reproduce the familiar Hawking formula $T_H = \frac{1}{8\pi M}$ in the large mass limit. After evaporating for a time $t_l \sim M_0^3$, the mass will have reached scales of minimal mass $M \gtrsim M_{\min}$ and the behaviour drastically changes: After reaching a maximal temperature $T_{\mathsf{max}} \sim 10^{-2}\kappa_0$ at the critical mass $M = M_c \approx 1.32 M_{\min}$ where the negative "heat capacity" $\partial M/\partial T$ diverges and changes sign, instead of diverging as $M \to 0$, the temperature vanishes as $M \to M_{\min}$. Eventually, using the Stefan-Boltzmann law $\frac{\mathrm{d}M}{\mathrm{d}t} \propto -T_H^4 A$ for the rate of energy loss of a radiating body, where $A = 4\pi r_+^2$ is the horizon area, the mass will asymptotically approach the minimal mass according to

$$M(t) - M_{\min} \propto \frac{1}{t}. \tag{71}$$

This means that the minimal mass and zero temperature can only be reached after infinite time. This statement can be seen as a third law of black hole thermodynamics. Interestingly, the exact solution also satisfies the following modified first law

$$G(\kappa_+)\,\mathrm{d}M = T_H \mathrm{d}S, \tag{72}$$

where $G(\kappa_+)$ is evaluated at the exterior horizon and the Bekenstein entropy is given by the horizon area as $S = A/4$.

In summary, we have presented a concrete theory where asymptotic freedom of gravity on a classical level resolves the singularity at the "center" of a black hole and replaces it by a static patch of de Sitter spacetime. This leads to the existence of a minimal black hole mass given by the inverse limiting curvature scale. The final product of evaporation is a remnant of minimal mass with vanishing temperature of Hawking radiation. The causal structure of non-minmal black holes features two horizons, a Schwarzschild horizon and a de Sitter horizon. Unlike the interior horizon of Reissner-Nordström and Kerr solutions, which is known to be the source of instabilities [69], this interior horizon is not a Cauchy horizon. The termination of evaporation and formation of a remnant is maybe the simplest resolution of the so-called information paradox, cf. [70]. In this case all information that falls into the black hole will remain inaccessible to exterior observers forever.

Slow rotation. In [4] the above exact solution was extended to rotating black holes to first order in angular momentum $j := J/M$. It was found that in this case the solution (66) in Schwarzschild coordinates gets corrected by a frame dragging term which takes the form

$$\mathrm{d}s^2 = \left(1 - A^2\left(r\right)\right)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 - A^2\left(r\right)} - r^2\mathrm{d}\Omega^2 + 2jA^2(r)\sin^2\vartheta\,\mathrm{d}t\mathrm{d}\varphi, \tag{73}$$

where $A^2(r)$ is the same function (67) as in the non-rotating case. In the case $\kappa_0 \to \infty$ or in the limit $r \gg r_*$ it holds that $A^2(r) = 2M/r$ and (73) is identical to the fist order expansion of the Kerr metric. It is a non-trivial result that the frame dragging function is still given by A^2, also in the modified theory.

The frame dragging function reaches its maximum at $r = r_*$ and then decreases before vanishing at $r = 0$. This suggests that the perturbative analysis in angular momentum is justified for the whole range of r provided that $j \ll 1$. Moreover, the spacetime close to $r = 0$ becomes similar to the non-rotating case and the singularity is resolved in the same way. All curvature invariants are bounded at $r = 0$,

$$R = 4H_l^2\left(H_l^2 j^2 - 3\right) + \mathcal{O}(r^2) \tag{74}$$

$$R^{\mu\nu}R_{\mu\nu} = 6H_l^4\left(H_l^4 j^4 - 4H_l^2 j^2 + 6\right) + \mathcal{O}(r^2) \tag{75}$$

$$R^{\mu\nu\sigma\lambda}R_{\mu\nu\sigma\lambda} = 8H_l^4\left(H_l^4 j^4 - 2H_l^2 j^2 + 3\right) + \mathcal{O}(r^2) \tag{76}$$

Note that to first order in $j = J/M$ the norm of the Killing vector field (62) is still given by $a^2 - 1$. Moreover, for the surface gravity it holds that

$$g_s = -\frac{1}{2}\frac{\mathrm{d}A^2}{\mathrm{d}r} + \mathcal{O}(j^2). \tag{77}$$

This shows that our above conclusions are robust even for slowly rotating black holes.

0.3.4 Higher spatial derivatives avert gradient instability

The scalar degree of freedom of vanilla mimetic gravity is dust-like and perturbations are non-propagating. It has been known for a while that minimal extensions of the model to make it dynamical, like addition of a term $\gamma(\Box\phi)^2$, will either lead to a ghost or a gradient instability [71]. In [58] the shift symmetry breaking theory $\widetilde{\mathcal{L}} = f(\Box\phi)R + \Lambda(\Box\phi) + U(\phi)$ with higher mixed derivatives in its modified Einstein equation has been shown to allow the adjustment of prefactors of both kinetic and gradient terms in a non-trivial way and thus linear instabilities can be prevented, at least in some regions. However, not only does this theory possess a hidden second scalar degree of freedom [59, 60], in the shift-symmetric case it even does not serve its purpose to prevent a gradient instability, because there is an instability induced by a higher order k^4 term, as shown in [5]. As a last resort for a stable, shift-symmetric theory, we explore the possibility to avert the gradient instability by inclusion of higher order spatial derivative terms.

In the following we will consider cosmological perturbations around the flat Friedmann background solutions described above in section 0.3.1 and review the linear stability analysis for the theory (8). The analysis of metric perturbations was performed in comoving gauge $\delta\phi = 0$, which for the background solution $\phi = t$ is equivalent to unitary gauge, as follows from the mimetic constraint (12). Note that “comoving” here refers to comoving with mimetic matter, i.e. the gauge in which the mimetic field is kept homogeneous. Vector and tensor perturbations do not cause any instabilities. While the propagation speed of gravitational waves in this theory deviates from the speed of light in the high curvature regime, already $t \geq 1\,\text{sec}$ after the end of the inflationary stage

$$1 - c_T \lesssim \left(\frac{\kappa_{pl}}{\kappa_0}\right)^2 \left(\frac{1\,\text{sec}}{t}\right)^2 \times 10^{-86}. \tag{78}$$

Late time experimental constraints $1 - c_T \lesssim \mathcal{O}(10^{-15})$ from multi-messenger events like GW170817 [72, 73] that happened around $t \sim 10^9$ years hence do not put any significant restrictions on the limiting curvature scale κ_0.

Let us now focus on the scalar degree of freedom of pure mimetic gravity. Expanding the action (7) to second order in the gauge invariant comoving curvature perturbation ζ, we find the second order action

$${}^{(2)}\mathcal{S}_\zeta = \frac{1}{8\pi}\int \mathrm{d}t\,\mathrm{d}^3x\,a^3\left\{\frac{3\ell'' f}{\ell'' - f}\,\dot{\zeta}^2 + \frac{1}{a^2}\,(\partial\zeta)^2 - \frac{\sigma_S^4}{a^6}\,\left(\partial^3\zeta\right)^2\right\}. \tag{79}$$

Here we considered only the scalar degree of freedom of mimetic gravity and matter fluctuations have been neglected. For details, see [5].

In the case $\sigma_S = 0$, i.e. without the addition of the higher spatial curvature term, we would read off the speed of sound

$$c_S^2 = -\frac{\ell'' - f}{3\ell'' f} \stackrel{*}{=} -\frac{G(1-G)}{3}, \tag{80}$$

where in the last equality the particular choice (46) was inserted. We will continue to use the name c_S^2 to refer to this quantity also in the case $\sigma_S \neq 0$, but it is important to note that in this case, due to the modified dispersion relation, the true speed of sound is in general different from c_S^2. The condition $c_S^2 < 0$ to avoid a ghost instability was discussed in section 0.3.1 and it is always satisfied for asymptotically free models.

To find the analogue of the Mukhanov-Sasaki variable in this theory, introduce the time coordinate τ and the expression z defined by

$$\mathrm{d}t = \frac{a}{\sqrt{-c_S^2}}\,\mathrm{d}\tau, \qquad z = \frac{a}{\sqrt{4\pi}}\left(-c_S^2\right)^{-1/4}. \tag{81}$$

The second order action written in terms of $v = z\,\zeta$ is then canonically normalized and performing a Fourier transformation, we find the modified Mukhanov-Sasaki equation

$$v_{k,\tau\tau} + \left(\frac{\sigma_S^4}{a^4}k^6 - k^2 - \frac{z_{\tau\tau}}{z}\right) v_k = 0. \tag{82}$$

As was to be expected, the k^2 term has the wrong sign that would lead to a gradient instability. However, in the case $\sigma_S > 0$ the gradient term is dominated by the k^6 term as long as the physical wavelength $\lambda_{\text{phys}} = a/k \ll \sigma_S$, which initially as $a \to 0$ is satisfied for all modes. On the other hand, in the late time limit the speed of sound (80) is vanishing as $G \to 1$ and the scalar degree of freedom becomes similar to the dust-like degree of freedom of vanilla mimetic gravity. This means that eventually, after a mode exits the “horizon”, i.e. after the term $z_{\tau\tau}/z$ is dominating (82), the mode will remain frozen forever and never re-enter the horizon. Thus, the region where the wrong sign k^2 term can lead to an instability due to exponential growth of modes according to $v \propto e^{k\tau}$ only lasts for a finite time and is sandwiched between regions without instabilities. As long as the duration $\Delta\tau_k$ of the gradient instability region for a mode k satisfies $k\Delta\tau_k \ll 1$, the mode cannot grow by any significant amount during this time. As found in [5], the condition to avoid a gradient instability in this way is $\sigma_S \gg 1/H_l$.

Figures 6 and 7 illustrate the different regions of domination of terms in (82) by plotting their dependence on the scale factor a for the two special cases of a background solution dominated by radiation and dust/mimetic dark matter, respectively.

The solution of (82) can be estimated by matching leading order solutions for the mode functions at the transition between regions of domination of different terms. Even though initially the solutions with quantum initial conditions determined from the k^6 term exhibit a scale invariant spectrum, one has to expect that the primordial spectra after “horizon” exit are in general far from scale invariant, because of the rapidly varying speed of sound at “horizon” exit. This shows that the endeavour to construct an inflationary scenario without inflaton fails in the case where the universe before the end of inflation is filled with only mimetic matter and nothing else. In this case the assumption to ignore matter fluctuations is exact and the long wavelength part of primordial spectra has a large blue tilt, see [5]. In contrast, any other spectator field, like a massless scalar, would still acquire a nearly scale invariant spectrum in the inflationary background. While this could lead one to speculate

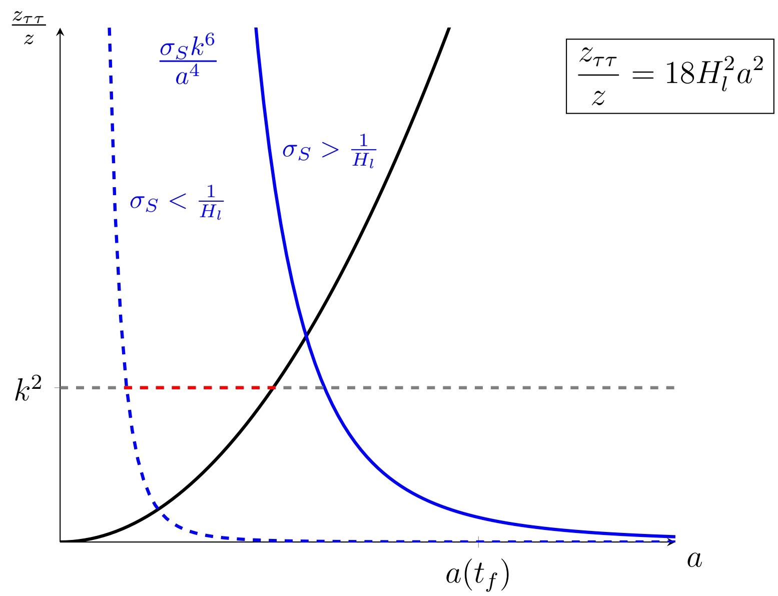
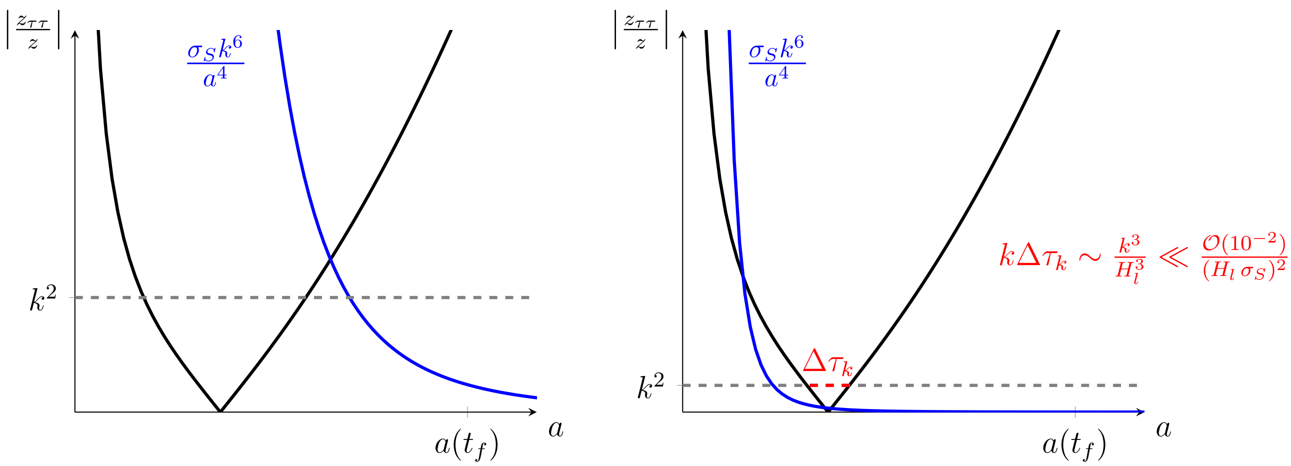


Figure 6: Comparison of terms in the modified Mukhanov-Sasaki equation (82) in a radiation dominated background. Provided that $\sigma_S \gg 1/H_l$ the "horizon exit" happens before the wrong sign gradient term gets any chance to cause an instability. Modes do not re-enter the horizon even after the end of inflation at t_f.

that a "curvaton"-like (cf. [74, 75]) extension of the asymptotically free mimetic model can likely be made to produce the correct spectra, this direction was not further pursued in this thesis.

It is clear that matter fluctuations and possibly also non-trivial matter couplings have to be included in any effort towards a more realistic model. Even if the primordial spectra would have come out right in a single component model, there would still be a gap: to explain how to transfer the nearly scale invariant spectrum from the dust-like mimetic degree of freedom to matter degrees of freedom. Note that this same gap has also been left unfilled by the shift-symmetry breaking models [58, 76] that have claimed to provide viable scenarios of "mimetic inflation".

$$\boxed{\frac{z_{\tau\tau}}{z} = \frac{3}{32} H_l^2 \left(154\, a^2 - \frac{1}{a} \right)}$$



(a) $k^2 \gg \mathcal{O}(10^{-1})(H_l/\sigma_S^2)^{2/3}$

(b) $k^2 \ll \mathcal{O}(10^{-1})(H_l/\sigma_S^2)^{2/3}$

Figure 7: Comparison of terms in the modified Mukhanov-Sasaki equation (82) in the case of a dust/mimetic matter dominated background, assuming $\sigma_S > 1/H_l$. Compared to the radiation dominated case, the situation is complicated by the initial divergence of $z_{\tau\tau}/z \sim 1/a$. For short wavelength modes $k^2 \gg \mathcal{O}(10^{-1})(H_l/\sigma_S^2)^{2/3}$ [subfigure (a)] the gradient instability is prevented just like in the radiation dominated case. However, for long wavelength modes $k^2 \ll \mathcal{O}(10^{-1})(H_l/\sigma_S^2)^{2/3}$ [subfigure (b)] there is an earlier intermediate super-“horizon” region followed by a gradient instability region (red) of duration $\Delta\tau_k$ before finally exiting the “horizon”. Provided that $\sigma_S \gg 1/H_l$, the duration of the gradient instability region is too short to lead to any actual instability.

0.4 Outlook

In this work we introduced a novel model of mimetic gravity as a candidate for a concrete realization of the idea of limiting curvature. On the level of background solutions, the goal of singularity resolution has been achieved in a variety of cosmological and black hole spacetimes. In comparison to other approaches to modified gravity, a remarkable feature of the mimetic model is its simplicity: the modified Einstein equations greatly simplify in backgrounds where a first integral can be found for the contribution of mimetic dark matter and the mimetic degree of freedom becomes trivial. This is also mirrored by the fact that exact solutions have been obtainable for most cases of interest studied in this thesis.

However, departing from highly symmetric backgrounds, one realizes that the mimetic degree of freedom is quite different from a run-of-the-mill scalar field: While not propagating at all in the low curvature GR-limit, without further measures, mimetic models going beyond vanilla mimetic gravity typically exhibit a wrong sign gradient term in cosmological perturbations. The first models that were introduced to treat this issue included higher couplings with spacetime curvature invariants. In these models, however, a hidden second scalar degree of freedom appears when perturbing around non-homogeneous mimetic field profiles and brings about new stability problems. In this work we explored a rather unconventional way to avert the exponential growth of modes that would be caused by the wrong sign gradient term: the addition of higher order couplings of *spatial* curvature invariants, which had already been introduced for the purpose of renormalizability in the context of mimetic Hořava gravity. Indeed, a modified dispersion relation including a k^6 term also for the scalar degree of freedom will sandwich the potential gradient instability region between two regions without instabilities and even allow for a region of parameterspace without any instability. To rigorously establish stability or instability of the theory, however, will require further work, starting with a full Hamiltonian analysis to check also for additional, hidden scalar degrees of freedom. This task remains a straightforward, but calculationally challenging open problem. Should this attempt at a simple, stable, shift-symmetric mimetic model fail, either the search for higher order couplings with non-pathological additional degrees of freedom would have to continue or one would have to settle for shift-symmetry breaking models. Another interesting candidate for a classical limiting curvature theory that so far was able to pass a number of stability tests could be found in Cuscuton gravity [77, 78].

Even though it was not our original intent to use this theory to build yet another inflationary scenario, an initial de Sitter stage was found to be a natural outcome and thus we felt obliged to answer the question whether this could provide a simple model of inflation without inflaton. The answer we found for the single component model that was analysed is negative. If one did not want to give up so easily on constructing an inflationary model, a curvaton-like extension of the single component model should be a promising direction.

Finally, another open question of great interest is the extension (likely only possible numerically) of our conclusions about regular black holes and remnants to the case of non-eternal black holes.

Paper 1

Asymptotically Free Mimetic Gravity*

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Abstract

The idea of "asymptotically free" gravity is implemented using a constrained mimetic scalar field. The effective gravitational constant is assumed to vanish at some limiting curvature. As a result singularities in spatially flat Friedmann and Kasner universes are avoided. Instead, the solutions in both cases approach a de Sitter metric with limiting curvature. We show that quantum metric fluctuations vanish when this limiting curvature is approached.

*published in: [1] A. H. Chamseddine, V. Mukhanov and T. B. Russ, *Asymptotically Free Mimetic Gravity,* Eur. Phys. J. **C79** (2019) 558. DOI: 10.1140/epjc/s10052-019-7075-y.

1.1 Introduction

In [35] mimetic matter was introduced utilizing reparametrization of the physical metric $g_{\mu\nu}$ in terms of an auxiliary metric $h_{\mu\nu}$ and a scalar field ϕ in the form

$$g_{\mu\nu} = h_{\mu\nu} h^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} \tag{1.1}$$

This definition implies that ϕ identically satisfies

$$g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} = 1. \tag{1.2}$$

Because the physical metric is invariant under Weyl transformations of $h_{\mu\nu}$, the trace of the equations obtained by variation of the Einstein action with respect to the metric vanishes identically. In the absence of matter these equations become

$$G^{\mu}_{\nu} - G\phi^{,\mu} \phi_{,\nu} = 0, \tag{1.3}$$

where $G^{\mu}_{\nu} = R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu} R$ is the Einstein tensor, and they do not imply that $R = 0$ even in vacuum. Therefore, equation (1.3) taken together with (1.2) has additional solutions imitating dust-like cold dark matter. The scalar field ϕ satisfies a first order differential equation (1.2) and hence has half a degree of freedom which, when combined with the non-dynamical longitudinal mode of gravity, provides an extra degree of freedom in the form of mimetic “dust”. Equivalently, the same theory is obtained by implementing equation (1.2) as a constraint added to the Einstein action:

$$S = \frac{1}{2} \int \mathrm{d}^4 x \sqrt{-g} \left(-\frac{1}{8\pi G} R + \lambda \left(g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - 1 \right) \right), \tag{1.4}$$

where λ is a Lagrange multiplier [37]. Unexpectedly, the concept of a mimetic field got a support in noncommutative geometry as a consequence of the volume quantization of compact three dimensional foliations of space time [79],[80],[81]. The mimetic field ϕ proved to be very robust. It could be used to modify Einstein Gravity in different possible ways. In particular, in [61] it was shown that adding appropriate potentials $V(\phi)$ to the action leads to many interesting cosmological solutions. Using instead gravity modification of the Born-Infeld type, where $\Box\phi$ is bounded by a limiting value, allowed to obtain bouncing solutions avoiding cosmological singularities [42] and to resolve black hole singularities [43]. Moreover, one can use the mimetic field to easily construct ghost free massive gravity with *non* Fierz-Pauli mass term [82],[83].

In this paper we will explore the possibility of a running gravitational constant assuming that it depends on $\Box\phi$, that is, $G = G(\Box\phi)$. As we shall see, this quantity is the only measure of curvature G can depend on without introducing higher time derivatives in the modified Einstein equation. Assuming that G vanishes at some limiting curvature characterized by $(\Box\phi)^2_L$ we will implement in this way the idea of “asymptotic freedom” for gravity and investigate its possible consequences.

1.2 Action and equations of motion

Let us consider the theory with action

$$S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(-f\left(\Box\phi\right)R - 2\Lambda\left(\Box\phi\right) + \lambda\left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - 1\right) + 2\mathcal{L}_m\right), \tag{1.5}$$

where

$$f\left(\Box\phi\right) = \frac{1}{8\pi G\left(\Box\phi\right)} \tag{1.6}$$

is the inverse running gravitational constant, $\mathcal{L}_m$ is the matter Lagrangian and for generality we also included a "cosmological-like term" $\Lambda\left(\Box\phi\right)$ †. Below we will use Planck units setting $8\pi G\left(\Box\phi = 0\right) = 8\pi G_0 = 1$. In these units $f\left(\Box\phi = 0\right) = 1$. Variation of the action with respect to the metric $g_{\mu\nu}$ gives

$$fG_{\mu\nu} + \left(\Box f - \Lambda + \tfrac{1}{2}\left(Z\,\phi^{;\alpha}\right)_{;\alpha}\right)g_{\mu\nu} - f_{;\mu\nu} - Z_{(,\mu}\phi_{,\nu)} = \lambda\phi_{,\mu}\phi_{,\nu} + T^{(\mathrm{m})}_{\mu\nu}, \tag{1.7}$$

where

$$Z := Rf' + 2\Lambda', \tag{1.8}$$

$T^{(\mathrm{m})}_{\mu\nu}$ is the energy momentum tensor for matter and the prime denotes derivative with respect to $\Box\phi$. The equation

$$\left(Z^{;\nu} + 2\lambda\phi^{;\nu}\right)_{;\nu} = 0 \tag{1.9}$$

follows from the variation of action (1.5) with respect to ϕ. Alternatively (1.9) can be obtained as a consequence of the Bianchi identities by taking the divergence of (1.7) and assuming that the energy momentum tensor $T^{(\mathrm{m})}_{\mu\nu}$ for ordinary matter is covariantly conserved. Taken together with the constraint

$$g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} = 1,$$

equation (1.9) allows to determine the Lagrange multiplier λ.

1.3 The synchronous coordinate system

The assumption of global solvability of (1.2) is of course a restriction on admissible spacetimes. As shown in [36], the existence of a function whose gradient is everywhere time-like implies stable causality, i.e. there are no closed time-like curves also for small perturbations of the metric. Since the norm of the gradient of ϕ is not just positive but everywhere equal to unity, $t := \phi$ even qualifies to be used as the time coordinate of a synchronous coordinate system (see [84])

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \gamma_{ik}\mathrm{d}x^i\mathrm{d}x^k \tag{1.10}$$

†Please note that we have changed the notations used in [42] and [43] to more convenient ones.

where the above equations greatly simplify. In this coordinate system, the mimetic field ϕ defines the space-like hypersurfaces of constant time. The extrinsic curvature of these hypersurfaces,

$$\kappa_{ik} = \frac{1}{2}\frac{\partial}{\partial t}\gamma_{ik} \tag{1.11}$$

can be expressed as $\kappa_{ik} = -\phi_{;ik}$, while $\phi_{;0\alpha} = 0$. Thus,

$$\Box\phi = g^{\alpha\beta}\phi_{;\alpha\beta} = \gamma^{ik}\kappa_{ik} = \kappa = \frac{\partial}{\partial t}\ln\sqrt{\gamma}, \tag{1.12}$$

that is, in this coordinate system $\Box\phi$ is simply equal to the trace of the extrinsic curvature of the hypersurfaces of constant ϕ. In this paper, for the sake of simplicity, we will only consider a homogeneous metric with vanishing spatial curvature. In this case γ_{ik} depends only on time t and equation (1.9) simplifies to

$$\frac{1}{\sqrt{\gamma}}\partial_t\left[\sqrt{\gamma}\left(\partial_t Z + 2\lambda\right)\right] = 0, \tag{1.13}$$

and can be easily integrated to give

$$\lambda = -\frac{1}{2}\dot{Z} + \frac{C}{\sqrt{\gamma}} \tag{1.14}$$

where the dot denotes derivative with respect to time t and the constant of integration C describes the contribution of mimetic matter.

Substituting the expression (1.14) for λ in (1.7) and calculating the covariant derivatives of f and Z we find that the $0-0$ component of the equation becomes

$$fG_{00} + \left(\dot{\kappa} + \frac{1}{2}R\right)\kappa f' - \Lambda + \kappa\Lambda' = \varepsilon, \tag{1.15}$$

where

$$\varepsilon \equiv T_{00} + \frac{C}{\sqrt{\gamma}}, \tag{1.16}$$

is the total energy density of mimetic and ordinary matter. Assuming that the spatial components of the energy-momentum tensor satisfy $T^i_k \propto \delta^i_k$, subtracting from the spatial components of equations (1.7) one third of their trace gives

$$f\left(G^i_k - \frac{1}{3}G^m_m\delta^i_k\right) - \left(f^{;i}_{;k} - \frac{1}{3}f^{;m}_{;m}\delta^i_k\right) = 0. \tag{1.17}$$

For the spatially flat metric γ_{ik}

$$R^0_0 = -\dot{\kappa} - \kappa^i_k\kappa^k_i, \quad R^i_k = -\tfrac{1}{\sqrt{\gamma}}\partial_0\left(\sqrt{\gamma}\kappa^i_k\right), \tag{1.18}$$

where $\kappa^i_k = \gamma^{im}\kappa_{mk}$ (see, for example, [84]). Using these expression, equations (1.15) and (1.17) become

$$\frac{1}{3}\left(f - 2\kappa f'\right)\kappa^2 - \Lambda + \kappa\Lambda' - \frac{1}{2}\left(f + \kappa f'\right)\tilde{\kappa}^i_k\tilde{\kappa}^k_i = \varepsilon \tag{1.19}$$

and

$$\partial_0 \left(f\sqrt{\gamma}\tilde{\kappa}^i_k\right) = 0, \tag{1.20}$$

correspondingly, where

$$\tilde{\kappa}^i_k = \kappa^i_k - \frac{1}{3}\kappa\delta^i_k, \tag{1.21}$$

is the traceless part of the extrinsic curvature.

The absence of higher time derivative terms in the modified Einstein equations can be understood by realizing that in the synchronous coordinate system

$$fR = f\left(-2\dot{\kappa} - \kappa^2 - \kappa^i_k\kappa^k_i - {}^3R\right) = -2\dot{F} - f\left(\kappa^2 + \kappa^i_k\kappa^k_i + {}^3R\right) \tag{1.22}$$

where f is assumed to be integrable with $f(\kappa) = F'(\kappa)$ and ${}^3R$ is the spatial curvature scalar. Hence the action contains, up to a total derivative only first order time derivatives of the metric.[‡] This is a distinguishing feature of the $f(\Box\phi)$-theory which would not be present if $\Box\phi$ is replaced by any other non-constant, covariant expression containing first time derivatives of the metric like e.g. $\phi^{;\mu\nu}\phi_{;\mu\nu} = \kappa^i_k\kappa^k_i$.

Note that if we choose f and Λ to be symmetric functions, then the time reversal invariance of the Einstein equation is maintained. Hence the expanding counterparts for all the contracting solutions presented in the following can be found simply by reversing the arrow of time.

1.4 Asymptotic freedom and the fate of a collapsing universe

Equation (1.19) can be further simplified by making the choice

$$\Lambda = \frac{2}{3}\kappa^2(f-1) \tag{1.23}$$

such that it becomes

$$\left(f - \frac{2}{3}\right)\kappa^2 - \frac{1}{2}\left(f + \kappa f'\right)\tilde{\kappa}^i_k\tilde{\kappa}^k_i = \varepsilon. \tag{1.24}$$

In our units the inverse gravitational constant f is normalized to unity for $\kappa^2 = 0$. To guarantee that at low curvatures the corrections to General Relativity will be in the next order in curvature we have to assume that for $\kappa^2 \ll 1,\ f = 1 + \mathcal{O}\left(\kappa^2\right);$ in this case $\Lambda = \mathcal{O}\left(\kappa^4\right)$. In addition we assume that the gravitational constant $G(\kappa^2) \propto 1/f$ vanishes at some limiting curvature κ_0^2 (cf. [85],[17],[18]) and thus take the simplest possible function for f, namely

$$f = \frac{1}{1 - (\kappa^2/\kappa_0^2)}, \tag{1.25}$$

‡ This argument can, however, only serve as a heuristic explanation. Strictly speaking, it is not allowed to use $\Box\phi = \kappa$ and impose gauge conditions in the action before variation.

where κ_0^2 is a free parameter of the theory and can be taken well below the Planckian value.

Friedmann Universe. First let us consider a flat contracting Friedmann universe with the metric

$$ds^2 = dt^2 - a^2(t)\,\delta_{ik}dx^i dx^k. \tag{1.26}$$

In this case

$$\kappa = 3\frac{\dot{a}}{a} \tag{1.27}$$

and $\tilde{\kappa}_k^i$ vanishes. Therefore, equation (1.21) is satisfied identically and equation (1.24) can be rewritten as

$$\frac{1}{3}\kappa^2\left(\frac{1+2\left(\kappa^2/\kappa_0^2\right)}{1-\left(\kappa^2/\kappa_0^2\right)}\right) = \varepsilon. \tag{1.28}$$

Before writing the exact solution for equation (1.28), we first consider some of its asymptotic limits. For $\kappa^2/\kappa_0^2 \ll 1$ it reduces in the leading order to the usual Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\varepsilon. \tag{1.29}$$

For a contracting universe dominated by matter with equation of state $p = w\varepsilon$ it has the solution

$$a \propto t^{\frac{2}{3(1+w)}}, \tag{1.30}$$

for large negative t. At the moment when the curvature approaches its limiting value, the gravitational constant begins to decrease and for $1-\left(\kappa^2/\kappa_0^2\right) \ll 1$, equation (1.28) can be approximated by

$$\kappa^2 = \kappa_0^2\left(1 - \frac{\kappa_0^2}{\varepsilon} + ...\right). \tag{1.31}$$

In a contracting universe the scale factor a decreases, while the energy density grows as $\varepsilon \propto a^{-3(1+w)}$. Hence, the solution of equation (1.31) approaches the contracting flat de Sitter universe with constant curvature where the scale factor decreases as

$$a \propto \exp\left(-\frac{\kappa_0 t}{3}\right) \tag{1.32}$$

for $\kappa_0 t \gg 1$. The gravitational constant $G \propto f^{-1}$ vanishes as $1/\varepsilon$ when $\varepsilon \to \infty$. The singularity is thus avoided as a result of the asymptotic freedom of gravity irrespective of the matter content of the universe.

For $\varepsilon \propto \gamma^{-\frac{1+w}{2}}$ the differential equation (1.28) can be integrated to obtain the exact implicit solution for $\kappa(t)$. In fact, differentiating the logarithm of equation (1.28) with respect to time and taking into account that $\partial \ln\gamma/\partial t = 2\kappa$, we obtain a first order differential equation which can be easily integrated to give

$$\frac{1+w}{2}\kappa_0 t = \frac{\kappa_0}{\kappa} - \operatorname{atanh}\frac{\kappa}{\kappa_0} - \sqrt{2}\arctan\left(\sqrt{2}\frac{\kappa}{\kappa_0}\right). \tag{1.33}$$

One can easily verify that the asymptotics (1.30) and (1.32) are smoothly connected in this solution. In particular, for large negative t the universe contracts according to (1.30). However, as it follows from (1.33), $\kappa\,(t=0) \simeq -0.6\kappa_0$ instead of blowing up as it would for solution (1.30) and for large positive t our solution approaches the de Sitter asymptotic (1.32).

In conclusion, the singularity is replaced by a smooth transition to a de Sitter metric. This qualitative behavior follows most naturally from our theory, independent of a specific choice of f and Λ. Note that the modified Friedmann equation is in general just a relation of the form $\kappa^2(\varepsilon)$. Demanding that this relation is smooth, one-to-one, bounded and has bounded slope, as it is necessary to ensure limiting curvature, the only remaining possibility is for κ to approach its constant limiting value as ε tends to infinity.

Kasner Universe. We now consider a contracting anisotropic Kasner universe to find out what happens when the curvature approaches its limiting value for which the gravitational constant vanishes. To simplify the formulae we will set the energy density of matter to zero although all our conclusions survive also in the presence of the matter. For an anisotropic universe

$$\gamma_{ik} = \gamma_{(i)}\,(t)\,\delta_{ik} \tag{1.34}$$

and $\gamma = \gamma_{(1)}\gamma_{(2)}\gamma_{(3)}$. The traceless part of the extrinsic curvature in this case is nonvanishing and is determined by intergrating equation (1.21):

$$\tilde{\kappa}^i_k = \frac{\lambda^i_k}{f\sqrt{\gamma}}, \tag{1.35}$$

where λ^i_k are constants of integration satisfying $\lambda^i_i = 0$. Substituting this expression in equation (1.24) and using (1.25) we obtain

$$\frac{1}{3}\kappa^2\left(\frac{1+2\left(\kappa^2/\kappa_0^2\right)}{1-\left(\kappa^2/\kappa_0^2\right)}\right) = \frac{1}{2}\frac{\left(1+\left(\kappa^2/\kappa_0^2\right)\right)\bar{\lambda}^2}{\gamma}, \tag{1.36}$$

where $\bar{\lambda}^2 = \lambda^i_k\lambda^k_i$. Because $\kappa = \dot{\gamma}/2\gamma$, this equation allows us to determine how the determinant of the metric depends on time. Knowing $\gamma\,(t)$, the components of the metric can be found in the following way: Without loss of generality we can diagonalize λ^i_k, so that, $\lambda^i_k = \lambda_{(i)}\delta^i_k$. Taking into account the definitions (1.11) and (1.21), equations (1.35) reduce to

$$\frac{\dot{\gamma}_{(i)}}{\gamma_{(i)}} - \frac{1}{3}\frac{\dot{\gamma}}{\gamma} = \frac{2\lambda_{(i)}}{f\sqrt{\gamma}}, \tag{1.37}$$

from which it follows that

$$\gamma_{(i)} = \gamma^{1/3}\exp\left(\int\frac{2\lambda_{(i)}}{f\sqrt{\gamma}}dt\right). \tag{1.38}$$

Before giving the exact solution of equation (1.36) it is more enlightening to study the asymptotic solutions. At low curvatures, that is, for $\kappa^2 \ll \kappa_0^2$, equation (1.36) simplifies to

$$\left(\frac{\dot{\gamma}}{\gamma}\right)^2 \simeq \frac{6\bar{\lambda}^2}{\gamma}, \tag{1.39}$$

and has the solution

$$\gamma = \frac{3}{2}\bar{\lambda}^2 t^2. \tag{1.40}$$

Taking into account that in this limit $f = 1$ and substituting this solution in (1.38) we find

$$\gamma_{(i)} = \left(\frac{3}{2}\bar{\lambda}^2\right)^{1/3} t^{2p_i}, \tag{1.41}$$

where

$$p_i = \frac{1}{3} \pm \sqrt{\frac{2}{3}}\frac{\lambda_{(i)}}{\bar{\lambda}}. \tag{1.42}$$

Since $\lambda_1 + \lambda_2 + \lambda_3 = 0$, the p_i satisfy the conditions

$$p_1 + p_2 + p_3 = 1, \qquad p_1^2 + p_2^2 + p_3^2 = 1,$$

and at low curvatures we have either an expanding or a contracting Kasner universe [84].

