Algebraic Constructions related to Quantum Field Theories

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Zusammenfassung

Die moderne theoretische Hochenergiephysik basiert im Wesentlichen auf einer Beschreibung der Natur mittels Quantenfeldtheorie. Nach ungefähr einhundert Jahren Forschung mit und an Quantenfeldtheorien bleibt eine mathematisch präzise und rigorose Beschreibung solcher Theorien jedoch ein ungelöstes Rätsel. Ein mathematisches Fundament für Quantenfeldtheorien erscheint vor dem Hintergrund moderner Ansätze in der Hochenergiephysik wie beispielsweise der Stringtheorie mit ihrer reichen mathematischen Struktur umso erstrebenswerter.

Zu den am besten verstandenen Quantenfeldtheorien gehören zweidimensionale, rationale konforme Quantenfeldtheorien. In dieser Dissertation wird eine mathematische Beschreibung solcher Theorien mittels sogenannter string-net Modelle erbracht. Heuristisch gesprochen ermöglichen string-net Modelle eine Herangehensweise mittels Feynmandiagramen auf Flächen allen Geschlechts. Es wird ein Eindeutigkeits- und Existenzresultat für offen-geschlossene rationale konforme Feldtheorien mit fixierter Randbedingung mittels kategorientherotischen Methoden der string-net Modelle gegeben. Zudem werden mit Hilfe der string-net Modelle konsistente Korrelatoren für rationale konforme Feldtheorien mit beliebigen topologischen Defekten sowie beliebigen symmetrieerhaltenden Randbedingungen konstruiert. Mittels dieser Konstruktion können Torus- und Kreisringzustandssumme berechnet werden, wobei bekannte Resultate reproduziert werden und die Theorie an frühere Beschreibungen angeknüpft wird. Im Vergleich zu anderen kategorientheoretischen Ansätzen benötigen string-net Modelle fast keine dreidimensionalen Objekte. Dadurch wird der Nutzen von kategorientheoretischen Prinzipien transparent und intuitiv.

Im zweiten Teil der Arbeit werden Homotopiealgebren und ihr Bezug zu Quantenfeldtheorien behandelt. Jede konsistente klassiche Feldtheorie mit einer gewissen Eichfreiheit produziert eine strong homotopy Lie algebra oder L_∞ Algebra über den Batalin Vilkovisky Formalismus. Somit können über das Studium von L_∞ Algebren Rückschlüsse auf Feldtheorien gezogen werden. Das erste Resultat in diese Richtung ist eine Erweiterung einer schiefsymmetrischen bilinearen Klammer auf einem Vektorraum zu einer endlichen L_∞ Algebra. Dies verallgemeinert die L_∞ Struktur des Courant Algebroiden. Als zweites Ergebnis wird ein Satz über den Bezug von L_∞ Algebra Quasiisomorphismen zu Seiberg Witten Abbildungen bewiesen, was zu einem besseren Verständins der Relation von mathematischen Begriffen der Homotopiealgebren und bekannten Begrifflichkeiten in der Physik beiträgt.

Abstract

Quantum field theory is the main technical tool in understanding modern theoretical high energy physics. After nearly a century of quantum field theory its mathematics remains mysterious, though. A firm grip on the mathematics of quantum field theory seems ever more desirable since the upcoming of string theory with its rich and challenging mathematical structure.

Among the best understood quantum field theories are two dimensional rational quantum field theories. In this work we contribute to a better mathematical understanding of such theories by providing a mathematically rigorous but intuitive description in terms of string-net models. Heuristically string-net models give a Feynman diagram framework for rational conformal field theories on all genus g surfaces. We prove a uniqueness and existence result for open-closed rational conformal field theories with fixed boundary condition making extensive use of the category theory underlying string-nets. Secondly, we give a construction of consistent correlators in rational conformal field theories with arbitrary topological defects and symmetry preserving boundary conditions using string-nets. As a proof of principle we compute torus and annulus partition functions in the string-net framework, thereby reproducing established results. Compared to earlier categorical approaches the use of string-nets almost completely avoids three dimensional considerations, rendering the use of categorical tools very intuitive.

The second part of the thesis deals with homotopy algebras and their appearance in quantum field theories. Roughly speaking every consistent classical field theory having some gauge freedom produces a strong homotopy Lie algebra through the Batalin Vilkovisky formalism. Hence by studying strong homotopy Lie algebras (or L_∞ algebras) one can learn something about field theories. The first result presented in that direction is a theorem closing every skewsymmetric bilinear bracket on a vector space into a finite term L_∞ algebra. This is a generalization of the L_∞ structure of the Courant algebroid. The second result is a theorem relating quasi-isomorphisms of L_∞ algebras to Seiberg-Witten maps, linking the mathematics of homotopy algebras closer to physical notions.

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Chapter 1

Introduction

In the course of the twentieth century theoretical high energy physics and mathematical physics have undergone an ever accelerating change. From the beginning of quantum theory dating back to Max Planck and Werner Heisenberg in the first half of the century to the groundbreaking development of quantum field theory in the second half, the description of nature at higher and higher energies has experienced fundamental changes. New models for phenomena at higher energies required new interesting mathematical tools. Without question, one of the greatest achievements was the invention of the standard model of particle physics giving a unified treatment of three of nature's four fundamental forces as a quantum field theory. Since then the question on how to include the fourth force, gravity, remains. Among others, the probably most developed theory in that direction is string theory dating back to the beginning of the 1980s. String theory shifts one crucial paradigm of fundamental descriptions of nature: The idea of particles as points. In string theory, fundamental particles are one dimensional objects, strings, giving the theory its name. To date most string theory constructions are rooted in a world volume formalism. As a string evolves in time it swipes out a two dimensional world volume W and string theory can be seen as quantum field theory for embeddings $X : W \to M$ of the world volume into a spacetime M. Such a theory is usually called a (non-linear) sigma model. Since the literature is vast and a historically accurate chronology of developments for string theory is way out of reach for this introduction we refer to the introduction of [25] or the review [40] where the reader can learn more about the development of string theory through the years. Maybe the most important point about sigma models for this thesis is the fact that the two dimensional sigma model describing a string in a flat space time is a *conformal field theory*.

After this historic account leading to string theory and before going into more details on two dimensional conformal field theories we pause a moment and discuss the essential question underlying all results presented in this thesis. This is the question of classification of quantum field theories. Of course this task is too broad to be treated in a closed or even precise manner. Nevertheless it can yield interesting results when narrowed to more specific situations. An example for such a classification is the recent swampland program in string phenomenology. Usually in string theory a top-down approach to phenomenological results is taken. One tries to derive the standard model in four dimensions from a higher dimensional string theory serving as its consistent quantum gravity extension. The swampland program maybe seen as bottom-up approach. Starting from a space of consistent quantum field theories in four dimensions at certain energy scales, what are general criteria for these theories to have a consistent ultra violet completion? Theories not meeting those criteria lie in the "swampland", whereas consistent theories are part of the so called "landscape". Thus this is nothing else than a very general attempt on classifying four dimensional quantum field theories wrt. to possible ultra violet completions. For further information and a historical account of the swampland program the reader can consolidate [124][19][29].

Besides phenomenological considerations as in the swampland program one can try to classify quantum field theories mathematically and this thesis is a very modest contribution to this subject. For a mathematically precise treatment one needs to start with a definition of quantum field theory (QFT). There are many attempts for a rigorous definition of QFTs. Based on the Haag Kastler axioms [70] there is a theory of algebraic quantum field theory (AQFT) giving a description of QFTs in terms of local algebras of observables (see e.g. [69] for a textbook account, [149] for a more modern approach and [12] for its relation to two dimensional conformal field theory). Another route is via the Wightman axioms [145], for which the reader can find an exhaustive discussion in [139]. Based on Atiyah's paper on topological quantum field theory (tft) [4] the field of functorial quantum field theory (FQFT) emerged. It aims at a formalization of the path integral. There are many references for the subject, all of which seem to be written for a mathematics rather than a physics audience, though. For a gentle introduction we refer to [32] and references therein. Lastly we list a rigorous approach which evolved out of the Batalin-Vilkovisky (BV) formalism [15][16][17][71], which is a description in terms of homotopy algebras [34]. In this thesis we mainly deal with QFTs in the form of the last two approaches.

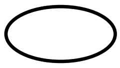
Coming back to the question of classification, as said before FQFT has its roots in topological quantum field theory. A tft with fields $\{\Phi_i\}$ on a spacetime M, roughly speaking is a field theory s.th. correlation functions

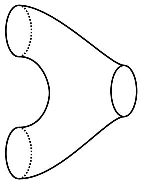
$$\langle \Phi_{i_1}(x_{i_1}) \cdots \Phi_{i_N}(x_{i_N}) \rangle \tag{1.1}$$

only dependent on the topology of M [147]. This is a very strong requirement. In particular there cannot be any local degrees of freedom in a tft since spacetime can be arbitrarily deformed as long as one doesn't rip holes into it without changing the content of the field theory. Its strong requirements make a mathematical treatment possible. A tft can be defined abstractly as a symmetric monoidal functor

$$F : \mathsf{Bordsf}_d \to \mathsf{Vect} \tag{1.2}$$

from a category of d-bordisms to the category of vector spaces. In two dimensions this gives an assignment

 $\longrightarrow$ $F(\bigcirc) =$ state space of the theory

 $\longrightarrow$ $\mu : F(\bigcirc) \otimes F(\bigcirc) \to F(\bigcirc)$

where the multiplication defined by the pair of pants only depends on the topology of three punctured sphere. The consistency requirements for a symmetric monoidal functor yield that any two dimensional tft is completely determined by a Frobenius algebra [101][112]. However, two dimensional topological field theories are far too restrictive to be really interesting.

This brings us back to two dimensional conformal field theories. The relaxation from topological to conformal field theory can be seen as a field theory analog of the step from merely constant functions to analytic functions. It is general enough to capture interesting cases but still restrictive enough to remain treatable. Interestingly enough the most accessible way for a topological field theory like treatment of CFTs is by relaxing the target Vect to more general categories rather than taking conformal structures on the source into account. So, what is conformal field theory? Assume we are given an oriented spacetime X and some open subset $U \hookrightarrow X$. A field theory can be conveniently built from representations of its preserved symmetries. In case of the standard model on Minkowski space (M, η) the basic symmetry is Poincaré invariance. This is invariance under all transformations preserving the Minkowski metric η. If the spacetime (X, g) is a smooth spacetime with Lorentzian metric g and U is small enough s.th. there exist local coordinates $x^i : U \to \mathbb{R}^{1,d-1}$ and $g = \eta$ is in Minkowski form in these coordinates, one may easily enhance this to conformal invariance. Conformal invariance is invariance under transformations $\xi^j(x^i)$ s.th.

$$\sum_{i,j=1}^{d} g_{ij;p}(\xi^j)\frac{\partial \xi^i}{\partial x^\ell}\frac{\partial \xi^j}{\partial x^k} = e^{\lambda(p)}\eta_{\ell k} \tag{1.3}$$

where $p \in U$ is a point and $\lambda : U \to \mathbb{R}$ is some smooth function. The case $\lambda = 1$ corresponds to local Poincaré invariance. Mathematically, we are allowing for all local diffeomorphisms $U \to U$ preserving the conformal equivalence class of the metric. We will be concerned with two dimensional field theories living on spacetimes with Riemannian metrics instead of Lorentzian ones. Analyzing the transformation rule (1.3) in this case yields, that there is an infinite set of local conformal transformations $f : U \to U$ given by all biholomorphic maps[1]. If we further assume that U is a punctured disk such a map has a Laurent extension

$$f(z) = \sum_{n\in\mathbb{Z}} f_n(-z^{n+1}) \tag{1.4}$$

[1]Equation (1.3) reduces to the Cauchy Riemann equations for a differentiable map $f : U \to U$.

where z is a local coordinate on U. Local conformal transformations are then generated by the *Witt algebra*

$$\ell_n = -z^{n+1}\partial_z \tag{1.5}$$

for which one easily computes

$$[\ell_n, \ell_m] = (n-m)\ell_{n+m} \quad . \tag{1.6}$$

In order to do quantum field theory the Witt algebra has to be centrally extended to the *Virasoro algebra* with generators $\{L_n\}_{n\in\mathbb{Z}}$ and commutator

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{m,-n} \tag{1.7}$$

with $c \in \mathbb{C}$ the *central charge*. The Virasoro algebra is at the heart of two dimensional conformal field theory. Fields transform in representations of it. The starting point for a discussion of conformal field theory is probably the groundbreaking paper [20]. For textbook accounts see e.g. [126][26]. Conformal invariance has far reaching consequences one of which is the existence of an operator product expansion of fields. Let $\{\phi_i\}$ be quasi primary fields transforming in representations of the Virasoro algebra of conformal weights $\{h_i\}$ and $z, w \in \mathbb{C}$ be points on the complex plane with $|w| > |z|$. Then the singular part of the operator product expansion is given by

$$\phi_i(w)\phi_j(z) \sim \sum_{k,n\geq 0} C_{ij}^k (w-z)^{-h_i-h_j+h_\ell+n}\partial^n\phi_\ell(z) \quad . \tag{1.8}$$

Another point one has to take into considerations are extended symmetries for conformal field theories. A string moving trough a target spacetime which is a Lie group G leads to a conformal field theory whose symmetry algebra contains the Kac Moody algebra $\hat{\mathfrak{g}}_k$ at a certain level k. This is the famous Wess Zumino Witten model [144][146]. Further examples of enhanced symmetry are models with so called $\mathscr{W}$-algebra symmetry. $\mathscr{W}$-algebras are extensions of the Virasoro algebra with higher spin currents. They were discovered in [150] an excellent review of the topic is [28]. So as a wish list for describing a conformal theory, at least on the sphere, one might come up with the following ingredients [140]:

I) Chiral and antichiral symmetry algebras V_L, V_R both containing the Virasoro algebra.

II) A state space $H = \bigoplus_{n,m} H_{n,m} U_n^L \otimes U_m^R$ splitting into representations of $V_L \otimes V_R$.

III) For $|z_1| > \cdots > |z_n|$ and $|\xi_1| > \cdots > |\xi_n|$ points in the complex plane $\mathbb{C}$ there should exist correlation function mappings

$$\begin{aligned} \mu_n(\bullet; z_1, \xi_1 \cdots, z_n, \xi_n) : H^{\otimes n} &\longrightarrow H \\ v_1 \otimes \cdots \otimes v_n &\longmapsto \mu_n(v_1, \cdots, v_n; z_1, \xi_1 \cdots, z_n, \xi_n) \end{aligned} \tag{1.9}$$

s.th. for all $\omega \in H$

$$\langle \omega, \mu_n(v_1, \cdots, v_n; z_1, \xi_1 \cdots, z_n, \xi_n)\rangle \tag{1.10}$$

converges absolutely as a function on $\{|z_1| > \cdots > |z_n| \cap |\xi_1| > \cdots > |\xi_n|\}$. In addition we demand the correlation function mappings to satisfy the following list of axioms:

III.1) For all $v_1, \cdots, v_n, \omega \in H$ the map $\langle \omega, \mu_n(v_1, \cdots, v_n; \bullet) \rangle$ has a multivalued analytic continuation to

$$\begin{aligned}\mathrm{Conf}^{2n}(\mathbb{C}) = \Big\{ (z_1, \xi_1 \cdots, z_n, \xi_n) \in \mathbb{C}^{2n} | z_i \neq z_j,\, \xi_i \neq \xi_j \,\forall i \neq j \\ z_i \neq \xi_j,\, \forall i, j \Big\}\end{aligned} \tag{1.11}$$

which restricts to a single valued function on $\xi_i = \overline{z}_i$.

III.2) Assume we have chiral and antichiral configurations on the punctured complex plane of the following type

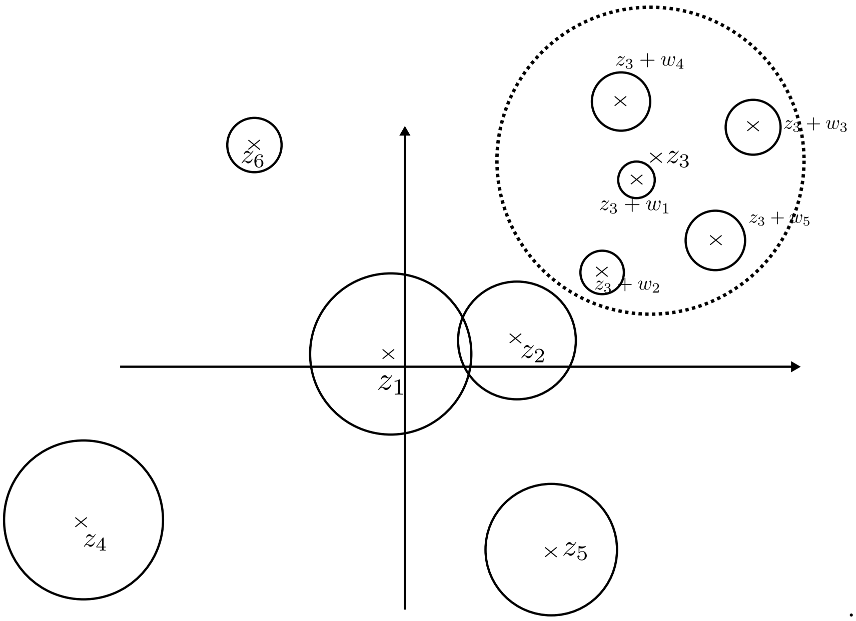


Figure 1.1

Then the map

$$\langle \omega, \mu_6(v_1, v_2, \mu_5(v_3, \cdots, v_7; w_1, \tau_1, \cdots, w_5, \tau_5), v_8, v_9, v_{10}; z_1, \xi_1, \cdots, z_5, \xi_5) \rangle \tag{1.12}$$

absolutely converges to

$$\langle \omega, \mu_{10}(v_1, \cdots, v_{10}; z_1, \xi_1, \cdots, z_3 + w_1, \xi_3 + \tau_1, \cdots, z_3 + w_5, \xi_3 + \tau_5, \cdots, z_6, \xi_6) \rangle \tag{1.13}$$

for all $\omega, v_1, \cdots, v_{10}$. Of course this should hold for all n-airy inputs. The specific case of $n = 10$ is chosen to exhibit the property at a concrete example.

Points I),II) of the wish list are clear, they tell that our theory is based on some symmetry containing local conformal invariance and fields in the theory transform in some representation of the symmetry. The symmetry algebras should be general enough to contain examples like the Kac-Moody symmetry or $\mathscr{W}$-algebras. The rest of the requirements are natural assumptions on correlation functions. The first of these corresponds to the requirement that on radially ordered (which in suitable coordinates on the punctured plane is time ordering) inputs of fields, correlation functions can be computed by subsequently taking operator product expansions (OPEs) between the fields. Then III.2) and III.3) tell that in fact the correlation function extends to all configurations and it doesn't matter in which order OPEs between fields are taken [2]. So how to realize the wish list mathematically? One possible route is via a Segal type definition as representations of the PROP of Riemann surfaces into topological vector spaces [134]. This is the conformal equivalent to the discussion of topological field theories, where two dimensional bordisms have a conformal structure and the symmetric monoidal functor has to respect the conformal structure in a suitable sense. Although formulated already in the 1980s this approach to CFTs is not very developed. A related program for genus zero was started by Huang in 1990 [77]. The central result of [77] is a formulation of vertex operator algebras in terms of representations of a partial operad of spheres with punctures and local holomorphic coordinates around these punctures. We arrived at the first point of our wish list: symmetry algebras. These can be given in the form of vertex operator algebras (VOAs). Vertex algebras were introduced by Borcherds [27] in order to prove the monstrous moonshine conjecture [33]. A vertex algebra is in some sense a holomorphic enhancement of an associative algebra. If one requires that a vertex algebra V has an embedded Virasoro algebra one speaks of a vertex operator algebra. They have a well-behaved representation theory (see e.g. [54]) and points I),II) can be settled by defining left and right moving symmetry algebras V_L, V_R to be VOAs and the state space to be a module of $V_L \otimes V_R$. For properly-behaved correlation functions one can introduce *chiral conformal blocks*. A configuration of field insertions in genus zero can be thought of as punctured sphere $(S, (p_1, z_1, H_1), \cdots, (p_n, z_n, H_n))$ where punctures are labeled with representations H_i of the underlying symmetry VOA V. Using the local coordinates one can define an action of V on such a labeled configuration. A *conformal block on the sphere* is defined as a map

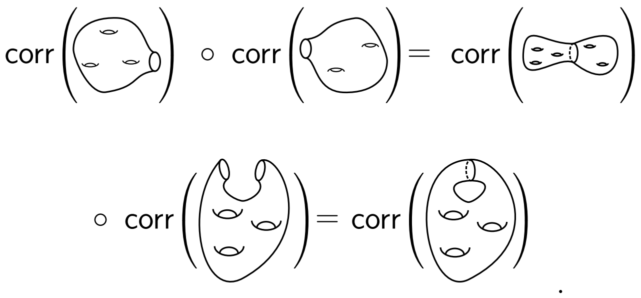
$$\langle \bullet \rangle : (S, (p_1, z_1, H_1), \cdots, (p_n, z_n, H_n)) \to \mathbb{C} \tag{1.14}$$

which is invariant under the action of V (see [52, section 9,10]). Therefore these are functionals satisfying the Ward identities for the symmetry algebra V. Conformal blocks on the sphere are equivalent to intertwining operators between representations of V [151]. Thus the theory is completely treatable with tools from VOA theory. Physically, intertwining operators correspond to all possible couplings between conformal families of fields in a given chiral CFT. A general correlation function is now a linear combination of products of chiral and antichiral conformal blocks. However, not all possible linear combinations are allowed. There are two further requirements on correlation functions which can be thought of as global symmetry statements [3]. Anticipating the extension of the genus zero theory to higher genus we give the requirements in general form. Assume the CFT at hand has

[2] Sometimes we will speak about field insertions on a sphere rather than on the plane. There is no essential difference since in the above the state ω is inserted at $\{\infty\}$ on the sphere.

[3] Local symmetries are already dealt with by conformal blocks.

an extension to all genus g surfaces. The first requirement is that a correlation function on a surface Σ_g is invariant under global orientation preserving diffeomorphisms of Σ_g. The second requirement is equivariance under sewings



Both together go under the name of *sewing constraints*. Sticking to genus zero the gluing part of the sewing constraints relates different ways of taking OPEs when computing correlation functions. Further analytic properties for correlation functions are taken care of by general VOA theory. We therefore might satisfy the wish list by

I) Left and right moving symmetry algebras are VOAs $V_L \otimes V_R$.

II) The state space is a module H over $V_L \otimes V_R$.

III) Correlation functions are linear combinations of conformal blocks satisfying the sewing constraints.

So the question is, can we classify all CFTs which are mathematically formulated as above? The answer is no. The basic obstacle is that things are too infinite dimensional. In particular, the space of intertwining operators between representations is infinite dimensional in general. Put differently, the space of conformal blocks is infinite dimensional. Therefore we make the simplifying assumption that the space of conformal blocks on every surface is finite dimensional and call a theory meeting this assumption *rational*. This implies that a conformal family present in the theory only couples to finitely many other conformal families. A classification of full rational CFTs in genus zero has appeared in [86] and it turns out that these are equivalent to Frobenius algebras in the representation category $\mathsf{R}_{V_L \otimes V_R}$ of $V_L \otimes V_R$. The discussion can be extended to include boundaries and full conformal field theories where possible boundaries have a fixed common boundary condition. Such open-closed ´RCFTs are classified by $(\mathsf{R}_V|\mathsf{Z}(\mathsf{R}_V))$-Cardy algebras [104]. The remaining question is, can this classification be extended to higher genus surfaces? The answer this time is an indefinite *almost*. The problem is an extension of conformal blocks to higher genus Riemann surfaces. For a fixed Riemann surface of genus g they can be defined similar as in the genus zero case. However, we should be able to compute their behavior as we move in the moduli space of Riemann surfaces of genus g. For a true theory of conformal blocks this is suspected to yield a flat vector bundle over moduli spaces of Riemann surfaces whose sections in addition satisfy factorization properties under the conformal sewing of Riemann surface. Let's call this a *complex analytic modular functor*.

Although there has been some recent progress in that direction [68] no such construction is available to date. But this is not the end of things we can do. Flat vector bundles over moduli space are equivalent to representations of the mapping class group and there is a construction of a *categorical modular functor* using this fact. Both types of modular functors are expected to give the same information [5] in the sense that there is an isomorphism between the two. Such an isomorphism would give an isomorphism between the actual complex analytic conformal blocks and their categorical counterparts. Modular functors in general describe the Moore-Seiberg monodromy data for a RCFT [121][120]. Given a *rational* VOA V, its representation category R_V is a modular tensor category and a categorical modular functor can be constructed from R_V. Spaces of conformal blocks are then given by morphism spaces in R_V (this almost equivalent to the Moore Seiberg data, see [5, section 5]). So we can switch to the categorical side and compute structure constants in an expansion of correlators in terms of categorical conformal blocks. Once an isomorphism between the categorical and complex analytic side is found these are exactly expansion coefficients of correlation functions, determining the latter completely. A treatment of RCFTs in this categorical setting has its roots probably in the seminal work of Witten [148].

The first result presented in this thesis deals with the question if a full RCFT with fixed boundary condition in genus zero and one, which is given in the form of a Cardy algebra actually determines the RCFT to all genera uniquely. The answer is yes. Such statements have been derived with the help of the Reshetikhin Turaev three dimensional topological field theory in the past [106]. We give a simplified treatment in terms of string-nets. In addition we give a construction of consistent correlators for RCFTs with arbitrary different boundary conditions and even topological defects in terms of string-nets.

The second part of this thesis deals with field theories in the form of homotopy algebras. One starts with a perturbative classical field theory with action S and field content Ψ. The fields are allowed to have a gauge freedom. By adjoining (anti-)ghost fields and antifields to the theory one can built up a classical master action $\widehat{S}$ living in the space of functionals on the ghost extended field content $\widehat{\Psi}$. The space of functionals is graded by ghost number and has a Poisson bracket $\{\bullet, \bullet\}$. The classical master action is an element of degree -1 satisfying

$$\left\{\widehat{S}, \widehat{S}\right\} = 0 \quad . \tag{1.15}$$

For a thorough discussion of all of these terms and implications we refer to [71]. Let $\mathcal{O}(\widehat{\Psi})$ be the space of observables for the theory. The space is again graded by ghost number and the classical master action induces a differential on it via

$$\mathrm{d}(O) = \left\{O, \widehat{S}\right\} \quad . \tag{1.16}$$

It turns out that the homology in ghost number zero is actually the space of all gauge invariant observables for the original theory given in terms of S, i.e.

$$H^0(\mathcal{O}(\widehat{\Psi}), \mathrm{d}) \simeq \{\text{gauge invariant observables}\} \quad . \tag{1.17}$$

The purpose of introducing all the ghosts and antifields is to make gauge invariance of quantities manifest. It turns out that information {classical master action} + {Poisson bracket} is equivalent to a L_∞ algebra defined on $\widehat{\Psi}$. L_∞ algebras, or *strong homotopy Lie algebras*, are generalizations of Lie algebras allowing for higher airy brackets instead of only a two-bracket. The relation between master action and homotopy algebras was first discovered by Zwiebach in his construction of closed bosonic string field theory [152]. On the other hand homotopy algebras themselves have a long history in mathematics dating back to Stasheff [137][138].

The upshot is that a classification of field theories with a classical master action can be turned into a question of classifications of L_∞ algebras. We employ the field theory $\leftrightarrow$ homotopy algebra relation by relating Seiberg-Witten maps between certain gauge theories to quasi-isomorphisms between L_∞ algebras. Secondly we show how to uniquely close a vector space X with antisymmetric bracket into a finite term L_∞ algebras. A field theory corollary of this theorem is the known L_∞ algebra structure on the Courant algebroid.

This thesis is structured as follows. In chapter 2 we recall how the representation category of a rational VOA attains the structure of a modular tensor category. This is the first instance where analytic properties of correlation functions get translated into categorical algebra. In chapter 3 a precise definition for the wish list of CFTs is given. This comes in the form of field algebras. We outline how the analytically defined field algebras are actually equivalent to purely categorical notions. No claim of originality is made on these chapters. They mostly settle the stage for the following chapter. Sometimes we give short proofs of the results, mostly in cases where proofs are somewhat hidden in the literature or the proofs themselves illustrate important properties of the theory. The first part of chapter 4 is based on the paper [141]. Its second part is an extension of the construction to include topological defects and arbitrary symmetry preserving boundary conditions. Furthermore we compute torus and annulus partition functions using string-nets relating the construction to known quantities in CFT and giving an a posteriori justification of some of its ingredients. The final chapter 5 deals with the results of the papers [74][21] starting with the construction of a L_∞ algebra from antisymmetric brackets. Its second half gives the relation between Seiberg-Witten maps and quasi-isomorphisms. Finally we included four appendices to make this thesis as self contained as possible and accessible to a broader audience. Appendix A is a lightning review of VOA theory. In appendix B we recall notions from category theory relevant for this thesis. Next, in appendix C string-net models are reviewed. In particular we included an outline of the three dimensional string-net tft. Finally in appendix D we simply list all fundamental open-closed world sheets and the 32 basic sewing relations relevant for the results in chapter 4.

Chapter 2

From VOAs to Modular Tensor Categories

In this chapter we recall the definition of the modular tensor category structure on the category of representations of a rational VOA. We start by reviewing the construction of the braided structure. This is originally due to [76][89][90][78] based on the notion of a vertex tensor category developed by the same authors in [88] [1]. Only the necessary existence and structure theorems will be shown and partially proven here. In particular the explicit construction of the monoidal product in terms of submodules is left out.

In appendix A.2 intertwining operators are discussed and we freely use the notation introduced there. For a tensor product intertwining *maps* are needed. The notions have a lot in common, the major difference being that the former takes values in some formal power series space with coefficients in the algebraic extension of a VOA module, whereas the later maps directly to the algebraic extension of a VOA module. In the following the algebraic tensor product of $\mathbb{C}$-vector spaces is denoted by $\otimes$.

Definition 2.0.1. Let V be a VOA, (M_1, Y_1), (M_2, Y_2), (M_3, Y_3) be V-modules and $z \in \mathbb{C}^\times$. An $P(z)$-*intertwining map* of type $\binom{M_3}{M_1\, M_2}$ is a $\mathbb{C}$-linear map $I : M_1 \otimes M_2 \to \overline{M}_3$ s.th. for formal variables x_1, x_0 and $u \in V$ as well as $m_i \in M_i$ the Jacobi identity

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - z}{x_0}\right) Y_3(u, x_1) I\left(m_1 \otimes m_2\right) =& z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) I\left(Y_1(u, x_0)m_1 \otimes m_2\right) \\ &+ x_1^{-1}\delta\left(\frac{z - x_1}{-x_0}\right) I\left(m_1 \otimes Y_2(u, x_1)m_2\right) \end{aligned} \tag{2.1}$$

holds. The vector space of $P(z)$-intertwining maps of type $\binom{M_3}{M_1\, M_2}$ is denoted by $I\left[P(z)\right]_{M_1 M_2}^{M_3}$.

Definition 2.0.2. A $P(z)$-*product* of V-modules M_1, M_2 is a V-module M_3 together with an intertwining map I of type $\binom{M_3}{M_1\, M_2}$.

[1] In fancy mathematics terms this is a holomorphic analogue of the fact that an E_2-algebra in Cat is a braided monoidal category. A vertex tensor category is an algebra over the partial operad of spheres with tubes in Cat. Since the E_2-operad is a suboperad of this partial operad, the braided monoidal structure is exactly the one induced by restriction.

As usual, the tensor product is characterized by a universal property.

Definition 2.0.3. The $P(z)$*-tensor product* of V-modules M_1, M_2 is a $P(z)$-product (M_3, I) s.th. if $(M_4, \tilde{I})$ is any other $P(z)$-product of M_1, M_2 there exists a unique up to isomorphism V-module map $\rho : M_3 \to M_4$ with $\tilde{I} = \overline{\rho} \circ I$.

The overline denotes the linear extension of ρ to a V-module map $\overline{M}_3 \to \overline{M}_4$. If a $P(z)$-tensor product exists it is obviously unique and it will be denoted by $\left(M_1 \boxtimes_{P(z)} M_2, Y_{P(z)}\right)$.

Intertwining maps have some technical advantages when introducing a tensor product on R_V. However, the following two lemmas show, that intertwining maps are completely determined by intertwining operators. Let $p \in \mathbb{Z}$ and ℓ_p be a branch of the complex logarithm, i.e. $\ell_p(z) = \log(|z|) + 2\pi \mathsf{i}\, p \arg(z)$, where $0 \leq \arg(z) < 2\pi$.

Lemma 2.0.4. *Let $\mathcal{Y} \in \mathcal{V}_{M_1 M_2}^{M_3}$, then for any $p \in \mathbb{Z}$ and $z \in \mathbb{C}^\times$*

$$\begin{aligned} I_{\mathcal{Y},p} : M_1 \otimes M_2 &\to \overline{M}_3 \\ m_1 \otimes m_2 &\mapsto \mathcal{Y}\left(m_1, e^{\ell_p(z)}\right) m_2 \end{aligned} \tag{2.2}$$

is in $I\left[P(z)\right]_{M_1 M_2}^{M_3}$.

Proof. As M_3 is $\mathbb{C}$-vector space the map is well defined. It suffices to check the Jacobi identity. Since $\mathcal{Y}$ is intertwining operator it satisfies the Jacobi identity

$$\begin{aligned} &x_0^{-1}\delta\left(\frac{x_1 - z}{x_0}\right) Y_3(u, x_1) \mathcal{Y}\left(m_1, e^{\ell_p(z)}\right) m_2 \\ &\qquad - x_0^{-1}\delta\left(\frac{z - x_1}{-x_0}\right) \mathcal{Y}\left(m_1, e^{\ell_p(z)}\right) Y_2(u, x_1) m_2 \\ =&z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) \mathcal{Y}\left(Y_1(u, x_0) m_1, e^{\ell_p(z)}\right) m_2 \end{aligned} \tag{2.3}$$

which by definition of $I_{\mathcal{Y},p}$ is equivalent to

$$\begin{aligned} &x_0^{-1}\delta\left(\frac{x_1 - z}{x_0}\right) Y_3(u, x_1) I_{\mathcal{Y},p}(m_1 \otimes m_2) \\ &\qquad - x_0^{-1}\delta\left(\frac{z - x_1}{-x_0}\right) I_{\mathcal{Y},p}\left(m_1 \otimes Y_2(u, x_1) m_2\right) \\ =&z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) I_{\mathcal{Y},p}\left(Y_1(u, x_0) m_1 \otimes m_2\right) \quad . \end{aligned} \tag{2.4}$$

□

In the other direction we have.

Lemma 2.0.5. *Given $I \in I\left[P(z)\right]_{M_1 M_2}^{M_3}$ and m_1, m_2 homogeneous elements in M_1 and M_2, define*

$$m_{1,(n)} m_2 \equiv e^{(n+1)\ell_p(z)} \pi_{|m_1|+|m_2|-n-1}\left(I(m_1 \otimes m_2)\right) \quad . \tag{2.5}$$

Linearly extending the map given on homogeneous by

$$\mathcal{Y}^I(m_1, x)m_2 \equiv \sum_{n\in\mathbb{C}} m_{1,(n)}m_2 x^{-n-1} \tag{2.6}$$

to $M_1 \otimes M_2$ gives an intertwining operator of type $\binom{M_3}{M_1 M_2}$.

Proof. Since $M_{3,n} = 0$ for $\mathrm{Re}(n) >> 0$, it holds $m_{1,(n)}m_2 = 0$ for $\mathrm{Re}(n) << 0$. Thus the truncation property holds. Next we show the Jacobi identity. Note that we could have equally defined the intertwining operator by

$$\mathcal{Y}^I(m_1, x)m_2 = e^{-\ell_p(z)L_0}x^{L_0} I\left(e^{\ell_p(z)L_0}x^{-L_0}m_1 \otimes e^{\ell_p(z)L_0}x^{-L_0}m_2\right) \tag{2.7}$$

as

$$I\left(m_1 \otimes m_2\right) = \sum_{n\in\mathbb{C}} \pi_{|m_1|+|m_2|-n-1}\left(I(m_1 \otimes m_2)\right) \quad . \tag{2.8}$$

In the Jacobi identity for I

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - z}{x_0}\right) Y_3(u, x_1) I\left(m_1 \otimes m_2\right) =& z^{-1}\delta\left(\frac{x_1 - x_0}{z}\right) I\left(Y_1(u, x_0)m_1 \otimes m_2\right) \\ &+ x_1^{-1}\delta\left(\frac{z - x_1}{-x_0}\right) I\left(m_1 \otimes Y_2(u, x_1)m_2\right) \end{aligned} \tag{2.9}$$

we re-scale $x_0 \to zx_2^{-1}x_0$ and $x_1 \to zx_2^{-1}x_1$. This gives, using the explicit expansion of the delta functions

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)&Y_3(u, zx_2^{-1}x_1) I\left(m_1 \otimes m_2\right) \\ &= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) I\left(Y_1(u, zx_2^{-1}x_0)m_1 \otimes m_2\right) \\ &\quad + x_1^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) I\left(m_1 \otimes Y_2(u, zx_2^{-1}x_1)m_2\right) \quad . \end{aligned} \tag{2.10}$$

Recall the adjoint action of the Virasoro algebra on module vertex operators. Since vertex operators take place in actual powers series of formal variables it holds

$$Y_i(u, zx_2^{-1}x_i) = e^{\ell_p(z)L_0}x_2^{-L_0} Y_i\left(e^{-\ell_p(z)L_0}x_2^{L_0}u, x_i\right) e^{-\ell_p(z)L_0}x_2^{L_0} \quad . \tag{2.11}$$

This yields

$$\begin{aligned} &x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) e^{\ell_p(z)L_0}x_2^{-L_0} Y_3(e^{-\ell_p(z)L_0}x_2^{L_0}u, x_1) e^{-\ell_p(z)L_0}x_2^{L_0} I\left(m_1 \otimes m_2\right) \\ &= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) I\left(e^{\ell_p(z)L_0}x_2^{-L_0} Y_1(e^{-\ell_p(z)L_0}x_2^{L_0}u, x_0) e^{-\ell_p(z)L_0}x_2^{L_0}m_1 \otimes m_2\right) \\ &\quad + x_1^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) I\left(m_1 \otimes e^{\ell_p(z)L_0}x_2^{-L_0} Y_2(e^{-\ell_p(z)L_0}x_2^{L_0}u, x_1) e^{-\ell_p(z)L_0}x_2^{L_0}m_2\right) \quad . \end{aligned} \tag{2.12}$$

Next, set $m_1 \to e^{-\ell_p(z)L_0} x_2^{L_0} m_1$, $m_2 \to e^{-\ell_p(z)L_0} x_2^{L_0} m_2$ and $u \to e^{-\ell_p(z)L_0} x_2^{L_0} u$ to get

$$\begin{aligned}
& x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) e^{\ell_p(z)L_0} x_2^{-L_0} Y_3(u, x_1) e^{-\ell_p(z)L_0} x_2^{L_0} I\left(e^{\ell_p(z)L_0} x_2^{-L_0} m_1 \otimes e^{\ell_p(z)L_0} x_2^{-L_0} m_2\right) \\
&= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) I\left(e^{\ell_p(z)L_0} x_2^{-L_0} Y_1(u, x_0) m_1 \otimes e^{\ell_p(z)L_0} x_2^{-L_0} m_2\right) \\
&\quad + x_1^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) I\left(e^{\ell_p(z)L_0} x_2^{-L_0} m_1 \otimes e^{\ell_p(z)L_0} x_2^{-L_0} Y_2(u, x_1) m_2\right) \quad .
\end{aligned} \tag{2.13}$$

Applying $e^{-\ell_p(z)L_0} x_2^{L_0}$ yields

$$\begin{aligned}
& x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_3(u, x_1) \mathcal{Y}^I(m_1, x_1) m_2 \\
&= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) \mathcal{Y}^I\left(Y_1(u, x_0) m_1, x_1\right) m_2 \\
&\quad + x_1^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) \mathcal{Y}^I(m_1, x_1) Y_2(u, x_1) m_2 \quad .
\end{aligned} \tag{2.14}$$

The Virasoro translation property can be proven along similar steps. A detailed derivation is left to the interested reader. □

Lemma 2.0.6. *The maps $\mathcal{V}^{M_3}_{M_1 M_2} \to I\left[P(z)\right]^{M_3}_{M_1 M_2}$ and $I\left[P(z)\right]^{M_3}_{M_1 M_2} \to \mathcal{V}^{M_3}_{M_1 M_2}$ from the above lemmas are inverse to each other.*

Proof. We compute

$$\begin{aligned}
\mathcal{Y}^{I_{\mathcal{Y},p}}(m_1, x) m_2 &= e^{-\ell_p(z)L_0} x^{L_0} I_{\mathcal{Y},p}\left(e^{\ell_p(z)L_0} x^{-L_0} m_1 \otimes e^{\ell_p(z)L_0} x^{-L_0} m_2\right) \\
&= e^{-\ell_p(z)L_0} x^{L_0} \mathcal{Y}\left(e^{\ell_p(z)L_0} x^{-L_0} m_1, e^{\ell_p(z)}\right) e^{\ell_p(z)L_0} x^{-L_0} m_2 \\
&= \mathcal{Y}(m_1, x) m_2
\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}
I_{\mathcal{Y}^I,p}(m_1 \otimes m_2) &= \mathcal{Y}^I\left(m_1, e^{\ell_p(z)}\right) m_2 \\
&= e^{-\ell_p(z)L_0} e^{\ell_p L_0} I_{\mathcal{Y},p}\left(e^{\ell_p(z)L_0} e^{-\ell_p L_0} m_1 \otimes e^{\ell_p(z)L_0} e^{-\ell_p L_0} m_2\right) \\
&= I(m_1 \otimes m_2) \quad .
\end{aligned} \tag{2.16}$$

□

We summarize the above results in the following proposition.

Proposition 2.0.7. *For V a VOA, M_1, M_2 and M_3 V-modules and any $p \in \mathbb{Z}$ there is an isomorphism of vector spaces*

$$\mathcal{V}^{M_3}_{M_1 M_2} \longleftrightarrow I\left[P(z)\right]^{M_3}_{M_1 M_2} \tag{2.17}$$

An immediate corollary of the proposition and the universal property of $P(z)$-tensor products is that morphism spaces including $P(z)$-tensor products are isomorphic to spaces of intertwining operators.

Corollary 2.0.8. *Assume $M_1 \boxtimes_{P(z)} M_2$ exists, then it holds*

$$\begin{array}{ccccc} \mathrm{Hom}(M_1 \boxtimes_{P(z)} M_2, M_3) & \xrightarrow{\simeq} & I\left[P(z)\right]^{M_3}_{M_1 M_2} & \xrightarrow{\simeq} & \mathscr{V}^{M_3}_{M_1 M_2} \\ f & \mapsto & I_f \equiv \overline{f} \circ \boxtimes_{P(z)} & \mapsto & \mathscr{Y}_{I_f,0} \end{array}$$

The existence of $P(z)$-tensor product can be readily established in case V is rational. In particular for any triple of V-modules, $\mathscr{V}^{M_3}_{M_1 M_2}$ and therefore $I\left[P(z)\right]^{M_3}_{M_1 M_2}$ is finite dimensional. One can easily check that for any vector space H, $H \otimes V$ is still a VOA with $Y_{H\otimes V} = \mathrm{id}_H \otimes Y$. Let $\{I_i\}_{i=1,\dots,N}$ be a basis for $I\left[P(z)\right]^{M_3}_{M_1 M_2}$ and $\{I^i\}$ its canonical dual basis.

Lemma 2.0.9. *Implicitly summing over double indices we define a map*

$$\begin{aligned} F^{M_3}_{M_1 M_2} : M_1 \otimes M_2 &\to \left(I\left[P(z)\right]^{M_3}_{M_1 M_2}\right)^* \otimes \overline{M}_3 \\ m_1 \otimes m_2 &\mapsto I^i \otimes I_i(m_1 \otimes m_2) \end{aligned} \tag{2.18}$$

$F^{M_3}_{M_1 M_2}$ is $P(z)$-intertwining map and independent of the chosen basis.

Proof. Independence and truncation property are clear. The Jacobi identity readily follows from triviality of the module vertex operator on $\left(I\left[P(z)\right]^{M_3}_{M_1 M_2}\right)^* \otimes \overline{M}_3$ in the first slot. □

This leads to the first major result in the construction of a modular structure, namely the existence of a monoidal product.

Proposition 2.0.10. *Let V be a rational VOA, M_1, M_2 V-modules and $\{U_i\}_{i\in\mathsf{I}}$ be the simple modules of V. Then $(M_1 \boxtimes_{P(z)} M_2, Y_{P(z)})$ exists and is canonically given by*

$$M_1 \boxtimes_{P(z)} M_2 = \coprod_{i\in\mathsf{I}} \left(I\left[P(z)\right]^{U_i}_{M_1 M_2}\right)^* \otimes U_i \quad . \tag{2.19}$$

Proof. The $P(z)$-intertwining map is given by $\coprod_{i\in\mathsf{I}} F^{U_i}_{M_1 M_2}$ and we are left to check the universal property. For this, let (M_4, I) be any other $P(z)$-product of M_1 and M_2. Since V is rational M_4 decomposes as a direct sum of simples. Let $\{b^i_\alpha\}$ be a basis of $\mathrm{Hom}(U_i, M_4)$ and $\left\{b^\beta_i\right\}$ its dual basis in $\mathrm{Hom}(M_4, U_i)$. Denote $\pi_\ell = \sum_\alpha b^\alpha_\ell : M_4 \to U_\ell$, then $I_\ell = \overline{\pi}_\ell \circ I$ is $P(z)$-intertwining map of type $\binom{U_\ell}{M_1 M_2}$ and therefore has an expansion $I_\ell = \sum_j I^j_\ell F_{j,\ell}$, where $\{F_{j,\ell}\}$ is a basis for $I\left[P(z)\right]^{U_\ell}_{M_1 M_2}$. To construct the universal map we set

$$\begin{aligned} \eta^j_\ell : F^j_\ell \otimes U_\ell &\to M_4 \\ F^j_\ell \otimes u &\mapsto \sum_\alpha b^\ell_\alpha(u) I^j_\ell \quad . \end{aligned} \tag{2.20}$$

Then $\eta \equiv \coprod_{i\in \mathsf{I}} \sum_j \eta_\ell^j$ is V-module map:

$$\begin{aligned} \eta_\ell^j \left(\pi_\ell \circ Y_{M_1 \boxtimes_{P(z)} M_2}(v,x) \left(F_\ell^j \otimes u\right)\right) &= \eta_\ell^j \left(F_\ell^j \otimes Y_{U_\ell}(v,x)u\right) \\ &= \sum_\alpha b_\alpha^\ell \left(Y_{U_\ell}(v,x)u\right) I_\ell^j \\ &= Y_{M_4}(v,x) \sum_\alpha b_\alpha^\ell(u) I_\ell^j \\ &= Y_{M_4}(v,x) \eta_\ell^j \left(F_\ell^j \otimes u\right) \end{aligned} \tag{2.21}$$

Finally we check that η intertwines the $P(z)$-intertwining maps

$$\begin{aligned} &\overline{\eta}_\ell \circ F_{M_1 M_2}^{U_\ell}(m_1 \otimes m_2) \\ =&\overline{\eta}_\ell \left(F_\ell^j \otimes F_{j,\ell}(m_1 \otimes m_2)\right) \\ =&\sum_\alpha \overline{b}_\alpha^\ell \left(F_{j,\ell}(m_1 \otimes m_2)\right) I_\ell^j \\ =&I_\ell(m_1 \otimes m_2) \quad . \end{aligned} \tag{2.22}$$

□

We are almost ready to define a monoidal product on R_V. We only need to get rid of the $z \in \mathbb{C}^\times \cup \{\infty\}$ dependence. In order to do so we need to be able to transport the monoidal product along any curve in $\mathbb{C}^\times \cup \{\infty\}$. This should give isomorphisms of $P(z)$-tensor products for different z's possibly depending on the curve. To ease our lives we first recall the following lemma.

Lemma 2.0.11. *[78, Lemma 14.9] As a V-module, $M_1 \boxtimes_{P(z)} M_2$ is spanned by homogeneous components of $\left\{m_1 \boxtimes_{P(z)} m_2\right\} \in \overline{M_1 \boxtimes_{P(z)} M_2}$ for all $m_1 \in M_1$ and $m_2 \in M_2$.*

Proof. Let $M_0 \subset M_1 \boxtimes M_2$ be the submodule spanned by homogeneous components $\left\{m_1 \boxtimes_{P(z)} m_2\right\}$ and

$$M = M_1 \boxtimes_{P(z)} M_2 \,/ M_0 \tag{2.23}$$

the quotient with projection map P. Then $\overline{P} \circ \boxtimes_{P(z)}$ is $P(z)$-intertwining map of type $\binom{M}{M_1 M_2}$. But the image of $\boxtimes_{P(z)}$ is in M_0, hence $\overline{P} \circ \boxtimes_{P(z)}$ is the zero map, which implies that P is the zero map. □

Let $\gamma \subset \mathbb{C}^\times$ be a curve from z_1 to z_z. One can analytically continue the logarithm starting from z_2 along γ and we denote the value of the analytic extension at z_1 by $\ell_\gamma(z_1)$. By the previous lemma we only need to define V-module maps from $M_1 \boxtimes_{P(z)} M_2$ on elements $\left\{m_1 \boxtimes_{P(z)} m_2\right\}$. Hence the following lemma makes sense.

Lemma 2.0.12. *Let*

$$\mathcal{T}_\gamma : M_1 \boxtimes_{P(z_1)} M_2 \to M_1 \boxtimes_{P(z_2)} M_2 \tag{2.24}$$

be defined by

$$\overline{\mathcal{T}}_\gamma(m_1 \boxtimes_{P(z_1)} m_2) = \mathcal{Y}_2\left(m_1, e^{\ell_\gamma(z_1)}\right) m_2 \tag{2.25}$$

where $\mathcal{Y}_2 = \mathcal{Y}_{\boxtimes_{P(z_2)},0}$ *is the intertwining operator corresponding to* $\boxtimes_{P(z_2)}$*. Then* $\mathcal{T}_Y$ *is an isomorphism of* V*-modules.*

Proof. First we check that it defines a map of V-modules. Let $e^{\ell_\gamma(z_1)} = e^{\ell_0(z_1)}c_\gamma$ and $m_1' = c_\gamma^{-L_0}m_1$, $m_2' = c_\gamma^{-L_0}m_2$, then

$$\begin{aligned}\mathcal{Y}_2\left(m_1, e^{\ell_\gamma(z_1)}\right) m_2 &= \mathcal{Y}_2\left((c_\gamma^{L_0}m_1', e^{\ell_0(z_1)}c_\gamma\right) m_2\\ &= c_\gamma^{L_0}\mathcal{Y}_2\left(m_1', e^{\ell_0(z_1)}\right) m_2'\\ &= c_\gamma^{L_0}m_1' \boxtimes_{P(z_2)} m_2'.\end{aligned} \tag{2.26}$$

This shows that $\overline{\mathcal{T}}_\gamma$ is well defined and furthermore by the spanning property of $\left\{m_1 \boxtimes_{P(z_1)} m_2\right\}$ a V-module map [2]. Let $\ell_\gamma(z_2)$ be the value of the logarithm determined by analytic continuation along γ starting at z_1. Then

$$\overline{\mathcal{T}}'_\gamma(m_1 \boxtimes_{P(z_2)} m_2) = c_\gamma^{L_0}\mathcal{Y}_1\left((m_1', e^{\ell_\gamma(z_2)}\right) m_2' \tag{2.27}$$

obviously defines an inverse map. □

Lastly we need an associativity morphism for rebracketing modules and left, respectively right, unit morphisms. Since the existence and well definedness of the associativity morphism very much depends on the partial-operad algebra structure we only state it, without giving any proofs. A heuristic characterization goes as follows. The tensor product of three modules corresponds to a map $M_1 \otimes M_2 \otimes M_3 \to \overline{M}_4$. In the operadic formulation of [79] such maps are determined by four punctured spheres. The associativity morphism then relates two different ways of gluing a four punctured sphere from two pair of pants. It is shown in [78, Proposition 14.10] that under the natural convergence properties for IOAs, the operadic representations of these different gluings and their isomorphism are equivalent to the statement[3] that for $|z_1| > |z_2| > 0$, $m_i \in M_i$,

$$\langle m_4', \mathcal{Y}_{I_1,0}(m_1, z_1)\mathcal{Y}_{I_2,0}(m_2, z_2)m_3\rangle \tag{2.28}$$

absolutely converges for all $P(z_1)$-intertwining maps I_1 and $P(z_2)$-intertwining maps I_2. This in turn is equivalent to an iterate of intertwining operators, which gives an associativity isomorphism. We summarize the result of [78] in the following proposition.

Proposition 2.0.13. *For* V *rational, there exists an isomorphism*

$$\mathcal{A}_{z_1,z_2}^{z_1-z_2,z_2} : \left(M_1 \boxtimes_{P(z_1)} \left(M_2 \boxtimes_{P(z_2)} M_3\right)\right) \to \left(M_1 \boxtimes_{P(z_1-z_2)} M_2\right) \boxtimes_{P(z_2)} M_3 \tag{2.29}$$

[2]This is analogous to defining a linear map on a basis and extending linearly to the whole vector space
[3]Plus some technical grading restriction property.

which on homogeneous components is given by

$$\overline{\mathcal{A}_{z_1,z_2}^{z_1-z_2,z_2}}(m_1 \boxtimes_{P(z_1)} (m_2 \boxtimes_{P(z_2)} m_3)) = (m_1 \boxtimes_{P(z_1-z_2)} m_2) \boxtimes_{P(z_2)} m_3 \quad . \tag{2.30}$$

In addition there are left and right unit isomorphisms λ_z, ρ_z defined by

$$\begin{aligned} \overline{\lambda}_z : V \boxtimes_{P(z)} M &\to \overline{M} \\ v \boxtimes_{P(z)} m &\mapsto Y_M(v,z)m \end{aligned} \tag{2.31}$$

and

$$\begin{aligned} \overline{\rho}_z : M \boxtimes_{P(z)} V &\to \overline{M} \\ m \boxtimes_{P(z)} v &\mapsto e^{zL_{-1}} Y_M(v,-z)m \end{aligned} \tag{2.32}$$

The first major outcome of [76, 89, 90, 78] is the following theorem, whose proof is sketched in [83].

Theorem 2.0.14. *Let V be a rational VOA and $\boxtimes = \boxtimes_{P(1)}$. Given V-modules M_1, M_2, M_3, and real numbers $z_1 > z_2 > z_1 - z_2 > 0$ let $\gamma_1 : [0,1] \to \mathbb{R}_+$ be s.th. $\gamma_1(0) = 1$ and $\gamma_1(1) = z_1$. Similarly let $1 \overset{\gamma_2}{\to} z_2$, $z_2 \overset{\gamma_3}{\to} 1$ and $z_1 - z_2 \overset{\gamma_{12}}{\to} 1$. Then*

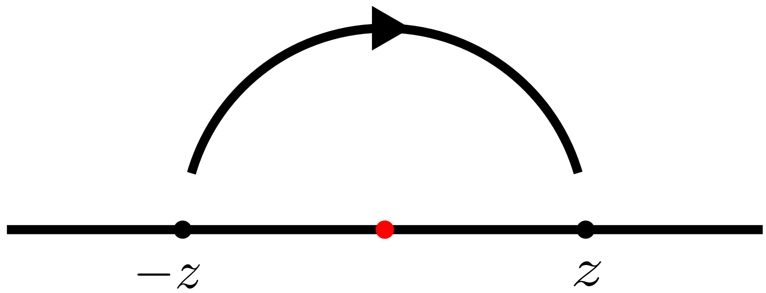
$$\mathcal{A} : M_1 \boxtimes (M_2 \boxtimes M_3) \to (M_1 \boxtimes M_2) \boxtimes M_3 \tag{2.33}$$

defined by

$$\begin{tikzcd} M_1 \boxtimes (M_2 \boxtimes M_3) \arrow[rr, "\mathcal{A}"] \arrow[d, "\left(\mathrm{id}_{M_1} \boxtimes_{P(z_1)} \mathcal{T}_{\gamma_2}\right) \circ \mathcal{T}_{\gamma_1}"'] & & (M_1 \boxtimes M_2) \boxtimes M_3 \\ M_1 \boxtimes_{P(z_1)} \left(M_2 \boxtimes_{P(z_2)} M_3\right) \arrow[rr, "\mathcal{A}_{z_1,z_2}^{z_1-z_2,z_2}"'] & & \left(M_1 \boxtimes_{P(z_1-z_2)} M_2\right) \boxtimes_{P(z_2)} M_3 \arrow[u, "\mathcal{T}_{\gamma_3} \circ \left(\mathcal{T}_{\gamma_{12}} \boxtimes \mathrm{id}_{M_3}\right)"'] \end{tikzcd}$$

is independent of z_1, z_2. Let $\lambda = \lambda_1$ and $\rho = \rho_1$, then the category $(\mathsf{R}_V, \boxtimes, \mathcal{A}, \lambda, \rho)$ is a monoidal category.

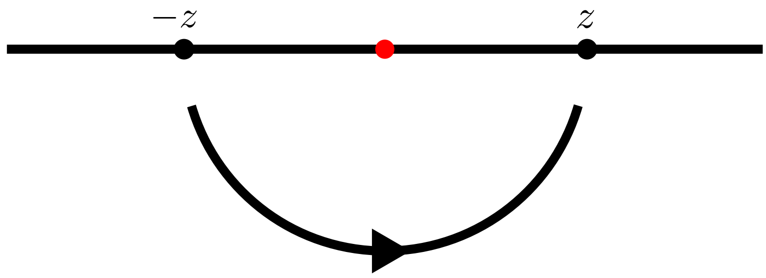
The proof is in fact very easy but quite lengthy to write down. One first shows pentagon and triangle diagrams on homogeneous elements and appropriate real numbers z_i. Using lemma 2.0.11 this gives pentagon and triangle diagrams for $P(z_i)$-products, which are transported to $\boxtimes$-diagrams using paths γ_i as in the example of the associativity morphism above. Note that naturality of all the isomorphisms directly follows from the universal property of $P(z)$-tensor products. Next we want to add a braiding onto $(\mathsf{R}_V, \mathcal{A}, \lambda, \rho)$. This is done similar to associativity and unit morphisms by going through $P(z)$-tensor products. Let $z \in \mathbb{C}^\times$ and γ_z^+ be the path



This gives a map $\beta_z^+ : M_1 \boxtimes_{P(z)} M_2 \to M_2 \boxtimes_{P(z)} M_1$ defined by

$$\overline{\beta^+}_z(m_1 \boxtimes_{P(z)} m_2) \equiv e^{zL_{-1}} \overline{\mathcal{T}}_{\gamma_z^+}(m_2 \boxtimes_{P(-z)} m_1) \quad . \tag{2.34}$$

Choosing the path γ_z^-



one gets a map $\overline{\beta^-}_z(m_1 \boxtimes_{P(z)} m_2) \equiv e^{zL_{-1}} \overline{\mathcal{T}}_{\gamma_z^-}$.

Lemma 2.0.15. *The maps β_z^- and β_z^+ are inverse to each other.*

Proof. We just compute $\beta_z^- \circ \beta_z^+$ as the other direction is the same. First note that

$$x^{L_0} L_{-1} x^{-L_0} = x L_{-1} \tag{2.35}$$

as $[L_0, L_{-1}] = L_{-1}$. Hence it holds.

$$x^{L_0} e^{zL_{-1}} x^{-L_0} = \sum_{k \geq 0} \frac{z^k}{k!} \left(x^{L_0} L_{-1} x^{-L_0} \right)^k = e^{xzL_{-1}} \tag{2.36}$$

Using this we compute

$$\begin{aligned}
&\overline{\beta^-}_z \circ \overline{\beta^+}_z(m_1 \boxtimes_{P(z)} m_2) \\
=&e^{zL_{-1}} \overline{\beta^-}_z(\mathcal{Y}_z \left(m_2, e^{\mathrm{i}\pi} e^{\log(z)} \right) m_1) \\
=&e^{zL_{-1}} \overline{\beta^-}_z \left(e^{\mathrm{i}\pi L_0} \left(e^{-\mathrm{i}\pi L_0} m_2 \boxtimes_{P(z)} e^{-\mathrm{i}\pi L_0} m_1 \right) \right) \\
=&e^{zL_{-1}} e^{\mathrm{i}\pi L_0} e^{zL_{-1}} \mathcal{Y}_z \left(e^{-\mathrm{i}\pi L_0} m_1, e^{-\mathrm{i}\pi} e^{\log(z)} \right) e^{-\mathrm{i}\pi L_0} m_2 \\
=&e^{zL_{-1}} e^{\mathrm{i}\pi L_0} e^{zL_{-1}} e^{-\mathrm{i}\pi L_0} \mathcal{Y}_z(m_1, e^{\log(z)}) m_2 \\
=&e^{zL_{-1}} e^{-zL_{-1}} (m_1 \boxtimes_{P(z)} m_2) \\
=&(m_1 \boxtimes_{P(z)} m_2)
\end{aligned} \tag{2.37}$$

□

Similar to theorem 2.0.14 one shows that $\beta_1^+ = \beta$ gives R_V the structure of a braided tensor category. The results so far are summarized in the following theorem.

Theorem 2.0.16. *Assuming the same conditions as in theorem 2.0.14, the natural isomorphism*

$$\beta : M_1 \boxtimes M_2 \to M_2 \boxtimes M_1 \tag{2.38}$$

gives $(\mathsf{R}_V, \mathcal{A}, \lambda, \rho, \beta)$ *the structure of a braided monoidal category.*

The next layer of structure to be added is a notion of duality in R_V. Unsurprisingly the contragradient module will be the dual object. A construction of a rigid structure on R_V first appeared in [83]. By construction for any three modules M_1, M_2 and M_3, a map $m : M_1 \boxtimes M_2 \to M_3$ is completely characterized by an intertwining operator $\mathcal{Y}_m \in \mathcal{V}_{M_1 M_2}^{M_3}$ via

$$\overline{m}(m_1, m_2) = \mathcal{Y}_m(m_1, 1)m_2 \quad . \tag{2.39}$$

Let $\{U_i\}_{i\in\mathsf{I}}$ be the finite list of simple V-modules. Recall that $\mathcal{V}_{0j}^{j} \simeq \mathbb{C}$ and an isomorphism is given by the module vertex operator $\mathcal{Y}_{0j}^{j} = Y_{U_j}$. This induces a basis $\mathcal{Y}_{j0}^{j} \equiv \mathcal{B}_{-1}\left(\mathcal{V}_{0j}^{j}\right)$ of $\mathcal{V}_{j0}^{j}$. Next we fix a basis for $\mathcal{V}_{ii'}^{0}$ by $\mathcal{Y}_{ii'}^{0} = e^{\pi i h_j} A_0 \circ \mathcal{B}_0\left(\mathcal{Y}_{0i}^{i}\right)$. Recall from appendix A.6.1 that $\frac{1}{d_j} \equiv F^{(jj'j)j}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \neq 0$.

Theorem 2.0.17. *[83, Theorem 3.8]*

1) Let

$$\begin{aligned} \mathrm{ev}_i : & U_i \boxtimes U_i' \to V \qquad u_i \boxtimes w_i' \mapsto m_{\mathcal{Y}_{ii'}^{e}}(u_i, v_i') \\ \widetilde{\mathrm{coev}}_i : & V \to U_i \boxtimes U_i', \ \textit{s.th.}\ \mathrm{ev}_i \circ \widetilde{\mathrm{coev}}_i = d_i \mathrm{id}_V \end{aligned} \tag{2.40}$$

where $\widetilde{\mathrm{coev}}_i$ *exists due to* $U_i \boxtimes U_i' = \coprod_j N_{ii'}^{j} U_j = V \sqcup \coprod_{j\neq 0} N_{ii'}^{j} U_j$.

2) Let

$$\begin{aligned} \widetilde{\mathrm{ev}}_i : & U_i' \boxtimes U_i \to V \qquad u_i' \boxtimes w_i \mapsto m_{\mathcal{Y}_{i'i}^{e}}(u_i', v_i) \\ \mathrm{coev}_i : & V \to U_i' \boxtimes U_i, \ \textit{s.th.}\ \widetilde{\mathrm{ev}}_i \circ \mathrm{coev}_i = d_i \mathrm{id}_V \ . \end{aligned} \tag{2.41}$$

Then $(U_i', \mathrm{ev}_i, \mathrm{coev}_i)$ *defines a right dual for* U_i *and* $(U_i', \widetilde{\mathrm{ev}}_i, \widetilde{\mathrm{coev}}_i)$ *is a left dual for* U_i*. Hence* R_V *is a rigid braided monoidal category.*

Theorem 2.0.18. $\mathcal{R}_V$ *is pivotal.*

Proof. This is essentially [54, Proposition 5.3.1]. As vector spaces $(M')' = M$ and its module structure is given by

$$\begin{aligned} \langle Y_{M''}(u,x)m_1'', m_2' \rangle &= \left\langle Y_M\left(e^{\frac{1}{x}L_1}(-x^2)^{L_0} e^{xL_1}\left(-\frac{1}{x^2}\right)^{L_0} u, x\right) m_1, m_2' \right\rangle \\ &= \langle Y_M(u,x)m_1, m_2' \rangle \end{aligned} \tag{2.42}$$

using the definition of the contragradient vertex operator twice in the first equality. In the second equality we used that

$$\begin{aligned} c^{L_0}e^{aL_1}c^{-L_0} &= \sum_{k\geq 0}\frac{a^k}{k!}\left(c^{L_0}L_1c^{-L_0}\right)^k \\ &= \sum_{k\geq 0}\frac{a^k}{k!}\left(\frac{1}{c}L_1\right)^k \\ &= e^{\frac{a}{c}L_1} \end{aligned} \tag{2.43}$$

which follows from $c^{L_0}L_1c^{-L_0} = e^{\log(c)\mathrm{ad}_{L_0}}L_1 = e^{-\log(c)}L_1$.

Thus the natural isomorphism $M \overset{\cong}{\Rightarrow} (M')'$ is just the identity map. This is obviously a monoidal natural isomorphism. □

The ribbon structure is in fact easy to prove.

Theorem 2.0.19. *[83, Theorem 4.1] The map $\Theta_M = e^{2\pi \mathrm{i} L_0}$ defines a twist in R_V.*

Proof. It suffices to show the balancing isomorphism for simple objects.

$$\begin{aligned} \overline{\Theta_{ij}}(u_i \boxtimes w_j) &= e^{2\pi\mathrm{i}L_0}\mathcal{Y}_1(u_i,1)w_j \\ &= \mathcal{Y}_1\left(e^{2\pi\mathrm{i}L_0}u_i, x\right)|_{x^n=e^{-2\pi\mathrm{i}n}}e^{2\pi\mathrm{i}L_0}w_j \\ &= \overline{\beta^2}(\Theta_i u_i \boxtimes \Theta_j w_j) \end{aligned} \tag{2.44}$$

where in the last steps we used that $e^{n\ell_{\gamma^2}} = e^{-2\pi\mathrm{i}n}$, where γ^2 is the path for the double braiding going once around zero clockwise. Next $\Theta_V = \mathrm{id}_V$ because VOAs are integer graded and finally $\Theta_{U_i'} = \Theta_{U_i}$ since U_i and its contragradient module have the same conformal weight. □

Finally one has to show that R_V, i.e. that the S-matrix is non-degenerate. This is the next theorem

Theorem 2.0.20. *[83, Corollary 4.4] The categorical trace of $\beta^2 \in \mathrm{Hom}(U_i \boxtimes U_j, U_i \boxtimes U_j)$ is given by*

$$\mathrm{tr}\left(\beta^2\right) = \frac{S_{ij}}{S_{00}} \tag{2.45}$$

As S_{ij} is invertible this shows that R_V is a modular tensor category.

Chapter 3

Full Open-Closed RCFT

In chapter 2 we recalled how chiral data of a RCFT induces the structure of a modular tensor category on the representation category of its symmetry algebra. Of course conformal field theory is more than just chiral data and this chapter aims at describing a rigorous construction of full RCFTs in genus zero and one based on the chiral data described before. One starts by defining certain objects called *field algebras*, whose definitions are essentially a list of ingredients and axioms one expects to hold for sensible full conformal field theories. Such a wish list was already outlined in the introduction. There will be open, closed and open-closed field algebras. Each of these comes with a state space, a vertex or field map and suitable conditions on convergences and associativity of correlation functions.

This chapter is structured as follows. We start in section 3.1 by recalling the analytic definitions of field algebras and their properties. This was developed by Huang and Kong in a series of papers [85][103][87][105][104]. In section 3.1.1 a detailed discussion of open field algebras is given. In particular we discuss how axioms on abstract correlation functions and symmetry algebra actions for open field algebras naturally lead to representations of a VOA and intertwining operators. The latter will give an algebra object in a suitable representation category. In sections 3.1.2, 3.1.3 a similar treatment for closed and open-closed field algebras is given. Since the theory is conceptually almost the same as in the open case, its presentation is not as detailed. The focus is on various consistency requirements. In section 3.2 the analytic results are reformulated in terms of categorical algebra, which leads to Cardy algebras, the object of our interest.

3.1 Analytic Field Algebras

3.1.1 Open Field Algebras

We start with the definition of what we call an open field algebra [1].

Definition 3.1.1. [85] An *open field algebra (OFA)* $(H, \mathbb{Y}, \mathbf{1}, D)$ is the data of:

[1] These are called *open-string vertex algebras* in [85]

(1) An $\mathbb{R}$-graded vector space $H \equiv \coprod_{n\in\mathbb{R}} H_n$, with grading operator $\mathbf{d}(v) = nv$ for $v \in H_n$, where H_n is said to be of *weight* n. The vector space is required to be degree-wise finite dimensional $\dim(H_n) < \infty$ and lower truncated: $H_n = 0$ for n small enough.

(2) An *open vertex operator*

$$\begin{aligned} \mathbb{Y} : (H \otimes H) \times \mathbb{R}_+ &\to \overline{H} \\ (v \otimes w) \times r &\mapsto \mathbb{Y}(v, r)w \quad . \end{aligned} \tag{3.1}$$

which is bilinear in V. Here $\overline{H} \equiv \prod_{n\in\mathbb{R}} H_n$ is the algebraic completion of H. Compared to H, $\overline{H}$ contains infinite sums over weight spaces.

(3) An element $\mathbf{1} \in H$, called *vacuum.*

(4) A linear operator $D : V \to V$ of weight one, i.e. $D : V_n \to V_{n-1}$.

These have to satisfy the following list of axioms.

(OFAI) *Vertex weight property:* For any $s, t \in \mathbb{R}$, there exists an $N(s, t) \subset \mathbb{R}$, s.th.

$$\mathrm{Im}\left(\mathbb{Y}\left(\coprod_{n\in s+\mathbb{Z}} H_n \otimes \coprod_{m\in t+\mathbb{Z}} H_m, r\right)\right) \subset \overline{\coprod_{k\in N(s,t)+\mathbb{Z}} H_k} \tag{3.2}$$

for any $r \in \mathbb{R}_+$.

(OFAII) *Identity/ creation of vacuum:*

$$\mathbb{Y}(\mathbf{1}, r) = \mathrm{id}_H \tag{3.3}$$

for any $r \in \mathbb{R}_+$ and

$$\lim_{r\to 0} \mathbb{Y}(v, r)\mathbf{1} = v \tag{3.4}$$

for any $v \in H$.

(OFAIII) *Lower truncation property:* Let $H' = \coprod_{n\in\mathbb{R}} H'_n$ be the graded dual and $D' : H' \to H'$ be the adjoint of D under the usual degree-wise evaluation pairing. For any $v' \in H'$ there exists a $N_{v'} \in \mathbb{N}$ s.th.

$$(D')^{N'_v} v' = 0 \quad . \tag{3.5}$$

(OFAIV) *Absolute convergence of correlation functions:* Let $\pi_n : H \to H_n$ be the natural projection operator of vector spaces. For any $w, v_1, \ldots, v_n \in H$ and $v' \in H'$ the power series

$$\begin{aligned} &\langle v', \mathbb{Y}(v_n, r_n) \cdots \mathbb{Y}(v_1, r_1)w\rangle \\ &\qquad \equiv \sum_{m_1,\ldots,m_{n-1}\in\mathbb{R}} \left\langle v', \mathbb{Y}(v_n, r_n)\pi_{m_{n-1}}\mathbb{Y}(v_{n-1}, r_{n-1}) \cdots \pi_{m_1}\mathbb{Y}(v_1, r_1)w\right\rangle \end{aligned} \tag{3.6}$$

converges absolutely for $r_n > r_{n-1} > \cdots > r_1 > 0$. Similarly for any $v_1, v_2, w \in H$ and $v' \in H'$ the series

$$\langle v', \mathbb{Y}\left(\mathbb{Y}(v_1, r_1)v_2, r_2)\right) w\rangle \equiv \sum_{n\in\mathbb{R}} \langle v', \mathbb{Y}\left(\pi_n \mathbb{Y}(v_1, r_1)v_2, r_2\right) w\rangle \tag{3.7}$$

converges absolutely for $r_1 > r_2 > 0$.

(OFAV) *Associativity of correlation functions:* For any $v_1, v_2, w \in H$ and $v' \in H'$, there is an equality of functions

$$\langle v', \mathbb{Y}(v_1, r_1)\mathbb{Y}(v_2, r_2)W\rangle = \langle v', \mathbb{Y}\left(\mathbb{Y}(v_1, r_1 - r_2)v_2, r_2\right) w\rangle \tag{3.8}$$

on the domain $r_1 > r_2 > r_1 - r_2 > 0$.

(OFAVI) **d**-*derivative:*

$$[\mathbf{d}, \mathbb{Y}(v, r)] = \mathbb{Y}(\mathbf{d}v, r) + r\frac{\mathrm{d}}{\mathrm{d}r}\mathbb{Y}(u, r) \quad . \tag{3.9}$$

(OFAVII) *D-derivative:* The map $\mathbb{Y} : \mathbb{R}_+ \to \mathrm{Hom}(H, \overline{H})$ is differentiable and the derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}r}\mathbb{Y}(u, r) = [D, \mathbb{Y}(v, r)] = \mathbb{Y}(Dv, r) \quad . \tag{3.10}$$

Definition 3.1.2. (1) A *homomorphism of OFAs* is a grading preserving linear map $f : H \to \widetilde{H}$ s.th. $f(D) = \tilde{D}$, $f(\mathbf{1}) = \mathbf{1}$ and its canonical extension $\overline{f} : \overline{H} \to \overline{\widetilde{H}}$ satisfies

$$\overline{f}\left(\mathbb{Y}(v, r)w)\right) = \mathbb{Y}((f(v), r)\, f(w) \quad . \tag{3.11}$$

It is an *isomorphism* if f is an isomorphism of vector spaces.

(2) A *subalgebra* of an open string vertex algebra is a sub-vector space $U \subset H$ s.th. $\mathbf{1} \in U$ and $(U, \mathbb{Y}|_U, \mathbf{1}, D_U)$ is an OFA.

In contrast to the familiar definition of VOAs and their modules, in definition 3.1.1 it is not required that $\mathbb{Y}$ has some kind of Laurent expansion. However such an expansion will follow from (OFAVI) and (OFAI)[2]. The **d**-derivative property can be integrated to a conjugation formula, showing the behavior of the vertex map under rescaling of r. Let $v, w \in H$ and $v' \in H'$ be homogeneous elements, then (3.9) gives

$$\begin{aligned} \langle v', [\mathbf{d}, \mathbb{Y}(v, r)]\, w\rangle &= \langle \mathbf{d}'v', \mathbb{Y}(v, r)w\rangle - |w|\, \langle v', \mathbb{Y}(v, r)w\rangle \\ &= (|v'| - |w|)\, \langle v', \mathbb{Y}(v, r)w\rangle \\ &= \left(|v| + r\frac{\mathrm{d}}{\mathrm{d}r}\right) \langle v', \mathbb{Y}(v, r)w\rangle \quad . \end{aligned} \tag{3.12}$$

[2](OFAI) merely ensures that we can really sum over a countable subset of modes in the expansion.

Thus $\langle v', \mathbb{Y}(v, \bullet)w\rangle$ is a function on $\mathbb{R}_+$ satisfying the differential equation

$$(|v'| - |w| - |v|)\,\langle v', \mathbb{Y}(v,r)w\rangle = r\frac{\mathrm{d}}{\mathrm{d}r}\,\langle v', \mathbb{Y}(v,r)w\rangle \tag{3.13}$$

which has universal solution $\langle v', \mathbb{Y}(v,r)w\rangle = c_1 r^{(|v'|-|w|-|v|)}$. This yields for $x \in \mathbb{R}_+$

$$\begin{aligned}\left\langle v', x^{\mathbf{d}}\mathbb{Y}(v,r)x^{-\mathbf{d}}w\right\rangle &= \left(x^{|v'|-|w|}\right)\left(cr^{|v'|-|w|-|v|}\right)\\ &= \left(c(xr)^{|v'|-|w|-|v|}\right)(xr)^{|v|}\\ &= \left\langle v', \mathbb{Y}(x^{\mathbf{d}}v, xr)w\right\rangle \quad .\end{aligned} \tag{3.14}$$

The dilation formula for the open string vertex operator thus reads

$$x^{\mathbf{d}}\mathbb{Y}(v,r)x^{-\mathbf{d}}w = \mathbb{Y}(x^{\mathbf{d}}v, xr)w \quad . \tag{3.15}$$

With the help of the dilation formula, a Laurent series expansion of $\mathbb{Y}$ can be derived. Let $v, w \in H$ be homogeneous elements, then define

$$v^+_{(n)}w \equiv \pi_{|v|+|w|-n-1}\mathbb{Y}(v,1)w \quad . \tag{3.16}$$

From this on derives

$$\begin{aligned}\mathbb{Y}(v,r)w = r^{-|v|}r^{\mathbf{d}}\mathbb{Y}(v,1)wr^{-|w|} &= r^{-|v|}\sum_{n\in\mathbb{R}} r^n\pi_n\left(\mathbb{Y}(v,1)w\right)r^{-|w|}\\ &= r^{-|v|}\sum_{n\in\mathbb{R}} r^{|v|+|w|-n-1}\pi_{|v|+|w|-n-1}\left(\mathbb{Y}(v,1)w\right)r^{-|w|}\\ &= \sum_{n\in R} v^+_{(n)}w\, r^{-n-1} \quad .\end{aligned} \tag{3.17}$$

Note that there are only finitely many terms in the series due to the vertex weight property. Therefore open string vertex operators have a natural power series expansion

$$\mathbb{Y}(v,r) = \sum_{n\in\mathbb{R}} v^+_{(n)}r^{-n-1} \tag{3.18}$$

with $v^+_{(n)} \in \mathrm{Hom}(H,H)$ of degree $|v| - n - 1$. The power series form of the open string vertex operator can be used to define a formal vertex operator, by replacing $r \in \mathbb{R}$ in (3.18) with any formal variable q.

Proposition 3.1.3. *Let*

$$\mathbb{Y}_f(v,q) \equiv \sum_{n\in\mathbb{R}} v^+_{(n)}q^{-n-1} \tag{3.19}$$

be the formal open string vertex operator. Let p be another formal variable. Then

(1)

$$p^{\mathbf{d}}\mathbb{Y}_f(v,q)p^{-\mathbf{d}} = \mathbb{Y}_f(p^{\mathbf{d}}v, pq) \tag{3.20}$$

(2)

$$\mathbb{Y}_f(v, p+q) = e^{pD}\mathbb{Y}_f(v,q)e^{-pD} = \mathbb{Y}_f\left(e^{pD}v, q\right) \tag{3.21}$$

(3)

$$\mathbb{Y}_f(v,q)\mathbf{1} = e^{qD}v \tag{3.22}$$

Proof. Let $w \in V$ be a homogeneous element. Then

(1)

$$\begin{aligned} p^{\mathbf{d}}\mathbb{Y}_f(v,q)p^{-\mathbf{d}}w &= \sum_{n\in\mathbb{R}} p^{|w|+|v|-n-1}v^+_{(n)}q^{-n-1}p^{-|w|}w \\ &= \sum_{n\in\mathbb{R}} (p^{\mathbf{d}}v)^+_{(n)}(pq)^{-n-1}w \\ &= \mathbb{Y}_f(p^{\mathbf{d}}v, pq)w \quad . \end{aligned} \tag{3.23}$$

(2) First note that

$$\begin{aligned} [D, \mathbb{Y}_f(v,q)] &= \sum_{n\in\mathbb{R}} \left[D, v^+_{(n)}\right] q^{-n-1} \\ &= \sum_{n\in\mathbb{R}} (Dv)^+_{(n)}q^{-n-1} \\ &= \mathbb{Y}_f(Dv, q) \end{aligned} \tag{3.24}$$

where we can use (3.10) in the second equality. This gives

$$\begin{aligned} e^{pD}\mathbb{Y}_f(v,q)e^{-pD} &= \sum_{n\in\mathbb{R}} e^{pD}v_{(n)}e^{-pD}q^{-n-1} \\ &= \sum_{n\in\mathbb{R}} e^{\mathrm{ad}_{pD}}v_{(n)}q^{-n-1} \\ &= \sum_{n\in\mathbb{R}} \left(e^{pD}v\right)_{(n)} q^{-n-1} \\ &= \mathbb{Y}_f(e^{pD}v, q) \end{aligned} \tag{3.25}$$

One the other hand it holds

$$\begin{aligned} [D, \mathbb{Y}_f(v,q)] &= \sum_{n\in\mathbb{R}} \left[D, v^+_{(n)}\right] q^{-n-1} \\ &= \sum_{n\in\mathbb{R}} v^+_{(n-1)}(-n)q^{-n-1} \end{aligned} \tag{3.26}$$

which again follows from (3.10) after expanding $\mathbb{Y}(v,r)$. With the same calculation as before we derive

$$\begin{aligned}
e^{pD}\mathbb{Y}_f(v,q)e^{-pD} &= \sum_{n\in\mathbb{R}} e^{pD}v^+_{(n)}e^{-pD}q^{-n-1} \\
&= \sum_{n\in\mathbb{R}}\sum_{m\in\mathbb{N}} \frac{1}{m!}\left[pD\left[\cdots\left[pD, v^+_{(n)}\right]\cdots\right]\right]q^{-n-1} \\
&= \sum_{n\in\mathbb{R}}\sum_{m\in\mathbb{N}} \frac{(-1)^m n\cdot(n-1)\cdots(n-m+1)}{m!}v^+_{(n-m)}p^m q^{-n-1} \\
&= \sum_{n\in\mathbb{R}}\sum_{m\in\mathbb{N}} v^+_{(n)}\frac{(-1)^m(n+m)(n+m-1)\cdots n+1}{m!}p^m q^{-n-m-1} \\
&= \sum_{n\in\mathbb{R}} v^+_{(n)}(q+p)^{-n-1} \\
&= \mathbb{Y}_f(v,p+q)
\end{aligned} \tag{3.27}$$

where the generalized binomial formula for real exponents is used.

(3) Lastly, from (3.4) we infer $v^+_{(n)}\mathbf{1} = 0$ for $n \leq 0$ and $v^+_{(-1)}\mathbf{1} = v$. This gives

$$\begin{aligned}
\mathbb{Y}_f(v,q)\mathbf{1} &= \sum_{n\in\mathbb{R}_{<-1}} v^+_{(n)}q^{-n-1}\mathbf{1} \\
&= \sum_{n\in\mathbb{R}_{>0}} v^+_{(-n-1)}\mathbf{1}q^n \\
&= \sum_{n\in\mathbb{R}_{>0}} \frac{1}{\lceil n\rceil!}(D^{\lceil n\rceil}v)^+_{(-n+\lceil n\rceil-1)}\mathbf{1}q^n \\
&= \sum_{n\in\mathbb{N}} \frac{q^n}{n!}D^n v \\
&= e^{qD}v
\end{aligned} \tag{3.28}$$

□

The formal vertex operator therefore satisfies all the properties expected for an intertwining operator. In addition it allows us to require an embedding of the Virasoro algebra in an OFA.

Definition 3.1.4. [85, Definition 1.10] An OFA is *conformal* if there exists $\omega \in H$ s.th. its open vertex operator is a power series expansion

$$\mathbb{Y}(\omega, r) = \sum_{n\in\mathbb{Z}} L_n r^{-n-2} \tag{3.29}$$

and its modes satisfy the Virasoro relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n,m} \tag{3.30}$$

for some central charge $c \in \mathbb{C}$. In addition, ω acts on any $v \in H$ in terms of an operator product expansion

$$[\mathbb{Y}_f(\omega, q_1), \mathbb{Y}_f(v, q_2)] = \mathrm{Res}_{q_0} \delta\left(\frac{q_1 - q_0}{q_2}\right) \mathbb{Y}_f\left(\mathbb{Y}_f(\omega, q_0)v, q_2\right) \tag{3.31}$$

and H is *conformally graded*, i.e. $\mathbf{d} = L_0$ and $D = L_{-1}$.

Note that the following proposition is immediate from analytic continuation A.4.1 together with absolute convergence and associativity of correlation functions.

