Supersymmetric theories, boundaries and quantum invariants

Alexander Tabler

München 2020

Supersymmetric theories, boundaries and quantum invariants

Alexander Tabler

Dissertation
an der Fakultät der Physik
der Ludwig–Maximilians–Universität
München

vorgelegt von
Alexander Tabler
aus Heraklion, Griechenland

München, den 04. November 2020

Erstgutachter: Prof. Dr. Peter Mayr
Zweitgutachter: Prof. Dr. Ilka Brunner
Tag der mündlichen Prüfung: 16. Dezember 2020

Contents

Abstract & Zusammenfassung

Abstract

Supersymmetric theories are an excellent playground in theoretical physics. Despite their phenomenological challenges, their high amount of symmetry often implies that the—otherwise ill-defined—infinite-dimensional path integral is well-defined and can in fact be computed exactly. The study of two-dimensional supersymmetric theories in particular has been propelled by string theory, leading among other things, to beautiful and surprising results both in physics as well as pure mathematics. In this thesis we want to study an "uplift" from two-dimensional to three-dimensional $\mathcal{N} = 2$ supersymmetric theories. Our discussion is focused on two related aspects of three-dimensional theories.

The first subject of study is a three-dimensional generalization of the correspondence between two-dimensional supersymmetric gauge theories at high energies and topological quantum invariants of the target space of these theories at low energies. We study a three-dimensional gauge theory on a non-trivial background, whose low energy limit is a non-linear sigma model with a Grassmannian $\mathrm{Gr}(M, N)$ target space, i.e., the space of complex M-planes in $\mathbb{C}^N$. In this three-dimensional gauge theory, we study supersymmetric Wilson loops. The algebra of Wilson loops describes, via the correspondence, the quantum K-theory of $\mathrm{Gr}(M, N)$ studied by mathematicians, i.e., classes of complex vector bundles with a deformed tensor product structure. The structure constants of this algebra are quantum invariants of $\mathrm{Gr}(M, N)$. We find agreement between results from the gauge theory on the one side, and results computed by mathematicians via different methods on the other side.

The second subject we study is flat three-dimensional supersymmetric theory in the presence of spacetime boundaries and its symmetric boundary conditions. In general, bulk supersymmetric theories admit so called supercurrent multiplets. These are supersymmetry representations, present in any local supersymmetric theory, whose components are the conserved currents of the bulk theory, such as the supercurrent and the energy-momentum tensor. The existence of a boundary means that we have to choose boundary conditions for the bulk fields and that some of the symmetries of the theory are broken. We generalize the notion of bulk supercurrent multiplets to supercurrent multiplets with bulk and boundary parts consistent with the unbroken symmetries and symmetric boundary conditions. We successfully test our definitions in a simple model with boundary conditions that are three-dimensional generalizations of matrix factorizations.

Zusammenfassung

Supersymmetrische Theorien sind ein spannendes Untersuchungsgebiet in der theoretischen Physik. Trotz ihrer phänomenologischen Herausforderungen impliziert ihr hoher Symmetriegrad oft, dass das—sonst schlecht definierte—unendlich dimensionale Pfadintegral wohldefiniert ist und tatsächlich genau berechnet werden kann. Insbesondere wurde die Untersuchung zweidimensionaler supersymmetrischer Theorien von der Stringtheorie vorangetrieben, was unter anderem zu schönen und überraschenden Ergebnissen sowohl in der Physik als auch in der reinen Mathematik führte. In dieser Arbeit wollen wir einen Übergang von zweidimensionalen supersymmetrischen Theorien zu dreidimensionalen $\mathcal{N} = 2$ supersymmetrischen Theorien untersuchen. Unsere Diskussion konzentriert sich auf zwei verwandte Aspekte dreidimensionaler Theorien.

Das erste Untersuchungsobjekt ist eine dreidimensionale Verallgemeinerung einer Korrespondenz zwischen zweidimensionalen supersymmetrischen Eichtheorien bei hohen Energien und topologischen Quanteninvarianten des Zielraums dieser Theorien bei niedrigen Energien. Wir untersuchen eine dreidimensionale Eichtheorie auf einem nicht trivialen Hintergrund, dessen Grenze bei niedrigen Energien ein nichtlineares Sigma-Modell mit einem Grassmann'schen $\mathrm{Gr}(M, N)$-Zielraum ist, also mit dem Raum komplexer M-Ebenen in $\mathbb{C}^N$. In dieser dreidimensionalen Eichtheorie untersuchen wir supersymmetrische Wilson-Loops. Die Algebra von Wilson-Loops beschreibt, über die Korrespondenz, die von Mathematikern untersuchte Quanten-K-Theorie von $\mathrm{Gr}(M, N)$, d.h. Klassen komplexer Vektorbündel mit einer deformierten Tensorproduktstruktur. Die Strukturkonstanten dieser Algebra sind Quanteninvarianten von $\mathrm{Gr}(M, N)$. Wir finden Übereinstimmung zwischen den Ergebnissen der Eichtheorie auf der einen Seite und den von Mathematikern mit verschiedenen Methoden berechneten Ergebnissen auf der anderen Seite.

Das zweite Thema, das wir untersuchen, ist die flache dreidimensionale supersymmetrische Theorie mit Raumzeitrand und ihre symmetrischen Randbedingungen. Im Allgemeinen kann man bei supersymmetrischen Bulktheorien ohne Ränder sogenannte Supercurrent-Multiplets definieren. Diese sind Supersymmetriedarstellungen, die in jeder lokalen supersymmetrischen Theorie vorhanden sind und deren Komponenten die konservierten Ströme der jeweiligen Theorie sind, wie z.B. der Superstrom und der Energie-Impuls-Tensor. Die Existenz eines Randes bedeutet, dass wir Randbedingungen für die Bulkfelder wählen müssen und dass einige Symmetrien der Theorie gebrochen sind. Wir verallgemeinern die Definition der Bulk-Supercurrent-Multiplets auf Supercurrent-Multiplets mit Bulk- und Randteilen, die mit den ungebrochenen Symmetrien und symmetrischen Randbedingungen konsistent bleiben. Wir testen unsere Definitionen erfolgreich in einem einfachen Modell mit Randbedingungen, die die dreidimensionale Verallgemeinerungen von Matrixfaktorisierungen sind.

Acknowledgements

First and foremost, I want to wholeheartedly thank my first supervisor, Prof. Peter Mayr and my second supervisor, Prof. Ilka Brunner for their guidance and support both in my master's thesis, as well during the Ph.D. years. Prof. Mayr has provided me with incredibly exciting research questions, but more importantly, has been a calm source of academic, professional and also emotional support. Prof. Brunner also carefully guided my first steps in Ph.D. research with another exciting topic, but again, more importantly, has provided me with nothing but positive academic and emotional reinforcement. As a graduate student with particular research interests, one often cannot be very "picky" with choosing a supervisor, let alone two. I consider myself extremely lucky for having two supervisors with which real communication was effortless, and both of whom I deeply respect and admire.

Secondly, I want to express my thanks to my collaborators on the different research projects behind this thesis: my friend and officemate Jonathan Schulz in Munich, as well as Prof. Dr. Hans Jockers and Dr. Urmi Ninad in Bonn. It is safe to say that this work would have been impossible without valuable inputs and long discussions with each of them and I thank them sincerely.

Working on a Ph.D. thesis is intimately tied to one's real life, and therefore many people implicitly or explicitly contribute to its successful completion. I feel deeply thankful towards Christianna, Giannis, Giorgos G., Juan, Juraj, Ismail, Katerina, Kerstin, Lazaros, Marios, May as well as the Greek "Meateater" group, Prof. Tasos Petkou and the "Kamaria" group, the 4th floor "Love Letters" mates and many other TMP alumni (and probably many more that I cannot list) for being one way or another part of my life. Very special thanks must go to my dear friends Pantelis Fragkos, Tomáš Procházka and Danu Thung, for learning with me, teaching me and making (not only) academic life much more enjoyable. I also with to thank Christianna, Danu, Ismail, Kerstin, Pantelis, Tomáš and Urmi for valuable comments and corrections on the manuscript.

Lastly, the pillar of wholehearted support and love without which none of this would have been achieved comes from my, by now, all-female family. Sofia, Mina, Inge, Tzeni, Iris, Marina and of course, *always*, Anna.

στις γυναίκες μου

Chapter 1

Introduction and background

In this chapter we give an introduction into the topics studied in this thesis. We very briefly describe motivation to study supersymmetric theories and sketch their connection to manifold invariants. We also discuss the relevance of boundaries and supercurrent multiplets in such theories. We present some of the mathematical notions behind some of this work, namely quantum cohomology and quantum K-theory. We end the chapter by a brief outline of the following two chapters, based on the publications of the author and a short overview of related topics. The publications [1, 2] relevant to chapter 2 are joint work with Prof. Dr. Hans Jockers, Prof. Dr. Peter Mayr and Dr. Urmi Ninad. The publication [3] relevant to chapter 3 is joint work with Prof. Dr. Ilka Brunner and Jonathan Schulz.

1.1 Supersymmetric field theories

1.1.1 Why supersymmetry?

Quantum field theory is the enormously successful[1] child of quantum mechanics and special relativity. However, an axiomatic, mathematically rigorous formulation of general quantum field theories (in more than two dimensions) is elusive: in short, the infinite dimensional integration of the path integral is in general not well-defined. Still, the interaction between physics and mathematics has been very fruitful for both fields.

The study of supersymmetric field theories was mainly initiated as a mathematically natural extension of the Standard Model, but phenomenologically speaking, such theories have fallen out of favor. However, supersymmetric theories are a natural playground for more tractable physical theories, as the high amount of symmetry severely constrains the dynamics. An incomplete list of features includes that supersymmetric theories enjoy non-renormalization theorems [4], are relevant in compactifications of superstring theory [5] and also play an important role in holography [6]. More importantly in the context of this thesis, in the last two decades there has been great progress in the computation of supersymmetric partition functions, the generating functions for correlators of observables, by techniques of so called supersymmetric localization (see [7, 8] for pioneering work and [9] for a review). These are powerful computational techniques that extend ideas of equivariant localization from mathematics to the infinite-dimensional setting. The techniques have lead to a plethora of *exact* computations (as opposed to perturbative computations

[1] https://pdg.lbl.gov

of partition functions of general field theories) in various dimensions, curved backgrounds, in the presence of boundaries or different amounts of supersymmetry. See [10–22] for an incomplete list of exact results.

In addition, supersymmetric field theories are often a surprising source of new ideas for mathematics. This is especially true for two-dimensional theories that are relevant in the study of string theory, with fields

$$f : \Sigma \to X \tag{1.1}$$

mapping from a two-dimensional surface Σ to some space X. A notable development is the discovery by physicists [23] that special classes of spaces used in string theory, six-dimensional Calabi-Yau manifolds, come in pairs connected by *mirror symmetry*. This discovery has been very fruitful; its precise formulation and implications is a subject of study by both physicists and mathematicians [24–26].

1.1.2 Twists and an example of invariants

Another (related) interesting direction is also that of *topological field theories* (TFTs), first introduced by Witten [27]. These are quantum field theories in which correlation functions of local operators do not depend on the insertion points and hence on the metric of the space on which they are defined. As such, the correlators are numbers that are invariant under continuous changes in the geometry of these spaces and describe *topological invariants*. Additionally, TFTs form a class of theories that can be treated axiomatically and rigorously [28], and the study of TFTs with *extended* excitations (i.e., not-pointlike operators) is now an evolving field of physics and pure mathematics related to higher category theory [29, 30].

General supersymmetric field theories are, despite the high amount of symmetry, not topological. Supersymmetric theories and topological theories are nevertheless intimately related. Namely, in many examples of supersymmetric theories, one can select a *nilpotent* supercharge Q, and restrict to the subsector of observables that lie in the Q-cohomology, i.e., observables corresponding to classes of operators where the representatives $\mathcal{O}$ satisfy

$$[Q, \mathcal{O}] = 0, , \tag{1.2}$$

where $[\cdot, \cdot]$ denotes a graded commutator and where we identify any two operators $\mathcal{O}, \mathcal{O}'$ that satisfy

$$\mathcal{O} = \mathcal{O}' + [Q, \mathcal{U}] \tag{1.3}$$

where $\mathcal{U}$ is any operator. In particular Q-exact operators $\mathcal{O} = [Q, \mathcal{U}]$ are equivalent to zero. This non-trivial restriction implies, among other things, that some or all (depending on the theory and choice of Q) translation generators P_μ become Q-exact, i.e., we have symmetry operators $Q, \mathcal{Q}_\mu, P_\mu$ with

$$Q^2 = [\mathcal{Q}_\mu, \mathcal{Q}_\nu] = [\mathcal{Q}_\mu, P_\nu] = [Q, P_\mu] = 0, \qquad [Q, \mathcal{Q}_\mu] \sim P_\mu. \tag{1.4}$$

This in turn means that in this subsector the dependence of the correlators on the insertion point in spacetime drops out and the theory of this restricted subsector is topological.

This is in fact how TFTs were introduced in [27, 31], through the procedure known as *topological twisting*. The procedure of twisting is roughly the following. One starts with a supersymmetry algebra on flat space, which is generated by fermionic supercharges, with an R-symmetry automorphism and one tries to extend its definition on some curved background. Mathematically, the curvature often comes with obstructions to defining necessary spinors for the description of the symmetry. The obstruction can be circumvented physically by 'replacing' some subgroup of the Lorentz rotation group by a *twisted* subgroup in the product of the Lorentz group with the R-symmetry group[2]. The choice of new Lorentz rotation group changes the spin of some fields, but does not change the fermion/boson statistics; in particular there is a *scalar fermionic* supercharge Q (a linear combination of charges of the original untwisted theory) which can be defined on the curved background.

A well-known example of twisting and invariants is given by the A-model. The model is determined by the so called A-twist of the $\mathcal{N} = (2,2)$ sigma model in [31]. The twist impies that in general only a subalgebra is preserved (and there is another twist where a different subalgebra is preserved leading to the mirror B-model). The sigma model is the quantum field theory where the (bosonic) fields are maps $\phi : \Sigma \to X$ from a Riemann surface Σ to a Kähler manifold X[3], while fermionic fields ψ are defined as certain sections of bundles over X. Mathematically, the "twisting" amounts to considering in fact a different theory. The new theory has the same local expression for the action pulled back on Σ, but the fields are now sections of *different* bundles, given by tensoring the original bundles with a square-root of the canonical bundle of X. In the twisted theory, we may identify odd-degree forms on X with the fermionic fields ψ and even forms to bispinors (bosons). In other words, in the A-model, for each differential form $\theta \in \Lambda^*(X)$ we associate an operator

$$\theta \mapsto \mathcal{O}_\theta. \tag{1.5}$$

The scalar supercharge Q defined from the original $\mathcal{N} = (2,2)$ supersymmetry by the twisting procedure now acts as

$$[Q, \mathcal{O}_\theta] = -\mathcal{O}_{d\theta}, \tag{1.6}$$

where d is the exterior derivative on X. We see that the nilpotent operator Q is mapped by this association to the exterior derivative d acting on differential forms. It is therefore easy to deduce that restricting to Q-cohomology corresponds to restricting to differential forms that are closed $d\theta = 0$, and closed differential forms that differ by exact forms are identified. In other words, Q-cohomology corresponds to de Rham cohomology and the correlators of operators representing Q-cohomology classes are *topological invariants* of X[4].

The wealth of results about (A- and B-twists of) two dimensional $\mathcal{N} = (2,2)$ sigma models has motivated a possible "uplift" to higher-dimensional theories with the same amount of supersymmetry, (4 real supercharges), in particular 3D $\mathcal{N} = 2$ theories and 4D $\mathcal{N} = 1$ theories. Half of this thesis (chapter 2) is dedicated to analogous aspects of supersymmetric field theories in three spacetime dimensions.

---

[2] In this discussion we assume that the twisting subgroups are Abelian.

[3] That is, a complex, Riemannian, symplectic manifold with compatible structures. The sigma model arose from (topological) string theory, where the target space X is a Calabi-Yau manifold, which is a Kähler manifold with vanishing canonical bundle, while Σ is the two-dimensional string worldsheet.

[4] The correlators are in fact computable; the equations of motion force the maps ϕ to be holomorphic, which is a strong enough constraint to make the path integral into a well-defined integral over the space of holomorphic maps $\Sigma \to X$. See next subsection 1.2 for some mathematical details.

1.1.3 Boundaries and supersymmetric boundary conditions

We now want to discuss theories where the worldsheet or worldvolume has a *boundary*. As a general rule, the boundary breaks the symmetry that generates translations perpendicular to the boundary. Hence it must also break *at least* part of the supersymmetry algebra generators, as otherwise they would anticommute to a broken generator. In fact, for a generic supersymmetric theory, introducing a boundary will break most of the symmetries, as divergence-free symmetry currents “leak” out of the boundary.

In any theory with boundary, we must choose sensible boundary conditions for the fields. Choosing boundary conditions for field theories is itself a rich subject, once we allow for purely boundary dynamics and couplings to bulk fields. However, for a very special class of boundary conditions, we may preserve *some* supersymmetry. These are called supersymmetric boundary conditions and are highly constrained compared to arbitrary boundary conditions, similarly to how supersymmetric theories are highly constrained compared to general field theories.

The A- and B-models were introduced in the previous subsections as examples of two-dimensional theories with some (smaller) supersymmetry algebra after some twisting procedure. The supersymmetry subalgebra can also be regarded as the remaining subalgebra after the introduction of a boundary on the world-sheet or world-volume as described above. The corresponding special classes of boundary conditions are referred to as A-type and B-type boundary conditions [32]. In the context of string theory, where boundary conditions define so called D-branes, these special boundary conditions are the A- and B-branes [33, 34].

The three-dimensional $\mathcal{N} = 2$ theories with boundary also possess boundary conditions related to the A- and B-types [35–37] that are classified by the 2D subalgebras that are preserved after the boundary is introduced. The choice of boundary conditions is a necessary step in the computation of path integrals via supersymmetric localization [22, 38] or supersymmetric half-indices via “sum over words” of classical UV operators [39].

Supercurrent multiplets

In any supersymmetric theory, the fields are (by definition) organized into representations of the supersymmetry algebra. Currents and charges of any symmetry are represented (classically) as function(al)s on those fields. One can study many universal features of supersymmetric theories by organizing these currents into representations of the supersymmetry algebra, i.e., into superfields (when the theory admits a superspace description) with symmetry currents as components. These are the so called supercurrent multiplets which will be discussed at length in chapter 3.

The identification of supercurrent multiplets has many advantages. Firstly, as a purely representation-theoretic construction, it is independent of a Lagrangian description and applies also to field theories that do not admit such a description. Secondly, the structure of supersymmetry algebras and its action on the superfields dictates that the supercurrents (the symmetry currents associated to supersymmetry) and the energy-momentum tensor are part of the supercurrent multiplets. The superfields satisfy further (defining) constraints that imply conservation of those currents. Since the multiplet contain the energy momentum tensor, one can use them to identify spacetime directions that can be twisted. Lastly, supercurrent multiplets are a technical tool for supersymmetric localization techniques on curved backgrounds [40] mentioned in the previous subsections[5].

Connecting to the topic of this subsection, one can then ask how to consistently extend supercurrent multiplets in theories with boundary. Their definitions change both because the supersymmetry algebra is broken to a smaller subalgebra, but also because the some currents that form their components are no longer conserved. The contents of the latter part of this thesis (chapter 3) are dedicated to discussing this question in three-dimensional theories with boundaries.

1.2 A brief introduction to quantum rings

In this section we will introduce the necessary notions for quantum cohomology and quantum K-theory. Our discussion is by no means rigorous or detailed, and we refer the reader to the mathematical literature for details. We begin with a brief introduction to quantum cohomology, as it is the precursor for quantum K-theory, which we describe afterwards. In addition, this relationship is deeply embedded in the physical descriptions related to supersymmetric theories from the previous section

$$\begin{aligned} \text{2D } \mathcal{N}=(2,2) \text{ theory with target space } X &\longleftrightarrow \text{ Quantum cohomology on } X \\ \text{3D } \mathcal{N}=2 \text{ theory with target space } X &\longleftrightarrow \text{ Quantum K-theory on } X \end{aligned}$$

Many results from the study of quantum K-theory from 3D supersymmetric gauge theories have their two-dimensional analog in quantum cohomology, which therefore serve as ‘guiding’ principles. The various features that do *not* have an analog are also subject of current research.

1.2.1 Quantum cohomology

Quantum cohomology is a deformation of the cup-product ring structure of classical cohomology of Kähler manifolds X[6]. The deformation is motivated by physical arguments [41–44], by counting so-called worldsheet instanton contributions. However its properties, e.g., associativity, have implications for the enumerative geometry of X and and the new deformed structure constants encode the Gromov-Witten invariants of X. By viewing classical cohomology of a X as the study of intersection theory of X, we may describe quantum cohomology as intersection theory of the space of marked holomorphic curves in X. For general X and general curves, these spaces are mathematically “wild”. For sufficiently “nice” X and under assumptions on the curves, these spaces become themselves possibly singular manifolds that admit an appropriate compactification. One such compactification, which we use in this introduction, is the *moduli spaces of stable maps* $\overline{\mathcal{M}_{g,n}}(X,\beta)$. Here, g is the genus of the curves, n is the number of marked points $p_1,\ldots,p_n$ on the curve and $\beta \in H_2^{\text{eff.}}$, where $H_2^{\text{eff.}} \subset H_2(X,\mathbb{Z})$ are the *effective* curve classes, controls the degree of the curve. The cohomology of X is recovered in the limit $\beta = 0$ of constant curves.

[5] One extends the flat space supersymmetry to supergravity, including (background) metric fields and superpartners. The metric is then coupled in an automatically supersymmetric fashion to the stress-tensor, by using the supercurrent multiplet as opposed to components, and the variation of background fields is set to zero.

[6] The Kähler condition can be weakened.

Let us list some of the ingredients necessary in a loose fashion. For mathematical details we refer to [45–50] and we follow mostly [25]. The points of $\overline{\mathcal{M}_{g,n}}(X,\beta)$ are pointed holomorphic maps

$$f : \big(\Sigma_g, (p_1, \dots, p_n)\big) \to X, \tag{1.7}$$

where p_i's are marked points on the genus g curve Σ_g. The maps must "realize" the class $\beta \in H_2^{\text{eff.}}$, i.e.,

$$f_*[\Sigma_g] = \beta \tag{1.8}$$

and must satisfy further *stability* conditions we do not list here (see [25] for details). There is a natural *evaluation map*

$$\text{ev} : \overline{\mathcal{M}_{g,n}}(X,\beta) \to X^n, \qquad f \mapsto \big(f(p_1), \dots, f(p_n)\big), \tag{1.9}$$

with components ev_i. There is also a *stabilization map*

$$\text{st} : \overline{\mathcal{M}_{g,n}}(X,\beta) \to \overline{\mathcal{M}_{g,n}} \tag{1.10}$$

defined by forgetting the holomorphic maps, where on the right-hand side we have the (Deligne-Mumford compactified) moduli space of curves of genus g with n markings, of dimension $\dim \overline{\mathcal{M}_{g,n}} = 3g - 3 + n$. The *expected dimension* of $\overline{\mathcal{M}_{g,n}}(X,\beta)$ is

$$d = \dim \overline{\mathcal{M}_{g,n}}(X,\beta) = (1-g)(\dim X - 3) - \langle \beta, c_1(X) \rangle + n, \tag{1.11}$$

The moduli space admits a *virtual fundamental class* $[\overline{\mathcal{M}_{g,n}}(X,\beta)] \in H_d\big(\overline{\mathcal{M}_{g,n}}(X,\beta)\big)$ allowing us to integrate. Given classes $\gamma_i \in H^*(X)$, the *Gromov-Witten invariants* of X then look like

$$I_{g,n,\beta}(\gamma_1, \dots, \gamma_n) = \int_{[\overline{\mathcal{M}_{g,n}}(X,\beta)]} \text{ev}^*(\gamma_1) \cup \dots \cup \text{ev}^*(\gamma_n). \tag{1.12}$$

Roughly speaking, these numbers count the number of holomorphic curves $f(\Sigma_g) \subset X$ of genus g, homology class β that are intersecting the cycles $\tilde{\gamma}_i \in H_*(X)$ (Poincaré dual to γ_i) such that $f(p_1) \in \tilde{\gamma}_i$ (by picking representatives for the cycles). In other words, the refined intersection theory counts the ways that cycles can touch "modulo" degree β holomorphic curves in X.

The genus zero Gromov-Witten invariants with three arguments define a quantum product structure on $H^*(X)$ as follows. Denote the *intersection pairing*

$$(a,b)_X := \int_X a \cup b, \qquad a, b \in H^*(X), \tag{1.13}$$

and fix a basis ϕ_i for $H^*(X)$ (which also fixes $g_{ij} := (\phi_i, \phi_j)_X$, $\phi^i = \sum g^{ij}\phi_j$ with $g^{ij} = (g^{-1})_{ij}$) and denote by ω the complexified Kähler class of X. Then the formula

$$a * b = \sum_i \sum_{\beta \in H_2^{\text{eff.}}} I_{0,3,\beta}(a, b, \phi_i) Q^\beta \phi^i, \tag{1.14}$$

where the Novikov variable Q^β is defined as $Q^\beta = e^{2\pi i \int_\beta \omega}$, defines the multiplication $*$ in the *small quantum cohomology ring*. The ring $H^*(X) \otimes \mathbb{C}[[Q_1, \dots, Q_{h_2(X)}]]$[7] is a

[7] Given a ring R, $R[t]$ denotes the polynomial ring in t with coefficients in R. Similarly, $R[[t]]$ denotes formal power series in t with coefficients in R and $R(t)$ denotes rational functions in t over R.

deformation of classical cohomology by the variables Q^β. It is easy to see that the product with $\beta = 0$ is the classical cup product on cohomology. The product is also associative, and satisfies $(a * b, c)_X = (a, b * c)_X$ and $\int_X a * b = (a, b)_X$, i.e., it is a Frobenius algebra.

One can also "enlarge" the deformation by $\dim H^*(X)$ new paramemeters called *insertions* or *deformations*. Set

$$t = \sum_i t_i \phi_i, \tag{1.15}$$

and classes $a, b \in H^*(X)$. Then the formula[8]

$$a \star b = \sum_i \sum_{n=0}^{\infty} \sum_{\beta \in H_2^{\text{eff.}}} \frac{1}{n!} I_{0,n+3,\beta}(a, b, \phi_i, t^n) Q^\beta \phi^i \tag{1.16}$$

defines the multiplication $\star$ in the *big quantum cohomology ring*. The ring $H^*(X) \otimes \mathbb{C}[[Q_1, \ldots, Q_{h_2(X)}, t_0, \ldots, t_{\dim H^*(X)-1}]]$[9] is a deformation of classical cohomology. Restricting to $t_i = 0$ yields the small quantum deformation (and product) defined above. The big quantum multiplication still satisfies $(a \star b, c)_X = (a, b \star c)_X$ and forms a Frobenius algebra. The structure of the big quantum ring is often described in terms of the so called *Gromov-Witten potential*, which is the formal sum

$$\Phi_{GW,0}(t) = \sum_{n=0}^{\infty} \sum_{\beta \in H_2^{\text{eff.}}} \frac{1}{n!} I_{0,n,\beta}(t^n) Q^\beta. \tag{1.17}$$

It can be thought of as the generating function for the Gromov-Witten invariants; furthermore we can show that (or equivalently define)

$$\phi_i \star \phi_j = \sum_k \frac{\partial^3 \Phi_{GW,0}}{\partial t_i \, \partial t_j \, \partial t_k}(t) \phi^k \tag{1.18}$$

More generally, we are interested in the *gravitational correlators*[10]

$$\alpha \langle \tau_{d_1} \gamma_1, \ldots, \tau_{d_n} \gamma_n \rangle_{g,n,\beta} = \int_{[\overline{\mathcal{M}_{g,n}}(X,\beta)]} \mathrm{st}^*(\alpha) \cup \bigcup_{i=1}^{n} \mathrm{ev}^*(\gamma_i) \cup c_1(\mathcal{L}_i)^{\cup d_i}, \tag{1.19}$$

where $\alpha \in H^*(\overline{\mathcal{M}_{g,n}})$ (chosen to be trivial for the following) and $\mathcal{L}_i$'s are the *universal cotangent line bundles* over $\overline{\mathcal{M}_{g,n}}(X, \beta)$, (loosely) defined by requiring the fiber over $f \in \overline{\mathcal{M}_{g,n}}(X, \beta)$ to be the cotangent space $T^*_{p_i} \Sigma_g$.

For $d_i = 0$ and trivial class α, we obtain the Gromov-Witten invariants defined above. One usually also studys the richer correlators with insertions t (1.15) of genus g by the formal sum

$$\langle\!\langle \tau_{d_1} \gamma_1, \ldots, \tau_{d_k} \gamma_k \rangle\!\rangle_g = \sum_{n=0}^{\infty} \sum_{\beta \in H_2^{\text{eff.}}} \frac{1}{n!} \langle \tau_{d_1} \gamma_1, \ldots, \tau_{d_k} \gamma_k, t^n \rangle_{g,n+k,\beta} Q^\beta \tag{1.20}$$

[8] Here, $I_{0,n,\beta}(t^n) = I_{0,n,\beta}(t, \ldots, t)$ with n-insertions.

[9] See footnote 7 for notation.

[10] The correlators deserve their name as they are identified with

$$\langle \tau_{d_1} \gamma_1, \ldots, \tau_{d_n} \gamma_n \rangle_{g,n,\beta} = \frac{\langle \mathcal{O}_{d_1,\gamma_1} \cdots \mathcal{O}_{d_n,\gamma_n} \rangle}{d_1! \cdots d_n!},$$

where $\mathcal{O}_{d,\gamma}$, $d = 0, 1, \ldots$, are operators obtained from the operator $\mathcal{O}_\gamma$ associated to $\gamma \in H^*(X)$ as in (1.5), via the so called *gravitational descent* [44].

where t^n denotes n insertions $t, \dots, t$.

1.2.2 Givental's cohomological J-function

The correlators and correlators with insertions t can be nicely "packaged" into the so-called Givental J-function [50] which we describe now. We consider the trivial $H^*(X)$-bundle over the space T of t-parameters with coordinates (1.15), and the *Givental connection* $\nabla^\hbar$ over it, acting as

$$\nabla^\hbar_{\frac{\partial}{\partial t_i}} = \hbar \frac{\partial}{\partial t_i} - \phi_i \star, \tag{1.21}$$

where $\hbar$ is a parameter[11]. Associativity of $\star$ is in fact equivalent to flatness of $\nabla^\hbar$, (any of) which is equivalent to $\frac{\partial^3 \Phi_{GW,0}}{\partial t_i \, \partial t_j \, \partial t_k}$ satisfying the Witten-Dijkgraaf-Verlinde-Verlinde equation [51, 52]. We then define the formal sections $S : H^*(X) \to H^*(X)$ by the endomorphism

$$S_{ij} = g_{ij} + \sum_{d=0}^{\infty} \hbar^{-d-1} \langle\!\langle \tau_d \phi_i, \phi_j \rangle\!\rangle_0, \tag{1.22}$$

where g_{ij} is defined by (1.13) in the basis $\phi_0, \dots, \phi_{\dim H^*(X)}$ and we conveniently abbreviate the infinite summation as

$$S_{ij} = g_{ij} + \left\langle\!\!\left\langle \frac{\phi_i}{\hbar - c}, \phi_j \right\rangle\!\!\right\rangle_0 = g_{ij} + \sum_{n=0}^{\infty} \sum_{\beta \in H_2^{\text{eff.}}} \frac{1}{n!} \left\langle \frac{\phi_i}{\hbar - c}, \phi_j, t^n \right\rangle_{0,n+2,\beta} Q^\beta, \tag{1.23}$$

by interpreting c^n as τ_n, representing an insertion of $c_1(\mathcal{L}_i)^n$ as in (1.19). Clearly, the sections are complicated functions of Q and t_i's. However, one can show that these are in fact $\nabla^\hbar$-flat sections, i.e., they satisfy

$$\hbar \frac{\partial S_{jk}}{\partial t_i} = \phi_i \star S_{jk}. \tag{1.24}$$

Givental's cohomological J-function is then defined as

$$J(t) = \sum_{ij} (S_{ij} \phi^j, \phi_0)_X \cdot \phi^i = \phi_0 + \sum_i \left\langle\!\!\left\langle \frac{\phi_i}{\hbar - c}, \phi_0 \right\rangle\!\!\right\rangle_0 \phi^i. \tag{1.25}$$

As with the endomorphisms S_{ij}, the dependence on Q and t_i's is complicated. However, the flatness equations (1.24) imply that certain relations must hold. In particular, let $D(\hbar \frac{\partial}{\partial t}, e^t, \hbar)$ represent a formal differential operator defined as a formal power series in the $\hbar \frac{\partial}{\partial t_i}$'s, e^{t_i}'s and $\hbar$, and let $D(\phi_i, Q_i, 0)$ the cohomology-valued formal power series obtained by setting $\hbar \frac{\partial}{\partial t_i} \to \phi_i$, $e^{t_i} \to Q_i$, replacing product with (small) quantum product and $\hbar = 0$ afterwards. Then the flatness equations imply [25] that

$$D(\hbar \tfrac{\partial}{\partial t}, e^t, \hbar) \cdot J_X = 0 \implies D(\phi_i, Q_i, 0) = 0 \text{ in small quantum cohomology.} \tag{1.26}$$

The operator D annihilating J_X is called a *quantum differential operator*.

[11] More precisely, a generator of $H^2_{\mathbb{C}^*}(\mathrm{pt})$, implying we have "refined" our treatment to $\mathbb{C}^*$-*equivariant* Gromov-Witten invariants. See [50] for details.

1.2.3 Quantum K-theory

Quantum K-theory and the study of K-theoretic Gromov-Witten invariants was first discussed by Givental [53] and Lee [54]. In this subsection we give a brief non-technical description of relevant notions following also [55, 56].

As discussed in the previous subsection, we may refine the study of intersection theory on a Kähler manifold X by looking at the intersection theory on $\mathcal{M}_{g,n}(X,\beta)$ instead. A similar refinement can be considered for K-theory, by viewing classical K-theory as the study of sheaves, or equivalently for smooth X, vector bundles. From this point of view, quantum K-theory is the study of complex vector bundles over the spaces of holomorphic curves into X. The various ingredients are defined in parallel to (quantum) cohomology. The (classical) product is now given by the tensor product of vector bundles. The virtual fundamental class is replaced by the *virtual structure sheaf* $[\mathcal{O}_{\overline{\mathcal{M}_{g,n}}(X,\beta)}]^{\text{vir}} \in K\big(\overline{\mathcal{M}_{g,n}}(X,\beta)\big)$, and the intersection pairing on X is replaced by the holomorphic Euler characteristic

$$g(a,b) \coloneqq \chi(X;a\otimes b) = \sum_k (-1)^k \dim H^k\big(X;\Gamma(a\otimes b)\big) = \int_X \operatorname{td}(X)\operatorname{ch}(a)\operatorname{ch}(b) \quad (1.27)$$

where on the right-hand side we have used the Hirzebruch-Riemann-Roch theorem for the sheaves a, b. Given classes $\gamma_i \in K(X)$, the *K-theoretic Gromov-Witten invariants* then look like

$$I_{g,n,\beta}(\gamma_1,\dots,\gamma_n) = \chi\big(\overline{\mathcal{M}_{g,n}}(X,\beta);\mathcal{O}_{\overline{\mathcal{M}_{g,n}}(X,\beta)}\otimes \operatorname{ev}_1^*(\gamma_1)\otimes\cdots\otimes\operatorname{ev}_n^*(\gamma_n)\big). \quad (1.28)$$

More generally, given the line bundles $\mathcal{L}_i \in K\big(\overline{\mathcal{M}_{g,n}}(X,\beta)\big)$ as in (1.19) and a class $\alpha \in K(\overline{\mathcal{M}_{g,n}})$ (chosen trivial henceforth) the *ordinary* K-theoretic *graviational correlators* are

$$\alpha\langle\tau_{d_1}\gamma_1,\dots,\tau_{d_n}\gamma_n\rangle_{g,n,\beta} = \chi\Big(\overline{\mathcal{M}_{g,n}}(X,\beta);\mathcal{O}_{\overline{\mathcal{M}_{g,n}}(X,\beta)}\otimes\operatorname{st}^*(\alpha)\otimes\bigotimes_{i=1}^n \mathcal{L}_i^{\otimes d_i}\otimes\operatorname{ev}_i^*(\gamma_i)\Big) \quad (1.29)$$

The quantum K-theory ring may then be deformed by the genus zero invariants, analogously to cohomology. In particular, we consider the *K-theoretic Gromov-Witten potential*

$$\mathcal{F}(Q,t) = \sum_{\beta\in H_2^{\text{eff.}}}\sum_{n=0}^{\infty}\frac{1}{n!}\langle t^n\rangle_{0,n,\beta}Q^\beta, \quad (1.30)$$

where Q^β is defined as in the previous subsection, with $H_2^{\text{eff.}} \subset H_2(X,\mathbb{Z})$ denoting the subgroup of *effective* curve classes, and the *deformation* t now denotes a general point in $K(X)$, upon picking a basis $\Phi_0,\dots,\Phi_{\dim K(X)-1}$:

$$t = \sum_i t_i\Phi_i. \quad (1.31)$$

The GW potential is valued in $\mathbb{C}[[Q,t]] = \mathbb{C}[[Q_1,\dots,Q_{h_2(X)},t_0,\dots,t_{\dim K(X)-1}]]$[12], the extension of the Novikov ring $\mathbb{C}[[Q]]$ by the deformations t. The (big) deformed ring

[12]See footnote 7 for notation.

structure can be defined using the *non-constant pairing* $G : K(X) \times K(X) \to \mathbb{C}[[Q,t]]$, acting on basis elements as

$$G_{ij} \coloneqq G(\Phi_i, \Phi_j) = \frac{\partial^3}{\partial t_0\, \partial t_i\, \partial t_j}\mathcal{F} = g_{ij} + \partial_i \partial_j \mathcal{F}. \tag{1.32}$$

One can check that

$$G_{ij} = g_{ij} + \sum_{\beta \in H_2^{\text{eff.}}} \sum_{n=0}^{\infty} \frac{Q^\beta}{n!} \langle \Phi_i, \Phi_j, t^n \rangle_{0,n+2,\beta}, \qquad G_{ij}|_{t=0,Q=0} = g_{ij}, \tag{1.33}$$

where $g_{ij} = \chi(X; \Phi_i \otimes \Phi_j)$. The *big quantum product* $\star$ on K-theory of X is then defined implicitly in the basis Φ_i by the non-constant pairing G as

$$G(\Phi_i \star \Phi_j, \Phi_k) = \frac{\partial^3}{\partial t_i\, \partial t_j\, \partial t_k}\mathcal{F} = \sum_{\beta \in H_2^{\text{eff.}}} \sum_{n=0}^{\infty} \frac{1}{n!} \langle \Phi_i, \Phi_j, \Phi_k, t^n \rangle_{0,n+3,\beta} Q^\beta. \tag{1.34}$$

$K(X) \otimes \mathbb{C}[[Q,t]]$ is a deformation of the classical K-theory ring, and reduces to it upon setting $Q = 0$ (the dependence on t drops in when $Q = 0$). The element $\Phi_0 = 1 = [\mathcal{O}_X]$ remains the identity in the quantum product. The product is associative and commutative by virtue of (1.34) and the K-theoretic WDVV equation [53, 54] and forms a Frobenious algebra, i.e., $G(\Phi_i \star \Phi_j, \Phi_k) = G(\Phi_i, \Phi_j \star \Phi_k)$. The restriction to $t = 0$ yields the *small quantum K-ring* $K(X) \otimes \mathbb{C}[[Q]]$ of X with product $\star|_{t=0} = *$.

1.2.4 K-theoretic J-function

Just as in the cohomological case, the K-theoretic correlators and quantum products are equipped with interesting structures. Let us consider the *symplectic loop space*[13] $\mathcal{K}$ of X [55, 57]

$$\mathcal{K} \coloneqq K(X) \otimes \mathbb{C}[[Q]] \otimes \mathbb{C}(q, q^{-1}), \tag{1.35}$$

where q is an indeterminate[14]. The symplectic structure is determined by the symplectic form

$$\Omega(f,g) \coloneqq \frac{1}{2\pi i} \oint \chi\big(X; f(q) \otimes g(q^{-1})\big) \frac{dq}{q}, \qquad f, g \in \mathcal{K}. \tag{1.36}$$

The loop space $\mathcal{K}$ admits a Lagrangian polarization

$$\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-, \tag{1.37}$$

where

$$\begin{aligned} \mathcal{K}_+ &= \mathbb{C}[q, q^{-1}] \otimes K(X) \otimes \mathbb{C}[[Q]], \\ \mathcal{K}_- &= \{ r \in \mathbb{C}(q, q^{-1}) | r(0) \neq \infty,\ r(\infty) = 0 \} \otimes K(X) \otimes \mathbb{C}[[Q]], \end{aligned} \tag{1.38}$$

i.e., $\mathcal{K}_+$ contains only Laurent polynomials in q and $\mathcal{K}_-$ rational functions in q where the denominators are regular at 0 and have strictly higher degree than their numerators.

[13] See footnote 7 for notation.

[14] In fact, q can be thought of as a $K_{\mathbb{C}^*}(\text{pt})$-equivariant parameter analogously to $\hbar$ in quantum cohomology. The 'Chern map' relates $q = e^{-\hbar}$.