In a contracting universe, at $|t| \simeq 1/\kappa_0$ the curvature becomes of the order of limiting curvature and for $1 - (\kappa^2/\kappa_0^2) \ll 1$, equation (1.36) is well approximated by

$$\kappa_0^2 \left(\frac{1}{1 - (\kappa^2/\kappa_0^2)}\right) = \frac{\bar{\lambda}^2}{\gamma}, \tag{1.43}$$

from which it follows that

$$\frac{\dot{\gamma}}{\gamma} = -2\kappa_0 \left(1 - \frac{\kappa_0^2\gamma}{\bar{\lambda}^2}\right)^{1/2} \tag{1.44}$$

in a contracting universe and for $\gamma \ll \bar{\lambda}^2/\kappa_0^2$ we have

$$\gamma \propto \exp\left(-2\kappa_0 t\right). \tag{1.45}$$

As follows from (1.44), in this limit

$$f = \frac{\bar{\lambda}^2}{\kappa_0^2\gamma} \tag{1.46}$$

and the integrals in (1.38) fast converge to some constants for $t \gg 1/\kappa_0$. These constants can be absorbed by redefinition of the spatial coordinates to give the asymptotic solution

$$\gamma_{(1)} = \gamma_{(2)} = \gamma_{(3)} = \gamma^{1/3} \propto \exp\left(-\frac{2}{3}\kappa_0 t\right), \tag{1.47}$$

that describes a contracting flat de Sitter universe with constant curvature.

The exact implicit solution of equation (1.36) for $\kappa\left(t\right)$ is given by

$$\kappa_0 t = \frac{\kappa_0}{\kappa} - \operatorname{atanh}\frac{\kappa}{\kappa_0} - \sqrt{2}\arctan\left(\sqrt{2}\frac{\kappa}{\kappa_0}\right) + \arctan\frac{\kappa}{\kappa_0}. \tag{1.48}$$

Note that in the anisotropic case we are forced to use asymptotic freedom if we want to obtain a non-singular modification where κ tends to its constant limiting value. Only in this way the anisotropy can disappear during contraction.

1.5 Quantum fluctuations

Now we look at what happens with quantum fluctuations of the gravitational field as we approach the limiting curvature where the gravitational constant vanishes. As it is well known (see, for example, [86]), in General Relativity the typical amplitude of quantum fluctuations of gravitational waves in Minkowski space and on curved background at scales l much smaller than the curvature scale is about

$$\delta h_l \simeq \frac{\sqrt{G}}{l}, \tag{1.49}$$

where G is the gravitational constant. Therefore, in our theory where this gravitational constant vanishes on the background with limiting curvature, one could expect that the quantum metric fluctuations must also vanish. We will now show that this is what really happens. Consider a slightly perturbed flat Friedmann Universe with metric

$$ds^2 = a^2(\eta)\left(d\eta^2 - (\delta_{ik} + h_{ik})\, dx^i dx^k\right), \tag{1.50}$$

where we have introduced conformal time $\eta = \int \frac{dt}{a(t)}$ and h_{ik} is the traceless ($h^i_i = 0$) and transverse ($h^i_{k,i} = 0$) part of the metric perturbations. Substituting this metric in action (1.5) and expanding it to second order in h we obtain the following action for the gravitational waves:

$$S = \frac{1}{8}\int f a^2 \left(h^{i\prime}_k h^{k\prime}_i - h^i_{k,m} h^{k,m}_i\right) d\eta d^3x, \tag{1.51}$$

where prime denotes the derivative with respect to conformal time η and the spatial indices are raised and lowered with δ_{ik}. This precisely coincides with the action for gravitational waves in a Friedmann universe with the "scale factor"

$$\tilde{a} := a\sqrt{f}.$$

In this case the quantization procedure is well known and there is no need to repeat all the steps here. Referring to section 8.4 in [86] we find that the typical amplitude squared for the quantum fluctuations is

$$\delta h^2(k,\eta) \simeq \frac{|v_k|^2 k^3}{\tilde{a}^2} = \frac{|v_k|^2 k^3}{f a^2}, \tag{1.52}$$

where k is the co-moving wave number and the mode function v_k satisfies the equation

$$v_k'' + \omega_k^2 v_k = 0, \qquad \omega_k^2 \equiv k^2 - \frac{\tilde{a}''}{\tilde{a}} \tag{1.53}$$

with initial conditions $v_k(\eta_{in}) = 1/\sqrt{\omega_k}$, $v_k'(\eta_{in}) = i\sqrt{\omega_k}$ for quantum fluctuations. When the solution approaches the limiting curvature we have $f \propto \varepsilon \propto a^{-3(1+w)}$ and $\tilde{a} \propto a^{-\frac{1}{2}(1+3w)}$.

Taking into account that in contracting de Sitter $a(\eta) = 3/\kappa_0\eta,$ where η grows, equation (1.53) becomes

$$v_k'' + \left(k^2 - \frac{(9w^2-1)}{4\eta^2}\right) v_k = 0. \tag{1.54}$$

We can define quantum fluctuations only for short wave gravitational waves satisfying $k\eta \gg 1,$ that is, for physical scales $l = a/k \ll \kappa_0^{-1}$. In this case $v_k \simeq \exp(ik\eta)/\sqrt{k}$ and, as follows from (1.52)

$$\delta h(l) \simeq \frac{1}{\sqrt{f}l} \simeq \frac{\sqrt{G(\kappa)}}{l}. \tag{1.55}$$

Hence, quantum fluctuations in a given physical scale $l \ll \kappa_0^{-1}$ vanish as $\kappa \to \kappa_0$ and correspondingly $G(\kappa) \to 0$. This is in complete agreement with our expectations. The perturbations with $k\eta \ll 1$, which were outside the horizon κ_0^{-1} finally come inside because η grows in a contracting de Sitter space-time. The amplitude of metric perturbations h is constant before horizon crossing, but after entering the horizon it decays as $\tilde{a}^{-1} \propto a^{\frac{1}{2}(1+3w)}$. Thus we have shown that the de Sitter space-time with limiting curvature is completely classical, with no quantum metric fluctuations present.

1.6 Conclusions

The simple observation that the conformal part of the metric in General Relativity can be extracted covariantly via a constrained scalar field ϕ has proven to be very fruitful. The resulting modified gravity theory does not induce any additional degrees of freedom for the graviton, but at the same time makes the longitudinal mode dynamical even in the absence of matter. This mode can serve as a viable candidate for dark matter in our Universe. Moreover the constrained scalar field allows us to build invariants which in synchronous coordinates can be expressed exclusively in terms of first order time derivatives of the metric. This opens the possibility to modify General Relativity in a simple way avoiding problematic higher order time derivative terms which generically lead to ghost degrees of freedom. Such a generalization of Einstein theory happens to be very interesting and allows us for example to implement the idea of limiting curvature and resolve spacelike singularities in Friedmann and Kasner universes as well as in black holes. The limiting curvature, which is a parameter of the theory, can be taken well below the Planckian curvature. Potentially, this would make the di□cult unresolved problem of non-perturbative quantum gravity obsolete for all practical purposes.

In this paper we have investigated the possibility of implementing the idea of classical asymptotic freedom just assuming that the gravitational constant vanishes at the limiting curvature. As it was shown, in this case the singularities in flat contracting Friedmann and Kasner universes are resolved and close to the limiting curvature the de Sitter solution is approached. Moreover, quantum metric fluctuations asymptotically vanish and the spacetime becomes fully classical at this limiting curvature. This opens an interesting possibility to resolve the longstanding singularity problem in General Relativity via a simple modification of Einstein theory at large curvatures without referring this problem to a yet unknown non-perturbative theory of quantum gravity.

For the sake of simplicity and to highlight the most important aspects first, in this paper we focused mainly on the homogeneous, spatially flat sector of the theory proposed above. In another soon to appear paper we will extend our analysis and consider applications to spatially non-flat spacetimes, including Black Holes.

Paper 2

Mimetic Hořava Gravity*

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Abstract

We show that the scalar field of mimetic gravity could be used to construct diffeomorphism invariant models that reduce to Hořava gravity in the synchronous gauge. The gradient of the mimetic field provides a timelike unit vector field that allows to define a projection operator of four-dimensional tensors to three-dimensional spatial tensors. Conversely, it also enables us to write quantities invariant under space diffeomorphisms in fully covariant form without the need to introduce new propagating degrees of freedom.

*published in: [2] A. H. Chamseddine, V. Mukhanov and T. B. Russ, *Mimetic Hořava gravity,* Physics Letters B, **798** (2019) 134939, DOI: 10.1016/j.physletb.2019.134939

It has been recognized for some time that in order to improve the UV behaviour of the graviton propagator and, thus, the renormalizability of gravity, it is necessary to add higher spatial derivatives to its Lagrangian but no higher time derivatives. Because this seems to contradict the relativistic local Lorentz invariance, it was thought necessary to break the symmetry between space and time. The most notable attempt is the one by Hořava [50], who constructed a model of quantum gravity with explicitly broken Lorentz symmetry, which allowed him to add to the action terms dependent on the spatial Ricci tensor and curvature scalar and their space derivatives (see e.g. [87] and references therein). This is a high price to pay because, although the Hořava model is renormalizable when projected into the product space $\mathbb{R} \times \Sigma_3$, this property is lost when the model is made covariant by adding one new field [51]. Various attempts were made to keep renormalizability of the models while restoring Lorentz invariance by adding a dynamical scalar or vector [54]. Such models exhibit additional propagating degrees of freedom, which limited their acceptance as a solution to the problem of renormalizability of gravity.

Mimetic gravity was proposed as a way of separating the scale factor from the metric and resulted in reproducing Einstein gravity in addition to half a degree of freedom which could be used to mimic dark matter [35]. The main observation is that one can define the metric tensor $g_{\mu\nu}$ in terms of an auxiliary metric $\widetilde{g}_{\mu\nu}$ by the relation

$$g_{\mu\nu} = \widetilde{g}_{\mu\nu} \left(\widetilde{g}^{\kappa\lambda} \partial_\kappa \phi \partial_\lambda \phi \right), \tag{2.1}$$

where ϕ is a scalar field. The metric $g_{\mu\nu}$ is invariant under the scale transformation $\widetilde{g}_{\mu\nu} \to \Omega^2 \widetilde{g}_{\mu\nu}$ and, as can be easily shown, satisfies the constraint

$$g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = 1, \tag{2.2}$$

governing the evolution of ϕ. Thus, instead of introducing the mimetic field ϕ through the reparametrization (2.1), it is easier to consider directly the physical metric $g_{\mu\nu}$ together with a constrained scalar field, enforcing (2.2) through a Lagrange multiplier [37]. This implies that out of the 11 variables $g_{\mu\nu}$ and ϕ there are only 10 independent fields. In the ADM decomposition of $g_{\mu\nu}$,

$$ds^2 = N^2 dt^2 - \gamma_{ij} \left(dx^i + N^i dt \right) \left(dx^j + N^j dt \right), \qquad i = 1, 2, 3 \tag{2.3}$$

where N is the lapse function, N^i is the shift vector, and $\gamma_{ij} = -g_{ij}$ is the metric on the spatial 3d hypersurface, the constraint (2.2) can be solved for N in terms of the 10 variables N_i, γ_{ij} and ϕ, yielding

$$N^2 = \frac{\left(\partial_0 \phi - N^i \partial_i \phi \right)^2}{\left(1 + \gamma^{ij} \partial_i \phi \partial_j \phi \right)}. \tag{2.4}$$

In the synchronous gauge $N = 1$, $N_i = 0$, a solution of (2.2) is given by

$$\phi = t + A, \tag{2.5}$$

where A is a constant. Since there exists a whole family of synchronous coordinate systems, corresponding to the freedom of choice of an initial hypersurface of constant time, this solution is not unique. On the other hand, ϕ can always be used as one particular synchronous time coordinate, fixing a unique $3+1$ slicing that we will use from now on. The timelike unit vector $n_\mu = \partial_\mu \phi$ points in this time direction. In particular, we can define the projection operator

$$P^\nu_\mu = \delta^\nu_\mu - \partial_\mu \phi \partial_\kappa \phi g^{\nu\kappa}, \tag{2.6}$$

satisfying the relations

$$P^\rho_\mu P^\nu_\rho = P^\nu_\mu, \qquad P^\nu_\mu \partial_\nu \phi = 0. \tag{2.7}$$

In the synchronous slicing from above we have

$$P^0_0 = 0, \qquad P^i_0 = 0, \qquad P^0_i = 0, \qquad P^j_i = \delta^j_i, \tag{2.8}$$

showing that P^ν_μ projects space-time vectors to space vectors. It is then clear that in mimetic gravity, using the projection operator and the vector $n_\mu = \partial_\mu \phi$, it is possible to construct four-dimensional tensors whose only non-zero components in the synchronous gauge are along space directions. For example, as we will show in the following, the expression

$$\widetilde{R} := 2R^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - R - (\Box\phi)^2 + \nabla_\mu \nabla_\nu \phi \nabla^\mu \nabla^\nu \phi \tag{2.9}$$

coincides with the spatial curvature scalar ${}^3R$ of synchronous slices.

In previous works we have shown that in mimetic gravity, without the need to introduce any additional fields, we can build cosmological models [61] and solve the singularity problem for Friedmann, Kasner [42] and Black hole [43] solutions by using the idea of limiting curvature. More recently we have shown that the idea of asymptotic freedom can be implemented in mimetic gravity by introducing a $\Box\phi$ dependent effective gravitational constant which vanishes at the limiting curvature [1]. Moreover, it was shown that such a dependence does not introduce higher time derivatives.

The purpose of this letter is to show that within mimetic gravity we can construct all the terms needed in Hořava gravity using four-dimensional tensors that reduce to the desired form in the synchronous gauge. We will thus show that in mimetic gravity it is possible to formulate Hořava gravity in a diffeomorphism invariant way without introducing ghost-like degrees of freedom.

The basic fields that appear in Hořava gravity are the three-dimensional tensors and scalars κ_{ij}, κ, ${}^3R_{ij}$, ${}^3R$, $D_k {}^3R_{ij}$, and their contractions needed to form space diffeomorphism invariant expressions. The extrinsic curvature of the synchronous slices $\phi = const.$ is given by

$$\kappa_{ij} = \frac{1}{2}\dot{\gamma}_{ij}, \qquad \kappa^j_i = \gamma^{jl}\kappa_{il}, \qquad \kappa = \kappa^i_i = (\ln\sqrt{\gamma})^{\cdot}, \tag{2.10}$$

where dot denotes t derivative and γ is the metric determinant. Using ϕ, it can be expressed covariantly as

$$\nabla_i \nabla_j \phi = -\kappa_{ij}, \quad \nabla_i \nabla^j \phi = \kappa^j_i, \quad \Box\phi = \kappa. \tag{2.11}$$

The non-vanishing components of the four-dimensional Riemann tensor are determined by

$$R^0_{\ kij} = D_i\kappa_{kj} - D_j\kappa_{ki}, \tag{2.12}$$
$$R^0_{\ k0j} = \dot{\kappa}_{jk} - \kappa_{jn}\kappa^n_k, \tag{2.13}$$
$$R^l_{\ kij} = {}^3R^l_{\ kij} + \kappa^l_i\kappa_{jk} - \kappa^l_j\kappa_{ik}, \tag{2.14}$$

where D_i and ${}^3R^l_{\ kij}$ are the covariant derivative and the Riemann tensor belonging to the metric γ_{ij}. With the help of the above identities, we can construct the four-dimensional tensor

$$\widetilde{R}^\sigma_{\ \rho\mu\nu} := P^\sigma_\delta P^\gamma_\rho P^\alpha_\mu P^\beta_\nu R^\delta_{\ \gamma\alpha\beta} + \nabla_\mu\nabla^\sigma\phi\nabla_\rho\nabla_\nu\phi - \nabla_\nu\nabla^\sigma\phi\nabla_\rho\nabla_\mu\phi \tag{2.15}$$

whose only non-zero components are ${}^3R^l_{\ kij}$ in the synchronous gauge. Next, we compute the Ricci tensor components

$$R_{00} = -\dot{\kappa} - \kappa_{ij}\kappa^{ij} \tag{2.16}$$
$$R_{0i} = D_l\kappa^l_i - D_i\kappa \tag{2.17}$$
$$R_{ij} = {}^3R_{ij} + \kappa\kappa_{ij} - \kappa^n_i\kappa_{nj} + R^0_{\ i0j}. \tag{2.18}$$

These relations allow us to define the tensor

$$\widetilde{R}_{\mu\nu} := P^\alpha_\mu P^\beta_\nu R_{\alpha\beta} + \Box\phi\nabla_\mu\nabla_\nu\phi - \nabla_\mu\nabla^\rho\phi\nabla_\nu\nabla_\rho\phi - R^\gamma_{\ \mu\delta\nu}\nabla^\delta\phi\nabla_\gamma\phi, \tag{2.19}$$

whose non-zero components coincide with ${}^3R_{ij}$ in the synchronous gauge. Contracting with $g^{\mu\nu}$, we arrive at (2.9).

We note in passing that the total derivative $\frac{1}{\sqrt{\gamma}}\partial_0\left(\sqrt{\gamma}\kappa\right)$ can be easily eliminated from the Lagrangian of Einstein-Hilbert gravity, leaving us with

$$-R - 2\nabla_\mu\left(\Box\phi\nabla^\mu\phi\right) = \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi - \left(\Box\phi\right)^2 + \widetilde{R}. \tag{2.20}$$

For manifolds with boundary $\partial M = \{\phi = \phi_i\}\cup\{\phi = \phi_f\}$ consisting of closed spatial hypersurfaces of constant ϕ, this has precisely the same effect as adding the Gibbons-Hawking boundary term.

Space derivatives of the above tensors can be obtained by applying the operator $P^\rho_\mu\nabla_\rho$. Note that the spatial components of $P^\gamma_\rho\nabla_\gamma\widetilde{R}_{\alpha\beta}$ coincide with $D_k{}^3R_{ij}$ in the synchronous gauge. To obtain a purely spatial tensor, one still must project all four-dimensional indices, i.e. one has to use $P^\gamma_\rho P^\alpha_\mu P^\beta_\nu\nabla_\gamma\widetilde{R}_{\alpha\beta}$. Thus, we can now define the analogue of the three-dimensional Cotton tensor

$${}^3C^i_j = \frac{1}{\sqrt{\gamma}}\epsilon^{ikl}D_k\left({}^3R_{jl} - \frac{1}{4}\gamma_{jl}\,{}^3R\right) \tag{2.21}$$

by writing

$$\widetilde{C}^\mu_\nu := -\frac{1}{\sqrt{-g}}\epsilon^{\mu\rho\kappa\lambda}\nabla_\lambda\phi\,\nabla_\rho\left(\widetilde{R}_{\nu\kappa} - \frac{1}{4}g_{\nu\kappa}\widetilde{R}\right), \tag{2.22}$$

whose only non-vanishing components in the synchronous gauge are ${}^3C^i_j$.

Another object that could be constructed is the Chern-Simons three form ω_P related to the Pontryagin topological invariant

$$R^\sigma_\rho \wedge R^\rho_\sigma = \mathrm{d}\omega_P\,, \tag{2.23}$$

$$\omega_P = \Gamma^\nu_\mu \wedge \mathrm{d}\Gamma^\mu_\nu + \frac{2}{3}\,\Gamma^\mu_\nu \wedge \Gamma^\nu_\rho \wedge \Gamma^\rho_\mu\,, \tag{2.24}$$

where $\Gamma^\nu_\mu = \mathrm{d}x^\rho \Gamma^\nu_{\rho\mu}$ and $R^\sigma_\rho = \frac{1}{2} R^\sigma_{\rho\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ are the Christoffel connection one-form and the curvature two form, respectively. The four-form $\mathrm{d}\phi \wedge \omega_P$ is not parity invariant. Up to a boundary term, its integral is given by

$$\int \mathrm{d}\phi \wedge \omega_P = -\int \phi\, R^\sigma_\rho \wedge R^\rho_\sigma. \tag{2.25}$$

This shows that such a contribution to the action is covariant and invariant under global shifts of ϕ. In the synchronous gauge the integrand reduces to

$$\epsilon^{ijk}\left(\Gamma^\nu_{i\mu}\partial_j\Gamma^\mu_{k\nu} + \frac{2}{3}\Gamma^\mu_{i\nu}\Gamma^\nu_{j\rho}\Gamma^\rho_{k\mu}\right) = {}^3\omega_P + 2\epsilon^{ijk}\kappa^n_i D_j\kappa_{kn}, \tag{2.26}$$

where

$${}^3\omega_P = \epsilon^{ijk}\left(\lambda^n_{im}\partial_j\lambda^m_{kn} + \frac{2}{3}\lambda^m_{in}\lambda^n_{jl}\lambda^l_{km}\right). \tag{2.27}$$

and λ^k_{ij} are the Christoffel symbols calculated for γ_{ij}. The term $2\epsilon^{ijk}\kappa^n_i D_j\kappa_{kn}$ can be written as

$$\epsilon^{ijk}\nabla_i\nabla^n\phi R^0_{njk}\,, \tag{2.28}$$

which coincides in the synchronous gauge with

$$\epsilon^{\mu\nu\rho\sigma}\nabla_\mu\phi\nabla_\nu\nabla^\lambda\phi R^\tau_{\lambda\rho\sigma}\nabla_\tau\phi. \tag{2.29}$$

Thus, the purely three-dimensional Chern-Simons form can be incorporated in the action by adding the term

$$\int \mathrm{d}\phi \wedge \widetilde{\omega}_P := \int \mathrm{d}\phi \wedge \left(\omega_P - \nabla^\lambda \mathrm{d}\phi \wedge R^\tau_\lambda \nabla_\tau\phi\right). \tag{2.30}$$

All of these manipulations illustrate that any expression invariant under spatial diffeomorphisms can be written as a combination of four-dimensional tensors that reduces to it in the synchronous gauge.

We conclude by writing an exemplary Hořava action in mimetic gravity, in terms of four-dimensional tensors and thus completely preserving diffeomorphism invariance, without the need for new degrees of freedom:

$$\begin{aligned} S = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}\,\Big(&\nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi - c_1\left(\Box\phi\right)^2 + c_2\,\widetilde{R} \\ &+ c_3\,\widetilde{R}^2 + c_4\,\widetilde{R}_{\mu\nu}\widetilde{R}^{\mu\nu} + c_5\,\widetilde{C}^\mu_\nu\widetilde{C}^\nu_\mu + c_6\,\eta^{\mu\nu\rho\sigma}\nabla_\mu\phi\left(\widetilde{\omega}_P\right)_{\nu\rho\sigma} \\ &+ c_7\,\eta^{\mu\nu\rho\sigma}\nabla_\sigma\phi\widetilde{R}_{\mu\alpha}\nabla_\nu\widetilde{R}^\alpha_\rho + \ldots + \lambda\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1\right)\Big)\,, \end{aligned} \tag{2.31}$$

where $\eta^{\mu\nu\rho\sigma} = \frac{1}{\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma}$. The case where $c_1 = c_2 = 1$ and all other couplings vanish is just a rewritten form of General Relativity with mimetic matter. The constants $c_1, \ldots, c_7$ could also be taken as functions of $\Box\phi$ in such a way as to reproduce General Relativity in the low curvature limit.

There is no need to repeat calculations done for the Hořava models, as those could be thought of as a gauge fixed version of a diffeomorphism invariant theory in the synchronous gauge.

In the projectable Hořava models, the lapse function N is assumed to depend on time only, $N = N(t)$. These models coincide with the above family of actions in the synchronous gauge. Their renormalization analysis was carried out in references [51], [52], where they were shown to be renormalizable. When the same analysis was applied to the non-projectable case where the lapse function is $N = N(x^i, t)$, so that terms dependent on the vector $a_i = \frac{\partial_i N}{N}$ can occur, it was found that these models become non-renormalizable. Attempts were made to construct diffeomorphism invariant Hořava models by adding a unit vector field u_μ, subject to the hypersurface orthogonality condition $u_{[\mu}\nabla_\nu u_{\rho]} = 0$. These models, however, have a spin-1 and spin-0 degree of freedom in addition to the graviton.

The synchronous gauge belongs to the family of temporal gauge, which for fluctuations of the metric takes the form $n^\mu h_{\mu\nu} = 0$, where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and $n_\mu = (1, 0, 0, 0)$. The main advantage of working in this gauge is that the model proposed above will be power counting renormalizable and that the ghosts associated with gauge fixing will decouple from the physical S-matrix. The disadvantage is the need to have an unambiguous prescription for the unphysical singularities of the form $(q.n)^{-\alpha}$, $\alpha = 1, 2$ (cf. [88]) and the difficulty in performing higher loop calculations. It is a challenging problem to perform a detailed analysis of the renormalization program, either in the synchronous gauge or in a covariant gauge, using the background field method and integrating out the mimetic constraint, along the lines of [51]. Even though an actual proof could be quite demanding, we expect the mimetic Hořava model presented here to be renormalizable.

Paper 3

Black Hole Remnants*

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Abstract

We show that in asymptotically free mimetic gravity with limiting curvature the black hole singularity can be resolved and replaced by a static patch of de Sitter space. As a result of Hawking evaporation of these non-singular black holes, there remain stable remnants with vanishing Hawking temperature.

*published in: [3] A. H. Chamseddine, V. Mukhanov and T. B. Russ, *Black Hole Remnants,* JHEP **1910** (2019) 104. DOI: 10.1007/JHEP10(2019)104

3.1 Introduction

The two unresolved issues of black hole physics actively discussed during the last 50 years are: *a)* the singularity problem, *b)* the final state of black hole evaporation and, closely related to it, the information paradox (see, for instance, [70]). Both issues are usually referred to some, as yet unknown, non-perturbative theory of quantum gravity. The nature of Planck mass black holes, their stability and the resolution of the space-like singularity remain unclear. In this letter we suggest a completely different approach to these problems. Namely, we assume that classical General Relativity is modified at curvatures close to, *but still below Planckian curvature*, in such a way that some limiting curvature, which is a free parameter of the theory, can never be exceeded (cf. [11], [13]). Moreover, we propose that at this limiting curvature the gravitational constant vanishes. These two assumptions allow us to avoid all problems related to non-perturbative quantum gravity effects and study in a fully controllable way the final stage of non-singular evaporating black holes. To implement the above ideas, we use the mimetic field introduced in [35] and further exploited in references [61], [42], [43], [82], [83], [1]. We show that in asymptotically free theories with limiting curvature, black holes generically have stable remnants with mass determined by the inverse limiting curvature value, thus exceeding Planck mass. These remnants have vanishing Hawking temperature and, by the arguments shown in [1], metric quantum fluctuations never become relevant for them.

3.2 The Lemaître coordinates

The metric of both black hole and de Sitter universe in "static" coordinates can be written as

$$\mathrm{d}s^2 = \left(1 - a^2\left(r\right)\right)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{\left(1 - a^2\left(r\right)\right)} - r^2\mathrm{d}\Omega^2, \tag{3.1}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2$. For a black hole of mass M the function $a^2\left(r\right) = r_g/r$, where $r_g = 2M$ and for the de Sitter universe $a^2 = \left(Hr\right)^2$ with H^{-1} being the radius of curvature. Throughout this paper we use Planck units where all fundamental constants are set to unity. In both cases the static coordinate system is incomplete. Moreover, at the horizon corresponding to $a^2 = 1$, there is a coordinate singularity. Therefore, in both cases it is more convenient to use the synchronous Lemaître coordinates

$$T = t + \int \frac{a}{1 - a^2}\mathrm{d}r, \quad R = t + \int \frac{\mathrm{d}r}{a\left(1 - a^2\right)}, \tag{3.2}$$

which are non-singular on the horizons (cf. [89]). In these coordinates metric (3.1) becomes

$$\mathrm{d}s^2 = \mathrm{d}T^2 - a^2\left(x\right)\mathrm{d}R^2 - b^2(x)\mathrm{d}\Omega^2, \tag{3.3}$$

where a^2 and $b^2 = r^2$ must be expressed in terms of Lemaître coordinates T and R using the relation

$$x \equiv R - T = \int \frac{\mathrm{d}r}{a\left(r\right)}, \tag{3.4}$$

which follows from (3.2). Note that the norm of the Killing vector field $\partial/\partial t = \partial/\partial R + \partial/\partial T$ vanishes wherever $a = 1$. The black hole metric in these new coordinates becomes

$$\mathrm{d}s^2 = \mathrm{d}T^2 - (x/x_+)^{-2/3}\,\mathrm{d}R^2 - (x/x_+)^{4/3}\,r_g^2\,\mathrm{d}\Omega^2, \tag{3.5}$$

and it is regular at the horizon $x = x_+ = 4M/3$. The region $x > 0$ covers both interior and exterior of the black hole and $x = 0$ corresponds to the physical space-like singularity where curvature invariants blow up. Hence, in General Relativity this metric is not extendable to negative x.

Correspondingly, the de Sitter metric takes the form

$$\mathrm{d}s^2 = \mathrm{d}T^2 - \exp\left(2H(x - x_-)\right)\left(\mathrm{d}R^2 + H^{-2}\mathrm{d}\Omega^2\right), \tag{3.6}$$

where x_- is a constant of integration in (3.4) and the de Sitter horizon occurs at $x = x_-$. The region $x < x_-$ corresponds to the patch of size $r = H^{-1}$ covered by static coordinates, which on larger scales do not exist.

Calculating the spatial curvature of constant T hypersurfaces of metric (3.3), one finds that it vanishes if $a^2 = (\mathrm{d}b/\mathrm{d}x)^2$. Hence, both solutions (3.5) and (3.6) are spatially flat in the Lemaître slicing.

The Schwarzschild metric has a Kasner type space-like singularity. In the previous paper [1] we have found that in a theory where the ideas of limiting curvature and asymptotic freedom of gravity are realized via the mimetic field, Kasner singularities are avoided and replaced by a de Sitter region at limiting curvature. In this paper we implement these same ideas for black holes. Using ansatz (3.3) for the metric, we will find an explicit solution describing a non-singular black hole whose metric approaches asymptotics (3.5) and (3.6) at low and high curvatures, respectively.

3.3 Modified Mimetic Gravity

Introducing the mimetic field ϕ through a Lagrange multiplier constraint, consider the theory of gravity defined by the action

$$S = \frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}\left(-\mathcal{L} + \lambda\left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - 1\right)\right), \tag{3.7}$$

with Lagrangian density

$$\mathcal{L} = f(\Box\phi)R + \left(f(\Box\phi) - 1\right)\widetilde{R} + 2\Lambda(\Box\phi), \tag{3.8}$$

where

$$\widetilde{R} \equiv 2\phi^{;\mu}\phi^{;\nu}G_{\mu\nu} - (\Box\phi)^2 + \phi^{;\mu\nu}\phi_{;\mu\nu}, \tag{3.9}$$

and $G_{\mu\nu}$ is the Einstein tensor. In [1] we argued that $\Box\phi$ is the unique measure of curvature that f can depend on without introducing higher time derivatives to the modified Einstein equation. The extension of the action presented in [1] by the $\widetilde{R}$-term is done with the purpose to remove higher spatial and mixed derivatives. Thus, (3.7) defines a theory of gravity free of any higher derivatives.

Since the Lemaître coordinates are synchronous, the generic solution of the constraint equation $\phi^{,\alpha}\phi_{,\alpha} = 1$ satisfied by the mimetic field, which is compatible with the symmetries of ansatz (3.3), is $\phi = T + const.$ In this coordinate system

$$\Box\phi = \kappa = \frac{\partial}{\partial T} \ln \sqrt{\gamma} = -\frac{\mathrm{d}}{\mathrm{d}x} \ln \left(ab^2\right) \tag{3.10}$$

represents the trace of extrinsic curvature $\kappa = \kappa_a^a$ of synchronous slices of constant T. The expression $\widetilde{R}$, given in (3.9), is nothing but the spatial curvature scalar ${}^3R$ of these slices, expressed in covariant form.

Variation of (3.7) with respect to the metric $g_{\mu\nu}$ yields the modified vacuum Einstein equations. Solving the equation obtained by varying (3.7) with respect to ϕ for the Lagrange multiplier λ and substituting the metric ansatz (3.3), after lengthy but straightforward calculations we find that the spatial components of the modified Einstein equations have the first integral (see [1], †)

$$\frac{\dot{b}}{b} - \frac{\dot{a}}{a} = \frac{3M}{fab^2}, \tag{3.11}$$

where dot denotes the derivative with respect to x and a constant of integration has been fixed to match the Schwarzschild solution with mass M in the limit $x \to \infty$. In deriving (3.11) we have assumed that the spatial curvature remains negligible everywhere for solutions matching the two spatially flat asymptotics (3.5) and (3.6). Later on we will justify this assumption.

Accordingly, the temporal modified Einstein equation becomes

$$\frac{\kappa^2 \left(f - 2\kappa f'\right) - 3\left(\Lambda - \kappa\Lambda'\right)}{\left(f + \kappa f'\right)} = \left(\frac{3M}{fab^2}\right)^2, \tag{3.12}$$

where prime denotes the derivative with respect to κ. The equations (3.11) and (3.12) determine the two unknown functions $a\left(x\right)$ and $b\left(x\right)$.

3.4 Asymptotic Freedom at Limiting Curvature

The inverse running gravitational constant is represented by

$$f\left(\Box\phi\right) = \frac{1}{G\left(\Box\phi\right)}, \tag{3.13}$$

normalized as $f\left(\Box\phi = 0\right) = 1$ in Planck units. Asymptotic freedom is characterized by a divergence of f as $\Box\phi \to |\kappa_0|$ approaches the limiting curvature κ_0, which is a free

† The more general and detailed calculations will be presented in a forthcoming publication by the authors.

parameter of the theory. In [1] we have shown that in a contracting Kasner universe the vanishing gravitational constant very e□ciently suppresses anisotropies and the solution close to the limiting curvature becomes isotropic, approaching a de Sitter universe with $H = \kappa_0/3$. Since black hole and Kasner singularities are similar, one can expect that the black hole singularity can be resolved in the same way. Namely, the asymptotic solution far away from the black hole, which is still given by (3.5), will start to approach (3.6) as soon as the curvature approaches its limiting value. The full solution extends for the entire range $-\infty < x < +\infty$.

For the Schwarzschild solution (3.5) the function $a \propto b^{-1/2}$ increases as $b \to 0$, while for the de Sitter solution $a \propto b$. It follows that $\dot{a}$ has to vanish at some intermediate point x_* where a reaches its maximum value before starting to decrease as we go deeper into the black hole. If $a(x_*) > 1$, there exist two Killing horizons where $a(x_\pm) = 1$ at $x_+ > x_*$ and $x_- < x_*$, named in analogy with (3.5) and (3.6). In the limiting case where $a(x_*) = 1$, the two horizons merge at $x_+ = x_- = x_*$ and the region where $\partial/\partial t = \partial/\partial R + \partial/\partial T$ is spacelike disappears. This is the case of a minimal black hole with mass $M_{\min} \sim \kappa_0^{-1}$ which exceeds the Planck mass if the limiting curvature is below the Planckian value. For $M < M_{\min}$, a is always smaller than unity, no horizon occurs and thus no black holes with mass smaller than $M_{\min}$ exist.

One can easily find that for the metric (3.3) the surface gravity of the Killing horizons is given by

$$g_s = -\dot{a}(x_\pm), \tag{3.14}$$

and it hence vanishes for the minimal black hole. Because the Hawking radiation temperature is proportional to g_s, it is equal to zero for these minimal mass black holes and as a result of evaporation there must remain stable remnants of mass $M_{\min}$. Thus, the existence of limiting curvature combined with asymptotic freedom of gravity generically leads to the existence of minimal stable black hole remnants.

To demonstrate this in a concrete theory, below we will find an explicit spatially flat, exact solution describing a non-singular black hole with stable remnant[†].

3.5 Exact Solution

Let us take

$$f(\tilde{\kappa}) = \frac{1 + 3\tilde{\kappa}^2}{(1+\tilde{\kappa}^2)(1-\tilde{\kappa}^2)^2}, \tag{3.15}$$

where $\tilde{\kappa} \equiv \kappa/\kappa_0$ and chose $\Lambda(\kappa)$ in such a way that the square root of the branch $\kappa < 0$ of (3.12) becomes

$$\frac{-\tilde{\kappa}}{1-\tilde{\kappa}^4} = \frac{3M/\kappa_0}{ab^2}. \tag{3.16}$$

Taking the x derivative of the logarithm of this equation and using (??), we obtain a first order di□erential equation for $\tilde{\kappa}(x)$ with the implicit solution

$$-\kappa_0 x = \frac{1}{\tilde{\kappa}} + 2\left(\arctan\tilde{\kappa} - \tanh^{-1}\tilde{\kappa}\right). \tag{3.17}$$

This provides a one-to-one relation between $x \in (-\infty, \infty)$ and $\tilde{\kappa} \in (-1, 0)$ and hence $\tilde{\kappa}$ can be used to parametrize the solution for a and b. After some algebra, we find that the exact solution of equations (3.11) and (3.16) parametrized in terms of $\tilde{\kappa}$ is given by

$$a^3(\tilde{\kappa}) = \frac{4M\kappa_0}{3}\,|\tilde{\kappa}|\left(1-\tilde{\kappa}^4\right)\left(\frac{1+\tilde{\kappa}^2}{1+3\tilde{\kappa}^2}\right)^2, \tag{3.18}$$

$$b^3(\tilde{\kappa}) = \frac{9M}{2\kappa_0^2\tilde{\kappa}^2}\left(1-\tilde{\kappa}^2\right)\left(1+3\tilde{\kappa}^2\right). \tag{3.19}$$

Using (3.17) to express κ in terms of x, we can easily verify that in the far exterior limit $x \to \infty$, $\tilde{\kappa}^2 \to 0$ and the above solution tends to (3.5), describing a black hole of mass M. On the other hand, deep inside the black hole at $x \to -\infty$, $\tilde{\kappa}^2 \to 1$ and we obtain asymptotic (3.6) corresponding to the de Sitter space with $H = \kappa_0/3$. Thus, the obtained exact solution smoothly matches the desired asymptotics, in agreement with our general consideration above. The function a reaches its maximum at $\tilde{\kappa}_* = -1/\sqrt{5}$. The horizons, which occur at $a = 1$, exist only if $a(\tilde{\kappa}_*) \geq 1$. This condition is satisfied only if

$$M \geq M_{\min} = \frac{5^{5/2}}{18\kappa_0}. \tag{3.20}$$

Otherwise, no horizon exists and the solution (3.17), (3.18), (3.19) describes solitonic-like objects whose metric is completely static and approaches the de Sitter metric in the center. For black holes with the minimum mass $M_{\min}$, there is only one horizon with vanishing surface gravity and hence these minimal black holes represent the stable remnants of evaporating black holes.

