Gravitational Black Hole Hair

Artem Averin

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Artem Averin

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Artem Averin
aus Barnaul

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Erstgutachter: Prof. Dr. Dieter Lüst
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Zusammenfassung

Die Fragestellung dieser Dissertation ist, ob die Allgemeine Relativitätstheorie ausreichend ist, um die statistische Mechanik schwarzer Löcher zu erklären. Hierzu zeigen wir auf, dass sich dies auf die Frage reduziert, wie der Hamilton'sche Phasenraum von reiner Einstein'scher Gravitation in der Nähe desjenigen Zustandes aussieht, der ein Schwarzschild'sches schwarzes Loch beschreibt.

Aufbauend auf den Eindeutigkeits- und Kein-Haar-Theoremen für schwarze Löcher, könnten erste Erwartungen hierzu in absolutem Gegensatz dazu stehen, was man aus der Thermodynamik schwarzer Löcher erwarten würde. Letztere suggerieren gravitatives Haar für schwarze Löcher, d.h. energetisch weiche Anregungen des Gravitationsfeldes, die im Phasenraum als Verschiebungen nahe des schwarzen Loch Zustandes sichtbar sein sollten. Unsere Hauptfrage ist daher, ob Einstein'sche Gravitation solches gravitatives Haar enthält?

Wir entwickeln neue Methoden zur systematischen Analyse des Hamilton'schen Phasenraumes einer gegebenen Theorie. Eine erste Anwendung unterstützt dabei die These der Existenz gravitativen Haars für schwarze Löcher. Wir sind in der Lage einen ersten Vorschlag für eine duale Theorie zu geben, die durch ihre Observablen und deren Algebra gegeben ist, die denjenigen Teil des Phasenraumes beschreiben soll, der für die Mikrozustände des schwarzen Lochs verantwortlich ist. Bemerkenswerter Weise wird der Vorschlag durch die Beobachtung unterstützt, dass das gravitative Haar, welches durch die duale Theorie beschrieben und mittels symplektischer Methoden hergeleitet wurde, mit demjenigen übereinstimmt, welches man in natürlicher Weise mittels geometrischer Herangehensweisen vermutet hätte.

Unabhängig von diesem konkreten Vorschlag argumentieren wir, warum zu erwarten ist, dass der für die Mikrozustände verantwortliche Teil des Phasenraumes eine konforme Invarianz besitzt. Eine duale Theorie, die diesen Teil im Phasenraum beschreiben soll, liefert daher eine Schwarzschild/konforme Feldtheorie (CFT)-Korrespondenz.

Ebenso unabhängig von unserem konkreten Schwarzschild/CFT-Vorschlag sind wir auf Basis allgemeiner Argumente in der Lage die Gestalt der Symmetrie-Generatoren dieser zu erwartenden konformen Symmetrie anzugeben. Dieses Wissen über die Symmetrie liefert bereits genug Informationen um die Zustandsentartung zu zählen. Bemerkenswerter Weise finden wir exakte Übereinstimmung mit der Bekenstein-Hawking Entropie.

Weiterhin zeigen wir, wie diese Resultate genutzt werden, um heraus zu finden, ob das durch einen gegebenen Schwarzschild/CFT-Vorschlag beschriebene gravitative Haar das

Korrekte ist, welches für Mikrozustände und Entropie verantwortlich ist. Während eine solche Analyse immer noch ausstehend ist, damit die Rede von einem standfesten Beweis unseres Schwarzschild/CFT-Vorschlags sein kann, können wir dennoch schon folgenden Schluss ziehen, der kurz das Hauptresultat der vorliegenden Dissertation zusammenfasst.

Die Allgemeine Relativitätstheorie scheint die richtigen Mechanismen und Freiheitsgrade bereitzustullen, um die thermodynamischen Eigenschaften schwarzer Löcher mikrokanonisch erklären zu können.

Abstract

In this thesis, we ask the question whether general relativity is enough to understand the statistical mechanical properties of black holes. To this end, we explain that this can be reduced to the question as to how the Hamiltonian phase space of pure Einstein gravity in the vicinity of a Schwarzschild black hole state does look like?

Naive expectations coming from black hole uniqueness and no-hair theorems might seem to be in absolute contrast to expectations based on black hole thermodynamics. The latter suggest the existence of gravitational black hole hair, i.e. energetically soft excitations of the gravitational field, that must be visible as shifts in phase space near the black hole state. Our main question is therefore whether Einstein gravity does provide such gravitational black hole hair?

We develop new methods to systematically analyze the Hamiltonian phase space of a given theory. A first application supports the existence of gravitational black hole hair. We are able to give a first proposal of a dual theory, given in terms of its observables and their algebra, that is supposed to describe the part of phase space responsible for black hole microstates. Remarkably, this proposal finds support in the fact that the gravitational hair described by the dual theory and inferred using symplectic methods coincides with the most natural guess based on purely geometric reasoning.

Independent of this particular proposal, we argue why the part of phase space responsible for black hole microstates is expected to be conformally invariant. A dual theory supposed to describe this part of phase space is thus giving rise to a Schwarzschild/conformal field theory (CFT)-correspondence.

Also independent of our concrete Schwarzschild/CFT proposal, we are able to infer by general arguments the form of the symmetry generators of this expected conformal symmetry. This knowledge about the symmetry provides already enough information to count the state degeneracy. Notably, we find precise agreement with the Bekenstein-Hawking entropy.

Furthermore, we show how these findings are used to check whether the gravitational black hole hair provided by any Schwarzschild/CFT proposal is the correct one responsible for microstates and entropy. While such an analysis is still remaining in order to speak of a rigorous proof of our Schwarzschild/CFT proposal, we can already draw the following conclusion which shortly summarizes the main finding of this thesis.

General relativity does seem to provide the right mechanisms and degrees of freedom to microcanonically explain the thermodynamic properties of black holes.

Declaration of own contribution

This dissertation is based on the following papers [1, 2, 3, 4]:

- A. Averin, G. Dvali, C. Gomez, and D. Lust, *"Gravitational Black Hole Hair from Event Horizon Supertranslations,"* JHEP 06 (2016) 088, arXiv:1601.03725 [hep-th].
- A. Averin, G. Dvali, C. Gomez, and D. Lust, *"Goldstone origin of black hole hair from supertranslations and criticality,"* Mod. Phys. Lett. A31 no. 39, (2016) 1630045, arXiv:1606.06260 [hep-th]. (not reviewed in this thesis)
- A. Averin, *"Schwarzschild/CFT from soft black hole hair?,"* JHEP 01 (2019) 092, arXiv:1808.09923 [hep-th].
- A. Averin, *"Entropy Counting from Schwarzschild/CFT and Soft Hair,"* Phys. Rev. D101 no. 4, (2020) 046024, arXiv:1910.08061 [hep-th].

The chapters 2.2, 2.3 and 3 of [1] were invented and written by me. [2] is a review of [1] containing a slightly refined and simpler (but equivalent) way to estimate the entropy. It is not reviewed in this thesis. [3, 4] were invented and written by me. The papers [1, 3, 4] are reviewed in this thesis. Chapter 4 is a review of [1]. The review in chapter 5 is a slightly edited version of [3]. The review in chapter 6 is a slightly edited version of [4].

Chapter 1

Introduction

Our current understanding of theoretical physics is build on two main blocks. Although both are now over 100 years old and great advances in their understanding have been achieved, there are still simple questions concerning the interplay between these two blocks that are still not understood. It is clear that such a fundamental lack of understanding represents both a problem as well as an opportunity for the further progress in understanding the laws of nature. To use this opportunity is the main motivation and starting point for this thesis.

What are these two main building blocks? On the one hand, we have general relativity. It is the theory that governs the dynamics of the gravitational field. On macroscopic scales its theoretical predictions are in excellent agreement with experiment. In fact, the perihelion precession of mercury, deflection of light by the sun, gravitational redshift of light, existence and recent direct observation of black holes and recent detection of gravitational waves are among the observed phenomena in the past years that provide strong support for general relativity.

On the other hand, we have quantum mechanics. It was one of the most important discoveries in the last century that nature appears at microscopic scales to be quantized. The classical theory of electromagnetism fails in explaining the spectrum of black body radiation. Planck observed in 1900 that the spectrum appears as if the electromagnetic field can only be excited in discrete "jumps" - contrary to what is predicted from the classical theory. The spacing of these jumps is controlled by Planck's famous constant $\hbar$ and this observation is considered to be the starting point of the quantum theory. Planck's idea that the observables of a classical theory appear quantized in microscopic phenomena was indeed successful in explaining heuristically several further experimental observations. In the 1920s a set of rules was worked out that determine how the quantization of the observables of a given classical theory is done in a formal mathematical way. These rules are what is today called quantum mechanics.

The theoretical predictions of quantum mechanics are in excellent agreement with experimental observations. Quantizing the system of a charged particle moving in a Coulomb-potential, quantum mechanics is able to explain the energy levels of the hydrogen atom - a milestone in the development of quantum mechanics. Quantizing the electromagnetic field by the same set of few simple rules, one is able to explain the spectrum of black body radiation that was derived heuristically by Planck. The list of further success of quantum mechanics is long and ranges from scattering experiments up to condensed matter physics.

In fact, one gets the impression that this set of few simple quantization rules - if applied to the proper classical system - can explain all the laws of nature.

There are however some problems with this impression. First, what is the classical system that one has to quantize in order to capture the laws of our universe? The answer to this question is unfortunately not completely known. Nevertheless, one has a theory that describes *some* amount of the universe's content. This is the famous standard model of elementary particle physics. It consists of a set of fields and their interactions that were worked out in a large series of theoretical and experimental investigations over the last 100 years. Quantization of the standard model leads to a theory that fits the experimental observations incredibly well. The direct observation of the last missing piece of the standard model was achieved quite recently with the discovery of the Higgs boson at the LHC. Despite of its success, the standard model does not take into account gravity and due to the existence of dark matter and dark energy it is known that there is more content in the universe than listed in the standard model.

A second problem is that despite the simplicity of the rules for quantizing a theory, it can be very hard to "solve" the theory in order to obtain predictions. This is of course no surprise. Already solving a classical theory, which means to solve its equations of motion, can amount to find solutions of very complicated differential equations. Therefore, it is clear that solving the equations of motion of a quantum theory can be even harder.

However, in order to get an understanding of the laws of nature, both problems have to be attacked. In the first problem, a possibility would be to consider the case of pure gravity first. The gravitational field is not incorporated in the standard model but since it is the oldest known force, we know for sure that it exists and has to be taken into account. Indeed, the attempt to quantize general relativity is known to lead to several puzzles as we will review during the next chapters. Even the most simplest but at the same time most safe predicted quantum gravitational effects may seem to lead to several serious consistency problems. It is thus clear that understanding the proper quantization of gravity is an important outstanding problem whose solution will teach us important lessons about the laws of nature.

The second problem is more subtle. Focusing as mentioned on pure Einstein gravity, we have a concrete theory whose quantization we want to study. However, to get predictions of quantum theories of much simpler field theories is already quite involved. It is thus important to understand what the phrase "quantization of Einstein gravity" precisely means and as an overlying goal to develop tools that allow to gain predictions of this quantized theory.

In order to do so, we will in the next chapter explain, what the quantization of a given classical theory precisely means. We will see that a misunderstanding of this step can especially in the case for Einstein gravity be the source of several inconsistencies that seem overlooked in the literature.

Chapter 2

Quantum Mechanics and Gravity

What does it precisely mean to quantize a given classical theory? Consider a general system given by generalized coordinates q_i and associated generalized momenta p_i with a Hamiltonian function $H(q,p)$. Applying the rules of canonical quantization, in the associated quantum theory one is in general interested in matrix elements of the time-evolution operator

$$\langle q_b | e^{-\frac{i}{\hbar}HT} | q_a \rangle \tag{2.1}$$

which give the probability amplitude for the system to evolve from the position eigenstates $|q_a\rangle$ to $|q_b\rangle$ in the time T. By q or p without an index we denote the set of all generalized coordinates $q = \{q_i\}$ or momenta $p = \{p_i\}$. (For further details see [5].) This probability amplitude is obtained by solving the Schrödinger equation and the solution is easily written down (for the discrete version and further explanations see [5])

$$\langle q_b | e^{-\frac{i}{\hbar}HT} | q_a \rangle = \left(\prod_i \int_{q(t=0)=q_a}^{q(t=T)=q_b} \mathcal{D}q_i(t)\mathcal{D}p_i(t) \right) \exp\left[\frac{i}{\hbar} \int_0^T dt \left(\sum_i p_i \dot{q}_i - H(q,p) \right) \right] \tag{2.2}$$

where in the path integration the endpoints of $q(t)$ are constrained as indicated but the momentum $p(t)$ is free. For a fixed time t the measure in (2.2) is given by

$$\prod_i \frac{dq_i dp_i}{2\pi\hbar}. \tag{2.3}$$

(2.2) is the most general starting point to extract all further predictions (partion function, correlation functions, scattering amplitudes,...) from the quantum system under consideration. From this equation, we can infer what it precisely means to quantize a given classical theory. The quantization procedure basically consists of two steps.

The first step consists in writing down the integral (2.2). The latter integral is build out of integrals over the Hamiltonian phase space. Thus, one has to identify the Hamiltonian phase space correctly, that is, what is the set of its coordinates $(q,p) = (q_i, p_i)$?

The second step is the actual evaluation of the integral (2.2) (for instance in the form of computation of quantities that are derived from (2.2)).

2.1 Hamiltonian Phase Space

As we have seen from equations (2.2) and (2.3), in order to quantize a given classical system, it is extraordinarily important to correctly identify the Hamiltonian phase space. After clarification of what the coordinates $(q,p) = (q_i, p_i)$ of the phase space are, the integral (2.2) is easily written down. However, if the phase space is identified incorrectly, the equation (2.2) can of course lead to wrong results. We will argue in this thesis for the possibility, that it is precisely an improper identification of the Hamiltonian phase space that leads to the well-known problems that are associated with the quantization of gravity.

Typically, the step of determining the Hamiltonian phase space of a given classical theory is "simple." In most of the cases the theory under consideration is given by an action functional of a set of fields that play the role of the generalized coordinates q_i. Using the standard procedure known from classical mechanics to switch from the Lagrangian to the Hamiltonian framework, one infers the associated generalized momenta p_i together with the Hamiltonian function $H(q,p)$. With this, one has all ingredients to write down the path integral (2.2). For instance, in the classical example of φ^4-theory in a Minkowski-spacetime, the phase space is parameterized by $(q,p) = (q_i, p_i) = (\varphi(t, \mathbf{x}), \partial_t \varphi(t, \mathbf{x}))$, i.e. by the field amplitude and its time-derivative at a fixed moment of time t. These are the coordinates that uniquely select a state in the phase space. At the same time, this is the free Cauchy-data that uniquely determines the time-evolution of the system.

However, the things get more involved if the system is constrained, i.e. if the theory under consideration is a gauge theory. In that case, the coordinates (q_i, p_i) that are derived with the above mentioned procedure are subject to gauge constraints. As a consequence, the physical phase space is only a submanifold of the space that is spanned by these (q_i, p_i). This submanifold has to be carefully determined as it is the domain over which is integrated in (2.2).

What are the implications of this for the case of gravity as given by the Einstein-Hilbert action? Although the Hamiltonian formulation, the famous ADM-formulation, is long known [6], a detailed analysis of the Hamiltonian phase space in gravity is still an open problem. That is, it is not known which set of canonical coordinates (q_i, p_i) parametrize the phase space. In order to quantize Einstein-gravity, according to (2.2), this is precisely what one needs to know. That the structure of complete phase space is not known is already reflected in the fact that there is still an ongoing debate which gauge transformations in gravity constitute redundancies and which constitute physical excitations, i.e. shifts in phase space [7]. It is clear that a wrong identification of the phase space can lead to over/undercounting in the integral (2.2). As a consequence of such a wrong counting, it might be that quantities derived from (2.2) (e.g. scattering amplitudes, renormalized parameters) might appear more UV-divergent or UV-sensitive than they in reality are.

Especially in the case of gravity the correct determination of the Hamiltonian phase space (and thus the quantization procedure) is quite subtle. The reason for this subtlety has a clear physical origin: General relativity contains black holes. To illustrate the point, remember the example of φ^4-theory on a Minkowski-spacetime. The available Cauchy-data parametrizing the Hamiltonian phase space enables us to freely choose the field amplitude

$\varphi(t, \mathbf{x})$ for a fixed moment of time t. In Einstein gravity the analog is not true. Suppose we were to prepare a wave packet out of gravitational waves at a fixed moment of time t. Then we are not able to freely choose the size and the amplitude of the wave packet. Decreasing the wave packet's size and increasing its amplitude, we will hit the point where the gravitational radius of the wave packet is larger than its formal size, i.e. we will hit the point of black hole formation. Mathematically, this is reflected in the fact that the configuration we were to prepare at fixed time t is still subject to the gauge constraints. As a consequence, the phase space is smaller than what one may naively expect. The correct phase space has to be taken into account as the domain of integration in (2.2). It might be the case that doing so resolves the problems typically associated with the quantization of Einstein gravity. The idea that Einstein gravity might be "self-complete" was already mentioned in [8, 9, 10].

Let us summarize what we have learned from our discussion so far. To quantize Einstein gravity according to (2.2), we have to carefully determine the Hamiltonian phase space as it constitutes the domain of integration. Therefore, to develop methods to systematically analyze the structure of the Hamiltonian phase space seems crucial. It would be even better, if we were able to develop tools for the actual evaluation of integrals like (2.2). To find such tools will be a central point of this thesis. Furthermore, we have seen that the structure of Hamiltonian phase space is especially intricate for Einstein gravity due to the existence of black holes. It appears that understanding the structure of phase space near a black hole state is absolutely crucial in order to understand the structure of the complete phase space. The latter problem is an important open problem in the literature. To address this particular problem will be the main focus of this thesis.

In the next chapter, we will explain how the latter problem is conected to the main problems of black holes that are discussed in the literature.

Chapter 3

Black Holes

The most robust predictions of non-perturbative quantum gravity concern the behavior of black holes. Interestingly, the classical solution of general relativity describing the simplest black hole - the Schwarzschild solution - is known for approximately 100 years. In fact, it was the first discovered non-trivial solution of the field equations of general relativity. Surprisingly, after 100 years, it is still not understood - to the full extent even not in the classical theory as we will explain in this chapter. It appears even more surprising given the fact that by now black holes are known to exist. In 2019, the Event Horizon Telescope even gave the first picture of a black hole. Nevertheless, it is still not known whether general relativity is enough to understand the object present on this picture. What are the problems with black holes?

As was observed starting in the 1970s, following the laws of thermodynamics, some general predictions about the behavior of black holes can be said [11, 12, 13]. By now there are very many independent derivations and arguments for this behavior such that it can be considered without doubt as the most robust prediction of what has to happen in non-perturbative quantum gravity (see [14] for a review also of the statements that follow in this chapter and references therein). The formation of a black hole is followed by its subsequent evaporation through thermal radiation. Its temperature is given by the famous Hawking-temperature. Accordingly, the black hole is assigned an entropy that is given by the famous Bekenstein-Hawking area-entropy law $S = \frac{A}{4\hbar G}$ with the horizon area A. In the classical $\hbar \to 0$ limit, the entropy gets infinite. According to Boltzmann this means that there have to be infinitely many points in the Hamiltonian phase space of pure Einstein gravity that correspond to the microstates of a black hole of fixed mass and angular momentum parameter.

On the other hand, the black hole uniqueness theorems are well-known in general relativity: Asymptotically flat and stationary solutions of the field equations in 4d pure Einstein gravity are diffeomorphic to the Kerr-metric - a solution describing a black hole of certain mass and angular momentum [15]. This creates the impression of black holes being completely characterized by few parameters and this missing of features is commonly phrased as black holes having no hair.

At first glance, this seems to lead to a contradiction. Where are the infinitely many microstates in classical Hamiltonian phase space in accordance with Bekenstein-Hawking entropy, given the fact that due to uniqueness theorems black holes are essentialy featureless? This open problem is one aspect of the black hole information paradox.

In the traditional formulation of the black hole information paradox the question is how to retrieve the information about what has formed a black hole after it has evaporated through Hawking radiation. In Hawking's analysis an exactly thermal spectrum for the Hawking radiation is obtained and one might conclude that it is as a matter of principle impossible to retrieve the information about the black hole's formation since the thermal radiation always looks the same. This information loss is of course in contradiction with the unitarity of quantum mechanics. However, it was proposed by Dvali and Gomez [16, 17, 18, 19] that exact thermality of Hawking radiation is an artifact of the semiclassical limit in which Hawking's analysis is performed. Going beyond this limit, the Hawking radiation is expected to obtain corrections to exact thermality which then could contain the missing information about the black hole's formation. If so, the black hole has nevertheless to possess a sufficient number of microstates that evaporate leading to the various different spectra of Hawking radiation. This brings us again back to the question where these necessary microstates are which still has to be clarified.

The black hole information paradox is one of the biggest open problems in theoretical physics. Note that on the one hand, black holes possess a huge number of microstates and appear to be very complicated - in fact, the Bekenstein bound tells us that they are in this respect even the most complicated objects in nature. On the other hand, by uniqueness theorems, there is the impression that black holes are featureless and hence are the most simplest objects in nature. Which of the two statements is true for the black holes that are seen in the sky?

As we explained, the statement that black holes possess a huge number of microstates can be considered as safe. Then, how to explain this state degeneracy and bring it in accordance with the uniqueness theorems of general relativity? In other words, how to microcanonically explain the Bekenstein-Hawking entropy using general relativity as the theory of gravity and the basic rules to quantize a given theory described in the last chapters?

At this point it might be argued that the source of the mentioned paradox is due to the need of a modification of either general relativity or quantum mechanics. So maybe without such a drastic modification it is even not possible to find a resolution for the mentioned tension. Let us briefly comment on that point.

What about a necessary modification of quantum mechanics (in the presence of gravity)? Such a step seems very radical. The set of few simple rules of quantum mechanics can be applied to systems as simple as the hydrogen atom up to systems as complicated as the entire standard model. As we have explained during the last chapters, theoretical predictions are in excellent agreement with experimental observations giving so far no sign of doubt on the laws of quantum mechanics. So why should things change with the inclusion of gravity? Furthermore, with the discovery of the AdS/CFT-correspondence in the end of the 1990s, it became clear that there are consistent quantum theories containing gravity in accordance with conventional quantum mechanics. So the need to modify quantum mechanics seems by now very unlikely and its laws are taken for safe.