Proposition 3.1.5. *For $v_1, v_2, w \in H$, $v' \in H'$ and $z_1, z_2 \in \mathbb{C}$ the functions*

$$\begin{aligned} &\langle v', \mathbb{Y}_f(v_1, z_1)\mathbb{Y}_f(v_2, z_2)w\rangle \\ &\langle v', \mathbb{Y}_f\left(\mathbb{Y}_f(v_1, z_1)v_2, z_2\right) w\rangle \end{aligned} \tag{3.32}$$

are holomorphic functions in the regions $|z_1| > |z_2| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively, which agree in the domain $|z_1| > |z_2| > |z_1 - z_2| > 0$.

In order to describe a sensible physical theory correlation functions, however, should have analytic continuations to

$$\mathscr{D} = \left\{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 \neq 0 \neq z_2,\ z_1 \neq z_2\right\} \quad . \tag{3.33}$$

The idea to this result is showing that an OFA H has a certain VOA $C_0(H)$ with it, s.th. the formal open vertex operator becomes an intertwining operator of type $\binom{H}{HH}$. Assuming there exists an injective embedding $V \hookrightarrow C_0(H)$ of a rational VOA V, the existence of the analytic continuation follows from the general theory of intertwining operator algebras (see appendix A.4). $C_0(H)$ is called *meromorphic center* and as a vector space it is defined as

$$\begin{aligned} C_0(H) \equiv \Big\{ v \in H \mid \forall u \in H :\ &\mathbb{Y}_f(v, x)u \in \overline{H}[\![x, x^{-1}]\!], \text{ and} \\ &\mathbb{Y}_f(v, x)u = e^{xL_{-1}}\mathbb{Y}_f(u, -x)v \Big\} \end{aligned} \tag{3.34}$$

where x is some formal variable. Thus elements of $C_0(H)$ have formal operator valued Laurent series. The key result is that four point functions involving at least one element of the meromorphic center can be analytically continued.

Theorem 3.1.6. *[85, Proposition 2.2] Let H be an OFA. For $v \in C_0(H)$, $u, w_1 \in H$, $w_2' \in H'$ there exists an analytic function $\phi(v, u, w_1, w_2'; z_1, z_2)$ on $\mathrm{Conf}_2(\mathbb{C}^\times)$ s.th.*

$$\begin{aligned} \phi(v, u, w_1, w_2'; z_1, z_2)\big|_{|z_1|>|z_2|>0} &= \langle w_2', \mathbb{Y}_f(v, z_1)\mathbb{Y}_f(u, z_2)w_1\rangle \\ \phi(v, u, w_1, w_2'; z_1, z_2)\big|_{|z_2|>|z_1|>0} &= \langle w_2', \mathbb{Y}_f(u, z_2)\mathbb{Y}_f(v, z_1)w_1\rangle \\ \phi(v, u, w_1, w_2'; z_1, z_2)\big|_{|z_2|>|z_1-z_2|>0} &= \langle w_2', \mathbb{Y}_f\left(\mathbb{Y}_f(v, z_1 - z_2)u, z_2\right) w_1\rangle \\ \phi(v, u, w_1, w_2'; z_1, z_2)\big|_{|z_1|>|z_1-z_2|>0} &= \langle w_2', \mathbb{Y}_f\left(\mathbb{Y}_f(u, z_2 - z_1)v, z_1\right) w_1\rangle \quad . \end{aligned} \tag{3.35}$$

Moreover, $\phi(v, u, w_1, w_2'; z_1, z_2)$ is single valued as a function of z_1 and has only poles at $z_1, z_2 = 0$, $z_1 = z_2$.

The proof is not that hard and mainly uses the skew-symmetry property for elements in $C_0(H)$ to relate the first two cases of restrictions above.

The major result about conformal OFAs is the following theorem.

Theorem 3.1.7. *[85, Theorem 2.3, Proposition 2.5] Let H be a conformal OFA. Then*

1) $C_0(H)$ is a VOA.

2) H is a $C_0(H)$-module.

3) $\mathbb{Y}_f$ is a $C_0(H)$-intertwining operator of type $\binom{H}{HH}$.

Proof. Point 1) is explicitly proven in [85]. Property 2) directly follows from theorem 3.1.6. In order to show 3), one only needs to check the Jacobi-identity, all other properties of an intertwining operator are automatically satisfied. The Jacobi-identity is a formal variable analog of a contour integral argument. For the reader's convenience we give a proof here. Let $g_{m,k,\ell}(z_1,z_2) = z_1^m z_2^k (z_1-z_2)^\ell$ for $m,k,\ell \in \mathbb{Z}$. In addition, let $R_1, R_2, r_1 \in \mathbb{R}_+$ be s.th. $R_1 > R_2 > r_1 > 0$ and $R_2 > R_1 - r_1 > 0$. Then for any $v \in C_0(H)$, $u, w_1 \in H$, $w_2' \in H'$ it holds

$$\begin{aligned}\int_{|z_2|=R_2}\int_{|z_1|=R_1} &\phi(v,u,w_1,w_2';z_1,z_2) g_{m,k,\ell}(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2 \\ &- \int_{|z_2|=R_2}\int_{|z_1|=r_1} \phi(v,u,w_1,w_2';z_1,z_2) g_{m,k,\ell}(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2 \\ = \int_{|z_2|=R_2}\int_{|z_1-z_2|=R_{12}} &\phi(v,u,w_1,w_2';z_1,z_2) g_{m,k,\ell}(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2\end{aligned} \tag{3.36}$$

where $R_{12} = R_1 - r_1$. In each of the integrals we can insert the restrictions of $\phi(v,u,w_1,w_2;z_1,z_2)$ on the respective domains. Since w_1, w_2' were arbitrary this in fact yields

$$\begin{aligned}\int_{|z_2|=R_2}\int_{|z_1|=R_1} &\mathbb{Y}_f(v,z_1)\mathbb{Y}_f(u,z_2) g_{m,k,\ell}(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2 \\ &- \int_{|z_2|=R_2}\int_{|z_1|=r_1} \mathbb{Y}_f(u,z_2)\mathbb{Y}_f(v,z_1) g_{m,k,\ell}(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2 \\ = \int_{|z_2|=R_2}\int_{|z_1-z_2|=R_{12}} &\mathbb{Y}_f\left(\mathbb{Y}_f(u,z_1-z_2)v,z_2\right) g_{m,k,\ell}(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2\end{aligned} \tag{3.37}$$

Inserting the Laurant expansion of the fields and $g_{m,k,\ell}$ and using that for $s \in \mathbb{C}$

$$\int_{|z|=1} z^s \mathrm{d}z = \int_{|z|=1} e^{\log(z)s}\mathrm{d}z = \begin{cases} \int_{|z|=1} \frac{1}{s+1}\frac{\mathrm{d}}{\mathrm{d}z} e^{\log(z)s}\mathrm{d}z = 0,\ s \neq -1 \\ 2\pi\mathrm{i},\ s = -1 \end{cases} \tag{3.38}$$

yields for (3.37)

$$\begin{aligned}\sum_{j\geq 0} (-1)^j \binom{\ell}{j} v_{(\ell-j+k)} u_{(j+m)} + (-1)^{\ell+j)} \binom{\ell}{j} u_{(\ell-j+k)} v_{(j+m)} \\ = \sum_{j\geq 0} \binom{m}{j} \left(v_{(j-\ell)} u\right)_{(m-j+k)} \quad .\end{aligned} \tag{3.39}$$

Equation (3.39) is also known as *Borcherd's identity.* On the other hand, multiplying the terms of the Jacobi identity with $g_{m,k,\ell}$ and applying $\mathrm{Res}_{z_1}\mathrm{Res}_{z_2}$ also reproduces (3.39). Since the Cauchy pairing (3.38) is non degenerate this proves the claim. $\square$

In general $C_0(H)$ is not a rational VOA and the tensor product theory and analytic properties of rational VOA theory cannot be used. Thus a refined version of an OFA is needed.

Definition 3.1.8. Let V be a rational VOA. An *OFA over* V is an OFA $(H, \mathbb{Y})$ together with an injective map of VOAs: $\iota : V \hookrightarrow C_0(H)$.

Due to theorem 3.1.7, the embedding ι yields a V-module structure on H and $\mathbb{Y}_f$ becomes a V-intertwining operator. Thus the naturally existing analytic continuations

$$\langle v', \mathbb{Y}_f(v_1, z_1) \cdots \mathbb{Y}_f(v_n, z_1) w \rangle \tag{3.40}$$

defined on $|z_1| > \cdots > |z_n| > 0$ have multivalued analytic continuations to $\mathrm{Conf}_n(\mathbb{C}^\times)$ by the general theory of intertwining operator algebras.

The following is immediate from theorem 3.1.7 and the properties of an IOA.

Corollary 3.1.9. *For V a rational VOA there is an isomorphism of categories*

Conformal OFAs $(H, \mathbb{Y}, \mathbf{1}, D)$, s.th. there is an injective homomorphism $\iota : V \hookrightarrow C_0(H)$ of VOAs.	$\xleftrightarrow{\simeq}$	*V-modules H with an intertwining operator $\mathbb{Y}$ of type $\binom{H}{HH}$ satisfying the creation and identity property together with an injective embedding $V \hookrightarrow C_0(H)$*

Later we will construct full conformal field theories from symmetric Frobenius algebras in R_V and conformal OFAs are particular instances for such algebras. In this interpretation the meromorphic center gets a natural interpretation. It is the maximally extended chiral symmetry algebra of the theory. Since OFAs give modules over the meromorphic center, they correspond to a maximally symmetry preserving boundary condition. This requirement can be dropped and boundary conditions may only preserve the fixed chiral symmetry V not the maximally extended symmetry algebra.

Lastly for defining Frobenius algebras later we need the notion of an invariant bilinear form on H.

Definition 3.1.10. [104, section 1.3] Let $r < 0$, then define

$$\mathbb{Y}(v, r)w \equiv e^{rL_{-1}} \mathbb{Y}(w, -r)v \tag{3.41}$$

where $w, v \in H$ and $(H, \mathbb{Y})$ is an OFA. An *invariant bilinear form* on $(H, \mathbb{Y})$ is a bilinear map

$$(\,\bullet\,,\,\bullet\,) : H \times H \to \mathbb{C} \tag{3.42}$$

s.th.

$$\begin{aligned} (u, \mathbb{Y}(v, r)w) &= \left(\mathbb{Y}\left(e^{-rL_1}(r^{-2})^{L_0} v, -\frac{1}{r}\right) u, w\right) \\ (\mathbb{Y}(v, r)u, w) &= \left(u, \mathbb{Y}\left(e^{-rL_1}(r^{-2})^{L_0} v, -\frac{1}{r}\right) w\right) \quad . \end{aligned} \tag{3.43}$$

If $(H, \mathbb{Y})$ is an OFA over a rational VOA V this induces a V-module map $A : H \to H'$, where H' is equipped with the contragradient module structure. A is an isomorphism precisely when the bilinear form is non-degenerate.

3.1.2 Closed Field Algebras

In this section we give the definition of a closed field algebra (CFA) over a VOA V. We give a definition in terms of a V-module C and an intertwining operator of type $\binom{H}{HH}$. The main theorem of [86] is that this is equivalent to a conformal full field algebra which is the data of a state space plus some correlation function mappings obeying some convergence properties to be expected from full two dimensional conformal field theory. Since we don't need the technical details of full field algebras we will be terse in their description. Nevertheless, they serve as the main physical motivation for the whole construction.

Full conformal field theory has left and right moving modes. Thus chiral objects such as intertwining operators need to be extended to include anti-chiral data. First note that for V^L, V^R rational VOAs, the tensor product VOA $V^L \otimes V^R$ is also rational. By [54, Theorem 4.7.4] it in fact follows that any $V^L \otimes V^R$-module M decomposes as a direct sum

$$M \simeq \bigoplus_{i \in \mathsf{I}_L, j \in \mathsf{I}_R} M_{ij}\, U_i^L \otimes U_j^R = \bigoplus_{\ell=1}^{K} U^L_{\nu^L(\ell)} \otimes U^R_{\nu^R(\ell)} = \bigoplus_n M_n^L \otimes M_n^R \tag{3.44}$$

where $\nu^{L,R} : 1, 2, \cdots, K \to \mathsf{I}^{L,R}$ are some functions. In addition for $i = 1, 2, 3$ and $M_i = M_i^L \otimes M_i^R$ being $V^L \otimes V^R$-modules it was shown in [44] that $\mathcal{V}^{M_3}_{M_1 M_2} \simeq \mathcal{V}^{M_3^L}_{M_1^L M_2^L} \otimes \mathcal{V}^{M_3^R}_{M_1^R M_2^R}$. We call this the *splitting property.*

Definition 3.1.11. Let V^L, V^R be rational VOAs, M a $V^L \otimes V^R$-module and $\mathcal{Y}$ a $V^L \otimes V^R$-intertwining operator of type $\binom{M}{MM}$. The *full vertex operator* associated to $\mathcal{Y}$ is given by

$$\begin{aligned} \mathbb{Y}^{\mathcal{Y}} : M \otimes M &\to M\{x, y\} \\ m_1 \otimes m_2 &\mapsto \mathbb{Y}^{\mathcal{Y}}(m_1; x, y) m_2 \end{aligned} \tag{3.45}$$

with

$$\mathbb{Y}^{\mathcal{Y}}(m_1; x, y) m_2 \equiv x^{L_0^L} y^{L_0^R} \mathcal{Y}(m_1, 1) x^{-L_0^L} y^{-L_0^R} \tag{3.46}$$

By the splitting property for $m_1^L \otimes m_1^R \in M_{i_1}^L \otimes M_{i_1}^R$ and $m_2^L \otimes m_2^R \in M_{i_2}^L \otimes M_{i_2}^R$ there exist intertwining operators $\mathcal{Y}_k^L \in \mathcal{V}^{M_k^L}_{M_{i_1}^L M_{i_2}^L}$ and $\mathcal{Y}_k^R \in \mathcal{V}^{M_k^R}_{M_{i_1}^R M_{i_2}^R}$ s.th.

$$\mathbb{Y}^{\mathcal{Y}}(m_1^L \otimes m_1^R; x, y)(m_2^L \otimes m_2^R) = \sum_k \mathcal{Y}_k^L(m_1^L, x) m_2^L \otimes \mathcal{Y}_k^R(m_1^R, y) m_2^R \quad . \tag{3.47}$$

Definition 3.1.12. [86] Let V^L, V^R be rational VOAs. A *closed field algebra (CFA)* over $V^L \otimes V^R$ is a triple $(C, \mathcal{Y}, \iota)$, where C is a $V^L \otimes V^R$-module, $\mathcal{Y}$ is an intertwining operator of type $\binom{C}{CC}$ and $\iota : V^L \otimes V^R \hookrightarrow C$ is an injective map of vector spaces. This data has to satisfy

CFAI) *Identity and creation property:*

$$\boldsymbol{\mathcal{Y}}(\iota(\mathbf{1}_L \otimes \mathbf{1}_R), x) = \mathrm{id}_C \tag{3.48}$$

and for all $c \in C$

$$\lim_{x \to 0} \boldsymbol{\mathcal{Y}}(c, x)\iota(\mathbf{1}_L \otimes \mathbf{1}_R) = c \tag{3.49}$$

CFAII) *Associativity:* Let c_1, c_2, $w \in C$ and $v' \in C'$. Then

$$\left\langle v', \mathbb{Y}^{\mathcal{Y}}(c_1; z_1, \xi_1)\mathbb{Y}^{\mathcal{Y}}(c_2; z_2, \xi_2)w \right\rangle = \left\langle v', \mathbb{Y}^{\mathcal{Y}}(\mathbb{Y}^{\mathcal{Y}}(c_1; z_1 - z_2, \xi_1 - \xi_2)c_2; z_2, \xi_2)w \right\rangle \tag{3.50}$$

for $|z_1| > |z_2| > |z_1 - z_2| > 0$ and $|\xi_1| > |\xi_2| > |\xi_1 - \xi_2| > 0$.

CFAIII) *Single valuedness:*

$$e^{2\pi \mathrm{i}(L_0^L - L_0^R)} = \mathrm{id}_C \tag{3.51}$$

CFAIV) *Skew symmetry:* For c_1, c_2 in C it holds

$$\mathbb{Y}^{\mathcal{Y}}(c_1; x, y)c_2 = e^{L_{-1}^L} e^{L_{-1}^R} \mathbb{Y}^{\mathcal{Y}}(c_1; e^{\pi \mathrm{i}}x, e^{-\pi \mathrm{i}}y)c_2 \tag{3.52}$$

The axioms for a CFA follow the by now familiar pattern of having a vertex operator map and an associativity condition. Note that no convergence property needs to be assumed since by the splitting property any product correlation functions for $v' = (v^L)' \otimes (v^R)'$, $c_i = c_i^L \otimes c_i^R$, $w = w^L \otimes w^R$ decomposes

$$\begin{aligned} &\left\langle v', \mathbb{Y}^{\mathcal{Y}}(c_1; z_1, \xi_1) \cdots \mathbb{Y}^{\mathcal{Y}}(c_n; z_n, \xi_n)w \right\rangle \\ &= \sum \left\langle (v^L)', \mathcal{Y}_{k_1}^L(c_1^L, z_1) \cdots \mathcal{Y}_{k_n}(c_n^L, z_n)w^L \right\rangle \left\langle (v^R)', \mathcal{Y}_{\ell_1}^R(c_1^R, \xi_1) \cdots \mathcal{Y}_{\ell_n}(c_n^R, \xi_n)w^R \right\rangle \ . \end{aligned} \tag{3.53}$$

The products in (3.53) are absolutely convergent for $|z_1| > \cdots > |z_n| > 0$ and $|\xi| > \cdots > |\xi_1| > 0$ by the properties of an IOA. What is new about the definition is the single valuedness. It corresponds to the fact that correlation functions should not have monodromies for transporting field insertions. Let us elaborate on this point. The goal is to derive from (3.53) single valued functions

$$\begin{aligned} \mu_n : C^{\otimes n} \times \mathrm{Conf}_n\left(\mathbb{C}^\times\right) &\to \overline{C} \\ (c_1, \cdots, c_n; z_1, \cdots, z_n) &\mapsto \mu_n(c_1, \cdots, c_n; z_1, \overline{z}_1, \cdots, z_n, \overline{z}_n) \end{aligned} \tag{3.54}$$

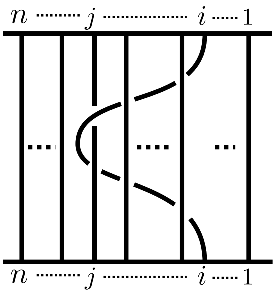
which are smooth in $(z_i, \overline{z}_i)$ for $i = 1, \cdots, n$. Pairing (3.54) with any element $v' \in C'$ would give a true $n+1$-point function in a full conformal field theory with state space C and couplings among fields described by $\boldsymbol{\mathcal{Y}}$. The absolutely convergent function (3.53) has a multivalued analytic continuation

$$\langle v', \widetilde{\mu_n}(c_1, \cdots, c_n; \bullet, \bullet)w \rangle \tag{3.55}$$

to $\mathrm{Conf}_n(\mathbb{C}^\times) \times \mathrm{Conf}_n(\mathbb{C}^\times)$. Let $(n, n-1, \cdots, 1) \in \mathrm{Conf}_n(\mathbb{C}^\times)$ and $w = \mathbf{1}$. The analytic extension $\widetilde{\mu_n}$ evaluated at $(n, n-1, \cdots 1)$ is uniquely fixed by (3.53) and $\widetilde{\mu_n}(c_1, \cdots, c_n; z_1, \xi_1, \cdots, z_n, \xi_n)$ can be obtained from paths $\gamma_z : (n, \cdots, 1) \to (z_1, \cdots, z_n)$ and $\gamma_\xi : (n, \cdots, 1) \to (\xi_1, \cdots, \xi_n)$ in $\mathrm{Conf}(\mathbb{C}^\times)$. Choosing $\gamma_\xi = \gamma_{\overline{z}}$ one defines

$$\mu_n(c_1, \cdots, c_n; z_1, \overline{z}_1, \cdots, z_n, \overline{z}_n) = \widetilde{\mu_n}^{\gamma_z \times \gamma_{\overline{z}}}(c_1, \cdots, c_n; z_1, \xi_1, \cdots, z_n, \xi_n) \tag{3.56}$$

where the rhs is the analytic continuation determined by $\gamma_z \times \gamma_{\overline{z}}$. In order for this to be well defined the analytic continuation has to be independent of the path γ_z. In [86, Theorem 2.11] it is shown that this is the case if and only if the single valuedness property holds. Since analytic continuations only depend on homotopy classes of paths and the fundamental group $\pi_1(\mathrm{Conf}_n(\mathbb{C}^\times), (n, \cdots, 1))$ is generated by braids[3]



this is invariance under monodromies. Furthermore, this shows that μ_n is invariant under permutations, i.e. for $\sigma \in \Sigma_n$ it holds

$$\mu_n(c_{\sigma(1)}, \cdots, c_{\sigma(n)}; z_{\sigma(1)}, \overline{z_{\sigma(1)}}, \cdots, z_{\sigma(n)}, \overline{z_{\sigma(n)}}) = \mu_n(c_1, \cdots, c_n; z_1, \overline{z}_1, \cdots, z_n, \overline{z}_n) \quad . \tag{3.57}$$

Finally by the associativity property of IOA one proves by induction that the correlation functions satisfy

$$\begin{aligned} \mu_n(c_1, \cdots, c_{n-1}, \mu_\ell(c_1^n, \cdots, c_\ell^n; w_1, \overline{w_1}, \cdots, w_\ell, \overline{w_\ell}); z_1, \overline{z_1}, \cdots, z_n, \overline{z_n}) \\ = \mu_{n+\ell-1} \end{aligned} \tag{3.58}$$

It turns out that arguments also work in the other direction [86, Theorem 2.11], i.e. giving a closed state space C and correlation functions (3.54) which satisfy the natural convergence property stated in the introduction and have an action of a VOA symmetry algebra produces a CFA. So a physically reasonable list of requirements for full RCFTs lead to the mathematically precise notion of a CFA, for which all the rigorous technical tools of VOA theory can be applied.

Note that due to corollary 3.1.9 and the splitting property, conformal full field algebras are closely related to OFAs.

Corollary 3.1.13. *[103, Corollary 3.3] For V^L, V^R rational VOAs there is an isomorphism of categories*

[3] These are exactly loops in configuration space.

$$\begin{array}{lcl} \textit{Conformal OFAs } (H, \mathbb{Y}, \mathbf{1}, D) & & \\ \textit{over } V^L \otimes V^R & \stackrel{\simeq}{\longleftrightarrow} & \textit{CFAs } (H, \mathcal{Y}, \iota). \end{array}$$

For later purposes we need non-degenerate bilinear forms on C. These will be the crucial ingredient for lifting associative algebras in representation categories to Frobenius algebras.

Definition 3.1.14. [103] Let $(C, \mathcal{Y}, \iota)$ be a CFA. An *invariant bilinear form* is a bilinear pairing

$$(\bullet, \bullet) : C \times C \to \mathbb{C} \tag{3.59}$$

s.th.

$$\left(\mathbb{Y}^{\mathcal{Y}}(c; e^{\pi \mathrm{i}}x, e^{-\pi \mathrm{i}}y)v, w\right) = \left(v, \mathbb{Y}^{\mathcal{Y}}\left(\left(e^{xL_1^L}(x^{-2})^{L_0^L} \otimes e^{yL_1^R}(y^{-2})^{L_0^R}\right)c; \frac{1}{x}, \frac{1}{y}\right)\right) \tag{3.60}$$

holds for all $v, w, c \in C$.

As in the open case this induces a $V^L \otimes V^R$-module map $A_{cl} : C \to C'$ which is an isomorphism if and only if the bilinear form is non-degenerate.

Modular Invariance of CFAs

As discussed in the previous section, correlation functions of CFA are monodromy invariant. Put differently, they are invariant under Dehn twist around punctures of spheres, which is tantamount to saying they are invariant under the mapping class group of an n-punctured sphere. When extending CFA to tori, correlation functions should be invariant under the action of the modular group. The construction of the genus one extension was done in [87]. It is closely related to the discussion of chiral genus one correlation functions recalled in section A.6.1. We freely use the notation introduced there.

Definition 3.1.15. Let $(C, \mathcal{Y}, \iota)$ be CFA over $V^L \otimes V^R$. The *genus one formal correlation functions* are defined as

$$\begin{aligned} &\mu_n^1(c_1, \cdots, c_n; x_1, y_1, \cdots, x_n, y_n; \tau_L, \tau_R) \\ &\equiv \mathrm{tr}_C \Big(\mathbb{Y}^{\mathcal{Y}}(\rho_x^L \rho_{-y}^R e^{-\pi \mathrm{i} L_0^R} c_1; e^{2\pi \mathrm{i} x_1}, e^{-2\pi \mathrm{i} y_1}) \cdots \\ &\qquad \mathbb{Y}^{\mathcal{Y}}(\rho_x^L \rho_{-y}^R e^{-\pi \mathrm{i} L_0^R} c_n; e^{2\pi \mathrm{i} x_n}, e^{-2\pi \mathrm{i} y_n}) q_{\tau_L}^{L_0^L - \frac{c_L}{24}} q_{\tau_R}^{L_0^R - \frac{c_R}{24}}\Big) \end{aligned} \tag{3.61}$$

where $\{x_i, y_i\}$ are formal variables, $c_i \in C$ and $\tau_L, \tau_R \in \mathbb{H}$ [4].

Let $1 > |q_{z_1}| > \cdots > |q_{z_n}| > |q_\tau| > 0$. The splitting property and the results presented in section A.6.1 immediately give that

$$\mu_n^1(c_1, \cdots, c_n; z_1, \overline{z}_1, \cdots, z_n, \overline{z}_n; \tau, \overline{\tau}) \tag{3.62}$$

[4] One may wonder about the extra factor of $e^{-\pi \mathrm{i} L_0^R}$ in (3.61). In case $y = \overline{z}$ it holds $\overline{\rho_z} = \rho_{-\overline{z}} e^{-\pi \mathrm{i} L_0^R}$, which explains the extra factor in the general situation.

absolutely converges and has a multivalued analytic extension M_n^1 to $\mathrm{Conf}_n(\mathsf{Teich}_1) \times \mathrm{Conf}_n(\mathsf{Teich}_1)$. Here $\mathrm{Conf}_n(\mathsf{Teich}_1)$ is the fiber bundle over $\mathsf{Teich}_1 = \mathbb{H}$ with fiber over a point $\tau \in \mathbb{H}$ given by $\mathrm{Conf}_n(T_\tau)$, the configuration space of n-points on a torus with complex structure τ.

Note that the Virasoro rescaling property of intertwining operators, permutation invariance (3.57) and genus zero uniqueness readily imply the following.

Theorem 3.1.16. *[87, Corollary 3.4] For any $c_1, \cdots, c_n \in C$ the multivalued analytic extension M_n^1 restricts to a unique smooth function*

$$M_n^1(c_1, \cdots, c_n; z_1, \overline{z_1}, \cdots, z_n, \overline{z_n}; \tau, \overline{\tau}) \tag{3.63}$$

on $\mathrm{Conf}_n(\mathsf{Teich}_1)$.

In particular for all $a, b \in \mathbb{Z}$ it holds

$$\begin{aligned} M_n^1(c_1, \cdots, c_n; z_1, \overline{z_n}, \cdots, z_i + a\tau + b, \overline{z_i} + a\overline{\tau} + b, \cdots, z_n, \overline{z_n}; \tau, \overline{\tau}) \\ = M_n^1(c_1, \cdots, c_n; z_1, \overline{z_1}, \cdots, z_n, \overline{z_n}; \tau, \overline{\tau}) \quad . \end{aligned} \tag{3.64}$$

There is an obvious definition of what it means for a CFA to be modular invariant.

Definition 3.1.17. [87, Definition 3.5] A CFA $(C, \mathcal{Y}, \iota)$ is *modular invariant* if for any $c_1, \ldots, c_n \in C$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \in \quad \mathrm{SL}(2, \mathbb{Z}) \tag{3.65}$$

it holds

$$\begin{aligned} M_n^1\Bigg(\left(\frac{1}{c\tau + d}\right)^{L_0^L}\left(\frac{1}{c\overline{\tau} + d}\right)^{L_0^R} c_1, \cdots, \left(\frac{1}{c\tau + d}\right)^{L_0^L}\left(\frac{1}{c\overline{\tau} + d}\right)^{L_0^R} c_n; \\ \frac{z_1}{c\tau + d}, \frac{\overline{z_1}}{c\overline{\tau} + d}, \cdots, \frac{z_n}{c\tau + d}; \frac{\overline{z_n}}{c\overline{\tau} + d}; \frac{a\tau + b}{c\tau + d}, \frac{a\overline{\tau} + b}{c\overline{\tau} + d}\Bigg) \\ = M_n^1(c_1, \cdots, c_n; z_1, \overline{z_1}, \cdots, z_n, \overline{z_n}; \tau, \overline{\tau}) \ . \end{aligned} \tag{3.66}$$

The above definition is of course very unhandy. But the splitting property tells that there should be a way to express modular invariance in terms of chiral and antichiral data. First we decompose the state space into left and right movers

$$C = \bigoplus_{\ell=1}^{K} U^L_{\nu^L(\ell)} \otimes U^R_{\nu^R(\ell)} \tag{3.67}$$

which by the splitting property yields

$$\begin{aligned} \mathbb{Y}^{\mathcal{Y}}(\bullet^L \otimes \bullet^R; x, y) = \sum_{\ell_1, \ell_2, \ell_3 = 1}^{K} \sum_{\alpha=1}^{N^{\nu^L(\ell_3)}_{\nu^L(\ell_1)\nu^L(\ell_2)}} \sum_{\alpha_2=1}^{N^{\nu^L(\ell_3)}_{\nu^L(\ell_1)\nu^L(\ell_2)}} Y^{\ell_3}_{\ell_1\ell_2}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \\ \mathcal{Y}^{\nu^L(\ell_3);L}_{\nu^L(\ell_1)\nu^L(\ell_2);\alpha_1}(\bullet^L, x) \otimes \mathcal{Y}^{\nu^R(\ell_3);R}_{\nu^R(\ell_1)\nu^R(\ell_2);\alpha_2}(\bullet^R, y) \end{aligned} \tag{3.68}$$

for complex numbers $\left\{ Y_{\ell_1 \ell_2}^{\ell_3} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \right\}$ and basis elements $\mathcal{Y}_{ij}^{k;L,R} \in \mathcal{V}_{ij}^{k;L,R}$, where the basis for left and right movers can be different. The reader may compare this to (A.90), the expansion of a conformal field in terms of intertwining operators. One can interpret (3.68) as a formal expansion of an arbitrary conformal field, which reduces to an actual field upon inserting states in the bullet blanks. By theorem A.6.6 and Theorem A.6.8 in the appendix for any $c_i^L \in U_i^L$ there exist complex numbers $\left\{ S_{jk}^L \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\}$ s.th.

$$\Phi^1_{\mathcal{Y}^{j,L}_{ij;\alpha}}(\bullet; -\frac{z}{\tau}; q_{-\frac{1}{\tau}}) = \sum_{k \in \mathsf{I}} \sum_{\beta=1}^{N_{ik}^k} S_{jk}^L \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \Phi^1_{\mathcal{Y}^{k;L}_{ik;\beta}}(\bullet; z; q_\tau) \quad . \tag{3.69}$$

Note that the second sum was absent in the discussion of the appendix since there $i = 0$ was chosen. The numbers $\left\{ S_{jk} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\}$ form an invertible matrix in the sense that there exist $\left\{ (S^L)^{-1}_{i\ell} \begin{bmatrix} \gamma \\ \delta \end{bmatrix} \right\}$ s.th.

$$\sum_{k \in \mathsf{I}^L} \sum_{\beta=1}^{N_{jk}^k} S_{jk}^L \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (S^L)^{-1}_{k\ell} \begin{bmatrix} \beta \\ \gamma \end{bmatrix} = \delta_{j,\ell} \delta_{\alpha,\gamma} \tag{3.70}$$

Analogous expressions hold for right movers.

Since correlation functions M_n^1 are uniquely determined by their restrictions on $1 > |q_{z_1}| > \cdots |q_{z_n}| > |q_\tau| > 0$ where they equal products of formal full vertex operators one can use the associativity property of intertwining operators to reduce modular invariance for all $n \geq 1$ to modular invariance for $n = 1$. Chasing through the splittings and taking extra factors of $e^{\pi \mathrm{i} L_0^R}$ for right movers into account one arrives at the follow.

Theorem 3.1.18. *[87, Theorem 3.6,3.8] A CFA over $V^L \otimes V^R$ is modular invariant if $c^L - c^R = 0 \bmod 24$ and*

$$\sum_{k=1}^{K} \sum_{\beta_1=1}^{N^{\nu^R(k}_{\nu^R(\ell)\nu^R(k)}} \sum_{\alpha_1=1}^{N^{\nu^L(k)}_{\nu^L(\ell)\nu^L(k)}} Y_{\ell k}^k \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} S^L_{\nu^L(k) i^L} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} (S^R)^{-1}_{\nu^r(k) j^R} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \sum_{m \in (\nu^L)^{-1}(i^L) \cap (\nu^R)^{-1}(j^R)} Y_{\ell m}^m \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} \tag{3.71}$$

holds for any $\ell \in \mathsf{I}$.

The condition on central charges ensures modular invariance under the T-transformation, which can be easily seen by shifting $\tau \to \tau + 1$ in (3.62). The second condition is invariance under the S-transformation.

3.1.3 Open-Closed Field Algebras

The final object to discuss are open-closed field algebras. They are mathematical formulations of rational full conformal field theories with boundary fields. We again start with a vertex operator type definition and discuss how this leads to a structure of closed and open state spaces H_{cl}, H_{op} together with a collection of correlation function mappings $\{\mu_{m,n}\}$ for all possible open-closed interactions. These correlation functions satisfy some natural consistency requirements.

Open-closed field algebras were introduced in [105]. The definition we give here is spread out over several definitions in [105].

Definition 3.1.19. Let V be a rational VOA. An *open-closed field algebra (OCFA) over* V is a tuple $(H_{cl}, \mathcal{Y}, \iota, H_{op}, \mathbb{Y}, \mathbb{Y}_{cl-op})$, where $(H_{cl}, \mathcal{Y}, \iota)$ is a CFA over $V \otimes V$, $(H_{op}, \mathbb{Y})$ is an OFA over V and the closed-open vertex operator $\mathbb{Y}_{cl-op}$ is a map[5]

$$\begin{aligned} \mathbb{Y}_{cl-op} : H_{cl} \otimes H_{op} \times \mathbb{H} \times \overline{\mathbb{H}} &\to \overline{H_{op}} \\ (c, o; \zeta, \omega) &\mapsto \mathbb{Y}_{cl-op}(c; \zeta, \omega)o \quad . \end{aligned} \tag{3.72}$$

This data has to satisfy

OCFAI) *Convergence 1:* For any $c_1, \cdots, c_n \in H_{cl}$, $o_1, \cdots, o_n \in H_{op}$ and $v' \in H'_{op}$, $\omega \in H_{op}$

$$\langle v', \mathbb{Y}_{cl-op}(c_1; \zeta_1, \omega_1)\mathbb{Y}(o_1, r_1) \cdots \mathbb{Y}_{cl-op}(c_n; \zeta_n, \omega_n)\mathbb{Y}(o_n, r_n)\omega\rangle \tag{3.73}$$

converges absolutely if $|\zeta_1|, |\omega_1| > |r_1| > \cdots > |\zeta_n|, |\omega_n| > |r_n| > 0$ and has a multivalued analytic extension to $\mathrm{Conf}_{3n}(\mathbb{C}^\times)$.

OCFAII) *Convergence 2:* For any $v' \in H'_{op}$, $o_1, o_2 \in H_{op}$ and $c \in H_{cl}$

$$\langle v', \mathbb{Y}(\mathbb{Y}_{cl-op}(c; \zeta, \omega)o_1; r)o_2\rangle \tag{3.74}$$

converges absolutely if $r > |\zeta|, |\omega| > 0$.

OCFAIII) *Convergence 3:* For any $v' \in H'_{op}$, $o \in H_{op}$ and $c_1, c_2 \in H_{cl}$

$$\left\langle v', \mathbb{Y}_{cl-op}(\mathbb{Y}^{\mathcal{Y}}(c_1; z_1, \xi_1)c_2; \zeta_2, \omega_2)o\right\rangle \tag{3.75}$$

converges absolutely if $|\zeta_2| > |z_1| > 0$, $|\omega_2| > |\xi_1| > 0$ and $|z_1| + |\xi_1| < |\zeta_2 - \omega_2|$.

OCFAIV) *Associativity 1:* For any $c_1, \in H_{cl}$, $o_1, o_2 \in H_{op}$ and $v' \in H'_{op}$ it holds

$$\langle v', \mathbb{Y}_{cl-op}(c_1; \zeta_1, \omega_1)\mathbb{Y}(o_1, r_1)o_2\rangle = \langle v', \mathbb{Y}(\mathbb{Y}_{cl-op}(c; \zeta, \omega)o_1; r)o_2\rangle \tag{3.76}$$

on $|\zeta|, |\omega| > r > 0$ and $r > |r - \zeta|, |r - \omega| > 0$.

OCFAV) *Associativity 2:* For $v\prime \in H_{op}$, $o \in H_{op}$ and $c_1, c_2 \in H_{cl}$ it holds

$$\langle v', \mathbb{Y}_{cl-op}(c_1; \zeta_1, \omega_1)\mathbb{Y}_{cl-op}(c_2; \zeta_2, \omega_2)o\rangle = \left\langle v', \mathbb{Y}_{cl-op}\left(\mathbb{Y}^{\mathcal{Y}}(c_1; \zeta_1 - \zeta_2, \omega_1 - \omega_2)c_2; \zeta_2, \omega_2\right) o\right\rangle \tag{3.77}$$

for $|\zeta_1|, |\omega_1| > |\zeta_2|, |\omega_2|$, $|\zeta_2| > |\zeta_1 - \zeta_2|$, $|\omega_2| > |\omega_1 - \omega_2|$ as well as $|\zeta_1 - \zeta_2| + |\omega_1 - \omega_2| < |\zeta_2 - \omega_2|$.

[5] The use of horizontal bars in the formula can be confusing. On $\mathbb{H}$ it means complex conjugation, whereas for H_{op} it is the algebraic completion.

OCFAVI) *Commutativity 1:* Let $\widehat{\mathbb{H}} = \mathbb{H} \cup \mathbb{R}_+$, $\widehat{\overline{\mathbb{H}}} = \overline{\mathbb{H}} \cup \mathbb{R}_+$, from OCFA1) it in fact follows that $\mathbb{Y}_{cl-op}$ has an extension to $\widehat{\mathbb{H}} \cup \widehat{\overline{\mathbb{H}}}$. Then for any $v' \in H'_{op}$, $o_1, o_2 \in H_{op}$ and $c \in H_{cl}$ and $\zeta, \omega \in \mathbb{R}_+$

$$\langle v', \mathbb{Y}_{cl-op}(c; \zeta, \omega)\mathbb{Y}(o_1, r)o_2\rangle \tag{3.78}$$

for $\zeta > \omega > r > 0$ and

$$\langle v', \mathbb{Y}(o_1, r)\mathbb{Y}_{cl-op}(c; \zeta, \omega)o_2\rangle \tag{3.79}$$

for $r > \zeta > \omega > 0$ are analytic extensions of each other along a path $\gamma \subset \widehat{\mathbb{H}} \cup \widehat{\overline{\mathbb{H}}}$.

OCFAVII) Let $H_{cl} = \bigoplus_{\ell=1}^{K} U^L_{\nu^L(\ell)} \otimes U^R_{\nu^R(\ell)}$ be a splitting. For $c = c^L \otimes c^R \in U^L_{\nu^L(\ell)} \otimes U^R_{\nu^R(\ell)}$ there exist V-modules I_1, I_2 and intertwining operators $\mathcal{Y}^L_1 \in \mathcal{V}^{H_{op}}_{\nu^L(\ell) I_1}$ and $\mathcal{Y}^R_2 \in \mathcal{V}^{I_1}_{\nu^R(\ell) H_{op}}$, $\mathcal{Y}^L_4 \in \mathcal{V}^{I_2}_{\nu^L(\ell) H_{op}}$, $\mathcal{Y}^R_3 \in \mathcal{V}^{H_{op}}_{\nu^R(\ell) I_2}$ s.th. for any $o \in H_{op}$, $v' \in H'_{op}$

$$\langle v', \mathbb{Y}_{cl-op}(c; \zeta, \omega)o\rangle = \Big\langle v', \mathcal{Y}^L_1(c^L, \zeta)\mathcal{Y}^R_2(c^R, \omega)o\Big\rangle \tag{3.80}$$

for $|\zeta| > |\omega| > 0$ and

$$\langle v', \mathbb{Y}_{cl-op}(c; \zeta, \omega)o\rangle = \Big\langle v', \mathcal{Y}^R_3(c^R, \omega)\mathcal{Y}^L_4(c^L, \zeta)o\Big\rangle \tag{3.81}$$

for $|\omega| > |\zeta| > 0$. In addition it holds

$$\mathbb{Y}_{cl-op}(\mathbf{1}; \zeta, \overline{\zeta})o \equiv \mathcal{Y}^L_1(\mathbf{1}, \zeta)\mathcal{Y}^R_2(\mathbf{1}, \overline{\zeta})o = o \tag{3.82}$$

As in the case of OFA and CFA let us discuss, how this definition leads to the notion of an open-closed field theory as discussed in the introduction. Let

$$\mathbb{R}^\ell_> \equiv \Big\{(r_1, \cdots, r_\ell) \in \mathbb{R}^\ell_+ \,|\, r_1 > r_2 > \cdots, r_n > 0\Big\} \quad . \tag{3.83}$$

Recall that one wants to derive correlation functions

$$\begin{aligned} \mu^{cl-op}_{n,\ell} &: H^{\otimes n}_{cl} \times H^{\otimes \ell}_{op} \times \mathrm{Conf}_n(\mathbb{H}) \times \mathbb{R}^\ell_+ \to \overline{H_{op}} \\ (c_1, \cdots, c_n; o_1, \cdots, o_\ell; z_1, \cdots, z_n; r_1, \cdots, r_\ell) &\mapsto \\ &\mu^{cl-op}_{n,\ell}(c_1, \cdots, c_n; o_1, \cdots, o_\ell; z_1, \overline{z_1}, \cdots, z_n, \overline{z_n}; r_1, \cdots, r_\ell) \end{aligned} \tag{3.84}$$

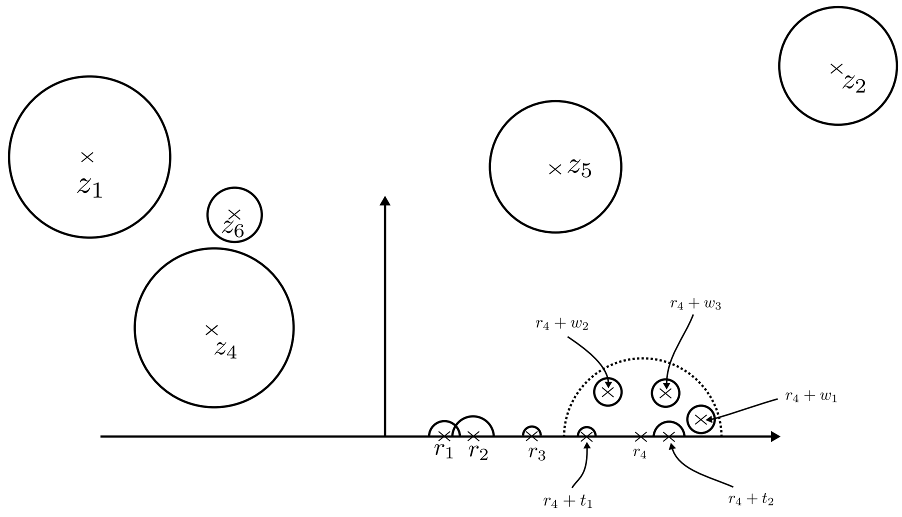
which are smooth and have two associativity properties: First, the concatenation

$$\begin{aligned} \mu^{cl-op}_{n,\ell}(c_1, \cdots, c_n; o_1, \cdots, o_{i-1}, \mu^{cl-op}_{r,s}(c^i_1, \cdots, c^i_r; o^i_1, \cdots, o^i_s; w_1, \cdots, w_r; t_1, \cdots, t_s), o_{i+1}, \\ \cdots, o_\ell; z_1, \cdots, z_n; r_1, \cdots, r_\ell) \end{aligned} \tag{3.85}$$

absolutely converges to

$$\begin{aligned} &\mu^{cl-op}_{n+r,\ell+s-1}(c_1, \cdots, c_n, c^i_1, \cdots, c^i_r; o_1, \cdots, o_{i-1}, o^i_1, \cdots, o^i_s, o_{i+1}, \cdots, o_\ell; \\ &z_1, \cdots, z_n, w_1 + r_i, \cdots, w_r + r_i; r_1, \cdots, r_{i-1}, r_i + t_1, \cdots, r_i + t_s, r_{i+1}, \cdots, r_\ell) \end{aligned} \tag{3.86}$$

for $|z_p - r_i|, |r_q - r_i| > |w_a|, |t_b|$ for any $p \in \{1, \cdots, n\}$, $q \in \{1, \cdots, i-1, i+1, \cdots \ell\}$, $a = \{1, \cdots r\}$ and $b = \{1, \cdots, s\}$ This corresponds to configuration of the following type



.

Second, for $j=1,\dots,n$ closed composition

$$\mu^{cl-op}_{n,\ell}(c_1,\cdots,c_{j-1},\mu_m(c^j_1,\cdots,c^j_m;o_1,\cdots,o_\ell;w_1,\overline{w_1},\cdots,w_m,\overline{w_m});c_{j-1},\\ \cdots,c_n;z_1,\overline{z_1},\cdots,z_n,\overline{z_n};r_1,\cdots,r_\ell) \tag{3.87}$$

absolutely converges to

$$\mu^{cl-op}_{n+m-1,\ell}(c_1,\cdots,c_{j-1},c^j_1,\cdots,c^j_m,c_{j+1},\cdots,c_n;o_1,\cdots,o_\ell;\\ z_1,\overline{z_1},\cdots,z_j+w_1,\overline{z_j}+\overline{w_1},\cdots,z_j+w_m,\overline{z_j}+\overline{w_m},\cdots,z_n,\overline{z_n};r_1,\cdots,r_\ell) \tag{3.88}$$

for $|z_p-z_j|$, $|r_q-z_j|>|w_s|$ for all $p=1,\cdots,n$, $p\neq j$ and $s=1,\cdots,\ell$ (see figure 1.1 in the introduction).

The procedure is similar to the closed case and we recall it from [105]. Let $\zeta_1>\omega_1>r_1>\cdots>\zeta_n>\omega_n>r_n>0$ be points on $\mathbb{R}_+$, then

$$\langle v',\mathbb{Y}_{cl-op}(c_1;\zeta_n,\omega_n)\mathbb{Y}(o_1,r_1)\cdots\mathbb{Y}_{cl-op}(c_n;\zeta_n,\omega_n)\mathbb{Y}(o_n,r_n)v\rangle \tag{3.89}$$

has an analytic continuation to $\mathrm{Conf}_{2n}(\mathbb{C}^\times)\times\mathbb{R}^n_>$ by axiom OCFAI). Since (3.89) uniquely fixes the analytic continuation on the simply connected domain

$$|\zeta_1|>|\omega_1|>r_1>\cdots>|\zeta_n|>|\omega_n|>r_n>0 \tag{3.90}$$

one can choose an arbitrary point in that domain as the initial point for analytic continuation. The natural pick is $(3n,3n-1,\cdots,1)$ and a path

$$(\mathbb{H}\cup\mathbb{R}_+)^n\supset\gamma_1:(3n,3(n-1),\cdots,3)\mapsto(z_1,\cdots,z_n)\quad . \tag{3.91}$$

Composing with the straight path

$$\mathbb{R}^n_>\supset\gamma_2:(3n-1,3(n-1)-1,\cdots,2)\mapsto(3n,3(n-1),\cdots,3) \tag{3.92}$$

and taking the complex conjugate of γ_1 gives a path

$$(\mathbb{H} \cup \mathbb{R}_+)^n \times \left(\overline{\mathbb{H}} \cup \mathbb{R}_+\right)^n \supset \gamma_{\mathbf{z}} \times \gamma_{\overline{\mathbf{z}}} : (3n, 3n-1, 3(n-1), 3(n-1)-1, \cdots, 3, 2) \mapsto (z_1, \overline{z_1}, \cdots, z_n, \overline{z_n}) \quad . \tag{3.93}$$

In addition let γ_{r} be the straight path

$$\mathbb{R}^n_{>} \supset \gamma_{\mathsf{r}} : (3n-2, 3(n-1)-2, \cdots, 1) \mapsto (r_1, \cdots, r_n) \quad . \tag{3.94}$$

Analytically continuing (3.89) from $(3n, 3n-1, \cdots, 1)$ along the combined path $\gamma_{\mathbf{z}} \times \gamma_{\overline{\mathbf{z}}} \times \gamma_{\mathsf{r}}$ gives a correlation function

$$\left\langle v', \mu^{cl-op}_{n,n}(c_1, \cdots, c_n; o_1, \cdots, o_n; z_1, \overline{z_1}, \cdots, z_n, \overline{z_n}; r_1, \cdots, r_n) v \right\rangle \quad . \tag{3.95}$$

for $(n, \ell) \neq (n, n)$ one defines $\mu^{cl-op}_{n,\ell}$ by appropriate insertions of either the closed or open unit. Since there is a CFA sitting inside the OCFA, this correlation function is independent of the choice of path $\gamma_{\mathbf{z}}$. The path γ_{r} is unique up to homotopy, thus this yields a well defined, smooth, single valued function

$$\mu^{cl-op}_{n,\ell} : \mathrm{Conf}_n(\mathbb{H}) \times \mathbb{R}^\ell_{>} \times H_{cl}^{\otimes n} \otimes H_{op}^{\otimes \ell} \to \overline{H_{op}} \quad . \tag{3.96}$$

The above stated associativity properties of $\mu^{cl-op}_{n,\ell}$ follow from the associativity assumptions OCFAIV) and OCFAV) in the definition of an OCFA.

The alert reader may wonder about open loop-channel closed tree-channel equivalence, i.e. the Cardy condition. Similar to modularity of CFA this involves a gluing operation. Unfortunately the true analytic description is cumbersome and relies a fair bit on operad theory. A detailed discussion is given in [104, sections 1-3]. We refrain from stating the full analytic expression, since explaining all ingredients in the necessary formula (see [104, equation (3.73)]) takes some effort and deviates from the main theme of the thesis which is categorical in nature. We just note that the procedure involves gluing a cylinder in two different ways which is described analytically in terms of corresponding coordinate transformations and taking traces similar to the torus case.

Definition 3.1.20. An *analytic Cardy algebra* is an OCFA whose CFA is modular and s.th. [104, equation (3.73)] is satisfied.

The categorical content of the definition will be explained in the subsequent sections.

3.2 Categorical Algebra of Field Algebras

3.2.1 Associative Algebras in R_V

Theorem 3.2.1. *[85, Theorem 4.3] Let V be a rational VOA. Then there is an isomorphism of categories*

Conformal OFAs $(H, \mathbb{Y}, \mathbf{1}, D)$, *s.th. there is an injective homomorphism* $\iota : V \hookrightarrow C_0(H)$ *of VOAs.*	$\xleftrightarrow{\simeq}$	*Associative Algebras* (A, m, η) *in* R_V

Since this theorem is central when transporting the analytic properties to categorical algebra we present the proof. This will hopefully help to understand the inner workings of the notion of a Cardy algebra later.

Proof. First a functor from conformal OFAs to associative algebras in R_V is constructed. Let $(H, \mathbb{Y}, \mathbf{1}, D)$ be a conformal OFA with an injective homomorphism $\iota : V \hookrightarrow C_0(H)$. By theorem 3.1.7 H is $C_0(H)$-module and $\mathbb{Y}_f$ is a $C_0(H)$-intertwining operator of type $\binom{H}{HH}$. The map ι makes H into a V-module and $\mathbb{Y}$ into a V-intertwining operator. Thus H is an object of R_V. The tensor product construction given in chapter 2 yields that there exists a morphism

$$m^r_{\mathbb{Y}_F} : H \boxtimes_{P(r)} H \to H \tag{3.97}$$

uniquely determined by

$$\overline{m^r_{\mathbb{Y}_f}}(v \boxtimes_{P(r)} w) = \mathbb{Y}_f(v, r)w \tag{3.98}$$

which for $r = 1$ gives a multiplication $m : H \boxtimes H \to H$. We have to show that this is associative. Let $r_1 > r_2 > r_1 - r_2 > 0$ and γ_1, γ_2, γ_{12} be paths in $\mathbb{R}_+$ from 1 to r_1, r_2 and $r_1 - r_2$. If γ_r is the path from 1 to r in $\mathbb{R}_+$, by the definition of the operator $\mathscr{T}_{\gamma_r}$ it holds

$$\begin{aligned} \overline{m^r_{\mathbb{Y}_f}} \circ \overline{\mathscr{T}_{\gamma_r}}(u \boxtimes v) =& \overline{m^r_{\mathbb{Y}_f}} \left(r^{-L_0} \left(r^{L_0} u \boxtimes_{P(r)} r^{L_0} v \right) \right) \\ =& r^{-L_0} \overline{m^r_{\mathbb{Y}_f}} \left(r^{L_0} u \boxtimes_{P(r)} r^{L_0} v \right) \\ =& r^{-L_0} \mathbb{Y}_f \left(r^{L_0}, r \right) r^{L_0} v \\ =& \mathbb{Y}_f(u, 1)v \\ =& \overline{m_{\mathbb{Y}_f}}(u \boxtimes v) \end{aligned} \tag{3.99}$$

where the first equality is the definition of $\mathscr{T}_{\gamma_r}$, the second equality uses that $\overline{m_{\mathbb{Y}_f}}$ is a module map. This gives a commutative diagram

$$\begin{tikzcd} H \boxtimes (H \boxtimes H) \arrow[r, "\mathscr{A}"] \arrow[d, "(\mathrm{id} \boxtimes_{P(r)} \mathscr{T}_{\gamma_2}) \circ \mathscr{T}_{\gamma_1}"'] & (H \boxtimes H) \boxtimes H \arrow[d, "(\mathscr{T}_{12} \boxtimes_{P(r_2)} \mathrm{id}) \circ \mathscr{T}_{\gamma_2}"] \\ H \boxtimes_{P(r_1)} (H \boxtimes_{P(r_2)} H) \arrow[r, "\mathscr{A}^{r_1 - r_2, r_2}_{r_1, r_2}"] \arrow[d, "m^{r_1}_{\mathbb{Y}_f} \circ (\mathrm{id} \boxtimes_{P(r_1)} m^{r_2}_{\mathbb{Y}_f})"'] & (H \boxtimes_{P(r_1 - r_2)} H) \boxtimes_{P(r_2)} H \arrow[d, "m^{r_2}_{\mathbb{Y}_f} \circ (m^{r_1 - r_2}_{\mathbb{Y}_f} \boxtimes_{P(r_2)} \mathrm{id})"] \\ H \arrow[r, "\mathrm{id}"] & H \end{tikzcd}$$

because vertical compositions are equal to $m \circ (\mathrm{id} \boxtimes m)$ and $m \circ (m \boxtimes \mathrm{id})$ respectively and

$$\begin{aligned} \overline{m^{r_1}_{\mathbb{Y}_f}} \circ (\mathrm{id} \boxtimes_{P(r_1)} \overline{m^{r_2}_{\mathbb{Y}_f}})(u \boxtimes_{P(r_1)} v) \boxtimes_{P(r_2)} &= \mathbb{Y}_f(u, r_1)\mathbb{Y}_f(v, r_2)w \\ \overline{m^{r_2}_{\mathbb{Y}_f}} \circ (\overline{m^{r_1 - r_2}_{\mathbb{Y}_f}} \boxtimes_{P(r_2)} \mathrm{id})(u \boxtimes_{P(r_1 - r_2)} v) \boxtimes_{P(r_2)} w &= \mathbb{Y}_f(\mathbb{Y}_f(u, r_1 - r_2)v, r_2)w \quad . \end{aligned} \tag{3.100}$$

The two expressions in (3.100) agree due to the associativity requirements of an OFA. This shows associativity.

The unit $\eta : V \to H$ is defined to be the map ι. Then

$$\begin{tikzcd} V \boxtimes H \arrow[r, "\eta \boxtimes \mathrm{id}"] \arrow[dr, "\lambda"'] & H \boxtimes H \arrow[d, "m_{\mathbb{Y}_f}"] \\ & H \end{tikzcd}$$

commutes tautologically. The right unit triangle diagrams

$$\begin{tikzcd} H \boxtimes H \arrow[d, "m_{\mathbb{Y}_f}"'] & H \boxtimes V \arrow[l, "\mathrm{id} \boxtimes \eta"'] \arrow[dl, "\rho"] \\ H & \end{tikzcd}$$

commutes as ι embeds V into $C_0(H)$ and the defining property of the meromorphic center shows commutativity.

In the other direction, assume (A, m, η) is an associative algebra in R_V. Then by definition A is an $\mathbb{R}$-graded vector space. Associated to the product is an intertwining operator $\mathbb{Y}_m$ of type $\binom{H}{HH}$ defined by

$$\overline{m}(u \boxtimes v) = \mathbb{Y}_m(u, 1)v \quad . \tag{3.101}$$

For $r > 0$ this gives an open vertex operator $\mathbb{Y}_m(\bullet, r)$. Since $\mathbb{Y}_m$ is an intertwining operator the vertex weight property follows by definition. In addition V is rational, hence $\coprod_{i \in \mathsf{I}} U_i$ is an IOA. Decomposing A into simple modules induces a decomposition of $\mathbb{Y}_m = \sum_{i,j,k \in \mathsf{I}} m_{ijk} \mathcal{Y}^k_{ij}$, where $m_{ijk} \in \mathbb{C}$. By the properties of an IOA correlation functions of products of intertwining operators $\mathcal{Y}^k_{ij}$ converge. Thus

$$\langle m_2', \mathbb{Y}_m(u_1, r_1) \cdots \mathbb{Y}_m(u_n, r_n) m_1 \rangle \tag{3.102}$$

converges for $r_1 > \cdots > r_n > 0$. By the left identity axiom for algebras it holds

$$\mathbb{Y}_m(\mathbf{1}, 1)u = u, \quad \forall U \in H \quad . \tag{3.103}$$

Applying r^{L_0} to (3.103) and noting that $\eta : V \to H$ has to be grading preserving gives

$$\mathbb{Y}_m(\mathbf{1}, r) r^{L_0} u = r^{L_0} u \tag{3.104}$$

which upon replacing $u \to u' \equiv r^{L_0} u$ gives the identity property. By a similar argument the right identity axiom gives

$$\mathbb{Y}_m(u, r)\mathbf{1} = e^{rL_{-1}} u \tag{3.105}$$

which in the limit $r \to 1$ yields the creation property. **d**- and D-derivative properties follow from the Virasoro action on intertwining operators. For associativity we note that there is again a commutative diagram like (3.2.1). By the definition of the open vertex operator from m this immediately gives associativity. The final thing to prove is, that η maps V injectively into the meromorphic center. We start by showing that it indeed maps to $C_0(H)$. $\mathbb{Y}_{m,f}(v,x) \in \mathrm{End}(H)[\![x, x^{-1}]\!]$ follows from the fact that η is a V-module map. The second requirement follows easily from the right unit property.

The two constructions are obviously inverse to each other. $\square$

Note that the proof doesn't use the braiding in R_V. The following is clear from corollary 3.1.13.

Corollary 3.2.2. *A CFA $(H, \mathcal{Y}, \iota)$ over $V^L \otimes V^R$ gives an associative algebra (C, μ, η) in $\mathsf{R}_{V^L \otimes V^R}$.*

In order to describe the single valuedness and skew-symmetry properties of CFAs in categorical language a closer look on $\mathsf{R}_{V^L \otimes V^R}$ is needed. The splitting property easily yields:

Lemma 3.2.3. *[103, Lemma 3.5] Let $M = M^L \otimes M^R$ and $N = N^L \otimes N^R$ be $V^L \otimes V^R$-modules. Then*

$$M \otimes N \longmapsto \left(M^L \boxtimes N^L\right) \otimes \left(M^R \boxtimes N^R\right) \tag{3.106}$$

is a $P(1)$-intertwining map and therefore a monoidal product in $R_{V^L \otimes V^R}$.

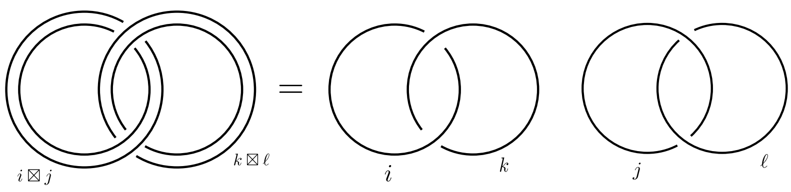
By the universal property of $P(1)$-intertwining maps there exists a unique $V^L \otimes V^R$-module map β_{ou} s.th.

$$\begin{tikzcd}
\left(M^L \boxtimes N^L\right) \otimes \left(M^R \boxtimes N^R\right) \arrow[r, "\simeq"] \arrow[d, "\beta_+^L \otimes \beta_-^R"'] & M \boxtimes N \arrow[d, "\beta_{ou}"] \\
\left(N^L \boxtimes M^L\right) \otimes \left(N^R \boxtimes M^R\right) \arrow[r, "\simeq"] & N \boxtimes M
\end{tikzcd}$$

commutes. Since β_+^L and β_-^R define braidings in R_{V^L} and R_{V^R} by the splitting property β_{ou} is a braiding in $(\mathsf{R}_{V^L \otimes V^R}, \boxtimes)$. The natural isomorphism

$$\Theta_{M^L \otimes M^R} \equiv \Theta_{M^L} \otimes \Theta_{M^R}^{-1} \tag{3.107}$$

with $\Theta_{M^R}^{-1} = e^{-2\pi \mathrm{i} L_0^R}$ defines a twist in $(\mathsf{R}_{V^L \otimes V^R}, \boxtimes, \beta_{ou})$ since Θ_{M^L} and $\Theta_{M^R}^{-1}$ define twists in $(\mathsf{R}_{V^L}, \boxtimes, \beta_+^L)$ and $(\mathsf{R}_{V^R}, \boxtimes, \beta_-^R)$. Clearly the S-matrix defined by



is non degenerate. In formulas this just reads

$$s_{i\boxtimes j,k\boxtimes\ell} = s_{ij} \otimes s_{\ell k} \tag{3.108}$$

In conclusion the following proposition holds.

Proposition 3.2.4. $(\mathsf{R}_{V^L\otimes V^R}, \boxtimes, \beta_{ou}, \Theta)$ *is a modular tensor category. It is naturally equivalent to* $\mathsf{R}_{V^L} \boxtimes \overline{\mathsf{R}_{V^R}}$*, the Deligne tensor product of* R_{V^L} *with the reversed category* $\overline{\mathsf{R}_{V^R}}$*.*

If $V^L = V^R$ by [136, Theorem 3.3] it follows that

$$(\mathsf{R}_{V\otimes V}, \boxtimes, \beta_{ou}, \Theta) \cong \mathsf{Z}\left(\mathsf{R}_V\right) \quad . \tag{3.109}$$

By the same computation showing that β_+, β_- are inverse to each other one shows that for a V^L-intertwining operator $\mathcal{Y}^L$ and a V^R-intertwining operator $\mathcal{Y}^R$ it holds

$$m_{\mathcal{B}_0(\mathcal{Y}^L)} = m_{\mathcal{Y}^L} \circ \beta_+^L, \qquad m_{\mathcal{B}_{-1}(\mathcal{Y}^R)} = m_{\mathcal{Y}^R} \circ \beta_-^R \quad . \tag{3.110}$$

The splitting property gives that for $\mathcal{Y} \in \mathcal{V}_{M_1M_2}^{M_3}$ a $V^L \otimes V^R$-intertwining operator the action of the braiding on the associated module map reads

$$m_{\mathcal{Y}} \circ \beta_{ou} = m_{\mathcal{B}_0\otimes\mathcal{B}_{-1}(\mathcal{Y})} \quad . \tag{3.111}$$

Theorem 3.2.5. *[103, Theorem 3.11] Let V^L, V^R be rational VOAs. Then there is an isomorphism of categories*

CFAs $(C, \mathcal{Y}, \iota)$ over $V^L \otimes V^R$	$\xleftrightarrow{\simeq}$	*Associative and commutative Algebras (C, m, η) in $(\mathsf{R}_{V^L\otimes V^R}, \boxtimes, \beta_{ou}, \Theta)$ with trivial twist.*

By definition of braiding and twist in $\mathsf{R}_{V^R\otimes V^L}$ single valuedness and skew-symmetry directly translate to trivial twist and commutativity for the associative algebra for a CFA given by corollary 3.2.2 and vice versa.

Finally we want to formulate OCFAs categorically. The main obstacle is that closed and open state space live in different categories and cannot be compared directly. It is a well known fact that monoidal functors map commutative algebra objects to commutative algebra objects. Let

$$F : (\mathsf{R}_{V\otimes V}, \boxtimes, \beta_{ou}, \Theta) \to \mathsf{R}_V \tag{3.112}$$

be the composition of the functor in (3.109) with the forgetful functor. $(F(H_{cl}), F(m), F(\eta))$ is a commutative algebra in R_V. By OCFAVII) and associativity of IOA for any homogeneous element $c^L \otimes c^R \in U_{\nu^L(\ell)} \otimes U_{\nu^R(\ell)} \hookrightarrow H_{cl}$ there exists a V-module I_ℓ and intertwining operators $\mathcal{Y}_\ell^1 \in \mathcal{V}_{I_\ell H_{op}}^{H_{op}}$ and $\mathcal{Y}_\ell^2 \in \mathcal{V}_{U_{\nu^L(\ell)}U_{\nu^R(\ell)}}^{I_\ell}$ s.th.

$$\left\langle v', \mathbb{Y}_{cl-op}(c^L \otimes c^R; \zeta, \omega)v \right\rangle = \left\langle v', \mathcal{Y}_\ell^1 \left(\mathcal{Y}_\ell^2(c^L, \zeta - \omega)c^R, \omega\right) v \right\rangle \tag{3.113}$$

holds for $\zeta > \omega > \zeta - \omega > 0$. Hence for $\mathcal{Y}^1 = \coprod_{\ell=1}^N \mathcal{Y}_\ell^N$ and $\mathcal{Y}^2 = \coprod_{\ell=1}^N \mathcal{Y}_\ell^2$ we have

$$\left\langle v', \mathbb{Y}_{cl-op}(w^L \otimes w^R; \zeta, \omega) v \right\rangle = \left\langle v', \mathcal{Y}^1 \left(\mathcal{Y}^2(c^L, \zeta - \omega) c^R, \omega \right) v \right\rangle \tag{3.114}$$

for any $v' \in H'_{op}$, $v \in H_{op}$ and $w^L \otimes w^R \in H_{cl}$. By the same arguments used several times now, the intertwining operators on the rhs of (3.114) give a V-module map

$$\rho : F(H_{cl}) \boxtimes H_{op} \to H_{op} \quad . \tag{3.115}$$

In addition let

$$\kappa_{cl-op} = \left[F(H_{cl}) \overset{\rho_V^{-1}}{\to} F(H_{cl}) \boxtimes V \overset{\mathrm{id} \boxtimes \iota_V}{\to} F(H_{cl}) \boxtimes H_{op} \overset{\rho}{\to} H_{op} \right] \quad . \tag{3.116}$$

The categorical description of OCFA given in [105, section 3] can be summarized as follows.

Theorem 3.2.6. *An OCFA* $(H_{cl}, \mathcal{Y}, \iota, H_{op}, \mathbb{Y}, \mathbb{Y}_{cl-op})$ *is equivalent to the data of*

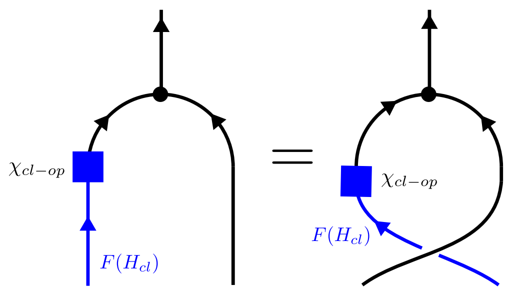
1. *a commutative associative algebra with trivial twist* $(H_{cl}, \mu_{cl}, \eta_{cl})$ *in* $\mathsf{Z}(\mathsf{R}_V)$.
2. *an associative algebra* $(H_{op}, \mu_{op}, \eta_{op})$ *in* R_V.

where in addition the map $\rho : F(H_{cl}) \boxtimes H_{op} \to H_{op}$ *gives* H_{op} *the structure of a* $F(H_{cl})$*-module s.th. the multiplication on* H_{cl} *is* $F(H_{cl})$*-homogeneous in both slots.*

Equivalently an OCFA $(H_{cl}, \mathcal{Y}, \iota, H_{op}, \mathbb{Y}, \mathbb{Y}_{cl-op})$ *is the same as*

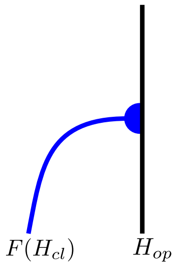
1. *a commutative associative algebra with trivial twist* $(H_{cl}, \mu_{cl}, \eta_{cl})$ *in* $\mathsf{Z}(\mathsf{R}_V)$.
2. *an associative algebra* $(H_{op}, \mu_{op}, \eta_{op})$ *in* R_V.

and an algebra homomorphism $\chi_{cl-op} : F(H_{cl}) \to H_{op}$ *s.th. the center condition*

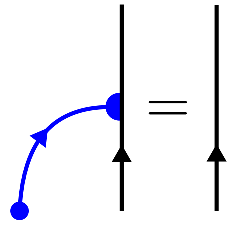


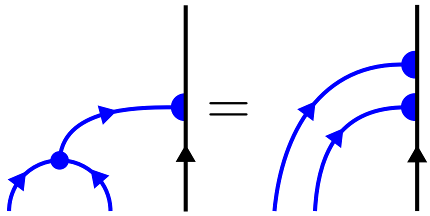
holds.

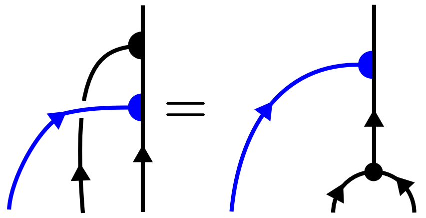
Note that we switched from a commutative algebra in $\mathsf{R}_{V\otimes V}$ to one in $\mathsf{Z}(\mathsf{R}_V)$ using (3.109). The proof goes by showing that axioms of an OCFA have nice categorical representations. Denoting the $F(H_{cl})$-module map ρ by

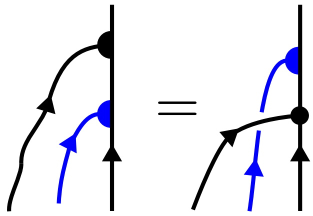
$\rho =$ 

the equivalence is given by the correspondence

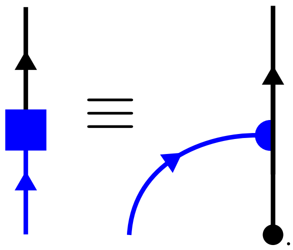
OCFAVII) (3.82) $\longleftrightarrow$ 

OCFAV) $\longleftrightarrow$ 

OCFAIV) $\longleftrightarrow$ 

OCFAVI) $\longrightarrow$ 

The map χ_{cl-op} is defined from ρ as

χ_{cl-op} 

That the two notions are equivalent can be easily shown using just category theory.

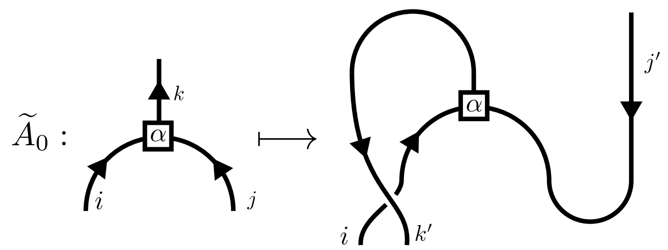
3.2.2 Frobenius Algebras in R_V

In the last section we described how OFA over a rational VOA are equivalent to associative algebras in the representation category of the VOA. Here we assume that the OFA has a non-degenerate bilinear form and show how this enhances the associative algebra to a Frobenius algebra. As before we present some details in the open case, the closed case is almost the same.

The key step in relating invariant bilinear forms to categorical notions is a description of the morphisms $\mathscr{B}_0$ and $\widetilde{A}_0$ in terms of string diagrams in R_V.

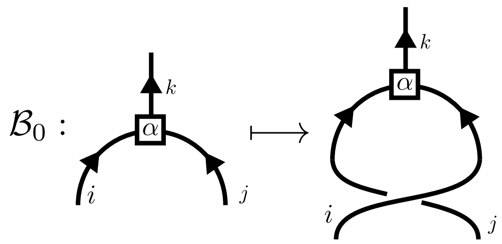
Lemma 3.2.7. *[103, Proposition 4.9] Let $\{\mathcal{Y}_\alpha\}$ be a basis in $\mathcal{V}_{ij}^{k}$.*

i) The action of $\widetilde{A}_0$ on the basis reads



.

ii) The action of $\mathscr{B}_0$ on the basis reads



.

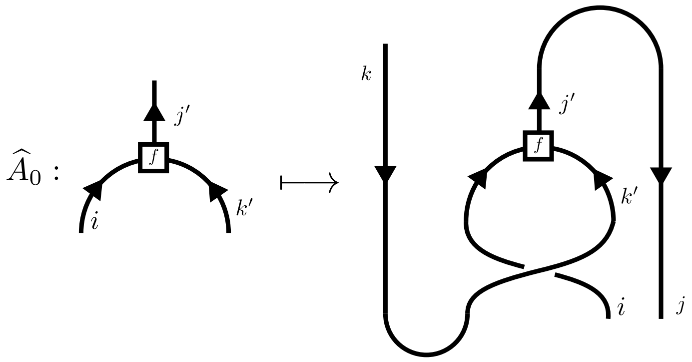
Let us comment on the proof. The second statement holds by definition of the braiding in R_V. For the first statement on picks a basis $\left\{\widetilde{\mathcal{Y}_\alpha}\right\}$ of $\mathcal{V}_{k'i}^{j'}$ s.th.

$$\widetilde{\mathrm{ev}}_k \circ m_{\mathcal{Y}_\alpha} \circ (\mathrm{id}_i \boxtimes \mathrm{coev}_j) = m_{\widetilde{\mathcal{Y}_\alpha}} \quad . \tag{3.117}$$

Then it suffices to show $\widetilde{\mathcal{Y}_\alpha} = \mathscr{B}_0 \circ \widetilde{A}_0(\mathcal{Y}_\alpha)$. This follows from associativity of IOA and a computation very similar to the proof of lemma 3.2.8. Recall that $\mathscr{B}_{-1}$ is inverse to $\mathscr{B}_0$. Therefore it is represented by the inverse crossing. $\widehat{A}_0$ is obtained by the same argument and is given by

$$\widehat{A}_0(f) = \widetilde{\mathrm{ev}}_j \circ \mathscr{B}_0 \circ f \circ (\mathrm{coev}_k \boxtimes \mathrm{id}_i) \quad . \tag{3.118}$$

where $f \in \mathrm{Hom}_{\mathsf{R}_V}(U_i \boxtimes U'_k, U'_i)$. Graphically this reads



. To relate bilinear forms to the above notion we prove the following result, which is originally stated in [104].

Lemma 3.2.8. *Invariance of bilinear forms* (3.43) *on OFAs over a rational VOA V is equivalent to the existence of a V-module map $\psi : H_{op} \to H'_{op}$ s.th.*

$$\begin{aligned} \psi \circ \mathbb{Y} &= \mathscr{B}_0 \circ \widetilde{A}_0 \circ (\psi \otimes \mathrm{id}_{H_{op}}) \\ \psi \circ \mathbb{Y} &= \widehat{A}_0 \circ \mathscr{B}_{-1} \circ (\mathrm{id}_{H_{op}} \otimes \psi) \end{aligned} \tag{3.119}$$

Proof. We only show the first one, the second one goes the same. Let $\mathcal{Y} \in \mathcal{V}^{H_{op}}_{H_{op}H_{op}}$, then for any formal variable x and $v, w \in H_{op}$, $u \in H'_{op}$, we compute

$$\begin{aligned} &\left\langle e^{-\frac{1}{x}L_{-1}} \mathscr{B}_0 \circ \widetilde{A}_0(\mathcal{Y})(u, \frac{1}{x}) e^{-xL_1} x^{-2L_0} v, w \right\rangle \\ =&\left\langle \widetilde{A}_0(\mathcal{Y}) \left(e^{-xL_1} x^{-2L_0} v, e^{\mathrm{i}\pi} \frac{1}{x} \right) u, w \right\rangle \\ =&\langle u, \mathcal{Y}(v, x) w \rangle \end{aligned} \tag{3.120}$$

where the first equation is just the definition of $\mathscr{B}_0$ and the second is the definition of $\widetilde{A}_0$ plus equation (2.43). Next define ψ via

$$(u, v) = \langle \psi(u), v \rangle \quad . \tag{3.121}$$

Then the invariance (3.43) is equivalent to

$$\langle \psi(u), \mathbb{Y}(v, r) w \rangle = \left\langle \psi \left(\mathbb{Y} \left(e^{-rL_1} r^{-2L_0} v, -\frac{1}{r} \right) u \right), w \right\rangle \tag{3.122}$$

By (3.41) this equals

$$\left\langle \psi \left(e^{-\frac{1}{r}L_{-1}} \mathbb{Y}(u, \frac{1}{r}) e^{-rL_1} r^{-2L_0} v \right), w \right\rangle \quad . \tag{3.123}$$

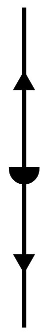
Using that ψ is a V-module map gives the claim. ∎

Theorem 3.2.9. *[104, Theorem 5.10] There is an isomorphism of categories*

OFAs $(H, \mathbb{Y}, \mathbf{1}, D)$ *over a rational VOA* V *with a non-degenerate bilinear form*	$\xleftrightarrow{\simeq}$	*Symmetric Frobenius Algebras* $(H, m, \eta, \Delta, \epsilon)$ *in* R_V

We give some parts of the proof with the aim of showing that the theorem can be obtained by mere category theory after having Lemma 3.2.7.

Proof. We will follow the standard graphical representations for Frobenius algebras presented in appendix B. Let $\psi : H_{op} \to H'_{op}$ be an isomorphism which will be depicted as

$\psi \equiv$  $\psi^{-1} \equiv$  .

In order for ψ to induce an invariant bilinear form it has to satisfy

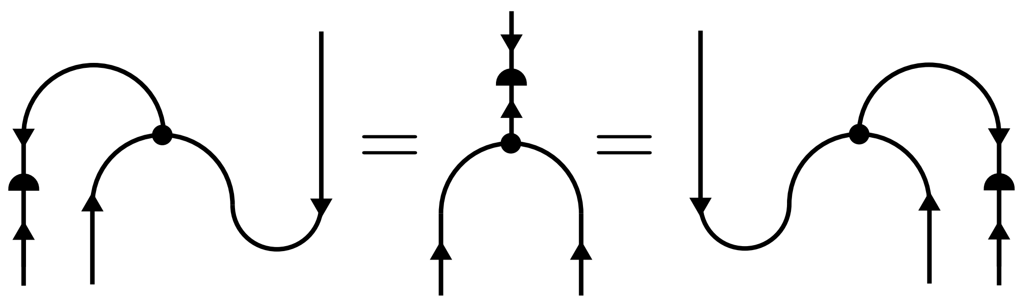
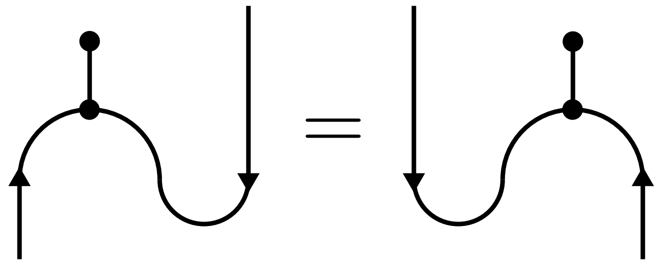


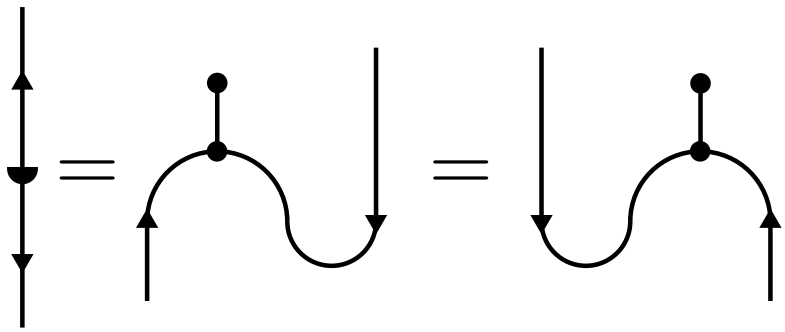
Figure 3.1: Graphical relations for ψ.

. This is just the graphical representation of (3.119). Assume $(H_{op}, m, \eta, \Delta, \epsilon)$ is a symmetric Frobenius algebra. By [59, Lemma 3.7] the morphism ϕ

$\phi =$



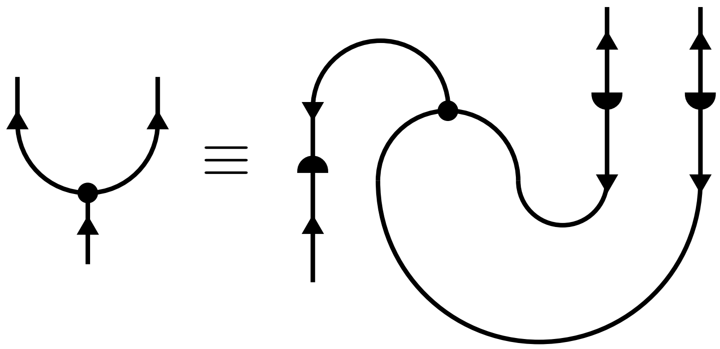
is invertible.
Setting



and using associativity once shows that ϕ satisfies 3.1. Hence a symmetric Frobenius algebra gives a non-degenerate invariant bilinear form.

Conversely, assume $(H_{op}, \mathbb{Y})$ has a non-degenerate invariant bilinear form with associated isomorphism ψ. The coproduct is defined by

$\Delta =$



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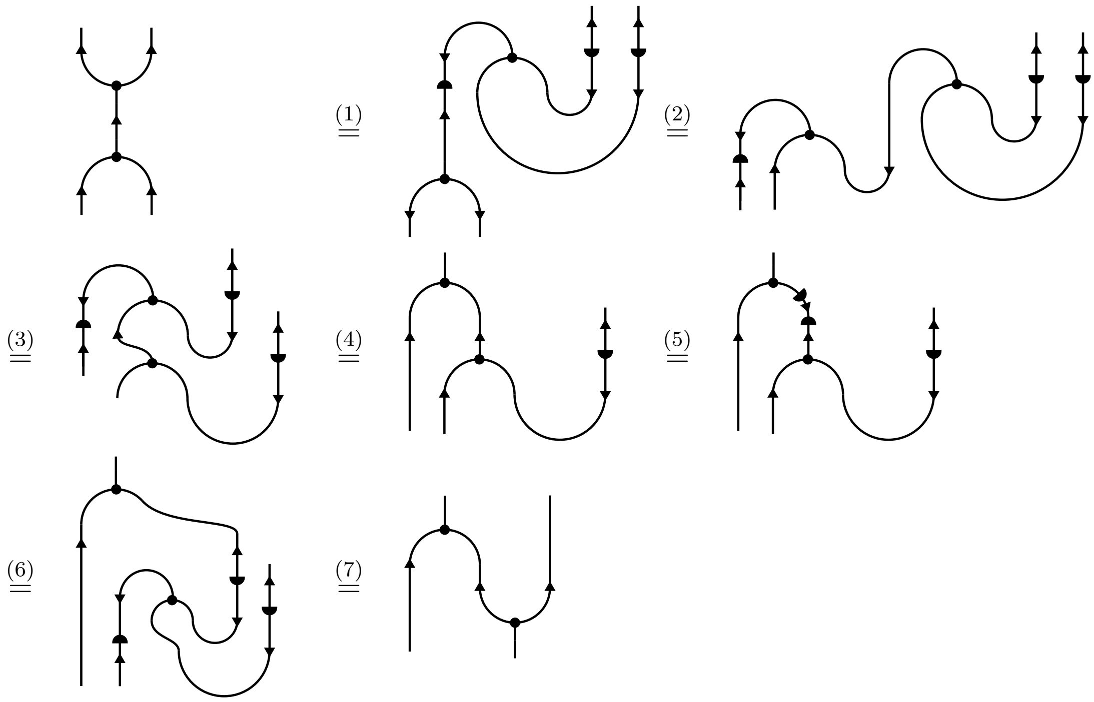
One easily checks that this is coassociative. The counit is defined similarly. We show one of the Frobenius properties. The other goes the same.



The steps are as follows: (1) and (7) are the definition of the coproduct, (2), (4) and (6) are the defining property of ψ, (3) is associativity of the product, (5) follows from ψ being an isomorphism. $\square$

With the same proof the analogous result for CFAs is shown.