One can check that solution (3.17), (3.18), (3.19) satisfies $a^2 = \dot{b}^2$ and hence the hypersurfaces $T =$ const. are exactly spatially flat, in complete agreement with the assumption under which it was derived. To better understand the properties of this solution, it is more illuminating to go back to the familiar singular static coordinates (3.1). From (3.4) it follows that $dr = adx$ and therefore $a^2 = \dot{b}^2$ implies that $b = r$ everywhere. Setting $b = r$ in equation (3.19t), we obtain an algebraic equation for $\tilde{\kappa}(r)$, which can be solved perturbatively and the obtained result can be substituted in (3.18) to determine $a^2(r)$ in the “static coordinates”, where the metric is given by (3.1).

For $r \to \infty$ where $\tilde{\kappa}^2 \ll 1$ we find the expansion

$$1 - a^2 = 1 - \frac{2M}{r}\left[1 - \mathcal{O}\left(\left(\frac{r_*}{r}\right)^3\right)\right], \tag{3.21}$$

where $r_* = \left(144M/5\kappa_0^2\right)^{1/3}$ is the “radial” coordinate at which $\tilde{\kappa}_* = -1/\sqrt{5}$ and the curvature becomes comparable to the limiting curvature. For large black holes with $M \gg M_{\min}$ the outer horizon, defined by $a(r_+) = 1$, is located at

$$r_+ = 2M\left[1 - \mathcal{O}\left(\left(\frac{M_{\min}}{M}\right)^2\right)\right], \tag{3.22}$$

and the deviations from the Schwarzschild solution become significant only deeply "inside" the black hole at $r \ll r_+$. Close to the limiting curvature $(\tilde{\kappa}^2 \to 1)$, that is for $r \ll r_*$, the metric is well approximated by

$$1 - a^2 = 1 - (Hr)^2 \left[1 - \mathcal{O}\left(\left(\frac{r}{r_*}\right)^3\right)\right] \tag{3.23}$$

where $H = \kappa_0/3$ and the inner de Sitter horizon occurs at

$$r_- = H^{-1}\left[1 + \mathcal{O}\left(\frac{M_{\min}}{M}\right)\right]. \tag{3.24}$$

If the mass M is comparable to the minimal mass $M_{\min}$, these two asymptotics fail to describe the region close to the horizons. In the minimal case $M = M_{\min}$ inner and outer horizon coincide. Expanding $a(\tilde{\kappa})$ around its maximum at $\tilde{\kappa}_*$ and using (3.19) to express $\tilde{\kappa}$ in terms of r, we find that the near horizon metric of such a minimal black hole is given by

$$1 - a^2 \approx \frac{10}{7}\left(1 - \frac{r}{r_*}\right)^2, \tag{3.25}$$

where $r_+ = r_- = r_* = 2\sqrt{5}/\kappa_0$. Note the similarity to the near horizon metric of an extremal charged Reissner-Nordström black hole.

3.6 Black hole thermodynamics

The Hawking temperature T_H is determined by the surface gravity (3.14) at the exterior horizon x_+. For solution (15) we find

$$T_H = \frac{g_s}{2\pi} = \frac{\kappa_0}{6\pi}\,|\tilde{\kappa}_+|\,\frac{1 - 5\tilde{\kappa}_+^2}{1 + 3\tilde{\kappa}_+^2}, \tag{3.26}$$

where $\tilde{\kappa}_+ = \tilde{\kappa}(x_+) \in (-1/\sqrt{5}, 0)$. Since $a(\tilde{\kappa}_+) = 1$, we can use (3.18) to express M also through $\tilde{\kappa}_+$ as

$$M = \frac{3}{4\kappa_0\,|\tilde{\kappa}_+|\,(1 - \tilde{\kappa}_+^4)}\left(\frac{1 + 3\tilde{\kappa}_+^2}{1 + \tilde{\kappa}_+^2}\right)^2. \tag{3.27}$$

The formulae (3.26) and (3.27) implicitly define the relation $T_H(M)$. In particular, at large mass we reproduce in leading order the familiar Hawking formula

$$T_H = \frac{1}{8\pi M}\left[1 + \mathcal{O}\left(\left(\frac{M_{\min}}{M}\right)^2\right)\right]. \tag{3.28}$$

Instead of diverging as $M \to 0$, the temperature reaches its maximum value $T_{max} \sim 10^{-2}\kappa_0$ at $|\tilde{\kappa}_+| \approx 0.23$ which corresponds to $M = M_c \approx 1.32 M_{\min}$. At this point the negative heat capacity diverges and becomes positive for $M < M_c$. Close to the minimal mass the temperature decreases as

$$T_H \propto \sqrt{M - M_{\rm min}}. \tag{3.29}$$

According to the Stefan-Boltzmann law, the rate of energy loss of a radiating body is determined by $\frac{\mathrm{d}M}{\mathrm{d}t} \propto -T_H^4 A$ where $A = 4\pi r_+^2$ is the horizon area. For an evaporating black hole close to minimal mass $A \sim M_{\rm min}^2$, and hence it will eventually approach $M_{\rm min}$ according to $M(t) - M_{\rm min} \propto t^{-1}$. That is, the final product of black hole evaporation is a stable minimal remnant with $M = M_{\rm min}$ and vanishing Hawking temperature.

Finally, taking into account that the Bekenstein entropy of the black hole is equal to $S = A/4 = \pi r_+^2 = \pi b^2\left(\tilde{\kappa}_+\right)$ it is straightforward to verify that the modified first law of thermodynamics becomes

$$G(\tilde{\kappa}_+)\mathrm{d}M = T_H\,\mathrm{d}S, \tag{3.30}$$

where $G(\tilde{\kappa}_+) = f^{-1}(\tilde{\kappa}_+)$ is the value of the gravitational constant at the outer horizon.

3.7 Conclusions

We have shown that a modification of classical Einstein theory at very high curvatures, implementing asymptotic freedom at limiting curvature, can spare us from having to deal with non-perturbative quantum gravity (at least in application to cosmological and black hole problems). The existence of a sub-Planckian limiting curvature at which the gravitational constant vanishes can resolve the black hole singularity and replace it with a patch of de Sitter space (similar to [15]). Moreover, in this theory the final result of black hole evaporation are remnants whose near horizon geometry is similar to the extremal Reissner-Nordström geometry. Therefore, the Hawking temperature of these remnants is equal to zero. In distinction from extremal Reissner-Nordström black holes, they do not exhibit a singularity and, because they have no charge, they are stable. As becomes clear from the maximal extension of the solution obtained above†, these remnants can store an unlimited amount of information. This information lies in the absolute future of external observers and remains forever inaccessible for them. Hence, their degeneracy should not lead to any paradoxes in calculating physical processes observed by external observers. This suggests one of the possible ways for a resolution of the information paradox (see [70]). Moreover, the stable remnants can serve as well as Dark Matter candidates.

Acknowledgments. The work of A. H. C is supported in part by the National Science Foundation Grant No. Phys-1518371. The work of V.M. and T.B.R. is supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy – EXC-2111 – 390814868.

Paper 4

Non-flat Universes and Black Holes in Asymptotically Free Mimetic Gravity*

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Abstract

The recently proposed theory of "Asymptotically Free Mimetic Gravity" is extended to the general non-homogeneous, spatially non-flat case. We present a modified theory of gravity which is free of higher derivatives of the metric. In this theory asymptotic freedom of gravity implies the existence of a minimal black hole with vanishing Hawking temperature. Introducing a spatial curvature dependent potential, we moreover obtain non-singular, bouncing modifications of spatially non-flat Friedmann and Bianchi universes.

*published in: [4] A. H. Chamseddine, V. Mukhanov and T. B. Russ, *Non-Flat Universes and Black Holes in Asymptotically Free Mimetic Gravity,* Fortsch. Phys. **68** (2020) 1. DOI: 10.1002/prop.201900103

4.1 The Theory

4.1.1 Introduction

General relativity is a quintessential example of a successful physical theory. At the same time as its conceptional simplicity is striking, it has to this day not failed a single experimental test. However, one theoretical prediction of GR that contradicts physical intuition is the formation of singularities from physically realistic initial conditions, as was shown in the singularity theorems of Hawking and Penrose [9]. The standard approach to argue away these singularities is to delegate their removal to a hypothetical theory of quantum gravity which should begin to take over as the curvature approaches its Planckian value. For a variety of reasons it is clear that GR is bound to fail at this scale. Moreover, at this point even the description of our physical world as a smooth manifold is no longer justified, complicating drastically any approach to quantum gravity.

A different approach to singularity resolution is to allow deviations from GR already at curvature scales some orders below the Planck curvature where the smooth manifold description of spacetime is still a sensible concept. If we can manage to modify gravity at these scales in such a way as to implement an upper bound on all curvatures, we could get rid of singularities on a classical level. Moreover, we would never enter the Planck regime and thus potentially avoid the practical need for a theory of quantum gravity altogether.

Einstein gravity is distinguished among all local, covariant theories of metric gravity in four spacetime dimensions by the fact that it has equations of motion which are only second order. It seems that the only chance to alter this theory and not end up with higher derivatives is to bring something else other than the metric into the game. In the case of the mimetic field, however, this is not an entirely new physical entity. Rather, it represents a reshuffling of the degrees of freedom of the metric itself. The starting point of so-called "mimetic gravity" was in [35] to reparametrize the physical metric $g_{\mu\nu}$ in the form

$$g_{\mu\nu} = h_{\mu\nu} h^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} \tag{4.1}$$

in terms of an auxiliary metric $h_{\mu\nu}$ and a scalar field ϕ, called the mimetic field. Since the physical metric is invariant under Weyl transformations of $h_{\mu\nu}$, the mimetic field takes over the job of representing the conformal degree of freedom of gravity. By definition, ϕ identically satisfies

$$g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} = 1, \tag{4.2}$$

and we can impose the nature of the mimetic field also by adding this condition as a constraint to the gravity action [37], giving it the general form

$$S = \frac{1}{16\pi} \int \mathrm{d}^4 x \sqrt{-g} \left(-\mathcal{L} \left[g_{\mu\nu}, \phi \right] + \lambda \left(g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - 1 \right) \right),$$

where λ is a Lagrange multiplier. This formulation is advantageous over simply applying the reparametrization because at the end of the day we are interested in a theory of the physical metric rather than a theory of the auxiliary metric. Note that constraint (4.2) can be derived as a consequence of 3d volume quantization in noncommutative geometry [79], [81].

Inserting the Einstein-Hilbert Lagrangian $\mathcal{L} = R[g_{\mu\nu}]$ just reproduces standard GR with an additional contribution of mimetic matter (cf. [35], [37]). Different choices for $\mathcal{L}$ with added potentials depending on ϕ or $\Box\phi$ have been considered in [61], [42], [43] and the mimetic field has also been used successfully to define ghost-free massive gravity [82], [83].

Recently, in [1], it was realized that the mimetic field can be used to implement in a covariant manner the idea of a running gravitational and cosmological constant by means of a Lagrangian of the form

$$\mathcal{L} = f[\phi]R[g_{\mu\nu}] + 2\Lambda[\phi] \tag{4.3}$$

where the "inverse gravitational constant" f and "cosmological constant" Λ can depend on ϕ and its derivatives in a way to be determined. To this end, let us find out which covariant quantities can be constructed from ϕ. First, note that by virtue of (4.2), $t := \phi$ qualifies to be used as the time coordinate of a synchronous coordinate system (cf. appendix A),

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \gamma_{ab}\mathrm{d}x^a\mathrm{d}x^b. \tag{4.4}$$

Hence, a simple ϕ dependence of f and Λ would resemble the introduction of a time dependent background. Moreover, the only covariant quantity constructed from first derivatives of ϕ is identically constant by (4.2). Gratifyingly, the second covariant derivatives of ϕ, however, represent measures of the curvature related to the conformal degree of freedom of the gravitational field. More precisely,

$$-\phi_{;ab} = \kappa_{ab} = \frac{1}{2}\frac{\partial}{\partial t}\gamma_{ab}$$

is the extrinsic curvature of the slices of constant ϕ, while $\phi_{;0\alpha} = 0$. The Ricci scalar written out in this synchronous slicing given by ϕ reads

$$-R = 2\dot{\kappa} + \kappa^2 + \kappa^a_b\kappa^b_a + {}^3R, \tag{4.5}$$

where dot denotes t-derivatives, $\kappa^a_b = \gamma^{ac}\kappa_{cb}$, ${}^3R$ is the 3-curvature of the spatial slices and

$$\kappa := \gamma^{ab}\kappa_{ab} = g^{\alpha\beta}\phi_{;\alpha\beta} = \Box\phi$$

is the trace of extrinsic curvature. From expression (4.5) we can read off the reason why the Einstein equation is only second order: because second derivatives of the metric appear only linearly in R and thus only contribute as total derivatives to the action. The only chance to introduce a curvature dependence of the gravitational constant and not spoil this property is $f[\phi] = f(\Box\phi)$. In this case

$$-f(\Box\phi)R = 2\dot{F}(\kappa) + f(\kappa)\left(\kappa^2 + \kappa^a_b\kappa^b_a + {}^3R\right) \tag{4.6}$$

where f is assumed to be integrable with $f(\kappa) = F'(\kappa) \equiv \partial F/\partial\kappa$. Since up to a total derivative such a Lagrangian still contains only first time derivatives of the metric, we can expect the modified Einstein equation of such a theory to be second order in time. While the arguments used to arrive at this result were rather heuristic[†], we can of course explicitly verify this statement. Indeed, the theory defined by

$$\mathcal{L} = f(\Box\phi)R + 2\Lambda(\Box\phi) \tag{4.7}$$

that has been studied in [1] turned out to be free of higher time derivatives in the synchronous frame. In the general spatially non-flat case, however, higher spatial and mixed derivatives will appear. The origin of these terms can be traced back to the presence of $f(\Box\phi)$ in front of ${}^3R$ in (4.6). Luckily, we can use ϕ to dissect this term in a covariant way as

$$\widetilde{R} = 2\phi^{,\mu}\phi^{,\nu}G_{\mu\nu} - (\Box\phi)^2 + \phi^{;\mu\nu}\phi_{;\mu\nu} \doteq {}^3R,$$

where $G_{\mu\nu}$ is the Einstein tensor and $\doteq$ means equality under the condition that (4.2) is satisfied. Subtracting the term that was added involuntarily, we will find the theory defined by

$$\mathcal{L} = f(\Box\phi)R + (f(\Box\phi) - 1)\widetilde{R} + 2\Lambda(\Box\phi)$$

to be free of higher derivatives of all sorts. The addition of the second summand is hence motivated by the same argument that made Einstein gravity unique.

For generality, we will still find it useful to include also a non-linear, spatial curvature dependent potential $h(\widetilde{R})$. It is clear that thereby higher spatial derivatives will reappear, but no higher time or mixed derivatives. While higher time derivatives would typically introduce additional (potentially ghost-like) degrees of freedom, higher spatial derivatives could actually be useful to improve the renormalizability of gravity, along the lines of Hořava gravity [2].

For this extended Lagrangian the interpretation of the functions f and Λ in terms of gravitational and cosmological constant might look less transparent than for (4.7). However, we would like to stress that the homogeneous, spatially flat sectors of both theories are identical.

In the context of a theory like (4.7) it is natural to realize the concept of limiting curvature (cf. [85],[17],[18]) by limiting the measure of curvature provided by $\Box\phi$. Motivated by the analysis of the anisotropic sector made in [1], the concept of "asymptotic freedom" of gravity gains special importance. This name is awarded to modifications where such a limiting curvature is implemented by a vanishing of the gravitational constant at some limiting value $\Box\phi = \kappa_0$ which is a free parameter of the theory and can be chosen well below the Planckian curvature.

In this paper, after presenting the general form of the theory motivated above, we will consider applications to spatially non-flat Friedmann and Bianchi universes and black holes, providing more detailed calculations for the results presented in [3].

[†]Strictly speaking, it is not allowed to use $\Box\phi = \kappa$ and impose gauge conditions in the action before variation.

4.1.2 Action and equations of motion

Let us consider the theory defined by the action

$$S = \frac{1}{16\pi} \int \mathrm{d}^4x \sqrt{-g} \left(-\mathcal{L} + \lambda \left(g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - 1 \right) \right), \tag{4.8}$$

where

$$\mathcal{L} = f(\Box\phi) R + (f(\Box\phi) - 1) \widetilde{R} + 2\Lambda(\Box\phi) + h(\widetilde{R}) \tag{4.9}$$

and

$$\widetilde{R} = 2\phi^{,\mu}\phi^{,\nu} G_{\mu\nu} - (\Box\phi)^2 + \phi^{;\mu\nu}\phi_{;\mu\nu}. \tag{4.10}$$

This action contains two free functions f and Λ of $\Box\phi$, representing the inverse running gravitational constant $G(\Box\phi)^{-1}$ and cosmological constant[‡] $\bar{\Lambda}(\Box\phi)$, respectively. For generality, we also included a spatial curvature dependent potential $h(\widetilde{R})$. In the following we will use Planck units setting $G\left(\Box\phi = 0\right) = G_0 = 1$, such that $f\left(\Box\phi = 0\right) = 1$.

Variation of the action with respect to the metric $g_{\mu\nu}$ yields the modified Einstein equation

$$\begin{aligned}(1 - h')R_{\mu\nu} &- \left(\tfrac{1}{2}\mathcal{L} + (\tilde{Z}\phi^{,\alpha})_{;\alpha} + \Box h' \right) g_{\mu\nu} + \left(\phi_{,\mu}\phi_{,\nu}\tilde{f}^{,\alpha} - \phi_{;\mu\nu}\tilde{f}\phi^{,\alpha} \right)_{;\alpha} \\ &+ 2\tilde{f}\phi^{,\alpha}\phi_{(,\mu}R_{\nu)\alpha} + 2\phi_{(,\mu}\tilde{Z}_{,\nu)} + h'_{;\mu\nu} = (\lambda + \tilde{f}R)\phi_{,\mu}\phi_{,\nu} + 8\pi T^{(\mathrm{m})}_{\mu\nu},\end{aligned} \tag{4.11}$$

where

$$\tilde{f} := f - 1 + h', \qquad Z := \tfrac{1}{2} f' \left((\Box\phi)^2 + \phi^{;\mu\nu}\phi_{;\mu\nu} \right) - \Lambda', \qquad \tilde{Z} := Z - \phi^{,\alpha} h'_{,\alpha},$$

$f' := \mathrm{d}f/\mathrm{d}\Box\phi$, $\Lambda' := \mathrm{d}\Lambda/\mathrm{d}\Box\phi$, $h' := \mathrm{d}h/\mathrm{d}\widetilde{R}$ and $T^{(\mathrm{m})}_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S^{\mathrm{m}}}{\delta g^{\mu\nu}}$ is the matter energy momentum tensor. For the detailed calculations in the variation of the action the interested reader is referred to appendix B. While at first glance this modified Einstein equation looks more involved than the one presented in [1], calculating its components in the synchronous frame will reveal that in the general case, apart from the new terms due to h, the theory defined by (4.9) is in fact the simpler one.

The evolution of the mimetic field is already completely determined by the constraint (4.2), which we obtain from variation with respect to the Lagrange multiplier. The equation obtained by varying (4.8) with respect to ϕ hence can only return the favor and provide a condition to determine λ. Conveniently written in terms of the quantity

$$\Xi := \lambda + \tilde{f}\left(R - R_{\mu\nu}\phi^{,\mu}\phi^{,\nu} \right) - \Box f - \phi^{,\mu} Z_{,\mu} - \phi^{,\mu} {h'}_{,\mu} \Box\phi, \tag{4.12}$$

this "equation of motion" of ϕ reads

$$\begin{aligned}\left(\Xi\, \phi^{,\nu} \right)_{;\nu} = \Big[&(f - h')^{,\mu}\phi^{;\nu}_{\mu} + Z^{,\nu} - \phi^{,\nu}\phi^{,\mu} Z_{,\mu} + \Box\phi \left(h'^{,\nu} - \phi^{,\nu}\phi^{,\mu} h'_{,\mu} \right) \\ &+ \tilde{f}\left(R^{\mu\nu}\phi_{,\mu} - R^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}\phi^{,\nu} \right) \Big]_{;\nu}.\end{aligned} \tag{4.13}$$

[‡]Note that for this interpretation we still need to factor out the gravitational constant such that $\Lambda = f\bar{\Lambda}$.

In the synchronous frame the right hand side turns out to be just the 3-divergence of a 3-vector (denoted by $X^a_{|a}$) and we find the solution

$$\Xi = \tfrac{1}{\sqrt{\gamma}} \int \mathrm{d}t \sqrt{\gamma} \left(\kappa^a_b f^{,b} + Z^{,a} + \tilde{f} R^a_0 + \kappa h'^{,a} - \kappa^a_b h'^{,b} \right)_{|a} . \tag{4.14}$$

Let us now evaluate (4.11) in the synchronous frame $t = \phi$ where it takes its simplest form. Using (4.12), the $0-0$ component of the modified Einstein equation becomes

$$\frac{1}{3}\left(f - 2\kappa f'\right)\kappa^2 - \Lambda + \kappa\Lambda' - \frac{1}{2}\left(f + \kappa f'\right)\tilde{\kappa}^a_b \tilde{\kappa}^b_a = \frac{1}{2}\left(h - {}^3R\right) + \Xi + 8\pi T^{(\mathrm{m})}_{00}, \tag{4.15}$$

where $\tilde{\kappa}^a_b := \kappa^a_b - \frac{1}{3}\kappa\delta^a_b$ is the traceless part of the extrinsic curvature. Note that inserting the solution (4.14) for Ξ, this equation becomes in general an integro-differential equation. Suitably taking another time derivative $(\%\phi^\mu)_{;\mu}$ of (4.15) yields a differential equation containing second time derivatives of the metric. This is a manifestation of the fact that in mimetic gravity the conformal degree of freedom of the gravitational field becomes dynamical.

Note that for spaces where ${}^3R = 0$ and the integrand in Ξ vanishes by homogeneity, (4.15) is precisely the same as the $0-0$ component of the modified Einstein equation presented in [1].

The spatial components of (4.11) after raising one index read

$$-\frac{1}{\sqrt{\gamma}}\,\partial_t\left(\sqrt{\gamma}\,\left(f\kappa^a_b + Z\delta^a_b\right)\right) - \tfrac{1}{2}\mathcal{L}\,\delta^a_b = S^a_b + 8\pi T^{(\mathrm{m})a}{}_b, \tag{4.16}$$

where

$$S^a_b := (1-h'){}^3R^a_b + h'{}^{|a}_{\ b} - \Delta h'\delta^a_b \tag{4.17}$$

contains spatial curvature terms. Subtracting from this equation one third of its (spatial) trace removes all isotropic terms proportional to δ^a_b and we obtain

$$-\frac{1}{\sqrt{\gamma}}\,\partial_t\left(\sqrt{\gamma}\,f\,\tilde{\kappa}^a_b\right) = \tilde{S}^a_b + 8\pi\tilde{T}^{(\mathrm{m})a}{}_b \tag{4.18}$$

where the right hand side consists of the traceless parts of S^a_b and $T^{(\mathrm{m})a}{}_b$. The spatial components of the modified Einstein equation are hence second order in time. A non-linear potential $h({}^3R)$ introduces higher spatial derivatives of up to forth order.

Finally, the mixed components of the modified Einstein equation (4.11) read

$$fR_{0a} + Z_{,a} + \kappa^b_a f_{,b} = 8\pi T^{(\mathrm{m})}_{0a}. \tag{4.19}$$

These equations, as in standard GR, contain only first time derivatives of the metric and could be thought of as a constraint that needs to be satisfied on an initial hypersurface $\phi = \phi_{\mathrm{i}}$ and then continues to hold by virtue of validity of the other components of the modified Einstein equation. Note that h does not appear in the mixed equations. Moreover, (4.19) can be used to simplify (4.14) to

$$\Xi = \tfrac{1}{\sqrt{\gamma}} \int \mathrm{d}t \sqrt{\gamma} \, \left(8\pi T^{(\mathrm{m})a}{}_{0} - (1-h')R^a_0 + \kappa h'^{,a} - \kappa^a_b h'^{,b}\right)_{|a} . \tag{4.20}$$

Note that time reversal invariance of general relativity is maintained in our modification if we choose f and Λ as symmetric functions of κ. Moreover, if we require

$$f = 1 + \mathcal{O}\left(\kappa^2\right), \quad \Lambda = \mathcal{O}\left(\kappa^4\right), \quad h = \mathcal{O}\left(\widetilde{R}^2\right),$$

then in the limit of low curvatures (4.15), (4.16) and (4.19) are just the components of the usual Einstein equation with a contribution of mimetic matter, given by the constant of integration in Ξ.

4.2 Non-flat Universes

4.2.1 Friedmann Universes

The metric of a homogeneous, isotropic universe with cosmological time t is given by

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left(\frac{\mathrm{d}r^2}{1-\varkappa r^2} + r^2 \mathrm{d}\Omega^2\right), \tag{4.21}$$

where $\varkappa \in \{-1, 0, +1\}$ and $\mathrm{d}\Omega^2 = \mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2$. Note that the unique solution of constraint (4.2) in such a spacetime which is compatible with homogeneity is $\phi = t + const$. Hence the quantities in the synchronous frame (4.4) are given by

$$\kappa = \frac{\dot{u}}{u} = 3\,\frac{\dot{a}}{a}, \qquad\qquad {}^3\!R = \frac{6\varkappa}{a^2},$$

where we introduced $u := a^3$. Moreover, by isotropy, $\tilde{\kappa}^a_b = 0$ and by homogeneity $\Xi \propto 1/u$ describes only the dust-like contribution of mimetic matter. For the sake of simplicity and because the remaining equation is still general enough for all our purposes we make again the simplifying choice

$$\Lambda = \frac{2}{3}\kappa^2(f-1), \tag{4.22}$$

familiar from [1]. The $0-0$ modified Einstein equation (4.15) then becomes

$$\left(f - \frac{2}{3}\right)\kappa^2 = \frac{1}{2}\left(h({}^3\!R) - {}^3\!R\right) + \varepsilon \tag{4.23}$$

where $\varepsilon := \Xi + 8\pi T^{(\mathrm{m})}_{00}$ is the total energy density of mimetic and ordinary matter. Note that this modified Friedmann equation is still formulated in terms of the same variables as the original Friedmann equation. Only the relation between curvature and energy density is changed at large curvatures. While the left hand side of (4.23) depends on κ^2 only, the right hand side is in general some function of u. Such a relation can be thought of as an integral curve in the "phase space" spanned by u and $\kappa = \mathrm{d}\ln u/\mathrm{d}t$. Drawing this phase portrait for a specific modification will allow us to understand its qualitative behaviour without the need to obtain explicit solutions.

Spatially flat universe: Let us use this phase portrait technique to show that in the case $\varkappa = 0$ there is essentially only one possibility for the behaviour of a non-singular modification. Since the total energy density ε is in general a monotonically decreasing function in u, in this case (4.23) can be understood as a relation of the form $u(\kappa^2)$. Furthermore, if this relation is to describe a non-singular modified Friedmann equation, it must be one-to-one. Otherwise, if at some point $\mathrm{d}u/\mathrm{d}\kappa^2 = 0$, then

$$\dot{\kappa} = \dot{u}\frac{\mathrm{d}\kappa}{\mathrm{d}u} = u\,\kappa\frac{\mathrm{d}\kappa}{\mathrm{d}u} = \frac{u}{2}\,\frac{\mathrm{d}\kappa^2}{\mathrm{d}u} \tag{4.24}$$

would diverge. Since in the general non-vacuum case we cannot expect divergences of $\dot{\kappa}$ and κ^2 to cancel out exactly in the Ricci scalar (4.5), both quantities have to be bounded separately to avoid a curvature singularity. Hence, in the following we will assume that the relation provided by (4.23) is of the form $\kappa^2(u)$ and it is one-to-one in the case $\varkappa = 0$.

By (4.24), boundedness of $\dot{\kappa}$ implies that the domain of definition of the relation $\kappa^2(u)$ can be extended to $u \in [0, \infty)$. Boundedness of κ implies that

$$\int_0^\infty \mathrm{d}u\frac{\mathrm{d}\kappa}{\mathrm{d}u} = -\kappa(u=0) \tag{4.25}$$

has a finite value, where we made use of the low curvature limit $\kappa^2(u \to \infty) = 0$. At the lower bound, the integral can only converge if $u\frac{\mathrm{d}\kappa}{\mathrm{d}u} \to 0$. By (4.24) it follows that $\dot{\kappa} \to 0$. Hence, in this limit κ must be asymptotically constant at some limiting value $\pm\kappa_0$. Recalling that κ is the logarithmic derivative of u, this means that asymptotically

$$u = a^3 \propto \exp\left(\pm\kappa_0 t\right) \tag{4.26}$$

as $t \to \mp\infty$. In conclusion, the most natural modifications generically replace Big Bang/Big Crunch singularities by a smooth transition to a de Sitter-like initial/final state with limiting curvature. In [1] we provided a concrete example of a non-singular, spatially flat universe using the simple choice

$$f = \frac{1}{1-(\kappa^2/\kappa_0^2)}. \tag{4.27}$$

Assuming $\varepsilon \propto (1/u)^{1+w}$, we found the implicit solution

$$\frac{1+w}{2}\kappa_0 t = \frac{\kappa_0}{\kappa} - \operatorname{atanh}\frac{\kappa}{\kappa_0} - \sqrt{2}\arctan\left(\sqrt{2}\frac{\kappa}{\kappa_0}\right) \tag{4.28}$$

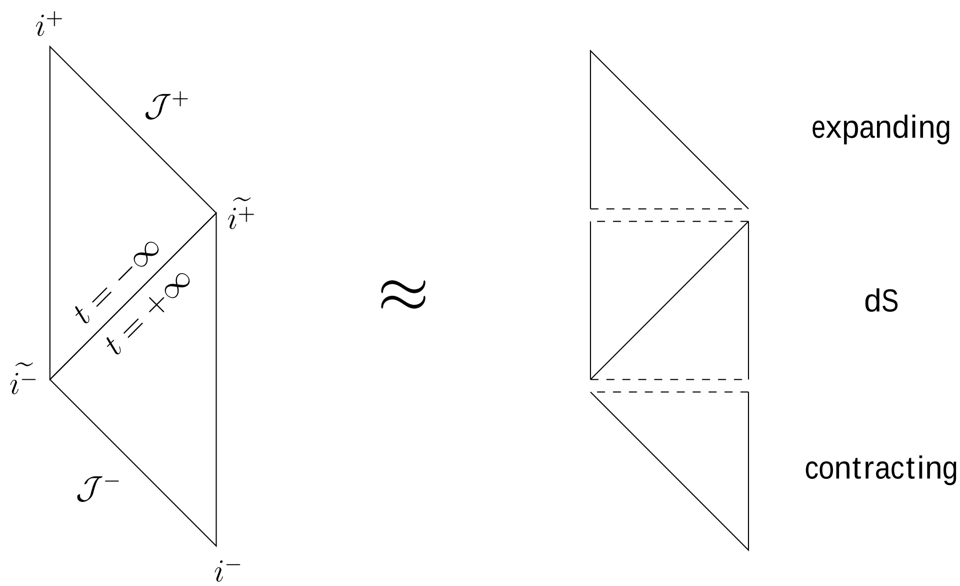

Figure 4.1: Conformal diagram of a modified spatially flat Friedmann universe.

for $\kappa(t)$. The expanding branch $\kappa > 0$ describes a smooth transition from an expanding de Sitter universe to an expanding Friedmann universe with $a \propto t^{2/3(1+w)}$. Its conformal diagram is given by the upper triangle of the left diagram of figure (4.1), cf. [86]. Comoving geodesics start from $\widetilde{i^-}$ and reach i^+ after infinite proper time. Other causal geodesics, however, are past incomplete in the same way as in an expanding de Sitter space in the flat slicing. At the line $t = -\infty$ all curvature invariants are bounded and hence we can complete the diagram by gluing the contracting solution $\kappa < 0$, which is related to the expanding solution simply by time reversal. Albeit $\phi = t$ is obviously discontinuous at the junction, the metrics can be joined smoothly just like the expanding and contracting de Sitter space. In this way we obtain a geodesically complete, non-singular spacetime.

Non-flat universe: Let us now extend our analysis to the spatially non-flat case $\varkappa = \pm 1$. In this case the relation $\kappa^2(u)$ provided by (4.23) does not have to be bijective. In fact, if it were one-to-one, the same arguments as above would apply and we would run into a curvature singularity as ${}^3R(u \to 0) \to \infty$. This can only be avoided by a bounce at some finite $u_{\min}$.

A zero of the relation $\kappa^2(u)$ at $u_{\min}$ can connect the contracting half-plane $\kappa < 0$ to the expanding half-plane $\kappa > 0$. If at this point also $\dot{\kappa}$ would vanish, there would be a static fixed point solution at $(u, \kappa) = (u_0, 0)$. Thus, by (4.24), only a zero of first order of $\kappa^2(u)$ can describe an actual bounce.

For concreteness, consider the case where $\varepsilon = 3c/u^{1+w}$ and make the simple power law ansatz

$$h({}^3R) = -6\left(\frac{\delta}{6}{}^3R\right)^{2n} = -6\left(\frac{\delta}{a^2}\right)^{2n}, \tag{4.29}$$

where the prefactors were chosen for later convenience and we restrict to even powers in order to obtain the same high curvature modifications for both $\varkappa = \pm 1$. The right hand side of the modified Friedmann equation (4.23) then becomes proportional to

$$\frac{c}{a^{3(1+w)}} - \frac{\varkappa}{a^2} - \left(\frac{\delta}{a^2}\right)^{2n}.$$

Since the left hand side of (4.23) should be a one-to-one function of κ^2 which is linear in the low curvature limit, a zero of this right hand side means that $\kappa = 0$ at such a point. In standard GR, setting $\delta = 0$, this right hand side has just one non-trivial zero, namely in the case $\varkappa = 1$ at

$$a_{\text{max}} = c^{1/(1+3w)}, \tag{4.30}$$

corresponding to the moment of recollapse of a closed universe.

In our modification, by the choice of sign of h, there is now also another first order zero describing a bounce. At the moment of bounce the linear contribution of ${}^3R$ is negligible compared to h and we find

$$a_{\text{min}} = \left(\frac{\delta^{2n}}{c}\right)^{1/(4n-3(1+w))}. \tag{4.31}$$

Assuming that $a_{\max} \gg a_{\min}$, there is some intermediate region where the energy density ε dominates both spatial curvature terms. In this region (4.23) becomes like in the spatially flat case and we expect a stage where $\kappa^2 \sim \kappa_0^2$ is approximately constant at the limiting curvature. A contracting universe will hence undergo a stage of exponential contraction before going through a bounce followed by exponential expansion.

We can estimate the duration of this inflationary stage which is, by time reversal symmetry, equal to the duration of the stage of exponential contraction. Inflation will end at the moment when κ^2 and hence also the left hand side of (4.23) drops below the order of κ_0^2. This will happen around $a = a_f$ where

$$a_f^{3(1+w)} \sim \frac{c}{\kappa_0^2}. \tag{4.32}$$

The value a_i at the beginning of inflation will not be much different from a_{min}. The number of e-folds expressed through the dimensionless quantities $\tilde{c}$ and $\tilde{\delta}$ defined by

$$\tilde{c} = c\,\kappa_0^{3(1+w)-2}, \qquad \tilde{\delta} = \delta\,\kappa_0^{2-1/n},$$

can hence be approximated by

$$N \sim \frac{2n}{4n-3(1+w)} \ln\left(\tilde{c}^{\frac{2}{3(1+w)}}\,\tilde{\delta}^{-1}\right). \tag{4.33}$$

A necessary number of e-folds can be achieved e.g. by an exponentially small value of $\tilde{\delta}$ or an exponentially large value of $\tilde{c}$. Note that for radiation with $w = 1/3$ we have to choose

$n > 1$ in order to still have a bounce. On the other hand, for $n = 1$, a value smaller than, but close to $1/3$,

$$w = \frac{1}{3}(1 - \epsilon), \quad \epsilon \ll 1, \tag{4.34}$$

can also explain a large number of e-folds even when all other dimensionless parameters are of the order of unity. In this case

$$N \sim \frac{\ln(\tilde{c}/\tilde{\delta}^2)}{\epsilon}. \tag{4.35}$$

Let us illustrate this in a simple concrete example for a closed universe ($\varkappa = 1$) and with the familiar choice (4.27) for the function f. In this case (4.23) becomes

$$\frac{1}{9}\kappa^2 \left(\frac{1 + 2\left(\kappa^2/\kappa_0^2\right)}{1 - \left(\kappa^2/\kappa_0^2\right)} \right) = \frac{1}{a^2} \left[\left(\frac{a_{\text{max}}}{a} \right)^{2-\epsilon} \left(1 - \left(\frac{a_{\text{min}}}{a} \right)^{\epsilon} \right) - 1 \right], \tag{4.36}$$

where $a_{\max}$ and $a_{\min}$ are defined by (4.30) and (4.31). Let us analyze the asymptotics of this equation starting at the moment of recollapse where we set $t = 0$. At this point $a \sim a_{\max} \gg a_{\min}$ and $\kappa \ll \kappa_0$. Moreover, the deviation from $w = 1/3$ is irrelevant for the behaviour in this region and we set $\epsilon = 0$ for simplicity. Recalling that $\kappa = 3\dot{a}/a$, (4.36) in this case becomes

$$\dot{a}^2 = \left(\frac{a_{\text{max}}}{a} \right)^2 - 1 \tag{4.37}$$

and has the solution

$$a(t) = \sqrt{a_{\text{max}}^2 - t^2}. \tag{4.38}$$

This would be the full exact solution of (4.36) in standard GR, i.e. if we would set $a_{\min} = 0$ and $\kappa_0 \to \infty$. It describes a closed universe starting from a Big Bang at $t = -a_{\max}$, expanding until $t = 0$, then recollapsing until finally reaching a Big Crunch at $t = a_{\max}$. Since for this solution

$$\kappa(t) = \frac{3t}{t^2 - a_{\text{max}}^2}, \tag{4.39}$$

both Big Bang and Big Crunch represent curvature singularities. In our theory, however, κ becomes of order of the limiting curvature at $t - a_{\max} \sim 1/\kappa_0$ and the modification starts to take over.