What about the need to modify general relativity? To think about this question, we should clarify once again what we are seeking for and thus what such a possible modification should provide in order to solve the mentioned problem. As described, we are seeking for an explanation of the black hole's Bekenstein-Hawking entropy. According to this entropy, a black hole of fixed mass and angular momentum parameter possesses a huge number of microstates. These microstates are then (nearly-)degenerate in energy and hence provide a huge number of *soft* excitation modes to a given black hole. Where are these soft modes? At first glance, the uniqueness theorems seem to provide no space for such soft modes in pure Einstein gravity. One might think that a modification of general relativity is needed which then should provide the missed soft modes predicted by Bekenstein-Hawking entropy. At low-energies general relativity is in perfect agreement with experimental observations and thus seems to provide space for modification only in the UV. Indeed, one might think that in the UV (typically expected to be the Planck scale), one has to include additional degrees of freedom (typically expected to UV-complete Einstein-gravity). These additional degrees of freedom could then also give rise to the soft modes that are missed in Einstein-gravity.

Although we cannot rule out this possibility, there seem to be nevertheless some problems associated to this. Naively, the physics of large black holes with a size $R \gg L_P$ much larger than the Planck length L_P should be independent of the UV-completion of Einstein-gravity. That is, the Bekenstein-Hawking entropy should be explainable solely with the degrees of freedom that general relativity already provides. But there is also another argument that stresses the problem associated with the possible need for a modification of gravity to explain black hole entropy more concretely.

The argument relies on a phenomenon in *soft* physics observed in [20].[1] There, a general field-theoretical mechanism was conjectured, which provides soft excitation modes for a special class of field configurations. These field configurations are called critical and the conditions for criticality were pointed out. Examples of this phenomenon are known in much simpler field theories, but in the gravity case it explains the physical origin of the black hole entropy: Black holes satisfy the conditions for criticality. Being critical field configurations, by the described conjectured mechanism, black holes must possess soft excitation modes. These soft excitation modes provide the physical origin of black hole entropy: They provide energetically nearly-degenerate excitations which are responsible for the black hole microstates and entropy.

With this general phenomenon in mind, the following problem associated with a possible need to modify gravity (even independently at which energy scale) emerges immediately: Within pure Einstein-gravity the stationary black hole solutions - i.e. the Kerr-family - satisfy the criteria of criticality of [20]. According to the phenomenon described in the last paragraph, the theory, i.e. pure Einstein-gravity alone, is expected to provide these field configurations with soft modes. These soft modes then have to be visible in the Hamiltonian phase space of pure Einstein-gravity.

[1] For a formulation of the phenomenon that is more closely to the one given here and for an explanation as to why and how it is expected to happen see [21].

It might be that the latter soft modes do not give rise to the correct degeneracy as required by the Bekenstein-Hawking entropy. And it could still be that a modification is needed (for instance, by integrating in new degrees of freedom in the UV) to account for the Bekenstein-Hawking entropy. Independent of this, by the argument of the last paragraph, the Kerr-family should possess at least some soft modes that are visible in the Hamiltonian phase space of pure Einstein-gravity solely. This seems to contradict the uniqueness theorems of general relativity. Where are these soft modes?

3.1 Statement of the Main Research Question

It seems that we have to take a closer look at the Hamiltonian phase space of pure Einstein-gravity in the vicinity of the Kerr-family. Are there soft modes? If there should be really no such modes, then we have to carefully understand why not. If yes, do they account for the Bekenstein-Hawking entropy? This would then mean that Einstein-gravity is indeed enough to understand the physics of black holes (and no additonal degrees of freedom that might come from a possible UV-completion are needed). To simplify the task, we specify to the simplest solution of the Kerr-family - the Schwarzschild solution.

Let us summarize what we have learned so far. We have arrived at a concrete mathematical question to be posed in Einstein-gravity. To understand its answer is as we have explained of crucial importance for the further understanding of black holes. This question will be the main question of this thesis.

How does the Hamiltonian phase space of pure Einstein-gravity look like in the vicinity of the Schwarzschild solution? Does it contain soft modes?

If there are really such soft modes, it should be clarified whether they account for the black hole's Bekenstein-Hawking entropy. Furthermore, the statement of black holes possessing no hair would need to be reformulated. If such soft modes do exist, they provide hair for the black hole. Since the hair would originate from pure Einstein-gravity, it would be entirely of gravitational nature, the black hole would possess *gravitational black hole hair*.

Having now formulated the main question of this thesis, we are facing immediately a problem. As we have seen, the black hole uniqueness theorems in general relativity suggest that the answer of the upper question seems to be that such soft modes are not existent. Although a careful analysis of the Hamiltonian phase space might indeed result in this answer, there are still several subtleties that have to be carefully taken into account. These subtleties thus provide indeed the possibility of gravitational black hole hair.

What are these subtleties? Since the uniqueness theorems single out the Kerr solutions up to diffeomorphisms, they already themselves suggest a possible loophole that may provide space for gravitational black hole hair. It may be that some of the diffeomorphisms are physical, i.e. shifts in phase space rather than gauge redundancies. This phenomenon is known to happen in gauge theories typically when the gauge parameters are non-vanishing in some asymptotic region. These kind of physical gauge transformations are called asymptotic symmetries. Such asymptotic symmetries could in gravity then potentially provide gravitational hair for the Kerr black hole and thus be responsible for its microstates and entropy.

To conclude, we hence see that our above mentioned main question about the existence of gravitational hair of a Schwarzschild black hole is not that simple to answer. As we said, the remaining part of this thesis is concerned to develop tools to answer this question. Therefore, we will show how to analyze the Hamiltonian phase space of a theory in a structured way. Independent of gravity and black holes, the developed tools are of possible interest for the quantization of any theory as it allows to rewrite (and evaluate) integrals of the type (2.2) in a systematic way (remember also the explanations given there about the correct determination of the Hamiltonian phase space).

However, using the developed tools, we will focus on analyzing the phase space of pure Einstein-gravity in the vicinity of a Schwarzschild black hole. To state the main result in short, we will see that gravitational black hole hair does indeed exist and that it can account for the Bekenstein-Hawking entropy. This means that the Hamiltonian phase space of pure Einstein-gravity seems to provide enough place to accommodate precisely enough gravitational hair that is required by the Bekenstein-Hawking entropy. In other words, we will see substantial evidence that general relativity seems to be enough to understand the quantum gravity properties of black holes!

In the next chapters, we will illustrate the statements of the last paragraph by reviewing the papers on which this dissertation is built. For more details, we refer to the original papers.

Chapter 4

Gravitational Black Hole Hair from Event Horizon Supertranslations

We provide in this chapter a very brief overview of the main aspects of the paper [1]. For more details, we refer to the original paper.

To approach the problem stated in 3.1, we have seen that asymptotic symmetries in gravity might provide place for gravitational black hole hair in the Hamiltonian phase space in the vicinity of, for instance, a Schwarzschild black hole. In gravity, asymptotic symmetries appeared already in the study by Bondi-Metzner-Sachs (BMS) of 4d spacetimes which are asymptotically flat at null infinity [22, 23]. The asymptotic symmetry algebra of those spacetimes was proposed to be the $\mathfrak{bms}_4$-algebra (see [7, 24, 25] for the various discussed definitions of $\mathfrak{bms}_4$ in the literature). In its simplest definition, the $\mathfrak{bms}_4$-algebra contains in addition to the expected Poincare-transformations an infinite dimensional algebra of transformations that generalizes spacetime-translations to the $\mathfrak{bms}_4$-supertranslations. Although their presence might at first sight appear to be surprising, their appearance is tied to the fact that the spacetime is asymptotically flat at null infinity as can be seen, for instance, in a definition that uses only the intrinsic structure at null infinity (see for instance [26] for a review of these issues).

Do the $\mathfrak{bms}_4$-supertranslations provide gravitational hair for a Schwarzschild black hole? The answer is yes, in the sense that it leads to shifts in the Hamiltonian phase space (rather than being gauge redundancies). However, these shifts reflect the degeneracy of the gravitational vacuum (see, for instance, [27]). Hence, the states obtained by acting with $\mathfrak{bms}_4$-supertranslations on a Schwarzschild black hole are not expected to be the microstates responsible for the black hole entropy.

The degeneracy of the gravitational vacuum is a feature that is present for any state independent of whether the state describes a black hole or some other configuration. In other words, we expect *additional* gravitational black hole hair that is present *precisely* if the state is a black hole. The excitations that would be provided by these new type of hair were termed $\mathcal{A}$-modes in [1] and - if existent - they would then be expected to reflect the criticality of the black hole and to be responsible for microstate degeneracy as required by the Bekenstein-Hawking entropy.

The question then is whether the Hamiltonian phase space contains such $\mathcal{A}$-modes? The idea of [1] is that there is a very natural way to quickly *guess* what the $\mathcal{A}$-modes might be. Remembering, that the $\mathfrak{bms}_4$-supertranslations have an interpretation that is tied to the intrinsic structure of null infinity, the same intrinsic structure is present at the event horizon of spacetimes containing black holes. Thus, the presence of an event horizon leads to (potentially) *additional* (candidates for) asymptotic symmetries: the event horizon supertranslations.

Using gauge-fixing conditions that can cover the part from past null infinity up to the future event horizon of a Schwarzschild black hole, the $\mathfrak{bms}_4$-supertranslations as well as the event horizon supertranslations were constructed in [1]. The observation is that both are indeed different. Their difference provides then candidates for the described $\mathcal{A}$-modes that are supposed to *enhance* the asymptotic symmetry algebra of the Schwarzschild black hole due to the presence of an event horizon as compared to spacetimes without black holes.

Having these candidates for the $\mathcal{A}$-modes, they are then expected to be responsible for the microstates of a Schwarzschild black hole. Indeed, it was checked in [1], that they keep the ADM-mass invariant and are hence candidates for the soft modes that we were searching for in 3.1.

Furthermore, the exact degeneracy of the soft modes associated to a critical system is typically lifted in the corresponding quantum theory [20]. An estimation of this lifting in [1] for the $\mathcal{A}$-modes which are supposed to reflect the criticality of the black hole leads to a state degeneracy that is in qualitative agreement with the expected Bekenstein-Hawking entropy (see also [2] for a slighty refined and simpler but equivalent argument).

To summarize, in an attempt to address the main problem stated in 3.1, we have given a first try in [1]. We have seen that asymptotic symmetries might indeed give rise to the missed soft modes. In fact, in a sense, already the simplest possible guess seems to provide the gravitational black hole hair that could account for the expected Bekenstein-Hawking entropy.

That excitations of the gravitational field provided by asymptotic symmetries could play a role in accounting for black hole microstates was already proposed by Carlip [28] in an approach to account microcanonically for the Bekenstein-Hawking entropy. Due to several new insights relating asymptotic symmetries and soft physics (see [27] and citations thereof) this proposal recently gained new attention as the "soft hair on black holes" proposal [29] (see also [30, 31] for various other aspects and different proposals). However, as reviewed here, although $\mathfrak{bms}_4$-supertranslations provide soft (gravitational) hair, it was one of the main aspects of [1] to point out that a different type of gravitational hair is needed that accounts for the microstate degeneracy and black hole entropy. This statement is in agreement with, and provides the resolution to, the criticism on the soft hair proposal stated later correctly in [32, 33, 34, 35].

Although the $\mathcal{A}$-modes lead to a promising candidate for the gravitational black hole hair that might account for the black hole entropy, this candidate was not obtained from a detailed phase space analysis as we asked for in 3.1. So, following our main problem stated in 3.1, the motivation for the next chapters is to see whether such an analysis would reproduce the result obtained here.

Chapter 5

Schwarzschild/CFT from soft black hole hair?

We provide in this chapter a review of the paper [3]. This chapter is a slightly edited version of [3] and for more details, we refer to the original paper.

We start with a very brief overview of the main aspects and later provide the technical details.

As we have motivated throughout the last chapters, an analysis of the Hamiltonian phase space in the vicinity of the Kerr solution in 4d Einstein gravity is still an open problem. This includes singling out - if existent - the relevant gauge transformations being both physical and responsible for the microstates and entropy counting. As we have seen in the last chapter, the phase space in pure Einstein gravity might indeed contain enough such gravitational hair to account for the expected Bekenstein-Hawking entropy. In [3] we start with a detailed phase space analysis and review in the following first results.

In particular, we argued in [3] that the conventional Bondi fall-off conditions imposed on the gravitational field at null infinity are too restrictive in the presence of an event horizon. Thus, precisely in the presence of a black hole, some would-be redundancies are supposed to become physical degrees of freedom (which we called $\mathcal{A}$-modes). It is this effect of enhancement of degrees of freedom in the presence of event horizons that puts new gravitational hair on the black hole. The new hair provided by the $\mathcal{A}$-modes is expected to be responsible for black hole microstates and goes beyond the BMS hair provided by the standard $\mathfrak{bms}_4$ asymptotic symmetry algebra as discussed e.g. in [36].

Precursors of this statement, as reviwed in the last chapter, we already made in [1], where it was emphasized that BMS hair cannot be responsible for microstates in agreement with, and providing a resolution to, the criticism on the soft hair proposal correctly stated later in [32, 33, 34, 35]. Instead, the $\mathcal{A}$-modes, which are provided by the above described effect of enhancement, are supposed to be responsible for microstates and can give rise to correct entropy counting as we explained in [2].

As explained in the last chapter, this effect matches a different phenomenon in soft physics observed in [20]. There, a general field-theoretical mechanism was conjectured, which provides soft excitation modes for a special class of field configurations. These field configurations are called critical and the conditions for criticality were pointed out. Examples of this phenomenon are known in much simpler field theories, but in the gravity case it explains the physical origin of the black hole entropy: Black holes satisfy the conditions for criticality. Being critical field configurations, by the described conjectured mechanism, black holes should possess soft excitation modes. These soft excitation modes provide the physical origin of black hole entropy: They provide energetically nearly-degenerate excitations which are responsible for the black hole microstates and entropy. In [20] there was furthermore made the proposal that the nearly-soft modes should be described by some sort of conformal field theory. The latter claim would imply that independently of what the degrees of freedom responsible for the nearly-soft modes are, they would admit a description through some type of black hole/conformal field theory correspondence.

As we remarked in [3], the presence of these soft modes, which are induced by criticality, provides a further argument that they must be visible in a Hamiltonian phase space analysis and thus they are matched precisely with the $\mathcal{A}$-modes. The scale-invariance even suggests that part of the phase space containing the black hole and its accompanied soft modes can be described by a conformally invariant theory (CFT).

Such a Kerr/CFT-correspondence was indeed proposed in [37, 38] being supported by studies of scattering off a Kerr black hole. However, neither the physical origin of why such a correspondence should exist was understood nor a concrete formulation of the dual CFT was given. The former point, as we explained in [3], is due to the criticality of the Kerr black hole which is reflected in the presence of an event horizon. As described above, its presence provides the $\mathcal{A}$-modes which are identified with the soft modes necessary by criticality and for black hole entropy.

The latter point, the explicit construction of the dual CFT, was the main goal of [3]. For the case of a Schwarzschild black hole, using covariant phase space methods [39, 40], we determined which gauge transformations constitute physical excitations and the algebra of their Hamiltonian generators under the Poisson bracket. This determines the gauge degrees of freedom of a Schwarzschild black hole and their algebra giving rise to a 2D dual theory.

Remarkably, the degrees of freedom obtained by symplectic methods and in particular the resulting $\mathcal{A}$-modes, which constitute microstates, match precisely the proposal of [1] (and also the expectations of the last chapter) inferred from geometrical reasoning.

The technical details of the statements made in this chapter will be concretely explained in the remaining part of this chapter.

5.1 Introduction to the problem

5.1.1 The information paradox for black holes

One of the most robust predictions of quantum gravity is that black hole formation is accompanied by its subsequent evaporation via Hawking radiation [12, 13]. Hawking's calculation predicts that this radiation has a unique thermal spectrum. This observation leads to the information paradox: Letting the black hole evaporate and observing its radiation, it seems as a matter of principle impossible to retrieve information about how the black hole was formed. Unitarity seems to be violated (see [14] for a review).

Hawking's calculation is done by treating the background metric as a classical field (on top of which additional fields are quantized). This approximation receives of course corrections and it was proposed in [16, 17, 18, 19] that they are sufficient to resolve the paradox.

In an arbitrary quantum field theory, there can be quantum states, in which the approximation of working with classical fields and using classical equations of motion is a good approximation (also known as the mean-field approximation in several contexts). This approximation receives corrections which are suppressed by a factor of some power of (*size of system*)$^{-1}$. Remembering the analogy of quantum field theory and statistical mechanics, they are the analog of the statistical fluctuations of an observable around its expectation value in an ensemble. These fluctuations are also suppressed by some power of (*size of system*)$^{-1}$. In [16, 17, 18, 19] these corrections were termed $\frac{1}{N}$-corrections (N being a parameter describing the size of the system) and their meaning for the Hawking-effect was stressed.

The thermal spectrum of emitted quanta gets corrected by these $\frac{1}{S}$-effects (the size N can be measured by the black hole entropy S). These corrections provide observable features from which (in principle) the information can be retrieved how the black hole was formed. After the half-life time of the black hole the $\frac{1}{S}$-corrections accumulate, so that the spectrum is far from thermality and information recovery starts to get efficient in accordance with Page's time [41]. Ignoring $\frac{1}{S}$-corrections (this is the limit in which Hawking's calculation is performed), one is left with the information paradox.

However, even if the Hawking spectrum is corrected by $\frac{1}{S}$-effects, the different $\frac{1}{S}$-effects must be sourced by different black hole microstates in order to be able to contain information about black hole formation. In other words, there must be a huge number of states in the Hilbert space, that correspond to the microstates of a suited black hole in agreement with the Bekenstein-Hawking entropy [11]. In pure Einstein gravity, the entropy is infinite in the classical ($\hbar \to 0$) limit. Thus, in the Hamiltonian phase space, there must be an infinite number of points corresponding to the microstates of a particular black hole. Where are these points in phase space? This is the question, that will be the subject of our investigations.

5.1.2 Kerr/CFT from Criticality

That black hole microstates have to be visible in the Hamiltonian phase space of Einstein gravity can be motivated also from another direction. In [20] the appearance of microstates and thus of black hole entropy is explained as to have its physical origin due to the following general field-theoretic phenomenon:

Suppose a theory with a bosonic field and attractive self-interaction. A field-configuration, which is right at the point of being self-sustained, that is, to be stationary and localized in space by its own attractive self-interaction, is accompanied by the appearance of gapless excitation modes.

The latter point is called a critical point and gapless here is meant with respect to the classical Hamiltonian (i.e. degeneracy in energy). Examples of this phenomenon are well-known in much simpler field theories from condensed matter physics (see [20, 42, 43] and references therein). The excitation modes of such field-configurations are in several contexts also called Bogoliubov-modes. The critical point described is thus accompanied with the appearance of gapless Bogoliubov-modes. The degeneracy is in the quantum theory lifted by $\frac{1}{N}$-effects. This implies that $\frac{1}{N}$-corrections can accumulate over a time-scale set by the size of the system N and deviate significantly with the predictions of a mean-field analysis. Therefore, quantum corrections can not be neglected at a critical point (even if the system is large). The critical point is a quantum critical point.

Now, the case of pure Einstein gravity provides a special case to what we have said. The stationary, asymptotically flat solutions are given by the Kerr-family [15]. These are critical field configurations and as such must possess gapless Bogoliubov-modes. These gapless modes are the physical origin of the black hole entropy. Thus, the Hamiltonian phase space Γ of Einstein gravity has to contain a region $S \subseteq \Gamma$ containing the Kerr-family and its gapless Bogoliubov-excitations.

Due to this scale-invariance, it is tempting to expect that the part S of Hamiltonian phase space has a conformal invariance. The lifting of the degeneracy of the Bogoliubov-modes by the $\frac{1}{S}$-effects in the quantum theory is reflected by conformal anomaly of this invariance.

Indeed, such a Kerr/CFT-correspondence was proposed [38] as an extrapolation of the extremal Kerr/CFT-correspondence [37]. By analysis of scattering of a non-extremal Kerr black hole, some data of the dual CFT could be obtained and were shown to be in agreement with Bekenstein-Hawking entropy. However, neither a formulation of the dual CFT has so far been obtained nor was it understood why there is a Kerr/CFT-correspondence. We notice, that the physical origin of a possible Kerr/CFT-correspondence is due to the criticality of the Kerr solutions.

5.1.3 Kerr/CFT from soft black hole hair

We have argued from various directions that the Hamiltonian phase space of pure Einstein gravity has to contain an infinite number of gapless excitations of the Kerr-family. But then, there is a problem. Where are these excitations that are among other things responsible for black hole microstates? According to the black hole uniqueness theorems, all solutions of the field equations that are asymptotically flat and stationary are given by the Kerr-metric up to diffeomorphisms. The hope is then that not all of these diffeomorphisms are gauge redundancies. Some of them should be physical excitations, i.e. shifts in the phase space, providing the necessary gapless excitation modes. This idea goes for the case of four-dimensional black holes already back to Carlip [28], has later on been one of the main motivations in the study of asymptotic symmetries [7] and has recently gained attention as the soft hair on black holes proposal [29]. However, a satisfactory analysis of the phase space so far has not been given in the literature.

In this chapter, we want to make a first step in this direction. Using mainly covariant phase space methods, we want to analyze the phase space near a Schwarzschild black hole solution. More specifically, we look at its gauge excitations and single out its surface degrees of freedom (chapter 5.3). These are found to violate the conventionally used Bondi fall-off conditions for the gravitational field. We explain in general, why these fall-off conditions are too restrictive in the presence of an event horizon (chapter 5.2). Calculating the surface charge algebra (chapter 5.4), we are able to propose a two-dimensional theory for the surface degrees of freedom of a Schwarzschild black hole. Remarkable is the appearance of central terms which supports the conjecture that the dual theory, if indeed being a CFT (Schwarzschild/CFT-correspondence) as suggested by the above reasoning, has a conformal anomaly.