We can now consider the K-theoretic version of the Givental connection on the trivial $K(X)$-bundle over $\mathcal{K}[[t]]$[15]

$$\nabla^q_{\frac{\partial}{\partial t_i}} = (1-q)\frac{\partial}{\partial t_i} + \Phi_i \star, \qquad i = 0, \ldots, \dim K(X) - 1. \tag{1.39}$$

Once again, this is a *flat* connection [53]. We can also write an explicit fundamental solution matrix $S \in \operatorname{End} K(X) \otimes \mathbb{C}(q, q^{-1}) \otimes \mathbb{C}[[Q, t]]$ for flat sections

$$S_{ij} = g_{ij} + \sum_{\beta \in H_2^{\text{eff.}}} \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \Phi_i, t^n, \frac{\Phi_j}{1 - q\mathcal{L}} \right\rangle_{0,n+2,\beta} Q^\beta, \tag{1.40}$$

where again $\Phi_j/(1-q\mathcal{L}) = \Phi_j \sum_{k\geq 0}(q\mathcal{L})^k$ with $q^k \mathcal{L}^k$ is interpreted as $\tau_k \Phi_j$, i.e., an insertion of $\mathcal{L}_{n+2}^{\otimes k}$ in the holomorphic Euler number. The endomorphism $S(Q,q,t)$, along with another endomorphism $T(Q,q,t)$, can be introduced in an equivalent way by

$$G(\Phi_i, S\Phi_j) = \overline{S}_{ij}, \qquad g(T\Phi_i, \Phi_j) = S_{ij}, \tag{1.41}$$

where $\overline{f} \in \mathcal{K}$ denotes $f(q^{-1}) \in \mathcal{K}$. With these definitions S, T satisfy the differential equations

$$\begin{aligned} &(1-q)\partial_i S + \Phi_i \star S = 0, \\ &(1-q)\partial_i T = T(\Phi_i \star), \end{aligned} \tag{1.42}$$

and furthermore also satisfy [56]

$$S = T^{-1}, \quad S|_{q\to\infty} = T|_{q\to\infty} = \mathrm{id}, \quad S|_{Q=0,t=0} = T|_{Q=0,t=0} = \mathrm{id}, \tag{1.43}$$

showing that T is also a fundamental solution matrix. The endomorphisms fulfill the *compatibility* equations with the pairings g and G:

$$g(\overline{T}\Phi_i, T\Phi_j) = G(\Phi_i, \Phi_j), \qquad G(\overline{S}\Phi_i, S\Phi_j) = g(\Phi_i, \Phi_j). \tag{1.44}$$

The endomorphisms S, T can be used to equip $\mathcal{K}$ with a q-difference module structure[16], generated by the q-shift operator q^θ, $\theta = Q\frac{\partial}{\partial Q}$. The operator preserves the space $T \cdot \mathcal{K}_+$ [55], and we can introduce the q-difference equations[17]

$$Sq^{\theta_r} = A_r q^{\theta_r} S, \qquad q^{\theta_r} T = T A_r q^{\theta_r}, \quad r = 1, \ldots, h_2, \tag{1.45}$$

where the endomorphism $A_r(Q,q,t) \in \operatorname{End} K(X) \otimes \mathbb{C}[[Q,t]]$ is (for now, tautologically) defined by

$$A_r = T^{-1}(Q_r)T(qQ_r) = S(Q_r)S^{-1}(qQ_r). \tag{1.46}$$

[15] Note a slight convention change compared to cohomological versions, leading to some relative (exponent) signs. We follow the notation of [56].

[16] This is a new development compared to quantum cohomology, where the extended rings form a differential quantum D-module structure (see e.g., [58]).

[17] We adopt the correction by 'logarithms of K-theoretic Chern roots' $P_i^{\log Q_i/\log q}$ (see [56]), which in chapter 2 corresponds to terms of the form $Q_a^{\epsilon_a}$ 'correcting' the I-function.

The appearance of a difference module structure in quantum K-theory was interpreted via deck transformations on the loop space LX in [59, p. V].
The Givental J-function for *ordinary* quantum K-theory on X is then defined as

$$\begin{aligned} J(Q,q,t) &= (1-q)T\Phi_0 \\ &= (1-q)\Phi_0 + t(q) + \sum_{\beta\in H_2^{\text{eff.}}}\sum_i\sum_{n=0}^{\infty}\frac{Q^\beta}{n!}\Phi^i\Big\langle t(q)^n, \frac{\Phi_i}{1-q\mathcal{L}}\Big\rangle_{0,n+1,\beta}, \end{aligned} \tag{1.47}$$

where $t(q)\in K(X)[q,q^{-1}]$ is called the *input* and in the last line we have used an implication of the string equation [54]. Note that previously $t\in K(X)$; the extension to Laurent polynomials in q implies the input $t(q)=\sum_a t_a q^a$ can now come with line bundle insertions $\mathcal{L}^{\otimes a}$. The above expansion is a decomposition into $\mathcal{K}_+\oplus\mathcal{K}_-$, where $J=(1-q)+t(q) \mod \mathcal{K}_-$. In other words, the J-function (minus the *dilaton shift* $1-q$) is the graph of a function

$$J_{\text{corr.}}:\mathcal{K}_+\to\mathcal{K}_-,\qquad t\mapsto \sum_{\beta\in H_2^{\text{eff.}}}\sum_i\sum_{n=0}^{\infty}\frac{Q^\beta}{n!}\Phi^i\Big\langle t^n, \frac{\Phi_i}{1-q\mathcal{L}}\Big\rangle_{0,n+1,\beta}. \tag{1.48}$$

Refinements

The study of K-theoretic Gromov-Witten invariant can be refined by considering the action of the symmetric group S_n (or subgroups thereof) on $\overline{\mathcal{M}_{g,n}}(X,\beta)$ by the automorphisms that permute the marked points. The equivariant n-insertion correlators are organized into representations labeled by Young diagrams μ of the symmetric group (see [59, p. I] for definitions)

$$\langle\tau_1(q),\dots,\tau_m(q);t(q),\dots,t(q)\rangle^{S_n}_{0,m+n,\beta} = \sum_{\mu\in\text{Irreps}(S_n)}\chi^{S_n;\mu}_{\overline{\mathcal{M}_{g,n}}(X,\beta)}\big(\tau_1(q),\dots,\tau_m(q);t(q)\big)\cdot\mu. \tag{1.49}$$

The *ordinary*, non-equivariant correlators are recovered by

$$\langle\tau_1(q),\dots,\tau_m(q);t(q),\dots,t(q)\rangle_{0,k+n,\beta} = \sum_{\mu\in\text{Irreps}(S_n)}\chi^{S_n;\mu}_{\overline{\mathcal{M}_{g,n}}(X,\beta)}\big(\tau_1(q),\dots,\tau_m(q);t(q)\big)\cdot\dim\mu. \tag{1.50}$$

The restriction to the symmetric representations, labeled by the trivial partition $\mu=(n)$, leads to *symmetric correlators* $\chi^{S_n;\text{sym}}_{\overline{\mathcal{M}_{g,n}}(X,\beta)}\big(\tau_1(q),\dots,\tau_m(q);t(q)\big)$.

The refined correlators can be collected into refined J-functions, just as in the ordinary case (1.47). In particular we have the *permutation-equivariant J-function*

$$J^{\text{eq}}(Q,q,t) = (1-q)\Phi_0\cdot\sigma_0 + t(q)\cdot\sigma_1 + \sum_{\beta\in H_2^{\text{eff.}}}\sum_{i,\mu}\sum_{n=0}^{\infty}Q^\beta\Phi^i\chi^{S_n;\mu}_{\overline{\mathcal{M}_{g,n}}(X,\beta)}\Big(\frac{\Phi_i}{1-q\mathcal{L}};t(q)\Big)\cdot\sigma_\mu, \tag{1.51}$$

where $\sigma_\mu\in\Lambda$ denotes a Schur polynomial in formal variables $\{x_1,\dots,\}$, labeling the symmetric representations, where $\Lambda=\mathbb{C}[[x_1,\dots]]^{\text{Sym}}$ is an extension of the Novikov ring. The equivariant input $t(q)\cdot\sigma_1$ is now valued in $\mathcal{K}_+\otimes\Lambda$.
The *permutation-symmetric J-function* is the restriction

$$J^{\text{sym}}(Q,q,t) = (1-q)\Phi_0 + t(q) + \sum_{\beta\in H_2^{\text{eff.}}}\sum_i\sum_{n=0}^{\infty}Q^\beta\Phi^i\chi^{S_n;\text{sym}}_{\overline{\mathcal{M}_{g,n}}(X,\beta)}\Big(\frac{\Phi_i}{1-q\mathcal{L}};t(q)\Big), \tag{1.52}$$

with $t(q) \in \mathcal{K}_+$. There can also be *mixed* versions with two kinds of input,

$$J^{\text{mix}}(Q,q,t) = (1-q)\Phi_0 + t_{\text{sym}}(q) + t_{\text{ord}}(q) \\ + \sum_{\beta \in H_2^{\text{eff.}}} \sum_i \sum_{n=0}^{\infty} Q^\beta \Phi^i \chi_{\mathcal{M}_{g,n}(X,\beta)}^{S_n;\text{sym}} \Big(\frac{\Phi_i}{1-q\mathcal{L}}, t_{\text{ord}}(q); t_{\text{sym}}(q) \Big), \tag{1.53}$$

As we will see, the correspondence between 3D gauge theories with target space X and quantum K-theory asserts that the path integral determines the permutation-symmetric quantum K-theoretic J-function for X. At zero input, all the J-functions coincide and are referred to as the "unperturbed" J-function

$$J(Q,q,0) = J^{\text{eq}}(Q,q,0) = J^{\text{sym}}(Q,q,0) = (1-q) + \sum_{\beta \in H_2^{\text{eff.}}} \sum_i \frac{Q^\beta}{n!} \Phi^i \Big\langle \frac{\Phi_i}{1-q\mathcal{L}} \Big\rangle_{0,1,\beta}. \tag{1.54}$$

1.3 Outline and outlook

1.3.1 Summary of chapter 2

We begin the chapter by the computation of the partition function of a non-Abelian gauge theory defined in the UV, the three-dimensional version [60–64] of Witten's gauged linear sigma model [65, 66] (GLSM). As described in [67], there's is an uplift of the correspondence between two-dimensional GLSMs and quantum cohomology (a deformation of classical cohomology) [41, 43, 44]: the 3D GLSM/quantum K-theory correspondence for GLSMs on $\Sigma \times S^1$, where $\Sigma = D^2$ or $\Sigma = S^2$.

The theory is a non-trivial extension of results in [67, 68] to gauge theories with a non-Abelian gauge group. The GLSM in question has a geometric phase (i.e., a region in its parameter space) where the low-energy description of the GLSM reduces to a non-linear sigma model with target space $\text{Gr}(M,N)$, the complex Grassmannian. The result of the path-integral computation is of the schematic form

$$\mathcal{Z}_{D^2 \times_q S^1} = \oint d\epsilon\, f_{\text{Gr}(M,N)}(\epsilon,q) I_{\text{Gr}(M,N)}(Q,\epsilon,q), \tag{1.55}$$

where the function I, called the I-function, captures K-theoretic invariants for $\text{Gr}(M,N)$. The physical incarnations of K-theory generators are the BPS Wilson loops wrapping the S^1 factor. In the remainder of the chapter we study the algebra of Wilson loops by studying the properties of this function. In accordance to the 3D GLSM/quantum K-theory correspondence we identify versions of these algebras with quantum rings. We find agreement with the rings computed by mathematicians by different methods, whenever a comparison is possible. We end chapter 2 by discussing perturbations of the theory that capture the IR parameter space, and mathematically correspond to further extensions into the so called big quantum K-ring for $\text{Gr}(M,N)$.

1.3.2 Summary of chapter 3

We start the chapter with a thorough discussion of Noether's theorem and generalize the classical statement to the cases with boundary. We then recall the structure and features of supercurrent multiplets in three-dimensional bulk theories, following [69]. Then we discuss at length how these structures must be modified in the presence of two-dimensional boundaries (focusing on B-type boundary conditions preserving an $\mathcal{N} = (0, 2)$ subalgebra), and formulate an Ansatz for their new definitions, check those for consistency and formulate so called *integrated* supercurrent multiplets. The integrated multiplets retain certain nice features that are present in two-dimensional supercurrent multiplets but are "lost" for general cases with boundary. We end the chapter with a detailed application on a simple example: a 3D chiral field with a superpotential. We compute its supercurrent multiplets, discuss supersymmetric boundary conditions and the three-dimensional analog of the Warner problem [70] and its relation to 3D matrix factorizations.

1.3.3 Related topics and outlook

Let us list some interesting topics that are relevant to this work and some open directions, in no particular order.

- Large part of the work in chapter 2 is deeply related to the so called gauge/Bethe correspondence between supersymmetric theories and integrable systems [71–73]. The connection of quantum K-theory to integrable systems and quantum groups is also relevant in [74–77], and in particular for Grassmannians in [78, 79].

- The GLSM/quantum ring correspondence from two-dimensional models on Σ and three-dimensional models on $\Sigma \times S^1$ is expected to also have another uplift to four-dimensional models on $\Sigma \times T^2$. The associated ring is expected to be a quantum deformation of elliptic cohomology [80] of the target space. The overarching structure of the quantum rings is governed by so called *formal group laws* of cobordism invariants [81].

- The results from chapter 3 can also be generalized to other dimensions and other supersymmetries that allow a superspace description. In addition it would be interesting to generalize the results into curved backgrounds, where a supergravity-analog of the statements is expected. Another interesting venture is the explicit physical description of additional non-trivial examples of 3D matrix factorizations.

Chapter 2

Quantum invariants of *Gr(M,N)* from supersymmetric theories

In this chapter we will describe the computation of quantum K-theory invariants for complex Grassmannians $\mathrm{Gr}(M,N)$ from supersymmetric gauge theories. The computation is an implication of the GLSM/quantum K-theory correspondence [67, 68], briefly mentioned in the introduction 1. The theory in question is a three-dimensional version of the non-Abelian gauged linear sigma model [66]. In the geometric phase [65], its low energy limit is a non-linear sigma model with $\mathrm{Gr}(M,N)$ target space. The contents of the chapter closely follow the results of the papers [1] and [2] by Prof. Dr. Hans Jockers, Prof. Dr. Peter Mayr, Dr. Urmi Ninad and the author.
Notation note: In this chapter, Greek letters $\mu, \nu, \ldots$ denote Young diagrams.

2.1 The K-theoretic *I*-function from gauge theory

2.1.1 Partition function on $D^2 \times_q S^1$

In this subsection we describe the computation of the K-theoretic I-function for the complex Grassmannian $X = \mathrm{Gr}(M,N)$ from the partition function of an $\mathcal{N}=2$ $U(M)$ gauge theory, defined in the UV with N fundamental chiral multiplets on an $D^2 \times_q S^1$ background. The parameter q here is a geometric twisting parameter, in the sense that following a loop around S^1, a point $z \in D^2$ is sent to qz (see [10] for details). Both vector and chiral multiplets are assigned Neumann boundary conditions $(\mathcal{N}, \mathrm{N})$ as in [22]. The computation closely follows the results of [2], to produce the result stated also in [1] (see also [82] for a similar result).

Collecting supersymmetric localization results

The partition function $\mathcal{Z}_{D^2\times_q S^1}$ of the gauge theory has been computed via supersymmetric localization on the Coulomb branch [10, 22, 38]. We follow mostly the explicit last reference. Given that the spacetime background $D^2 \times_q S^1$ has a T^2-boundary, we must first choose appropriate boundary conditions for all fields, as well as for Chern-Simons terms, in order to preserve (some) supersymmetry. This can be done consistently by possibly adding boundary terms to cancel supersymmetry violating terms [22] (see also [3] for a detailed, flat-space discussion). Then by deforming the action by a supersymmetrically exact term, the original infinite-dimensional path integral reduces to a finite-dimensional integral over Coulomb branch parameters $\mathrm{Lie}(G) = \mathfrak{u}(M)$, implemented by Wilson loops. The form of the result is

$$\mathcal{Z}_{D^2\times_q S^1} = \frac{1}{M!}\int\prod_{a=1}^{M}\frac{dz_a}{2\pi i z_a}e^{-S_{\text{class.}}}\mathcal{Z}_{\text{1-loop}}. \tag{2.1}$$

where $z_a = e^{\sigma_a}$ for $a = 1,\dots,M$ are Wilson loops associated each of the Cartan generators of $U(1)^M \subset U(M)$. The classical contribution is given by Chern-Simons and Fayet-Iliopoulos contributions

$$\mathcal{Z}_{\text{class.}} = \mathcal{Z}_{\text{CS}}\cdot\mathcal{Z}_{\text{FI}}, \tag{2.2}$$

Here, the contribution from Chern-Simons action is[1]

$$Z_{\text{CS}} = e^{\frac{\kappa_S}{2\log q}\operatorname{tr}_{SU(M)}(\sigma^2)+\frac{\kappa_A}{2\log q}\operatorname{tr}_{U(1)}(\sigma^2)-\kappa_R\operatorname{tr}_R(\sigma)}, \tag{2.3}$$

where κ_S measures the bare $SU(M)$ gauge/gauge level, κ_A measures the bare diagonal $U(1)$ gauge/gauge level, while κ_R captures the bare diagonal $U(1)$-gauge/$U(1)_R$-flavor level. In principle one can add further gauge/flavor or background flavor/flavor levels in the theory, which will affect the final result only by overall factors. Here the variable

$$q = e^{-\beta\hbar}, \tag{2.4}$$

is the $U(1)$ weight of rotation in $U(1)_R\times U(1)_{\text{spin}}$ generated by a rotation around the S^1 factor in $D^2\times_q S^1$, where β is tha radius of the S^1 factor. We set $\beta = 1$ for most of the following computations. The bare levels are 'corrected' by Chern-Simons-like terms coming from integrating out charged fermions [61], leading to different effective levels captured by $(\hat\kappa_S,\hat\kappa_A,\hat\kappa_R)$, to be determined later in (2.24). Invariance under large gauge transformations (determined in 3D by $\pi_3(G)$) and three-dimensional parity anomaly cancelation [83–85] impose (half-)integrality conditions on the effective levels[2]

$$\hat\kappa_S\in\mathbb{Z},\quad \frac{\hat\kappa_A-\hat\kappa_S}{M}\in\mathbb{Z},\quad \hat\kappa_R\in\mathbb{Z}/2. \tag{2.5}$$

The Fayet-Iliopoulos contribution to the finite integral is

$$\mathcal{Z}_{\text{FI}} = e^{2\pi\zeta\operatorname{tr}(\sigma)} = \tilde{Q}^{-\frac{\operatorname{tr}(\sigma)}{\log q}},\quad \tilde{Q} = q^{-2\pi\zeta}, \tag{2.6}$$

where ζ is the real Fayet-Iliopoulos constant.
The 1-loop corrections contain the 1-loop determinants for the field theory content, namely 3D vector and chiral multiplets, as well as possible 2D ($\mathcal{N} = (0,2)$) chiral and Fermi multiplets

$$\mathcal{Z}_{\text{1-loop}} = \mathcal{Z}_{\text{vect.}}\cdot\mathcal{Z}_{\text{chir.}}\cdot\mathcal{Z}_{\text{2D matter}}. \tag{2.7}$$

[1] For a matrix $\sigma = \mathrm{diag}(\sigma_1,\dots,\sigma_M)$ the trace symbols are defined as $\operatorname{tr}_{U(M)}(\sigma^2) = \sum_a\sigma_a^2$, $\operatorname{tr}_{U(1)}(\sigma^2) = \frac{1}{M}(\sum_a\sigma_a)^2$, $\operatorname{tr}_{SU(M)}(\sigma^2) = \operatorname{tr}_{U(M)}(\sigma^2)-\operatorname{tr}_{U(1)}(\sigma^2)$ and $\operatorname{tr}_R(\sigma) = \operatorname{tr}(\sigma) = \sum_a\sigma_a$.

[2] Note that when Neumann boundary conditions $\mathcal{N}$ are used for the gauge fields on the T^2-boundary, there is a non-trivial boundary anomaly captured by these effective Chern-Simons levels. Strictly speaking, the computation of this subsection is valid only when the bare CS levels are chosen such that the effective ones are zero, otherwise the theory is 'sick'. We will nevertheless keep an explicit dependence on the effective levels, to produce an I-function depending on three (half-)integer parameters. In the next subsection 2.1.2 we will again extract the I-function using Dirichlet boundary conditions $\mathcal{D}$ for the gauge fields, allowing us — from a physical perspective — to keep the effective levels to non-trivial values.

In this work we do not consider contributions from boundary matter, and the corresponding 1-loop factor is set to 1. The three-dimensional contributions have been computed carefully in [22]:

$$
\begin{aligned}
\mathcal{Z}_{\text{vect. }\mathcal{N}} &= \prod_{\text{roots }\alpha\neq 0} e^{\frac{\alpha(\sigma)^2}{\log q}} (e^{-\alpha(\sigma)}; q)_\infty, \\
\mathcal{Z}_{1\times\text{chir. N}} &= \prod_{\text{flavor }\ell} \prod_{\text{weights }\rho} e^{\mathcal{E}\left(\rho(\sigma) - r\frac{\log q}{2} + c_\ell \log y_\ell\right)} \frac{1}{(e^{\rho(\sigma) - r\frac{\log q}{2} + c_\ell \log y_\ell}; q)_\infty}, \\
\mathcal{Z}_{1\times\text{chir. D}} &= \prod_{\text{flavor }\ell} \prod_{\text{weights }\rho} e^{-\mathcal{E}\left(-\rho(\sigma) - (2-r)\frac{\log q}{2} - c_\ell \log y_\ell\right)} (e^{-\rho(\sigma) - (2-r)\frac{\log q}{2} - c_\ell \log y_\ell}; q)_\infty.
\end{aligned}
\tag{2.8}
$$

where

$$
\mathcal{E}(z) = -\frac{\log q}{24} + \frac{1}{4} z - \frac{1}{4 \log q} z^2. \tag{2.9}
$$

Here, r is the R-charge of the chiral fields (and $q^{1/2}$ the corresponding fugacity), while c_ℓ and y_ℓ are the flavor charges and fugacities respectively for any present (Abelian) flavor symmetries. See appendix A for our definitions of basic functions. For a $U(M)$ gauge theory, the roots α are indexed all pairs $a \neq b$ with $a, b \in 1, \ldots, M$ so that

$$
\alpha_{ab}(\sigma) = \sigma_a - \sigma_b =: \sigma_{ab}, \tag{2.10}
$$

while the weights ρ of the fundamental representation are simply labeled by a's with

$$
\rho_a(\sigma) = \sigma_a. \tag{2.11}
$$

We consider N chiral multiplets with Neumann boundary conditions, R-charge zero, and charges c_ℓ, $\ell = 1, \ldots, N$ for each chiral. We find that 1-loop contributions for our $U(M)$ gauge theory are

$$
\begin{aligned}
Z_{\text{vect.}} &= \prod_{a\neq b}^{M} e^{\frac{\sigma_{ab}^2}{4\log q}} (e^{\sigma_{ab}}; q)_\infty \ , \\
Z_{\text{chir}} &= \prod_{\ell=1}^{N} \prod_{a=1}^{M} e^{-\frac{\log q}{24} + \frac{\sigma_a}{4} - \frac{\sigma_a^2}{4\log q}} \frac{1}{(y_\ell^{c_\ell} e^{\sigma_a}; q)_\infty} \ .
\end{aligned}
\tag{2.12}
$$

where in the last line we have omitted irrelevant constant factors depending on the flavor fugacities[3]. For the remainder of the work we set the flavor charges of all chiral fields to zero, so that $y_\ell = 0$ for all ℓ (see [2] for further discussion).

Summing the poles

For large ζ and Q, we have $|q| < 1$ and the integral picks up a pole from the chiral contributions coming from the q-Pochhammer symbol in the denominator. Under the variable transformation defined implicitly by

$$
\tilde{d}_a := d_a - \epsilon_a = -\frac{\sigma_a}{\log q}, \tag{2.13}
$$

[3]We have also disregarded mixed terms $\sim \frac{\sigma_a \log y_\ell}{2 \log q}$ in the exponent, as they correspond shifts in mixed gauge/flavor Chern-Simons terms, which we do not consider.

with $\epsilon_a \in \mathbb{C}$ and $d_a \in \mathbb{Z}_{\geq 0}$, there is a pole at $\epsilon_a = 0$ for each $d_a \geq 0$. Collecting all the factors, the partition function (2.1) becomes the following sum over poles

$$Z_{D^2\times_q S^1} = \frac{(\log q)^M}{M!} q^{-\frac{MN}{24}} \oint \prod_{a=1}^{M} \frac{d\epsilon_a}{2\pi i} \sum_{\vec{d}\in\mathbb{Z}_{\geq 0}} \tilde{Q}^{\sum_{a=1}^M \tilde{d}_a} q^{\overline{CS}(\tilde{d})} \frac{\prod_{a\neq b}^M (q^{\tilde{d}_{ab}};q)_\infty}{\prod_{a=1}^M (q^{-\tilde{d}_a};q)_\infty^N}\,, \tag{2.14}$$

where the explicit q-exponent terms are collected into

$$\overline{CS}(\tilde{d}) = \frac{\kappa_S + M - \frac{N}{2}}{2} \operatorname{tr}_{SU(M)}(\tilde{d}^2) + \frac{\kappa_A - \frac{N}{2}}{2} \operatorname{tr}_{U(1)}(\tilde{d}^2) + (\kappa_R - \tfrac{N}{4}) \operatorname{tr}_R(\tilde{d}), \tag{2.15}$$

which contains bare Chern-Simons terms, as well as shifts from the 1-loop determinants (2.12), after using the identity

$$\sum_{a\neq b}^{M} \tilde{d}_{ab}^2 = 2M \operatorname{tr}_{SU(M)}(\tilde{d}^2). \tag{2.16}$$

Again, $d_{ab} = d_a - d_b$ and similarly for other doubly indexed variables. The infinite q-Pochhammer in the denominator may be rewritten using (A.3), (A.7c) and $d_a \geq 0$:

$$\begin{aligned}\prod_{a=1}^{M} \frac{1}{(q^{-\tilde{d}_a};q)_\infty^N} &= \frac{q^{-\frac{N}{2}\sum_{a=1}^M \epsilon_a(\epsilon_a-1)}}{\prod_{a=1}^M (q^{\epsilon_a};q)_\infty^N} \cdot \frac{(-1)^{N\sum_{a=1}^M d_a} q^{\frac{N}{2}\sum_{a=1}^M \tilde{d}_a(\tilde{d}_a+1)}}{\prod_{a=1}^M \prod_{r=1}^{d_a} (1-q^{r-\epsilon_a})^N} \\ &= \frac{q^{-\frac{N}{2}\operatorname{tr}_{SU(M)}(\epsilon^2) - \frac{N}{2}\operatorname{tr}_{U(1)}(\epsilon^2) + \frac{N}{2}\operatorname{tr}_R(\epsilon)}}{\prod_{a=1}^M (q^{\epsilon_a};q)_\infty^N} \times \\ &\quad \times \frac{(-1)^{N\sum_{a=1}^M d_a} q^{\frac{N}{2}\operatorname{tr}_{SU(M)}(\tilde{d}^2) + \frac{N}{2}\operatorname{tr}_{U(1)}(\tilde{d}^2) + \frac{N}{2}\operatorname{tr}_R(\tilde{d})}}{\prod_{a=1}^M \prod_{r=1}^{d_a} (1-q^{r-\epsilon_a})^N}\end{aligned} \tag{2.17}$$

and similarly using (A.7b) the infinite q-Pochhammer in the numerator can be expressed

$$\begin{aligned}\prod_{a<b}^{M} (q^{\tilde{d}_{ab}};q)_\infty (q^{-\tilde{d}_{ab}};q)_\infty &= \prod_{a<b}^{M} \theta_q(q^{-\epsilon_{ab}}) q^{\frac{1}{2}\epsilon_{ab}(\epsilon_{ab}+1)} \cdot \prod_{a<b}^{M} (-1)^{d_{ab}} q^{-\frac{1}{2}\tilde{d}_{ab}^2} (q^{\tilde{d}_{ab}/2} - q^{-\tilde{d}_{ab}/2}) \\ &= q^{M\operatorname{tr}_{SU(M)}(\epsilon^2)} \prod_{a<b}^{M} \frac{(q^{\epsilon_{ab}};q)_\infty (q^{-\epsilon_{ab}};q)_\infty}{q^{\frac{1}{2}\epsilon_{ab}^2}(q^{-\frac{1}{2}\epsilon_{ab}} - q^{\frac{1}{2}\epsilon_{ab}})} \times \\ &\quad \times q^{-M\operatorname{tr}_{SU(M)}(\tilde{d}^2)} (-1)^{(M+1)\sum_{a=1}^M d_a} \prod_{a<b}^{M} q^{\frac{1}{2}\tilde{d}_{ab}^2} (q^{\frac{1}{2}\tilde{d}_{ab}} - q^{-\frac{1}{2}\tilde{d}_{ab}})\,,\end{aligned} \tag{2.18}$$

where we have also used that

$$\sum_{a<b}^{M} d_{ab} = (M+1)\sum_{a=1}^{M} d_a \bmod 2. \tag{2.19}$$

We substitute the massaged q-Pochhammer expressions back in (2.14) and define

$$Q = (-1)^{N+M}\tilde{Q} = (-1)^{N+M} q^{-2\pi\zeta}, \tag{2.20}$$

and we obtain

$$Z_{D^2\times_q S^1} = \frac{1}{M!}\oint \prod_{a=1}^{M}\frac{d\epsilon_a}{2\pi i}\, f_{\mathrm{Gr}(M,N)}(q,\epsilon)\cdot I^{SQK}_{\mathrm{Gr}(M,N)}(Q,q,\epsilon), \tag{2.21}$$

where we have collected all Q-dependent terms and summed over the d_a-dependent terms in the *normalized K-theoretic I-function*

$$I^{SQK}_{\mathrm{Gr}(M,N)}(Q,q,\epsilon) = c_0 \sum_{\vec{d}\in\mathbb{Z}^M_{\geq 0}} (-Q)^{\sum_{a=1}^M \tilde{d}_a} q^{CS(\tilde{d})}\,\frac{\prod_{a<b}^M q^{\frac{1}{2}\tilde{d}^2_{ab}}(q^{\frac{1}{2}\tilde{d}_{ab}} - q^{-\frac{1}{2}\tilde{d}_{ab}})}{\prod\limits_{a=1}^{M}\prod\limits_{r=1}^{d_a}(1-q^{r-\epsilon_a})^N}\,. \tag{2.22}$$

Some further explanation is in order; the q-exponent is given by

$$CS(\tilde{d}) = \frac{1}{2}\hat{\kappa}_S\,\mathrm{tr}_{SU(M)}(\tilde{d}^2) + \frac{1}{2}\hat{\kappa}_A\,\mathrm{tr}_{U(1)}(\tilde{d}^2) + \hat{\kappa}_R\,\mathrm{tr}_R(\tilde{d}) \tag{2.23}$$

where the triple $(\hat{\kappa}_S,\hat{\kappa}_A,\hat{\kappa}_R)$ are the *effective* Chern-Simons levels satisfying the integrality conditions (2.5), given in terms of the bare levels (2.3) by

$$\hat{\kappa}_S = \kappa_S - M + \frac{N}{2} \quad , \quad \hat{\kappa}_A = \kappa_A + \frac{N}{2} \quad , \quad \hat{\kappa}_R = \kappa_R + \frac{N}{4}. \tag{2.24}$$

The normalization factor is

$$c_0(q,\epsilon,\hat{\kappa}_S,\hat{\kappa}_A,\hat{\kappa}_R) = \frac{1-q}{q^{CS(-\epsilon)}\prod_{a<b}^M q^{\frac{1}{2}\epsilon^2_{ab}}(q^{-\frac{1}{2}\epsilon_{ab}} - q^{\frac{1}{2}\epsilon_{ab}})}. \tag{2.25}$$

The statement of 3d gauge theory/quantum K-theory correspondence from [67] in this case reads:

> The generalized q-series $I^{SQK}_{\mathrm{Gr}(M,N)}(Q,q,\epsilon)$ computes the I-function of the permutation-symmetric quantum K-theory of $X = \mathrm{Gr}(M,N)$ defined in ref. [59].

For $\hat{\kappa}_S,\hat{\kappa}_A,\hat{\kappa}_R = 0$, it corresponds to the level-zero I-function [86]. When the effective CS levels are non-trivial, we obtain a three-parameter family of I-functions. One can check (cf. discussion in page 26) that on the one-parameter slices $(\hat{\kappa}_S,\hat{\kappa}_A,\hat{\kappa}_R) = (\ell_\square,\ell_\square,-\ell_\square/2)$ and $(\hat{\kappa}_S,\hat{\kappa}_A,\hat{\kappa}_R) = (0, M\ell_{\mathrm{det}}, -\ell_{\mathrm{det}}/2)$ the above result respectively reproduces the I-functions at level $\ell_\square$ in the fundamental representation of $U(M)$ and at level ℓ_{det} in the determinantal representation of $U(M)$ in quantum K-theory with level structure, as defined in ref. [87] (see also refs. [82, 88–90]).

All remaining Q- and d_a-independent terms have been collected in the so called *folding factor* [2, 67, 68]

$$\begin{aligned}
f_{\mathrm{Gr}(M,N)}(q,\epsilon) &= (\log q)^M q^{-\frac{MN}{24}} q^{\left(M-\frac{N}{2}\right)\mathrm{tr}_{SU(M)}(\epsilon^2) - \frac{N}{2}\mathrm{tr}_{U(1)}(\epsilon^2) + \frac{N}{2}\mathrm{tr}_R(\epsilon)} \times \\
&\quad\times \frac{(-1)^{(1+M+N)\sum_{a=1}\epsilon_a} c_0^{-1}}{\prod_{a<b}^M q^{\frac{1}{2}\epsilon^2_{ab}}(q^{-\frac{1}{2}\epsilon_{ab}} - q^{\frac{1}{2}\epsilon_{ab}})}\,\frac{\prod_{a<b}^M (q^{\epsilon_{ab}};q)_\infty (q^{-\epsilon_{ab}};q)_\infty}{\prod_{a=1}^M (q^{\epsilon_a};q)^N_\infty} \\
&= (\log q)^M q^{-\frac{MN}{24}} q^{\left(M-\frac{N}{2}\right)\mathrm{tr}_{SU(M)}(\epsilon^2) - \frac{N}{2}\mathrm{tr}_{U(1)}(\epsilon^2) + \frac{N}{2}\mathrm{tr}_R(\epsilon) + CS(-\epsilon)} \times \\
&\quad\times \frac{(-1)^{(1+M+N)\sum_{a=1}\epsilon_a}}{1-q}\,\frac{\prod_{a<b}^M (q^{\epsilon_{ab}};q)_\infty (q^{-\epsilon_{ab}};q)_\infty}{\prod_{a=1}^M (q^{\epsilon_a};q)^N_\infty}\,.
\end{aligned} \tag{2.26}$$

Contour integrals as integrals on X

The path integral (2.21) can be recast as an integral over $X = \mathrm{Gr}(M,N)$. To illustrate this, we must identify the remaining ϵ_a-variables with *cohomological* elements: setting $\tilde{\epsilon}_a = \log q\ \epsilon_a$ we *identify*

$$\tilde{\epsilon}_a = \beta \cdot x_a \tag{2.27}$$

where x_a denote the Chern roots of the dual tautological bundle S^* over $X = \mathrm{Gr}(M,N)$. In other words, ϵ_a's are *rescaled* cohomological elements.[4]

With this identification, for a symmetric polynomial $F(\tilde{\epsilon})$ in the $\tilde{\epsilon}_a$'s we have that [92]

$$\frac{1}{M!}\oint \prod_{a=1}^{M} \frac{d\tilde{\epsilon}_a}{2\pi i}\mathcal{E}_{\mathrm{Gr}(M,N)}(\tilde{\epsilon})F(\tilde{\epsilon}) = \int_{\mathrm{Gr}(M,N)} F^H(\sigma_\mu) = F_{(N-M)^M}, \tag{2.28}$$

where the 'Euler class'-type integration kernel

$$\mathcal{E}_{\mathrm{Gr}(M,N)}(\tilde{\epsilon}) = \frac{\prod_{a\neq b}^{M} \tilde{\epsilon}_{ab}}{\prod_{a=1}^{M} \tilde{\epsilon}_a^N}, \quad \tilde{\epsilon}_{ab} = \tilde{\epsilon}_a - \tilde{\epsilon}_b, \tag{2.29}$$

transforms a contour integration to an integration on X. The formula can be interpreted as follows. Any symmetric polynomial F defines a cohomology class F^H on X since we can always decompose a symmetric polynomial in terms of Schur polynomials

$$F(\tilde{\epsilon}) = \sum_{\mu} F_\mu \sigma_\mu(\tilde{\epsilon}), \tag{2.30}$$

where the right-hand side is F^H after identifying Schur polynomials inside the $M \times (N-M)$-box with generators of $H^*(X)$, associated to Poincaré duals of Schubert cells [95]. The integration (2.28) then simply "picks out" the coefficient $F_{(N-M)^M}$ of the top cohomology class $\sigma_{(N-M)^M} \in H^{2\dim X}(X)$ given by a "full" $M \times (N-M)$-box.

A similar useful identity also follows directly from the definition of Schur polynomials (B.6)

$$F_\mu = \oint \prod_{a=1}^{M} \frac{d\tilde{\epsilon}_a}{2\pi i}\mathcal{G}_{\mathrm{Gr}(M,N),\mu}(\tilde{\epsilon})F(\tilde{\epsilon}), \quad F(\tilde{\epsilon}) = \sum_{\mu} F_\mu \sigma_\mu(\tilde{\epsilon}), \tag{2.31}$$

where

$$\mathcal{G}_{\mathrm{Gr}(M,N),\mu}(\tilde{\epsilon}) = \frac{\prod_{a<b} \tilde{\epsilon}_{ab}}{\tilde{\epsilon}^{\mu+1}}, \quad \tilde{\epsilon}^{\mu} = \prod_{a=1}^{M} \tilde{\epsilon}_a^{\mu_a}. \tag{2.32}$$

Equations (2.28) and (2.31) are in fact equivalent (see [93], Theorem 1.8).

[4] The Chern roots x_a are of course not elements of $H^*(X)$, as S^* does not split. However, by construction the elementary symmetric polynomials $(-1)^a\sigma_{1^a}$ in the x_a (expressed here in terms of Schur polynomials) are the Chern classes $c_a(x)$ of S and therefore represent elements of $H^*(X)$. Together with the Chern classes of the quotient bundle Q, which are expressible in terms of x_a as $c_i(Q) = \sigma_i(x)$, they provide in fact a valid presentation of $H^*(X)$ with the relation $c(S)c(Q) = 1$, It follows that $H^*(X)$ is generated as a vector space by Schur polynomials σ_μ in the x_a's, with $\mu \subseteq \mathcal{B}_{M\times(N-M)}$ inside a box $\mathcal{B}_{M\times(N-M)}$ of height M and length $N-M$. One may also identify x_a as the generators of $H^*(Y)$, $Y = (\mathbb{P}^{N-1})^M$, where we view Y as the "intermediate" symplectic quotient by the maximal torus $U(1)^M \subset U(M)$. Both points of view (the S^* bundle and the symplectic quotient construction) are equally good for our purposes. See [91–93] for the cohomological symplectic quotient construction, and [94] for related K-theoretic versions. For more details on $H^*(X)$ and the characteristic classes of X, we refer the reader to the appendix B.

The partition function as a geometric integral

With the identification of ϵ_a's as rescaled Chern roots of S^* on $X = \mathrm{Gr}(M,N)$ from the previous subsubsection, we may now express the folding factor (2.26) in terms of characteristic classes.

Note first that we may rewrite the factor

$$\frac{\prod_{a<b}^{M}(q^{\epsilon_{ab}};q)_\infty(q^{-\epsilon_{ab}};q)_\infty}{\prod_{a=1}^{M}(q^{\epsilon_a};q)_\infty^N} = (q;q)_\infty^{2\binom{M}{2}-MN}(1-q^{-1})^{N\sum_{a=1}^{M}\epsilon_a}\,\mathrm{td}_q(X)\cdot\Gamma_q(X)\cdot\mathcal{E}_X(\tilde\epsilon) \tag{2.33}$$

where $\Gamma_q(X)$ is the q-Gamma class (B.62) of X and td_q stands for the Todd class (B.61) with rescaled Chern roots $x_a \mapsto \tilde\epsilon_a = \log q\ \epsilon_a$ and $\mathcal{E}_X$ is the 'Euler class' (2.29). The q-Gamma class $\Gamma_q(X)$ is the proposed [67] q-generalization of the ordinary Γ-class as discussed e.g., in [96, 97]. In the small radius limit $\beta \to 0$, where $q \to 1$, the q-Gamma class is mapped to the Gamma class of X upon the identification (2.27) [98]. In two-dimensional theories, the Γ-class defines an *integral structure* in quantum cohomology [99], and is related to the central charge of D-branes on $\partial D^2 = S^1$. The q-uplift Γ_q is expected to play analogous roles for quantum K-theory and elliptic boundary conditions on $\partial(D^2\times_q S^1) = T^2$.