In the region where the energy density is dominating both spatial derivative terms, (4.36) becomes like for a flat Friedmann universe. The exact implicit solution of an equation of this form was obtained in [1] and for the case at hand it reads

$$\frac{2}{3}\kappa_0(t - a_{\max}) = \frac{\kappa_0}{\kappa} - \operatorname{atanh}\frac{\kappa}{\kappa_0} - \sqrt{2}\arctan\left(\sqrt{2}\frac{\kappa}{\kappa_0}\right), \tag{4.40}$$

where the constant of integration was fixed such that the $-\kappa \ll \kappa_0$ asymptotic

$$\kappa = \frac{3}{2(t - a_{\max})} \tag{4.41}$$

of (4.40) matches the $t \sim a_{\max}$ asymptotic of (4.39). The contracting branch of (4.40) describes a smooth transition between the asymptotic (4.41) at $t \ll a_{\max}$ and the asymptotically constant solution $\kappa \sim -\kappa_0$ at $t \gg a_{\max}$. Hence, the scale factor does not vanish at $t = a_{\max}$ but after $t \gg a_{\max}$ starts to decrease exponentially as

$$a \propto \exp\left(-\tfrac{1}{3}\kappa_0 t\right). \tag{4.42}$$

Contrary to the spatially flat case, the exponential contraction now cannot continue until $t \to \infty$ because at some point the spatial curvature dependent potential will start to counteract this contraction and thus prevent an unbounded growth of the spatial curvature. Expanded around $a \sim a_{\min}$ (4.36) becomes

$$\kappa^2/\kappa_0^2 \left(\frac{1 + 2\left(\kappa^2/\kappa_0^2\right)}{1 - \left(\kappa^2/\kappa_0^2\right)}\right) = \tilde{\epsilon}\left(\frac{a}{a_{\mathsf{min}}} - 1\right), \tag{4.43}$$

where

$$\tilde{\epsilon} := \epsilon \, \frac{9}{\kappa_0^2 \, a_{\min}^2} \left(\frac{a_{\mathsf{max}}}{a_{\min}}\right)^{2-\epsilon}. \tag{4.44}$$

Isolating a from this equation on one side and taking the time derivative of the logarithm of this equation, we obtain a separable first order differential equation for κ with the implicit solution

$$\frac{1}{6}\kappa_0(t - t_b) = \operatorname{atanh}\frac{\kappa}{\kappa_0} + \tilde{\epsilon}_- \arctan\left(\tilde{\epsilon}_- \frac{\kappa}{\kappa_0}\right) + \tilde{\epsilon}_+ \arctan\left(\tilde{\epsilon}_+ \frac{\kappa}{\kappa_0}\right), \tag{4.45}$$

where the constant of integration t_b corresponds the moment of bounce and

$$\tilde{\epsilon}_\pm^2 := \frac{4}{\tilde{\epsilon} - 1 \pm \sqrt{1 - 10\tilde{\epsilon} + \tilde{\epsilon}^2}}.$$

The $t \ll t_b$ asymptotic of the contracting branch of this solution is $\kappa \sim -\kappa_0$, in agreement with the late time asymptotic of (4.40). Note that the sign in front of the atanh term in (4.45) is opposite to that in (4.40). This means that $-\kappa$ is now decreasing again until $\kappa = 0$ at $t = t_b$. Expansion of (4.45) around $\kappa \sim 0$ yields

$$\kappa = \frac{\tilde{\epsilon}}{6}\kappa_0(t - t_b). \tag{4.46}$$

Integrating again, we find a smooth bounce described by the leading order solution

$$a = a_{\mathsf{min}} \left(1 + \frac{\tilde{\epsilon}}{12}\kappa_0 \left(t - t_b\right)^2\right)^{1/3}. \tag{4.47}$$

At $t = t_b$ we pass from the contracting into the expanding halfplane. By time reversal invariance, the solution in the expanding plane will be just a mirror image of the solution in the contracting plane. Hence, after the bounce κ will grow until reaching κ_0 as described by the expanding branch of (4.45). It will stay approximately constant for the number of e-folds given by (4.35) followed by a smooth graceful exit described by the expanding branch of (4.40). Finally, after $\kappa \ll \kappa_0$, the term linear in spatial curvature will dominate again and cause a recollapse according to (4.38), restarting the whole cycle anew. The solution is hence eternally oscillating clockwise in the phase space (u, κ) on the closed trajectory described by (4.36).

4.2.2 Bianchi Universes

The general§ form of a homogeneous but not necessarily isotropic spatial metric is given by

$$\gamma_{ab} = \gamma_{AB}\, e^A_a e^B_b\,, \tag{4.48}$$

where γ_{AB} is spatially constant and the frame one-forms e^A_a are constant in time and satisfy

$$\left(\frac{\partial e^C_a}{\partial x^b} - \frac{\partial e^C_b}{\partial x^a}\right) e^a_A e^b_B = \mathcal{C}^C_{AB}. \tag{4.49}$$

Here e^a_A is the inverse of e^A_a and $\mathcal{C}^C_{AB}$ are the structure constants of the group of motion of the three-dimensional homogeneous space under consideration [84], [90]. The inverse metric and the metric determinant are given by

$$\gamma^{ab} = \gamma^{AB}\, e^a_A e^b_B, \qquad \sqrt{\gamma} = uv, \tag{4.50}$$

where γ^{AB} is the inverse of γ_{AB}, v is the determinant of e^A_a and u^2 is the determinant of γ_{AB} and hence depends only on time.

The extrinsic curvature of these homogeneous spatial slices is given by

$$\kappa_{ab} = \frac{1}{2}\dot{\gamma}_{AB}\, e^A_a e^B_b =: \kappa_{AB}\, e^A_a e^B_b, \qquad \kappa^a_b = \frac{1}{2}\gamma^{AC}\dot{\gamma}_{CB}\, e^a_A e^B_b =: \kappa^A_B\, e^a_A e^B_b \tag{4.51}$$

and we see that

$$\kappa = \kappa^a_a = \kappa^A_A = \frac{\dot{u}}{u}. \tag{4.52}$$

The spatial connection coe□cients are given by

$$\lambda^c_{ab} = \frac{\partial e^C_a}{\partial x^b}\, e^c_C - \mathcal{A}^C_{AB}\, e^c_C e^A_a e^B_b, \tag{4.53}$$

with the spatially constant coe□cients

$$\mathcal{A}^C_{AB} = \frac{1}{2}\left(\mathcal{C}^C_{AB} - \mathcal{C}^E_{DB}\gamma_{EA}\gamma^{DC} - \mathcal{C}^E_{DA}\gamma_{EB}\gamma^{DC}\right). \tag{4.54}$$

§Excluding Kantowski-Sachs type models which bear more similarity with the interior of black holes than with cosmological models, cf. section 4.3.2.

Note that $\gamma^{AB}\mathcal{C}^C_{AB} = 0$ and $\mathcal{A}^C_{DC} = \mathcal{C}^C_{DC}$. The mixed components of the $4-$Ricci tensor are given by

$$R^0_a = \left(\kappa^D_C \mathcal{C}^C_{AD} - \kappa^D_A \mathcal{C}^E_{DE}\right) e^A_a, \tag{4.55}$$

and the spatial Ricci curvature is

$$^3R_{ab} = {}^3R_{AB}\, e^A_a e^B_b, \qquad {}^3R_{AB} = \mathcal{A}^C_{DC}\mathcal{A}^D_{AB} - \mathcal{A}^D_{BC}\mathcal{A}^C_{AD}. \tag{4.56}$$

The spatial Bianchi identity reads

$$^3R^D_C \mathcal{C}^C_{AD} - {}^3R^D_A \mathcal{C}^E_{DE} = 0. \tag{4.57}$$

Note that $^3R^a_a = {}^3R^A_A = {}^3R$ is constant in space. Hence we can immediately solve (4.14) to find that

$$\Xi \propto \frac{1}{u}, \tag{4.58}$$

and the spatial curvature contribution (4.17) to the spatial modified Einstein equation is just

$$S^a_b = (1-h')^3 R^a_b. \tag{4.59}$$

The whole modified Einstein equation can be expressed independent of the frame vectors. Let us restrict to the case of isotropic, comoving matter, e.g. dust or mimetic matter, with total energy density ε. Then the temporal equation (4.15) reads

$$\frac{1}{3}\left(f - 2\kappa f'\right)\kappa^2 - \Lambda + \kappa\Lambda' = \frac{1}{2}\left(f + \kappa f'\right)\tilde{\kappa}^A_B\tilde{\kappa}^B_A + \frac{1}{2}\left(h - {}^3R\right) + \varepsilon. \tag{4.60}$$

The trace subtracted spatial equations (4.18) become

$$-\frac{1}{u}\,\partial_t\left(u\, f\, \tilde{\kappa}^A_B\right) = (1-h')\left({}^3R^A_B - \frac{1}{3}{}^3R\,\delta^A_B\right). \tag{4.61}$$

Contracting with $\tilde{\kappa}^B_A$ we find the useful equation

$$\frac{1}{fu^2}\partial_t\left(u^2 f^2 \tilde{\kappa}^A_B\tilde{\kappa}^B_A\right) = -2\left(1-h'\right){}^3R^A_B\,\tilde{\kappa}^B_A. \tag{4.62}$$

Finally, the mixed component equation is simply

$$\tilde{\kappa}^D_C \mathcal{C}^C_{AD} - \tilde{\kappa}^D_A \mathcal{C}^E_{DE} = 0. \tag{4.63}$$

Let us in the following assume that the frame metric γ_{AB} is diagonal. This additional assumption is an expression of non-rotating Kasner axes, cf. [90].

Bianchi type I.

This is the case where all structure constants vanish and the spatial slices are hence Euclidean. (4.61) is easily integrated and yields

$$\tilde{\kappa}^A_B = \frac{\lambda^A_B}{fu}, \tag{4.64}$$

with constants of integration λ^A_B and (4.60) becomes

$$\frac{1}{3}\left(f - 2\kappa f'\right)\kappa^2 - \Lambda + \kappa\Lambda' = \frac{f + \kappa f'}{2f^2}\frac{\bar{\lambda}^2}{u^2} + \varepsilon, \tag{4.65}$$

where $\bar{\lambda}^2 := \lambda^A_B \lambda^B_A$. Like in the Friedmann case, this equation defines an integral curve in the phase space spanned by u and κ. Provided that $(f + \kappa f')/f^2$ is bounded, it will look qualitatively similar to (4.23) for the Friedmann universe. At low curvatures where $\kappa \ll \kappa_0$ and $u \to \infty$, the right hand side will be dominated by any contribution of matter with equation of state $w < 1$. The evolution of u is then given by

$$u \propto t^{2/(1+w)} \tag{4.66}$$

like in a Friedmann universe. Moreover, according to (4.64), the anisotropic extrinsic curvature contribution $\tilde{\kappa}^A_B \propto u^{-1}$ will decay at $t \to \infty$ faster than $\kappa \propto t^{-1}$. This means that the presence of any kind of isotropic matter with $w < 1$ will eventually lead to isotropy of an expanding universe.

On the other hand, approaching $u \to 0$, we see that the term u^{-2} coming from curvature due to anisotropy will dominate any such matter contribution. This is why in order to understand the behaviour close to singularities it is su□cient to study the vacuum case. Without modifications, the vacuum solution is given by the Kasner metric, featuring a singularity. In an attempt to remove this anisotropic singularity, we will find that the property of asymptotic freedom is an inevitable condition for any non-singular modification:

Just as for a modified flat Friedmann universe, (4.65) establishes a one-to-one relation between κ and u. By the same argument as in the last section, the only way it can be non-singular is for κ to tend to its constant limiting value as $u \to 0$. While in this case κ^2 as well as $\dot{\kappa}$ are bounded,

$$\tilde{\kappa}^A_B \tilde{\kappa}^B_A = \frac{\bar{\lambda}^2}{f^2 u^2} \tag{4.67}$$

will become singular, unless that $f(\kappa)$ diverges fast enough as $\kappa \to \kappa_0$. It follows that the only way to avoid a curvature singularity in (4.5) is a fast enough divergence of f at the limiting curvature. Moreover, if $1/(fu) \to 0$, then the anisotropic Kasner solution will even become isotropic close to the limiting curvature. A concrete illustration of this was presented in [1].

Bianchi type V.

As an example for an anisotropic, spatially non-flat universe where we can still understand the modified Einstein equation in terms of a two dimensional phase portrait, we consider Bianchi type V. Here the non-vanishing structure constants are determined by

$$\mathcal{C}^{\bar{2}}_{\bar{1}\bar{2}} = 1, \qquad \mathcal{C}^{\bar{3}}_{\bar{1}\bar{3}} = 1. \tag{4.68}$$

We calculate

$$\mathcal{A}^{\bar{2}}_{\bar{1}\bar{2}} = \mathcal{A}^{\bar{3}}_{\bar{1}\bar{3}} = 1, \qquad \mathcal{A}^{\bar{1}}_{\bar{2}\bar{2}} = -\gamma^{\bar{1}\bar{1}}\gamma_{\bar{2}\bar{2}}, \qquad \mathcal{A}^{\bar{1}}_{\bar{3}\bar{3}} = -\gamma^{\bar{1}\bar{1}}\gamma_{\bar{3}\bar{3}}, \tag{4.69}$$

and find that the spatial curvature components in the frame are given by

$${}^3R^{\bar{1}}_{\bar{1}} = {}^3R^{\bar{2}}_{\bar{2}} = {}^3R^{\bar{3}}_{\bar{3}} = -2\gamma^{\bar{1}\bar{1}}. \tag{4.70}$$

Hence the spatial curvature is still isotropic and (4.61) has the first integral

$$\tilde{\kappa}^A_B = \frac{\lambda^A_B}{fu}. \tag{4.71}$$

By the mixed components (4.63) it follows that

$$\tilde{\kappa}^{\bar{1}}_{\bar{1}} = 0, \qquad \tilde{\kappa}^{\bar{2}}_{\bar{2}} = -\tilde{\kappa}^{\bar{3}}_{\bar{3}} =: \frac{\tilde{\lambda}}{fu}, \tag{4.72}$$

where we can assume without loss of generality that the constant of integration $\tilde{\lambda} \geq 0$. Integrating again yields the frame metric

$$\gamma_{AB} = u^{2/3}\,\mathrm{diag}\left(1/\alpha^2, b^2, \alpha^2 b^{-2}\right), \tag{4.73}$$

where α is a constant of integration and b is a function of time. The differential equations (4.49) are solved by the frame vectors

$$e^{\bar{1}}_a = \alpha\,\delta^1_a, \quad e^{\bar{2}}_a = e^{\alpha x}\,\delta^2_a, \quad e^{\bar{3}}_a = e^{\alpha x}/\alpha\,\delta^3_a \tag{4.74}$$

and, fixing overall constant factors, this frame metric corresponds to the spacetime metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - u^{2/3}\left[\mathrm{d}x^2 + e^{2\alpha x}\left(b^2\mathrm{d}y^2 + b^{-2}\mathrm{d}z^2\right)\right]. \tag{4.75}$$

Note that the spatial slices are spaces of constant negative curvature. The equation (4.60) for this class becomes

$$\frac{1}{3}\left(f - 2\kappa f'\right)\kappa^2 - \Lambda + \kappa\Lambda' = \frac{f + \kappa f'}{f^2}\frac{\tilde{\lambda}^2}{u^2} + \frac{1}{2}\left(h(-6\alpha^2 u^{-2/3}) + 6\alpha^2 u^{-2/3}\right) + \varepsilon. \tag{4.76}$$

It is again just an integral curve in the phase space (u, κ) which can be treated as for the non-flat Friedmann universe. If h includes a term $\propto ({}^3R)^n$ with $n \geq 3$ and has the right sign, it will describe a bounce in the same way as (4.36). The solution for b is determined by

$$\frac{\dot{b}}{b} = \tilde{\kappa}^{\bar{2}}_{\bar{2}} = \frac{\tilde{\lambda}}{fu} \geq 0. \tag{4.77}$$

Hence b is monotonically increasing and the moment of greatest slope of b is at the bounce, where u assumes its minimum, $\kappa = 0$ and $f(\kappa) = 1$.

Long before/long after the bounce, i.e. at $u \gg u_{\min}$, the linear contribution of spatial curvature $\propto u^{-2/3}$ is dominating both h and $\tilde{\lambda}^2/u^2$. Moreover, in this region $\kappa^2 \ll \kappa_0^2$. In the case of vacuum, the asymptotic solution of (4.76) is hence given by

$$u(t) = (\alpha\,|t|)^3\,. \tag{4.78}$$

Integrating (4.77) then yields

$$b \propto \exp\left(\frac{-\tilde{\lambda}}{2\alpha^3 t^2}\right) \xrightarrow[t\to\pm\infty]{} b_0^{\pm}. \tag{4.79}$$

Fixing the constant of integration, we can achieve that $b_0^+ = 1/b_0^-$. Hence, starting at $t \to -\infty$ from a contracting spacetime, after the bounce we obtain the time reversed expanding spacetime where the directions y and z are interchanged. Since $\kappa^a_b\kappa^b_a$ is everywhere bounded and in the early/late time asympotic it holds that $\kappa^a_b\kappa^b_a \propto \kappa^2 \propto 1/t^2$, the condition for causal completeness from appendix A is satisfied.

Bianchi types II, VI$_0$, VII$_0$, VIII, IX.

These five Bianchi types can be treated on a common footing by labeling the non-vanishing structure constants as

$$\mathcal{C}^{\bar{1}}_{\bar{2}\bar{3}} = \lambda, \qquad \mathcal{C}^{\bar{2}}_{\bar{3}\bar{1}} = \mu, \qquad \mathcal{C}^{\bar{3}}_{\bar{1}\bar{2}} = \nu. \tag{4.80}$$

The individual classes can then be read off from the following table:

	II	VI$_0$	VII$_0$	VIII	IX
λ	1	1	1	1	1
μ	0	-1	1	1	1
ν	0	0	0	-1	1

Corresponding solutions for the frame vectors can be found in [90]. Taking a diagonal metric, the mixed components (4.63) are trivially satisfied. Parametrizing the frame metric as

$$\gamma_{AB} = u^{2/3}\mathrm{diag}(a^2, b^2, c^2), \tag{4.81}$$

where $abc = 1$, it holds that

$$\tilde{\kappa}^{\bar{1}}_{\bar{1}} = \frac{\dot{a}}{a}, \qquad \tilde{\kappa}^{\bar{2}}_{\bar{2}} = \frac{\dot{b}}{b}, \qquad \tilde{\kappa}^{\bar{3}}_{\bar{3}} = \frac{\dot{c}}{c}. \tag{4.82}$$

The spatial curvature components are given by

$$ {}^3R^{\bar{1}}_{\bar{1}} = \frac{1}{2u^{2/3}} \left(\lambda^2 a^4 - (\mu b^2 - \nu c^2)^2\right), \tag{4.83} $$

$$ {}^3R^{\bar{2}}_{\bar{2}} = \frac{1}{2u^{2/3}} \left(\mu^2 b^4 - (\nu c^2 - \lambda a^2)^2\right), \tag{4.84} $$

$$ {}^3R^{\bar{3}}_{\bar{3}} = \frac{1}{2u^{2/3}} \left(\nu^2 c^4 - (\lambda a^2 - \mu b^2)^2\right). \tag{4.85} $$

Finally, the traceless spatial modified Einstein equations (4.61) become

$$ \partial_t \left(uf\frac{\dot{a}}{a}\right) = \frac{(h'-1)u^{1/3}}{3}\left(\lambda a^2\left(2\lambda a^2 - \mu b^2 - \nu c^2\right) - \left(\mu b^2 - \nu c^2\right)^2\right), \tag{4.86} $$

$$ \partial_t \left(uf\frac{\dot{b}}{b}\right) = \frac{(h'-1)u^{1/3}}{3}\left(\mu b^2\left(2\mu b^2 - \nu c^2 - \lambda a^2\right) - \left(\nu c^2 - \lambda a^2\right)^2\right), \tag{4.87} $$

$$ \partial_t \left(uf\frac{\dot{c}}{c}\right) = \frac{(h'-1)u^{1/3}}{3}\left(\nu c^2\left(2\nu c^2 - \lambda a^2 - \mu b^2\right) - \left(\lambda a^2 - \mu b^2\right)^2\right). \tag{4.88} $$

In general, (4.60) describes a hypersurface in a six-dimensional phase space parametrized e.g. by $(u, a, b, \kappa, \tilde{\kappa}^{\bar{1}}_{\bar{1}}, \tilde{\kappa}^{\bar{2}}_{\bar{2}})$. Clearly, the general analysis of this system becomes intractable analytically. Let us hence restrict to the case where the conformal degree of freedom decouples from the rest. For general λ, μ, ν we look for special solutions where (4.60) contains only u and κ and can be decoupled from the other equations. By (4.62), the condition for this to be possible is

$$ {}^3R^A_B\,\tilde{\kappa}^B_A = 0 \quad \Leftrightarrow \quad \tilde{\kappa}^A_B\tilde{\kappa}^B_A \propto \frac{1}{u^2 f^2}. \tag{4.89} $$

After a short calculation, we find that for the Bianchi types under consideration

$$ {}^3R^A_B\,\tilde{\kappa}^B_A = -\frac{1}{2u^{2/3}}\,\partial_t\left(u^{2/3}\,{}^3R\right). \tag{4.90} $$

Hence the condition (4.89) is equivalent to

$$ {}^3R = \frac{-d}{2u^{2/3}}, \tag{4.91} $$

where

$$ d := \lambda^2 a^4 + \mu^2 b^4 + \nu^2 c^4 - 2\left(\mu\nu b^2 c^2 + \nu\lambda a^2 c^2 + \lambda\mu a^2 b^2\right) \stackrel{!}{=} const. \tag{4.92} $$

The temporal modified Einstein equation is in this case just the same as (4.76) for Bianchi type V. Again, a bounce can be implemented by a term $\propto ({}^3R)^n$ with $n \geq 3$ in h. Such a bounce ensures that $\kappa^A_B\kappa^B_A$ is bounded.

In the case of negative spatial curvature ($d > 0$) there is only one bounce and no recollapse and hence

$$ \kappa^A_B\kappa^B_A \propto \frac{1}{t^2} \qquad \text{as} \quad t \to \pm\infty, \tag{4.93} $$

like in Bianchi type V. It follows that in the case $d > 0$ such a bounce is already enough to ensure causal geodesic completeness, as one finds by slightly modifying the theorem presented in [91], cf. appendix A.

In the case of positive spatial curvature ($d < 0$) the solution for $u(t)$ and $\kappa(t)$ will be cyclic, similar to the closed Friedmann universe. Note that

$$d + \frac{4\mu\nu}{a^2} = \left(\lambda a^2 - \mu b^2 - \nu c^2\right)^2 , \tag{4.94}$$

which holds also for simultaneous cyclic permutations of (a, b, c) and (λ, μ, ν). This can be used to express the right hand side of (4.86) solely through a and u. For Bianchi type IX in particular we find that

$$\partial_t \left(uf \frac{\dot{a}}{a} \right) = (1 - h')u^{1/3} \left(a^2 \sqrt{d + \frac{4}{a^2}} + \frac{d}{3} \right) . \tag{4.95}$$

By symmetry, the same equation must also hold if a is replaced with b or c. Since both u and κ, which appear as sources in this equation, are periodic in time, we expect that also the solutions for a, b and c are oscillating between their minimal and maximal values and the corresponding spacetimes will be non-singular.

4.3 Modified Black Hole

4.3.1 Black hole in synchronous coordinates

In GR the metric of a non-rotating, eternal black hole in the synchronous Lemaître coordinates [89] is given by

$$\mathrm{d}s^2 = \mathrm{d}T^2 - (x/x_+)^{-2/3}\,\mathrm{d}R^2 - (x/x_+)^{4/3}\,r_g^2\,\mathrm{d}\Omega^2, \tag{4.96}$$

where $x = R - T$, and $r_g = 2M$. These coordinates are regular at the horizon

$$x = x_+ := \frac{4}{3}M,$$

and the region $x > 0$ covers both interior and exterior of the Schwarzschild black hole. For comoving observers with $R, \vartheta, \varphi = const.$, T represents proper time. In the Schwarzschild radial coordinate $r = r_g\,(x/x_+)^{2/3}$ the paths followed by these synchronous observers correspond to radially infalling geodesics. They start from rest at $r \to \infty$ at proper time $T \to -\infty$ and reach the singularity at $r = 0$ at the finite proper time $T = R$.

To see how (4.96) is modified in our theory, we consider in the synchronous coordinates (4.4) provided by $T = \phi$ the ansatz

$$\mathrm{d}s^2 = \mathrm{d}T^2 - a^2\,(x)\,\mathrm{d}R^2 - b^2\,(x)\,\mathrm{d}\Omega^2, \tag{4.97}$$

where the functions a and b still depend only on $x = R - T$. The transformation to Schwarzschild coordinates t and r is given by

$$t = T - \int \mathrm{d}x\,\frac{a^2}{1-a^2}, \qquad r = b(R-T), \tag{4.98}$$

which brings the metric to the form

$$\mathrm{d}s^2 = (1-a^2)\mathrm{d}t^2 - \frac{a^2}{b'^{\,2}(1-a^2)}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2. \tag{4.99}$$

The dependence of a and b' on r has to be found by inverting $r = b(x)$. The spatial metric determinant of (4.97) is

$$\gamma = a^2 b^4 \sin^2\vartheta =: u^2(x)\sin^2\vartheta,$$

and the non-vanishing components of the extrinsic curvature are given by

$$\kappa^R_R = \frac{\dot a}{a} = -\frac{a'}{a}, \qquad \kappa^\vartheta_\vartheta = \kappa^\varphi_\varphi = \frac{\dot b}{b} = -\frac{b'}{b},$$

where the prime denotes x-derivatives.

The spatial Ricci curvature components for the class of metrics (4.97) are given by

$$ {}^3R^R_R = R^R_T = 2\left(\gamma^{RR}\left(\kappa^\vartheta_\vartheta\right)^2 - \gamma^{\vartheta\vartheta} + {}^3R^\vartheta_\vartheta\right), \tag{4.100} $$

$$ {}^3R^\vartheta_\vartheta = {}^3R^\varphi_\varphi = \frac{1}{2\kappa^\vartheta_\vartheta}\left(\gamma^{RR}\left(\kappa^\vartheta_\vartheta\right)^2 - \gamma^{\vartheta\vartheta}\right)' - 2\left(\gamma^{RR}\left(\kappa^\vartheta_\vartheta\right)^2 - \gamma^{\vartheta\vartheta}\right). \tag{4.101} $$

The condition for spatial flatness hence amounts to the single equation

$$ \gamma^{RR}\left(\kappa^\vartheta_\vartheta\right)^2 - \gamma^{\vartheta\vartheta} = 0 \qquad \Leftrightarrow \quad a^2 = b'^{\,2}. \tag{4.102} $$

In this case the metric in Schwarschild coordinates takes the form

$$ \mathrm{d}s^2 = (1-a^2)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{(1-a^2)} - r^2\mathrm{d}\Omega^2, \tag{4.103} $$

and we see that the Schwarzschild metric (4.96) is spatially flat in Lemaître coordinates.

Note that in the direction of the vector field

$$ k^\mu\frac{\partial}{\partial x^\mu} := \frac{\partial}{\partial R} + \frac{\partial}{\partial T} = \frac{\partial}{\partial t} \tag{4.104} $$

the Lie derivative of (4.97) vanishes. In other words, k^μ is a Killing vector field with norm

$$ k^\mu k_\mu = 1 - a^2(x). $$

It follows that a Killing horizon occurs wherever $a^2(x) = 1$. Let us, in analogy with (4.96), denote the largest value of x where this happens, i.e. the most exterior horizon, by x_+. We can also calculate the surface gravity g_s of this Killing horizon which is defined by the equation [92]

$$ k^\nu_{\;;\mu}k^\mu = g_s\, k^\nu, \tag{4.105} $$

evaluated at the horizon. We find that it is related to the extrinsic curvature of the synchronous slices by

$$ g_s = \kappa^R_R(x_+) = -a'(x_+). \tag{4.106} $$

4.3.2 On generality of the solution $\phi = T$

In the last section we were making ansatz (4.97) from the beginning in the synchronous coordinates provided by $T = \phi$. One could, however, ask the question how some other synchronous time coordinate would be related to this specific one, i.e. if there is a more general solution of the constraint (4.2) in a metric given by (4.97). Such a solution still has to be consistent with the isometries of (4.97), e.g. it should be independent of the angular coordinates by spherical symmetry. Moreover, applying the Lie-derivative $\mathcal{L}_{k^\mu\frac{\partial}{\partial x^\mu}}$ to the constraint equation (4.2) or to the modified Einstein equation (4.11), we find the consistency condition

$$ \left[k^\mu\frac{\partial}{\partial x^\mu}\,,\,\phi^{,\nu}\frac{\partial}{\partial x^\nu}\right] = \left(k^\mu\frac{\partial}{\partial x^\mu}\phi^{,\nu} - \phi^{,\mu}\frac{\partial}{\partial x^\mu}k^\nu\right)\frac{\partial}{\partial x^\nu} = 0, \tag{4.107} $$

from which it follows that

$$\phi = c\,T + \xi(R - T), \tag{4.108}$$

where c is a constant and ξ is an arbitrary function. Reinserting into the constraint equation (4.2), we find the family of solutions

$$\xi_{\pm}^{(c)}(x) = \int \mathrm{d}x \frac{c\,a^2 \pm a\sqrt{c^2 - 1 + a^2}}{a^2 - 1}. \tag{4.109}$$

Introducing the radial coordinate

$$\tilde{r} := c\,R - \varrho(R - T) \tag{4.110}$$

and requiring the coordinates $(\tilde{t} := \phi, \tilde{r}, \vartheta, \varphi)$ to be synchronous, i.e. $\tilde{g}^{\tilde{t}\tilde{r}} = 0$, yields the corresponding solution

$$\varrho_{\pm}^{(c)}(x) = -c \int \mathrm{d}x \frac{1 \pm c\,a/\sqrt{c^2 - 1 + a^2}}{a^2 - 1}. \tag{4.111}$$

Note that the combination

$$\tilde{x} := \tilde{r} - \tilde{t} = \mp \int \mathrm{d}x \frac{a}{\sqrt{c^2 - 1 + a^2}} \tag{4.112}$$

is only a function of $x = R - T$ and hence all functions of x can be expressed as functions of $\tilde{x}$. The full metric ansatz in these coordinates will thus be again of the form

$$\mathrm{d}s^2 = \mathrm{d}\tilde{t}^{\,2} - \tilde{a}^2\,(\tilde{x})\,\mathrm{d}\tilde{r}^2 - \tilde{b}^2\,(\tilde{x})\,\mathrm{d}\Omega^2, \tag{4.113}$$

where now $\phi = \tilde{t}$ and

$$\tilde{a}^2(\tilde{x}) = \frac{c^2 - 1 + a^2(x(\tilde{x}))}{c^2}, \qquad \tilde{b}^2(\tilde{x}) = b^2(x(\tilde{x})). \tag{4.114}$$

Hence, the ansatz (4.97) made above (corresponding to the case $c = 1$) is fully general. The Killing vector field $\partial/\partial t$ in these coordinates has the expression

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial R} + \frac{\partial}{\partial T} = c\left(\frac{\partial}{\partial \tilde{r}} + \frac{\partial}{\partial \tilde{t}}\right). \tag{4.115}$$

Suppose that the metric is spatially flat in the original coordinates T, R, i.e. $b'(x)^2 = a^2(x)$. Then the relation in the new coordinates is

$$\left(\frac{\mathrm{d}\tilde{b}}{\mathrm{d}\tilde{x}}\right)^2 = c^2\tilde{a}^2(\tilde{x}). \tag{4.116}$$

It follows that there can only be at most one synchronous coordinate system in which such a metric is spatially flat.

The case $c = 0$, where even $k^\mu \phi_{,\mu} = 0$, is exceptional and deserves special attention. In this case the corresponding synchronous coordinates $(\tilde{t} = \phi, \tilde{r})$ are defined by

$$\tilde{t} = -\int \mathrm{d}x \frac{a}{\sqrt{a^2 - 1}}, \qquad \tilde{r} = R + \int \mathrm{d}x \frac{1}{a^2 - 1}, \tag{4.117}$$

and the metric is

$$\mathrm{d}s^2 = \mathrm{d}\tilde{t}^{\,2} - \left(a^2 - 1\right) \mathrm{d}\tilde{r}^2 - b^2 \mathrm{d}\Omega^2, \tag{4.118}$$

where a and b can now be expressed as functions of $\tilde{t}$ only. This is a Kantowski-Sachs type metric [93] and hence homogeneous. In these coordinates the metric can never be spatially flat. The condition for spatial flatness in the original $c = 1$ coordinates translates to

$$\left(\frac{\mathrm{d}b}{\mathrm{d}\tilde{t}}\right)^2 = a^2 - 1. \tag{4.119}$$

Note that $\phi = \tilde{t}$ is not a global solution of (4.2) because it is only defined where $a^2 > 1$ and becomes singular at the horizon. This is the ansatz that was first considered and then discarded in [43] before it was realized that the global solution $\phi = T$ has to be used.

4.3.3 Modified Einstein equations

Considering for simplicity the theory where $h = 0$, we want to derive the modified Einstein equation for a metric of the form (4.97). By virtue of the fact that all relevant quantities depend on R and T only through the quantity $x = R - T$, we can replace $\partial_T = -\partial_R$ and reduce partial differential equations to ordinary differential equations.

For example, for the vacuum case at hand (4.20) yields

$$\Xi = -\tfrac{1}{\sqrt{\gamma}} \int \mathrm{d}T\, \partial_R \left(\sqrt{\gamma} R^R_T\right) = R^R_T = {}^3R^R_R,$$

where we set the constant of integration to zero to be consistent with the asymptotic exterior vacuum solution (4.96). Hence the temporal modified Einstein equation (4.15) becomes

$$\frac{1}{3}\left(f - 2\kappa f'\right)\kappa^2 - \Lambda + \kappa\Lambda' - \frac{1}{2}\left(f + \kappa f'\right)\tilde{\kappa}^a_b\tilde{\kappa}^b_a = \gamma^{RR}\left(\kappa^\vartheta_\vartheta\right)^2 - \gamma^{\vartheta\vartheta}. \tag{4.120}$$

The trace subtracted spatial equations (4.18) read

$$\frac{1}{u}\left(uf\tilde{\kappa}^a_b\right)' = {}^3R^a_b - \frac{1}{3}{}^3R\delta^a_b. \tag{4.121}$$

By spherical symmetry and tracelessness they contribute only one independent equation. Subtracting the $\vartheta - \vartheta$ equation from the $R - R$ equation and inserting (4.100), (4.101) it can be written as

$$\frac{1}{u}\left(u\, f\left(\frac{b'}{b} - \frac{a'}{a}\right)\right)' = \frac{1}{2\kappa^\vartheta_\vartheta}\left(\gamma^{RR}(\kappa^\vartheta_\vartheta)^2 - \gamma^{\vartheta\vartheta}\right)'. \tag{4.122}$$

For the Schwarzschild solution (4.96) it holds that $\kappa = -1/x$. Hence, for large mass black holes with
$$M \gg \frac{1}{\kappa_0} \tag{4.123}$$
the extrinsic curvature at the horizon $x = x_+ \approx 4M/3$ is much lower than the limiting curvature scale κ_0 and we can still expect the exterior solution to be given by (4.96) and modifications to restrict themselves to the interior region. As we have seen, the Schwarzschild solution (4.96) is exactly spatially flat in the given slicing. Let us assume that the spatial curvature will remain negligible also for some range of x after the modification has taken over. In fact, we will find that the linear contribution of spatial curvature is irrelevant for the region close to the horizon even in the case $M \sim \kappa_0^{-1}$.