We want to warn that the present work is just a first step and there are still a lot of things to be understood. An analysis of the phase space structure in the region of the Kerr-family is beyond our present scope. However, we explain which assumptions entered in the derivation of our dual theory (chapter 5.5) and give an outlook what is at our current investigation. Especially, Carlip's approach to entropy counting is in our approach a Sugawara-construction of a 2D stress-energy tensor for our dual theory. It is then tempting to expect that this dual theory is a CFT describing the phase space of the whole Kerr-solutions (Kerr/CFT-correspondence).

In the following, we use units in which we set the speed of light to 1 but we keep Newton's constant G and Planck's constant $\hbar$ explicit. Latin letters $a, b, \ldots = 0, \ldots, 3$ denote spacetime indices.

5.2 Cauchy-Data for asymptotically flat 4d spacetimes

We start by asking what is a possible set of Cauchy-data (gauge-fixed solution space) to specify a solution describing a particular state in phase space in Einstein gravity? This question already appeared in the study of gravitational waves starting with the analysis by Bondi-Metzner-Sachs [22, 23] and we adopt the answer which is reviewed in [7]. We denote coordinates by $(x^0, x^1, x^A) = (u, r, \vartheta, \varphi)$ with $A, B, \ldots = 2, 3$. The Bondi gauge-fixing conditions read

$$\begin{aligned} g_{rr} &= g_{rA} = 0 \\ \det g_{AB} &= r^4 \sin^2 \vartheta. \end{aligned} \tag{5.1}$$

Imposing Bondi fall-off conditions, the metric is written as

$$ds^2 = e^{2\beta}\frac{V}{r}du^2 - 2e^{2\beta}dudr + g_{AB}\left(dx^A - U^A du\right)\left(dx^B - U^B du\right) \tag{5.2}$$

with

$$g_{AB}dx^A dx^B = r^2\gamma_{AB}dx^A dx^B + O(r), \tag{5.3}$$

where

$$\gamma_{AB}dx^A dx^B = d\vartheta^2 + \sin^2\vartheta d\varphi^2 \tag{5.4}$$

is the metric on the unit 2-sphere. The remaining fall-off conditions are

$$\begin{aligned} \beta &= O(r^{-2}) \\ \frac{V}{r} &= -1 + O(r^{-1}) \\ U^A &= O(r^{-2}). \end{aligned} \tag{5.5}$$

The Bondi-gauge with required fall-offs is suited to describe the gravitational field of asymptotically flat spacetimes near null infinity J. (In this chapter, the required fall-offs correspond to retarded Bondi-gauge and cover the region near future null infinity J^+.)

A metric in Bondi-gauge and with Bondi fall-off conditions that is further satisfying vacuum Einstein field equations is fully determined by the set of functions

$$\begin{aligned} \mathcal{X} = \{&N_{AB}(u, x^C); M(u_0, x^A); N_A(u_0, x^B); C_{AB}(u_0, x^C); \\ &D_{AB}(x^C); E_{AB}(u_0, r, x^C)\}, \end{aligned} \tag{5.6}$$

for a fixed retarded time u_0. That means, to specify a concrete solution, one has to specify the Bondi-News $N_{AB}(u, x^C)$, which characterize the gravitational radiation passing through null infinity. The remaining part of the Cauchy-data consists of functions on S^2, which we will collectively denote boundary Cauchy-data (BCD). Among these are the mass and angular momentum aspects $M(u_0, x^A)$, $N_A(u_0, x^B)$ for fixed time, as well as leading BCD $C_{AB}(u_0, x^C)$, $D_{AB}(x^C)$ and subleading (in r) BCD summarized in the function $E_{AB}(u_0, r, x^C)$.

For the conditions on the functions appearing in $\mathcal{X}$ and their detailed connections to the metric (5.2), we refer to [7]. For the purpose of illustration, we give here the conventional leading large-r expansion near J^+ of (5.2) in terms of (5.6)

$$
\begin{aligned}
ds^2 = & - du^2 - 2dudr + r^2\gamma_{AB}dx^A dx^B \\
& + \frac{2M}{r}du^2 + rC_{AB}dx^A dx^B + D^B C_{AB} dudx^A \\
& + \frac{1}{16r^2}C_{AB}C^{AB}dudr \\
& + \frac{1}{r}\left(\frac{4}{3}N_A - \frac{4}{3}D_A M + \frac{1}{3}C_{AB}D_C C^{BC} + \frac{1}{4}C^{BC}D_A C_{BC}\right. \\
& \left. + \frac{4}{3}uD_A M - \frac{1}{8}D_A\left(C_{BD}C^{BD}\right)\right) dudx^A \\
& + \frac{1}{4}\gamma_{AB}C_{CD}C^{CD}dx^A dx^B \\
& + \dots .
\end{aligned}
\tag{5.7}
$$

We point out that the Bondi fall-off conditions are also imposed in the determination of the asymptotic symmetry algebra. That means, the asymptotic symmetries are defined as the residual gauge transformations preserving the Bondi gauge-fixing (5.1) as well as Bondi fall-offs (5.2)-(5.5). This results in the $\mathfrak{bms}_4$-algebra (see [22, 23] [24, 7, 25] for the various definitions and realization on gauge-fixed solution space (5.6)). However, our point is that in the presence of an event horizon the Bondi fall-offs (5.2)-(5.5) are too restrictive. As a consequence, precisely in the presence of a black hole, there is an *enhancement* in (5.6) in the required Cauchy-data by additional BCD.

As is already evident from the derivation of the gauge-fixed solution space (5.6) in [7], after relaxing the Bondi fall-offs, there are solutions with additional terms in (5.2) violating Bondi fall-offs. However, gravitational radiation passing through J^+ as characterized by the Bondi-News N_{AB} has no effect on them. In other words, there is no associated memory effect. Any additional Cauchy-data is seen as a redundancy.

The situation is different in the presence of an event horizon. There is *a priori* the possibility, that gravitational radiation passing the event horizon can leave an imprint on the additional terms in (5.2) that violate the Bondi fall-offs. This is the possibility that we want to advocate here. The additional BCD labels the different black hole microstates. Choosing different BCD corresponds to exciting different microstates. Imposing Bondi fall-offs (and thus ignoring the additional BCD), one encounters a sort of black hole information paradox: Looking at the solution space (5.6), there is no space for the black hole microstates.

What is then the additional BCD that has to be included in (5.6) in the presence of an event horizon? In the next chapter, we try to answer this question for the example of a Schwarzschild black hole, in which case the data (5.6) reads

$$\begin{aligned} N_{AB} = C_{AB} = D_{AB} = E_{AB} = 0 \\ N_A = 0 \\ M = \frac{r_S}{2G}, \end{aligned} \tag{5.8}$$

where r_S is the Schwarzschild radius.

5.3 Surface degrees of freedom of a Schwarzschild black hole

The well-known Schwarzschild metric is in Schwarzschild coordinates given by

$$ds^2 = -\left(1 - \frac{r_S}{r}\right) dt^2 + \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \tag{5.9}$$

Defining the tortoise coordinate

$$r^* = r + r_S \ln \left| \frac{r}{r_S} - 1 \right|, \tag{5.10}$$

one has

$$\frac{dr^*}{dr} = \left(1 - \frac{r_S}{r}\right)^{-1}. \tag{5.11}$$

Choosing ingoing Eddington-Finkelstein coordinates $(v, r, \vartheta, \varphi)$ with

$$v = t + r^*, \tag{5.12}$$

the metric reads

$$\begin{aligned} ds^2 &= -\left(1 - \frac{r_S}{r}\right) dv^2 + 2dvdr + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \\ &= -\left(1 - \frac{r_S}{r}\right) dv^2 + 2dvdr + r^2\gamma_{AB}dx^A dx^B. \end{aligned} \tag{5.13}$$

In this coordinates, the metric satisfies the Bondi-gauge conditions. However, note that from now on, we are working in advanced Bondi-gauge, in which the $r \to \infty$ limit, describes the region near past null infinity J^-.

We now fix a Schwarzschild radius r_S, then (5.13) provides a concrete reference point g_{ab} in gauge-fixed solution space. Our task in this chapter is to find nearby points in (gauge-fixed) solution space $g_{ab} + h_{ab}$, which are candidates for the microstates of this particular Schwarzschild black hole with mass parameter $\frac{r_S}{2G}$. As already noted at the end oft the last chapter, $g_{ab} + h_{ab}$ has to satisfy Bondi gauge-fixing conditions, but we expect it to violate Bondi fall-offs.

How do we then find the relevant excitations h_{ab} potentially responsible for black hole microstates?

Our strategy is that we insist on the existence of a consistent Hamiltonian description of the phase space at least in the neighboorhood of g_{ab}. To analyze the Hamiltonian phase space near g_{ab}, we use covariant phase space methods [39, 40] although at some points the direct Hamiltonian approach is employed. A review of the covariant phase space approach (including the relevant references) is given in [44], whereas the Hamiltonian approach is reviewed in [45].

A helpful observation comes from the black hole uniqueness theorems, which state that every asymptotically flat and stationary solution of the vacuum field equations in four dimensions is *diffeomorphic* to the Kerr-solution. Therefore, there is the possibility that the black hole microstates could be hidden in the form of excitations $h_{ab} = \mathcal{L}_\xi g_{ab}$ which take the form of gauge transformations. Of course, most of these excitations will correspond to gauge redundancies. However, there could be a subclass corresponding to the excitations of real physical degrees of freedom, i.e. a shift in the Hamiltonian phase space. This possibility was recently proposed in [29] and termed “soft hair on black holes.” (See also the earlier work of Carlip [28].) Nevertheless, a determination of the relevant degrees of freedom responsible for microstates is still missing. We want to make a proposal in this direction.

As explained, the candidate excitations h_{ab} should preserve Bondi-gauge (5.1) and must take the form of a gauge transformation $h_{ab} = \mathcal{L}_\zeta g_{ab}$ for a vectorfield ζ. However, we do not impose any fall-off conditions. These residual gauge transformations are found to be [7, 36]

$$\zeta = \zeta\left(X, X^A\right) = X\partial_v - \frac{1}{2}\left(rD_AX^A + D^2X\right)\partial_r + \left(X^A + \frac{1}{r}D^AX\right)\partial_A. \tag{5.14}$$

Here, $X = X(v, x^A)$ is an arbitrary scalar and $X^A = X^A(v, x^B)$ an arbitrary vectorfield on S^2. Indices $A, B, \ldots = 2, 3$ labeling coordinates on the sphere are raised and lowered with γ_{AB}. D_A denotes the associated covariant derivative and D^2 the Laplace-operator. The corresponding non-zero shifts in the metric components $h_{ab} = \mathcal{L}_\zeta g_{ab}$ are (with $V := 1 - \frac{r_S}{r}$)

$$\begin{aligned}
h_{vv} &= \frac{GM}{r}D_BX^B + \frac{GM}{r^2}D^2X - 2V\partial_vX - r\partial_vD_BX^B - D^2\partial_vX \\
h_{Av} &= -\frac{r}{2}D_AD_BX^B - \frac{1}{2}D_AD^2X - VD_AX + r^2\partial_vX_A + r\partial_vD_AX \\
h_{AB} &= r^2(D_AX_B + D_BX_A - \gamma_{AB}D_CX^C) + r(2D_AD_BX - \gamma_{AB}D^2X) \\
h_{vr} &= -\frac{1}{2}D_BX^B + \partial_vX.
\end{aligned} \tag{5.15}$$

To investigate, which of the excitations (5.15) are physical, we inspect the Hamiltonian generators of these excitations. The relevant formulas of the covariant phase space approach are reviewed in [44] on which we refer. We use also some formulas summarized in [36]. The covariant phase space $\bar{\mathcal{F}}$ is given by the (not gauge-fixed) solution space of the theory (set of field configurations satisfying equations of motion). After gauge-fixing, we obtain the gauge-fixed solution space Γ, which can be taken up to residual symplectic zero-modes as the phase space. Since we are only interested in the gauge excitations of a Schwarzschild black hole, we will consider the fixed point $g_{ab} \in \Gamma$ and gauge excitations in the tangent space $T_{g_{ab}}\Gamma$. In general, the Hamiltonian generator H of a gauge transformation $\mathcal{L}_\xi g_{ab}$ over a Cauchy-surface Σ is determined by

$$\delta H[h_{ab}; g_{ab}] = \int_\Sigma \omega[h_{ab}, \mathcal{L}_\xi g_{ab}; g_{ab}], \tag{5.16}$$

where δH denotes the variation of H between the points g_{ab} and $g_{ab} + h_{ab}$. On-shell (5.16) reduces to a boundary integral

$$\delta H[h_{ab}; g_{ab}] = -\frac{1}{16\pi G} \oint_{\partial\Sigma} *F, \tag{5.17}$$

for a well-known 2-Form F over the spacetime. We will consider the expression (5.17), where $\partial\Sigma$ is a cross-section from the event horizon. Thus $\partial\Sigma$ has fixed v and $r = r_S$ and has the topology of an S^2 parametrized by the remaining coordinates x^A. In this case, we have

$$\delta H[h_{ab}; g_{ab}] = -\frac{r_S^2}{16\pi G} \oint_{\partial\Sigma} d^2x \sqrt{\gamma} F_{rv}, \tag{5.18}$$

where $\gamma = \det \gamma_{AB}$ and

$$\begin{aligned} F_{rv}|_{r=r_S} = {} & \xi^A \left(\partial_r h_{Av} - \frac{2}{r_S} h_{Av}\right) + \xi^v \left(-\frac{1}{r_S^2} D^A h_{Av} + \frac{1}{r_S^2}\partial_v h^A{}_A - \frac{2}{r_S} h_{vv}\right. \\ & \left. - \frac{1}{2r_S^3} h^A{}_A\right) + \partial_r \xi^v h_{vv} + \frac{1}{r_S^2} D^A \xi^v h_{vA} - \frac{1}{2r_S^2}\partial_v \xi^v h^A{}_A + \xi^r \left(\frac{2}{r_S} h_{vr}\right. \\ & \left. + \frac{1}{r_S^3} h^A{}_A\right) + \frac{1}{2r_S^2}\partial_r \xi^r h^A{}_A. \end{aligned} \tag{5.19}$$

Here, the vectorfield ξ is the gauge transformation to be implemented by H and h_{ab} satisfies linearized field equations around the fixed g_{ab} but for later purposes h_{ab} need not to be gauge fixed in (5.19). (Therefore, (5.19) contains terms which vanish for h_{ab} respecting Bondi-gauge.)

The change of the Hamiltonian generator $\delta H_{(Y,Y^A)}$ implementing a gauge excitation $\zeta = \zeta(Y, Y^A)$ (see (5.14)) under a gauge excitation $h_{ab} = h_{ab}(X, X^A)$ (see (5.15)) is then given by

$$\begin{aligned} & \delta H_{(Y,Y^A)}[h_{ab}; g_{ab}] \\ & = \frac{r_S}{16\pi G} \oint_{\partial\Sigma} d^2x \sqrt{\gamma} \left(Y(1 - D^2) D_B X^B + D_B Y^B (D^2 - 1) X\right). \end{aligned} \tag{5.20}$$

From (5.20), we infer that excitations with

$$X = X(x^A) \\ X^A = X^A(x^B) \tag{5.21}$$

with non-vanishing divergence $D_A X^A$ change the on-shell values of the Hamiltonian generators (5.20).[1] They are non-zero modes of the presymplectic form and thus constitute physical excitations of the Schwarzschild black hole. Furthermore, we see that any v-dependence which would be allowed in the residual gauge transformation (5.14) does not constitute any new physical excitation other than (5.21).[2] At least from the point of view of the generators (5.20), all physical gauge excitations of g_{ab} are given by (5.21). In other words, the physical gauge excitations (which form a subspace of $T_{g_{ab}}\Gamma$) can be parametrized (in Bondi-gauge) by the coordinates

$$X = X(x^A) \\ D_A X^A, \tag{5.22}$$

where $X = X(x^A)$ is a scalar on S^2 and $X^A = X^A(x^B)$ is a vectorfield on S^2. These excitations are physical in the sense that they are shifts in the phase space. They form the gauge or surface degrees of freedom of the Schwarzschild black hole. We will refer to the coordinates (5.22) as the *gauge aspects.*

After having identified the gauge degrees of freedom of a Schwarzschild black hole (5.22), we make some comments on their geometry and physics.

The choice

$$X = f(x^A) \\ X^A = 0 \tag{5.23}$$

for a function f on S^2 in (5.14) corresponds to the usual $\mathfrak{bms}_4$-supertranslations [7]. These excitations respect Bondi fall-offs and are thus contained in the (gauge-fixed) solution-space spanned by the Cauchy-data (5.6). As explained in the last chapter, $\mathfrak{bms}_4$-supertranlations are thus not expected to be responsible for black hole microstates. Indeed, they just reflect the degeneracy of the gravitational vacuum [27]. It was already stated in [1] that ordinary $\mathfrak{bms}_4$-supertranslations[3] are not responsible for the microstates of a Schwarzschild black hole, but instead it was proposed that there is an enhanced asymptotic symmetry algebra.

[1]Note that the differential operator $D^2 - 1$ is invertible on S^2 as it has no zero eigenvalues.

[2]Note that all dependence on v-derivatives of X and X^A cancels in (5.20). (5.20) depends only on $X = X(v_0, x^A)$ and $X^A = X^A(v_0, x^B)$ with v_0 being the retarded time of $\partial\Sigma$.

[3]By the ordinary $\mathfrak{bms}_4$-supertranslations, we mean the supertranslations that are part of the asymptotic symmetry algebra $\mathfrak{bms}_4$ defined in [7]. As explained, for our case of the Schwarzschild spacetime g_{ab}, they take the form (5.14) with the choice (5.23).

It is the *enhancement* (which were called $\mathcal{A}$-modes)[4], which were proposed to be responsible for the microstates and correct entropy counting [2].[5] This reasoning resolves the criticism on the soft hair proposal correctly stated in [32, 33, 34, 35].

We are therefore left with the question whether (5.22) contains additional excitations (that are not part of $\mathfrak{bms}_4$) which then would by definition constitute the $\mathcal{A}$-modes.

What are then the additional excitations contained in (5.22) besides $\mathfrak{bms}_4$-supertranslations? For the vectorfield X^A on S^2 we have a Helmholtz theorem, i.e. we can decompose

$$X^A = Y^A - D^A g, \tag{5.24}$$

where Y^A is divergence-free $D_A Y^A = 0$ (and thus a gauge redundancy) and g is a scalar function on S^2. A proof of (5.24) is given in the Appendix. The gauge aspects (5.22) are thus parametrized by two scalars on S^2

$$\begin{aligned} X &= f \\ X^A &= -D^A g \end{aligned} \tag{5.25}$$

and this parametrization is unique up to constant shifts in g, which constitute gauge redundancies. As noted, f describes $\mathfrak{bms}_4$-supertranslations. What is the meaning of g? Out of the excitations (5.25), precisely the choice $\zeta = \zeta(X, X^A)$ with

$$\begin{aligned} X &= f \\ X^A &= -\frac{1}{r_S} D^A f \end{aligned} \tag{5.26}$$

keeps the induced metric on the event horizon invariant for an arbitrary scalar f on S^2. One has $\zeta|_{r=r_S} = f\partial_v$. Due to these similarities with the ordinary $\mathfrak{bms}_4$-supertranslations at null infinity, the excitations (5.26) are identified as event horizon supertranslations. In the limit $r_S \to \infty$ the future event horizon tends to past null infinity and indeed the event horizon supertranslations (5.26) converge to the $\mathfrak{bms}_4$-supertranslations at past null-infinity. We arrive at the conclusion, that the degrees of freedom of a Schwarzschild black hole are given by $\mathfrak{bms}_4$-supertranslations and the event horizon supertranslations (5.26). The latter contain a pure $\mathfrak{bms}_4$-supertranslation part. As these excitations reflect the

[4] By the enhancement, we mean any additional physical gauge excitations which are not part of $\mathfrak{bms}_4$ (as defined in [7]). These additional excitations were previously called $\mathcal{A}$-modes [1]. In the following it is analyzed whether such excitations ($\mathcal{A}$-modes) exist.

[5] This statement has to be understood as a proposal. So far, we have not proven that $\mathcal{A}$-modes are responsible for the microstates. However, in our opinion the proposal is justified by the following reasoning. If the phase space contains (soft, i.e. degenerate in energy) physical gauge excitations, which are not part of $\mathfrak{bms}_4$, then what is their physical meaning? It would in our opinion be strange if the phase space would contain such excitations which neither reflect the degeneracy of the gravitational vacuum nor the microstate degeneracy.

degeneracy of the gravitational vacuum, we subtract them to obtain the candidates for the black hole microstates

$$\begin{aligned} X &= 0 \\ X^A &= -D^A g \end{aligned} \tag{5.27}$$

with a scalar function g on S^2. Therefore, the physical gauge excitations (5.22) consist of the $\mathfrak{bms}_4$-supertranslations (5.23) and the additional excitations (5.27), which do not lie in $\mathfrak{bms}_4$, and therefore constitute the $\mathcal{A}$-modes. Thus, the asymptotic symmetries of the Schwarzschild solution g_{ab} are enhanced by the $\mathcal{A}$-modes (5.27) with respect to the asymptotic symmetry algebra $\mathfrak{bms}_4$ present also in the case without event horizon. Already in [1] the $\mathcal{A}$-modes were by this pure geometric reasoning (although in a different gauge) proposed as candidates for the microstates. It is nice to see, that a symplectic reasoning tends to the same answer.[6]

In addition, the $\mathcal{A}$-modes (5.27) violate the Bondi fall-off conditions as expected in chapter 5.2 for potential candidates for black hole microstates. That is, the set of data (5.6) is not enough to specify the excitations of g_{ab} given by (5.8). At the point g_{ab} in phase space the gauge aspect g provides additional Cauchy-data as it is a physical degree of freedom.

To summarize, in this chapter we have analyzed the Hamiltonian phase space near the point g_{ab} (5.8) (5.13) describing a Schwarzschild spacetime. Precisely, we analyze the tangent space $T_{g_{ab}}\Gamma$ of the phase space right at the point $g_{ab} \in \Gamma$. Motivated by black hole uniqueness theorems/soft hair proposal, we further restricted to tangent vectors h_{ab} that have the form of gauge transformations, i.e. that correspond to gauge excitations of g_{ab}. Gauge-fixing to Bondi-gauge (5.14), we constructed the Hamiltonian generators of these gauge excitations (5.20). We inferred that all physical gauge excitations of g_{ab} (i.e. those which are not gauge redundancies) are parametrized by the gauge aspects (5.25). They consist of $\mathfrak{bms}_4$-supertranslations reflecting the degeneracy of gravitational vacua. In addition, there are $\mathcal{A}$-modes (5.27) violating Bondi fall-offs and thus giving rise to additional BCD in (5.6) as expected in chapter 5.2 for excitations describing microstates. Thus, we propose the $\mathcal{A}$-modes (5.27) to be responsible for black hole microstates of g_{ab}.