Secondly, the explicit sums over ϵ_a's in (q-)exponents in (2.26) can be expressed as in terms of Chern classes and characters of X using (B.60), (B.64). Finally, writing infinite q-Pochhammer symbols using the Dedekind eta function

$$\eta(q) = q^{1/24}(q;q)_\infty, \quad |q|<1, \tag{2.34}$$

we can collect everything into

$$f_{\mathrm{Gr}(M,N)}(q,\epsilon) = \mathrm{td}_q(X)\cdot\Gamma_q(X)\cdot\mathcal{E}_X(\tilde\epsilon)\cdot\frac{q^{\mathbf{A}(\epsilon)}N(q,\epsilon)}{\eta(q)^{MN-M(M-1)}}, \tag{2.35}$$

where

$$\begin{aligned}\mathbf{A}(\epsilon) &= \left(M-\tfrac{N}{2}\right)\mathrm{tr}_{SU(M)}(\epsilon^2) - \tfrac{N}{2}\,\mathrm{tr}_{U(1)}(\epsilon^2) - \tfrac{N}{2}\kappa_R\,\mathrm{tr}_R(\epsilon) + CS(-\epsilon)\\ &= -\frac{1}{2}c_1(\epsilon) - \mathrm{ch}_2(\epsilon) + CS(-\epsilon),\end{aligned} \tag{2.36}$$

captures the total anomaly term. Here $CS(-\epsilon)$ is given by the restriction $d_a = 0$ in (2.23). The remaining factor is

$$N(q,\epsilon) = (\log q)^M q^{-\frac{1}{12}\binom{M}{2}}(-1)^{\frac{M+1}{N}c_1(\epsilon)}(1-q)^{c_1(\epsilon)-1}. \tag{2.37}$$

This concludes the expression of $f_{\mathrm{Gr}(M,N)}$ in terms of characteristic classes. We refer the reader to the summary of this section in 2.1.3 for further discussion.

2.1.2 From half-index to I-function

The half-index

Apart from the localization computation of the previous section, one may also compute the I-function using the *half-index*

$$\mathbb{I}_{\mathcal{B}} = \mathrm{tr}_{\mathrm{Ops}_{\mathcal{B}}}(-1)^F q^{J+\frac{R}{2}}x^f\ , \tag{2.38}$$

where F is the fermion operator, J is generated two-dimensional rotations on the boundary torus, R is the R-charge and x and f are general flavor fugacities and charges respectively. The half-index counts half-BPS boundary operators in the cohomology of the supercharge operator preserved by a supersymmetric boundary condition $\mathcal{B}$ [36, 38, 39].

In the case of Neumann boundary conditions $\mathcal{N}$ for the gauge fields, the computation of $\mathbb{I}_{\mathcal{N}}$ involves an integration over $\mathfrak{u}(M)$ Coulomb branch parameters, to project to gauge-invariant observables. Furthermore, it is only valid for zero effective CS levels as well, and $\mathbb{I}_{\mathcal{N}}$ is thus a priori only indirectly sensitive to them. Both the integration and the indirect sensitivity to bare CS levels, is completely analogous to the partition function calculation via localization from the previous subsection.

On the other hand, we may also choose Dirichlet boundary conditions $\mathcal{D}$ for the gauge fields. In that case, the boundary condition sets $A_\mu|_\partial = 0$ and the only remaining gauge transformations on the boundary are the ones that are constant along the boundary directions. The $U(M)$ gauge symmetry is thus broken, and the constant gauge transformation build a remnant, global $U(M)_\partial$ symmetry.

The result for $\mathbb{I}_{\mathcal{D}}$ was computed as a non-perturbative sum over boundary monopole sectors in [39]. The presence of the monopoles shifts the charges of contributions of matter and gauge fields to the index due to the presence of magnetic flux counting the monopole sectors. In addition, CS levels explicitly enter the sum over sectors, and the index is directly sensitive to them. However, the global nature of the $U(M)_\partial$ symmetry means that one does not have to choose bare Chern-Simons levels that cancel anomalous terms. The CS parameters in the I-function extracted from the $\mathbb{I}_{\mathcal{D}}$ index can then be chosen freely.

The general proposal from [39]

For our $U(M)$ gauge theory with N chiral multiplets with $(\mathcal{D}, \mathrm{N})$ boundary conditions, the proposal in [39] for the half-index is

$$\mathbb{I}_{(\mathcal{D},\mathrm{N})}(z,q,\tau) = \frac{1}{(q)_\infty^M} \sum_{\vec{m}\in\mathbb{Z}^M} \frac{q^{\frac{1}{2}\vec{m}\cdot K\vec{m} + \vec{m}\cdot K\vec{\tau} - \tilde{\kappa}_R \sum_{a=1}^M m_a} z^{\sum_{a=1}^M m_a}}{\prod_{a\neq b}^M (q^{1+\tau_{ab}+m_{ab}};q)_\infty} \mathbb{I}^N_{\text{chir, N}}(q^{m+\tau}) \ , \tag{2.39}$$

where $\vec{m} \in \mathrm{cochar}\big(U(M)\big) = \mathbb{Z}^M$ counts the monopole sectors, q^{τ_a} with $a = 1, \ldots, M$ are the fugacities for the boundary $U(M)_\partial$ flavor symmetry, z is the fugacity for the topological $U(1)$ flavor symmetry corresponding to the FI terms and again $m_{ab} = m_a - m_b$, $\tau_{ab} = \tau_a - \tau_b$. The effective (boundary) gauge/gauge Chern-Simons levels as *proposed* in [39] are captured by the $M \times M$-matrix K with entries

$$K_{ab} = \tilde{\kappa}_S \delta_{ab} + \frac{\tilde{\kappa}_A - \tilde{\kappa}_S}{M} \ , \tag{2.40}$$

where $\tilde{\kappa}_S$ is the $SU(M)$-level, $\tilde{\kappa}_A$ is the Abelian $U(1)$-level and $\tilde{\kappa}_R$ captures the gauge/R-symmetry CS level. The denominator accounts for the contribution of the vector multiplet. The contributions from chiral fields with Neumann boundary conditions N is

$$\mathbb{I}_{\text{chir, N}}(q^{m+\tau}) = \prod_{a=1}^M \frac{1}{(q^{m_a+\tau_a};q)_\infty} \ . \tag{2.41}$$

Extracting the I-function with general CS levels

To rewrite the proposed half-index (2.39), note that we may write the chiral contribution

$$\mathbb{I}_{\text{chir, N}}(q^{m+\tau}) = \prod_{a=1}^{M} \frac{1}{(q^{m_a+\tau_a};q)_\infty} = \prod_{a=1}^{M} \frac{(q^{\tau_a};q)_{m_a}}{(q^{\tau_a};q)_\infty}. \tag{2.42}$$

where in the last equation we have used (A.3). The finite product in the numerator is

$$(q^{\tau_a};q)_{m_a} = \begin{cases} (1-q^{\tau_a})\prod_{r=1}^{m_a-1}(1-q^{r+\tau_a}) & , \text{ when } m_a > 0 \\ \prod_{r=1}^{-m_a} \frac{1}{1-q^{\tau_a-r}} & , \text{ when } m_a \leq 0 \end{cases} . \tag{2.43}$$

The overall factor $(1-q^{\tau_a}) = \tau_a + \mathcal{O}(\tau_a^2)$ for m_a positive implies that the leading order in τ_a is in the contributions with $m_a \leq 0$. Note that the chemical potentials τ_a associated to $U(M)_\partial$ play a similar role as ϵ_a's in the Neumann computation of the partition function (2.21): they are interpreted as (rescaled) Chern roots of the dual tautological bundle S^* over $X = \mathrm{Gr}(M,N)$. In particular, they are nilpotent cohomological elements and terms of order $\mathcal{O}(\tau_a^N)$ will be set to zero in the I-function. Sending $m_a \mapsto -m_a$ for convenience, eq. (2.39) becomes

$$\mathbb{I}_{(\mathcal{D},\text{N})} = \frac{1}{(q)_\infty^M}\left[\sum_{\vec{m}\in\mathbb{Z}_{\geq 0}^M} \frac{q^{\frac{1}{2}\vec{m}\cdot K\vec{m} - \vec{m}\cdot K\vec{\tau} + \tilde{\kappa}_R\sum_{a=1}^M m_a} z^{-\sum_{a=1}^M m_a}}{\prod_{a\neq b}^M (q^{1-\tilde{m}_{ab}};q)_\infty \prod_{a=1}^M (q^{-\tilde{m}_a};q)_\infty^N} + \mathcal{O}(\tau^N)\right] , \tag{2.44}$$

where $\tilde{m}_a = m_a - \tau_a$ and $\tilde{m}_{ab} = m_{ab} - \tau_{ab}$. The vector contribution may be expressed by the identity

$$\begin{aligned} \prod_{a<b}^{M} \frac{1}{(q^{1+\tilde{m}_{ab}};q)_\infty (q^{1-\tilde{m}_{ab}};q)_\infty} &= \prod_{a<b}^{M} \theta_q(q^{-\epsilon_{ab}}) q^{\frac{1}{2}\epsilon_{ab}(\epsilon_{ab}+1)} \cdot \prod_{a<b}^{M} (-1)^{d_{ab}} q^{-\frac{1}{2}\tilde{d}_{ab}^2} (q^{\tilde{d}_{ab}/2} - q^{-\tilde{d}_{ab}/2}) \\ &= \prod_{a<b}^{M} \frac{1}{(q^{1+\tau_{ab}};q)_\infty (q^{1-\tau_{ab}};q)_\infty} \frac{1}{q^{\frac{1}{2}\tau_{ab}^2}(q^{-\frac{1}{2}\tau_{ab}} - q^{\frac{1}{2}\tau_{ab}})} \times \\ &\quad \times (-1)^{(M+1)\sum_{a=1}^M m_a} \prod_{a<b}^{M} q^{\frac{1}{2}\tilde{m}_{ab}^2} (q^{\frac{1}{2}\tilde{m}_{ab}} - q^{-\frac{1}{2}\tilde{m}_{ab}}) \, . \end{aligned} \tag{2.45}$$

which follows from equations (A.2) and (2.18). The numerator in the sum (2.44) becomes[5]

$$q^{-\frac{\tilde{\kappa}_S}{2}\mathrm{tr}_{SU(M)}(\tau^2) - \frac{\tilde{\kappa}_A}{2}\mathrm{tr}_{U(1)}(\tau^2) + \tilde{\kappa}_R \mathrm{tr}_R(\tau)} \cdot q^{\frac{\tilde{\kappa}_S}{2}\mathrm{tr}_{SU(M)}(\tilde{m}^2) + \frac{\tilde{\kappa}_A}{2}\mathrm{tr}_{U(1)}(\tilde{m}^2) + \tilde{\kappa}_R \mathrm{tr}_R(\tilde{m})} z^{-\sum_{a=1}^M m_a} \tag{2.46}$$

[5]With K as in (2.40), in the notation of footnote 1, we have

$$\frac{1}{2}\vec{m}\cdot K\vec{m} = \frac{\tilde{\kappa}_S}{2}\mathrm{tr}_{SU(M)}(m^2) + \frac{\tilde{\kappa}_A}{2}\mathrm{tr}_{U(1)}(m^2)$$

The chiral contribution is rewritten using (2.17). Collecting everything we find

$$
\begin{aligned}
\mathbb{I}_{(\mathcal{D},\mathrm{N})} =& \frac{1}{(q)_\infty^M} \frac{1}{\prod_{a<b}^M (q^{1+\tau_{ab}};q)_\infty (q^{1-\tau_{ab}};q)_\infty \prod_{a=1}^M (q^{\tau_a};q)_\infty^N} \\
&\times \frac{1}{q^{\widetilde{CS}(-\tau)} \prod_{a<b}^M q^{\frac{1}{2}\tau_{ab}^2}(q^{-\frac{1}{2}\tau_{ab}} - q^{\frac{1}{2}\tau_{ab}})} \\
&\times \left[\sum_{\vec{m}\in\mathbb{Z}^M_{\geq 0}} q^{\widetilde{CS}(\tilde{m})} ((-1)^{(N+M+1)} z^{-1})^{\sum_{a=1}^M m_a} \frac{\prod_{a<b}^M q^{\frac{1}{2}\tilde{m}_{ab}^2}(q^{\frac{1}{2}\tilde{m}_{ab}} - q^{-\frac{1}{2}\tilde{m}_{ab}})}{\prod_{a=1}^M \prod_{r=1}^{m_a} (1-q^{r-\tau_a})^N} + \mathcal{O}(\tau^N) \right]
\end{aligned}
\tag{2.47}
$$

We can rewrite the last line into the final result

$$
\begin{aligned}
\mathbb{I}_{\mathcal{D},\mathrm{N}}(z,q,\tau) =& \frac{1}{(q)_\infty^M (1-q)} \frac{((-1)^{(N+M+1)} z^{-1})^{\sum_{a=1}^M \tau_a}}{\prod_{a<b}^M (q^{1+\tau_{ab}};q)_\infty (q^{1-\tau_{ab}};q)_\infty \prod_{a=1}^M (q^{\tau_a};q)_\infty^N} \times \\
&\times \left[I^{SQK}_{\mathrm{Gr}(M,N)}((-1)^{M+N} z^{-1}, q, \tau) + \mathcal{O}(\tau^N) \right] ,
\end{aligned}
\tag{2.48}
$$

where the CS levels in I, defined as in (2.22), are[6]

$$
\widetilde{CS}(\tilde{m}) = \frac{\tilde{\kappa}_S + N}{2} \operatorname{tr}_{SU(M)}(\tilde{m}^2) + \frac{\tilde{\kappa}_A + N}{2} \operatorname{tr}_{U(1)}(\tilde{m}^2) + (\tilde{\kappa}_R + \frac{N}{2}) \operatorname{tr}_R(\tilde{m}) \,. \tag{2.49}
$$

2.1.3 Results of this section

The path integral of an $\mathcal{N} = 2$ $U(M)$ gauge theory with N fundamental chirals on $D^2 \times_q S^2$ and Neumann boundary conditions $(\mathcal{N}, \mathrm{N})$ for vector and chiral fields on T^2_q respectively computes the permutation-symmetric K-theoretic I-function for the Higgs branch $X = \mathrm{Gr}(M,N)$. The result (2.22) from subsection 2.1.1 as in integral over X (see (2.28)) is

$$
\mathcal{Z}_{D^2\times_q S^1} = \int_{\mathrm{Gr}(M,N)} f^H_{\mathrm{Gr}(M,N)}(q) \cdot I^{H,SQK}_{\mathrm{Gr}(M,N)}(Q,q). \tag{2.50}
$$

The K-theoretic I-function is K-theory-valued, i.e.,

$$
I^{SQK}_{\mathrm{Gr}(M,N)}(Q,q,P) = \sum_{\mu\subseteq\mathcal{B}_{M\times(N-M)}} I^\mu(Q,q)\mathcal{O}_\mu(1-P). \tag{2.51}
$$

The I^H-function given above is the image of I via the Chern character isomorphism $\mathrm{ch}: K(X) \to H^*(X)$[7], and is cohomology-valued. It is derived from (2.22) by expanding in terms of Schur polynomials

$$
I^{SQK}_{\mathrm{Gr}(M,N)}(Q,q,\epsilon) = \sum_{\mu\subseteq\mathcal{B}_{M\times(N-M)}} I^\mu(Q,q)\sigma_\mu(\epsilon) \tag{2.52}
$$

and interpreting the admissible Schur polynomials as rescaled $H^*(\mathrm{Gr}(M,N))$ generators associated to the (Poincaré duals to) Schubert cells [95] (see the appendix B for details).

[6]See footnote 1 for our sum conventions.

[7]Explicitly the ch sends K-theoretic Chern roots P_a to $q^{-\epsilon_a}$, see page 26 and appendix B.2.

The *folding factor* computed in (2.35) produces the following cohomological integration kernel

$$f_X^H = \mathrm{td}_q(X) \cdot \Gamma_q(X) \cdot \frac{q^{\mathbf{A}(\epsilon)} N_X(q)}{\eta(q)^{MN-M(M-1)}} \tag{2.53}$$

with $\mathbf{A}$ from (2.36) and $N_X(q)$ as in (2.37).

The I-function depends on the Chern-Simons levels $\hat{\kappa}$ appearing in the $\mathcal{N}=2$ gauge theory. Strictly speaking, when using $\mathcal{N}$ boundary conditions for the gauge fields these have to be chosen such that the effective levels are zero, otherwise the physical theory is anomalous. In subsection 2.1.2 we computed the same I-function using $\mathcal{D}$ boundary conditions for the gauge fields. The global nature of the remnant gauge symmetry on the boundary means that the effective CS levels can be freely chosen here, up to integrality constraints (see (2.5)). The dependence on the CS levels is important, as for levels outside a certain "window" the I-function computed by the field theory will have a non-zero so called *input* (see [1] and the discussion following in 2.2.2).

The two computations for Neumann and Dirichlet boundary conditions indicate that $I^{SQK}_{\mathrm{Gr}(M,N)}$ does *not* to depend on the boundary condition chosen; choosing different boundary conditions amounts to inserting in (2.50) non-trivial *brane factors* $\mathfrak{f}_{\mathcal{B}}(q^\epsilon, q)$ [67, 68] (see also [100, 101] for two-dimensional analogs). After resumming the poles (which sets $z \sim q^{-\epsilon}$), these become *elliptic* factors

$$\mathfrak{f}_{\mathcal{B}}(qz, q) = \mathfrak{f}_{\mathcal{B}}(z, q) \tag{2.54}$$

and are constants with respect to q-difference equations satisfied by I. The factors can be chosen in such a way to "pick out" a certain (linear combination of) I-function coefficients I^μ from some (undetermined) basis of operators. In the language of [102, 103], the partition function computes the overlap between the boundary state on T^2, labeled by $\mathcal{B}$, and the state determined by inserting an operator labeled by μ in the center of the disk D^2 via the state-operator map. See [67] for a detailed discussion.

2.2 Wilson loops, q-difference equations and small quantum K-theory

The K-theoretic I-function for Grassmannians $\mathrm{Gr}(M,N)$ computed via an $\mathcal{N}=2$ $U(M)$ gauge theory with N chiral multiplets from the previous section is

$$I^{SQK}_{\mathrm{Gr}(M,N)}(Q,q,\epsilon) = c_0 \sum_{\vec{d}\in\mathbb{Z}^M_{\geq 0}} (-Q)^{\sum_{a=1}^M \tilde{d}_a} q^{CS(\tilde{d})} \frac{\prod_{a<b}^M q^{\frac{1}{2}\tilde{d}_{ab}^2}(q^{\frac{1}{2}\tilde{d}_{ab}} - q^{-\frac{1}{2}\tilde{d}_{ab}})}{\prod_{a=1}^M \prod_{r=1}^{d_a} (1-q^{r-\epsilon_a})^N} \tag{2.55}$$

where $CS(\tilde{d})$, depending on three CS parameters $(\hat{\kappa}_s, \hat{\kappa}_A, \hat{\kappa}_R)$, is given in (2.23) and c_0 is chosen such that $I^{SQK}_{\mathrm{Gr}(M,N)}(0,q,\epsilon) = 1-q$ in (2.25). This function is of generalized basic hypergeometric type [104, 105] and we will study its rich structure in this section.

2.2.1 Various expressions for the I-function

First we will collect various technical results: we will re-write the function $I^{SQK}_{\mathrm{Gr}(M,N)}$ from (2.55) in a few more ways to study its properties.

Sum over multiplicative genera

Let us write the factors in (2.55) in yet another form. Using (2.18) backwards and combining with (A.7b) we find after some calculation

$$\prod_{a<b} \frac{q^{\frac{1}{2}\tilde{d}_{ab}^2}(q^{\frac{\tilde{d}_{ab}}{2}} - q^{-\frac{\tilde{d}_{ab}}{2}})}{q^{\frac{1}{2}\epsilon_{ab}^2}(q^{-\frac{\epsilon_{ab}}{2}} - q^{\frac{\epsilon_{ab}}{2}})} = (-1)^{(M+1)\sum_{a=1}^M d_a} \prod_{a\neq b}^{M} (q^{1-\epsilon_{ab}};q)_{d_{ab}}. \tag{2.56}$$

Furthermore, since $d_a \geq 0$ we have

$$\frac{1}{\prod_{a=1}^M \prod_{r=1}^{d_a}(1-q^{r-\epsilon_a})^N} = \frac{1}{\prod_{a=1}^M (q^{1-\epsilon_a};q)_{d_a}^N}. \tag{2.57}$$

Lastly, the CS factor can be written

$$CS(\tilde{d}) = CS(d) + CS(-\epsilon) - \hat{\kappa}_S \sum_{a=1}^M d_a\epsilon_a - \frac{\hat{\kappa}_A - \hat{\kappa}_S}{M} \sum_{a,b} d_a\epsilon_b. \tag{2.58}$$

Substituting in (2.55), we find

$$\begin{aligned} I^{SQK}_{\mathrm{Gr}(M,N)}(Q,q,\epsilon) = (1-q)(-Q)^{-\sum_{a=1}^M \epsilon_a} \times \\ \times \sum_{\vec{d}\in\mathbb{Z}^M_{\geq 0}} ((-1)^M Q)^{\sum_{a=1}^M d_a} q^{CS(d)-\hat{\kappa}_S \vec{d}\cdot\vec{\epsilon} - \frac{\hat{\kappa}_A-\hat{\kappa}_S}{M}\sum_{a,b=1}^M d_a\epsilon_b} \frac{\prod_{a\neq b}^M (q^{1-\epsilon_{ab}};q)_{d_{ab}}}{\prod_{a=1}^M (q^{1-\epsilon_a};q)_{d_a}^N}. \end{aligned} \tag{2.59}$$

In this form, the I-function for zero effective CS levels is a sum over multiplicative genera in the sense of (B.59).

Genuinely K-theory-valued

We can now also write the I-function as a genuinely $K(X)$-valued function, as opposed to its ch image in $H^*(X)$. The K-theoretic Chern roots P_a, $a = 1,\ldots,a$ of S have images $\mathrm{ch}(P_a) = e^{-x_a}$ and with setting $\beta = 1$ in the rescaling (2.27) and abusing notation we write (see appendix B.2 for details)

$$P_a = q^{-\epsilon_a}. \tag{2.60}$$

Then, (2.59) becomes

$$I^{SQK}_{\mathrm{Gr}(M,N)}(Q,q,\epsilon) = (1-q)(-Q)^{-\sum_{a=1}^M \epsilon_a} \mathcal{I}^{\hat{\kappa}_S,\hat{\kappa}_A,\hat{\kappa}_R}_{\mathrm{Gr}(M,N)}(Q,q,P), \tag{2.61}$$

where

$$\mathcal{I}^{\hat{\kappa}_S,\hat{\kappa}_A,\hat{\kappa}_R}_{\mathrm{Gr}(M,N)}(Q,q,P) = \sum_{\vec{d}\in\mathbb{Z}^M_{\geq 0}} \Big((-1)^M Q \prod_{b=1}^M P_b^{\frac{\hat{\kappa}_A-\hat{\kappa}_S}{M}}\Big)^{\sum_{a=1}^M d_a} q^{CS(d)} \cdot \prod_{a=1}^M P_a^{\hat{\kappa}_S d_a} \cdot \frac{\prod_{a\neq b}^M (q\frac{P_a}{P_b};q)_{d_{ab}}}{\prod_{a=1}^M (qP_a;q)_{d_a}^N}. \tag{2.62}$$

As mentioned in page 19, the I-function for various one-parameter slices of CS levels matches expressions in the literature. In the above form this is easy to verify.

Via Abelian symplectic quotient

We now pass to an *Abelianized* version of the I-function, which amounts to refining the Q-dependence

$$Q^{\sum_{a=1}^M \tilde{d}_a} \mapsto \prod_{a=1}^M Q_a^{\tilde{d}_a}. \tag{2.63}$$

Physically, this means we are analyzing a $U(1)^M$ gauge theory instead of a $U(M)$, which leads to M different Fayet-Iliopoulos parameters ζ_a, thus defining M different Q_a's as in (2.20), one in each vortex sector d_a. Mathematically, this means we're analyzing the symplectic quotient $(\mathbb{P}^{N-1})^M = \mathrm{Hom}(\mathbb{C}^M, \mathbb{C}^N)//U(1)^M$ by the maximal torus instead of $\mathrm{Gr}(M,N) = \mathrm{Hom}(\mathbb{C}^M, \mathbb{C}^N)//U(M)$. Since $h_2\big((\mathbb{P}^{N-1})^M\big) = M$ we obtain M parameters Q_a. The Abelianized I-function becomes

$$\hat{I}^{SQK}_{\mathrm{Gr}(M,N)}(\vec{Q}, q, \epsilon) = c_0 \sum_{\vec{d}\in\mathbb{Z}^M_{\geq 0}} q^{CS(\tilde{d})} \prod_{a<b}^M q^{\frac{1}{2}\tilde{d}^2_{ab}}(q^{\frac{1}{2}\tilde{d}_{ab}} - q^{-\frac{1}{2}\tilde{d}_{ab}}) \prod_{a=1}^M \frac{(-Q_a)^{\tilde{d}_a}}{\prod_{r=1}^{d_a}(1-q^{r-\epsilon_a})^N}, \tag{2.64}$$

where of course the "de-Abelianization" is

$$I^{SQK}_{\mathrm{Gr}(M,N)}(Q, q, \epsilon) = \hat{I}^{SQK}_{\mathrm{Gr}(M,N)}(\vec{Q}, q, \epsilon)|_{Q_a = Q}. \tag{2.65}$$

After easy manipulations we can write the numerator in (2.55) coming from the gauge bosons as

$$\prod_{a<b}^M q^{\frac{1}{2}\tilde{d}^2_{ab}}(q^{\frac{1}{2}\tilde{d}_{ab}} - q^{-\frac{1}{2}\tilde{d}_{ab}}) = q^{(M-1)\sum_{a=1}^M \binom{\tilde{d}_a}{2}} q^{-\sum_{a<b}^M \tilde{d}_a\tilde{d}_b} \prod_{a<b}(q^{\tilde{d}_a} - q^{\tilde{d}_b}). \tag{2.66}$$

Similarly, using definitions from footnote 1 we can write the CS factor (2.23)

$$q^{CS(\tilde{d})} = q^{\frac{1}{2}(\hat{\kappa}_S + \frac{\hat{\kappa}_A - \hat{\kappa}_S}{M})\sum_{a=1}^M \tilde{d}_a^2 + \hat{\kappa}_R \sum_{a=1}^M \tilde{d}_a + \frac{\hat{\kappa}_A - \hat{\kappa}_S}{M}\sum_{a<b}^M \tilde{d}_a\tilde{d}_b}. \tag{2.67}$$

We can now write (2.64) as

$$\hat{I}^{SQK}_{\mathrm{Gr}(M,N)}(\vec{Q}, q, \epsilon) = \Delta \cdot \tilde{I}^{SQK}_{\mathrm{Gr}(M,N)}(\vec{Q}, q, \epsilon), \tag{2.68}$$

where the first factor is a Vandermonde q-difference operator

$$\Delta = \prod_{a<b}(\hat{p}_a - \hat{p}_b), \quad \hat{p}_a = q^{Q_a \frac{\partial}{\partial Q_a}}, \tag{2.69}$$

acting on the Abelianized Q_a's and $\tilde{I}^{SQK}_{\mathrm{Gr}(M,N)}$ takes the quasi-factorized form

$$\tilde{I}^{SQK}_{\mathrm{Gr}(M,N)}(\vec{Q}, q, \epsilon) = c_0 \sum_{\vec{d}\in\mathbb{Z}^M_{\geq 0}} q^{\gamma \sum_{a<b}^M \tilde{d}_a\tilde{d}_b} \prod_{a=1}^M I^{\alpha,\beta;d_a}_{\mathrm{Gr}(1,N)}(Q_a, q, \epsilon_a), \tag{2.70}$$

where the factors in the product are the summands in the generalized $\mathbb{P}^{N-1} = \mathrm{Gr}(1,N)$ I-function:

$$I^{\alpha,\beta;d}_{\mathrm{Gr}(1,N)}(Q, q, \epsilon) := \frac{q^{\frac{\alpha}{2}\tilde{d}^2 + \beta\tilde{d}}(-Q)^{\tilde{d}}}{\prod_{r=1}^d (1-q^{r-\epsilon})^N}, \qquad I^{SQK}_{\mathrm{Gr}(1,N)}(Q, q, \epsilon) = \sum_{d\geq 0} I^{\alpha,\beta;d}_{\mathrm{Gr}(1,N)}(Q, q, \epsilon). \tag{2.71}$$

Equation (2.68) can be thought of as a three-dimensional/K-theoretic analog of the proposal in the appendix of [106], later proved in [107]. The three CS parameters $(\hat{\kappa}_S, \hat{\kappa}_A, \hat{\kappa}_R)$ are shuffled into

$$\begin{aligned}\alpha &= \hat{\kappa}_S + \frac{\hat{\kappa}_A - \hat{\kappa}_S}{M} + M - 1,\\ \beta &= \hat{\kappa}_R - \frac{M-1}{2},\\ \gamma &= \frac{\hat{\kappa}_A - \hat{\kappa}_S}{M} - 1.\end{aligned} \tag{2.72}$$

with inverse mapping

$$\begin{aligned}\hat{\kappa}_S &= \alpha - \gamma - M,\\ \hat{\kappa}_A &= \alpha + \gamma(M-1),\\ \hat{\kappa}_R &= \beta + \frac{1}{2}(M-1).\end{aligned} \tag{2.73}$$

It is evident in this form, that $I^{SQK}_{\mathrm{Gr}(M,N)}$ admits a series expansion in integral powers of q only if the CS levels further satisfy the q-integrality congruence condition

$$\alpha \equiv 2\beta \mod 2 \quad \Big(\Leftrightarrow \hat{\kappa}_S + \frac{\hat{\kappa}_A - \hat{\kappa}_S}{M} \equiv 2\hat{\kappa}_R \mod 2\Big), \tag{2.74}$$

where the original integrality condition (2.5) has been used.

2.2.2 Putting bounds on the permutation-symmetric input

Let us make a small digression and discuss the mapping between the UV and IR of our 3D gauge theory on $\Sigma \times S^1$, as well as subtleties concerning J-functions and I-functions.

The gauge theory we start with is defined in the UV as a gauged linear sigma model. It flows in the IR (in the "geometric phase" [65]) to some non-linear sigma model with a Kähler target space X, in our case a complex Grassmannian $\mathrm{Gr}(M,N)$. The non-trivial computations of chiral rings and other BPS objects like Wilson lines, associated to quantum rings of X, are performed in the IR. In the two-dimensional theories, e.g., in the limit of vanishing S^1 radius of the $\Sigma \times S^1$ theory, the twisted chiral ring of the $\mathcal{N} = (2,2)$ theory corresponds to quantum cohomology. The IR correlators of the 2D A-model can be identified with intersection products on the moduli space of stable maps $\Sigma \to X$, which in turn can be nicely described in terms of cohomological J-functions (see section 1.2 for some details). The UV theory involves *different* of maps $\Sigma \to X$, the quasi-maps (see for example [108–111] and references therein). The intersection products on the moduli space of quasi-maps are then collected into the cohomological I-functions. The 3D lift of a 2D theory on Σ to a theory on $\Sigma \times S^1$ with non-zero radius leads to quantum K-theoretic analogs of these constructions.

The description of the UV and IR regimes is therefore different. Nevertheless, there are 'protected' quantities: the supersymmetric indices (e.g., the half-index from the previous section) are invariant under the renormalization group flow and can then be used to study the IR regime, for a given UV theory. The caveat is of course, that there is in general a non-trivial mapping between UV and IR observables, known in 2D as the *mirror map* [23, 112]. The mirror map relates I-functions to J-functions. However, when the mirror map is trivial, the I- and J-functions *coincide*.

The I-function computed by the gauge theory according to the proposal of [67] has, in general, a non-trivial permutation-symmetric *input* $t(q) \in \mathcal{K}_+$ (1.52), where $\mathcal{K}_+$ is the Lagrangian subspace of the symplectic loop space $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ (1.35) as discussed in the introduction. In other words, we can expand the (ch-image of the) I-function computed by the gauge theory as

$$I(Q,q,\epsilon) = (1-q) + t(Q,q,\epsilon) + I_{\text{corr.}}\big(t(Q,q,\epsilon)\big), \quad t(Q,q,\epsilon) \in \mathcal{K}_+, \quad I_{\text{corr.}}(t) \in \mathcal{K}_-. \tag{2.75}$$

The non-trivial symmetric input can be removed by integrating in appropriate massive fields in the UV. A non-zero input thus precisely indicates a non-trivial UV-IR *mirror map* between parameters of the theory in the UV and IR: an unperturbed theory (i.e., without the massive fields) defined in the UV, leads in general to a perturbed theory in the IR. The non-zero permutation-symmetric $\mathcal{K}_+$-part must be first removed for the $\mathcal{K}_-$ to encode IR correlators "on the nose". We will discuss this point in more detail in section 2.3.

For certain "window" of Chern-Simons levels $(\hat{\kappa}_S, \hat{\kappa}_A, \hat{\kappa}_R)$ or (α, β, γ) (2.72), the I-function computed by the gauge theory has trivial input $t(Q,q,\epsilon) = 0$. In that case the mirror map is trivial and the I-function computed by the gauge theory coincides with Givental's unperturbed J-function discussed in the introduction 1.2.4.

On a technical level, in view of the definition of $\mathcal{K}_-$, a zero input requires that the I-function (e.g., in the form of (2.70) together with the Vandermonde factor (2.69)) to only contain summands whose coefficients rational functions in q, where the denominators are regular at 0 and have strictly higher degree than the numerators. This puts further constraints on the allowed CS levels (α, β, γ), in addition to the integrality constraints (2.5) and (2.74). While a precise, closed condition on α, β, γ is not easy to detect, inspection of the "Abelian" $\mathbb{P}^{N-1}$-factor (2.71) shows that the window requires at least the following condition to hold

$$0 \leq \alpha \leq N. \tag{2.76}$$

As an example, we find computationally that the following (α,β,γ)-triples lead to zero input up to order Q^{10} in $I^{(\alpha,\beta,\gamma)}_{\mathrm{Gr}(2,4)}(Q,q,\epsilon)$:

$$(\alpha,\beta,\gamma) \in \left\{ \begin{array}{lllll}
(0,0,0), & (0,1,3), & (2,-1,-1), & (3,-\frac{3}{2},-1), & (4,-2,-1), \\
(0,0,1), & \boxed{(1,-\frac{1}{2},-1)}, & (2,-1,0), & (3,-\frac{3}{2},0), & (4,-2,0), \\
(0,0,2), & (1,-\frac{1}{2},0), & (2,-1,1), & (3,-\frac{3}{2},1), & (4,-1,-3), \\
(0,0,3), & (1,-\frac{1}{2},1), & (2,-1,2), & (3,-\frac{1}{2},-3), & (4,-1,-2), \\
(0,0,4), & (1,-\frac{1}{2},2), & (2,0,-2), & (3,-\frac{1}{2},-2), & (4,-1,-1), \\
(0,1,0), & (1,-\frac{1}{2},3), & (2,0,-1), & (3,-\frac{1}{2},-1), & (4,-1,0), \\
(0,1,1), & (1,\frac{1}{2},-1), & (2,0,0), & (3,-\frac{1}{2},0), & \\
(0,1,2), & (1,\frac{1}{2},0), & (2,0,1), & (3,-\frac{1}{2},1), & \\
(0,1,3), & (1,\frac{1}{2},1), & (2,0,2), & & \\
 & (1,\frac{1}{2},2), & & & \\
 & (1,\frac{1}{2},3), & & &
\end{array} \right\} \tag{2.77}$$

The canonical levels $\hat{\kappa}_S, \hat{\kappa}_A, \hat{\kappa}_R = 0$ correspond to $\alpha = M-1 = 1$, $\beta = -\frac{M-1}{2} = -\frac{1}{2}$, $\gamma = -1$, which is the boxed element in the list. Canonical levels for other Grassmannians $\mathrm{Gr}(M,N)$ lead to I-functions with zero input for all $M < N$ as shown in [89].

2.2.3 An ideal of q-difference equations for the Abelianized theory

The Abelianization (2.68) of the I-function (2.55) is very useful when including half-BPS Wilson loop operators in our physical theory, wrapping the S^1 factor of $D^2 \times_q S^1$ at the origin of $p \in D^2$. Wilson loops in the $U(M)$ gauge theory are

$$W_R = \frac{1}{\dim R} \operatorname{tr}_R \mathcal{P} \exp\Big(\oint_{S^1} iA^\mu x_\mu + \sigma|\dot{x}|\Big), \tag{2.78}$$

where R labels a representation of the gauge group, A^μ is the (representation of the) gauge field and σ is the (representation of the) scalar in the vector multiplet. In an Abelianized theory with gauge group is $U(1)^M$, a Wilson loop $W_{\vec{n}} = \prod_{a=1}^M W_a^{n_a}$ is labeled by the negative charges n_a, $a = 1, \dots, M$. It has an expectation value [18, 20, 22, 113]

$$\langle W_{\vec{n}} \rangle_{\text{Ab.}} \sim \frac{1}{M!} \int \prod_{a=1}^M \frac{dz_a}{2\pi i z_a} e^{-S_{\text{class.}}} \mathcal{Z}_{\text{1-loop}} \cdot e^{-\vec{n}\cdot \log \vec{z}}. \tag{2.79}$$

A computation along the lines of (2.1) with an insertion $e^{-\vec{n}\cdot\log\vec{z}}$ as above leads to a vortex sum $\hat{I}_{\vec{n}}$ with an insertion of $q^{n_a \tilde{d}_a}$ in the vortex sector counted by d_a:

$$\langle W_{\vec{n}} \rangle_{\text{Ab.}} \sim I_{\vec{n}}(\vec{Q}, q, \epsilon) = c_0 \cdot \Delta \sum_{\vec{d} \in \mathbb{Z}^M_{\geq 0}} q^{\gamma \sum_{a<b}^M \tilde{d}_a \tilde{d}_b} \prod_{a=1}^M q^{n_a \tilde{d}_a} I^{\alpha,\beta;d_a}_{\text{Gr}(1,N)}(Q_a, q, \epsilon_a) \tag{2.80}$$

with Δ as in (2.69), and the remaining factors as in (2.70). The effect of inserting a Wilson loop operator $W_{\vec{n}}$ can therefore reproduced by the action of a q-difference operator on the Abelianized $\hat{I}$-function:

$$I_{\vec{n}}(\vec{Q}, q, \epsilon) = \Delta \cdot D_{\vec{n}}(\hat{p}) \cdot \tilde{I}^{SQK}_{\text{Gr}(M,N)}(\vec{Q}, q, \epsilon), \qquad D_{\vec{n}}(\hat{p}) = \prod_{a=1}^M \hat{p}_a^{n_a}. \tag{2.81}$$

Classically, the Wilson loop algebra of the Abelianized theory is freely generated as a ring by such Wilson loop operators modulo the relation

$$D_{\vec{n}} \cdot D_{\vec{m}} = D_{\vec{n}+\vec{m}}. \tag{2.82}$$

Quantum mechanically however, the basic hypergeometric nature of $\tilde{I}^{SQK}_{\text{Gr}(M,N)}$ implies non-trivial relations. The relations manifest as q-difference equations satisfied by $\hat{I}^{SQK}_{\text{Gr}(M,N)}$ (2.64). In particular, for $r \in \{1, \dots, M\}$ fixed, the operator $(1 - \hat{p}_r)^N$ acts on the r-th $\mathbb{P}^{N-1}$-summand (2.71) as

$$(1 - \hat{p}_r)^N I^{\alpha,\beta;d_r}_{\text{Gr}(1,N)}(Q_r, q, \epsilon) = (-Q_r) q^{\alpha d'_r + \frac{\alpha}{2} + \beta} I^{\alpha,\beta;d'_r}_{\text{Gr}(1,N)}(Q_r, q, \epsilon), \quad d'_r = d_r - 1. \tag{2.83}$$

Using also that

$$q^{\gamma \sum_{a<b}^M \tilde{d}_a \tilde{d}_b} = q^{\gamma \sum_{a<b}^M \tilde{d}'_a \tilde{d}'_b} \cdot \prod_{a \neq r} q^{\gamma \tilde{d}'_a}, \qquad d'_a = d_a - \delta_{ar}, \tag{2.84}$$

the shift can be absorbed into a redefinition of the summation over d_a's, disregarding terms of order $\mathcal{O}(\epsilon^N)$. We find that the Abelianized I-function from (2.64) satisfies the system of q-difference equations

$$\begin{aligned}\mathfrak{L}_{Q_r,r}(\alpha,\beta,\gamma)\cdot\hat{I}^{SQK}_{\mathrm{Gr}(M,N)}(\vec{Q},q,\epsilon) &= \mathcal{O}(\epsilon_r^N), \qquad r=1,\dots,M,\\ \mathfrak{L}_{Q_r,r}(\alpha,\beta,\gamma) &= (1-\hat{p}_r)^N + q^{\frac{\alpha}{2}+\beta}Q_r\hat{p}_r^{\alpha}\prod_{a\neq r}\hat{p}_a^{\gamma}.\end{aligned} \tag{2.85}$$

The q-difference system depends on the Chern-Simons levels through α,β,γ (2.72). In particular, since $\hat{p}_r$ is invertible, we see that the highest degree in $\hat{p}_r^{\pm 1}$ is

$$D = \deg(\mathfrak{L}_{Q_r,r}) = \begin{cases} N & \text{if } 0\leq\alpha\leq N\\ N-\alpha & \text{if } \alpha<0\\ \alpha & \text{if } \alpha>N\end{cases}. \tag{2.86}$$

We see that the Chern-Simons window condition (2.76) implies that the first term in the q-difference operator is in fact the leading term. When $0\leq\alpha\leq N$ and $\gamma<0$ (e.g., for canonical levels $\hat{\kappa}=0$), the q-difference system contains denominators; these can be cleared however, by repeated use of these ideal relations. In particular we can substitute both denominators *and* high powers of $\hat{p}_r$ by using the following two (equivalent) useful forms of the relations:

$$\hat{p}_r^N = (-1)^{N+1}q^{\frac{\alpha}{2}+\beta}Q_r\hat{p}_r^{\alpha}\prod_{a\neq r}\hat{p}_a^{\gamma} + (-1)^{N+1}\sum_{k=0}^{N-1}\binom{N}{k}(-\hat{p}_r)^k \tag{2.87a}$$

$$\frac{1}{\hat{p}_r} = -q^{\frac{\alpha}{2}+\beta}Q_r\hat{p}_r^{\alpha-1}\prod_{a\neq r}\hat{p}_a^{\gamma} + \sum_{k=0}^{N-1}\binom{N}{k+1}(-\hat{p}_r)^k \tag{2.87b}$$

2.2.4 Wilson loop algebras

In this subsection we will describe the algebras of Wilson loop operators in our $\mathcal{N}=2$ $U(M)$ gauge theory. We will do so by describing the algebra as a quotient by an *ideal quotient*[8] of q-difference operators acting on the Abelianized I-function from the previous subsection.