In this spatial flatness approximation (4.121) is easily integrated and yields
$$\tilde{\kappa}^R_R = \frac{2M}{fu}, \qquad \tilde{\kappa}^\vartheta_\vartheta = -\frac{M}{fu}, \tag{4.124}$$
where the constants of integration have been fixed to match the Schwarzschild solution in the limit $x \to \infty$. Accordingly, (4.120) becomes
$$\frac{\kappa^2 (f - 2\kappa f') - 3 (\Lambda - \kappa \Lambda')}{f + \kappa f'} = \left(\frac{3M}{fu}\right)^2, \tag{4.125}$$
which is formally the same equation as (4.65) for a modified Kasner universe and can be used to determine $u(x)$. We can integrate (4.124) again to obtain the solutions for $a(x)$ and $b(x)$ as
$$a = u^{1/3} \left(\frac{2}{3}\,\kappa_0\, e^H\right)^{2/3}, \qquad\qquad b = u^{1/3} \left(\frac{2}{3}\,\kappa_0\, e^H\right)^{-1/3}, \tag{4.126}$$
where the prefactors have been chosen for dimensionality and later convenience and
$$H := \int \mathrm{d}T\, \frac{3M}{fu}. \tag{4.127}$$
Note that the integrand can be expressed entirely through κ from (4.125). Moreover, applying the same technique as before and taking the time derivative of the logarithm of (4.125) yields a first order differential equation for κ where M drops out. The dependence of a, b and u on the mass parameter M can hence only come from a constant of integration which needs to be fixed to match (4.96). Solutions of the spatially flat approximation hence generically scale as
$$a, b \propto M^{1/3}. \tag{4.128}$$

For the analogous case of a contracting Kasner universe we have seen in [1] that a fast enough divergence of f at the limiting curvature can make anisotropies disappear during contraction. Under the condition that $\dot{H} \propto 1/fu \to 0$ while fu^2 remains finite when the limiting curvature is approached, also for the case at hand it follows that $\tilde{\kappa}_R^R$ and $\tilde{\kappa}_\vartheta^\vartheta$ will vanish as $x \to -\infty$. Hence, the functions a and b become alike, up to some finite constant factor

$$\zeta := \lim_{x \to -\infty} \frac{a(x)}{b(x)}. \tag{4.129}$$

In the original Schwarzschild solution the function a is increasing as $a \propto b^{-1/2}$ as we go towards $r = b \to 0$. Since the modification smoothly connects this solution to $a \propto b$, it is clear that a' has to change sign at some point where a reaches a maximum value before starting to decrease as we go deeper inside the black hole to $x \to -\infty$. If this maximum value of a is greater than one, we hence expect two Killing horizons $x_\pm$, one on each side of the maximum. In the limiting case where the maximum is exactly equal to one, these two horizons merge and the region where the Killing vector k^μ is spacelike (region II in the conformal diagram, cf. section 4.3.5) shrinks to a single horizon. Decreasing the mass of the black hole even further, we find that no horizon occurs at all. Hence there is a minimal mass of order

$$M_{\min} \sim \frac{1}{\kappa_0} \tag{4.130}$$

below which no black hole solution exists. Moreover, by (4.106) it follows that the surface gravity of this minimal black hole vanishes. This indicates that the final product of black hole evaporation will approach a minimal mass remnant for which Hawking radiation stops [3], [94].

Before illustrating this in a concrete example, let us discuss the fate of the singularity of (4.96) at the "center" $x = 0$. The asymptotic solution of the spatially flat approximation in the limit $x \to -\infty$ becomes

$$\mathrm{d}s^2 = \mathrm{d}T^2 - u^{\frac{2}{3}}(x) \left(\zeta^{\frac{4}{3}} \mathrm{d}R^2 + \zeta^{-\frac{2}{3}} \mathrm{d}\Omega^2 \right), \tag{4.131}$$

where

$$u = u_0 \exp(\kappa_0 x). \tag{4.132}$$

The spatial curvature components of this asymptotic solution are given by

$$^3R_R^R = 0, \qquad ^3R_\vartheta^\vartheta = {}^3R_\varphi^\varphi = \left(1 - \left(\frac{\kappa_0}{3\zeta} \right)^2 \right) \left(\frac{\zeta}{u} \right)^{2/3}. \tag{4.133}$$

Note that for modifications where it happens that $\zeta = \kappa_0/3$, the prefactor in the spatial curvature exactly cancels and the asymptotic solution is hence spatially flat. As a consequence, the full Ricci scalar of this asymptotic solution has a constant value and it describes a part of de Sitter spacetime. This can be seen also by defining the new radial coordinate $\tilde{R} := (u_0 e^{\kappa_0 R}/\zeta)^{1/3}$ which brings (4.131) to the form

$$\mathrm{d}s^2 = \mathrm{d}T^2 - e^{-\frac{2}{3}\kappa_0 T} \left(\left(\frac{3\zeta}{\kappa_0} \right)^2 \mathrm{d}\tilde{R}^2 + \tilde{R}^2 \mathrm{d}\Omega^2 \right). \tag{4.134}$$

Since inside the inner horizon x_- the Killing vector field k^μ is timelike again, we know that the metric in this region of the modified solution has to be static. Transforming to static Schwarzschild coordinates (4.98), we find the relations $a = \zeta r$, $b' = (\kappa_0/3)r$ and the asymptotic metric

$$\mathrm{d}s^2 = (1-\zeta^2 r^2)\mathrm{d}t^2 - \frac{(3\zeta/\kappa_0)^2}{(1-\zeta^2 r^2)}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2. \tag{4.135}$$

In the case $\zeta = \kappa_0/3$, the solution in the region $x \in (-\infty, x_-)$ (region IIa in figures 4.2 and 4.3) hence approaches the static patch of the de Sitter spacetime and has the same causal structure. The singularity is hence replaced by a smooth transition to a part of de Sitter space [15].

The above shows that in principle it is possible to find a modification for which the spatial curvature and hence also the potential $h({}^3R)$ will never become important and even exactly vanish in both limits $x \to \pm\infty$.

If $\zeta \neq \kappa_0/3$, the spatial curvature in the asymptotic region is of order

$$\gamma^{RR}\left(\tilde{\kappa}^\vartheta_\vartheta\right)^2 - \gamma^{\vartheta\vartheta} \to \frac{\kappa_0/3\zeta - 1}{b^2} \propto \frac{1}{u^{2/3}}. \tag{4.136}$$

Contracting (4.121) with κ^b_a we find that

$$\frac{1}{2u^2 f}\left(u^2 f^2 \tilde{\kappa}^a_b \tilde{\kappa}^b_a\right)' = \kappa^a_b\, {}^3\widetilde{R}^b_a = \frac{1}{3}\left(\frac{\kappa^R_R}{\kappa^\vartheta_\vartheta} - 1\right)\left(\gamma^{RR}\left(\kappa^\vartheta_\vartheta\right)^2 - \gamma^{\vartheta\vartheta}\right)'. \tag{4.137}$$

By asymptotic freedom and isotropy of the asymptotic solution, the right hand side remains negligible for the whole range of x even if $\zeta \neq \kappa_0/3$. As a consequence it still holds approximately that

$$\tilde{\kappa}^a_b \tilde{\kappa}^b_a \propto \frac{1}{u^2 f^2}, \tag{4.138}$$

and thus the linear spatial curvature contribution $\propto u^{-2/3}$ to (4.125) is dominated by u^{-2}. Hence, we expect the solution in this case to remain qualitatively unchanged compared to the spatially flat approximation except for the fact that now $|{}^3R| \to \infty$ as $x \to -\infty$. Naively treating (4.120) in the region where a and b are alike as a formal analogue of a modified non-flat Friedmann universe, one would come to the conclusion that in order to achieve a bounce and prevent this blowing up of spatial curvature, it would take a spatial curvature dependent potential including a term

$$h = -\left|{}^3R\right|^n, \qquad n > 3. \tag{4.139}$$

Of course this argument is purely heuristic and a rigorous verification would require an analysis of the full system of equations given by the analogues of (4.120) and (4.121) with $h \neq 0$. Without simplifying approximations, these constitute a highly coupled system of non-linear differential equations for a and b, or alternatively, u and H. To thoroughly verify our above speculation would hence require further investigation (perhaps numerical) beyond the scope of this paper.

4.3.4 A spatially flat exact solution

If we can find a modification such that the solution of the modified Einstein equation in the spatial flatness approximation everywhere exactly satisfies the spatial flatness condition (4.102), this would be an exact solution of the full modified Einstein equation, even in the case $h \neq 0$. The following modification provides a concrete (perhaps not the simplest) example where this possibility is realized.

Consider the asymptotically free modification given by

$$f(\kappa) = \frac{1 + 3\left(\kappa/\kappa_0\right)^2}{\left(1 + \left(\kappa/\kappa_0\right)^2\right)\left(1 - \left(\kappa/\kappa_0\right)^2\right)^2}, \tag{4.140}$$

$$\begin{aligned}\Lambda(\kappa) = \kappa^2 \left(\frac{\frac{4}{3}\left(\kappa/\kappa_0\right)^2}{\left(1 - \left(\kappa/\kappa_0\right)^2\right)^2} - \frac{1 + 2\left(\kappa/\kappa_0\right)^2}{1 + 4\left(\kappa/\kappa_0\right)^2 + 3\left(\kappa/\kappa_0\right)^4} \right) + \\ - \frac{\kappa_0}{6}\kappa \left(\arctan\frac{\kappa}{\kappa_0} - 3\sqrt{3}\arctan\left(\sqrt{3}\,\frac{\kappa}{\kappa_0}\right) + 2\,\mathrm{atanh}\frac{\kappa}{\kappa_0} \right).\end{aligned} \tag{4.141}$$

With this choice (4.125) becomes

$$\frac{\kappa^2}{\left(1 - \left(\kappa/\kappa_0\right)^4\right)^2} = \left(\frac{3M}{u}\right)^2. \tag{4.142}$$

Taking the time derivative of the logarithm of this equation we find that

$$\dot{\kappa} = -\kappa^2 \frac{1 - \left(\kappa/\kappa_0\right)^4}{1 + 3\left(\kappa/\kappa_0\right)^4}, \tag{4.143}$$

which has the implicit solution

$$-\kappa_0\, x = \frac{\kappa_0}{\kappa} - 2\,\mathrm{atanh}\frac{\kappa}{\kappa_0} + 2\arctan\frac{\kappa}{\kappa_0}. \tag{4.144}$$

Evaluating (4.127) as an integral over κ yields

$$H(\kappa) = \ln\left(-\left(\kappa/\kappa_0\right)\, \frac{1 + \left(\kappa/\kappa_0\right)^2}{1 + 3\left(\kappa/\kappa_0\right)^2} \right), \tag{4.145}$$

where the constant of integration was fixed to match the Schwarzschild solution. It follows that

$$\frac{a}{b} = \frac{2}{3}\kappa_0 e^H = \frac{2}{3}\left(-\kappa\, \frac{1 + \left(\kappa/\kappa_0\right)^2}{1 + 3\left(\kappa/\kappa_0\right)^2} \right) = -\frac{1}{3}\left(\kappa - \frac{3M}{fu}\right) = \left|\kappa^\vartheta_\vartheta\right|, \tag{4.146}$$

which shows that this solution is spatially flat and hence an exact solution of the full modified Einstein equation.

The solutions (4.126) for a and b expressed through κ are given by

$$a^3(\kappa) = \frac{4M}{3}\,|\kappa|\,\left(1-(\kappa/\kappa_0)^4\right)\left(\frac{1+(\kappa/\kappa_0)^2}{1+3\,(\kappa/\kappa_0)^2}\right)^2, \tag{4.147}$$

$$b^3(\kappa) = \frac{9M}{2\kappa^2}\,\left(1-(\kappa/\kappa_0)^2\right)\left(1+3\,(\kappa/\kappa_0)^2\right). \tag{4.148}$$

For this particular solution a assumes its maximum value at $\kappa = \kappa_* = -\kappa_0/\sqrt{5}$. At this point

$$a(\kappa_*) = \left(\frac{18\kappa_0}{25\sqrt{5}}M\right)^{1/3} =: \left(\frac{M}{M_{\min}}\right)^{1/3}, \tag{4.149}$$

and we find that the minimal possible black hole mass in this specific modification is given by

$$M_{\min} = \frac{25\sqrt{5}}{18\kappa_0}. \tag{4.150}$$

This solution was studied already in [3]. To aid our intuition, let us transform to Schwarzschild coordinates (4.98). By virtue of spatial flatness, it takes there the form (4.103). The location of the maximum of a in the Schwarzschild r-coordinate is given by

$$r_* = b(\kappa_*) = \left(144M/5\kappa_0^2\right)^{1/3}. \tag{4.151}$$

Far away from the black hole, in the limit $r\to\infty$, where $(\kappa/\kappa_0)^2 \ll 1$, we find the expansion

$$1-a^2 = 1-\frac{2M}{r}\left[1-\frac{5}{16}\left(\frac{r_*}{r}\right)^3 + \mathcal{O}\left(\left(\frac{r_*}{r}\right)^6\right)\right]. \tag{4.152}$$

It follow that the location of the outer horizon of a large mass black hole is given by

$$r_+ = 2M\left[1-\frac{729}{6250}\left(\frac{M_{\min}}{M}\right)^2 + \mathcal{O}\left(\left(\frac{M_{\min}}{M}\right)^4\right)\right]. \tag{4.153}$$

On the other hand, close to the limiting curvature $\kappa^2\to\kappa_0^2$ we find the expansion

$$1-a^2 = 1-(\zeta r)^2\left[1-\frac{4}{5}\left(\frac{r}{r_*}\right)^3 + \mathcal{O}\left(\left(\frac{r}{r_*}\right)^6\right)\right], \tag{4.154}$$

where $\zeta = \kappa_0/3$ and the inner horizon ($\sim$ de Sitter horizon) occurs at

$$r_- = \zeta^{-1}\left[1+\frac{27\sqrt{5}}{1600}\frac{M_{\min}}{M} + \mathcal{O}\left(\left(\frac{M_{\min}}{M}\right)^2\right)\right]. \tag{4.155}$$

Both asymptotics fail to describe the region between the two horizons. Expanding the solution around the maximum of a at r_* we find that

$$1-a^2 \approx 1-\left(\frac{M}{M_{\min}}\right)^{2/3}\left(1-\frac{10}{7}\,(1-r/r_*)^2\right). \tag{4.156}$$

For the minimal Black Hole $M = M_{\min}$ inner and outer horizon coincide, i.e. $r_* = r_+ = r_-$, and the metric close to this single horizon is given by

$$1 - a^2 \approx \frac{10}{7}\left(1 - r/r_*\right)^2 . \tag{4.157}$$

Note the similarity to the near horizon metric of an extremal Reissner-Nordström black hole.

4.3.5 Conformal diagrams

The conformal diagrams of the family of solutions found in the last section can be obtained by standard methods by gluing the diagrams of the individual regions separated by horizons, [94]. For the case of a non-minimal black hole $M > M_{\min}$ with two separate horizons, the solution with range $x \in (-\infty, \infty)$, i.e. $\kappa \in (-\kappa_0, 0)$, covers the three regions of the eternal black hole solution, the exterior I, the region between horizons II and the region IIa between the inner horizon and $r = 0$ which is essentially a static de Sitter patch. By time reversal invariance of our theory, the corresponding white hole solution can be found simply by reversing the arrow of time. Identifying the black and white hole exterior regions, we find the new regions IV and IVa which are just time reversed versions of II and IIa. Note that the static regions I and IIa are identical to their time reversed version. The conformal diagrams encompassing these regions for the three cases $M > M_{\min}$, $M = M_{\min}$, $M < M_{\min}$ are shown in figure 4.2.

Note that a static de Sitter patch is not geodesically complete and hence neither are the diagrams $M \geq M_{\min}$ in figure 4.2. Synchronous observers with $R = const.$ start at i^- ($T = -\infty$) from rest, pass outer and inner horizon and after infinite proper time reach $\widetilde{i^+}$ ($T = \infty, r = 0$). Hence these comoving geodesics are complete and fully contained in the union of regions I, II, IIa. However, light rays and massive particles with negative initial radial velocity at i^- will reach $r = 0$ at finite synchronous time. Since no singularity occurs at this point, they will simply be reflected towards the upper horizon $r = r_-$ of region IIa where also $T = \infty$. The diagram can be easily completed by identifying the black hole region IIa with the region IVa' of another white hole and continuing this procedure ad infinitum. The conformal diagram of the maximally extended eternal black hole solution is shown in figure 4.3.

The maximally extended solution shows that all non-comoving particles falling through the event horizon will eventually escape to another universe [15]. It follows that no information is “trapped” inside the finite region IIa and there is no upper limit on the amount of information that can fall into the black hole.

Even though figure 4.3 bears similarity with the conformal diagram of the Reissner-Nordström and Kerr spacetimes [95], there is a crucial difference: There is no Cauchy horizon at $r = r_-$ and the regions IIa, IIb, IVa, IVb etc. represent static patches of de Sitter space. Moreover, there is no singularity at $r = 0$ and geodesics reaching this point will simply be reflected.

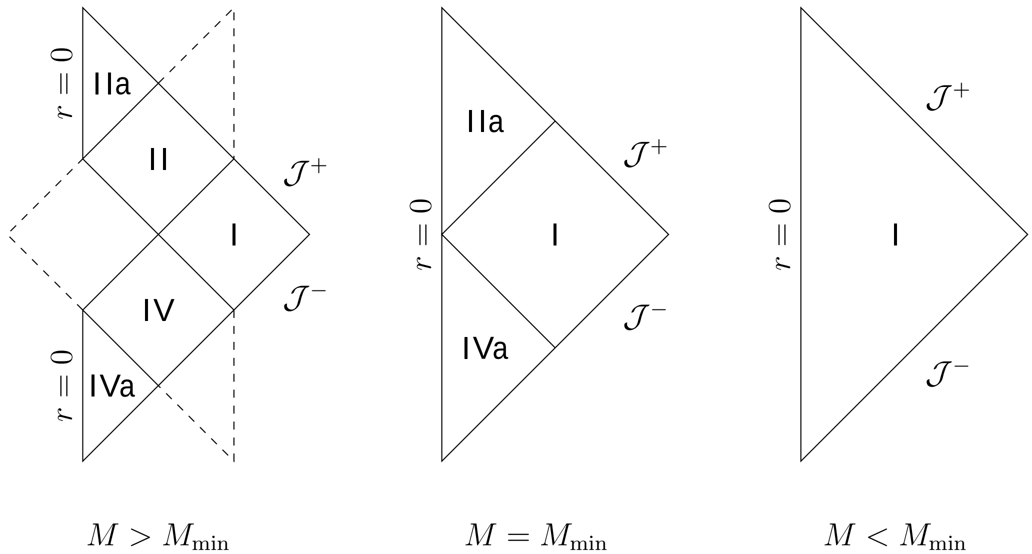

Figure 4.2: Conformal diagrams of the solution found in section 4.3.4 in the three cases $M > M_{\rm min}$, $M = M_{\rm min}$, $M < M_{\rm min}$.

$M > M_{\rm min}$: The eternal black hole solution is covered by the regions I, II, IIa. Regions I, IV, IVa describe the time reversed white hole solution. Dashed lines indicate the mirror symmetric extension.

$M = M_{\rm min}$: The outer and inner horizon coincide and the regions II and IV shrink to a single horizon $r_+ = r_- = r_*$.

$M < M_{\rm min}$: No horizon occurs and the causal structure is just like Minkowski spacetime. Close to $r = 0$ the solution approaches a static de Sitter metric replacing the singularity.

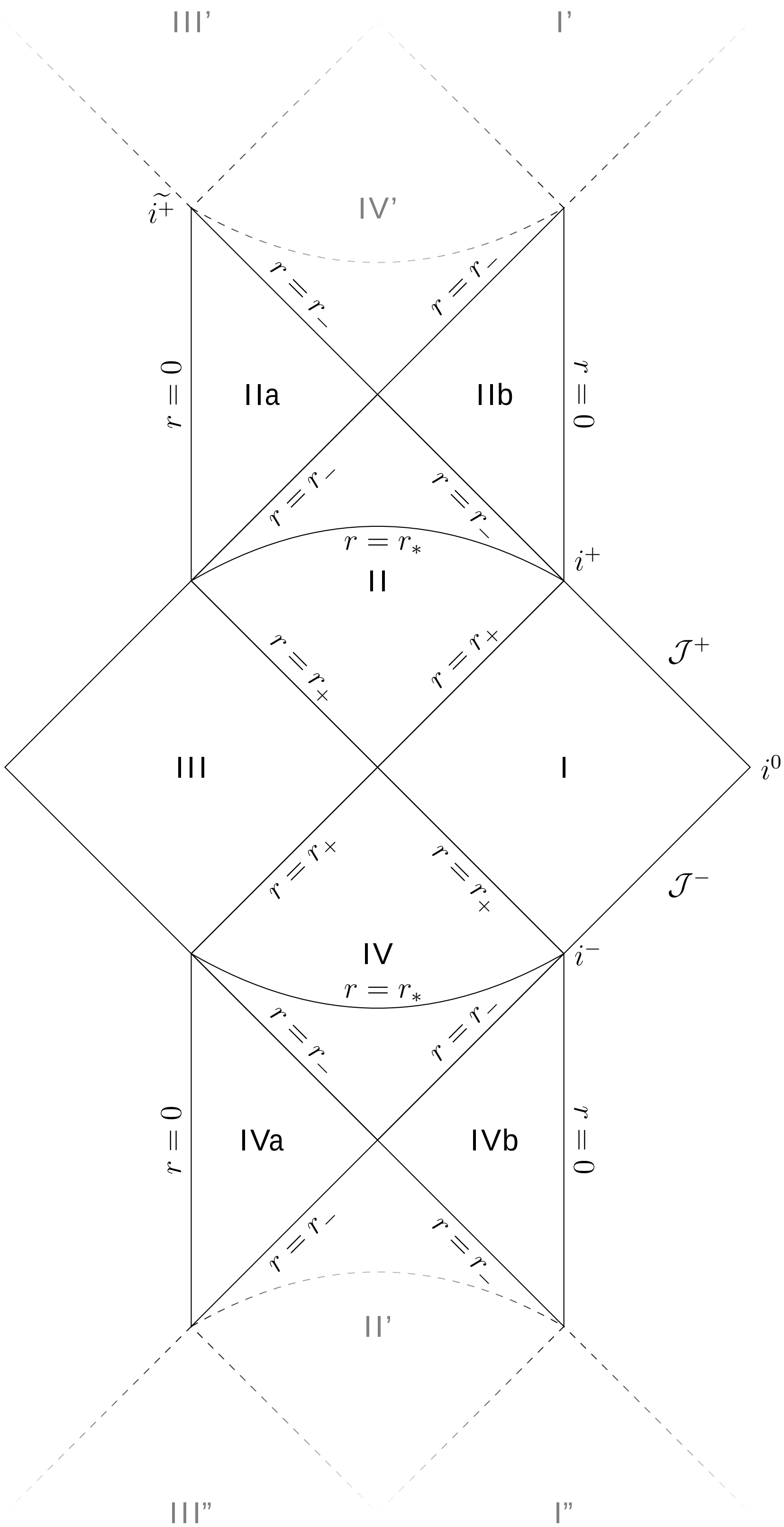

Figure 4.3: Conformal diagram of the maximally extended black hole solution from section 4.3.4 in the case $M > M_{\min}$.

4.3.6 Electric charge

The Reissner-Nordström metric in Schwarzschild coordinates takes the form (4.103), where now

$$1 - a^2 = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \tag{4.158}$$

Using the same transformation to Lemaître coordinates as in [3], we find that in this case they can only cover the part $r > \frac{Q^2}{2M}$ of this spacetime which contains both horizons but not the singularity. The expressions for $a(x)$ and $b(x)$ in (4.97) can be found from the relation

$$r(x) = \frac{Q^2}{2M}\left(\theta(\bar{x}) + \theta^{-1}(\bar{x}) - 1\right), \tag{4.159}$$

where

$$\theta^3(\bar{x}) = 1 + 2\bar{x}^2\left(1 + \sqrt{1 + \bar{x}^{-2}}\right), \qquad \bar{x} := \frac{3M^2\, x}{Q^3}. \tag{4.160}$$

More generally, also the synchronous coordinates associated to the solution $\xi_+^{(c>1)}$ (cf. section 4.3.2) cover only the region

$$r > \frac{M}{c-1}\left(\sqrt{1 + (c-1)\frac{Q^2}{M^2}} - 1\right). \tag{4.161}$$

Hence, even though one can obtain synchronous coordinates covering regions arbitrarily close to $r = 0$, there is no global synchronous coordinate system covering the Reissner-Nordström metric. The constraint (4.2) does not have a global solution and no synchronous Cauchy hypersurface exists. This can be taken as an indicator of the pathologies associated to the unstable interior of this spacetime which exhibits a Cauchy horizon [95].

Searching for a modification of this metric in our modified theory of gravity, we, however, have to care only about matching this GR solution in the low curvature limit in the exterior. Since this metric still respects spherical symmetry and the Killing vector field $\partial/\partial t$, the ansatz (4.97) is general enough to cover also modifications of charged black holes. Note that for the Reissner-Nordström metric in Lemaître coordinates the trace of extrinsic curvature expanded around $x = 0$ reads

$$\kappa = -\frac{1}{x} - \frac{16M^4}{3Q^2}x + \mathcal{O}(x^3). \tag{4.162}$$

It is hence singular at the point $x = 0$ corresponding to $r = Q^2/2M$ already before the actual curvature singularity¶ at $r = 0$ is reached. The modification must hence anyway take over well before this point is reached.

Since the metric is of the form (4.103), we know already that the Reissner-Nordström solution is spatially flat in Lemaître coordinates, just like the Schwarzschild solution. The main difference in going from the uncharged to the charged case is that now we are no

¶ $R^{\mu\nu}R_{\mu\nu} = 4Q^4/r^8$, $R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} = 8\left(Q^4 + 6\left(Q^2 - Mr\right)^2\right)/r^8$

longer looking for vacuum solutions but for *electro*vacuum solutions. In the exterior we expect the static observers defined by the Killing vector field $\partial/\partial t$ to observe only an electric but no magnetic field. Moreover, due to spherical symmetry, this electric field should only depend on the Schwarzschild r-coordinate. We hence use the ansatz

$$A_\mu \mathrm{d}x^\mu = \Phi(r(x))\mathrm{d}t = \Phi(x)\frac{\mathrm{d}T - a^2\mathrm{d}R}{1-a^2} \tag{4.163}$$

for the electromagnetic potential 1-form. In Lemaître coordinates, its components with raised index are

$$A^T = A^R = \frac{\Phi(x)}{1-a^2}. \tag{4.164}$$

It follows that the only non-vanishing components of the Faraday tensor are

$$F^{TR} = -F^{RT} = \frac{2}{a^2}\frac{\partial}{\partial x}A^T. \tag{4.165}$$

The vacuum Maxwell equation amounts to

$$F^{T\mu}_{\ ;\mu} = \frac{1}{ab^2}\frac{\partial}{\partial R}\left(ab^2 F^{TR}\right) = 0, \tag{4.166}$$

from which it follows that

$$F^{TR} = \frac{Q}{ab^2} \qquad \text{and} \qquad \frac{\partial}{\partial x}A^T = 2Q\frac{a}{b^2}, \tag{4.167}$$

where a constant of integration corresponding to charge was fixed. The energy momentum tensor of the electromagnetic field is

$$T^\alpha_\beta = \frac{1}{4\pi}\left(F^{\alpha\mu}F_{\mu\beta} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\,\delta^\alpha_\beta\right) = \frac{Q^2}{8\pi b^4}\,\mathrm{diag}\left(-1,-1,1,1\right). \tag{4.168}$$

Inserting this into the modified Einstein equation, the temporal equation takes the form

$$\frac{1}{3}\left(f - 2\kappa f'\right)\kappa^2 - \Lambda + \kappa\Lambda' - \frac{1}{2}\left(f + \kappa f'\right)\tilde{\kappa}^a_b\tilde{\kappa}^b_a = \gamma^{RR}\left(\kappa^\vartheta_\vartheta\right)^2 - \gamma^{\vartheta\vartheta} - \frac{Q^2}{b^4}, \tag{4.169}$$

and the analogue of (4.122) becomes

$$\frac{1}{ab^2}\left(ab^2 f\left(\frac{b'}{b} - \frac{a'}{a}\right)\right)' = \frac{b}{2b'}\left(\gamma^{RR}(\kappa^\vartheta_\vartheta)^2 - \gamma^{\vartheta\vartheta}\right)' - \frac{2Q^2}{b^4}. \tag{4.170}$$

Assuming spatial flatness, i.e. $a = \pm b'$, this equation has the first integral

$$\frac{b'}{b} - \frac{a'}{a} = \frac{1}{ab^2 f}\left(3M \pm \frac{2Q^2}{b}\right). \tag{4.171}$$

Moreover, in this case

$$A^T = \mp\frac{2Q}{b}, \tag{4.172}$$

and the temporal equation (4.169) becomes

$$\kappa^2 (f - 2\kappa f') - 3(\Lambda - \kappa\Lambda') = (f + \kappa f')\left(\frac{3M \pm 2Q^2/b}{fu}\right)^2 - \frac{3Q^2}{b^4}. \tag{4.173}$$

We see that the new terms coming from charge can become dominant only close to $b = r \to 0$. They remain negligible until well after the modifications have taken over at r_*, provided that

$$\left(M^4 M_{\min}^2\right)^{1/3} \gg Q^2. \tag{4.174}$$

Since we can expect the Schwinger effect to discharge a black hole much faster than the timescales of Hawking radiation, this condition should be satisfied by realistic black holes close to the end of their evolution. This suggests that our conclusions drawn for the fate of evaporating uncharged black holes in [3] should remain valid also in the charged case.

Noting the sign of the charge contribution to (4.170), there is the possibility that charge can lead to a bounce even without a spatial curvature dependent potential. As we can check explicitly by inserting the modified solution from section 4.3.4, the right hand side of (4.170) in this case exhibits a zero. Such a bounce would also prevent a blow up of the electromagnetic field energy at $r = 0$. Of course, a rigorous verification of this speculation would require a more extensive analysis beyond the scope of this paper.

4.3.7 First order rotation

To first order in angular momentum J the Kerr metric in Boyer-Lindquist coordinates reads [95]

$$\mathrm{d}s^2 = \left(1 - \frac{r_g}{r}\right)\mathrm{d}t^2 - \frac{\mathrm{d}t^2}{\left(1 - \frac{r_g}{r}\right)} - r^2\mathrm{d}\Omega^2 + \frac{4J}{r}\sin^2\vartheta\,\mathrm{d}t\mathrm{d}\varphi \tag{4.175}$$

and is still spherically symmetric. In the Lemaître coordinates

$$\mathrm{d}T = \mathrm{d}t + \sqrt{\frac{r_g}{r}}\left(1 - \frac{r_g}{r}\right)^{-1}\mathrm{d}r, \qquad \mathrm{d}R = \mathrm{d}t + \sqrt{\frac{r}{r_g}}\left(1 - \frac{r_g}{r}\right)^{-1}\mathrm{d}r,$$

it becomes

$$\mathrm{d}s^2 = \mathrm{d}T^2 - a^2(x)\mathrm{d}R^2 - b^2(x)\mathrm{d}\Omega^2 + \frac{2j\,\omega(x)\sin^2\vartheta}{1 - a^2(x)}\left(\mathrm{d}T - a^2(x)\mathrm{d}R\right)\mathrm{d}\varphi, \tag{4.176}$$

where $x = R - T$, $j := J/M^2$ and

$$a(x) = \left(\frac{x}{x_+}\right)^{-1/3}, \quad b(x) = \left(\frac{x}{x_+}\right)^{2/3} r_g, \quad \omega(x) = M\left(\frac{x}{x_+}\right)^{-2/3}. \tag{4.177}$$

Let us take (4.176) as an ansatz where we consider a, b and ω as independent functions. Note that even though these coordinates are not synchronous, to first order in the angular momentum it still holds that

$$g^{\mu\nu}T_{,\mu}T_{,\nu} = 1 + \mathcal{O}\left(j^2\right). \tag{4.178}$$

Making the expansion

$$\phi = T + j\,\phi_0(x) + \mathcal{O}\left(j^2\right), \tag{4.179}$$

where $\phi_0(x)$ should depend on x to preserve spherical symmetry, we find the condition $\phi_0' = 0$. Hence, in first order perturbation theory we can still use the approximate solution

$$\phi = T + \mathcal{O}\left(j^2\right). \tag{4.180}$$

However, now $\phi^{;\varphi} = g^{T\varphi} \neq 0$ and

$$\phi_{;T\varphi} = \frac{j\,\omega \sin^2\vartheta}{(1-a^2)}\frac{b'}{b} + \mathcal{O}\left(j^3\right) \neq 0. \tag{4.181}$$

Hence, we cannot directly use the components of the modified Einstein equation that were derived in the synchronous coordinates above. Instead, we have to expand the full equation (4.11) in j. Expanding the Ricci tensor

$$R_{\mu\nu} = {}^{(0)}R_{\mu\nu} + j\,{}^{(1)}R_{\mu\nu} + \mathcal{O}\left(j^2\right), \tag{4.182}$$

we find that the only new non-vanishing contibutions to first order in angular momentum are given by

$${}^{(1)}R^T_\varphi = {}^{(1)}R^R_\varphi = \frac{\sin^2\vartheta}{a^2}\left[\frac{1}{2}a\left(\frac{\omega'}{a}\right)' - \omega\left(\frac{b''}{b} + \left(\frac{b'}{b}\right)^2 - \frac{a'}{a}\frac{b'}{b}\right)\right]. \tag{4.183}$$

In first order perturbation theory it still holds that

$$\phi^{;T}_{\ \ T} = 0, \quad \phi^{;R}_{\ \ R} = -\frac{a'}{a}, \quad \phi^{;\vartheta}_{\ \ \vartheta} = \phi^{;\varphi}_{\ \ \varphi} = -\frac{b'}{b}, \quad \Box\phi = -\frac{a'}{a} - 2\frac{b'}{b}, \tag{4.184}$$

where corrections would appear in order $\mathcal{O}(j^2)$. Moreover,

$$\phi^{;T}_{\ \ \varphi} = 0, \quad \phi^{;T\ \ T}_{\ \ \varphi} = 0, \quad \phi^{;R}_{\ \ \varphi} = \frac{j\sin^2\vartheta}{2a^2}\,b^2\left(\frac{\omega}{b^2}\right)' \tag{4.185}$$

where corrections would appear in order $\mathcal{O}\left(j^3\right)$.

The functions a and b are hence still determined by the zeroth order equations (4.120), (4.122). The function ω has to be obtained from one of the new off-diagonal modified Einstein equations, e.g. the $T-\varphi$ equation which, making use of (4.185), becomes

$$fR^T_\varphi + \phi^{;R}_{\ \ \varphi} f_{,R} = 0. \tag{4.186}$$

More explicitely, we have to determine ω from the equation

$$f\left(a\left(\frac{\omega'}{a}\right)' - 2\left(\frac{b''}{b} + \left(\frac{b'}{b}\right)^2 - \frac{a'}{a}\frac{b'}{b}\right)\omega\right) + f_{,R}\,b^2\left(\frac{\omega}{b^2}\right)' = 0. \tag{4.187}$$

Assuming spatial flatness of the slices $T = const.$ (i.e. $a = \pm b'$) this simplifies to

$$f\left(a\left(\frac{\omega'}{a}\right)' - 2\left(\frac{b'}{b}\right)^2 \omega\right) + f_{,R}\, b^2 \left(\frac{\omega}{b^2}\right)' = 0. \tag{4.188}$$

It is easy to see that one solution of this linear ODE is given by $\omega \propto b^2$. However, this solution does not agree with the asymptotic solution (4.177). Multiplying (4.188) by b^2/a and using (4.122) in the spatially flat case, it is straightforward to verify that another independent solution is given by

$$\omega = Ma^2, \tag{4.189}$$

where the constant of integration was already fixed to match (4.177). Note that this identity for the GR solution continues to hold in the modification in the spatially flat case.

It follows that the frame dragging function ω assumes a maximum at the same location as a at $r = r_*$ and after that decreases until it vanishes at $r = 0$ according to $\omega \propto r^2$. Since ω is bounded by a maximum value, we can expect that for a small enough value of $j \ll 1$ our perturbative analysis is justified for the whole range of x. Moreover, this suggests that the spacetime structure of a rotating black hole close to $r = 0$ is in fact not different from the non-rotating case.

Note that in first order perturbation theory the norm of the Killing vector field (and hence the location of horizons) is still given by $a^2 - 1$. Moreover, for the surface gravity it holds that

$$g_s = -a'(x_+) + \mathcal{O}(j^2). \tag{4.190}$$

This shows that our above conclusions are robust even for slowly rotating black holes.

4.4 Conclusions

The introduction of the mimetic field ϕ allowed us to find a remarkably simple high-curvature modification of GR, where a scale dependence of gravitational and cosmological constant can be implemented covariantly. We found that $\Box\phi$ is the unique measure of curvature on which the gravitational constant can depend such that the resulting modified Einstein equation is still second order in time. This modified theory of gravity hence does not exhibit any additional degrees of freedom except that the conformal degree of freedom of the metric becomes dynamical.

As a first application, we found that the most natural class of modified Friedmann universes arising from this theory generically feature a de Sitter-like initial state replacing the Big Bang singularity. To resolve also the anisotropic Kasner singularity in the same way, we found that we have to require "asymptotic freedom" of gravity, i.e. the vanishing of the gravitational constant at limiting curvature.

Taking on the task of singularity resolution in general, spatially non-flat spacetimes, it is clear that this is too much to ask of a theory where only the conformal degree of freedom is modified. Gratifyingly, we found that the mimetic field also permits to introduce in a covariant manner a potential depending on spatial curvature. In fact, adding such higher order terms to the action could even improve the renormalizability of gravity, along the lines of Hořava gravity. We showed that in spatially non-flat Friedmann and certain Bianchi universes a simple power law potential is already enough to replace the singularity with a bounce.