5.4 Surface Charge Algebra

In the last chapter, we figured out the surface degrees of freedom of a Schwarzschild black hole. They are elements of the tangent space at g_{ab} describing gauge-fixed gauge excitations. In order to find their surface charge algebra in this chapter, we need first to make some

[6] In addition, our results are in line with the recent work [46]. There, it was also observed, using a different gauge, that $\mathfrak{bms}_4$-supertranslations of a Schwarzschild black hole are superpositions of event horizon supertranslations and an additional part. Since our approach uses Bondi-gauge, we see that this additional part (5.27) violates Bondi fall-offs.

technical considerations about how gauge-fixing takes place in the covariant phase space formalism.

What does gauge-fixing technically mean? Let $h_{ab} = \mathcal{L}_\xi g_{ab} \in T_{g_{ab}}\bar{\mathcal{F}}$ be a gauge excitation, which need not be gauge-fixed. That means, the vectorfield ξ has not to be a residual gauge transformation with respect to Bondi-gauge. By subtracting symplectic zero-modes from h_{ab}, we can construct a gauge-fixed excitation $\tilde{h}_{ab} \in T_{g_{ab}}\Gamma$. Since the non-gauge-fixed h_{ab} and the gauge-fixed $\tilde{h}_{ab}$ differ only by symplectic zero-modes, they are equal excitations in phase space. The mapping $h_{ab} \to \tilde{h}_{ab}$ is a projection operator, which performs the gauge-fixing of h_{ab}. We will construct in (5.32) such a projection operator which maps h_{ab} onto a gauge-fixed excitation of surface degrees of freedom in $T_{g_{ab}}\Gamma$ by dividing out symplectic zero-modes.

This will enable us to finally obtain the surface charge algebra given in (5.48) along with (5.41), (5.42), (5.49).

To derive (5.32), we consider the Hamiltonian generators $\delta H_{(X,X^A)}[h_{ab}; g_{ab}]$ of the surface degrees of freedom $(X, X^A) = (f, -D^A g)$ for scalar functions f, g on S^2 as found in (5.25). These generators are given by (5.18) and (5.19) with the vectorfield ξ being of the form (5.14) with the given functions (X, X^A).

These Hamiltonian generators define linear forms on the tangent space $T_{g_{ab}}\bar{\mathcal{F}}$. For an arbitrary vectorfield ξ on the spacetime, the linear forms have for the gauge excitation $h_{ab} = \mathcal{L}_\xi g_{ab}$ the form

$$\begin{aligned}
&\delta H_{(X,X^A)}[h_{ab}, g_{ab}] = -\frac{r_S^2}{16\pi G}\oint_{\partial\Sigma} d^2x\sqrt{\gamma}\,\Big(X^A r_S^2 \gamma_{AB}\partial_r\partial_v\xi^B \\
&+ D_A X^A \left(\frac{1}{r_S}\xi^v - 2\partial_r\xi^r - \frac{1}{r_S}\xi^r - \partial_v\xi^v - \frac{3}{2}D_B\xi^B\right) \\
&+ X\Big(-r_S\partial_r\partial_v D_B\xi^B + \frac{1}{r_S^2}D^2\xi^v - \frac{2}{r_S}\partial_r D^2\xi^r - \frac{2}{r_S^2}D^2\xi^r \\
&-\frac{1}{r_S}D_B\xi^B - \frac{1}{r_S}\partial_v D^2\xi^v - \frac{1}{r_S}D^2 D_B\xi^B\Big)\Big).
\end{aligned} \tag{5.28}$$

Performing on the vectorfield ξ^A on S^2 the decomposition (5.62)

$$\xi^A = \tilde{\xi}^A + D^A h, \tag{5.29}$$

where $\tilde{\xi}^A$ is divergence-free $D_A\tilde{\xi}^A = 0$ and h is a scalar on S^2, (5.28) is rewritten

$$\begin{aligned}
&\delta H_{(X,X^A)}[h_{ab}, g_{ab}] = -\frac{r_S^2}{16\pi G}\oint_{\partial\Sigma} d^2x\sqrt{\gamma}\,\Big(\\
&D_A X^A\left(-r_S^2\partial_r\partial_v h + \frac{1}{r_S}\xi^v - 2\partial_r\xi^r - \frac{1}{r_S}\xi^r - \partial_v\xi^v - \frac{3}{2}D_B\xi^B\right) \\
&+X\Bigg(-r_S\partial_r\partial_v D_B\xi^B + \frac{1}{r_S^2}D^2\xi^v - \frac{2}{r_S}\partial_r D^2\xi^r - \frac{2}{r_S^2}D^2\xi^r \\
&-\frac{1}{r_S}D_B\xi^B - \frac{1}{r_S}\partial_v D^2\xi^v - \frac{1}{r_S}D^2 D_B\xi^B\Bigg)\Big).
\end{aligned} \tag{5.30}$$

If $\xi = \xi(Y, Y^A)$ is itself chosen to be an excitation of the surface degrees of freedom with gauge aspects Y, Y^A (see (5.14) and (5.25)), we get as in (5.20)

$$\begin{aligned}
&\delta H_{(X,X^A)}[h_{ab} = h_{ab}(Y, Y^A); g_{ab}] \\
&= -\frac{r_S^2}{16\pi G}\oint_{\partial\Sigma} d^2x\sqrt{\gamma}\left(D_A X^A\cdot\frac{1}{r_S}(1-D^2)Y + X\cdot\frac{1}{r_S}(D^2-1)D_B Y^B\right).
\end{aligned} \tag{5.31}$$

That means, an arbitrary gauge-excitation ξ (not satisfying Bondi-gauge) excites (up to zero-modes of the symplectic form) the gauge aspects (Y, Y^A) determined by

$$\begin{aligned}
&\frac{1}{r_S}(1-D^2)Y \\
&= -r_S^2\partial_r\partial_v h + \frac{1}{r_S}\xi^v - 2\partial_r\xi^r - \frac{1}{r_S}\xi^r - \partial_v\xi^v - \frac{3}{2}D_B\xi^B\Big|_{\partial\Sigma} \\
&\frac{1}{r_S}(D^2-1)D_B Y^B \\
&= -r_S\partial_r\partial_v D_B\xi^B + \frac{1}{r_S^2}D^2\xi^v - \frac{2}{r_S}\partial_r D^2\xi^r - \frac{2}{r_S^2}D^2\xi^r \\
&-\frac{1}{r_S}D_B\xi^B - \frac{1}{r_S}\partial_v D^2\xi^v - \frac{1}{r_S}D^2 D_B\xi^B\Big|_{\partial\Sigma}.
\end{aligned} \tag{5.32}$$

The right hand side of (5.32) has to be evaluated at the coordinates $(v, r = r_S)$, where $\partial\Sigma$ is located. Since $D^2 - 1$ is an invertible operator on S^2, (5.32) defines uniquely the gauge aspects $(Y, D_A Y^A)$ as functions on S^2. The gauge excitation ξ can excite additional degrees of freedom corresponding to shifts of other Cauchy-data in (5.6). For example, ξ can excite also radiative degrees of freedom describing radiation passing through the event horizon or null-infinity. To determine the correct shifts in phase space the symplectic form (5.28) has to be evaluated both also with respect to all others than the surface degrees of freedom (X, X^A) and the location of $\partial\Sigma$ has to be varied across a whole Cauchy-surface. However, rather than doing a complete analysis of the phase space, we restrict ourselves to the surface degrees of freedom. Their excitations are are given (up to zero-modes of the linear forms (5.28), i.e. up to gauge redundancies) by (5.32).

In other words, (5.32) defines a projection operator, which maps the subspace of gauge-excitations $h_{ab} = \mathcal{L}_\xi g_{ab} \in T_{g_{ab}}\bar{\mathcal{F}}$ in the tangent space $T_{g_{ab}}\bar{\mathcal{F}}$ to an excitation in $T_{g_{ab}}\Gamma$ of the surface degrees of freedom with gauge aspects $(Y, D_A Y^A)$.

Let now

$$\begin{aligned}
\xi_1(X_1, X_1^A) &= X_1\partial_v - \frac{1}{2}\left(rD_A X_1^A + D^2 X_1\right)\partial_r + \left(X_1^A + \frac{1}{r}D^A X_1\right)\partial_A \\
\xi_2(X_2, X_2^A) &= X_2\partial_v - \frac{1}{2}\left(rD_A X_2^A + D^2 X_2\right)\partial_r + \left(X_2^A + \frac{1}{r}D^A X_2\right)\partial_A
\end{aligned} \tag{5.33}$$

be two gauge excitations with gauge aspects

$$\begin{aligned}
X_1 &= f_1 \\
X_1^A &= -D^A g_1 \\
X_2 &= f_2 \\
X_2^A &= -D^A g_2.
\end{aligned} \tag{5.34}$$

What are the gauge aspects (according to the projector (5.32)) of the Lie-bracket $[\xi_1, \xi_2]$? We have

$$[\xi_1, \xi_2]^v = X_1^A D_A X_2 - (1 \longleftrightarrow 2) \tag{5.35}$$

as well as

$$\begin{aligned}
[\xi_1, \xi_2]^r &= r\left(-\frac{1}{2}X_1^A D_A D_B X_2^B\right) \\
&+ \left(\frac{1}{4}D^2 X_1 D_B X_2^B - \frac{1}{2}X_1^A D_A D^2 X_2 - \frac{1}{2}D^A X_1 D_A D_B X_2^B\right) \\
&+ \frac{1}{r}\left(-\frac{1}{2}D^A X_1 D_A D^2 X_2\right) \\
&- (1 \longleftrightarrow 2)
\end{aligned} \tag{5.36}$$

and

$$\begin{aligned}
[\xi_1, \xi_2]^A &= \left(X_1^B D_B X_2^A\right) \\
&+ \frac{1}{r}\left(\frac{1}{2}D_B X_1^B D^A X_2 + X_1^B D_B D^A X_2 + D^B X_1 D_B X_2^A\right) \\
&+ \frac{1}{r^2}\left(\frac{1}{2}D^2 X_1 D^A X_2 + D^B X_1 D_B D^A X_2\right) \\
&- (1 \longleftrightarrow 2)\,.
\end{aligned} \tag{5.37}$$

From this, we infer for the gauge aspects

$$(Y, D_A Y^A) = (\hat{f}, -D^2\hat{g}) \tag{5.38}$$

of the Lie-bracket $[\xi_1, \xi_2]$ from (5.32)

$$\begin{aligned}
&\frac{1}{r_S}(1-D^2)Y \\
&= \frac{1}{r_S^2}\left(-\frac{5}{4}D^A X_1 D_A D^2 X_2\right) \\
&+ \frac{1}{r_S}\Big(X_1^A D_A X_2 - X_1^A D_A D^2 X_2 \\
&-\frac{1}{2}D_A X_1^A D^2 X_2 + \frac{1}{4}D_A D_B X_1^B D^A X_2\Big) \\
&- (1 \longleftrightarrow 2)
\end{aligned} \tag{5.39}$$

and

$$\begin{aligned}
&\frac{1}{r_S}(D^2-1)D_B Y^B \\
&= \frac{1}{r_S}(D^2-1)D_A\left(X_1^A D_B X_2^B\right) \\
&+ \frac{1}{r_S^2}\Big(-X_1^B D_B D^2 X_2 - \frac{1}{2}D_B X_1^B D^2 X_2 + \frac{1}{2}D_A D_B X_1^B D^A X_2 \\
&+D^2\left(-\frac{1}{2}D^A X_2 D_A D_B X_1^B + X_1^A D_A X_2\right)\Big) \\
&+ \frac{1}{r_S^3}\left(\frac{1}{2}D_A D^2 X_1 D^A X_2 + D^2\left(\frac{1}{2}D_A D^2 X_1 D^A X_2\right)\right) \\
&- (1 \longleftrightarrow 2)\,.
\end{aligned} \tag{5.40}$$

On the surface degrees of freedom (5.34), the conventional spacetime Lie-bracket is realized through the algebra[7]

$$\begin{aligned}
&(1-D^2)\hat{f} \\
&= \frac{1}{r_S}\left(-\frac{5}{4}D^A f_1 D_A D^2 f_2\right) \\
&+ \Big(- D^A g_1 D_A f_2 + D^A g_1 D_A D^2 f_2 \\
&+ \frac{1}{2}D^2 g_1 D^2 f_2 - \frac{1}{4}D_A D^2 g_1 D^A f_2\Big) \\
&- (1 \longleftrightarrow 2)
\end{aligned} \tag{5.41}$$

[7] Remember that the differential operator $D^2 - 1$ is invertible on S^2. Thus, $\hat{f}$ and $\hat{g}$ are uniquely determined by (5.41) and (5.42).

and

$$
\begin{aligned}
&(1-D^2)D^2\hat{g} \\
&= (D^2-1)D_A\left(D^A g_1 D^2 g_2\right) \\
&+ \frac{1}{r_S}\left(D^B g_1 D_B D^2 f_2 + \frac{1}{2}D^2 g_1 D^2 f_2 - \frac{1}{2}D_A D^2 g_1 D^A f_2\right. \\
&\left.+ D^2\left(\frac{1}{2}D^A f_2 D_A D^2 g_1 - D^A g_1 D_A f_2\right)\right) \\
&+ \frac{1}{r_S^2}\left(\frac{1}{2}D_A D^2 f_1 D^A f_2 + D^2\left(\frac{1}{2}D_A D^2 f_1 D^A f_2\right)\right) \\
&- (1 \longleftrightarrow 2)\,.
\end{aligned}
\tag{5.42}
$$

It is known, that the Hamiltonian generators form a representation (with respect to the Poisson-bracket) of the Lie-algebra of symplectic symmetries up to central extensions. That is,

$$\{H_X, H_Y\} = H_{[X,Y]} + K_{X,Y} \tag{5.43}$$

for symplectic symmetries X, Y and their generators H_X, H_Y. The central extension $K_{X,Y}$ is a c-number, which is a constant over path-connected parts of the phase space. $[X, Y]$ is the Lie-bracket of X and Y as vectorfields on the phase space. If $X = \delta_{\xi_1}$ and $Y = \delta_{\xi_2}$ are gauge transformations, we assume that (5.43) takes on-shell the form

$$\{H_{\xi_1}, H_{\xi_2}\} = H_{[\xi_1,\xi_2]} + K_{\xi_1,\xi_2} \tag{5.44}$$

with $[\xi_1, \xi_2]$ being the Lie-bracket of vectorfields on the spacetime manifold. That means, on shell $[X, Y] = \delta_{[\xi_1,\xi_2]}$ up to gauge redundancies.[8]

Choosing in (5.44) for the gauge transformations the surface degrees of freedom (5.33), we get

$$\left\{H_{(X_1,X_1^A)}, H_{(X_2,X_2^A)}\right\} = H_{(Y,Y^A)} + K_{(X_1,X_1^A),(X_2,X_2^A)}. \tag{5.45}$$

Remembering $\left\{H_{(X_1,X_1^A)}, H_{(X_2,X_2^A)}\right\} = \delta_{(X_2,X_2^A)} H_{(X_1,X_1^A)}$ we get the central term from (5.20)

$$
\begin{aligned}
K_{(X_1,X_1^A),(X_2,X_2^A)} &= \frac{r_S}{16\pi G}\oint_{\partial\Sigma} d^2x\sqrt{\gamma}X_1(-D^2+1)D_A X_2^A \\
&- \frac{r_S}{16\pi G}\oint_{\partial\Sigma} d^2x\sqrt{\gamma}X_2(-D^2+1)D_A X_1^A \\
&- H_{(Y,Y^A)}[g_{ab}].
\end{aligned}
\tag{5.46}
$$

[8]Although (5.43) is often used in the form (5.44) [47, 48, 49], we do not know a proof of that. We further comment on this assumption in the next chapter. For now, in this chapter we justify the use of (5.44) by being able to reproduce known results.

Hamiltonian generators are determined only up to a constant. We use this freedom to set all surface charges to 0 at the reference solution g_{ab}

$$H_{(X,X^A)}[g_{ab}] = 0. \tag{5.47}$$

This choice fixes uniquely all generators and the central terms (5.46).

To summarize, for the surface degrees of freedom (5.25) of a Schwarzschild black hole, the surface charge algebra is given by

$$\{H_{f_1,g_1}, H_{f_2,g_2}\} = H_{\hat{f},\hat{g}} + K(f_1, g_1; f_2, g_2). \tag{5.48}$$

Here, the gauge aspects $\hat{f}$ and $\hat{g}$ are given by the algebra (5.41) (5.42) and the central term follows from (5.46) (with the choice (5.47))

$$\begin{aligned} K(f_1, g_1; f_2, g_2) = \frac{r_S}{16\pi G} \oint_{\partial\Sigma} d^2x \sqrt{\gamma} f_1 (D^2 - 1) D^2 g_2 \\ - (1 \longleftrightarrow 2)\,. \end{aligned} \tag{5.49}$$

We comment on some implications of this algebra. First, we have for the choice $f_1 = r_S, g_1 = 0$ and $f_2 = f, g_2 = g$ the bracket

$$\{H_{r_S,0}, H_{f,g}\} = 0. \tag{5.50}$$

The charge $H_{r_S,0}$ is (up to constant shift set by (5.47) and normalization) equal to the ADM-energy subtracted of by the energy passing through future null infinity and the portion of the event horizon between the location of $\partial\Sigma$ and the horizon's future end point. Thus, if there is no radiation passing through these regions, $H_{r_S,0}$ coincides with the ADM-energy. (5.50) then states that the surface degrees of freedom are gapless excitations, i.e. they keep the ADM-energy invariant. They provide soft black hole hair. As mentioned, the $\mathfrak{bms}_4$-supertranslations f reflect degeneracy of the gravitational vacuum. The $\mathcal{A}$-modes g are the gapless Bogoliubov-modes associated with the criticality of the Schwarzschild black hole.

Furthermore, as a consistency check, we find that the Poisson-bracket between event horizon supertranslations (5.26) (i.e. choosing $g_i = \frac{1}{r_S} f_i$ for $i = 1, 2$ and arbitrary f_i in (5.48)) vanishes. This is in agreement with [50, 51].

We have identified the surface degrees of freedom of a Schwarzschild black hole as the gauge aspects, which are functions on S^2. The algebra with respect to the Poisson-bracket of the gauge aspects is given by (5.48). We thus arrived at a lower dimensional theory describing part of the phase space near the Schwarzschild solution g_{ab}. This aims to be a new and concrete realization of the holographic principle [52, 53] for the case of a Schwarzschild black hole.

5.5 Assumptions, Limitations and Outlook

After having found the surface degrees of freedom of a Schwarzschild black hole as well as their algebra under the Poisson-bracket, which aims at providing a dual theory for the Schwarzschild black hole, it is interesting to analyze its consequences. However, we want to warn that in our path, we made several assumptions. These assumptions may cause corrections to our results. In this chapter, we want to list these assumptions and give an outlook. Further investigation of these issues will be left for future research.

5.5.1 Choice of symplectic form, integrability vs. Gibbons-Hawking-York term

Given the Lagrangian of a theory, the covariant phase space formalism starts by prescribing a presymplectic potential. Unfortunately, this prescription is affected by adding a boundary term to the action and has a further ambiguity on its own (see [44]). These ambiguities affect the definition of the presymplectic form and therefore also the Hamiltonian generators. As commonly done in the literature, we used in our derivations of formulas like (5.20) the canonical presymplectic potential as derived from the Einstein-Hilbert action.

On the other hand, in the Hamiltonian approach (see [45]) any ambiguity in the definition of the Hamiltonian generators is fixed (of course up to a constant) by the requirement of differentiability in the sense of Regge-Teitelboim [54]. Having found a candidate for a Hamiltonian generator of a symplectic symmetry, a suited boundary term has to be added to make the generator a differentiable functional over phase space. This fixes any ambiguity.

Having a theory with a well-defined action, that means, an action that is added a suited boundary term to ensure Regge-Teitelboim differentiability in the variational principle, there is the following version of Noether's theorem incorporating boundary effects:

For a symmetry of a well-defined action, the canonical Noether-procedure assigns a charge which is a differentiable Hamiltonian generator of that symmetry (see [45] for the details).

The derivation of black hole entropy in [55] using Euclidean methods suggests that the variation of the Gibbons-Hawking-York boundary term S_{GHY} vanishes

$$\delta_\xi S_{GHY}\big|_{g_{ab}} = 0 \tag{5.51}$$

for the physical gauge excitations ξ of the black hole geometry g_{ab} that are responsible for the microstates. That means first, that for the construction of the Hamiltonian generators of the ξs, the boundary term in the action does not affect the presymplectic potential. Second, the above Noether-theorem guarantees the existence of differentiable Hamiltonian generators constructed by the canonical Noether-procedure.

In summary, the canonical choice of the presymplectic potential (that we used throughout) is justified for the problem. However, it has to be checked that for our surface degrees of freedom (5.51) is indeed satisfied

$$\delta_{f,g} S_{GHY}|_{g_{ab}} = 0 \tag{5.52}$$

for all gauge aspects f, g and the reference metric g_{ab}.

Note that the above Noether-theorem also guarantees integrability of the Hamiltonian generators (5.20) over a suited region in phase space near g_{ab}. Note also that over the last chapter, we assumed integrability, which is in general not guaranteed.

Our physical interpretation of (5.52) is that gauge excitations f, g do not excite gravitational radiation passing through boundaries of spacetime. It was already noted in [40] that integrability of Hamiltonian generators is spoiled by flux terms.