Wilson loops in the non-Abelian gauge theory

In the $U(M)$ gauge theory, there is only one exponentiated Fayet-Iliopoulos parameter Q. However, we still find the q-difference structure for the I-function, generated by Wilson loops. Wilson loops are now labeled by a representation R of $U(M)$, determined by some Young diagram with height up to M, corresponding to symmetrized $U(1)^M$ Wilson loops. Computing the expectation value of a Wilson loop will, as opposed to (2.79),

[8] Recall, an ideal quotient $I:J$ of two ideals in a ring $I,J\subset R$ is itself an ideal defined as

$$I:J=\{r\in R|rJ\subset I\}.$$

amount to inserting a symmetrized sum of q-factors in the path integral, which will lead to a symmetrized sum of q-shifts in the vortex sum

$$\sum_{\vec{n}} c_{\vec{n}} e^{-\vec{n}\cdot\log\vec{z}} \mapsto \sum_{\vec{n}} c_{\vec{n}} q^{\vec{n}\cdot\vec{d}}, \tag{2.88}$$

for some $c_{\vec{n}}$ symmetric. The shift can again be implemented by q-difference equations acting on an Abelianized version $\hat{I}$ of the I-function. In other words, we may pick a basis of Wilson loops by Schur polynomials in the q-shift operators $\hat{p}_a = q^{Q_a \frac{\partial}{\partial Q_a}}$ and associate

$$W_\mu \mapsto \sigma_\mu(\hat{p}). \tag{2.89}$$

In view of the the q-difference structure satisfied by the (Abelianized) I-function, given in (2.85), it is useful to also rather use the basis of *shifted* Wilson loops $\hat{W}_\mu$, defined by Schur polynomials in the q-difference operators

$$\hat{\delta}_a = 1 - \hat{p}_a, \tag{2.90}$$

so we associate

$$\hat{W}_\mu \mapsto \sigma_\mu(\hat{\delta}). \tag{2.91}$$

Shifted and unshifted Wilson loops have been studied as alternative bases for quantum K-theory also in [114]. In the following, we will also choose a basis defined by *Grothendieck* polynomials (B.29)

$$\hat{W}'_\mu \mapsto \mathcal{O}_\mu(\hat{\delta}), \tag{2.92}$$

to match the basis of Schubert structure sheaves in K-theory (see appendix B.2 for details).

To summarize, in the basis of Wilson loops given by (for example) $\hat{W}'_\mu$, we find that[9]

$$\langle \hat{W}_\mu \rangle \sim I_\mu(Q,q,\epsilon) \coloneqq \left[\Delta \cdot \mathcal{O}_\mu(\hat{\delta}) \cdot \tilde{I}^{SQK}_{\mathrm{Gr}(M,N)}(\vec{Q},q,\epsilon)\right]_{Q_a=Q}. \tag{2.93}$$

with Δ as in (2.69) and $\tilde{I}^{SQK}_{\mathrm{Gr}(M,N)}$ as in (2.70). We now address the question of *multiplying* two such Wilson loops in our theory.

The ideal from Abelian q-difference equations

Just as in the Abelian case (2.82), the q-difference operator algebra associated to Wilson loops is *classically* additively freely generated, and multiplicatively obeys the relation (see (B.29) in the appendix)

$$\mathcal{O}_\mu(\hat{\delta}) \cdot \mathcal{O}_\nu(\hat{\delta}) = \sum_\rho D^\rho_{\mu\nu} \mathcal{O}_\rho(\hat{\delta}), \tag{2.94}$$

where $D^\rho_{\mu\nu}$ are the K-theoretic Littlewood-Richardson coefficients [115]. However, viewed as operators acting on $\tilde{I}^{SQK}_{\mathrm{Gr}(M,N)}$, as soon as operators $\hat{p}^D_a$ act on $\tilde{I}^{SQK}_{\mathrm{Gr}(M,N)}$, where D is given by (2.86), we must employ the quantum relations (2.85). Taking the Vandermonde factor Δ into account, we must quotient out any q-difference operators $\mathfrak{D}$, whose product $\Delta \cdot \mathfrak{D}$ with Δ lies in $\langle \mathfrak{L}_{Q_a,a} \rangle_{a=1,\dots,M}$. In other words, we are taking the quotient by the ideal

[9] $I_\mu(Q,q,\epsilon)$ is not to be confused with $I_\mu(Q,q)$ from $I(Q,q,\epsilon) = \sum_\mu I_\mu(Q,q)\sigma_\mu(\epsilon)$ or related notions.

quotient $\langle \mathfrak{L}_{Q_a,a} \rangle_{a=1,\ldots,M} : \langle \Delta \rangle$ (see footnote 8). Since the Vandermonde factor Δ is of degree $M-1$ in each $\hat{p}_a$, we find that the Grothendieck (or other) polynomials must have individual powers up to $D-M$ in each operator $\hat{p}_a$ or $\hat{\delta}_a$. The generating set of Wilson loops $\hat{W}'_\mu$ is thus restricted to lie in the box

$$\mu \subseteq \mathcal{B}_{M\times(D-M)}, \tag{2.95}$$

that is, the Wilson loops associated to larger partitions are set to zero. In total we have a (Chern-Simons level dependent) number of independent Wilson loops given by

$$\binom{D}{M} = \operatorname{tr}(-)^F, \tag{2.96}$$

which is expected to coincide with the 3D Witten index [116, 117]. Furthermore, the structure constants are modified so that

$$\mathcal{O}_\mu(\hat{\delta}) \star \mathcal{O}_\nu(\hat{\delta}) = \sum_{\rho\subseteq\mathcal{B}_{M\times(D-M)}} D^\rho_{\mu\nu}(Q,q,\vec{\kappa})\, \mathcal{O}_\rho(\hat{\delta}), \tag{2.97}$$

as operators acting on $\tilde{I}^{SQK}_{\mathrm{Gr}(M,N)}$, where we indicate by $\star$ the multiplication of q-difference operators upon taking the quotient by the relations $\mathfrak{L}_{Q_a,a}$ (2.85), and setting $Q_a = Q$ in the end.

Extracting the Wilson loop algebra

The algebra of q-difference operators is not the same as the algebra of Wilson loops: the Wilson loops "should" only act on the I-function, while the q-difference operators in $\hat{\delta}_a$'s will also act on the Q_a-factors coming from $\mathfrak{L}_{Q_a,a}$. To extract the action on $\hat{I}$ only, note that

$$\hat{\delta}\big(f(Q)\hat{I}\big) = (\hat{\delta}f)\hat{I} + f(qQ)\hat{\delta}\hat{I} = f(Q)\hat{\delta}\hat{I} + \mathcal{O}(1-q), \tag{2.98}$$

for a meromorphic function f. This computation shows that the structure constants of the Wilson loop algebra

$$\hat{W}'_\mu * \hat{W}'_\nu = \sum_{\rho\subseteq\mathcal{B}_{M\times(D-M)}} d^\rho_{\mu\nu}(Q,\vec{\kappa})\, \hat{W}'_\rho(\hat{\delta}) \tag{2.99}$$

are given simply by the leading coefficient $(1-q)^0$ in the constants from (2.97), so that

$$d^\rho_{\mu\nu}(Q,\vec{\kappa}) = D^\rho_{\mu\nu}(Q,1,\vec{\kappa}). \tag{2.100}$$

2.2.5 Algebras for different Chern-Simons levels

In this subsection we specialize the general discussion from the previous subsection to specific Chern-Simons levels $\hat{\kappa}_S, \hat{\kappa}_A, \hat{\kappa}_R$. In the last example, $\hat{\kappa}_S, \hat{\kappa}_A, \hat{\kappa}_R = 0$, to be discussed in the next subsection, we will argue that the algebra of the theory with canonical effective Chern-Simons levels (2.24) is isomorphic to the ordinary small quantum K-theory [53, 54] of complex Grassmannians computed e.g., in [118].

Factorized case: $\gamma = 0$

When $\gamma = 0$ (2.72), the Abelianized ideal (2.82) of q-difference operators factorizes,

$$\mathfrak{L}_{Q_r,r}(\alpha,\beta,0) = (1-\hat{p}_r)^N + q^{\frac{\alpha}{2}+\beta} Q_r \hat{p}_r^{\alpha}, \qquad r = 1,\dots,M, \tag{2.101}$$

reflecting that also the I-function (2.70) factorizes

$$I^{SQK}_{\mathrm{Gr}(M,N)}(\vec{Q},q,\epsilon) = c_0 \Big[\Delta \cdot \prod_{a=1}^{M} I^{\alpha,\beta}_{\mathrm{Gr}(1,N)}(Q_a,q,\epsilon_a)\Big]_{Q_a=Q}. \tag{2.102}$$

The relations match those obtained in [20] on an S^3 background.

Quantum cohomology/Verlinde algebra: $\alpha = \beta = \gamma = 0$

We now consider the subcase of the factorized case with CS levels $\alpha = \beta = \gamma = 0$, or

$$\hat{\kappa}_S = -M, \qquad \hat{\kappa}_A = 0, \qquad \hat{\kappa}_R = \frac{M-1}{2}. \tag{2.103}$$

These levels are in the Chern-Simons window (2.76) discussed earlier, and the I-function is unperturbed. The ideal relations (2.82) become, expressed in the operator $\hat{\delta}_a$:

$$\hat{\delta}_a^N = -Q_a. \tag{2.104}$$

These relations are the same relations as in the "symplectic quotient" presentation of quantum cohomology $QH^*\big(\mathrm{Gr}(M,N)\big)$ (B.17) studied also in [42, 43, 119] (up to a sign convention, and $Q_a = Q$). According to [120], this algebra is also isomorphic to the Verlinde algebra [121] for the gauged Wess-Zumino-Novikov-Witten model $U(M)/U(M)$ at level $N-M$. In this model we find a member of the family deformed Wilson loop algebras isomorphic to this algebra (see also [90] for a connection between K-theory correlators and Verlinde numbers).

2D limit of all Wilson loop algebras

We can also consider the 2D limit of our three-dimensional gauge theory by shrinking tha radius $\beta \to 0$ of the S^1 factor. Taking the limit carefully, $q = e^{-\hbar\beta}$ will be sent to 1, the (rescaled) K-theory Chern roots $P_a = 1 - q^{-\epsilon_a} \to \beta\epsilon$. The Fayet-Iliopoulos parameter Q gets renormalized to $Q_a \to Q_a^{2D}\beta^N$ [67]. For *any* Chern-Simons levels α,β,γ the q-difference relations (2.82) become in the above 2D limit the *differential* relations

$$(\beta Q_a \tfrac{\partial}{\partial Q_a})^N = -\beta^N Q_a^{2D}. \tag{2.105}$$

This reproduces once again the relation (B.17) of quantum cohomology, so the three-parameter family of Wilson loop algebras collapse in the 2D limit to a single algebra.

2.2.6 Wilson loop algebra for level zero is quantum K-theory

Finally, we disuss the "canonical" case with zero effective levels $\hat{\kappa}_S, \hat{\kappa}_A, \hat{\kappa}_R = 0$ or equivalently $\alpha = M-1$, $\beta = -\frac{M-1}{2}$, $\gamma = -1$. Again, the I-function (2.55) takes the form

$$I^{SQK}_{\mathrm{Gr}(M,N)}(Q,q,\epsilon) = \big[\Delta \cdot \tilde{I}^{SQK}_{\mathrm{Gr}(M,N)}(\vec{Q},q,\epsilon)\big]_{Q_a=Q} \tag{2.106}$$

with $\tilde{I}^{SQK}_{\mathrm{Gr}(M,N)}$ as in (2.70), $\Delta = \prod_{a<b}(\hat{p}_a - \hat{p}_b)$ and effective CS levels zero. Furthermore, it precisely matches the K-theoretic small J-functions for Grassmannians from the mathematics literature [86, 88, 89] modulo an overall $(-Q)^{-\sum_{a=1}^{M}\epsilon_a}$ factor. We assert that

> The algebra of Wilson loops at zero effective Chern-Simons levels is isomorphic to small quantum K-theory of $\mathrm{Gr}(M,N)$.

Let us spell out the computation of the product $*$ (2.99) of two Wilson loops $\hat{W}'_\mu, \hat{W}'_\nu$ as defined in (2.92). The q-difference ideal (2.85) that annihilates $\hat{I}^{SQK}_{\mathrm{Gr}(M,N)}$ is generated by

$$\mathfrak{L}^{\vec{\kappa}=0}_{Q_a,a} = (1-\hat{p}_a)^N + Q_a \frac{\hat{p}_a^M}{\prod_{b=1}^{M}\hat{p}_b}, \qquad a = 1,\dots,M. \tag{2.107}$$

As discussed around (2.87) we write this relation into two equivalent, but useful forms:

$$\hat{p}_a^N = (-1)^{N+1} Q_a \frac{\hat{p}_a^M}{\prod_{b=1}^{M}\hat{p}_b} + (-1)^{N+1}\sum_{k=0}^{N-1}\binom{N}{k}(-\hat{p}_a)^k, \tag{2.108a}$$

$$\frac{1}{\hat{p}_a} = -Q_a \frac{\hat{p}_a^{M-1}}{\prod_{b=1}^{M}\hat{p}_b} + \sum_{k=0}^{N-1}\binom{N}{k+1}(-\hat{p}_a)^k. \tag{2.108b}$$

These allow us to apply the "*ideal replacement algorithm*"

1. replace monomials of order N or higher in $\hat{p}$ with a polynomial in $\hat{p}$ with lower degree, possibly with denominators and

2. replace denominators that don't cancel with a polynomial in $\hat{p}$ with lower degree, possibly with new denominators.

3. repeat as necessary and set $Q_a \mapsto Q$ in the end,

until both "high" powers and denominators have been cleared. The product $\hat{W}'_\mu * \hat{W}'_\nu$ is then computed as follows

Wilson loop product algorithm

1. Assign $\hat{W}'_\mu * \hat{W}'_\nu \mapsto \mathcal{O}_\mu(\hat{\delta}) \cdot \mathcal{O}_\nu(\hat{\delta}) = P_{\mu\nu}(\hat{p})$, $\hat{\delta}_a = 1 - \hat{p}_a$ where on the right classical multiplication $\cdot$ of Grothendieck polynomials is used. Note that $P_{\mu\nu}$ is a symmetric polynomial in $\hat{p}$'s (and hence also $\hat{\delta}$'s).
2. Multiply $P_{\mu\nu}(\hat{p})$ by $\Delta = \prod_{a<b}(\hat{p}_a - \hat{p}_b)$.
3. Apply the *ideal replacement algorithm*, resulting in a totally antisymmetric polynomial $A_{\mu\nu}(\hat{p})$.
4. Decompose
$$\frac{A_{\mu\nu}(\hat{p})}{\Delta} = \sum_{\rho \subseteq \mathcal{B}_{M\times(N-M)}} D^\rho_{\mu\nu}(Q)\mathcal{O}_\rho(\hat{\delta}), \tag{2.109}$$
to obtain the structure constants
$$\hat{W}'_\mu * \hat{W}'_\nu = \sum_{\rho \subseteq \mathcal{B}_{M\times(N-M)}} D^\rho_{\mu\nu}(Q)\hat{W}'_\rho. \tag{2.110}$$

Notice that the ordering subtlety discussed around (2.98) is already resolved in the replacement algorithm: difference operators are treated like formal commuting variables and do not act on the Q_a's.

An example in $\mathrm{Gr}(2,4)$

Let us work through an example in the simplest non-trivial complex Grassmannian $\mathrm{Gr}(2,4)$. The basis of Wilson loops is given by the Grothendieck polynomials

$$\begin{aligned} &\mathcal{O}_0 = 0, \quad \mathcal{O}_1(\hat{\delta}) = \hat{\delta}_1 + \hat{\delta}_2 - \hat{\delta}_1\hat{\delta}_2, \quad \mathcal{O}_2(\hat{\delta}) = \hat{\delta}_1^2 + \hat{\delta}_2^2 + \hat{\delta}_1\hat{\delta}_2 - \hat{\delta}_1\hat{\delta}_2^2 - \hat{\delta}_1^2\hat{\delta}_2, \\ &\mathcal{O}_{1,1}(\hat{\delta}) = \hat{\delta}_1\hat{\delta}_2, \quad \mathcal{O}_{2,1}(\hat{\delta}) = \hat{\delta}_1\hat{\delta}_2^2 + \hat{\delta}_1^2\hat{\delta}_2 - \hat{\delta}_1^2\hat{\delta}_2^2, \quad \mathcal{O}_{2,2}(\hat{\delta}) = \hat{\delta}_1^2\hat{\delta}_2^2 \end{aligned} \tag{2.111}$$

Computing e.g., the product $\hat{W}'_1 * \hat{W}'_{2,1}$ we have with $\hat{\delta}_a = 1 - \hat{p}_a$:

1. $\mathcal{O}_1(\hat{\delta}) \cdot \mathcal{O}_{2,1}(\hat{\delta}) = P_{(1),(2,1)}(\hat{p}) = 1 - \hat{p}_1 - \hat{p}_2 - \hat{p}_1\hat{p}_2 + 2\hat{p}_1^2\hat{p}_2 + 2\hat{p}_1\hat{p}_2^2 - \hat{p}_1^2\hat{p}_2^2 - \hat{p}_1^3\hat{p}_2^2 - \hat{p}_1^2\hat{p}_2^3 + \hat{p}_1^3\hat{p}_2^3$.
2. $\Delta \cdot P_{(1),(2,1)} = -\hat{p}_1 + \hat{p}_1^2 + \hat{p}_1^2\hat{p}_2 - 2\hat{p}_1^3\hat{p}_2 + \hat{p}_1^3\hat{p}_2^2 + \hat{p}_1^4\hat{p}_2^2 - \hat{p}_1^4\hat{p}_2^3 - (1 \leftrightarrow 2)$.
3. Ideal replacement $\hat{p}_{1/2}^4 = -1 + 4\hat{p}_{1/2} - 6\hat{p}_{1/2}^2 + 4\hat{p}_{1/2}^3 - Q\frac{\hat{p}_{1/2}}{\hat{p}_{2/1}}$ and for denominators that do not cancel: $\frac{\hat{p}_{2/1}}{\hat{p}_{1/2}} = -Q + 4\hat{p}_{2/1} - 6\hat{p}_1\hat{p}_2 + 4\hat{p}_{1/2}^2\hat{p}_{2/1} - \hat{p}_{1/2}^3\hat{p}_{2/1}$. We get

$$\begin{aligned} \Delta \cdot P_{(1),(2,1)}(\hat{p}) &= -\hat{p}_1 + \hat{p}_1^2 + \hat{p}_1^2\hat{p}_2 - 2\hat{p}_1^3\hat{p}_2 + \hat{p}_1^3\hat{p}_2^2 + (\hat{p}_2^2 - \hat{p}_2^3)\hat{p}_1^4 - (1 \leftrightarrow 2) \\ &= -\hat{p}_1 + \hat{p}_1^2 + \hat{p}_1^2\hat{p}_2 - 2\hat{p}_1^3\hat{p}_2 + \hat{p}_1^3\hat{p}_2^2 \\ &\quad + (\hat{p}_2^2 - \hat{p}_2^3)(-1 + 4\hat{p}_1 - 6\hat{p}_1^2 + 4\hat{p}_1^3 - Q\tfrac{\hat{p}_1}{\hat{p}_2}) - (1 \leftrightarrow 2) \\ &= -\hat{p}_1 + \hat{p}_1^2 - \hat{p}_2^2 + \hat{p}_1^2\hat{p}_2 + 4\hat{p}_1\hat{p}_2^2 + \hat{p}_2^3 - 6\hat{p}_1^2\hat{p}_2^2 - 2\hat{p}_1^3\hat{p}_2 - 4\hat{p}_1\hat{p}_2^3 \\ &\quad + 5\hat{p}_1^3\hat{p}_2^2 + 6\hat{p}_1^2\hat{p}_2^3 - 4\hat{p}_1^3\hat{p}_2^3 + (-\hat{p}_1\hat{p}_2 + \hat{p}_1\hat{p}_2^2)Q - (1 \leftrightarrow 2) \\ &= A_{(1),(2,1)}(\hat{p}) \end{aligned}$$

4. We write $A_{(1),(2,1)}(1-\hat\delta) = (\hat\delta_1 - \hat\delta_2)\big(\hat\delta_1^2\hat\delta_2^2 + Q(1-\hat\delta_1-\hat\delta_2+\hat\delta_1\hat\delta_2)\big)$, so

$$\frac{A_{(1),(2,1)}(1-\hat\delta)}{\Delta} = \mathcal{O}_{2,2}(\hat\delta) + Q\big(\mathcal{O}_0(\hat\delta) - \mathcal{O}_1(\hat\delta)\big) \tag{2.112}$$

so we find in $\mathrm{Gr}(2,4)$ that

$$\mathcal{O}_1 * \mathcal{O}_{2,1} = \mathcal{O}_{2,2} + Q(\mathcal{O}_0 - \mathcal{O}_1). \tag{2.113}$$

In a similar fashion, we obtain the multiplication table for all generators of $\mathrm{Gr}(2,4)$:

$$\begin{aligned}
&\mathcal{O}_1 * \mathcal{O}_1 = \rho, && \mathcal{O}_1 * \mathcal{O}_2 = \mathcal{O}_{2,1},\\
&\mathcal{O}_1 * \mathcal{O}_{1,1} = \mathcal{O}_{2,1}, && \mathcal{O}_1 * \mathcal{O}_{2,1} = \mathcal{O}_{2,2} + Q(\mathcal{O}_0 - \mathcal{O}_1),\\
&\mathcal{O}_1 * \mathcal{O}_{2,2} = Q\mathcal{O}_1, && \mathcal{O}_2 * \mathcal{O}_2 = \mathcal{O}_{2,2},\\
&\mathcal{O}_2 * \mathcal{O}_{1,1} = Q\mathcal{O}_0, && \mathcal{O}_2 * \mathcal{O}_{2,1} = Q\mathcal{O}_1,\\
&\mathcal{O}_2 * \mathcal{O}_{2,2} = Q\mathcal{O}_{1,1}, && \mathcal{O}_{1,1} * \mathcal{O}_{1,1} = \mathcal{O}_{2,2},\\
&\mathcal{O}_{1,1} * \mathcal{O}_{2,1} = Q\mathcal{O}_1, && \mathcal{O}_{1,1} * \mathcal{O}_{2,2} = Q\mathcal{O}_2,\\
&\mathcal{O}_{2,1} * \mathcal{O}_{2,1} = Q\rho, && \mathcal{O}_{2,1} * \mathcal{O}_{2,2} = Q\mathcal{O}_{2,1},\\
&\mathcal{O}_{2,2} * \mathcal{O}_{2,2} = Q^2\mathcal{O}_0, && \rho = \mathcal{O}_{1,1} - \mathcal{O}_{2,1} + \mathcal{O}_2.
\end{aligned} \tag{2.114}$$

This multiplication table matches the ones obtained from the the "quantum" Pieri and Giambelli formulas defined by Buch and Mihalcea in [118], which use quite different arguments. We print the multiplication tables for $\mathrm{Gr}(2,5) \cong \mathrm{Gr}(3,5)$ and $\mathrm{Gr}(2,6) \cong \mathrm{Gr}(4,6)$ in the appendix B.3.

2.2.7 q-difference structure for level zero

In this subsection we discuss the q-difference equations that the I-function (2.55) satisfies. As discussed in the introduction 1.2.4, the big quantum multiplication and the flatness equations for sections (1.39) imply some differential and q-difference operators annihilate the endomorphism T implicitly defining the J-function (1.47). In case the J-function is unperturbed, it coincides with the I-function, and the small quantum multiplication $*$ is enough to determine the q-difference equations for $J(0) = I(0)$. To see this, given the equation (1.45)

$$q^\theta T = TAq^\theta, \qquad \theta = Q\frac{\partial}{\partial Q}, \tag{2.115}$$

we may compare against

$$\begin{aligned}
(1-q)q^\theta T|_{t=0}\cdot\Phi_0 &= q^\theta J(0)\\
&= [q^{\sum_{a=1}^M \theta_a}\hat I]_{Q_a=Q}\\
&= [\big(1-\mathcal{O}_1(\hat\delta)\big)\hat I]_{Q_a=Q}\\
&= \big(1-\Phi_1\big) * J(0),
\end{aligned} \tag{2.116}$$

where we have used $\theta_a = Q_a\frac{\partial}{\partial Q_a}$, the Abelianization $\hat I$ from (2.64) and the identity of Grothendieck polynomials $1-\mathcal{O}_1(x) = \prod_a(1-x_a)$. This shows that, as an element of $\mathrm{End}\, K(X)\otimes\mathbb{C}[[Q,t]]$[10]

$$A_{t=0} = \big((1-\mathcal{O}_1)\bullet\big)_{t=0} = (1-\mathcal{O}_1)*. \tag{2.117}$$

[10]See footnote 7 for notation.

In other words, a relation of the form

$$\sum_{k=0}^{d} b_k \mathcal{D}_q^k \mathcal{O}_0 = 0, \qquad \mathcal{D}_q = \big((1-\mathcal{O}_1)\,*\,\big)q^{\theta} \tag{2.118}$$

for some degree d, implies by (2.115) that the J-function satisfies the *quantum q-difference equation*

$$\Big(\sum_{k=0}^{d} b_k (q^{\theta})^k\Big) J(0) = 0. \tag{2.119}$$

This is in complete analogy to quantum cohomology, where every quantum differential equation implies relations in quantum cohomology [25] (but not the other way around).

The case with effective CS levels zero $\hat{\kappa}_S, \hat{\kappa}_A, \hat{\kappa}_R = 0$ from the previous subsection is "firmly" in the CS window (2.76) and the I-function (2.55) for these levels is unperturbed (see also direct proof of the last assertion in [89]). We may thus compute sufficiently high powers of $\mathcal{D}_q = (1-\mathcal{O}_1) * q^{\theta}$, where $*$ is the product of Wilson loops/quantum K-theory product from the previous subsection. Relations between the powers will then lead to q-difference relations for $J(0)$.

$\mathrm{Gr}(2,4)$ as an example and its 2D limit

As an example, let us consider the simplest non-trivial case of $\mathrm{Gr}(2,4)$. We can compute from (2.114) that

$$\begin{aligned}
\mathcal{D}_q^0 \cdot \mathcal{O}_0 &= 1,\\
\mathcal{D}_q^1 \cdot \mathcal{O}_0 &= 1-\mathcal{O}_1,\\
\mathcal{D}_q^2 \cdot \mathcal{O}_0 &= 1-2\mathcal{O}_1+\mathcal{O}_2+\mathcal{O}_{1,1}-\mathcal{O}_{2,1},\\
\mathcal{D}_q^3 \cdot \mathcal{O}_0 &= 1+Q-(3+Q)\mathcal{O}_1+3\mathcal{O}_2+3\mathcal{O}_{1,1}-5\mathcal{O}_{2,1}+\mathcal{O}_{2,2},\\
\mathcal{D}_q^4 \cdot \mathcal{O}_0 &= 1+(5+q)Q-\big(4+(6+2q)Q\big)\mathcal{O}_1+(6+qQ)\mathcal{O}_2\\
&\quad +(6+qQ)\mathcal{O}_{1,1}-(14+qQ)\mathcal{O}_{2,1}+6\mathcal{O}_{2,2},\\
\mathcal{D}_q^5 \cdot \mathcal{O}_0 &= 1+(14+5q+q^2)Q+q^2Q^2-\big(5+(20+11q+3q^2)Q+q^2Q^2\big)\mathcal{O}_1\\
&\quad +\big(10+(6q+3q^2)Q\big)\mathcal{O}_2+\big(10+(6q+3q^2)Q\big)\mathcal{O}_{1,1}\\
&\quad -\big(30+(6q+5q^2)Q\big)\mathcal{O}_{2,1}+(20+q^2Q)\mathcal{O}_{2,2}.
\end{aligned} \tag{2.120}$$

With some computer assistance we find that

$$\big[(1-\hat{p})^5+Q(1+q\hat{p})(q\hat{p}^2-1)\big] I^{SQK}_{\mathrm{Gr}(2,4)}(Q,q,\epsilon) = 0 \tag{2.121}$$

with $\hat{p} = q^{\theta}$, up to terms of order $\mathcal{O}(\epsilon^4)$. In a similar fashion we obtain quantum q-difference equations for low-dimensional Grassmannians, which are printed in the appendix (B.3).

We can also take the 2D limit of the q-difference equations as discussed in 2.2.5, by $q = e^{-\beta}$ (setting $\hbar = 1$), $\hat{p} = e^{-\beta\theta}$ and $Q = Q^{2D}\beta^N$ and taking the leading term in β. We find

$$\mathcal{D}_{\mathrm{Gr}(2,4)} = \theta^5 - 2Q^{2D}(1+2\theta). \tag{2.122}$$

This is precisely quantum differential operator annihilating the cohomological I-function for $\mathrm{Gr}(2,4)$, computed in [122].

2.3 Perturbed theory and the big quantum K-theory

In this section we will discuss the perturbation of the original theory by massive fields, as well as pairings in quantum K-theory and their interpretation.

2.3.1 Perturbed theory and reconstruction

We recall our discussion from subsection 2.2.2: an unperturbed theory defined in the UV leads, in general, to a perturbed theory in the IR. The non-trivial identification between UV and IR observables is the 3D analog of the *mirror map* between coupling parameters.

The mirror map is non-trivial precisely when the input $t = I|_{\mathcal{K}_+} - (1-q)$ of the permutation-symmetric I-function is non-zero. The input of the I-function computed by the gauge theory may be modified, e.g., to zero, by integrating in appropriate massive fields in the UV. In addition, the different choices of massive fields can be used to completely characterize the space of deformations in the IR. Let us briefly sketch this procedure following the proposals in [67].

The I-function (2.55) is of the form (ignoring the obstruction to factorization (2.70), and a global $(-Q)^{-\sum \epsilon_a}$-factor)

$$I(Q,q,\epsilon) \sim \Big[\Delta \cdot \sum_{\vec{d}=0}^{\infty} I_{\vec{d}}\,(-\vec{Q})^{\vec{d}}\Big]_{Q_a=Q}. \tag{2.123}$$

It is computed, for example when the gauge fields satisfy Neumann boundary conditions on the T^2 boundary, via supersymmetric localization as described in subsection 2.1.1. The summation over $\vec{d}$ comes from the multi-dimensional Jeffrey-Kirwan integral over Wilson lines z_a, to which the infinite dimensional path integral reduces (see (2.1)). Now we consider deforming the original gauge theory, by integrating in a collection of massive Neumann chiral fields in some representation $U(M)$ labeled by a diagram μ, and mass parameter ζ. The computation of the partition function of the deformed theory follows similar steps as the ones outlined in section 2.1.1. The chiral fields will schematically contribute a factor

$$\frac{1}{(z^{-\mu}\zeta_\mu;q)_\infty}, \tag{2.124}$$

to the integral, where the notation $f(z^\mu)$ for a Young diagram μ implies an appropriate symmetrization $\sum_s \prod_a f(z_a^{\mu_{s(a)}})$. As with the insertion of Wilson line operators (2.81), the factor from the massive fields may be reproduced by acting with a q-difference operator on the unperturbed, Abelianized I-function, and taking the limit $Q_a = Q$ in the end. In other words a collection of insertions will act as

$$\begin{aligned} I(\zeta,Q,q,\epsilon) &= \Big[\frac{1}{(\zeta_\mu P^\mu q^{\mu\theta};q)_\infty} I(Q_a,q,\epsilon)\Big]_{Q_a=Q} \\ &= \Big[\exp\Big(\sum_{r=1}^{\infty} \frac{\zeta_\mu^r (P^\mu q^{\mu\theta})^r}{r(1-q^r)}\Big)\cdot I(Q_a,q,\epsilon)\Big]_{Q_a=Q} \end{aligned} \tag{2.125}$$

where $P_a = q^{-\epsilon_a}$ accounts for the 'missing' $(-Q)^{-\sum \epsilon_a}$-factor and on the right-hand side we have used (A.5). This deformation, after a refinement of ζ_μ to a matrix of mass parameters Z_μ, corresponds to so called *multi-trace deformations* [67], generated by a q-difference operator

$$\mathcal{R}_{\mathrm{MT}}(\zeta) = \exp\Big(\sum_{r=1}^{\infty} \frac{\operatorname{tr} Z(\zeta_\mu)^r (P^\mu q^{\mu\theta})^r}{r(1-q^r)}\Big). \tag{2.126}$$

The specialization to the $r = 1$ term in the exponential is referred to as a *single-trace deformation*, generated by a q-difference operator depending on twisted masses ρ

$$\mathcal{R}_{\mathrm{ST}}(\rho) = \exp\Big(\frac{\operatorname{tr} R(\rho_\mu)(P^\mu q^{\mu\theta})}{1-q}\Big). \tag{2.127}$$

In the context of the 3D gauge theory/quantum K-theory correspondence, these difference operators were identified in [67] with Givental's reconstruction operators [59, p. VIII] (see also other reconstruction results in [56, 123]). In particular, integration of multi-trace operators reproduces the permutation-equivariant reconstruction operator $\mathcal{R}_{\mathrm{eq}}\big(\zeta(t_{\mathrm{eq}})\big)$, while integration of single-trace operators reproduces the ordinary reconstruction operator $\mathcal{R}_{\mathrm{ord}}\big(\rho(t_{\mathrm{ord}})\big)$. The operators 'reconstruct' the permutation-equivariant or ordinary J-function respectively at non-zero (equivariant or ordinary) input[11] from the unperturbed J-function. For example, for the ordinary J-function

$$J(Q,q,t) = \mathcal{R}_{\mathrm{ord}}\big(\rho(t)\big) J(Q,q,0), \tag{2.128}$$

and similarly for the equivariant case.

To sum up this subsection, we reiterate the most important points. The partition function $Z(\zeta,\rho)$ of a general UV gauge theory with multi- and single-trace deformations computes the I-function with non-trivial *mixed* permutation-symmetric and ordinary input $t_{\mathrm{ord}} + t_{\mathrm{eq}}$. Choosing different deformations amounts to acting with appropriate reconstruction difference operators on the I-functions, and we can "map out" the spaces of these deformations. For special initial theories, namely for those for which the CS terms are inside the "CS window" described in subsection 2.2.2, the input is zero. In that case we can map-out the space of ordinary deformations solely by acting with ordinary reconstruction operators. We describe this in the following subsection.

2.3.2 A technical description for ordinary reconstruction

Starting from an unperturbed J-function $J(0)$ we may perturb it to a J-function with an *ordinary* non-zero input $J(t)$, where

$$t = \sum_{\mu \in \mathcal{B}_{M\times(N-M)}} t_\mu \mathcal{O}_\mu \ \in \mathcal{K}_+, \tag{2.129}$$

The coefficients t_μ a priori lie in $\mathbb{C}[q,q^{-1}] \otimes \mathbb{C}[[Q]]$; however, the physical correlators are encoded in the J-function with input $t \in K(X)$ (or more generally, $K(X)[q,q^{-1}]$, which we do not consider here) and hence we must only perturb such that $t_\mu \in \mathbb{C}$. This can be done recursively in powers of Q, by acting with the *ordinary* reconstruction operator $\hat{\mathcal{R}}$

$$J(t) = \big[\hat{\mathcal{R}}\big(\rho(Q,q,t)\big) \cdot J_{Q_a}(0)\big]_{Q_a=Q}, \tag{2.130}$$

[11] We are omitting a more detailed discussion of the mathematical reconstruction theorems, which would require discussing also Givental's *ruling spaces*, *fake* quantum K-theory, *overruled cones* and their *adelic* characterization.

where

$$\hat{\mathcal{R}}\big(\rho(Q,q,t)\big) := \exp\Big[(1-q)^{-1}\sum_{\alpha\subseteq\mathcal{B}^*}\rho_\alpha\hat{\mathcal{O}}_\alpha\Big]. \tag{2.131}$$

The dependence of $\hat{\mathcal{R}}$ on t is implicit and complicated but can be computed recursively. Let us explain how this operator acts, determine the dependence on t and elaborate on important technical details in the process.

Firstly, the operators $\hat{\mathcal{O}}_\alpha$ are q-difference operators

$$\hat{\mathcal{O}}_\alpha = \mathcal{O}_\alpha(1-Pq^\theta), \tag{2.132}$$

where on the right-hand side we have the Grothendieck polynomial (see (B.29)) in the q-difference operators $1-P_aq^{\theta_a}$. The action of the operators is defined through the Abelianized I-function (2.64), but this is a mere technicality. As explained in [2], the procedure of integrating in massive fields can be performed in the non-Abelian $U(M)$ theory, with no need to "factor through" the Abelianized $U(1)^M$ version.

Secondly, note that we do not restrict the Young diagrams $\alpha\subseteq\mathcal{B}^*$ to lie in $\alpha\subseteq\mathcal{B}_{M\times(N-M)}$, as one would expect since $\dim K(X) = \binom{N}{M}$ = number of diagrams in $\mathcal{B}_{M\times(N-M)}$; in fact for a recursive computation it will suffice to take $\alpha\subseteq\mathcal{B}_{M\times N}$[12]. This is a technicality: the operators act on an Abelianized version J_{Q_a} of the J-function, matching (2.64) without the overall $\prod_{a=1}^M(-Q_a)^{-\epsilon_a}$-factor. The missing factor is important as the operation by $\hat{\mathcal{O}}_\mu$ is identical to the operation of $\mathcal{O}_\mu(\hat\delta)$ ($\hat\delta = 1-q^\theta$), when they are acting on $\hat{I}(Q_a)$ (2.64). To see this, note that for $P=q^{-\epsilon}$ and $\tilde d = d-\epsilon$, we have that

$$\begin{aligned}\mathcal{O}(\hat\delta)\cdot\sum_d(-\vec{Q})^{\vec{\tilde d}} &= \sum_d\mathcal{O}(1-q^{\tilde d})(-\vec{Q})^{\vec{\tilde d}}\\ &= (-\vec{Q})^{-\vec\epsilon}\cdot\Big(\sum_d\mathcal{O}(1-Pq^d)(-\vec{Q})^{\vec d}\Big)\\ &= (-\vec{Q})^{-\vec\epsilon}\cdot\mathcal{O}(1-Pq^\theta)\cdot\sum_d(-\vec{Q})^{\vec d},\end{aligned} \tag{2.133}$$

where $\vec{Q}^{\vec d} = \prod_{a=1}^M Q_a^{d_a}$. This sketch would suggest that the q-difference operators $\hat{\mathcal{O}}_\alpha$ may be substituted by (fewer) K-theory generators $\mathcal{O}_\mu *$ with $\mu\subseteq\mathcal{B}_{M\times(N-M)}$, where $*$ is the small quantum multiplication from (2.110). However, using q-difference operators as opposed to formal K-theory generators is more convenient: due to the relations (2.107) it is clear that the q-difference operators acting on $\hat I$ (2.64) identically satisfy

$$\mathcal{O}_\mu(\hat\delta)\cdot\mathcal{O}_\nu(\hat\delta)\hat{I} = \sum_\rho D^\rho_{\mu\nu}(Q)\mathcal{O}_\rho(\hat\delta)\hat{I} + \cdots\,. \tag{2.134}$$

Here, the suppressed higher order terms come from the repeated action of q-difference operators acting on Q_a-dependent coefficients. The labels are unrestricted and Grothendieck polynomials of "too large" Young diagrams are in fact expressible in terms of polynomials labeled by smaller, admissible diagrams with coefficients that involve higher powers of Q, so there is no contradiction with (2.110) (see proposal in [2] for details).

[12] In fact even fewer generators are needed: Working in the basis of "K-theoretic" Schur polynomials $\sigma_\beta(x^K)$ (B.27), we only need diagrams with $|\beta|\le\dim\mathrm{Gr}(M,N) = M(N-M)$. Expressing these in terms of Grothendieck polynomials $\mathcal{O}_\alpha$, we find we must allow labels up to $\alpha\subseteq\mathcal{B}_{M\times N}$ as above (but of course only $\dim\mathrm{Gr}(M,N)$-linearly independent combinations of these).