In application to non-rotating black holes, we found that our modification of GR generically leads to a lower bound on the black hole mass. Minimal black holes have vanishing Hawking temperature and the final product of black hole evaporation is hence a stable remnant of minimal mass. Moreover, we found that this result is also robust when putting small amounts of charge or angular momentum. An inner horizon is already present in the non-rotating, uncharged case and the causal structure resembles those of Reissner-Nordström and Kerr, except that the region inside the inner horizon is replaced with a static de Sitter patch. Furthermore, since the mere assumption of existence of a global solution to the mimetic constraint already implies stable causality, we expect no Cauchy horizon in the interior even for arbitrary charge and rotation. Hence the instabilities present in Reissner-Nordström and Kerr solutions could be cured in such a modification.

Appendix

A: Synchronous coordinates.

Variation of the action (4.8) with respect to the Lagrange multiplier λ yields the constraint (4.2)

$$g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} = 1.$$

We will see that the existence of a global scalar field with this porperty already has some far reaching consequences, e.g. on the causal structure of admissible spacetimes.[‖] Taking a covariant derivative of this equation shows that

$$\phi^{,\mu}\nabla_\mu\phi^{,\nu} = \frac{1}{2}\nabla^\nu\left(\phi^{,\mu}\phi_{,\mu}\right) = 0, \tag{4.191}$$

and hence the vector field $\phi^{,\mu}$ is tangent to a congruence of timelike geodesics. Through every point of a hypersurface of constant ϕ passes a unique geodesic in the congruence. Choosing coordinates[∗∗] x^a on some initial $3-$hypersurface $\{\phi = \phi_i = const.\}$ then defines coordinates on any other hypersurface $\{\phi = \phi_0 = const.\}$ by traveling along these unique geodesics. Since the congruence is hypersurface orthogonal and its normal vector field $\phi^{,\mu}$ has unit norm, $(t := \phi, x^a)$ defines a synchronous coordinate system in which the metric takes the form [84]

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \gamma_{ab}\mathrm{d}x^a\mathrm{d}x^b,$$

where latin indices run over spatial coordinates. The whole spacetime is sliced into into spatial hypersurfaces $\{\phi = \text{const.}\}$ with extrinsic curvature

$$\kappa_{ab} = \frac{1}{2}\frac{\partial}{\partial t}\gamma_{ab}, \qquad \kappa^{ab} := \gamma^{ac}\gamma^{bd}\kappa_{cd} = -\frac{1}{2}\frac{\partial}{\partial t}\gamma^{ab}, \qquad \kappa := \gamma^{ab}\kappa_{ab} = \frac{\partial}{\partial t}\ln\sqrt{\gamma}$$

and metric determinant $\gamma = \det\gamma_{ab} = -\det g_{\mu\nu}$. In this splitting the non-vanishing connection coefficients are given by

$$\Gamma^0_{ab} = \kappa_{ab}, \qquad \Gamma^a_{0b} = \kappa^a_b := \gamma^{ac}\kappa_{cb}, \qquad \Gamma^a_{bc} = \lambda^a_{bc},$$

where λ^a_{bc} are the connection coefficients of the Levi-Civita connection D belonging to the Riemannian $3-$metric γ_{ab}. Note that

$$\phi_{;0\alpha} = 0, \qquad \phi_{;ab} = -\kappa_{ab}, \tag{4.192}$$

and the expansion and shear of the geodesic congruence $\phi^{,\mu}$ are given by $\Box\phi = \kappa$ and $\tilde{\kappa}^a_b := \kappa^a_b - \frac{1}{3}\kappa\delta^a_b$, respectively.

[‖] For example, as shown in [36], the existence of a function whose gradient is everywhere time-like implies stable causality.

[∗∗] More generally: an atlas

The covariant 4-divergence of a vector X^μ is given by

$$\nabla_\mu X^\mu = \partial_0 X^0 + \kappa X^0 + D_a X^a = \tfrac{1}{\sqrt{\gamma}}\partial_0\left(\sqrt{\gamma}X^0\right) + D_a X^a$$

and the d'Alembertian of a scalar S is

$$\Box S = \tfrac{1}{\sqrt{\gamma}}\,\partial_0\Big(\sqrt{\gamma}\,\dot{S}\Big) - \Delta S = \ddot{S} + \kappa\dot{S} - \Delta S$$

where the dot denotes t-derivatives and Δ is the Laplacian belonging to the Riemannian metric γ.

The non-vanishing components of the four-dimensional Riemann tensor in this splitting are determined by

$$R^0_{\ abc} = \kappa_{ac|b} - \kappa_{ab|c} \qquad\qquad R^0_{\ a0b} = \dot{\kappa}_{ab} - \kappa^c_a\kappa_{bc}$$
$$R^d_{\ abc} = {}^3R^d_{\ abc} + \kappa^d_b\kappa_{ca} - \kappa^d_c\kappa_{ab}$$

where ${}^3R^d_{\ abc}$ is the Riemann tensor of the spatial metric γ_{ab} and the notation $\%_{|b} := D_b\%$ was used. We find the useful identity

$$\phi^{,\mu}\phi_{,\nu}R^\nu_{\alpha\mu\beta} = -\phi^{,\mu}\nabla_\mu\left(\phi_{;\alpha\beta}\right) - \phi^{;\mu}_{\ \alpha}\phi_{;\mu\beta}. \tag{4.193}$$

The Ricci tensor R^μ_ν splits into extrinsic and intrinsic curvature as

$$R^0_0 = R_{00} = -\dot{\kappa} - \kappa^{ab}\kappa_{ab} \qquad\qquad R^0_a = R_{0a} = \kappa^b_{a|b} - \kappa_{,a}$$
$$-R^a_b = \gamma^{ac}R_{cb} = \tfrac{1}{\sqrt{\gamma}}\partial_0\left(\sqrt{\gamma}\kappa^a_b\right) + {}^3R^a_b.$$

The Ricci scalar is thus given by

$$-R = 2\dot{\kappa} + \kappa^2 + \kappa^{ab}\kappa_{ab} + {}^3R.$$

The $0-0$ component of the Einstein tensor $G^\mu_\nu = R^\mu_\nu - \frac{1}{2}R\delta^\mu_\nu$ is hence

$$G_{00} = G^{00} = G^0_0 = \tfrac{1}{2}\left(\kappa^2 - \kappa^{ab}\kappa_{ab} + {}^3R\right).$$

This allows to isolate the spatial curvature scalar as

$${}^3R = 2G_{\mu\nu}\phi^{,\mu}\phi^{,\nu} - (\Box\phi)^2 + \phi^{;\mu\nu}\phi_{;\mu\nu}.$$

For evaluation of the modified Einstein equation in the synchronous frame it will be useful to note that

$$\nabla_\mu\left(\phi^{;\alpha}_{\ \beta}\tilde{f}\phi^{,\mu}\right) = \tfrac{1}{\sqrt{\gamma}}\partial_0\left(\sqrt{\gamma}\tilde{f}\kappa^a_b\right)\delta^\alpha_a\delta^b_\beta. \tag{4.194}$$

A causal completeness condition

In this section we will find a su□cient (but not necessary) condition for causal geodesic completeness of a metric of the form (4.4). We follow mainly the steps taken in [91], applied to the $3+1$ splitting at hand. Consider the velocity vector u^μ of a geodesic parametrized by an a□ne parameter s,

$$u^0 = \frac{\mathsf{d}t}{\mathsf{d}s}, \qquad u^a = \frac{\mathsf{d}x^a}{\mathsf{d}s} = u^0 \frac{\mathsf{d}x^a}{\mathsf{d}t} =: u^0 v^a.$$

The temporal component of the geodesic equation reads

$$0 = u^\mu \nabla_\mu u^0 = \frac{\mathsf{d}}{\mathsf{d}s} u^0 + \kappa_{ab} u^a u^b = u_0^2 \left(\frac{1}{u_0} \frac{\mathsf{d}}{\mathsf{d}t} u^0 + \kappa_{ab} v^a v^b \right)$$

and we can integrate to find

$$\ln u^0 = - \int \mathrm{d}t\, \kappa_{ab} v^a v^b. \tag{4.195}$$

Timelike geodesics with $u^\mu = \phi^{,\mu}$ describe "comoving" observers with $v^a = 0$. They are freely falling and the synchronous time t measures proper time for these observers. Their a□ne parameter extension is hence infinite, if t can be extended to the range $(-\infty, \infty)$. Assuming this to be the case, the a□ne parameter extension of a general causal geodesic is given by

$$\int \mathsf{d}s = \int_{-\infty}^{\infty} \frac{\mathsf{d}t}{u^0}, \tag{4.196}$$

and thus it is future resp. past complete, if this integral diverges at $t \to \infty$ resp. $t \to -\infty$. Using the Cauchy-Schwarz inequality for the scalar product $\langle A, B \rangle := \gamma^{ab}\gamma^{cd} A_{ac} B_{bd}$ of spatial tensors A and B on the right hand side of (4.195), we see that

$$\ln u^0 \le \int \mathrm{d}t\, \sqrt{\kappa^{ab}\kappa_{ab}} (v^c v_c) \le \int \mathrm{d}t\, \sqrt{\kappa^{ab}\kappa_{ab}},$$

where the second inequality is an equality for light-like geodesics. It hence follows that if

$$\int_{-\infty}^{\infty} \mathrm{d}t\, \sqrt{\kappa^{ab}\kappa_{ab}} < \infty, \tag{4.197}$$

then $1/u^0$ will be uniformly bounded from 0 and hence all causal geodesics are both past- and future complete. Note that also the weaker condition

$$\sqrt{\kappa^{ab}\kappa_{ab}} \le \frac{1}{|t|} \quad \text{asymptotically as} \quad t \to \pm\infty, \tag{4.198}$$

su□ces to have $u^0 \le |t|$ and hence logarithmic divergence of (4.196). Thus also (4.198) is a su□cient condition for causal completeness.

B: Explicit calculations in the variation of the action

Variation with respect to the mimetic field. Let us vary (4.8) with respect to ϕ. To this end, calculate

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta\Box\phi} &= f'\left(R + 2G_{\mu\nu}\phi^{,\mu}\phi^{,\nu} - (\Box\phi)^2 + \phi^{;\mu\nu}\phi_{;\mu\nu}\right) - 2(f-1+h')\Box\phi + 2\Lambda' \\ &\doteq -2\left[(\tilde{f}-h')_{,\alpha}\phi^{,\alpha} + \tilde{f}\,\Box\phi + \tfrac{1}{2}f'\left((\Box\phi)^2 + \phi^{;\mu\nu}\phi_{;\mu\nu}\right) - \Lambda'\right] \\ &=: -2\left[(\tilde{f}\phi^{,\alpha})_{;\alpha} + \tilde{Z}\right] \qquad (4.199)\end{aligned}$$

where we introduced the useful notations

$$\tilde{f} := f - 1 + h', \qquad Z := \tfrac{1}{2}f'\left((\Box\phi)^2 + \phi^{;\mu\nu}\phi_{;\mu\nu}\right) - \Lambda', \qquad \tilde{Z} := Z - \phi^{,\alpha}h'_{,\alpha}$$

and $\doteq$ means equality if the constraint (4.2) is satisfied. It follows that

$$\begin{aligned}\tfrac{1}{2}\,\delta_\phi\mathcal{L} &= -\left[(\tilde{f}\phi^{,\mu})_{;\mu} + \tilde{Z}\right]\Box\delta\phi + \tilde{f}\left(2G_{\mu\nu}\phi^{,\mu}\,\delta\phi^{,\nu} + \phi^{;\mu\nu}\delta\phi_{;\mu\nu}\right) \\ &= -\left[(\tilde{f}\phi^{,\mu})_{;\mu} + \tilde{Z}\right]\delta\phi^{;\nu}_{\nu} + \tilde{f}\left(\phi^{;\mu}_{\nu}\,\delta\phi^{,\nu}\right)_{;\mu} - \tilde{f}\left[\phi^{;\mu}_{\nu\mu} - 2G_{\mu\nu}\phi^{,\mu}\right]\,\delta\phi^{,\nu}\end{aligned}$$

Thus the variation of (4.8) yields

$$\begin{aligned}-8\pi\delta_\phi S &= \int \mathrm{d}^4x\sqrt{-g}\,\delta\phi^{,\nu}\left\{(\tilde{f}\phi^{,\mu})_{;\mu\nu} - (\tilde{f}\phi^{;\mu}_{\nu})_{;\mu} + \tilde{Z}_{;\nu} + \tilde{f}2G_{\mu\nu}\phi^{,\mu} - \lambda\phi_{,\nu}\right\} \\ &= \int \mathrm{d}^4x\sqrt{-g}\,\delta\phi^{,\nu}\left\{(\tilde{f}_{,\nu}\,\phi^{,\mu})_{;\mu} + \tilde{Z}_{,\nu} + \tilde{f}\left(2G_{\mu\nu} - R_{\mu\nu}\right)\phi^{,\mu} - \lambda\phi_{,\nu}\right\}\end{aligned}$$

where covariant partial integration and the commutator of covariant derivatives were used. Here and in the following section we ignore boundary terms in the variation. Integrating by parts once again, we find the equation of motion

$$\nabla_\nu\left[(\lambda + \tilde{f}R)\phi^{,\nu} - (\tilde{f}^{,\nu}\,\phi^{,\mu})_{;\mu} - \tilde{Z}^{,\nu} - \tilde{f}R^{\mu\nu}\phi_{,\mu}\right] = 0$$

which will be used to determine λ.

Variation with respect to the metric. Next we have to vary (4.8) with respect to $g_{\mu\nu}$. In the course of this undertaking the following identities for the varations of the metric determinant, connection coe□cients and Ricci tensor will be put to good use:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta g^{\mu\nu}, \qquad \delta\Gamma^\lambda_{\mu\nu} = -g_{\alpha(\mu}\nabla_{\nu)}\delta g^{\alpha\lambda} + \tfrac{1}{2}g_{\mu\alpha}g_{\nu\beta}\nabla^\lambda\delta g^{\alpha\beta}$$

$$\delta R_{\mu\nu} = \nabla_\lambda\delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu\delta\Gamma^\lambda_{\lambda\mu}$$

Combining the latter two yields

$$\delta R_{\mu\nu} = \tfrac{1}{2}\left[g_{\mu\alpha}g_{\nu\beta}\Box + g_{\alpha\beta}\nabla_\nu\nabla_\mu - g_{\mu\beta}\nabla_\alpha\nabla_\nu - g_{\nu\beta}\nabla_\alpha\nabla_\mu\right]\delta g^{\alpha\beta}.$$

In the variation of the usual Einstein action one only encounters the term

$$g^{\mu\nu}\delta R_{\mu\nu} = \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)\delta g^{\mu\nu},$$

which turns out to be a total covariant derivative, provided that it appears with a constant prefactor. Using, in a first step,

$$\delta_g\left(2G_{\mu\nu}\phi^{,\mu}\phi^{,\nu}\right) = 2\delta R_{\mu\nu}\phi^{,\mu}\phi^{,\nu} + 4R_{\alpha\mu}\phi^{,\alpha}\phi_{,\nu}\delta g^{\mu\nu} - \delta R - R\phi_{,\mu}\phi_{,\nu}\delta g^{\mu\nu},$$

the expression $\delta\mathcal{L}/\delta\Box\phi$ from (4.199) and ignoring boundary terms we find that

$$\begin{aligned}
&-16\pi\,\delta_g S = \\
&\int \mathrm{d}^4x\,\sqrt{-g}\Bigg\{\left[R_{\mu\nu} - \frac{1}{2}\mathcal{L}\,g_{\mu\nu} + 4\tilde{f}\phi^{,\alpha}R_{\alpha(\mu}\phi_{,\nu)} - \left(\lambda + \tilde{f}R\right)\phi_{,\mu}\,\phi_{,\nu}\right]\delta g^{\mu\nu} \\
&\qquad - \left[(\tilde{f}\phi^{,\alpha})_{;\alpha} + \tilde{Z}\right]\underbrace{2\,\delta_g\Box\phi}_{\text{I}} + \tilde{f}\,\underbrace{2\,\phi^{,\mu}\phi^{,\nu}\delta R_{\mu\nu}}_{\text{II}} + \tilde{f}\,\underbrace{\delta_g\left(\phi^{;\mu\nu}\phi_{;\mu\nu}\right)}_{\text{III}} - h'\,\underbrace{\delta R}_{\text{IV}}\Bigg\}
\end{aligned}$$

The modified Einstein equation hence reads

$$\begin{aligned}
G_{\mu\nu} - \Lambda g_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\left[(f-1)(R + {}^3R) + h\right] + 4\phi^{,\alpha}\phi_{(,\mu}R_{\nu)\alpha} + \ldots \\
\cdots - T^{\text{I}}_{\mu\nu} + T^{\text{II}}_{\mu\nu} + T^{\text{III}}_{\mu\nu} - T^{\text{IV}}_{\mu\nu} = (\lambda + \tilde{f}R)\phi_{,\mu}\phi_{,\nu} + 8\pi T^{(\mathrm{m})}_{\mu\nu},
\end{aligned}$$

where we still have to figure out the contribution of the terms $\text{I} - \text{IV}$.

Starting with term I, we first have to calculate

$$2\,\delta_g\Box\phi = -\phi^{,\mu}\nabla_\mu\left(g_{\alpha\beta}\delta g^{\alpha\beta}\right) + 2\nabla_\mu\left(\delta g^{\mu\nu}\phi_{,\nu}\right)$$

where only the variation of the metric determinant and the identity

$$\Gamma^\nu_{\nu\mu} = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}$$

were used. The contribution to the variation of the action of a term like I multiplied by an arbitrary spacetime function $\mathcal{F}$ is thus

$$\begin{aligned}
\int \mathrm{d}^4x\sqrt{-g}\,\mathcal{F}\,2\,\delta_g\Box\phi &= \int \mathrm{d}^4x\sqrt{-g}\,\mathcal{F}\left(-\nabla_\alpha\left(g_{\mu\nu}\delta g^{\mu\nu}\right)\phi^{,\alpha} + 2\nabla_\mu\left(\delta g^{\mu\nu}\phi_{,\nu}\right)\right) \\
&= \int \mathrm{d}^4x\sqrt{-g}\left(g_{\mu\nu}(\mathcal{F}\phi^{,\alpha})_{;\alpha} - 2\mathcal{F}_{(,\mu}\phi_{,\nu)}\right)\delta g^{\mu\nu}
\end{aligned}$$

where covariant partial integration and the symmetry of $\delta g^{\mu\nu}$ were used. The contribution of term I to the modified Einstein equation is thus

$$T^{\mathrm{I}}_{\mu\nu} = g_{\mu\nu}\nabla_\beta\left[\nabla_\alpha(\tilde{f}\phi^{,\alpha})\,\phi^{,\beta}\right] - 2\phi_{(,\mu}\nabla_{\nu)}\nabla_\alpha(\tilde{f}\phi^{,\alpha}) + g_{\mu\nu}\nabla_\beta(\tilde{Z}\phi^{,\beta}) - 2\tilde{Z}_{(,\mu}\phi_{,\nu)}$$

Next, let us turn to term II. Using $\delta R_{\mu\nu}$ from above, the fact that $\phi^{,\mu}$ is geodesic (4.191) and the commutator of covariant derivatives acting on a 2-tensor, we can express

$$\begin{aligned}2\phi^{,\mu}\phi^{,\nu}\delta R_{\mu\nu} \doteq{}& \phi_{,\alpha}\phi_{,\beta}\Box\delta g^{\alpha\beta} + \phi^{,\mu}\nabla_\mu\left[\phi^{,\nu}\nabla_\nu\left(g_{\alpha\beta}\delta g^{\alpha\beta}\right) - 2\phi_{(,\alpha}\nabla_{\beta)}\delta g^{\alpha\beta}\right] + \\ &+ 2\phi^{,\mu}\left(\phi_{,\nu}R^{\nu}{}_{\alpha\mu\beta} - \phi_{(,\alpha}R_{\beta)\mu}\right)\delta g^{\alpha\beta}\end{aligned}$$

The second term is in a form ready for covariant partial integration and the second line does not contain derivatives of $\delta g^{\alpha\beta}$. The first and the third term, however, still need some rewriting. To this end, calculate

$$\phi_{,\alpha}\phi_{,\beta}\Box\delta g^{\alpha\beta} = \Box\left(\phi_{,\alpha}\phi_{,\beta}\delta g^{\alpha\beta}\right) - 2\nabla_\mu\left(\nabla^\mu\left(\phi_{,\alpha}\phi_{,\beta}\right)\delta g^{\alpha\beta}\right) + \Box\left(\phi_{,\alpha}\phi_{,\beta}\right)\delta g^{\alpha\beta}$$

and

$$\phi_{(,\alpha}\nabla_{\beta)}\delta g^{\alpha\beta} = \nabla_{(\alpha}\left(\phi_{,\beta)}\delta g^{\alpha\beta}\right) - \phi_{;\alpha\beta}\delta g^{\alpha\beta}.$$

Thus, in summary, we find that

$$\begin{aligned}2\phi^{,\mu}\phi^{,\nu}\delta R_{\mu\nu} \doteq{}& \Box\left(\phi_{,\alpha}\phi_{,\beta}\delta g^{\alpha\beta}\right) + \phi^{,\mu}\nabla_\mu\left[\phi^{,\nu}\nabla_\nu\left(g_{\alpha\beta}\delta g^{\alpha\beta}\right) - 2\nabla_{(\alpha}\left(\phi_{,\beta)}\delta g^{\alpha\beta}\right)\right] + \\ &- 2\nabla_\mu\left(\nabla^\mu\left(\phi_{,\alpha}\phi_{,\beta}\right)\delta g^{\alpha\beta}\right) + 2\phi^{,\mu}\nabla_\mu\left(\phi_{;\alpha\beta}\,\delta g^{\alpha\beta}\right) + \\ &+ \left(2\phi^{,\mu}\phi_{,\nu}R^{\nu}{}_{\alpha\mu\beta} - 2\phi^{,\mu}\phi_{(,\alpha}R_{\beta)\mu} + \Box\left(\phi_{,\alpha}\phi_{,\beta}\right)\right)\delta g^{\alpha\beta}\end{aligned}$$

Applying covariant partial integration, we find that the contribution to the modified Einstein equation of term II is given by

$$\begin{aligned}T^{\mathrm{II}}_{\alpha\beta} ={}& \phi_{,\alpha}\phi_{,\beta}\Box\tilde{f} + g_{\alpha\beta}\nabla_\nu\left(\nabla_\mu(\tilde{f}\phi^{,\mu})\phi^{\nu}\right) - 2\phi_{(,\alpha}\nabla_{\beta)}\left(\nabla_\mu(\tilde{f}\phi^{,\mu})\right) + \\ &+ 2\nabla^\mu\left(\phi_{,\alpha}\phi_{,\beta}\right)\tilde{f}_{,\mu} - 2\phi_{;\alpha\beta}\nabla_\mu(\tilde{f}\phi^{,\mu}) + \\ &+ \left(2\phi^{,\mu}\phi_{,\nu}R^{\nu}{}_{\alpha\mu\beta} - 2\phi^{,\mu}\phi_{(,\alpha}R_{\beta)\mu} + \Box\left(\phi_{,\alpha}\phi_{,\beta}\right)\right)\tilde{f}\end{aligned}$$

Note that the second and third term in the first line cancel the two terms containing $\tilde{f}$ in $T^{\mathrm{I}}_{\alpha\beta}$.

Going on to term III, calculate

$$\delta_g\left(g^{\mu\alpha}g^{\nu\beta}\phi_{;\mu\nu}\phi_{;\alpha\beta}\right) = 2\phi^{;\mu}_{\alpha}\phi_{;\beta\mu}\,\delta g^{\alpha\beta} - 2\phi^{;\mu\nu}\,\delta\Gamma^{\lambda}_{\mu\nu}\phi_{,\lambda}$$

Inserting the variation of the connection coefficients from above, the second term becomes

$$\begin{aligned}
-2\phi^{;\mu\nu}\,\delta\Gamma^{\lambda}_{\mu\nu}\phi_{,\lambda} &= 2\phi^{;\mu}_{\alpha}\,\phi_{,\beta}\nabla_{\mu}\delta g^{\alpha\beta} - \phi_{;\alpha\beta}\,\phi^{,\lambda}\nabla_{\lambda}\left(\delta g^{\alpha\beta}\right)\\
&= 2\nabla_{\mu}\left(\phi^{;\mu}_{\alpha}\,\phi_{,\beta}\delta g^{\alpha\beta}\right) - 2\nabla_{\mu}\left(\phi^{;\mu}_{\alpha}\,\phi_{,\beta}\right)\delta g^{\alpha\beta} +\\
&\quad - \phi^{,\lambda}\nabla_{\lambda}\left(\phi_{;\alpha\beta}\delta g^{\alpha\beta}\right) + \phi^{,\lambda}\nabla_{\lambda}\left(\phi_{;\alpha\beta}\right)\delta g^{\alpha\beta}
\end{aligned}$$

where in the second step the expression was brought to a form ready for covariant partial integration. Summarizing, we find that

$$\begin{aligned}
\delta_g\left(\phi^{;\mu\nu}\phi_{;\mu\nu}\right) &= 2\nabla_{\mu}\left(\phi^{;\mu}_{\alpha}\,\phi_{,\beta}\delta g^{\alpha\beta}\right) - \phi^{,\lambda}\nabla_{\lambda}\left(\phi_{;\alpha\beta}\delta g^{\alpha\beta}\right) +\\
&\quad + \left(\phi^{,\mu}\phi_{;\alpha\beta\mu} - 2\phi_{,\beta}\phi^{;\mu}_{\alpha\mu}\right)\delta g^{\alpha\beta}
\end{aligned}$$

Applying covariant partial integration, the contribution of term III to the modified Einstein equation is hence

$$T^{\mathrm{III}}_{\alpha\beta} = -\nabla^{\mu}\left(\phi_{\alpha}\phi_{,\beta}\right)\tilde{f}_{,\mu} + \phi_{;\alpha\beta}\nabla_{\mu}(\phi^{,\mu}\tilde{f}) + \left(\phi^{,\mu}\phi_{;\alpha\beta\mu} - \phi_{,\beta}\phi^{;\mu}_{\alpha\mu} - \phi_{,\alpha}\phi^{;\mu}_{\beta\mu}\right)\tilde{f}.$$

Using (4.191) and the commutator of covariant derivatives we can bring the last term into a form more similar to terms in T^{II} as

$$\phi^{,\mu}\phi_{;\alpha\beta\mu} - \phi_{,\beta}\phi^{;\mu}_{\alpha\mu} - \phi_{,\alpha}\phi^{;\mu}_{\beta\mu} = \phi^{;\mu}_{\alpha}\phi_{;\beta\mu} - R^{\nu}_{\alpha\mu\beta}\phi^{,\mu}\phi_{,\nu} - \Box\left(\phi_{,\alpha}\phi_{,\beta}\right).$$

Note that the appearing Riemann tensor components can be rewritten purely in terms of covariant derivatives of ϕ according to (4.193). Combining all our results, we find that the sum of contibutions to the modified Einstein equation is

$$\begin{aligned}
-T^{\mathrm{I}}_{\alpha\beta} + T^{\mathrm{II}}_{\alpha\beta} + T^{\mathrm{III}}_{\alpha\beta} &= \phi_{,\alpha}\phi_{,\beta}\Box\tilde{f} - \nabla_{\mu}(\tilde{Z}\phi^{,\mu})g_{\alpha\beta} + 2\phi_{(,\alpha}\tilde{Z}_{,\beta)} +\\
&\quad + \nabla^{\mu}\left(\phi_{,\alpha}\phi_{,\beta}\right)\tilde{f}_{,\mu} - \nabla_{\mu}(\phi_{;\alpha\beta}\tilde{f}\phi^{,\mu}) - 2\phi^{,\mu}\phi_{(,\alpha}R_{\beta)\mu}\tilde{f}\,.
\end{aligned}$$

Finally, term IV is easily found to be given by

$$T^{\mathrm{IV}}_{\alpha\beta} = \left(g_{\alpha\beta}\Box - \nabla_{\alpha}\nabla_{\beta} + R_{\alpha\beta}\right)h'.$$

Acknowledgments

The work of A. H. C is supported in part by the National Science Foundation Grant No. Phys-1518371 and Phys-5912998. The work of V.M. and T.B.R. is supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy – EXC-2111 – 390814868. V.M. is grateful to Korea Institute for Advanced Study, where the part of this work was completed, for hospitality.

Paper 5

On Stability of Asymptotically Free Mimetic Hořava Gravity*

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Abstract

Asymptotically free mimetic gravity has been introduced as a proposal for a classical limiting curvature theory with the purpose of singularity resolution. It was found that in a spatially flat universe an initial stage of exponential expansion with graceful exit is a generic consequence, regardless of the matter content. In this work I will analyze linear stability of cosmological perturbations in such a model, considering only the degrees of freedom of pure mimetic gravity. I show that the addition of Hořava-gravity-like higher order spatial curvature terms can lift the gradient instability of scalar perturbations, even when the gradient term has the wrong sign throughout. Calculating the primordial spectra of tensor and scalar perturbations in the simplest single component model, I find that the initially scale invariant spectra turn out to be destroyed later by the rapidly varying speed of sound at horizon exit.

*preprint available at: [5] T. B. Russ, *On Stability of Asymptotically Free Mimetic Hořava Gravity,* (2021), arXiv:210312442 [gr-qc]

5.1 Introduction

By the unique status of general relativity, any alternative theory of metric gravity usually either has to allow for higher derivatives of the metric, higher dimensions of spacetime or to introduce new fields, separate from the metric. Another option is to reparametrize the degrees of freedom of the physical metric itself, e.g. by a disformal transformation [33]. "Mimetic gravity" stems from the reparametrization of the physical metric $g_{\mu\nu}$ in terms of an auxiliary metric $\tilde{g}_{\mu\nu}$ and the "mimetic field" ϕ as

$$g_{\mu\nu} = \tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}. \tag{5.1}$$

This particular disformal transformation is special for two reasons: *1.)* It is singular, explaining how a simple reparametrization can actually lead to new physics [34], [96], *2.)* The physical metric is invariant under Weyl transformations of the auxiliary metric. This means that the new degree of freedom introduced by ϕ represents what was called a "conformal degree of freedom of gravity". Soon after this reparametrization was first introduced in [35], it was shown in [37] that the mimetic field can be introduced equivalently as a constrained scalar field, subject to the constraint

$$g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} = 1. \tag{5.2}$$

Apart from the dust-like component called "Mimetic Dark Matter" that emerges as a constant of integration in the modified Einstein equation of mimetic gravity, the introduction of the mimetic field also enables a wealth of possible new terms in the gravity action. By breaking shift symmetry in ϕ, i.e. introducing a ϕ dependent potential, one can produce an extremely flexible theory where essentially any conceivable background solution can be realized in a Friedmann universe, cf. [61].

Conversely, if we restrict to shift symmetric theories without higher derivatives of the metric in the modified Einstein equation, the range of possibilities for a non-singular universe becomes very narrow. For the most natural (and arguably only viable) class of modified flat Friedmann universes, it was shown in [1], [4] that the only thing that can replace the Big Bang singularity is a smooth transition to a piece of de Sitter spacetime at limiting curvature. In this work I will show that this class of modifications also happens to coincide with the class of models that avoid a ghost instability of scalar metric perturbations.

The mimetic field, by definition (5.2), provides a global time function whose gradient is everywhere timelike. In [2] it was shown that this can be used to covariantly dissect any scalar quantity that is invariant under spatial diffeomorphisms in the slicing given by ϕ. In this way it is easy to write down a Hořava-gravity-like theory with only higher spatial derivatives but no higher time derivatives or mixed derivatives. In Hořava gravity [50] such an "asymmetry" between space and time is used to improve the UV behaviour of the graviton propagator for the purpose of renormalizability. Projectable Hořava models have been shown to be renormalizable in [51], [52]. Compared to other covariantized version of Hořava gravity like [54], mimetic Hořava gravity has the advantage of not having any additional propagating degrees of freedom in a Minkowski background. Interestingly, also the following connections between mimetic gravity and Hořava gravity can be drawn: In [55] it was explored how a dust-like component emerges as a constant of integration in Hořava-Lifshitz gravity. In [56] an equivalence between the IR limit of projectable Hořava gravity and a mimetic matter scenario has been found. Another Hořava-like mimetic model has been presented in [57].

In this paper I will show that higher spatial derivative terms of sixths order can not only render a power counting renormalizable theory, they also serve to alleviate the gradient instability of the scalar degree of freedom of mimetic gravity in an expanding universe.

The paper is organised as follows: In section 5.2, I merge parts of the Lagrangians from [4] and [2] to introduce the theory that will be used in the rest of the paper. In section 5.3, I re-derive a simplified version of the background solutions found in [1]. Introducing modified conformal time, these solutions can be written in closed form. In section 5.4, I analyse metric perturbations in a flat Friedmann universe in comoving gauge. I discuss stability issues and calculate the primordial spectra of tensor and scalar perturbations for the particular case of a radiation dominated background. The analysis of the Mukhanov-Sasaki equations with modified dispersion relations follows similar steps as [97]. In section 5.5, I summarize my results and give a brief outlook on possible extensions. In Appendix 5.5, I present the second order actions for a more general mimetic theory and for the more general case of perturbations around a non-flat Friedmann universe. In Appendix 5.5, I perform the linear stability analysis for bouncing solutions driven by higher order spatial curvature terms in a non-flat universe, as found in [4]. Throughout this paper I use Planck units where $G_0 = G(\Box\phi = 0) = 1$, $\hbar = 1$, $c = 1$, $k_B = 1$.

5.2 The theory

Consider the shift-symmetric theory of mimetic gravity defined by

$$\mathcal{S}_g = -\frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}\left\{\mathcal{L}_{\mathsf{nhd}} + \mathcal{L}_{\mathsf{hd}} + \lambda\left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - 1\right)\right\}, \tag{5.3}$$

where the Lagrangian $\mathcal{L} = \mathcal{L}_{\mathsf{nhd}} + \mathcal{L}_{\mathsf{hd}}$ is divided into a part $\mathcal{L}_{\mathsf{nhd}}$ without higher derivatives and a part $\mathcal{L}_{\mathsf{hd}}$ with higher spatial derivatives in the corresponding second order actions.

In [4] we found that the Lagrangian

$$\mathcal{L}_{\mathsf{nhd}} := f(\Box\phi)R + (f(\Box\phi) - 1)\widetilde{R} + 2\Lambda(\Box\phi) \tag{5.4}$$

with

$$\widetilde{R} := 2\phi^{;\mu}\phi^{;\nu}G_{\mu\nu} - (\Box\phi)^2 + \nabla^\mu\nabla^\nu\phi\nabla_\mu\nabla_\nu\phi,$$

leads to a modified Einstein equation that is free of all higher derivatives of the metric. Using the mimetic constraint, a slight generalization of a calculation from [2] shows that (apart from considerations involving boundary terms) it is equivalent to consider the

Lagrangian

$$\mathcal{L}_{\mathsf{nhd}} \doteq \tfrac{4}{3}\ell(\Box\phi) - f(\Box\phi)\left(\nabla^\mu\nabla^\nu\phi\nabla_\mu\nabla_\nu\phi - \tfrac{1}{3}(\Box\phi)^2\right) - \widetilde{R}, \tag{5.5}$$

where $\doteq$ denotes equality up to a total covariant derivative and

$$\ell(\Box\phi) = \tfrac{3}{2}\Box\phi F(\Box\phi) - (\Box\phi)^2 f(\Box\phi) + \tfrac{3}{2}\Lambda(\Box\phi). \tag{5.6}$$

The function $F(\Box\phi)$ is definded by $F'(\Box\phi) \equiv \mathrm{d}F/\mathrm{d}\Box\phi = f(\Box\phi)$. In the following, the choice

$$\Lambda(\Box\phi) = \tfrac{2}{3}\,\Box\phi\,(\Box\phi f(\Box\phi)\, - F(\Box\phi)) \tag{5.7}$$

will prove to be particularly simple.

Motivated by the goal of renormalizability along the lines of Hořava gravity [50], as suggested in [2], I include the sixth order higher spatial derivative terms

$$\mathcal{L}_{\mathsf{hd}} := \sigma_T^4\,\widetilde{C}^\mu_\nu\widetilde{C}^\nu_\mu - \frac{\sigma_S^4}{8}P^\mu_\nu\nabla_\mu\widetilde{R}\nabla^\nu\widetilde{R}, \tag{5.8}$$

where the projector $P^\nu_\mu = \delta^\nu_\mu - \phi_{,\mu}\phi^{,\nu}$ and the covariant analogues of the spatial Cotton tensor $\widetilde{C}^\mu_\nu$ and the spatial Ricci tensor $\widetilde{R}_{\mu\nu}$ have been introduced in [2] as

$$\widetilde{C}^\mu_\nu := -\tfrac{1}{\sqrt{-g}}\epsilon^{\mu\rho\kappa\lambda}\nabla_\lambda\phi\,\nabla_\rho\left(\widetilde{R}_{\nu\kappa} - \frac{1}{4}g_{\nu\kappa}\widetilde{R}\right), \tag{5.9}$$

$$\widetilde{R}_{\mu\nu} := P^\alpha_\mu P^\beta_\nu R_{\alpha\beta} + \nabla_\alpha\left(\phi^{,\alpha}\nabla_\mu\nabla_\nu\phi\right). \tag{5.10}$$

Note that $g^{\mu\nu}\widetilde{R}_{\mu\nu} = -\widetilde{R}$, using the mimetic constraint.