5.5.2 Lie-bracket vs. surface deformation bracket

As noted in the last chapter, the algebra (5.43) was assumed to take the form (5.44) on-shell. Although (5.44) is often used [47, 49, 48], we are not aware of a general proof. In the Hamiltonian approach [56] a known result states that for spacetime vectorfield ξ_1, ξ_2 one has the relation

$$\{\Gamma_{\xi_1}, \Gamma_{\xi_2}\} = \Gamma_{\{\xi_1,\xi_2\}_{\mathrm{SD}}} + K_{\xi_1,\xi_2} \tag{5.53}$$

if differentiable Hamiltonian generators $\Gamma_{\xi_1}, \Gamma_{\xi_2}$ are existent. Here, $\{\xi_1, \xi_2\}_{\mathrm{SD}}$ is the surface deformation bracket which is in general different from the Lie-bracket $[\xi_1, \xi_2]$. The difference is calculated in [56] and it is argued why it often happens (but not has to happen) that on-shell

$$\Gamma_{\{\xi_1,\xi_2\}_{\mathrm{SD}}} = \Gamma_{[\xi_1,\xi_2]}. \tag{5.54}$$

(5.54) has to be checked and this was the assumption made in the derivation of the surface charge algebra in the last chapter.

5.5.3 Sugawara-construction of 2D stress-tensor and entropy counting

In the last chapter, we found a lower-dimensional theory on S^2 with the gauge aspects as degrees of freedom and their Poisson-brackets given by (5.48). This theory describes part of the phase space near the Schwarzschild solution g_{ab}. Note that so far, we did not specify how the word “near” has to be understood.

Strictly speaking, we performed our calculations right at the reference point g_{ab} in phase space and in the tangent space thereof (see formulas like (5.20)). As explained in chapter 5.5.1 the algebra (5.48) is derived under the assumption of integrability. That is, for the generators of gauge aspects, (5.16) defines a 1-form $\delta H_{f,g}$ over phase space Γ which can over a suited region $S \subseteq \Gamma$ be integrated to obtain generators $H_{f,g}$ satisfying the algebra (5.48) over this region $S \subseteq \Gamma$. Our analysis in $T_{g_{ab}}\bar{\mathcal{F}}$ was powerful enough to obtain the algebra (5.48). However, only at the point g_{ab}, we know how the excitation of the gauge aspects generated by $H_{f,g}$ looks like (see (5.15) with (5.25)). The action of $H_{f,g}$ at other points in S, we do not know in general. Of course, the residual gauge transformations at other points in S look different than in (5.14). Neither, we know how large the region $S \subseteq \Gamma$ is. We want to argue for a reasonable S by asking what the theory obtained actually describes?

Since we showed, that the gauge aspects are gapless excitations of a Schwarzschild black hole, S should contain these points. As already explained in chapter 5.1, this scale invariance suggests that our two-dimensional theory is a conformal field theory. This Schwarzschild/CFT-correspondence would then deliver a two-dimensional CFT which describes the part of the phase space S of the full four-dimensional Einstein-gravity. S at least contains the gapless excitations of the Schwarzschild black hole.

A conformal anomaly (as suggested by the appearance of central terms in (5.48)) would then reflect the quantum mechanical lifting of gapless modes by $\frac{1}{S}$-corrections as explained in the introduction.

If the dual theory of the last chapter is indeed conformally invariant, it has to posess a 2D stress-tensor with the Virasoro-algebra being compatible with (5.48). Since we know the algebra (5.48), it is natural to search for the stress-tensor via a Sugawara-construction. That is, we construct the Virasoro-generators out of the surface degrees of freedom under the requirement of validity of the Virasoro-algebra. As an ansatz for the Virasoro-generators, we motivate ourselves with the cases of the Brown-Henneaux analysis [57] or the case of extremal Kerr/CFT [37, 58]. There, the Virasoro-generators themselves are the generators of suited gauge transformations. Following 5.5.2, we search for spacetime vectorfields satisfying a Witt-algebra with respect to the Lie-bracket. The associated generators from the gauge aspects (obtained with the projection operator (5.32)) then satisfy via (5.48) a Virasoro-algebra and thus are candidates for the Virasoro-generators building the stress-tensor.

To this end, we define the spacetime vectorfields

$$\xi^a_n = \begin{pmatrix} \xi^v_n \\ \xi^r_n \\ \xi^\vartheta_n \\ \xi^\varphi_n \end{pmatrix} = \begin{pmatrix} 2r_S A\left(1 - \frac{A}{A+B}\left(1 + \frac{in}{A}\right)\right) \\ r_S\left(\frac{r_S}{r} - 1\right) in \\ 0 \\ \frac{A}{A+B}\left(1 + \frac{in}{A}\right) \end{pmatrix} e^{\frac{in}{2r_S A}v} e^{in\varphi} \tag{5.55}$$

and

$$\bar{\xi}^a_n = \begin{pmatrix} \bar{\xi}^v_n \\ \bar{\xi}^r_n \\ \bar{\xi}^\vartheta_n \\ \bar{\xi}^\varphi_n \end{pmatrix} = \begin{pmatrix} -\frac{2r_S AB}{A+B}\left(1 + \frac{in}{B}\right) \\ -inr_S\left(1 - \frac{r_S}{r}\right) \\ 0 \\ \frac{B}{A+B}\left(1 + \frac{in}{B}\right) \end{pmatrix} (-1)^{\frac{in}{B}} e^{\frac{inr^*}{r_S B}} e^{-\frac{inv}{2r_S B}} e^{in\varphi} \tag{5.56}$$

for $n \in \mathbb{Z}$. The vectorfields are given in infalling Eddington-Finkelstein coordinates used in chapter 5.2. The constants $A, B \in \mathbb{R}$ are arbitrary. We then have $(\xi^a_n)^* = \xi^a_{-n}$ and $(\bar{\xi}^a_n)^* = \bar{\xi}^a_{-n}$. They fulfill two copies of the Witt-algebra

$$\begin{aligned}[\xi_m, \xi_n] &= -i(m-n)\xi_{m+n} \\ \left[\bar{\xi}_m, \bar{\xi}_n\right] &= -i(m-n)\bar{\xi}_{m+n} \\ \left[\xi_m, \bar{\xi}_n\right] &= 0.\end{aligned} \tag{5.57}$$

The choice is motivated by similar vectorfields appearing in Carlip's approach to entropy counting in [28, 59] but changed in such a way as to satisfy Witt-algebra (5.57) and treat future and past horizon equally. Similar vectorfields appear in [48]. Let (f_n, g_n) be the associated gauge aspects to (5.55). Furthermore, let

$$H_n := H_{(f_n, g_n)} \tag{5.58}$$

be the associated Hamiltonian generators under the choice (5.47) $H_n[g_{ab}] = 0$ for the fixed reference solution g_{ab}. Since H_n has dimension of an action, we can define dimensionless generators

$$\hbar L_n := H_n + \frac{r_S^2}{4G} \frac{2A^2B + B - A}{(A+B)^2} \delta_n \tag{5.59}$$

for $n \in \mathbb{Z}$ and with $\delta_n = \delta_{n,0}$ being the Kronecker delta.

Computing the central terms from the algebra (5.48) and under the assumptions of this chapter, we get the classical Virasoro-algebra

$$\{L_m, L_n\} = -\frac{i}{\hbar}(m-n)L_{m+n} - \frac{i}{\hbar^2}\frac{r_S^2}{2G}\frac{B-A}{(A+B)^2} m(m^2-1)\delta_{m+n}. \tag{5.60}$$

Canonical quantization yields a Virasoro-algebra with (using standard conventions)

$$\begin{aligned} c &= \frac{6r_S^2}{\hbar G}\frac{B-A}{(A+B)^2} \\ L_0[g_{ab}] &= \frac{r_S^2}{4\hbar G}\frac{2A^2B + B - A}{(A+B)^2}. \end{aligned} \tag{5.61}$$

We note that our computation of surface charges in (5.18) and thus of gauge aspects use $\partial\Sigma$ to be located on the future event horizon at a particular time v. Whereas the gauge aspects of (5.55) (f_n, g_n) depend on the choice of v, the result (5.61) does not. Unfortunately, the computation of gauge aspects of (5.56) contains divergences. This is due to the fact, that whereas (5.55) is regular at the future event horizon, (5.56) is at the past event horizon but are singular vice versa. Performing the computation of the gauge aspects $(\bar{f}_n, \bar{g}_n)$ of (5.56) at the past event horizon, the anti-chiral analog of (5.61) $\bar{c}, \bar{L}_0$

does not depend on the location of $\partial\Sigma$ and thus the limit of taking $\partial\Sigma$ to the bifurcation of the horizons is for the evaluation of the Virasoro-algebras well-defined. Unfortunately, the projection formulas (5.32) are not suited to determine the anti-chiral gauge aspects $(\bar{f}_n, \bar{g}_n)$. This is due to the fact, that their derivation has to be refined in that (working in the advanced Bondi-gauge) the limit where $\partial\Sigma$ goes to the past horizon has to be taken carefully. We note that these issues are under current investigation. The hope then is, that counting the degeneracy with the Cardy-formula matches Bekenstein-Hawking entropy. However, there must be a finite result for the anti-chiral analog of (5.61) as we could have also performed the calculation in retarded Bondi-gauge. The gauge aspects would then have to be matched by a similar matching condition as the one in [27].

Wheras there are still issues under current investigation, our approach sheds new light on Carlip's approach to a microcanonical counting of entropy [28, 47]. In Carlip's approach, the choice of vectorfields giving rise to Virasoro-algebra seems ad-hoc. The near-horizon asymptotic symmetry algebra has to be unnaturally reduced to yield a Virasoro-algebra with central terms for the generators [49]. In our approach, such a reduction is first due to dividing out zero-modes by projecting arbitrary gauge excitations onto the surface degrees of freedom via (5.32). That is, different gauge excitations can correspond to the same excitations of the gauge aspects. Second, only the very special generators (5.59) correspond to Virasoro-generators out of the full set of generators of surface degrees of freedom.

On the other hand, note that in (5.61) r_S is the Schwarzschild-radius of the reference solution g_{ab}. It is a fixed parameter for our dual theory. Also note the appearance of the two arbitrary parameters A, B in (5.61). Such an ambiguity was already present in Carlip's approach, although it was canceled in the entropy counting giving consistent result.[9] This ambiguity reflects the fact that Hamiltonian generators are only defined up to constant. Had we chosen in (5.47) a different reference solution, we would have obtained a different theory (5.48) with other central terms and this would affect the associated Virasoro-algebra. This ambiguity is reflected in the presence of the parameters A, B in (5.61).

After all, it is tempting to expect that $S \subseteq \Gamma$ covers the whole Kerr-family. That is, we conjecture our dual theory describes part of the phase space containing the Kerr-family and its gapless excitations. Such a Kerr/CFT-correspondence was already conjectured in [38] from the study of scattering off a non-extremal Kerr black hole. Comparing with (5.55) (5.56) the Virasoro-modes $L_0, \bar{L}_0$ "measure" the mass and angular momentum parameter of a particular Kerr black hole.

Whether this is the right way to think about the problem has still to be understood. We have given an outlook of what is currently at our investigation.

[9] See the sentence before and after equation (3.5) in [28].

Appendix: Proof of (5.24)

In this chapter, we want to prove that a vectorfield X^A on S^2 has a Helmholtz-Hodge decomposition

$$X^A = Y^A + D^A f, \tag{5.62}$$

where Y^A is a divergence-free vectorfield on S^2 $D_A Y^A = 0$ and f is a scalar function on S^2. Proving (5.62), we have proven (5.24).

Let X^A be a vectorfield on S^2. According to the Hodge-decomposition, we can write the 1-form X_A as

$$X = df + \delta\beta + \gamma, \tag{5.63}$$

where f is a scalar on S^2, β is a 2-form on S^2 and γ is a harmonic 1-form. d denotes the exterior derivative and δ the codifferential. On S^2, there are no harmonic 1-forms, since the first de Rham cohomology-group vanishes. Thus, $\gamma = 0$. Defining the vectorfield

$$Y^A = (\delta\beta)^A \tag{5.64}$$

(5.62) follows immediately from (5.63)

$$X^A = D^A f + Y^A,$$

if we can show $D_A Y^A = 0$. For a generic vectorfield V^A, we have for the associated 1-form V_A

$$\delta V = - * d(*V) = - * *(D_A V^A) = -D_A V^A. \tag{5.65}$$

Using this identity, we conclude

$$-D_A Y^A = \delta Y = \delta^2 \beta = 0.$$

Y^A is indeed divergence-free and this shows (5.62).

Chapter 6

Entropy Counting from Schwarzschild/CFT and Soft Hair

We provide in this chapter a review of the paper [4]. This chapter is a slightly edited version of [4] and for more details, we refer to the original paper.

We start with a very brief overview of the main aspects and later provide the technical details.

After having obtained a candidate theory that is supposed to describe the part of phase space responsible for the microstates of a Schwarzschild black hole, the next step is clear: To provide further evidence of the correctness of the 2D dual theory that we obtained in [3] and explore its consequences.

The work of [3] has to be considered as a first step in analyzing the Hamiltonian phase space of black holes. There, we applied new tools to analyze the phase space in a systematic way for the specific case of a Schwarzschild black hole in Einstein-gravity. These general tools are of possible interest on their own right. We briefly give a first presentation of those tools themselves for the general case of an arbitrary theory in [4].

In the application of those general tools in [3], the calculations were limited to a minimum in order to be feasible but powerful enough to provide first non-trivial results. For example, we analyzed the phase space restricted to the point right at the Schwarzschild solution and tangent vectors that have the form of gauge transformations in order to obtain a 2D dual theory describing the Schwarzschild black hole. However, the obtained dual theory is expected to describe the phase space over the whole Kerr-family. In fact, in [3] we gave a complete list of the assumptions and computational limitations. Furthermore, we provided an outlook of necessary consistency checks and how a relaxation of the assumptions and enlargeing computations might possibly affect the obtained 2D dual theory.

Carrying out this program is in essence the main part that would naturally constitute the next step following the last chapter. The most important guiding principle for this is the correct reproduction of the Bekenstein-Hawking entropy by the dual 2D theory. Therefore, the main question of this chapter is whether such a reproduction is possible?

We conjectured in [3] that the obtained 2D dual theory is a CFT (by this we mean, that it possesses an invariance under the 2D local conformal algebra) as suggested by the criticality of the black hole. From this perspective, Carlip's approach to entropy counting [28, 47] reemerges as a Sugawara construction of a 2D stress-tensor out of the degrees of freedom present in the 2D theory. This provides an opportunity for both a substantial consistency check for the correctness of the 2D theory and support for Carlip's approach including a microcanonical explanation of entropy.

The main point of [4] is to revisit Carlip's approach to entropy counting. There, we first explain why such a revision is necessary. Although motivated by the Schwarzschild/CFT proposal of [3], such a revision is also of interest on its own right and can be discussed independently of the issues in [3]. In the revision [4], for the example of a Schwarzschild black hole, we show how to single out diffeomorphisms forming in contrast to Carlip's analysis the full 2D local conformal algebra. We provide arguments as to why their Hamiltonian generators are expected to be the symmetry generators of a possible conformal field theory describing the part of phase space responsible for black hole microstates. Then, we can infer central charges and temperatures of this CFT by inspecting the algebra of these Hamiltonian generators. Using these data in the Cardy formula, precise agreement with the Bekenstein-Hawking entropy is found.

Hence, we have shown that after all diffeomorphisms *do* exist whose generators provide reasonable candidates for the Virasoro-generators of a *possible* CFT that might describe the part of phase space responsible for black hole microstates. While the idea itself is not new, other approaches like Carlip's original approach [47] or the more recent one by Haco, Hawking, Perry, Strominger [60] differ in proposing different choices of diffeomorphisms. However, we comment on other choices and observe that they suffer from various subtleties. For instance, in the original approach [47] only one chiral half of a Virasoro-algebra of diffeomorphisms is found whereas two copies are expected for the symmetry algebra of the anticipated CFT. On the other hand, the choice in [60] is only able to reproduce the Bekenstein-Hawking entropy after non-canonical corrections of the symplectic structure that is used to study the algebra of diffeomorphism generators.

Contrary to that, we have shown in [4] that there exists a "preferred" choice of diffeomorphisms forming two commuting copies of a Virasoro-algebra and providing reasonable candidates for the symmetry generators of a possible CFT that is supposed to describe the microcanonical physics of the black hole. Indeed, we have shown in [4] that this choice exactly reproduces the Bekenstein-Hawking entropy with a canonical choice of the symplectic structure.

To summarize, *if* there is a CFT describing the part of phase space of e.g. a Schwarzschild black hole, then we have shown that reasonable candidates for the associated conformal generators *do* exist and can be explicitly found. That is, we know what the conformal generators should be. However, in this analysis - this is a drawback of all approaches following Carlip's idea - it is not known *what* this CFT explicitly is or whether it even exist.

Nevertheless, this provides the opportunity for a substantial consistency check of any e.g. Schwarzschild/CFT proposal that one might have under consideration. As we explain in [3, 4], we have to directly project the above-mentioned conformal generators onto the observables present in the dual theory given by the Schwarzschild/CFT proposal. If counting the state degeneracy after projection still reproduces the Bekenstein-Hawking entropy, this would provide substantial consistency check that the considered Schwarzschild/CFT proposal contains exactly the right black hole degrees of freedom or otherwise needs refinement.

To carry out the step for the explicit Schwarzschild/CFT propsal of [3] is still under investigation. The purpose of [4] was to find out *what* the proper conformal generators are that have to be projected onto potential black hole degrees of freedom.

The steps to appear in future research are therefore twofold. Using covariant phase space techniques, we have seen there is a natural way to parametrize the Hamiltonian phase space of a general theory. These are the tools that we used to obtain the Schwarzschild/CFT proposal in the last chapter. To understand these general tools further and apply in simpler theories than gravity is of course necessary to finally understand any Schwarzschild/CFT proposal and will appear somewhere else. Furthermore, we have to carry out the above-mentioned consistency procedure to arrive at the proper Schwarzschild/CFT-correspondence capturing the right gravitational hair responsible for microstates and Bekenstein-Hawking entropy.

The technical details of the statements made in this chapter will be concretely explained in the remaining part of this chapter.

6.1 Introduction to the Problem

It is one of the main problems in quantum gravity to explain the microcanonical origin of the Bekenstein-Hawking entropy $S = \frac{A}{4\hbar G}$ of black holes [11, 13, 12]. In the classical $\hbar \to 0$ limit the entropy becomes infinite. Therefore, the Hamiltonian phase space of pure Einstein gravity has to contain infinitely many points corresponding to the microstates of a black hole for fixed mass and angular momentum parameter. On the other hand, the black hole uniqueness theorems [15] tell that asymptotically flat and stationary solutions of Einstein's field equations are given by the Kerr-family up to diffeomorphisms. This can create the impression of an arising paradox: Due to the uniqueness theorems, it may naively seem that there is no place in Hamiltonian phase space that can accomodate the infinitely many microstates as required by the classically infinite entropy. Therefore, one can ask: How to reconcile the black hole uniqueness theorems with the classically infinite Bekenstein-Hawking entropy?

Since the uniqueness theorems single out the Kerr solutions up to diffeomorphisms, they already themselves suggest a possible solution. It may be that some of the diffeomorphisms are physical, i.e. shifts in phase space rather than gauge redundancies. This phenomenon is known to happen in gauge theories typically when the gauge parameters are non-vanishing in some asymptotic region. Such asymptotic symmetries could in gravity then be responsible for microstates of a Kerr black hole.

Indeed, the study of asymptotic symmetries in 3D gravity [57] brought some success in understanding the BTZ black hole. It is found that the asymptotic symmetry algebra contains the 2D local conformal algebra. Conformal field theory techniques can then be used to count the state degeneracy [61] and agreement with the Bekenstein-Hawking entropy is found. Carlip raised the idea [28, 47, 62, 63] to mimic this in the higher-dimensional case. Although it is not clear which gauge transformations are responsible for microstates, Carlip was able to single out a Witt-algebra of diffeomorphisms in the presence of a black hole event horizon. The Hamiltonian generators of these diffeomorphisms are then candidates for the generators of a possible conformal symmetry that may govern the part of phase space responsible for black hole microstates. Hamiltonian methods can then be used to study the conformal algebra of the diffeomorphism generators and CFT techniques then to count the state degeneracy. Indeed, agreement with the expected Bekenstein-Hawking entropy is found.

Although Carlip's approach is universal, it tells nothing about what the possible underlying CFT describing the relevant part in phase space really is. To understand this part in phase space was always one of the main motivations in the study of asymptotic symmetries (see for instance [7] and references therein). The idea recently gained new interest as the proposal of "soft black hole hair" [29].[1] To analyze the structure of Hamiltonian

[1] The most simplest choice, the $\mathfrak{bms}_4$-supertranslations contained in the asymptotic symmetry algebra of spacetimes that are asymptotically flat at null infinity [7], does not work. It was already explained in [1] that they can not be responsible for microstates. Instead, it was proposed there that the presence of an event horizon enhances the asymptotic symmetry algebra and it is the *enhancement* that is responsible for microstates and entropy counting [2]. This provides the resolution to the criticism on the soft hair

phase space in the vicinity of a black hole state is nevertheless still a necessary and open problem.

To improve the situation is the overall goal of our investigations and in [3] we described, how the Hamiltonian phase space can be analyzed in a systematic way. For a Schwarzschild black hole and assuming in a sense the application of the simplest possible scenario, we proposed a concrete candidate of a dual theory describing the part of phase space responsible for microstates. This theory was given in terms of its observables and their Poisson-bracket algebra. If conformally invariant, as it is expected from several directions, Carlip's approach to entropy counting reemerges at this point as a Sugawara-construction of the conformal generators of this Schwarzschild/CFT correspondence out of its observables. Accordingly, one has to find a suited choice of Witt-algebra of diffeomorphisms. Carlip presented a general construction of such an algebra in the presence of a black hole.

Here, we want to revisit Carlip's approach. In Carlip's construction only one copy of a Witt-algebra of diffeomorphisms and associated Hamiltonian generators are found. The two-dimensional conformal algebra consists however of two commuting copies. Since there seems to be no reason, why black holes should be described by chiral CFTs, it is natural to seek for diffeomorphisms building two Witt-algebra copies. Is such a choice possible and does it lead to something maybe even more appropriate?

That such a choice is possible follows directly from [3]. There, we have provided a $Vir \oplus \overline{Vir}$-algebra of diffeomorphisms. Using this choice, the entropy counting procedure in the context of the proposed Schwarzschild/CFT-correspondence was discussed. The main idea in the construction of this $Vir \oplus \overline{Vir}$-diffeomorphisms was to still follow Carlip's construction [47] closely. But whereas Carlip singles out Vir-diffeomorphisms in the presence of a local Killing horizon, we insist in treating both the future and past event horizon of the black hole on the same level. This then leads to two copies of Witt-algebra diffeomorphisms.