Theorem 3.2.10. *[103, Theorem 4.15] There is an isomorphism of categories*

CFAs $(H_{cl}, \mathcal{Y}, \iota)$ *over a rational VOA* $V^L \otimes V^R$ *with non-degenerate invariant bilinear form*	$\xleftrightarrow{\simeq}$	*Symmetric, commutative Frobenius algebras in* $(H, m, \eta, \Delta, \epsilon)$ *in* $\mathsf{R}_{V^L \otimes V^R}$

In [57, Proposition 2.25] it is shown that a symmetric, commutative Frobenius algebra in a braided tensor category automatically has trivial twist and vice versa. Therefore this extra property doesn't need to be listed on the rhs of the equality.

3.2.3 $(\mathsf{R}_V|\mathsf{Z}(\mathsf{R}_V))$ - Cardy Algebras

So far we related analytic open and closed field algebras to Frobenius algebras in representation categories of rational VOAs. In addition the open-closed structure was expressed through an algebra homomorphism. In this section we give the categorical requirement for modular invariance of CFAs and state the Cardy condition. This will give a Cardy algebra in $\mathsf{R}_{V \otimes V}$. Unfortunately the definition of a Cardy algebra we want to use later takes place in $\mathsf{Z}(\mathsf{R}_V)$. The latter notion is obtained by using the adjoint functor to the forgetful functor.

Similar to section 3.2.2 we state one key result relating analytic expressions to categorical ones. Recall that in the appendix A.6 a and b moves on torus two point conformal blocks

$$\Psi^1_{\mathcal{Y}^j_{0j},\mathcal{Y}^0_{ii'}}(\bullet,\bullet;z_1,z_2;q_\tau) : U_i \otimes U_i' \to G^1_{2,0} \tag{3.124}$$

were used to derive the Verlinde formula. It was shown in [87, section 4] that a, b can be extended to two point conformal blocks with arbitrary intermediate propagating family. With the same proof one can show that this further extends to arbitrary simple inputs. We summarize this in the following proposition.

Proposition 3.2.11. *The space of genus one two point conformal blocks*

$$\Psi^1_{\mathcal{Y}^k_{\ell k},\mathcal{Y}^\ell_{ij}}(\bullet,\bullet;z_1,z_2;q_\tau) : U_i \otimes U_j :\to G^1_2 \tag{3.125}$$

is closed under the action of a- *and* b-*moves*

$$\begin{aligned}\mathsf{a}\left(\Psi^1_{\mathcal{Y}^k_{\ell k},\mathcal{Y}^\ell_{ij}}(\bullet,\bullet;z_1,z_2;q_\tau)\right) &= \Psi^1_{\mathcal{Y}^k_{\ell k},\mathcal{Y}^\ell_{ij}}(\bullet,\bullet;z_1,z_2-1;q_\tau)\\ \mathsf{b}\left(\Psi^1_{\mathcal{Y}^k_{\ell k},\mathcal{Y}^\ell_{ij}}(\bullet,\bullet;z_1,z_2;q_\tau)\right) &= \Psi^1_{\mathcal{Y}^k_{\ell k},\mathcal{Y}^\ell_{ij}}(\bullet,\bullet;z_1,z_2+\tau;q_\tau) \quad .\end{aligned} \tag{3.126}$$

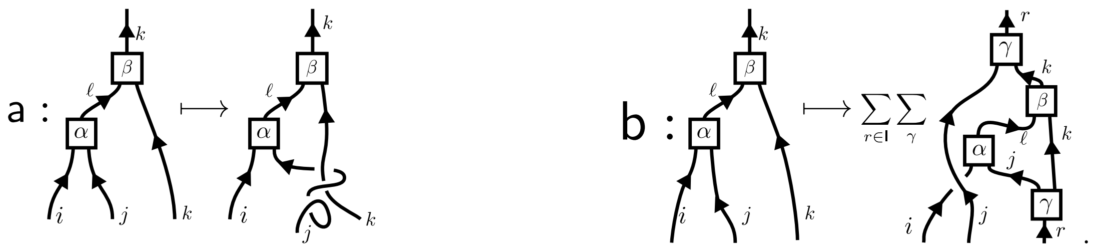
The corresponding transformation matrices still satisfy

$$S \circ \mathsf{a} = \mathsf{b} \circ S \quad . \tag{3.127}$$

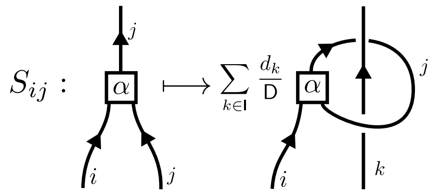
Since a, b and S are part of a projective representation of the mapping class group, equation (3.127) fixes S up to a constant. Therefore, given a categorical presentation for a and b one can derive S by means of category theory. Note that a, b induce maps

$$\bigoplus_{\ell,k\in \mathsf{I}} \mathcal{V}^k_{\ell k} \otimes \mathcal{V}^\ell_{ij} \to \bigoplus_{\ell,k\in \mathsf{I}} \mathcal{V}^\ell_{\ell k} \otimes \mathcal{V}^k_{ij} \tag{3.128}$$

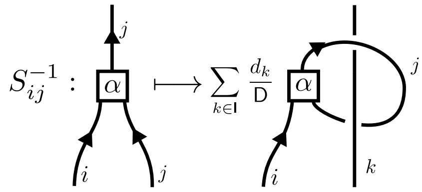
which we still denote by a, b. It is shown in [104, Lemma 4.3,4.4] that their graphical representation reads



Using the $SL(2,\mathbb{Z})$ relations between T and S transformations one can fix the remaining constant after solving (3.127). With the graphical calculus in ribbon categories it is not hard to show, that



will do the job. Using relation B.4 from the appendix one easily shows



Recall the splitting of the full vertex operator given by (3.68). In genus one due to the trace the only surviving terms are

$$\sum_{\ell_1=1}^{K} \mathbb{Y}^1_{\ell_1}(\bullet^L \otimes \bullet^R; x, y) = \sum_{\ell_1=1}^{K}\sum_{\ell_2=1}^{K}\sum_{\alpha=1}^{N^{\nu^L(\ell_3)}_{\nu^L(\ell_1)\nu^L(\ell_2)}}\sum_{\alpha_2=1}^{N^{\nu^L(\ell_3)}_{\nu^L(\ell_1)\nu^L(\ell_2)}} Y^{\ell_2}_{\ell_1\ell_2}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} \mathcal{Y}^{\nu^L(\ell_2);L}_{\nu^L(\ell_1)\nu^L(\ell_2);\alpha_1}(\bullet^L, x) \otimes \mathcal{Y}^{\nu^R(\ell_2);R}_{\nu^R(\ell_1)\nu^R(\ell_2);\alpha_2}(\bullet^R, y) \tag{3.129}$$

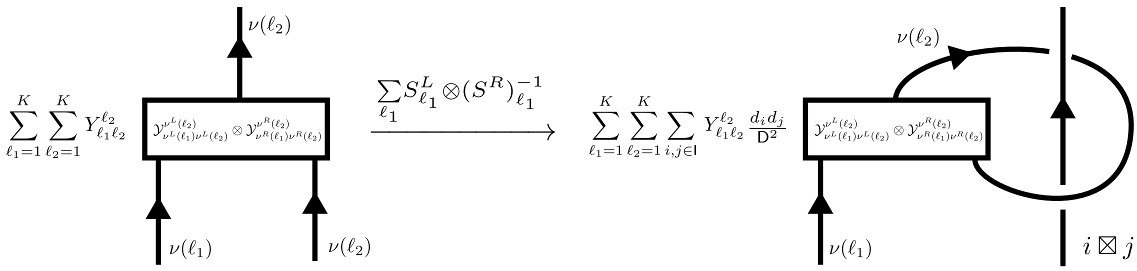
If we denote for the S-transformation

$$S^L_{\ell_1} = \sum_{\ell_2=2}^{K} S^L_{\nu^L(\ell_1)\nu^L(\ell_2)} \tag{3.130}$$

and let m_{ℓ_1} be the morphism in $\mathsf{R}_{V^L\otimes V^R}$ corresponding to $\mathbb{Y}^1_{\ell_1}$, the modular invariance property can be rewritten as

$$\sum_{\ell_1=1}^{K} S^L_{\ell_1} \otimes (S^R)^{-1}_{\ell_1} m_{\ell_1} = \sum_{\ell_1=1}^{K} m_{\ell_1} \quad . \tag{3.131}$$

Note that these are maps, whereas (3.71) is the same equation expanded into coefficients wrt. to a basis of intertwining operators. Invoking the graphical representation for S, (3.131) in fact reads



where we abbreviated $\nu(\ell_1) = \nu^L(\ell_1) \otimes \nu^R(\ell_1)$ and the sum over basis elements in the space of intertwining operators is implicit. Comparing coefficients one derives the equation

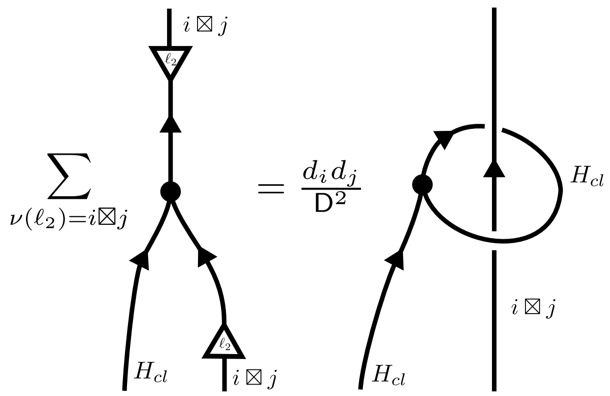
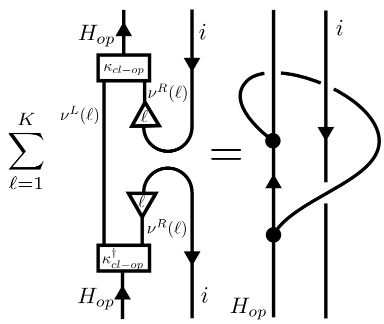


Figure 3.2: Modular Invariance Property

The sums over intertwining operators exactly give the product of the Frobenius algebra and summing over ℓ_1 gives the object H_{cl}. The upshot of the discussion is, that one can distill categorical expressions from analytic transformation properties of q-traces with the exact same content. In the case of modular invariance these are the expressions for a and b moves, which enable us to give modular invariance of CFA (3.71) in the form of figure 3.2. In [104] a similar discussion is made for the Cardy condition. We only state the result.

Definition 3.2.12. An OCFA $(H_{cl}, \mathcal{Y}, \iota, H_{op}, \mathbb{Y})$ over a rational VOA V satisfies the Cardy condition if the corresponding Frobenius algebras satisfy

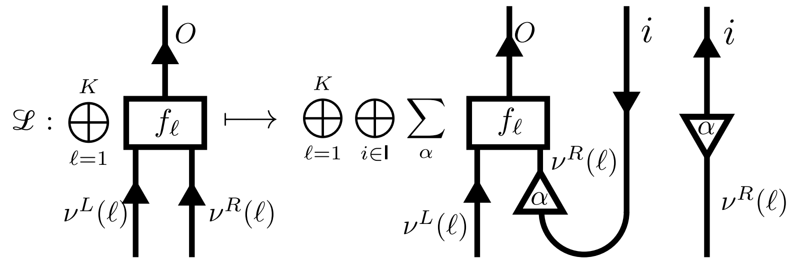


for all $i \in \mathsf{I}$.

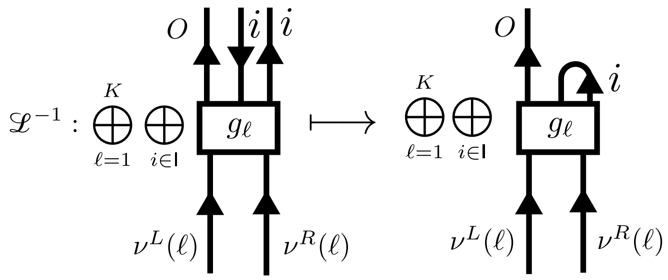
Recall the definition of the functor $L : \mathsf{C} \to \mathsf{Z}(\mathsf{C})$ left and right adjoint to the forgetful functor. This is a Frobenius functor by [107, Proposition 2.23], thus for $(H_{op}, m_{op}, \eta_{op}, \Delta_{op}, \epsilon_{op})$ the Frobenius algebra corresponding to an OFA, $L(H_{op})$ is a Frobenius algebra in $\mathsf{Z}(\mathsf{R}_V)$. In [107] it was shown how to transport the notion of a Cardy algebra to the Drinfeld center using the natural isomorphisms of the adjunction. We quickly recall the basic arguments to close the line of reasoning starting from true correlation functions of genus zero full RCFTs towards the notion of a Cardy algebra we use in the main part of this thesis.

Let $O \in \mathsf{R}_V$ and $C = \bigoplus_{\ell=1}^{K} U_{\nu^L(\ell)} \otimes U_{\nu^R(\ell)} \in \mathsf{Z}(\mathsf{R}_V)$. The natural isomorphisms of the adjunction are given by

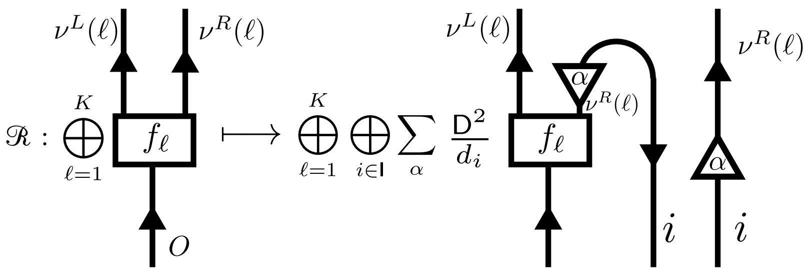
$$\mathscr{L} : \mathrm{Hom}_{\mathsf{R}_V}(F(C), O) \to \mathrm{Hom}_{\mathsf{Z}(\mathsf{R}_V)}(C, L(O))$$



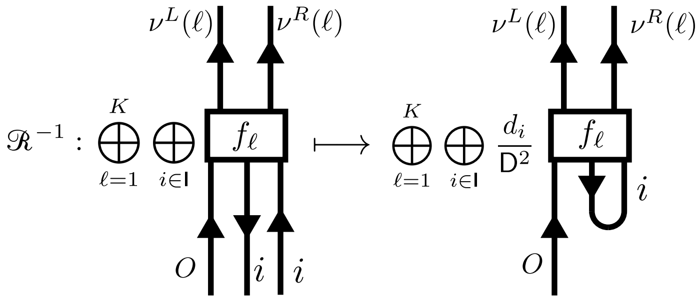
$$\mathscr{L}^{-1} : \mathrm{Hom}_{\mathsf{Z}(\mathsf{R}_V)}(C, L(O)) \to \mathrm{Hom}_{\mathsf{R}_V}(F(C), O)$$



$$\mathscr{R} : \mathrm{Hom}_{\mathsf{R}_V}(O, F(C)) \to \mathrm{Hom}_{\mathsf{Z}(\mathsf{R}_V)}(L(O), C)$$



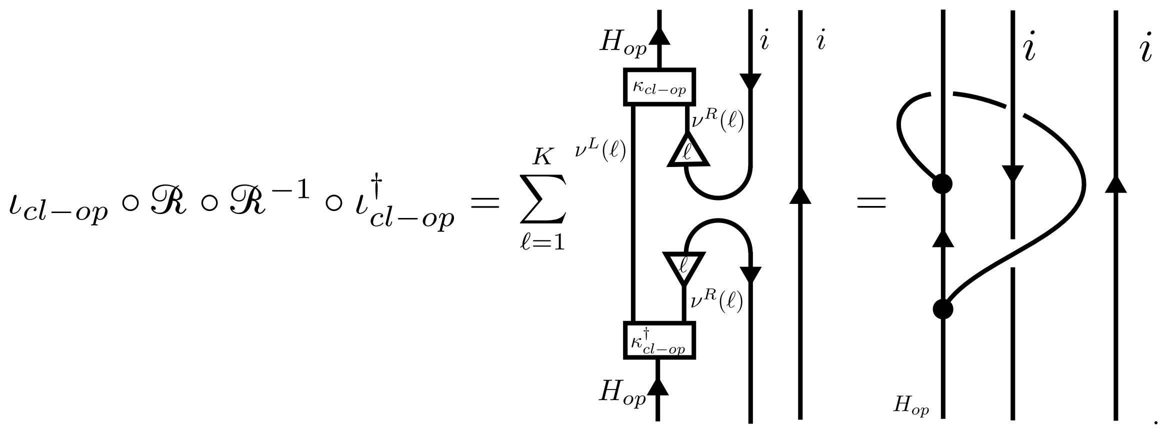
$$\mathscr{R}^{-1} : \mathrm{Hom}_{\mathsf{Z}(\mathsf{R}_V)}(L(O), C) \to \mathrm{Hom}_{\mathsf{R}_V}(O, F(C))$$



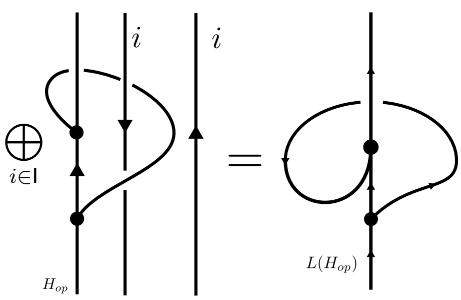
In the definition of a Cardy algebra in R_V there is the algebra homomorpism $\kappa_{cl-op} : F(C) \to O$. One defines

$$\iota_{cl-op} \equiv \mathscr{L}(\kappa_{cl-op}) : C \to L(O) \tag{3.132}$$

To show that ι_{cl-op} is still an algebra homomorphism one just applies $\mathscr{L}^{-1}$ to both sides of the defining equation for being an algebra homomorphism and recovers the respective equation for κ_{cl-op}. The argument for the center condition is the same. The most non obvious one is the Cardy condition. Spelling out the string diagrams for $\iota_{cl-op} \circ \iota^{\dagger}_{cl-op}$ one can easily see that the following is true



In [133] it was shown that $L(\mathbf{1})$ is a commutative Frobenius algebra in $\mathsf{Z}(\mathsf{C})$. With the completeness of the basis elements in morphism spaces it in addition directly follows that $L(\mathbf{1})$ is a special Frobenius algebra. Using this one shows



We summarize the above in a definition of a Cardy algebra in the Drinfeld center. The preceding paragraph shows that this is equivalent to the previous one.

Definition 3.2.13. [108, Definition 3.7] Let C be a modular tensor category. A $(\mathsf{C}|\mathsf{Z}(\mathsf{C}))$-*Cardy algebra* $(\mathcal{H}_{cl}, \mathcal{H}_{op}, \iota_{cl-op})$ is the data of

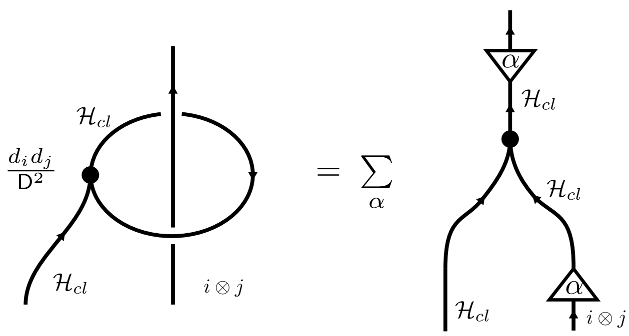
A) a commutative symmetric Frobenius algebra $(\mathcal{H}_{cl}, m_{cl}, \eta_{cl}, \Delta_{cl}, \epsilon_{cl})$ in $\mathsf{Z}(\mathsf{C})$.

B) a symmetric Frobenius algebra $(\mathcal{H}_{op}, m_{op}, \eta_{op}, \Delta_{op}, \epsilon_{op})$ in C.

C) a morphism $\iota_{cl-op} \in \mathrm{Hom}_{\mathsf{Z}(\mathsf{C})}(\mathcal{H}_{cl}, L(\mathcal{H}_{op}))$.

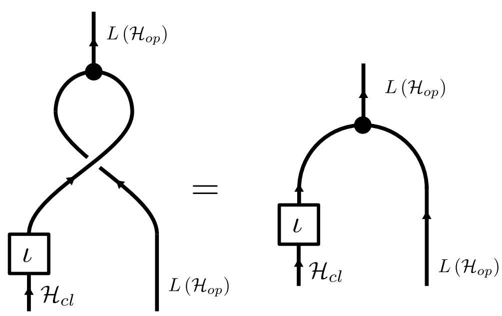
This has to satisfy the following conditions

I) $\mathcal{H}_{cl}$ has to be *modular*, i.e. there is the equality

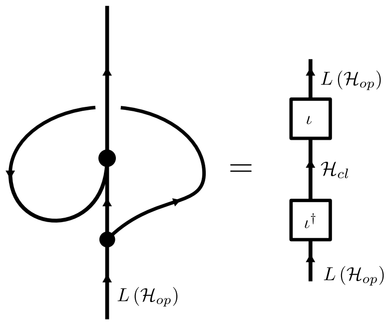


II) ι_{cl-op} is an algebra homomorphism.

III) The *center condition* holds:



IV) The *Cardy condition* holds:



Definition 3.2.14. A morphism between $(\mathsf{C}|\mathsf{Z}(\mathsf{C}))$-Cardy algebras $(\mathscr{H}_{cl}, \mathscr{H}_{op}, \iota_{cl-op})$ and $(\mathscr{G}_{cl}, \mathscr{G}_{op}, \iota'_{cl-op})$ is a pair of maps $f_{cl} \in \mathrm{Hom}_{\mathsf{Z}(\mathsf{C})}(\mathscr{H}_{cl}, \mathscr{G}_{cl})$, $f_{op} \in \mathrm{Hom}_{\mathsf{C}}(\mathscr{H}_{op}, \mathscr{G}_{op})$ s.th.:

I) Both, f_{cl} and f_{op}, are homomorphisms of Frobenius algebras.

II) The following diagram commutes

$$\begin{tikzcd}
\mathscr{H}_{cl} \arrow[r, "f_{cl}"] \arrow[d, "\iota_{cl-op}"] & \mathscr{G}_{cl} \arrow[d, "\iota'_{cl-op}"] \\
L(\mathscr{H}_{op}) \arrow[r, "L(f_{op})"'] & L(\mathscr{G}_{op})\ .
\end{tikzcd}$$

Chapter 4

Consistent Correlators from String-Nets

This chapter is mostly based on [141], but contains some additional new results. We start with a review of the category of open-closed world sheets. Conformal blocks in their categorical appearance are determined by a symmetric monoidal functor from open-closed world sheets to vector spaces and the sewing constraints are neatly cast into a monoidal natural transformation from the trivial functor to the conformal blocks functor. This gives a list of 32 relations , recalled in appendix D, which need to be checked in order to define a consistent set of correlators. Using string-nets we define a functor of conformal blocks and Cardy algebra colored string-nets will solve the sewing constraints and vice versa, i.e. solving the sewing constraints for the string-net conformal blocks functor will give a Cardy algebra.

4.1 Open-closed World Sheets

The definition of the category of open-closed world sheets we give here is originally due to [49]. It uses the orientation double of two dimensional manifolds to properly define open boundary components. Although we will exclusively work with string-nets on quotient surfaces, this doubling seems unavoidable for a rigorous definition of open field insertions. An *open-closed world sheet* is defined as a tuple

$$\tilde{S} \equiv \left(\widehat{S}, \iota_S, B_S^i, B_D^o, \mathrm{or}_S, \mathrm{ord}\right) \tag{4.1}$$

consisting of

1. an oriented compact surface $\widehat{S}$ with boundary components $\pi_0\left(\partial\widehat{S}\right) = B_S^i \sqcup B_S^o$. The set B_S^i is the set of incoming boundary components, whereas B_S^o are outgoing boundaries.

2. an orientation reversing involution $\iota_S : \widehat{S} \to \widehat{S}$ s.th. $S \equiv \widehat{S}/\langle\iota_S\rangle$ is a two dimensional manifold with corners, called the *quotient surface* and $\widehat{S} \to S$ is a $\mathbb{Z}_2$-bundle. The

partition of boundary components into sets B_S^i, B_S^o is fixed under ι_S. In addition boundary components $\pi_0\left(\widehat{S}\right)$ are parametrized s.th. ι_S acts as complex conjugation on the parametrization ψ_b of any boundary component b. Fixed points B_{op} of ι_S in $\pi_0(\widehat{S})$ are called *open boundaries*, all other are called *closed boundary components* B_{cl}.

3. global section $\mathrm{or}_S : S \to \widehat{S}$. If $b \in \pi_0\left(\widehat{S}\right)$ is fixed by ι_S it holds $\psi_b^{-1}\left(S^1 \cap \mathbb{H}\right) = \mathrm{Im}(\mathrm{or}_S)$.

4. an ordering function $\mathrm{ord}_S : \pi_0\left(\widehat{S}\right) \to \left|\pi_0\left(\widehat{S}\right)\right|$ which satisfies $\mathrm{ord}_S(o) < \mathrm{ord}_S(c)$ for $o \in B_{op}$ and $c \in B_{cl}$. Furthermore for a subset N of B_{op} s.th. its projection under the quotient map $\widehat{S} \to S$ sits on a single connected component of the boundary of the quotient surface there exists an $n \in \left\{1, \cdots, \left|\pi_0\left(\widehat{S}\right)\right|\right\}$ s.th. $\mathrm{ord}_S(N) = \{n, n+1, \cdots, n+|N|-1\}$.

The classification of boundary components into closed and open on the orientation double $\widehat{S}$ induces a division of the boundary of S. A point $p \in \partial S$ is on a *closed boundary* if it has two preimages p_0, p_1 on disjoint connected components of $\partial\widehat{S}$. It is on an *open boundary* if it has two distinct preimages in $\widehat{S}$ which are on the same connected component of $\partial\widehat{S}$. Finally there are *physical boundary components* on S. A point $p \in \partial S$ is on a physical boundary if its preimage is a fixed point under ι_S. Note that preimages of physical boundary points don't need to be on boundary components of $\widehat{S}$.

Open-closed world sheets can of course be sewn. The data for a *sewing of world sheets* is a set G of pairs of incoming and outgoing boundary components $(i, o) \in B_S^i \sqcup B_S^o$ s.th. $(\iota_S(i), \iota_S(o)) \in G$, either $(i, o) \in B_{op}^i \sqcup B_{op}^o$ or $(i, o) \in B_{cl}^i \sqcup B_{cl}^o$ and there is no $o' \in B_S^o$ s.th. $(i, o') \in G$ and no $i' \in B_S^i$ s.th. $(i', o) \in G$. The *sewn world sheet* $\widetilde{\mathcal{S}(S)}$ has the orientation double $\widehat{\mathcal{S}(S)} = \widehat{S}/\sim$ with $\psi_i^{-1}(z) \sim \psi_o^{-1}(-\overline{z})$. Let $P_S : \widehat{S} \to \widehat{\mathcal{S}(S)}$ be the projection map. Then there exists an orientation reversing involution $\iota_{\mathcal{S}(S)} \circ P_S = P_S \circ \iota_S$ and $B_{\mathcal{S}(S)}^i = \{i \in B_S^i | (i, \bullet) \notin G\}$, $B_{\mathcal{S}(S)}^o = \{o \in B_S^o | (\bullet, o) \notin G\}$. The global section is given by $\mathrm{or}_{\mathcal{S}(S)} = P_S \circ \mathrm{or}_S$. There is an additional definition for the ordering function which is a bit cumbersome to write down and since we don't need it we refrain from actually stating it.

Finally a *homeomophism* $F : \widetilde{S} \to \widetilde{T}$ *of open-closed world sheets* is a homeomorphism $F : \widehat{S} \to \widehat{T}$ mapping

$$F \circ \iota_S = \iota_T \circ F, \quad \psi_T \circ F = \psi_S, \quad F(B_S^{i,o}) = B_T^{i,o}, \quad F \circ \mathrm{Im}(\mathrm{or}_S) = \mathrm{Im}(\mathrm{or}_T) \quad . \tag{4.2}$$

Both, sewings and homeomorphisms of open-closed world sheets are combined into morphisms of open-closed world sheets. This gives a category of open-closed world sheets as defined in [106].

Definition 4.1.1. The category of open-closed world sheets WS has objects open-closed world sheets and morphisms $\widetilde{S} \to \widetilde{T}$ are pairs $(F, \mathcal{S})$, where $\mathcal{S}$ is a sewing of $\widetilde{S}$ and F is a homeomorphism $\widetilde{\mathcal{S}(S)} \simeq \widetilde{T}$.

Two morphisms in WS are *homotopic* if the their sewing part agrees and their homeomorphisms of world sheets are isotopic as maps of two dimensional manifolds.

Together with the disjoint union this defines a symmetric monoidal category. It is fairly obvious that WS has a nice presentation in terms of a generating set $\mathcal{G}$ of so called fundamental world sheets. For any object $S \in \mathsf{WS}$ there exists a list of fundamental objects $\left\{\widetilde{S}_{i_1}, \cdots, \widetilde{S}_{i_{N_S}}\right\}$ and a morphism $(\mathcal{S}, F) : \widetilde{S}_{i_1} \otimes \cdots \otimes \widetilde{S}_{i_{N_S}} \to S$. Neither the list of fundamental world sheets, nor the morphism is unique. These facts just correspond to cutting a two dimensional compact manifold into smaller pieces. An over-complete generating set of world sheets is given in appendix D. Let $\Phi, \Psi : \mathsf{WS} \to \mathsf{Vect}$ be symmetric monoidal functors, which are constant on homotopy classes of morphisms. Furthermore, Ψ is required to be invertible, i.e. $\Psi(\mathcal{S}, F)$ is an invertible linear map for any morphism of world sheets. A monoidal natural transformation $g : \Psi \Rightarrow \Phi$ can be uniquely constructed from linear maps

$$g_i : \Psi(\widetilde{S}_i) \to \Phi(\widetilde{S}_i), \quad i \in \mathcal{G} \tag{4.3}$$

by defining

$$g_S = \Phi(\mathcal{S}, F) \circ (g_{i_1} \otimes \cdots \otimes g_{i_\ell}) \circ \Psi(\mathcal{S}, F)^{-1} \tag{4.4}$$

where $(\mathcal{S}, F) : \widetilde{S}_{i_1} \otimes \cdots \otimes \widetilde{S}_{i_\ell} \to S$ is sewing of S from fundamental world sheets. One of the key steps in showing that a certain collection of correlators defined on simple world sheets actually solves the sewing constraints is the following theorem.

Theorem 4.1.2. *[106, Theorem 2.8] Given functors Ψ, Φ as above and a collection of linear maps $g_i : \Psi(\widetilde{S}_i) \to \Phi(\widetilde{S}_i)$, then g extends to a monoidal natural transformation if and only if*

$$g(R_{i,l}) = g(R_{i,r}) \tag{4.5}$$

for $\{R_{i,l},\ R_{i,r}\}$ the left and right hand side of the 32 fundamental world sheet sewings given in appendix D.

Theorem 4.1.2 in particular applies to the trivial symmetric monoidal functor

$$\Delta : \mathsf{WS} \to \mathsf{Vect} \tag{4.6}$$

which maps any open-closed world sheet to the ground field and any morphism just to the identity.

Definition 4.1.3. Let $\Phi : \mathsf{WS} \to \mathsf{Vect}$ be a symmetric monoidal functor which is constant on homotopy classes of morphisms. A *solution to the sewing constraints for* Φ, or *consistent set of correlators for* Φ is a monoidal natural transformation

$$\mathsf{corr} : \Delta \Rightarrow \Phi \quad . \tag{4.7}$$

This definition together with theorem 4.1.2 breaks the construction of a correlator mapping down to checking the 32 relations for a monoidal natural transformation. Let us disentangle what a natural transformation from the trivial functor to Φ does. First of all for any open-closed world sheet there is a linear map

$$\mathsf{corr}_S : \mathbb{C} \to \Phi(S) \tag{4.8}$$

which is nothing but a singled out vector in the vector space $\Phi(S)$. In case Φ is a functor of conformal blocks, this corresponds to the fact that correlators are specific vectors in the space of conformal blocks on a surface. Let $F \equiv (0, F) : S_1 \to S_2$ be a homeomophism of world sheets. By the definition of a natural transformation there is a commutative diagram

$$\begin{tikzcd} \Delta(S_1) = \mathbb{C} \arrow[r, "\mathrm{id}"] \arrow[d, "\mathsf{corr}_{S_1}"'] & \Delta(S_2) = \mathbb{C} \arrow[d, "\mathsf{corr}_{S_2}"] \\ \Phi(S_1) \arrow[r, "\Phi(F)"] & \Phi(S_1) \end{tikzcd}$$

where $\Phi(F)$ is the action of the mapping class group element F on the space of conformal blocks. Commutativity tells that corr gives vectors which are invariant under the action of the mapping class group. The same diagram for a sewing morphism of world sheets yields

$$\Phi(\mathcal{S}) \circ \mathsf{corr}_S = \mathsf{corr}_{\mathcal{S}(S)} \quad . \tag{4.9}$$

Hence gluing correlators for simpler pieces of world sheets gives the correlator on the glued surface.

In summary, a monoidal natural transformation from the trivial functor to a functor of conformal blocks yields consistent correlators for the RCFT described by the conformal blocks functor.

4.2 Cardy Algebras and Consistent Correlators

In this section we show how a Cardy algebra solves the sewing constraints and vice versa. We start with the construction of a functor of conformal blocks from string-nets. Using the Drinfeld center instead of the Deligne double allows us to work directly on the quotient surface of an open-closed surface instead of going through the orientation double. However, in order to properly separate the open sector, the definition of the symmetric monoidal functor is a bit cumbersome. The reason being that $L : \mathsf{C} \to \mathsf{Z}(\mathsf{C})$ is a Frobenius functor, but not a tensor functor. Throughout this section we fix a modular tensor category C and a $(\mathsf{C}|\mathsf{Z}(\mathsf{C}))$-Cardy algebra $(\mathscr{H}_{cl}, \mathscr{H}_{op}, \iota_{cl-op})$. The reader may have $\mathsf{R}_V = \mathsf{C}$ in mind, but the results apply to an arbitrary modular tensor category. In addition, given a compact surface Σ with possibly non-empty boundary, we denote g_Σ for the genus of the closed surface obtained from Σ by gluing disks to its boundary components.

We separate open-closed world sheets into three types: disks (type I), spheres with non-empty open and closed boundaries but no physical boundary (type II) and all other world sheets (type III). This classification is not closed under sewing of world sheets and we have to discuss how the sewing operation acts in different cases. We start by a precise definition of the different types of world sheets and the vector spaces the block functor assigns to them.

I) By a disk we mean an object $\widetilde{S} \in \mathsf{WS}$ s.th. $g_{\widehat{S}} = 0$ and for any $b_i \in \pi_0\left(\partial\widehat{S}\right)$ it holds $\iota_S(b_i) = b_i$, i.e. there are only open boundary components, whose images in the quotient surface sit on a single connected component of the boundary, which is given by the fixed point set of ι_S. Therefore the quotient surface is just a disk $S_{n,m}$ with n incoming and m outgoing open boundaries. Open boundaries are ordered by the ordering function ord_S and we denote

$$\overline{\mathcal{H}_{op}} = \underbrace{\widetilde{\mathcal{H}_{op}} \otimes \cdots \widetilde{\mathcal{H}_{op}}}_{n+m} \tag{4.10}$$

with

$$\widetilde{\mathcal{H}_{op}} = \begin{cases} \mathcal{H}_{op}, \text{ for outgoing boundary} \\ \mathcal{H}_{op}^*, \text{ for incoming boundary} \end{cases} \tag{4.11}$$

and the tensor factors of incoming and outgoing components appear according to ord_S. We define

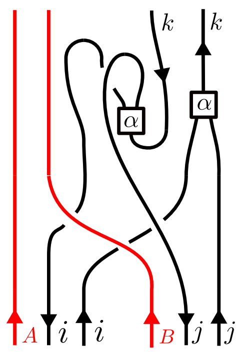
$$\mathcal{B}\ell(\widetilde{S}) = H^s(S, \overline{\mathcal{H}_{op}}) \quad . \tag{4.12}$$

That is, the functor assigns the string-net space on the disk with boundary value the tensor product $\overline{\mathcal{H}_{op}}$.

II) A sphere with non empty open and closed boundary components corresponds to an object $\widetilde{S} \in \mathsf{WS}$ with $g_S = 0$ and $B_{op} \neq \emptyset$, $B_{cl} \neq \emptyset$. In addition we require that the fixed point set of ι_S is connected and has non-empty intersection with B_{op}. Thus there is no connected component of the boundary on the quotient surface which is purely a physical boundary. In this case we are going to consider a certain subspace of the string-net space $H^s(S, \overline{\mathcal{H}_{cl}}, \overline{L(\mathcal{H}_{op})}) = \mathrm{Hom}_{\mathsf{Z(C)}}(\mathbf{1}, \overline{\mathcal{H}_{cl}} \otimes \overline{L(\mathcal{H}_{op})})$, where closed boundary components of the quotient surface are labeled with $\mathcal{H}_{cl}$ and open ones by $L(\mathcal{H}_{op})$ and notation indicates the same assignments as in case I). Since L is lax and colax tensor functor for any A, $B \in \mathsf{C}$ there are morphisms

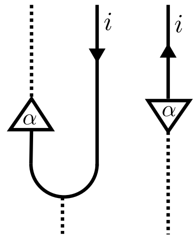
$$\phi^L : L(A) \otimes L(B) \to L(A \otimes B),$$

$$\phi^L = \bigoplus_{i,j,k \in \mathsf{I}} \sum_{\alpha}$$



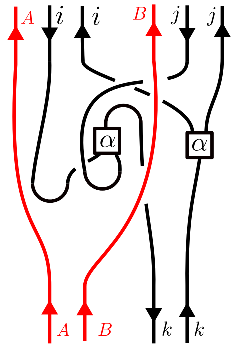
$$\phi^L_{\mathbf{1}} : \mathbf{1}_{\mathsf{Z(C)}} \to L(\mathbf{1}_{\mathsf{C}}),$$

$$\phi^L_{\mathbf{1}} = \bigoplus_{i\in\mathsf{I}} \sum_{\alpha}$$



$$\psi^L : L(A\otimes B) \to L(A)\otimes L(B),$$

$$\psi^L = \bigoplus_{i,j,k\in\mathsf{I}} \sum_{\alpha} \frac{d_i d_j}{d_k \mathsf{D}^2}$$



$$\psi^L_{\mathbf{1}} : L(\mathbf{1}_{\mathsf{C}}) \to \mathbf{1}_{\mathsf{Z(C)}}, \qquad \psi^L_{\mathbf{1}} = \mathsf{D}^2 \mathrm{id}_{\mathsf{ZC}}.$$

. The graphical presentation for $\psi^L_{\mathbf{1}}$ is similar to $\phi^L_{\mathbf{1}}$ but we don't need in the following, thus we just gave the formula. Recall the map $\mathscr{L}^{-1} : \mathrm{Hom}_{\mathsf{Z(C)}}(C, L(O)) \to \mathrm{Hom}_{\mathsf{C}}(F(C), O)$ from section 3.2.3. We define two maps

$$\begin{aligned} Z : \mathrm{Hom}_{\mathsf{Z(C)}}(\mathbf{1}, \overline{\mathscr{H}_{op}}) &\to \mathrm{Hom}_{\mathsf{Z(C)}}(\mathbf{1}, \overline{L(\mathscr{H}_{op})}) \\ f &\mapsto \left(\psi^L \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}\right) \circ \cdots \circ \left(\psi^L \otimes \mathrm{id}\right) \circ \psi^L \circ L(f) \circ \phi^L_{\mathbf{1}} \end{aligned} \tag{4.13}$$

and

$$Y : \mathrm{Hom}_{\mathsf{Z(C)}}\left(\mathbf{1}, \overline{L(\mathscr{H}_{op})}\right) \to \mathrm{Hom}_{\mathsf{C}}\left(\mathbf{1}, \overline{\mathscr{H}_{op}}\right) \simeq \mathrm{Hom}_{\mathsf{Z(C)}}\left(\mathbf{1}, \overline{\mathscr{H}_{op}}\right) \tag{4.14}$$

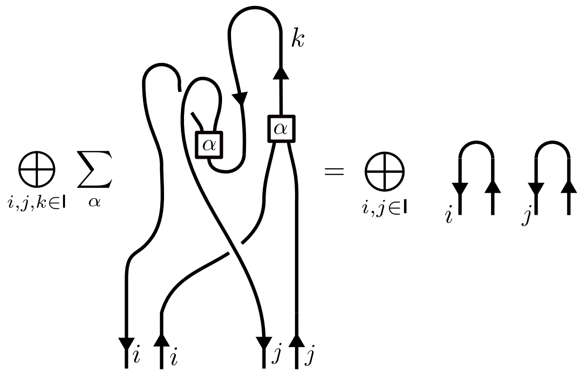
mapping

$$g \mapsto Y(g) \equiv \mathscr{L}^{-1} \circ F\left[\phi^L \circ (\mathrm{id}\otimes\phi^L) \circ \cdots \circ (\mathrm{id}\otimes\cdots\otimes\mathrm{id}\otimes\phi^L) \circ g\right] \tag{4.15}$$

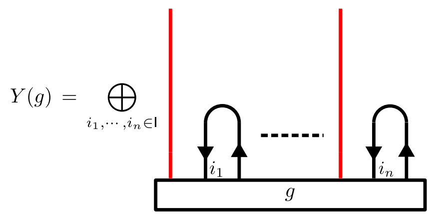
Lemma 4.2.1. *Y is left inverse to Z, i.e. $Y \circ Z = \mathrm{id}_{\mathrm{Hom}_{\mathsf{Z(C)}}(\mathbf{1},\overline{\mathscr{H}_{op}})}$.*

Proof. The proof is done using the graphical representation of the morphisms. In some of the pictures we skip the red strands corresponding to the objects in C in the definition of L. They are freely braided above black strands and clutter arguments unnecessary.

The first thing to note is that by completeness the following holds

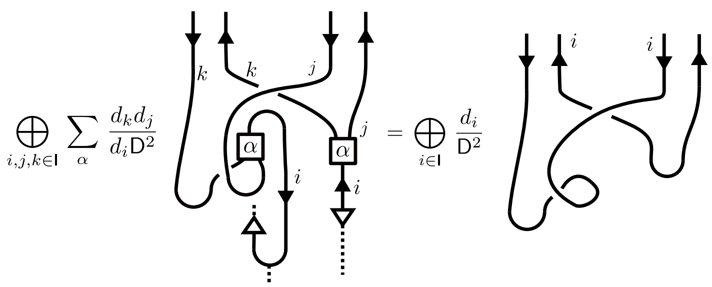


. Thus the definition of the map $\mathscr{L}^{-1}$ and induction on the number of tensor factors in $\mathrm{Hom}_{\mathsf{Z(C)}}\left(\mathbf{1}, \overline{L(\mathscr{H}_{op})}\right)$ gives



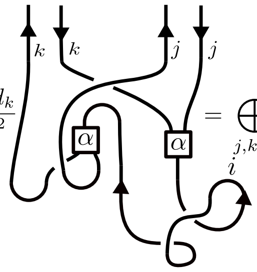
.

Here we don't give an orientation of red strands since they either correspond to $\mathscr{H}_{op}$ or $\mathscr{H}_{op}^{*}$, but the precise appearance of factors is irrelevant. Note that this is just $\mathscr{L}^{-1}$ applied to any tensor factor individually. Next, the composition of the black strands part of $L(f) \circ \phi_{\mathbf{1}}^{L}$ reads

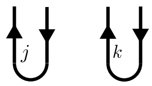


Thus the black strand part of $(\mathrm{id} \otimes \mathscr{L}^{-1}) \circ (\psi^{L} \circ \mathrm{id}) \circ L(f) \circ \phi_{\mathbf{1}}^{L}$ gives

$$\bigoplus_{i,j,k\in \mathsf{I}} \sum_{\alpha} \frac{d_i d_j d_k}{d_i \mathsf{D}^2}$$



$$= \bigoplus_{j,k\in \mathsf{I}} \frac{d_j d_k}{\mathsf{D}^2}$$



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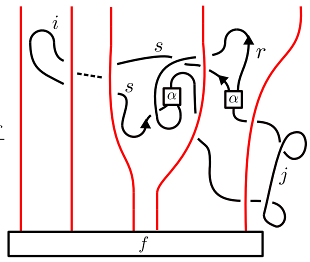
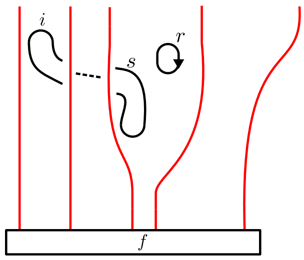
Induction gives

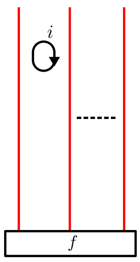
$$Y \circ Z(f) \quad = \quad \bigoplus_{i,\cdots,s,r,j\in \mathsf{I}} \sum_{\alpha_1,\cdots} C_1 \frac{d_s d_r}{\mathsf{D}^2}$$



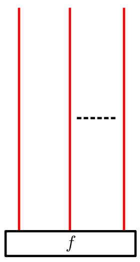
$$= \quad \bigoplus_{i,\cdots,s,r,j\in \mathsf{I}} \sum_{\cdots} C_2 \frac{d_r}{\mathsf{D}^2}$$



$$= \quad \bigoplus_{i\in \mathsf{I}} \frac{d_i}{\mathsf{D}^2}$$



$$=$$



.

□

Let $\overline{\mathscr{H}_{cl}}^*$ be the tensor product of closed boundary labels for S were $\mathscr{H}_{cl}$ is assigned to an *incoming* closed boundary and $\mathscr{H}_{cl}^*$ to an *outgoing* closed boundary. It is clear that with possible intermediate application of braiding morphisms there is map

$$\tilde{\circ} : \mathrm{Hom}_{\mathsf{Z(C)}}\left(\mathbf{1}, \overline{\mathscr{H}_{cl}^*}\right) \otimes \mathrm{Hom}_{\mathsf{Z(C)}}\left(\mathbf{1}, \overline{\mathscr{H}_{cl}} \otimes \overline{L(\mathscr{H}_{op})}\right) \to \mathrm{Hom}_{\mathsf{Z(C)}}\left(\mathbf{1}, \overline{L(\mathscr{H}_{op})}\right) \quad . \tag{4.16}$$

The vector space assigned to a type II) world sheet by the conformal block functor is now defined as

$$\begin{aligned} \mathscr{B}\ell(\tilde{S}) = \Big\{ g \in \mathrm{Hom}_{\mathsf{Z(C)}}\left(\mathbf{1}, \overline{\mathscr{H}_{cl}} \otimes \overline{L(\mathscr{H}_{op})}\right) \Big| & \\ \forall f \in \mathrm{Hom}_{\mathsf{Z(C)}}\left(\mathbf{1}, \widetilde{\mathscr{H}_{cl}}\right) \otimes \cdots\cdots \otimes \mathrm{Hom}_{\mathsf{Z(C)}}\left(\mathbf{1}, \widetilde{\mathscr{H}_{cl}}\right), & \\ \exists h \in \mathrm{Hom}_{\mathsf{Z(C)}}\left(\mathbf{1}, \overline{\mathscr{H}_{op}}\right) \text{ s.th. } f \tilde{\circ} g = Z(h) \Big\} & \quad . \end{aligned} \tag{4.17}$$

III) Let $\tilde{S}$ be any world sheet not of type I) or II). The block space on $\tilde{S}$ is defined as

$$\mathscr{B}\ell(\tilde{S}) = H^s(S, \overline{L(\mathscr{H}_{op})} \otimes \overline{\mathscr{H}_{cl}}) \tag{4.18}$$

where again tensor factors appear according to ord_S and incoming boundaries are labeled with the dual object.

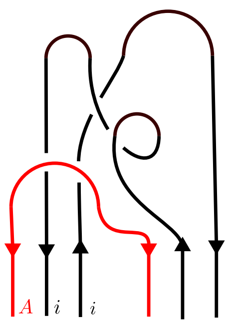
Next we have to define $\mathscr{B}\ell$ on morphisms. The action of the mapping class group part F of a morphism $(\mathcal{S}, F)$ in WS is readily given. By the definition of F it steps down to a homeomorphism of the quotient surface. This homeomorphism acts on the graph of a string-net inducing a linear map on string-net space. This linear map preserves the subspace of the string-net space for type II) world sheets, as these are string-nets on spheres and the only non-trivial elements of the mapping class group are Dehn twists around homology cycles corresponding to boundary components. These can be reversed using the projector circles. Defining the action of sewings is more involved. First note that for any $A \in \mathsf{C}$ there is an isomorphism

$$L(A^*) \simeq L(A)^* \quad . \tag{4.19}$$

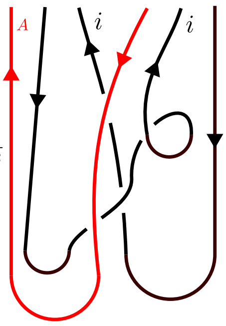
This readily follows by defining evaluation and coevaluation morphisms

$$\begin{aligned} \mathrm{ev}_{L(A)} &= \psi_{\mathbf{1}}^L \circ L(\mathrm{ev}_A) \circ \phi^L : L(A^*) \otimes L(A) \to \mathbf{1} \\ \mathrm{coev}_{L(A)} &= \psi^L \circ L(\mathrm{coev}_A) \circ \phi_{\mathbf{1}}^L : \mathbf{1} \to L(A) \otimes L(A^*) \end{aligned} \tag{4.20}$$

and their graphical representation reads

$$\mathrm{ev}_{L(A)} = \bigoplus_{i \in \mathsf{I}} \frac{\mathsf{D}^2}{d_i}$$


$$\mathrm{coev}_{L(A)} = \bigoplus_{i \in \mathsf{I}} \frac{d_i}{\mathsf{D}^2}$$



The straightening identities follow from the fact that C is ribbon and rigid. Just as for objects we distinguish different types of sewings. Whenever we talk about string-nets on world sheets of type I),II) or III) in the following we actually refer to string-nets on their quotient surfaces.

i) *[I)+I)→ I)]* First consider sewing two world sheets of type I) s.th. the result is again a type I) world sheet. In this case we just concatenate string-nets.

ii) *[I)+I)→ III)]* The second case is sewing two type I) world sheet resulting in a type III) world sheet. First we apply the map Z to the source string-nets on the type I) world sheets and then concatenate. In addition projector circles are a added to possibly new boundary components of the quotient surface.

iii) *[III)+II)→ I)]* This is sewing a type III) world sheet to a type II) world sheet giving a type I) world sheet. In this case the type III) world sheet has quotient surface a disjoint union of disks each having a single closed boundary component. The result is necessarily a disk with solely open boundary components. The action on the string-nets is giving by first concatenating the string-nets and then applying Y to the result mapping it to the correct vector space.

iv) *[I)+II)/III)→ II)/III)]* This is gluing disks with only open insertions to other world sheets using open sewings. First we apply Z to the string-net on the disk and then concatenate.

v) Any other type of sewing is just concatenation of string-nets.

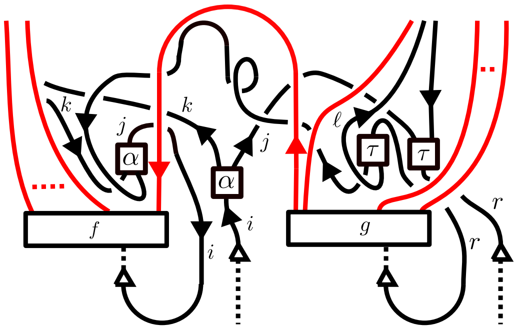
The first major result is the following theorem.

Theorem 4.2.2. *The definitions given for $\mathcal{B}\ell$ on objects of* WS *and on morphisms according to the last paragraph yield a symmetric monoidal functor $\mathcal{B}\ell : \mathsf{WS} \to \mathsf{Vect}$.*

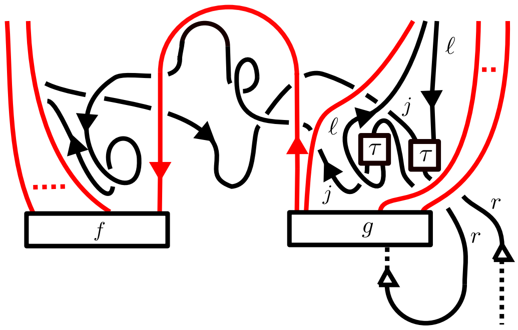
Proof. The only non-trivial part of the theorem is checking that composition of morphisms is well defined. For this we first check that concatenating string-nets on type I) world sheet and then applying Z gives the same as first applying Z to the individual string-nets and then concatenate. Concatenation of string-nets uses evaluation morphisms as coupons are labeled by elements in the vector spaces $\langle \bullet \rangle = \mathrm{Hom}(\mathbf{1}, \bullet)$. Thus let $f \in \mathrm{Hom}_{\mathsf{Z(C)}}(\mathbf{1}, \overline{\mathcal{H}_{op}} \otimes$

$\mathscr{H}_{op}$) and $g \in \mathrm{Hom}_{\mathsf{Z(C)}}(\mathbf{1}, \mathscr{H}_{op}^{*} \otimes \overline{\mathscr{H}_{op}})$, then it holds $Z(f) \in \mathrm{Hom}_{\mathsf{Z(C)}}(\mathbf{1}, \overline{L(\mathscr{H}_{op})} \otimes L(\mathscr{H}_{op})^{*})$, $Z(g) \in \mathrm{Hom}_{\mathsf{Z(C)}}(\mathbf{1}, L(\mathscr{H}_{op}) \otimes \overline{L(\mathscr{H}_{op})})$. For the concatenation we compute

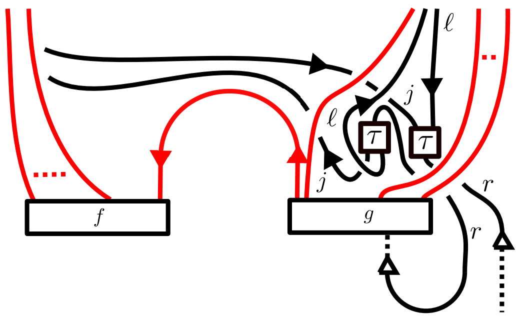
$$\bigoplus_{i,k,j,\ell,r\in \mathsf{I}} \sum_{\alpha,\tau} \frac{d_k d_\ell d_j}{\mathsf{D}^2 d_i d_r}$$



$$\overset{(1)}{=} \bigoplus_{j,\ell,r\in \mathsf{I}} \sum_{\alpha,\tau} \frac{d_\ell d_j}{\mathsf{D}^2 d_r}$$



$$\overset{(2)}{=} \bigoplus_{j,\ell,r\in \mathsf{I}} \sum_{\alpha,\tau} \frac{d_\ell d_j}{\mathsf{D}^2 d_r}$$



where in (1) the definition of the basis elements $\left\{\theta_{\alpha}^{(ij);k}\right\}$ is used and (2) uses the ribbon structure of C. The last picture is nothing but Z applied to the concatenation of the string-nets. Next we note that compositions of gluing disks to closed boundary components of type II) world sheets and gluing along open boundary components of the same world sheet is well defined due to (4.19) . Hence compositions including open boundary components are well defined. Compositions of gluing along closed boundary compositions are obviously well defined. $\square$

To summarize, we defined a functor of conformal blocks by assigning to an object in WS the string-net space with boundary values from the Cardy algebra $(\mathscr{H}_{cl}, \mathscr{H}_{op}, \iota_{cl-op})$ modulo technicalities in case of type II) world sheets. The restriction to the subspace in this case is necessary to get a well defined functor. The conformal block space for type II) world sheets contains all interesting string-nets. In particular it contains all string-nets which are obtained from gluing type I) world sheets to the open boundary component of a type II) world sheet, i.e. it contains all open $\leftrightarrow$ open-closed correlator sewings.

Furthermore, the string-net spaces reproduce state spaces from the Reshetikhin-Turaev tft. Therefore it is sensible to call this functor really a functor of conformal blocks. As at the end of the day all correlators are determined by elements on simple world sheets by factorization, we quickly recall the state spaces of $\mathscr{B}\ell$ in some simple cases.

First of all , to any disk $D(n, m)$ with n incoming open boundary components and m outgoing boundary components it assigns the vector space

$$\mathscr{B}\ell(D(n,m)) = \hat{H}^s(D, \overline{\mathscr{H}_{op}}) = \mathrm{Hom}_{\mathsf{Z(C)}}(\mathbf{1}, \overline{\mathscr{H}_{op}}) \simeq \mathrm{Hom}_{\mathsf{C}}(\mathbf{1}, \overline{\mathscr{H}_{op}}) \quad . \tag{4.21}$$

The genus zero part of the closed theory embedded in the Cardy algebra in the form of the Frobenius algebra $\mathscr{H}_{cl}$ has conformal block spaces

$$\mathrm{Hom}_{\mathsf{Z(C)}}(\mathbf{1}, \overline{\mathscr{H}_{cl}}) \tag{4.22}$$

which are exactly the vector spaces the functor $\mathscr{B}\ell$ gives for spheres with only closed boundary components. Finally in the open closed sector, fundamental world sheets have a single open boundary component. Thus the spaces of conformal blocks read

$$\begin{aligned} \mathscr{B}\ell(I) &= \hat{H}^s(I, L_{op}(\mathscr{H}_{op}) \otimes \mathscr{H}_{cl}^*) \simeq \mathrm{Hom}_{\mathsf{Z(C)}}(\mathbf{1}, L_{op}(\mathscr{H}_{op}) \otimes \mathscr{H}_{cl}^*) \\ \mathscr{B}\ell(I^\dagger) &= \hat{H}^s(I^\dagger, L_{op}(\mathscr{H}_{op})^* \otimes \mathscr{H}_{cl}) \simeq \mathrm{Hom}_{\mathsf{Z(C)}}(\mathbf{1}, L_{op}(\mathscr{H}_{op})^* \otimes \mathscr{H}_{cl}) \quad . \end{aligned} \tag{4.23}$$

This can be easily seen by noting that L not being a tensor functor is irrelevant when having a single open tensor factor, i.e. a single open boundary component. Therefore the subspace (4.17) is in fact the whole space $\mathrm{Hom}_{\mathsf{Z(C)}}\left(\mathbf{1}, \overline{\mathscr{H}_{cl}} \otimes \widetilde{L(\mathscr{H}_{op})}\right)$.

Having a sensible symmetric monoidal functor of conformal blocks the first task is to construct a complete set of consistent correlators on all genus g surfaces. The route we take is to start with a set of fundamental correlator string-nets which define the structure maps of a monoidal natural transformation. For these string-nets we show that the 32 sewing relations hold. As the reader may suspect from the 32 relations, the consistency requirements of a $(\mathsf{C}|\mathsf{Z(C)})$-Cardy algebra will solve the sewing relations.

In the following we don't display the $\mathsf{Z(C)}$-colorings of the string-nets explicitly. Instead, red edges are always colored with $\mathscr{H}_{op}$, orange edges are colored with $\mathscr{H}_{cl}$ and purple edges are colored with $L(\mathscr{H}_{op})$. Though $\mathscr{H}_{op}$ is an object of C, we view it as an object of $\mathsf{Z(C)}$ through the fully faithful embedding $\mathsf{C} \hookrightarrow \mathsf{Z(C)}$ given by $A \mapsto (A, \beta_{A,\bullet})$ using the fact that C itself is braided. In addition we don't give morphism labels. Instead disk shaped vertices of the string-net graph correspond to structure morphisms of Frobenius algebras. The type of structure morphism, i.e. multiplication, comultiplication, unit or counit is determined by the number of incident edges and their orientations. In addition the open-closed structure map ι_{cl-op} and its Frobenius adjoint $\iota^{\dagger}_{cl-op}$ are displayed as rectangular orange boxes, where again the orientation of incident edges determines if the box corresponds to ι_{cl-op} or $\iota^{\dagger}_{cl-op}$. The morphisms for string-nets are always elements in $\mathrm{Hom}(\mathbf{1}, \bullet)$. All structure morphisms can be brought to such vector spaces using evaluation and coevaluation maps. Since concatenation of string-nets is performed using evaluation maps this doesn't matter and we can pretend to draw the actual morphism on the surface. Finally, some of the fundamental string-nets miss projector circles. This is due to the fact that on the corresponding topologies the circles can be homotoped to a point and vanish. Equivalently, the circle can be freely homotoped into an embedded disk, which doesn't intersect the rest of the string-net and according to the graphical calculus in C this evaluates to the identity. We make the following definition.

I) **Open World Sheets:**

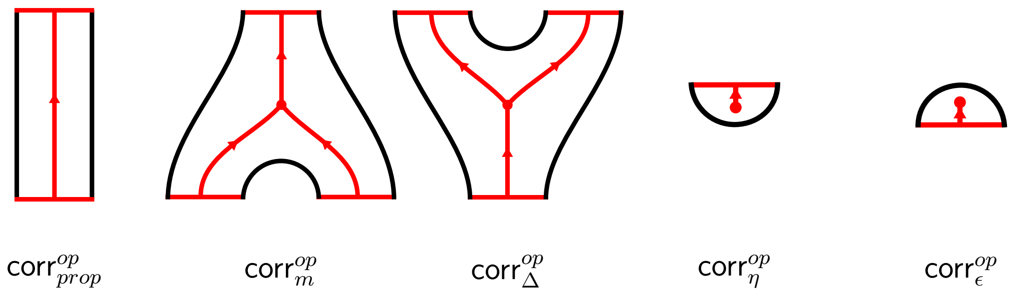


Figure 4.1: Open fundamental correlators.

II) **Closed World Sheets:**

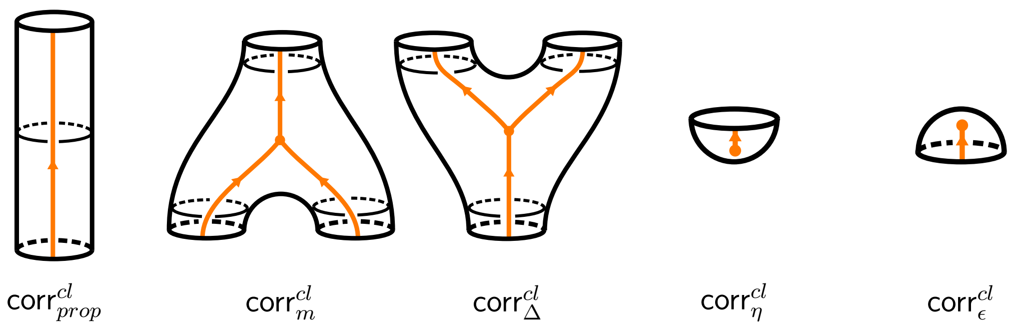
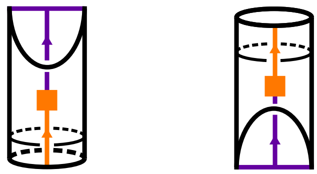


Figure 4.2: Closed fundamental correlators.

III) **Open-Closed World Sheets:**



corr_I $\mathsf{corr}_{I^\dagger}$

Figure 4.3: Open-closed fundamental correlators.

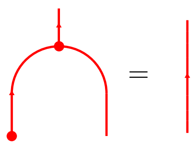
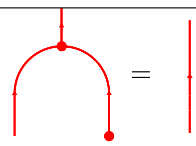
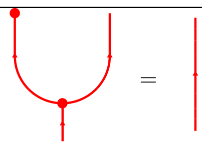
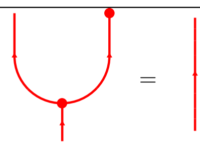
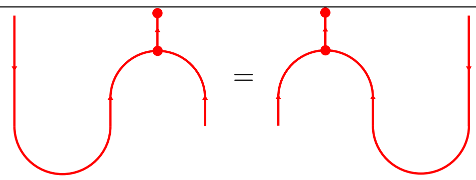
We split the proof for consistent correlators into three separate lemmas.

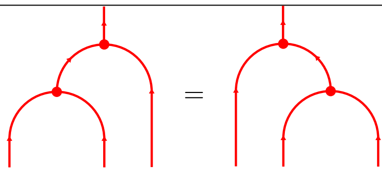
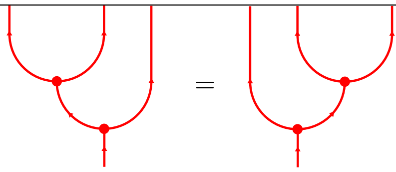
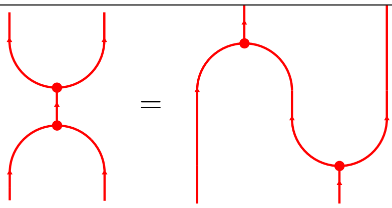
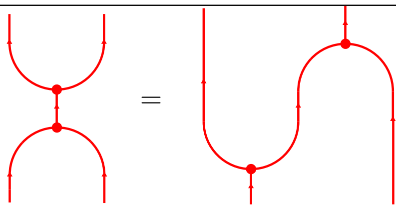
Lemma 4.2.3. *The maps*

$$\left\{\mathsf{corr}^{op}_{prop}, \mathsf{corr}^{op}_{m}, \mathsf{corr}^{op}_{\Delta}, \mathsf{corr}^{op}_{\eta}, \mathsf{corr}^{op}_{\epsilon}\right\} \tag{4.24}$$

satisfy all open sewing constraints.

Proof. This essentially immediately follows from the Frobenius properties. Since the string-nets are embedded into disks the graphical calculus directly applies. In the following table we list the non-trivial open sewing relations in the left column and the property of the Frobenius algebra $\mathcal{H}_{op}$ which ensures the respective sewing relation to hold in the right column.

R1)	$\longleftrightarrow$	
R2)	$\longleftrightarrow$	
R3)	$\longleftrightarrow$	
R4)	$\longleftrightarrow$	
R5)	$\longleftrightarrow$	

R6)	$\longleftrightarrow$	
R7)	$\longleftrightarrow$	
R8)	$\longleftrightarrow$	
R9)	$\longleftrightarrow$	

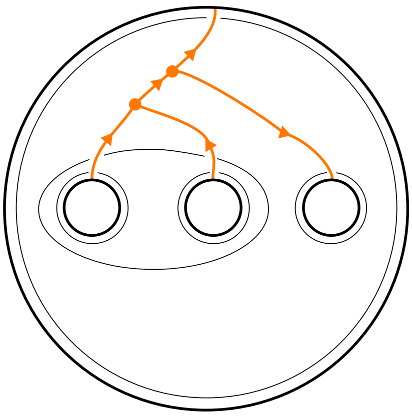
. Relations R10)-R13) just follow from the fact that $\mathsf{corr}^{op}_{prop}$ is the identity on $\mathscr{H}_{op}$. $\square$

Lemma 4.2.4. *The maps*

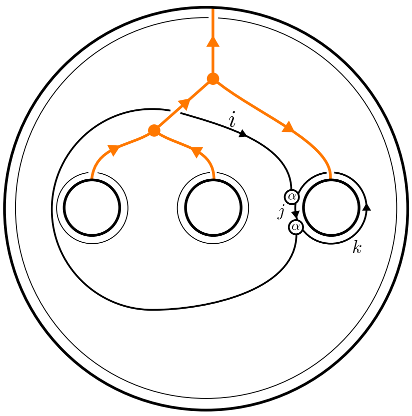
$$\left\{\mathsf{corr}^{cl}_{prop}, \mathsf{corr}^{cl}_{m}, \mathsf{corr}^{cl}_{\Delta}, \mathsf{corr}^{cl}_{\eta}, \mathsf{corr}^{cl}_{\epsilon}\right\} \tag{4.25}$$

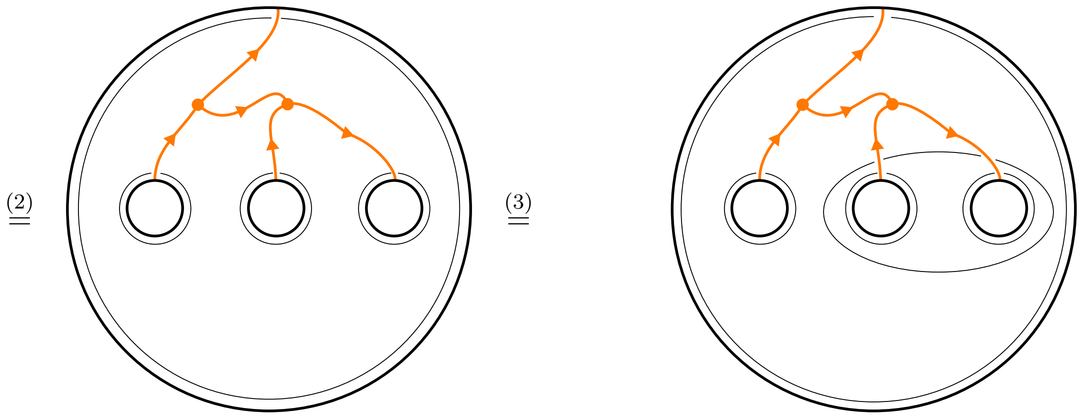
satisfy all closed sewing constraints and the genus 1 relation.

Proof. Relations R14)-R23) follow by almost the same reasons as in the corresponding open case. The only difference is the treatment of the projector circles. We show the argument once for Frobenius relation R22), all other circle deformations follow by the same argument.



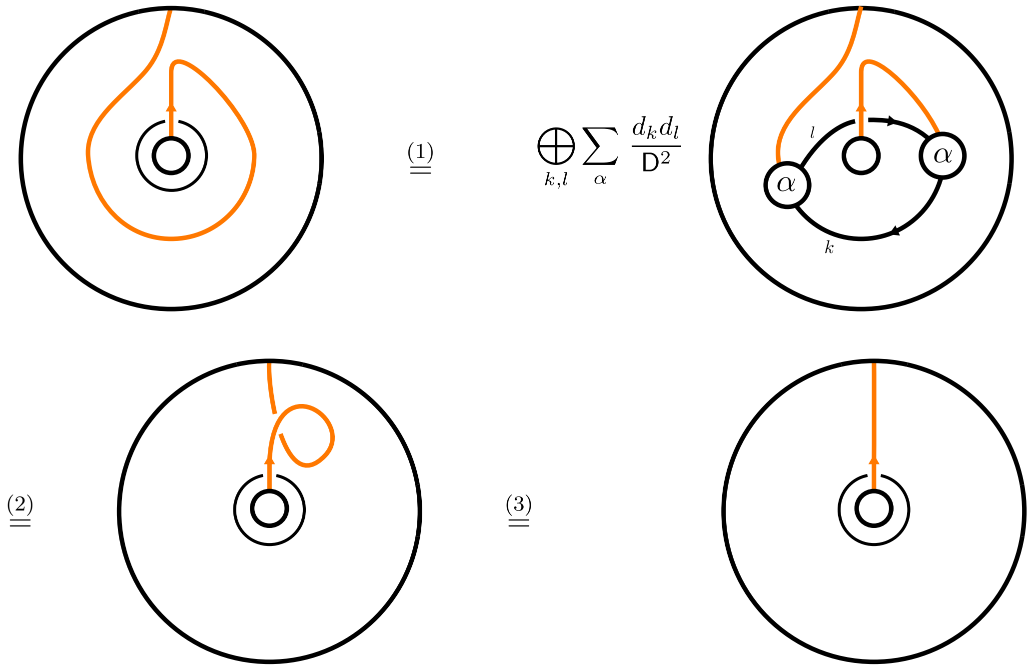
$$\overset{(1)}{=} \bigoplus_{i,j,k\in \mathsf{I}} \sum_{\alpha} \frac{d_i}{\mathsf{D}^2}\frac{d_k}{\mathsf{D}^2} d_j$$





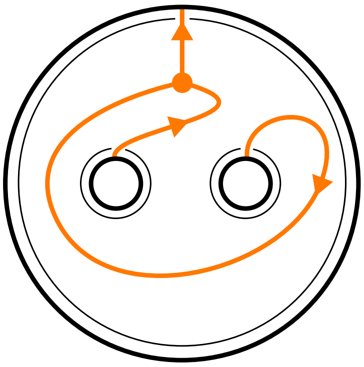
Equality (1) is the completeness relation applied to projector circles. In (2) the completeness relation is performed along the k-labeled circle and in addition one of the Frobenius properties is used to change the $\mathscr{H}_{cl}$-colored string-net. In (3) finally the same argument for circle dragging as in (1) and (2) is performed after doubling the outer projector circle using the projector property.

Next we show the Dehn twist relation R24).

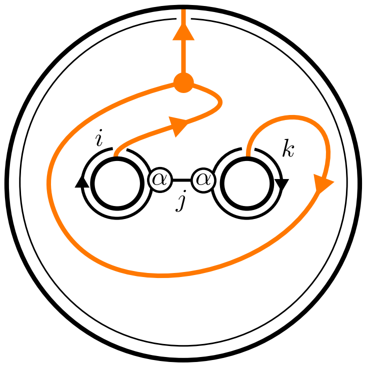


In (1) and (2) the completeness relation is used, which is possible, since $\mathsf{Z}(\mathsf{C})$ has simple objects $(U_i \otimes U_j, \beta^{ou}_{ij,\bullet})$. Then this relation is expanding $\mathscr{H}_{cl}$ into its simple summands and using the completeness relation on the summands, seen as tensor products in C. Since drawing these steps one by one is cumbersome and not enlightening we summarize the procedure in one step. Finally in (3) we use [57, Proposition 2.25] saying that a commutative, symmetric Frobenius algebra in a ribbon category has trivial twist.

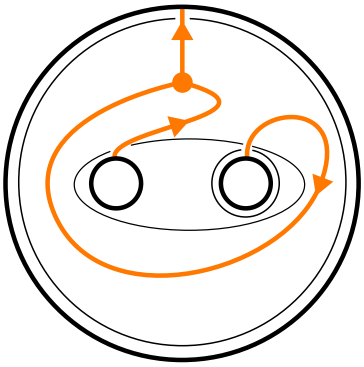
The braiding relation R25) follows from



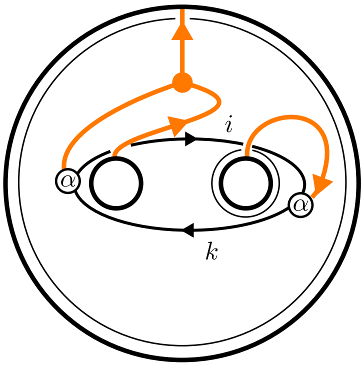
$$\overset{(1)}{=} \bigoplus_{i,j,k\in \mathsf{I}} \sum_{\alpha} \frac{d_i d_k}{\mathsf{D}^4} d_j$$



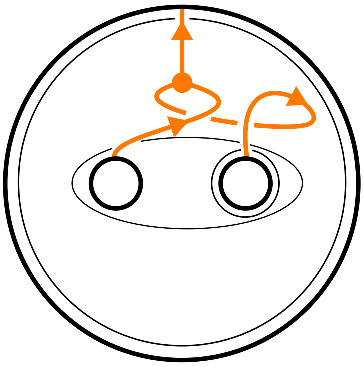
$$\overset{(2)}{=}$$



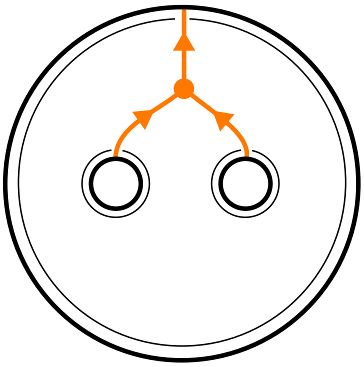
$$\overset{(3)}{=} \bigoplus_{i,k\in \mathsf{I}} \sum_{\alpha} \frac{d_i}{\mathsf{D}^2} \frac{d_k}{\mathsf{D}^2}$$



$$\overset{(4)}{=}$$

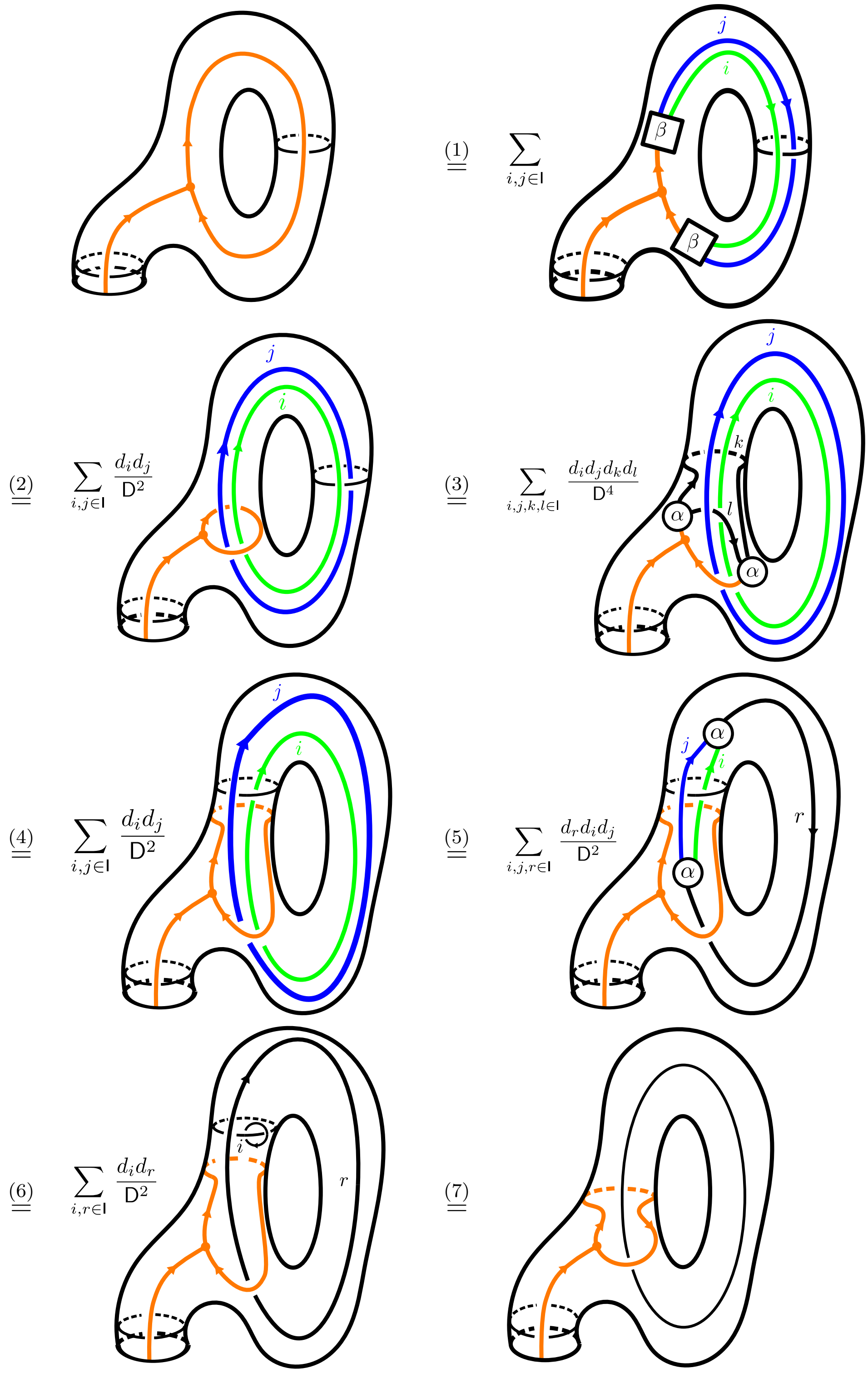


$$\overset{(5)}{=}$$



Again (1), (2) is completeness for projector circles, (3) and (4) uses completeness to drag the $\mathscr{H}_{cl}$-colored edge along the circle and in (5) we first use that $\mathscr{H}_{cl}$ has trivial twist and braiding, as well as a circle dragging on projector circles.

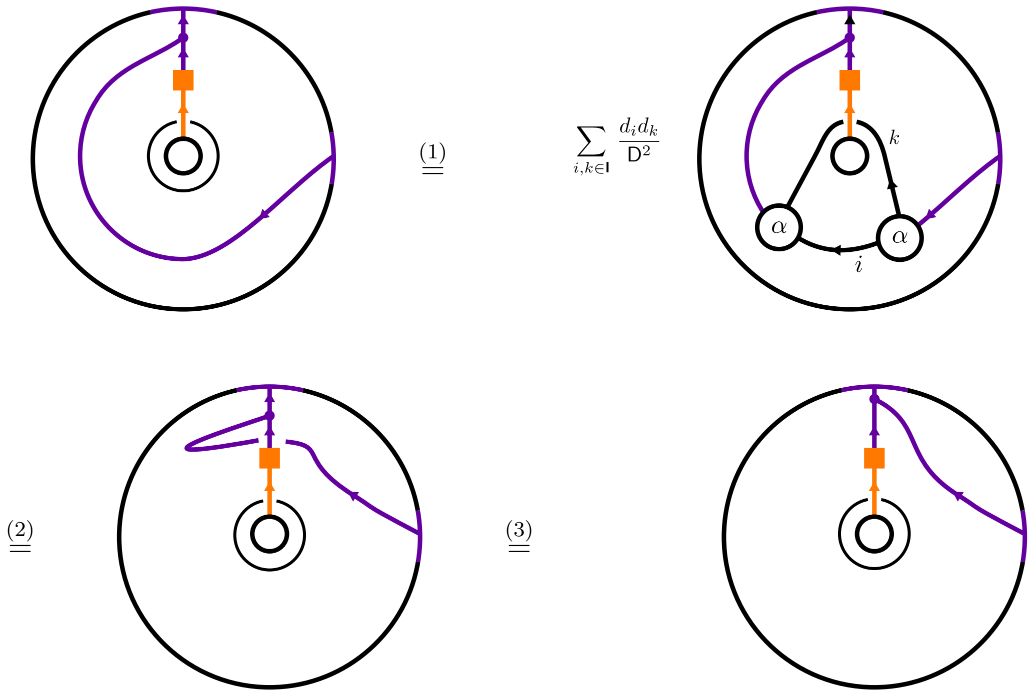
So we are left with showing the genus 1 property which is satisfied by modular invariance of $\mathscr{H}_{cl}$.



Here (1) uses completeness in a different normalization, therefore we replaced coupons with rectangles in the basis summation. Equation (2) is the modular property of $\mathscr{H}_{cl}$. In (3), (4), (5) and (6) we again use completeness to manipulate the string-net graph. Finally in (7) equation B.3 is used. □

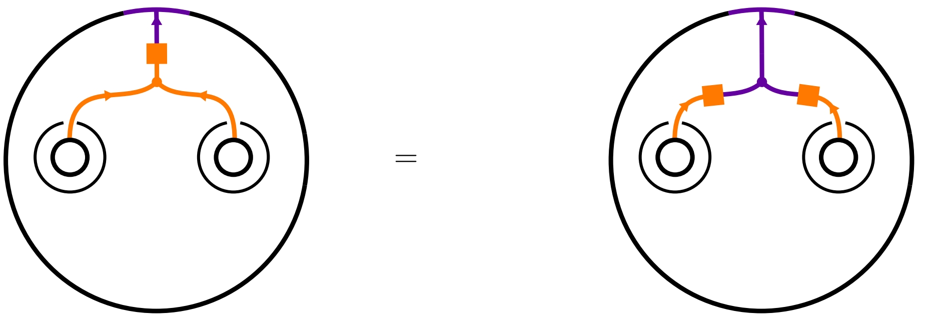
Lemma 4.2.5. *All open-closed sewing relations R26)-R31) are satisfied.*

Proof. The first relation to show is R26).

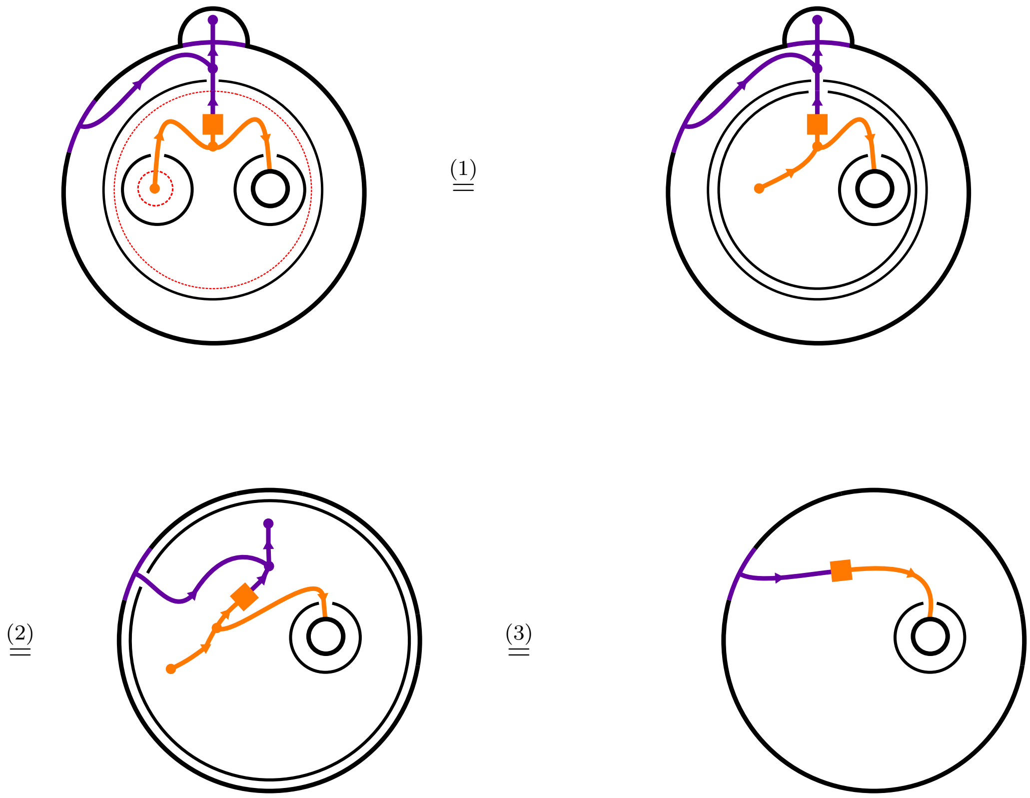


First we drag the $L(\mathscr{H}_{op})$-colored string-net along the projector circle in (1) and (2), then we use the center condition in (3).