Lastly, ρ_α is valued

$$\rho_\alpha = \rho_\alpha(Q,q,t) \in \mathbb{C}[q,q^{-1}] \otimes \mathbb{C}[[Q]] = \mathcal{K}_+/K(X), \tag{2.135}$$

and we refer to it as 'testinput'. It depends on t in the sense that it depends on the t_μ's; furthermore it satisfies

$$\rho_\mu|_{Q^0} = t_\mu \text{ for } \mu \subseteq \mathcal{B}_{M\times(N-M)}, \quad \rho_\alpha|_{Q^0} = 0 \text{ for } \alpha \nsubseteq \mathcal{B}_{M\times(N-M)}. \tag{2.136}$$

This is the initial condition for the recursive computation in powers of Q for $\rho(Q,q,t)$ which we now describe.
To compute $\rho(Q,q,t)$ such that (2.130) holds for a given perturbation $t \in K(X)$ we have the *first step*

1. Set a testinput $\rho_\mu^{(0)} = t_\mu \in \mathbb{C}$ for $\mu \subseteq \mathcal{B}_{M\times(N-M)}$ and $\rho_\alpha^{(0)} = 0$ otherwise.

2. Compute
$$\left[\hat{\mathcal{R}}\big(\rho^{(0)}\big) \cdot J_{Q_a}(0)\right]_{Q_a=Q} = J(o^{(1)}), \tag{2.137}$$

3. Decompose
$$J(o^{(1)}) = 1 - q + o^{(1)} \mod \mathcal{K}_-, \tag{2.138}$$
to determine $o^{(1)} \in \mathcal{K}_+$. The "output" $o^{(1)}$ can be expanded in powers of Q and it satisfies
$$o^{(1)} = \sum_\mu t_\mu \mathcal{O}_\mu + \sum_\alpha \tilde{c}_\alpha^{(1)} \mathcal{O}_\alpha Q + \mathcal{O}(Q^2), \tag{2.139}$$
for some 'correction' coefficients $\tilde{c}_\alpha^{(1)} \in \mathbb{C}[q,q^{-1}]$[13].

4. The correction coefficients are subtracted from the new testinput: set
$$\rho_\alpha^{(1)} = \delta_\alpha{}^\mu t_\mu - \tilde{c}_\alpha^{(1)} Q. \tag{2.140}$$

The process can now be repeated for a *general step* labeled i, $i > 0$:

5. Compute
$$J(o^{(i+1)}) = \left[\hat{\mathcal{R}}\big(\rho^{(i)}\big) \cdot J_{Q_a}(0)\right]_{Q_a=Q}. \tag{2.141}$$
where the testinput is
$$\rho_\alpha^{(i)} = \delta_\alpha{}^\mu t_\mu - \sum_{k=1}^{i-1} c_\alpha^{(k)} Q^k + \tilde{c}_\alpha^{(i)} Q^i, \tag{2.142}$$
with correction coefficients $c_\alpha^{(k)}, \tilde{c}_\alpha^{(i)} \in \mathbb{C}[q,q^{-1}]$. The coefficient of Q^i is not fully corrected yet.

[13] The correction coefficients might be labeled by "too large" diagrams α. As explained in the previous paragraph, the 'generators' $\mathcal{O}_\alpha$ (expressed here as Grothendieck polynomials in $1-q^{-\epsilon_a}$'s) of such large diagrams are expressible (in quantum K-theory) in terms of admissible generators $\mathcal{O}_\mu$ (possibly increasing the powers of Q), but it is more convenient keep the output in this form.

6. Decompose
$$J(o^{(i+1)}) = 1 - q + o^{(i+1)} \mod \mathcal{K}_-, \tag{2.143}$$
where the output is 'corrected' up to Q^{i-1} in the sense that
$$o^{(i+1)} = \sum_\mu t_\mu \mathcal{O}_\mu + 0 + \sum_\alpha \tilde{c}'^{(i)}_\alpha Q^i \mathcal{O}_\alpha + \mathcal{O}(Q^{i+1}). \tag{2.144}$$

7. (Internal step at order Q^i:)
We set $\tilde{c}^{(i)}_\alpha \to \tilde{c}^{(i)}_\alpha - \tilde{c}'^{(i)}_\alpha$ and repeat step 5. and 6. until the output is corrected up to Q^i:
$$o^{(i+1)} = \sum_\mu t_\mu \mathcal{O}_\mu + 0 + \sum_\alpha \tilde{c}^{(i+1)}_\alpha Q^{i+1} \mathcal{O}_\alpha + \mathcal{O}(Q^{i+2}), \tag{2.145}$$

8. We set $c^{(i)}_\alpha = \tilde{c}^{(i)}_\alpha$, and then
$$\rho^{(i+1)}_\alpha = t_\mu - \sum_{k=1}^{i} c^{(k)}_\alpha Q^k + \tilde{c}^{(i+1)}_\alpha Q^i, \tag{2.146}$$
and we move to step $i+1$.

The process is terminated at some chosen order Q^λ. It is also useful to rescale $t_\mu \mapsto \tau \cdot t_\mu$, with τ a formal variable tracking the number of insertions: the coefficients of τ^n in $J(t)$ will be (linear combinations of) n insertions of t.

Example: Gr$(2,4)$

The level zero I-function matches the unperturbed J-function, i.e the function
$$J_{\mathrm{Gr}(2,4)}(0) = c_0 \sum_{d_1,d_2\geq 0} (-Q)^{\tilde{d}_1+\tilde{d}_2} \frac{q^{\frac{1}{2}\tilde{d}_{12}^2}(q^{\frac{1}{2}\tilde{d}_{12}} - q^{-\frac{1}{2}\tilde{d}_{12}})}{\prod_{a=1}^{2}\prod_{r=1}^{d_a}(1-q^{r-\epsilon_a})^4}, \tag{2.147}$$
where $\tilde{d}_{12} = \tilde{d}_1 - \tilde{d}_2$ and c_0 is given by (2.25), satisfies
$$J_{\mathrm{Gr}(2,4)}(0)|_{\mathcal{K}_+} = 1 - q. \tag{2.148}$$
We perturb it to a non-zero *ordinary* input $J(t)$,
$$t = t_1\mathcal{O}_1 + t_2\mathcal{O}_2 + t_{1,1}\mathcal{O}_{1,1} + t_{2,1}\mathcal{O}_{2,1} + t_{2,2}\mathcal{O}_{2,2} \;\in K(X), \tag{2.149}$$
with $t_\mu \in \mathbb{C}$ by applying the ordinary Givental reconstruction operator, with testinput $\rho(t)$ computed recursively: up to orders Q^1 and τ^2 (τ counts homogenous order it t_μ's):
$$\begin{aligned}\rho(Q,q,t) = {}& Q\tau^2\left(-t_2t_{2,1} - \tfrac{1}{2}t_2^2\right)\mathcal{O}_0 + \left(\tau t_1 + Q\tau^2(t_2(t_{2,1}-t_{2,2}) - \tfrac{1}{2}t_{2,1}^2 + \tfrac{t_2^2}{2})\right)\mathcal{O}_1\\ &+ \left(\tau t_2 + Q\tau^2(\tfrac{1}{2}t_{2,1}^2 - t_{2,1}t_{2,2})\right)\mathcal{O}_2 + \left(\tau t_{1,1} + Q\tau^2(\tfrac{1}{2}t_{2,1}^2 + t_2t_{2,2})\right)\mathcal{O}_{1,1}\\ &+ \left(\tau t_{2,1} + Q\tau^2(t_{2,1}t_{2,2} - \tfrac{1}{2}t_{2,1}^2)\right)\mathcal{O}_{2,1} + \tau t_{2,2}\mathcal{O}_{2,2}\\ &- \tfrac{1}{2}Q\tau^2t_{2,2}^2(\mathcal{O}_3 + \mathcal{O}_4 - \mathcal{O}_{3,1} - \mathcal{O}_{3,2} - \mathcal{O}_{3,3} + \mathcal{O}_{4,1} + \mathcal{O}_{4,2} + \mathcal{O}_{4,3} + \mathcal{O}_{4,4})\end{aligned} \tag{2.150}$$

Indeed, with this testinput we find up to terms above $Q^1\tau^2$ that

$$J_{\mathrm{Gr}(2,4)}(t) = \left[\hat{\mathcal{R}}(\rho)\cdot J_{\mathrm{Gr}(2,4),Q_a}(0)\right]_{Q_a=Q}, \tag{2.151}$$

where the perturbed J-function enjoys the expansion

$$J_{\mathrm{Gr}(2,4)}(t) = 1-q+t+J^{\mathrm{corr.}}_{\mathrm{Gr}(2,4)}(t)+\mathcal{O}(Q^2,\tau^3), \tag{2.152}$$

with

$$J^{\mathrm{corr.}}_{\mathrm{Gr}(2,4)}(t) = \sum_{\mu\subseteq\mathcal{B}_{M\times(N-M)}} J^{\mathrm{corr}\ \mu}_{\mathrm{Gr}(2,4)}\mathcal{O}_\mu. \tag{2.153}$$

We find, suppressing unnecessary labels, the correlator coefficients for the Grassmannian $\mathrm{Gr}(2,4)$:

$$\begin{aligned}
J^0 &= Q\Big[\tfrac{-q-1}{(q-1)^3}+\tau\big(\tfrac{(-q-1)t_1}{(q-1)^3}+\tfrac{t_2}{(q-1)^2}+\tfrac{t_{1,1}}{(q-1)^2}+\tfrac{t_{2,1}}{1-q}\big)\\
&\quad+\tau^2\big(\tfrac{(-q-1)t_1^2}{2(q-1)^3}+\big(\tfrac{t_2}{(q-1)^2}+\tfrac{t_{1,1}}{(q-1)^2}+\tfrac{t_{2,1}}{1-q}\big)t_1+\tfrac{t_2t_{1,1}}{1-q}\big)\Big],\\
J^1 &= Q\Big[-\tfrac{2q(q+2)}{(q-1)^4}+\tau\big(-\tfrac{q(q+3)t_1}{(q-1)^4}+\tfrac{qt_2}{(q-1)^3}+\tfrac{qt_{1,1}}{(q-1)^3}+\tfrac{t_{2,2}}{1-q}\big)\\
&\quad+\tau^2\big(-\tfrac{qt_1^2}{(q-1)^4}+\big(\tfrac{qt_{2,1}}{(q-1)^2}+\tfrac{t_{2,2}}{1-q}\big)t_1+\tfrac{t_2^2}{2(q-1)^2}+\tfrac{t_{1,1}^2}{2(q-1)^2}+\tfrac{t_2t_{2,1}}{1-q}+\tfrac{t_{1,1}t_{2,1}}{1-q}\big)\Big],\\
J^2 &= -\tfrac{\tau^2t_1^2}{2(q-1)}+Q\Big[-\tfrac{q(3q^2+8q-1)}{(q-1)^5}+\tau\big(-\tfrac{q(q^2+4q-1)t_1}{(q-1)^5}+\tfrac{qt_2}{(q-1)^3}+\tfrac{qt_{1,1}}{(q-1)^3}+\tfrac{qt_{2,1}}{(q-1)^3}-\tfrac{qt_{2,2}}{(q-1)^2}\big)\\
&\quad+\tau^2\big(-\tfrac{qt_1^2}{2(q-1)^4}+\big(-\tfrac{qt_2}{(q-1)^4}-\tfrac{qt_{1,1}}{(q-1)^4}+\tfrac{qt_{2,1}}{(q-1)^3}\big)t_1+\tfrac{qt_2^2}{2(q-1)^3}+\tfrac{qt_{1,1}^2}{2(q-1)^3}-\tfrac{t_{2,1}^2}{2(q-1)}+\tfrac{t_{1,1}t_{2,2}}{1-q}\big)\Big],\\
J^{1,1} &= -\tfrac{\tau^2t_1^2}{2(q-1)}+Q\Big[-\tfrac{q(3q^2+8q-1)}{(q-1)^5}+\tau\big(-\tfrac{q(q^2+4q-1)t_1}{(q-1)^5}+\tfrac{qt_2}{(q-1)^3}+\tfrac{qt_{1,1}}{(q-1)^3}+\tfrac{qt_{2,1}}{(q-1)^3}-\tfrac{qt_{2,2}}{(q-1)^2}\big)\\
&\quad+\tau^2\big(-\tfrac{qt_1^2}{2(q-1)^4}+\big(-\tfrac{qt_2}{(q-1)^4}-\tfrac{qt_{1,1}}{(q-1)^4}+\tfrac{qt_{2,1}}{(q-1)^3}\big)t_1+\tfrac{qt_2^2}{2(q-1)^3}+\tfrac{qt_{1,1}^2}{2(q-1)^3}-\tfrac{t_{2,1}^2}{2(q-1)}+\tfrac{t_2t_{2,2}}{1-q}\big)\Big],\\
J^{2,1} &= \big(\tfrac{t_1^2}{2(q-1)}+\big(\tfrac{t_2}{1-q}+\tfrac{t_{1,1}}{1-q}\big)t_1\big)\tau^2\\
&\quad+Q\Big[-\tfrac{q(5q^3+19q^2-3q-1)}{(q-1)^6}\\
&\quad+\tau\big(\tfrac{q(q^2-4q-1)t_2}{(q-1)^5}-\tfrac{q(q^2+6q+1)t_1}{(q-1)^5}+\tfrac{q(q^2-4q-1)t_{1,1}}{(q-1)^5}+\tfrac{q(3q+1)t_{2,1}}{(q-1)^4}-\tfrac{q(q+1)t_{2,2}}{(q-1)^3}\big)\\
&\quad+\tau^2\big(\tfrac{q(q^2+6q+1)t_1^2}{2(q-1)^6}-\tfrac{q(3q+1)t_1t_2}{(q-1)^5}-\tfrac{q(3q+1)t_1t_{1,1}}{(q-1)^5}+\tfrac{q(q+1)t_1t_{2,1}}{(q-1)^4}\\
&\qquad+\tfrac{q(q+1)t_2^2}{2(q-1)^4}+\tfrac{q(q+1)t_{1,1}^2}{2(q-1)^4}+\tfrac{t_{2,1}^2}{2(q-1)}+\tfrac{t_{2,1}t_{2,2}}{1-q}\big)\Big],\\
J^{2,2} &= \tau^2\big(-\tfrac{t_2^2}{2(q-1)}-\tfrac{t_{1,1}^2}{2(q-1)}+\tfrac{t_1t_{2,1}}{1-q}\big)\\
&\quad+Q\Big[-\tfrac{2(3q^2+14q+3)q^2}{(q-1)^6}\\
&\quad+\tau\big(-\tfrac{(q^3+3q^2-19q-5)t_1q^2}{(q-1)^7}+\tfrac{(q^2-8q-3)t_2q^2}{(q-1)^6}+\tfrac{(q^2-8q-3)t_{1,1}q^2}{(q-1)^6}+\tfrac{2(2q+1)t_{2,1}q^2}{(q-1)^5}-\tfrac{(q+1)t_{2,2}q^2}{(q-1)^4}\big)\\
&\quad+\tau^2\big(\tfrac{q^2(q^2+7q+2)t_1^2}{(q-1)^7}-\tfrac{2(2q+1)t_1t_2q^2}{(q-1)^6}-\tfrac{2(2q+1)t_1t_{1,1}q^2}{(q-1)^6}+\tfrac{(q+1)t_1t_{2,1}q^2}{(q-1)^5}+\tfrac{q^2(q+1)t_2^2}{2(q-1)^5}+\tfrac{q^2(q+1)t_{1,1}^2}{2(q-1)^5}\big)\Big].
\end{aligned}$$

2.3.3 Pairings in quantum K-theory and sphere partition functions

In this subsection we want to compute the *non-constant pairing* (1.32), its variations and recall its relation to sphere partition functions. We will first discuss the CS level zero case $\hat{\kappa}=0$. Afterwards we will discuss how the pairing can be modified when the levels are non-trivial.

At level zero

In the K-theory basis of Grassmannians $\{\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_{1,1}, \ldots, \mathcal{O}_{N-M,\ldots,N-M}\}$ the pairing can be defined as

$$G_{\mu\nu} = g_{\mu\nu} + \sum_{\beta \in H_2^{\text{eff.}}} \sum_{n=0}^{\infty} \frac{1}{n!} \langle \mathcal{O}_\mu, \mathcal{O}_\nu, t^n \rangle_{0,n+2,\beta} Q^\beta, \tag{2.154}$$

where $g_{\mu\nu} = \chi(X; \mathcal{O}_\mu \otimes \mathcal{O}_\nu)$ is the Euler pairing in K-theory.

Let us recall from section 1.2.4 how the pairing G can be computed from the reconstructed $J(t)$-function. The J-function is

$$J(t) = (1-q) T \mathcal{O}_0, \tag{2.155}$$

where $T = T(Q, q, t) \in \operatorname{End} K(X) \otimes \mathbb{C}(q, q^{-1}) \otimes \mathbb{C}[[Q, t]]$ is a matrix that satisfies

$$(1-q)\partial_\mu T = T(\mathcal{O}_\mu \star), \tag{2.156}$$

where $\partial_\mu = \frac{\partial}{\partial t_\mu}$.

The above equation has a natural interpretation from the point of view of the gauge theory. As mentioned before in the end of section 2.1, the partition function can be interpreted as computing the overlap between two states, one determined by the boundary condition on $\partial(D^2 \times_q S^1) = T_q^2$, and another determined by inserting an operator (e.g., a Wilson line) on the tip of D^2 (wrapping the S^1-factor). The partition function computed in subsection 2.1.1 corresponds to the insertion of the trivial operator $1 = \mathcal{O}_0$, and computes the overlap of the vacuum state and the boundary state, tacitly also labeled by $\mathcal{O}_\mu$. The overlaps may thus be identified the matrix elements of T. Now, insertions of non-trivial operators in the deformed theory can be generated by single-trace deformations (2.127), taking the derivative at zero with respect to the twisted mass ρ. Schematically, this is

$$(1-q)\frac{\partial}{\partial \rho_\mu} \int \prod \frac{dz_a}{z_a} \exp\left(\frac{\sum_\nu \rho_\nu z^\nu}{1-q}\right) f_X(q,z) I_X(Q,q,z)\Big|_{\rho=0}, \tag{2.157}$$

corresponding to the left-hand side of (2.156). Since $\frac{\partial}{\partial \rho_\mu} \exp\left(\frac{\sum_\nu \rho_\nu z^\nu}{1-q}\right)|_{\rho=0} = \frac{z^\mu}{1-q}$ (the notation again implying some symmetrization), this computation amounts to an insertion of a Wilson loop W_μ as explained in subsection 2.2.4. In view the identification of the Wilson loop algebra with small quantum K-theory, we arrive at the right-hand side of (2.156).

The endomorphism T is related to the non-constant pairing G by the compatibility equation

$$G(\mathcal{O}_\mu, \mathcal{O}_\nu) = g(\overline{T}\mathcal{O}_\mu, T\mathcal{O}_\nu), \tag{2.158}$$

where $\overline{T} = T(Q, q^{-1}, t)$. Note that while the right-hand side depends a priori on q, this equation shows that it is in fact q-independent. This is a non-trivial check of the results computed by the gauge theory.

Note that T can be recovered from the $J(t)$-function. We have by definition $T\mathcal{O}_0 = \frac{1}{1-q} J(t)$, which determines a row of matrix elements for T. For the other rows

$$T\mathcal{O}_\mu = T(\mathcal{O}_\mu \star \mathcal{O}_0) = (1-q)\partial_\mu T \mathcal{O}_0 = \partial_\mu J(t) \tag{2.159}$$

by the flatness equation (2.156). Expanding left- and right-hand sides in the basis of $\mathcal{O}_\mu$'s we find the matrix elements of T as functions of Q, q, t

$$\sum_\nu T^\nu{}_\mu \mathcal{O}_\nu = \sum_\nu \partial_\mu J^\nu \mathcal{O}_\nu. \tag{2.160}$$

Using the T-matrix, we may compute the matrix elements of the non-constant pairing

$$G_{\mu\nu} = \sum_{\rho,\lambda} \overline{T}^\rho{}_\mu g_{\rho\lambda} T^\lambda{}_\nu. \tag{2.161}$$

The non-trivial check of q-independence has a natural interpretation on the gauge theory side, as did the flatness equation (2.156). By viewing the matrix elements of T as the overlaps between boundary states on $\partial(D^2 \times_q S^1)$ and operators inserted at the "tip", the above equation (seen as a definition of the left-hand side) computes the *sphere* partition function[14] with insertions $\mathcal{O}_\mu, \mathcal{O}_\nu$ at the poles, by summing over pairs of products disk partition functions [103]. The sums-of-products are interpolated by the classical pairing g, interpreted as the annulus partition function. The two disks have opposite orientation (a 'left'-boundary becomes a 'right' boundary), reflected in the inversion of the geometric twisting parameter q in one of them, while the partition function on the sphere as expected does not depend on q. This is an instance of a general phenomenon for disk partition functions known as factorization [10]. See also [67] for a more detailed discussion.

Example: Gr(2, 4) We can set $T^\nu{}_\mu = \sum_{k,n\geq 0} (T^\nu{}_\mu)_{k,n} Q^k \tau^n$, where τ counts the homogeneous order of t_μ's. In the ordered basis

$$\{\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_{1,1}, \mathcal{O}_{2,1}, \mathcal{O}_{2,2}\}, \tag{2.162}$$

for the first few (printable) orders, we find the coefficients of the T-matrix as follows $(T^\nu{}_\mu)_{0,0} = \mathrm{id}$, and ($\nu$ determines the row)

$$(T^\nu{}_\mu)_{1,0} = \begin{pmatrix} \frac{q+1}{(q-1)^4} & \frac{-q-1}{(q-1)^3} & \frac{1}{(q-1)^2} & \frac{1}{(q-1)^2} & \frac{1}{1-q} & 0 \\ \frac{2q(q+2)}{(q-1)^5} & -\frac{q(q+3)}{(q-1)^4} & \frac{q}{(q-1)^3} & \frac{q}{(q-1)^3} & 0 & \frac{1}{1-q} \\ \frac{q(3q^2+8q-1)}{(q-1)^6} & -\frac{q(q^2+4q-1)}{(q-1)^5} & \frac{q}{(q-1)^3} & \frac{q}{(q-1)^3} & \frac{q}{(q-1)^3} & -\frac{q}{(q-1)^2} \\ \frac{q(3q^2+8q-1)}{(q-1)^6} & -\frac{q(q^2+4q-1)}{(q-1)^5} & \frac{q}{(q-1)^3} & \frac{q}{(q-1)^3} & \frac{q}{(q-1)^3} & -\frac{q}{(q-1)^2} \\ \frac{q(5q^3+19q^2-3q-1)}{(q-1)^7} & -\frac{q(q^2+6q+1)}{(q-1)^5} & \frac{q(q^2-4q-1)}{(q-1)^5} & \frac{q(q^2-4q-1)}{(q-1)^5} & \frac{q(3q+1)}{(q-1)^4} & -\frac{q(q+1)}{(q-1)^3} \\ \frac{2q^2(3q^2+14q+3)}{(q-1)^7} & -\frac{q^2(q^3+3q^2-19q-5)}{(q-1)^7} & \frac{q^2(q^2-8q-3)}{(q-1)^6} & \frac{q^2(q^2-8q-3)}{(q-1)^6} & \frac{2q^2(2q+1)}{(q-1)^5} & -\frac{q^2(q+1)}{(q-1)^4} \end{pmatrix}, \tag{2.163}$$

$$(T^\nu{}_\mu)_{0,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{t_1}{1-q} & 0 & 0 & 0 & 0 & 0 \\ \frac{t_2}{1-q} & \frac{t_1}{1-q} & 0 & 0 & 0 & 0 \\ \frac{t_{1,1}}{1-q} & \frac{t_1}{1-q} & 0 & 0 & 0 & 0 \\ \frac{t_{2,1}}{1-q} & \frac{-t_1+t_2+t_{1,1}}{1-q} & \frac{t_1}{1-q} & \frac{t_1}{1-q} & 0 & 0 \\ \frac{t_{2,2}}{1-q} & \frac{t_{2,1}}{1-q} & \frac{t_2}{1-q} & \frac{t_{1,1}}{1-q} & \frac{t_1}{1-q} & 0 \end{pmatrix}. \tag{2.164}$$

[14] The resulting 'sphere' obtained from gluing two solid tori $D^2 \times_q S^1$ by Dehn surgery can be $S^2 \times S^1$ or S^3 or generalizations of such spaces. The field theoretic constructions are known to be compatible with such topological operations in a variety of examples. See [10, 124] and references therein.

Furthermore, we set

$$\sum_{k,n\geq 0}(\tilde{G}_{\mu\nu})_{k,n}Q^k\tau^n = (1-Q)e^{-t_0}G_{\mu\nu}, \tag{2.165}$$

in order to get simpler expressions. Note that the exponential term captures the entire dependence on t_0. At order τ^0 we find

$$(\tilde{G}_{\mu\nu})_{0,0} = g_{\mu\nu} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\tilde{G}_{\mu\nu})_{1,0} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}, \tag{2.166}$$

and $(\tilde{G}_{\mu\nu})_{2,0} = \delta_{\mu,(2,2)}\delta_{\nu,(2,2)}$, while $(\tilde{G}_{\mu\nu})_{k,0} = 0$ for $k > 2$. These results agree with results from [82] computed by directly evaluating the unperturbed $S^2 \times S^1$ partition function. At order τ^1 we find

$$(\tilde{G}_{\mu\nu})_{0,1} = \begin{pmatrix} U(t) & U(t)-t_{2,2} & t_1+t_2 & t_{1,1}+t_1 & t_1 & 0 \\ U(t)-t_{2,2} & t_{1,1}+t_1+t_2 & t_1 & t_1 & 0 & 0 \\ t_1+t_2 & t_1 & 0 & 0 & 0 & 0 \\ t_{1,1}+t_1 & t_1 & 0 & 0 & 0 & 0 \\ t_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.167}$$

$$(\tilde{G}_{\mu\nu})_{2,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & t_{2,2} \\ 0 & 0 & 0 & 0 & 0 & t_{2,2} \\ 0 & 0 & t_{2,2} & 0 & t_{2,2} & t_{2,1}+t_{2,2}+t_2 \\ 0 & 0 & 0 & t_{2,2} & t_{2,2} & t_{1,1}+t_{2,1}+t_{2,2} \\ 0 & 0 & t_{2,2} & t_{2,2} & t_{2,1}+t_{2,2} & U(t)-t_1 \\ t_{2,2} & t_{2,2} & t_{2,1}+t_{2,2}+t_2 & t_{1,1}+t_{2,1}+t_{2,2} & U(t)-t_1 & U(t) \end{pmatrix}, \tag{2.168}$$

and

$$(\tilde{G}_{\mu\nu})_{1,1} = \begin{pmatrix} U(t) & U(t)-t_{2,2} & t_1+t_2 & t_{1,1}+t_1 & t_1 & t_{2,2} \\ U(t)-t_{2,2} & t_{1,1}+t_1+t_2 & t_1 & t_1 & 0 & t_{2,2} \\ t_1+t_2 & t_1 & t_{2,2} & 0 & t_{2,2} & t_{2,1}+t_{2,2}+t_2 \\ t_{1,1}+t_1 & t_1 & 0 & t_{2,2} & t_{2,2} & t_{1,1}+t_{2,1}+t_{2,2} \\ t_1 & 0 & t_{2,2} & t_{2,2} & t_{2,1}+t_{2,2} & U(t)-t_1 \\ t_{2,2} & t_{2,2} & t_{2,1}+t_{2,2}+t_2 & t_{1,1}+t_{2,1}+t_{2,2} & U(t)-t_1 & U(t) \end{pmatrix}, \tag{2.169}$$

where $U(t) = t_1 + t_2 + t_{1,1} + t_{2,1} + t_{2,2}$.

We can also compute the K-theoretic Gromov-Witten potential. As before, the t_0-dependence can be dropped, as it can easily be reinstated and we have the relation

$$\partial_\mu\partial_\nu F = G_{\mu\nu} - g_{\mu\nu}, \tag{2.170}$$

evaluated at $t_0 = 0$. It can be integrated order-by-order in Q's and τ's, starting at τ^2. We set

$$F(Q,t) = \frac{1}{1-Q}\sum_{k,n\geq 0}\tilde{F}_{k,n}Q^k\tau^n, \tag{2.171}$$

and we find at order Q^0

$$\tilde{F}_{0,2} = 0, \quad \tilde{F}_{0,3} \;\; = \tfrac{t_1^3}{6} + \tfrac{1}{2}t_1^2 t_{1,1} + \tfrac{1}{2}t_2 t_1^2, \quad F_{0,4} = \tfrac{t_1^4}{12}. \tag{2.172}$$

At order Q^1 (modulo the overall factor)

$$\begin{aligned} \tilde{F}_{1,2} &= \tfrac{1}{2!}\big(U(t)^2 - t_{2,2}^2\big), \\ \tilde{F}_{1,3} &= \tfrac{1}{3!}\big[U(t)^3 - \big(t_1^3+t_{2,1}^3+t_{2,2}^3+6(t_2t_{2,1}t_{2,2}+t_{1,1}t_{2,1}t_{2,2}) \\ &\qquad +3(t_1^2t_{1,1}+t_1t_{2,2}^2+t_2t_{2,2}^2+t_{1,1}t_{2,2}^2 \\ &\qquad +t_{2,1}t_{2,2}^2+t_2^2t_{2,2}+t_{1,1}^2t_{2,2}+t_{2,1}^2t_{2,2}+t_2t_1^2)\big)\big], \end{aligned} \tag{2.173}$$

where once more $U(t) = t_1 + t_2 + t_{1,1} + t_{2,1} + t_{2,2}$. At order Q^2:

$$\begin{aligned} \tilde{F}_{2,2} &= \tfrac{1}{2!} t_{2,2}^2, \\ \tilde{F}_{2,3} &= \tfrac{1}{6}\big(t_{2,1}^3 + t_{2,2}^3 + 6(t_2 t_{2,2} t_{2,1} + t_{1,1} t_{2,2} t_{2,1}) \\ &\quad + 3(t_{2,2} t_{2,1}^2 + t_{2,2}^2 t_{2,1} + t_1 t_{2,2}^2 + t_2 t_{2,2}^2 + t_{1,1} t_{2,2}^2 + t_2^2 t_{2,2} + t_{1,1}^2 t_{2,2})\big). \end{aligned} \tag{2.174}$$

At non-zero level

The endomorphism T and the non-constant pairing G were computed from the $J = I$-function of the canonical theory with Chern-Simons levels $\hat{\kappa} = 0$. For non-zero levels the I-function computed by the gauge theory corresponds to the "quantum K-theory with level structure", first studied in [87]. The correlators (1.29), i.e., the Euler characteristics of sheaves on (some) moduli space of curves to $\mathrm{Gr}(M, N)$, are now twisted by further insertions $\mathcal{D} \in K\big(\overline{\mathcal{M}_{g,n}}(X, \beta)\big)$ (see [87] for details)

$$\langle \tau_{d_1}\gamma_1, \ldots, \tau_{d_n}\gamma_n \rangle^{\mathcal{D}}_{g,n,\beta} = \chi\Big(\overline{\mathcal{M}_{g,n}}(X,\beta); \mathcal{O}_{\overline{\mathcal{M}_{g,n}}(X,\beta)} \otimes \mathcal{D} \otimes \bigotimes_{i=1}^{n} \mathcal{L}_i^{\otimes d_i} \otimes \mathrm{ev}_i^*(\gamma_i)\Big) \tag{2.175}$$

If the CS levels are inside the "window" from subsection 2.2.2, the I-function computed by the gauge theory has zero permutation symmetric input, and can be identified with the J-function.

$$I_{\hat{\kappa}}(0) = J_{\hat{\kappa}}(0). \tag{2.176}$$

After reconstructing the function at non-zero ordinary input t, the endomorphism $T_{\hat{\kappa}}$ is computed as

$$(T_{\hat{\kappa}})^{\nu}{}_{\mu} = \partial_\mu (J_{\hat{\kappa}})^{\nu}. \tag{2.177}$$

As noted in [87], the classical pairing g must be modified. With a modified classical pairing $g_{\hat{\kappa}}$, we may compute the analog of the non-constant pairing (1.32) by

$$G_{\hat{\kappa}}(\mathcal{O}_\mu, \mathcal{O}_\nu) = g_{\hat{\kappa}}(\overline{T}_{\hat{\kappa}}\mathcal{O}_\mu, T_{\hat{\kappa}}\mathcal{O}_\nu) \tag{2.178}$$

or in components

$$(G_{\hat{\kappa}})_{\mu\nu} = \sum_{\rho,\lambda} (\overline{T_{\hat{\kappa}}})^{\rho}{}_{\mu} (g_{\hat{\kappa}})_{\rho\lambda} (T_{\hat{\kappa}})^{\lambda}{}_{\nu}. \tag{2.179}$$

For non-zero levels (but still inside the window) we find the right-hand side is q-independent if the classical pairing is modified as

$$g^{\hat{\kappa}}_{\mu\nu} = \chi\big(\,\mathrm{Gr}(M,N); \mathcal{O}_\mu \otimes \mathcal{O}_\nu \otimes \omega_{\mathrm{Gr}(M,N)}^{\otimes 2\hat{\kappa}_R}\big) \tag{2.180}$$

where $\omega_{\mathrm{Gr}(M,N)}$ is the canonical line bundle of $\mathrm{Gr}(M, N)$. In terms of Schur polynomials in the Chern roots of S^* (see appendix B.1)

$$\mathrm{ch}(\omega_{\mathrm{Gr}(M,N)}) = \exp(-\sigma_1) = \exp(-\epsilon_1 - \ldots - \epsilon_M). \tag{2.181}$$

For example, for $\mathrm{Gr}(2, 4)$ we have

$$g^{\hat{\kappa}}_{\mu\nu} = \begin{pmatrix} \frac{1}{3}(\hat{\kappa}_R-1)^2(2\hat{\kappa}_R-3)(2\hat{\kappa}_R-1) & -\frac{1}{3}(\hat{\kappa}_R-1)(2\hat{\kappa}_R-1)(4\hat{\kappa}_R-3) & (\hat{\kappa}_R-1)(2\hat{\kappa}_R-1) & (\hat{\kappa}_R-1)(2\hat{\kappa}_R-1) & 1-2\hat{\kappa}_R & 1 \\ -\frac{1}{3}(\hat{\kappa}_R-1)(2\hat{\kappa}_R-1)(4\hat{\kappa}_R-3) & (2\hat{\kappa}_R-1)^2 & 1-2\hat{\kappa}_R & 1-2\hat{\kappa}_R & 1 & 0 \\ (\hat{\kappa}_R-1)(2\hat{\kappa}_R-1) & 1-2\hat{\kappa}_R & 1 & 0 & 0 & 0 \\ (\hat{\kappa}_R-1)(2\hat{\kappa}_R-1) & 1-2\hat{\kappa}_R & 0 & 1 & 0 & 0 \\ 1-2\hat{\kappa}_R & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.182}$$

The dependence of $T_{\hat{\kappa}}$, and hence also of $G_{\hat{\kappa}}$, on the levels $\hat{\kappa}$ is more complicated: the levels $\hat{\kappa}$ influence the decomposition into $\mathcal{K}_+ \oplus \mathcal{K}_-$ by adding powers of q in the numerators of summands of $J_{\hat{\kappa}}(0)$. The reconstruction to $J_{\hat{\kappa}}(t)$ is then performed recursively on a case-by-case basis for the CS levels. The q-independence of $G_{\hat{\kappa}}$ has been established for $\mathrm{Gr}(2,4)$ for all levels (2.77) leading to a zero permutation-symmetric input.

Chapter 3

Supercurrent multiplets in three dimensions with boundaries

In this chapter we will study supercurrent multiplets in three-dimensional theories with boundaries, following mostly the results from the joint work [3] with Jonathan Schulz and Prof. Dr. Ilka Brunner.
Notation note: In this chapter, greek letters $\mu, \nu, \ldots$ and $\alpha, \beta, \ldots$ denote spacetime and spinor indices respectively, and not Young diagrams as in chapter 2.

3.1 Currents, charges and Noether's theorem in boundary theories

We start off by discussing how the classical Noether's theorem is modified in the presence of a flat boundary on the Minkowski background. The modification of the theorem into curved bulk and boundary spaces involves introducing appropriate bulk and induced boundary metrics that we do not deal with here. While we keep the dimension general in this subsection, we will specialize to the case of three spacetime dimensions in the next section.

3.1.1 Set-up: bulk and boundary actions

We thus consider the Minkowski half-space M with spacelike boundary ∂M

$$M = \{x^\mu = (x^0, x^1, \ldots, x^\perp, \ldots, x^{N-1}) | x^\perp \leq 0\}, \qquad \partial M = \{x^\perp = 0\} = \{x^{\hat{\mu}}\}. \tag{3.1}$$

Indices with hats take all values except the normal direction $\perp$. A Lagrangian theory with bulk and boundary dynamics will have the an action of the form

$$S = S^B + S^\partial = \int_M \mathcal{L}^B + \int_{\partial M} \mathcal{L}^\partial. \tag{3.2}$$

Here, the Lagrangians are functionals

$$\mathcal{L}^B[\phi, \partial_\mu \phi], \qquad \mathcal{L}^\partial[\pi, \partial_{\hat{\mu}}\pi, \phi|_\partial, \partial_{\hat{\mu}}\phi|_\partial], \tag{3.3}$$

where ϕ denotes bulk fields, π denotes boundary fields and we suppress the integration measure. Note that we also assume that the boundary Lagrangian $\mathcal{L}^\partial$ contains only tangential derivatives along $x^{\hat{\mu}}$-directions and that all derivative terms are of first order (otherwise 'higher-order' versions of Noether's theorem must be used later).

To study such a theory, one has to impose *boundary conditions*; in this language, these will be general relations of the form

$$\mathcal{B}(\text{fields}|_\partial, \text{derivatives of fields}|_\partial) = 0. \tag{3.4}$$

One then would like to consider variations of the fields $\delta\phi$, $\delta\pi$, such that these (or their boundary restrictions $\delta\phi|_\partial$ for the case of bulk fields) are *compatible* with the chosen boundary conditions in the sense that

$$\delta\mathcal{B}|_{\mathcal{B}=0} = 0, \tag{3.5}$$

holds. Under such a compatible, rigid[1] variation, the action transforms

$$\begin{aligned}\delta S &= \int_M \delta\mathcal{L}^B + \int_{\partial M} \delta\mathcal{L}^\partial \\ &= \int_M \underbrace{\Big[\frac{\partial\mathcal{L}^B}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}^B}{\partial(\partial_\mu\phi)}\Big]}_{\text{Bulk EoM}^B[\phi]}\delta\phi + \int_{\partial M}\Big[\underbrace{\Big(\frac{\partial\mathcal{L}^\partial}{\partial\pi} - \partial_{\hat\mu}\frac{\partial\mathcal{L}^\partial}{\partial(\partial_{\hat\mu}\pi)}\Big)}_{\text{Boundary EoM}^\partial[\pi]}\delta\pi + \underbrace{\Big(\frac{\partial\mathcal{L}^\partial}{\partial\phi|_\partial} - \partial_{\hat\mu}\frac{\partial\mathcal{L}^\partial}{\partial(\partial_{\hat\mu}\phi|_\partial)} + \frac{\partial\mathcal{L}^B}{\partial(\partial_n\phi|_\partial)}\Big)}_{=:\mathcal{A}}\delta\phi|_\partial\Big].\end{aligned} \tag{3.6}$$

modulo $\partial_{\hat\mu}(\cdot)$-terms in the last integral. Minimizing the action under the assumption of boundary conditions $\mathcal{B}$ means imposing the bulk and boundary equations of motion, as well as

$$[\mathcal{A}\cdot\delta\phi|_\partial]_{\mathcal{B}=0} = \partial_{\hat\mu}A^{\hat\mu}, \tag{3.7}$$

for some boundary vector $A^{\hat\mu}$.

Throughout this chapter will work with a special class of boundary conditions, also appearing in [39, 125], the *dynamical boundary conditions*. This is determined by the choice

$$\mathcal{B} \coloneqq \mathcal{A}. \tag{3.8}$$

Then (3.7) is automatically satisfied with zero right-hand side.

3.1.2 What constitutes a symmetry in a theory with boundary?

Let us spell out what is meant by a symmetry in the case of a Lagrangian theory with a boundary. Analogously to the case of a bulk theory, a symmetry is an off-shell transformation δ_{sym} of bulk and boundary fields ϕ, π that leaves the action invariant, *possibly* after using boundary conditions $\mathcal{B}$:

$$\delta_{\text{sym}}S|_{\mathcal{B}} = (\delta_{\text{sym}}S^B + \delta_{\text{sym}}S^\partial)|_{\mathcal{B}=0} = 0. \tag{3.9}$$

However, in the presence of boundaries, we must additionally have that boundary conditions are invariant with respect to the symmetry action [3, 126]:

$$\delta_{\text{sym}}\mathcal{B}|_{\mathcal{B}=0} = 0. \tag{3.10}$$

These boundary conditions are called *symmetric boundary conditions* with respect to δ_{sym}. This condition parses well with the requirement (3.5): it simply means that δ_{sym} is an allowed variation, or rather, that $\mathcal{B}$ has been chosen compatibly with a given δ_{sym}.

[1] I.e., no variation of the manifold M.