Our purpose here is to report further on our investigations whether this choice of $Vir \oplus \overline{Vir}$-diffeomorphisms is a proper one. This question is in principle independent of the issues discussed in [3]. Our $Vir \oplus \overline{Vir}$-diffeomorphisms are of interest as they provide a novel choice of diffeomorphisms to be used in Carlip's approach. Inspecting the algebra of Hamiltonian generators, we will in this work infer central charges and Virasoro zero-modes (or equivalently CFT temperatures) that reproduce via Cardy-formula the expected Bekenstein-Hawking entropy. There appears no need to correct the canonically-derived Hamiltonian generators by any counterterms. Furthermore, the derived CFT temperatures are in agreement with the temperatures obtained directly from the $Vir \oplus \overline{Vir}$-vectorfields by thermodynamic considerations.

The recent work by Haco, Hawking, Perry, Strominger [60] also follows Carlip's approach to black hole entropy counting. There, an alternative choice of $Vir \oplus \overline{Vir}$-diffeomorphisms for the case of a Kerr black hole is proposed. We comment on that choice throughout our investigations.

This chapter is organized as follows. In chapter 6.2 we briefly review Carlip's approach

proposal stated later correctly in [32, 33, 34, 35].

to black hole entropy counting. To inspect the algebra of the Hamiltonian generators of diffeomorphisms, we derive the relevant formulas. Especially, we explain how the CFT data needed in the Cardy-formula is derived once a choice of $Vir \oplus \overline{Vir}$-diffeomorphisms has been made. In chapter 6.3, for the example of a Schwarzschild black hole, we explain how to single out a "preferred" $Vir \oplus \overline{Vir}$-algebra of spacetime diffeomorphisms. Alternatively to the considerations in chapter 6.2, we fix the associated CFT temperatures by some thermodynamic considerations. The role of the Casimir-operators of the global conformal algebra for scattering off a black hole is briefly discussed. In chapter 6.4, we use the derived $Vir \oplus \overline{Vir}$-diffeomorphisms in the framework of chapter 6.2 to infer the relevant CFT data for entropy counting. We find agreement with the Bekenstein-Hawking entropy. In chapter 6.5, we connect these findings with [3].

In the following, we use units in which we set the speed of light to 1 but we keep Newton's constant G and Planck's constant $\hbar$ explicit. Latin letters $a, b, \ldots = 0, \ldots, 3$ denote spacetime indices.

6.2 General Argument and Realization

6.2.1 General Argument

Here, we give a brief review of Carlip's approach to explain the statistical mechanical origin and counting of the black hole entropy especially in dimensions higher than 3. For a more detailed discussion and references, we refer to the original papers [28, 47, 62, 63]. The interpretation of this approach in light of a recently proposed Schwarzschild/CFT-correspondence [3] was already given in that reference and will also be discussed in chapter 6.5.

Consider an arbitrary diffeomorphism-invariant theory of gravity given by some action, which possibly can contain black hole solutions. For a diffeomorphism given by some vectorfield ξ over the spacetime manifold, we denote by H_ξ the associated Hamiltonian generator. The Hamiltonian generator H_ξ - if it exists - is a function over the phase space Γ of the theory and implements the diffeomorphism ξ. The generator H_ξ is given as the sum of a bulk integral over suited gauge constraints and a suited boundary integral. On-shell H_ξ is therefore given by a boundary integral. If H_ξ is non-constant over phase space, the diffeomorphism ξ constitutes a physical excitation, i.e. H_ξ implements a shift in the Hamiltonian phase space, otherwise a gauge redundancy.

The algebra of the Hamiltonian generators H_ξ with respect to the Poisson-bracket forms on-shell a representation of the algebra of the associated diffeomorphisms with respect to the ordinary Lie-bracket of vectorfields over the spacetime manifold up to central extensions. That is, for spacetime vectorfields ξ_1, ξ_2 we have on-shell the relation

$$\{H_{\xi_1}, H_{\xi_2}\} = H_{[\xi_1,\xi_2]} + K_{\xi_1,\xi_2} \tag{6.1}$$

where K_{ξ_1,ξ_2} are constant c-numbers.

So far, we have reviewed general statements of the Hamiltonian mechanics for gravity theories. What happens if we have a black hole solution? In this case the idea is to treat the event horizon as a boundary of the spacetime manifold. The presence of such a boundary can render some diffeomorphisms H_ξ from would-be gauge redundancies to physical excitations which could be important for the statistical mechanics of the black hole. The presence of the boundary can furthermore give rise to non-vanishing central extensions in the algebra (6.1) of the aforementioned generators H_ξ. Carlip's observation was that for black hole event horizons there are "natural" ways to find diffeomorphisms ξ_n ($n \in \mathbb{Z}$) which form a Witt-algebra

$$[\xi_m, \xi_n] = -i(m-n)\xi_{m+n}. \tag{6.2}$$

The subalgebra of (6.1) of the associated generators H_{ξ_n} then forms a Virasoro-algebra.

Virasoro-algebras constitute the symmetry algebras of two dimensional conformal field theories. The assumption then is, that there is a 2D CFT which describes the part of the phase space that is responsible for black hole microstates and whose conformal generators are provided by the H_{ξ_n}. At this stage, it is of course not clear *whether* such a theory exists or *what* this theory is. However, accepting this assumption one has fortunately the luxury that a lot of information about a given 2D CFT can be gained from its Virasoro-algebra - for our black hole case this would then be the algebra of the generators H_{ξ_n}.

For instance, the degeneracy of states in a 2D CFT is (often) fixed through the Cardy formula by the central charge which is read directly from the Virasoro-algebra. Therefore, it is tempting to perform the following sort of consistency check of the aforementioned assumption. One can determine the central charge of the Virasoro-algebra formed by the H_{ξ_n} by calculating the extensions K_{ξ_m,ξ_n} in (6.1). The associated degeneracy of states of the would-be CFT is then compared with the black hole's Bekenstein-Hawking entropy. Carlip's result was that both of them agree.

As already mentioned, there are still several remaining open questions. For example, why is it appropriate to treat the event horizon as a boundary in the evaluation of the boundary integrals that determine the extensions K_{ξ_m,ξ_n}? To put it differently, this can be phrased as *what* the CFT governing the black hole microstates in phase space is and *how* to obtain it. To answer these questions is the overlying goal of our investigations and we will briefly come back to these issues in chapter 6.5 where we emphasize the connection with previous work.

However, our point in this work is to revisit Carlip's approach to entropy counting in several directions. Why is such a revision necessary?

The symmetry algebra of a 2D CFT contains *two* mutually commuting copies of Virasoro-algebras. The associated chiral and anti-chiral central charges are usually equal (CFTs in curved background with different chiral and anti-chiral central charges are even known to be inconsistent [64]). In Carlip's approach [47] instead, the entire contribution to black hole entropy comes from a chiral half of a would-be CFT.

In this work, we therefore want to present a way to construct diffeomorphisms ξ_n and

$\overline{\xi}_n$ ($n \in \mathbb{Z}$) satisfying two copies of the Witt-algebra

$$\begin{aligned} [\xi_m, \xi_n] &= -i(m-n)\xi_{m+n} \\ \left[\overline{\xi}_m, \overline{\xi}_n\right] &= -i(m-n)\overline{\xi}_{m+n} \\ \left[\xi_m, \overline{\xi}_n\right] &= 0. \end{aligned} \tag{6.3}$$

The choice of diffeomorphisms should be such that the associated Hamiltonian generators H_{ξ_n} and $H_{\overline{\xi}_n}$ are reasonable candidates for the symmetry algebra of a possible CFT governing the statistical mechanics of the black hole under consideration.

6.2.2 Realization

In order to be as simple and as concrete as possible, we consider the case of a Schwarzschild black hole in pure Einstein gravity. We denote spacetime coordinates by $x^a = (x^0, x^1, x^A)$ with angular coordinates indexed by $A, B, \ldots = 2, 3$. In ingoing Eddington-Finkelstein coordinates $(v, r, x^A) = (v, r, \vartheta, \varphi)$ the Schwarzschild-metric reads

$$\begin{aligned} ds^2 &= g_{ab} dx^a dx^b \\ &= -\left(1 - \frac{r_S}{r}\right) dv^2 + 2dvdr + r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right) \\ &= -\left(1 - \frac{r_S}{r}\right) dv^2 + 2dvdr + r^2 \gamma_{AB} dx^A dx^B. \end{aligned} \tag{6.4}$$

They are related to the ordinary Schwarzschild-coordinates $(t, r, \vartheta, \varphi)$ by

$$v = t + r^*, \tag{6.5}$$

where the tortoise coordinate is given by

$$r^* = r + r_S \ln\left|\frac{r}{r_S} - 1\right|. \tag{6.6}$$

We work here in Eddington-Finkelstein coordinates because then the metric (6.4) satisfies (advanced) Bondi-gauge just to be compatible with our previous conventions in [3]. However, for what follows this choice is arbitrary and we could work in any coordinates that cover the event horizon. γ_{AB} denotes the metric on the unit 2-sphere. In what follows, we consider the metric (6.4) as a fixed reference point $g_{ab} \in \Gamma$ in Hamiltonian phase space describing a Schwarzschild black hole with mass parameter $\frac{r_S}{2G}$. Our task is then to look at the behavior of Hamiltonian generators of suited diffeomorphisms in the vicinity of this point.

To study the behavior of these generators, we have to refer to Hamiltonian mechanics. In order to do so, several approaches exist. One way would be to use the direct Hamiltonian approach to general relativity as it was done in [28]. However, we will use the covariant phase space formalism [39, 40] which is manifestly covariant as used in [47]. The formalism is for example reviewed in [44, 36, 65] and we will use some formulas collected there.

In the covariant phase space formalism the Hamiltonian phase space Γ is given by the solution space of the theory under consideration (i.e. the set of all field configurations satisfying the equations of motion). In principle, one has to divide out symplectic zero-modes by appropriately fixing the gauge but in our present context this step is not relevant. For a diffeomorphism ξ the infinitesimal change of the associated Hamiltonian generator between points $g_{ab} + h_{ab} \in \Gamma$ and $g_{ab} \in \Gamma$ in phase space is determined by

$$\delta H_\xi \left[h_{ab}; g_{ab}\right] = -\frac{1}{16\pi G} \oint_{\partial\Sigma} *F. \tag{6.7}$$

Here, Σ is a Cauchy-surface in the spacetime manifold and the 2-form F_{ab} is well-known [44, 36, 65]

$$\begin{aligned} F_{ab} = &\frac{1}{2}\left(\nabla_a \xi_b - \nabla_b \xi_a\right) h^c{}_c + \left(\nabla_a h^c{}_b - \nabla_b h^c{}_a\right) \xi_c \\ &+ \left(\nabla_c \xi_a h^c{}_b - \nabla_c \xi_b h^c{}_a\right) - \left(\nabla_c h^c{}_b \xi_a - \nabla_c h^c{}_a \xi_b\right) \\ &- \left(\nabla_a h^c{}_c \xi_b - \nabla_b h^c{}_c \xi_a\right). \end{aligned} \tag{6.8}$$

We will take $\partial\Sigma$ for our case of a Schwarzschild-background to be a cross-section of the event horizon, so it is given by the coordinates $(v = \text{const.}, r = \text{const.}, x^A)$ and thus has topology of S^2 parameterized by the angular coordinates x^A. In that case, (6.7) takes the form

$$\delta H_\xi \left[h_{ab}; g_{ab}\right] = -\frac{r^2}{16\pi G} \oint_{\partial\Sigma} d^2x \sqrt{\gamma} F_{rv}. \tag{6.9}$$

Evaluating (6.8) for our Schwarzschild-metric g_{ab} in (6.4), one obtains for (6.9)

$$
\begin{aligned}
&\delta H_\xi \left[h_{ab}; g_{ab}\right] = -\frac{r^2}{16\pi G}\oint_{\partial\Sigma} d^2x\sqrt{\gamma}\,\Big(\\
&\xi^v \left(-\frac{r_S}{2r^2}h^A{}_A - r^{-2}D^A h_{Av} - \frac{2}{r}h_{vv} - \frac{4}{r}\left(1-\frac{r_S}{r}\right)h_{vr} + \partial_v h^A{}_A\right. \\
&-\left(1-\frac{r_S}{r}\right)r^{-2}D^A h_{Ar} - \frac{2}{r}\left(1-\frac{r_S}{r}\right)^2 h_{rr} + \frac{1}{r}\left(1-\frac{r_S}{r}\right)h^A{}_A \\
&\left.+\left(1-\frac{r_S}{r}\right)\partial_r h^A{}_A\right) \\
&+\partial_r\xi^v\left(\frac{1}{2}\left(1-\frac{r_S}{r}\right)^2 h_{rr} - \frac{1}{2}\left(1-\frac{r_S}{r}\right)h^A{}_A + h_{vv} + \left(1-\frac{r_S}{r}\right)h_{vr}\right) \\
&+\partial_r\xi^r\left(-\frac{1}{2}\left(1-\frac{r_S}{r}\right)h_{rr} + \frac{1}{2}h^A{}_A\right) \\
&+\partial_v\xi^v\left(\frac{1}{2}\left(1-\frac{r_S}{r}\right)h_{rr} - \frac{1}{2}h^A{}_A\right) \\
&+\xi^r\left(\frac{r_S}{2r^2}h_{rr} + r^{-2}D^A h_{Ar} + \frac{2}{r}h_{vr} + \frac{2}{r}\left(1-\frac{r_S}{r}\right)h_{rr}\right. \\
&\left.-\frac{1}{r}h^A{}_A - \partial_r h^A{}_A\right) \\
&+\xi^A\left(\partial_r h_{Av} - \frac{2}{r}h_{Av} - \partial_v h_{Ar}\right) \\
&-\partial_v\xi^r h_{rr} + r^{-2}D^A\xi^v\left(h_{Av} + \left(1-\frac{r_S}{r}\right)h_{Ar}\right) - r^{-2}D^A\xi^r h_{Ar}\Big).
\end{aligned}
\tag{6.10}
$$

D_A and D^A denote the covariant derivative on the unit 2-sphere where the index is raised and lowered with γ_{AB}.

Now we have derived the theoretical ground to accomplish our task. If we can find "natural" diffeomorphisms satisfying the two copies of Witt-algebra (6.3), we are able to provide candidates for the Virasoro-generators of the black hole at the point $g_{ab} \in \Gamma$ in phase space.[2] Since the Hamiltonian generators H_ξ have the dimension of an action, we

[2] Contrary to Carlip's approach, we do not impose the constraint of integrability on the diffeomorphisms ξ_n and their generators (6.7). This is because we allow the diffeomorphisms generated by the conformal generators to be field-dependent. Fortunately, the knowledge of those diffeomorphisms at the reference point $g_{ab} \in \Gamma$ given by (6.4) is sufficient to determine their generator algebra at this point. The purpose of the present analysis is to find out whether after all suited diffeomorphisms ξ_n exist that could give rise to the conformal generators at the reference point $g_{ab} \in \Gamma$. Away from $g_{ab} \in \Gamma$, the diffeomorphisms generated by the conformal generators may look different and we will not determine them here. As a consequence, we are able to infer information about the conformal symmetry only right at the reference point g_{ab}. For instance, we are not able to determine the temperature dependence in (6.42) away from g_{ab}. We will come back to these issues in chapter 6.5.

can define dimensionless generators by

$$\begin{aligned} H_{\xi_n} &=: \hbar L_n \\ H_{\bar{\xi}_n} &=: \hbar \overline{L}_n \end{aligned} \tag{6.11}$$

at $g_{ab} \in \Gamma$ and for $n \in \mathbb{Z}$. Since the generators satisfy according to (6.1) and (6.3) two centrally extended Witt-algebras

$$\begin{aligned} \{H_{\xi_m}, H_{\xi_n}\} &= -i(m-n)H_{\xi_{m+n}} + K_{\xi_m,\xi_n} \\ \left\{H_{\bar{\xi}_m}, H_{\bar{\xi}_n}\right\} &= -i(m-n)H_{\bar{\xi}_{m+n}} + K_{\bar{\xi}_m,\bar{\xi}_n} \\ \left\{H_{\xi_m}, H_{\bar{\xi}_n}\right\} &= 0, \end{aligned} \tag{6.12}$$

we find after canonical quantization $\{\cdot,\cdot\} \to \frac{1}{i\hbar}[\cdot,\cdot]$ of (6.12) that the Virasoro-generators L_n and $\overline{L}_n$ fulfill two copies of the Virasoro-algebra in the standard form

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n} \\ \left[\overline{L}_m, \overline{L}_n\right] &= (m-n)\overline{L}_{m+n} + \frac{\overline{c}}{12}m(m^2-1)\delta_{m+n} \\ \left[L_m, \overline{L}_n\right] &= 0. \end{aligned} \tag{6.13}$$

$\delta_{m+n} = \delta_{m+n,0}$ denotes the Kronecker-delta. Thus, the central charges $c, \overline{c}$ and Virasoro-generators $L_0[g_{ab}], \overline{L}_0[g_{ab}]$ can be inferred from

$$\begin{aligned} \delta_{\xi_{-m}} H_{\xi_m}\Big|_{g_{ab}} &= -2i\hbar m L_0[g_{ab}] - i\frac{\hbar c}{12}m(m^2-1) \\ \delta_{\bar{\xi}_{-m}} H_{\bar{\xi}_m}\Big|_{g_{ab}} &= -2i\hbar m \overline{L}_0[g_{ab}] - i\frac{\hbar \overline{c}}{12}m(m^2-1) \end{aligned} \tag{6.14}$$

for $m \in \mathbb{Z}$.

Equation (6.14) already fixes the data needed for the counting of state degeneracy in a CFT. The computation of the left hand side of (6.14) can be done by (6.10).

To summarize, the task left for the next chapter is to find a "preferred" Witt-algebra of diffeomorphisms (6.3). We will have to argue in what sense these diffeomorphisms will be preferred. But if this can be accomplished, the associated Hamiltonian generators will provide natural Virasoro-generators of a possible CFT describing part of the phase space responsible for the microstates of the black hole.

Nevertheless, the choice of diffeomorphisms is at this stage only a guess. It might be that chosen diffeomorphisms have nothing to do with the symmetry generators of the aforementioned CFT. Within this approach, it is even not clear that such a CFT exists. However, in this chapter we have shown that the choice of diffeomorphisms (6.3) fixes via equations (6.14) and (6.10) the central charges and conformal weights (or equivalently the temperatures) of a would-be CFT. This is already enough data to determine the degeneracy of states in this CFT in order to see whether it agrees with the Bekenstein-Hawking entropy of the black hole.

6.3 Searching for Virasoro-Algebra

The goal of this chapter is to find out, whether the presence of a black hole event horizon singles out a Witt-algebra of diffeomorphisms (6.3) in a natural way and what natural in this context might mean. In the last chapter, we have explained how such diffeomorphisms could be related to the generators of a conformal symmetry governing the black hole's phase space and provided formulas to extract information of this CFT directly from the diffeomorphisms.

6.3.1 Virasoro-Vectorfields

Given the Schwarzschild black hole $g_{ab} \in \Gamma$ in (6.4), what diffeomorphisms forming a Witt-algebra might be singled out? Remember, that a Witt-algebra (6.2) is isomorphic to the algebra $\mathfrak{Diff}\ S^1$ of all diffeomorphisms on S^1. The question can thus be rephrased as whether there are preferred directions in a Schwarzschild-spacetime. If so, periodic reparameterizations along these directions provide a Witt-algebra $\mathfrak{Diff}\ S^1$ and the periodicities would then fix the temperatures of a possible CFT. In [47] Carlip provided for the general case of a local Killing horizon a candidate for such a preferred direction. Although the associated Hamiltonian generators can be shown to generate central extensions and to give rise to the correct entropy, only one copy of a Virasoro-algebra is found.

Instead, we want to give a somewhat different proposal for constructing diffeomorphisms forming the algebra (6.3). We will see that the associated Hamiltonian generators will indeed form two copies of Virasoro-algebra with equal central charges as one would expect for the symmetry algebra of a CFT. The diffeomorphisms we are going to construct were already given in [3] up to cosmetic changes. We now explain how they are singled out.

We keep the philosophy of [47] that a local Killing horizon singles out a preferred direction which provides the basis for the construction of a $\mathfrak{Diff}\ S^1$ algebra. However, a Schwarzschild geometry has in its maximal extension two event horizons and we propose to treat both on the same footing in the search for an algebra (6.3).

In the well-known Kruskal-coordinates (U, V, x^A) which cover the entire maximal extension of the Schwarzschild spacetime, the metric takes the form

$$ds^2 = -\frac{4r_S^3}{r}e^{-\frac{r}{r_S}}dUdV + r^2\gamma_{AB}dx^A dx^B. \tag{6.15}$$

The future (past) event horizon is located at $U = 0$ ($V = 0$). Indeed, the Schwarzschild geometry (6.15) provides the *two* preferred lightlike directions ∂_U and ∂_V.