Relation R27) is almost trivial and obviously is satisfied as ι_{cl-op} is an algebra homomorphism. In pictures this is

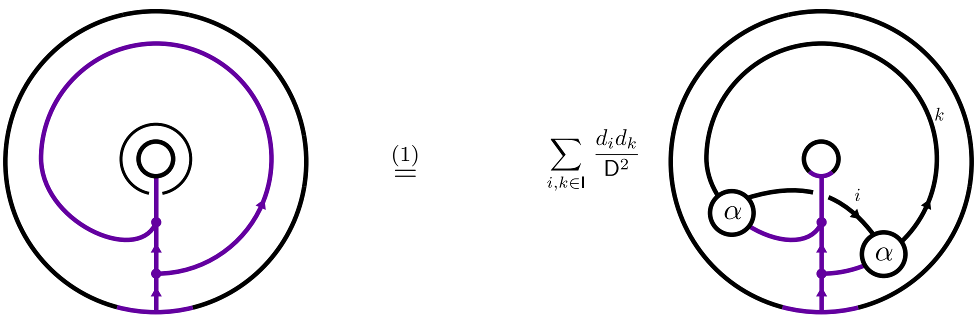


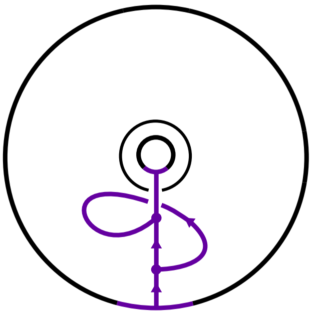
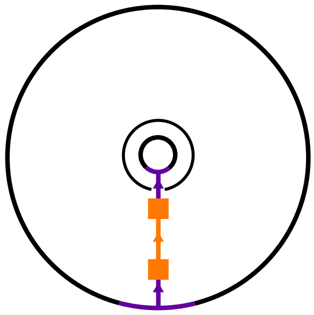
The relations R28) and R29) are consistency requirements for ι_{cl-op} and $\iota^{\dagger}_{cl-op}$. We only show R28), the other follows by a similar computation.



In the first picture the red dotted lines indicate the gluing of the world sheet and (3) is just the definition of $\iota_{cl-op}^{\dagger}$.

The relation R30) is the unit part of ι_{cl-op} being an algebra homomorphism. Since we had related arguments several times by now and there is really not much to show we don't give the graphical representation here. The last relation is R31) and since the only condition we haven't used so far is the Cardy condition one might already guess that this is the central point of the argument.



$\overset{(2)}{=}$  $\overset{(3)}{=}$ 

One last time we note that (1) and (2) is the completeness relation dragging the $L(\mathcal{H}_{op})$-colored string-net along the projector circle. Equality (3) is the Cardy condition. $\square$

So we see that each property of a Cardy algebra corresponds to a specific sewing relation. One might say that the procedure given here realizes the higher genus conformal bootstrap in an abstract, categorical way since we showed that any correlator for an open-closed rational conformal field theory has a unique expansion in terms of some fundamental correlators. It is also noteworthy that it suffices to have consistency conditions in genus zero and a single requirement in genus one in order to have an all genus description. No further constraints appear for higher genus correlators. We summarize the results obtained so far in a theorem.

Theorem 4.2.6. *Given the fundamental correlators*

$$\left\{\mathsf{corr}^{op}_{prop}, \mathsf{corr}^{op}_{m}, \mathsf{corr}^{op}_{\Delta}, \mathsf{corr}^{op}_{\eta}, \mathsf{corr}^{op}_{\epsilon}, \mathsf{corr}^{cl}_{prop}, \mathsf{corr}^{cl}_{m}, \mathsf{corr}^{cl}_{\Delta}, \mathsf{corr}^{cl}_{\eta}, \mathsf{corr}^{cl}_{\epsilon}, \mathsf{corr}_{I}, \mathsf{corr}_{I^\dagger}\right\} \tag{4.26}$$

on generating world sheets all 32) sewing relations are satisfied. Therefore there is a monoidal natural transformation

$$\mathsf{corr} : \Delta \Rightarrow \mathcal{B}\ell \tag{4.27}$$

giving consistent all genus correlators for the open-closed RCFT described by the $(\mathsf{C}|\mathsf{Z}(\mathsf{C}))$-*Cardy algebra* $(\mathcal{H}_{cl}, \mathcal{H}_{op}, \iota_{cl-op})$.

One might wonder if the other direction of the theorem holds, i.e. given a consistent set of correlators for the conformal blocks functor, does this determine a Cardy algebra? By methods involving the use of the Reshetikhin-Turaev tft this was shown in [106]. Here we give a somewhat easier proof in terms of string-nets. The main simplification is that one doesn't need to go through three dimensional bordisms for whom proofs tend to become difficult as there is no easy way of picturing three dimensional objects on a sheet of paper.

We start by recalling that the first homology class $H_1(\Sigma, \mathbb{Z})$ of a compact surface Σ of genus g with n boundary components $\{\partial\Sigma\}_i$ is generated by circles around a and b-cycles of the g-handles and circles which can be freely homotoped to a boundary component. An example showing the homology cycles is given in figure 4.4.

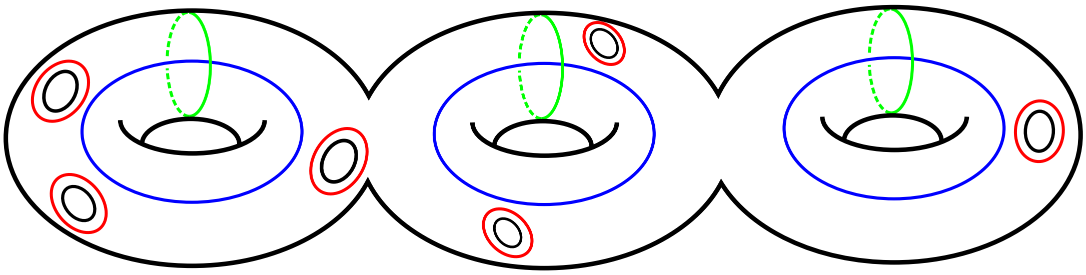
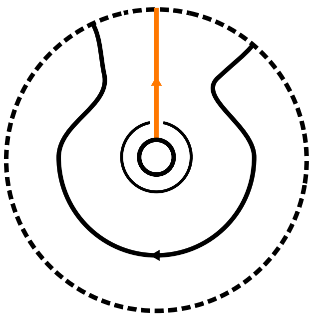
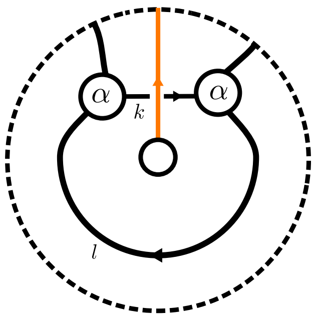


Figure 4.4: A genus 3 surface $S_{3,6}$ with a-cycles respectively b-cycles shown in green and blue. Red circle show boundary generators in $H_1(S_{3,6})$.

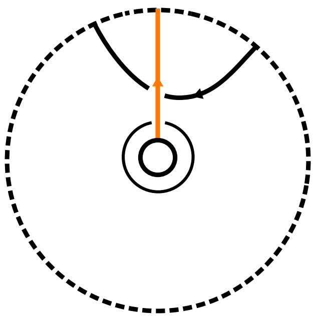
String-nets on any compact surface have projector circles winding once around boundary generators of the first homology class. Assume that a string-net winds around such a generator. In an annular neighborhood of the boundary we can perform the manipulation of string-nets



$$\overset{(1)}{=} \sum_{k,l\in \mathsf{I}} \frac{d_k d_l}{\mathsf{D}^2}$$

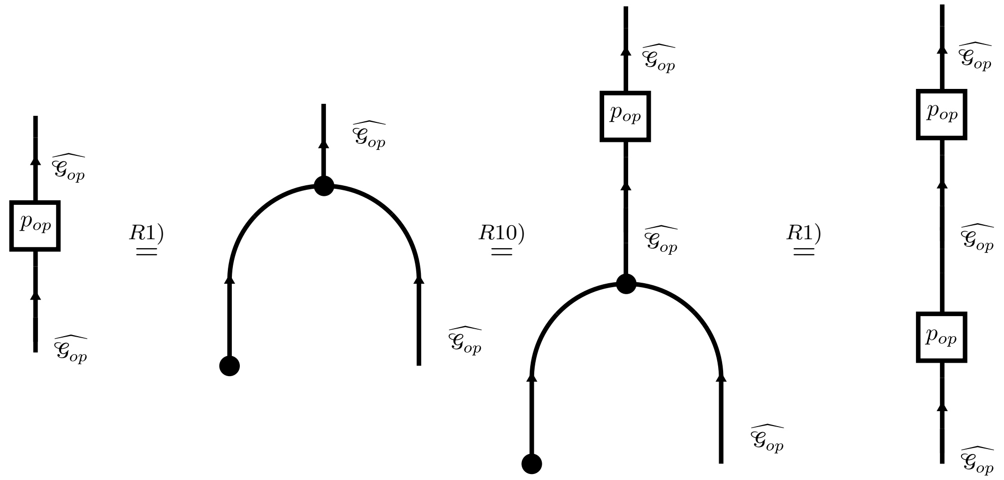


$$\overset{(2)}{=}$$



with (1), (2) the usual circle dragging. Orientations of string-nets shown in the picture are arbitrary and by induction on the winding number one might replace any string-net by an equivalent string-net which doesn't wind around boundary homology cycles. In particular string-nets on spheres are all equivalent to string-nets with a single coupon, as they can be unwind leading to a string-net where all edges not intersecting the boundary are localized in a disk. These internal edges together with their incident vertices can be composed into a single morphism. Hence one might conclude that string-nets on fundamental world sheets are all of the form shown in figures 4.1, 4.2 and 4.3, with vertices labeled by arbitrary morphisms. This is almost true. The only complication comes from open and closed propagator world sheets, whose correlators evaluated to the identity map before. A general string-net on O_{prop}, C_{prop} can of course have an arbitrary morphism. Thus we have to enlarge the setup.

Assume we are given objects $\widehat{\mathcal{G}_{op}} \in \mathsf{C}$ and $\widehat{\mathcal{G}_{cl}} \in \mathsf{Z}(\mathsf{C})$ and we color open boundaries with $\widehat{\mathcal{G}_{op}}$, closed boundaries with $\widehat{\mathcal{G}_{cl}}$. A string-net on O_{prop} is determined by a morphism $p_{op} : \widehat{\mathcal{G}_{op}} \to \widehat{\mathcal{G}_{op}}$. Furthermore, suppose that the morphisms on all other fundamental world sheets are chosen s.th. we get a solution to the sewing constraints. Then relation R10) and R1) give



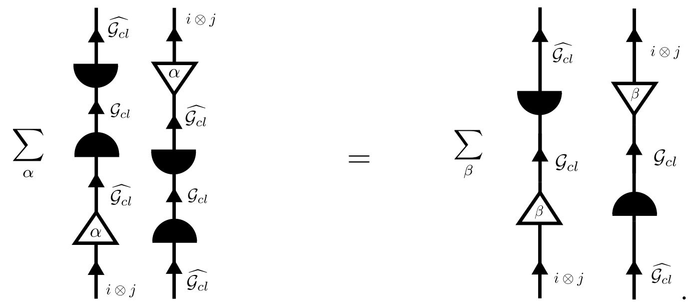
showing that p is in fact an idempotent. By the same argument based on relations R14) and R16) one shows that the closed propagator map $p_{cl} : \widehat{\mathcal{G}_{cl}} \to \widehat{\mathcal{G}_{cl}}$ is also an idempotent. Since C and $\mathsf{Z}(\mathsf{C})$ are abelian categories one can split the idempotents

$$\begin{aligned} r_{op} : \widehat{\mathcal{G}_{op}} \to \mathcal{G}_{op}, \quad e_{op} : \mathcal{G}_{op} &\to \widehat{\mathcal{G}_{op}} \\ r_{op} \circ e_{op} = \mathrm{id}_{\mathcal{G}_{op}}, \quad e_{op} \circ r_{op} &= p_{op} \end{aligned} \tag{4.28}$$

$$\begin{aligned} r_{cl} : \widehat{\mathcal{G}_{cl}} \to \mathcal{G}_{cl}, \quad e_{cl} : \mathcal{G}_{cl} &\to \widehat{\mathcal{G}_{cl}} \\ r_{cl} \circ e_{cl} = \mathrm{id}_{\mathcal{G}_{cl}}, \quad e_{cl} \circ r_{cl} &= p_{cl} \quad . \end{aligned} \tag{4.29}$$

In physics terms this corresponds to a situation where the state space of the theory is overly large leading to states with non-invertible two point function. We may put invertibility of the two point functions in by hand, i.e. we require p_{op} and p_{cl} to be invertible. Then they are invertible idempotents on finite dimensional vector spaces, thus they are the identity. But choosing instead the retracts (splitting) (4.28), (4.29) corresponds to restricting the theory to smaller state spaces $\mathcal{G}_{op}$, $\mathcal{G}_{cl}$ with invertible two point functions. For the morphisms exhibiting the retracts the graphical representation

is chosen. Next note that similar to [106, Lemma 4.4] it holds



It is in fact not hard to show this. Both side are elements in

$$\bigoplus_{i,j\in \mathsf{I}} \mathrm{Hom}_{\mathsf{Z}(\mathsf{C})}(U_i\otimes U_j, \widehat{\mathcal{G}_{cl}})\otimes \mathrm{Hom}_{\mathsf{Z}(\mathsf{C})}(\widehat{\mathcal{G}_{cl}}, U_i\otimes U_j) \quad . \tag{4.30}$$

Since $\left\{(U_i\otimes U_j, \beta^{ou}_{ij,\bullet})\right\}$ are the simple objects of $\mathsf{Z}(\mathsf{C})$ the map

$$\begin{aligned} \bigoplus_{i,j\in \mathsf{I}} \mathrm{Hom}_{\mathsf{Z}(\mathsf{C})}(U_i\otimes U_j, \widehat{\mathcal{G}_{cl}})\otimes \mathrm{Hom}_{\mathsf{Z}(\mathsf{C})}(\widehat{\mathcal{G}_{cl}}, U_i\otimes U_j) &\to \mathrm{Hom}_{\mathsf{Z}(\mathsf{C})}(\widehat{\mathcal{G}_{cl}}, \widehat{\mathcal{G}_{cl}}) \\ \sum_{i,j\in\mathsf{I}} v_{ij}\otimes w_{ij} &\mapsto \sum_{i,j\in\mathsf{I}} v_{ij}\circ w_{ij} \end{aligned} \tag{4.31}$$

is bijective. Applying the map to both sides and using completeness of the basis as well as the splitting properties of r_{cl}, e_{cl} shows that both sides map to the same element in $\mathrm{Hom}_{\mathsf{Z}(\mathsf{C})}(\widehat{\mathcal{G}_{cl}}, \widehat{\mathcal{G}_{cl}})$.

From the discussion of string-nets on genus zero surfaces it is clear that there are morphisms .

O_m	O_Δ	O_η	O_ϵ	C_m	C_Δ	C_η	C_η	I	$I^\dagger$
$\widehat{m_{op}}$	$\widehat{\Delta_{op}}$	$\widehat{\eta_{op}}$	$\widehat{\epsilon_{op}}$	$\widehat{m_{cl}}$	$\widehat{\Delta_{cl}}$	$\widehat{\eta_{cl}}$	$\widehat{\epsilon_{cl}}$	$\widehat{\iota}$	$\widehat{\iota}^\dagger$

Table 4.7: The first row states the type of world sheet and the second row the corresponding maps.

These morphisms have source and target the big state spaces $\widehat{\mathcal{G}_{op}}$ and $\widehat{\mathcal{G}_{op}}$. In addition one can use the same proofs as before to establish certain properties for the hatted morphisms

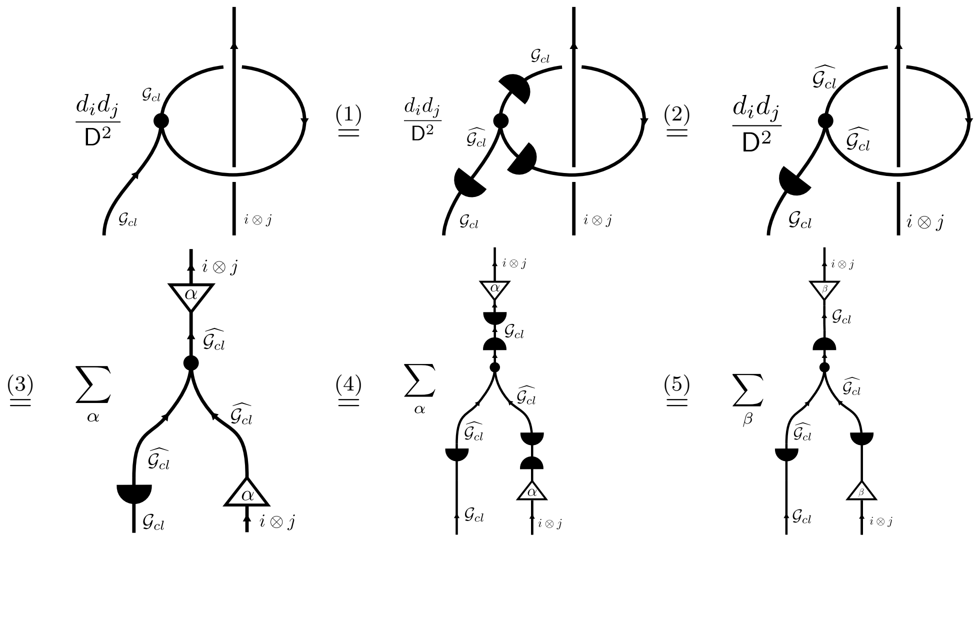
R6)	$\Rightarrow$	$\widehat{m_{op}}$ is associative product on $\widehat{\mathcal{G}_{op}}$
R7)	$\Rightarrow$	$\widehat{\Delta_{op}}$ is coassociative coproduct on $\widehat{\mathcal{G}_{op}}$
R8)	$\Rightarrow$	$\widehat{\Delta_{op}} \circ \widehat{m_{op}} = (\widehat{m_{op}} \circ \mathrm{id}) \circ (\mathrm{id} \circ \widehat{\Delta_{op}})$
R9)	$\Rightarrow$	$\widehat{\Delta_{op}} \circ \widehat{m_{op}} = (\widehat{\Delta_{op}} \circ \mathrm{id}) \circ (\mathrm{id} \circ \widehat{m_{op}})$
R9)	$\Rightarrow$	$p_{op} \circ \widehat{m_{op}} = \widehat{m_{op}}$
R10)	$\Rightarrow$	$\widehat{\Delta_{op}} \circ p_{op} = \widehat{\Delta_{op}}$
R10)	$\Rightarrow$	$p_{op} \circ \widehat{\eta_{op}} = \widehat{\eta_{op}}$
R11)	$\Rightarrow$	$\widehat{\epsilon_{op}} \circ p_{op} = \widehat{\epsilon_{op}}$
R16)	$\Rightarrow$	$p_{cl} \circ \widehat{m_{cl}} = \widehat{m_{cl}}$
R17)	$\Rightarrow$	$\widehat{\Delta_{cl}} \circ p_{cl} = \widehat{\Delta_{cl}}$
R18)	$\Rightarrow$	$p_{cl} \circ \widehat{\eta_{cl}} = \widehat{\eta_{cl}}$
R19)	$\Rightarrow$	$\widehat{\epsilon_{cl}} \circ p_{cl} = \widehat{\epsilon_{cl}}$
R20)	$\Rightarrow$	$\widehat{m_{cl}}$ is associative product on $\widehat{\mathcal{G}_{cl}}$
R21)	$\Rightarrow$	$\widehat{\Delta_{cl}}$ is coassociative coproduct on $\widehat{\mathcal{G}_{cl}}$
R22)	$\Rightarrow$	$\widehat{\Delta_{cl}} \circ \widehat{m_{cl}} = (\widehat{m_{cl}} \circ \mathrm{id}) \circ (\mathrm{id} \circ \widehat{\Delta_{cl}})$
R23)	$\Rightarrow$	$\widehat{\Delta_{cl}} \circ \widehat{m_{cl}} = (\widehat{\Delta_{cl}} \circ \mathrm{id}) \circ (\mathrm{id} \circ \widehat{m_{cl}})$
R24)	$\Rightarrow$	$\widehat{\mathcal{G}_{cl}}$ has trivial twist.
R25)	$\Rightarrow$	$\widehat{m_{cl}}$ is commutative.
R26)	$\Rightarrow$	$\widehat{\iota_{cl-op}}$ satisfies the center condition for $L(\widehat{m_{op}})$.
R27) & R30)	$\Rightarrow$	$\widehat{\iota_{cl-op}}$ is an algebra homomorphism for $L(\widehat{m_{op}})$.
R31)	$\Rightarrow$	$\widehat{\iota_{cl-op}}$ satisfies the Cardy condition for $L(\widehat{m_{op}})$

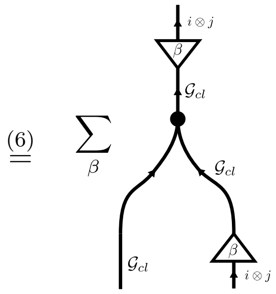
R32) $\Rightarrow$ $\widehat{m_{cl}}$ satisfies the modularity condition.

But relations R1-R4) and R15),R16) don't give that $\widehat{\eta_{op}}$/ $\widehat{\epsilon_{op}}$ or their closed equivalents are unit/ counit maps. But it is readily checked that

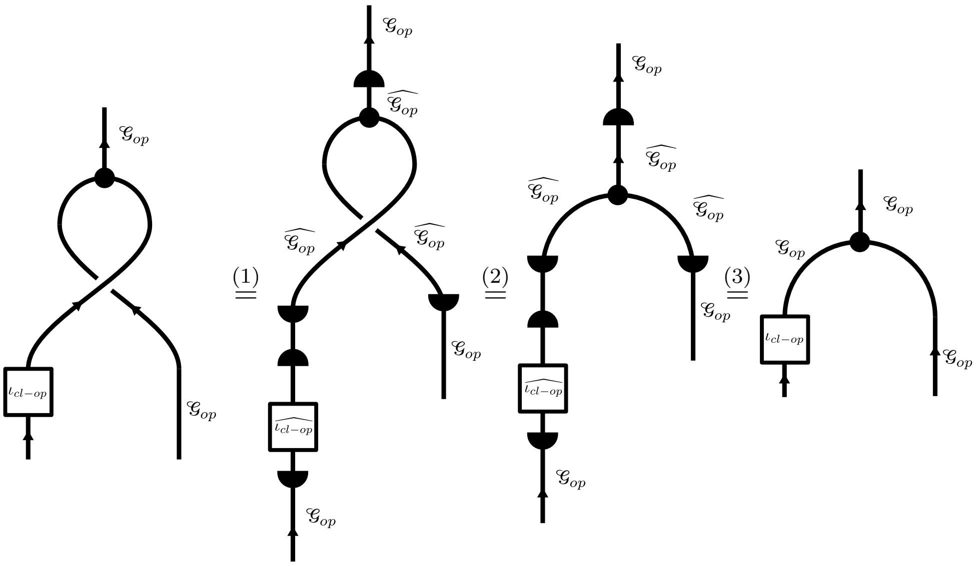
$$\begin{aligned}
m_{op} &= r_o \circ \widehat{m_{op}} \circ (e_o \otimes e_o) : \mathcal{G}_{op} \otimes \mathcal{G}_{op} \to \mathcal{G}_{op} \\
\Delta_{op} &= (r_o \otimes r_o) \circ \widehat{\Delta_{op}} \circ e_o : \mathcal{G}_{op} \to \mathcal{G}_{op} \otimes \mathcal{G}_{op} \\
\eta_{op} &= r_o \circ \widehat{\eta_{op}} : \mathbf{1} \to \mathcal{G}_{op} \\
\epsilon_{op} &= \widehat{\epsilon_{op}} \circ e_o : \mathcal{G}_{op} \to \mathbf{1} \\
m_{cl} &= r_{cl} \circ \widehat{m_{cl}} \circ (e_{cl} \otimes e_{cl}) : \mathcal{G}_{cl} \otimes \mathcal{G}_{cl} \to \mathcal{G}_{cl} \\
\Delta_{cl} &= (r_{cl} \otimes r_{cl}) \circ \widehat{\Delta_{cl}} \circ e_{cl} : \mathcal{G}_{cl} \to \mathcal{G}_{cl} \otimes \mathcal{G}_{cl} \\
\eta_{cl} &= r_{cl} \circ \widehat{\eta_{cl}} : \mathbf{1} \to \mathcal{G}_{cl} \\
\epsilon_{cl} &= \widehat{\epsilon_{cl}} \circ e_{cl} : \mathcal{G}_{cl} \to \mathbf{1} \\
\iota_{cl-op} &= L(r_o) \circ \widehat{\iota_{cl-op}} \circ e_{cl} : \mathcal{G}_{cl} \to L(\mathcal{G}_{op}) \\
\iota^{\dagger}_{cl-op} &= r_{cl} \circ \widehat{\iota^{\dagger}_{cl-op}} \circ L(e_o) : L(\mathcal{G}_{op}) \to \mathcal{G}_{cl}.
\end{aligned} \tag{4.32}$$

yield a symmetric Frobenius algebra structure on $\mathcal{G}_{op}$ and a commutative, symmetric Frobenius algebra structure on $\mathcal{G}_{cl}$. Intermediate propagators vanish due to relations in the table involving the propagator morphisms. Less trivial is the fact that center condition, Cardy condition and modularity still hold for the unhatted morphisms. We start with modularity.

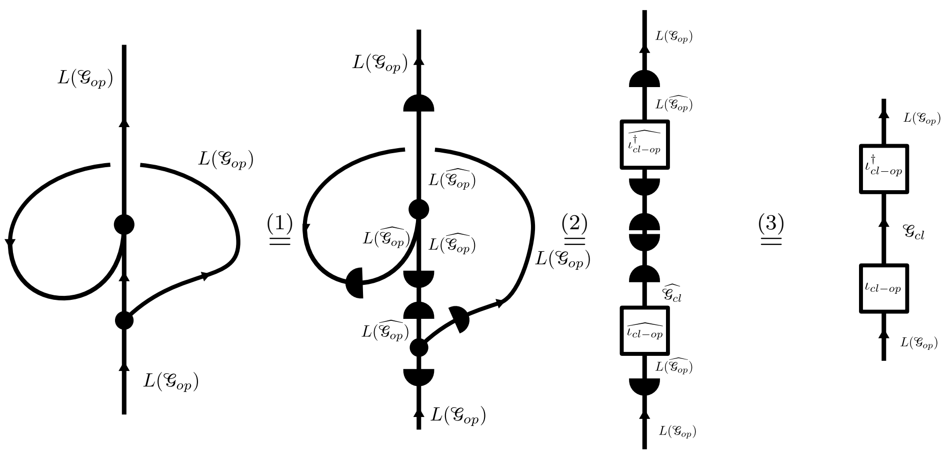




In (1) the definition of m_{cl} is inserted, (2) and (4) are $\widehat{m_{cl}} \circ (\mathrm{id} \otimes p_{cl}) = \widehat{m_{cl}}$, (3) is modularity for $\widehat{m_{cl}}$, (5) uses 4.6 and (6) is again the definition of m_{cl}. The center condition follows from



where (1) is the definition of ι_{cl-op} and $m_{L(\mathscr{G}_{op})}$, (2) is the center condition for $\widehat{\iota_{cl-op}}$ where we inserted an intermediate propagator morphism without changing the diagram. (3) is again insertion of definitions. Finally the Cardy condition is the computation



with all steps following the by now familiar arguments from modularity and center condition.

This shows that if we start with a solution of the sewing constraints for a $\widehat{\mathscr{G}_{op}}/\ \widehat{\mathscr{G}_{cl}}$-colored conformal blocks functor from string-nets, we in fact get the structure of a Cardy algebra on the retracts $\mathscr{G}_{op}/\ \mathscr{G}_{cl}$. But even more is true. Sewing constraints give a Cardy algebra unique up to isomorphism. Assume that for a given solution we choose another set of retracts $(\mathscr{G}'_{op}, r'_{op}, e'_{op})$, $(\mathscr{G}'_{cl}, r'_{cl}, e'_{cl})$.

Lemma 4.2.7. *The maps*

$$\begin{aligned} f_{op} : r_{op} \circ e'_{op}\mathscr{G}'_{op} &\to \mathscr{G}_{op} \\ f_{cl} : r_{cl} \circ e'_{cl}\mathscr{G}'_{cl} &\to \mathscr{G}_{cl} \end{aligned} \tag{4.33}$$

are isomorphisms of Frobenius algebras.

Proof. It suffices to show that both are algebra as well as coalgebra maps. We only present the proof for the open maps, the closed case goes exactly the same.

$$\begin{aligned} m_{op} \circ (f_{op} \otimes f_{op}) &= r_{op} \circ \widehat{m_{op}} \circ (e_{op} \circ f_{op} \otimes e_{op} \circ f_{op}) \\ &= r_{op} \circ \widehat{m_{op}} \circ (p_{op} \circ e'_{op} \otimes p_{op} \circ e'_{op}) \\ &= r_{op} \circ \widehat{m_{op}} \circ (e'_{op} \otimes e'_{op}) \\ &= r_{op} \circ e'_{op} \circ r'_{op} \circ \widehat{m_{op}} \circ (e'_{op} \otimes e'_{op}) \\ &= f_{op} \circ m'_{op} \end{aligned} \tag{4.34}$$

$$f_{op} \circ \eta' = f_{op} \circ r'_{op} \circ \widehat{\eta_{op}} = r_{op} \circ \widehat{\eta_{op}} = \eta \tag{4.35}$$

This shows that f_{op} is an algebra map.

$$\begin{aligned} \Delta_{op} \circ f_{op} &= (r_{op} \otimes r_{op}) \circ \widehat{\Delta_{op}} \circ e_{op} \circ r_{op} \circ e'_{op} \\ &= (r_{op} \otimes r_{op}) \circ \widehat{\Delta_{op}} \circ e'_{op} \\ &= (r_{op} \circ e_{op} \otimes r_{op} \circ e_{op}) \circ (r'_{op} \otimes r'_{op})\widehat{\Delta_{op}} \circ e'_{op} \\ &= (f_{op} \otimes f_{op}) \circ \Delta'_{op} \end{aligned} \tag{4.36}$$

$$\epsilon_{op} \circ f_{op} = \epsilon_{op} \circ e_{op} \circ r_{op} \circ e'_{op} = \epsilon_{op} \circ e'_{op} = \epsilon'_{op} \tag{4.37}$$

This shows that f_{op} is a coalgebra map. □

Lemma 4.2.8. *The diagram*

$$\begin{array}{ccc} \mathscr{G}'_{cl} & \xrightarrow{f_{cl}} & \mathscr{G} \\ {\scriptstyle \iota'_{cl-op}}\downarrow & & \downarrow{\scriptstyle \iota_{cl-op}} \\ L(\mathscr{G}'_{op}) & \xrightarrow{L(f_{op})} & L(\mathscr{G}_{op}) \end{array}$$

commutes.

Proof.

$$\begin{aligned}\iota_{cl-op} \circ f_{cl} &= L(r_{op}) \circ \widehat{\iota_{cl-op}} \circ e_{cl} \circ r_{cl} \circ e'_{cl} \\ &= L(r_{op}) \circ \widehat{\iota_{cl-op}} \circ e'_{cl} \\ &= L(r_{op}) \circ L(e'_{op} \circ r'_{op}) \widehat{\iota_{cl-op}} \circ e'_{cl} \\ &= L(f_{op}) \circ \iota'_{cl-op}\end{aligned} \tag{4.38}$$

□

We summarize the above lemmas in a theorem.

Theorem 4.2.9. *Given any set of fundamental string-nets on generating world sheets with closed boundary values $\widehat{\mathcal{G}_{cl}}$ and open boundary values $\widehat{\mathcal{G}_{op}}$, $L(\widehat{\mathcal{G}_{op}})$ which satisfy the sewing constraints defines a $(\mathsf{C}|\mathsf{Z}(\mathsf{C}))$-Cardy algebra $(\mathcal{G}_{cl}, \mathcal{G}_{op}, \iota_{cl-op})$, which is unique up to isomorphism.*

4.3 Structure Constants and Defect Fields

There exists a convenient way to compute structure constants and prove modular invariance as well as factorization of correlators under sewing of world sheets in a more explicit fashion. The procedure is related to the FRS formalism [58][60][61][62][63][48], but instead of the Reshetikhin-Turaev tft we use the string-net tft outlined in appendix C. This allows us to compute structure constants of correlators for highest weight fields in an expansion of elementary two and three point functions in terms of conformal blocks. The spaces of conformal blocks are still given by Hom-spaces in a modular tensor category. Thus the missing ingredient for true correlation functions is an isomorphism of modular functors $\mathcal{M}_{Cat} \xrightarrow{\simeq} \mathcal{M}_{an}$ where $\mathcal{M}_{Cat}$ is the modular functor in its categorical form as used in this thesis and $\mathcal{M}_{an}$ is a modular functor given as a vector bundle with flat connection over the moduli space of Riemann surfaces. The genus zero and one part of the latter functor is given by correlation functions of intertwining operators and their q-traces. A higher genus analytic construction is missing though.

Before giving the construction, we take our time and discuss boundary states and defects in RCFTs. In a Cardy algebra there is a fixed boundary condition in the form of a Frobenius algebra. Hence there are no boundary condition changing operators in the theory. The categorical description of boundary fields and conditions is due to [60]. Its central object is a special, symmetric Frobenius algebra F in a modular tensor category C. The reader may think of C as the representation category of a rational VOA. One can construct a full RCFT with arbitrary topological defects and symmetry preserving boundary conditions just from F. This may come as a surprise since F is purely chiral data, bulk objects, however, require the inclusion of antichiral fields. In section 4.4 we compute the bulk partition function using the string-net formalism. This computation shows, that indeed, the full content of the bulk theory can be derived from F. So, how to arrive at F? In chapter 3.1.1 we discussed conformal open field algebras $(H, \mathbb{Y}, \mathbf{1})$ over a rational VOA V. In the course of section 3.1.1 we explained that the state space H is a module over V. Hence we may decompose it in terms of simple modules $\{U_i\}$ for V

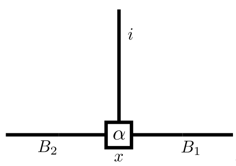
$$H = \bigoplus_{i \in \mathsf{I}} n_i U_i \tag{4.39}$$

where $n_i \in \mathbb{Z}_{\geq 0}$ are multiplicity coefficients. The open vertex operator $\mathbb{Y}$ became a V-intertwining operator of type $\binom{H}{HH}$. Splitting it into simple objects yields

$$\mathbb{Y} = \sum_{i,j,k \in \mathsf{I}} \sum_{\alpha=1}^{n_i} \mathcal{Y}_{ij,\alpha}^{k} = \sum_{i \in \mathsf{I}} \sum_{\alpha=1}^{n_i} \Psi_{i,\alpha}^{H} \quad . \tag{4.40}$$

The summands $\Psi_{i,\alpha}^{H}$ give the coupling of the primary field corresponding to the representation U_i to the boundary. Since there are n_i such primaries, there are n_i possibly different couplings. We have seen that the product of intertwining operators gave $(H, \mathbb{Y}, \mathbf{1})$ the structure of an algebra in R_V. If there is in addition a non-degenerate bilinear form on H this algebra becomes a symmetric Frobenius algebra. For it to be *special* one has to make some additional assumptions. For a discussion of how the assumptions lead to specialness of the Frobenius algebra see [60, section 3.2]. The first assumption is that there are no primaries of negative conformal weight in the theory. Recall that simple representations $\{U_i\}$ of a VOA come with fixed lowest weights $\{h_i\}$ and the statement translates to the fact that in all categorical construction non of the representations with $h_i \leq 0$ appear. Recall that a necessary condition for a unitary highest weight representation of the Virasoro algebra is a positive conformal dimension of the highest weight state. The second assumption is that the *bulk theory* H_{cl} adjacent to the boundary theory H has a unique vacuum state $|0\rangle$ which is normalized $\langle 0|0\rangle = 1$. Categorically this means that $\dim_{\mathbb{C}}[\mathrm{Hom}(\mathbf{1}, H_{cl})] = 1$. When computing the partition function later, this in turn is the requirement $Z_{00} = 1$. However, from the formalism one very well computes partition functions having $Z_{00} > 0$. As discussed in [60, section 3.2] this corresponds to the superposition of several CFTs whose correlation functions completely decouple. Hence the assumption still applies in the sense that it applies in each individual summand. One could circumvent this problem by demanding F to be *indecomposable* [123], i.e. there don't exist algebras $F_1, F_2 \in \mathsf{C}$ s.th. $F = F_1 \oplus F_2$ as an algebra.

Assume that we are in the more general situation of having different boundary conditions B_1, B_2 which preserve the underlying chiral symmetry, i.e. they are representations of the chosen underlying VOA V. In such a theory one can have *boundary changing operators* $\left\{\Psi_{i,\alpha}^{B_2 B_1}(x)\right\}$ [31][115][125][47] corresponding to a situation modeled on the upper half plane by



.

These are fields changing the boundary conditions from B_1 to B_2 inserted at the point x on the real axis. In addition they transform in the representation U_i of V and have a multiplicity label α. Similar to the previous case, when interpreting $\Psi^{B_1 B_2}_{i,\alpha}$ as an intertwining operator, this gives the coupling of the primary corresponding to a U_i summand of the theory to the two boundaries. Conformal invariance and sewing constraints [125][115] naturally lead to the fact that there are OPEs

$$\left\{ \begin{matrix} \text{boundary changing} \\ \text{fields}\, B_2 \to B_3 \end{matrix} \right\} \times \left\{ \begin{matrix} \text{boundary changing} \\ \text{fields}\, B_1 \to B_2 \end{matrix} \right\} \to \left\{ \begin{matrix} \text{boundary changing} \\ \text{fields}\, B_1 \to B_3 \end{matrix} \right\} \tag{4.41}$$

which in case $B_1 = B_2 = F$, $B_3 \equiv B$ can be interpreted as having a morphism

$$\rho : F \otimes B \to B \tag{4.42}$$

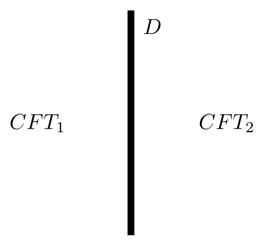
in C (see [60, section 4.4]). The sewing constraints give that ρ equips B with the structure of a left F-module. The upshot is, that other boundary conditions are described by left F-modules in C and field insertions are elements of $\mathrm{Hom}_F(B_1 \otimes U_k, B_2)$. Being a left F-module has some nice consequences, e.g. as on object of C, B decomposes as

$$B = \bigoplus_{i \in \mathsf{I}} A^B_{i,F} U_i \tag{4.43}$$

where $A^B_{i,F} = \dim \left[\left\{ \text{boundary changing operators } \Psi^{B,F}_{i,\alpha} \right\} \right]$. On the other hand $A^B_{i,F}$ should be expansion coefficients of the annulus partition function with one boundary condition given by B and the other by F. We will compute these integers using string-nets in section 4.4.

A similar discussion can be made for defects. The naturally expected outcome is that defect conditions correspond to $F_1|F_2$-bimodules in C if the defect separates CFTs build from boundary algebras F_1, F_2. Field insertions are naturally elements of $\mathrm{Hom}_{F_1|F_2}(U_k \otimes^+ D_1 \otimes^- U_l, D_2)$ describing a defect changing field $D_1 \to D_2$. In case $D_1 = D_2 = F$ these are just *bulk fields*.

However, we make the simplifying assumption that the defects should be *topological*. Assume the following situation



where D is the defect. Let $T_1, \overline{T}_1$ and $T_2, \overline{T}_2$ be the holomorphic and anti-holomorphic parts of the energy momentum tensors for CFT_1 and CFT_2. The defect should preserve conformal symmetry. By the *folding trick* one can view D as boundary for the conformal field theory $CFT_1 \otimes \overline{CFT_2}$, where $\overline{CFT_2}$ has left and right movers interchanged. Hence preservation of conformal symmetry can be given in terms of the gluing condition

$$T_1(x) + \overline{T}_2(x) - \overline{T}_1(x) - T_2(x) = 0 \tag{4.44}$$

with x a coordinate running along the defect. Being a topological defect means that (4.44) is solved by

$$T_1(x) = T_2(x), \quad \overline{T}_1(x) = \overline{T}_2(x) \quad . \tag{4.45}$$

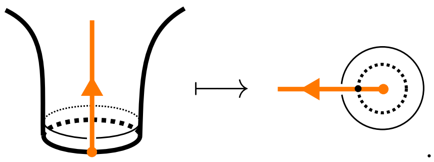
Note that this in particular implies that CFT_1 and CFT_2 have the same central charge and that defect lines can be freely moved as long as they don't cross a field insertion. Hence the name *topological.*

Before we start with the construction a word about choices. Notice that one chooses a fixed but arbitrary symmetry preserving boundary condition F for a VOA V at the beginning. The corresponding special symmetric Frobenius algebra in R_V fixes the whole RCFT. A good question is how much arbitrariness is involved in that choice, i.e under which conditions do different choices of boundary conditions lead to different RCFTs. Since the description is purely categorical this question can be stated as, when do different special symmetric Frobenius algebras in a modular tensor category C lead to different theories? Since all ingredients of the theory are given in form of modules for the chosen Frobenius algebra it comes not as a surprise that different choices F_1, F_2 lead to the same theory if they are *Morita equivalent* as algebras in C [60][123]. An algebra F in C has a category of left modules ${}_F\mathsf{M}$ with objects $(M, \rho) \in \mathsf{C}$ left modules and morphisms are maps $\mathrm{Hom}_{\mathsf{C}}(M, N) \ni \Phi : (M, \rho_M) \to (N, \rho_N)$ intertwining the module maps. Roughly speaking F_1, F_2 are Morita equivalent if there is an equivalence of categories ${}_{F_1}\mathsf{M} \simeq {}_{F_2}\mathsf{M}$.

In this section it will be convenient to change the model for extended surfaces. So far we used compact surfaces Σ with parametrized boundaries and implicitly we also assumed that there is a given marked point on each component of the boundary, which for convenience can be taken as the preimage of $(0, 1) \in \mathbb{C}$ under the boundary parametrization. An equivalent description is given by gluing disks to each boundary component resulting in a closed surface. Let D_i be the disk glued to the i-th boundary component. Denote p_0 for the image of $0 \in D_i$ under the gluing. Then an additional germ of an arc, or equivalently a non-zero vector $v \in T_{p_0}\Sigma$ is chosen. That the two descriptions of the surface, one with parametrized boundary and marked point on the boundary and the other as a closed surface with marked points and a choice of non-vanishing tangent vectors at marked points, are the same is shown e.g. in [5, Proposition 5.18]. Let $(\Sigma, \{\psi_i\}, \{p_i\})$ be a compact surface Σ with $\partial\Sigma = \bigsqcup_i \partial_i\Sigma$ boundaries, boundary parametrizations $\psi_i : \partial_i\Sigma \to S^1$ and marked points $p_i \in \partial_i\Sigma$. The corresponding closed surface is denoted by $(\widehat{\Sigma}, \{q_i\}, \{v_i\})$, with $\widehat{\Sigma}$ the closed surface obtained by gluing in disks, q_i the center of the i-th glued disk and v_i the non vanishing tangent vector at q_i. There is a well defined string-net space on $(\widehat{\Sigma}, \{q_i\}, \{v_i\})$, where univalent vertices of the embedded graph have to be marked points on the surface and the edge incident to a univalent graph has to agree in an open neighborhood of the point with the germ of an arc induced by the non vanishing tangent vector. Denote the string-net space with boundary value $\{A_i \in \mathsf{Z}(\mathsf{C})\}$ by $H^s(\widehat{\Sigma}, \mathbf{A})$. In [98, Theorem 7.3] it is shown that there is an isomorphism of string-net spaces

$$H^s(\Sigma, \mathbf{A}) \simeq H^s(\widehat{\Sigma}, \mathbf{A}) \tag{4.46}$$

which, roughly speaking, is given by the map



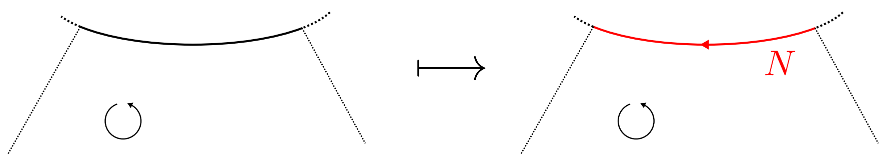
On the rhs the dotted circle corresponds to the former boundary with marked point and the solid circle is still a projector circle.

The advantage of this description in this section is that boundaries of surfaces will correspond to boundary field insertion whereas closed field insertions are just marked points in the bulk. In the following C will be a fixed modular tensor category with list of simple objects $\{U_i\}_{i\in\mathsf{I}}$ and $F \in \mathsf{C}$ is a symmetric, special Frobenius algebra. In addition in this section a *world sheet* is an oriented compact surface Σ with possibly non-empty boundary and a finite set of marked points $\{(p_i, v_i)\}_{i=1,\cdots,N}$ in the bulk and a finite set of marked points $\{(q_j, \pm)_{j=1,\cdots,M}\}$ on the boundary. The extra label $\pm$ for boundary points keeps track of incoming insertions $(-)$ and outgoing ones $(+)$. Starting from a similar setup, in [50] a correlator for the surface with arbitrary bulk, boundary and defect fields is constructed. We give a construction in terms of string-net graphs.

Recall that a string-net has an underlying isotopy class of an embedded graph. The starting point is an isotopy class of an oriented, finite graph $\Gamma \hookrightarrow \Sigma$, s.th. its image on the surface Σ gives a cell decomposition. In addition each vertex has an arbitrarily chosen distinguished edge, s.th. the orientation of Σ induces a cyclic order on its incident edges. In the following we will not distinguish between the abstract graph Γ and its image under the embedding. Let $V_2 \subset V(\Gamma)$ be the subset of two-valent vertices. Let C_2 be the set of 2-cells determined by Γ. To define a string-net we label each 2-cell c with a fixed special, symmetric Frobenius algebra F_c in C. A string-net is built on Σ with given 2-cell labels along the following steps:

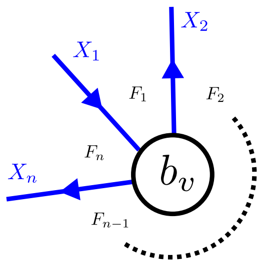
I) *(Bulk edges)* An edge E in the bulk is adjacent to exactly two 2-cells c_l, c_r, one to the left and one to the right according to the orientation of the graph. The edge gets colored by a $F_{c_l}|F_{c_r}$-bimodule.

II) *(Boundary edges I)* An edge running along a boundary component without two valent vertex is colored as



where the oriented circle shows the orientation of Σ and N is a left F_{c_l}-module.

III) *(Bulk vertex I)* For $n \geq 3$, an n-valent vertex v in the bulk gets colored as

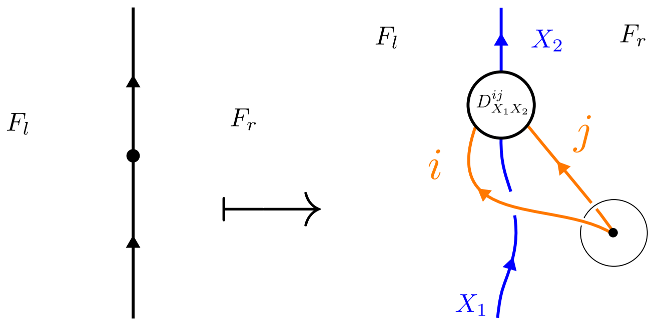


with $b_v \in \left\langle X_1^* \otimes_{F_1} X_2 \otimes_{F_2} \cdots \otimes_{F_{n-1}} X_n \right\rangle$. The first edge is the distinguished edge here and b_v is in the image of the map

$$\mathrm{Hom}_{F_1|F_n}(X_1, X_2 \otimes_{F_2} \cdots \otimes_{F_{n-1}} X_n) \to \mathrm{Hom}(\mathbf{1}, X_1^* \otimes_{F_1} X_2 \otimes_{F_2} \cdots \otimes_{F_{n-1}} X_n) \quad . \tag{4.47}$$

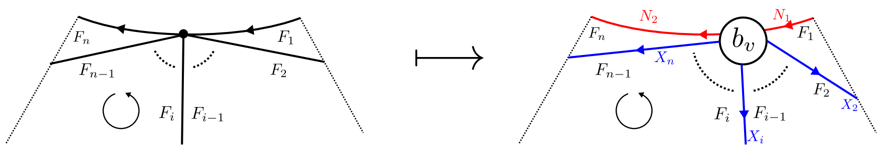
If one of the edges is colored by a Frobenius algebra F, i.e. is a transparent defect, the map is required to factor through the module morphism. In case the transparent defect edge is oriented towards the vertex the module map is precomposed with ϕ_F^{-1}.

IV) *(Bulk vertex II)* For a two valent vertex in the bulk we get a string-net



Note that it doesn't matter if the insertion of the simple object $(U_i \otimes U_j, \beta^{ou}_{ij}) \in \mathsf{Z}(\mathsf{C})$ is to the left or right of the edge entering the $D^{ij}_{X_1X_2}$-colored coupon. The edge can be dragged along the projector circle mapping one choice to the other.

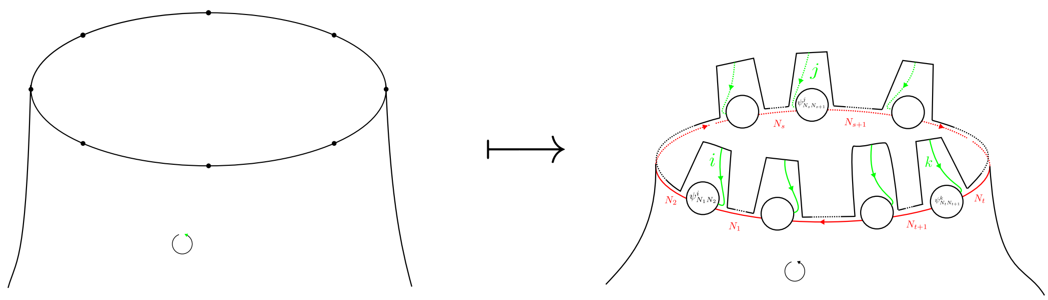
V) *(Boundary vertex I)* For $n \geq 3$ a n-valent vertex on the boundary is replaced with the string-net



where

$$b_v \in \mathrm{Hom}_{F_1|\mathbf{1}}(N_1, X_2 \otimes_{F_2} \cdots \otimes_{F_n-1} X_n \otimes_{F_n} N_2) \quad . \tag{4.48}$$

VI) *(Boundary vertex II)* Two valent vertices on the boundary are replaced by string-nets in strips. The general situation looks as follows



where we only show the two valent vertices of Γ in a neighborhood of the boundary. As before N_ℓ is left F-module and $\psi^{\ell}_{PQ} \in \mathrm{Hom}_F(P \otimes U_\ell, Q)$. In addition we labeled only some field insertions and dotted black lines running along the boundary indicate possible additional field insertions not shown here. We also suppressed the $\pm$-label of field insertions. The manifold with string-net is a manifold with corners and in case the boundary two valent vertex has label $-$ we think of the line segment to which the corresponding green curve is attached as oriented opposite to the orientation induced from the bulk.

The following is immediate from the properties of string-net spaces.

Lemma 4.3.1. *Given a world sheet Σ with graph Γ, a fixed label for 2-cells, bulk marked points $\left\{(p_i, v_i, D^{k_i\ell_i}_{X_iY_i})\right\}_{i=1,\cdots N}$, boundary marked points $\left\{(q_j, \pm, \psi^{s_j}_{N_jN_{j+1}})\right\}_{j=1,\cdots,M}$ and a fixed choice of boundary conditions $\{N_f\}$ for boundary components without field insertion and fixed coloring of $n \geq 3$-valent vertices the above assignments give a well defined element*

$$\mathsf{corr}(\mathbf{D}, \Psi) \in H^s(\Sigma, \bigotimes_{i=1}^{N} U_{k_i\ell_i} \otimes \bigotimes_{j=1}^{M} U_{s_j}) \tag{4.49}$$

where we denoted $U_{pq} = (U_p \otimes U_q, \beta^{ou}_{pq,\bullet})$ and $\mathbf{D} = (D^{k_1\ell_1}_{X_1Y_1}, \cdots, D^{k_N\ell_N}_{X_NY_N})$, $\Psi = (\psi^{s_1}_{N_1N_2}, \cdots, \psi^{s_M}_{N_MN_{M+1}})$.

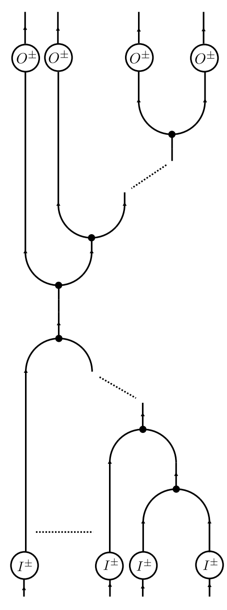
Note that we implicitly choose a specific order for open and closed insertions.

If two adjacent 2-cells are labeled with the same Frobenius algebra F, an edge incident to both can be labeled with F itself. Such edges are usually called *transparent*, as they don't correspond to an actual defect. Let $E_T(\Gamma) \subset E(\Gamma)$ be the set of transparent edges and $\Gamma_T \subset \Gamma$ the *transparent subgraph* given by the union of all transparent half-edges and vertices incident to transparent half-edges. Its complement $\Gamma_D = \Gamma \backslash \Gamma_T$ is called *defect subgraph*. It includes all true defect lines as well as all field insertions. Let C_D be the union of all two cells whose boundaries are unions of true defect edges. We require that the transparent edges of Γ give a triangulation of C_D. A sensible isomorphism of world sheets should leave the defect graph invariant. The right notion of morphism was formulated in [50, Definition 5.1] and it says that a morphism $f : \Sigma \to \widetilde{\Sigma}$ is an element of the mapping class group of Σ, which maps $f(\Gamma_D) = \widetilde{\Gamma}_D$ and in addition preserves all 2-cell, edge and vertex labels.

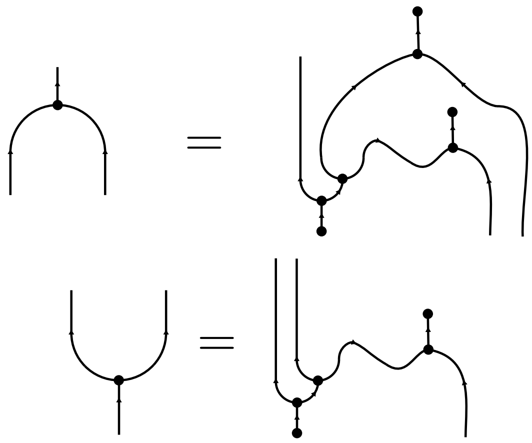
Proposition 4.3.2. *Let $f : (\Sigma, \Gamma) \to (\widetilde{\Sigma}, \widetilde{\Gamma})$ be an isomorphism of world sheets with defect graph. Then*

$$f_{\#}(\mathsf{corr}_{\Sigma}(\mathbf{D}, \Psi)) = \mathsf{corr}_{\widetilde{\Sigma}}(\mathbf{D}, \Psi) \quad . \tag{4.50}$$

Proof. Since f preserves the isotopy class of Γ_D we can assume $\Gamma = \widetilde{\Gamma}_D$. Hence it suffices to show that action on the transparent subgraph doesn't change the correlator. First of all note that by [50, Lemma A.2, Lemma A.1] any $n \geq 3$-valent vertex of the transparent subgraph with only F-labeled edges incident to it factors through three valent vertices and is given by



with $I^+ = O^+ = \mathrm{id}_F$ and $I^- = \phi_F^{-1}$, $O^- = \phi_F$. Using the Frobenius properties it is not hard to see that it holds

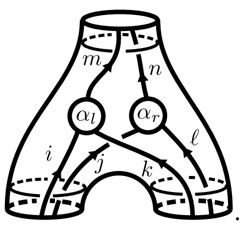


In addition if $v \in \Gamma_T$ has defect edges incident to it, its morphism label factors through left module maps as described in [50, Definition 3.1]. The morphism f fixes each 2-cell of C_D as it fixes the defect graph. It only changes the transparent subgraph inside a 2-cell of C_D. Furthermore Γ_T is a dual triangulation of such a 2-cell which is labeled according to [48, Appendix B]. Thus by [48, Lemma 3.3-3.6] this is independent of the triangulation. Since f maps any triangulation to another triangulation, this shows (4.50). □

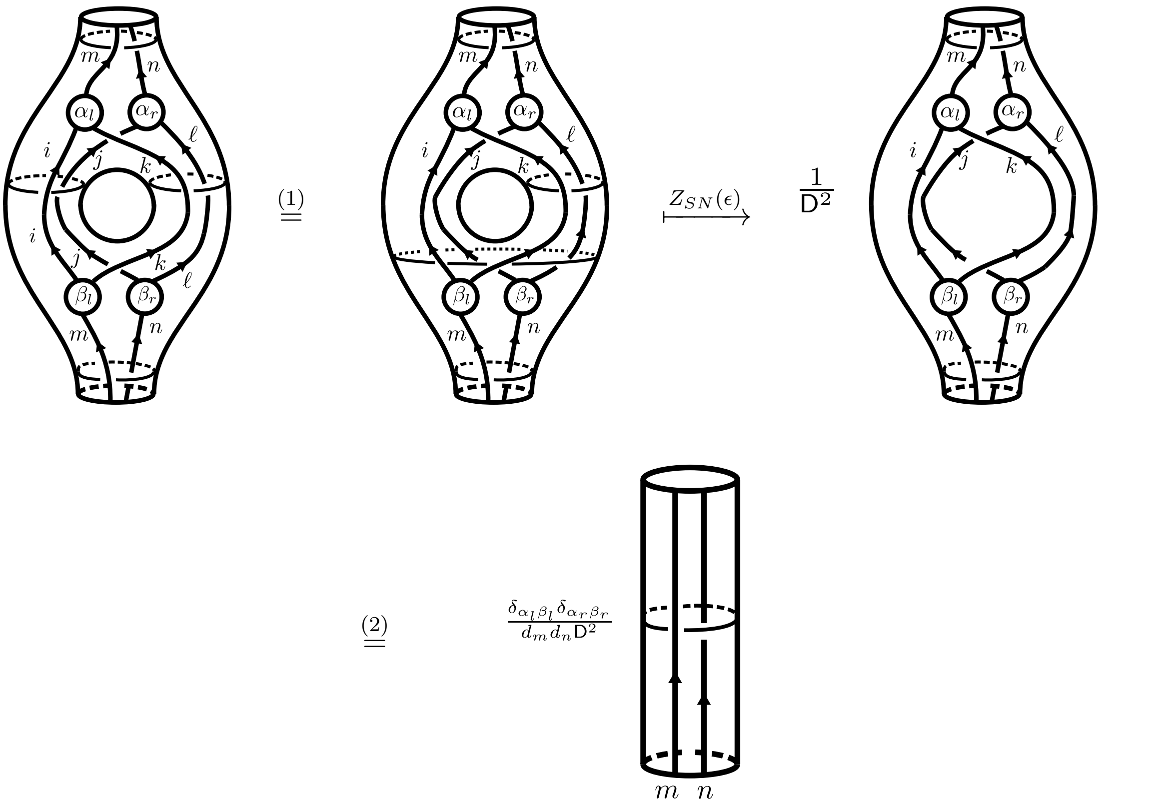
In order to compute structure constants we have to give a basis for string-net spaces on the sphere. The boundary values of our string-nets consist entirely of simple objects in $\mathsf{Z(C)}$ and

$$H^s(S^2, \bigotimes_{i=1}^{K} U_{s_i r_i}) = \mathrm{Hom}_{\mathsf{Z(C)}}(\mathbf{1}, \bigotimes_{i=1}^{K} U_{s_i r_i}) \quad . \tag{4.51}$$

Let $\left\{\theta_\alpha^{((ij)(kl));(mn)}\right\}$ be basis in $\mathrm{Hom}_{\mathsf{Z(C)}}(U_{mn}, U_{ij} \otimes U_{kl})$. Since left and right movers in the Drinfeld center separate, the corresponding string-net on a three punctured sphere reads

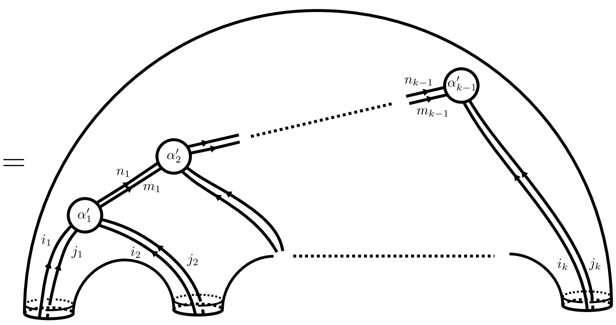


The ϵ-move of the string-net yields

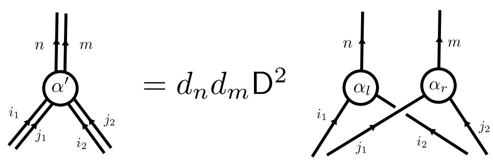


In the first step the projector circle is dragged and (2) is duality of the basis elements. This shows that string-nets

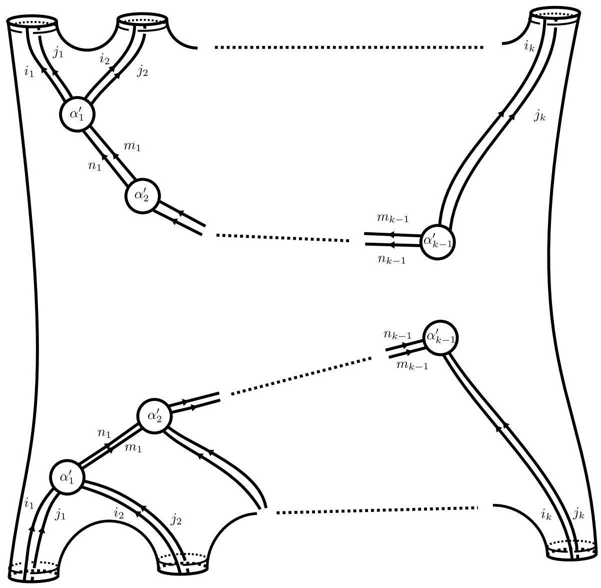
$\Theta_{\alpha_1,\cdots,\alpha_{k-1}} =$



are a basis in $H^s(S^2, \bigotimes_{\ell=1}^k U_{i_\ell j_\ell})$ where

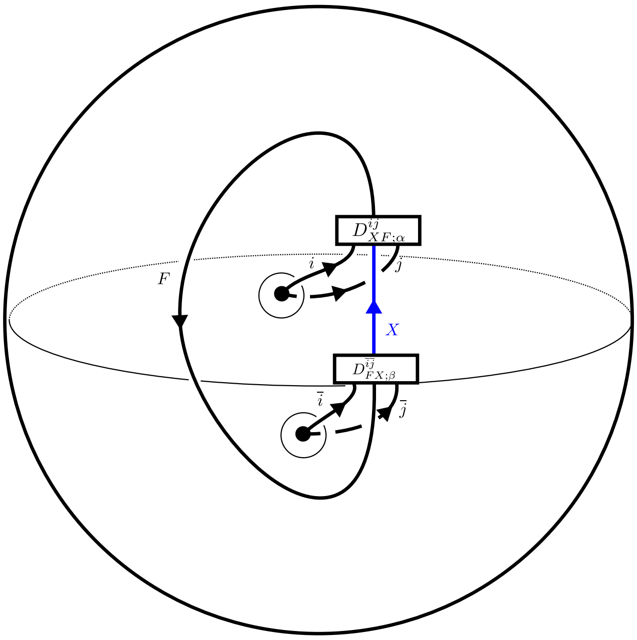


and



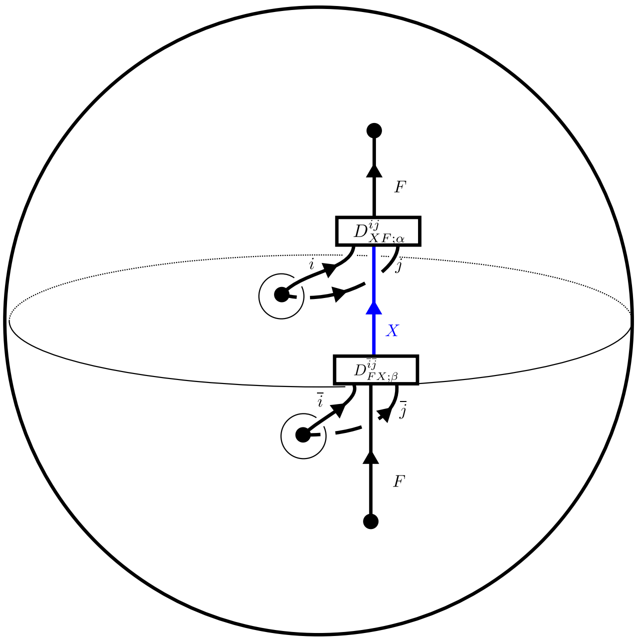
is a projector onto span $\left\{\Theta_{\alpha_1,\cdots,\alpha_{k-1}}\right\}$. Note that the projector acts by concatenation of string-nets followed by $k-1$ applications of the ϵ-move.

The goal of this section is to derive a formula for factorization of correlators involving defects. Since defects can be fused, the only non-trivial cutting is given by cutting along a circle, which is crossed by a single defect line. Alternatively we may glue world sheets at defect generating field insertions using a suitable insertion of $1 = \sum_v |v\rangle \langle v|$. As usual the complete basis of states should be given by inverses of two point functions. Since we only want to factor crossing a defect line, the only surviving terms in the complete basis will correspond to two point functions of defect generating fields. So we start by computing structure constants of defect two point functions on the sphere. We start with a cell decomposition of the sphere with two 2-cells, which are glued together along a single 1-cell (the equator). Having two field insertions generating a defect with label X, the string-net then reads



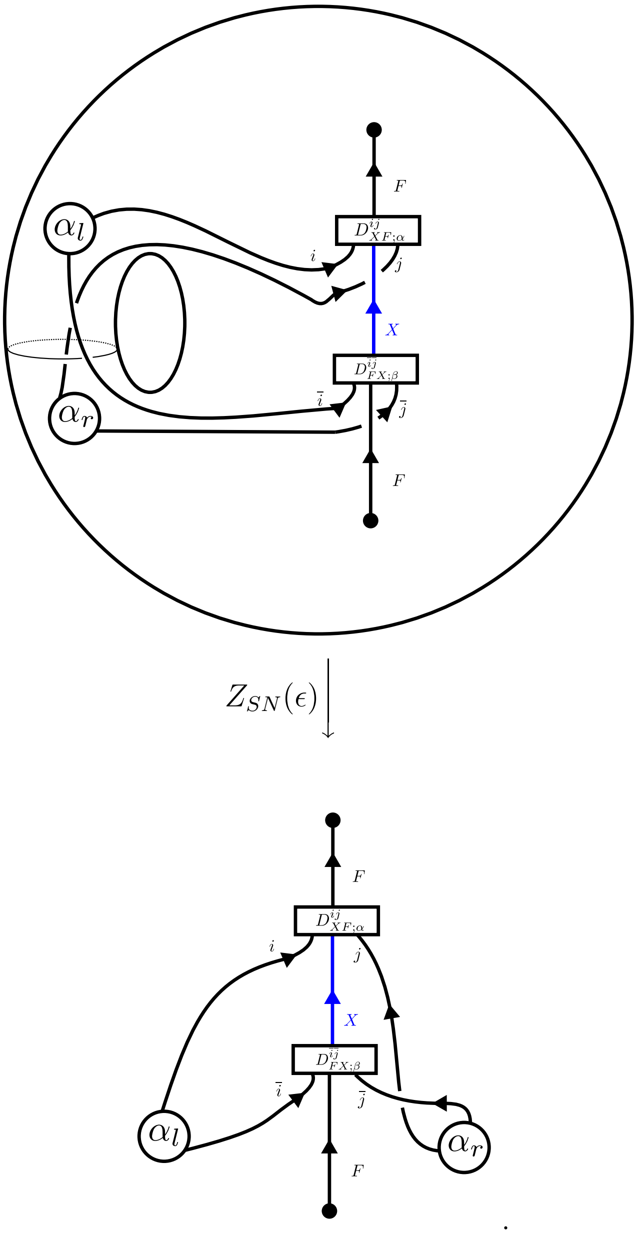
.

In the figure we pulled the F-labeled part of the 1-cell to the front. The thin horizontal circle is not an element of the defect graph. It is included for presentation purposes only and highlights the fact that we are working on a sphere. Using that F is symmetric special and $D^{ij}_{XF;\alpha} \in \mathrm{Hom}_{F|F}(U_i \otimes^+ X \otimes^- U_j, F)$, $D^{\bar{i}\bar{j}}_{FX;\beta} \in \mathrm{Hom}_{F|F}(U_i^* \otimes^+ F \otimes^- U_j^*, X)$ intertwine left F-actions it is easy to show that this is equivalent to



.

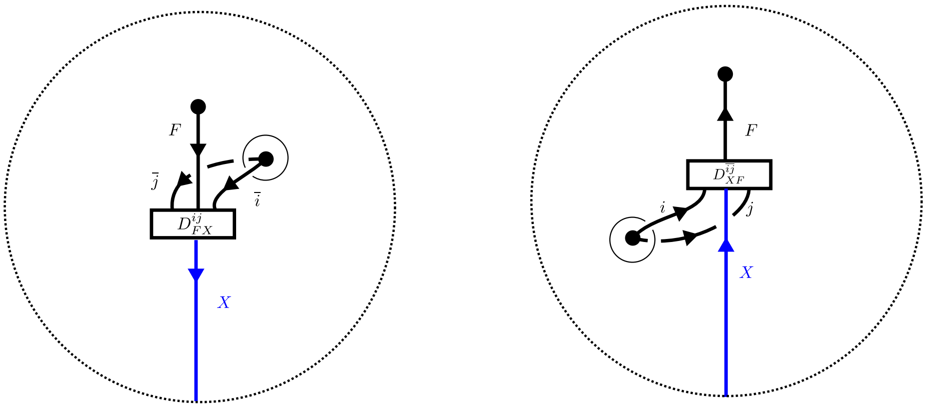
In the following we compute the structure constant wrt. to the basis chosen in the previous paragraph. In the graphical proof we omit the lower part of the string-net projector and only show the upper part giving the proportionality constant. Applying the appropriate projector yields



In the second figure the underlying sphere is not shown. Since this is a morphism in $\mathrm{Hom}_{\mathsf{C}}(\mathbf{1},\mathbf{1}) \simeq \mathbb{C}$ it is just a number. We denote this number by

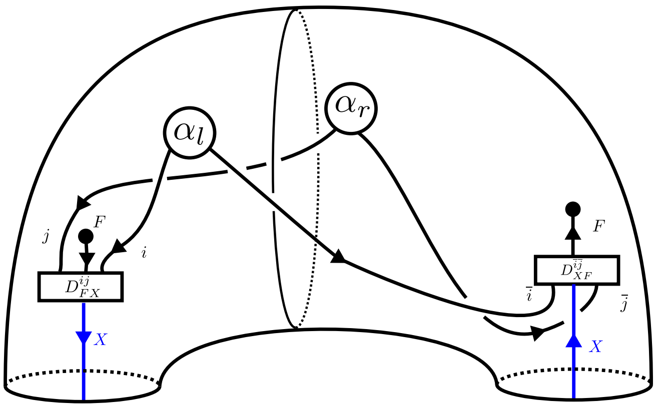
$$K^{ij}_{XF,\alpha\beta} \quad . \tag{4.52}$$

Thus we can glue world sheets crossing defect lines starting from insertions



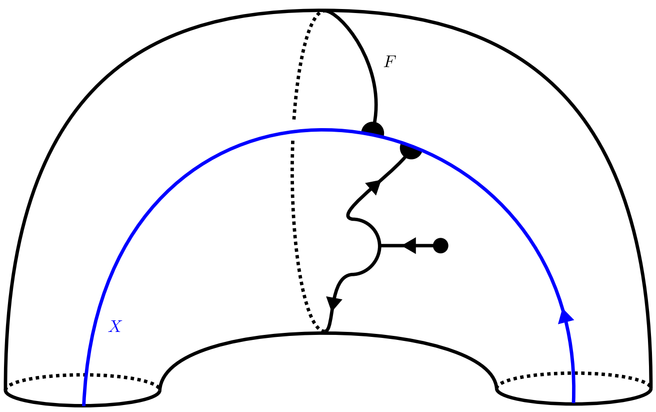
where the interior of the dashed circles show local disk shaped neighborhoods on world sheets. The regions may correspond to disjoint world sheets or a single one. Gluing is now given by gluing in the respective basis element, resulting in a cylinder shaped region with string-net

$g\ell_X^{ij} =$



On the other hand it is not hard to check, that the correlator on the world sheet with the same topology as the glued world sheet a cell decomposition from a defect graph with defect edge labeled by X running along the cylinder gives

$\mathsf{c}_X =$



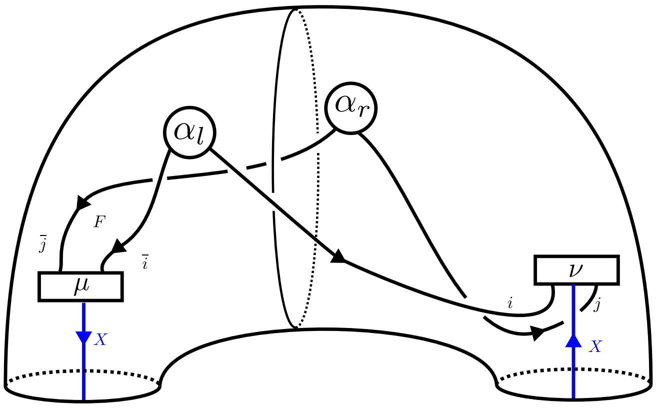
Both local string-nets are elements in the string-net space

$$H^s(S^2, X^* \otimes X) \simeq \mathrm{Hom}_{\mathsf{Z(C)}}(X, X). \tag{4.53}$$

By semi-simplicity this is equal to

$$\mathrm{Hom}_{\mathsf{Z(C)}}(X, X) \simeq \bigoplus_{i,j \in \mathsf{I}} \mathrm{Hom}_{\mathsf{Z(C)}}(X, U_i \otimes U_j) \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathsf{Z(C)}}(U_i \otimes U_j, X) \tag{4.54}$$

and a basis element has the string-net presentation

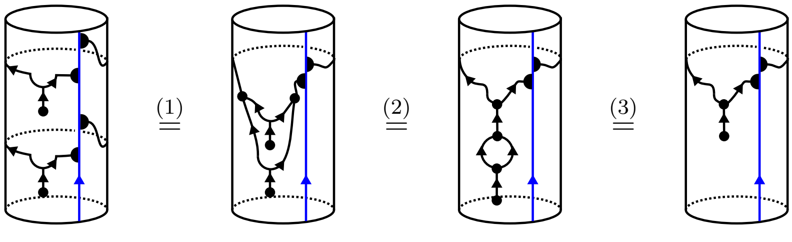
$\mathsf{b}^{ij}_{\mu\nu} =$ 

Before we continue, we have to give a quick technical remark. The string-net c_X is not really part of a string-net space since its boundaries are not decorated. We will encounter this situation several times in the following. In all situations this is due to the fact, that we look at a cut out part of a string-net on a bigger surface. In [66, Lemma 69] it is shown that these parts can be replaces by string-nets which are true elements of string-net spaces. Thus all manipulations to come are fine and the pedantic reader may replace the string-nets appropriately when performing calculations.

The procedure is now similar to the one in [48][50]. We start by showing that c_X is a projector. We then show that its image is spanned by string-nets of the form $\mathcal{g}\ell^{ij}_X$. This shows that c_X has an expansion in terms of $\mathcal{g}\ell^{ij}_X$ and we only have to compute the structure constants.

Lemma 4.3.3. c_X *is a projector.*

Proof. As most of the time, the proof is graphical.



Step (1) is the bimodule property, (2) is twice the Frobenius property and (3) is specialness of F. □

Pre- and postcomposing $b^{ij}_{\mu\nu}$ with c_X yields

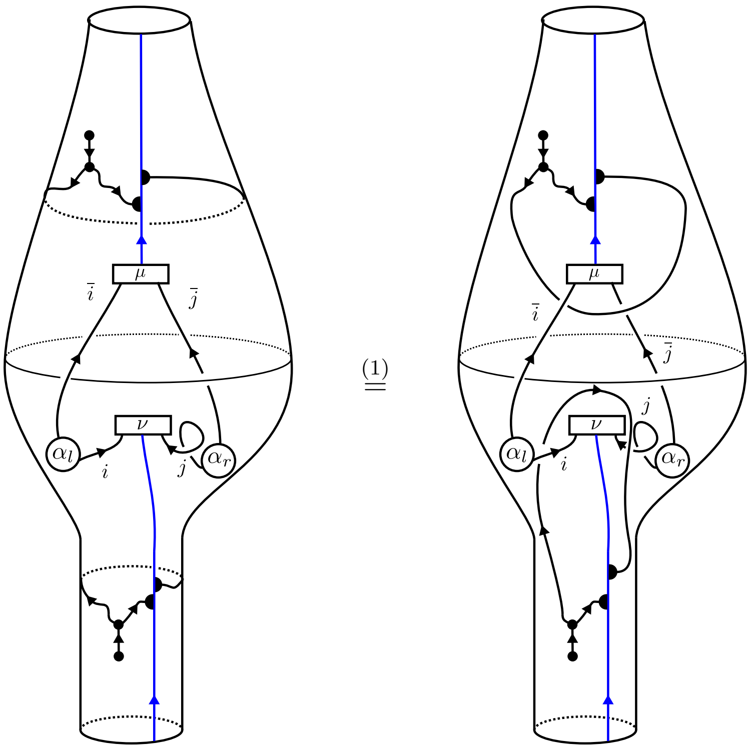


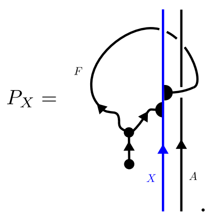
Figure 4.5

where (1) is dragging along the projector circle. We need a small lemma to relate the above string-nets to $g\ell^{ij}_X$.

Lemma 4.3.4. *Let X be a F-bimodule and A, $B \in \mathsf{C}$. Define the map*

$$P_X : X \otimes A \to X \otimes A \tag{4.55}$$

by



Then P_X is an idempotent. In addition, defining

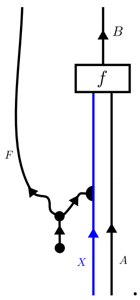
$$\mathrm{Hom}_{P_X}(X \otimes A, B) = \{ f \in \mathrm{Hom}(X \otimes A, B) \,|\, f \circ P_X = f \} \quad . \tag{4.56}$$

it holds

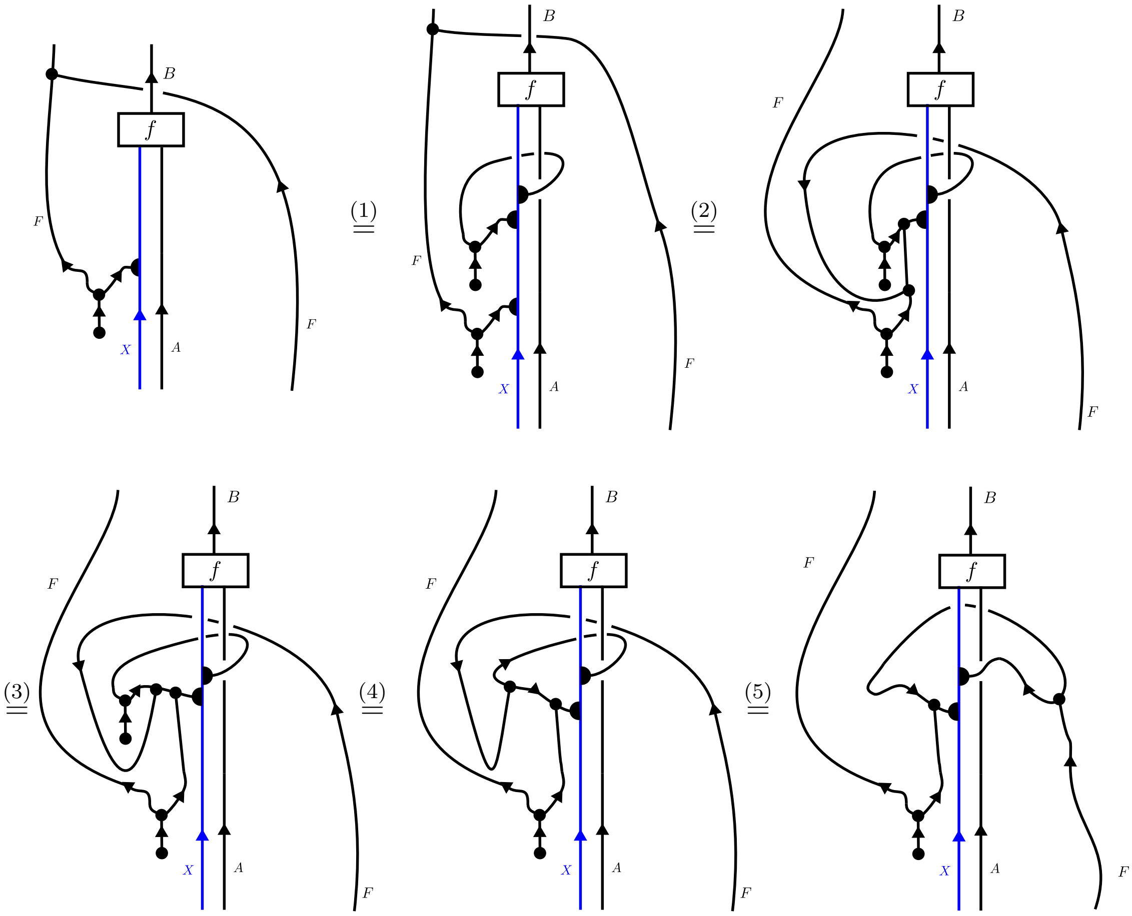
$$\mathrm{Hom}_{P_X}(X \otimes A, B) \simeq \mathrm{Hom}_{F|F}(X \otimes^- A, F \otimes^+ B) \tag{4.57}$$

Proof. To show that P_X is an idempotent is the same computation as in the proof of lemma 4.3.3.

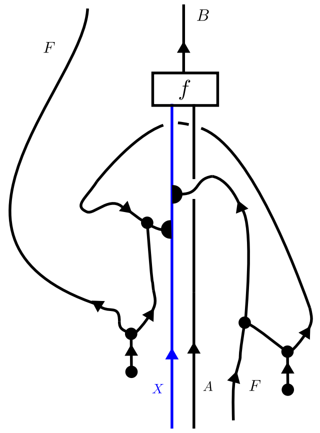
For the second point, let $f \in \mathrm{Hom}_{P_X}(X \otimes A, B)$. Define the map I as

$I(f) =$ 

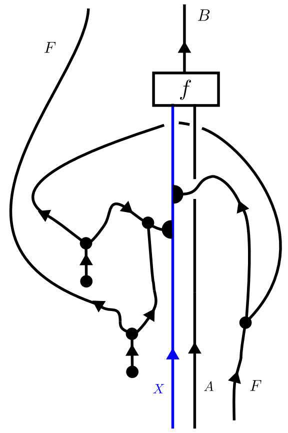
We have to check that $I(f) \in \mathrm{Hom}_{F|F}(X \otimes^- A, F \otimes^+ B)$. One easily checks that $I(f)$ intertwines left F-actions by using the properties of the Frobenius algebra F. The proof for intertwining of right actions is more interesting and uses invariance under the projector.