One often wants to study (and we do in this work) possible symmetry transformations δ^B_{sym} that are 'inherited' from a pure bulk theory, i.e., where $\delta^B_{\text{sym}}\mathcal{L}^B = \partial_\mu V^\mu$ holds, for some bulk vector field V^μ. Applying this transformation to the total action we obtain

$$\delta^B_{\text{sym}} S|_{\mathcal{B}=0} = \int_M \delta^B_{\text{sym}}\mathcal{L}^B + \int_{\partial M} \delta^B_{\text{sym}}\mathcal{L}^\partial|_{\mathcal{B}=0} = \int_{\partial M} (V^\perp + \delta^B_{\text{sym}}\mathcal{L}^\partial)|_{\mathcal{B}=0} \tag{3.11}$$

If the boundary condition $\mathcal{B} = 0$ can be chosen such that the integrand of the right-hand side vanishes, then we say that $\mathcal{B}$ *preserves* the bulk symmetry δ^B_{sym}. More interestingly, if the integrand can be set to zero by choosing an appropriate $\mathcal{L}^\partial$, we say that the symmetry is preserved *without a choice of specific boundary condition* $\mathcal{B}$ [125, 126].

3.1.3 Currents and charges: general structure

When a δ^B_{sym}-transformation is a symmetry of a bulk theory, i.e., $\delta^B_{\text{sym}}\mathcal{L}^B = \partial_\mu V^\mu$, there exists a current J^μ_B, $\partial_\mu J^\mu_B = 0$ associated to that symmetry. The corresponding charge $Q_B = \int_\Sigma J^0_B$, where Σ is a spacelike constant-time slice of M, is preserved in the sense that $\partial_0 Q_B = 0$ due to the divergence-freeness of J^μ_B.

It is clear that when a boundary is introduced, Q_B is in general no longer time-independent "on the nose", since now Σ has a non-trivial boundary $\partial\Sigma$: the bulk "leaks" from the boundary. In fact, when a δ_{sym}-transformation is a symmetry of a theory with boundary, then there must exist bulk and boundary currents J^μ_B and $J^{\hat\mu}_\partial$ that satisfy

$$\partial_\mu J^\mu_B = 0, \qquad \partial_{\hat\mu} J^{\hat\mu}_\partial = J^\perp_B|_\partial, \tag{3.12}$$

where the second equation holds *possibly* after using a boundary condition $\mathcal{B} = 0$. The "boundary piece" $J^{\hat\mu}_\partial$ compensates the "leakage" of the bulk current through the boundary. These may be combined into the total current

$$J^\mu_{\text{full}} = J^\mu_B + \delta(x^\perp)\mathcal{P}^\mu{}_{\hat\mu} J^{\hat\mu}_\partial, \qquad \partial_\mu J^\mu_{\text{full}} = \delta(x^\perp) J^\perp_{\text{full}}|_\partial \tag{3.13}$$

where $\mathcal{P}^n{}_{\hat\mu} = 0$, $\mathcal{P}^{\hat\nu}{}_{\hat\mu} = \delta^{\hat\nu}{}_{\hat\mu}$. The time-independent charge can be then defined as

$$Q = \int_\Sigma J^0_B + \int_{\partial\Sigma} J^0_\partial = \int_\Sigma J^0_{\text{full}}. \tag{3.14}$$

3.1.4 Improvements and their geometric interpretation

Currents associated to a charge Q are note unique: they are determined up to *improvement transformations*. For a pure bulk theory, an improvement is a transformation of the currents that locally takes the form

$$J^\mu_B \mapsto \widetilde{J}^\mu_B = J^\mu_B + \partial_\nu \Omega^{[\mu\nu]}. \tag{3.15}$$

It is clear that the divergence-freeness equation $\partial_\mu \widetilde{J}^\mu_B = 0$ and the charge $Q_B = \int_\Sigma \widetilde{J}^0_B$ are preserved.

For a theory with boundary, the appropriate extension of the notion is a transformation that sends

$$\begin{Bmatrix} J^\mu_B \\ J^{\hat\mu}_\partial \end{Bmatrix} \mapsto \begin{Bmatrix} \widetilde{J}^\mu_B = J^\mu_B + \partial_\nu \Omega^{[\mu\nu]} \\ \widetilde{J}^{\hat\mu}_\partial = J^{\hat\mu}_\partial + \Omega^{\perp\hat\mu}|_\partial + \partial_{\hat\nu}\omega^{[\hat\mu\hat\nu]} \end{Bmatrix}, \tag{3.16}$$

which preserves the generalized conservation equation (3.12) and the charge (3.14). Here, the bulk improvement via $\Omega^{[\mu\nu]}$ induces a boundary improvement; the boundary current may still be further improved by $\omega^{[\hat{\mu}\hat{\nu}]}$.

The currents, their improvements and the corresponding improvement-invariant charges have a topological interpretation[2]. Let us first restrict to pure bulk theories. Implicitly in this discussion, we have assumed we have a family of embeddings $i_t : \Sigma_t \hookrightarrow M$ foliating our manifold M by $(N-1)$-dimensional constant-time slices Σ_t. Furthermore, there are maps $\Sigma_t \to \Sigma_{t'}$ which "time-evolve" the constant-time slices. As discussed in the previous subsection, to each symmetry of the theory we associate a current with local expression J_μ, and more generally, a one-form J with local expression $J = J_\mu \mathrm{d}x^\mu$. The divergence-freeness equation is written in coordinate-free manner as

$$\mathrm{d} \star J = 0 \tag{3.17}$$

where $\star$ is the Hodge-star operator on M, while the charge may be written as

$$Q_t = \int_{\Sigma_t} i_t^*(\star J) \tag{3.18}$$

Thus, (3.17) means that $\star J$ is a *closed* $(N-1)$-form, and hence Q_t depends only on the homology class $(i_t)_*[\Sigma_t] \in H_{N-1}(M)$, i.e., $Q_t = \langle i_t^*(\star J) \smile 1, [\Sigma_t]\rangle = [\Sigma_t] \frown i_t^*(\star J)$, where the pairing is integration, which we rewrote as a cap product in de Rham cohomology. The resulting charge is an element of $H_0(M) \cong \mathbb{R}$. The statement of conservation of the charge can be rephrased as a topological statement

The slices $\{\Sigma_t\}_t$ are homologous and hence Q_t does not depend on t. (3.19)

Furthermore, the improvements follow naturally: the transformation

$$\star J \mapsto \star\widetilde{J} = \star J + \mathrm{d} \star \Omega \tag{3.20}$$

for some 2-form Ω corresponds to a change of $\star J$ by exact terms and thus does not change the cohomology class $[\star J] \in H^{N-1}(M)$.

Analogously, for the case with boundary we have the following (family of) embeddings, forming a commutative diagram:

$$\begin{tikzcd} \partial\Sigma_t \arrow[r, hook, "i_t^\partial"] \arrow[d, hook, "b_t"] & \partial M \arrow[d, hook, "b"] \\ \Sigma_t \arrow[r, hook, "i_t"] & M \end{tikzcd} \tag{3.21}$$

To a symmetry we associate one forms J_B and J_∂ on M and ∂M respectively. The generalized conservation equations (3.12) are written as

$$\mathrm{d} \star J_B = 0, \qquad \mathrm{d}_\partial \star_\partial J_\partial = b^*(\star J_B) \tag{3.22}$$

where $\mathrm{d}_\partial, \star_\partial$ are the operations on ∂M. The charge is written as

$$Q_t = \int_{\Sigma_t} i_t^*(\star J_B) + \int_{\partial\Sigma_t} i_t^{\partial\,*}(\star_\partial J_\partial) \tag{3.23}$$

[2] We are tacitly assuming that M is connected, compact for these arguments to apply without asterisks.

Improvements then take the form

$$\begin{aligned}\star J_B &\mapsto \star J_B + \mathrm{d}\star\Omega,\\ \star_\partial J_\partial &\mapsto \star_\partial J_\partial - b^*(\star\Omega) + \mathrm{d}_\partial \star_\partial \omega\end{aligned} \tag{3.24}$$

for some 2-form Ω on M and a 2-form ω on ∂M, and they preserve Q_t by the commutativity of the diagram (3.21).

One can extend these definitions to accommodate for "charges with indices" or *brane currents* [69], corresponding to higher-dimensional charged objects, using the cap product $\frown$ in de Rham cohomology. Let us spell out some structure for the case of a pure bulk theory. Instead of a current one-form, we associate to such a higher dimensional charged object a current d-form C on M. The corresponding charge Q_t will be a closed $(d-1)$-form on M defined $Q_t = u(W_t)$ where $W_t \in H_{d-1}(M)$ is a homology class defined by the cap product $W_t = [\Sigma_t] \frown i_t^*(\star C)$ and $u : H_m(M) \xrightarrow{\cong} (H^m(M))^* \cong H^m(M)$ is the isomorphism defined by the universal coefficient theorem. Given a local expression $C_{\mu_1\cdots\mu_d}$ for C, Q_t is locally

$$(Q_t)_{\mu_1\cdots\mu_{d-1}} = \int_{\Sigma_t} {C^0}_{\mu_1\cdots\mu_{d-1}}\, \mathrm{d}^{N-1}x. \tag{3.25}$$

3.1.5 Currents and charges: computation via modified Noether's theorem

Now let us turn to actually computing the currents and charges with the structure outlined in the previous subsection. To this end we must modify Noether's theorem from bulk Lagrangian theories to theories with boundary, following [3, 126].

By assumption, we have a symmetry of the *full* theory, meaning that

$$0 = \delta_{\mathrm{sym}} S|_{\mathcal{B}=0} = \int_M \partial_\mu V^\mu + \int_{\partial M} \delta_{\mathrm{sym}}\mathcal{L}^\partial|_{\mathcal{B}=0} = \int_{\partial M} [V^\perp + \delta_{\mathrm{sym}}\mathcal{L}^\partial]_{\mathcal{B}=0}, \tag{3.26}$$

which implies that we must have

$$[V^\perp + \delta_{\mathrm{sym}}\mathcal{L}^\partial]_{\mathcal{B}=0} = \partial_{\hat\mu} K^{\hat\mu} \tag{3.27}$$

for a vector field $K^{\hat\mu}$ on the boundary ∂M. The same variation can be performed "on-shell", yielding on the boundary Lagrangian

$$\delta_{\mathrm{sym}}\mathcal{L}^\partial = \left[\tfrac{\partial\mathcal{L}^\partial}{\partial\phi|_\partial} - \partial_{\hat\mu}\tfrac{\partial\mathcal{L}^\partial}{\partial(\partial_{\hat\mu}\phi|_\partial)}\right]\delta_{\mathrm{sym}}\phi|_\partial + \mathrm{EoM}^\partial[\pi]\delta_{\mathrm{sym}}\pi + \partial_{\hat\mu}\big(\tfrac{\partial\mathcal{L}^\partial}{\partial(\partial_{\hat\mu}\pi)}\delta_{\mathrm{sym}}\pi + \tfrac{\partial\mathcal{L}^\partial}{\partial(\partial_{\hat\mu}\phi|_\partial)}\delta_{\mathrm{sym}}\phi|_\partial\big), \tag{3.28}$$

where ϕ and π denotes bulk and boundary fields, respectively. We rewrite the first term of the right-hand side using the stationarity condition (3.7) (assuming that the equation of motion is satisfied)

$$\partial_{\hat\mu} A^{\hat\mu} \overset{\text{on-shell}}{=} \left[\tfrac{\partial\mathcal{L}^\partial}{\partial\phi|_\partial} - \partial_{\hat\mu}\tfrac{\partial\mathcal{L}^\partial}{\partial(\partial_{\hat\mu}\phi|_\partial)}\right]\delta_{\mathrm{sym}}\phi|_\partial + [V^\perp - J_B^\perp]_\partial, \tag{3.29}$$

where $J_B^\mu = -\frac{\partial\mathcal{L}^B}{\partial(\partial_\mu\phi)}\delta_{\mathrm{sym}}\phi + V^\mu$ is the bulk Noether current. With this we obtain that on-shell, assuming $\mathcal{B}=0$ we have that

$$\delta_{\mathrm{sym}}\mathcal{L}^\partial \overset{\text{on-shell}}{=} [J_B^\perp - V^\perp]_\partial + \partial_{\hat\mu}\big(A^{\hat\mu} + \tfrac{\partial\mathcal{L}^\partial}{\partial(\partial_{\hat\mu}\pi)}\delta_{\mathrm{sym}}\pi + \tfrac{\partial\mathcal{L}^\partial}{\partial(\partial_{\hat\mu}\phi|_\partial)}\delta_{\mathrm{sym}}\phi|_\partial\big). \tag{3.30}$$

Together with (3.27) we obtain the *boundary Noether current* in the sense of (3.12).

$$J_\partial^{\hat\mu} = K^{\hat\mu} - A^{\hat\mu} - \tfrac{\partial\mathcal{L}^\partial}{\partial(\partial_{\hat\mu}\pi)}\delta_{\mathrm{sym}}\pi - \tfrac{\partial\mathcal{L}^\partial}{\partial(\partial_{\hat\mu}\phi|_\partial)}\delta_{\mathrm{sym}}\phi|_\partial, \quad \partial_{\hat\mu} J_\partial^{\hat\mu} = J_B^\perp|_\partial. \tag{3.31}$$

where the $A^{\hat\mu}$ is defined by (3.7) and $K^{\hat\mu}$ through (3.27).

3.2 Supercurrent multiplets in bulk theories

3.2.1 Generalities and ancient history

We may now focus on the goal of this chapter: supercurrent multiplets. As already outlined in the first section of this chapter, supercurrent multiplets are supermultiplets, a.k.a. representations of a supersymmetry algebra, whose components contain (Noether) currents of a given (local) field theory. The list of Noether currents included contains *at least* the *supercurrents*, i.e., the Noether currents associated to supersymmetry, and the energy-momentum tensor of the local field theory[3].

The study of supercurrent multiplets is an old story [127–129]. As in [3], we refer the reader to the literature [69, 130, 131] for a general, comprehensive review of supercurrent multiplets. We will spell out *some* general structure of these objects in this subsection, and we will focus on the case we're interested in in the next subsections: three-dimensions and (initially) $\mathcal{N} = 2$ supersymmetry. We will follow [3] and [69] throughout this exposition.

To start, a *supercurrent multiplet* must satisfy the following points:

a. The energy-momentum tensor $(T^B)^{\mu\nu}$ is a component of the multiplet; in fact, it is the only component with spin 2.

b. The supercurrents, i.e., conserved currents associated to supersymmetry, are components of the multiplet; in fact, they are the only components with spin 3/2. No component other than the supercurrents and the energy-momentum tensor are allowed to have spin larger than 1.

c. The supercurrent multiplet is not unique; it can be transformed by (supersymmetrically complete) improvements.

d. The multiplet is *indecomposable*, i.e., it may have non-trivial submultiplets, but it may not be decomposed into two independent decoupled multiplets.

As stated in point c, the conserved currents and other components of a supercurrent multiplet are note unique: they can be improved arbitrarily by "exact" terms, leading to the same conserved charges. The structure of the multiplet, however, restricts these improvements: for the resulting, improved components to form a multiplet, the "exact" piece by which we improve must also form a multiplet. Conversely, given two components (say, conserved currents) that initially do not form a multiplet under the SUSY algebra, one may find improvements such that the resulting components do form a multiplet. In [3], we introduced the notion of an *improvement frame*, to describe representatives of currents (and other components) that do in fact form a supersymmetry multiplet.

3.2.2 Intermezzo: conventions and (spinor) notation

Let us first establish notation and conventions from [3].

[3]The nomenclature is somewhat degenerate: the three notions of supermultiplets, supercurrent multiplets and supercurrents are all related, but different. *Supermultiplets* are abstract representations of the SUSY algebra; *supercurrent multiplets* are specific examples of those, which furthermore contain the *supercurrents* as components.

Spacetime The three-dimensional (half-)spacetime is given by

$$M = \{(x^0, x^1, x^2) | x^1 \leq 0\}, \tag{3.32}$$

with a mostly-plus Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1)$. Light-cone coordinates are

$$x^{\pm} = x^0 \pm x^2, \quad x_{\pm} = \tfrac{1}{2}(x_0 \pm x_2), \quad x^1 = x_1 = x^{\perp}, \tag{3.33}$$

and the metric in these coordinates reads

$$\eta_{\mu\nu} = \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \eta^{\mu\nu} = \begin{pmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.34}$$

The reader is urged to some caution not to confuse $\pm$-spacetime indices with $\pm$-spinor indices. The Levi-Civita symbol is defined by $\epsilon_{012} = -1$, $\epsilon^{012} = 1$. In light-cone coordinates it is $\epsilon_{+-\perp} = -\frac{1}{2}$, $\epsilon^{+-\perp} = 2$. It satisfies

$$\begin{aligned} \epsilon_{\mu\nu\lambda}\epsilon^{\sigma\rho\lambda} &= \delta_\mu{}^\rho \delta_\nu{}^\sigma - \delta_\mu{}^\sigma \delta_\nu{}^\rho, \\ \epsilon_{\mu\rho\lambda}\epsilon^{\nu\rho\lambda} &= -2\delta_\mu{}^\nu. \end{aligned} \tag{3.35}$$

Spinors Spinors in 3D two component spinors ψ_α, $\alpha \in \{1, 2\} = \{-, +\}$. Their indices are raised and lowered by $\epsilon_{\alpha\beta}, \epsilon^{\alpha\beta}$, where $\epsilon_{12} = -1, \epsilon^{12} = 1$ according to the rule

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta, \ \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta. \tag{3.36}$$

Explicitly, we have

$$\psi^\alpha = \begin{pmatrix} \psi^- \\ \psi^+ \end{pmatrix} = \begin{pmatrix} \psi_+ \\ -\psi_- \end{pmatrix}, \quad \psi_\alpha = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} = \begin{pmatrix} -\psi^+ \\ \psi^- \end{pmatrix}. \tag{3.37}$$

Indices that are contracted with "north-west to south-east" convention are omitted:

$$\psi\chi \coloneqq \psi^\alpha \chi_\alpha = \psi^- \chi_- + \psi^+ \chi_+. \tag{3.38}$$

Note that $\psi\chi = \chi\psi$ holds. Since Hermitian conjugation flips the order of spinors without flipping index position (unlike 4D notation), we have that $\overline{\psi\chi} = -\overline{\psi}\overline{\chi}$.
Some useful identities are given by

$$\begin{aligned} \psi\psi &= 2\psi^+\psi^-, \\ \psi^\alpha\psi^\beta &= -\tfrac{1}{2}(\psi\psi)\epsilon^{\alpha\beta}, \\ \psi_\alpha\psi_\beta &= \tfrac{1}{2}(\psi\psi)\epsilon_{\alpha\beta}. \end{aligned} \tag{3.39}$$

Clifford algebra We use a real, symmetric basis for gamma matrices

$$\gamma^\mu_{\alpha\beta} = (\gamma^0_{\alpha\beta}, \gamma^1_{\alpha\beta}, \gamma^2_{\alpha\beta}) = (-\mathbb{1}, \sigma^1, \sigma^3). \tag{3.40}$$

In light-cone coordinates these are

$$\gamma^\mu_{\alpha\beta} = (\gamma^+_{\alpha\beta}, \gamma^-_{\alpha\beta}, \gamma^\perp_{\alpha\beta}) = \left(\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right). \tag{3.41}$$

They are symmetric $\gamma^\mu_{\alpha\beta} = \gamma^\mu_{\beta\alpha}$, real and satisfy the Clifford algebra

$$(\gamma^\mu\gamma^\nu)_\alpha{}^\beta = \eta^{\mu\nu}\delta_\alpha{}^\beta + \epsilon^{\mu\nu\rho}(\gamma_\rho)_\alpha{}^\beta. \tag{3.42}$$

A useful list of identities follows from these:

$$\begin{aligned}(\gamma^\mu)_{\alpha\beta}(\gamma_\mu)_{\gamma\delta} &= \epsilon_{\alpha\gamma}\epsilon_{\delta\beta} + \epsilon_{\alpha\delta}\epsilon_{\gamma\beta},\\ (\gamma^\mu\gamma^\rho\gamma_\mu)_{\alpha\beta} &= -(\gamma_\rho)_{\alpha\beta},\\ A_\alpha{}^\beta &= \tfrac{1}{2}(\operatorname{tr} A)\delta_\alpha{}^\beta + \tfrac{1}{2}\operatorname{tr}(\gamma_\mu A)(\gamma^\mu)_\alpha{}^\beta.\end{aligned} \tag{3.43}$$

The symmetry allows us to nicely decompose bispinors into subspaces. In particular, we may map vectors to symmetric bispinors and vice versa by

$$v_{\alpha\beta} = -2\gamma^\mu_{\alpha\beta}v_\mu, \quad v_\mu = \tfrac{1}{4}\gamma_\mu^{\alpha\beta}v_{\alpha\beta}. \tag{3.44}$$

These imply in particular that

$$\begin{aligned}v_\pm &= \tfrac{1}{4}v_{\pm\pm},\\ v^\pm &= -\tfrac{1}{2}v^{\pm\pm},\\ v^\perp &= v_\perp = -\tfrac{1}{2}v_{+-} = -\tfrac{1}{2}v^{-+}.\end{aligned} \tag{3.45}$$

These rules establish how to transform space-time vectors (tensors) to symmetric bispinors (multispinors).

Integration We establish integration conventions by the requirements

$$\int \mathrm{d}^2\theta\,\theta^2 = 1, \quad \int \mathrm{d}^2\overline{\theta}\,\overline{\theta}^2 = -1 \quad \text{and} \quad \int \mathrm{d}\theta^+\mathrm{d}\overline{\theta}^+\overline{\theta}^+\theta^+ = 1 \tag{3.46}$$

i.e., adjacent symbols cancel in $(0,2)$-integration. These imply

$$\int \mathrm{d}^2\theta = \frac{1}{2}\int \mathrm{d}\theta^-\mathrm{d}\theta^+, \quad \int \mathrm{d}^2\overline{\theta} = \frac{1}{2}\int \mathrm{d}\overline{\theta}^+\mathrm{d}\overline{\theta}^-, \tag{3.47}$$

and

$$\int \mathrm{d}^4\theta = \frac{1}{4}\int \mathrm{d}^2\theta^+\mathrm{d}^2\theta^-, \tag{3.48}$$

where we have defined

$$\int \mathrm{d}^2\theta^\pm = \int \mathrm{d}\theta^\pm\mathrm{d}\overline{\theta}^\pm. \tag{3.49}$$

Supercharges and superderivatives The supercharges of 3D $\mathcal{N} = 2$ supersymmetry as super-differential operators in terms of $(x^\mu, \theta, \overline{\theta})$-coordinates are

$$\begin{aligned}Q_\alpha &= \frac{\partial}{\partial\theta^\alpha} + i(\gamma^\mu\overline{\theta})_\alpha\partial_\mu = \begin{pmatrix}\frac{\partial}{\partial\theta^-} + 2i\overline{\theta}^-\partial_- - i\overline{\theta}^+\partial_\perp\\ \frac{\partial}{\partial\theta^+} + 2i\overline{\theta}^+\partial_+ - i\overline{\theta}^-\partial_\perp\end{pmatrix},\\ \overline{Q}_\alpha &= -\frac{\partial}{\partial\overline{\theta}^\alpha} - i(\gamma^\mu\theta)_\alpha\partial_\mu = \begin{pmatrix}-\frac{\partial}{\partial\overline{\theta}^-} - 2i\theta^-\partial_- + i\theta^+\partial_\perp\\ -\frac{\partial}{\partial\overline{\theta}^+} - 2i\theta^+\partial_+ + i\theta^-\partial_\perp\end{pmatrix}.\end{aligned} \tag{3.50}$$

The covariant derivatives are

$$\begin{aligned}D_\alpha &= \frac{\partial}{\partial\theta^\alpha} - i(\gamma^\mu\overline{\theta})_\alpha\partial_\mu = \begin{pmatrix}\frac{\partial}{\partial\theta^-} - 2i\overline{\theta}^-\partial_- + i\overline{\theta}^+\partial_\perp \\ \frac{\partial}{\partial\theta^+} - 2i\overline{\theta}^+\partial_+ + i\overline{\theta}^-\partial_\perp\end{pmatrix},\\ \overline{D}_\alpha &= -\frac{\partial}{\partial\overline{\theta}^\alpha} + i(\gamma^\mu\theta)_\alpha\partial_\mu = \begin{pmatrix}-\frac{\partial}{\partial\overline{\theta}^-} + 2i\theta^-\partial_- - i\theta^+\partial_\perp \\ -\frac{\partial}{\partial\overline{\theta}^+} + 2i\theta^+\partial_+ - i\theta^-\partial_\perp\end{pmatrix}.\end{aligned} \tag{3.51}$$

They satisfy the $\mathcal{N}=2$ supersymmetry algebra

$$\{Q_\alpha, \overline{Q}_\beta\} = 2i\gamma^\mu_{\alpha\beta}\partial_\mu, \quad \{D_\alpha, \overline{D}_\beta\} = -2i\gamma^\mu_{\alpha\beta}\partial_\mu. \tag{3.52}$$

The action of physical supercharges via commutators is represented by the action via the super-differential operators by

$$[\xi^\alpha\mathcal{Q}_\alpha - \overline{\xi}^\alpha\overline{\mathcal{Q}}_\alpha, X] = i(\xi^\alpha Q_\alpha - \overline{\xi}^\alpha\overline{Q}_\alpha)X =: i\delta^{\xi,\bar{\xi}}_{\mathrm{sym}}X$$

where $\mathcal{Q}$ denotes the physical supercharge.

3.2.3 Supercurrent multiplet in bulk 3D $\mathcal{N}=2$

In this subsection recall, following [69] some structure of the case of interest: bulk 3D $\mathcal{N}=2$ theories.

The most general supercurrent multiplet satisfying the conditions (a)–(d) (called the *$\mathcal{S}$-multiplet*) consists of three superfields, $\mathcal{S}_{\alpha\beta}, \chi_\alpha, \mathcal{Y}_\alpha$ with $\mathcal{S}_{\alpha\beta}$ real symmetric (equivalently, $\mathcal{S}_\mu$ real), $\chi_\alpha, \mathcal{Y}_\alpha$ fermionic, and a complex constant C. They must satisfy the defining superfield relations:

$$\begin{aligned}\overline{D}^\beta\mathcal{S}_{\alpha\beta} &= \chi_\alpha + \mathcal{Y}_\alpha,\\ \overline{D}_\alpha\chi_\beta &= \tfrac{1}{2}C\epsilon_{\alpha\beta},\\ D^\alpha\chi_\alpha + \overline{D}^\alpha\overline{\chi}_\alpha &= 0,\\ D_\alpha\mathcal{Y}_\beta + D_\beta\mathcal{Y}_\alpha &= 0,\\ \overline{D}^\alpha\mathcal{Y}_\alpha + C &= 0.\end{aligned} \tag{3.53}$$

These defining relations are solved by the following expansions (using bispinor relations (3.44)):

$$\begin{aligned}\mathcal{S}_\mu &= j_\mu - i\theta(S_\mu + \tfrac{i}{\sqrt{2}}\gamma_\mu\overline{\omega}) - i\overline{\theta}(\overline{S}_\mu - \tfrac{i}{\sqrt{2}}\gamma_\mu\omega) + \tfrac{i}{2}\theta^2\overline{Y}_\mu + \tfrac{i}{2}\overline{\theta}^2 Y_\mu\\ &\quad - (\theta\gamma^\nu\overline{\theta})\big(2T_{\nu\mu} - \eta_{\mu\nu}A - \tfrac{1}{4}\epsilon_{\mu\nu\rho}H^\rho\big) - i\theta\overline{\theta}\big(\tfrac{1}{4}\epsilon_{\mu\nu\rho}F^{\nu\rho} + \epsilon_{\mu\nu\rho}\partial^\nu j^\rho\big)\\ &\quad + \tfrac{1}{2}\theta^2\overline{\theta}\big(\gamma^\nu\partial_\nu S_\mu - \tfrac{i}{\sqrt{2}}\gamma_\mu\gamma_\nu\partial^\nu\overline{\omega}\big) + \tfrac{1}{2}\overline{\theta}^2\theta\big(\gamma^\nu\partial_\nu\overline{S}_\mu + \tfrac{i}{\sqrt{2}}\gamma_\mu\gamma_\nu\partial^\nu\omega\big)\\ &\quad - \tfrac{1}{2}\theta^2\overline{\theta}^2\big(\partial_\mu\partial^\nu j_\nu - \tfrac{1}{2}\partial^2 j_\mu\big),\end{aligned} \tag{3.54a}$$

$$\begin{aligned}\chi_\alpha &= -i\lambda_\alpha(y) + \theta_\beta\big[\delta_\alpha{}^\beta D(y) - (\gamma^\mu)_\alpha{}^\beta\big(H_\mu(y) - \tfrac{i}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho}(y)\big)\big]\\ &\quad + \tfrac{1}{2}\overline{\theta}_\alpha C - \theta^2(\gamma^\mu\partial_\mu\overline{\lambda})_\alpha(y),\end{aligned} \tag{3.54b}$$

$$\begin{aligned}\mathcal{Y}_\alpha &= \sqrt{2}\omega_\alpha + 2\theta_\alpha B + 2i\gamma^\mu_{\alpha\beta}\overline{\theta}^\beta Y_\mu + \sqrt{2}i(\theta\gamma^\mu\overline{\theta})\epsilon_{\mu\nu\rho}(\gamma^\nu\partial^\rho\omega)_\alpha\\ &\quad + \sqrt{2}i\theta\overline{\theta}(\gamma^\mu\partial_\mu\omega)_\alpha + i\theta^2\gamma^\mu_{\alpha\beta}\overline{\theta}^\beta\partial_\mu B - \overline{\theta}^2\theta_\alpha\partial_\mu Y^\mu + \tfrac{1}{2\sqrt{2}}\theta^2\overline{\theta}^2\partial^2\omega_\alpha.\end{aligned} \tag{3.54c}$$

Here $(S^{\mu})_{\alpha}, (\overline{S}^{\mu})_{\alpha}$ are conserved supercurrents, $T_{\mu\nu}$ is a symmetric energy-momentum tensor in the same improvement frame (see page 56) as the supercurrents, and

$$\begin{aligned} \lambda_{\alpha} &= -2(\gamma^{\mu}\overline{S}_{\mu})_{\alpha} + 2\sqrt{2}i\omega_{\alpha}, \\ D &= -4T^{\mu}{}_{\mu} + 4A, \\ B &= A + i\partial_{\mu}j^{\mu}, \\ \mathrm{d}H = 0, &\quad \mathrm{d}Y = 0, \quad \mathrm{d}F = 0, \end{aligned} \tag{3.55}$$

where H, F, Y are forms with components $H_{\mu}, F_{\mu\nu}, Y_{\mu}$. Additionally, y is the "chiral" coordinate $y^{\mu} = x^{\mu} - i\theta\gamma^{\mu}\overline{\theta}$. If the forms Y or H are exact, the superfields $\mathcal{Y}_{\alpha}$ or χ_{α} may be written as covariant derivatives: If $Y_{\mu} = \partial_{\mu}x$, then $\mathcal{Y}_{\alpha} = D_{\alpha}X$ where $X|_{\theta^0} = x$, and if $H_{\mu} = \partial_{\mu}g$, then $\chi_{\alpha} = i\overline{D}_{\alpha}G$ where $G|_{\theta^0} = g$.

Improvements

As discussed in earler, the solutions (3.54) with constraints (3.55) and current conservation equations $\partial_{\mu}(S^{\mu})_{\alpha} = 0$, $\partial^{\mu}T_{\mu\nu} = 0$ are not the unique solution to defining constraints (3.53). We may improve without violating the constraints

$$\begin{aligned} \mathcal{S}_{\mu} &\mapsto \mathcal{S}_{\mu} + \tfrac{1}{4}\gamma_{\mu}^{\alpha\beta}[D_{\alpha}, \overline{D}_{\beta}]U, \\ \chi_{\alpha} &\mapsto \chi_{\alpha} - \overline{D}^{2}D_{\alpha}U, \\ \mathcal{Y}_{\alpha} &\mapsto \mathcal{Y}_{\alpha} - \tfrac{1}{2}D_{\alpha}\overline{D}^{2}U, \end{aligned} \tag{3.56}$$

where $U = u + \theta\eta - \overline{\theta}\overline{\eta} + \theta^2 N - \overline{\theta}^2\overline{N} + (\theta\gamma^{\mu}\overline{\theta})V_{\mu} - i\theta\overline{\theta}K + \ldots$ is an arbitrary real superfield. On a level of components, the improvement transforms

$$\begin{aligned} (S_{\mu})_{\alpha} &\mapsto (S_{\mu})_{\alpha} + \epsilon_{\mu\nu\rho}(\gamma^{\nu}\partial^{\rho}\eta)_{\alpha}, \\ T_{\mu\nu} &\mapsto T_{\mu\nu} + \tfrac{1}{2}(\partial_{\mu}\partial_{\nu} - \eta_{\mu\nu}\partial^{2})u, \\ H_{\mu} &\mapsto H_{\mu} - 4\partial_{\mu}K, \\ F_{\mu\nu} &\mapsto F_{\mu\nu} - 4(\partial_{\mu}V_{\nu} - \partial_{\nu}V_{\mu}), \\ Y_{\mu} &\mapsto Y_{\mu} - 2\partial_{\mu}\overline{N}. \end{aligned} \tag{3.57}$$

The general multiplet $\mathcal{S}_{\mu}$ encompasses other, more restricted versions of supercurrent multiplets. This is reflected in the fact that $\mathcal{S}_{\mu}$ may be improved into smaller multiplets. In particular:

1. When $C = 0$, $\chi_{\alpha} = i\overline{D}_{\alpha}G$ (in other words, H is exact) where there exists a well-defined real U such that $G = 2i\overline{D}^{\alpha}D_{\alpha}U$, then an improvement by U as above sends χ_{α} to zero and we obtain a *Ferrara-Zumino multiplet* [127] (we relabel $\mathcal{S}_{\alpha\beta}$ to $\mathcal{J}_{\alpha\beta}$):

$$\begin{aligned} \overline{D}^{\beta}\mathcal{J}_{\alpha\beta} &= \mathcal{Y}_{\alpha}, \\ D_{\alpha}\mathcal{Y}_{\beta} + D_{\beta}\mathcal{Y}_{\alpha} = 0, &\quad \overline{D}^{\alpha}\mathcal{Y}_{\alpha} = 0. \end{aligned} \tag{3.58}$$

2. When $C = 0$, $\mathcal{Y}_{\alpha} = D_{\alpha}X$ (equivalently, when Y is exact) where there exists a well-defined real U such that $X = \frac{1}{2}\overline{D}^{2}U$, then an improvement by U as above sends $\mathcal{Y}_{\alpha}$ to zero and we obtain an *$\mathcal{R}$-multiplet* [129]:

$$\begin{aligned} \overline{D}^{\beta}\mathcal{R}_{\alpha\beta} &= \chi_{\alpha}, \\ \overline{D}_{\alpha}\chi_{\beta} = 0, &\quad D^{\alpha}\chi_{\alpha} + \overline{D}^{\alpha}\overline{\chi}_{\beta} = 0. \end{aligned} \tag{3.59}$$

In this case, the lowest component j_μ of the multiplet $\mathcal{R}_\mu$ (we relabel $\mathcal{S}_\mu$ to $\mathcal{R}_\mu$) is a conserved R-current (in the general $\mathcal{S}_\mu$-multiplet, j^μ is not conserved; however, we still call it a "non-conserved R-current").

3. When $C = 0$ and the improvements from 1. and 2. coincide, we can improve both superfields $\chi_\alpha, \mathcal{Y}_\alpha$ to zero *simultaneously*. In that case we obtain a *superconformal multiplet*

$$\overline{D}^\beta \mathcal{S}_{\alpha\beta} = 0. \tag{3.60}$$

Note that even if smaller multiplets exist, they enjoy some remaining "improvement gauge invariance". In particular, we may further improve the smaller multiplets without violating the respective additional constraints. For example, in the case of the $\mathcal{R}$-multiplet, the improvements that preserve the defining constraints (3.59) are transformations

$$\begin{aligned} \mathcal{R}_\mu &\mapsto \mathcal{R}_\mu + \tfrac{1}{4}\gamma_\mu^{\alpha\beta}[D_\alpha, \overline{D}_\beta]U, \\ \chi_\alpha &\mapsto \chi_\alpha - \overline{D}^2 D_\alpha U. \end{aligned} \tag{3.61}$$

where U is not not arbitrary but must satisfy

$$D_\alpha \overline{D}^2 U = 0 \tag{3.62}$$

Brane currents

We may associate to the closed forms F, H, Y, C the *brane currents* defined by taking their Hodge dual in M:

$$C_\mu \sim \epsilon_{\mu\nu\rho} F^{\nu\rho}, \quad C_{\mu\nu} \sim \epsilon_{\mu\nu\rho} H^\rho, \quad C'_{\mu\nu} \sim \epsilon_{\mu\nu\rho} \overline{Y}^\rho, \quad C_{\mu\nu\rho} \sim \epsilon_{\mu\nu\rho} \overline{C}. \tag{3.63}$$

and their conservation follows by construction and the *brane charges* $Z_{\mu_1 \ldots \mu_k} = \int_\Sigma C^0{}_{\mu_1 \ldots \mu_k}$ are conserved and are invariant under the improvements (3.57) (cf. subsection 3.1.4). The brane charges, if they are non-trivial, are central charges of the supersymmetry algebra (but not of the Poincaré algebra). Indeed, the multiplet structure of $\mathcal{S}_\mu$ implies a (more general) local version of the supersymmetry algebra:

$$\begin{aligned} \{\overline{\mathcal{Q}}_\alpha, (S_\mu)_\beta\} &= \gamma^\nu_{\alpha\beta}(2T_{\nu\mu} - \tfrac{1}{4}\epsilon_{\mu\nu\rho}H^\rho) + i\epsilon_{\alpha\beta}\tfrac{1}{4}\epsilon_{\mu\nu\rho}F^{\nu\rho} + \text{total derivatives}, \\ \{\mathcal{Q}_\alpha, (S_\mu)_\beta\} &= \tfrac{1}{4}(\gamma_\mu)_{\alpha\beta}\overline{C} + i\epsilon_{\mu\nu\rho}\gamma^\nu_{\alpha\beta}\overline{Y}^\rho, \end{aligned} \tag{3.64}$$

One may find non-trivial central charges in the supersymmetry algebra upon integration of both sides of each equation (to map $(S_\mu)_\beta$ to Q_β). Each current $C_{\mu\mu_1 \ldots \mu_k}$ and the corresponding charge $Z_{\mu_1 \ldots \mu_k}$ is associated to a k-brane and non-zero brane charges form a physical obstruction to improvements into smaller multiplets. In particular, a non-zero charge associated to F or H obstructs the existence of a Ferrara-Zumino multiplet, and a non-zero charge associated to Y obstructs the existence of an $\mathcal{R}$-multiplet.

3.3 Supercurrent multiplets in 3D with boundary

In this section we will now layout the appropriate extension of supercurrent multiplets to the case with boundary.

3.3.1 General remarks and observations

Let us first discuss how the introduction of a boundary influences the structure of bulk supercurrent multiplets.

Firstly, since at least one momentum generator is broken, supersymmetry is broken to a subalgebra. The main focus of this work and of [3], is the case where subalgebra isomorphic to 2D $\mathcal{N}=(0,2)$ is preserved. We will also partly discuss the case where 2D $\mathcal{N}=(1,1)$ is preserved. Now, the bulk supercurrent multiplets, previously 3D $\mathcal{N}=2$ superfields, must decompose to corresponding subalgebra-superfields. We spell out these decompositions in appendix C.2.1. In addition, the constraints (3.53), which are written in 3D $\mathcal{N}=2$ language, must now decompose into constraints of the smaller subalgebra superfields. We will exhibit the decompositions of these superfield equations in the next subsection.

Secondly, as we saw in section 3.1, when a boundary is introduced, the conserved bulk currents of any *remaining* symmetries (since some can be in principle broken) must be supplemented by appropriate with boundary currents (3.12) satisfying appropriate conservation equations. The complete conservation equations of currents that are included in supercurrent multiplets must follow from the constraints that define said supercurrent multiplets, just like in bulk theories. The *full* supercurrent multiplets must therefore consist of bulk and boundary pieces. The schematic form of full supercurrent multiplets reads

$$\begin{aligned}\mathcal{S}_\mu^{\text{full}} &= \mathcal{S}_\mu^B + \delta(x^n)\mathcal{P}^{\hat{\mu}}{}_\mu \mathcal{S}^\partial_{\hat{\mu}},\\ \chi_\alpha^{\text{full}} &= \chi_\alpha^B + \delta(x^n)\chi^\partial_\alpha,\\ \mathcal{Y}_\alpha^{\text{full}} &= \mathcal{Y}_\alpha^B + \delta(x^n)\mathcal{Y}^\partial_\alpha,\end{aligned} \tag{3.65}$$

where once again $\mathcal{P}^\mu{}_{\hat{\mu}}$ is an embedding.

These two points imply, among other things, that conditions (a)–(d) are modified. It is clear that the new superfields should contain the *full* conserved currents of unbroken symmetries (conditions (a), (b)). Furthermore, improvements of the full conserved currents in the sense of (3.16) that form consistent multiplets under the smaller subalgebra, will now form the improvements new supercurrent multiplets under the smaller subalgebra (condition (c)). Lastly, under the smaller symmetry algebra, the previously indecomposable (bulk) multiplet in general decomposes into several indecomposable multiplets. Therefore condition (d) is not preserved in general.