However, the Cardy-formula provides the degeneracy of states $S = S(L_0, \overline{L}_0)$ in a CFT at particular values of the Virasoro-generators L_0 and $\overline{L}_0$. Since the black hole entropy gives the state degeneracy at fixed mass and angular momentum, the candidates for the Virasoro zero-modes H_{ξ_0} and $H_{\bar{\xi}_0}$ should therefore "measure" the mass and angular momentum parameter of the black hole. Therefore, it seems that reparameterizations along the direction U (V) are not enough. The diffeomorphisms ξ_n and $\overline{\xi}_n$ to form (6.3) should also contain components in the direction ∂_φ which is conjugated to the angular momentum.[3]

Thus, in Kruskal-coordinates $(U, V, \vartheta, \varphi)$, we make the ansatz for the vectorfields (6.3)

$$\begin{aligned} \xi_n &= f_n \partial_V + g_n \partial_\varphi \\ \overline{\xi}_n &= \overline{f}_n \partial_U + \overline{g}_n \partial_\varphi \end{aligned} \tag{6.16}$$

for $n \in \mathbb{Z}$. Here, the functions

$$\begin{aligned} f_n &= f_n(V, \varphi) \\ g_n &= g_n(V, \varphi) \\ \overline{f}_n &= \overline{f}_n(U, \varphi) \\ \overline{g}_n &= \overline{g}_n(U, \varphi) \end{aligned} \tag{6.17}$$

need to be determined from the requirement (6.3). The first equation in (6.3) yields two conditions on the functions (6.17), namely

$$f_m \partial_V f_n + g_m \partial_\varphi f_n - (m \longleftrightarrow n) = -i(m-n) f_{m+n} \tag{6.18}$$

and

$$f_m \partial_V g_n + g_m \partial_\varphi g_n - (m \longleftrightarrow n) = -i(m-n) g_{m+n} \tag{6.19}$$

for $m, n \in \mathbb{Z}$. Analogous equations follow from the second equation in (6.3) for the anti-chiral functions $\overline{f}_n, \overline{g}_n$. The last equation of (6.3) then yields the conditions

$$\begin{aligned} \partial_\varphi \overline{f}_n &= 0 \\ \partial_\varphi f_n &= 0 \\ g_m \partial_\varphi \overline{g}_n - \overline{g}_n \partial_\varphi g_m &= 0. \end{aligned} \tag{6.20}$$

The last conditions of (6.20) can be fulfilled by choosing the product ansatz

$$\begin{aligned} g_m(V, \varphi) &= \Phi(\varphi) G_m(V) \\ \overline{g}_m(U, \varphi) &= \Phi(\varphi) \overline{G}_m(U). \end{aligned} \tag{6.21}$$

[3] One can also check that although reparameterizations along U (V) form the algebra (6.3), the Hamiltonian generators do not develop a central charge in (6.14).

With these restrictions on f_n and g_n, equation (6.18) becomes

$$f_m\partial_V f_n - (m \longleftrightarrow n) = -i(m-n)f_{m+n} \tag{6.22}$$

and (6.19) yields

$$f_m\partial_V G_n - (m \longleftrightarrow n) = -i(m-n)G_{m+n}. \tag{6.23}$$

Choosing $f_n = f_n(V)$ to satisfy (6.22), equation (6.23) is fulfilled with the choice

$$G_n = \partial_V f_n. \tag{6.24}$$

Therefore, the vectorfields

$$\begin{aligned} \xi_n &= f_n(V)\partial_V + \Phi(\varphi)\partial_V f_n \partial_\varphi \\ \overline{\xi}_n &= \overline{f}_n(U)\partial_U + \Phi(\varphi)\partial_U \overline{f}_n \partial_\varphi \end{aligned} \tag{6.25}$$

provide an algebra (6.3) if the functions $f_n = f_n(V)$ are chosen to satisfy (6.22) and $\overline{f}_n$ are chosen analogously. $\Phi = \Phi(\varphi)$ is at this stage arbitrary. A legal choice is then

$$\begin{aligned} f_n(V) &= \frac{1}{A}V^{1+inA} \\ \overline{f}_n(U) &= \frac{1}{B}U^{1+inB} \end{aligned} \tag{6.26}$$

where $A, B \in \mathbb{R}\backslash\{0\}$ are so far arbitrary parameters.

Unfortunately, the constructed vectorfields (6.25) are still not satisfactory. In order to give rise to independent Virasoro generators L_n, the ξ_n have to be linearly independent functions of the angular coordinates. One possibility is that a factor $e^{in\varphi}$ appears in (6.25) instead of a fixed function $\Phi(\varphi)$. However, this is now easy to achieve. Since (6.25) satisfies a Witt-algebra (6.3), we can generate such vectorfields out of (6.25) by applying an active coordinate transformation. The new vectorfields then still satisfy (6.3). We choose the coordinate transformation[4]

$$\begin{aligned} U' &= Ue^{\frac{1}{B}\varphi} \\ V' &= Ve^{-\frac{1}{A}\varphi} \\ \vartheta &= \vartheta \\ \varphi &= \varphi. \end{aligned} \tag{6.27}$$

[4]The vectorfields of [3] are obtained by putting an additional minus sign in the exponential of the first equation in (6.27). This was omitted here in order to make the frequencies and temperatures positive that are going to appear later. In addition, we replaced A, B with their inverse values as compared to [3].

The procedure yields

$$\xi_n^a = \begin{pmatrix} \xi_n^U \\ \xi_n^V \\ \xi_n^\vartheta \\ \xi_n^\varphi \end{pmatrix} = \begin{pmatrix} \frac{1}{AB} U(1+inA)V^{inA} e^{in\varphi}\Phi(\varphi) \\ \frac{1}{A} V^{1+inA} e^{in\varphi} - \frac{1}{A^2}(1+inA)V^{1+inA}e^{in\varphi}\Phi(\varphi) \\ 0 \\ \frac{1}{A}(1+inA)V^{inA}e^{in\varphi}\Phi(\varphi) \end{pmatrix} \tag{6.28}$$

and

$$\overline{\xi}_n^a = \begin{pmatrix} \frac{1}{B} U^{1+inB} e^{-in\varphi} + \frac{1}{B^2}(1+inB)U^{1+inB}e^{-in\varphi}\Phi(\varphi) \\ -\frac{1}{AB} V(1+inB)U^{inB}e^{-in\varphi}\Phi(\varphi) \\ 0 \\ \frac{1}{B}(1+inB)U^{inB}e^{-in\varphi}\Phi(\varphi) \end{pmatrix}. \tag{6.29}$$

In order to meet the conventions of the last chapter, we formulate (6.28) and (6.29) in Eddington-Finkelstein coordinates $(v, r, \vartheta, \varphi)$ getting

$$\xi_n^a = \begin{pmatrix} \xi_n^v \\ \xi_n^r \\ \xi_n^\vartheta \\ \xi_n^\varphi \end{pmatrix} = \begin{pmatrix} 2r_S\left(\frac{1}{A} - \frac{1}{A^2}(1+inA)\Phi(\varphi)\right) \\ \left(1-\frac{r_S}{r}\right) r_S \left(\frac{1}{A} + \frac{1}{A}\left(-\frac{1}{A}+\frac{1}{B}\right)(1+inA)\Phi(\varphi)\right) \\ 0 \\ \frac{1}{A}(1+inA)\Phi(\varphi) \end{pmatrix} \times e^{inA\frac{v}{2r_S}} e^{in\varphi} \tag{6.30}$$

and

$$\overline{\xi}_n^a = \begin{pmatrix} -2r_S \frac{1}{AB}(1+inB)\Phi(\varphi) \\ r_S\left(1-\frac{r_S}{r}\right)\left(\frac{1}{B} + \frac{1}{B}\left(-\frac{1}{A}+\frac{1}{B}\right)(1+inB)\Phi(\varphi)\right) \\ 0 \\ \frac{1}{B}(1+inB)\Phi(\varphi) \end{pmatrix} \times (-1)^{inB} e^{inB\frac{r^*}{r_S}} e^{-inB\frac{v}{2r_S}} e^{-in\varphi} \tag{6.31}$$

for $n \in \mathbb{Z}$ and with r^* from (6.6). These vectorfields fulfill $(\xi_n^a)^* = \xi_{-n}^a$ and $\left(\overline{\xi}_n^a\right)^* = \overline{\xi}_{-n}^a$ and form two copies of Witt-algebra (6.3) as required. Formulas (6.30) and (6.31) are the main result of this chapter and will be used in the next chapter for the entropy counting within the framework developed in chapter 6.2. In what follows, we will try to fix the remaining arbitrary function $\Phi(\varphi)$ and $A, B \in \mathbb{R}\backslash\{0\}$. On the road, we will also comment on different approaches made to find such Virasoro-vectorfields.

6.3.2 Temperatures

Although our construction of (6.30) and (6.31) is motivated by [47], we note that our final result is really different. Our vectorfields violate the horizon boundary conditions proposed in [47] and thus the construction is genuinely different. This is mainly due to the appearance of a ∂_φ-component in our choice of diffeomorphisms. Indeed, we wanted this component to appear in order for the Virasoro zero-modes to "measure" the black hole's mass and angular momentum parameter. The Virasoro zero-modes are according to chapter 6.2 induced by the $\mathfrak{u}(1) \oplus \overline{\mathfrak{u}(1)}$-subalgebra of (6.3) spanned by ξ_0 and $\bar{\xi}_0$. These vectorfields hence should be - in Schwarzschild-coordinates $(t, r, \vartheta, \varphi)$ - linear combinations of ∂_t and ∂_φ. For this to be fulfilled, in (6.30) and (6.31) the r-component has to vanish for $n = 0$ which requires the choice

$$\Phi(\varphi) = \frac{AB}{B - A}. \tag{6.32}$$

With this choice the $\mathfrak{u}(1) \oplus \overline{\mathfrak{u}(1)}$-subalgebra is given by

$$\begin{aligned} \xi_0 &= -2r_S \frac{1}{B-A}\partial_t + \frac{B}{B-A}\partial_\varphi \\ \bar{\xi}_0 &= -2r_S \frac{1}{B-A}\partial_t + \frac{A}{B-A}\partial_\varphi . \end{aligned} \tag{6.33}$$

By some thermodynamic considerations, (6.33) provides enough information to fix the temperatures of the CFT described in chapter 6.2 that would be associated with the full Witt-algebra (6.3). However, note that these considerations provide rather a consistency check as the temperatures are already fixed through (6.14) by $L_0[g_{ab}]$ and $\bar{L}_0[g_{ab}]$: The temperatures inferred from the algebra (6.14) can be compared to the temperatures that are thermodynamically obtained from (6.33) by a procedure to be explained in the following.

In [60], a different approach to find Virasoro-vectorfields was made but the latter consistency check was not done. There, the im-terms which would determine the temperature are neither displayed in equation (5.15) nor in the counterterm-correction (5.16). It would be interesting to see whether in [60] the temperatures obtained algebraically via (6.14) agree with the temperatures inferred thermodynamically.

In order to thermodynamically infer the CFT temperatures, we introduce a scalar field to be put in thermal contact with the black hole. Consider a free massless Klein-Gordon field on the Schwarzschild-background. Its eigenmodes are of the form $F(r, \vartheta)e^{-i\omega t + im\varphi}$ with frequeny ω and angular momentum m. These are then also eigenfunctions of (6.33) with eigenvalues

$$\begin{aligned} \xi_0 &= -in = -i\left(\frac{2r_S\omega}{A-B} - m\frac{B}{B-A}\right) \\ \bar{\xi}_0 &= -i\bar{n} = -i\left(\frac{2r_S\omega}{A-B} - m\frac{A}{B-A}\right). \end{aligned} \tag{6.34}$$

If we would allow for backreaction, the scalar field can exchange energy and angular momentum with the gravitational field under the constraint that both are conserved in total. In thermal equilibrium, for the Schwarzschild black hole, the scalar eigenmodes (ω, m) are thermally distributed weighted by a Boltzmann-factor

$$e^{-\frac{\hbar\omega}{T_H}}$$

with the Hawking-temperature $T_H = \frac{\hbar}{4\pi r_S}$. This is rewritten in terms of the eigenfrequencies $(n, \overline{n})$ as $e^{-\frac{n}{T}-\frac{\overline{n}}{\overline{T}}}$ with the temperatures

$$\begin{aligned} T &= \frac{1}{2\pi}\frac{1}{A} \\ \overline{T} &= -\frac{1}{2\pi}\frac{1}{B}. \end{aligned} \tag{6.35}$$

Due to the zeroth law of thermodynamics, (6.35) are also the temperatures of the CFT governing the black hole if associated to the diffeomorphisms (6.30) (6.31).

6.3.3 $SL(2;\mathbb{R})$-Casimir and Conformal Symmetry in Scattering

In this chapter, we have presented one particular way to single out Virasoro-vectorfields (6.30),(6.31) which we will use for entropy counting via Carlip's approach in the next chapter. The recent work [60] also follows Carlip's approach but a different philosophy is used to find a conformal algebra of vectorfields.[5]

The Witt-algebra (6.3) has a $\mathfrak{sl}(2,\mathbb{R}) \oplus \overline{\mathfrak{sl}(2,\mathbb{R})}$-subalgebra spanned by ξ_{-1}, ξ_0, ξ_1 and its anti-chiral counterpart. Associated to this global conformal algebra are the Casimir-operators

$$\mathcal{H}^2 = -\mathcal{L}_{\xi_0}\mathcal{L}_{\xi_0} + \frac{1}{2}\left(\mathcal{L}_{\xi_1}\mathcal{L}_{\xi_{-1}} + \mathcal{L}_{\xi_{-1}}\mathcal{L}_{\xi_1}\right) \tag{6.36}$$

and an analogous anti-chiral expression. Now, one can try to find a preferred $\mathfrak{sl}(2,\mathbb{R})$-algebra of diffeomorphisms by studying the form of the associated differential operator (6.36). In [38] vectorfields forming an $\mathfrak{sl}(2,\mathbb{R}) \oplus \overline{\mathfrak{sl}(2,\mathbb{R})}$-algebra were given. It was further shown there, that for a free massless Klein-Gordon field in a Kerr-background - in a suited regime - eigenfunctions of (6.36) give rise to eigenmodes of the Klein-Gordon equation. As a consequence of the $\mathfrak{sl}(2,\mathbb{R}) \oplus \overline{\mathfrak{sl}(2,\mathbb{R})}$-invariance of (6.36), scattering - in a suited in regime - behaves as being invariant under a "hidden" 2D global conformal symmetry (see [38] for further details).

The idea of [60] is then to find a full local conformal $Vir \oplus \overline{Vir}$-algebra (6.3) of diffeomorphisms which realizes the latter hidden conformal symmetry and then can be used for entropy counting.

However, to our understanding the $Vir \oplus \overline{Vir}$-vectorfields proposed in [60] form only an enhancement of the $\mathfrak{u}(1) \oplus \overline{\mathfrak{u}(1)}$-algebra given in [38] (spanned by $\xi_0, \overline{\xi}_0$). They seem

[5] The analysis there is done for a Kerr black hole (Kerr-Newman in [66]) and diverges in the Schwarzschild-limit but this will be not important here.

not to contain the global conformal $\mathfrak{sl}(2,\mathbb{R}) \oplus \overline{\mathfrak{sl}(2,\mathbb{R})}$-algebra of [38] (spanned by $\xi_n, \overline{\xi}_n$ for $n = -1, 0, 1$).

But then, we do not understand in what way the Kerr-geometry singles out the $Vir \oplus \overline{Vir}$-vectorfields of [60] so that their Hamiltonian generators could govern a possible CFT of the Kerr black hole. Indeed, in [60], the expected central charges are only obtained from the generator algebra after non-canonical counterterm-corrections.

Nevertheless, it might still be useful to have the $\mathfrak{sl}(2,\mathbb{R})$-Casimir (6.36) in mind. What is its meaning for our $Vir \oplus \overline{Vir}$ choice in (6.30),(6.31)? Inspired by [38] and our construction of these diffeomorphisms, a natural expectation would be that $\mathcal{H}^2$ and $\overline{\mathcal{H}}^2$ possibly govern the scalar scattering in the vicinity of the bifurcation of the horizons in a suited regime of parameters. We do not enter an analysis of these questions further at this place. However, we want to note that a computation reveals that the expression for (6.36) with the vectorfields (6.30)(6.31) indeed greatly simplyfies for the choice

$$B = -A. \tag{6.37}$$

This could be a hint that the Vir-algebra (6.30) and $\overline{Vir}$-algebra (6.31) can belong to the Virasoro-algebra of the same CFT only with the choice (6.37). However, we will leave the parameters A and B unspecified. We will then see further evidence for this conjecture from the fact, that the Cardy-entropy will get extremized precisely for the choice (6.37).

Using (6.32) and (6.37), the last unspecified parameter in (6.30) and (6.31) is then A. We will see that it will cancel out of the entropy counting. Such an ambiguity parameter was already present in [28]. As explained there, euclidean quantum gravity suggests the choice $A = 1$ together with (6.37) since the wavenumber for v is then given by the surface gravity $\kappa = \frac{1}{2r_S}$. However, we will leave A unspecified since this ambiguity can have a mathematical meaning as can be seen in chapter 6.5.

6.4 Entropy Counting

With the vectorfields (6.30) and (6.31) of the last chapter we are now ready to apply the framework of chapter 6.2 for entropy counting.

6.4.1 Schwarzschild-Entropy

Our goal is to determine the data on the right hand side of (6.14). To this end, we evaluate the left hand side of (6.14) using the vectorfields determined in (6.30) and (6.31).

A computation yields

$$
\begin{aligned}
&\delta_{\xi_{-m}} H_{\xi_m}\Big|_{g_{ab}} \\
=&-im\frac{r^2}{4G}\left(\frac{r_S^2}{r^2}\frac{1}{A}\Phi\left(\frac{2}{A}-\frac{2}{A}\left(\frac{1}{A}+\frac{1}{B}\right)\Phi\right)\right. \\
&\left.+\frac{r_S}{r}\left(1-\frac{r_S}{r}\right)\left(-\frac{4}{A}-\frac{4}{AB}\Phi+\frac{4}{A^2}\Phi+\frac{8}{A^2B}\Phi^2\right)\right) \\
&-im^3\frac{r^2}{4G}\left(2\Phi-2\frac{r_S^2}{r^2}\Phi^2\left(\frac{1}{A}+\frac{1}{B}\right)+\frac{r_S}{r}\left(1-\frac{r_S}{r}\right)\frac{8}{B}\Phi^2\right) \\
&+im^3\frac{r_S^2}{8G}\int_0^\pi d\vartheta\sin^{-1}(\vartheta)\left(1-\frac{r_S}{r}\right)\frac{8}{AB}\Phi.
\end{aligned}
\tag{6.38}
$$

In the limit $r \to r_S$ the expression is well-defined and we get from (6.14)

$$
\begin{aligned}
L_0[g_{ab}]-\frac{c}{24} &= \frac{r_S^2}{4\hbar G}\left(\frac{1}{A}\Phi\left(\frac{1}{A}-\frac{1}{A}\left(\frac{1}{A}+\frac{1}{B}\right)\Phi\right)\right) \\
c &= \frac{3r_S^2}{\hbar G}\left(2\Phi-2\Phi^2\left(\frac{1}{A}+\frac{1}{B}\right)\right).
\end{aligned}
\tag{6.39}
$$

Using (6.32) and applying the Cardy-formula[6]

$$
S_{chiral} = 2\pi\sqrt{\frac{c}{6}\left(L_0-\frac{c}{24}\right)} = \frac{\pi r_S^2}{\hbar G}\frac{-2AB}{(B-A)^2}. \tag{6.40}
$$

Consistently, the canonical version of the Cardy-formula with the temperatures (6.35) yields the same result

$$
S_{chiral} = \frac{\pi^2}{3}cT = \frac{\pi r_S^2}{\hbar G}\frac{-2AB}{(B-A)^2}. \tag{6.41}
$$

As conjectured in chapter 6.3.3, these expressions are maximized if (6.37) holds. In this case, one has

$$
\begin{aligned}
L_0[g_{ab}]-\frac{c}{24} &= \frac{r_S^2}{8\hbar G}\frac{1}{A} \\
c &= \frac{3r_S^2}{\hbar G}A
\end{aligned}
\tag{6.42}
$$

and

$$
S_{chiral} = \frac{1}{2}\frac{\pi r_S^2}{\hbar G}. \tag{6.43}
$$

[6]We take the convention $A > 0$ and $B < 0$ in the following in order for the temperatures (6.35) to be positive.

The anti-chiral contribution is determined in the same way

$$\begin{aligned}
&\delta_{\bar{\xi}_{-m}} H_{\bar{\xi}_m}\Big|_{g_{ab}} \\
=& - im\frac{r^2}{4G}\left(\frac{r_S^2}{r^2}\left(\frac{2}{B^2}\Phi + 2\left(\frac{1}{A}+\frac{1}{B}\right)\frac{1}{B^2}\Phi^2\right)\right. \\
&\left.-\frac{r_S}{r}\left(1-\frac{r_S}{r}\right)\left(2\Phi\left(\frac{2}{AB}-\frac{2}{B^2}\right)-\frac{4}{B}+\Phi^2\frac{8}{AB^2}\right)\right) \\
&- im^3\frac{r^2}{4G}\left(2\Phi + 2\frac{r_S^2}{r^2}\left(\frac{1}{A}+\frac{1}{B}\right)\Phi^2 - \frac{r_S}{r}\left(1-\frac{r_S}{r}\right)\Phi^2\frac{8}{A}\right) \\
&+ im^3\frac{r_S^2}{8G}\int_0^\pi d\vartheta \sin^{-1}(\vartheta)\left(1-\frac{r_S}{r}\right)\frac{8\Phi}{AB}.
\end{aligned} \tag{6.44}$$

In the limit $r \to r_S$ the expression is well-defined and we get from (6.14)

$$\begin{aligned}
\overline{L}_0[g_{ab}] - \frac{\bar{c}}{24} &= \frac{r_S^2}{4\hbar G}\left(\frac{1}{B^2}\Phi + \left(\frac{1}{A}+\frac{1}{B}\right)\frac{1}{B^2}\Phi^2\right) \\
\bar{c} &= \frac{3r_S^2}{\hbar G}\left(2\Phi + 2\left(\frac{1}{A}+\frac{1}{B}\right)\Phi^2\right).
\end{aligned} \tag{6.45}$$

Using (6.32) and applying the Cardy-formula

$$S_{anti-chiral} = 2\pi\sqrt{\frac{\bar{c}}{6}\left(\overline{L}_0 - \frac{\bar{c}}{24}\right)} = \frac{\pi r_S^2}{\hbar G}\frac{-2AB}{(B-A)^2}. \tag{6.46}$$

Consistently, the canonical version of the Cardy-formula with the temperatures (6.35) yields the same result

$$S_{anti-chiral} = \frac{\pi^2}{3}\bar{c}\overline{T} = \frac{\pi r_S^2}{\hbar G}\frac{-2AB}{(B-A)^2}. \tag{6.47}$$

These expressions coincide with the chiral contribution and are also maximized if (6.37) is fulfilled. For that case, one has

$$\begin{aligned}
\overline{L}_0[g_{ab}] - \frac{\bar{c}}{24} &= \frac{r_S^2}{8\hbar G}\frac{1}{A} \\
\bar{c} &= \frac{3r_S^2}{\hbar G}A
\end{aligned} \tag{6.48}$$

and

$$S_{anti-chiral} = \frac{1}{2}\frac{\pi r_S^2}{\hbar G}. \tag{6.49}$$

Thus, the total Cardy-entropy is

$$S = S_{chiral} + S_{anti-chiral} = \frac{\pi r_S^2}{\hbar G}. \tag{6.50}$$

This matches precisely the Bekenstein-Hawking entropy of a Schwarzschild black hole.