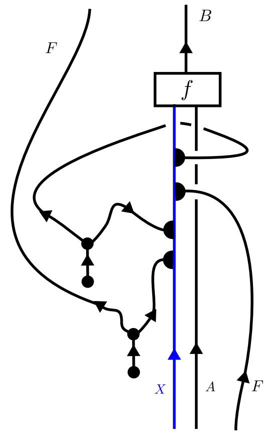
$\overset{(6)}{=}$



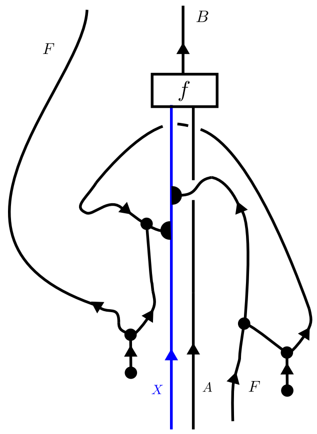
$\overset{(7)}{=}$



$\overset{(8)}{=}$



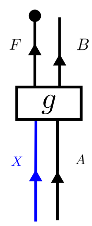
$\overset{(9)}{=}$



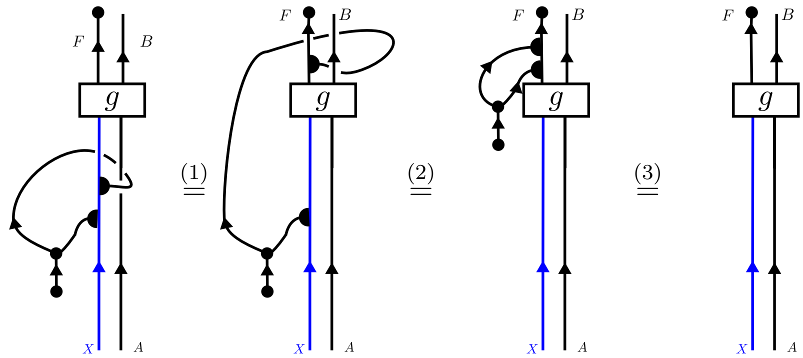
In (1) the projector is inserted, (2) uses symmetry of F to move the product to the right and employs the left representation property, (3) is associativity, (4) holds by the Frobenius property and unitality of the product, (5) is just a deformation of the string diagram, (6) is again Frobenius property and unitality, (7) is a deformation, (8) uses the left and right representation property and finally (9) is invariance under the projector.

In the other direction define the map J as

$J(g) =$



and we check that it is actually invariant under precomposition with P_X.

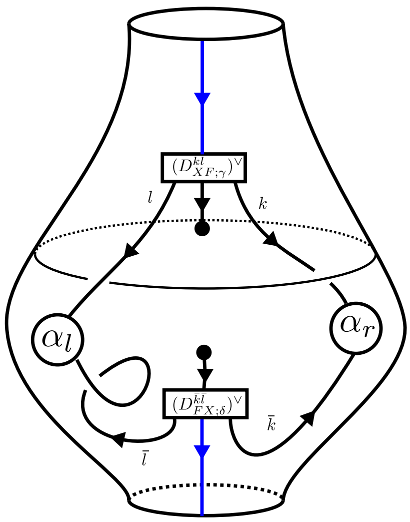


The first step follows from g being a $F-F$-bimodule map, (2) is symmetry of F and (3) is associativity of F followed by specialness and unitality.

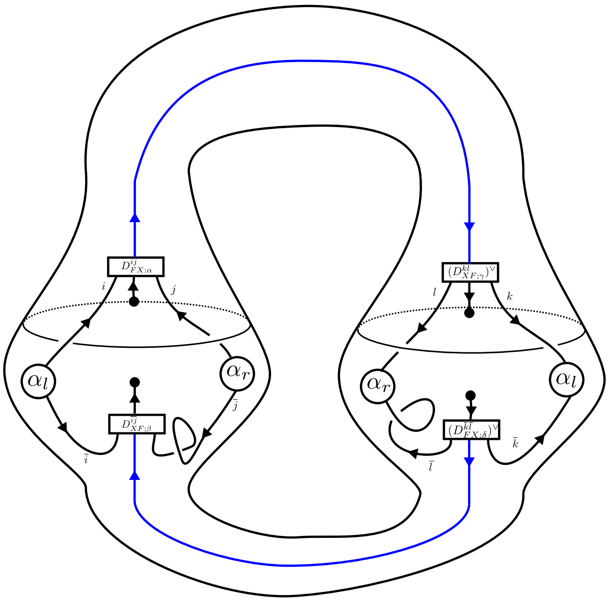
With the help of the Frobenius properties and representation properties it is very easy to show that J is left and right inverse to I. $\square$

The lemma together with [63, Lemma 2.2] shows that the rhs of figure 4.5 has an expansion in terms of $g\ell_X^{ij}$. To compute the structure constants, we show that the string-nets

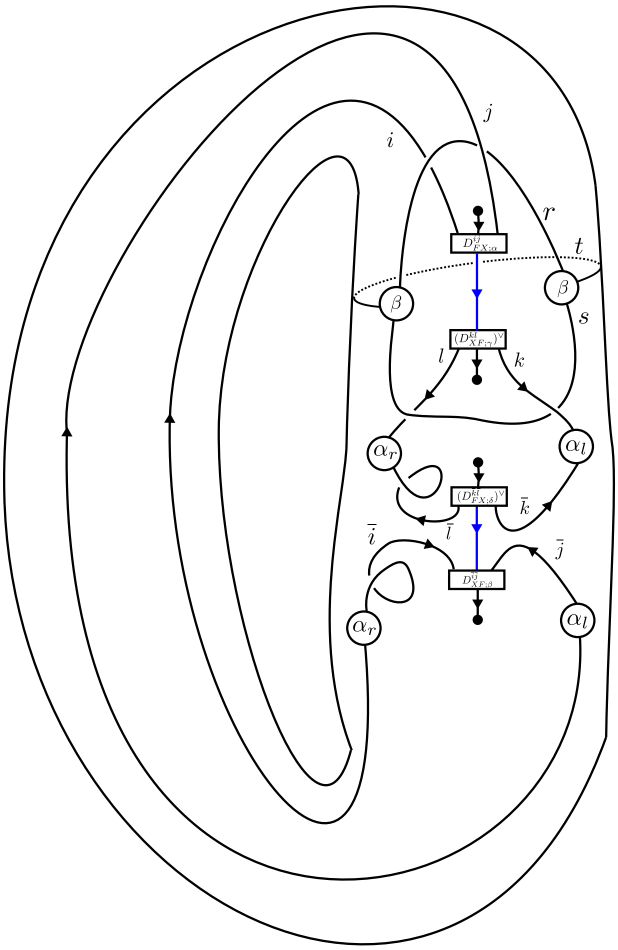
$$\widetilde{\left(g_X^{ij}\right)}^{\vee}_{\delta\gamma} =$$



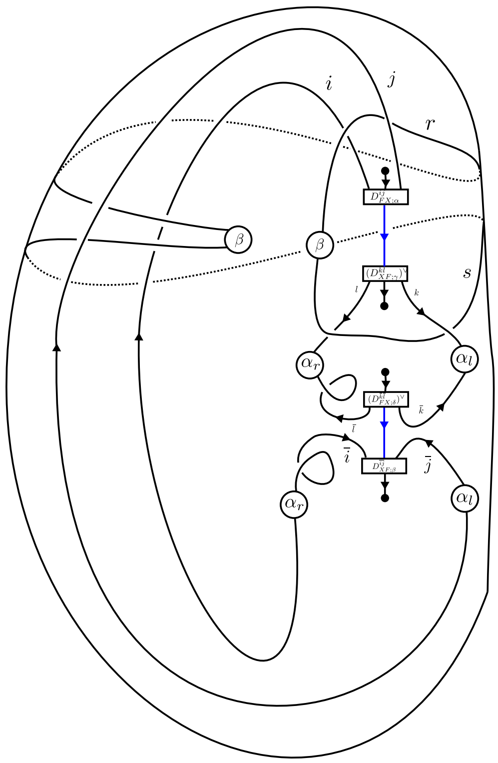
for $\left(D^{lk}_{XF;\gamma}\right)^\vee \in \mathrm{Hom}_{F|F}(X, U_k \otimes^+ F \otimes^- U_l)$, $\left(D^{\bar{l}\bar{k}}_{FX;\delta}\right)^\vee \in \mathrm{Hom}_{F|F}(F, U_k^* \otimes^+ X \otimes^- U_l^*)$ give a dual basis to the formal vector space spanned by the $\left\{g\ell^{ij}_{X,\alpha\beta}\right\}$.



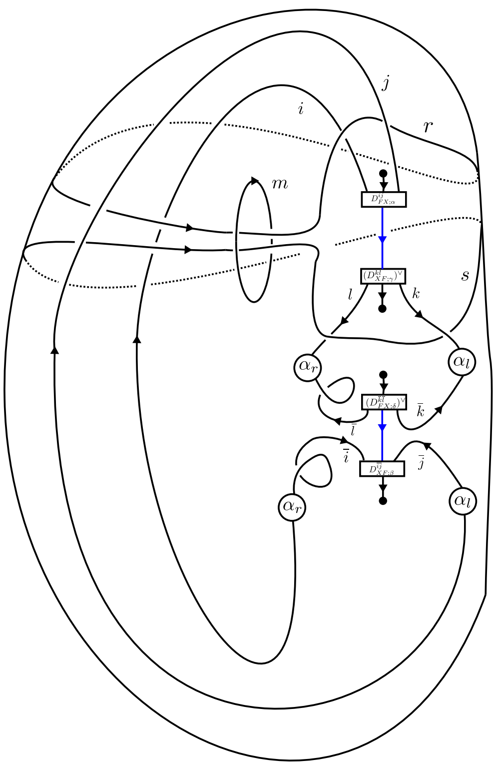
$$\stackrel{(1)}{=} \quad \bigoplus_{r,s,t\in \mathsf{I}} \sum_{\beta} \frac{d_r d_s d_t}{\mathsf{D}^4}$$



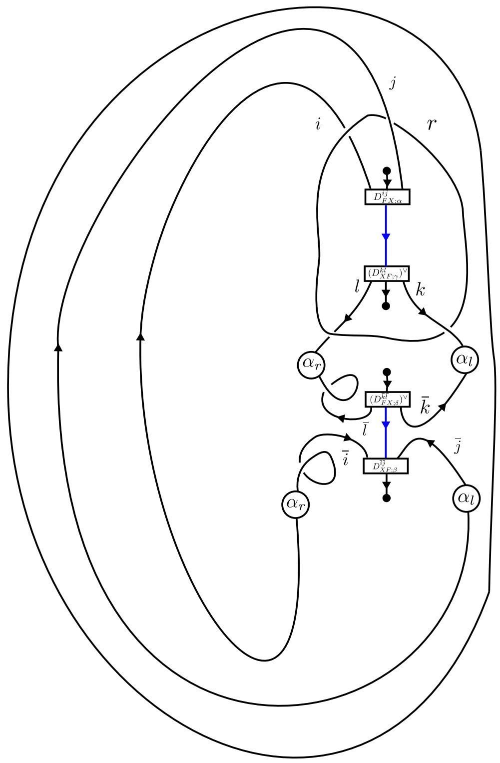
$$\xmapsto{Z_{SN}(\epsilon)} \quad \bigoplus_{r,s\in \mathsf{I}} \sum_{\beta} \frac{d_r d_s}{\mathsf{D}^4}$$



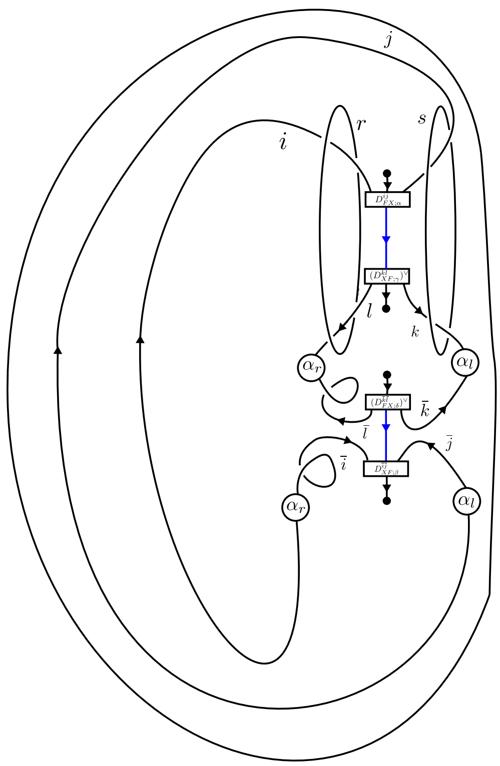
$$\overset{(2)}{=} \quad \bigoplus_{r,s,m\in \mathsf{I}} \frac{d_m d_r d_s}{\mathsf{D}^6}$$



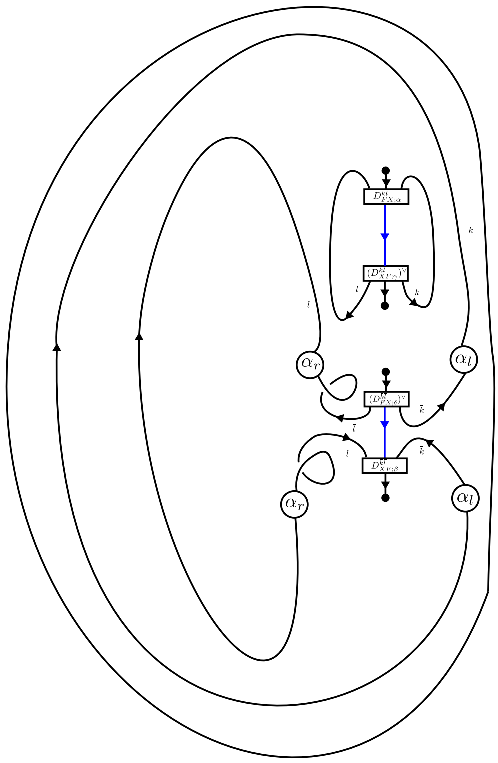
$$\overset{(3)}{=} \quad \bigoplus_{r\in\mathsf{I}} \frac{d_r}{\mathsf{D}^4}$$



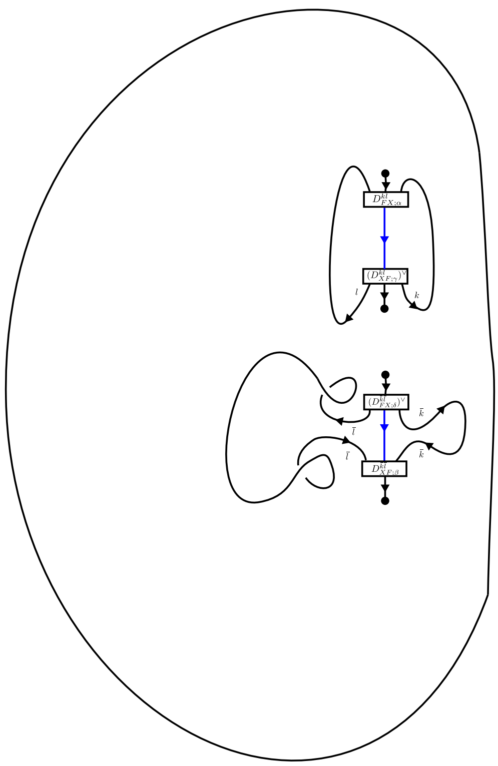
$$\overset{(4)}{=} \quad \bigoplus_{r,s\in\mathsf{I}} \frac{d_r d_s}{\mathsf{D}^6}$$



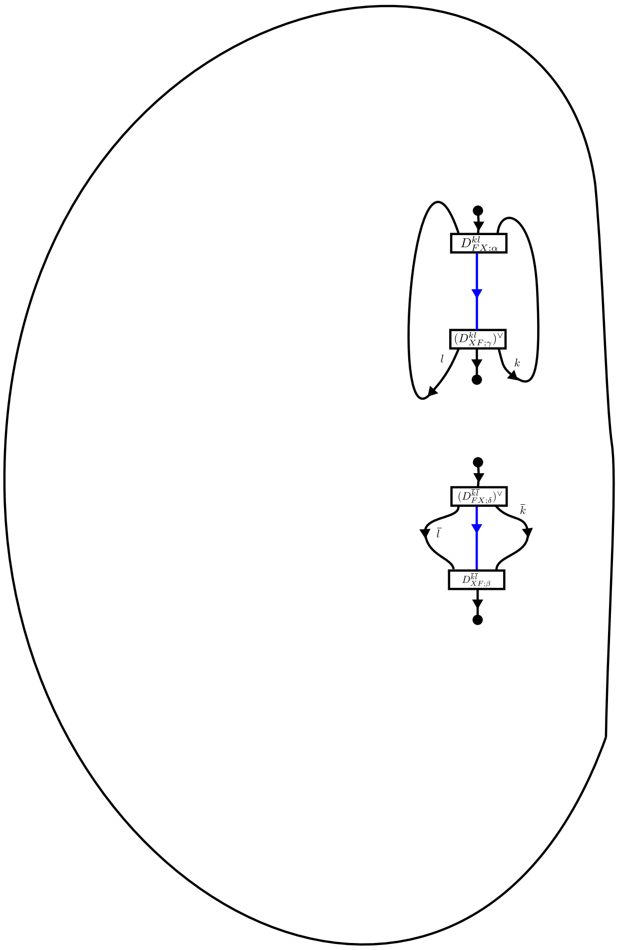
$$\stackrel{(5)}{=} \quad \frac{1}{d_l\, d_k\, \mathsf{D}^2}$$



$$\stackrel{(6)}{=} \quad \frac{1}{d_l^2\, d_k^2\, \mathsf{D}^2}$$

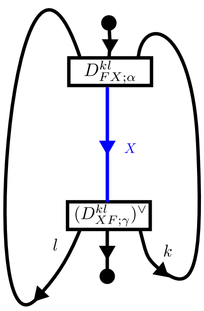


$$\overset{(7)}{=} \quad \frac{1}{d_l^2 d_k^2 \mathsf{D}^2}$$

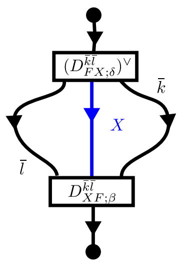


In (1) the string-net in the left tube is moved to the right, followed by a splitting of the projector circles. The second step is application of $Z_{SN}(\epsilon)$ and a deformation of a β-coupon around the sphere. Steps (2) and (3) is application of identities B.4. In (4) we doubled the projector circle and (5) is again B.4. Step (6) is immediate from the definition of the coupon morphisms and (7) is just a deformation of the string-nets. For the upper string-net on the sphere is computed in [50, (A.10)]. The value of the second one follows from duality of the basis and specialness of F.

$\delta_{\alpha,\gamma} d_X =$



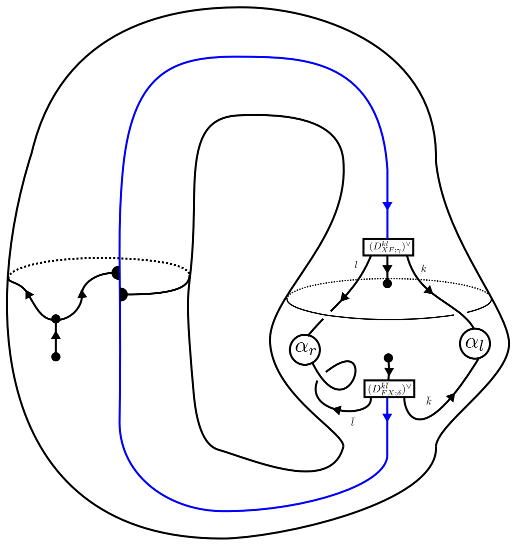
$\delta_{\beta,\delta} d_F =$



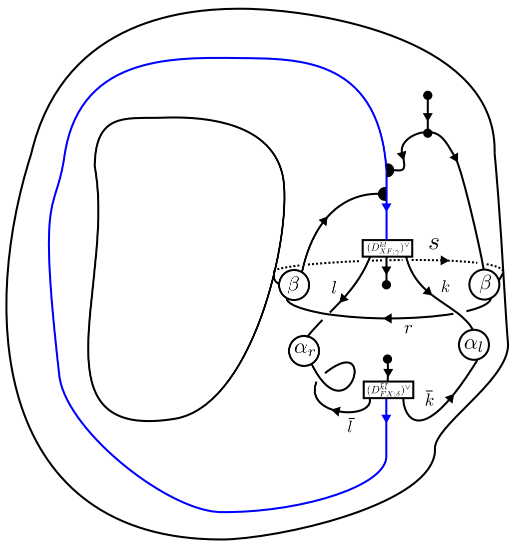
We absorb the proportionality constant by defining duals

$$\left(g_X^{ij}\right)^{\vee}_{\delta\gamma} \equiv \frac{d_i^2 d_j^2 \mathsf{D}^2}{d_X d_F} \widetilde{\left(g_X^{ij}\right)^{\vee}_{\delta\gamma}} \quad . \tag{4.58}$$

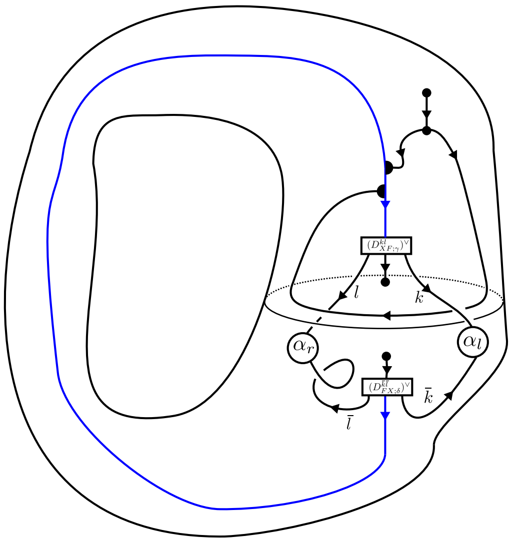
We are ready to compute the expansion constants.



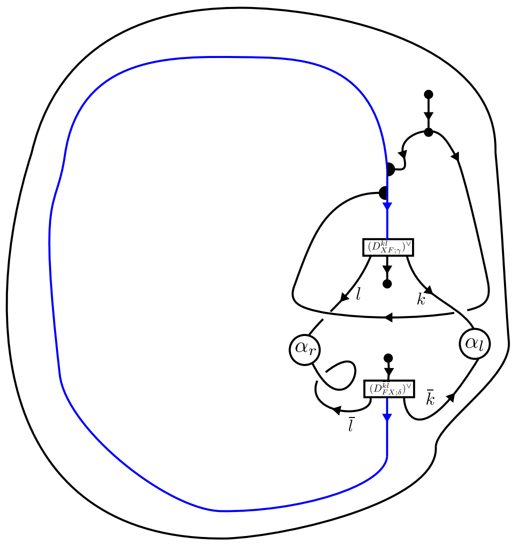
$$\overset{(1)}{=} \quad \bigoplus_{r,s\in\mathsf{I}} \sum_{\beta} \frac{d_r d_s}{\mathsf{D}^2}$$



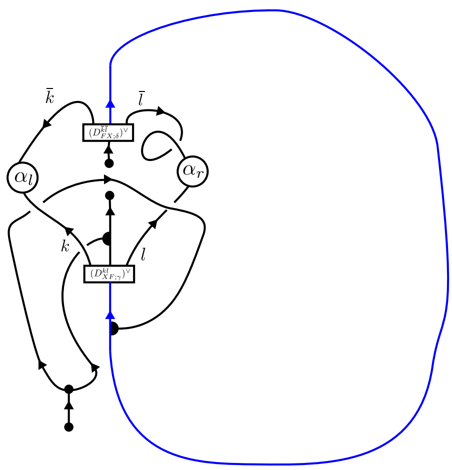
$\overset{(2)}{=}$



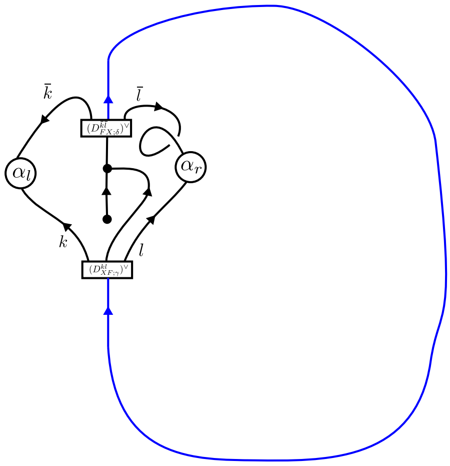
$\xmapsto{Z_{SN}(\epsilon)}$ $\frac{1}{\mathsf{D}^2}$



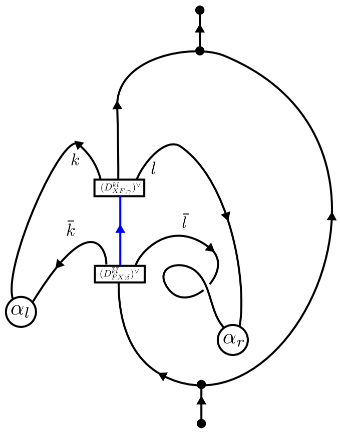
$\overset{(3)}{=}$ $\frac{1}{\mathsf{D}^2}$



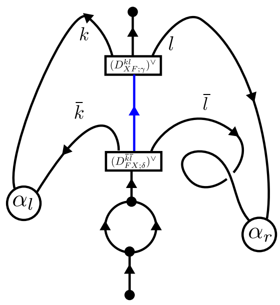
$$\overset{(4)}{=} \quad \frac{1}{\mathsf{D}^2}$$



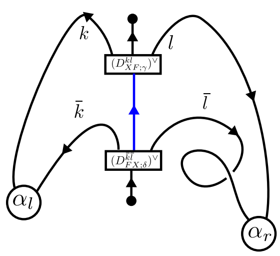
$$\overset{(5)}{=} \quad \frac{1}{\mathsf{D}^2}$$



$$\overset{(6)}{=} \quad \frac{1}{\mathsf{D}^2}$$



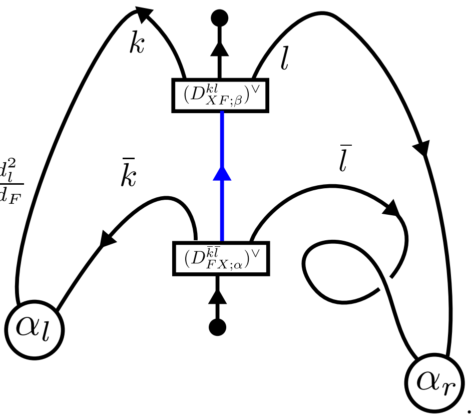
$$\overset{(7)}{=} \quad \frac{1}{\mathsf{D}^2}$$



In the computation step (1) is deformation of the string-net and the completeness relation, (2) is again completeness. In step (3) we just deformed the string-net and used that $\left(D^{kl}_{XF}\right)^{\vee}$ intertwines left F-actions. In addition we omitted to draw the sphere the string-net is located on. (4) first uses the Frobenius property followed by unitality and counitality of F as well as the right F-action intertwining property for $\left(D^{\bar{k}\bar{l}}_{FX}\right)^{\vee}$. Next, (5) first is unitality of F to get a straight F-colored edge. Then the upper coupon is moved along the X-colored edge to the bottom and the $\bar{k}$-colored edge is swiped around the sphere. Finally symmetry of F is used to insert the product and coproduct. (6) is right action intertwining used twice to move the product to the bottom and lastly (7) is just specialness of F.

Recalling the normalization factor of the dual element $\left(\not{g}\ell^{ij}_X\right)^{\vee}$, the expansion coefficient reads

$$\left\langle \not{g}\ell^{kl}_{X;\alpha\beta} \middle| \mathsf{c}_X \right\rangle = \frac{d_k^2 d_l^2}{d_X d_F}$$



The last step towards a closed formula for the gluing is to relate the above constant to the defect two point function.

Lemma 4.3.5.

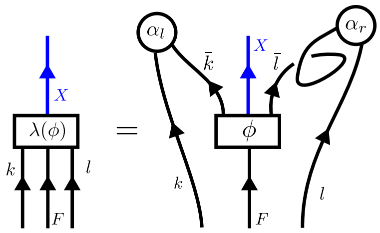
$$\left\langle \not{g}^{kl}_{X;\alpha\beta} \middle| \mathsf{c}_X \right\rangle = d_k d_l \left(K^{kl}_{X;\alpha\beta}\right)^{-1} \tag{4.59}$$

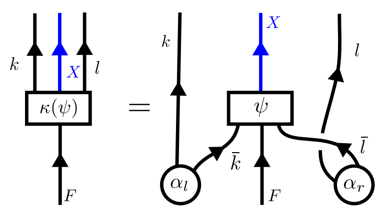
The proof is almost the same as the one presented in [50, Lemma A.5]. The major difference is that our normalization may be different, thus we quickly give a proof.

Proof. Let $\phi \in \mathrm{Hom}_{F|F}(F, U_k^* \otimes^+ X \otimes^- U_l^*)$ and $\psi \in \mathrm{Hom}_{F|F}(U_k^* \otimes^+ F \otimes^- U_l^*, X)$. Define maps

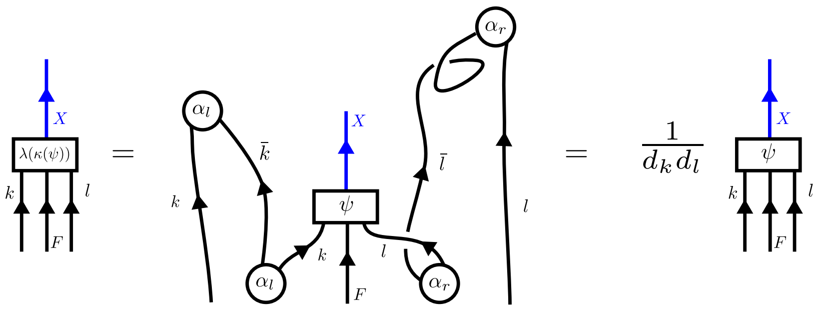
$$\begin{aligned} \lambda &: \mathrm{Hom}_{F|F}(F, U_k^* \otimes^+ X \otimes^- U_l^*) \to \mathrm{Hom}_{F|F}(U_k \otimes^+ F \otimes^- U_l, X) \\ \kappa &: \mathrm{Hom}_{F|F}(U_k^* \otimes^+ F \otimes^- U_l^*, X) \to \mathrm{Hom}_{F|F}(F, U_k \otimes^+ X \otimes^- U_l) \end{aligned} \tag{4.60}$$

by





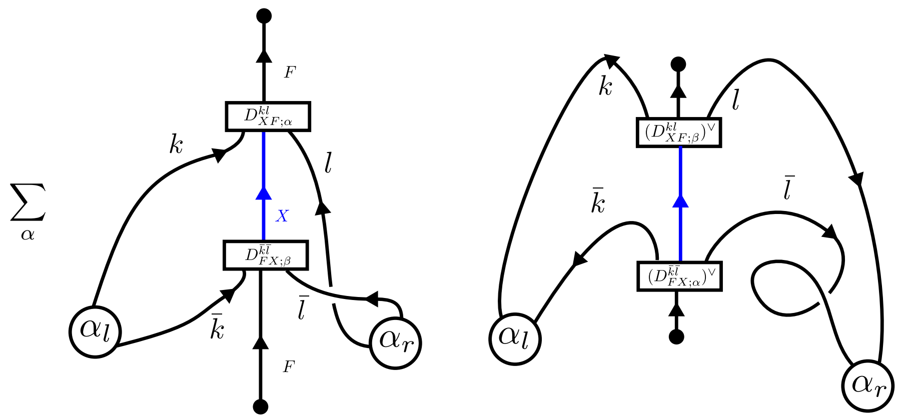
One quickly computes

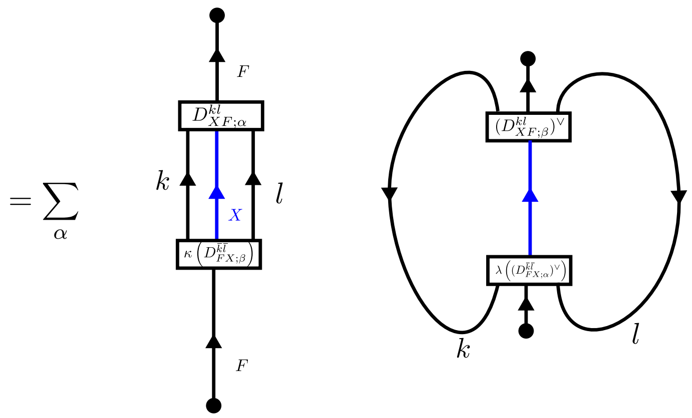


and similarly $\lambda(\kappa(\phi)) = \phi$. Thus if we pick a basis $\left\{\phi^{kl}_{\alpha}\right\}$ in $\mathrm{Hom}_{F|F}(F, U_k^* \otimes^+ X \otimes^- U_l^*)$ and a basis $\left\{\psi^{kl}_{\beta}\right\}$ in $\mathrm{Hom}_{F|F}(U_k^* \otimes^+ F \otimes^- U_l^*, X)$ the representing matrices of λ and κ satisfy

$$\sum_{\beta} \lambda_{\alpha\beta}\kappa_{\beta\gamma} = \frac{\delta_{\alpha\gamma}}{d_k d_l} = \sum_{\beta} \kappa_{\gamma\beta}\lambda_{\beta\alpha} \tag{4.61}$$

Thus we compute





Expanding λ, κ in basis finally gives

$$\sum_{\sigma\omega\alpha} \kappa_{\beta\sigma}\delta_{\sigma\alpha} d_F \lambda_{\alpha\omega}\delta_{\omega\gamma} d_X = \frac{d_F d_X}{d_k d_l}\delta_{\beta\gamma} \quad . \tag{4.62}$$

Putting all things together yields

$$\sum_{\alpha} K^{kl}_{X;\beta\alpha} \left\langle \mathscr{g}\ell^{kl}_{X;\alpha\gamma} \middle| \mathsf{c}_X \right\rangle = d_k d_l \delta_{\beta\gamma} \tag{4.63}$$

which is equivalent to

$$\left\langle \mathscr{g}\ell^{kl}_{X;\alpha\gamma} \middle| \mathsf{c}_X \right\rangle = d_k d_l \left(K^{kl}_{X;\gamma\alpha}\right)^{-1} \quad . \tag{4.64}$$

□

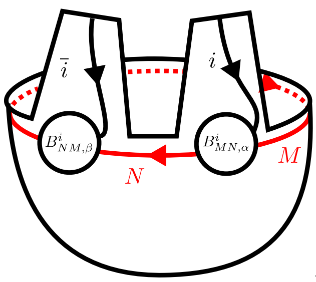
We summarize the section so far in a theorem.

Theorem 4.3.6. *The set of correlators is closed under defect crossing gluings. More precisely,* $(\Sigma, \mathbf{D}, D^{kl}_{XF;\alpha}, D^{\bar{k}\bar{l}}_{FX,\beta}, \mathbf{\Psi})$ *is a world sheet with defect and boundary field insertions* $\mathbf{D}$*,* $\mathbf{\Psi}$ *which can be glued according to the procedure described above and* $(\widetilde{\Sigma}, \mathbf{D}, \mathbf{\Psi})$ *is the world sheet with the same topology as* $\mathscr{g}\ell(\Sigma)$ *and concurring field insertions. Then there is an expansion*

$$\mathsf{corr}_{\widetilde{\Sigma}}(\mathbf{D}, \mathbf{\Psi}) = \sum_{k,l\in\mathsf{I}} \sum_{\alpha,\beta} d_k d_l \left(K^{kl}_{X;\beta\alpha}\right)^{-1} \mathscr{g}\ell\left(\mathsf{corr}_{\Sigma}(\mathbf{D}, D^{kl}_{XF;\alpha}, D^{\bar{k}\bar{l}}_{FX,\beta}, \mathbf{\Psi})\right) \tag{4.65}$$

Note that this is the same formula as in [50, Proposition 2.3], which is to be expected as we are working with the same chiral data of a defect RCFT. Nonetheless we derived the formula using the string-net tft instead of the Reshetikhin-Turaev tft, which required quite some different technical methods.

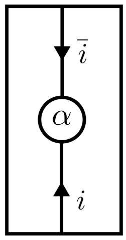
Besides bulk factorization, world sheets can also factor through the boundary. We will be more terse in that discussion. Since the crucial computations in [48] are in terms of string diagrams in C we can use these results to manipulate string-nets. Recall that boundary insertions are given by elements in $\mathrm{Hom}_F(M \otimes U_i, N)$ for left F-modules M, N. First we outline the computation of a disk two point function. As a disk is a 2-cell from the start there may be a transparent subgraph inside of the disk. However, using the Frobenius properties it is easy to check that is can be completely removed (see [48, section C]), thus the string-net simply reads



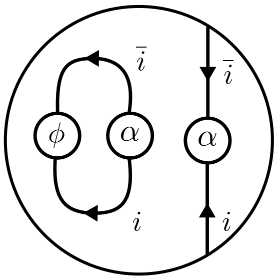
.

This is a string-net on a strip with boundary value $U_i \otimes U_i^*$. A basis for this string-net space is given by

$$C_i =$$

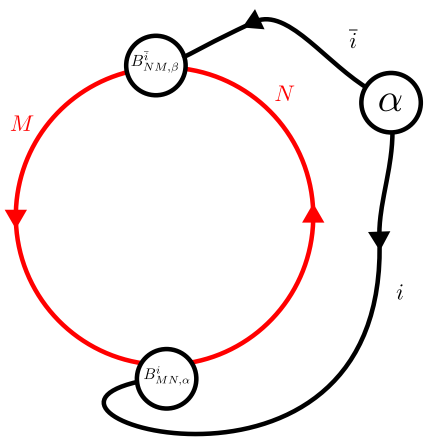


An arbitrary string-net on the U_i-colored strip is determined by a single morphism $\phi = K_\phi \mathrm{id}_{U_i}$ and its projection onto C_i is computed by applying the projector



inside a disk (there is some obvious cutting and gluing of string-diagrams involved). Computing the projection of the two point function we find

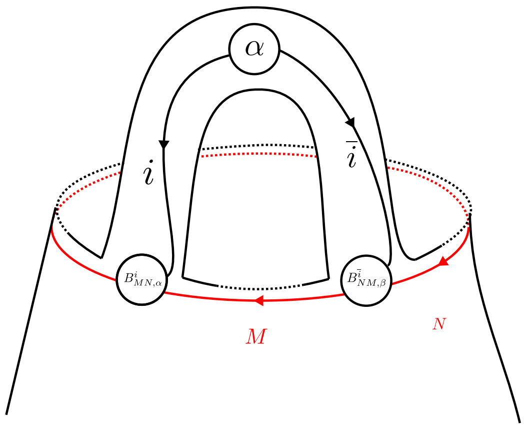
$$K^i_{MN,\alpha\beta} = \mathsf{D}^2$$



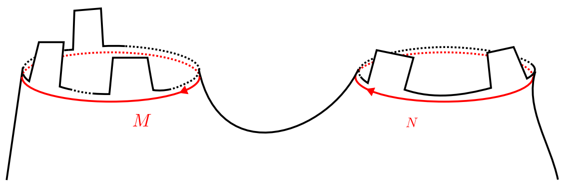
.

Note that this is the same number as [48, (C.3)]. Two cases have to be distinguished for boundary factorization.

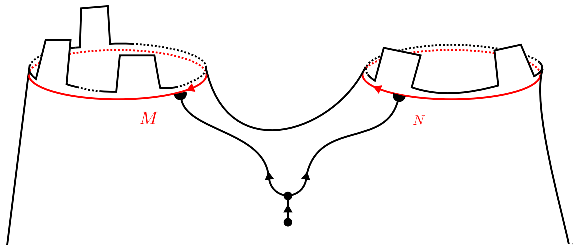
I) The first case is gluing along boundary insertions on the same boundary. One of the boundary segments needs to have label $-$, the other is labeled $+$ (we are gluing incoming to outgoing boundaries). This we define as follows



. In words, we glue in a strip whose upper surface is facing towards the reader in the picture. The dotted black line around the boundary indicates, that there might in addition be other field insertions not shown in the picture. Locally the string-net on the surface Σ' (the surface with the same topology as the glued surface) looks as follows



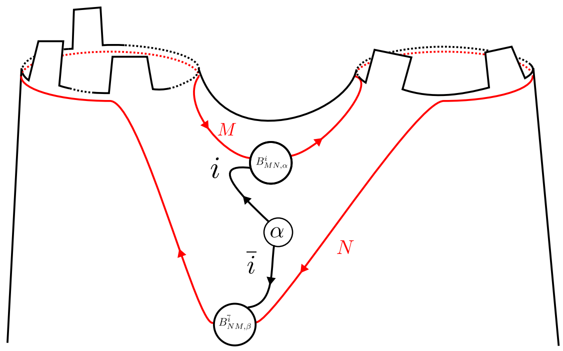
where the boundary field insertions on the left boundary component are the insertions encircled by the i and $\bar{i}$-colored edge in the gluing picture. We have to show that the defect graph defining the correlator on Σ' decomposes in terms of the one obtained from Σ plus the local part from the gluing formula for all possible boundary field insertions. We can use the proof given in [48, section 4] almost verbatim. First note that given a fixed defect graph $\Gamma_{\Sigma'}$ one can add a transparent edge connecting the two boundary components. By the definition of the gluing the boundary components bound world sheet faces colored by the same Frobenius algebra F. In addition we assume that the defect graphs agree outside the local gluing area, hence no non-trivial defect edge can run between the boundary components. The respective string-net gets an extra part



.

By [48, (4.22)] this equals

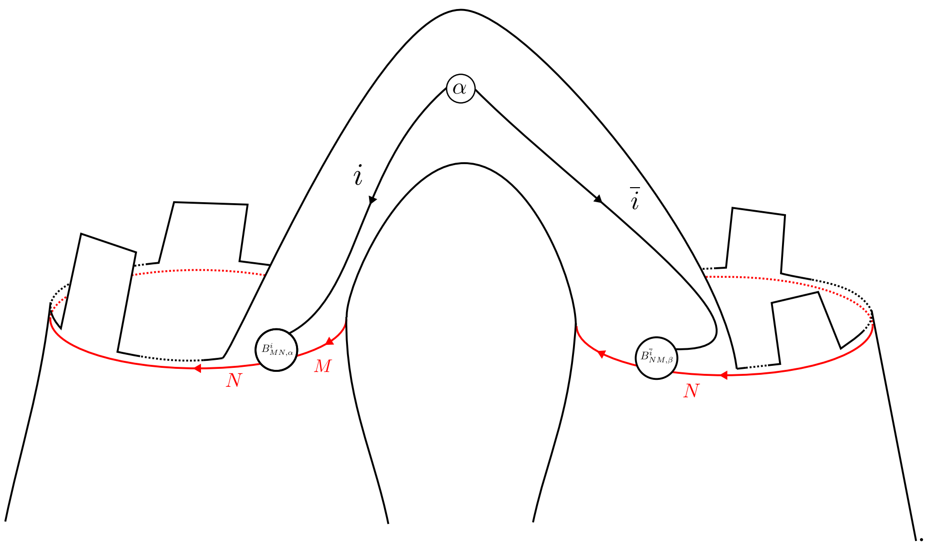
$$\sum_{i\in \mathsf{I}} \sum_{\alpha,\beta} d_i \, (K^i_{MN})^{-1}_{\alpha\beta}$$



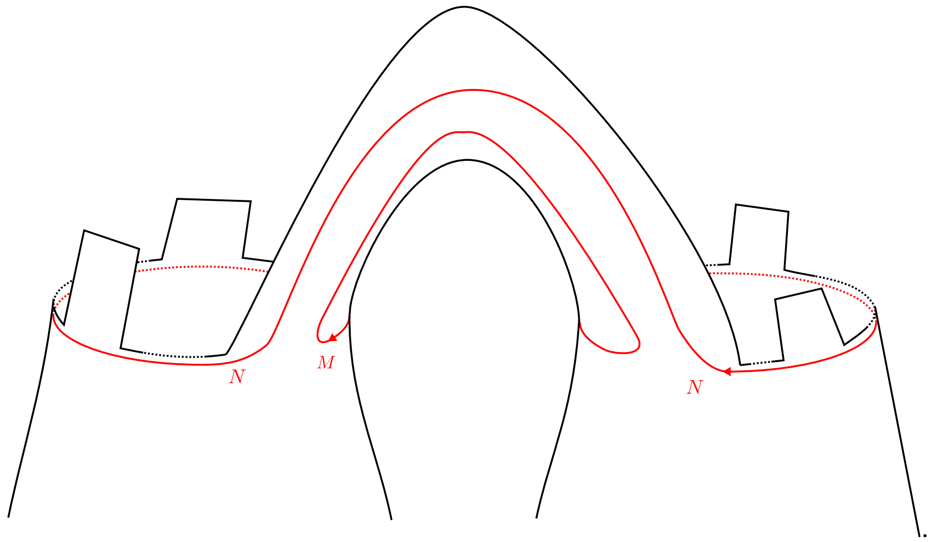
This shows that the string-net on Σ' is a summation over the string-nets obtained from gluing. Therefore factorization along the boundary holds in this case.

II) The second case to consider is gluing of boundary components lying on different boundary components. Note that this intertwines topologies of world sheets in the opposite direction to the previous case. Here we start with two different connected components of the world sheet, which get glued to a single one. In case I, the gluing split a single connected component into two.

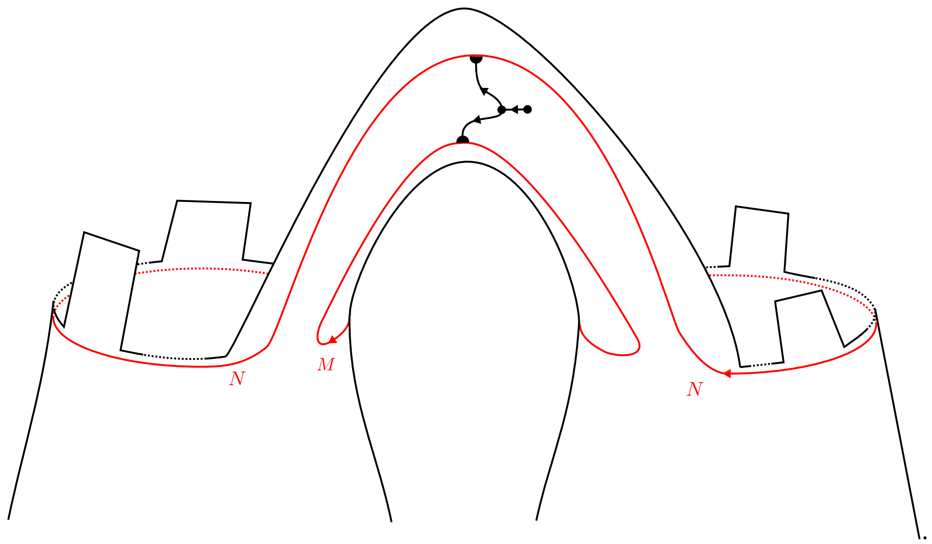
The relevant gluing now looks as follows



On the other hand on the topology of the glued surface the string-net defining the correlator is given by

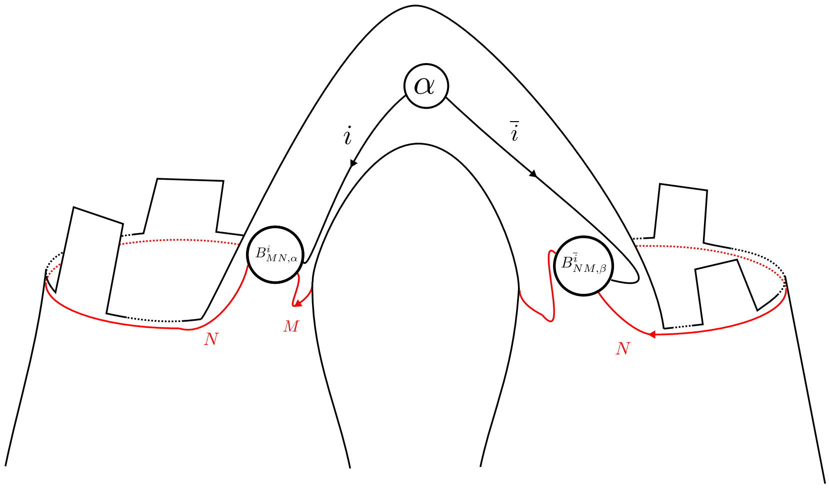


By the same argument as in case I) we may add an extra edge. This yields



Again by [48, (4.22)] this equals

$$\sum_{i\in \mathsf{I}}\sum_{\alpha,\beta} d_i (K^i_{MN})^{-1}_{\alpha\beta}$$



which is again a sum over world sheets with defect graph obtained from gluing.

The results for boundary factorization or open gluing can be summarized in the following theorem.

Theorem 4.3.7. *The set of correlators defined via defect graphs is closed under boundary factorization. That is, let* $(\Sigma, \mathbf{D}, \mathbf{\Psi}, B^i_{MN,\alpha}, B^{\bar{i}}_{NM,\beta})$ *be a world sheet which can be glued along boundary insertions according to the rules above. If we denote* $(\Sigma', \mathbf{D}, \mathbf{\Psi})$ *the world sheet with the same topology as* $g\ell(\Sigma)$ *and concurring field insertions. The correlator for* $(\Sigma', \mathbf{D}, \mathbf{\Psi})$ *can be expanded as follows*

$$\mathsf{corr}_{\Sigma'}(\mathbf{D}, \mathbf{\Psi}) = \sum_{i\in \mathsf{I}}\sum_{\alpha,\beta} d_i (K^i_{MN})^{-1}_{\alpha\beta}\, g\ell \left(\mathsf{corr}_{\Sigma}(\mathbf{D}, \mathbf{\Psi}, B^i_{MN,\alpha}, B^{\bar{i}}_{NM,\beta})\right) \tag{4.66}$$

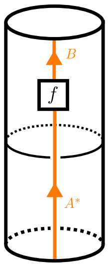
4.4 Partition Function and Examples of Cardy Algebras

In section 3.2 Cardy algebras were derived from analytic correlation functions in genus zero and one. However, no example was presented. In this section we take another route and present examples for Cardy algebras which are derived by categorical methods. We start by computing the bulk partition function for an RCFT with given symmetric Frobenius algebra F in a modular tensor category C. In other words we compute the zero point correlator on the torus using string-net methods. The result unsurprisingly reproduces the partition function derived in [60]. The computation a posteriori justifies the definition of bulk field insertions in case the defect is the transparent defect F. As a pure motivational result we then recall a result from [107] telling that special symmetric Frobenius algebras and the bulk partition function assemble into a Cardy algebra.

Before computing the partition function we discuss string-net spaces on the torus. Recall [66, Theorem 70] telling that for a given list of boundary values $\mathbf{A} \in \mathsf{Z}(\mathsf{C})$, the string-net space on a surface Σ can be computed by cutting Σ into generators $\{\sigma\}$ and sum over all possible internal boundary decorations at cutting circle modulo the equivalence relation that cylinder pieces can be freely changed from one generator to another. It is not hard to show that for a cylinder this boils down to the following isomorphism [66, Theorem 68]

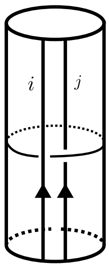
$$\mathrm{Hom}_{\mathsf{Z}(\mathsf{C})}(\mathbf{1}, A \otimes B) \longrightarrow H^s(S^2, A \otimes B) \tag{4.67}$$

$$f \longmapsto$$



.

Recall that a list of simple objects in $\mathsf{Z}(\mathsf{C})$ is given by $\left\{(U_i \otimes U_j, \beta_{ij}^{ou})\right\}$. Thus any string-net on a cylinder factors through string-nets



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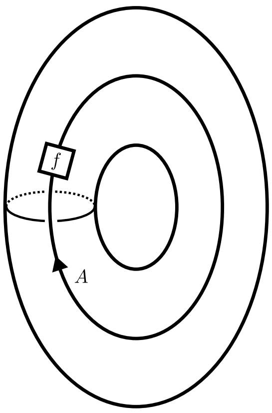
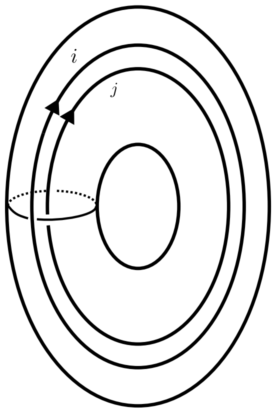
Decomposing a torus into two cylinders along two a-cycle cuts and employing the equivalence relation under gluing yields that a general string-net on a torus is in an equivalence class of string-nets of type



which decompose into simple objects

$T_{ij} =$

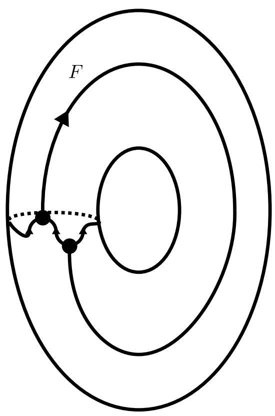


.

Obviously $\{T_{ij}\}_{i,j\in \mathsf{I}}$ constitute a basis for $H^s(T) \simeq \bigoplus_{i,j\in \mathsf{I}} \mathrm{Hom}_{\mathsf{Z(C)}}(\mathbf{1}, (U_i \otimes U_j) \otimes_{\mathsf{Z(C)}} (U_i^* \otimes U_j^*)) \simeq \bigoplus_{i,j\in \mathsf{I}} \mathrm{Hom}_{\mathsf{Z(C)}}(U_i \otimes U_j, U_i \otimes U_j)$.

To compute the torus partition function we start with a suitable transparent graph. Picking an F-colored triangulation it is not hard to check, that we end up with a string-net

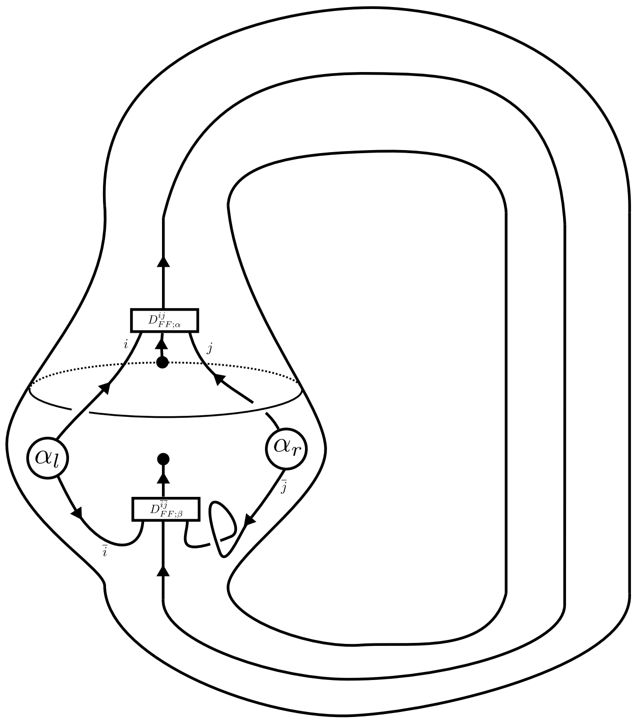
$Z(F) =$



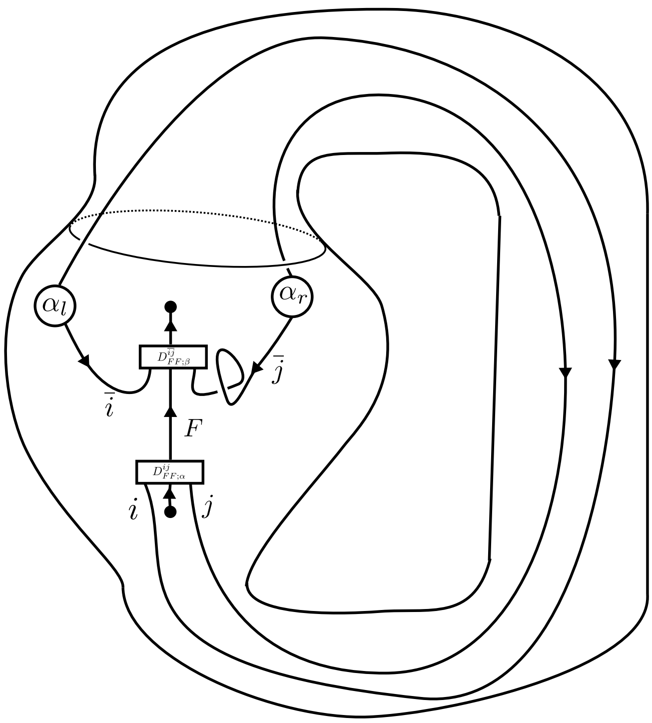
.

The factorization formula derived in the previous chapter yields

$$Z(F) = \sum_{i,j\in \mathsf{I}} \sum_{\alpha,\beta} d_j d_j \left(K^{ij}_{F;\alpha\beta}\right)^{-1}$$



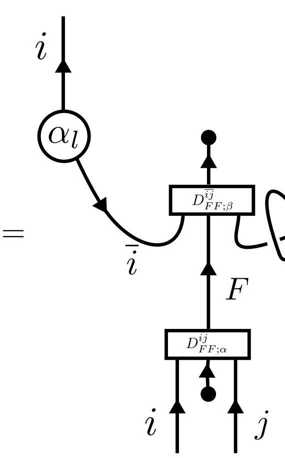
$$= \sum_{i,j\in \mathsf{I}} \sum_{\alpha,\beta} d_j d_j \left(K^{ij}_{F;\alpha\beta}\right)^{-1}$$



where in the equality we just dragged the string-net around the b-cycle. The non-trivial string-net in the left cylinder is a morphism in $\mathrm{Hom}_{\mathsf{C}}(U_i \otimes U_j, U_i \otimes U_j)$. Since the trace is a non-degenerate bilinear form on hom-space in a semi-simple category, we can pick an

ONB $\{e_1, \cdots, e_N\}$ in $\mathrm{Hom}_{\mathsf{C}}(U_i \otimes U_j, U_i \otimes U_j)$ for which $e_1 = \frac{1}{d_i d_j}\mathrm{id}_{U_i \otimes U_j}$. The coefficients of an expansion of $f \in \mathrm{Hom}_{\mathsf{C}}(U_i \otimes U_j, U_i \otimes U_j)$ in terms of the basis are given by $tr(f \circ e_i)$. Let

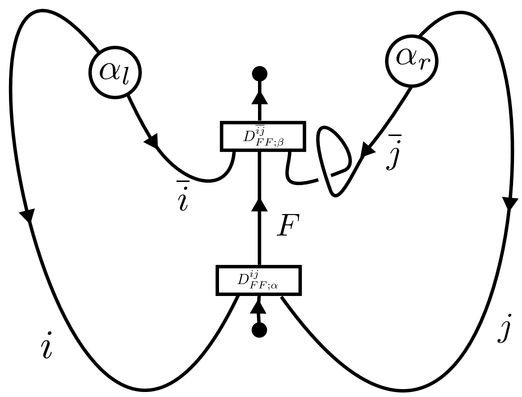
$P^{ij} =$



,

then it holds

$tr(e_1 \circ P^{ij}) =$



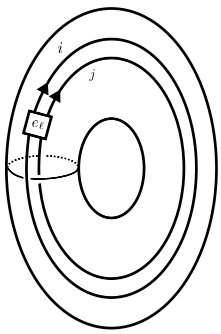
$= \frac{1}{d_i d_j} K^{ij}_{F;\beta\alpha}$

Thus we have a decomposition of $Z(F) = Z(F)_1 + Z(F)_2$ into two terms:

$$Z(F)_1 = \sum_{i,j \in \mathsf{I}} Z_{i,j}(F) T_{i,j} \tag{4.68}$$

and

$$Z(F)_2 = \sum_{i,j \in \mathsf{I}} \sum_{\alpha,\beta} d_j d_j \left(K^{ij}_{F;\alpha\beta} \right)^{-1} \sum_{\ell=2}^{N} C_\ell$$



.

By the discussion of basis element for torus string-net spaces $Z(F)_2 = 0$. Its string-net part needs to have an expansion in terms of e_1 in order to being non-vanishing, but these are exactly the terms collected in $Z(F)_2$.

We summarize

Theorem 4.4.1. *Given a special symmetric Frobenius algebra F in a modular tensor category C, its partition function is given by*

$$Z(F) = \sum_{i,j\in \mathsf{I}} Z_{i,j} T_{i,j} \tag{4.69}$$

with expansion coefficients

$$Z_{i,j}(F) \equiv \dim_{\mathbb{C}} \Big[\mathrm{Hom}_{F|F}(U_i \otimes^+ F \otimes^- U_j, F) \Big] \quad . \tag{4.70}$$

Coming back to the Grothendieck ring of $\mathsf{Z}(\mathsf{C})$, we note that the string-net basis elements are in one-to-one correspondence with objects $\{U_i \otimes U_j\}$. Identifying these with characters of simple representations $\chi_i \overline{\chi_j}$ we see that (4.70) gives an expansion

$$Z(F) = \sum_{i,j\in \mathsf{I}} Z_{i,j} \chi_i \overline{\chi_j} \tag{4.71}$$

in terms of left and right moving characters of simple representations, in case C is the representation category of a rational VOA.

As a concrete example one can consider the ADE classification of modular invariant partition functions for $\widehat{\mathfrak{sl}(2)}_k$. It is possible to compute the respective modular invariant partition functions using the categorical setup. We are not going to perform the calculations, but lay out the necessary steps. The ADE classification first appeared in [30][39]. A rigorous construction was given in [46][100][123]. Let R_k be the modular tensor category of representations of $\widehat{\mathfrak{sl}(2)}_k$. It contains $k+1$ equivalence classes of simple objects $\{U_0, \cdots, U_k\}$, which are the integrable highest weight representations. According to [123] the relation between Dynkin diagrams and algebras in R_k reads

Dynkin diagram	level ℓ	Algebra F
A_n	$n-1$	$F = U_0$
D_{2n+2}	$4n$	$F = U_0 \oplus U_{4n}$
E_6	10	$F = U_0 \oplus U_6$
E_7	16	$F = U_0 \oplus U_8 \oplus U_{16}$
E_8	28	$F = U_0 \oplus U_{10} \oplus U_{18} \oplus U_{28}$

The algebra structure in each case is derived from the branching rules (which are the fusion rules)

$$[U_i] \star [U_j] = \sum_{\ell=|i-j|}^{\min(i+j,2k-i-j)} [U_\ell] \quad . \tag{4.72}$$

Having the components $\left\{\mu_{ij}^{k}\right\}$ of the algebra map $\mu : F \otimes F \to F$ one can use [60, (3.83)] to express components $\left\{\Delta_{k}^{ij}\right\}$ of the coalgebra map $\Delta : F \to F \otimes F$ in terms of the $\left\{\mu_{ij}^{k}\right\}$ and other categorical data. Finally [60, (5.85)] gives an expression for $Z_{\bar{m}n}$ in terms of $\left\{\mu_{ij}^{k}\right\}$, fusion matrices of R_k and their inverses, braiding morphisms in R_k as well as components of twists of simple objects in R_k. This gives in principle an algorithm for computing Z_{mn}, though going through all the steps (in particular working out the structure constants of the algebra map) is a cumbersome and lengthy task. In case of the E_7 invariant this was done in [60] and the result for the partition function is

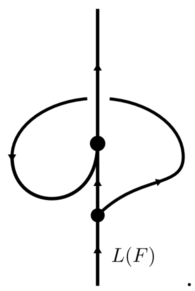
$$\begin{aligned} Z(F) &= \sum_{i,j=0}^{16} Z_{ij}\chi_i\overline{\chi_j} \\ &= |\chi_0+\chi_{16}|^2 + |\chi_4+\chi_{12}|^2 + |\chi_6+\chi_{10}|^2 + |\chi_8|^2 \\ &\qquad + \chi_8(\overline{\chi_3}+\overline{\chi_{14}}) + (\chi_2+\chi_{14})\overline{\chi_8} \end{aligned} \tag{4.73}$$

which is of course the result known in physics (see e.g. [26, table 4.1]).

Finally we come back to Cardy algebras. It turns out that there exists a $(\mathsf{C}|\mathsf{Z}(\mathsf{C}))$-Cardy algebra $(F, Z(F), \iota)$, where $Z(F) = \sum_{i,j\in \mathsf{I}(\mathsf{C})} Z_{i,j}(F) U_i \otimes U_j$. The crucial step is the realization of $Z(F)$ as the image of a retract

$$r : L(F) \to Z(F), \quad e : Z(F) \to L(F), \quad e \circ r = P_{L(F)} \tag{4.74}$$

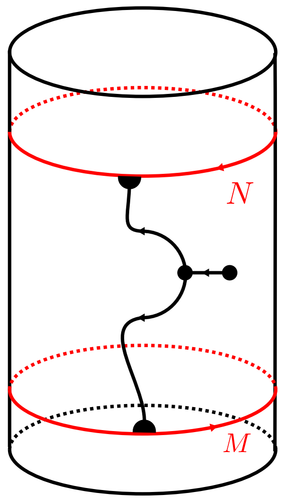
of the idempotent

$P_{L(F)} =$ 

This is shown e.g. in [128]. The precise relation to Cardy algebras is the theorem

Theorem 4.4.2. *[107, Theorem 3.18] Given a special, symmetric Frobenius algebra F in C. The triple $(F, Z(F), \iota_{cl-op} = e)$ is a $(\mathsf{C}|\mathsf{Z}(\mathsf{C}))$-Cardy algebra.*

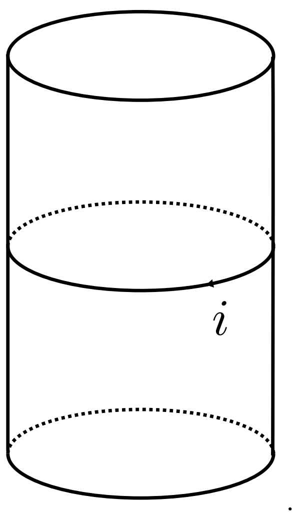
With the considerations for the bulk partition function it is not hard to compute the annulus partition for two boundary modules M, N. The defect graph consists of two boundary edges winding around the boundary components of the cylinder plus a F-colored triangulation of the cylinder. Upon using the Frobenius properties of F one easily checks that this can always be reduced to

$$A(F)^N_M =$$ 

Note that this a string-net without projector circles, i.e. with empty boundary value. By [98, Theorem 8.4,6.4] it holds

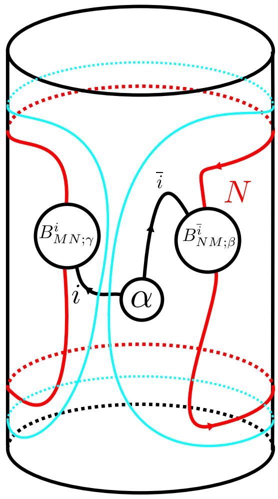
$$\begin{aligned} H^s(A) \simeq \mathrm{Hom}_{\mathsf{Z(C)}}(L(\mathbf{1}), L(\mathbf{1})) &\simeq \bigoplus_{i,j \in \mathsf{I}} \mathrm{Hom}_{\mathsf{C}}(U_i^*, U_j^*) \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathsf{C}}(U_i, U_j) \\ &\simeq \bigoplus_{i \in \mathsf{I}} \mathrm{Hom}_{\mathsf{C}}(U_i, U_i) \end{aligned} \tag{4.75}$$

but this is a vector space of dimension $|\mathsf{I}|$ spanned by the identity morphisms. Going through Kirillov's construction it is not hard to see that a string-net basis is given by

$$B_i =$$ 

Using boundary factorization we can rewrite $A(F)^N_M$ as

$$A(F)_M^N = \sum_{i \in \mathsf{I}} \sum_{\beta,\gamma} d_i \left(K^i_{MN}\right)^{-1}_{\beta\gamma}$$



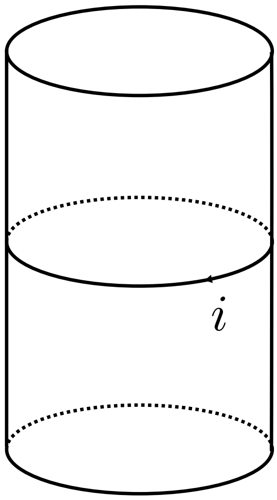
.

In the figure we added a contractible circle shown in turquoise which is not part of the string-net but highlights an embedded disk. The string-net inside this disk is an element in $\mathrm{Hom}_{\mathsf{C}}(U_i, U_i)$. To find its projection onto the identity morphism one take its categorical trace and divides by d_i. The result is easily seen to be

$$\frac{K^i_{MN:\gamma\beta}}{d_i} \quad . \tag{4.76}$$

Thus the partition function has an expansion

$$A(F)_M^N = \sum_{i \in \mathsf{I}} \sum_{\beta,\gamma} d_i \left(K^i_{MN}\right)^{-1}_{\beta\gamma} \frac{K^i_{MN:\gamma\beta}}{d_i}$$



.

or in plain formulas

$$A(F)_M^N = \sum_{i \in \mathsf{I}} \dim\left[\mathrm{Hom}_F(M \otimes U_i, N)\right] \, B_i \tag{4.77}$$

With the one to one map between simple objects and characters of a VOA, this is the formula

$$A(F)_M^N = \sum_{i \in \mathsf{I}} A(F)^N_{i,M} \chi_i \tag{4.78}$$

with $A(F)_{i,M}^{N} = \dim\left[\mathrm{Hom}_F(M\otimes U_i, N)\right]$. This justifies the use of $\mathrm{Hom}_F(M\otimes U_i, N)$ as the space of boundary fields. As a first consistency remark we note that $A(F)_{i,M}^{N} \in \mathbb{Z}_{\geq 0}$ sine they are dimensions of vector spaces. This is of course expected for multiplicity spaces of fields. Furthermore we rederived formulas [60, (5.135),(5.64)] for annulus and torus partition function. In addition in [60, section 5] various checks for the expansion coefficients were performed. As an example it was shown that the trace formula [129] holds

$$\sum_{M\in \mathsf{I}_F} A(F)_{i,M}^{M} = \sum_{k,\ell,j\in \mathsf{I}} N_{\ell i}^{\bar{k}} Z(F)_{\ell k} \tag{4.79}$$

where the sum one the lhs runs over all simple left F-modules[1]. As another proof of principle one can easily derive that the annulus partition function gives a non-negative integral matrix representation (NIM-rep) of the fusion rules. As the left module structure on $M\otimes U_i$ reads

$$\rho_{M\otimes U_i} = \rho_M \otimes \mathrm{id}_{U_i} \tag{4.80}$$

it holds that

$$\mathrm{Hom}_F(M\otimes U_i, N) \simeq \mathrm{Hom}_F(M, N\otimes U_i^*) \tag{4.81}$$

by applying evaluation and coevaluation morphisms for U_i. Let M, K be simple left F-modules. It holds

$$\begin{aligned}
\sum_{N\in \mathsf{I}_F} A(F)_{i,M}^{N} A(F)_{j,N}^{K} &\overset{(1)}{=} \sum_{N\in \mathsf{I}_F} \dim\left[\mathrm{Hom}_F(M\otimes U_i, N)\right] \dim\left[\mathrm{Hom}_F(N\otimes U_j, K)\right] \\
&\overset{(2)}{=} \sum_{N\in \mathsf{I}_F} \dim\Big[\mathrm{Hom}_F(M\otimes U_i, N) \otimes_{\mathbb{C}} \mathrm{Hom}_F(N, K\otimes U_j^*)\Big] \\
&\overset{(3)}{=} \dim\Big[\mathrm{Hom}(M\otimes U_i, K\otimes U_j^*)\Big] \\
&\overset{(4)}{=} \sum_{k\in \mathsf{I}} N_{ij}^{k} \dim\left[\mathrm{Hom}(M\otimes U_k, K)\right] \\
&\overset{(5)}{=} \sum_{k\in \mathsf{I}} N_{ij}^{k} A(F)_{k,M}^{K} \quad .
\end{aligned} \tag{4.82}$$

In step (1) only definitions are inserted, in (2) it is used that the product of dimensions of vector spaces is the dimension of the tensor product and (4.81) is applied. (3) is semi-simplicity of ${}_F M$. In (4) again (4.81) is used followed by the fusion rules $U_i\otimes U_j = \bigoplus_{k\in \mathsf{I}} N_{ij}^{k} U_k$. Lastly (5) is just the definition again.

During the final stage of writing this thesis it was brought to our attention that independently from our work, in [132] a construction related to the one presented in this chapter is performed.

[1]It was shown on [65, section 5] that for a special Frobenius algebra F in a modular tensor category C the category of left modules ${}_F\mathsf{M}$ is semi-simple.

4.5 Conclusions

In this chapter we have shown that $(\mathsf{C}|\mathsf{Z}(\mathsf{C}))$-Cardy algebras uniquely determine consistent correlators for all genus open-closed RCFTs. The proof is via string-net models on surfaces. One way of saying this is that we give a hands on proof where the categorical string-diagrams really correspond to Feynman diagrams of the RCFTs. Furthermore we extended the construction to include symmetry preserving boundary conditions as well as topological defects. We showed that the extended construction still solves the sewing constraints by showing precise factorization formulas for correlators under bulk and boundary gluing. Using this we computed the bulk and boundary partition functions and showed that the bulk computation reproduces known partition functions. Furthermore, the categorical prescriptions give easy proofs of some facts for RCFTs, e.g. we could easily show that annulus partition functions constitute a NIM-rep. of the fusion rules.

A natural question at this point is if the string-net construction can expanded to an equivariant setting. By this we mean that one might look at situations where a discrete group G acts on the surface and the correlators should be equivariant under the action of G. Quotiening the group action out, one should obtain correlators of the G-orbifold theory. Another loose end is an extension beyond rational CFTs. In contrast to the G-equivariant setting where natural generalizations of modular tensor categories exist, the categorical setting in the non-rational case is less developed. For logarithmic CFTs there is an extended notion of modularity, but the construction given in this thesis heavily depends on semi-simplicity of all categories in question. Lastly one might ask about the relation to other "curves on surfaces" approaches to open-closed interactions. One related example might be the Arc-complex approach of [96][97]. An obvious loose end is the missing connection between the true complex analytic and the categorical world. In an algebraic geometry setting recent progress towards an understanding of vector bundles of conformal blocks for rational VOAs (in particular the behavior under factorization) has been made in the papers [37][38]. In the analytic setting articles [67][68] contain a construction of the vector bundle and a proof for factorization. Using the tools of [73] it seems possible to get a true complex analytic modular functor from [68]. This could then be compared to the categorical notions.

Chapter 5

Homotopy Algebras and Field Theories

This chapter deals with the second thematic part, homotopy algebras and their role as an organizing structure for classical gauge theories.The results presented in this chapter are based on the publications [74][21].

5.1 Basics of L_∞ algebras

The main technical tool will be strong homotopy Lie algebras also known as L_∞ algebras. There are by now many sources for basic L_∞-theory, since they turn out to play a role in various different areas in mathematics and physics. Classic sources are [109][72][110]. A textbook account via operads can be found in [119]. Mathematical descriptions of field theories using local L_∞-algebras appeared in [35][36]. More physics minded introductions are given e.g. in [75][95][152][23][24].

This introductory section is structured as follows. We begin with a quick conceptual approach to L_∞ algebras using tree shaped graphs. This will lead to a definition in terms of higher skew symmetric brackets on a graded vector space. The bar construction will give an alternative definition with symmetric higher brackets on the degree shifted graded vector space. This step is mainly a technical one. The determined reader could stick with the first definition and work out all results along the same lines as presented here. But as a word of warning, signs and degrees become very unhandy very quickly. The degree shifted version is better suited for describing morphisms and quasi-isomorphisms of L_∞ algebras, whose definition we give next. Their definition becomes conceptually clearer in the shifted version of L_∞ algebras.

We begin with a conceptual approach to L_∞ algebras. The reader familiar with operads will recognize the cofibrant replacement of the Lie operad, but we will not use these terms. The take away from these introductory words may be the results of [118]. An algebra over such a cofibrant replacement is really a structure up to homotopy, meaning in this specific case, that given an L_∞ algebra on a chain complex $(A_\bullet, d_A)$, and given another chain complex $(B_\bullet, d_B)$, which is chain homotopy equivalent to $(A_\bullet, d_A)$, the L_∞-structure can be transported to $(B_\bullet, d_B)$ via the chain homotopy equivalence. This justifies the "homotopy" in strong homotopy Lie algebra. Recall that a *Lie algebra* on a vector space V is a skewsymmetric bilinear bracket

$$[\,\bullet\,,\,\bullet\,] : V \times V \to V, \quad [v, w] = -[w, v] \tag{5.1}$$

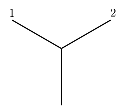
satisfying the Jacobi identity

$$[u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0 \quad . \tag{5.2}$$

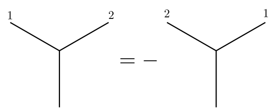
In case the vector space is graded $V = \bigoplus_{n\in\mathbb{Z}} V_n$ the bracket and Jacobi identity has to respect the internal grading, i.e. for homogeneous elements v, w

$$[v, w] = (-1)^{1+|v||w|}[w, v], \quad [v, w] \in V_{|v|+|w|} \tag{5.3}$$

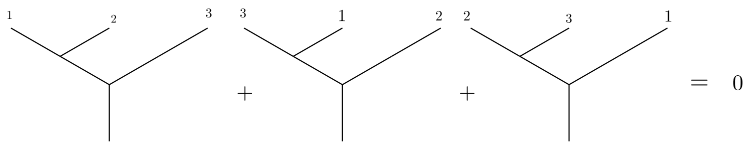
and similar signs appear in the Jacobi-identity. The essential information behind the definition can be nicely encoded in planar trees. The bracket corresponds to a binary tree



which satisfies skew symmetry



and Jacobi-relation



A bit more formal we consider the S_2-module $\Sigma(2)$, generated by $\curlyvee$, which is just the sign-representation. Here S_n denotes the symmetric group on n elements. The corollas can be grafted by gluing roots to leaves as we did for the Jacobi identity relation. Given a planar tree T with n-leaves one can form a S_n-module by decorating vertices with S_i-modules. To be more precise, consider a collection of S_i-modules $\{K(i)\}$, then one sets

$$K(T) = \bigoplus_{v\in v(T)} K(|in(v)|) \tag{5.4}$$

where $in(v)$ is the set of incoming edges at v. All our trees are oriented top to bottom, i.e. from leaves to root. In case we consider the collection

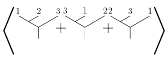
$$\Sigma(n) = \begin{cases} \Sigma_2 \simeq \mathbb{C}[\curlyvee], n = 2 \\ 0, \text{ else} \end{cases} \tag{5.5}$$

the only surviving contributions in $\Sigma(T)$ are trees with trivalent vertices only. Of course this just corresponds to all possible ways to compose the bracket. Modulo technicalities, the $S = \bigsqcup_{n\geq 2} S_n$ -module $F(K)$ given by

$$F(K)(n) \equiv \bigoplus_{T\in PRT, L(T)=n} K(T) \tag{5.6}$$

with PRT the finite set of all rooted planar trees and $L(T)$ the number of leaves, is the *freely generated* **S**-*module.* E.g. the trees in the Jacobi-relation are elements in $F(\mathbb{C}[\curlyvee])(3)$. Since we want the Jacobi-identity to hold, we quotient the free module by these relations, i.e

$$\mathscr{L}\!\mathit{ie} = F(\mathbb{C}[\curlyvee])/\left\langle \quad \right\rangle . \tag{5.7}$$



To get a Lie algebra on a graded vector space V we consider the **S**-module $\{\mathrm{Hom}(V^{\otimes n}, V)\}_{n\geq 2}$ with $\mathrm{Hom}(V^{\otimes n}, V) = \bigoplus_{m\in\mathbb{Z}} \mathrm{Hom}_m(V^{\otimes n}, V)$ the vector space of n-airy bilinear maps of all degrees m. The symmetric group acts by permuting inputs of bilinear maps. A Lie algebra on V is nothing else then a **S**-equivariant, degree preserving map

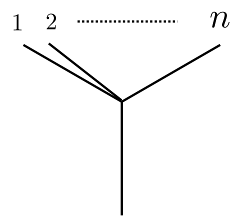
$$L : \{\mathscr{L}\!\mathit{ie}(n)\} \to \left\{\mathrm{Hom}(V^{\otimes n}, V)\right\} \quad . \tag{5.8}$$

In addition, both sides have a natural notion of composition. On the lhs this is grafting of trees, where on the rhs it is simply composition of maps. We require L to respect compositions, in other words, it should be a map of $\mathbb{C}$-linear operads. That this indeed defines a Lie algebra is easily checked. Since the left hand side is concentrated in internal degree 0, there will be only degree preserving linear maps in the image of L. The only generating element is the binary tree, which gets mapped to a skew-symmetric bracket due to equivariance. The quotient enforces the Jacobi identity for the bracket. All higher airy operations are given by compositions of the bracket. Note that in this construction the defining ingredient of a Lie algebra, the bracket, is separated from the vector space it acts on. This allows for an independent study of such operations. This is the whole essence of *operads.*

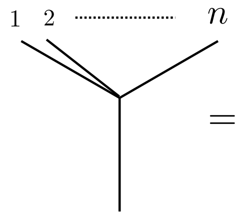
The notion of a L_∞ algebra resolves Lie algebras. This means, that one sets up a chain complex $(\mathscr{L}\!\mathit{ie}_\infty, \partial)$ and taking $\mathscr{L}\!\mathit{ie}$ as a chain complex with trivial differential concentrated in degree 0 there should be quasi-isomorphism

$$(\mathscr{L}\!\mathit{ie}_\infty, \partial) \xrightarrow{\simeq} \mathscr{L}\!\mathit{ie} \quad . \tag{5.9}$$

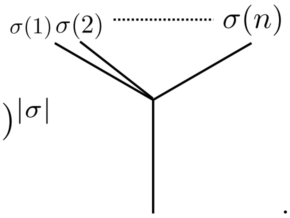
We are working in homological grading, i.e differentials are of degree -1. A concise definition of $\mathscr{L}\!\mathit{ie}_\infty$ can be found in [117]. Let V be a graded vector space. We denote $V[k] = \bigoplus_{n\in\mathbb{Z}} V[k]_n$ with $V[k]_n \equiv V_{n-k}$ for the k-shifted vector space. Consider the natural higher airy generalization $L(n) = sgn_n$ of the generating module in the Lie algebra case. One might think of this as the $\mathbb{C}$-vector space generated by n-corolla



subject to the skewsymmetry relation



$$= (-1)^{|\sigma|}$$



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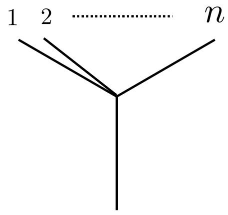
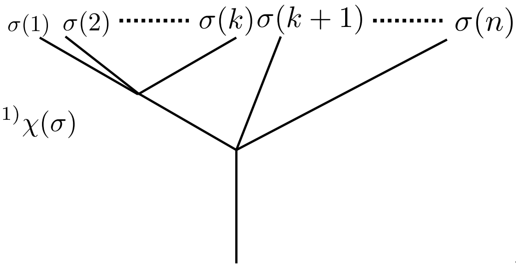
As before consider the **S**-module $F(L)$, freely generated by $\{L(n)[n-2]\}_{n\geq 2}$ via grafting of trees. We turn this into a chain complex by defining the differential on generators

$$(-1)^n\partial$$



$$= \sum_{\substack{j+k=n+1\\ j,k\geq 2}} \sum_{\sigma\in US(k,n)} (-1)^{k(j-1)}\chi(\sigma)$$



.

The second sum runs over all (k,n)-unshuffles, which are permutations satisfying

$$\sigma(1) < \cdots \sigma(k), \quad \sigma(k+1) < \cdots < \sigma(n) \tag{5.10}$$

and $\chi(\sigma)$ is the Koszul sign of the permutation, which in this case is just $(-1)^{|\sigma|}$. The differential can be extended to all of $F(L)$ by requiring it to be a derivation wrt. to grafting of trees. Note that the differential is minimal. It has no linear term, rather it is a sum over quadratic terms. Since ∂ is extended as a derivation, its homology is the free operad on the homology of the generators. One easily checks, that in degree 0, the differential is the zero map. The image of the three corolla is exactly the Jacobi-identity. Hence the homology in degree zero exactly reproduces $\mathscr{L}ie$. One has to show that higher homologies vanish, which is lengthy but straightforward. The alert reader recognizes the structure of an L_∞-algebra in the differential. Indeed an L_∞ *algebra* is a **S**-equivariant, degree preserving map

$$(\mathscr{L}ie_\infty, \partial) \longrightarrow \left(\bigoplus_{n\geq 2} \mathrm{Hom}_\bullet(V^{\otimes n}, V), d\right) \tag{5.11}$$

of chain complexes, where (V,d) is a chain complex and the differential is extended to multilinear maps as

$$(df)(v_1,\cdots,v_n) = d(f(v_1,\cdots,v_n)) + \sum_{i=1}^{n}(-1)^{|v_1|+\cdots+|v_{i-1}|} f(v_1,\cdots,d(v_i),\cdots,v_n) \quad . \tag{5.12}$$

This means the data of an L_∞-algebra consists of a chain complex with higher airy brackets $(V, \{\mu_n\}_{n\geq 1})$ of degree $|\mu_n| = n-2$, which are skew symmetric

$$\mu_n(\cdots, v_i, v_{i+1}, \cdots) = (-1)^{1+|v_i||v_{i-1}|}\mu_n(\cdots, v_{i+1}, v_i, \cdots) \quad . \tag{5.13}$$

and we set $\mu_1 = d$. Being a map of chain complexes in addition leads to the L_∞-relations

$$\mathcal{J}_n = \sum_{\substack{j+k=n+1\\ j,k\geq 1}} \sum_{\sigma\in US(k,n)} (-1)^{k(j-1)}(-1)^\sigma \chi(\sigma)\mu_j(\mu_k(v_{\sigma(1)}, \cdots, v_{\sigma(k)}), v_{\sigma(k+1)}, \cdots, v_{\sigma(n)}) = 0 \tag{5.14}$$

for any $v_1, \cdots, v_n \in V$ and $n \geq 1$. The Koszul sign now takes the internal grading of V into account, i.e.

$$\chi(\sigma) v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)} = v_1 \wedge \cdots \wedge v_n \tag{5.15}$$

for $v \wedge w = (-1)^{|v||w|} w \wedge v$. Let us spell out the first few defining equations to get a better feeling for L_∞ algebras. In the following we denote $u, v, w \in V$.