Let us state the structure of (maximal) subalgebras of the 3D $\mathcal{N}=2$ algebra which may be preserved after the introduction of a boundary. The remaining, unbroken symmetry algebra is generated by tangential translations $P_{\hat{\mu}}$, Lorentz transformations $M_{\hat{\mu}\hat{\nu}}$ in the directions tangential to the boundary, as well as one of the following sets:

1. the supercharges $Q_+, \overline{Q}_+$, which correspond to a 2D $(0,2)$-subalgebra satisfying

$$(Q_+)^2 = 0, \quad \{Q_+, \overline{Q}_+\} = -4i\partial_+, \tag{3.66}$$

2. the supercharges $Q_-, \overline{Q}_-$, i.e., the left-moving $(2,0)$ counterparts,

3. (real) supercharges $\mathbb{Q}_-, \mathbb{Q}_+$ corresponding to a 2D $(1,1)$-subalgebra satisfying

$$(\mathbb{Q}_\pm)^2 = -i\partial_\pm, \quad \{\mathbb{Q}_-, \mathbb{Q}_+\} = 0. \tag{3.67}$$

We focus mostly on the first case. In particular, we want to determine constraint equations that define $\mathcal{R}$-supercurrent multiplets in a 3D theory with boundary and 2D $\mathcal{N} = (0, 2)$ supersymmetry. To do so, we first decompose the 3D $\mathcal{N} = 2$ bulk constraints that define the $\mathcal{R}$-multiplet into $\mathcal{N} = (0, 2)$ bulk constraints, and then investigate $\mathcal{N} = (0, 2)$ boundary constraints.

As a technical step, we supplement the superspace derivative operators $Q_+, \overline{Q}_+$ with covariant derivatives $D_+^{(0,2)}$, $\overline{D}_+^{(0,2)}$ defined in appendix C.1.1. These operators are *not* to be confused with the operators D_α, $\overline{D}_\alpha$ acting on 3D $\mathcal{N} = 2$ superspace; the two sets of operators are related by (C.4). We will omit the label $(0, 2)$ from now on.

3.3.2 Bulk 3D $\mathcal{N} = 2$ constraints decomposed into $\mathcal{N} = (0, 2)$

In this subsection, we will decompose the superfields[4] $(\mathcal{R}_\mu, \chi_\alpha)$ and the constraints (3.59) by the $\mathcal{N} = (0, 2)$ subalgebra. The decomposition can be performed using features of superspace: we define a *branching coordinate* ξ^μ that will essentially reduce the decomposition into a Taylor expansion. It has the defining property that in the coordinates $(\xi^\mu, \theta^+, \theta^-)$, the preserved supercharges Q_+ and $\overline{Q}_+$ commute with θ^- and $\overline{\theta}^-$; another property is that Q_+, $\overline{Q}_+$ do not involve a derivative in $\perp$-direction; for precise details, see appendix C.1.1. In terms of ξ we can decompose

$$
\begin{aligned}
\mathcal{R}^B_\mu(x, \theta, \overline{\theta}) &= \mathcal{R}^{B(0)}_\mu + \theta^- \mathcal{R}^{B(1)}_\mu - \overline{\theta}^- \overline{\mathcal{R}^{B(1)}_\mu} + \theta^- \overline{\theta}^- \mathcal{R}^{B(2)}_\mu, \\
\chi^B_\alpha(x, \theta, \overline{\theta}) &= \chi^{B(0)}_\alpha + \theta^- \chi^{B(1a)}_\alpha + \overline{\theta}^- \chi^{B(1b)}_\alpha + \theta^- \overline{\theta}^- \chi^{B(2)}_\alpha,
\end{aligned}
\tag{3.68}
$$

where we now denote bulk fields by a sub-/superscript B, and boundary fields (to appear later) with a sub-/superscript ∂. The bracketed number superscripts refer to the order in $\theta^-, \overline{\theta}^-$ we have expanded in. Here, each field on each right-hand side is a function of $(\xi, \theta^+, \overline{\theta}^+)$, and we have suppressed the dependence for readability. The attractive feature of ξ is the following: because Q_+, $\overline{Q}_+$ commute with θ^-, $\overline{\theta}^-$, the coefficient at each order in $\theta^-, \overline{\theta}^-$ is in fact a $(0, 2)$-submultiplet — the remaining supersymmetry group acts independently on each of them. This is a constructive way to decompose 3D $\mathcal{N} = 2$ superfields with respect to the 2D $\mathcal{N} = (0, 2)$ subalgebra. The above decomposition, e.g., for $\mathcal{S}$ is written explicitly in the appendix (C.26)–(C.28). The decomposition for $\mathcal{R}$-multiplet then follows immediately by setting appropriate terms to zero.

In terms of the $(0, 2)$-submultiplets, the constraints (3.59) are then written as the following collection of equations, where we use coordinates $\xi^+ = \xi^0 + \xi^1$, $\xi^- = \xi^0 - \xi^1$ and $\xi^\perp = x^\perp + i(\theta^+ \overline{\theta}^- - \theta^- \overline{\theta}^+)$. Explicitly:
From $\overline{D}_- \chi_\alpha = 0$:

$$
\chi^{B(1b)}_\alpha = 0, \tag{3.69a}
$$

$$
\chi^{B(2)}_\alpha + 2i\partial_- \chi^{B(0)}_\alpha = 0. \tag{3.69b}
$$

[4]The more general case of the $\mathcal{S}$-multiplet is quite similar and is given in the appendix, see (C.32)–(C.37).

From $\overline{D}_+\chi_\alpha = 0$:

$$\overline{D}_+\chi_\alpha^{B(0)} = 0, \tag{3.70a}$$
$$\overline{D}_+\chi_\alpha^{B(1a)} + 2i\partial_\perp\chi_\alpha^{B(0)} = 0, \tag{3.70b}$$
$$\overline{D}_+\chi_\alpha^{B(2)} = 0. \tag{3.70c}$$

From $\operatorname{Im} D^\alpha\chi_\alpha = 0$:

$$\operatorname{Im}(D_+\chi_-^{B(0)} - \chi_+^{B(1a)}) = 0, \tag{3.71a}$$
$$\overline{D}_+\overline{\chi_-^{B(1a)}} + \chi_+^{B(2)} - 2i\partial_-\chi_+^{B(0)} - 2i\partial_\perp\chi_-^{B(0)} = 0, \tag{3.71b}$$
$$\operatorname{Im}(D_+\chi_-^{B(2)} - 2i\partial_-\chi_+^{B(1a)} - 2i\partial_\perp\chi_-^{B(1a)}) = 0. \tag{3.71c}$$

Finally, the relation $\overline{D}^\beta\mathcal{R}_{\alpha\beta} = \chi_\alpha$ yields:[5]

$$\chi_\alpha^{B(0)} = \overline{D}_+\mathcal{R}_{\alpha-}^{B(0)} - \overline{\mathcal{R}_{\alpha+}^{B(1)}}, \tag{3.72a}$$
$$-\chi_\alpha^{B(1a)} = \overline{D}_+\mathcal{R}_{\alpha-}^{B(1)} + \mathcal{R}_{\alpha+}^{B(2)} + 2i\partial_\perp\mathcal{R}_{\alpha-}^{B(0)} + 2i\partial_-\mathcal{R}_{\alpha+}^{B(0)}, \tag{3.72b}$$
$$0 = \overline{D}_+\overline{\mathcal{R}_{\alpha\beta}^{B(1)}}, \tag{3.72c}$$
$$\chi_\alpha^{B(2)} = \overline{D}_+\mathcal{R}_{\alpha-}^{B(2)} + 2i\partial_\perp\overline{\mathcal{R}_{\alpha-}^{B(1)}} + 2i\partial_-\overline{\mathcal{R}_{\alpha+}^{B(1)}}. \tag{3.72d}$$

Note that we have not introduced *any* new structure here: component-wise, equations (3.59) have identical content as (3.69a)–(3.72d), simply "packaged" in $\mathcal{N} = (0,2)$ language. In particular, the bulk conservation equations follow from these constraints. Let us explicitly recover for example the conservation of the R-current j_μ^B. This might seem like a pointless exercise, but it will elucidate the analogous computation necessary to "guess" the boundary constraints later.

We start with equation (3.72b) setting $\alpha = +$ and taking the imaginary part. Using the reality of $\mathcal{R}_{\alpha\beta}$ (which implies the reality of $\mathcal{R}_{\alpha\beta}^{B(0)}$ and $\mathcal{R}_{\alpha\beta}^{B(2)}$), we arrive at

$$-\operatorname{Im}(\chi_+^{B(1a)}) = \operatorname{Im}(\overline{D}_+\mathcal{R}_{+-}^{B(1)}) + 2\partial_\perp\mathcal{R}_{+-}^{B(0)} + 2\partial_-\mathcal{R}_{++}^{B(0)}. \tag{3.73}$$

Now consider equation (3.72a); setting $\alpha = -$, conjugating, applying $\overline{D}_+$ on both sides and finally taking the imaginary part we obtain

$$\operatorname{Im}(\overline{D}_+\overline{\chi_-^{B(0)}}) = \operatorname{Im}(\overline{D}_+D_+\mathcal{R}_{--}^{B(0)}) - \operatorname{Im}(\overline{D}_+\mathcal{R}_{-+}^{B(1)}). \tag{3.74}$$

From the reality of $\mathcal{R}_{\alpha\beta}^{B(0)}$ we get $\operatorname{Im}(\overline{D}_+D_+\mathcal{R}_{--}^{B(0)}) = 2\partial_+\mathcal{R}_{--}^{B(0)}$. Finally, we use (3.71a) to combine (3.73) and (3.74) into the bulk conservation equation for the R-current:

$$2\partial_+\mathcal{R}_{--}^{B(0)} + 2\partial_\perp\mathcal{R}_{+-}^{B(0)} + 2\partial_-\mathcal{R}_{++}^{B(0)} = 0. \tag{3.75}$$

This equation also implies the bulk conservation of $(S_\mu^B)_+, (\overline{S}_\mu^B)_+$ and $T_{\mu+}^B$, as can be verified by the expansions (C.26)–(C.28).

The bulk conservation for $\mathcal{R}_{\alpha\beta}^{B(1)}$ and $\mathcal{R}_{\alpha\beta}^{B(2)}$ follows similarly, while the conservation of $\mathcal{R}_{\alpha\beta}^{B(2)}$ follows from (3.71c), (3.72b) and (3.72d), and implies the conservation of the bulk

[5] These have already been simplified by some relations already derived, e.g., (3.69a).

tensor $T^B_{\mu-}$. More interestingly, the conservation of $\mathcal{R}^{B(1)}_{\alpha\beta}$ follows from (3.72a)–(3.72d) together with (3.71b), and implies the conservation of bulk supercurrents $(S^B_\mu)_-, (\overline{S}^B_\mu)_-$ and the tensor $T^B_{\mu\perp}$. As expected, the decomposition "singles out" these components, indicating that they are components in "broken" directions (once a boundary is introduced).

3.3.3 Boundary constraints in $\mathcal{N} = (0, 2)$

In this subsection we want to essentially guess the correct structure for the boundary pieces of the supercurrent multiplet. In the case of the $\mathcal{R}$-multiplet we have restricted ourselves to, the boundary parts are (collections of) superfields $\mathcal{R}^\partial_\mu$ and χ^∂_α (and $\mathcal{Y}^\partial_\alpha$ in the case of the $\mathcal{S}_\mu$-multiplet). These must satisfy constraints, i.e., $\mathcal{N} = (0, 2)$-superfield equations, such that the boundary conservation equations follow for their components. To extract an educated guess, we note again that the bulk constraints (3.69)–(3.72) can be rewritten: bulk and boundary superfields are combined to our *total* supercurrent multiplet

$$\mathcal{R}^{\text{full}}_\mu = \mathcal{R}^B_\mu + \delta(\xi^\perp)\mathcal{P}_\mu{}^{\hat\mu}\mathcal{R}^\partial_{\hat\mu}, \tag{3.76}$$

where ξ is the "branching coordinate" (see discussion in the appendix around (C.1)) and where both bulk and boundary pieces can be decomposed into $(0, 2)$-multiplets:[6]

$$\begin{aligned} \mathcal{R}^B_\mu(x, \theta, \overline{\theta}) &= \mathcal{R}^{B(0)}_\mu + \theta^- \mathcal{R}^{B(1)}_\mu - \overline{\theta}^- \overline{\mathcal{R}^{B(1)}_\mu} + \theta^-\overline{\theta}^- \mathcal{R}^{B(2)}_\mu, \\ \mathcal{R}^\partial_{\hat\mu}(x, \theta, \overline{\theta}) &= \mathcal{R}^{\partial(0)}_{\hat\mu} + \theta^-\overline{\theta}^- \mathcal{R}^{\partial(2)}_{\hat\mu}, \end{aligned} \tag{3.77}$$

and for auxiliary fields

$$\begin{aligned} \chi^B_\alpha(x, \theta, \overline{\theta}) &= \chi^{B(0)}_\alpha + \theta^- \chi^{B(1a)}_\alpha + \overline{\theta}^- \chi^{B(1b)}_\alpha + \theta^-\overline{\theta}^- \chi^{B(2)}_\alpha, \\ \chi^\partial_\alpha(x, \theta, \overline{\theta}) &= \chi^{\partial(0)}_\alpha + \theta^- \chi^{\partial(1a)}_\alpha + \overline{\theta}^- \chi^{\partial(1b)}_\alpha + \theta^-\overline{\theta}^- \chi^{\partial(2)}_\alpha. \end{aligned} \tag{3.78}$$

Here, again, the fields on the right-hand sides are functions of $(\xi, \theta^+, \overline{\theta}^+)$. The boundary part is motivated by the $(0, 2)$-expansions (C.26)–(C.28) of the bulk multiplet, which imply that

$$\begin{aligned} \mathcal{R}^{B(0)}_\mu &= j^B_\mu + \dots, \\ \mathcal{R}^{B(1)}_\mu &= -i(S^B_\mu)_- + \dots, \\ \mathcal{R}^{B(2)}_\mu &= -2K_{\mu-} + \dots, \end{aligned} \tag{3.79}$$

where again j^B_μ is the R-symmetry current, (S^B_μ) the supercurrent and, (for the $\mathcal{R}$-multiplet) $K_{\mu\nu} = 2T_{\nu\mu} - \frac{1}{4}\epsilon_{\mu\nu\rho}H^\rho$. We therefore simply complete these pairs by setting

$$\begin{aligned} \mathcal{R}^{\partial(0)}_{\hat\mu} &= j^\partial_{\hat\mu} + \dots, \\ \mathcal{R}^{\partial(2)}_{\hat\mu} &= -2K^\partial_{\hat\mu-} + \dots, \end{aligned} \tag{3.80}$$

as the bulk conserved currents have to be paired with their respective boundary currents. Note that we do not consider a boundary contribution to the "broken" $(S^B_\mu)_-$ currents, as we have no guiding principle in this framework. The constraints that the boundary

[6] This is essentially an embedding into 3D $\mathcal{N} = 2$ superspace, see [132, 133].

pieces must satisfy, must be of similar form as (3.69a)–(3.72d), but instead of imposing divergence-freeness, they should impose (3.12) on the *remaining, conserved* boundary currents.

We postulate the following adjustments on (3.69)–(3.72), now applied to boundary multiplets $\mathcal{R}^{\partial(*)}_{\alpha\alpha}$ and we check their validity directly after:

1. There are only $+,-$ spacetime directions for a two-dimensional boundary (x_{++}, x_{--} in bispinor notation, cf. (3.44)). Hence, we only have superfields $\mathcal{R}^{\partial(*)}_{++}, \mathcal{R}^{\partial(*)}_{--}$, and no superfield $\mathcal{R}^{\partial(*)}_{+-}$.
2. The supersymmetry associated to $(S^B_\mu)_-$ is broken in the presence of a boundary; therefore, we do not consider boundary contributions to this component and as such no $\mathcal{R}^{\partial(1)}_{\alpha\alpha}$ should appear.
3. Lastly, in order to transform the bulk conservation equations to boundary conservation equations for boundary components, we must replace terms of the form $\partial_\perp A^\partial$ with $-A^B|_\partial$ whenever such terms appear, as motivated by the form of (3.12). This transformation parses well with the fact that derivatives in the $\perp$-direction make little sense when they act on boundary currents, in particular when the boundary currents are functions of purely boundary fields.

Using these principles, we can obtain the following set of constraints on the postulated boundary multiplets:
Analogs to (3.69):

$$\chi^{\partial(1b)}_\alpha = 0, \tag{3.81a}$$
$$\chi^{\partial(2)}_- + 2i\partial_-\chi^{\partial(0)}_- = 0. \tag{3.81b}$$

Analogs to (3.70):

$$\overline{D}_+\chi^{\partial(0)}_- = 0, \tag{3.82a}$$
$$\overline{D}_+\chi^{\partial(1a)}_+ - 2i\chi^{B(0)}_+|_\partial = 0, \tag{3.82b}$$
$$\overline{D}_+\chi^{\partial(2)}_\alpha = 0. \tag{3.82c}$$

Analogs to (3.71):

$$\mathrm{Im}(D_+\chi^{\partial(0)}_- - \chi^{\partial(1a)}_+) = 0, \tag{3.83a}$$
$$\mathrm{Im}(D_+\chi^{\partial(2)}_- - 2i\partial_-\chi^{\partial(1a)}_+ + 2i\chi^{B(1a)}_-|_\partial) = 0. \tag{3.83b}$$

Lastly, the analogs to (3.72):

$$\chi^{\partial(0)}_+ = 0, \tag{3.84a}$$
$$\chi^{\partial(0)}_- = \overline{D}_+\mathcal{R}^{\partial(0)}_{--}, \tag{3.84b}$$
$$\chi^{\partial(1a)}_+ = -\mathcal{R}^{\partial(2)}_{++} + 2i\mathcal{R}^{B(0)}_{+-}|_\partial - 2i\partial_-\mathcal{R}^{\partial(0)}_{++}, \tag{3.84c}$$
$$\chi^{\partial(1a)}_- = 2i\mathcal{R}^{B(0)}_{--}|_\partial, \tag{3.84d}$$
$$\chi^{\partial(2)}_+ = -2i\overline{\mathcal{R}^{B(1)}_{+-}}|_\partial, \tag{3.84e}$$
$$\chi^{\partial(2)}_- = \overline{D}_+\mathcal{R}^{\partial(2)}_{--} - 2i\overline{\mathcal{R}^{B(1)}_{--}}|_\partial. \tag{3.84f}$$

Note that, a priori by applying our guiding rules 1.–3. we obtain three further relations, which we have intentionally omitted. More precisely, these are: the analog of (3.69b) for $\alpha = +$, which reads

$$\chi_+^{\partial(2)} + 2i\partial_- \chi_+^{\partial(0)} = 0, \tag{3.85}$$

the analog of (3.70b) for $\alpha = -$, which reads

$$\overline{D}_+ \chi_-^{\partial(1a)} - 2i\chi_-^{B(0)}|_\partial, \tag{3.86}$$

and lastly, the analog of (3.71b) reads

$$\overline{D}_+ \overline{\chi_-^{\partial(1a)}} + \chi_+^{\partial(2)} - 2i\partial_- \chi_+^{\partial(0)} + 2i\chi_-^{B(0)}|_\partial = 0. \tag{3.87}$$

To see why these must be omitted, we have the following argument from [3]: the first relation (3.85) is compatible with equations (3.84a) and (3.84e) *only if* $\mathcal{R}_{+-}^{B(1)}|_\partial = 0$. The second relation (3.86) is consistent with equations (3.84d) and (3.72a) again *only if* $\mathcal{R}_{+-}^{B(1)}|_\partial = 0$. Lastly, the third relation (3.87) is consistent with (3.84a), (3.84d), (3.84e) and (3.72a), once more *only if* $\mathcal{R}_{+-}^{B(1)}|_\partial = 0$. Hence, including any of the three relations in the constraints of boundary $(0,2)$-supermultiplets implies that we must have

$$\mathcal{R}_{+-}^{B(1)}|_\partial = -i(S_\perp^B)_-|_\partial + \ldots = 0, \tag{3.88}$$

which is equivalent to the conservation of the "broken" charge Q_- (and similarly for $\overline{Q}_-$, with trivial boundary components (3.12) with a trivial boundary part. However, the conservation of $Q_-, \overline{Q}_-$ would imply, by the supersymmetry algebra, the conservation of of $P_\perp$, which is obviously inconsistent with the presence of the boundary.

Let us now verify that the boundary conservation equation follows from these constraints, in the example of $\mathcal{R}_{\alpha\alpha}^{\partial(0)}$: Taking the imaginary part of (3.84c) and using the reality of the multiplet, we have

$$\mathrm{Im}(\chi_+^{\partial(1a)}) = 2\mathcal{R}_{+-}^{B(0)}|_\partial - 2\partial_- \mathcal{R}_{++}^{\partial(0)}. \tag{3.89}$$

Now we take equation (3.84b), conjugate it, apply $\overline{D}_+$ on both sides and finally take imaginary part again and we get:

$$\mathrm{Im}(\overline{D}_+ \overline{\chi_-^{\partial(0)}}) = \mathrm{Im}(\overline{D}_+ D_+ \mathcal{R}_{--}^{\partial(0)}). \tag{3.90}$$

Again, due to the reality of $\mathcal{R}_{--}^{\partial(0)}$, we have that $\mathrm{Im}(\overline{D}_+ D_+ \mathcal{R}_{--}^{\partial(0)}) = 2\partial_+ \mathcal{R}_{--}^{\partial(0)}$. Finally, we can combine equations (3.89) and (3.90) using (3.83a) into the conservation equation for the boundary R-current:

$$2\mathcal{R}_{+-}^{B(0)}|_\partial - 2\partial_- \mathcal{R}_{++}^{\partial(0)} - 2\partial_+ \mathcal{R}_{--}^{\partial(0)} = 0. \tag{3.91}$$

Like its bulk counterpart, this superfield equation also implies the boundary conservation of $(S_{\hat\mu}^\partial)_+, (\overline{S}_{\hat\mu}^\partial)_+$ and $T_{\hat\mu+}^\partial$.

In a similar fashion, we may derive boundary conservation of $\mathcal{R}_{\alpha\alpha}^{\partial(2)}$ using equations (3.84f), (3.83b), (3.84c) and (3.84d). Component-wise it implies the conservation of the boundary tensor $T_{\hat\mu-}^\partial$. As we would expect, no boundary analog to bulk conservation of $(S_\mu^B)_-$ follows from the boundary constraints.

3.3.4 Integrated supercurrent multiplets

Comparison to pure 2D theories

The supercurrent multiplets defined by (3.69)–(3.72) for the bulk part and (3.81)–(3.84) for the boundary part are $(0,2)$-multiplets of a three-dimensional theory with boundary, with supersymmetry algebra isomorphic to that of a 2D $(0,2)$-theory. Since 2D $(0,2)$-theories come with a lot of inherent structure [134, 135], it is interesting to compare these pure 2D $(0,2)$ supercurrent multiplets with the exising structure of this theory. Let us first recall some generalities about the pure 2D supercurrent multiplets following [69, 134].

In a 2D theory with $\mathcal{N}=(0,2)$ supersymmetry, the general $\mathcal{S}$-multiplet is given by the following superfields $(\mathcal{S}_{++}^{(0,2)}, \mathcal{W}_{-}^{(0,2)}, \mathcal{T}_{----}^{(0,2)}, C)$ along with defining constraints, such that conditions (a)–(d) are satisfied. For details on their structure see [69, 134]. Similarly to three and four dimensions, if the $(0,2)$-model we consider has an R-symmetry, we may improve the $\mathcal{S}_{++}^{(0,2)}$-multiplet into a smaller multiplet set $(\mathcal{R}_{\hat{\mu}}^{(0,2)}, \mathcal{T}_{----}^{(0,2)})$ whose lowest components are those of the R-current. In addition to the R-current components, these multiplets contain an improved energy momentum tensor $T_{\mu\nu}$. Furthermore, the structure of the multiplet guarantees that we can define the so-called *half-twisted* energy-momentum tensor $\widetilde{T}_{\mu\nu}$

$$\begin{aligned}\widetilde{T}_{++} &= T_{++} + \tfrac{i}{2}\partial_{+} j_{+},\\ \widetilde{T}_{+-} &= T_{+-} - \tfrac{i}{2}\partial_{-} j_{+},\\ \widetilde{T}_{--} &= T_{--} - \tfrac{i}{2}\partial_{-} j_{-},\end{aligned} \tag{3.92}$$

which satisfies

$$\begin{aligned}&\{\overline{Q}_{+}, \cdots\} = \widetilde{T}_{+\hat{\mu}},\\ &\{\overline{Q}_{+}, \widetilde{T}_{--}\} = 0, \quad \text{but } \{\overline{Q}_{+}, \cdots\} \neq \widetilde{T}_{--},\end{aligned} \tag{3.93}$$

simply by virtue of the multiplet structure. In other words, the components of the twisted energy-momentum tensor are $\overline{Q}_{+}$-cohomology elements, and $\widetilde{T}_{--}$ is a non-trivial element. The representation in $\overline{Q}_{+}$-cohomology is important, because one can show that the chiral algebra in the $\overline{Q}_{+}$-cohomology of observables is invariant under renormalization group flow and therefore 'knows' about possible IR fixed points of the model under consideration [134]. In particular, there is an emergent conformal symmetry on the level of cohomology.

As a technical step, note that the $\overline{Q}_{+}$-cohomology is isomorphic to the $\overline{D}_{+}$-cohomology (as an operator acting on fields). We can therefore establish the existence of the "half-twisted" stress tensor on the level of superfields by examining the defining constraints of the 2D $(0,2)$ supercurrent multiplet: the non-trivial components of the twisted energy-momentum tensor must arise as a $\overline{D}_{+}$-closed linear combination of superfields from the supercurrent multiplet. Indeed, one can easily check [134] that the following relation holds:

$$\overline{D}_{+}(\mathcal{T}_{----}^{(0,2)} + 2i\partial_{-}\mathcal{R}_{--}^{(0,2)}) = 0. \tag{3.94}$$

In the remaining of this subsection we will perform a similar analysis to conclude whether some linear combination of fields (corresponding to a "half-twisted" stress tensor) is in the $\overline{Q}_{+}$-cohomology.

An energy-momentum tensor (not) in the cohomology

Our three-dimensional theory with boundary has the same supersymmetry algebra as a 2D $\mathcal{N} = (0, 2)$ theory; hence, we might expect a similar structure as far as $\overline{Q}_+$-cohomology is concerned. Indeed, we can identify the relations similar to (3.94) in 3D by combining bulk equations (3.69b),(3.72a) and (3.72d) for $\alpha = -$ to

$$\overline{D}_+(\mathcal{R}^{B(2)}_{--} + 2i\partial_-\mathcal{R}^{B(0)}_{--}) = -2i\partial_\perp \overline{\mathcal{R}^{B(1)}_{--}}, \tag{3.95}$$

as well as boundary equations (3.81b), (3.84b) and (3.84f) to

$$\overline{D}_+(\mathcal{R}^{\partial(2)}_{--} + 2i\partial_-\mathcal{R}^{\partial(0)}_{--}) = 2i\overline{\mathcal{R}^{B(1)}_{--}}|_\partial. \tag{3.96}$$

In addition, we can also use (3.70b), (3.72a) and (3.72b) for $\alpha = +$ to conclude

$$\overline{D}_+(\mathcal{R}^{B(2)}_{++} + 2i\mathcal{R}^{B(0)}_{++}) = -2i\partial_\perp \overline{\mathcal{R}^{B(1)}_{++}}. \tag{3.97}$$

Similarly, we combine boundary equations (3.82b), (3.83b) and (3.72a) into

$$\overline{D}_+(\mathcal{R}^{\partial(2)}_{++} + 2i\partial_-\mathcal{R}^{\partial(0)}_{++}) = 2i\overline{\mathcal{R}^{B(1)}_{++}}|_\partial. \tag{3.98}$$

Using the full $\mathcal{R}$-multiplet $\mathcal{R}^{\text{full}(*)}_{\alpha\beta} = \mathcal{R}^{B(*)}_{\alpha\beta} + \delta(\xi^\perp)\mathcal{R}^{\partial(*)}_{\alpha\beta}$ these relations are written

$$\overline{D}_+(\mathcal{R}^{\text{full}(2)}_{\pm\pm} + 2i\partial_-\mathcal{R}^{\text{full}(0)}_{\pm\pm}) = -2i\partial_\perp \overline{\mathcal{R}^{B(1)}_{\pm\pm}} + 2i\delta(\xi^\perp)\overline{\mathcal{R}^{B(1)}_{\pm\pm}}|_\partial, \tag{3.99}$$

Integrated currents and multiplets

From equation (3.99) we deduce we cannot simply apply the pure 2D argument to extract a local energy-momentum tensor twisted by the R-symmetry such that it is a $\overline{Q}_+$-cohomology element. However, some similar structure still remains which we discuss now.

We may view our three-dimensional quantum field theory with boundary with a finite number of fields as a two-dimensional quantum field theory with an infinite number of fields. More precisely, instead of viewing bulk fields, loosely speaking, as maps $\partial M \times \mathbb{R}_{\leq 0} \to T$, we view them as maps $\partial M \to \{\text{maps: } \mathbb{R}_{\leq 0} \to T\}$ [39, 136]. Now, instead of considering separate bulk and boundary actions, we can write a single Lagrangian for the full theory:

$$\mathcal{L}^{\text{int.}} := \mathcal{L}^\partial + \int_{\mathbb{R}_{\leq 0}} \mathrm{d}x^\perp \mathcal{L}^B; \quad S = \int_{\partial M} \mathrm{d}x^{N-1}\mathcal{L}^{\text{int.}}. \tag{3.100}$$

The action is unchanged, but integration along $x^\perp$ is now an operation on the new target space, as opposed to an integral on spacetime. The integration along $x^\perp$ also translates to conserved currents: as the theory is now formally two-dimensional, applying Noether's theorem to the above Lagrangian yields a two-dimensional current of the form

$$J^{\text{int.}}_{\hat{\mu}} = J^\partial_{\hat{\mu}} + \int_{\mathbb{R}_{\leq 0}} \mathrm{d}x^n J^B_{\hat{\mu}}. \tag{3.101}$$

Its conserved charge

$$Q = \int_{\partial\Sigma} J^0_{\text{int.}} \tag{3.102}$$

is identical to the one belonging to the local current: To find the conserved charge of a current, one has to integrate all spatial directions, and the integrated current is merely an "intermediate step" of this integration. The conservation equations (3.12) now take the familiar form

$$\partial^{\hat{\mu}} J^{\text{int.}}_{\hat{\mu}} = 0, \tag{3.103}$$

where boundary conditions are possibly used.

In a similar fashion, we introduce the *integrated supercurrent multiplets*

$$\mathcal{R}^{\text{int.}}_{\alpha\alpha} = \mathcal{R}^{\partial}_{\alpha\alpha} + \int_{\mathbb{R}_{\leq 0}} \mathrm{d}x^{\perp} \mathcal{R}^{B}_{\alpha\alpha} \tag{3.104}$$

These have then the desirable property that the right-hand side of (3.99) cancels exactly due to the integral:

$$\begin{aligned} \overline{D}_{+}(\mathcal{R}^{\text{int.}(2)}_{--} + 2i\partial_{-}\mathcal{R}^{\text{int.}(0)}_{--}) &= 0, \\ \overline{D}_{+}(\mathcal{R}^{\text{int.}(2)}_{++} + 2i\partial_{-}\mathcal{R}^{\text{int.}(0)}_{++}) &= 0. \end{aligned} \tag{3.105}$$

The general arguments from [134] presented in section 3.3.4 then imply that the lowest components, $\sim (T^{\text{int.}}_{--} - \frac{i}{2}\partial_{-} j^{\text{int.}}_{-})$ appearing in the first equation and $\sim (T^{\text{int.}}_{-+} - \frac{i}{2}\partial_{-} j^{\text{int.}}_{+})$ appearing in the second equation, are in fact $\overline{Q}_{+}$-cohomology elements in the integrated 2D theory. In fact, a stronger statement holds: the integrated multiplets are genuine 2D $\mathcal{N} = (0,2)$ supersymmetry multiplets. Setting

$$\begin{aligned} \mathcal{R}^{(0,2)}_{\alpha\alpha} &:= \mathcal{R}^{\text{int.}(0)}_{\alpha\alpha}, \\ \mathcal{T}^{(0,2)}_{----} &:= -\mathcal{R}^{\text{int.}(2)}_{--}, \end{aligned} \tag{3.106}$$

the constraints (3.69)–(3.72) and (3.81)–(3.84) imply that these are proper 2D $(0,2)$-supercurrent multiplets in the sense of [69, 134].

3.4 Case study: Landau-Ginzburg model

In this section we want to test the structure outlined in the previous section 3.3 on a simple theory, namely one with a chiral superfield and a superpotential, following [3]. First we recall some structure of this model from kindergarden, including explicit field content in components, the Lagrangian and equations of motion. Then we introduce a boundary and discuss preservation of ($\mathcal{N} = (0,2)$) supersymmetry. In that regard we discuss adding boundary matter and possible boundary conditions. Lastly, we explicitly identify examples of supercurrent multiplets accommodating for the existence of the boundary.

3.4.1 Set-up: Field content and boundaries

Bulk theory for one chiral with superpotential

We now study a 3D $\mathcal{N} = 2$ Landau-Ginzburg model, i.e., a theory of chiral superfields with superpotential, which lives on three-dimensional Minkowski space, which will be modified later to the half-space $M = \{x \in \mathbb{R}^{1,2} | x^{\perp} := x^{1} \leq 0\}$ with boundary.

The chiral field is given by

$$\Phi_{3D}(x,\theta,\overline{\theta}) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y), \qquad y^{\mu} = x^{\mu} - i\theta\gamma^{\mu}\overline{\theta}, \tag{3.107}$$

where ϕ is a complex scalar field, ψ_α is a complex fermion, and F is a complex auxiliary field. Under the 3D $\mathcal{N}=2$ supersymmetry, the components transform:

$$\begin{aligned}\delta_{\text{sym}}\phi &= \sqrt{2}\epsilon\psi,\\ \delta_{\text{sym}}\psi_\alpha &= \sqrt{2}\epsilon_\alpha F - \sqrt{2}i(\gamma^\mu\bar{\epsilon})_\alpha\partial_\mu\phi,\\ \delta_{\text{sym}}F &= -\sqrt{2}i\bar{\epsilon}\gamma^\mu\partial_\mu\psi.\end{aligned} \tag{3.108}$$

The simplest Landau-Ginzburg model consists of a single chiral superfield with Kähler term

$$\mathcal{L}_{\text{kin.}} = \int \mathrm{d}^4\theta\, \Phi_{3D}\overline{\Phi}_{3D} = -\partial_\mu\bar{\phi}\partial^\mu\phi + \tfrac{i}{2}(\bar{\psi}\gamma^\mu\partial_\mu\psi) - \tfrac{i}{2}(\partial_\mu\bar{\psi}\gamma^\mu\psi) + \overline{F}F + \tfrac{1}{4}\partial^2(\bar{\phi}\phi), \tag{3.109}$$

where we keep total derivative terms explicitly as they become important when a boundary is introduced.

The superpotential is determined in superspace and component form:

$$\mathcal{L}_W = \int \mathrm{d}^2\theta\, W(\Phi_{3D}) + \text{cc.} = W'(\phi)F - \tfrac{1}{2}W''(\phi)\psi\psi + \text{cc.} \tag{3.110}$$

The bulk equations of motion are given by

$$\overline{D}^2\overline{\Phi} = -4W'(\Phi) \Leftrightarrow \left\{\begin{aligned}0 &= \overline{F} + W'(\phi)\\ 0 &= \partial_\mu\partial^\mu\bar{\phi} + W''(\phi)F - \tfrac{1}{2}W'''(\phi)\psi\psi\\ 0 &= i(\gamma^\mu\partial_\mu\bar{\psi})_\alpha - W''(\phi)\psi_\alpha\end{aligned}\right\}. \tag{3.111}$$

Adding a boundary and breaking to $\mathcal{N}=(0,2)$

Now let's consider that the theory is on the half-space M. As discussed in subsection 3.3.2, we must first decompose bulk fields into representations of the remaining $\mathcal{N}=(0,2)$ subalgebra. Under the subalgebra, the chiral field Φ_{3D} decomposes into a $(0,2)$ chiral multiplet and a Fermi multiplet. More details about decompositions are written in appendix C.1.1. We obtain

$$\begin{aligned}\Phi &= \phi + \sqrt{2}\theta^+\psi_+ - 2i\theta^+\bar{\theta}^+\partial_+\phi,\\ \Psi &= \psi_- - \sqrt{2}\theta^+F - 2i\theta^+\bar{\theta}^+\partial_+\psi_- + \sqrt{2}i\bar{\theta}^+\partial_\perp\phi - 2i\theta^+\bar{\theta}^+\partial_\perp\psi_+.\end{aligned} \tag{3.112}$$

The chirality condition reads

$$\overline{D}_+\Phi = 0, \tag{3.113}$$

while the "chirality" condition for the Fermi superfield is

$$\overline{D}_+\Psi = \sqrt{2}E_\Psi, \quad E_\Psi = -i\partial_\perp\Phi, \tag{3.114}$$

where E_Ψ is the E-potential. The $(0,2)$ supersymmetry variation (generated by $\delta_{\text{sym}} := \epsilon Q_+ - \bar{\epsilon}\overline{Q}_+$, cf. equation (C.2)) of the component fields is

$$\begin{aligned}&\delta_{\text{sym}}\phi = \sqrt{2}\epsilon\psi_+, && \delta_{\text{sym}}\psi_+ = -2\sqrt{2}i\bar{\epsilon}\partial_+\phi,\\ &\delta_{\text{sym}}F = 2\sqrt{2}i\bar{\epsilon}\partial_+\psi_- + \sqrt{2}i\bar{\epsilon}\partial_\perp\psi_+, && \delta_{\text{sym}}\psi_- = -\sqrt{2}\epsilon F + \sqrt{2}i\bar{\epsilon}\partial_\perp\phi,\end{aligned} \tag{3.115}$$

i.e., simply the restriction of (3.108) to the $(0,2)$-subalgebra, given by choosing $\epsilon^\alpha = \left(\begin{smallmatrix}0\\ \epsilon\end{smallmatrix}\right)$. The Lagrangian in terms of $(0,2)$-superspace is written as

$$
\begin{aligned}
\mathcal{L}_{\text{kin.}} &= \frac{1}{2}\int \mathrm{d}^2\theta^+ \Big[i\overline{\Phi}\partial_-\Phi - i\partial_-\overline{\Phi}\Phi + \overline{\Psi}\Psi \\
&\qquad\qquad + \partial_\perp \left(\tfrac{1}{2}\theta^+\overline{\theta}^+\partial_\perp(\overline{\Phi}\Phi) + \tfrac{i}{\sqrt{2}}\theta^+\overline{\Phi}\Psi + \tfrac{i}{\sqrt{2}}\overline{\theta}^+\overline{\Psi}\Phi \right) \Big], \\
\mathcal{L}_W &= -\tfrac{1}{\sqrt{2}}\int \mathrm{d}\theta^+ \Psi W'(\Phi) + \text{cc.}
\end{aligned}
\tag{3.116}
$$

Note that the first line in $\mathcal{L}_{\text{kin.}}$ is invariant under $(0,2)$-supersymmetry even in the presence of a boundary, as its $(0,2)$-variation is just a total x^+-derivative. The second line is an $x^\perp$-derivative, so it manifestly breaks $(0,2)$-supersymmetry in the presence of a boundary, and hence dictates part of the "boundary compensating term" that must be compensated to preserve supersymmetry in "a boundary condition-independent way" [126].

The equations of motion may again be written as $\mathcal{N}=(0,2)$ superfield equations:

$$
\begin{aligned}
0 &= 2i\partial_-\overline{D}_+\overline{\Phi} + \sqrt{2}i\partial_\perp\overline{\Psi} - \sqrt{2}W''(\Phi)\Psi, \\
0 &= \overline{D}_+\overline{\Psi} + \sqrt{2}W'(\Phi).
\end{aligned}
\tag{3.117}
$$

3.4.2 Preserving some supersymmetry: boundary fields, factorization, boundary conditions

The bulk action (3.116) is not $(0,2)$-supersymmetric in the presence of a boundary:

$$
\delta_{\text{sym}} S = \int_M \delta_{\text{sym}}(\mathcal{L}_{\text{kin.}} + \mathcal{L}_W) = \int_M \partial_\mu (V^\mu_{\text{kin.}} + V^\mu_W) = \int_{\partial M} (V^\perp_{\text{kin.}} + V^\perp_W), \tag{3.118}
$$

which is non-zero in general. To preserve at least $\mathcal{N}=(0,2)$ supersymmetry, we must compensate these bulk variations.