6.4.2 Extrapolation to the General Case

The CFT data derived in (6.42), (6.48) and (6.35) can be written in the form

$$\begin{aligned} c = \bar{c} &= \frac{3\mathcal{A}}{4\pi\hbar G}A \\ T = \overline{T} &= \frac{1}{2\pi}\frac{1}{A} \end{aligned} \tag{6.51}$$

with the horizon area $\mathcal{A}$. In [62] one copy of a Witt-algebra of vectorfields was presented to reproduce central charges and temperatures similar to the chiral half of (6.51) for the general case of a stationary black hole of dimension $3+1$. However, there are some differences. In [62] these quantities contain divergences which cancel out in entropy counting and the temperature is derived only by thermodynamic considerations and not from a computation of Virasoro zero-modes. In addition, the chiral Virasoro-algebra in [62] is only able to account for half of the expected Bekenstein-Hawking entropy.

For a Schwarzschild black hole, we have managed to provide the missing second copy of Virasoro-algebra accounting for the second missing half of the entropy. In addition, our choice of $Vir \oplus \overline{Vir}$-vectorfields leads to well-defined quantities (6.51) that contain no divergences. Furthermore, the temperatures in (6.51) are consistently in agreement with their derivation from Virasoro zero-modes $L_0[g_{ab}]$ and $\overline{L}_0[g_{ab}]$ using covariant phase space methods.

We derived (6.51) for the case of a Schwarzschild black hole. However, our methods employed allow for canonical generalization. The strategy to pick out a $Vir \oplus \overline{Vir}$-algebra of vectorfields can be analogously applied in the general case. Due to the similarities of (6.51) and the general analysis of [62], we conjecture (6.51) to apply also in this general case leading to the entropy

$$S_{Cardy} = \frac{\pi^2}{3}cT + \frac{\pi^2}{3}\bar{c}\overline{T} = \frac{\mathcal{A}}{4\hbar G} \tag{6.52}$$

as required. Note, that we have provided a proof of (6.51) and (6.52) only for the Schwarzschild case and left the general case as a conjecture. Checking the conjecture would now require a straightforward computation that we do not enter at this place.

6.5 Discussion and Interpretation

In the preceding chapters, we have revisited Carlip's approach to entropy counting. We have provided a $Vir \oplus \overline{Vir}$-algebra of diffeomorphisms and analyzed the algebra of the associated Hamiltonian generators. We found that the latter give rise to a Virasoro-algebra such that counting the state degeneracy of the would-be CFT is in agreement with Bekenstein-Hawking entropy.

So far, this approach does not tell much about this would-be CFT that possibly governs the part of phase space responsible for black hole microstates. What is needed, is to analyze the Hamiltonian phase space in the vicinity of a black hole state in a systematic fashion. In [3] a systematic way was proposed to analyze the Hamiltonian phase space of general relativity and to find a dual theory describing the relevant part of phase space responsible for black hole microstates.

Here, we want to briefly sketch how and why such a systematic treatment works in order to show how the entropy counting presented here fits into this procedure. An application of this treatment including the role of Carlip's entropy counting was already given in [3]. A more detailed description of the treatment itself including applications to simpler theories than gravity will be provided somewhere else. Instead, here we will just sketch the main ideas.

6.5.1 Holography in Covariant Phase Space

Consider an arbitrary field theory over some n-dimensional manifold M given by an action $S = S[\Phi]$. We denote the fields in the theory collectively by Φ. The goal is to analyze the Hamiltonian phase space in a structured way. Due to its flexibility, we use the covariant phase space approach for our explanations [39, 40] (see [44] for a review). The main idea of the covariant phase space approach is the observation that the Hamiltonian phase space is isomorphic to the set of all field configurations satisfying the field equations. This solution space $\overline{\mathcal{F}}$ is equipped with a suited (pre)symplectic form and (after dividing out symplectic zero-modes through suitable gauge-fixing) then gives rise to the covariant phase space Γ equivalent to the Hamiltonian phase space.

We denote the coordinates on the covariant phase space by $[\Phi]^A$ with $A, B, \ldots$ being the indexes. The action S is assigned a differential form $\omega = \omega[\delta_1\Phi, \delta_2\Phi; \Phi]$ of degree $n-1$ over M which is in addition a closed 2-form over the space of all field configurations. On shell, that is for $\Phi \in \overline{\mathcal{F}}$ and $\delta_1\Phi, \delta_2\Phi \in T_\Phi\overline{\mathcal{F}}$, ω is exact $\omega = dk$ for a form $k = k[\delta_1\Phi, \delta_2\Phi; \Phi]$ of degree $n-2$ over M.

Let $\Sigma \subseteq M$ be a hypersurface with the boundary $\partial\Sigma = B_1 \cup B_2$, where B_1 and B_2 are disconnected codimension 2 surfaces. The symplectic flow passing through Σ is then on-shell given by a boundary integral

$$\int_\Sigma \omega[\delta_1\Phi, \delta_2\Phi; \Phi] = \oint_{B_1} k[\delta_1\Phi, \delta_2\Phi; \Phi] - \oint_{B_2} k[\delta_1\Phi, \delta_2\Phi; \Phi]. \tag{6.53}$$

Each of the boundary integrals in (6.53) can be used to define a symplectic form over Γ.

$$\oint_{B_1} k[\delta_1\Phi, \delta_2\Phi; \Phi] = \Omega_{AB}\,[\delta_1\Phi]^A\,[\delta_2\Phi]^B \tag{6.54}$$

defines the symplectic form $\Omega_{AB} = \Omega_{AB}^{(B_1)}$ over Γ (relative to B_1). This can then be used to define the Poisson-bracket in the usual way. We denote quantities sometimes with the superscript (B_1) to remember that they are defined with B_1 as the chosen reference. For a vectorfield X over Γ corresponding to field variations $\delta_X \Phi$, the expression

$$\delta H_X\left[\delta\Phi; \Phi\right] = \oint_{B_1} k\left[\delta\Phi, \delta_X\Phi; \Phi\right] \tag{6.55}$$

defines a 1-form over the phase space. If X is a symplectic symmetry $\mathcal{L}_X \Omega_{AB}^{(B_1)} = 0$, the 1-form (6.55) is exact and can be integrated over phase space to provide the scalar $H_X = H_X^{(B_1)}[\Phi]$. The role of this scalar is to generate $\delta_X\Phi$ via the Poisson-bracket.

Due to the expression (6.55) the value $H_X[\Phi]$ contains information about the field configuration Φ over the surface B_1. In fact, for the linearly independent symplectic symmetries X, we can think of $H_X[\Phi]$ as part of the Cauchy-data required to specify $\Phi \in \Gamma$. These values $H_X[\Phi]$ can therefore be thought of forming part of the coordinates of a chart for the phase space Γ. Due to their holographic nature (6.55), we termed them in [3] as boundary Cauchy-data (BCD).

The BCD is defined with respect to the codimension 2 surface B_1. What would have been if we had wanted to define it with repect to a different surface B_2 of codimension 2? In that case, we have to choose a hypersurface Σ connecting B_1 and B_2 and correct the BCD relative to B_1 by the symplectic current passing through Σ. Due to (6.53) and (6.55) the BCD of a symplectic symmetry X are related by

$$\delta H_X^{(B_1)}\left[\delta\Phi; \Phi\right] - \delta H_X^{(B_2)}\left[\delta\Phi; \Phi\right] = \int_\Sigma \omega\left[\delta\Phi, \delta_X\Phi; \Phi\right]. \tag{6.56}$$

That is, the change of the BCD from a surface B_1 to a surface B_2 is dictated by the symplectic current $\omega[\delta\Phi, \delta_X\Phi; \Phi]$ passing through the hypersurface in between. The specification of these symplectic currents along an entire Cauchy-surface Σ forms the remaining Cauchy-data (in addition to the BCD for a particular codimension 2 cross-section of Σ) that uniquely determines a point in phase space Γ.

For the case of 4D Einstein-gravity, one can push Σ towards null infinity. In that case, the latter currents reduce to the Bondi-news whereas the BCD is essentially given by the mass-aspect, angular momentum-aspect and additional functions on S^2 that provide the Cauchy-data for the solution space (for a review of the solution space in that case see[7]). This example is meant to illustrate the way of thinking. As already mentioned, more detailed explanations and examples in simpler settings will be provided somewhere else.

In summary, so far we have said that the phase space Γ can be parametrized by the BCD associated to the symplectic symmetries over a codimension 2 surface together with their associated symplectic currents. While the BCD will be of our main interest in the following, we want briefly explain that already at this point we are able to learn something.

(6.56) describes the change of the BCD from B_1 to B_2 caused by the symplectic current passing through a hypersurface Σ connecting them. In this way, (6.56) reflects a *memory effect.* The independence of the particular choice of Σ connecting B_1 and B_2 in (6.56) is a consequence of the *constraint* $d\omega[\delta\Phi, \delta_X\Phi; \Phi] = 0$. In this way, each *symplectic symmetry* X

gives rise to a memory effect along an arbitrary hypersurface in M and also to a constraint which altogether reflect the equations of motion. The relation between the concepts symmetry, memory and constraints was recently emphasized in a variety of examples starting with [27, 67, 68, 69] and references thereof. Here we see, that in the covariant phase space language the equivalence between these concepts becomes obvious and is just reflecting the equations of motion.

In the remaining part of the chapter, we will explain that the particular way to parametrize the phase space Γ can actually be indeed useful to approach various problems.

Choose a particular point $\Phi \in \Gamma$ by specifying its coordinates, i.e. the BCD $H_X[\Phi] = H_X^{(B_1)}[\Phi]$ for the symplectic symmetries X and their associated symplectic currents. Now, take the latter fixed and vary the BCD. This spans an entire subspace $S \subseteq \Gamma$ on which the BCD then can be seen as coordinates. Thus, S is a submanifold in the phase space Γ. However, S has an additional structure. The Poisson-bracket algebra of the generators $H_X = H_X^{(B_1)}[\Phi]$ forms a representation of the Lie-bracket algebra of symplectic symmetries up to central extension. That means, for symplectic symmetries X, Y one has

$$\{H_X, H_Y\} = H_{[X,Y]} + K_{X,Y} \tag{6.57}$$

for c-numbers $K_{X,Y} = K_{X,Y}^{(B_1)}$. Therefore, the submanifold S is a symplectic manifold on its own. Its coordinates are given by the BCD H_X and their Poisson-bracket algebra is given by (6.57). The part S in phase space Γ can therefore be described by a theory on its own right, a "holographic dual" associated to the chosen codimension 2 surface $B_1 \subseteq M$.

To summarize, we see that to a chosen codimension 2 surface $B_1 \subseteq M$, a holographic dual theory describing a suited part $S \subseteq \Gamma$ can be associated. Choosing a different surface $B_2 \subseteq M$ or different gauge will in general affect the form of (6.57) describing the same part $S \subseteq \Gamma$. Choosing a different $B_2 \subseteq M$ can also lead to a different submanifold in phase space.

The hope is that the construction of these submanifolds is useful to approach some problems. Usually, the above constructed submanifold $S \subseteq \Gamma$ is too large. However, subalgebras of the algebra of symplectic symmetries will due to (6.57) lead to lower-dimensional submanifolds S' in S. Choosing this S' small enough, one is left with a theory that covers a small part of the phase space that might be of interest for a particular problem under consideration.

Can this be useful?

6.5.2 A Microscopic Theory for the Schwarzschild Black Hole

To apply the ideas of the last chapter 6.5.1 to a Schwarzschild black hole in Einstein-gravity was essentially the content of [3]. We recap very briefly the steps. Working in Bondi-gauge, the Schwarzschild-metric fixes a particular point $g_{ab} \in \Gamma$ in covariant phase space. The goal is to find the part of phase space that is responsible for the microstates. The hope is, that the submanifolds constructed in the last section are candidates for this. For the codimension 2 surface B_1, it is natural to take a cross-section of the event horizon in the hope that the algebra (6.57) will get especially simple.

The next step is then to study closed algebras of symplectic symmetries and their associated submanifolds in Γ. For the Schwarzschild black hole g_{ab}, there is a simplest choice to start with. Due to the black hole uniqueness theorems, one expects microstate excitations to have the form of residual gauge transformations $\delta g_{ab} = \mathcal{L}_\xi g_{ab} \in T_{g_{ab}}\Gamma$ for suited vectorfields ξ. Therefore, one is interested in symplectic symmetries X which at $g_{ab} \in \Gamma$ take the form of a residual gauge transformation $X|_{g_{ab}} = \delta_\xi \in T_{g_{ab}}\Gamma$. Symplectic symmetries of such form and their associated BCD H_X were called gauge aspects in [3]. *Under the assumption,* that the symplectic symmetries due to the gauge aspects cover enough of the phase space relevant for the Schwarzschild black hole microstates, the BCD parametrizing this submanifold $S \subseteq \Gamma$ as well as its algebra (6.57) was determined in [3]. As explained in the last section, this symplectic submanifold $S \subseteq \Gamma$ provides a theory in its own right and is a candidate for the holographic dual theory of the Schwarzschild black hole. Since this procedure determines the BCD, one is able to infer the form of the residual gauge transformations at g_{ab} which are the candidates for the black hole microstates.

6.5.3 Counting Degrees of Freedom

In the last section, we have explained the construction of a symplectic submanifold $S \subseteq \Gamma$ that is a candidate for the part of the phase space relevant for the microstates of a Schwarzschild black hole. Its coordinates given by the BCD provide observables with the Poisson-bracket algebra (6.57). In this way, we have an explicit theory that provides a candidate for the dual theory governing the Schwarzschild black hole. How can we check whether our candidate theory is correct?

The first check would be to see whether one can deduce the correct black hole entropy from $S \subseteq \Gamma$. As explained in [3], there are arguments from several directions indicating that the part of phase space responsible for black hole microstates should possess a 2D local conformal symmetry. Therefore, one is tempted to ask whether $S \subseteq \Gamma$ is compatible with this conformal invariance. If so, the observables of S must give rise to a 2D stress-tensor such that its Virasoro-generators fulfill a Virasoro-algebra. Since we know the algebra of observables (6.57), we can search for a Sugawara-construction of these Virasoro-generators out of the BCD over S. This is precisely Carlip's approach to entropy counting in disguise as we will explain in the following.

In [3] a projection operator $T_{g_{ab}}\overline{\mathcal{F}} \to T_{g_{ab}}S$ was given, that maps an arbitrary (possibly not gauge-fixed) excitation of a Schwarzschild black hole g_{ab} onto the relevant microstate excitation of g_{ab}. In this way, an arbitrary gauge-excitation $\mathcal{L}_\xi g_{ab}$ (that in general also contains components that are not tangential to S at g_{ab}) is mapped to the relevant symplectic symmetry X tangential to S. It is this mapping $\xi \mapsto X$ from spacetime diffeomorphisms to the vectorfields over S that makes the connection with Carlip's approach clear. In the above mentioned Sugawara-construction, we are searching for symplectic symmetries, i.e. vectorfields $X_n, \overline{X}_n$ over S such that their generators H_{X_n} and $H_{\overline{X}_n}$ satisfy via (6.57) a $Vir \oplus \overline{Vir}$-algebra. Instead, we can look for diffeomorphisms ξ_n and $\overline{\xi}_n$ giving rise to a

$Vir \oplus \overline{Vir}$-algebra with respect to the spacetime Lie-bracket. This is precisely what we did in chapter 6.2-6.4. We furthermore inspected the algebra of the Hamiltonian generators of ξ_n and $\overline{\xi}_n$ to see that the emerged Virasoro-algebra gives indeed rise to the expected entropy. However, in chapters 6.2-6.4 we did not employ the mapping $\xi \mapsto X$. Therefore, so far we only know that the Hamiltonian generators of the $Vir \oplus \overline{Vir}$-diffeomorphisms provide candidates for the Virasoro-generators of a possible would-be CFT governing the black hole microstates. The approach is not sensitive to the details of what this CFT might be.

As already proposed in [3], the situation is different once we have figured out our candidate theory $S \subseteq \Gamma$. We can use the projection operator $\xi_n \mapsto X_n$ and $\overline{\xi}_n \mapsto \overline{X}_n$ to obtain with H_{X_n} and $H_{\overline{X}_n}$ candidates for the Virasoro-generators in S. Precisely this step is sensitive to the choice of S. That means, X_n and $\overline{X}_n$ and their generators would change if the space S were different. Since we project on S, we are directly probing the degrees of freedom covered by S. Up from here, we can proceed the same way as in the indirect approach. Inspecting the Virasoro-algebra formed by H_{X_n} and $H_{\overline{X}_n}$ via (6.57), we can count the degeneracy of states and compare it to the expected entropy. If the result were to agree, this would provide substantial consistency check that the theory given by $S \subseteq \Gamma$ is correct and covers all degrees of freedom of the Schwarzschild black hole. Furthermore, it would support that S is indeed conformally invariant thus providing a concrete realization of a Schwarzschild/CFT-correspondence. In case that disagreement is found, one has to enlarge S successively by allowing larger algebras of symplectic symmetries in its construction, up until the procedure is going to converge.

To summarize, with (6.30) and (6.31) we have given the needed $Vir \oplus \overline{Vir}$-algebra of diffeomorphisms that is needed in the above procedure of projecting directly onto black hole degrees of freedom and counting entropy. These vectorfields were already given in [3]. Here, we have given their systematic construction. Furthermore, we have provided arguments what singles out the presented $Vir \oplus \overline{Vir}$-diffeomorphisms. Most importantly, we have seen that inspection of the Hamiltonian generators (without projecting directly onto degrees of freedom), we were able to show that the Poisson-bracket algebra consistently leads to the expected Bekenstein-Hawking entropy. Therefore, the $Vir \oplus \overline{Vir}$-diffeomorphisms seem to be the right candidates for the approach described in [3] and reviewed here. Performing this approach, we leave for future investigations. The purpose of this work here was to provide convincing arguments that the $Vir \oplus \overline{Vir}$-vectorfields are the appropriate diffeomorphisms to use.

Chapter 7

Summary and Outlook

Concerning our main question in chapter 3.1, what have we learned?

We have seen that it is possible to find candidates for reasonable gravitational black hole hair of the Schwarzschild black hole that might account for its thermodynamic properties. In fact, in a sense, already the most simplest guess for the gravitational black hole hair leads to field excitations that might account for the microstates and black hole entropy.

Indeed, a first application of newly developed methods to analyze the Hamiltonian phase space supports this guess. Within those tools, we were able to propose a dual theory, given by its observables and their algebra, that is supposed to describe the part of Hamiltonian phase space which is responsible for the microstates of the Schwarzschild black hole. We have provided arguments why such a dual theory is expected to be conformally invariant, therefore our obtained candidate dual theory provides a first proposal for a Schwarzschild/CFT-correspondence.

Note that in [16, 17, 18, 19, 20] a quantum N-portrait of black holes was developed which also leads to a proposal for a microscopic description of microstates and origin of entropy as well as a black hole/conformal field theory correspondence.

Independent of our particular Schwarzschild/CFT proposal, the part of phase space responsible for the microstates and entropy is expected to possess a conformal invariance. On the particular example of the Schwarzschild black hole, we showed how the associated conformal generators are expected to look like by various arguments. This information is a already enough to infer the state degeneracy and indeed precise agreement with the Bekenstein-Hawking entropy is found.

We explained how the conformal generators should manifest themselves in a Schwarzschild/CFT proposal that one might write down. Moreover, we showed how this leads to a consistency check to find out whether a given Schwarzschild/CFT proposal precisely contains the right gravitational hair that is responsible for the microstates. While carrying this out for our concrete proposal derived during the last chapters is left for future investigations, we can draw already the following conclusion. We can shortly summarize the main result of this thesis as follows:

General Relativity seems to provide the right degrees of freedom to account for the sta-

tistical mechanical properties of black holes.

We want to briefly describe what the next steps are. While it is important to further check the Schwarzschild/CFT proposal in order to finally give a proof of its correctness (and the gravitational hair that it provides), we should more properly understand the mentioned general tools to analyze the Hamiltonian phase space of a given theory. Finally, this will of course help to fix the right Schwarzschild/CFT-correspondence (and also the equivalent for more general black holes). As an outlook, we want to draw the emergent picture that these tools provide. A detailed analysis of those tools with checks on theories simpler than gravity will appear somewhere else.

Consider a theory given by some action that depends on local fields living on some manifold M. Then, we are suggesting that the Hamiltonian phase space can be parametrized by a very natural class of coordinate systems. These coordinate systems can be used in parametrizing functional integrals of the form (2.2) or in the quantum mechanical partition function Z that is derived thereof.

Note that by switching between these coordinate systems the symplectic form, which can be derived out of Z by covariant phase space methods, will take different forms and in some cases can be simplified by singling out a special coordinate system in this class.

Furthermore, these coordinate systems in this class come with a very remarkable property. Part of its coordinates parametrize submanifolds of the phase space which have a symplectic structure on its own. Analogously they have a quantum mechanical partition function Z' on its own which can be derived out of Z by integrating out appropriate paths in the functional integral. In this way, these parts of coordinates lead to well-defined theories on their own right. The choice of these parts of coordinates is tied to a choice of closed, connected codimension 2 surface in M. In this sense, the obtained submanifolds (or equivalently partition functions Z') can be called holographic.

What does this has to do with black holes in Einstein-gravity? We propose that the part of phase space responsible for the statistical mechanical properties of a black hole is given by a suited submanifold of the above-mentioned type. Conversely, by searching and studying such submanifolds we can obtain e.g. a Schwarzschild/CFT proposal and this is how our first application of these methods led to the first Schwarzschild/CFT proposal of this thesis.

Moreover, within this language, we explain the universal form of the black hole entropy as follows. If a state in the Hamiltonian phase space describes a black hole, i.e. contains an event horizon, then there exists a natural coordinate system around this state. In this coordinate system a conformal symmetry of the symplectic form becomes manifest that is present over a whole suited ensemble containing the microstate excitations of our black hole state. The action of the conformal symmetry becomes especially simple at the black hole state we started with. However, knowing the action at this particular point is due to CFT techniques sufficient to infer the degeneracy of the entire ensemble. The inferred degeneracy is in precise agreement with the Bekenstein-Hawking entropy. On the example of a Schwarzschild black hole, we have in this thesis shown that this proposal works.

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