1)
$$\mathcal{J}_1 = \mu_1 \circ \mu_1 = 0 \tag{5.16}$$

The first equation is of course just the fact that μ_1 is a differential.

2)
$$\mathcal{J}_2(u, v) = -\mu_1(\mu_2(u, v)) + \mu_2(\mu_1(u), v) + (-1)^{|u|}\mu_2(u, \mu_1(v)) = 0 \tag{5.17}$$

This is the information that the differential is a derivation of the two bracket.

3)

$$\begin{aligned}\mathcal{J}_3(u, v, w) =& \mu_1(\mu_3(u, v, w)) + \mu_3(\mu_1(u), v, w) + \\ & (-1)^{|u|}\mu_3(u, \mu_1(v), w) + (-1)^{|u|+|v|}\mu_3(u, v, \mu_1(w)) + \mu_2(\mu_2(u, v), w) \\ & + (-1)^{(|u|+|v|)|w|}\mu_2(\mu_2(w, u), v) + (-1)^{(|v|+|w|)|u|}\mu_2(\mu_2(v, w), u) \\ =& 0\end{aligned} \tag{5.18}$$

The third equation captures the failure of μ_2 being a Lie bracket. It gives that the Jacobi identity is satisfied up to derivative terms of a higher homotopy μ_3. Note that relation $\mathcal{J}_n$ is a map of degree $n-3$.

This pattern continues for higher equations. For example $\mathcal{J}_4$ gives that a generalized Jacobi identity among μ_3,μ_2 holds up to derivative terms of a μ_4 homotopy.

For some purposes it is more convenient to give a definition of an L_∞ algebra in terms of symmetric maps. Mathematically this is the bar construction. The construction assigns to an L_∞ algebra $(V, \{\mu_n\}_{n\geq 1})$ a codifferential D of degree -1 on the cofree coalgebra $\bigwedge V[1] = \bigoplus_{n\geq} \bigwedge^n V[1]$. We neither discuss coderivations, nor the terms cofree or coalgebra.

Instead we decompose the coderivation in terms of linear maps. D decomposes naturally into

$$D = \sum_{n,k\geq 1} D^{n,k}, \quad D^{n,k} : \bigwedge^n V[1] \to \bigwedge^k V[1], k \leq n \tag{5.19}$$

and in fact being a coderivation ensures that there are linear and symmetric maps $\left\{d_i : \bigwedge^i V[1] \to V[1]\right\}$, s.th.

$$D^{n,n-k}(w_1, \cdots, w_n) = \sum_{\sigma\in US(k,n)} d_{k+1}(w_{\sigma(1)}, \cdots, w_{\sigma(k+1)}) \wedge w_{\sigma(k+2)} \wedge \cdots \wedge w_{\sigma(n)} \tag{5.20}$$

Being a codifferential, i.e $D^2 = 0$ produces an infinite set of equations for the linear maps $\{d_i\}$, just as there were infinitely many consistency equations for an L_∞-algebra. Figuring out these equations is just a matter of algebra and the result is

$$0 = \sum_{\substack{k+\ell=n+1\\ k,\ell\geq 1}} \sum_{\sigma\in US(k,n)} \chi(\sigma)\, d_\ell(d_k(w_{\sigma(1)}, \cdots, w_{\sigma(k)}), w_{\sigma(k+1)}, \cdots, w_{\sigma(n)})), \tag{5.21}$$

for any $w_1, \cdots, w_n \in V[1]$ and $n \geq 1$. Hence one defines a 1-*shifted* L_∞-*algebra* to be a graded vector space W with symmetric multilinear maps $\{d_i\}_{i\geq 1}$ satisfying (5.21). Given a L_∞ algebra $(V, \{\mu_i\})$ one gets a 1-shifted L_∞ algebra on $V[1]$ by setting

$$d_n(sv_1, \cdots, sv_n) = (-1)^{\sum_{i=1}^n |v_i|(n-i)} s\mu_n(v_1, \cdots, v_n) \tag{5.22}$$

where $s : V_n \to V[1]_{n+1}$ is the shifting isomorphism. One easily checks that d_n is symmetric and of degree -1. That the maps indeed give a 1-shifted L_∞ algebra is shown in [110]. The reason for introducing 1-shifted L_∞ algebras is their easier handling. All maps have the same degree, there are no extra signs from skew symmetry and in addition the defining equation (5.21) has less signs. Moreover the definition of a L_∞-morphism becomes quite handy in the shifted case. Being really terse one might define a morphism of L_∞ algebras $F : (V, \{\mu_n\}) \to (W, \{\nu_n\})$ to be a map of codifferential, cofree coalgebras $(\bigwedge V[1], D_V) \to (\bigwedge W[1], D_W)$. Of course this definition is not very enlightening in terms of doing computations. Spelled out in terms of the component maps of the codifferentials this is equivalent to a collection of symmetric multilinear maps $\left\{F_k : V[1]^{\otimes k} \to W[1]\right\}_{k\geq 1}$ of degree 0 satisfying the defining equation

$$\begin{aligned} &\sum_{k+l=n+1} \sum_{\sigma\in US(k,n)} \chi(\sigma)\, F_l\Big(d_k^V(x_{\sigma(1)}, \ldots, x_{\sigma(k)}), x_{\sigma(k+1)}, \ldots, x_{\sigma(n)}\Big) \\ &= \sum_{j=1}^n \sum_{\substack{k_1+\cdots+k_j=n\\ k_i\geq 1}} \sum_{\sigma\in US(k_1,\cdots,k_n)} \frac{\chi(\sigma)}{j!}\, d_j^W(F_{k_1}\otimes\cdots\otimes F_{k_j})(x_{\sigma(K)})\ . \end{aligned} \tag{5.23}$$

A morphism $\left\{F_k : V[1]^{\otimes k} \to W[1]\right\}_{k\geq 1}$ is a *quasi-isomorphism* if the induced map $F_1 : H_\bullet(V[1], d_1^V) \to H_\bullet(W[1], d_1^W)$ is an isomorphism. Note that being a quasi-isomorphism is an equivalence relation as shown in [131, Corollary 5.11]. In addition in [131] it is proven that every L_∞ algebra splits uniquely up to quasi-isomorphism

$$(L, \{\mu_i\}) \simeq (F, \mu_1) \oplus (H, \{\nu_i\}_{i\geq 2}) \tag{5.24}$$

where $H \equiv H_\bullet(L, \mu_i)$ is the homology of the underlying chain complex and (F, μ_1) its complement in (L, μ_1). The first summand is called *linear contractible* and the second *minimal model*. Uniqueness follows now from the fact, that there exist a quasi-isomorphism

$$(L, \{\mu_i\}) \xrightarrow{\simeq} (H, \{\nu_j\}_{j\geq 2}) \quad . \tag{5.25}$$

5.2 Construction of L_∞ algebras

In this section we give the main result of [74] and outline its proof. A corollary of the theorem will be e.g. the L_∞ structure for the Courant algebroid.

In [10] it was shown that for a given vector space H and a homological resolution

$$\cdots \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} \cdots \xrightarrow{d_1} X_0 \simeq B_0 \oplus H \xrightarrow{pr_H} H \longrightarrow 0$$

one can recursively construct a L_∞-structure on (X, μ_1), where we collected all the differentials of the resolution in μ_1 and threw away the degree -1 term H, as it is inessential to the construction. The construction is by no means unique nor canonical. It crucially depends on the choice of a homotopy inverse $\Upsilon : H \to (X_\bullet, \mu_1)$ to the quasi-isomorphism $\Sigma : (X_\bullet, \mu_1) \to H$[1]. In particular the construction depends on a choice of two bracket

$$\mu_2 : X_0 \otimes X_0 \to X_0 \tag{5.26}$$

satisfying

$$\mu_2(\mathrm{Im}(d_1), x_0) \in \mathrm{Im}(d_1) \tag{5.27}$$

and

$$\mathrm{Jac}^{\mu_2}(u, v, w) \in \mathrm{Im}(d_1) \tag{5.28}$$

with

$$\mathrm{Jac}^{\mu_2}(u, v, w) = \mu_2(u, \mu_2(v, w)) - \mu_2(v, \mu_2(u, w)) + \mu_2(w, \mu_2(u, v)) \tag{5.29}$$

the *Jacobiator* for the bracket μ_2 and elements $u, v, w \in X_0$. Assume that $[\,\bullet\,,\,\bullet\,] : H \otimes H \to H$ is Lie bracket. It is not hard to show that the following is true

Lemma 5.2.1. *[10, section 2] Given a resolution $(X_\bullet, \mu_1) \xrightarrow{\simeq} H$ the triple $(\mu_2, \Upsilon, [\,\bullet\,,\,\bullet\,])$ satisfies the two-out-of-three property, i.e. given any two elements of the triple determines the third one.*

[1]This means that $\Sigma \circ \Upsilon = \mathrm{id}_H$ and $\Upsilon \circ \Sigma \sim \mathrm{id}_{X_\bullet}$, where the two maps are homotopic as maps of chain complexes.

The construction of the L_∞ algebra on $(X_\bullet, \mu_1)$ in [10] starts with the assumption of the map μ_2, which by the previous lemma is equivalent to the existence of a Lie algebra structure on H and the choice of a homotopy inverse. In this general setup, the result in [10] merely guarantees existence of an L_∞ algebra structure, but says nothing about its form. In particular the L_∞ resolution may have infinitely many brackets, rendering the construction possibly quite unhandy. The main result of [74] is that in situations where μ_2 is canonically given, the L_∞ resolution has a minimally truncated solution with only finitely many non-vanishing brackets and no choices are required (up to L_∞-quasi-isomorphism). This in particular applies to situations where $(H, [\,\bullet\,,\,\bullet\,])$ as a Lie algebra is known solely as a quotient $H = (X_0, \mu_2)/B_0$.

Theorem 5.2.2. *[74, Theorem 2] Given a three term complex*

$$\cdots \longrightarrow 0 \longrightarrow X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \longrightarrow 0$$

with $H_1(X_\bullet) = H_2(X_\bullet) = 0$ and X_0 carries a skew symmetric bilinear map $\mu_2 : X_0 \otimes X_0 \to X_0$ s.th. $\mathrm{Im}(d_1)$ *is an ideal*

$$\mu_2(\mathrm{Im}(d_1), x) \in \mathrm{Im}(d_1) \tag{5.30}$$

and

$$\mathrm{Jac}(x, y, z) \in \mathrm{Im}(d_1) \tag{5.31}$$

for $x, z, y \in X_0$, then there exists an L_∞ structure on $(X_\bullet, \mu_1)$ with highest non-trivial bracket $\mu_4 : X_0^{\otimes 4} \to X_0$.

Proof. For the proof it will be convenient to indicate the vector space an element lives in by a superscript, i.e. $u^i \in X_i$. The strategy of the proof is to solve the defining equations $\mathcal{J}_n$ iteratively starting with $\mathcal{J}_1$, which holds by definition. Some of the computations, especially in higher degree become quite lengthy. Since this is straightforward algebra we sometimes refer to the respective equation in [74], where the full details are spelled out.

1) $\mathcal{J}_2$: $\mathcal{J}_2$ is of degree -1 hence it is only non-vanishing on inputs of total degrees $1, 2, 3$. Thus we go through possible inputs one by one.

$X_1 X_0$:

$$d_1(\mu_2(u^1, v^0)) = \mu_1(\mu_2(u^1, v^0)) = \mu_2(d_1(u^1), v^0) \tag{5.32}$$

where the second term on the rhs vanishes due to degree reasons. By assumption there exists an element $f^1(u^1, v^0) \in X_1$ s.th.

$$d_1(f^1(u^1, v^0)) = \mu_2(d_1(u^1), v^0) \tag{5.33}$$

and we set

$$\mu_2(u^1, v^0) = f^1(u^1, v^0) = -\mu_2(v^0, u^1) \quad . \tag{5.34}$$

X_1X_1 :

$$\begin{aligned} d_2(\mu_2(u^1,v^1)) = \mu_1(\mu_2(u^1,v^1)) &= \mu_2(\mu_1(u^1),v^1) - \mu_2(u^1,\mu_1(v^1)) \\ &= -\mu_2(v^1,d_1(u^1)) - \mu_2(u^1,d_1(v^1)) \end{aligned} \tag{5.35}$$

Note that $X_2 \simeq \ker(d_1)$ and $d_2 = \iota : \ker(d_1) \hookrightarrow X_1$. Hence (5.35) has a solution if

$$-\mu_2(v^1,d_1(u^1)) - \mu_2(u^1,d_1(v^1)) = -f^1(v^1,d_1(u^1)) - f^1(u^1,d_1(v^1)) \in \ker(d_1) \tag{5.36}$$

Note that

$$d_1(f^1(d_1(u^1),v^1)) = \mu_2(d_1(u^1),d_1(v^1)) = d_1(f^1(u^1,d_1(v^1)) \quad . \tag{5.37}$$

Using the symmetry properties of the two bracket this yields

$$d_1(f^1(v^1,d_1(u^1)) + f^1(u^1,d_1(v^1))) = 0 \quad . \tag{5.38}$$

Thus we set

$$\mu_2(v^1,u^1) \equiv -(f^1(v^1,d_1(u^1)) + f^1(u^1,d_1(v^1)) \quad . \tag{5.39}$$

X_0X_2 : One easily checks that this equation reduces to

$$\iota(\mu_2(v^2,u^0)) = f^1(\iota(v^2),u^0) \quad . \tag{5.40}$$

One solves this by setting

$$\mu_2(v^2,u^0) \equiv f^1(\iota(v^2),u^0) \tag{5.41}$$

which is possible since

$$d_1(f^1(\iota(v^2),u^0)) = \mu_1(\mu_2(\iota(v^2),u^0)) = 0 \;\Rightarrow\; f^1(\iota(v^2),u^0) \in \mathrm{Im}(\iota) \quad . \tag{5.42}$$

We suppressed the inverse of the inclusion map ι in our notation.

X_2X_1 : By the previous relation we know

$$\mu_1(\mu_2(v^2,u^0)) = d_1(f^1(\iota(v^2),u^0)) = 0 \tag{5.43}$$

hence the relation reduces to

$$0 = \mu_2(\iota(v^2),u^1) + \mu_2(v^2,d_1(u^1)) = -f^1(\iota(v^2),d_1(u^1)) + f^1(\iota(v^2),d_1(u^1)) \tag{5.44}$$

where we used (5.39) and (5.41). Hence the brackets defined so far are consistent as $X_3 = 0$ and therefore no non-zero two bracket with inputs of degree three can exist.

2) $\mathscr{J}_3$: $\mathscr{J}_3$ is of degree 0, hence non-trivial equations only appear for inputs of total degree $0,1,2$. Again, we go through the relations degreewise.

$X_0X_0X_0$: Obviously this is the equation

$$0 = d_1(\mu_3(u^0, v^0, w^0)) + \mathrm{Jac}(u^0, v^0, w^0) \tag{5.45}$$

which has a solution

$$\mu_3(u^0, v^0, w^0) \equiv -f^1(u^0, v^0, w^0) \tag{5.46}$$

by assumption.

$X_1X_0X_0$: This is the equation

$$\begin{aligned} 0 =& \iota(\mu_3(u^1, v^0, w^0)) - f^1(d_1(u^1), v^0, w^0) \\ &+ f^1(f^1(u^1, v^0), w^0) - f^1(f^1(u^1, w^0), v^0) - f^1(u^1, \mu_2(v^0, w^0)) \quad . \end{aligned} \tag{5.47}$$

This has a solution if

$$\begin{aligned} g^1(u^1, v^0, w^0) \equiv& f^1(d_1(u^1), v^0, w^0) - f^1(f^1(u^1, v^0), w^0) \\ &+ f^1(f^1(u^1, w^0), v^0) + f^1(u^1, \mu_2(v^0, w^0)) \end{aligned} \tag{5.48}$$

is in $\ker(d_1)$. To show this is an easy task using (5.45) and (5.30). Omitting once more the inverse of ι from the notation we can therefore set

$$\mu_3(u^1, v^0, w^0) \equiv g^1(u^1, v^0, w^0) \quad . \tag{5.49}$$

$X_1X_1X_0$: This is a consistency equation, since a potential new bracket μ_3 solving it were to take values in $X_3 = 0$. Putting all the terms together is a bit cumbersome but straightforward. We refer to [74, (4.20)] for the explicit computation.

$X_2X_0X_0$: This is again a consistency equation. In the definition (5.49) for $u^1 = \iota(u^2)$ the terms from the three bracket vanishes. Inserting this in $\mathrm{Jac}(u^2, v^0, w^0)$ and using the definition (5.41) on readily checks that this equation is satisfied.

3) $\mathscr{J}_4$: These equations are of degree 1, hence only inputs of degrees $0,1$ may lead to non-trivial equations.

$X_0X_0X_0X_0$: We split this equation and suppress the superscript for elements for better readability

$$\mathscr{J}_4(u_1, u_2, u_4, u_4) = \mu_1(\mu_4(u_1, u_2, u_3, u_4)) + \mathscr{L}(u_1, u_2, u_3, u_4) \tag{5.50}$$

with

$$\begin{aligned} \mathscr{L}(u_1, u_2, u_3, u_4) =& \sum_{\sigma \in US(3,1)} (-1)^{|\sigma|+1} \mu_2(\mu_3(u_{\sigma(1)}, u_{\sigma(2)}, u_{\sigma(3)}), u_{\sigma(4)})) \\ &+ \sum_{\sigma \in US(2,2)} (-1)^{|\sigma|} \mu_3(\mu_2(u_{\sigma(1)}, u_{\sigma(2)}), u_{\sigma(3)}, u_{\sigma(4)}) \\ =& \sum_{\sigma \in US(3,1)} (-1)^{|\sigma|} \mu_2(f^1(u_{\sigma(1)}, u_{\sigma(2)}, u_{\sigma(3)}), u_{\sigma(4)})) \\ &+ \sum_{\sigma \in US(2,2)} (-1)^{|\sigma|+1} f^1(\mu_2(u_{\sigma(1)}, u_{\sigma(2)}), u_{\sigma(3)}, u_{\sigma(4)}) \quad . \end{aligned} \tag{5.51}$$

As before, this equation has a solution if $\mathscr{L}(u_1, u_2, u_3, u_4) \in \ker(d_1)$. Applying the differential we find

$$\begin{aligned} d_1(\mathscr{L}(u_1, u_2, u_3, u_4)) = &\sum_{\sigma \in US(3,1)} (-1)^{|\sigma|} \mu_2(\mathrm{Jac}(u_{\sigma(1)}, u_{\sigma(2)}, u_{\sigma(3)}), u_{\sigma(4)})) \\ &+ \sum_{\sigma \in US(2,2)} (-1)^{|\sigma|+1} \mathrm{Jac}(\mu_2(u_{\sigma(1)}, u_{\sigma(2)}), u_{\sigma(3)}, u_{\sigma(4)}) \quad . \end{aligned} \tag{5.52}$$

Inserting all the terms of the Jacobi identity a lengthy calculation shows, that this indeed vanishes. By the same argument as before we can define

$$h^1(u_1, u_2, u_3, u_4) = -\mathscr{L}(u_1, u_2, u_3, u_4) \quad . \tag{5.53}$$

Note that there cannot be any further non-trivial brackets due to degree.

$X_1X_0X_0X_0$: For this equation $u, v, w \in X_0$ and $z \in X_1$. Note that $\mathscr{J}_4(u, v, w, z)$ is totally anti-symmetric in its first three entries. For $F(x_1, \cdots, x_N)$ a linear function we write

$$\sum_{anti} F(x_1, \cdots, x_N) \equiv \sum_{\sigma \in S_N} \frac{(-1)^{|\sigma|}}{N!} F(x_{\sigma(1)}, \cdots, x_{\sigma(N)}) \tag{5.54}$$

for the totally anti-symmetrized sum. We then find

$$\begin{aligned} \mathscr{L}(u, v, w, z) = &\sum_{anti} -\mu_2(\mu_3(u, v, w), z) + 3\mu_2(\mu_3(z, u, v), w) \\ &+ 3\mu_3(\mu_2(u, v), w, z) + 3\mu_3(u, v, \mu_2(w, z)) \end{aligned} \tag{5.55}$$

where the sum runs over the anti-symmetrization of u, v, w. On the other hand writing $h^1(u, v, w, d_1(z))$ as an anti-symmetrized sum yields

$$\begin{aligned} h^1(u, v, w, d_1(z)) = &\sum_{anti} 3f^1(u, f^1(v, w, d_1(z)) - f^1(d_1(z), f^1(u, v, w)) \\ &+ 3f^1(\mu_2(u, v), w, d_1(z)) - 3f^1(\mu_2(d_1(z), u), v, w) \end{aligned} \tag{5.56}$$

and it is not hard to show that (5.55) and (5.56) exactly cancel. For further details we refer to [74, (4.27)].

4) $\mathscr{J}_5$: For degree reasons, this is the final non-trivial equation and it is possibly non zero for inputs $X_0X_0X_0X_0X_0$. Since the computation is a mere matter of algebra we only lay out the strategy. First of all, no μ_5 exists, simplifying the equation considerably. Next, using total anti-symmetry of $\mathscr{J}_5$ on $X_0X_0X_0X_0X_0$ one rewrites the equation as a totally anti-symmetrized sum as follows

$$\begin{aligned} \mathscr{J}_5(u, v, w, x, y) = &\sum_{anti} 10\mu_4(\mu_2(u, v), w, x, y) + 5\mu_2(\mu_4(u, v, w, x), y) \\ &+ 10\mu_3(\mu_3(u, v, w), x, y) \end{aligned} \tag{5.57}$$

Inserting the relevant brackets it is straightforward to show that this vanishes.

□

The paradigm example for the above theorem is the Courant algebroid. Recall that a *Lie algebroid* is a $\mathscr{C}^\infty$-linear extension of a Lie algebra. More precisely, given a smooth vector bundle $E \to M$ over a smooth manifold M, the sheaf of sections $\Gamma(E)$ is a Lie algebroid if it has a skew symmetric $\mathbb{R}$-bilinear pairing $[\bullet\,,\,\bullet] : \Gamma(E) \otimes \Gamma(E) \to \Gamma(E)$ giving it the structure of a Lie algebra. In addition there should exist an *anchor map* $\rho : \Gamma(E) \to TM$ satisfying

$$\rho([e_1, e_2]) = [\rho(e_1), \rho(e_2)] \tag{5.58}$$

and

$$[e_1, f e_2] = f[e_1, e_2] + (\rho(e_1)f)e_2 \tag{5.59}$$

for any $e_1, e_2 \in \Gamma(E)$ and $f \in \mathscr{C}^\infty(M)$. A Courant algebroid is a relaxation of the above ingredients. In particular the skew symmetric bracket is not required to define a Lie algebra.

Definition 5.2.3. A *Courant algebroid* is a smooth vector bundle $E \to M$ over a smooth manifold M, which has a smooth $\mathbb{R}$-bilinear, non-degenerate form $\langle \bullet\,,\,\bullet \rangle : E \otimes E \to \underline{\mathbb{R}}$[2], a skew symmetric $\mathbb{R}$-bilinear bracket $[\bullet\,,\,\bullet] : \Gamma(E) \otimes \Gamma(E) \to \Gamma(E)$ and an anchor map $\rho : \Gamma(E) \to TM$. Defining

$$T(e_1, e_2, e_3) = \frac{1}{3}\left(\langle [e_1, e_2], e_3\rangle + \langle [e_3, e_1], e_2\rangle + \langle [e_2, e_3], e_1\rangle\right) \in \mathscr{C}^\infty(M) \tag{5.60}$$

for $e_1, e_2, e_3 \in \Gamma(E)$ and a map $\mathscr{D}(M) : \mathscr{C}^\infty \to \Gamma(E)$ via

$$\langle \mathscr{D}(f), e\rangle = \frac{1}{2}\rho(e)f \tag{5.61}$$

the above data has to satisfy for $e_1, e_2, e_3,\ g, f \in \mathscr{C}^\infty(M)$

1) $\mathrm{Jac}(e_1, e_2, e_3) = \mathscr{D}(T(e_1, e_2, e_3))$.
2) $\rho([e_1, e_2]) = [\rho(e_1), \rho(e_2)]$.
3) $[e_1, f e_2] = f[e_1, e_2] + (\rho(e_1)f)e_2 - \langle e_1, e_2\rangle\, \mathscr{D}(f)$
4) $\langle \mathscr{D}(f), \mathscr{D}(g)\rangle = 0$
5) $\rho(e_1)\,\langle e_2, e_3\rangle = \langle [e_1, e_2] + \mathscr{D}(\langle e_1, e_2\rangle), e_3\rangle + \langle e_2, [e_1, e_3] + \mathscr{D}(\langle e_1, e_3\rangle)\rangle$.

The failure of a Courant algebroid to be a Lie algebroid can be cast into a L_∞ algebra using theorem 5.2.2. One starts with the chain complex

$$\cdots \longrightarrow 0 \longrightarrow X_2 \equiv \ker(\mathscr{D}) \xrightarrow{\ \iota\ } X_1 \equiv \mathscr{C}^\infty(M) \xrightarrow{\ \mathscr{D}\ } X_0 \equiv \Gamma(E) \longrightarrow 0$$

[2] $\underline{\mathbb{R}}$ denotes the trivial line bundle over M.

which is merely a chain complex of infinite dimensional vector spaces. By [127, Proposition 4.2] it holds

$$[e, \mathscr{D}(f)] = \mathscr{D}(\langle e, \mathscr{D}(f)\rangle) \tag{5.62}$$

for all $e \in \Gamma(E)$ and $f \in \mathscr{C}^\infty(M)$. In particular $[e, \mathrm{Im}(\mathscr{D})] \in \mathrm{Im}(\mathscr{D})$. Since by 1) in the axioms for a Courant algebroid we can apply theorem 5.2.2.

Corollary 5.2.4. *The complex*

$$\cdots \longrightarrow 0 \longrightarrow X_2 \equiv \ker(\mathscr{D}) \xrightarrow{\iota} X_1 \equiv \mathscr{C}^\infty(M) \xrightarrow{\mathscr{D}} X_0 \equiv \Gamma(E) \longrightarrow 0$$

has a L_∞ structure with only non trivial higher airy brackets

$$\begin{aligned} \mu_2(e_1, e_2) &= [e_1, e_2], \qquad e_1, e_2 \in X_0 \\ \mu_2(e, f) &= \langle e, \mathscr{D}(f)\rangle\,, \qquad e \in \Gamma(E),\ f \in \mathscr{C}^\infty(M) \\ \mu_3(e_1, e_2, e_3) &= -T(e_1, e_2, e_3), \qquad e_1, e_2, e_3 \in X_0 \end{aligned} \tag{5.63}$$

Proof. By the theorem there for sure exists a L_∞ structure, we only have to check that non-trivial maps from the theorem not listed in (5.63) indeed vanish. First note that for $f \in \ker(\mathscr{D})$ it holds

$$\mu_2(f, e) = \mu_2(\iota(f), e) = -\langle e, \mathscr{D}(\iota(f))\rangle = 0 \tag{5.64}$$

and for $f, g \in X_1$ one gets

$$\mu_2(f, g) = 2\,\langle \mathscr{D}(f), \mathscr{D}(g)\rangle = 0 \quad . \tag{5.65}$$

Hence the two bracket with values in X_2 vanishes. Next for $f \in \mathscr{C}^\infty(M)$ and $e_1, e_2 \in \Gamma(E)$ one has

$$\begin{aligned} \mu_3(f, e_1, e_2) = &- T(\mathscr{D}(f), e_1, e_2) - \langle e_2, \mathscr{D}(\langle e_1, \mathscr{D}(f)\rangle)\rangle \\ &+ \langle e_1, \mathscr{D}(\langle e_2, \mathscr{D}(f)\rangle)\rangle - \langle [e_1, e_2], \mathscr{D}(f)\rangle = 0 \end{aligned} \tag{5.66}$$

by [127, Lemma 5.1]. Finally [127, Lemma 5.2] gives that

$$\mu_4(e_1, e_2, e_3, e_4) = 0 \quad . \tag{5.67}$$

□

Originally the L_∞ algebra for a Courant algebroid was discovered in [127]. Here we have shown that it is a Corollary of the more general theorem 5.2.2.

5.3 Seiberg-Witten maps and L_∞-qisms

We start this section with a lightning review of L_∞ algebras as an organizing structure for classical field theories. There are many formulations of this, all of which boil down to a translation of the Batalin-Vilkovisky formalism to the world of homotopy algebras. The mathematically inclined reader should consult the books by Costello and Gwilliam [36][35], we take a down to earth route outlined in [75]. The advantage of the approach given in [75] is that it strips all ghost, antifield, antighost etc. from the BV formalism, leaving the essentials of the theory in form of a L_∞ algebra. One might start with the question of how to figure out how many interacting extensions a free field theory with a given gauge freedom can have. So we start with a free theory for some gauge field Υ, which can be a sum of various fields, having some gauge freedom described by gauge parameters Λ. If one has Yang-Mills theories in mind the story stops there, but thinking of a field theory with a two form with values in some semi-simple Lie algebra as a fundamental field one sees that gauge parameters for gauge parameters should be included. This already gets a bit cumbersome, thus we introduce a chain complex

$$\cdots \longrightarrow X_1 \xrightarrow{\ell_1} X_0 \xrightarrow{\ell_1} X_{-1} \xrightarrow{\ell_1} X_{-2}$$

where $\Lambda \in X_0$ is the vector space of gauge parameters, X_1 is a potentially non zero space of gauge parameters for gauge parameters, $\Upsilon \in X_0$ is the space of fields in the theory and X_{-1} is the space where the equations of motion of the theory take place. The chain map $X_{-1} \xrightarrow{\ell_1} X_{-2}$ is the differential operator defining the equations of motion of the linear theory. The map $X_0 \xrightarrow{\ell_1} X_{-1}$ is the linear part of the gauge transformation and being a chain complex at degree -1 just reflects the fact that pure gauge configurations should trivially satisfy the equations of motion.

At this point it is good to introduce an example to make the general considerations more palpable. The paradigm example is Yang-Mills theory in four dimensions. To ease our lives we make some simplifying assumptions on pure Yang-Mills, all of which prevent further mathematical technicalities. Let (M, g) be a four dimensional, closed manifold with Lorentzian metric g being of signature $(1, 3)$. In addition let G be a Lie group with Lie algebra $\mathfrak{g}$ and $P \simeq M \times G \to M$ the trivial G-principal bundle over M. There is no essential difference if we drop the assumption on M being compact and P being trivial. A good source for the general case is e.g. [18, chapter 7]. Let $\star : \Omega^k(M) \to \Omega^{4-k}(M)$ be the Hodge operator wrt. to g which we choose in such a way that $\star\star|_{\Omega^k(M)} = (-1)^{k(4-k)+1}$. Differential forms with values in the Lie algebra $\mathfrak{g}$ are defined as $\Omega^k(M, \mathfrak{g}) \equiv \Omega^k(M) \otimes \mathfrak{g}$ and the Hodge operator can be naturally extended to $\Omega^k(M, \mathfrak{g})$ by acting solely on the differential form part. We further assume that $\mathfrak{g}$ has an inner product $\langle \bullet, \bullet \rangle_{\mathfrak{g}}$ which is invariant under the adjoint action of G on $\mathfrak{g}$. In case $\mathfrak{g}$ is semi-simple one might choose the Killing form. For two k-forms $\omega, \sigma \in \Omega^k(M, \mathfrak{g})$ there exists a G-invariant inner product

$$\langle \bullet, \bullet \rangle_{\mathfrak{g}} : \Omega^k(M, \mathfrak{g}) \otimes \Omega^k(M, \mathfrak{g}) \to \mathscr{C}^\infty(M) \quad . \tag{5.68}$$

On homogeneous elements $\omega = \alpha \otimes X$, $\sigma = \beta \otimes Y$ the inner product is defined as

$$\langle \alpha \otimes X, \beta \otimes Y \rangle = \alpha \wedge \star\beta \cdot \langle X, Y \rangle_{\mathfrak{g}} \quad . \tag{5.69}$$

Choosing an orthonormal basis (e_1, e_2, e_3, e_4) in (T_pM, g_p) locally around a point $p \in M$ and an orthonormal basis $(Z_1, \cdots, Z_N)$ in $\mathfrak{g}$ this is equivalent to

$$\begin{aligned} \langle \omega, \sigma \rangle_p &= \frac{1}{k!} \sum_{\substack{\mu_1, \cdots, \mu_k = 1 \\ \nu_1, \cdots, \nu_k = 1}}^{4} g_{\mu_1\nu_1} \cdots g_{\mu_l\nu_k} \omega^{\mu_1 \cdots \mu_k; A} \sigma_A^{\nu_1 \cdots \nu_k} \mathrm{dvol}_g \\ &= \frac{1}{k!} \, \omega^A_{\mu_1 \cdots \mu_k} \sigma_A^{\mu_1 \cdots \mu_k} \mathrm{dvol}_g \end{aligned} \tag{5.70}$$

where we employed the Einstein summation convention in the last step. Greek indices are spacetime indices running in $\{1, \cdots, 4\}$ and capital Latin letters are Lie algebra indices running in $\{1, \cdots, N\}$. The basic field of the theory is a connection 1-form $A \in \Omega^1(M, \mathfrak{g})$. Gauge parameters are $\mathfrak{g}$-valued functions $\lambda \in \mathscr{C}^\infty(M, \mathfrak{g}) = \Omega^0(M, \mathfrak{g})$ acting on fields via

$$\delta_\lambda A = \mathrm{d}\lambda + [\lambda, A] \tag{5.71}$$

where d is the usual deRham differential. The curvature of the connection reads

$$F(A) \equiv \mathrm{d}A + \frac{1}{2}[A, A] \quad . \tag{5.72}$$

Given the connection A, there exists a differential operator

$$\mathrm{d}_A : \Omega^k(M, \mathfrak{g}) \to \Omega^{k+1}(M, \mathfrak{g}) \tag{5.73}$$

whose adjoint operator wrt to $\langle \bullet \, , \bullet \rangle$ is given by $\delta_A|_{\Omega^{k+1}(M,\mathfrak{g})} = \star \, \mathrm{d}_A \star$. Note that $F(A) = d_A A$. The Yang-Mills action functional is defined to be

$$\begin{aligned} S_{YM}(A) &= \int_M \langle F(A), F(A) \rangle = \int_M \langle A, \delta_A F(A) \rangle \\ &= \int_M \left\langle A, \star\mathrm{d} \star \mathrm{d}A + \star\mathrm{d} \star \frac{1}{2}[A, A] + \star[A, \star\mathrm{d}A] + \frac{1}{2} \star [A, \star[A, A]] \right\rangle \quad . \end{aligned} \tag{5.74}$$

The first term in the last formula is the free theory, i.e. it is quadratic in the fundamental field. All other terms are self interactions of the gauge field, where the second and third term contribute to a trivalent vertex in a Feynman diagram expansion and the last term gives a quartic vertex.

We have all the ingredients to setup the complex for the Yang-Mills L_∞ algebra:

$$\begin{tikzcd} \cdots \arrow[r] & X_1 \arrow[r] \arrow[d, equal] & X_0 \arrow[r, "\ell_1"] \arrow[d, equal] & X_{-1} \arrow[r, "\ell_1"] \arrow[d, equal] & X_{-2} \arrow[d, equal] \\ \cdots \arrow[r] & 0 \arrow[r] & \mathscr{C}^\infty(M, \mathfrak{g}) \arrow[r, "\mathrm{d}"] & \Omega^1(M, \mathfrak{g}) \arrow[r, "\star\mathrm{d}\star\mathrm{d}"] & \Omega^1(M, \mathfrak{g}) \end{tikzcd}$$

Higher brackets in Yang-Mills capturing the full gauge transformation of the field, the self interaction terms as well as as the non-abelian gauge algebra can now be found as follows. First of all the gauge algebra is a usual Lie algebra, thus the only non-trivial contribution from the gauge algebra in degree zero is the Lie bracket

$$\ell_2(\lambda_1, \lambda_2) = [\lambda_1, \lambda_2] \quad . \tag{5.75}$$

Next the transformation of the field (5.71) can be written as

$$\delta_\lambda A = \mathrm{d}\lambda + [\lambda, A] = \ell_1(A) + \ell_2(\lambda, A) \tag{5.76}$$

thereby defining a non-trivial two bracket in degree -1. These are all brackets in degree 0 and -1. The dynamics, i.e. the action of the theory can be written in the form

$$S_{YM}(A) = \int_M \frac{1}{2}\langle A, \ell_1(A)\rangle - \frac{1}{3!}\langle A, \ell_2(A, A)\rangle - \frac{1}{4!}\langle A, \ell_3(A, A, A)\rangle \tag{5.77}$$

giving higher brackets in degree -2. Note that in this formulation the equation of motions become

$$\mathcal{F}(A) = \ell_1(A) - \frac{1}{2}\ell_2(A, A) - \frac{1}{3!}\ell_3(A, A, A) \quad . \tag{5.78}$$

Lastly, equations of motion should be covariant under gauge transformations. Applying a gauge transformation to the equations of motion in the case of Yang-Mills one easily finds that there is one higher bracket

$$\delta_\lambda \mathcal{F}(A) = [\mathcal{F}, \lambda] = \ell_2(\lambda, \mathcal{F}(A)) \quad . \tag{5.79}$$

For the list of non-vanishing brackets in components and a proof that these really define a L_∞ algebra (at least in the case of the Minkowski metric, or one could say locally) we refer to [75, section 4.3].

The example of Yang-Mills theory gives a cooking recipe on how to derive the L_∞ algebra of a gauge theory. The procedure is due to [75] and we refer to [75, sections 2 and 3] for a thorough discussion. Given a gauge theory with field content Υ, gauge parameters Λ and equation of motions $\mathcal{F}$ one sets up a 1-shifted L_∞ algebra $(X_\bullet, \{b_n\}_{n\geq 1})$ with underlying chain complex

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & X_2 & \xrightarrow{b_1} & X_1 & \xrightarrow{b_1} & X_0 & \xrightarrow{b_1} & X_{-1} \\ & & & & & & & & \\ & & & & \Lambda & & \Upsilon & & \mathcal{F} \end{array}$$

as follows. First of all we note that the theory is more naturally described by 1-shifted L_∞ algebra, as the field lives in degree zero in that case. In other words it has ghost number zero as it should be. The field equations can be cast in the form

$$\mathcal{F}(\Upsilon) = \sum_{n\geq 1} \frac{1}{n!}\, b_n(\Upsilon, \cdots, \Upsilon) \;\in\; X_{-1} \tag{5.80}$$

which via polarization identities leads to a definition of all brackets $\ell_n(\Upsilon_1, \cdots, \Upsilon_n)$ for $\Upsilon_i \in X_0$. Next the gauge variation of the field gets

$$\delta_\Lambda \Upsilon = \sum_{n\geq 1} \frac{1}{n!} b_{n+1}(\Lambda, \Upsilon, \cdots, \Upsilon) \ \in \ X_0 \tag{5.81}$$

which sets all brackets $\ell_{n+1}(\Lambda, \Upsilon_1, \cdots, \Upsilon_n)$. Gauge covariance of the equations of motion naturally leads to

$$\delta_\Lambda \mathcal{F} = \sum_{n\geq 0} \frac{1}{n!} b_{n+2}(\Lambda, \mathcal{F}, \Upsilon \cdots \Upsilon) \ \in \ X_{-1} \tag{5.82}$$

giving brackets $b_{n+2}(\Lambda, \mathcal{F}, \Upsilon_1 \cdots \Upsilon_n)$. In order to read off higher degree brackets one has to consider the algebra of gauge transformations acting on the fields. Given two gauge parameters Λ_1, Λ_2 the commutator of gauge transformations acting on a field

$$[\delta_{\Lambda_1}, \delta_{\Lambda_2}]\,\Upsilon \tag{5.83}$$

should be equal to a transformation

$$\delta_{G(\Lambda_1,\Lambda_2)} \Upsilon \tag{5.84}$$

at least on shell. Here $G(\Lambda_1, \Lambda_2) \in X_1$ is a function which has to be linear in Λ_1, Λ_2 since (5.83) is linear in Λ_1, Λ_2. However, it is allowed to be a powerseries in Υ as gauge transformation (5.81) is a powerseries in the field. Hence the reasonable expression reads

$$G(\Lambda_1, \Lambda_2) = \sum_{n\geq 1} \frac{1}{n!} b_{n+2}(\Lambda_1, \Lambda_2, \Upsilon, \cdots, \Upsilon) \ \in \ X_1 \tag{5.85}$$

leading to brackets $b_{n+2}(\Lambda_1, \Lambda_2, \Upsilon_1, \cdots, \Upsilon_n)$. Since the gauge theory at hand may have a closed gauge algebra only on shell there might be a term

$$H(\Lambda_1, \Lambda_2, \mathcal{F}) \tag{5.86}$$

being linear in Λ_1, Λ_2 for the same reasons as before and depending linearly on the equation of motions. Again, since the gauge transformations are power series in the fields this may also be a powerseries in the fields. Hence we naturally arrive at a function of the form

$$H(\Lambda_1, \Lambda_2, \mathcal{F}) = \sum_{n\geq 1} \frac{1}{n!} b_{n+3}(\Lambda_1, \Lambda_2, \mathcal{F}, \Upsilon, \cdots, \Upsilon) \ \in \ X_1 \tag{5.87}$$

Giving brackets $b_{n+3}(\Lambda_1, \Lambda_2, \mathcal{F}, \Upsilon_1, \cdots, \Upsilon_n)$. Summarizing the last paragraph, the commutator of two gauge transformations acting on a field may be written in the form

$$[\delta_{\Lambda_1}, \delta_{\Lambda_2}]\,\Upsilon = \delta_{G(\Lambda_1,\Lambda_2)} \Upsilon + \delta^T_{H(\Lambda_1,\Lambda_2,\mathcal{F})} \Upsilon \tag{5.88}$$

with

$$\delta^T_{H(\Lambda_1,\Lambda_2,\mathcal{F})} \Upsilon = H(\Lambda_1, \Lambda_2, \mathcal{F}) \quad . \tag{5.89}$$

Amazingly, though the gauge algebra may only close on shell, the gauge Jacobiator always vanishes

$$[\delta_{\Lambda_1}, [\delta_{\Lambda_2}, \delta_{\Lambda_3}]] + [\delta_{\Lambda_3}, [\delta_{\Lambda_1}, \delta_{\Lambda_2}]] + [\delta_{\Lambda_2}, [\delta_{\Lambda_3}, \delta_{\Lambda_1}]] = 0 \tag{5.90}$$

as shown in [75, section 2.4]. In all of the above it is implicit but indispensable that the consistency of the gauge theory exactly corresponds to the defining relations of the 1-shifted L_∞ algebra. Take for example the closure of the gauge algebra (5.88). In a reasonable gauge theory, a formula like this holds. After making the ansatz (5.81) for the gauge transformation and the ansatz (5.85) for G as well as the ansatz (5.87) for H, the formula (5.88) is equivalent to the 1-shifted L_∞ relations on inputs $(\Lambda_1, \Lambda_2, \Upsilon_1, \Upsilon_2, \cdots)$. Higher order brackets and relations correspond to higher consistency requirements for further redundancies in the gauge theory.

This equivalence allows to turn the logic of the argument around. One might start with a free theory and linear gauge transformation and ask how many interacting theories are there which have the given free theory. Note that this can be asked at various levels of strictness, e.g. a plausible route is to fix the non-linear gauge behavior of the field at hand, thereby fixing some brackets in the 1-shifted L_∞ algebra, and trying to derive all dynamics by completing the so defined brackets to a full 1-shifted L_∞ algebra. But in the most general case, one might even bootstrap the gauge behavior of a field by building up a L_∞ algebra. This is the idea of the L_∞ *bootstrap* given in [22]. In [22] the focus was on non-associative generalizations of gauge theories, where the gauge algebra is not known a priori. We don't go into detail of non-associative deformations of the algebra of functions on a given spacetime, but stick to the easiest case of the Moyal star product as a motivating example. Before going into detail, we have to address the motivation of why looking into the computations about to come. Bootstrapping a 1-shifted L_∞ algebra from the linear theory gives a consistent theory, but not all different solutions of the bootstrap lead to different physical theories. The most natural idea is to look at equivalence classes of solutions under quasi-isomorphism instead. Hence the question is, do quasi-isomorphic L_∞ algebras give equivalent physical theories. This question is addressed in [21] by relating quasi-isomorphisms to Seiberg-Witten maps. Approaches in that direction in terms of local BRST (Becchi, Rouet, Stora and Tyutin) cohomology [9] appeared in [8][11].

We start with an example on how vastly different physically equivalent gauge theories can look. The example is non-commutative Yang-Mills theory as formulated by Seiberg and Witten in their seminal paper [135]. Non-commutative Yang-Mills (NCYM) on $\mathbb{R}^N$ uses the Moyal star product

$$\begin{aligned} \star : \mathscr{C}^\infty(\mathbb{R}^N) \times \mathscr{C}^\infty(\mathbb{R}^N) &\to \mathscr{C}^\infty(\mathbb{R}^N) \\ (f, g) &\mapsto f \star g \end{aligned} \tag{5.91}$$

with

$$(f \star g)(x) \equiv \exp\left[\frac{\mathrm{i}}{2}\theta^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\right] f(y)g(z)|_{y=z=x} \tag{5.92}$$

where $\theta^{ij} = -\theta^{ji} \in \mathbb{R}$ and $|\theta^{ij}| << 1$ for all $i, j \in \{1, \cdots, N\}$. This defines a non-commutative, but still associative algebra $(\mathscr{C}^\infty, \star) \equiv \mathscr{M}^N$. In addition the algebra of differential forms gets deformed. For this, pick a local basis $\{\mathrm{d}x^i\}_{i=1,\cdots,N}$ of 1-forms. The deformed wedge product of differential forms is given by

$$A \overset{\star}{\wedge} B = \sum_{I,J} (A_I \star B_J)\, \mathrm{d}x^I \wedge \mathrm{d}x^J \tag{5.93}$$

where $A = \sum_I A_I \mathrm{d}x^I$ is an expansion of the differential form in terms of wedge products of basis one forms. Formally the ingredients of NCYM and usual Yang-Mills look alike. One just replaces pointwise products in component expressions by their star analog. Assume $\mathfrak{g}$ is some matrix Lie algebra and $\widehat{A} \in \Omega^1(\mathbb{R}^N, \mathfrak{g})$, $\widehat{\lambda} \in \mathscr{M}^N$. Then the gauge transformation reads

$$\widehat{\delta_{\widehat{\lambda}}} \widehat{A}_j = \partial_j \widehat{\lambda} + \mathrm{i}\widehat{\lambda} \star \widehat{A}_j - \mathrm{i}\widehat{A}_j \star \widehat{\lambda} \tag{5.94}$$

with field strength

$$\widehat{F}_{ij} = \partial_i \widehat{A}_j - \partial_j \widehat{A}_i - \mathrm{i}\widehat{A}_i \star \widehat{A}_j + \mathrm{i}\widehat{A}_j \star \widehat{A}_i \quad . \tag{5.95}$$

Based on a regularization argument Seiberg and Witten argued that the NCYM with Lie algebra $\mathfrak{g}$ should be equivalent to usual Yang-Mills theory based on the same Lie algebra. In the following we distinguish between ordinary Yang-Mills and NCYM based on the same matrix Lie algebra by putting hats on all ingredients of NCYM. The equivalence is performed in terms of a Seiberg-Witten map, which relates the data of the theories as follows

$$\widehat{\lambda} = \widehat{\lambda}(\lambda, A), \quad \widehat{A} = \widehat{A}(A) \quad . \tag{5.96}$$

Computing at fixed orders of θ at a time in [135] it was then shown that for NCYM such a map indeed exists and it maps gauge orbits onto gauge orbits, i.e.

$$\widehat{A}(A + \delta_\lambda A) = \widehat{A}(A) + \widehat{\delta}_{\lambda(\lambda,A)} \widehat{A}(A) \quad . \tag{5.97}$$

As a remark on how vastly different the two theories appear, note that for $\mathfrak{g} = \mathfrak{u}(1)$ the NCYM theory has a non-abelian gauge Lie algebra, whereas usual Yang-Mills is of course abelian. If one tries to relate the closure of the gauge algebra on both sides one has to take this into account. Inspired by the L_∞ discussion of field theories one can make the ansatz

$$\widehat{A}(A + \delta_{[\lambda_1,\lambda_2]} A) = \widehat{A}(A) + \widehat{\delta}_{\widehat{\lambda}([\lambda_1,\lambda_2],A)} \widehat{A}(A) \tag{5.98}$$

and try and compute the function $\widehat{\lambda}([\lambda_1, \lambda_2], A)$. Using the explicit expression of the Seiberg-Witten map for NCYM one easily computes up to first order [21]

$$\widehat{\lambda}([\lambda_1, \lambda_2], A)\Big|_{\mathcal{O}(\theta)} = \left[\widehat{\lambda}_1, \widehat{\lambda}_2\right]_\star\Big|_{\mathcal{O}(\theta)} + \left(\widehat{\lambda}(\lambda_1, \delta_{\lambda_2} A) - \widehat{\lambda}(\lambda_2, \delta_{\lambda_1} A)\right)\Big|_{\mathcal{O}(\theta)} \quad . \tag{5.99}$$

One may conjecture that this formula continues to hold to all orders of θ giving

$$\begin{aligned} \widehat{A}\left(A + \delta_{[\lambda_1,\lambda_2]} A\right) &= \widehat{A}(A) + \widehat{\delta}_{[\widehat{\lambda}_1,\widehat{\lambda}_2]_\star} \widehat{A}(A) \\ &\quad + \widehat{\delta}_{\widehat{\lambda}(\lambda_1,\delta_{\lambda_2} A)} \widehat{A}(A) - \widehat{\delta}_{\widehat{\lambda}(\lambda_2,\delta_{\lambda_1} A)} \widehat{A}(A) \end{aligned} \tag{5.100}$$

where in all formulas $[\bullet, \bullet]_\star$ is the commutator in the algebra $\mathscr{M}^N$. So, there are two questions from the point of NCYM. First how to investigate Seiberg-Witten maps for other gauge theories and how to get closed formulas to all orders. Both can be dealt with the help of 1-shifted L_∞ algebras. The set up are 1-shifted L_∞ algebras of the form $(X_1 \oplus X_0 \oplus X_{-1}, \{b_i\})$ and $(Z_1 \oplus Z_0 \oplus Z_{-1}, \left\{\tilde{b}_i\right\})$ where the vector space X_i has degree i and the same holds for Z_i. Hence we are considering gauge theories where gauge parameters don't have any further redundancies. However we allow for the full dynamics of the theory by including the vector spaces X_{-1}, Z_{-1}. We use the letter λ to denote elements in X_1, A will be elements in X_0 and E are general elements in X_{-1}. The hatted letters refer to the respective elements in Z_i, i.e. $\hat{\lambda} \in Z_1$, $\hat{A}(A) \in Z_0$ and $\hat{E} \in Z_{-1}$.

Definition 5.3.1. [21] Given two gauge theories $(X_1 \oplus X_0 \oplus X_{-1}, \{b_i\})$ and $(Z_1 \oplus Z_0 \oplus Z_{-1}, \left\{\tilde{b}_i\right\})$ a *Seiberg-Witten map* is map

$$\hat{\lambda} = \hat{\lambda}(\lambda, A), \quad \hat{A} = \hat{A}(A), \quad \hat{\mathscr{F}} = \hat{\mathscr{F}}(\mathscr{F}, A) \tag{5.101}$$

s.th.

1)
$$\hat{A}(A + \delta_\lambda A) = \hat{A}(A) + \hat{\delta}_{\hat{\lambda}(\lambda,A)}\hat{A}(A) \tag{5.102}$$

2)
$$\begin{aligned}&\hat{A}(A + \delta_{C(\lambda_1,\lambda_2,A)}\, A + \delta^T_{C(\lambda_1,\lambda_2,\mathscr{F})}A)\\ &\quad= \hat{A}(A) + \hat{\delta}_{\hat{C}(\hat{\lambda}_1,\hat{\lambda}_2,\hat{A})+\hat{\lambda}(\lambda_2,\delta_{\lambda_1}A)-\hat{\lambda}(\lambda_1,\delta_{\lambda_2}A)}\,\hat{A}(A) + \hat{\delta}^T_{\hat{C}(\hat{\lambda}_1,\hat{\lambda}_2,\hat{\mathscr{F}})}\hat{A}(A)\end{aligned} \tag{5.103}$$

3)
$$\hat{\mathscr{F}}(\mathscr{F} + \delta_\lambda\mathscr{F}, A + \delta_\lambda A) = \hat{\mathscr{F}}(\mathscr{F}, A) + \hat{\delta}_{\hat{\lambda}(\lambda,A)}\hat{\mathscr{F}}(\mathscr{F}, A) \tag{5.104}$$

hold. In addition we want this to be reflexive, i.e. having a map $\hat{\lambda} = \hat{\lambda}(\lambda, A)$, $\hat{A} = \hat{A}(A)$ and $\hat{\mathscr{F}} = \hat{\mathscr{F}}(\mathscr{F}, A)$ implies that there exists a map $\lambda = \lambda(\hat{\lambda}, \hat{A})$ and $A = A(\hat{A})$ and $\mathscr{F} = \mathscr{F}(\hat{\mathscr{F}}, \hat{A})$ with the same properties. In other words we require the relation to be invertible.

The extra terms in the closure formula are defined as in the general discussion of gauge transformations in terms of L_∞ algebras.

Having this, we can turn to quasi-isomorphisms $\mathbf{F} \equiv \{F_n\}$ $(X_1 \oplus X_0 \oplus X_{-1}, \{b_i\}) \to (Z_1 \oplus Z_0 \oplus Z_{-1}, \left\{\tilde{b}_i\right\})$. Recall that the maps F_n are of degree zero. Assume $\hat{\lambda}$ is in the image of $\mathbf{F}$. Then it has an expansion of the form

$$\hat{\lambda} \sim \sum_{k\geq 0}\left(\sum_{n_k\geq 0} F_{n_k+1+2k}(\lambda, \lambda_1, \cdots, \lambda_k, E_1, \cdots, E_k, A^{n_k})\right) \tag{5.105}$$

The $k = 0$ term is what we expect from the point of view of Seiberg-Witten maps. For the other terms note that the whole formalism is for *infinitesimal* gauge transformations.

One way of making this explicit is equipping gauge parameters with a small parameter $0 < \epsilon << 1$. Hence all $k \geq 2$ terms are suppressed by these parameters and we can discard them. Thus the general formula can be further simplified assuming natural physical assumptions. A similar discussion holds for fields. In general a field $\hat{A}$ in the image of $\mathbf{F}$ may be expanded as

$$\hat{A} \sim \sum_{k\geq 0}\left(\sum_{n_k\geq 0} F_{n_k+2k}(\lambda_1,\cdots,\lambda_k,E_1,\cdots,E_k,A^{n_k})\right) \quad . \tag{5.106}$$

Again, the $k=0$ is to be expected, and terms $k \geq 2$ are suppressed. The only mysterious term is the $k=1$ term. Recalling the physical interpretation of the vector spaces at hand, such terms would correspond to a field depending explicitly on a gauge parameter. This is not too sensible as the former is a dynamical variable, whereas the latter is not. Hence from a physical point of view we can safely ignore this term. Finally an element $\hat{E}$ in the image of $\mathbf{F}$ may have an expansion

$$\hat{E} \sim \sum_{k\geq 0}\left(\sum_{n_k\geq 0} F_{n_k+1+2k}(\lambda_1,\cdots,\lambda_k,E_1,\cdots,E_k,E,A^{n_k})\right) \tag{5.107}$$

by the same arguments as before we only keep the $k=0$ term and through away the others as they are either physically unreasonable or suppressed. With these consideration in the back of our minds we can give the main result of [21].

Theorem 5.3.2. *Given to 1-shifted L_∞ algebras $(X_1 \oplus X_0 \oplus X_{-1}, \{b_i\})$ and $(Z_1 \oplus Z_0 \oplus Z_{-1}, \left\{\tilde{b}_i\right\})$ thought of as underlying two classical gauge theories, there exists a Seiberg-Witten map*

$$\hat{\lambda} = \hat{\lambda}(\lambda, A), \quad \hat{A} = \hat{A}(A) \tag{5.108}$$

if and only if there exist graded symmetric maps $\{F_n\} : X_1 \oplus X_0 \oplus X_{-1} \to Z_1 \oplus Z_0 \oplus Z_{-1}$ with

$$F_{n+k+l}(\lambda_1,\cdots,\lambda_k,E_1,\cdots,E_l,A^n) = 0, \quad \textit{for all}\, k,l \in \{1,2\}\,,\ n\geq 0 \tag{5.109}$$

satisfying the L_∞ quasi-isomorphism equations on inputs (A^n), (λ, A^n), $(\lambda_1, \lambda_2, A^2)$ and $(\lambda_1, \lambda_2, E, A^n)$, and (λ, E, A^n) for any $n \in \mathbb{Z}_{\geq 0}$.

Proof. The proof of the theorem can be split into several lemmas. We begin with a set of multilinear, graded symmetric maps $\{F_n\} : X_1 \oplus X_0 \oplus X_{-1} \to Z_1 \oplus Z_0 \oplus Z_{-1}$ which satisfy (5.109). Following the discussion before the theorem we make the ansatz

$$\begin{aligned}\hat{A}(A) &= \sum_{n=1}^{\infty}\frac{1}{n!}\,F_n(A^n)\\ \hat{\lambda}(\lambda,A) &= \sum_{k=0}^{\infty}\frac{1}{k!}\,F_{k+1}(\lambda,A^k)\\ \hat{\mathscr{F}}(A) &= \sum_{n=0}^{\infty}\frac{1}{n!}\,F_{n+1}\left(\mathscr{F},A^n\right)\end{aligned} \tag{5.110}$$

for a Seiberg-Witten map.

Lemma 5.3.3. *The gauge orbit condition*

$$\hat{A}(A+\delta_\lambda A) = \hat{A}(A) + \hat{\delta}_{\hat{\lambda}(\lambda,A)}\,\hat{A}(A) \tag{5.111}$$

holds if and only if the maps $\{F_n\}$ *satisfy the* L_∞ *morphism equations on inputs* (λ, A^n) *for all* $n \geq 0$.

Proof. We start with a straightforward computation of the lhs using the definition of the map $\hat{A}$ as well as the gauge transformation in terms of 1-shifted L_∞ algebras.

$$\begin{aligned}
\hat{A}(A+\delta_\lambda A) &\overset{(1)}{=} \hat{A}\left(A + \sum_{k=0}^{\infty}\frac{1}{k!}\,b_{k+1}(\lambda, A^k)\right)\\
&\overset{(2)}{=} \sum_{n=1}^{\infty}\frac{1}{n!}\,F_n\Bigg(A+\sum_{k=0}^{\infty}\frac{1}{k!}\,b_{k+1}(\lambda,A^k),\ldots,A+\sum_{k=0}^{\infty}\frac{1}{k!}\,b_{k+1}(\lambda,A^k)\Bigg)\\
&\overset{(3)}{=} \sum_{n=1}^{\infty}\frac{1}{n!}\left[F_n\left(A^n\right) + n\,F_n\left(A^{n-1},\sum_{k=0}^{\infty}\frac{1}{k!}\,b_{k+1}(\lambda,A^k)\right) + \mathcal{O}(\lambda^2)\right]\\
&\overset{(4)}{=} \hat{A}(A) + \sum_{n=0}^{\infty}\frac{1}{n!}\left[F_{n+1}\left(\sum_{k=0}^{\infty}\frac{1}{k!}\,b_{k+1}(\lambda,A^k),A^n\right)\right]\\
&\overset{(5)}{=} \hat{A}(A) + \sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k+n=m}\frac{m!}{n!k!}\,F_{n+1}\left(b_{k+1}(\lambda,A^k),A^n\right)\,.
\end{aligned} \tag{5.112}$$

In (1) we inserted the definition of the gauge transformation, (2) is the ansatz for the map $\hat{A}$. In (3) we collected all terms which are of order $\mathcal{O}(\lambda^2)$. These are suppressed as before and we can leave them out. There is an additional combinatorial factor n for the second summand as A is of degree 0 and F_n's are graded symmetric. In (4) we shifted the outer infinite sum and changed order of arguments causing no extra signs as every input is of degree 0. The final step is a mere rewriting of sums in a more handy form. We want to relate these terms to the defining equation for a L_∞ morphism. The lhs of the defining equation (5.23) reads

$$\sum_{k+l=n+1}\sum_{\sigma\in US(k,l)}\chi(\sigma)\,F_{l+1}\Big(b_{k+1}(x_{\sigma(1)},\ldots,x_{\sigma(k+1)}),x_{\sigma(k+2)},\ldots,x_{\sigma(n+1)}\Big)\quad. \tag{5.113}$$

For inputs $x_1 = \lambda$, $x_2 = \cdots = x_{n+1} = A$ the sum simplifies considerably. Note that interchanging any two inputs of F_{l+1} will not cause extra minus signs as inputs are of degree 0. We can split the summation into two terms. The first has the gauge parameter in the first slot, the second has it in the $k+2$-th slot. Finally there are $\frac{n!}{k!l!}$ unshuffles partitioning the gauge field inputs into two set of order k and l. These simplifications yield

$$\begin{aligned}
\sum_{k+l=n}\frac{n!}{k!l!}\,F_{l+1}&\Big(b_{k+1}(\lambda,A^k),A^l\Big)\\
&+\frac{n!}{(k+1)!(l-1)!}\,F_{l+1}\Big(b_{k+1}(A^{k+1}),\lambda,A^{l-1}\Big)\,.
\end{aligned} \tag{5.114}$$

So we see that the second term of (5.112) appears in the defining relation. What about the second term in (5.114). Since the second term of (5.112) is a sum over all orders $n \geq 0$ we sum over all orders and get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k+l=n}\frac{n!}{(k+1)!(l-1)!}\,F_{l+1}\left(b_{k+1}(A^{k+1},\lambda,A^{l-1}\right)\\
=&\sum_{n=0}^{\infty}\sum_{k+l=n+1}\frac{1}{k!(l-1)!}\,F_{l+1}\left(b_k(A^k),\lambda,A^{l-1}\right)\\
=&\sum_{l=0}^{\infty}\frac{1}{(l-1)!}F_{l+1}\left(\sum_{k=1}^{\infty}\frac{1}{k!}\,b_k(A^k),\lambda,A^{l-1}\right)\\
=&\sum_{l=0}^{\infty}\frac{1}{(l-1)!}F_{l+1}\left(\mathscr{F},\lambda,A^{l-1}\right)=0\ .
\end{aligned}\tag{5.115}$$

In the last step we used the assumption (5.109) to set the sum to zero. So in general there are terms linear in the equations of motion. Either one says that the Seiberg-Witten condition holds on shell making these term vanish or the additional assumptions (5.109) have to be made.

Next we consider the rhs of (5.23) for inputs (λ, A^n). To simplify the equation note that

$$\Big|\mathrm{Unsh}(k_1+\cdots+k_j)\Big| = \binom{n}{k_1,\ldots,k_j}\tag{5.116}$$

for $k_1+\cdots+k_j=n$. This follows from the definition of the multinomial coefficient as the number of possibilities of splitting a set with n into j subsets of length k_i irrespective of the order of elements in the subsets. Since permutations don't cause extra signs this gives for the rhs of (5.23)

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k_1+\cdots+k_j=n+1}\frac{1}{j!}&\Big[\tilde b_j\Big((F_{k_1}(\lambda,A^{k_1-1}),F_{k_2}(A^{k_2}),\ldots,F_{k_j}(A^{k_j})\Big)\tbinom{n}{(k_1-1),k_2,\ldots,k_j}\\
&+\tilde b_j\Big(F_{k_1}(A^{k_1}),F_{k_2}(\lambda,A^{k_2-1}),\ldots,F_{k_j}(A^{k_j}\Big)\tbinom{n}{k_1,(k_2-1),\ldots,k_j}\\
&\vdots\\
&+\tilde b_j\Big(F_{k_1}(A^{k_1}),\ldots,F_{k_j}(\lambda,A^{k_j})\Big)\tbinom{n}{k_1,\ldots,(k_j-1)}\Big]\ .
\end{aligned}\tag{5.117}$$

By graded symmetry terms in square brackets are all equivalent. This gives

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k_1+\cdots+k_j=n+1}\tfrac{1}{(j-1)!}\,\tfrac{1}{(k_1-1)!\cdots k_j!}\,\tilde b_j\Big(F_{k_1}(\lambda,A^{k_1-1}),F_{k_2}(A^{k_2}),\ldots,F_{k_j}(A^{k_j})\Big)\\
\overset{(1)}{=}&\sum_{n=0}^{\infty}\sum_{k_0+\cdots+k_j=n}\frac{1}{j!}\,\frac{1}{k_0!\cdots k_j!}\,\tilde b_{j+1}\Big(F_{k_0+1}(\lambda,A^{k_1}),F_{k_1}(A^{k_1}),\ldots,F_{k_j}(A^{k_j})\Big)\\
\overset{(2)}{=}&\sum_{j=0}^{\infty}\frac{1}{j!}\sum_{k_0\geq 0}\sum_{k_1,\ldots,k_j\geq 1}\frac{1}{k_0!\cdots k_j!}\,\tilde b_{j+1}\Big((F_{k_0+1}(\lambda,A^{k_0}),F_{k_1}(A^{k_1}),\ldots,F_{k_{j+1}}(A^{k_j})\Big)
\end{aligned}\tag{5.118}$$

At the beginning we just inserted the definition of the multinomial coefficient. In (1) we shifted the summation indices j and k_1 and (2) is a convenient rewriting of the sum. We want to compare this expression to the rhs of (5.111). For this we compute

$$\begin{aligned}\hat{\delta}_{\hat{\lambda}}\,\hat{A}(A) &\overset{(1)}{=} \sum_{j=0}^{\infty}\frac{1}{j!}\,\tilde{b}_{j+1}\Big(\hat{\lambda}(\lambda,A),\hat{A}(A)^j\Big)\\ &\overset{(2)}{=} \sum_{j=0}^{\infty}\frac{1}{j!}\,\tilde{b}_{j+1}\left(\sum_{n=0}^{\infty}\frac{1}{n!}\,F_{n+1}(\lambda,A^n),\Big(\sum_{k=1}^{\infty}\frac{1}{k!}\,F_k(A^k)\Big)^j\right)\\ \overset{(3)}{=} \sum_{j=0}^{\infty}\frac{1}{j!}\sum_{n=0}^{\infty}&\sum_{k_1,\ldots,k_j\geq 1}\frac{1}{n!}\frac{1}{k_1!\cdots k_j!}\,\tilde{b}_{j+1}\Big(F_{n+1}(\lambda,A^n),F_{k_1}(A^{k_1}),\ldots,F_{k_j}(A^{k_j})\Big)\,.\end{aligned} \tag{5.119}$$

Equality (1) is the definition of the gauge transformation in terms of the 1-shifted L_∞ structure, (2) is the ansatz for the Seiberg-Witten map and (3) is the explicit expression of the product of sums. We see that this is nothing but (5.118). Hence (5.111) gives the defining equations of a L_∞ morphism and vice versa, proving the claim of lemma 5.3.3. □

We discussed the proof of this lemma quite thoroughly since it serves as the paradigm example for computations proving lemmas about to come. All steps in the proofs are similar to the ones we performed above and mostly consist of combinatorial considerations taking grading of inputs and symmetry of maps into account. The interested reader may work out the details of the equations.

For the closure statement we start with ignoring the potential term involving the equations of motion. The equation to show then reads

$$\hat{A}(A+\delta_{C(\lambda_1,\lambda_2,A)}\,A) = \hat{A}(A) + \hat{\delta}_{\hat{C}(\hat{\lambda}_1,\hat{\lambda}_2,\hat{A})+\hat{\lambda}(\lambda_2,\delta_{\lambda_1}A)-\hat{\lambda}(\lambda_1,\delta_{\lambda_2}A)}\,\hat{A}(A) \tag{5.120}$$

which upon using lemma 5.3.3 is equivalent to showing

$$\hat{\lambda}(C(\lambda_1,\lambda_2,A),A) = \hat{C}(\hat{\lambda}_1,\hat{\lambda}_2,\hat{A}) + \hat{\lambda}(\lambda_2,\delta_{\lambda_1}A) - \hat{\lambda}(\lambda_1,\delta_{\lambda_2}A)\,. \tag{5.121}$$

As stated before, to show this one performs steps very similar to the steps pf the previous proof. For the additional term we compute

$$\begin{aligned}\hat{A}\Big(A+\delta_{C(\lambda_1,\lambda_2,A)}\,A+\delta^T_{C(\lambda_1,\lambda_2,\mathscr{F})}A\Big)&\\ &\overset{(1)}{=}\sum_{n=1}^{\infty}\frac{1}{n!}\,F_n\Big((A+\delta_{C(\lambda_1,\lambda_2,A)}\,A+\delta^T_{C(\lambda_1,\lambda_2,\mathscr{F})}A)^n\Big)\\ &\overset{(2)}{=}\sum_{n=1}^{\infty}\left[\frac{1}{n!}\,F_n(A^n)+\frac{1}{(n-1)!}\,F_n\Big(\delta_{C(\lambda_1,\lambda_2,A)}\,A,A^{n-1}\Big)\right.\\ &\qquad\left.+\frac{1}{(n-1)!}\,F_n\Big(\delta^T_{C(\lambda_1,\lambda_2,\mathscr{F})}A,A^{n-1}\Big)+\mathcal{O}(\lambda^3)\right]\,,\end{aligned} \tag{5.122}$$

(1) is the ansatz for the map $\hat{A}$ and (2) separates terms in orders of gauge parameters and takes symmetry into account. Throwing away terms of order $\mathcal{O}(\lambda^3)$ we are left with

computing the third term in the final expression. By definition it holds

$$\delta^T_{C(\lambda_1,\lambda_2,\mathscr{F})} A = \sum_{n=0}^{\infty} \frac{1}{n!} b_{n+3}(\lambda_1, \lambda_2, \mathscr{F}, A^n) \tag{5.123}$$

thus we get

$$\begin{aligned} &\sum_{n\geq 1} \frac{1}{(n-1)!} F_n\Big(\delta^T_{C(\lambda_1,\lambda_2,\mathscr{F})} A, A^{n-1}\Big) + \mathcal{O}(\lambda^3) \\ =&\sum_{n\geq 1} \frac{1}{(n-1)!} \sum_{k\geq 0} \frac{1}{k!} F_n\left(b_{k+3}(\lambda_1, \lambda_2, \mathscr{F}, A^k), A^{n-1}\right) \quad . \end{aligned} \tag{5.124}$$

Again we start with the lhs of the defining equation (5.23) on inputs $(\lambda_1, \lambda_2, E, A^n)$

$$\begin{aligned} \sum_{k+l=n+1} &\frac{(n-3)!}{(k-3)!(l-1)!} F_l\Big(b_k(\lambda_1, \lambda_2, E, A^{k-3}), A^{l-1}\Big) \\ &+ \frac{(n-3)!}{(k-2)!(l-2)!} F_l\Big(b_k(\lambda_1, \lambda_2, A^{k-2}), E, A^{l-2}\Big) \\ &- \frac{(n-3)!}{(k-2)!(l-2)!} F_l\Big(b_k(\lambda_1, E, A^{k-2}), \lambda_2, A^{l-2}\Big) \\ &+ \frac{(n-3)!}{(k-2)!(l-2)!} F_l\Big(b_k(\lambda_2, E, A^{k-2}), \lambda_1, A^{l-2}\Big) \\ &+ \frac{(n-3)!}{(k-1)!(l-3)!} F_l\Big(b_k(\lambda_1, A^{k-1}), \lambda_2, E, A^{l-3}\Big) \\ &- \frac{(n-3)!}{(k-1)!(l-3)!} F_l\Big(b_k(\lambda_2, A^{k-1}), \lambda_1, E, A^{l-3}\Big) \\ &+ \frac{(n-3)!}{(k-1)!(l-3)!} F_l\Big(b_k(E, A^{k-1}), \lambda_1, \lambda_2, A^{l-3}\Big) \\ &+ \frac{(n-3)!}{k!(l-4)!} F_l\Big(b_k(A^k), \lambda_1, \lambda_2, E, A^{l-4}\Big) \ . \end{aligned} \tag{5.125}$$

Due to assumption (5.109) all terms except the first one vanish. Changing the summation indices and summing this term over $n \geq 1$ including the factorial prefactor we get (5.124) for $E = \mathscr{F}$. As we did before we turn to the rhs of (5.23) evaluated on $(\lambda_1, \lambda_2, E, A^n)$ and summed over all n. This yields

$$\begin{aligned} \sum_{n\geq 1} \sum_{j=1}^{n} \frac{1}{(n-1)!} &\sum_{\substack{k_1+\dots+k_j=n \\ k_i\geq 1}} \frac{1}{(j-3)!} \frac{(n-3)!}{(k_1-1)!(k_2-1)!(k_3-1)!\cdots k_j!} \\ &\tilde{b}_j(F_{k_1}(\lambda_1, A^{k_1-1}, F_{k_2}(\lambda_2, A^{k_2-1}), F_{k_3}(E, A^{k_3-1}), \dots, F_{k_j}(A^{k_j})) \quad . \end{aligned} \tag{5.126}$$

To arrive at (5.126) we used (5.109) to drop terms and used symmetry relations very similar to the ones used in (5.119). Shifting summation indices as we did before we get that this equals

$$\sum_{n=1}^{\infty} \frac{1}{n!}\, \tilde{b}_n(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\mathscr{F}}, \hat{A}^n) = \hat{\delta}^T_{\hat{C}(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\mathscr{F}})}\, \hat{A} \tag{5.127}$$

Putting things together we see that

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{1}{(n-1)!}\, F_n\Big(\delta^T_{C(\lambda_1,\lambda_2,\mathscr{F})} A, A^{n-1}\Big) \\ &\qquad = \sum_{n=1}^{\infty} \frac{1}{n!}\, \tilde{b}_n(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\mathscr{F}}, \hat{A}^n) = \hat{\delta}^T_{\hat{C}(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\mathscr{F}})}\, \hat{A} \end{aligned} \tag{5.128}$$

holds if the maps F_n satisfy the defining equations for inputs $(\lambda_1, \lambda_2, E, A^n)$. This proves the equivalence of the gauge closure condition of Seiberg-Witten maps and the corresponding subset of L_∞ algebra morphisms.

Next we turn to dynamics of the theories.

Lemma 5.3.4.

$$\sum_{n=0}^{\infty} \frac{1}{n!}\, F_{n+1}\left(\mathscr{F}, A^n\right) = \sum_{n=1}^{\infty} \frac{1}{n!}\, \tilde{b}_n\left(\hat{A}^n\right) \tag{5.129}$$

Assuming that the maps F_n define a L_∞ quasi-isomorphism, this is the known statement that quasi-isomorphisms are isomorphisms on the sets of Maurer-Cartan elements of the two L_∞ algebras. There are surely proofs of this fact in the literature but we weren't able to pinpoint a concrete instance where an explicit proof was given. Since proving this by hand is not difficult we quickly do the math.

Proof.

$$\sum_{n=0}^{\infty} \frac{1}{n!}\, F_{n+1}\left(\mathscr{F}, A^n\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k\geq 1} \frac{1}{k!}\, F_{n+1}\left(b_k(A^k), A^n\right) \tag{5.130}$$

This is again the lhs (5.23) evaluated on field input only. The rhs of (5.23) evaluated on fields only and summed over n plus taking prefactors into account gives

$$\begin{aligned} &\sum_{n\geq 1} \frac{1}{n!} \sum_{j=1}^{n} \sum_{\substack{k_1+\cdots+k_j=n \\ k_i\geq 1}} \frac{1}{(j-1)!} \frac{n!}{k_1!\cdots k_j!} \tilde{b}_j(F_{k_1}(A^{k_1}), \ldots, F_{k_j}(A^{k_j})) \\ &= \sum_{j\geq 1} \frac{1}{j!} \tilde{b}_j(\hat{A}^j) \end{aligned} \tag{5.131}$$

where we performed the same steps as in (5.119). □

So this Seiberg-Witten equation is equivalent to the L_∞ morphisms evaluated on fields only.

We are left with the Seiberg-Witten equation (5.104). Again we start with computing the lhs of it.

$$\begin{aligned}
&\hat{\mathcal{F}}(\mathcal{F}+\delta_\lambda\mathcal{F}, A+\delta_\lambda A)\\
&\overset{(1)}{=}\sum_{n=0}^{\infty}\frac{1}{n!}\Big[F_{n+1}\left(\mathcal{F},A^n\right)+F_{n+2}\left(\mathcal{F},\delta_\lambda A,A^n\right)+F_{n+1}\left(\delta_\lambda\mathcal{F},A^n\right)\Big]+\mathcal{O}(\lambda^2)\\
&\overset{(2)}{=}\hat{\mathcal{F}}(\mathcal{F},A)+\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{n+k=m}\frac{m!}{n!k!}\Big[F_{n+2}\left(b_{k+1}(\lambda,A^k),\mathcal{F},A^n\right)\\
&\qquad\qquad+F_{n+1}\left(b_{k+2}(\lambda,\mathcal{F},A^k),A^n\right)\Big]\,.
\end{aligned} \tag{5.132}$$

The equality (1) is merely insertion of the definitions of the maps and separating terms of different orders in the gauge parameters. In (2) we threw away terms of too high order in λ, inserted the definition of the gauge transformations in terms of 1-shifted L_∞ algebras and rewrote the summation. In order to relate this to (5.23) we evaluate the lhs of (5.23) on inputs (λ, E, A^n). This gives

$$\begin{aligned}
&\sum_{k+l=n+1}\frac{(n-2)!}{(k-2)!(l-1)!}F_l\Big(b_k(\lambda,E,A^{k-2}),A^{l-1}\Big)+\frac{(n-2)!}{(k-1)!(l-2)!}F_l\Big(b_k(\lambda,A^{k-1}),E,A^{l-2}\Big)\\
&\qquad-\frac{(n-2)!}{(k-1)!(l-2)!}F_l\Big(b_k(E,A^{k-1}),\lambda,A^{l-2}\Big)+\frac{(n-2)!}{k!(l-3)!}F_l\Big(b_k(A^k),\lambda,E,A^{l-3}\Big)\,.
\end{aligned} \tag{5.133}$$

The last two terms vanish and changing summation indices accordingly for the first two terms as well as summing over n gives the second term in the last equality of (5.132). Hence we need to evaluate the rhs of (5.23) on (λ, E, A^n). This gives

$$\begin{aligned}
&\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{j=0}^{\infty}\sum_{k_1+\dots+k_{j+2}=m}\frac{1}{j!k_1!\dots k_{j+2}!}\,\tilde{b}_{j+2}\Big(F_{k_1+1}(\lambda,A^{k_1}),F_{k_2+1}(\mathcal{F},A^{k_2}),\\
&\qquad\qquad F_{k_3}(A^{k_3}),\dots,F_{k_{j+2}}(A^{k_{j+2}})\Big)\\
&=\sum_{j=0}^{\infty}\frac{1}{j!}\,\tilde{b}_{j+2}\left(\hat{\lambda}(\lambda,A),\hat{\mathcal{F}}(\mathcal{F},A),\hat{A}(A)^j\right)\\
&=\hat{\delta}_{\hat{\lambda}(\lambda,A)}\,\hat{\mathcal{F}}(\mathcal{F},A)\,.
\end{aligned} \tag{5.134}$$

where the equalities are due to the by now familiar manipulations performed in previous computations.

A final word about the existence of the maps F_n. Though they are defined on equal elements one can use polarization identities to get maps on different input elements. Though we are using the defining equations of a L_∞ morphism, the fact that the F_n should correspond to a quasi-isomorphism is guaranteed by the requirement of invertibility for Seiberg-Witten maps. □

5.4 Conclusions

In the first part of this chapter we gave a L_∞ algebra closure theorem for vector spaces with arbitrary skewsymmetric brackets. The L_∞ extension is finite in the sense that the underlying complex is a three term complex and the L_∞ has non-trivial higher n-brackets only up to and including $n = 4$. In addition we presented how the L_∞ structure of the Courant algebroid is corollary of the construction. In the second part of the chapter we proved a theorem relating Seiberg Witten maps to L_∞ quasi-isomorphisms.

Appendix A

Vertex Operator Algebra Basics

A.1 Vertex Operator Algebras and Operator Product Expansions

Vertex operator algebras (VOAs) can be seen as a holomorphic enhancement of associative algebras, where the product depends on a complex variable. They are a mathematical tool for describing the chiral symmetries of two dimensional conformal fields theories (CFTs). Since we are aiming at a more mathematical treatment of CFTs we stick with mathematics conventions for notations and expansions. When giving the definitions of the objects involved, we comment on the relation to the physics conventions. The following definitions and remarks can be found in any textbook on VOAs, classic sources are [54][113][53][93][52].

Definition A.1.1. A *vertex operator algebra* $(V, \mathbf{1}, Y, \omega)$ is the data of

(1) a $\mathbb{Z}$-graded vector space $V = \bigoplus_{n\in\mathbb{Z}} V_{(n)}$ satisfying grading restrictions $\dim V_{(n)} < \infty$ and there exists an $N \in \mathbb{Z}$ s.th. $V_{(n)} = 0$ for $n \leq N$.

(2) a *vacuum* $\mathbf{1} \in V_{(0)}$.

(3) a *vertex operator* $Y(\bullet, z) : V \to \mathrm{End}(V)[\![z, z^{-1}]\!]$ for $z \in \mathbb{C}$. These can be expanded in operator valued Laurent series

$$Y(v, z) = \sum_{m\in\mathbb{Z}} u_{(n)} z^{-n-1} \tag{A.1}$$

where for $u \in V_{(k)}$ the modes satisfy $u_{(n)} : V_{(a)} \to V_{(a+k-n-1)}$.

(4) a *conformal vector* $\omega \in V_{(2)}$.

The data has to satisfy the following axioms:

(i) *Units:* For any $z \in \mathbb{C}$ the vertex operator of the vacuum evaluates to the unit map

$$Y(\mathbf{1}, z) = \mathrm{id}_V \quad . \tag{A.2}$$

In addition applying vertex operators to the vacuum gives an analytic function

$$Y(v,z)\mathbf{1} = v + zV[\![z]\!] \quad . \tag{A.3}$$

(ii) *Virasoro property:* The modes in the vertex operator Laurent expansion of the conformal vector

$$Y(\omega,z) = \sum_{n\in\mathbb{Z}} L_{(n)} z^{-n-2} \tag{A.4}$$

define a copy of the Virasoro algebra, i.e.

$$\left[L_{(n)}, L_{(m)}\right] = (n-m)L_{(m+n)} + \frac{c}{12}(n^3-n)\delta_{m,n} \tag{A.5}$$

where $c \in \mathbb{C}$ is the *central charge* of the VOA. Furthermore the zero mode is a grading operator

$$L_{(0)}v = nv, \quad \text{for } v \in V_{(n)} \tag{A.6}$$

and $L_{(-1)}$ acts as a translation operator in the sense that

$$\frac{\mathrm{d}}{\mathrm{d}z} Y(v,z) = Y(L_{(-1)}v, z) \quad . \tag{A.7}$$

Due to (A.6) we refer to the grading as the *conformal weight* of an element.

(iii) *Convergence:* Let $V^* = \bigoplus_{n\in\mathbb{Z}} V^*_{(n)}$ the direct sum of linear dual spaces and $\pi_n : V \to V_{(n)}$ the projection operator onto the conformal weight n space. For $w \in V^*$, $v_1, v_2, v_3 \in V$ there exists a function

$$C_1(z_1,z_2) = \frac{G(z_1,z_2)}{z_1^a z_2^b (z_1-z_2)^c}, \quad a,b,c \in \mathbb{N},\ G \in \mathbb{C}[z_1,z_2] \tag{A.8}$$

s.th.

$$\langle w, Y(v_1,z_1)Y(v_2,z_2)v_3\rangle \equiv \sum_{n\in\mathbb{Z}} \langle w, Y(v_1,z_1)\pi_n Y(v_2,z_2)v_3\rangle = \iota_{12} C_1(z_1,z_2) \quad . \tag{A.9}$$

for $|z_1| > |z_2| > 0$. On the right hand side the natural evaluation pairing between dual elements is used. Defined like that absolute convergence of $\langle w, Y(v_1,z_1)Y(v_2,z_2)v_3\rangle$ is implied, being the power series expansion of a holomorphic function.

Similarly there exists a function

$$C_2(z_1,z_2) = \frac{H(z_1,z_2)}{z_1^o z_2^p (z_1-z_2)^q}, \quad o,p,q \in \mathbb{N},\ H \in \mathbb{C}[z_1,z_2] \tag{A.10}$$

s.th.

$$\langle w, Y(Y(v_1,z_1-z_2)v_2,z_2)v_3\rangle \equiv \sum_{n\in\mathbb{Z}} \langle w, Y(\pi_n Y(v_1,z_1-z_2)v_2,z_2)v_3\rangle = \iota_{z_2,z_1-z_2} C_2(z_1,z_2) \tag{A.11}$$

for $|z_2| > |z_1 - z_2| > 0$.

(iv) *Operator Product Expansion (OPE):* The above expressions agree in the sense that $C_1(z_1, z_2) = C_2(z_1, z_2)$. Thus they are just different expansions of the same rational function in different regions.