NB. In a flat Friedmann universe, the Lagrangian studied in [1],

$$\begin{aligned}\widetilde{\mathcal{L}} &= f(\Box\phi)R + 2\Lambda(\Box\phi)\\ &\doteq \tfrac{4}{3}\ell(\Box\phi) - f(\Box\phi)\left(\nabla^\mu\nabla^\nu\phi\nabla_\mu\nabla_\nu\phi - \tfrac{1}{3}(\Box\phi)^2 + \widetilde{R}\right),\end{aligned} \tag{5.11}$$

leads to the same background dynamics as (5.3). However, in general its modified Einstein equation contains higher mixed derivatives of the metric in the synchronous frame $t = \phi$. While it was found in [58] that these mixed derivatives can change the sign of the gradient term of scalar metric perturbations and help to prevent a gradient instability, it was later realized in [59], [60] that this comes at the price of introducing an additional hidden degree of freedom. Even tough this second scalar degree of freedom does not show up when perturbing around a homogeneous background in unitary gauge, it was found in [60] that already for perturbations around Minkowski spacetime with non-homogeneous mimetic field profile it can lead to instabilities. Since this type of higher mixed derivatives is not present in $\mathcal{L}_{\mathsf{nhd}}$, one could hope that no such additional degree of freedom will appear in this theory. A full Hamiltonian analysis similar to [59], [60], [98], [99] would of course be a more involved task beyond the scope of this paper.

5.3 Background dynamics

In the homogeneous, isotropic background given by the flat Friedmann metric

$$\mathrm{d}s^2 = dt^2 - a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{5.12}$$

the only consistent background solution of the mimetic constraint up to shifts is

$$\phi = t, \tag{5.13}$$

for which we find

$$\Box\phi \equiv \kappa = 3H \equiv 3\tfrac{\dot{a}}{a}. \tag{5.14}$$

Either from the equation of motion given in [4] or from analysis of the zeroth order action (see appendix 5.5), we arrive at the modified Friedmann equation

$$\tfrac{2}{3}\left(\kappa\ell'(\kappa) - \ell(\kappa)\right) = \frac{c_{\mathsf{MDM}}}{a^3} + 8\pi\varepsilon^m =: 8\pi\varepsilon. \tag{5.15}$$

The constant of integration c_{MDM} describes the contribution of mimetic matter and ε^m is some general homogeneous, isotropic matter energy density. Note that this background equation is the same for all the theories (5.3), (5.5) and (5.11). Using the simplifying choice (5.7) and the suggestive notation $G = 1/f$ familiar from [1], the modified Friedmann equation becomes

$$H^2 = \frac{8\pi}{3}G(\kappa)\,\varepsilon. \tag{5.16}$$

Assuming a monotonically decreasing dependence of ε on the scale factor, such a modified Friedmann equation can be understood as an integral curve of the form $a(H^2)$ in the phase space spanned by a and H, cf. [4], [48]. The only possible relations of this form which *a)* are one-to-one *b)* have limiting curvature $\kappa < \kappa_0$ and *c)* obey the GR-limit $G(\kappa) = 1+\mathcal{O}\left((\kappa/\kappa_0)^2\right)$, generically replace the Big Bang singularity by a smooth transition to an initial de Sitter stage. In (5.16) such a behaviour can only be realized by "asymptotic freedom"†, i.e. $G(\kappa \to \kappa_0) \to 0$. For a concrete example with limiting curvature κ_0, I will take the simple choice

$$G(\kappa) = f(\kappa)^{-1} = 1 - \left(\frac{\kappa}{\kappa_0}\right)^2, \tag{5.17}$$

familiar from [1]. Note that in this case $\ell'' = f^2$. Assuming a single matter component with constant equation of state $w = p/\varepsilon$, taking a time derivative of (5.16) we find that during the inflationary stage

$$-\frac{\dot{H}}{H^2} = \frac{3(1+w)}{2}G(\kappa) \ll 1, \tag{5.18}$$

†Note that in (5.15) without using (5.7), such a background behaviour could be equally well implemented by a choice of Λ without asymptotic freedom. However, as shown in [1], [4], asymptotic freedom becomes unavoidable for singularity resolution in an anisotropic universe.

where $\dot{}$ denotes t derivatives. However, the second "slow-roll" parameter

$$\frac{\ddot{H}}{2H\dot{H}} = \frac{3(1+w)}{2} = \mathcal{O}(1) \tag{5.19}$$

is constant and of order of unity, showing that this background solution does not fit into a "slow-roll" description.

It is straightforward to obtain the following implicit solution of (5.16) for $\kappa(t)$:

$$\frac{1+w}{2}\kappa_0\, t = \frac{\kappa_0}{\kappa} - \operatorname{atanh}\frac{\kappa}{\kappa_0} \tag{5.20}$$

Modified conformal time. Introducing the modified conformal time coordinate $\tilde{\eta}$ by

$$\mathrm{d}\tilde{\eta} = \frac{\mathrm{d}t}{a\sqrt{f}}, \tag{5.21}$$

the modified Friedmann equation (5.16) in modified conformal time $\tilde{\eta}$ looks exactly like the usual Friedmann equation in usual conformal time:

$$(a_{\tilde{\eta}})^2 = \frac{8\pi}{3}\varepsilon\, a^4 \tag{5.22}$$

A subscript $a_{\tilde{\eta}} = \partial_{\tilde{\eta}} a$ denotes $\tilde{\eta}$ derivatives. Assume that the total energy density is dominated by a component with equation of state w and parametrized as

$$\frac{8\pi}{3}\varepsilon = \left(\frac{c}{H_l^2}\right)^{\frac{1+3w}{2}} \left(\frac{1+3w}{2}\, a\right)^{-3(1+w)}, \tag{5.23}$$

where the prefactors were introduced for later convenience and $H_l = \kappa_0/3$ accounts for dimensions such that c is dimensionless. In this case we find the solution

$$a(\tilde{\eta}) = \frac{2}{(1+3w)}\frac{\sqrt{c}}{H_l}\,\tilde{\eta}^{\frac{2}{1+3w}}, \tag{5.24}$$

where the initial condition $a(\tilde{\eta}=0)=0$ was used. The range of modified conformal time is then $0 < \tilde{\eta} < \infty$.

For the choice (5.17) the solution for $\kappa(\tilde{\eta})$ is

$$\left(\frac{\kappa_0}{\kappa}\right)^2 = \left(\frac{H_l}{H}\right)^2 = 1 + c\,\tilde{\eta}^{\frac{6(1+w)}{1+3w}}. \tag{5.25}$$

It describes a smooth transition from exponential expansion at limiting curvature with $H \sim H_l$ to the late time stage dominated by the matter component with equation of state w where $H \propto t^{-1}$. The end of the inflationary stage, i.e. the end of accelerated expansion, happens at

$$\left(\frac{H_f}{H_l}\right)^2 = \frac{1+3w}{3(1+w)}, \tag{5.26}$$

or, in modified conformal time at

$$\tilde{\eta}_f = \left(\frac{2}{(1+3w)c}\right)^{\frac{1+3w}{6(1+w)}}. \tag{5.27}$$

5.4 Metric perturbations in comoving gauge

In the same way as the modified Einstein equation of mimetic gravity takes its simplest form with the choice of time coordinate $t = \phi$, the analysis of metric perturbations is most readily performed in comoving gauge $\delta\phi = 0$. Metric perturbations in different versions of mimetic gravity and also in different gauges have been analysed already a number of times, e.g. in [40, 57, 58, 60, 71, 76, 98, 100, 101]. See appendix 5.5 for the calculation of second order actions around a non-flat Friedmann background in a more general theory.

Starting with a general ADM metric,

$$\mathrm{d}s^2 = N^2\mathrm{d}t^2 - \gamma_{ij}\left(\mathrm{d}x^i + N^i\mathrm{d}t\right)\left(\mathrm{d}x^j + N^j\mathrm{d}t\right), \tag{5.28}$$

the mimetic constraint fixes the lapse function as

$$N^2 = \frac{\left(\partial_0\phi - N^i\partial_i\phi\right)^2}{\left(1 + \gamma^{ij}\partial_i\phi\partial_j\phi\right)}. \tag{5.29}$$

In comoving gauge where $\partial_i\phi = 0$ this implies $N = \dot{\phi}$. Thus, the spatial slices coincide with slices of constant ϕ and the homogeneous lapse N is determined only by the background. Using the background solution (5.13) amounts to setting $N = 1$. In this gauge it holds that the quantities

$$\Box\phi = \kappa, \qquad \nabla^\mu\nabla^\nu\phi\nabla_\mu\nabla_\nu\phi = \kappa^{ij}\kappa_{ij}, \qquad \widetilde{R} = {}^3R \tag{5.30}$$

are given in terms of the extrinsic curvature $\kappa_{ij} = \frac{1}{2}\left(\dot{\gamma}_{ij} - N_{i,j} - N_{j,i}\right)$ and intrinsic Ricci curvature ${}^3R_{ij}$ of spatial slices and their traces. Similar, straightforward calculations show that the higher order terms from (5.8) take the form

$$P^\mu_\nu\nabla_\mu\widetilde{R}\nabla^\nu\widetilde{R} = -\gamma^{kl}\partial_k({}^3R)\,\partial_l({}^3R), \tag{5.31}$$

$$\widetilde{C}^\mu_\nu\widetilde{C}^\nu_\mu = {}^3C^i_j\,{}^3C^j_i. \tag{5.32}$$

The spatial Cotton tensor is defined as in [2] by

$${}^3C^i_j = \frac{1}{\sqrt{\gamma}}\epsilon^{ikl}\bar{\nabla}_k\left({}^3R_{jl} - \frac{1}{4}\gamma_{jl}\,{}^3R\right), \tag{5.33}$$

where $\bar{\nabla}$ denotes the covariant derivative associated to γ_{ij} and indices on spatial tensors are raised with γ^{ij}.

Perturbing around the flat Friedmann background (5.12) in comoving gauge $\phi = t$, the metric perturbations can be further decomposed as

$$\gamma_{ij} = a^2(t)\left(e^{-2\psi}\delta_{ij} - 2E_{,ij} - 2F_{(i,j)} + (e^h)_{ij}\right), \qquad N_i = \chi_{,i} - aS_i \tag{5.34}$$

where

$$F^i_{,i} = 0, \qquad S^i_{,i} = 0, \qquad h^i_i = 0, \qquad h^i_{j,i} = 0. \tag{5.35}$$

Indices are raised with δ^{ij} and comma denotes partial derivative. While the temporal coordinate is completely fixed, there is still a remaining freedom of choice in the spatial coordinates. In contrast to the synchronous gauge condition $N_i = 0$, the choice $E = 0$, $F_i = 0$ fixes the coordinates uniquely, cf. [86].

5.4.1 Vector perturbations

In mimetic gravity, like in GR, the vector perturbations parametrized by F_i and S_i as

$$\gamma_{ij} = a^2(t)\left(\delta_{ij} - F_{i,j} - F_{j,i}\right), \qquad N_i = -aS_i, \tag{5.36}$$

are non-dynamical in the absence of sources. Note that the spatial Ricci tensor ${}^3R_{ij} = \mathcal{O}(F^2)$ is second order in vector perturbations when perturbing around a spatially flat Friedmann universe. Hence, higher order spatial curvature terms like in (5.8) do not contribute to the linearized equation of motion for vector perturbations. Choosing the gauge $S_i = 0$, we can use the equations of motion derived in synchronous coordinates in [4] and express them in terms of the gauge invariant variable $V_i = S_i - a\dot{F}_i$. Assuming hydrodynamical matter of the perfect fluid type, the only non-vanishing vector components of the perturbed energy momentum tensor $\delta T_{\mu\nu}$ are of the form $\delta T_{0i} = (\varepsilon^m + p)\,\delta u_{\perp i}$, [86].

The spatial modified Einstein equations become

$$\partial_t\left(a^2 f\left(V_{i,j} + V_{j,i}\right)\right) = 0, \tag{5.37}$$

and have the solution

$$V_i = \frac{C_{\perp i}}{f a^2}. \tag{5.38}$$

Note that the gauge invariant vector perturbation V_i decays both in the late time limit, like in GR, as well as in the early time limit, as $f \to \infty$. This is completely in line with the previous results from [4] that due to asymptotic freedom anisotropies decay during contraction at limiting curvature. Provided that the constant of integration $C_{\perp i}$ and thus the maximum value of V_i is bounded, we see that in this model vector perturbations never become important. Reinserting this solution into the $0-i$ modified Einstein equation

$$f\Delta V_i = 16\pi\left(\varepsilon^m + p\right) a\,\delta u_{\perp i}, \tag{5.39}$$

with $\Delta = \delta^{kl}\partial_k\partial_l$, shows that for the physical velocity $\delta v^i = -a^{-1}\delta u_{\perp i}$ it holds that

$$\delta v^i = -\frac{\Delta C_{\perp i}}{16\pi a^4\left(\varepsilon^m + p\right)}, \tag{5.40}$$

where f cancels and which is thus the same equation that one finds in standard GR. Note that only the matter energy density and pressure are appearing since mimetic matter does not source vector perturbations.

5.4.2 Tensor perturbations

Tensor perturbations are parametrized by

$$\gamma_{ij} = a^2 \left(\eta_{ij} + h_{ij} + \frac{1}{2} h_i^k h_{kj} \right), \qquad N_i = 0, \tag{5.41}$$

where indices on h_{ij} are raised with δ^{ij} and $h_i^i = 0$, $h_{j,i}^i = 0$. Expanding (5.66) to second order in h_{ij} yields the second order action (see appendix 5.5)

$$^{(2)}\mathcal{S}_h = \frac{1}{64\pi} \int \mathrm{d}t\, \mathrm{d}^3x\, a^3 \left\{ f\,(\dot{h}_{ij})^2 - \frac{1}{a^2}(\partial h_{ij})^2 - \frac{\sigma_T^4}{a^6}(\partial^3 h_{ij})^2 \right\}, \tag{5.42}$$

where $\dot{}$ denotes t derivatives, $(\partial h_{ij})^2 \equiv \delta^{kl}\partial_k h^{ij}\partial_l h_{ij}$ and $(\partial^3 h_{ij})^2 \equiv (\partial\Delta h_{ij})^2$. Since $f > 0$, tensor perturbations do not exhibit any instabilities provided that $\sigma_T^4 \geq 0$.

We can read o the propagation speed of gravitational waves,

$$c_T^2 = f^{-1}(\kappa) = G(\kappa), \tag{5.43}$$

and find that it deviates from unity in the early time/high curvature regime. In particular, note that c_T is vanishing at limiting curvature $\kappa = \kappa_0$ in the case of asymptotic freedom. Using (5.17), right at the end of inflation $c_T^2 = \frac{2}{3(1+w)}$. Already one second later, in the late time regime where $\kappa \propto t^{-1}$ and t roughly corresponds to the time since the end of inflation, the propagation speed of gravitational waves is approximately

$$1 - c_T \approx 1 - \sqrt{1 - \left(\frac{1}{\kappa_0\, t}\right)^2} \lesssim \left(\frac{\kappa_{pl}}{\kappa_0}\right)^2 \left(\frac{1\,\text{sec}}{t}\right)^2 \times 10^{-86}. \tag{5.44}$$

The limiting curvature κ_0 could naturally be taken to lie a few orders of magnitude below the Planck curvature $\kappa_{pl} = 1/t_{pl}$ for the sake of singularity resolution. Thus, late time experimental constraints like $1 - c_T < \mathcal{O}(10^{-15})$, which stems from the multi messenger event GW170817 ([72], [73]) that happened around $t \sim 10^9$ years after the conjectured inflationary period, are not touched in the slightest by this deviation of c_T from the speed of light in the limiting curvature regime. They do, however, tightly constrain low curvature / IR modifications due to additional terms in the Lagrangian like $(\Box\phi)^2$ or in particular $\nabla^\mu\nabla^\nu\phi\nabla_\mu\nabla_\nu\phi$, see [102], cf. [103].

Mode expansion. Introducing $y = a f^{1/4}$ and substituting the expansion

$$h_j^i(\tilde{\eta}, \mathbf{x}) = \sqrt{32\pi} \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \frac{u_k(\tilde{\eta})\, e^{i\mathbf{k}\mathbf{x}}}{y\sqrt{\mathrm{e}_n^m \mathrm{e}_m^n}}\, \mathrm{e}_j^i(\mathbf{k}), \tag{5.45}$$

where the mode function u_k and the contraction $\mathrm{e}_n^m\mathrm{e}_m^n$ of the polarization tensor only depend on the magnitude of $\mathbf{k}$ by isotropy, the second order action in modified conformal time $\mathrm{d}\tilde{\eta} = \frac{\mathrm{d}t}{a\sqrt{f}}$ becomes

$$^{(2)}\mathcal{S}_h = \frac{1}{2} \int \mathrm{d}\tilde{\eta}\, \mathrm{d}^3k \left\{ u_{k,\tilde{\eta}}^2 - \left(k^2 + \frac{\sigma_T^4}{a^4} k^6 - \frac{y_{\tilde{\eta}\tilde{\eta}}}{y} \right) u_k^2 \right\}. \tag{5.46}$$

A subscript $u_{k,\tilde{\eta}} \equiv \partial u_k/\partial\tilde{\eta}$ denotes derivatives. The modified Mukhanov-Sasaki equation for u_k reads

$$u_{k,\tilde{\eta}\tilde{\eta}} + \left(\frac{\sigma_T^4}{a^4}k^6 + k^2 - \frac{y_{\tilde{\eta}\tilde{\eta}}}{y}\right)u_k = 0, \tag{5.47}$$

and the power spectrum of gravitational waves is given by [86]

$$\delta_h^2(k,\tilde{\eta}) = \frac{8}{\pi}\frac{k^3\left|u_k\right|^2}{y^2}. \tag{5.48}$$

In the late time limit $f \to 1$, $\tilde{\eta} \to \eta$ is identical to regular conformal time and (5.47) becomes just like in GR,

$$u_{k,\eta\eta} + \left(k^2 - \frac{a_{\eta\eta}}{a}\right)u_k = 0. \tag{5.49}$$

Initially, at $\tilde{\eta} \to 0$ the k^6 term is dominating. The requirement of an initial minimal level of quantum fluctuations determines the initial conditions at $\tilde{\eta} \to 0$ (up to an unimportant phase factor, see [86]) as

$$u_k = \frac{a}{\sigma_T k^{3/2}}, \qquad u_{k,\tilde{\eta}} = i\,\frac{\sigma_T k^{3/2}}{a}. \tag{5.50}$$

In the initial region where the k^6 term is dominating and $a_{\tilde{\eta}}\,a \ll k^3\sigma_T^2$, the solution of (5.47) with initial conditions (5.50) is well described by the WKB approximation

$$u_k = \frac{a(\tilde{\eta})}{\sigma_T k^{3/2}}\exp\left(-i\int \mathrm{d}\tilde{\eta}\frac{\sigma_T^2 k^3}{a(\tilde{\eta})^2}\right). \tag{5.51}$$

Note that the initial spectrum is scale invariant, by virtue of the same higher order k^6 term that is needed for a power counting renormalizable theory. After "horizon" exit, i.e. when $y_{\tilde{\eta}\tilde{\eta}}/y$ becomes dominating, the solution of (5.47) is

$$u_k(\tilde{\eta}) = y(\tilde{\eta})\left(A_k^T + B_k^T\int_{\tilde{\eta}}^{\infty}\frac{\mathrm{d}\tilde{\eta}'}{y^2(\tilde{\eta}')}\right), \tag{5.52}$$

where the second term is decaying compared to the first term and will be ignored for the following estimates. The primordial spectrum of tensor perturbations is hence given by

$$\delta_h^2(k,\tilde{\eta}) \approx \frac{8}{\pi}k^3\left|A_k^T\right|^2. \tag{5.53}$$

The initially scale invariant spectrum would only be preserved for modes which exit the "horizon" before the k^2 term starts do dominate and in a region where $y \propto a$, i.e. where the propagation speed c_T is almost constant. This is certainly not satisfied for modes exiting the horizon during the inflationary stage of an asymptotically free model where f is rapidly changing. Hence we can expect a primordial spectrum of tensor perturbations which is far from being scale invariant on large scales.

Radiation dominated background. For a concrete example with the simplest possible background evolution, consider the case where the total energy density is dominated by a component with equation of state $w = 1/3$. In this case, fixing c by setting $a(\tilde{\eta}_f) = 1$ at the end of inflation, the solution (5.24) is $a(\tilde{\eta}) = H_l \tilde{\eta}$ and we find

$$y = \sqrt[4]{1 + (H_l \tilde{\eta})^4}, \qquad \frac{y_{\tilde{\eta}\tilde{\eta}}}{y} = \frac{3 H_l^4 \tilde{\eta}^2}{(1 + (H_l \tilde{\eta})^4)^2}. \tag{5.54}$$

Before the end of inflation at $\tilde{\eta}_f = 1/H_l$, the modified Mukhanov-Sasaki equation (5.47) is well approximated by

$$u_{k,\tilde{\eta}\tilde{\eta}} + \left(\frac{\tilde{\sigma}_T^4}{H_l^8 \tilde{\eta}^4} k^6 + k^2 - 3 H_l^4 \tilde{\eta}^2 \right) u_k = 0, \tag{5.55}$$

with the dimensionless $\tilde{\sigma}_T := \sigma_T H_l$. The “horizon” exit of the mode k happens at $\tilde{\eta}_k = k / H_l^2 \sqrt{D}$ where D solves

$$\tilde{\sigma}_T^4 D^3 + D = 3. \tag{5.56}$$

Note that by the assumption made before, this is only valid for modes $k^2 \ll D H_l^2$ which exit the “horizon” before the end of inflation, i.e. $\tilde{\eta}_k \ll \tilde{\eta}_f$.

In the initial region where the k^6 term is dominating, the solution of (5.55) is given by

$$u_k(\tilde{\eta}) = \frac{H_l^2 \, \tilde{\eta}}{k^{3/2} \tilde{\sigma}_T} \exp\left(-i \frac{\tilde{\sigma}_T^2 k^3}{H_l^4 \tilde{\eta}} \right), \tag{5.57}$$

where the initial conditions (5.50) have been taken into account.

It depends only on $\tilde{\sigma}_T$ if modes will exit the “horizon” before or after the gradient term is dominating over the k^6 term. For $\tilde{\sigma}_T \gg 1$ the gradient term will never become important and $D = (3/\tilde{\sigma}_T^4)^{1/3}$. By matching the absolute value of $|u_k|$ from (5.57) and (5.52) we can estimate

$$|A_k^T| \approx \frac{H_l}{k^{3/2} \tilde{\sigma}_T} \frac{a(\tilde{\eta}_k)}{y(\tilde{\eta}_k)} \approx \frac{1}{\sqrt{D} \, \tilde{\sigma}_T \, k^{1/2}}. \tag{5.58}$$

On the other hand, if $\tilde{\sigma}_T \ll 1$ then there is an intermediate region where the k^2 term is dominating and $D = 3$. In this region, starting at

$$\tilde{\eta}_* = \frac{\tilde{\sigma}_T}{H_l^2} k < \tilde{\eta}_k = \frac{k}{\sqrt{3} H_l^2}, \tag{5.59}$$

the absolute value $|u_k|$ is approximately constant and we can estimate

$$|A_k^T| \approx \frac{H_l}{k^{3/2} \tilde{\sigma}_T} \frac{a(\tilde{\eta}_*)}{y(\tilde{\eta}_k)} \approx \frac{1}{k^{1/2}}. \tag{5.60}$$

Note that this is identical to the result that one would get if $\tilde{\sigma}_T = 0$ and the initial conditions were determined from the k^2 term.

In summary, the primordial spectrum of large wavelength modes $k^2 \ll DH_l^2$ becomes

$$\delta_h^2(k,\tilde{\eta}) \approx \begin{cases} \frac{8}{\pi}\, k^2 & \text{if } \tilde{\sigma}_T \ll 1 \\ \frac{8}{\pi}\, \frac{k^2}{(\sqrt{3}\,\tilde{\sigma}_T)^{2/3}} & \text{if } \tilde{\sigma}_T \gg 1 \end{cases} \tag{5.61}$$

Note that here $1/k$ is the physical wavelength at the end of inflation. In both cases the spectrum is indeed far from being scale invariant, with a large blue tilt and a spectral index of

$$n_T = 2. \tag{5.62}$$

For a different background, e.g. $w = 0$ the situation becomes more complicated because $y_{\tilde{\eta}\tilde{\eta}}/y$ initially diverges as $\sim -1/\tilde{\eta}^2$. Hence, after an initial sub-“horizon” k^6 domination region, there can be an earlier intermediate super-“horizon” region, followed again by a sub-“horizon” region where k^2 dominates before finally exiting the horizon. A similar situation occurs for scalar perturbations in a dust dominated background, see below.

5.4.3 Scalar perturbations

The scalar metric perturbations in comoving gauge are characterized through ζ and χ by

$$\gamma_{ij} = a^2(t)e^{2\zeta}\delta_{ij}, \qquad N_i = \chi_{,i}, \tag{5.63}$$

where

$$\zeta = -\psi - \frac{H}{\dot{\phi}}\delta\phi. \tag{5.64}$$

is the gauge invariant curvature perturbation in comoving gauge $\delta\phi = 0$.

In contrast to standard GR, even in the absence of any matter fluctuations the conformal degree of freedom of mimetic gravity can become dynamical. In this case the action (5.66) expanded to second order in scalar perturbations (see appendix 5.5) becomes

$$\begin{aligned} {}^{(2)}\mathcal{S} = \frac{1}{8\pi}\int \mathrm{d}^4x\sqrt{\eta}\,a^3\Big\{ & -3\ell''\dot{\zeta}^2 + \tfrac{1}{a^2}(\partial\zeta)^2 - \tfrac{\sigma_S^4}{a^6}\,(\partial^3\zeta)^2 \\ & - \Big[\tfrac{1}{3}(\ell''-f)\Delta\chi - 2\ell''a^2\dot{\zeta}\Big]\,\tfrac{1}{a^4}\Delta\chi\Big\}, \end{aligned} \tag{5.65}$$

where $\dot{}$ denotes t derivatives, $(\partial\zeta)^2 \equiv \delta^{kl}\partial_k\zeta\partial_l\zeta$ and $(\partial^3\zeta)^2 \equiv (\partial\Delta\zeta)^2$. In the case $\ell''-f = 0$, variation with respect to χ shows that $\dot{\zeta} = 0$ and thus the conformal degree of freedom is frozen. In the case $\ell''-f \neq 0$, the second order action after integrating out χ becomes

$${}^{(2)}\mathcal{S}_\zeta = \frac{1}{8\pi}\int \mathrm{d}t\,\mathrm{d}^3x\,a^3\Big\{\frac{3\ell''f}{\ell''-f}\,\dot{\zeta}^2 + \frac{1}{a^2}\,(\partial\zeta)^2 - \frac{\sigma_S^4}{a^6}\,\big(\partial^3\zeta\big)^2\Big\}. \tag{5.66}$$

No ghost instability. Since $f > 0$, the condition to have no ghost instability reads

$$\frac{\ell''}{\ell'' - f} > 0 \qquad \Rightarrow \quad \ell'' > f. \tag{5.67}$$

The other possible case $\ell'' < 0$ was excluded because for a smooth low curvature GR limit it must hold that $\ell'' \to 1$, $f \to 1$ as $\kappa \to 0$.

The condition (5.67) constrains the slope of the modified Friedmann equation (5.15) by

$$\frac{8\pi}{3}\frac{\partial\varepsilon}{\partial H^2} = \frac{1}{\kappa}\frac{\partial}{\partial\kappa}\left(\kappa\ell' - \ell\right) = \ell'' \overset{!}{>} f > 0, \tag{5.68}$$

which can be rewritten as

$$0 < \frac{\partial H^2}{\partial\varepsilon} < \frac{8\pi}{3}G(\kappa). \tag{5.69}$$

Note that this condition is fully general and does not make use of the simplifying choice of Λ given by (5.7). Using (5.7), the condition becomes $\kappa G' < 0$ and we see that the running gravitational constant can only decrease when going to higher curvatures. This condition is always satisfied for the asymptotically free background solutions from section 5.3.

Instability of spatially flat bouncing solutions. In a previous mimetic model [42], limiting curvature was realized by a bounce. In order to achieve a bouncing background solution in a spatially flat universe, the modified relation $a(H^2)$ cannot be one-to-one. In fact, the generic background evolution described in section 5.3 can only be circumvented by including multi-valued functions with intricate branch changes in the Lagrangian, cf. [47]. Moreover, it was discussed in [48] that the bouncing solution from [42] is unstable under perturbations. From the above it is easy to see that this is an unavoidable feature of any bouncing solution of (5.15): In order to obtain a bounce there must be a region where $H^2(\varepsilon)$ decreases until it eventually reaches a zero at some finite $a_{\min} > 0$. In this region the condition (5.69) is violated and hence there is a ghost instability.

The stability analysis for bounces in non-flat universes found in [4], driven by higher order spatial curvature terms, is performed in appendix 5.5.

Gradient instability? As has been noticed in [71], mimetic models without any higher derivatives typically exhibit a gradient instability. In fact, also for models which include higher derivatives, a negative square of the speed of sound $c_S^2 < 0$ is something we have to deal with, at least in the low curvature regime, in any mimetic model with a well behaved GR limit. However, in this same limit also $\ell'' \to f$ and thus the scalar degree of freedom of mimetic gravity stops to propagate.

In a theory with higher mixed derivatives, the sign of the gradient term can be made negative in the high curvature regime. In fact, for the Lagrangian (5.11) the second order action is

$${}^{(2)}\widetilde{\mathcal{S}}_\zeta = \frac{1}{8\pi}\int \mathrm{d}t\,\mathrm{d}^3x\, a^3\left\{\frac{3\ell'' f}{\ell'' - f}\,\dot\zeta^2 + \left[f + \tfrac{1}{a}\partial_0\left(a\frac{6f}{\kappa}\right)\right]\tfrac{1}{a^2}(\partial\zeta)^2 + \frac{6f'}{\kappa}\tfrac{1}{a^4}\left(\partial^2\zeta\right)^2\right\}. \tag{5.70}$$

A similar second order action has been used in [58] and in [76]. However, the models considered in these works were not symmetric under shifts of ϕ. Using a ϕ dependent potential essentially amounts to the introduction of a time dependent background. In this way it is easy to produce any background evolution one could wish for, including an inflationary stage, cf. [61]. In such a highly flexible model, indeed the sign of the gradient term in (5.70) can be adjusted more or less independent of the background. However, note that such an adjustment must restrict to the high curvature regime. Otherwise the GR limit is violated, as it happens in [76].

Sticking to shift-symmetric mimetic models, the background evolution and the evolution of perturbations are no longer decoupled, but must be driven by the same dynamics. Trying to use asymptotic freedom and the background solution from section 5.3 in the model (5.70), we find that the gradient term would actually become negative during the inflationary stage, however the sign of the $(\partial^2\zeta)^2$ term is strictly positive and diverges as $f' \to \infty$. This shows that even though higher mixed derivatives can make c_S^2 positive at high curvatures, in shift symmetric models they actually lead to a higher order instability that could be removed only by a high amount of tuning.

Instead, let us return to our original theory (5.3) without higher mixed derivatives but with higher spatial derivatives coming from (5.8). In the corresponding second order action (5.66) the gradient term comes with the constant prefactor $+1$ and in the case $\sigma_S = 0$ we would read off the speed of sound

$$c_S^2 = -\frac{\ell'' - f}{3\ell'' f} < 0. \tag{5.71}$$

We will continue to use the name c_S^2 to refer to this quantity also in the case $\sigma_S \neq 0$, but it is important to note that in this case due to the modified dispersion relation the true speed of sound is in general different from c_S^2. Note that c_S^2 has to be negative throughout by (5.67). Using (5.17), it can be rewritten as

$$c_S^2 = -\frac{G(1-G)}{3}. \tag{5.72}$$

Note that c_S^2 goes to zero in the late time limit as required by the GR limit $G \to 1$, but it also goes to zero in the early time limit as $G \to 0$. The minimal value $c_S^2 = -1/12$ is reached when $G = 1/2$ at $\kappa = \kappa_0/\sqrt{2}$. At the earliest times, the gradient term in (5.66) will be dominated by the higher order spatial derivative term $\frac{1}{a^6}(\partial^3\zeta)^2$. In this region there is no instability, provided that $\sigma_S^4 > 0$. Thus, the potential gradient instability region is sandwiched between the region of domination of the higher order term and the late time region where the scalar degree of freedom is “frozen”. These two other regions are without instabilities and hence the gradient instability gets to act, if at all, only during a limited time. As I will show below in two concrete examples, provided that $\sigma_S \gg 1/H_l$, the wrong sign gradient term cannot lead to any dangerous instability.

Mode expansion. Introducing the time coordinate τ and the expression z,

$$\mathrm{d}t = a\sqrt{f}\,\mathrm{d}\tilde{\eta} = a\sqrt{\frac{3f\ell''}{\ell''-f}}\,\mathrm{d}\tau, \qquad z = \frac{a}{\sqrt{4\pi}}\left(\frac{3f\ell''}{\ell''-f}\right)^{1/4}, \tag{5.73}$$

the second order action (5.66) written in terms of the canonically normalized variable $v = z\zeta$ becomes

$$^{(2)}\mathcal{S}_\zeta = \frac{1}{2}\int \mathrm{d}\tau\,\mathrm{d}^3x \left\{ v_\tau^2 + (\partial v)^2 - \frac{\sigma_S^4}{a^4}(\partial^3 v)^2 + \frac{z_{\tau\tau}}{z}v^2 \right\}. \tag{5.74}$$

A subscript $v_\tau \equiv \partial v/\partial\tau$ denotes derivatives. Performing the mode expansion of v into Fourier modes v_k, the modified Mukhanov-Sasaki equation becomes

$$v_{k,\tau\tau} + \left(\frac{\sigma_S^4}{a^4}k^6 - k^2 - \frac{z_{\tau\tau}}{z}\right)v_k = 0. \tag{5.75}$$

The power spectrum of the curvature perturbation ζ is given by [86]

$$\delta_\zeta^2(k) = \frac{k^3\left|v_k\right|^2}{2\pi^2 z^2}. \tag{5.76}$$

In the initial region where the k^6 term is dominating and $a_{\tilde{\eta}}\,a \ll k^3\sigma_S^2$, the solution of (5.75) with quantum vacuum initial conditions is well described by the WKB approximation

$$v_k = \frac{a(\tilde{\tau})}{\sigma_S k^{3/2}}\exp\left(-i\int \mathrm{d}\tau \frac{\sigma_S^2 k^3}{a(\tilde{\eta})^2}\right), \tag{5.77}$$

and it has a scale invariant spectrum. After “horizon” exit, when $z_{\tau\tau}/z$ is dominating, the solution of (5.75) is

$$v_k(\tau) = z(\tau)\left(A_k^S + B_k^S\int_\tau^\infty \frac{\mathrm{d}\tau'}{z^2(\tau')}\right), \tag{5.78}$$

where the second term is decaying compared to the first term and will be ignored in the following estimates. At late times the primordial spectrum after horizon exit is

$$\delta_\zeta^2 \approx \frac{k^3}{2\pi^2}\left|A_k^S\right|^2. \tag{5.79}$$

The wrong sign gradient term $-k^2$ will never get to dominate and cannot cause instability, provided that at “horizon” exit the condition

$$\frac{\sigma_S^4}{a^4}k^6 \sim \frac{z_{\tau\tau}}{z} \gg k^2 \tag{5.80}$$

holds. In other words, the physical wavelength $\lambda_{\mathsf{phys}} = a/k$ at horizon exit should satisfy

$$\lambda_{\mathsf{phys}}\Big|_{\frac{\sigma_S^4}{a^4}k^6 \sim \frac{z_{\tau\tau}}{z}} \ll \sigma_S. \tag{5.81}$$

The initial scale invariant spectrum would be preserved only if at horizon exit $z \propto a$, i.e.

$$-\frac{1}{c_S^2} = \frac{3f\ell''}{\ell'' - f} \approx const. \tag{5.82}$$

This condition will in general not be satisfied both at late times where $\ell'' - f \to 0$ and at early times where $f \to \infty$, $\ell'' \to \infty$. Thus, without a substantial amount of tuning of the functions f and Λ in order for (5.82) to be satisfied at least in some intermediate region where the relevant modes exit the "horizon", one can already expect a primordial spectrum that will be far from scale invariant.

Note that all modes ζ_k will at some point exit the "horizon" and never again re-enter, also after transitioning to the post-inflationary phase. This is a manifestation of the fact that in the GR limit the scalar degree of freedom of pure mimetic gravity is dust-like and non-propagating. It is clear that aiming for a more realistic model, matter perturbations would have to be taken into account.

Radiation dominated background. In modified conformal time $\tilde{\eta}$ the radiation dominated background solution from section 5.3 is given by $a(\tilde{\eta}) = H_l\tilde{\eta}$, where c was fixed by setting $a(\tilde{\eta}_f) = 1$, and we can find the explicit expressions

$$z = \sqrt{\tfrac{\sqrt{3}}{4\pi}\left(1 + (H_l\tilde{\eta})^4\right)}, \qquad \frac{\mathrm{d}\tilde{\eta}}{\mathrm{d}\tau} = \sqrt{3(1 + (H_l\tilde{\eta})^4)}, \qquad \frac{z_{\tau\tau}}{z} = 18H_l^4\tilde{\eta}^2. \tag{5.83}$$

The modified Mukhanov-Sasaki equation (5.75) becomes

$$v_{k,\tau\tau} + \left(\frac{\tilde{\sigma}_S^4}{H_l^8\tilde{\eta}^4}k^6 - k^2 - 18H_l^4\tilde{\eta}^2\right)v_k = 0, \tag{5.84}$$

where the dimensionless $\tilde{\sigma}_S = H_l\sigma_S$ was introduced. Assuming that $\tilde{\sigma}_S \gg \mathcal{O}(1)$, the "horizon" crossing happens at

$$\tilde{\eta}_k = \left(\frac{\tilde{\sigma}_S^4}{18}\right)^{1/6}\frac{k}{H_l^2} \tag{5.85}$$

and the condition (5.80) is satisfied for all modes.