(v) *Commutativity:* Again, assume the conditions for convergence hold, then

$$\langle w, Y(v_2, z_2)Y(v_1, z_1)v_3\rangle = \iota_{21}C_1(z_1, z_2) \quad . \tag{A.12}$$

For some of the proofs and applications it is more convenient to express rationality and the OPE property for VOAs in terms of a Jacobi identity for formal power series. The calculus of formal variables suits as a formal analog of the calculus of Laurent expansions of meromorphic functions. Let V be a vector space and $\{x_i\}$ will denote formal variables, of which the reader may secretly think of as coordinates in the complex planes. We will use formal power series of the following type

$$\begin{aligned}
V\,[x] &= \left\{\sum_{n=0}^{N} v_n x^n \,|v_n \in V \; N \in \mathbb{N}\right\} \\
V[\![x]\!] &= \left\{\sum_{n\in\mathbb{N}} v_n x^n \,|v_n \in V\right\} \\
V\left[x, x^{-1}\right] &= \left\{\sum_{n=-N}^{M} v_n x^n \,|\, N, M \in \mathbb{N}\right\} \\
V[\![x, x^{-1}]\!] &= \left\{\sum_{n\in\mathbb{Z}} v_n x^n \,|\, v_n \in V\right\} \\
V((x)) &= \left\{\sum_{n\geq -N} v_n x^n \,|\, v_n \in V,\; N \in \mathbb{N}\right\} \\
V\,\{x\} &= \left\{\sum_{n\in\mathbb{Q}} v_n x^n \,|\, v_n \in V\right\}
\end{aligned} \tag{A.13}$$

and their obvious multivariable analogs. In addition, the localization

$$V[\![x_1, x_2]\!]_S \tag{A.14}$$

at $S = \langle x_1, x_2, x_1 - x_2\rangle$ will play a prominent role. We define a map $\iota_{12} : V[\![x_1, x_2]\!]_S \to V[\![x_1, x_2, x_1^{-1}, x_2^{-1}]\!]$ from the expansion

$$\begin{aligned}
\iota_{12}\frac{1}{(z_1 - z_2)^n} = \iota_{12}\frac{1}{z_1^n(1 - \frac{z_2}{z_1})^n} &= \frac{1}{z_1^n}\left[\sum_{k\geq 0}\left(\frac{z_2}{z_1}\right)^k\right]^n \\
&= \sum_{k\geq 0}\binom{k+n-1}{n-1}\frac{z_2^k}{z_1^{k+n}} \\
&= \sum_{k\geq 0}(-1)^k\binom{-n}{k} z_2^k z_1^{-k-n}.
\end{aligned} \tag{A.15}$$

in the region $|z_1| > |z_2|$ on the complex plane. Hence ι_{12} on an homogeneous element of $V[\![x_1, x_2]\!]_S$ is given by

$$\iota_{12} \frac{P(x_1, x_2)}{x_1^\ell x_2^k (x_1 - x_2)^n} = \sum_{m \geq 0} (-1)^k \binom{-n}{k} P(x_1, x_2) x_1^{-\ell-n-m} x_2^{-k+m} \tag{A.16}$$

and is linearly extended to $V[\![x_1, x_2]\!]_S$. It is a power series expansion into only finitely many negative powers of x_2. For the Jacobi identity we need the formal delta function in $V[\![x, x^{-1}]\!]$

$$\delta(x) = \sum_{n \in \mathbb{Z}} x^n \tag{A.17}$$

which has its name due to the fact that for $f \in V[\![x, x^{-1}]\!]$ it holds

$$f(x)\delta(x) = \sum_{n,m} f_{(n)} x^{n+m} = f(1)\delta(x) \quad . \tag{A.18}$$

Since power series in this thesis will appear mostly as power series of linear operators on vector spaces we give a sensible definition of limits of such power series.

Definition A.1.2. A possibly infinite family $(F_{(i)})_{i \in I} \in \mathrm{End}(V)$ is *summable* if for any $v \in V$, $F_{(i)} v \neq 0$ for only finitely many i. Given $F \in \mathrm{End}(V)[\![x_1, x_1^{-1}, x_2, x_2^{-1}]\!]$, then

$$\lim_{x_1 \to x_2} F(x_1, x_2) = \lim_{x_1 \to x_2} \sum_{n,m \in \mathbb{Z}} F_{n,m} x_1^n x_2^m \tag{A.19}$$

exists if for any $m \in \mathbb{Z}$, the family $\left(F_{(n,m-n)}\right)_n$ is summable. The limit is defined to be

$$\lim_{x_1 \to x_2} F(x_1, x_2) = \sum_{n,m \in Z} F_{(n,m-n)} x_2^m \quad . \tag{A.20}$$

If $\lim_{x_1 \to x_2} F(x_1, x_2)$ exists we compute

$$\begin{aligned} F(x_1, x_2)\delta\left(\frac{x_1}{x_2}\right) &= \sum_{n,m,k \in \mathbb{Z}} F_{(n,m)} x_1^{n+k} x_2^{m-k} = \sum_{n,m,k \in \mathbb{Z}} F_{(n-k,m)} x_1^n x_2^{m-k} \\ &= \sum_{n,m,k \in \mathbb{Z}} F_{(k,m)} x_1^n x_2^{m-n-k} \\ &= \sum_{n,m,k \in \mathbb{Z}} F_{(k,m-k)} x_2^m x_1^n x_2^{-n} \\ &= F(x_2, x_2)\delta\left(\frac{x_1}{x_2}\right) \quad . \end{aligned} \tag{A.21}$$

Thus $\delta\left(\frac{x_1}{x_2}\right)$ acts by restricting to $x_1 = x_2$. The definition of the delta function can be extended to more general arguments by setting

$$\delta\left(\frac{x_1 - x_2}{x_3}\right) = \sum_{k \in \mathbb{N}} \sum_{\ell \in \mathbb{Z}} (-1)^k \binom{\ell}{k} x_3^{-\ell} x_1^{\ell-k} x_2^k \quad . \tag{A.22}$$

Lemma A.1.3.

$$x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right) = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right) \tag{A.23}$$

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right) - x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right) = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right) \tag{A.24}$$

The proof is a straightforward computation expanding delta functions into power series.

Let $C(x_0, x_1, x_2) = \frac{h(x_0,x_1,x_2)}{x_0^r x_1^s x_2^t}$ with $h(x_0, x_1, x_2) \in \mathbb{C}[x_0, x_1, x_2]$ and $r, s, t \in \mathbb{N}$. Along the same steps it is not hard to show

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\iota_{12}C|_{x_0=x_1-x_2} - x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\iota_{21}C|_{x_0=x_1-x_2} \\ = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\iota_{10}C|_{x_2=x_1-x_0} \end{aligned} \tag{A.25}$$

Recall that the definition of a VOA involved a rational function $C(z_0, z_1, z_2)$. Rationality, commutativity and associativity (OPE) can be phrased as

$$\begin{aligned} \langle v', Y(v_1, z_1)Y(v_2, z_2)v_3\rangle &= \iota_{12}C|_{z_0=z_1-z_2} \\ \langle v', Y(v_2, z_2)Y(v_1, z_1)v_3\rangle &= \iota_{21}C|_{z_0=z_1-z_2} \\ \langle v', Y(Y(v_1, z_0)v_2, z_2)v_3\rangle &= \iota_{20}C|_{z_0=z_1-z_2} \end{aligned} \tag{A.26}$$

from which one immediately deduces

Proposition A.1.4. *(Jacobi identity for VOAs)*

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(v_1, x_1)Y(v_2, x_2) - x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y(v_2, x_2)Y(v_1, x_1) \\ = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y(Y(v_1, x_0)v_2, x_2) \end{aligned} \tag{A.27}$$

The Jacobi identity is an elegant intermediate step for deriving the OPE formula known from the physics literature. Before continuing we note that the opposite line of reasoning also works, i.e. starting with the definition of a VOA, where all axioms are kept but convergence (rationality), commutativity and OPE (associativity) are replaced by the single axiom of the Jacobi identity nothing is lost. By a contour integration argument the Jacobi identity is equivalent to *Borcherd's identity* for modes of vertex operators. The Lie algebra structure on the modes (which is equivalent to the OPE) is a special case of Borcherd's identity. We give the relation between Jacobi identity and OPE in the following paragraph. Deriving rationality, commutativity and associativity for correlation functions is done e.g. in [54, section 3.3]. The proof of the equality is more generally applicable to situations with vertex operator maps and we freely switch between axiomatic presentations of structures involving correlation functions or the Jacobi identity, as both presentations have advantages in different situations.

Let

$$\mathrm{Res}_{x_0} x_0^n = \delta_{n,-1} \tag{A.28}$$

be the formal analog of the residue pairing and apply it to (A.27). This yields

$$\begin{aligned}
[Y(v_1,x_1),Y(v_2,x_2)] &= \mathrm{Res}_{x_0} x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right) Y(Y(v_1,x_0)v_2,x_2) \\
&= \sum_{m\in\mathbb{N}}\left(\sum_{n\in\mathbb{Z}}\binom{n}{m}(-1)^m x_2^{-n-1}x_1^{n-m}\,Y\left((v_1)_{(m)}v_2,x_2\right)\right) \qquad \text{(A.29)}\\
&= \sum_{m=0}^{N}\left(\sum_{n\in\mathbb{Z}}\binom{n}{m}(-1)^m x_2^{-n-1}x_1^{n-m}\,Y\left((v_1)_{(m)}v_2,x_2\right)\right)
\end{aligned}$$

where the lower truncation property for VOAs is used in the second step.

Proposition A.1.5. *Let*

$$\begin{aligned}
:Y(v,z_1)Y(w,z_2): &\equiv \sum_{n\in\mathbb{Z}}\sum_{m<0} v_{(n)}w_{(m)}z_2^{-n-1}z_1^{-m-1} + \sum_{n\in\mathbb{Z}}\sum_{m\geq 0} w_{(n)}v_{(m)}z_2^{-n-1}z_1^{-m-1} \\
&\equiv Y(v,z_1)_+Y(w,z_2) + Y(w,z_2)Y(v,z_1)_-
\end{aligned} \qquad \text{(A.30)}$$

be the normal ordered product of two vertex operators. For $z_1, z_2 \in \mathbb{C}$ s.th. $|z_1| > |z_2|$ it holds

$$Y(v,z_1)Y(w,z_2) = \sum_{m=0}^{N} Y(v_{(m)}w,z_2)\iota_{12}\frac{1}{(z_1-z_2)^{m+1}} + :Y(v,z_1)Y(w,z_2): \quad . \qquad \text{(A.31)}$$

Proof. First note that for any $z_1, z_2 \in \mathbb{C}$ there is the equation

$$Y(v,z_1)Y(w,z_2) = [Y(v,z_1)_-, Y(w,z_2)] + :Y(v,z_1)Y(w,z_2): \quad . \qquad \text{(A.32)}$$

The commutator can be computed from (A.30) by noting that the $-$ index just restricts the commutator formula to negative powers of z_1. Hence we have

$$\begin{aligned}
[Y(v,z_1)_-,Y(w,z_2)] &= \sum_{m=0}^{N}\left(\sum_{n<m}\binom{n}{m}(-1)^m z_2^{-n-1}z_1^{n-m}\,Y\left((v_1)_{(m)}v_2,z_2\right)\right) \\
&= \sum_{m=0}^{N}\left(\sum_{n<0}\binom{n}{m}(-1)^m z_2^{-n-1}z_1^{n-m}\,Y\left((v_1)_{(m)}v_2,z_2\right)\right) \\
&= \sum_{m=0}^{N}\left(\sum_{n\geq 0}\binom{-n-1}{m}(-1)^m z_2^{n}z_1^{-n-1-m}\,Y\left((v_1)_{(m)}v_2,z_2\right)\right) \qquad \text{(A.33)}\\
&= \sum_{m=0}^{N} Y\left((v_1)_{(m)}v_2,z_2\right)\iota_{12}\frac{1}{(z_1-z_2)^{-m-1}}
\end{aligned}$$

where in the second equality we use $\binom{n}{m} = 0$ for $0 \leq n < m$. For the last equality the following chain of binomial coefficient identities is used

$$\binom{-n-1}{m}(-1)^m = \binom{n+m}{m} = \binom{n+m}{n} = \binom{-m-1}{n}(-1)^n \quad . \qquad \text{(A.34)}$$

□

Note that if we define in analogy to the usual Cauchy integration formula the normal ordered product at $z_1 = z_2$ as

$$:Y(w,z_2)\partial^n Y(v,z_2):\equiv \oint_{\mathscr{C}(z_2)} \frac{\mathrm{d}z_1}{2\pi\mathrm{i}} \frac{Y(v,z_1)Y(w,z_2)}{(z_1-z_2)^{n+1}} \tag{A.35}$$

we recover the usual OPE formula

$$Y(v,z_1)Y(w,z_2) = \sum_{m=0}^{N} Y(v_{(m)}w,z_2)\iota_{12}\frac{1}{(z_1-z_2)^{m+1}} + \sum_{n\geq 0}\frac{(z_1-z_2)^n}{n!}:Y(w,z_2)\partial^n Y(v,z_2): \tag{A.36}$$

from (A.31).

The upshot of this section is that having fields in the form of vertex operators, s.th. the correlation functions among the fields produce rational functions and have a natural associativity property, produces an operator product expansion among the fields with only rational dependence on the location of the fields.

A.2 VOA Representations and Intertwining Operators

In this section we give a quick overview over the key points relevant for this thesis in VOA representation theory. This includes in particular rational VOAs and a discussion of intertwining operators. The later part will be a helpful primer for the discussion of modular tensor categories arising as representation categories of VOAs. Again there is a plethora of literature on the material presented here, textbook accounts can be found e.g. in [53][54][42][52][113].

Definition A.2.1. Given a VOA $(V, Y, \mathbf{1}, \omega)$ a *representation* is a triple (W, Y_W) consisting of a graded vector space $W = \amalg_{n\in\mathbb{R}} W_n$, a vertex operator map

$$\begin{aligned} Y_W : V &\to \mathrm{End}(W)[\![x, x^{-1}]\!] \\ v &\mapsto \sum_{n\in\mathbb{Z}} v_{(n)}x^{-n-1} \end{aligned} \tag{A.37}$$

and an identity element $\mathbf{1} \in W_0$. The data has to satisfy

(R1) *(Lower truncation:)* $\dim W_{(n)} < \infty$ and $v_{(n)}w = 0$ for n big enough.

(R2) *(Identity property:)* $Y(\mathbf{1}, z) = \mathrm{id}_W$.

(R3) *(Jacobi identity:)*

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_W(v_1,x_1)Y_W(v_2,x_2) - x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y_W(v_2,x_2)Y_W(v_1,x_1) \\ = x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y_W(Y(v_1,x_0)v_2,x_2) \end{aligned} \tag{A.38}$$

(R4) *(Virasoro embedding:)* The modes of the Virasoro vertex operator $Y_W(\omega, x) = \sum_{n\in\mathbb{Z}} L_{(n)} x^{-n-2}$ satisfy the Virasoro relations with the central charge given by the central charge of V.

(R5) *(Virasoro grading:)* The zeroth mode of $Y_W(\omega, x)$ acts the grading operator: $L_{(0)}w = nw$ for $w \in W_{(n)}$. In addition the Virasoro field is required to satisfy the usual translation property

$$\frac{\mathrm{d}}{\mathrm{d}x} Y_W(v, x) = Y_W(L_{(-1)}v, x) \tag{A.39}$$

Definition A.2.2. A *morphism* $f : W_1 \to W_2$ is a grading preserving linear map such that

$$f(Y_{W_1}(v, x)w_1) = Y_{W_2}(v, x)f(w_1) \tag{A.40}$$

for all $w_1 \in W_1$ and $v \in V$.

Note that the definition applies for any formal variable x. Again, choosing complex numbers for formal variable one could derive rationality properties of correlation functions on the two sphere from the Jacobi identity. The vertex operator map might be seen as a formal power series enhancement of a module map for an associative algebra, just as the vertex operator of a VOA is the enhancement of an associative algebra. The Jacobi identity then corresponds to the usual intertwining of module map and product. In addition one can start from the Jacobi identity and derive along the exact same steps as before an OPE of fields, now living in some representation. However, note one crucial difference: The underlying vector space of a VOA representation is not $\mathbb{Z}$-graded in general, though vertex operators are genuinely integer spaced. In addition OPEs will only be defined for elements in the underlying VOA, which now act on W, not for arbitrary elements of W.

Clearly any VOA is a module over itself.

Definition A.2.3. A VOA $(V, Y, \mathbf{1}, \omega)$ is *simple* if it is irreducible as a module over itself.

Given two representations (W_1, Y_{W_1}), (W_2, Y_{W_2}) over VOAs V_1 and V_2 one might also try and define a $V_1 \otimes V_2$-representation $W_1 \otimes W_2$ using in both cases the tensor product of vector spaces. Though for VOAs there is no problem in defining the tensor product, the general $\mathbb{R}$-grading of representations prevents the tensor product $W_1 \otimes W_2$ from having finite dimensional weight spaces.

Given a representation W, a natural task is to define an adjoint representation on $W^* = \amalg_{n\in\mathbb{R}} W_n^*$. For this to work one has to derive some easy statements about the transformation behavior of Y_W under the action of $\mathrm{SL}(2, \mathbb{C})$, which acts by exponentiating the action of the $\mathfrak{sl}(2, \mathbb{C})$ generators $\left\{L_{(-1)}, L_{(0)}, L_{(1)}\right\}$.

Lemma A.2.4. *Let* $(W, Y_W, \mathbf{1})$ *be a* V*-representation. Let* $w, z \in \mathbb{C}$*, global conformal* *transformations act as*

$$\begin{aligned} e^{wL_{(-1)}}Y_W(v,z)e^{-wL_{(-1)}} &= Y_w\left(e^{wL_{(-1)}}v,z\right) = Y_W(v,z+w) \\ e^{wL_{(0)}}Y_W(v,z)e^{-wL_{(0)}} &= Y_W\left(e^{wL_{(0)}}v,e^w z\right) \\ e^{wL_{(1)}}Y_W(v,z)e^{-wL_{(1)}} &= Y_W\left(e^{w(1-wz)L_{(1)}}(1-wz)^{-2L_{(0)}}v,\frac{z}{1-wz}\right) \end{aligned} \tag{A.41}$$

Proof. Proving the first identity goes along the same steps as in the proof for (3.21). For the second identity note multiplying the Jacobi identity by x_1 and applying $\mathrm{Res}_{x_0}\mathrm{Res}_{x_1}$ gives

$$\begin{aligned} \left[v_{(1)},Y_W(w,x)\right] &= Y_W\left(v_{(1)}w,x\right) + xY_W(v_{(0)}w,x) \\ \Leftrightarrow \quad \left[v_{(1)},w_{(n)}\right] &= \left(v_{(1)}w\right)_{(n)} + \left(v_{(0)}w\right)_{(n+1)} \quad . \end{aligned} \tag{A.42}$$

Taking the degree shift in $Y_W(\omega,x) = \sum_{n\in\mathbb{Z}} L_{(n)}x^{-n-2}$ into account one finds for $v \in V$ homogeneous

$$\begin{aligned} \left[L_{(0)},v_{(n)}\right] &= |v|v_{(n)} + \left(L_{(-1)}v\right)_{(n+1)} \\ &= (|v|-n-1)v_{(n)} \end{aligned} \tag{A.43}$$

where we used (A.39) in the second equality. So $L_{(0)}$ acts as a grading operator on the modes. From this one easily derives

$$\begin{aligned} e^{wL_{(0)}}Y_W(v,z)e^{-wL_{(0)}} &= \sum_{n\in\mathbb{Z}} e^{w\mathrm{ad}_{L_{(0)}}}v_{(n)}z^{-n-1} \\ &= \sum_{n\in\mathbb{Z}}\sum_{m\geq 0}\frac{(w(|v|-n-1))^m}{m!}v_{(n)}z^{-n-1} \\ &= \sum_{n\in\mathbb{Z}} e^{w|v|}v_{(n)}\left(e^w z\right)^{-n-1} \\ &= Y_W\left(e^{wL_{(0)}}v,e^w z\right) \quad . \end{aligned} \tag{A.44}$$

Finally the proof of the third identity is conceptually the same as the previous one, but significantly more cumbersome. We omit it here and refer to [54, Proposition 5.2.3] for a detailed exposition. □

We remark that the proofs for the identities only use the Jacobi identity and the Virasoro properties in the definition of a representation. This will allow to state similar transformation properties in situations where a Jacobi identity and Virasoro actions hold later. A consequence of the transformation properties is the following necessary condition on irreducible modules.

Lemma A.2.5. *Let* $(W,Y_W,\mathbf{1})$ *be a simple* V*-representation. Then there exists* $h_W \in \mathbb{R}$ *s.th.*

$$W = \coprod_{n\in\mathbb{Z}} W_{h_W+n} \quad . \tag{A.45}$$

Proof. A VOA module is irreducible if for a sub-vector space $U \subset W$ from $Y_W(v,x)u \in U((x))$ for all $v \in V$ it follows that $U = W$ or $U = 0$. Assume now that there exist homogeneous elements w_1, w_2 in W with $|w_1| - |w_2| \in \mathbb{R}\backslash\mathbb{Z}$. Let W_1, W_2 be the V-modules inside W, generated by w_1 and w_2 respectively. These are by definition the smallest subspaces s.th. $w_1 \in W_1$, $w_2 \in W_2$ and $Y(v,x)u_1 \in W_1((x))$ for $v \in V$ and $u_1 \in W_1$ and similar for W_2. Since

$$|(w_1)_{(n)}| = |w_1| - n - 1 \tag{A.46}$$

it is clear that for $u \in W_1$ it holds $|u| \in |w_1| + \mathbb{Z}$. But then $w_2 \notin W_1$, thus W_1 is a non-trivial submodule contradicting irreducibility. $\square$

As promised one can use the transformation properties to define an adjoint or dual module.

Theorem A.2.6. *[54, Theorem 5.2.1] Let (W, Y_W) be an $(V, Y, \mathbf{1}, \omega)$ representation. The graded dual W^* carries a vertex operator*

$$\begin{aligned} Y_{W^*} : V &\to \operatorname{End}(W^*) [\![x, x^{-1}]\!] \\ v &\mapsto Y_{W^*}(v,x) = \sum_{n\in\mathbb{Z}} v^*_{(n)} x^{-n-1} \end{aligned} \tag{A.47}$$

defined by setting

$$\langle Y_{W^*}(v,x)w', w\rangle \equiv \left\langle v', Y_W \left(e^{xL_{(1)}} \left(-x^{-2}\right)^{L_{(0)}} v, x^{-1}\right) w \right\rangle \tag{A.48}$$

for all $w' \in W^*$, $w \in W$. $(W^*, Y_{W^*}, \mathbf{1})$ is called contragradient module for W.

Though we omit the proof here we still give the definition of the Virasoro action, which will allow us to move the action of the Virasoro algebra from a representation to its dual when paired. By the creation property of a VOA it holds $L_{(-2)}\mathbf{1} = \omega$, $L_{(n)}\mathbf{1} = 0$ for $n \geq -1$ and the Virasoro identity gives $L_{(1)}\omega = 0$. Equation (A.48) for ω therefore yields

$$\begin{aligned} \sum_{n\in\mathbb{Z}} \left\langle L^*_{(n)} w', w \right\rangle x^{-n-2} = \langle Y_{W^*}(\omega, x) w', w\rangle &= \left\langle w', Y_W \left(x^{-4}\omega, x^{-1}\right) w\right\rangle \\ &= \sum_{n\in\mathbb{Z}} \left\langle w', L_{(n)} w\right\rangle x^{n-2} \end{aligned} \tag{A.49}$$

which is equivalent to

$$\left\langle L^*_{(-n)} w', w\right\rangle = \left\langle w', L_{(n)} w \right\rangle \quad . \tag{A.50}$$

We will see, that contragradient modules naturally also serve as categorical duals of representations. The fact that a single dual module is defined here already hints at a rigid, braided monoidal category of representations. Finally, we want to be able to perform OPE-like expansions between elements of different representations. This is possible by introducing intertwining operators.

Definition A.2.7. Let W_1, W_2, W_3 be representations of $(V, Y, \mathbf{1}, \omega)$. An *intertwining operator of type* $\binom{W_3}{W_1 W_2}$ is a map

$$\begin{aligned} \mathcal{Y} : W_1 &\to \operatorname{Hom}(W_2, W_3)\{x\} \\ w &\mapsto \sum_{n\in\mathbb{Q}} w_{(n)} x^{-n-1} \end{aligned} \tag{A.51}$$

where $w_{(n)} \in \operatorname{Hom}(W_2, W_3)$ s.th.

IO1) *Lower truncation*: for $u \in W_2$ it holds $w_{(n)}u = 0$ for n big enough.

IO2) *Jacobi identity*:

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_{W_3}(v, x_1)\mathcal{Y}(w, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\mathcal{Y}(w, x_2)Y_{W_2}(v, x_1) \\ = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)\mathcal{Y}(Y_{W_1}(v, x_0)w, x_2) \end{aligned} \tag{A.52}$$

IO3) *Virasoro translation*: $\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{Y}(w, x) = \mathcal{Y}\left(L_{(-1)}w, x\right)$.

If we want to refer to intertwining operators between arbitrary modules $W_\alpha, W_\beta, W_\gamma$ we sometimes write $\binom{\gamma}{\alpha\,\beta}$ for the type of the intertwining operators and the space of intertwining operators of type $\binom{\gamma}{\alpha\,\beta}$ is denoted as $\mathcal{V}_{\alpha\beta}^{\gamma}$. Obviously the sum of two intertwining operators of the same type is again an intertwining operator, hence $\mathcal{V}_{\alpha\beta}^{\gamma}$ is a $\mathbb{C}$-vector space and its dimension is denoted by $N_{\alpha\beta}^{\gamma}$.

In section A.1 we gave the definition of a VOA in terms of associativity, rationality and commutativity properties of convergent correlation functions. A similar definition can be made for VOA modules. For intertwining operators of modules only a definition in terms of the Jacobi identity is possible, though. A definition in terms of rational functions will clearly fail as intertwining operators don't have power series expansions. In the next section we will discuss under which assumptions on V this can be remedied.

Lemma A.2.8. *Let $\mathcal{Y} \in \mathcal{V}_{ij}^{k}$ be an intertwining operator between simple modules, then*

$$\mathcal{Y}(w_i, x)w_j \in x^{h_k - h_i - h_j} W_k((x)) \tag{A.53}$$

where $W_i = \coprod_{n\in\mathbb{Z}} W_{h_i+n}$.

Proof. From the Virasoro properties one derives as before $|(w_i)_{(n)}| = |w_i| - n - 1$. Hence

$$|(w_i)_{(n)}w_j| = |w_i| + |w_j| - n - 1 = |h_i| + r + |h_j| + s - n - 1 = |h_k| + m, \quad r, s, m \in \mathbb{Z} \tag{A.54}$$

or equivalently

$$n + 1 \in |h_i| + |h_j| - |h_k| + \mathbb{Z} \tag{A.55}$$

which immediately implies the claim. □

The following is stated e.g. in [80, Proposition 1.11].

Lemma A.2.9. *Let* $\mathcal{Y} \in \mathcal{V}_{\alpha\beta}^{\gamma}$ *and* $r \in \mathbb{Z}$*. The map*

$$\begin{aligned}\mathcal{B}_r(\mathcal{Y}) : W_\beta \otimes W_\alpha &\to W_\gamma \{x\} \\ (w_2, w_1) &\mapsto \mathcal{B}_r(\mathcal{Y})(w_2, x)w_1 \equiv e^{xL_{(-1)}} \mathcal{Y}(w_1, e^{(2r+1)\pi i}x)w_2\end{aligned} \tag{A.56}$$

defines an isomorphism $\mathcal{V}_{\alpha\beta}^{\gamma} \simeq \mathcal{V}_{\beta\alpha}^{\gamma}$*.*

Proof. First we prove that $\mathcal{B}_r(\mathcal{Y})$ is in fact an intertwining operator of type $\binom{\gamma}{\beta\,\alpha}$. The lower truncation axiom is obvious. The expression $\mathcal{Y}(w_1, e^{(2r+1)\pi i}x)w_2$ only involves finitely many negative powers of x. Since $e^{xL_{(-1)}}$ only raises the powers, this is still true for $B_r(\mathcal{Y})(w_2, x)w_1$. One may define

$$\mathcal{B}_r(\mathcal{Y})(w_2, x)w_1 = \sum_{n \in \mathbb{R}} (w_2)^r_{(n)} w_1 x^{-n-1} \tag{A.57}$$

by equating coefficients in front of powers of x. By the previous sentence this immediately implies

$$(w_2)^r_{(n)} w_1 = 0, \qquad \text{n big enough} \quad . \tag{A.58}$$

Next the Virasoro translation property:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{B}_r(\mathcal{Y})(w_2, x)w_1 &= e^{xL_{(-1)}} L_{(-1)} \mathcal{Y}(w_1, e^{(2r+1)\pi i}x)w_2 + e^{xL_{(-1)}} \frac{\mathrm{d}}{\mathrm{d}x}\mathcal{Y}(w_1, e^{(2r+1)\pi i}x)w_2 \\ &= e^{xL_{(-1)}} L_{(-1)} \mathcal{Y}(w_1, e^{(2r+1)\pi i}x)w_2 - e^{xL_{(-1)}} \mathcal{Y}(L_{(-1)}w_1, e^{(2r+1)\pi i}x)w_2\end{aligned} \tag{A.59}$$

using

$$L_{(-1)}(w_1)_{(n)} - \left(L_{(-1)}w_1\right)_{(n)} = (w_1)_{(n)} L_{(-1)} \tag{A.60}$$

this equals

$$e^{xL_{(-1)}} \mathcal{Y}(w_1, e^{(2r+1)\pi i}x) L_{(-1)} w_2 = \mathcal{B}_r(\mathcal{Y})(L_{(-1)}w_2, x)w_1 \tag{A.61}$$

Next we check the Jacobi identity:

$$\begin{aligned}&x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_{W_3}(v, x_1)\mathcal{B}_r(\mathcal{Y})(w_2, x_2)w_1 \\ =&x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) e^{x_2L_{(-1)}} e^{-x_2L_{(-1)}} Y_{W_3}(v, x_1) e^{x_2L_{(-1)}} \mathcal{Y}(w_1, e^{(2r+1)\pi i}x_2)w_2 \\ =&x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) e^{x_2L_{(-1)}} Y_{W_3}(v, x_1 - x_2) \mathcal{Y}(w_1, e^{(2r+1)\pi i}x_2)w_2 \\ =&x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) e^{x_2L_{(-1)}} Y_{W_3}(v, x_0) \mathcal{Y}(w_1, e^{(2r+1)\pi i}x_2)w_2 \\ =&x_1^{-1}\delta\left(\frac{x_0 - (-x_2)}{x_1}\right) e^{x_2L_{(-1)}} Y_{W_3}(v, x_0) \mathcal{Y}(w_1, e^{(2r+1)\pi i}x_2)w_2\end{aligned} \tag{A.62}$$

$$
\begin{aligned}
=&x_1^{-1}\delta\left(\frac{-x_2-x_0}{-x_1}\right)e^{x_2L_{(-1)}}\mathcal{Y}(w_1,e^{(2r+1)\pi i}x_2)Y_{W_3}(v,x_0)w_2\\
&\qquad -x_2^{-1}\delta\left(\frac{x_0-x_1}{-x_2}\right)e^{x_2L_{(-1)}}\mathcal{Y}(Y_{W_2}(v,x_1)w,e^{(2r+1)\pi\mathrm{i}}x_2)w_2\\
=&x_1^{-1}\delta\left(\frac{-x_2-x_0}{-x_1}\right)\mathcal{B}_r(\mathcal{Y})(Y_{W_3}(v,x_0)w_2,x_2)w_1\\
&\qquad -x_2^{-1}\delta\left(\frac{x_0-x_1}{-x_2}\right)\mathcal{B}_r(\mathcal{Y})(w_2,x_2)Y_{W_2}(v,x_1)\\
=&x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\mathcal{B}_r(\mathcal{Y})(Y_{W_3}(v,x_0)w_2,x_2)w_1\\
&\qquad -x_2^{-1}\delta\left(\frac{x_0-x_1}{-x_2}\right)\mathcal{B}_r(\mathcal{Y})(w_2,x_2)Y_{W_2}(v,x_1)\\
=&x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)\mathcal{B}_r(\mathcal{Y})(Y_{W_3}(v,x_0)w_2,x_2)w_1\\
&\qquad x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\mathcal{B}_r(\mathcal{Y})(w_2,x_2)Y_{W_2}(v,x_1) \quad .
\end{aligned}
$$

We used relation (A.23) several times in the computation. In addition note that we cannot replace $e^{(2r+1)\pi\mathrm{i}}x$ with $-x$ in intertwining operators as its expansion includes real powers. However in module vertex operators or inside delta functions only integer powers appear and we are allowed to make the replacement. Finally, to prove that it is an isomorphism it is enough to give an inverse map. The claim is that $\mathcal{B}_{-r-1}$ is the inverse of $\mathcal{B}_r$.

$$
\begin{aligned}
\mathcal{B}_{-r-1}\left(\mathcal{B}_r(\mathcal{Y})\right)(w_1,x)w_2 &= e^{xL_{(-1)}}\mathcal{B}_r\left(\mathcal{Y}\right)(w_2,e^{(-2r-1)\pi\mathrm{i}}x)w_1\\
&= e^{xL_{(-1)}}e^{-xL_{(-1)}}\mathcal{Y}(w_1,x)w_2\\
&= \mathcal{Y}(w_1,x)w_2
\end{aligned}
\tag{A.63}
$$

The other direction goes the same. □

Lemma A.2.10. *Let $r\in\mathbb{Z}$ and*

$$A_r:\mathcal{V}_{ij}^{k}\to\mathcal{V}_{ik'}^{j'} \tag{A.64}$$

defined by

$$\langle A_r\left(\mathcal{Y}(m_i,x)\right)m_k',m_j\rangle=\left\langle m_k',\mathcal{Y}\left(e^{xL_1}e^{(2r+1)\pi\mathrm{i}L_0}x^{-2L_0}m_1,x^{-1}\right)m_j\right\rangle \quad . \tag{A.65}$$

Then A_r is well defined and an isomorphism with inverse A_{-r-1}. Setting

$$\hat{A}_r(\mathcal{Y})(\bullet,x)\bullet\equiv e^{-(2r+1)\pi\mathrm{i}L_0}A_{-r-1}(\mathcal{Y})(\bullet,x)e^{(2r+1)\pi\mathrm{i}L_0}\bullet \tag{A.66}$$

also defines an isomorphism $\mathcal{V}_{ij}^{k}\xrightarrow{\cong}\mathcal{V}_{ik'}^{j'}$. Its inverse is given by

$$\tilde{A}_r(\mathcal{Y})(\bullet,x)\bullet\equiv e^{-(2r+1)\pi\mathrm{i}L_0}A_r(\mathcal{Y})(\bullet,x)e^{(2r+1)\pi\mathrm{i}L_0}\bullet \quad . \tag{A.67}$$

The proof is almost the same as before, thus it is omitted here.

Lastly we state the definition of a rational VOA. There are some competing notions in the literature, since over the years the definition got refined. We settle with possibly the most restrictive one.

Definition A.2.11. A VOA V is called *rational* if:

1) It is simple (i.e. irreducible) as a module over itself.

2) It is of CFT type: $V_{(n)} = 0$ for $n < 0$, $V_{(0)} = \mathbb{C}\mathbf{1}$ and $V^* \simeq V$ as V-modules.

3) Every weak V-module is completely reducible [1].

The last point needs some explanation. A *weak module* has all the properties of a V-module except that it doesn't need to have a grading, but it satisfies all the conditions of a V-module which still make sense under this generalization. For a list of rational VOAs we refer to [91]. Examples include VOAs constructed from even lattices [41][43] or VOAs corresponding to integrable highest weight representations $L(k, 0)$ of affine Lie algebras at level k [55][92]. The last example will be discussed in the next section.

A.3 Examples and CFT Fields

After the mathematical treatment of VOAs and their representations we relate mathematical terms to quantities better known in physics. Usually in physics the state space H of a RCFT is decomposed into left and right moving conformal families $H = \bigoplus_{i,j\in \mathsf{I}} U_i \times \overline{U}_j$. The vector spaces $U_i, \overline{U}_j$ are Verma modules built from a primary ϕ_i of conformal weight h_i and similar for $\overline{U}_j$. In VOA terms this is nothing but a simple representation of a rational VOA V, which due to the unit properties in the definition are built by applying modes of module vertex operators to a lowest weight state $|h_i\rangle$, which is the state corresponding to a primary field. Therefore these representations are of Verma module type. Since the Virasoro algebra is embedded into a VOA this automatically contains a conformal family. Bur in addition the representation may be generated by application of other field modes, e.g. in case the CFT has Kac-Moody symmetry the representation should contain states from applications of Kac-Moody currents. As a non Lie algebra type example for a CFT with additional symmetry consider $\mathscr{W}$-algebras. In this case the representation should also be generated by modes of the higher spin currents. In both cases, though, the primary state spaces remain representations of the Virasoro algebra. VOAs are now just the mathematical object capturing all of these symmetry algebras and a representation of the VOA for a given symmetry is part of the usual chiral subspace of the state space of the physical theory. Note however that this doesn't exclude non-unitary representations in general. To

[1] Often this is stated as two points: That every $\mathbb{N}$-gradable weak module is completely reducible and that V is C_2-cofinite. That this is equivalent to the statement given in the definition follows from results in [116][1].

make this more palpable we discuss the case of Kac-Moody algebras[2]. The VOA structure for Kac-Moody algebras is discussed in many places, we follow [52, section 2]. Let $\mathfrak{g}$ be a semi-simple Lie algebra and

$$\hat{\mathfrak{g}}_k = \mathfrak{g}((t)) \oplus kC \tag{A.68}$$

its centrally extended loop algebra. Since $\mathfrak{g}$ is semi-simple its Killing form $(\bullet, \bullet)$ is a non-degenerate bilinear form. Let $h^\vee$ be the dual Coxeter number of $\mathfrak{g}$, the normalized inner product is defined as

$$\langle \bullet, \bullet \rangle = \frac{1}{2h^\vee} (\bullet, \bullet) \tag{A.69}$$

and the Lie bracket on generators of $\hat{\mathfrak{g}}_k$ reads

$$[Xt^n, Yt^m] = [X,Y]t^{n+m} + km \langle X, Y \rangle \, \delta_{n+m,0} C \quad . \tag{A.70}$$

Let $\hat{g}_+ = \mathfrak{g}[\![t]\!]t \oplus kC \subset \hat{\mathfrak{g}}$ be the positive mode sub Lie algebra and $\mathbb{C}_k$ be its one dimensional representation defined by the action of a single generator

$$\begin{aligned} X \otimes t^n \mathbf{1} &= 0, \quad \forall\, n \geq 0 \\ C \cdot \mathbf{1} &= k\mathbf{1} \quad . \end{aligned} \tag{A.71}$$

The *vacuum VOA* associated to $\hat{\mathfrak{g}}_k$ has underlying vector space the induced module

$$V_k(\hat{\mathfrak{g}}) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_+)} \mathbb{C}_k \tag{A.72}$$

where $U(\hat{\mathfrak{g}})$ is the universal enveloping algebra. As a vector spaces there is an isomorphism

$$V_k(\hat{\mathfrak{g}}) \simeq U(t^{-1}\mathfrak{g}[\![t^{-1}]\!]) \tag{A.73}$$

which is tantamount to saying that it is the following vector space

$$V_k(\hat{\mathfrak{g}}) \simeq \mathrm{span}_{\mathbb{C}} \left\{ j^{a_1}_{n_1} \cdots j^{a_m}_{n_m} \mathbf{1} \,\middle|\, n_1 \leq \cdots \leq n_m < 0, \text{ if } n_i = n_{i+1} \text{ then } a_i \leq a_{i+1} \right\} \tag{A.74}$$

for $\{j^a\}$ a basis in $\mathfrak{g}$ and we abbreviated $j^a_n = j^a t^n$. The vertex operator is defined as

$$\begin{aligned} Y(\mathbf{1}, z) &= \mathrm{id}_{V_k(\hat{\mathfrak{g}})} \\ Y(j^a_{-1}\mathbf{1}, z) &= j^a(z) = \sum_{n\in\mathbb{Z}} j^a_n z^{-n-1} \end{aligned} \tag{A.75}$$

and more generally

$$Y(j^{a_1}_{n_1} \cdots j^{a_m}_{n_m}\mathbf{1}, z) = \frac{1}{(-n_1-1)! \cdots (-n_m-1)!} :\partial_z^{-n_1-1} j^{a_1}(z) \cdots \partial_z^{-n_m-1} j^{a_m}(z): \quad . \tag{A.76}$$

[2]It turns out that representation theory of $\mathscr{W}$-algebras very much follows from this by Drinfeld Sokolov reduction. The interested reader may consult [2] for original results on this or [3] for an excellent review.

If we assume that $\{j^a\}$ is an ONB wrt $\langle \bullet , \bullet \rangle$ we can define the Sugawara energy momentum tensor as the vertex operator

$$T(z) \equiv Y\left(\frac{1}{2(k+h^\vee)} j^a_{-1} j^a_{-1} \mathbf{1}, z\right) = \frac{1}{2(k+h^\vee)} :j^a(z) j^a(z): \quad . \tag{A.77}$$

The embedded Virasoro algebra then has central charge

$$c(k) = \frac{k \dim \mathfrak{g}}{2(k+h^\vee)} \quad . \tag{A.78}$$

That this indeed defines a VOA is checked in [52, section 2.4]. Furthermore there is a relation between representations of $\hat{\mathfrak{g}}_k$ and VOA representations of $V_k(\hat{\mathfrak{g}})$. A representation U of $\hat{\mathfrak{g}}_k$ is called *smooth* if for any $u \in U$ there exists $N > 0$ s.th.

$$X t^n \cdot u = 0, \quad \forall X \in \mathfrak{g}, \, n \geq N \quad . \tag{A.79}$$

It is shown in [52, section 5] that smooth $\hat{\mathfrak{g}}_k$-modules are in one-to-one correspondence with $V_k(\hat{\mathfrak{g}})$-representations. This is not too surprising since the vertex operator or its module vertex operator may equally be described by its Lie algebra of modes, which in this case is the universal enveloping algebra of $\hat{\mathfrak{g}}_k$. The smoothness condition then just ensures the lower truncation property of modules. Examples of such modules are highest weight representations of $\hat{\mathfrak{g}}_k$ [3]. Given a weight λ of $\hat{\mathfrak{g}}_k$ its highest weight representation M_λ is generated from a highest weight stat $|\lambda\rangle$ by applying negative modes. That is, a vector in M_λ is a finite linear combination of vectors

$$m = \rho_\lambda(T^{a_1}_{n_1}) \cdots \rho_\lambda(T^{a_m}_{n_m}) \, |\lambda\rangle \tag{A.80}$$

where $\{T^a_n\}$ are elements of $\hat{\mathfrak{g}}_k$ in its Cartan Weyl basis. Here $\rho_\lambda : \hat{\mathfrak{g}}_k \to \mathrm{End}(M_\lambda)$ is the module map and similar to the previous case only modes

$$T^a_n = \begin{cases} E^{-\alpha_i}_{n_i}, \; \alpha_i \in \Phi_+ \, n_i \leq 0, \\ H^{a_i}_{n_i}, \; a_i = 1, \cdots, \mathrm{rk}(\mathfrak{g}), \, n_i < 0 \\ E^{\alpha_i}_{n_i}, \; \alpha_i \in \Phi_+ \, n_i < 0 \end{cases} \tag{A.81}$$

with Φ_+ the positive roots of $\mathfrak{g}$, appear. Using the explicit commutation relations (A.70) it is obvious that these are smooth modules. Let P_+ be the dominant integral weights of $\mathfrak{g}$ and $\theta \in \Phi_+$ the highest root of $\mathfrak{g}$. For weights

$$P^k_+ = \{\lambda = (\lambda_\mathfrak{g}, k, 0) \,|\, (\lambda_\mathfrak{g}, \theta) \leq k, \quad \lambda_\mathfrak{g} \in P_+\} \tag{A.82}$$

the representation M_λ is integrable, meaning that the induced representation of any $\mathfrak{sl}_2$ triple $\{E^\alpha_n, E^{-\alpha}_n, H^\alpha\} \subset \hat{\mathfrak{g}}_k$ for $\alpha \in \Phi_+$ is finite dimensional. The condition (A.82) ensures

[3] see e.g. [64] for an exhaustive treatment of the representation theory of affine Lie algebras

that at a fixed level there are only finitely many integrable highest weight representations. It turn out that the corresponding VOA modules for $V_k(\hat{\mathfrak{g}})$ are exactly the simple representations of $V_k(\hat{\mathfrak{g}})$. Unfortunately it is not true that $V_k(\hat{\mathfrak{g}}_k)$ is a rational VOA. Of course $V_k(\hat{\mathfrak{g}}_k)$ is a module over itself. Let N_k be the $\hat{\mathfrak{g}}_k$-submodule of $V_k(\hat{\mathfrak{g}})$ generated by $(E^{\theta}_{-1})^{k+1}\mathbf{1}$. It is the maximal submodule and the quotient

$$V^k(\hat{\mathfrak{g}}_k) \equiv V_k(\hat{\mathfrak{g}}_k)/N_k \tag{A.83}$$

inherits naturally a VOA structure as it is a submodule and the module map is the vertex operator. As a vector space there is an isomorphism $V^k(\hat{\mathfrak{g}}_k) \simeq L(k,0)$, with $L(k,0)$ the irreducible highest weight module of $\hat{\mathfrak{g}}_k$ with weight $\lambda = 0$. This is a rational VOA and its simple modules are still labeled by P^k_+. Highest weight modules for $\hat{\mathfrak{g}}_k$ will give $\mathbb{N}$-graded VOA modules similar to general simple modules for VOAs. The conformal weight of the corresponding $V^k(\hat{\mathfrak{g}})$-representation M_λ are not too hard to compute (see e.g. [39, chapter 15.3.3]) and given by

$$h_\lambda = \frac{(\lambda_{\mathfrak{g}}, \lambda_{\mathfrak{g}} + 2\rho)}{2(k + h^\vee)} \tag{A.84}$$

where $\rho = \frac{1}{2}\sum_{i \in \Phi^s_+} \alpha_i$ is the sum over simple positive roots of $\mathfrak{g}$ (usually called the Weyl vector).

This settles the state space, but what about fields? Though intertwining operators or vertex operators resemble conformal fields the two notions should not be confused. A field operator inserted at $0 \in \mathbb{C}$ in a CFT is usually given in the form of an operator valued series expansion

$$\Phi(z, \overline{z}) = \sum_{n,m \in Z} \phi_{[n,m]} z^{-n-h_l} \overline{z}^{-m-h_r} \tag{A.85}$$

where (h_l, h_r) are left and right moving conformal weights of the field. In this thesis we adopted the mathematics convention of expanding operator valued Laurent series with fixed exponents irrespective of conformal weights

$$\Phi(z, \overline{z}) = \sum_{n,m \in Z} \phi_{n,m} z^{-n-1} \overline{z}^{-m-1} \quad . \tag{A.86}$$

The coefficients are related by a simple index shift

$$\phi_{[n,m]} = \phi_{n+h_l-1, m+h_r-1} \quad . \tag{A.87}$$

Assume that $\Phi(z, \overline{z})$ is a field of a RCFT with state space $H = \bigoplus_{i,j} H_{i,j} U_i \times \overline{U}_j$. By the operator-state correspondence there exists a unique vector $|v\rangle \in H$ corresponding to Φ via

$$|\phi\rangle = \lim_{z,\overline{z}} \Phi(z, \overline{z})\mathbf{1} = \phi_{[-h_l, -h_r]}\mathbf{1} \quad . \tag{A.88}$$

The bare field itself is of course not very interesting. Only through its interactions, or couplings, to all other fields in the theory it becomes relevant. But these are exactly given by the intertwining operators! If we assume for simplicity that $|\phi\rangle = v^L \otimes v^R \in U_i \times \overline{U}_j \hookrightarrow H$ is a homogeneous element of H and rewrite H as

$$H = \bigoplus_{n=1}^{K} U^L_{\nu^L(n)} \times U^R_{\nu^R(n)} \tag{A.89}$$

the field has an expansion

$$\Phi(z, \overline{z}) = \sum_{\substack{k,\ell,n=1,\cdots,K \\ \nu^L(k)=i,\nu^R(k)=j}} \sum_{\alpha_1=1}^{N^{\nu(n)}_{\nu^L(k)\nu^L(\ell)}} \sum_{\alpha_2=1}^{N^{\nu^R(s)}_{\nu^R(k)\nu^R(\ell)}} C^n_{k\ell} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \mathcal{Y}^{\nu^L(n)}_{\nu^L(k)\nu^L(\ell);\alpha_1}(v^L, z) \mathcal{Y}^{\nu^R(n)}_{\nu^R(k)\nu^R(\ell);\alpha_2}(v^R, \overline{z}) \tag{A.90}$$

for some coefficients $C^n_{k\ell} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \in \mathbb{C}$. Hence computing a correlation function reduces to computing

$$\begin{aligned} &\langle \Phi_1(z_1, \overline{z}_1) \cdots \Phi_n(z_n, \overline{z}_n) \rangle \\ \sim &\sum \left\langle \phi^L_{i_1} \middle| \mathcal{Y}^{i_1}_{i_2 p_1}(\phi^L_{i_2}, z_2) \mathcal{Y}^{p_1}_{i_3 p_2}(\phi^L_{i_2}, z_3) \cdots \mathcal{Y}^{p_{n-1}}_{i_{n-1} i_n}(\phi^L_{i_{n-1}}, z_{n-1}) \middle| \phi^L_{i_n} \right\rangle \\ &\qquad \times \left\langle \phi^R_{k_1} \middle| \mathcal{Y}^{k_1}_{k_2 q_1}(\phi^R_{k_2}, \overline{z}_2) \mathcal{Y}^{q_1}_{k_3 q_2}(\phi^R_{k_3}, \overline{z}_3) \cdots \mathcal{Y}^{q_{n-1}}_{k_{n-1} k_n}(\phi^R_{k_{n-1}}, z_{n-1}) \middle| \phi^R_{k_n} \right\rangle \end{aligned} \tag{A.91}$$

where we secretly moved the insertion point of Φ_1 to ∞ and the insertion of Φ_n to 0. Let us make this more concrete by considering a three point function.

$$\langle \Phi_t(z_1, \overline{z}_1) \Phi_u(z_2, \overline{z}_2) \Phi_v(z_3, \overline{z}_3) \rangle \tag{A.92}$$

Wlog we may send $z_1 \to \infty$, $z_2 \to 1$, $z_3 \to 0$ to express the three point function as a matrix element

$$\lim_{z_2 \to 1} \langle \phi_t | \, \Phi_u(z_2, \overline{z}_2) \, | \phi_v \rangle = \lim_{z_2 \to 1} \lim_{z_1 \to \infty} z_1^{2h^L_t} \overline{z}_1^{2h^R_t} \langle \Phi_t(z_1, \overline{z}_1) \Phi_u(z_2, \overline{z}_2) \Phi_v(0,0) \rangle \quad . \tag{A.93}$$

This can be computed by inserting the OPE ansatz

$$\Phi_u(z_2, \overline{z}_2) \Phi_v(0,0) \sim \sum_{n=1}^{K} \sum_{\mathbf{m}^L, \mathbf{m}^R} C^{n;\mathbf{m}^L,\mathbf{m}^R}_{uv} z_2^{h^L_n - h^L_2 - h^L_3 - M^L} \overline{z}_2^{h^R_n - h^R_2 - h^R_3 - M^R} \Phi_n^{\mathbf{m}^L,\mathbf{m}^R}(0,0) \tag{A.94}$$

with $M^L = \sum_i m^L_i$, $M^R = \sum_j m^R_j$ and $\mathbf{m}^L, \mathbf{m}^R$ are multiindices running over descendants

$$\left\{ L^L_{-\mathbf{m}^L} L^R_{-\mathbf{m}^R} \Phi_n \equiv L^L_{-m^L_1} \cdots L^L_{-m^L_P} L^R_{-m^R_1} \cdots L^R_{-m^R_Q} \Phi_n \right\} \quad . \tag{A.95}$$

Inserting (A.94) into (A.93) and assuming for simplicity that all fields are highest weight primaries yields

$$\lim_{z_2 \to 1} \langle \phi_t | \, \Phi_u(z_2, \overline{z}_2) \, | \phi_v \rangle = C^t_{uv} d_{tt} \quad . \tag{A.96}$$

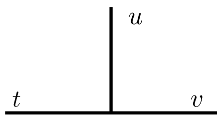
We may assume $d_{tt} = 1$ by picking an ONB. On the other hand we can compute (A.93) in the expansion of intertwining operators

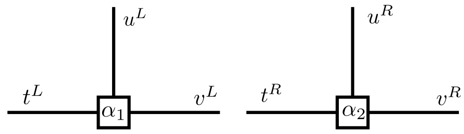
$$\begin{aligned}\lim_{z_2\to 1} \langle\phi_1|\,\Phi_2(z_2,\overline{z}_2)\,|\phi_3\rangle = \lim_{z_2\to 1} &\sum_{\substack{k,\ell,n=1,\cdots,K\\ \nu(k)\in u,\nu(l)\in v,\nu(n)\in t}} \sum_{\alpha_1=1}^{N^{\nu(n)}_{\nu^L(k)\nu^L(\ell)}} \sum_{\alpha_2=1}^{N^{\nu^R(s)}_{\nu^R(k)\nu^R(\ell)}} C^n_{k\ell}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} \\ &\langle\phi_1|\,\mathcal{Y}^{\nu^L(n)}_{\nu^L(k)\nu^L(\ell);\alpha_1}(\phi^L_2,z_2)\mathcal{Y}^{\nu^R(n)}_{\nu^R(k)\nu^R(\ell);\alpha_2}(\phi^R_2,\overline{z}_2)\,|\phi_3\rangle \quad .\end{aligned} \tag{A.97}$$

The first sum is restricted to indices $\nu(k) = \nu^L(k)\otimes\nu^R(k)$ in the simple representation the state ϕ_2 lives in and similar for $\nu(n)$, $\nu(l)$. Evaluating this gives

$$\lim_{z_2\to 1} \langle\phi_1|\,\Phi_2(z_2,\overline{z}_2)\,|\phi_3\rangle = \sum_{\substack{k,\ell,n=1,\cdots,K\\ \nu(k)\in u,\nu(l)\in v,\nu(n)\in t}} \sum_{\alpha_1=1}^{N^{\nu(n)}_{\nu^L(k)\nu^L(\ell)}} \sum_{\alpha_2=1}^{N^{\nu^R(s)}_{\nu^R(k)\nu^R(\ell)}} C^n_{k\ell}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} \mathcal{F}^t_{uv;\alpha_1}\overline{\mathcal{F}}^t_{uv,\alpha_2} \tag{A.98}$$

with $\mathcal{F}^1_{23,\alpha}$ three point chiral conformal blocks. Comparing (A.98) with (A.96) one get a relation between the expansion coefficients of a conformal field and the coefficients of the OPE, which is most conveniently given in a diagram

C^t_{uv} 

$= \sum_{\alpha_1=1}^{N^{t^L}_{u^L v^L}} \sum_{\alpha_2=1}^{N^{t^R}_{u^R v^R}} C^t_{uv}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}$ 

.

Having explained in detail the relation between conformal fields, their expansion in terms of intertwining operators and OPE coefficients we briefly discuss the four point case. In contrast to the three point discussion we suppress most of the indices in order to highlight the essentials. Given a four point function

$$\langle\phi_i|\,\Phi_j(z_1,\overline{z}_1)\Phi_k(z_2,\overline{z}_2)\,|\phi_\ell\rangle \tag{A.99}$$

for $|z_1| > |z_2| > 0$ we insert the expansion of Φ_j, Φ_k in terms of intertwining operators

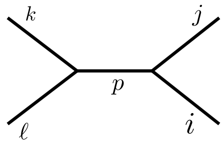
$$\begin{aligned}&\sum_p C^i_{jp}C^p_{k\ell}\,\langle\phi_i|\,\mathcal{Y}^{i^L}_{j^L p^L}(\phi^L_j,z_1)\mathcal{Y}^{p^L}_{k^L\ell^L}(\phi^L_k,z_2)\mathcal{Y}^{i^R}_{j^R p^R}(\phi^R_j,\overline{z}_1)\mathcal{Y}^{p^R}_{k^R\ell^R}(\phi^R_k,\overline{z}_2)\,|\phi_\ell\rangle \\ &= \sum_p C^i_{jp}C^p_{k\ell}\,\mathcal{F}^{ij}_{k\ell}(p|z_1,z_2)\overline{\mathcal{F}}^{ij}_{k\ell}(p|\overline{z}_1,\overline{z}_2)\end{aligned} \tag{A.100}$$

where we used the notation of [26, section 2.12] for conformal blocks and the explicit isomorphism between conformal blocks and intertwining operators. In case $|z_1| > |z_2| > |z_1 - z_2|$ we can use proposition A.4.5 *IOA3)* to equally expand this as

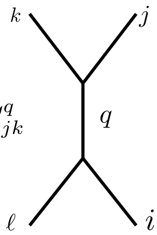
$$\begin{aligned}&\sum_q C^i_{q\ell}C^q_{jk}\,\langle\phi_i|\,\mathcal{Y}^{i^L}_{q^L\ell^L}\left(\mathcal{Y}^{q^L}_{j^L k^L}(\phi^L_j,z_1-z_2)\phi^L_k,z_2\right)\mathcal{Y}^{i^R}_{q^R\ell^R}\left(\mathcal{Y}^{q^R}_{j^R k^R}(\phi^R_j,\overline{z}_1-\overline{z}_2)\phi^R_k,\overline{z}_2\right)|\phi_\ell\rangle \\ &= \sum_q C^i_{q\ell}C^q_{jk}\,\mathcal{F}^{i\ell}_{jk}(q|z_2,z_1-z_2)\overline{\mathcal{F}}^{i\ell}_{jk}(q|\overline{z}_2,\overline{z}_1-\overline{z}_2) \quad .\end{aligned} \tag{A.101}$$

The essential information is again best given in terms of a diagram

$$\sum_p C^i_{jp} C^p_{k\ell}$$



$$= \sum_q C^i_{q\ell} C^q_{jk}$$



which is of course nothing but the usual diagram when discussing crossing symmetry of the four point function.

A.4 Intertwining Operator Algebras

Intertwining operator algebras are a mathematical tool to describe genus zero conformal blocks. They were introduced in [78]. The treatment given here follows [80]. Since results become gradually more complicated we refrain from presenting proofs, but give detailed reference where to find them in the literature. Most of the results only involve the basic statements about intertwining operators given previously plus some theorems about partial differential equations with regular singularities.Before we go into details we state a result from complex analysis which is secretly used in some places.

Proposition A.4.1. *Let $I \subset \mathbb{R}$ be an open subset. A function $f : I \to \mathbb{C}$ has a possibly multivalued analytic continuation $\tilde{f} : \mathbb{C} \to \mathbb{C}$ if and only if f is an analytic function. In addition on any branch the analytic continuation is uniquely determined by the identity principle.*

First we recall the major result of [81] and the logic of arguments leading to it.

Theorem A.4.2. *[81, Theorem 1.4,2.5] Let V be a rational VOA, M_1, M_2, M_3, M_4 and M be V-modules. Let $\mathcal{Y}_1 \in \mathcal{V}^{M_4}_{M_1 M}$ and $\mathcal{Y}_2 \in \mathcal{V}^{M}_{M_2 M_3}$. Then for $m_i \in M_i$ there exist N, L $\in \mathbb{N}$ and*

$$c_i(z_1, z_2),\ k_j(z_1, z_2) \ \in \mathbb{C}\left[z_1^{\pm 1}, z_2^{\pm}, (z_1 - z_2)^{-1}\right] \tag{A.102}$$

for $i = 1, \ldots, N$ and $j = 1, \ldots, L$, s.th.

$$\Psi(z_1, z_2) \equiv \langle m_4', \mathcal{Y}_1(m_1, z_1)\mathcal{Y}_2(m_2, z_2)m_3\rangle \tag{A.103}$$

satisfies

$$\begin{aligned} &\frac{\partial^m}{\partial z_1^m}\Psi + \sum_{i=1}^{N} c_i(z_1, z_2)\frac{\partial^{N-i}}{\partial z_1^{N-i}}\Psi = 0 \\ &\frac{\partial^m}{\partial z_2^m}\Psi + \sum_{j=1}^{L} k_j(z_1, z_2)\frac{\partial^{L-j}}{\partial z_1^{L-j}}\Psi = 0 \end{aligned} \tag{A.104}$$

for $|z_1| > |z_2| > 0$. Similar statements hold for iterates on $|z_2| > |z_1 - z_2| > 0$. Furthermore, singular points $z_1 = 0$, $z_1 = \infty$, $z_2 = 0$, $z_2 = \infty$ and $z_1 = z_2$ are regular.

The proof of the theorem relies on an intelligent filtration of $\mathbb{C}\left[z_1^{\pm 1}, z_2^{\pm 1}, (z_1 - z_2)^{-1}\right]$, but the result is not too surprising as the whole point of the construction is to describe chiral correlation functions, whose singular points are exactly at $\left\{z_1^{\pm 1}, z_2^{\pm 1}, (z_1 - z_2)^{-1}\right\}$. Regularity of the singular points tells that the solution has at most poles at singularities.

The consequence of theorem A.4.2 we are most interested in is the following theorem.

Theorem A.4.3. *[81, Theorem 3.5] Let V be a rational VOA. Then in the setup of theorem A.4.2, there exists $P \in \mathbb{Z}$ and for any $m_i \in M_i$ homogeneous there exist n_i, m_i s.th. for*

$$|m_1| + |m_2| + n_i > P \tag{A.105}$$

$\Psi(z_1, z_2)$ is absolutely convergent on $|z_1| > |z_2| > 0$ and there exist analytic functions f_i s.th. Ψ can be analytically extended to

$$\sum_{i=1}^{4} z_2^{m_i} (z_1 - z_2)^{n_i} f_i\left(\frac{z_1 - z_2}{z_2}\right) \tag{A.106}$$

on $|z_2| > |z_1 - z_2| > 0$. Furthermore, any product

$$\langle m', \mathcal{Y}_1(m_1, z_1) \cdots \mathcal{Y}_n(m_n, z_n) m_{n+1}\rangle \tag{A.107}$$

absolutely converges on $|z_1| > \cdots > |z_n| > 0$ and can be analytically extended to $\mathrm{Conf}_n(\mathbb{C}^\times)$.

As promised in the previous section this gives an associativity statement for correlation functions of intertwining operators.

Theorem A.4.4. *[78, Theorem 16.2,14.8] Let V be a rational VOA, then in the setup of theorem A.4.2, there exists a V-module $\widetilde{M}$ and intertwining operators $\mathcal{Y}_3 \in \mathcal{V}_{M_1 M_2}^{\widetilde{M}}$, $\mathcal{Y}_4 \in \mathcal{V}_{\widetilde{M} M_3}^{M_4}$ s.th.*

$$\langle m_4', \mathcal{Y}_4\left(\mathcal{Y}_3(m_1, z_1 - z_2) m_2, z_2\right) m_3\rangle \tag{A.108}$$

absolutely converges on $|z_2| > |z_1 - z_2| > 0$ and equals (A.103) *on $|z_1| > |z_2| > |z_1 - z_2| > 0$.*

This section can be conveniently summarized in the following proposition.

Proposition A.4.5. *Let V be a rational VOA with simple modules $\{U_i\}_{\in \mathsf{I}}$. Then $O = \coprod_{i \in \mathsf{I}} U_i$ is an intertwining operator algebra (IOA)[80, Definition 2.1], i.e. it satisfies the following list of axioms:*

IOA1) (Convergence 1) For any U_{i_ℓ}, $\ell = 1, \ldots, n+2$ and $\mathcal{Y}_k \in \mathcal{V}_{i_k j_k}^{j_{k-1}}$ for $k = 2, \ldots, n-1$ and $\mathcal{Y}_1 \in \mathcal{V}_{i_1 j_i}^{i_{n+2}}$, $\mathcal{Y}_n \in \mathcal{V}_{i_n i_{n+1}}^{j_n}$

$$\phi(m_1, \ldots, m_n; z_1, \ldots, z_n) \equiv \left\langle m_{i_{n+2}}', \mathcal{Y}_1(m_{i_1}, z_1) \cdots \mathcal{Y}_n(m_{i_n}, z_n) m_{i_{n+1}}\right\rangle \tag{A.109}$$

converges absolutely on $|z_1| > \cdots > |z_n| > 0$. In addition it has a multivalued analytic continuation $\Phi(m_1, \ldots, m_n; z_1, \ldots, z_n)$ to $\mathrm{Conf}_n(\mathbb{C}^\times)$.

IOA2) (Convergence 2) For any $\mathcal{Y}_1 \in \mathcal{V}^{j}_{i_1 i_2}$ *and* $\mathcal{Y}_2 \in \mathcal{V}^{i_4}_{j i_3}$

$$\langle m_4', \mathcal{Y}_2\left(\mathcal{Y}_1(m_1, z_1 - z_2)m_2, z_2\right) m_3\rangle \tag{A.110}$$

converges absolutely on $|z_2| > |z_1 - z_2| > 0$.

IOA3) (Associativity) For any $\mathcal{Y}_1 \in \mathcal{V}^{i_4}_{i_1 j}$, $\mathcal{Y}_2 \in \mathcal{V}^{j}_{i_2 i_3}$ *there exist* $\mathcal{Y}^{\ell}_{3,\alpha} \in \mathcal{V}^{\ell}_{i_1 i_2}$, $\mathcal{Y}^{\ell}_{4,\alpha} \in \mathcal{V}^{i_4}_{\ell i_3}$ *and* $Q \in \mathbb{N}$ *for all* $\ell \in \mathsf{I}$ *s.th.*

$$\begin{aligned}&\langle m_4', \mathcal{Y}_1(m_{i_1}, z_1)\mathcal{Y}_2(m_{i_2}, z_2)m_{i_3}\rangle \\ &\quad= \sum_{\ell\in\mathsf{I}}\sum_{\alpha=1}^{Q} \left\langle m_{i_4}', \mathcal{Y}^{\ell}_{4,\alpha}\left(\mathcal{Y}^{\ell}_{3,\alpha}(m_{i_1}, z_1 - z_2))m_{i_2}, z_2\right) m_{i_3}\right\rangle\end{aligned} \tag{A.111}$$

holds for $|z_1| > |z_2| > |z_1 - z_2| > 0$.

IOA4) (Skew symmetry) For any $\mathcal{Y}_1 \in \mathcal{V}^{i_4}_{i_1 j}$, $\mathcal{Y}_2 \in \mathcal{V}^{j}_{i_2 i_3}$, *there exist* $\mathcal{Y}_3 \in \mathcal{V}^{i_4}_{i_2 j}$, $\mathcal{Y}_2 \in \mathcal{V}^{j}_{i_1 i_3}$ *and* $Q \in \mathbb{N}$ *for all* $\ell = 1, \ldots, \mathsf{I}$ *s.th.*

$$\langle m_4', \mathcal{Y}_1(m_{i_1}, z_1)\mathcal{Y}_2(m_{i_2}, z_2)m_{i_3}\rangle \tag{A.112}$$

defined on $|z_1| > |z_2|$ *and*

$$\sum_{\ell\in\mathsf{I}}\sum_{\alpha=1}^{Q} \left\langle m_4', \mathcal{Y}^{\ell}_{3,\alpha}(m_{i_2}, z_2)\mathcal{Y}^{\ell}_{4,\alpha}(m_{i_1}, z_1)m_{i_3}\right\rangle \tag{A.113}$$

defined on $|z_2| > |z_1|$ *are analytic extensions of each other.*

Correlation functions (A.109), or better their analytic continuation to configuration space, exactly give conformal blocks as shown in [151].

A word of warning should be given, whenever we discuss correlation functions of intertwining operators. In general expressions like $\langle m_4', \mathcal{Y}(m_{i_1}, z_1)\mathcal{Y}_2(m_{i_2}, z_2)m_{i_3}\rangle$ are multivalued analytic functions on their domain of convergence. One can make them into single valued functions by specifying a branch of the logarithm. That is if we write

$$\langle m_4', \mathcal{Y}(m_{i_1}, x_1)\mathcal{Y}_2(m_{i_2}, x_2)m_{i_3}\rangle\,|_{x_i^n = e^{n\log(z_i)}} \tag{A.114}$$

it is uniquely determined and single valued. Throughout the whole thesis we always choose branches with cuts along the positive real axis and leave this choice implicit.

A.5 Fusion, Braiding and Pentagon Equation

In the previous section we gave an outline on how intertwining operators for rational VOAs give rise to chiral correlation functions in genus zero, which satisfy the expected properties of rationality and having the correct singular points. In order to prove rigidity and modularity of R_V for V rational, one needs the Verlinde formula. A rigorous proof appeared in [84], the key steps are reviewed in the next sections.

We start by introducing fusion and braiding matrices of genus zero conformal blocks. As before, let $\{U_i\}_{i\in\mathsf{I}}$ be the simple modules of a rational VOA V. For $A \in \mathbb{N}$, let $\left\{\mathcal{Y}^{i_3;A}_{i_1i_2;1},\ldots,\mathcal{Y}^{i_3;A}_{i_1i_2;N^{i_3}_{i_1i_2}}\right\}$ be a basis for $\mathcal{V}^{i_3}_{i_1i_2}$. Note that for different values of A, these are different bases.

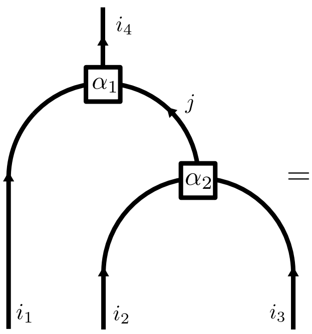
Definition A.5.1. Let $|z_1| > |z_2| > |z_1 - z_2| > 0$, the *fusion matrices*

$$\mathsf{F}^{(i_1i_2i_3);i_4}\begin{bmatrix}\alpha_1 & j & \alpha_2\\ \beta_3 & k & \beta_4\end{bmatrix} \tag{A.115}$$

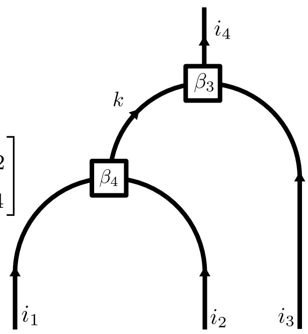
are defined by

$$\begin{aligned}&\left\langle u'_{i_4}\mathcal{Y}^{i_4;1}_{i_1j;\alpha_1}(u_{i_1},z_1)\mathcal{Y}^{j;2}_{i_2i_3;\alpha_2}(u_{i_2},z_2)u_{i_3}\right\rangle\\ &=\sum_{k\in\mathsf{I}}\sum_{\beta_3=1}^{N^{i_4}_{ki_3}}\sum_{\beta_4=1}^{N^{k}_{i_1i_2}}\mathsf{F}^{(i_1i_2i_3);i_4}\begin{bmatrix}\alpha_1 & j & \alpha_2\\ \beta_3 & k & \beta_4\end{bmatrix}\left\langle u'_{i_4},\mathcal{Y}^{i_4;3}_{ki_2;\beta_3}\left(\mathcal{Y}^{k;4}_{i_1i_2;\beta_4}(u_{i_1},z_1-z_2)u_{i_2},z_2\right)u_{i_3}\right\rangle\end{aligned} \tag{A.116}$$

As formulas are already overloaded, the dependence of fusion matrices on bases of intertwining operators is only implicit through an extra index at the index running through different basis elements. Fusion matrices are just an expansion of *IOA3)* in terms of basis elements. Graphically fusion matrices are represented by



$$=\sum_{k\in\mathsf{I}}\sum_{\beta_3=1}^{N^{i_4}_{ki_3}}\sum_{\beta_4=1}^{N^{k}_{i_1i_2}}\mathsf{F}^{(i_1i_2i_3);i_4}\begin{bmatrix}\alpha_1 & j & \alpha_2\\ \beta_3 & k & \beta_4\end{bmatrix}$$



.

Since single valued analytic continuations of products and iterates to the universal cover $\mathrm{Conf}_2(\mathbb{C}^\times)$ are uniquely determined by the functions on the simply connected domains $|z_1| > |z_2| > 0$ and $|z_2| > |z_1 - z_2| > 0$, fusion matrices also give coefficients for the analytic extensions. Denoting the analytic continuation of

$$\left\langle u'_{i_4},\mathcal{Y}^{i_4;3}_{ki_3;\beta_3}\left(\mathcal{Y}^{k;4}_{i_1i_2;\beta_4}(u_{i_1},z_1-z_2)u_{i_2},z_2\right)u_{i_3}\right\rangle \tag{A.117}$$

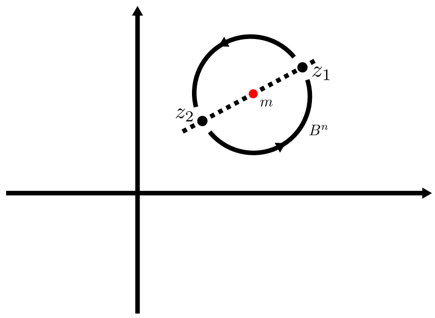
by

$$\Psi^{k}_{\beta_4\beta_3}(u_{i_1},u_{i_2},u_{i_3},u'_{i_4};z_1-z_2,z_2) \tag{A.118}$$

this reads

$$\begin{aligned}
&\Phi^{j}_{\alpha_1\alpha_2}(u_{i_1}, u_{i_2}, u_{i_3}, u'_{i_4}; z_1, z_2) \\
&\quad = \sum_{k\in\mathsf{I}} \sum_{\beta_3=1}^{N^{i_4}_{ki_3}} \sum_{\beta_4=1}^{N^{k}_{i_1i_2}} \mathsf{F}^{(i_1i_2i_3);i_4} \begin{bmatrix} \alpha_1 & j & \alpha_2 \\ \beta_3 & k & \beta_4 \end{bmatrix} \Psi^{k}_{\beta_4\beta_3}(u_{i_1}, u_{i_2}, u_{i_3}, u'_{i_4}; z_1 - z_2, z_2)
\end{aligned} \tag{A.119}$$

Similarly braiding matrices can be defined as coefficients in a basis expansion of *IOA4)*. Let $|z_1| > |z_2| > 0$ and $m = \frac{z_1 - z_2}{2}$. For any $n \in \mathbb{Z}$, we define a path B^n as sketched in the following figure



where B^n maps $z_1 \rightleftharpoons z_2$ along a circle of radius $|\frac{z_1 - z_2}{2}|$ and winds n times around m. It winds counterclockwise for $n \geq 0$ and clockwise for $n < 0$.

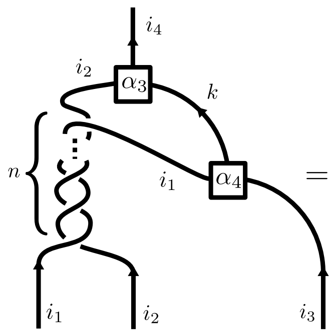
Definition A.5.2. Let $\widehat{B^n}$ denote the operation of analytically continuing an analytic function on $\widetilde{\mathrm{Conf}_2(\mathbb{C}^\times)}$ along the lift $\widetilde{B^n} \subset \widetilde{\mathrm{Conf}_2(\mathbb{C}^\times)}$ of the path $B^n : [0,1] \to \mathrm{Conf}_2(\mathbb{C}^\times)$. By *IOA4)* for any n there exist *braiding matrices*

$$\mathsf{B}_n^{i_1(i_2i_3)i_4} \begin{bmatrix} \alpha_1 & j & \alpha_2 \\ \alpha_3 & k & \alpha_4 \end{bmatrix} \tag{A.120}$$

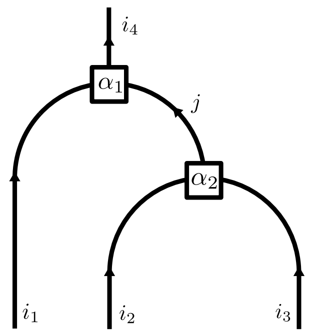
defined as

$$\begin{aligned}
&\widehat{B^n}\left(\Phi^{j}_{\alpha_1\alpha_2}(u_{i_2}, u_{i_1}, u_{i_3}, u'_{i_4}; z_2, z_1)\right) \\
&\quad = \sum_{k\in\mathsf{I}} \sum_{\alpha_3=1}^{N^{i_4}_{i_1k}} \sum_{\alpha_4=1}^{N^{k}_{i_2i_3}} \mathsf{B}_n^{i_1(i_2i_3)i_4} \begin{bmatrix} \alpha_1 & j & \alpha_2 \\ \alpha_3 & k & \alpha_4 \end{bmatrix} \Phi^{k}_{\alpha_3\alpha_4}(u_{i_1}, u_{i_2}, u_{i_3}, u'_{i_4}; z_1, z_2)
\end{aligned} \tag{A.121}$$

The graphical representation of braiding matrices reads



$$= \sum_{j\in\mathsf{I}} \sum_{\alpha_3=1}^{N^{i_4}_{i_1j}} \sum_{\alpha_4=1}^{N^{j}_{i_2i_3}} \mathsf{B}_n^{i_1(i_2i_3)i_4} \begin{bmatrix} \alpha_3 & k & \alpha_4 \\ \alpha_1 & j & \alpha_2 \end{bmatrix}$$



.

The stage is almost set for proving pentagon and hexagon equations for fusion and braiding matrices. Similar to the four point case we introduce the notation

$$\Phi^{kl}_{\alpha_1\alpha_2\alpha_3}(u_{i_1}, u_{i_2}, u_{i_3}, u_{i_1}, u_{i_5}; z_1, z_2, z_3) \tag{A.122}$$

for the single valued analytic continuation to $\widetilde{\mathrm{Conf}_3(\mathbb{C}^\times)}$ of

$$\left\langle u'_{i_5}, \mathcal{Y}^{i_5;1}_{i_1k,\alpha_1}(u_{i_1}, z_1)\mathcal{Y}^{k;2}_{i_2l;\alpha_2}(u_{i_2}, z_2)\mathcal{Y}^{l;3}_{i_3i_4;\alpha_3}(u_{i_3}, z_3)u_{i_4}\right\rangle \tag{A.123}$$

and

$$\Psi^{kl}_{\beta_1\beta_2\beta_3}(u_{i_1}, u_{i_2}, u_{i_3}, u_{i_4}, u'_{i_5}; z_1 - z_2, z_2 - z_3, z_3) \tag{A.124}$$

for the analytic continuation of pure iterates.

Proposition A.5.3. *[84, Proposition 1.2, 1.3]*

1) *Basis elements of four point functions are linearly independent. To be more precise the maps*

$$\begin{aligned} \Phi^j_{\alpha_1\alpha_2} : U'_{i_4} \otimes U_{i_1} \otimes U_{i_2} \otimes U_{i_3} &\to \mathrm{Hol}\left(\widetilde{\mathrm{Conf}_2(\mathbb{C}^\times)}\right) \\ u'_{i_4} \otimes u_{i_1} \otimes u_{i_2} \otimes u_{i_3} &\mapsto \Phi^j_{\alpha_1\alpha_2}(u_{i_1}, u_{i_2}, u_{i_3}, u'_{i_4}; z_1, z_2) \end{aligned} \tag{A.125}$$

are linearly independent for any $j \in \mathsf{I}$, $\alpha_1 = 1, \dots, N^{i_4}_{i_1 j}$, $\alpha_2 = 1, \dots, N^j_{i_2 i_3}$ and choice of basis. Analogously the maps

$$\begin{aligned} \Psi^k_{\beta_2\beta_1} : U'_{i_4} \otimes U_{i_1} \otimes U_{i_2} \otimes U_{i_3} &\to \mathrm{Hol}\left(\widetilde{\mathrm{Conf}_2(\mathbb{C}^\times)}\right) \\ u'_{i_4} \otimes u_{i_1} \otimes u_{i_2} \otimes u_{i_3} &\mapsto \Psi^k_{\beta_2\beta_1}(u_{i_1}, u_{i_2}, u_{i_3}, u'_{i_4}; z_1 - z_2, z_2) \end{aligned} \tag{A.126}$$

are linearly independent for any $k \in \mathsf{I}$, $\beta_1 = 1, \dots, N^{i_4}_{k i_3}$, $\beta_2 = 1, \dots, N^k_{i_1 i_2}$ and choice of basis.

2) *Basis elements of five point functions are linearly independent, e.g.*

$$\begin{aligned} \Phi^{kl}_{\alpha_1\alpha_2\alpha_3} : U'_{i_5} \otimes U_{i_1} \otimes U_{i_2} \otimes U_{i_3} \otimes U_{i_4} &\to \mathrm{Hol}\left(\widetilde{\mathrm{Conf}_3(\mathbb{C}^\times)}\right) \\ u'_{i_5} \otimes u_{i_1} \otimes u_{i_2} \otimes u_{i_3} \otimes u_{i_4} &\mapsto \Phi^{kl}_{\alpha_1\alpha_2\alpha_3}(u_{i_1}, u_{i_2}, u_{i_3}, u_{i_1}, u_{i_5}; z_1, z_2, z_3) \end{aligned} \tag{A.127}$$

are linearly independent for any $k, l \in \mathsf{I}$, $\alpha_1 = 1, \dots, N^{i_5}_{i_1 k}$, $\alpha_2 = 1, \dots, N^k_{i_2,l}$ and $\alpha_3 = 1, \dots, N^l_{i_3 i_4}$ and choice of basis. Statements for iterates of intertwining operators and combinations of iterates and products hold alike.

The proof of the proposition relies on the explicit presentation of $P(z)$-tensor products as $U_{i_1} \boxtimes_{P(z)} U_{i_2} \simeq \coprod_{j\in\mathsf{I}} \left(\mathcal{V}^j_{i_1 i_2}\right)^* \otimes U_j \simeq \coprod_{j\in\mathsf{I}} N^j_{i_1 i_2} U_j$, its universal property and lemma 2.0.11.

Corollary A.5.4. *The pentagon equation holds*

$$\sum_{s,t,r\in\mathsf{I}}\sum_{\beta_4=1}^{N^k_{ri_4}}\sum_{\beta_5=1}^{N^r_{i_2i_3}}\sum_{\gamma_7=1}^{N^s_{i_1r}}\mathsf{F}^{(i_1i_2i_3);s}\begin{bmatrix}\gamma_7 & r & \beta_5\\ \delta_8 & t & \delta_9\end{bmatrix}$$

$$\mathsf{F}^{(i_1ri_4);i_5}\begin{bmatrix}\alpha_1 & k & \beta_4\\ \gamma_6 & s & \gamma_7\end{bmatrix}\mathsf{F}^{(i_1i_2i_3);k}\begin{bmatrix}\alpha_2 & \ell & \alpha_3\\ \beta_4 & r & \beta_5\end{bmatrix} \tag{A.128}$$

$$=\sum_{p,q\in\mathsf{I}}\sum_{\epsilon_4=1}^{N^{i_5}_{p\ell}}\mathsf{F}^{(i_1i_2i_3);i_4}\begin{bmatrix}\epsilon_4 & \ell & \alpha_3\\ \gamma_6 & q & \delta_8\end{bmatrix}\mathsf{F}^{(i_1i_2\ell);i_5}\begin{bmatrix}\alpha_1 & k & \alpha_2\\ \epsilon_4 & p & \delta_9\end{bmatrix}$$

Proof. Since elements $\left\{\Phi^{kl}_{\alpha_1\alpha_2\alpha_3}\right\}$ and $\left\{\Psi^{ij}_{\beta_1\beta_2\beta_3}\right\}$ are all linearly independent, the above equation follows by applying fusion morphisms to $\Phi^{kl}_{\alpha_1\alpha_2\alpha_3}(u_{i_1},u_{i_2},u_{i_3},u_{i_1},u_{i_5};z_1,z_2,z_3)$ s.th. the result is a sum over elements $\Psi^{kl}_{\beta_1\beta_2\beta_3}(u_{i_1},u_{i_2},u_{i_3},u_{i_4},u'_{i_5};z_1-z_2,z_2-z_3,z_3)$ with coefficients exactly the left or right hand side of (A.128). Linear independence immediately gives the result. □

By the same arguments one derives the hexagon equation for B_1. Since we don't need the hexagon equation and writing it in full detail is very cumbersome we leave it to the enthusiastic reader to add the details on the hexagon equation.

A.6 Properties of genus one correlation functions

So far the discussion solely involved genus zero correlation functions. The treatment of genus 1 correlation functions follows a similar pattern. Though the technical details are different at some points, the line of reasoning is the same as in the genus zero case. One starts by gluing genus zero correlation functions to genus one correlation functions using traces. This defines analytic functions on certain domains in the complex plane. Again, they satisfy a nice partial differential equation and the general theory of holomorphic PDEs give that they have an analytic continuation to the configuration space of the torus. Since the functions are glued from genus zero correlation functions, associativity and commutativity properties hold and linear independence statements can be derived just as in the genus zero case. The only new feature is the action of the mapping class group of the torus on correlation functions. One needs to show that the space of multivalued analytic functions on a twice punctured torus obtained by taking traces of genus zero four point functions is closed under action of the mapping class group. This yields matrices corresponding to Dehn twists as well as the S-matrix. Finally one derives an equation relating S-matrix and braiding, which together with the pentagon equation will give the famous Verlinde formula. That the modular S-matrix diagonalizes fusion rules was first conjectured by Verlinde in [142]. A more rigorous treatment bases on highest weight representations of the Virasoro algebra and intertwining operators [4] was given in [121]. It took another 15 years and a

[4]Called "couplings" in [121].

series of papers mostly due to Huang and Lepowsky in order to arrive at a mathematical proof for all rational VOAs. The ideas of the proof still follow [121], the main steps were outlined above and we give the key results to the formula.

Again we mostly omit proofs for the statements, but point to the exact spot in the literature, where the reader can find proofs.

A.6.1 Genus one correlation functions

In order to properly lift genus zero correlation functions to the torus we have to discuss some basics of holomorphic coordinate transformations. This is due to the fact that intertwining operators live on the plane, but the gluing is best defined in terms of a cylinder. A thorough discussion of such transformations is given e.g. in [52, section 6], though technically it is only for transformations of infinitesimal or formal disks. The structure sheaf of a formal disk is just $\mathbb{C}[\![z]\!]$ where z is some chosen formal generator. Hence infinitesimal transformations act on formal power series. We will only need exponentials and logarithms, which are globally defined holomorphic functions and thus have power series expansions with infinite radius of convergence. Therefore we can expand the functions and apply the formulas derived in the infinitesimal setting to the respective power series.

Let $z = z_0 + \mathsf{i}z_1$ be a coordinate on a cylinder of radius one and infinite length, i.e. we make the identification $z \sim z + 2\pi\mathsf{i}$. Then $w = \rho(z) \equiv q_z = e^{2\pi\mathsf{i}z}$ is a coordinate for an annulus in the complex plane. Inserting an operator at a point z_i on the cylinder gives an operator insertion at q_{z_i} in the annulus. The standard local coordinate around z_i vanishing at z_i is given by $\xi = z - z_i$. Unfortunately q_ξ doesn't give a local coordinate around q_{z_i} vanishing at z_i. To remedy this, one considers

$$x \mapsto \frac{1}{2\pi i}\log(1+2\pi\mathsf{i}x) = \sum_{k=1}^{\infty}(2\pi\mathsf{i})^{k-1}\frac{x^k}{k} \tag{A.129}$$

for $x \in \mathbb{C}$. By comparing coefficients one finds $c_i \in \mathbb{C}$, $i > 0$ s.th.

$$e^{\sum_{i>0} c_i x^{i+1}\partial_x} x = \frac{1}{2\pi\mathsf{i}}\log(1+2\pi\mathsf{i}x) \quad . \tag{A.130}$$

Using the usual representation of the Virasoro algebra on holomorphic functions the lhs of (A.130) can be written as

$$e^{\mathbf{c}L_+}x, \qquad \text{with} \quad \mathbf{c}L_+ \equiv \sum_{i>0} c_i L_i \tag{A.131}$$

The inverse map to (A.129) is

$$y \mapsto \frac{1}{2\pi\mathsf{i}}\left(e^{2\pi\mathsf{i}y}-1\right) = e^{-\mathbf{c}L_+}y \quad . \tag{A.132}$$

A good local coordinate is now given by

$$\rho_{z_i}(x) = \rho(x+z_i) - \rho(z_i) = \frac{1}{2\pi\mathsf{i}}\left(e^{2\pi\mathsf{i}x}-1\right)q_{z_i}2\pi\mathsf{i} = e^{-\mathbf{c}L_+}q_{z_i}2\pi\mathsf{i}x = e^{-\mathbf{c}L_+}\left(q_{z_i}2\pi\mathsf{i}\right)^{L_0}x \quad . \tag{A.133}$$

This is the pullback of the local coordinate $\xi = z - z_i$ to the complex plane.

Definition A.6.1. Let V be a rational VOA and U_i a simple V-module. Recall that there exists $h_i \in \mathbb{C}$ s.th. $U_i = \coprod_{n\in\mathbb{N}} U_{h_i+n}$. Consider $f : U_i \to \overline{U}_i$, the *trace* of f is defined as

$$\operatorname{tr}(f) = \sum_{n\in\mathbb{N}} \operatorname{tr}_{U_{h_i+n}}(f) = \sum_{n\in\mathbb{N}} \sum_{i=1}^{\dim U_{h_i+n}} \left\langle (e_i^n)', f e_i^n \right\rangle \tag{A.134}$$

where $\{e_i^n\}$ is some basis of U_{h_i+n}.

The reader should take this definition with a grain of salt, since for arbitrary maps f, there is no reason why its trace should exist. The sum (A.134) is in general infinite, making it somewhat ill defined. However, we only need specific traces over intertwining operators, for which the definition turns out to be sensible (see Theorem A.6.5). Hence the definition should be read as a rule of computation, displayed once without having to deal with national overloaded formulas for intertwining operators.

Definition A.6.2. Let V be a rational VOA and $\mathcal{Y}_i \in \mathcal{V}_{M_i P_i}^{P_{i-1}}$ for $i = 1, \ldots, N$, where M_i, P_i are V-modules and $P_0 = P_N$. Let $\phi^1_{\mathcal{Y}_1,\ldots,\mathcal{Y}_N}(\bullet; z_1, \cdots, z_N; q)$ be defined as

$$\begin{aligned} m_1 \otimes \cdots \otimes m_N \mapsto & \phi^1_{\mathcal{Y}_1,\ldots,\mathcal{Y}_N}(m_1, \cdots, m_N; z_1, \cdots, z_N; q) \\ & \equiv \operatorname{tr}_{P_0} \left(\mathcal{Y}_1(\rho_{z_1} m_1, q_{z_1}) \cdots \mathcal{Y}_N(\rho_{z_N} m_N, q_{z_N}) q^{L_0 - \frac{c}{24}} \right) \end{aligned} \tag{A.135}$$

Lemma A.6.3. *Let $\mathbb{C}_{1>\cdots>N} = \{(z_1, \ldots, z_N) \in \mathbb{C}^\times \,|\, |z_1| > \cdots > |z_N| > 0\}$ and $A_{z_1>\cdots>z_N}$ be the space of analytic functions on $\mathbb{C}_{1>\cdots>N}$. Then*

$$\phi^1_{\mathcal{Y}_1,\ldots,\mathcal{Y}_N}(m_1, \cdots, m_N; z_1, \cdots, z_N; q) \in A_{q_{z_1}>\cdots>q_{z_N}}((q)) \tag{A.136}$$

for all $m_1, \cdots, m_N$.

Proof. Due to grading restriction it holds $\rho_{z_i} m_i \in M_i$. Hence the lemma immediately follows from the convergence property of products of intertwining operators at fixed degree. □

In order to show that $\Phi^1_{\mathcal{Y}_1,\cdots,\mathcal{Y}_N}$ is an absolutely convergent function and can be analytically extended to the whole torus it is shown in [82] that it satisfies a suitable differential equation. Before giving the differential equation, let's make some comments of what to expect from it. As described in [52] conformal blocks for a rational VOA should correspond to horizontal sections of a holomorphic vector bundle with (projectively) flat connection over the moduli space of Riemann surfaces. Thus the differential equation for genus 1 can be interpreted as the local expression of a (projectively) flat connection. Furthermore being a vector bundle over moduli space, the space of solutions should be invariant under the action of the mapping class group, hence for the torus, solution space is preserved under modular transformations. Therefore one might expect modular forms to play a significant role and this is exactly what happens.

For $k \in \mathbb{N}$ let $G_{2k+2}(\tau)$ be the k-th Eisenstein series. For $k = 0$ this is a pseudo modular form, whereas for $k \geq 1$ it is a modular form of weight $2k+2$, i.e. for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \in \mathrm{SL}(2, \mathbb{Z}) \tag{A.137}$$

it holds

$$G_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 G_2(\tau) - 2\pi \mathrm{i} c(c\tau + d) \tag{A.138}$$

and

$$G_{2k+2}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{2k+2} G_{2k+1}(\tau) \tag{A.139}$$

In addition, let $\wp_m(z; \tau)$ be the m-th Weierstrass zeta function, which is a modular form of weight m

$$\wp_m\left(\frac{z}{c\tau + d}; \frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^m \wp(z; \tau) \tag{A.140}$$

Both type of functions can be expressed in terms of q_τ instead of τ and we don't distinguish between the two presentations. The natural analog of (A.102) is the algebra

$$M = \mathbb{C}\left[G_4(q_\tau), G_6(q_\tau), \wp_2(z_i - z_j; q_\tau), \wp_3(z_i - z_j; q_\tau)\right]_{i<j=1,\ldots,N} \quad . \tag{A.141}$$

M has an obvious grading in terms of modular weight. Lastly there is following differential operator

$$D_j(w) \equiv -4\pi q_\tau \partial_{q_\tau} + wG_2(q_\tau) + G_2(q_\tau)\sum_{i=1}^{N} z_i \partial_{z_i} - \sum_{i \neq j} \wp_1(z_i - z_j; q_\tau)\partial_{z_i} \tag{A.142}$$

for $w \in \mathbb{C}$ and

$$\mathbf{D}_j^s(\mathbf{w}) \equiv \prod_{k=1}^{s} D_j(w_k), \quad \forall s \geq 1 \tag{A.143}$$

Note that elements of M are in particular analytic functions in $(z_1, \ldots, z_n; \tau)$. The main technical result about traces of products of intertwining operators now reads

Theorem A.6.4. *[82, Theorem 3.9] For V a rational VOA, $\mathcal{Y}_i \in \mathcal{V}_{M_i P_i}^{P_{i-1}}$ for $i = 1, \ldots, N$, where M_i, P_i are V-modules and $P_0 = P_N$ and $m_1, \cdots, m_N$ homogeneous there exist functions*

$$C_{m,j} \in M \quad K_{m,j} \in M, \qquad m = 1, \ldots, L,\ j = 1, \ldots, N \tag{A.144}$$

where $C_{m,j}$ has modular weight m and $K_{m,j}$ has modular weight $2m$, s.th.

$$\left(\partial_{z_j}^L + \sum_{m=1}^{L} C_{m,j}\partial_{z_j}^{L-m}\right)\phi^1_{\mathcal{Y}_1,\ldots,\mathcal{Y}_N}(m_1, \cdots, m_N; z_1, \cdots, z_N; q_\tau) = 0 \tag{A.145}$$

and

$$\left(\mathbf{D}_i^L(\mathbf{a}_1)+\sum_{m=1}^{L}K_{m,j}\mathbf{D}_i^{L-m}(\mathbf{a}_2)\right)\phi^1_{\mathcal{Y}_1,\ldots,\mathcal{Y}_N}(m_1,\cdots,m_N;z_1,\cdots,z_N;q_\tau)=0 \quad . \tag{A.146}$$

holds for $1>|q_{z_1}|>\cdots>|q_{z_N}|>|q_\tau|>0$. *In* (A.146) *we abbreviated* $a_{1,j}=\sum_{i=1}^N|m_i|+2(L-j)$ *and* $a_{2,j}=\sum_{i=1}^N|m_i|+2(L-m-j)$.

The differential equation looks fairly complicated. Its full technical derivation is quite involved but one can make some vague comments on its form, especially on the appearance of the differential operators (A.142). As noted before, conformal blocks are horizontal sections of a projectively flat vector bundle over the moduli space of Riemann surfaces. This point of view was pioneered by [56]. The connection on the conformal block bundle is related to the action of the energy momentum tensor (see e.g. [51, section 7],[52, chapter 17][68, equation 3.6.11]) and therefore to the action of the Virasoro algebra. The differential operators (A.142) are closely related to the Virasoro action on q-traces (A.135) as shown in [82, Lemma 3.8]. This at least slightly motivates the form of the differential equations (A.145),(A.146), a detailed derivation of the differential equations using the general theory of conformal block should be possible using the results of [68, section 5.5].

Two important statements follow from theorem A.6.4. The first addresses analytic continuations and the second the action of the modular group.

Theorem A.6.5. *[82, Theorem 4.1] For* $1>|q_{z_1}|>\cdots>|q_{z_N}|>|q_\tau|>0$*, q-traces* $\phi^1_{\mathcal{Y}_1,\ldots,\mathcal{Y}_N}(m_1,\cdots,m_N;z_1,\cdots,z_N;q)$ *are absolutely convergent. In addition they can be analytically continued to*

$$T_N\equiv\{(z_1,\ldots,z_N;\tau)\subset\mathbb{C}^n\times\mathbb{H}\,|\,z_i\neq z_j+p\tau+q,\ p,q\in\mathbb{Z}\} \tag{A.147}$$

The analytically continued single valued continuation of $\phi^1_{\mathcal{Y}_1,\ldots,\mathcal{Y}_N}(m_1,\cdots,m_N;z_1,\cdots,z_N;q)$ *on the universal cover* $\widetilde{T_N}$ *will be denoted as*

$$\Phi^1_{\mathcal{Y}_1,\ldots,\mathcal{Y}_N}(m_1,\cdots,m_N;z_1,\cdots,z_N;q) \quad . \tag{A.148}$$

Again single valuedness requires choosing a fixed branch for $\phi^1_{\mathcal{Y}_1,\ldots,\mathcal{Y}_N}(m_1,\cdots,m_N;z_1,\cdots,z_N;q)$ on the region $1>|q_{z_1}|>\cdots>|q_{z_N}|>|q_\tau|>0\subset\mathbb{C}^N\times\mathbb{H}$. The theorem immediately follows from the analyticity of the differential equations (A.145),(A.146). For modular invariance we introduce

$$\begin{aligned}G^1_N=\Big\{f\in\mathrm{Hol}\left(\widetilde{T_N}\right)\,|\,\exists\mathcal{Y}_1,\cdots,\mathcal{Y}_N,\ m_1,\cdots,m_N\ \text{s.th.}\\ f(z_1,\cdots,f_N;q)=\Phi^1_{\mathcal{Y}_1,\ldots,\mathcal{Y}_N}(m_1,\cdots,m_N;z_1,\cdots,z_N;q)\Big\}\ .\end{aligned} \tag{A.149}$$

Theorem A.6.6. *[82, Theorem 7.3] The space* G^1_N *is invariant under modular transformations, that is for*

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\quad\in\mathrm{SL}(2,\mathbb{Z}) \tag{A.150}$$

denote $\frac{a\tau+b}{c\tau+d} = \tau'$, *then it holds*

$$\Phi^1_{\mathcal{Y}_1,\ldots,\mathcal{Y}_N}\left(\left(\frac{1}{c\tau+d}\right)^{L_0} m_1,\cdots,\left(\frac{1}{c\tau+d}\right)^{L_0} m_N; \frac{z_1}{c\tau+d},\cdots,\frac{z_N}{c\tau+d}; q_{\tau'}\right) \in G^1_N \quad \text{(A.151)}$$

This statement requires considerably more work to prove and the proof relies on certain properties of Zhu's algebra. Genus one correlation functions satisfy associativity and commutativity properties, which due to the explicit form in terms of q-traces directly follow from their genus zero counterparts.

Theorem A.6.7. *[82, Theorem 4.2, 4.3] In the situation of definition A.6.2, for any* $i = 1,\ldots,N$ *there exist* $\widetilde{\mathcal{Y}}_{i+1} \in \mathcal{V}^{P_{i-1}}_{M_{i+1}iP_i}$ *and* $\widetilde{\mathcal{Y}}_i \in \mathcal{V}^{P_i}_{M_iP_{i+1}}$ *s.th.*

$$\begin{aligned}&\Phi^1_{\mathcal{Y}_1,\ldots,\mathcal{Y}_N}(m_1,\cdots,m_N;z_1,\cdots,z_N;q)\\ &\qquad = \Phi^1_{\mathcal{Y}_1,\ldots,\mathcal{Y}_{i-1},\widetilde{\mathcal{Y}}_{i+1},\widetilde{\mathcal{Y}}_i,\ldots,\mathcal{Y}_N}(m_1,\ldots,m_{i+1},m_i,\ldots,m_N;z_1,\cdots,z_N;q) \quad .\end{aligned} \quad \text{(A.152)}$$

Furthermore, for any $i = 1,\ldots,N$, *there exists a* V*-module* L_i *and* $\mathcal{Y}'_i \in \mathcal{V}^{L_i}_{M_iM_{i+1}}$, $\mathcal{Y}'_{i+1} \in \mathcal{V}^{P_{i-1}}_{L_iP_{i+1}}$ *s.th.*

$$\begin{aligned}&\Psi^1_{\mathcal{Y}_1,\ldots,\mathcal{Y}'_i,\mathcal{Y}'_{i+1},\ldots,\mathcal{Y}_N}(m_1,\ldots,m_N;z_1,\ldots,z_N;q)\\ &\equiv\Phi^1_{\mathcal{Y}_1,\ldots,\mathcal{Y}_{i-1},\mathcal{Y}'_{i+1},\ldots\mathcal{Y}_N}(m_1,\cdots,\mathcal{Y}'_i(m_i,z_i-z_{i+1})m_{i+1},\ldots,m_N;z_1,\cdots,z_{i-1},z_{i+1},\ldots,z_N;q)\end{aligned} \quad \text{(A.153)}$$

is absolutely convergent on

$$\left\{1 > |q_{z_1}| > \cdots > |q_{z_{i-1}}| > |q_{z_{i+1}}| > \cdots > |q| > 0\right\} \cap \left\{|q_{z_{i+1}}| > |q_{z_i} - q_{z_{i+1}}| > 0\right\} . \quad \text{(A.154)}$$

On $\left\{1 > |q_{z_1}| > \cdots > |q_{z_{i-1}}| > |q_{z_i}| > |q_{z_{i+1}}| > \cdots > |q| > 0\right\} \cap \left\{|q_{z_{i+1}}| > |q_{z_i} - q_{z_{i+1}}| > 0\right\}$ *it agrees with*

$$\phi^1_{\mathcal{Y}_1,\ldots,\mathcal{Y}_N}(m_1,\cdots,m_N;z_1,\cdots,z_N;q) \quad . \quad \text{(A.155)}$$

The final result which we recall in this section is linear independence of genus one correlation functions for simple modules. This is needed to prove identities involving Dehn twists and S-transformations, as their matrix representations are coefficients in front of such functions.

Theorem A.6.8. *[84, Proposition 2.2] In the setup of definition A.6.2, let* $N = 2$ *and* $\left\{\mathcal{Y}^{i_3;A}_{i_1i_2;\alpha}\right\}_{\alpha=1,\ldots,N^{i_3}_{i_1i_2}}$ *be a basis in* $\mathcal{V}^{i_3}_{i_1i_2}$. *Then, q-traces seen as maps*

$$\Phi^1_{\mathcal{Y}^{i_4;1}_{i_1\ell;\alpha_1},\mathcal{Y}^{\ell;2}_{i_2i_4;\alpha_2}}(\bullet,\bullet;z_1,z_2;q) : U_{i_1} \otimes U_{i_2} \to G^1_N \quad \text{(A.156)}$$

are linearly independent for any $\ell \in \mathsf{I}$ *and* $\alpha_1 = 1,\ldots,N^{i_4}_{i_1\ell}$, $\alpha_2 = 1,\ldots,N^{\ell}_{i_2i_4}$.

An analogous statement holds for Ψ^1.

From A.6.8 one defines the *S-matrix* corresponding to the S-transformation.

Definition A.6.9. Let $U_0 = V$, then the S-matrix is defined by

$$\Phi^1_{\mathcal{Y}^i_{0i;1}}\left(v; -\frac{z}{\tau}; q_{-\frac{1}{\tau}}\right) = \sum_{j\in \mathsf{I}} S_{ij}\Phi^1_{\mathcal{Y}^j_{0j;1}}(v; z; q_\tau) \quad . \tag{A.157}$$

The same matrix appears when considering two point functions instead of one point functions.

Lemma A.6.10. *[84, section 2]*

$$S\left(\Psi^1_{\mathcal{Y}^j_{ej;1},\mathcal{Y}^0_{ii';1}}(m, n'; z_1, z_2; q_\tau)\right) = \sum_{\ell\in \mathsf{I}} S_{\ell j}\Psi^1_{\mathcal{Y}^\ell_{e\ell;1},\mathcal{Y}^0_{ii';1}}(m_i, v; z_1, z_2; q_\tau) \tag{A.158}$$

Proof. We compute

$$\begin{aligned}
&S\left(\Psi^1_{\mathcal{Y}^j_{0j;1},\mathcal{Y}^0_{ii'}}(m, n'; z_1, z_2; q_\tau)\right)\\
&\overset{(1)}{=}\mathrm{tr}_{M_j}\left(\mathcal{Y}^j_{0j;1}\left(\rho_{-\frac{z_1}{\tau}}\left(\mathcal{Y}^0_{ii';1}\left(\left(-\frac{1}{\tau}\right)^{L_0} m, -\frac{1}{\tau}(z_1 - z_2)\right)\left(-\frac{1}{\tau}\right)^{L_0} n', q_{-\frac{z_2}{\tau}}\right)\right) q_{-\frac{1}{\tau}}^{L_0-\frac{c}{24}}\right)\\
&\overset{(2)}{=}\mathrm{tr}_{M_j}\left(\mathcal{Y}^j_{0j;1}\left(\rho_{-\frac{z_1}{\tau}}\left(-\frac{1}{\tau}\right)^{L_0}\left(\mathcal{Y}^0_{ii';1}\left(m, (z_1 - z_2)\right) n', q_{-\frac{z_2}{\tau}}\right) q_{-\frac{1}{\tau}}^{L_0-\frac{c}{24}}\right)\right)\\
&\overset{(3)}{=}\Phi^1_{\mathcal{Y}^j_{0j;1}}\left(\left(-\frac{1}{\tau}\right)^{L_0}\mathcal{Y}^0_{ii';1}(m; z_1 - z_2)n'; -\frac{z_2}{\tau}; q_{-\frac{1}{\tau}}\right)\\
&\overset{(4)}{=}\sum_{\ell\in \mathsf{I}} S_{j\ell}\Phi^1_{\mathcal{Y}^\ell_{0\ell;1}}\left(\mathcal{Y}^0_{ii';1}(m; z_1 - z_2)n'; z_2; q_\tau\right)\\
&\overset{(5)}{=}\sum_{\ell\in \mathsf{I}} S_{j\ell}\Psi^1_{\mathcal{Y}^\ell_{e\ell;1},\mathcal{Y}^0_{ii';1}}(m, n'; z_1, z_2; q_\tau)
\end{aligned} \tag{A.159}$$

In (1) the expansion of Ψ^1 in the suitable domain is inserted, (2) is the $\mathrm{SL}(2,\mathbb{C})$-action (A.41) for intertwining operators. In (3) we used the definition of Φ^1 and (4) is (A.157). Finally, (5) is again just the definition of Ψ^1. □

Let $R : \mathcal{V}^k_{ij} \to \mathcal{V}^{j'}_{k'i}$ be the isomorphism[5]

$$R(\mathcal{Y}) = e^{\pi \mathrm{i}(h_{i_3} - h_{i_2})}\mathcal{B}_0 \circ A_0(\mathcal{Y}) \quad . \tag{A.160}$$

The next step in order to prove the Verlinde formula is to analyze how two point functions behave under Dehn twists around generators of middle homology, i.e. transporting insertions around a and b-cycles of the torus.

[5]The letter R is not by accident. In the graphical representation, after rescaling, R rotates the trivalent vertex corresponding to $\mathcal{Y}$ by π.

Theorem A.6.11. *[84, Proposition 4.3] Let*

$$\begin{aligned}\mathsf{a}\left(\Psi^1_{\mathcal{Y}^j_{0j;1},\mathcal{Y}^0_{ii'}}(m_i,v;z_1,z_2;q_\tau)\right) &= P_0\left[\Psi^1_{\mathcal{Y}^j_{0j;1},\mathcal{Y}^0_{ii'}}(m_i,v;z_1,z_2-1;q_\tau)\right]\\ \mathsf{b}\left(\Psi^1_{\mathcal{Y}^j_{0j;1},\mathcal{Y}^0_{ii'}}(m_i,v;z_1,z_2;q_\tau)\right) &= P_0\left[\Psi^1_{\mathcal{Y}^j_{0j;1},\mathcal{Y}^0_{ii'}}(m_i,v;z_1,z_2+\tau;q_\tau)\right]\end{aligned} \tag{A.161}$$

where $P_0 : G^1_2 \to G^1_{2;0}$ is the projection to correlation functions, with intermediate propagating representation U_0. Then it holds

$$\mathsf{a}\left(\Psi^1_{\mathcal{Y}^j_{0j;1},\mathcal{Y}^0_{ii'}}(m_i,v;z_1,z_2;q_\tau)\right) = e^{-2\pi \mathrm{i} h_i}\left(\mathsf{B}^{(ji'i)j}_{-1}\begin{bmatrix}1 & 0 & 1\\ 1 & 0 & 1\end{bmatrix}\right)^2 \Psi^1_{\mathcal{Y}^j_{0j;1},\mathcal{Y}^0_{ii'}}(m_i,v;z_1,z_2;q_\tau) \tag{A.162}$$

$$\begin{aligned}\mathsf{b}\left(\Psi^1_{\mathcal{Y}^j_{0j;1},\mathcal{Y}^0_{ii';1}}(m_i,v;z_1,z_2;q_\tau)\right) = e^{-2\pi \mathrm{i} h_i}\sum_{\ell\in \mathsf{I}}\sum_{\alpha_1=1}^{N^\ell_{i'j}}\sum_{\alpha_1=1}^{N^j_{i\ell}} \mathsf{F}^{(ijj')j}\begin{bmatrix}1 & 0 & 1\\ \alpha_1 & \ell & \alpha_2\end{bmatrix}\\ \mathsf{F}^{(i\ell'\ell)i}\begin{bmatrix}\alpha_1 & j & R(\alpha_2)\\ \beta_1 & 0 & \beta_2\end{bmatrix}\Psi^1_{\mathcal{Y}^\ell_{0\ell;1},\mathcal{Y}^0_{ii';1}}(m_i,v;z_1,z_2;q_\tau)\end{aligned} \tag{A.163}$$

As noted in footnote 5, a rescaled version of R is related to a pivotal structure. The final step for the Verlinde Formula is a fact from topology. It is well known that the S-transformation interchanges a and b-cycle of a torus. Since genus one correlation functions are a representation of the mapping class group of the torus, a similar relation should hold among a, b and S-actions.

Theorem A.6.12. *[84, Proposition 4.5]*

$$S\circ \mathsf{a} = \mathsf{b}\circ S \tag{A.164}$$

as maps on $G^1_{2,0}$.

The proof is very similar to the proof of lemma A.6.10, using once the rescaling property of intertwining operators. Putting all things together, the final statement of the whole section is the next theorem.

Theorem A.6.13. *[84, Corollary 4.7] For all $i,j,k\in\mathsf{I}$ it holds*

$$\sum_{\ell\in\mathsf{I}} S_{i\ell}\left(\mathsf{B}^{(\ell j'j)\ell}_{-1}\right)^2\begin{bmatrix}1 & 0 & 1\\ 1 & 0 & 1\end{bmatrix} S^{-1}_{\ell k} = N^k_{ij}\,\mathsf{F}^{(jj'j)j}\begin{bmatrix}1 & 0 & 1\\ 1 & 0 & 1\end{bmatrix} \tag{A.165}$$

The theorem follows from (A.164) and the pentagon identity, which is used to show that the rhs of (A.163) modulo the twist equals the rhs of (A.165) (see [84, section 3]). This is almost the Verlinde formula. Note that $\mathsf{F}^{(jj'j)j}\begin{bmatrix}1 & 0 & 1\\ 1 & 0 & 1\end{bmatrix}\neq 0$, as otherwise the lhs of

(A.165) vanishes. But for $\ell = 0$ the braiding matrix in (A.165) is derived from correlation function

$$\langle m_k', Y_V(u; z_1) Y_{U_k}(v, z_2) n_k \rangle \tag{A.166}$$

where $n_k, m_k \in U_k$ and $u, v \in V$. Since both, the module map and the vertex operator are power series expansions, they are independent of the paths in the definition of the braiding matrix. Hence the braiding matrix in this case is constant one, giving a contradiction.

Corollary A.6.14.

$$N_{ij}^k = \sum_{\ell \in \mathsf{I}} \frac{S_{i\ell} S_{j\ell} S_{k\ell}^{-1}}{S_{0\ell}} \tag{A.167}$$

Proof. Let

$$\Upsilon_{kj} = \frac{\left(\mathsf{B}_{-1}^{(kj'j)k}\right)^2 \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}}{\mathsf{F}^{(jj'j)j} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}} \tag{A.168}$$

then (A.165) can be written as

$$\sum_{\ell \in \mathsf{I}} S_{i\ell} \Upsilon_{\ell j} S_{\ell k}^{-1} = N_{ij}^k \quad . \tag{A.169}$$

Since U_i, U_j and U_k are simple modules it holds $N_{0j}^k = \delta_{jk}$. This yields

$$S_{0m} \Upsilon_{mj} = \sum_{k \in \mathsf{I}} \delta_{jk} S_{km} \quad \Leftrightarrow \quad \Upsilon_{mj} = \frac{S_{jm}}{S_{0m}} \quad . \tag{A.170}$$

Inserting (A.170) in (A.169) gives (A.167). □

Appendix B

Category Theory Basics

In order to be as self contained as possible we recall some definitions and facts about categories. There is a plethora of literature on category theory. A classic source is e.g. [111]. Textbook accounts more oriented towards applications in this thesis can be found in [45] and [5].

A (small) *category* C consists of a set of objects $Ob(\mathsf{C})$ and for any two objects c, d a set of morphisms $\mathrm{Hom}_{\mathsf{C}}(c, d)$ together with a composition map

$$\circ : \mathrm{Hom}_{\mathsf{C}}(c,d) \times \mathrm{Hom}_{\mathsf{C}}(d,e) \to \mathrm{Hom}_{\mathsf{C}}(c,e) \quad . \tag{B.1}$$

A category is called $\mathbb{C}$-*linear* if $\mathrm{Hom}_{\mathsf{C}}(c, d)$ is a $\mathbb{C}$-vector space for any two objects c, d in C.

We will typically write $c \in \mathsf{C}$ for $c \in Ob(\mathsf{C})$. Let C, D be categories. A *functor* $F : \mathsf{C} \to \mathsf{D}$ is a map $Ob(\mathsf{C}) \to Ob(\mathsf{D})$ and $F : \mathrm{Hom}_{\mathsf{C}}(c, d) \to \mathrm{Hom}_{\mathsf{D}}(F(c), F(d))$ satisfying

$$F(\mathrm{id}_{\mathsf{C}}) = \mathrm{id}_{\mathsf{D}}, \qquad F(f \circ g) = F(f) \circ F(g) \quad . \tag{B.2}$$

Given two functors $F, G : \mathsf{C} \to \mathsf{D}$ a *natural transformation* $\eta : F \to G$ is a collection of maps $\eta_c : F(c) \to G(c)$ for any $c \in \mathsf{C}$ with commuting diagrams

$$\begin{tikzcd} F(c) \arrow[r, "F(f)"] \arrow[d, "\eta_c"'] & F(d) \arrow[d, "\eta_d"] \\ G(c) \arrow[r, "G(f)"] & G(d) \end{tikzcd}$$

for any $c, d \in \mathsf{C}$ and $f \in \mathrm{Hom}_{\mathsf{C}}(c, d)$. If η_c is an isomorphism for any $c \in \mathsf{C}$, η is a *natural isomorphism*. Given two functors

$$L : \mathsf{C} \rightleftharpoons \mathsf{D} : R \tag{B.3}$$

then L is *left adjoint* to R if there is a natural isomorphism

$$\mathrm{Hom}_{\mathsf{D}}(L(\bullet), \bullet) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{C}}(\bullet, R(\bullet)) \tag{B.4}$$

in which case R is *right adjoint* to L. Two categories C, D are *equivalent* if there is pair of adjoint functors $L : \mathsf{C} \rightleftharpoons \mathsf{D} : R$ and both functors are *fully faithful* meaning that

$$\mathrm{Hom}_{\mathsf{C}}(\bullet, \bullet) \to \mathrm{Hom}_{\mathsf{D}}(L(\bullet), L(\bullet)) \tag{B.5}$$

is a bijection of sets. A *monoidal category* is a category C together with a bifunctor $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ an object $\mathbf{1}$ and natural transformations $\mathscr{A} : \otimes \circ (\mathrm{id} \times \otimes) \to \otimes(\otimes \times \mathrm{id})$, λ, ρ satisfying

$$\rho : c \otimes \mathbf{1} \simeq c, \qquad \lambda : \mathbf{1} \otimes c \simeq c \tag{B.6}$$

and the usual pentagon and hexagon diagrams relating all different ways of bracketing tensor products of four elements commute.

The prototypical example of a monoidal category is of course the category of vector spaces or more general the category of R-modules for some ring R. A monoidal category is *strict* if associativity and unit morphisms are identities. Any monoidal category is equivalent to a strict monoidal category, thus when dealing with purely categorical terms we assume monoidal categories to be strict. In the graphical calculus the tensor product of two elements is displayed by placing them next to each other. A monoidal category is *braided* if there is a natural isomorphism $\beta_{A,B} : A \otimes B \to B \otimes A$. In pictures the braiding reads

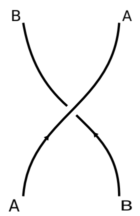


Figure B.1: $\beta_{A,B}$

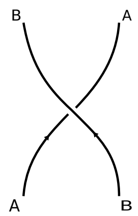
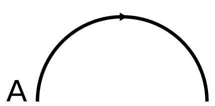
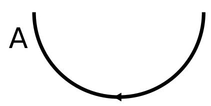


Figure B.2: $\beta^{-1}_{A,B}$

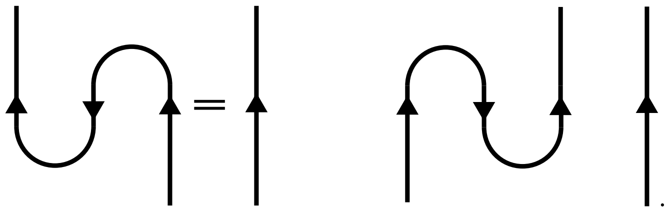
.

Following the example of vector spaces one can abstract the notion of dual objects to an arbitrary category. An object A has a *right dual* A^* if there are morphisms $\mathrm{ev}_A : A \otimes A^* \to \mathbf{1}$, $\mathrm{coev}_A : \mathbf{1} \to A^* \otimes A$ graphically given by





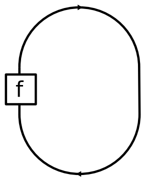
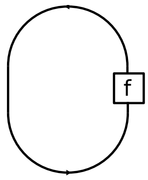
They have to satisfy the straightening properties



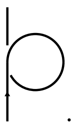
Similarly one defines a *left dual* for A as an object ${}^*A$ with evaluation $\widetilde{\mathrm{ev}}_A : {}^*A \otimes A \to \mathbf{1}$ and coevaluation morphisms $\widetilde{\mathrm{coev}}_A : \mathbf{1} \to A \otimes {}^*A$. Here the graphical calculus are the obvious cup and cap running in the other direction.

A category in which every object has a left and right dual is called *rigid*. Right and left dual objects in fact don't need to be the same or even isomorphic. However, given that C has a natural isomorphism $\pi : \mathrm{id}_{\mathsf{C}} \Rightarrow (\bullet)^{**}$ left and right dual objects are easily shown to be isomorphic. The map π is called *pivotal structure*[1]. It turns out that every category with a pivotal structure is equivalent to a category with strict pivotal structure, i.e. a pivotal structure whose component morphisms are just the identity. Thus we will assume that all our categories are strictly pivotal.

So far we equipped C with a (strict) monoidal structure, a braiding, a rigid structure and a strict pivotal structure. This allows us to define left and right traces for $f \in \mathrm{Hom}_{\mathsf{C}}(A, A)$ by

$tr_r(f) =$  $tr_l(f) =$ 

If $tr_r(\bullet) = tr_l(\bullet)$ the category is called *spherical*. The last bit of structure is a twist, i.e. a natural isomorphism $\theta : \mathrm{id}_{\mathsf{C}} \Rightarrow \mathrm{id}_{\mathsf{C}}$ satisfying $\theta_{A\otimes B} = \theta_A \otimes \theta_B \circ \beta_{B,A} \circ \beta_{A,B}$. Graphically the twist is represented by



A spherical category with a twist is called *ribbon*. The name is due to the fact, that in a ribbon category one might think of the lines in the graphical calculus as thickened to ribbons. The twist then corresponds to twisting the ribbon once around its core.

The next requirements concern the size of the category. Heuristically we want it as finite as possible while still being interesting. First of all C should be enhanced in Vect, the category of finite dimensional $\mathbb{C}$-vector spaces. Any Hom-space therefore is a finite dimensional vector space and the monoidal product is bilinear. Next an object A is called

[1] The name stems from the fact, that when employing the graphical calculus the pivotal structure rotates coupons by π.

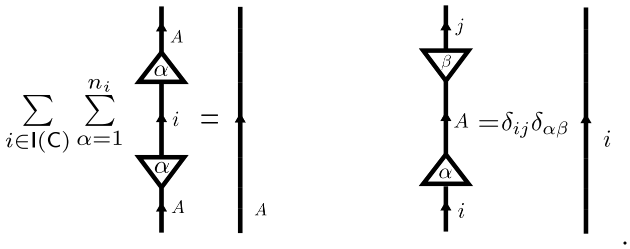
simple if $\mathrm{Hom}_{\mathsf{C}}(A,A) \simeq \mathbb{C}$. In Vect the only isomorphism class of simple objects is $[\mathbb{C}]$. Finally, the category is *semi-simple* if any object decomposes as a finite direct sum of simple objects. Even better it is *fusion* if the monoidal unit is simple. In a spherical fusion category an object has a number attached to it, its *dimension* $d_A = tr(\mathrm{id}_A) \in \mathrm{Hom}_{\mathsf{C}}(\mathbf{1},\mathbf{1})$. Typically we will denote the set of isomorphism classes of simple objects as $\mathsf{I}(\mathsf{C})$ or I if the category in question is clear from the context. Representatives of such an isomorphism class are usually denoted by U_i. Note that for a simple object its dual is also simple. For a fusion category there exist decomposition maps for an object A

$$b_i^\alpha : A \to U_i, \quad b_\alpha^i : U_i \to A, \quad \alpha = 1, \cdots, \dim_{\mathbb{C}}\left(\mathrm{Hom}_{\mathsf{C}}(A, U_i)\right) = n_i \tag{B.7}$$

which are dual to each other in the sense that

$$\sum_{i\in\mathsf{I}} \sum_{\alpha} b_\alpha^i \circ b_i^\alpha = \mathrm{id}_A, \quad b_i^\alpha \circ b_\beta^j = \delta_{ij}\delta_{\alpha\beta}\mathrm{id}_{U_j}. \tag{B.8}$$

In the graphical calculus the duality identities read



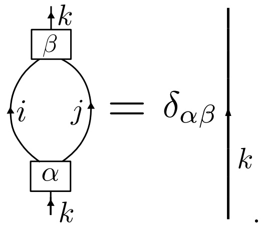
In addition in a semi-simple category simple objects have non-zero trace. The *global dimension* is the number

$$\mathsf{D}^2 = \sum_{i\in\mathsf{I}} d_i^2 \quad . \tag{B.9}$$

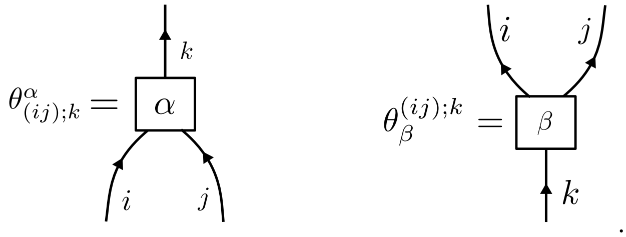
Furthermore we need the *fusion rules* $N_{ij}^k \equiv \dim_{\mathbb{C}}\left(\mathrm{Hom}_{\mathsf{C}}(U_i \otimes U_j, U_k)\right)$. The name is justified by taking the Grothendieck ring, i.e. the ring with elements isomorphism classes $[A]$ of objects. The sum is defined as $[A] + [B] = [A \oplus B]$ and the product is given by $[A] \cdot [B] = [A \otimes B]$. If C is the representation category of a rational VOA this recovers the ring of characters $\chi_A = [A]$ of representations, which has generators the characters of simple representations $\chi_i = [U_i]$ and fusion product

$$\chi_i \cdot \chi_k = \sum_{k\in\mathsf{I}} N_{ij}^k \chi_k \quad . \tag{B.10}$$

For $\mathrm{Hom}_{\mathsf{C}}(U_i \otimes U_j, U_k)$ we introduce a basis $\left\{\theta_{(ij),k}^{\alpha}\right\}_{\alpha=1,\cdots,N_{ij}^k}$ with dual basis $\left\{\theta_\beta^{k;(ij)}\right\}_{\beta=1,\cdots,N_{ij}^k}$, satisfying

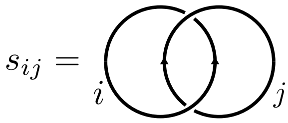


where



The final ingredient we want on our fusion, ribbon category is modularity.

Definition B.0.1. A fusion, ribbon category is *modular* if the $|I| \times |I|$-matrix s with entries



is invertible.

Again the name is derived from conformal field theory. For a representation category of a rational VOA, the matrix s is up to a normalization the modular S-matrix acting on the ring of characters.

Given two $\mathbb{C}$-linear categories C, D one can form their *Deligne tensor product* $\mathsf{C} \boxtimes \mathsf{D}$ which has objects given by formal direct sums $\bigoplus_i C_i \boxtimes D_i$, for objects $C_i \in \mathsf{C}$ and $D_i \in \mathsf{D}$. The morphism spaces are given by

$$\mathrm{Hom}_{\mathsf{C}\boxtimes\mathsf{D}}(C_1 \boxtimes D_1, C_2 \boxtimes D_2) = \mathrm{Hom}_{\mathsf{C}}(C_1, D_1) \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathsf{D}}(D_1, D_2) \tag{B.11}$$

linearly extended to formal sums. Given a modular tensor category (MTC) C, denote $\overline{\mathsf{C}}$ for the MTC with the same objects and hom-spaces as C but with reversed braiding and twist. The Deligne product $\mathsf{C} \boxtimes \overline{\mathsf{C}}$ can be seen as putting chiral and antichiral data together to describe full conformal field theory. Instead of the Deligne product we will work with the *Drinfeld center* $\mathsf{Z}(\mathsf{C})$ of a modular tensor category. For any monoidal category D, its Drinfeld center $\mathsf{Z}(\mathsf{D})$ has objects

$$(D, \beta_{D,\bullet}) \tag{B.12}$$

where $D \in \mathsf{D}$ and $\beta_{D,\bullet} : D \otimes \bullet \Rightarrow \bullet \otimes D$ is a natural isomorphism called *half-braiding.* Note that D itself doesn't need to have a braiding. In case it has a braiding, D is fully faithfully embedded into its center by $D \mapsto \beta_{D,\bullet}$. But in general an object of D can have many different embeddings given by different half-braidings. If C is semi-simple and spherical the Drinfeld center is a modular tensor category [122]. It was shown by Shimizu [136] that there is a braided equivalence

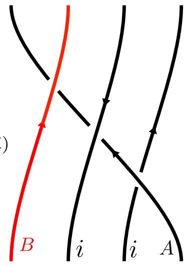
$$\mathsf{C} \boxtimes \overline{\mathsf{C}} \xrightarrow{\simeq} \mathsf{Z}(\mathsf{C}) \tag{B.13}$$

for C a MTC. Thus, in this thesis we work with the Drinfeld center instead of the Deligne product when describing closed sectors of conformal field theories. Of course there exists an obvious forgetful functor $F : \mathsf{Z(C)} \to \mathsf{C}$ mapping $(C, \beta_{C,\bullet}) \mapsto C$. It was shown in [107] that there exists a left and right adjoint functor to the forgetful functor

$$\begin{aligned} L : \mathsf{C} &\to \mathsf{Z(C)} \\ B &\mapsto \left(\bigoplus_{i \in \mathsf{I(C)}} (B \otimes U_i^*) \otimes U_i, \beta^{ou}_{L(B),\bullet} \right) \end{aligned} \tag{B.14}$$

with the over-under half-braiding

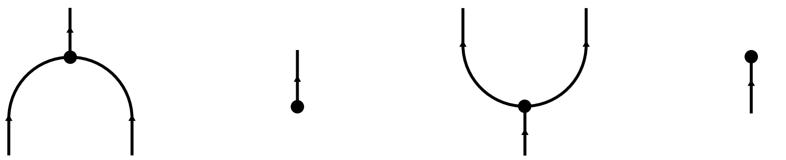
$$\beta_{L(B),A} = \bigoplus_{i \in \mathsf{I(C)}}$$



and its action on morphisms is given by $L(f) = f \otimes \mathrm{id}_{U_i^* \otimes U_i}$. The forgetful functor is easily seen to be a *strong monoidal functor*, i.e. there exist natural isomorphisms $\phi_{\bullet,\bullet} : F(\bullet \otimes \bullet) \Rightarrow F(\bullet) \otimes F(\bullet)$ and $\phi_{\mathbf{1}} : \mathbf{1}_{\mathsf{C}} \Rightarrow F(\mathbf{1}_{\mathsf{Z(C)}})$. Its adjoint functor on the other hand is just lax and colax monoidal, meaning that there exist similar natural transformations, which fail to be isomorphisms. However in [107] it was shown that L is a *Frobenius functor*, i.e. a functor preserving Frobenius algebra.

Lemma B.0.2. *[107, Lemma 2.25] If $F \in \mathsf{C}$ is a Frobenius algebra then $L(F) \in \mathsf{Z(C)}$ is Frobenius algebra. In a addition F is symmetric and special if and only if $L(F)$ is symmetric and special.*

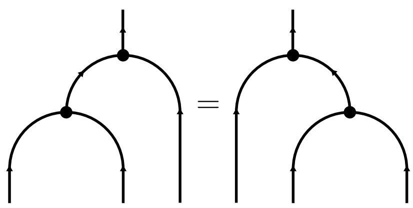
Recall that a *Frobenius algebra in a monoidal category* C is the data of an object $A \in \mathsf{C}$ with morphisms

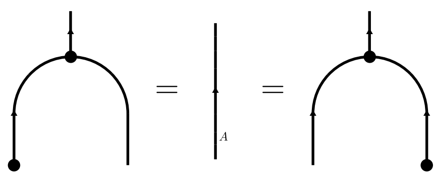


$m : A \otimes A \to A$ $\quad \eta : \mathbf{1} \to A$ $\quad \Delta : A \to A \otimes A$ $\quad \epsilon : A \to \mathbf{1}$

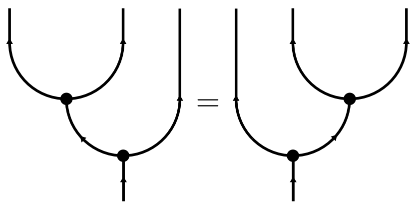
satisfying

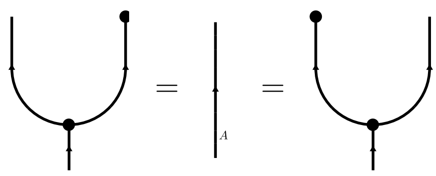
I) (m, η) define an associative algebra on A:



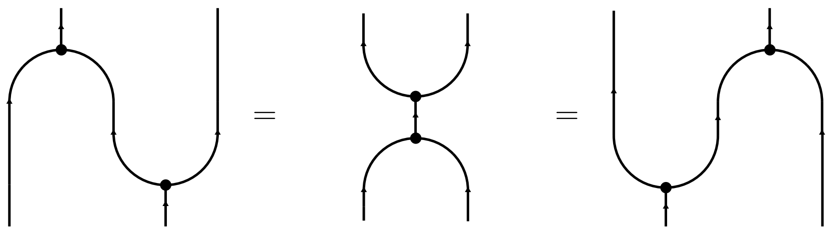


II) (Δ, ϵ) define a coassociative coalgebra on A:



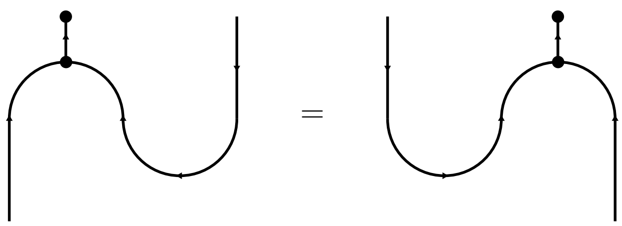


III) The *Frobenius properties* hold

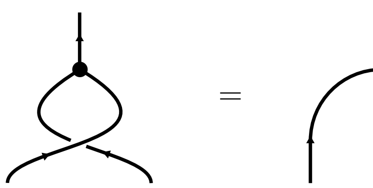


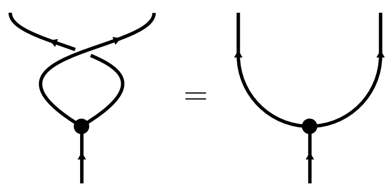
.

If A is in addition pivotal, we can ask for $(A, m, \eta, \Delta, \epsilon)$ to be *symmetric*, i.e. there is an equality of morphisms



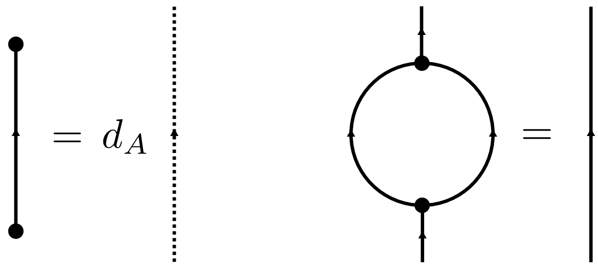
The above morphisms are easily seen to be isomorphisms $A \to A^*$. For C braided, we can require $(A, m, \eta, \Delta, \epsilon)$ to be (co-)commutative, i.e.





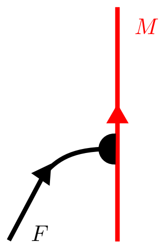
.

In addition, $(A, m, \eta, \Delta, \epsilon)$ a symmetric Frobenius algebra is called *special* if it holds



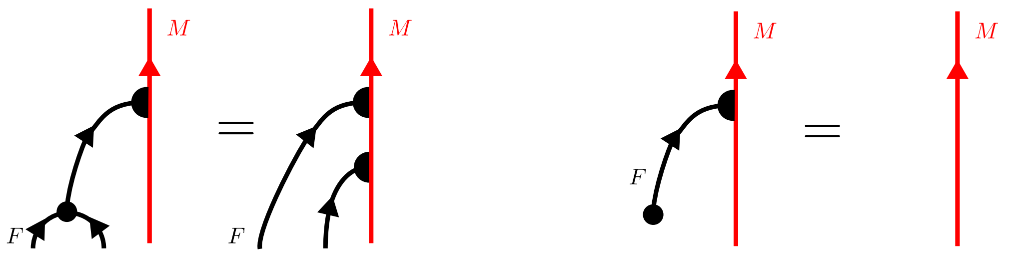
with dotted line in the first picture representing just the unit object.

Besides Frobenius algebras themselves, one can also have left-, right- and bimodule objects in C for a given Frobenius algebra F. A *left module* is an object $M \in \mathsf{C}$ with a morphism

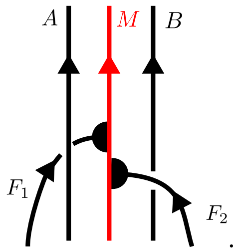


$$\rho_l : F \otimes M \to M$$

satisfying the obvious representation theory constraints



Right modules are define analogously and a bimodule is simply a left and right module s.th. left and right actions ρ_l, ρ_r commute. Note that modules are only required to be modules for the algebra structure of F. No consistency requirements for the coalgebra structure have to hold. Given two left F-modules M_1, M_2, the subspace of $\mathrm{Hom}_{\mathsf{C}}(M_1, M_2)$ intertwining the algebra action is denoted by $\mathrm{Hom}_F(M_1, M_2)$. For two Frobenius algebras F_1, F_2 there is the natural notion of a F_1-F_2-bimodule M, which has left F_1-action and right F_2-action. Spaces of morphisms between F_1-F_2-bimodules are denoted as $\mathrm{Hom}_{F_1|F_2}(M_1, M_2) \subset \mathrm{Hom}_{\mathsf{C}}(M_1, M_2)$. Given a F_1-F_2-bimodule M and objects A, B in C, the tensor product $A \otimes M \otimes B$ can be given the structure of a F_1-F_2-bimodule using the braiding. The left and right actions are defined to be



The object $A \otimes M \otimes B$ with this bimodule structure is denoted as $A \otimes^{+} M \otimes^{-} B$.

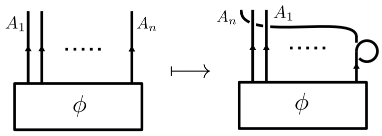
Lastly for the reader's convenience we recall some special morphism spaces used in the description of string-nets. We omit the simple proofs and refer to [99], where some of the proofs can be found. First of all in a fusion category C the symmetric bilinear pairing

$$\begin{aligned} \mathrm{Hom}_{\mathsf{C}}(C, D) \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathsf{C}}(D, C) &\to \mathrm{Hom}_{\mathsf{C}}(\mathbf{1}, \mathbf{1}) \simeq \mathbb{C} \\ f \otimes g &\mapsto tr(f \circ g) \end{aligned} \tag{B.15}$$

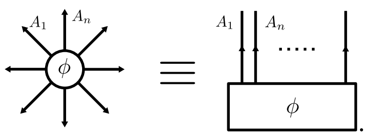
is non-degenerate. For a list of objects $A_1, \cdots, A_n \in \mathsf{C}$ we denote

$$\langle A_1, \cdots, A_n \rangle \equiv \mathrm{Hom}_{\mathsf{C}}(\mathbf{1}, A_1 \otimes \cdots \otimes A_n). \tag{B.16}$$

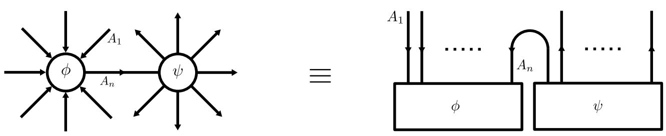
Since braiding and twist are natural isomorphisms the map



is an isomorphism $\langle A_1, \cdots, A_n \rangle \simeq \langle A_n, A_1, \cdots, A_{n-1} \rangle$. Hence only the cyclic order of elements matter in $\langle A_1, \cdots, A_n \rangle$ and we replace boxes for morphisms by round coupons in that case

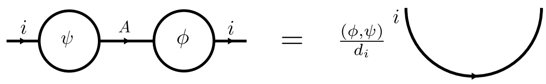


For an arrow oriented towards the coupon with label A the respective element gets replaced by A^*. Coupons can be composed with the help of the evaluation morphisms.

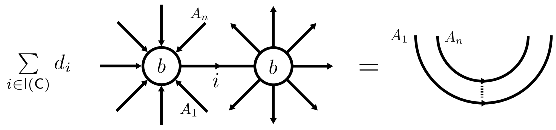


In addition the following identities hold: For any $A \in \mathsf{C}$ there are isomorphisms

a)



b) For $\{b^i_\alpha\}$ a basis in $\langle i, A^*_n, \dots A^*_1\rangle$ with dual basis $\{b^\alpha_i\}$ in $\langle i, A_1, \dots A_n\rangle$ in the sense that $(b^i_\alpha, b^\beta_j) = \delta_{ij}\delta_{\alpha\beta}$ it holds



where the *b*-*b* coupons indicate summation over basis and dual basis.

The second relation is called *completeness property* and will be used several times. Finally we list some useful identities when dealing with modular tensor categories. The interested reader can find proofs in [5, section 3]. The following identities hold

Figure B.3

and

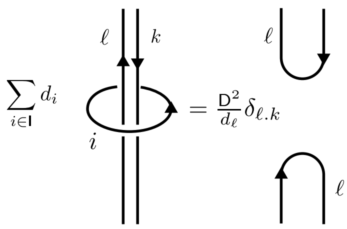


Figure B.4

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Appendix C

The String-Net Topological Field Theory

The main constructions in this thesis are performed in terms of so called *string-nets*. Originally introduced by Levin and Wen [114] in the realm of topological phases of matter (see also [102][94]), string-nets became a tool for constructing a 3-2-1 extension of the Turaev-Viro three dimensional topological field theory (tft). A first mathematical treatment of string-nets was given by Kirillov in [98] and an extension of the Turaev-Viro tft appeared in a series of papers [99][6][7]. This extension of the Turaev-Viro tft was based on the idea of string-nets, however an extended string-net tft was constructed by Goosen in his PhD thesis [66]. The construction of the string-net tft is based on a classification of 3-2-1 extended three dimensional tfts given in [14][13]. The classification goes through a generators and relation formulation of tfts as described by Walker in an unpublished manuscript [143].

This section is structured as follows. We start with recalling the definition of the string-net space assigned to a surface and state the main result of [98][99][6][7] that this in fact agrees with the state space of the Reshetikhin-Turaev tft. Next we give a very brief account of fully extended tft in three dimensions via generators and relations in order to present the fully extended string-net tft constructed by Goosen. Throughout this section C denotes a spherical fusion category.

C.1 The String-Net Space on Surfaces

Let Σ be an arbitrary two dimensional smooth, oriented manifold, possibly with boundary and not necessarily compact. A *string-net* on Σ is an isotopy class of an embedded, oriented finite graph $\Gamma \hookrightarrow \Sigma$. Finite means we require Γ to have finitely many edges and vertices. Furthermore the embedding has to be s.th. the intersection of the image of Γ with $\partial\Sigma$ are exactly the images of the univalent vertices of Γ. A C-*coloring* of Γ is an assignment of an object $C(\mathbf{e}) \in \mathsf{C}$ to any edge $\mathbf{e} \in E(\Gamma)$ and an assignment of a morphism $\phi \in \langle C(\mathbf{e}_1), \cdots, C(\mathbf{e}_n)\rangle$ to any vertex v. Here $\mathbf{e}_i$ are the edges incident to the vertex taken to be oriented away from the vertex. When flipping the orientation of an edge colored by C, its color changes to C^*. The *boundary value* of a C-colored string-net is a pair $(\mathbf{p}, \mathbf{C})$ consisting of intersection points $\mathbf{p} = \{\Gamma \cap \partial\Sigma\}$ and the colors $\mathbf{C}$ of edges intersecting the boundary.

Let $D \hookrightarrow \Sigma$ be an embedded disk for which we can assume that its boundary intersects Γ tranversally in edges $\mathbf{e}_1, \cdots, \mathbf{e}_m$. Then there is an injective *evaluation map*

$$\langle \bullet \rangle_D : \Gamma \cap D \to \langle C(\mathbf{e}_1), \cdots, C(\mathbf{e}_m) \rangle \tag{C.1}$$

given by composing all the morphisms from vertices in the interior of the disk according to the orientation of the edges. From now on all string-nets are assumed to be C colored. Two string-nets Γ_1, Γ_2 are *equivalent* if there exists a disk $D \hookrightarrow \Sigma$ s.th. Γ_1, Γ_2 agree on $\Sigma \backslash D$ and

$$\langle \Gamma_1 \cap D \rangle_D = \langle \Gamma \cap D \rangle_D \quad . \tag{C.2}$$

For a given boundary value $\mathbf{C}$ one defines a $\mathbb{C}$-vector space

$$V\mathrm{Graph}(\Sigma, \mathbf{C}) \equiv \text{formal finite } \mathbb{C}\text{-linear combinations of string-nets with boundary value } \mathbf{C} \tag{C.3}$$

Recall that C is a $\mathbb{C}$-linear spherical fusion category, hence its morphism spaces are finite dimensional vector spaces. An element $\mathbf{\Gamma} = \sum x_i \Gamma_i \in V\mathrm{Graph}(\Sigma, \mathbf{C})$ is *null* if exists a disk $D \hookrightarrow \Sigma$ s.th. all the Γ_i agree on $\Sigma \backslash D$ and

$$\langle \mathbf{\Gamma} \cap D \rangle_D = 0 \tag{C.4}$$

where the zero on the rhs is the zero in the respective morphism space of C. Denote the vector space of null string-nets with a given boundary value $\mathbf{C}$ by

$$N\mathrm{Graph}(\Sigma, \mathbf{C}) \quad . \tag{C.5}$$

The *pre string-net space* is defined as the quotient

$$\widehat{H^s}(\Sigma, \mathbf{C}) \equiv \frac{V\mathrm{Graph}(\Sigma, \mathbf{C})}{N\mathrm{Graph}(\Sigma, \mathbf{C})} \quad . \tag{C.6}$$

The quotient ensures that the graphical calculus of C holds locally on Σ, i.e. we can manipulate string-nets on open disks according to string diagrams for morphisms in C. For the true string-net space boundary values have to be taken into account properly. This is done defining a category of boundary values $\mathsf{B}(R)$ for any one dimensional manifold R. Its objects are pairs $(\mathbf{p}, \mathbf{C})$ of pairwise disjoint points $\mathbf{p} \equiv p_1, \cdots, p_k \in R$ which are labeled by objects $\mathbf{C} \equiv C_1, \cdots, C_k \in \mathsf{C}$, where C_i is the label of the i-th point. Morphism spaces in $\mathsf{B}(R)$ are defined as

$$\mathrm{Hom}_{\mathsf{B}(R)}\left((\mathbf{p}_1, \mathbf{C}_1), (\mathbf{p}_2, \mathbf{C}_2)\right) \equiv \widehat{H^s}(R \times I; \mathbf{C}_1^*, \mathbf{C}_2) \tag{C.7}$$

with $I = [0,1]$ and $\mathbf{C}_1^*$ is placed on $R \times \{0\}$.

Theorem C.1.1. *[98, Theorem 6.4] There is an equivalence of categories* $\mathsf{B}(S^1) \simeq \mathsf{Z}(\mathsf{C})$ *and* $\mathsf{B}(\mathbb{R}) \simeq \mathsf{C}$*.*

The *string-net space* on Σ is again a quotient of formal vector spaces. Let $(\mathbf{p}, \mathbf{A}) \in \mathsf{B}(\partial\Sigma)$, one defines

$$\begin{aligned} V\text{Graph}(\Sigma, \mathbf{A}) \equiv\ & \text{pairs } (f, \mathbf{\Gamma}) \text{ of formal } \mathbb{C}\text{-linear combinations of string-nets } \Gamma \\ & \text{with boundary value } \mathbf{C} \text{ and morphisms } f \in \text{Hom}_{\mathsf{B}(\partial\Sigma)}(\mathbf{C}, \mathbf{A}) \end{aligned} \tag{C.8}$$

and

$$\begin{aligned} N\text{Graph}(\Sigma, \mathbf{A}) \equiv\ & \text{subspace of null graphs under local relations as before} \\ & \quad \text{plus relation } (f\gamma, \mathbf{\Gamma}) = (f, \gamma\mathbf{\Gamma}) \text{ where } \gamma \in \text{Hom}_{\mathsf{B}(\partial\Sigma)}(\mathbf{C}', \mathbf{C}), \\ & \quad f \in \text{Hom}_{\mathsf{B}(\partial\Sigma)}(\mathbf{C}, \mathbf{A}) \text{ and } \mathbf{\Gamma} \text{ has boundary value } \mathbf{C}' \end{aligned} \tag{C.9}$$

The string-net space for boundary value $(\mathbf{p}, \mathbf{C}) \in \mathsf{B}(\partial\Sigma)$ is the quotient

$$H^s(\Sigma, \mathbf{C}) \equiv \frac{V\text{Graph}(\Sigma, \mathbf{C})}{N\text{Graph}(\Sigma, \mathbf{C})} \quad . \tag{C.10}$$

The main result of the series of papers [98][99][6][7] can be summarized in the following theorem.

Theorem C.1.2. *Let Σ be a compact oriented surface of genus g with parametrized boundary $\partial\Sigma$. Let $\mathbf{C} = \left\{C_1, \cdots, C_{|\pi_0(\partial\Sigma)|=C_n}\right\}$ be a list of objects in $\mathsf{Z}(\mathsf{C})$. There is an isomorphism of vector spaces*

$$H^s(\Sigma, \mathbf{C}) \simeq Z_{TV,\mathsf{C}}(\Sigma, \mathbf{C}) \simeq Z_{RT,\mathsf{Z}(\mathsf{C})}(\Sigma, \mathbf{C}) \simeq \text{Hom}_{\mathsf{Z}(\mathsf{C})}(\mathbf{1}, C_1 \otimes \cdots C_n \otimes L^g) \tag{C.11}$$

where $L = \bigoplus_{i \in \mathsf{I}(\mathsf{Z}(\mathsf{C}))} U_i^* \otimes U_i$ and $Z_{TV,\mathsf{C}}$, $Z_{RT,\mathsf{Z}(\mathsf{C})}$ are the state spaces of the Turaev-Viro and Reshetikhin-Turaev tfts.

The last equality is just the usual computation of the state space of the Reshetikhin-Turaev tft. How to incorporate the category of boundary values when drawing string-nets on a surface? This can be done by *projector circles*, which for $C \in \mathsf{Z}(\mathsf{C})$ is defined as the string-net

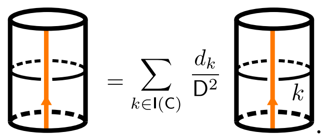


Figure C.1: Drinfeld-center projector

An element in string-net space is an equivalence class of an embedded graph with boundary value in the Drinfeld center and for generators in $H_1(\Sigma)$ corresponding to connected components of the boundary a projector circle is added. An example is given in figure C.2.

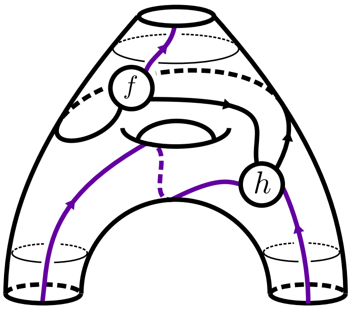


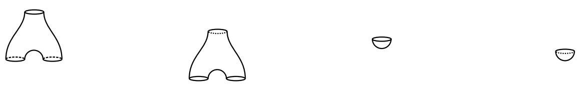
Figure C.2: Example of a projected string-net on a genus 1 surface.

C.2 The string-net TFT

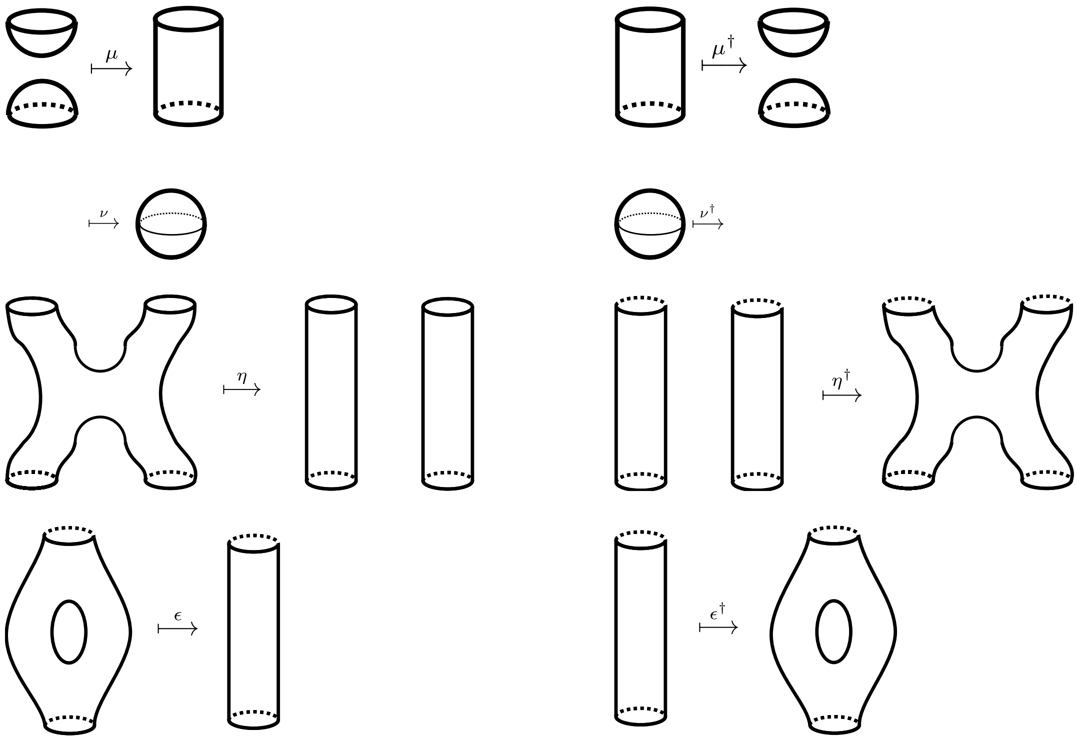
Based on classification results for three dimensional twice extended tfts [130][14][13] Goosen constructed a three dimensional string-net tft. Let Bord^{or}_{123} be the bicategory with objects disjoint unions of circles, 1-morphisms are oriented two dimensional cobordisms between disjoint unions of circles and 2-morphisms are three manifolds with corners whose boundaries are surfaces corresponding to 1-morphisms. This will be the source category for extended three dimensional tfts. The target category is the bicategory of Vect-enriched profunctors $F\mathsf{Vect}$. This is the bicategory with objects $\mathbb{C}$-linear categories, 1-morphisms $\mathsf{C} \to \mathsf{D}$ are functors $\mathsf{D}^{op} \boxtimes \mathsf{C} \to \mathsf{Vect}$ and 2-morphisms are natural transformations between such functors. A twice extended three dimensional tft is defined as functor of bicategories

$$Z : \mathsf{Bord}^{or}_{123} \to F\mathsf{Vect} \quad . \tag{C.12}$$

This definition is quite abstract, but it can be broken down to fairly concrete conditions. First of all Bord^{or}_{123} has a nice presentation as a freely generated bicategory, which means that there are finitely many generating objects, 1-morphisms and 2-morphisms plus some consistency requirements among them s.th. any other object, 1-morphism or 2-morphism factors through the generating elements. In [13, section 3] the *anomaly free modular presentation* was given. Its generating object is . Generating 1-morphisms are



and generating 2-morphisms are four invertible maps corresponding to associativity of pair of pants, left and right units of cylinders, braiding moves of pair of pants and Dehn twist of a cylinder. There are in addition eight non-invertible 2-morphisms

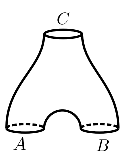


The ν, $\nu^\dagger$-morphism has source respectively target the zero object which corresponds to the empty set. The generating 2-morphisms have to satisfy a total of nine relations for which we refer to [13, section 3]. A twice extended three dimensional topological field theory is now a map Z with source the bicategory freely generated by the anomaly free modular presentation, assigning to the circle a $\mathbb{C}$-linear category $Z(\bigcirc)$. To generating 1-morphisms it assigns vector spaces, where boundary circles of the bordisms representing the 1-morphisms are colored with objects of $Z(\bigcirc)$ and the assignment is functorial in the coloring. Generating 2-morphisms are mapped to linear maps, which have to satisfy the nine consistency relations.

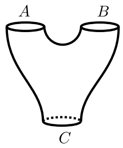
In [66, section 5.2] it is shown that there is a *string-net tft* Z_{SN} for any spherical fusion category C mapping

$$\xmapsto{Z_{SN}} \quad \mathsf{Z}(\mathsf{C})$$

and for $A, B, C \in \mathsf{Z}(\mathsf{C})$ it assigns



$$\xmapsto{Z_{SN}} \quad H^s(\triangle; A, B, C^*)$$



$$\xmapsto{Z_{SN}} \quad H^s(\triangledown; A^*, B^*, C)$$

$$\xmapsto{Z_{SN}} \quad H^s(\ominus; A)$$

$$\xmapsto{Z_{SN}} \quad H^s(\ominus; A^*)$$

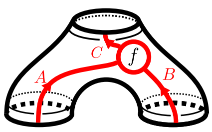
By the construction of a general tft functor, to an arbitrary 1-morphism Σ with incoming boundary colorings $A_1, \cdots, A_n$ and outgoing boundary colorings $B_1, \cdots, B_m$, decomposed into generating 1-morphisms $\sigma = \{\sigma_1, \cdots, \sigma_k\}$ the string-net tft gives

$$Z_{SN}(\Sigma; A_1^*, \cdots, A_n^*, B_1, \cdots, B_m) = \int^{\mathbf{L} \in \mathsf{Z}(\mathsf{C})} \bigotimes_{\sigma} Z_{SN}(\sigma_i; \mathbf{A}_i^*, \mathbf{B}_i, \mathbf{L}_i) \tag{C.13}$$

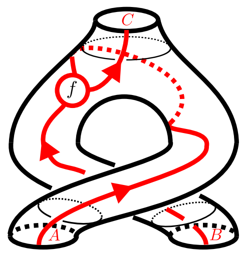
where the coend ranges over all internal colorings. It is shown in [66, Theorem 70] that there is an isomorphism

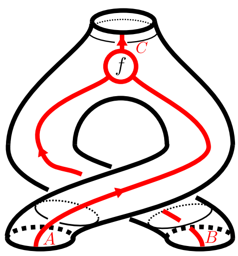
$$Z_{SN}(\Sigma; A_1^*, \cdots, A_n^*, B_1, \cdots, B_m) \simeq H^s(\Sigma; A_1^*, \cdots, A_n^*, B_1, \cdots, B_m) \quad . \tag{C.14}$$

Lastly we recall the non-trivial generating 2-morphisms of the string-net tft. The generating morphisms are given for string-nets on the source morphism being in a specific form. It is not hard to show that any string-net on the respective source bordism can be brought in that specific form. First the Dehn-twist winds a string-net once around cylinder. The braiding moves map

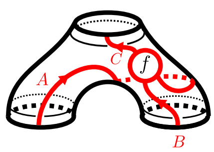


$$\xmapsto{Z_{SN}(\beta)}$$



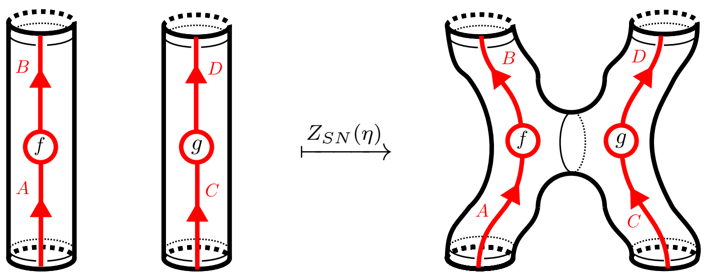


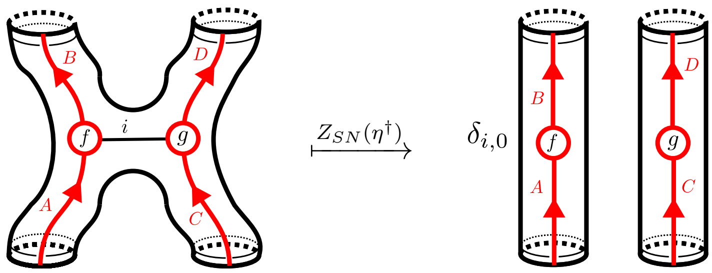
$$\xmapsto{Z_{SN}(\beta^{-1})}$$



.

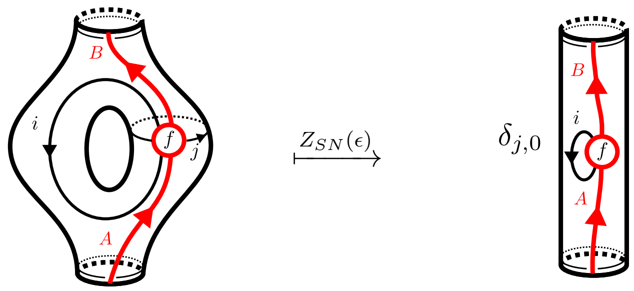
Next the η-moves map

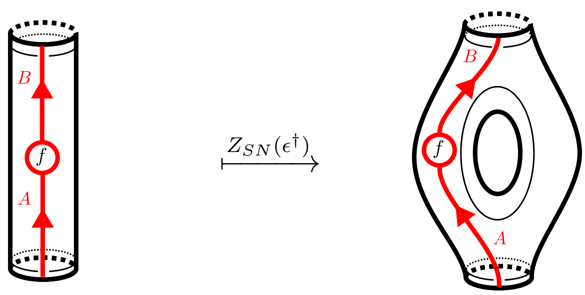




.

Finally ϵ-moves act as follows





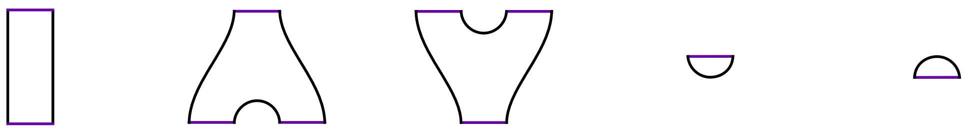
For all other generating 2-morphisms we refer to [66, section 5].

Appendix D

Generating World Sheets and Sewing Relations

In order to be as self-contained as possible we give the set of generating open-closed world sheets. The following figures display the quotients of the orientation double for generating world sheets.

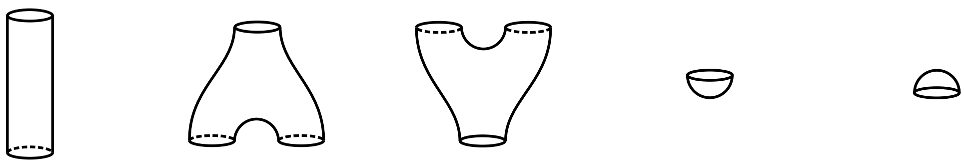
I) **Open World Sheets:**



O_{prop} O_m O_Δ O_η O_ϵ

Purple colored parts of the boundary correspond to open boundaries. Black boundaries are physical boundaries.

II) **Closed World Sheets:**



C_{prop} C_m C_Δ C_η C_ϵ

III) **Open-Closed World Sheets:**

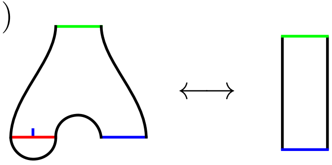
I $I^\dagger$

D.1 Sewing Relations

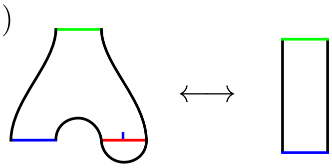
In the following figures red curves indicate how the world sheet displayed is glued from easier parts. The blue flag on gluing curves indicate the direction of gluing. For the part containing the flag an incoming boundary is glued. In the figures blue boundaries denote in-boundaries and green ones correspond to out-boundaries.

I) **Open Relations:**

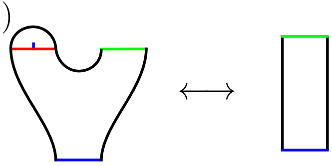
R1)



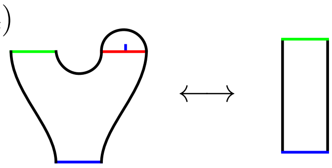
R2)



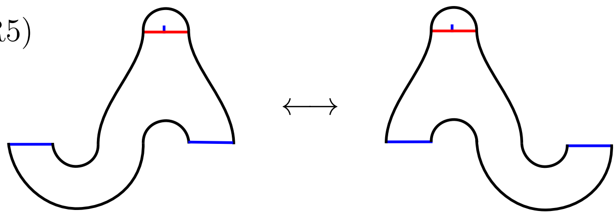
R3)



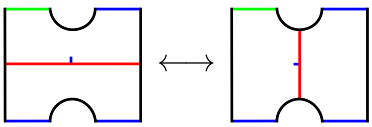
R4)



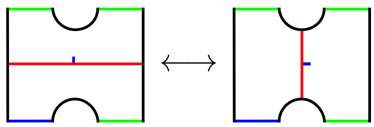
R5)



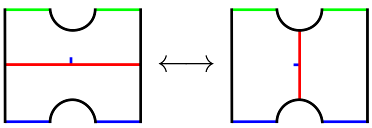
R6)



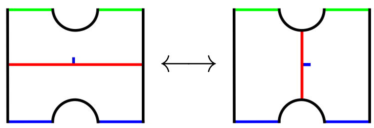
R7)



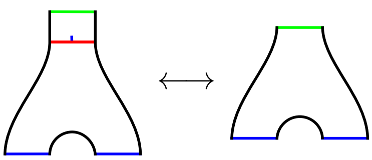
R8)



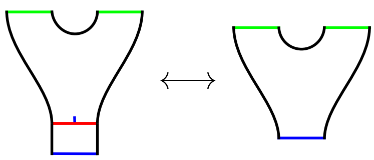
R9)



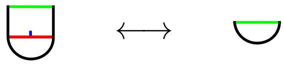
R10)



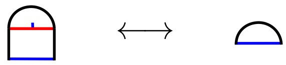
R11)



R12)

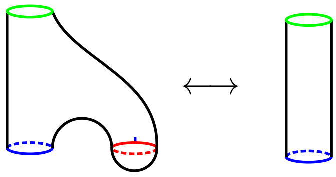


R13)

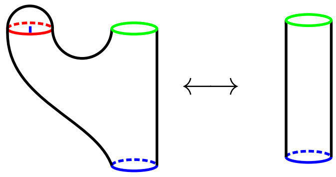


II) **Closed Relations:**

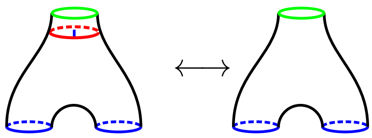
R14)



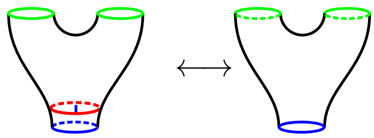
R15)



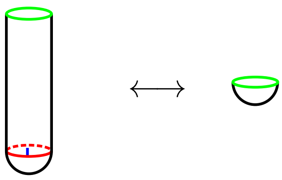
R16)



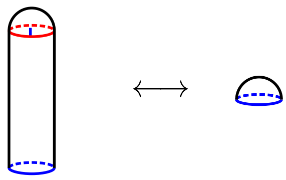
R17)



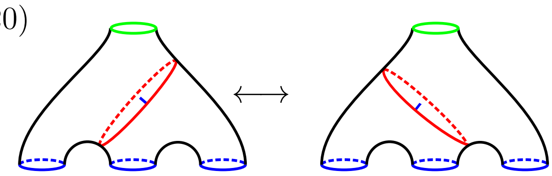
R18)



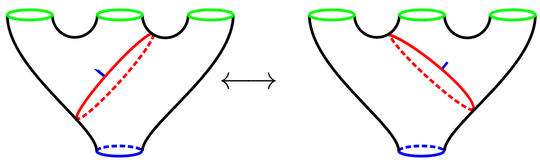
R19)



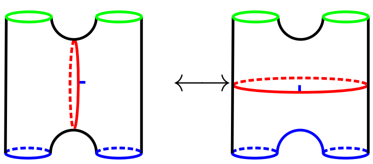
R20)



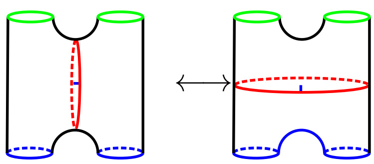
R21)



R22)

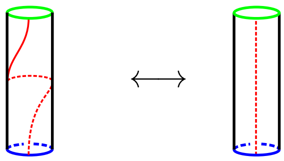


R23)

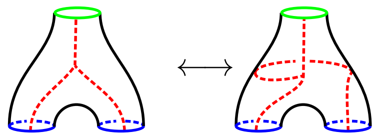


In the picture of the Dehn-twist and braid move the red dashed lines are not gluing lines, but auxiliary curves to display the action of the elements of the mapping class group corresponding to the moves.

R24)

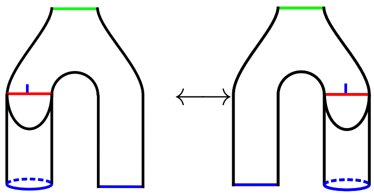


R25)

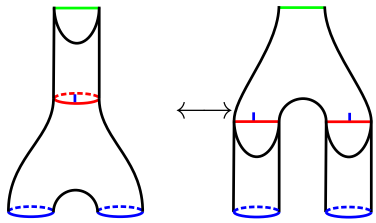


III) **Open-Closed Relations**

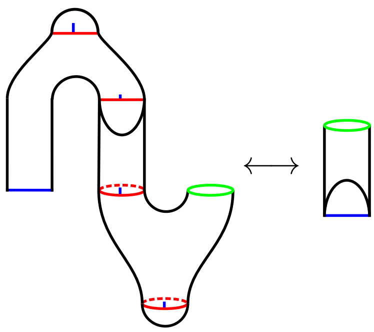
R26)



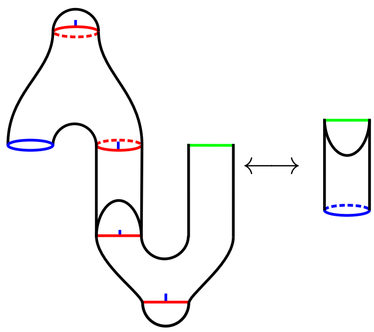
R27)



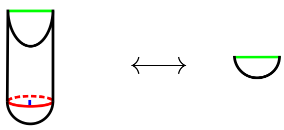
R28)



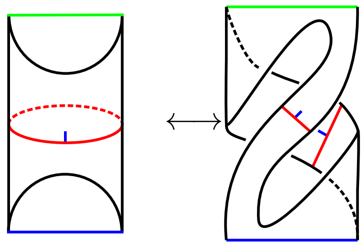
R29)



R30)

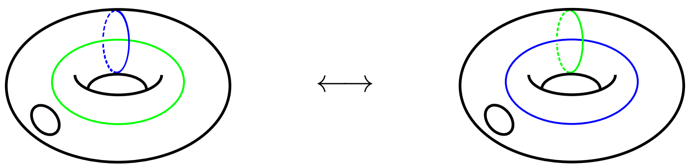


R31)



IV) **Genus 1 Relation:** The genus one move takes place on a torus with one boundary component and interchanges a- and b-cycle of the torus as indicated by the colors.

R32)



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