Accelerated expansion ends at $\tilde{\eta}_f = 1/H_l$ and before that we can approximate $\tau = \tilde{\eta}/\sqrt{3}$ and write the τ derivatives in (5.84) as $\tilde{\eta}$ derivatives. Matching the absolute value of the initial solution with quantum vacuum initial conditions

$$v_k(\tilde{\eta}) = \frac{H_l^2\,\tilde{\eta}}{3^{1/4}\tilde{\sigma}_S k^{3/2}}\exp\left(-i\frac{\sqrt{3}\tilde{\sigma}_S^2 k^3}{H_l^4\tilde{\eta}}\right), \tag{5.86}$$

to the solution (5.78) after horizon exit, we can estimate the late time spectrum of large wavelength modes $k^2 \ll H_l^2(18/\tilde{\sigma}_S^4)^{1/3}$ which exit the horizon before the end of inflation as

$$\delta_\zeta^2(k) \approx \frac{(2/3)^{2/3}}{3\pi}\frac{k^2}{\tilde{\sigma}_S^{2/3}}. \tag{5.87}$$

Note that here $1/k$ is the physical wavelength at the end of inflation. It is far from being scale invariant with a large blue tilt and a spectral index of

$$n_S - 1 = 2. \tag{5.88}$$

Combining with (5.61), the tensor to scalar ratio is given by

$$r = \frac{\delta_h^2}{\delta_\zeta^2} \approx \begin{cases} 24\left(\frac{3}{2}\tilde{\sigma}_S\right)^{2/3} & \text{if } \tilde{\sigma}_T \ll 1 \\ 24\left(\frac{\sqrt{3}}{2}\frac{\tilde{\sigma}_S}{\tilde{\sigma}_T}\right)^{2/3} & \text{if } \tilde{\sigma}_T \gg 1 \end{cases} \tag{5.89}$$

It can be small only if $\sigma_T \gg \sigma_S \gg 1/H_l$.

Dust/MDM dominated background. If we consider now the background solution dominated by dust or mimetic dark matter with equation of state $w = 0$, the situation gets complicated by an additional early intermediate super-“horizon” region. Even though in this case long wavelength modes go through a short gradient instability phase, we will find that the growth of modes during this stage is completely negligible if $\sigma_S \gg 1/H_l$.

Using the background solution in modified conformal time $\tilde{\eta}$ from section 5.3 with $w = 0$ and fixing c by $a(\tilde{\eta}_f) = 1$ at the end of inflation, the scale factor is given by $a(\tilde{\eta}) = H_l^2\tilde{\eta}^2/8$ and we calculate

$$z = \sqrt{\tfrac{\sqrt{3}}{16\pi} H_l \tilde{\eta}\,(1+2a^3)}, \qquad \frac{\mathrm{d}\tilde{\eta}}{\mathrm{d}\tau} = \sqrt{3(1+2a^3)}, \qquad \frac{z_{\tau\tau}}{z} = \frac{3(-1+154a^3)}{4\tilde{\eta}^2}. \tag{5.90}$$

The modified Mukhanov-Sasaki equation (5.75) now reads

$$v_{k,\tau\tau} + \left(\left(\frac{\tilde{\sigma}_S}{H_l\, a}\right)^4 k^6 + \frac{3}{4\tilde{\eta}^2} - k^2 - \frac{231}{16}\,(H_l\, a)^2\right) v_k = 0, \tag{5.91}$$

and we see that compared to (5.84) there is an additional term coming from $z_{\tau\tau}/z$.

Assuming $\tilde{\sigma}_S \gg 1$, modes with

$$k^2 \gg \mathcal{O}(10^{-1})\frac{H_l^2}{\tilde{\sigma}_S^{4/3}} \tag{5.92}$$

exit the horizon at

$$\tilde{\eta}_k \sim \mathcal{O}(1)\frac{\tilde{\sigma}_S^{1/3}}{H_l^{3/2}}\sqrt{k}, \tag{5.93}$$

and both the gradient term and the $1/\tilde{\eta}^2$ term never become important.

On the other hand, modes with wavelengths larger than (5.92) already exit the horizon for a first time at

$$\tilde{\eta}_{1,k} \sim \mathcal{O}(1)\frac{\tilde{\sigma}_S^{2/3}}{H_l^2}k. \tag{5.94}$$

Since $z_{\tau\tau}/z$ is changing sign at $\tilde{\eta}_0 = (2^8/77)^{1/6}/H_l$, modes will at some point shortly re-enter the horizon due to the gradient term before finally re-exiting again. Expanding $z_{\tau\tau}/z$ around $\tilde{\eta}_0$, we can estimate the duration $\Delta\tilde{\eta}_k$ of the gradient instability region as

$$k\Delta\tilde{\eta}_k \sim \mathcal{O}(1)\frac{k^3}{H_l^3} \ll \frac{\mathcal{O}(10^{-2})}{\tilde{\sigma}_S^2}. \tag{5.95}$$

During this short time span the mode function v_k can only grow by a factor $\sim \exp(k\Delta\tilde{\eta}_k)$ which is completely negligible for $\tilde{\sigma}_S \gg \mathcal{O}(1)$.

Ignoring the effects of the gradient instability region, one would estimate the primordial spectrum of the longest wavelength modes $k \ll H_l/\tilde{\sigma}_S^{2/3}$ which exit the horizon well before the end of inflation as

$$\delta_\zeta^2 \approx \mathcal{O}(1)\frac{k^3}{H_l}. \tag{5.96}$$

Again, this is far being scale invariant with a spectral index $n_S - 1 = 3$. Since in this dust/MDM dominated case large wavelength modes go through several intermediate regions lasting only for a short time, such an analysis like in the radiation dominated case where the leading order solutions in different regions were continuously matched at the crossing between regions should be taken with caution. A full calculation of primordial spectra would require a numerical study beyond the scope of this paper.

5.5 Conclusions

The initial idea of “Asymptotically Free Mimetic Gravity” was to find a concrete modified theory of gravity with limiting curvature to address the singularity problem of GR. It has been successful in achieving this goal in a variety of settings, including both cosmological as well as black hole spacetimes. It was found that the concept of “asymptotic freedom”, i.e. the vanishing of the $\Box\phi$ dependent gravitational constant at limiting curvature, becomes a crucial ingredient to resolve anisotropic singularities. Along the way it was realized that Hořava-like higher order spatial curvature terms can be added in a simple, covariant way to mimetic gravity with the goal of renormalizability.

Combining both ideas, in this paper I considered “Asymptotically Free Mimetic Hořava Gravity”. In this theory, an initial stage of exponential expansion with graceful exit is a necessary feature of any non-singular modified flat Friedmann universe. It is a natural question to ask whether it could provide a full-blown inflationary scenario without inflaton. Since the existence of the inflationary background solution is independent of the matter content, as it is anyway suppressed by asymptotic freedom, there is no need to assume vanishing matter density during inflation. Thus, the idea of inflation driven by asymptotic freedom of gravity could open an interesting possibility to avoid the necessity of a reheating stage for the sake of populating the universe with matter after inflation.

In this work I analyzed stability of the inflationary background solutions under metric perturbations, considering only the degrees of freedom of pure mimetic gravity. It was found that a ghost instability is naturally avoided by any model with asymptotic freedom.

Although the gradient term of scalar perturbations is of the wrong sign throughout, its short era of domination is sandwiched between the domination of higher order spatial curvature terms and the late time region where the scalar degree of freedom of mimetic gravity remains frozen forever. Thus, the gradient instability can be circumvented thanks to higher order spatial curvature terms. Under the condition that $\sigma_S \gg 1/H_l$, I showed that the inflationary background solutions of asymptotically free models are free of any dangerous instability.

After passing stability tests, we have to ask if the primordial spectra produced by such an inflationary model can agree with CMB observations. In this paper I showed that for the simplest one-component models the answer to this question is negative. The initially scale invariant spectra of both tensor and scalar perturbations can in general not be preserved until the horizon exit. For the concrete case of a radiation dominated inflationary background, the primordial spectra of the largest wavelength modes were found to be far from scale invariant with a large blue tilt and $n_T = n_S - 1 = 2$. Moreover, a small tensor-to-scalar ratio $r < \mathcal{O}(10^{-1})$ requires tuning of the higher order coe□cients such that $\sigma_T \gg \mathcal{O}(10^4)\sigma_S$.

However, it is clear that this analysis is incomplete, as any more realistic model would also have to include matter fluctuations. While the primordial spectrum of the dust-like scalar degree of freedom of mimetic gravity might be far from scale invariant, any other spectator field in the inflationary background solution would still acquire a nearly scale invariant spectrum. Depending on the details of the conversion process of perturbations between matter degrees of freedom and the conformal degree of freedom of mimetic gravity, one could thus speculate that in a “curvaton”-like extension of the asymptotically free mimetic model, a nearly scale invariant primordial matter power spectrum would be obtainable. It remains an interesting open question whether it is possible to construct a viable inflationary scenario from shift-symmetric mimetic gravity.

Acknowledgements

I would like to thank my supervisor Slava Mukhanov for his support and guidance throughout the years of my PhD.

The author’s work is supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy – EXC-2111 – 390814868.

Appendix

A: Calculation of second order actions

In this appendix I present the calculation of the second order actions used above. For generality, I will consider the action

$$\mathcal{S}_g = -\frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}\left\{L\left(X_i\right)+\lambda\left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}-1\right)\right\}, \tag{5.97}$$

where L is a general function of the quantities

$$\begin{aligned}
X_1 &= \Box\phi && \cong \quad \kappa \\
X_2 &= \nabla^\mu\nabla^\nu\phi\nabla_\mu\nabla_\nu\phi-\tfrac{1}{3}(\Box\phi)^2 && \cong \quad \tilde{\kappa}^{ij}\tilde{\kappa}_{ij} \\
X_3 &= \widetilde{R} && \cong \quad {}^3R \\
X_4 &= -\widetilde{R}^{\mu\nu}\left(\nabla_\mu\nabla_\nu\phi-\tfrac{1}{3}\Box\phi\, g_{\mu\nu}\right) && \cong \quad {}^3R^{ij}\tilde{\kappa}_{ij} \\
X_5 &= \widetilde{R}^{\mu\nu}\widetilde{R}_{\mu\nu}-\tfrac{1}{3}\widetilde{R}^2 && \cong \quad {}^3\widetilde{R}^{ij}\,{}^3\widetilde{R}_{ij} \\
X_6 &= -P^\mu_\nu\nabla_\mu\widetilde{R}\nabla^\nu\widetilde{R} && \cong \quad \gamma^{kl}\bar{\nabla}_k({}^3R)\bar{\nabla}_l({}^3R) \\
X_7 &= -P^\mu_\nu P^\alpha_\gamma P^\beta_\delta\nabla_\mu\widetilde{R}^{\gamma\delta}\nabla^\nu\widetilde{R}_{\alpha\beta}+\tfrac{1}{3}X_6 && \cong \quad \gamma^{kl}\bar{\nabla}_k({}^3\widetilde{R}^{ij})\,\bar{\nabla}_l({}^3\widetilde{R}_{ij})
\end{aligned} \tag{5.98}$$

where

$$P^\nu_\mu := \delta^\nu_\mu-\phi_{,\mu}\phi^{,\nu} \tag{5.99}$$

$$\widetilde{R}_{\mu\nu} := P^\alpha_\mu P^\beta_\nu R_{\alpha\beta}+\nabla_\alpha\left(\phi^{,\alpha}\nabla_\mu\nabla_\nu\phi\right) \tag{5.100}$$

$$\widetilde{R} := 2\phi^{,\mu}\phi^{,\nu}G_{\mu\nu}-(\Box\phi)^2+\nabla^\mu\nabla^\nu\phi\nabla_\mu\nabla_\nu\phi \tag{5.101}$$

The right column of (5.98), denoted by $\cong$, shows the quantities X_i evaluated in comoving gauge $\phi = t$ where $N = 1$, but the shift N_i is still arbitrary. They are given in terms of the trace and trace-less part $\kappa = \gamma^{ij}\kappa_{ij}$, $\tilde{\kappa}_{ij} = \kappa_{ij}-\frac{1}{3}\kappa\gamma_{ij}$, ${}^3\widetilde{R} = \gamma^{ij}\,{}^3R_{ij}$, ${}^3\widetilde{R}_{ij} = {}^3R_{ij}-\frac{1}{3}{}^3\widetilde{R}\gamma_{ij}$ of the extrinsic curvature $\kappa_{ij} = \frac{1}{2N}\left(\dot{\gamma}_{ij}-\bar{\nabla}_jN_i-\bar{\nabla}_iN_j\right)$ and of the intrinsic Ricci tensor ${}^3R_{ij}$ of the spatial slices of (5.28), respectively. $\bar{\nabla}_i$ denotes the covariant derivative with respect to the spatial metric γ_{ij}. Indices on spatial tensors are raised with γ^{ij}.

Note that terms depending on the square of the spatial Cotton tensor $\widetilde{C}^\mu_\nu$ as introduced in (5.9) are also covered by the general Lagrangian ansatz, since

$$\widetilde{C}^\mu_\nu\widetilde{C}^\nu_\mu \cong {}^3C^i_j\,{}^3C^j_i \doteq X_7+(\tfrac{1}{3}-\tfrac{3}{8})X_6+\tfrac{1}{2}X_3X_5+3\,{}^3\widetilde{R}^i_j\,{}^3\widetilde{R}^j_k\,{}^3\widetilde{R}^k_i, \tag{5.102}$$

where $\doteq$ now denotes equality up to a total covariant spatial derivative $\bar{\nabla}$. In an isotropic universe the trace-less part of ${}^3R_{ij}$ is first order in perturbations, hence the last term in (5.102) is always higher order and does not contribute to the second order action.

Consider now perturbations around the general non-flat Friedmann background

$$\mathrm{d}s^2 = dt^2-a^2(t)\eta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \qquad \eta_{ij} = \frac{\delta_{ij}}{\left(1+\frac{\varkappa}{4}(x^2+y^2+z^2)\right)^2} \tag{5.103}$$

in comoving gauge $\phi = t$. The metric perturbations of (5.28) are then further decomposed as

$$\gamma_{ij} = a^2(t)\left(e^{-2\psi}\eta_{ij} - 2D_iD_jE - 2D_{(i}F_{j)} + (e^h)_{ij}\right), \qquad N_i = \chi_{,i} - aS_i \tag{5.104}$$

where

$$\eta^{ij}D_jF_i = 0, \qquad \eta^{ij}h_{ij} = 0, \qquad \eta^{jk}D_kh_{ij} = 0, \qquad \eta^{ij}D_jS_i = 0 \tag{5.105}$$

and D_i denotes the covariant derivative associated with η_{ij}.

Background. Variation of the zeroth order action ${}^{(0)}\mathcal{S} = {}^{(0)}\mathcal{S}_g + {}^{(0)}\mathcal{S}_m$ with respect to a^3 yields

$$\delta_{a^3}{}^{(0)}\mathcal{S} = -\frac{1}{16\pi}\int \mathrm{d}^4x\,\sqrt{\eta}\left[L - \tfrac{1}{a^3}\partial_0\left(a^3L_1\right) - 4L_3\tfrac{\varkappa}{a^2} - 16\pi p\right]\,\delta(a^3) \tag{5.106}$$

where subscripts L_i denote derivatives of L with respect to X_i evaluated on the background. Here it was used that for homogeneous, isotropic matter

$$\delta\mathcal{S}_m = \tfrac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\,T_{\mu\nu}\delta g^{\mu\nu} = \int \mathrm{d}^4x\sqrt{\eta}\,p\,\delta(a^3). \tag{5.107}$$

The background equation of motion is hence

$$L - \frac{1}{a^3}\partial_0\left(a^3L_1\right) - 4L_3\frac{\varkappa}{a^2} = 16\pi p. \tag{5.108}$$

Using the continuity equation $\frac{1}{a^3}\partial_0\left(a^3\varepsilon\right) = -3\frac{\dot a}{a}p$, the first integral of (5.108) becomes the modified Friedmann equation

$$\frac{1}{2}\left(\kappa L_1 - L\right) = \frac{c_{\mathsf{MDM}}}{u} + 8\pi\varepsilon = 8\pi\varepsilon, \tag{5.109}$$

where the constant of integration c_{MDM} describes the contribution of mimetic matter.

Tensor perturbations. Tensor perturbations are parametrized by

$$\gamma_{ij} = a^2\left(\eta_{ij} + h_{ij} + \frac{1}{2}h_i^k h_{kj}\right), \qquad N_i = 0, \tag{5.110}$$

where indices on h_{ij} are raised with η^{ij}. With this parametrization the inverse spatial metric is

$$\gamma^{ij} = \frac{1}{a^2}\left(\eta^{ij} - h^{ij} + \frac{1}{2}h^{ik}h_k^j\right) + \mathcal{O}(h^3) \tag{5.111}$$

and it holds that $\sqrt{\gamma} = a^3\sqrt{\eta}$ and $\kappa = 3\frac{\dot{a}}{a}$ are still homogeneous up to $\mathcal{O}(h^3)$. The extrinsic curvature and intrinsic Ricci scalar up to second order are

$$\kappa_{ij} = \frac{\dot{a}}{a}\gamma_{ij} + \frac{1}{2}a^2\left(\dot{h}_{ij} + \frac{1}{2}\dot{h}_{ik}h^k{}_j + \frac{1}{2}h_{ik}\dot{h}^k{}_j\right), \tag{5.112}$$

$$^3R \doteq \frac{1}{a^2}\left(6\varkappa - \frac{1}{4}D_k h^{ij}D^k h_{ij} - \frac{1}{2}\varkappa\, h^{ij}h_{ij}\right), \tag{5.113}$$

where $\doteq$ now denotes equality up to a total covariant spatial derivative D. The spatial Ricci tensor and its trace-less part up to first order are given by

$$^3R_{ij} = 2\varkappa\eta_{ij} - \frac{1}{2}\Delta h_{ij} + 3\varkappa h_{ij}, \qquad ^3\widetilde{R}_{ij} = -\frac{1}{2}\Delta h_{ij} + \varkappa h_{ij}, \tag{5.114}$$

where $\Delta = \eta^{ij}D_iD_j$. The quantities (5.98) expanded to second order are

$$\begin{aligned}
X_1 &= 3\tfrac{\dot{a}}{a}\\
X_2 &= \tfrac{1}{4}\dot{h}^{ij}\dot{h}_{ij}\\
X_3 &\doteq \tfrac{1}{a^2}\left(6\varkappa - \tfrac{1}{4}D_k h^{ij}D^k h_{ij} - \tfrac{1}{2}\varkappa\, h^{ij}h_{ij}\right)\\
X_4 &\doteq \tfrac{1}{8a^2}\partial_0\left(D_k h^{ij}D^k h_{ij} + 2\varkappa h^{ij}h_{ij}\right)\\
X_5 &\doteq \tfrac{1}{a^4}\left(\tfrac{1}{4}\Delta h^{ij}\Delta h_{ij} + \varkappa D_k h^{ij}D^k h_{ij} + \varkappa^2 h^{ij}h_{ij}\right)\\
X_6 &= 0\\
X_7 &\doteq \tfrac{1}{a^6}\left(\tfrac{1}{4}D_k\Delta h^{ij}D^k\Delta h_{ij} + \varkappa\Delta h^{ij}\Delta h_{ij} + \varkappa^2 D_k h^{ij}D^k h_{ij}\right)
\end{aligned} \tag{5.115}$$

and we find the second order action

$$\begin{aligned}
^{(2)}\mathcal{S}_h = -\frac{1}{64\pi}\int \mathrm{d}^4x\sqrt{\eta}\,a^3\Big\{&L_2\,\dot{h}^{ij}\dot{h}_{ij} - \left(L_3 + \tfrac{1}{2a}\partial_0\left(aL_4\right) - \tfrac{2\varkappa}{a^2}L_5\right)\tfrac{2\varkappa}{a^2}\,h^{ij}h_{ij} +\\
&+ \left(L_3 + \tfrac{1}{2a}\partial_0\left(aL_4\right) - \tfrac{4\varkappa}{a^2}\left(L_5 + \tfrac{\varkappa}{a^2}L_7\right)\right)\tfrac{1}{a^2}h^{ij}\Delta h_{ij} +\\
&+ \frac{L_5 + \frac{4\varkappa}{a^2}L_7}{a^4}h^{ij}\Delta^2 h_{ij} - \frac{L_7}{a^6}h^{ij}\Delta^3 h_{ij}\Big\}.
\end{aligned} \tag{5.116}$$

Subscripts L_i denote derivatives of L with respect to X_i evaluated on the background.

Scalar perturbations. The scalar metric perturbations in comoving gauge are parametrized through ζ and χ by

$$\gamma_{ij} = a^2(t)e^{2\tilde{\zeta}}\delta_{ij}, \qquad N_i = \chi_{,i}, \tag{5.117}$$

where

$$\tilde{\zeta} = \zeta - \ln\left(1 + \tfrac{\varkappa}{4}(x^2 + y^2 + z^2)\right). \tag{5.118}$$

The metric determinant is given by $\sqrt{\gamma} = a^3 e^{3\tilde{\zeta}} = a^3\sqrt{\eta}\,e^{3\zeta}$ and the trace of extrinsic curvature is

$$\kappa = 3\frac{\dot{a}}{a} + 3\dot{\zeta} - \bar{\Delta}\chi, \tag{5.119}$$

where the Laplacian $\bar{\Delta} = \gamma^{ij}\bar{\nabla}_i\bar{\nabla}_j$ of the covariant derivative $\bar{\nabla}$ with respect to γ_{ij} is given by

$$\bar{\Delta}\chi = \tfrac{1}{a^2}\Delta\chi - \tfrac{2}{a^2}\zeta\Delta\chi + \tfrac{1}{a^2}\eta^{ij}\zeta_{,i}\chi_{,j} + \mathcal{O}\left(\chi\zeta^2\right). \tag{5.120}$$

To first order, the traceless part of κ_{ij} is given entirely through χ as

$$\tilde{\kappa}_{ij} = -\bar{\nabla}_j\bar{\nabla}_i\chi + \tfrac{1}{3}\bar{\Delta}\chi\gamma_{ij} = -D_jD_i\chi + \tfrac{1}{3}\Delta\chi\eta_{ij} + \mathcal{O}(\zeta\chi). \tag{5.121}$$

The spatial metric γ_{ij} is conformally flat which simplifies the calculation of

$$\begin{aligned}
{}^3R_{ij} &= -\left(\tilde{\zeta}_{,ij} - \tilde{\zeta}_{,i}\tilde{\zeta}_{,j}\right) - \left(\tilde{\zeta}^{,k}_{\ \ k} + \tilde{\zeta}^{,k}\tilde{\zeta}_{,k}\right)\gamma_{ij} && (5.122)\\
&= 2\varkappa\eta_{ij} - \left(D_jD_i\zeta + \Delta\zeta\eta_{ij}\right) + \left(\zeta_{,i}\zeta_{,j} - \eta^{kl}\zeta_{,k}\zeta_{,l}\,\eta_{ij}\right)\\
{}^3R &= \frac{e^{-2\zeta}}{a^2}\left(6\varkappa - 4\Delta\zeta - 2\eta^{ij}\zeta_{,i}\zeta_{,j}\right) && (5.123)\\
&= \frac{1}{a^2}\left(6\varkappa - 4\Delta\zeta - 12\varkappa\zeta + 12\varkappa\zeta^2 + 8\zeta\Delta\zeta - 2\eta^{ij}\zeta_{,i}\zeta_{,j}\right) + \mathcal{O}\left(\zeta^3\right)\\
{}^3\widetilde{R}_{ij} &= -\left(D_jD_i\zeta - \tfrac{1}{3}\Delta\zeta\eta_{ij}\right) + \mathcal{O}(\zeta^2) && (5.124)
\end{aligned}$$

where $\Delta = \eta^{ij}D_iD_j$. Combining these results, the quantities (5.98) expanded to second order read

$$\begin{aligned}
X_1 &= 3\tfrac{\dot{a}}{a} + 3\dot{\zeta} - \tfrac{1}{a^2}\Delta\chi + \tfrac{2}{a^2}\zeta\Delta\chi - \tfrac{1}{a^2}\left(D\zeta\right)^2\\
X_2 &\doteq \tfrac{1}{a^4}\left(\tfrac{2}{3}(\Delta\chi)^2 - 2\varkappa\left(D\chi\right)^2\right)\\
X_3 &= \tfrac{1}{a^2}\left(6\varkappa - 4\Delta\zeta - 12\varkappa\zeta + 12\varkappa\zeta^2 + 8\zeta\Delta\zeta - 2\left(D\zeta\right)^2\right)\\
X_4 &\doteq \tfrac{1}{a^4}\Delta\chi\left(\tfrac{2}{3}\Delta\zeta + 2\varkappa\zeta\right)\\
X_5 &\doteq \tfrac{1}{a^4}\left(\tfrac{2}{3}(\Delta\zeta)^2 - 2\varkappa\left(D\zeta\right)^2\right)\\
X_6 &\doteq \tfrac{16}{a^6}\left(\left(D\Delta\zeta\right)^2 - 6\varkappa\left(\Delta\zeta\right)^2 + 9\varkappa^2\left(D\zeta\right)^2\right)\\
X_7 &\doteq \tfrac{1}{a^6}\left(\tfrac{2}{3}\left(D\Delta\zeta\right)^2 - 6\varkappa\left(\Delta\zeta\right)^2 + 12\varkappa^2(D\zeta)^2\right)
\end{aligned} \tag{5.125}$$

with the notation $\left(D\zeta\right)^2 \equiv \eta^{ij}\zeta_{,i}\zeta_{,j}$.

Neglecting matter fluctuations of $\mathcal{S}_m = \int d^4x\sqrt{\eta}\,a^3\,e^{3\zeta}p$ and using the background equation of motion (5.108), the total second order action ${}^{(2)}\mathcal{S} = {}^{(2)}\mathcal{S}_g + {}^{(2)}\mathcal{S}_m$ becomes

$$\begin{aligned}
{}^{(2)}\mathcal{S} = -\frac{1}{32\pi}\int d^4x\sqrt{\eta}\,a^3\Big\{ &- \left[L_3 - \tfrac{3}{a}\partial_0\left(aL_{13}\right) - \tfrac{12\varkappa}{a^2}L_{33}\right]\tfrac{12\varkappa}{a^2}\zeta^2 + && (5.126)\\
&+ L_{11}9\dot{\zeta}^2 - \left[L_3 - \tfrac{3}{a}\partial_0\left(aL_{13}\right) - \tfrac{\varkappa}{a^2}(24L_{33} + L_5) + \tfrac{6\varkappa^2}{a^4}(12L_6 + L_7)\right]\tfrac{4}{a^2}\,\zeta\Delta\zeta +\\
&+ \left[16L_{33} + \tfrac{4}{3}L_5 - \tfrac{12\varkappa}{a^2}\left(16L_6 + L_7\right)\right]\tfrac{1}{a^4}\,\zeta\Delta^2\zeta - \left[32L_6 + \tfrac{4}{3}L_7\right]\tfrac{1}{a^6}\,\zeta\Delta^3\zeta\\
&+ \left[(8L_{13} + \tfrac{4}{3}L_4)\left(3\varkappa\zeta + \Delta\zeta\right) - 6L_{11}a^2\dot{\zeta} + (L_{11} + \tfrac{4}{3}L_2)\Delta\chi + 4\varkappa L_2\chi\right]\tfrac{1}{a^4}\Delta\chi\Big\}.
\end{aligned}$$

Subscripts L_i denote derivatives of L with respect to X_i evaluated on the background.

Variation with respect to $\Delta\chi$ yields

$$(L_{11} + \tfrac{4}{3}L_2)\Delta\chi + 4\varkappa L_2\chi = 3L_{11}a^2\dot{\zeta} - (4L_{13} + \tfrac{2}{3}L_4)\,(3\varkappa\zeta + \Delta\zeta)\,. \tag{5.127}$$

In the spatially flat case the second order action for ζ after integrating out χ, assuming $L_{11} + \frac{4}{3}L_2 \neq 0$, becomes

$$\begin{aligned} {}^{(2)}\mathcal{S}_\zeta = -\frac{1}{32\pi}\int \mathrm{d}^4x\, a^3 \Bigg\{ & \frac{12L_{11}L_2}{L_{11} + \frac{4}{3}L_2}\,\dot{\zeta}^2 - \left[L_3 - \tfrac{1}{a}\partial_0\left(a\frac{4L_2L_{13} - \frac{1}{2}L_{11}L_4}{L_{11} + \frac{4}{3}L_2}\right)\right]\tfrac{4}{a^2}\,\zeta\Delta\zeta \\ & + \left[16L_{33} + \tfrac{4}{3}L_5 - \frac{(4L_{13} + \frac{2}{3}L_4)^2}{L_{11} + \frac{4}{3}L_2}\right]\tfrac{1}{a^4}\,\zeta\Delta^2\zeta - \left[32L_6 + \tfrac{4}{3}L_7\right]\tfrac{1}{a^6}\,\zeta\Delta^3\zeta \Bigg\}. \end{aligned} \tag{5.128}$$

NB. Note that in the case $L_{33} = L_5 = L_6 = L_7 = 0$ it is tempting to make the choice, $L_4 = -6L_{13}$ such that

$$ {}^{(2)}\mathcal{S}_\zeta = -\frac{1}{32\pi}\int \mathrm{d}^4x\, a^3 \left\{ \frac{12L_{11}L_2}{L_{11} + \frac{4}{3}L_2}\,\dot{\zeta}^2 - \left[L_3 - 3\tfrac{1}{a}\partial_0\,(aL_{13})\right]\tfrac{4}{a^2}\,\zeta\Delta\zeta \right\}. \tag{5.129}$$

However, note that in this case a gradient instability can only be prevented at the expense of introducing a gradient instability for tensor perturbations.

B: Stability analysis of higher order spatial curvature bounces

As seen in [4], the goal of singularity resolution in spatially non-flat universes requires the introduction of a scalar spatial curvature dependent potential. By isotropy of the background, there is a degeneracy of higher order spatial curvature terms that will lead to the same background dynamics. In the following I will consider the generalization

$$\mathcal{L} = \mathcal{L}_{\mathsf{nhd}} + \mathcal{L}_{\mathsf{hd}} + V(\widetilde{R}) + \alpha^4\,\widetilde{R}\left(\widetilde{R}^{\mu\nu}\widetilde{R}_{\mu\nu} - \tfrac{1}{3}\widetilde{R}^2\right), \tag{5.130}$$

where now σ_T, σ_S and α can depend on $\widetilde{R}$. Note that in the spatially flat case this Lagrangian is equivalent to (5.3).

Background. The modified Friedmann equation with non-vanishing spatial curvature becomes

$$\tfrac{2}{3}\,(\kappa\ell' - \ell) = \tfrac{1}{2}\left(V\left({}^3\!R\right) - {}^3\!R\right) + 8\pi\varepsilon. \tag{5.131}$$

It is independent of the higher order terms σ_T, σ_S and α. This modified Friedmann equation has been discussed in [4]. A bounce is made possible if the term $V\left({}^3\!R\right)$ is negative and dominating over ε at small scale factor. Assuming a cubic potential,

$$V\left({}^3\!R\right) = -6\,\delta^4\left(\frac{\varkappa}{6}{}^3\!R\right)^3 = -\frac{6\,\delta^4}{a^6}, \tag{5.132}$$

this will be satisfied, provided that the matter equation of state satisfies $w < 1$. The bounce then happens at the minimal value of the scale factor

$$a_{\min} = \left(\frac{1+3w}{2}\right)^{\frac{1+w}{1-w}} \left(\frac{\sqrt{c}}{H_l}\right)^{-\frac{1+3w}{3(1-w)}} \delta^{\frac{4}{3(1-w)}}, \tag{5.133}$$

where (5.23) was assumed. In the case $\varkappa = 1$ a re-collapse happens at

$$a_{\max} = \left(\frac{1+3w}{2}\right)^{-\frac{3(1+w)}{1+3w}} \frac{\sqrt{c}}{H_l}. \tag{5.134}$$

Second order actions. The second order action for tensor perturbations is

$$\begin{aligned} {}^{(2)}\mathcal{S}_h = \frac{1}{64\pi} \int \mathrm{d}^4x \sqrt{\eta}\, a^3 \Big\{ & f\, \dot{h}^{ab}\dot{h}_{ab} - \Big(1 - V' + \tfrac{6\varkappa^2}{a^4}(\sigma_T^4 + 2\alpha^4)\Big) \tfrac{2\varkappa}{a^2}\, h^{ab}h_{ab} \\ & - \Big(1 - V' + \tfrac{8\varkappa^2}{a^4}(2\sigma_T^4 + 3\alpha^4)\Big) \tfrac{1}{a^2}(Dh_{ab})^2 \\ & - (7\sigma_T^4 + 6\alpha^4)\tfrac{\varkappa}{a^6}(D^2h_{ab})^2 - \frac{\sigma_T^4}{a^6}(D^3h_{ab})^2 \Big\}, \end{aligned} \tag{5.135}$$

where $(Dh_{ab})^2 \equiv \eta^{cd}D_c h^{ab} D_d h_{ab}$, $(D^2h_{ab})^2 \equiv \Delta h^{ab}\Delta h_{ab}$, $(D^3h_{ab})^2 \equiv \eta^{cd}D_c\Delta h^{ab}D_d\Delta h_{ab}$.

Neglecting matter perturbations, the second order action for scalar perturbations, after integrating out χ is

$$\begin{aligned} {}^{(2)}\mathcal{S}_\zeta = \frac{1}{8\pi} \int \mathrm{d}^4x \sqrt{\eta}\, a^3 \Big\{ & \dot{\zeta}\, \frac{3\ell'' f\,(\Delta + 3\varkappa)}{(\ell'' - f)\Delta - 3f\varkappa}\, \dot{\zeta} - \left[\tfrac{1}{3}(1 - V') + 4V''\tfrac{\varkappa}{a^2}\right] \tfrac{\varkappa}{a^2}\, 9\zeta^2 \\ & + \left[1 - V' + \tfrac{3\varkappa}{a^2}(8V'' + \tfrac{\varkappa}{a^2}(2\alpha^4 - 3\sigma_S^4)\right] \tfrac{1}{a^2}\,(D\zeta)^2 + \\ & - \left[4V'' + \tfrac{2\varkappa}{a^2}(\alpha^4 - 3\sigma_S^4)\right] \tfrac{1}{a^4}\,(D^2\zeta)^2 - \tfrac{\sigma_S^4}{a^6}\,(D^3\zeta)^2 \Big\}. \end{aligned} \tag{5.136}$$

Ghost instability? The condition for no ghost instability of the mode characterized by the eigenvalue $-k^2$ of the curved Laplacian Δ can be written as

$$\frac{\ell'' f\,(k^2 - 3\varkappa)}{(\ell'' - f)k^2 + 3f\varkappa} > 0. \tag{5.137}$$

In the case $\ell'' \neq f$, the condition for short wavelength modes $k^2 \gg \varkappa$ is the same as (5.67) in the spatially flat case. On the other hand, in the limit $\ell'' - f \to 0$, $f \to 1$ (which applies also in the region around the bounce where H vanishes) the condition becomes

$$\varkappa k^2 > 3. \tag{5.138}$$

In an open universe $\varkappa = -1$ the spectrum of the curved Laplacian is continuous and bounded by $k^2 \geq 1$. Hence all modes su er from a ghost instability. In a closed universe $\varkappa = 1$, however, the eigenvalues of Δ are discrete and given by [104], [105]

$$k^2 = n^2 - \varkappa, \qquad n \in \mathbb{N}_{\geq 1}. \tag{5.139}$$

The discreteness of the spectrum is due to the periodic boundary conditions that eigenfunctions have to satisfy. Note that only the modes $n = 1, 2$ would su er from a ghost instability. However, these two longest wavelength modes can be shown to correspond to pure gauge terms [106], [107]. In conclusion, in the closed case $\varkappa = 1$ there is no ghost instability.

Gradient instability? Let us now consider $\varkappa = 1$ and take for simplicity the cubic potential (5.132) and constant $\sigma_S^4 \geq 0$, $\sigma_T^4 \geq 0$, $\alpha^4 \geq 0$. In this case tensor perturbations do not exhibit any instability and the sixth order term in (5.136) does not lead to an instability of scalar perturbations. The condition following from the right sign of the forth order term reads

$$\alpha^4 \geq 3\sigma_S^4 + 2\delta^4. \tag{5.140}$$

From the gradient term we can read o that there is a gradient instability wherever

$$1 + \tfrac{3}{a^4}(2\alpha^4 - 3\sigma_S^4 - 7\delta^4) > 0. \tag{5.141}$$

Using (5.140), we see that if $\delta^4 \leq 2\sigma_S^4$ this condition is satisfied at all times and there is a gradient instability throughout. Conversely, if $\delta^4 > 2\sigma_S^4$ it is possible to avoid the gradient instability in the region around the bounce, provided that

$$a_{\min}^4 < 3\left(7\delta^4 + 3\sigma_S^4 - 2\alpha^4\right). \tag{5.142}$$

In order to avoid the gradient instability at all times, it would be necessary to have

$$a_{\max}^4 < 3\left(7\delta^4 + 3\sigma_S^4 - 2\alpha^4\right) \overset{!}{<} 9\left(\delta^4 - 2\sigma_S^4\right), \tag{5.143}$$

where for the last inequality (5.140) was used. Note, however, that in this case

$$\left(\frac{a_{\max}}{a_{\min}}\right) \sim \left(\frac{a_{\max}}{\delta}\right)^{\frac{4}{3(1+w)}} \overset{!}{<} \mathcal{O}(10), \tag{5.144}$$

and hence a gradient instability region is unavoidable for any universe that undergoes any significant amount of expansion.

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