For the kinetic term, we add a boundary compensating term $\widetilde{\Delta}_{\text{kin.}}$ to the boundary Lagrangian (in a boundary-condition-independent way) as in [126, 137]. The boundary term is determined to be minus the total $\perp$-derivative from the bulk Lagrangian in $(0,2)$-superspace (3.116):

$$
\widetilde{\Delta}_{\text{kin.}} := -\tfrac{1}{4}\partial_\perp(\overline{\phi}\phi) - \tfrac{i}{2}\overline{\psi}_+\psi_- + \tfrac{i}{2}\overline{\psi}_-\psi_+. \tag{3.119}
$$

This ensures that no boundary condition is necessary for cancelation. In addition, we note that the $-\frac{1}{4}\partial_\perp(\overline{\phi}\phi)$ of the proposed $\widetilde{\Delta}_{\text{kin.}}$ cancels the bulk total derivative in $x^\perp$ direction when it is rewritten as a bulk term. Hence, we may drop a $\frac{1}{4}\partial^2(\overline{\phi}\phi)$ term from the bulk and a$-\frac{1}{4}\partial_\perp(\overline{\phi}\phi)$ term from the boundary simultaneously, leaving us with bulk and boundary Lagrangians with first-order derivatives only (which makes our lives easier when applying Noether's theorem later on)

$$
\begin{aligned}
\mathcal{L}^B &= -\partial_\mu\overline{\phi}\partial^\mu\phi + \tfrac{i}{2}(\psi\gamma^\mu\partial_\mu\overline{\psi}) - \tfrac{i}{2}(\partial_\mu\psi\gamma^\mu\overline{\psi}) \\
&\qquad + \overline{F}F + W'(\phi)F + \overline{W}'(\overline{\phi})\overline{F} - \tfrac{1}{2}W''(\phi)\psi\psi + \tfrac{1}{2}\overline{W}''(\overline{\phi})\overline{\psi\psi}, \\
\Delta_{\text{kin.}} &= -\tfrac{i}{2}\overline{\psi}_+\psi_- + \tfrac{i}{2}\overline{\psi}_-\psi_+ = -\tfrac{i}{2}\overline{\psi}\psi.
\end{aligned}
\tag{3.120}
$$

For the bulk superpotential term, the supersymmetry variation yields

$$\delta_{\text{sym}}\mathcal{L}_W = \partial_\perp \left(-i\int \mathrm{d}\theta^+\bar{\epsilon}W(\Phi) + \text{cc.}\right) + \partial_+(\dots) = \partial_\perp(-i\bar{\epsilon}\psi_+ W'(\phi) + \text{cc.}) + \partial_+(\dots), \tag{3.121}$$

where the right-hand side needs to be canceled again somehow. To that end, we introduce a 2D boundary $\mathcal{N} = (0,2)$ Fermi multiplet with E- and J-potential terms, as done in [100, 137–139], where a 1D Fermi multiplet was used to compensate bulk 2D superpotential terms, or as in for three-dimensional cases [22, 36]. The Fermi multiplet in components is given by

$$H = \eta - \sqrt{2}\theta^+ G - 2i\theta^+\bar{\theta}^+\partial_+\eta - \sqrt{2}\bar{\theta}^+ E(\phi) + 2\theta^+\bar{\theta}^+ E'(\phi)\psi_+, \tag{3.122}$$

where η is a complex Weyl fermion, G is an auxiliary field (related to the J-potential on-shell), the E-potential is a holomorphic function of chiral fields — in our case the restriction of the bulk chiral fields to the boundary. The "chirality" condition is

$$\bar{D}_+ H = \sqrt{2}E(\Phi), \tag{3.123}$$

and the explicit $(0,2)$-supersymmetry variation of the components is given by

$$\begin{aligned} \delta_{\text{sym}}\eta &= -\sqrt{2}(\epsilon G + \bar{\epsilon}E), \\ \delta_{\text{sym}}G &= \sqrt{2}\bar{\epsilon}(2i\partial_+\eta - E'\psi_+). \end{aligned} \tag{3.124}$$

The boundary Lagrangian term associated to the Fermi field is

$$\begin{aligned} \mathcal{L}_H &= \int \mathrm{d}^2\theta^+ \tfrac{1}{2}\bar{H}H - \int \mathrm{d}\theta^+ \tfrac{i}{\sqrt{2}}J(\Phi)H + \int \mathrm{d}\bar{\theta}^+ \tfrac{i}{\sqrt{2}}\bar{J}(\bar{\Phi})\bar{H} \\ &= i\bar{\eta}\partial_+\eta - i\partial_+\bar{\eta}\eta - E'\bar{\eta}\psi_+ - \bar{E}'\bar{\psi}_+\eta + iJ'\eta\psi_+ - i\bar{J}'\bar{\psi}_+\bar{\eta} - |E|^2 - |J|^2, \end{aligned} \tag{3.125}$$

The boundary equations of motion are

$$\bar{D}_+\bar{H} + \sqrt{2}iJ(\Phi) = 0 \Leftrightarrow \left\{ \begin{matrix} G = i\bar{J} \\ 2i\partial_+\eta = E'(\phi)\psi_+ - i\bar{J}'(\bar{\phi})\bar{\psi}_+ \end{matrix} \right\}. \tag{3.126}$$

The $\mathcal{N} = (0,2)$ supersymmetry variation can be computed to be

$$\delta_{\text{sym}}\mathcal{L}_H = i\int \mathrm{d}\theta^+\bar{\epsilon}J(\Phi)E(\Phi) + \text{cc.} + \partial_+(\dots). \tag{3.127}$$

which is of the same form as the variation of (3.121). Hence, in case of a *matrix factorization*

$$W(\Phi)|_\partial = E(\Phi)J(\Phi)|_\partial, \tag{3.128}$$

the bulk term from the variation will be compensated exactly, and $(0,2)$-supersymmetry is preserved. This deviates from the folklore [65] that a pure 2D $\mathcal{N} = (0,2)$ theory must fulfill $E \cdot J = 0$ in order to preserve supersymmetry: the "failure" of the boundary Fermi multiplet to meet this condition cancels the failure of the bulk theory to preserve $\mathcal{N} = (0,2)$-supersymmetry at the boundary.

The full action of the factorized Landau-Ginzburg model then reads in $(0,2)$ superspace

$$\begin{aligned} S &= \int_M \mathcal{L}^B + \int_{\partial M} \mathcal{L}^\partial \\ &= \frac{1}{2}\int_M \left\{ \int \mathrm{d}^2\theta^+ \left[i\overline{\Phi}\partial_-\Phi - i\partial_-\overline{\Phi}\Phi + \overline{\Psi}\Psi + \partial_\perp\Delta \right] - \sqrt{2}\int \mathrm{d}\theta^+ \Psi W(\Phi) + \mathrm{cc.} \right\} \\ &\quad + \frac{1}{2}\int_{\partial M} \left\{ \int \mathrm{d}^2\theta^+ \left[-\Delta|_\partial + \overline{H}H \right] - \sqrt{2}i \int \mathrm{d}\theta^+ J(\Phi)H + \mathrm{cc.} \right\}, \end{aligned} \tag{3.129}$$

where $\frac{1}{2}\int \mathrm{d}^2\theta^+\Delta = \frac{i}{2\sqrt{2}}\int \mathrm{d}^2\theta^+(\theta^+\overline{\Phi}\Psi + \overline{\theta}^+\overline{\Psi}\Phi) = \frac{i}{2}(\overline{\psi}_+\psi_- - \overline{\psi}_-\psi_+)$ (cf. (3.120)). After using the algebraic (auxiliary) equations of motion, we get the following component expansions:

$$\begin{aligned} \mathcal{L}^B &= -\partial_\mu\overline{\phi}\partial^\mu\phi + \tfrac{i}{2}(\psi\gamma^\mu\partial_\mu\overline{\psi}) - \tfrac{i}{2}(\partial_\mu\psi\gamma^\mu\overline{\psi}) \\ &\quad - |W(\phi)|^2 - \tfrac{1}{2}W''(\phi)\psi\psi + \tfrac{1}{2}\overline{W}''(\overline{\phi})\overline{\psi\psi}, \\ \mathcal{L}^\partial &= i\overline{\eta}\partial_+\eta - i\partial_+\overline{\eta}\eta - |J|^2 - |E|^2 - \overline{E}'\overline{\psi}_+\eta - E'\overline{\eta}\psi_+ \\ &\quad - iJ'\psi_+\eta - i\overline{J}'\overline{\psi}_+\overline{\eta} - \tfrac{i}{2}(\overline{\psi}_+\psi_- - \overline{\psi}_-\psi_+)|_\partial. \end{aligned} \tag{3.130}$$

Assuming the factorization condition, the $(0,2)$-variation of the total action is zero, and ence $\mathcal{N} = (0,2)$ supersymmetry is preserved in a boundary-condition-independent way (although imposing factorization may restrict the possible boundary conditions). Let us however look at possible (classes of) explicit boundary condition in more detail.

Symmetric boundary conditions

We discuss the case with and without superpotential separately, following [3]

1. Without superpotential (discussed also in [39])

   - *(generalized) Dirichlet*: $\Phi = 0$ or more generally $\Phi = c$ (in components $\phi = c$ and $\psi_+ = 0$), where we may need to add boundary terms to the action to make it supersymmetric.
   - *Neumann*: $\Psi = 0$ (in components $\partial_\perp\phi = 0$ and $\psi_- = 0$. This is the dynamical boundary condition in the sense of (3.8) for the action (3.120) without superpotential. One can "flip" into a (generalized) Dirichlet as a dynamical boundary condition by adding appropriate boundary terms [39].
   - *Mixed conditions*: In models with more than one 3D chiral superfield, we may assign Dirichlet conditions to some and Neumann conditions to others [39].

2. With superpotential.

   - *(generalized) Dirichlet*: Setting $\Phi = c$ is symmetric and also statically cancels the supervariation of the potential (3.121), although in a boundary-condition-*dependent* way. However, if $W'(c)|_\partial \neq 0$, supersymmetry is broken spontaneously, as the vacuum expectation value of ψ_- then transforms non-trivially under supersymmetry.

- *Mixed conditions*: Setting $\Psi = 0$ (Neumann) is only symmetric if $W'(\phi)|_\partial = 0$. For one bosonic field, this holds only if $W = 0$, as ϕ is unconstrained on the boundary. If $W \neq 0$ and the theory has more than one chiral superfield, one can assign Dirichlet conditions to some and Neumann conditions to others while maintaining supersymmetry (a requirement the authors in [39] call "sufficiently Dirichlet").
- *Factorized Neumann*: This is the main case we want to focus on. It the dynamical boundary condition imposed if we introduce boundary Fermi multiplet and have a non-zero superpotential, as in this subsection (see page 73). It is thus the analogue to the Neumann boundary condition for zero superpotential:

$$\overline{\Psi} = -i\overline{H}E'(\Phi) - HJ'(\Phi) \Leftrightarrow \left\{ \begin{aligned} \overline{\psi}_- &= -i\overline{\eta}E' - \eta J', \\ \partial_\perp \overline{\phi} &= -\overline{E}E' - \overline{J}J' - (\overline{\eta}E'' - i\eta J'')\psi_+ \end{aligned} \right\}. \tag{3.131}$$

One can check that it is indeed symmetric if the factorization condition (3.128) is met. We use this boundary condition in our computations for currents and current multiplets. This choice of boundary condition in fact encodes a collection of boundary conditions labeled by the choices of *matrix factorizations* of W (since the boundary condition depends explicitly on E and J).

3.4.3 Current components

In this subsection we explicitly compute the Noether currents of the Landau-Ginzburg model with non-zero superpotential, a boundary Fermi multiplet and assuming factorization condition (3.128) and (factorized) Neumann boundary condition (3.131).

R-current

Suppose that the superpotentials W, E, J are (quasi-)homogeneous functions of Φ. In the case of one chiral field this means that

$$W = \Phi^{(1/\alpha)}, \quad E = \Phi^{\ell_E}, \quad J = \Phi^{\ell_J}, \tag{3.132}$$

where $1/\alpha, \ell_E, \ell_J$ are non-negative integers. For multiple chiral fields Φ_i, the condition for quasi-homogeneity reads $W(\Phi_1, \ldots, \Phi_k) = \sum_i \alpha_i \Phi_i \partial_{\Phi_i} W$ for some choice of R-charges α_i. Under these assumptions the action is invariant under the R-symmetry transformation:

$$\begin{aligned} \theta^+ &\mapsto e^{-i\tau}\theta^+, & \Phi &\mapsto e^{-2i\tau\alpha}\Phi, \\ \Psi &\mapsto e^{-i\tau(2\alpha-1)}\Psi, & H &\mapsto e^{-i\tau(\ell_E-\ell_J)\alpha}H, \end{aligned} \tag{3.133}$$

where τ is the symmetry variation parameter Note that factorization implies

$$\alpha(\ell_E + \ell_J) = 1. \tag{3.134}$$

The bulk contribution to the R-current is given by

$$j^B_\mu = 2i\alpha(\overline{\phi}\partial_\mu\phi - \partial_\mu\overline{\phi}\phi) + (1-2\alpha)\overline{\psi}\gamma_\mu\psi, \tag{3.135}$$

while the boundary contribution is given by

$$j^\partial_{\hat{\mu}} = \begin{pmatrix} j^\partial_+ \\ j^\partial_- \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha(\ell_E - \ell_J)\overline{\eta}\eta \end{pmatrix}. \tag{3.136}$$

Supercurrents

After introducing the boundary only $(0,2)$-supersymmetry is preserved; however we present th full 3D $\mathcal{N}=2$ supersymmetry in the bulk, as the $(0,2)$-restrictions of the bulk currents remain identical (and covariant notation can be conveniently used).

- *Noether frame ($\mathcal{S}$-frame).*
  The bulk supercurrent induced by $\delta_{\text{sym}}=\epsilon Q$ in the Noether frame is given by:
  $$(S^B_\mu)_\alpha=\sqrt{2}(\gamma^\nu\gamma_\mu\psi)_\alpha\partial_\nu\overline{\phi}-\sqrt{2}i(\gamma_\mu\overline{\psi})_\alpha\overline{W}'. \tag{3.137}$$
  Its $(0,2)$-restriction $\delta_{\text{sym}}=\epsilon^+Q_+$ is given by setting $\alpha=+$:
  $$(S^B_\mu)_+=\begin{pmatrix}(S^B_+)_+\\(S^B_-)_+\\(S^B_\perp)_+\end{pmatrix}=\begin{pmatrix}2\sqrt{2}\psi_+\partial_+\overline{\phi}\\-\sqrt{2}(\psi_-\partial_\perp\overline{\phi}+i\overline{\psi}_-\overline{W}')\\\sqrt{2}(\psi_+\partial_\perp\overline{\phi}-2\psi_-\partial_+\overline{\phi}+i\overline{\psi}_+\overline{W}')\end{pmatrix}. \tag{3.138}$$
  The boundary contribution induced by $\delta_{\text{sym}}=\epsilon^+Q_+$ reads in the Noether frame:
  $$(S^\partial_{\hat{\mu}})_+=\begin{pmatrix}(S^\partial_+)_+\\(S^\partial_-)_+\end{pmatrix}=\begin{pmatrix}0\\-\sqrt{2}(\overline{J}\eta-i\overline{E}\eta)\end{pmatrix}. \tag{3.139}$$

- *$\mathcal{R}$-frame.*
  If the Lagrangian has an R-symmetry (3.133), we may improve the above supercurrent to a supercurrent which is part of the $\mathcal{R}$-multiplet. We call this improvement frame the $\mathcal{R}$-frame. The bulk components are:
  $$\begin{aligned}(S^B_\mu)^{\mathcal{R}}_\alpha&=(S^B_\mu)^{\mathcal{S}}_\alpha-2\sqrt{2}\alpha\epsilon_{\mu\nu\rho}(\gamma^\nu\partial^\rho(\overline{\phi}\psi))_\alpha\\&=\sqrt{2}(1-2\alpha)\big((\gamma^\nu\gamma_\mu\psi)_\alpha\partial_\nu\overline{\phi}+i(\gamma_\mu\overline{\psi})_\alpha\overline{W}'\big)+2\sqrt{2}\alpha(\partial_\mu\overline{\phi}\psi_\alpha-\overline{\phi}\partial_\mu\psi_\alpha),\end{aligned} \tag{3.140}$$
  where $(S^B_\mu)^{\mathcal{S}}_\alpha$ denotes the supercurrent in the Noether frame, $\alpha=(\deg W)^{-1}$ and the last equality holds assuming equations of motion (3.111) and homogeneity of W.
  The boundary components are
  $$(S^\partial_{\hat{\mu}})^{\mathcal{R}}_+=(S^\partial_{\hat{\mu}})^{\mathcal{S}}_++2\sqrt{2}\alpha\epsilon_{\hat{\mu}\nu n}(\gamma^\nu\psi)_+\overline{\phi}=\begin{pmatrix}0\\\sqrt{2}\alpha(\ell_J-\ell_E)(\overline{J}\eta+i\overline{E}\eta)\end{pmatrix}, \tag{3.141}$$
  where the last equality uses boundary conditions (3.131).

Energy-momentum tensor

Similarly to the case of supercurrents, we stick to covariant notation for the bulk pieces, even though certain directions are no longer symmetries. Let us start by simplifying the Lagrangians (3.130) on-shell:[7]

$$\mathcal{L}^B\overset{\text{on-shell}}{=}-\partial_\rho\overline{\phi}\partial^\rho\phi-|W'|^2, \tag{3.142}$$
$$\mathcal{L}^\partial\overset{\text{on-shell}}{=}-|E|^2-|J|^2. \tag{3.143}$$

[7]Note that the second equation also uses boundary conditions (3.131). Without using them, we get
$$\mathcal{L}^\partial\overset{\text{on-shell}}{=}-|E|^2-|J|^2-\tfrac{1}{2}(\overline{E}'\overline{\psi}_+\eta+E'\overline{\eta}\psi_++iJ'\psi_+\eta+i\overline{J}'\overline{\psi}_+\overline{\eta})-\tfrac{i}{2}(\overline{\psi}_+\psi_--\overline{\psi}_-\psi_+)|_\partial.$$

- *Noether frame ($\mathcal{S}$-frame).*
 Using the Noether procedure, in the bulk we find the non-symmetric stress tensor

$$\widehat{T}^B_{\mu\nu} = \partial_\mu\overline{\phi}\partial_\nu\phi + \partial_\nu\overline{\phi}\partial_\mu\phi + \tfrac{i}{2}\partial_\mu\overline{\psi}\gamma_\nu\psi - \tfrac{i}{2}\overline{\psi}\gamma_\nu\partial_\mu\psi - \eta_{\mu\nu}(\partial_\rho\overline{\phi}\partial^\rho\phi + |W'|^2), \tag{3.144}$$

 and in the boundary we find (using equations of motion but *not* boundary conditions)

$$\begin{aligned}
\widehat{T}^\partial_{++} &= 0,\\
\widehat{T}^\partial_{--} &= \tfrac{i}{2}\overline{\eta}\partial_-\eta - \tfrac{i}{2}\partial_-\overline{\eta}\eta,\\
\widehat{T}^\partial_{+-} &= \tfrac{i}{4}\overline{\psi}_+\psi_- - \tfrac{i}{4}\overline{\psi}_-\psi_+ + \tfrac{1}{2}|E|^2 + \tfrac{1}{2}|J|^2\\
&\qquad + \tfrac{1}{2}\big(E'\overline{\eta}\psi_+ + \overline{E}'\overline{\psi}_+\eta - iJ'\eta\psi_+ + i\overline{J}'\overline{\psi}_+\overline{\eta}\big),\\
\widehat{T}^\partial_{-+} &= \tfrac{i}{4}\overline{\psi}_+\psi_- - \tfrac{i}{4}\overline{\psi}_-\psi_+ + \tfrac{1}{2}|E|^2 + \tfrac{1}{2}|J|^2\\
&\qquad + \tfrac{1}{4}\big(E'\overline{\eta}\psi_+ + \overline{E}'\overline{\psi}_+\eta - iJ'\eta\psi_+ + i\overline{J}'\overline{\psi}_+\overline{\eta}\big).
\end{aligned} \tag{3.145}$$

 If we utilize the boundary conditions (3.131), the expressions simplify to

$$\begin{aligned}
\widehat{T}^\partial_{++} &= 0,\\
\widehat{T}^\partial_{--} &= \tfrac{i}{2}\overline{\eta}\partial_-\eta - \tfrac{i}{2}\partial_-\overline{\eta}\eta,\\
\widehat{T}^\partial_{+-} &= \tfrac{i}{2}\overline{\eta}\partial_+\eta - \tfrac{i}{2}\partial_+\overline{\eta}\eta + \tfrac{1}{2}(|E|^2 + |J|^2),\\
\widehat{T}^\partial_{-+} &= \tfrac{1}{2}(|E|^2 + |J|^2).
\end{aligned} \tag{3.146}$$

- *Symmetrization.*
 The stress tensor can become symmetric using an improvement. In the bulk we find

$$\begin{aligned}
T^B_{\mu\nu} &= \widehat{T}^B_{\mu\nu} - \tfrac{1}{8}\epsilon_{\mu\nu\rho}H^\rho\\
&= (\partial_\mu\overline{\phi}\partial_\nu\phi + \partial_\nu\overline{\phi}\partial_\mu\phi) - \eta_{\mu\nu}(|\partial\phi|^2 + |W'|^2) + \tfrac{i}{2}(\partial_{(\mu}\overline{\psi}\gamma_{\nu)}\psi) - \tfrac{i}{2}(\overline{\psi}\gamma_{(\nu}\partial_{\mu)}\psi),
\end{aligned} \tag{3.147}$$

 where $H^\rho = -2i\partial^\rho(\overline{\psi}\psi)$.[8] The bulk improvement induces a boundary improvement is $T^\partial_{\hat{\mu}\hat{\nu}} = \widehat{T}^\partial_{\hat{\mu}\hat{\nu}} - \frac{i}{4}\epsilon_{\hat{\mu}\hat{\nu}n}\overline{\psi}\psi|_\partial$, so we find

$$\begin{aligned}
T^\partial_{++} &= 0,\\
T^\partial_{--} &= \tfrac{i}{2}\overline{\eta}\partial_-\eta - \tfrac{i}{2}\partial_-\overline{\eta}\eta,\\
T^\partial_{+-} &= \tfrac{i}{2}\overline{\eta}\partial_+\eta - \tfrac{i}{2}\partial_+\overline{\eta}\eta + \tfrac{1}{2}(|E|^2 + |J|^2) - \tfrac{i}{8}(\overline{\psi}_-\psi_+ - \overline{\psi}_+\psi_-)|_\partial,\\
T^\partial_{-+} &= \tfrac{1}{2}(|E|^2 + |J|^2) + \tfrac{i}{8}(\overline{\psi}_-\psi_+ - \overline{\psi}_+\psi_-)|_\partial.
\end{aligned} \tag{3.148}$$

 Note that using boundary conditions (3.131) and equations of motion for η (3.126) we find that

$$\tfrac{i}{2}(\overline{\psi}_-\psi_+ - \overline{\psi}_+\psi_-)|_\partial = i\overline{\eta}\partial_+\eta - i\partial_+\overline{\eta}\eta, \tag{3.149}$$

 which shows that the boundary components are symmetric modulo boundary conditions in this frame as well.

[8] This is precisely the brane current from the supercurrent multiplet, see appendix C.2.5. To obtain the desired form for $T^B_{\mu\nu}$ we use equations of motion and the Clifford algebra.

- *$\mathcal{R}$-frame.*
  Again, as for the supercurrent, there is an improved energy-momentum tensor in the $\mathcal{R}$-frame. We find

$$\begin{aligned}(T^B_{\mu\nu})^{\mathcal{R}} &= (T^B_{\mu\nu})^{\mathcal{S}} + \tfrac{1}{2}[\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2](-2\alpha\overline{\phi}\phi) \\ &= (1-\alpha)(\partial_\nu\phi\partial_\mu\overline{\phi} + \partial_\mu\phi\partial_\nu\overline{\phi}) - \alpha(\partial_\mu\partial_\nu\overline{\phi}\phi + \overline{\phi}\partial_\mu\partial_\nu\phi) + \tfrac{i}{2}(\partial_{(\nu}\overline{\psi}\gamma_{\mu)}\psi) \\ &\quad - \tfrac{i}{2}(\overline{\psi}\gamma_{(\mu}\partial_{\nu)}\psi) - (1-2\alpha)\eta_{\mu\nu}(|\partial\phi|^2 - |W'|^2) + \alpha\eta_{\mu\nu}(i\psi\gamma^\rho\partial_\rho\overline{\psi} - i\partial_\rho\psi\gamma^\rho\overline{\psi})\big),\end{aligned} \tag{3.150}$$

  where for the last equality we have used equations of motion. The boundary contributions are given by $(T^\partial)^R_{\hat{\mu}\hat{\nu}} = (T^\partial)^S_{\hat{\mu}\hat{\nu}} + \frac{1}{2}\eta_{\hat{\mu}\hat{\nu}}\partial_\perp(-2\alpha\overline{\phi}\phi)$, hence

$$\begin{aligned}(T^\partial_{++})^{\mathcal{R}} &= 0, \\ (T^\partial_{--})^{\mathcal{R}} &= \tfrac{i}{2}\overline{\eta}\partial_-\eta - \tfrac{i}{2}\partial_-\overline{\eta}\eta, \\ (T^\partial_{+-})^{\mathcal{R}} &= \tfrac{i}{2}\overline{\eta}\partial_+\eta - \tfrac{i}{2}\partial_+\overline{\eta}\eta + \tfrac{1}{2}(|E|^2+|J|^2) + \tfrac{\alpha}{2}\partial_\perp(\overline{\phi}\phi)|_\partial - \tfrac{i}{8}(\overline{\psi}_-\psi_+ - \overline{\psi}_+\psi_-)|_\partial, \\ (T^\partial_{-+})^{\mathcal{R}} &= \tfrac{1}{2}(|E|^2+|J|^2) + \tfrac{\alpha}{2}\partial_\perp(\overline{\phi}\phi)|_\partial + \tfrac{i}{8}(\overline{\psi}_-\psi_+ - \overline{\psi}_+\psi_-)|_\partial.\end{aligned} \tag{3.151}$$

  Note that the boundary stress tensor is still symmetric, assuming the factorized Neumann boundary condition.

3.4.4 Supercurrent multiplets of the LG model

In this subsection we will assemble the conserved current components into supercurrent multiplets. First we assemble the bulk parts into 3D $\mathcal{N}=2$ supercurrent multiplets. Then we break them into multiplets under the $\mathcal{N}=(0,2)$ subalgebra, and complete these into admissible supercurrent multiplets of theories with boundary, in the sense of section 3.3, following [3]. Finally, we discuss the *integrated* supercurrent multiplets for this theory.

In the bulk, in $\mathcal{N}=2$

Let us assume for the moment that the theory is a pure bulk theory and the Lagrangian is given by $\mathcal{L} = \mathcal{L}_{\text{kin.}} + \mathcal{L}_W$ where the summands are as in (3.109), (3.110). In such a theory a valid $\mathcal{S}$-multiplet is given by

$$\mathcal{S}_{\alpha\beta} = D_\alpha\Phi_{3D}\overline{D}_\beta\overline{\Phi}_{3D} + D_\beta\Phi_{3D}\overline{D}_\alpha\overline{\Phi}_{3D}. \tag{3.152}$$

It contains the supercurrent and energy-momentum tensor (in the $\mathcal{S}$-frame) in its components (cf. (C.44)). We may set

$$\mathcal{Y}_\beta \coloneqq -D_\alpha\Phi_{3D}\overline{D}^2\overline{\Phi}_{3D} = 4D_\alpha W(\Phi_{3D}), \qquad \chi_\alpha \coloneqq (-\tfrac{1}{2})\overline{D}^2 D_\alpha(\Phi_{3D}\overline{\Phi}_{3D}) \tag{3.153}$$

where in the first equation we have used the bulk equation of motion (3.111). The multiplet $\S_{\alpha\beta}$ then readily satisfies

$$\overline{D}^\alpha\mathcal{S}_{\alpha\beta} = \mathcal{Y}_\beta + \chi_\beta. \tag{3.154}$$

and the remaining defining equations in (3.53) can be verified easily, proving that this is indeed an $\mathcal{S}$-multiplet, with zero central charge C. This $\mathcal{S}$-multiplet can be improved to a Ferrara-Zumino multiplet (3.58) using the improvement $U_{\text{FZ}} = -\frac{1}{2}\overline{\Phi}_{3D}\Phi_{3D}$ in (3.56), yielding $\chi'_\alpha = -\frac{1}{2}\overline{D}^2 D_\alpha(\overline{\Phi}_{3D}\Phi_{3D}) - \overline{D}^2 D_\alpha U = 0$. The multiplet is explicitly

$$\mathcal{J}_{\alpha\beta} = \tfrac{1}{2}(D_\alpha\Phi_{3D}\overline{D}_\beta\overline{\Phi}_{3D} + D_\beta\Phi_{3D}\overline{D}_\alpha\overline{\Phi}_{3D}) + \tfrac{1}{2}(i\overline{\Phi}_{3D}\partial_{\alpha\beta}\Phi_{3D} - i\partial_{\alpha\beta}\overline{\Phi}_{3D}\Phi_{3D}). \tag{3.155}$$

If the theory has an R-symmetry (i.e., if $W(\Phi_{3D}) = \Phi_{3D}^{1/\alpha}$, cf. section 3.4.3) is present, we may improve the multiplet using $U_{\mathcal{R}} = -2\alpha\overline{\Phi}_{3D}\Phi_{3D}$, which sets $\mathcal{Y}_\alpha$ to zero modulo equations of motion:

$$\mathcal{Y}'_\alpha = 4D_\alpha W(\Phi_{3D}) - \tfrac{1}{2}D_\alpha\overline{D}^2 U = 4D_\alpha W(\Phi_{3D}) - 4\alpha D_\alpha(\Phi_{3D}W'(\Phi_{3D})) = 0. \tag{3.156}$$

Then $\mathcal{S}_{\alpha\beta}$ becomes

$$\mathcal{R}_{\alpha\beta} = (1-2\alpha)(D_\alpha\Phi_{3D}\overline{D}_\beta\overline{\Phi}_{3D} + D_\beta\Phi_{3D}\overline{D}_\alpha\overline{\Phi}_{3D}) + 2\alpha(i\overline{\Phi}_{3D}\partial_{\alpha\beta}\Phi_{3D} - i\partial_{\alpha\beta}\overline{\Phi}_{3D}\Phi_{3D}). \tag{3.157}$$

We see that the lowest component of this multiplet is exactly the R-current (3.135), and one can check that the remaining currents in the $\mathcal{R}$-multiplet are in the $\mathcal{R}$-frame.

In the "bulk", in $\mathcal{N} = (0,2)$

Let us revert back to our Landau-Ginzburg theory with a boundary and a boundary Fermi multiplets whose E- and J-potentials factorize the superpotential (3.128). We want to extend the above bulk supercurrent multiplet to a bulk and boundary supercurrent multiplet as described in subsection 3.3.2. Following [3], we do so for the Noether current in the R-frame, furnishing the R-multiplet.

The proposed structure of the full supercurrent multiplet takes the following form, written as an embedding [132] into 3D $\mathcal{N} = 2$ superspace:

$$\begin{aligned} \mathcal{R}^B_{\alpha\beta} &= \mathcal{R}^{B(0)}_{\alpha\beta} + \theta^-\mathcal{R}^{B(1)}_{\alpha\beta} - \overline{\theta}^-\overline{\mathcal{R}^{B(1)}_{\alpha\beta}} + \theta^-\overline{\theta}^-\mathcal{R}^{B(2)}_{\alpha\beta}, \\ \mathcal{R}^\partial_{\alpha\alpha} &= \mathcal{R}^{\partial(0)}_{\alpha\alpha} + \theta^-\underbrace{\mathcal{R}^{\partial(1)}_{\alpha\alpha}}_{=0} - \overline{\theta}^-\underbrace{\overline{\mathcal{R}^{\partial(1)}_{\alpha\alpha}}}_{=0} + \theta^-\overline{\theta}^-\mathcal{R}^{\partial(2)}_{\alpha\alpha}. \end{aligned} \tag{3.158}$$

The boundary components only have $(++)$ and $(--)$ indices as the indices $(+-)$ correspond to the $\perp$-direction. Each summand of the above expansion is a $(0,2)$-submultiplet of the left-hand side (for details, see the intermezzo in 3.2.2). Explicitly, these submultiplets of the bulk $\mathcal{R}$-multiplet are:

- The zeroth-order bulk $(0,2)$-superfields are

$$\begin{aligned} \mathcal{R}^{B(0)}_{++} &= 8\alpha(i\overline{\Phi}\partial_+\Phi - i\partial_+\overline{\Phi}\Phi) - 2(1-2\alpha)\overline{D}_+\overline{\Phi}D_+\Phi \\ &= 4j^B_+ + \dots, \\ \mathcal{R}^{B(0)}_{--} &= 8\alpha(i\overline{\Phi}\partial_-\Phi - i\partial_-\overline{\Phi}\Phi) - 4(1-2\alpha)\overline{\Psi}\Psi \\ &= 4j^B_- + \dots, \\ \mathcal{R}^{B(0)}_{+-} &= -4\alpha(i\overline{\Phi}\partial_\perp\Phi - i\partial_\perp\overline{\Phi}\Phi) - \sqrt{2}(1-2\alpha)(\overline{D}_+\overline{\Phi}\Psi + \overline{\Psi}D_+\Phi) \\ &= -2j^B_\perp + \dots. \end{aligned} \tag{3.159}$$

- The first-order bulk $(0,2)$-superfields are

$$
\begin{aligned}
\mathcal{R}^{B(1)}_{++} &= 4(1-2\alpha)\big(i\partial_\perp\overline{\Phi}D_+\Phi - \overline{D}_+\overline{\Phi}\overline{W}'(\overline{\Phi})\big) - 8i\sqrt{2}\alpha(\partial_+\overline{\Phi}\Psi - \overline{\Phi}\partial_+\Psi)\\
&= -4i(S^B_+)^{\mathcal{R}}_- + \dots,\\
\mathcal{R}^{B(1)}_{--} &= -8i\sqrt{2}(1-\alpha)\Psi\partial_-\overline{\Phi} + 8i\sqrt{2}\alpha\overline{\Phi}\partial_-\Psi\\
&= -4i(S^B_-)^{\mathcal{R}}_- + \dots,\\
\mathcal{R}^{B(1)}_{+-} &= 2\sqrt{2}i\big(\partial_\perp\overline{\Phi}\Psi - \overline{\Phi}\partial_\perp\Psi\big) + 2(1-2\alpha)\big(iD_+\Phi\partial_-\overline{\Phi} - \sqrt{2}\overline{\Psi}\overline{W}'(\overline{\Phi}) + \sqrt{2}i\overline{\Phi}\partial_\perp\Psi\big)\\
&= 2i(S^B_\perp)^{\mathcal{R}}_- + \dots .
\end{aligned}
\tag{3.160}
$$

- The second-order bulk $(0,2)$-superfields are[9]

$$
\begin{aligned}
\mathcal{R}^{B(2)}_{++} &= -16\big(\partial_+\overline{\Phi}\partial_-\Phi + \partial_-\overline{\Phi}\partial_+\Phi + \alpha\partial_+\partial_-(\overline{\Phi}\Phi) - \tfrac{\alpha}{2}\partial_\perp^2(\overline{\Phi}\Phi)\\
&\qquad - \tfrac{i}{4}\partial_-\overline{D}_+\overline{\Phi}D_+\Phi + \tfrac{i}{4}\overline{D}_+\overline{\Phi}\partial_-D_+\Phi - \tfrac{1}{2}\widetilde{\mathcal{L}^B}\big)\\
&= -16\big(\alpha\partial_+\partial_-(\overline{\Phi}\Phi) + \tfrac{1}{2}\partial_\perp\overline{\Phi}\partial_\perp\Phi - \tfrac{\alpha}{2}\partial_\perp^2(\overline{\Phi}\Phi) + \tfrac{1}{2}|W'(\Phi)|^2\\
&\qquad - \tfrac{i}{4}\partial_-\overline{D}_+\overline{\Phi}D_+\Phi + \tfrac{i}{4}\overline{D}_+\overline{\Phi}\partial_-D_+\Phi\big)\\
&= -16(T^B_{-+})^{\mathcal{R}} + 2(C^B_{+-})^{\mathcal{R}} + \dots,\\
\mathcal{R}^{B(2)}_{--} &= -16\big(2\partial_-\overline{\Phi}\partial_-\Phi - \alpha\partial_-^2(\overline{\Phi}\Phi) - \tfrac{i}{2}\partial_-\overline{\Psi}\Psi + \tfrac{i}{2}\overline{\Psi}\partial_-\Psi\big)\\
&= -16(T^B_{--})^{\mathcal{R}} + \dots,\\
\mathcal{R}^{B(2)}_{+-} &= 8\big(\partial_-\overline{\Phi}\partial_\perp\Phi + \partial_\perp\overline{\Phi}\partial_-\Phi - \alpha\partial_-\partial_\perp(\overline{\Phi}\Phi)\\
&\qquad + \tfrac{i}{2\sqrt{2}}(\partial_-\overline{D}_+\overline{\Phi}\Psi + \partial_-\Psi D_+\Phi - \overline{D}_+\overline{\Phi}\partial_\Psi - \overline{\Psi}\partial_-D_+\Phi)\big)\\
&= 8(T^B_{-\perp})^{\mathcal{R}} - (C^B_{\perp-})^{\mathcal{R}} \dots ,
\end{aligned}
\tag{3.161}
$$

where the lowest components are given by the energy-momentum tensor (3.150) and the brane current $(C^B_{\mu\nu})^{\mathcal{R}} = \epsilon_{\mu\nu\rho}(H^\rho)^{\mathcal{R}}$ in the $\mathcal{R}$-frame: $H^{\mathcal{R}}_\mu = -2i(1-4\alpha)\partial_\mu(\overline{\psi}\psi)$ where we have used (3.57) and the explicit improvement $U_{\mathcal{R}}$. Additionally, $\widetilde{\mathcal{L}^B}$ stands for the $(0,2)$-completed superfield starting from the bulk Lagrangian on-shell

$$
\widetilde{\mathcal{L}^B} = 2\partial_+\overline{\Phi}\partial_-\Phi + 2\partial_-\overline{\Phi}\partial_+\Phi - \partial_\perp\overline{\Phi}\partial_\perp\Phi - |W'(\Phi)|^2. \tag{3.162}
$$

Adding a boundary

We now complete the decomposed bulk pieces by appropriate boundary pieces, guided by the explicit current components from the previous section.
For the zeroth component $\mathcal{R}^{\partial(0)}_{\hat\mu}$, we simply $(0,2)$-supersymmetrically complete the boundary R-current (3.136), where again $\alpha = (\deg W)^{-1}$, $\ell_E = \deg E$ and $\ell_J = \deg J$:

$$
\begin{aligned}
\mathcal{R}^{\partial(0)}_{--} &= 4\alpha(\ell_E - \ell_J)\overline{H}H,\\
\mathcal{R}^{\partial(0)}_{++} &= 0.
\end{aligned}
\tag{3.163}
$$

This $(0,2)$-completion $(\mathcal{R}^{\partial(0)})_{\hat\mu}$ of $(j^\partial)_{\hat\mu}$ does not contain *all* the boundary contributions necessary: we need also T^∂_{--} to the energy-momentum tensor, which are not contained in

[9] Note the general expansions (C.26c),(C.27c) and (C.28c), in particular the definition of $K_{\mu\nu}$ (C.29).

our boundary multiplet $(\mathcal{R}^{\partial(0)})_{++}$.[10] The correction for the second-order terms $(\mathcal{R}^{\partial(2)})_{\alpha\alpha}$ is

$$\begin{aligned}
\mathcal{R}^{\partial(2)}_{++} &= 8\widetilde{\mathcal{L}^{\partial}} - 8\alpha\partial_\perp(\overline{\Phi}\Phi) + 4\sqrt{2}i\alpha(\overline{D}_+\overline{\Phi}\Psi - \overline{\Psi}D_+\Phi)|_\partial \\
&= -8|J(\Phi)|^2 - 8|E(\Phi)|^2 - 8\alpha\partial_\perp(\overline{\Phi}\Phi) + 4\sqrt{2}i\alpha(\overline{D}_+\overline{\Phi}\Psi - \overline{\Psi}D_+\Phi)|_\partial \\
&= -16(T^\partial_{-+})^{\mathcal{R}} + 2C^\partial_{+-} + \dots, \\
\mathcal{R}^{\partial(2)}_{--} &= 8i\partial_-\overline{H}H - 8i\overline{H}\partial_-H \\
&= -16(T^\partial_{--})^{\mathcal{R}} + \dots,
\end{aligned} \tag{3.164}$$

where the boundary contribution $(C^\partial_{\hat{\mu}\hat{\nu}})^{\mathcal{R}}$ to the brane current $(C^B_{\mu\nu})^{\mathcal{R}}$ in the $\mathcal{R}$-frame $(C^\partial_{+-})^{\mathcal{R}} = -i(1-4\alpha)\overline{\psi}\psi$ is induced by symmetrization of the energy-momentum tensor (cf. page 77), and $\widetilde{\mathcal{L}^\partial}$ stands for the $(0,2)$-completed the on-shell boundary Lagrangian:

$$\widetilde{\mathcal{L}^\partial} = -|J(\Phi)|^2 - |E(\Phi)|^2. \tag{3.165}$$

Integrated supercurrent multiplets

We now look at *integrated* supercurrent multiplets as in section 3.3.4. We integrate all the fields along $x^\perp \in (-\infty, 0)$, which renders the Landau-Ginzburg model effectively two-dimensional. By doing so, we recover the genuine 2D $\mathcal{N} = (0,2)$ (integrated) supercurrent multiplets.

We thus find, according to (3.104):

$$\begin{aligned}
\mathcal{R}^{\text{int.}(0)}_{++} &= \int \mathrm{d}x^\perp \left[8\alpha(i\overline{\Phi}\partial_+\Phi - i\partial_+\overline{\Phi}\Phi) - 2(1-2\alpha)\overline{D}_+\overline{\Phi}D_+\Phi\right], \\
\mathcal{R}^{\text{int.}(0)}_{--} &= 4\alpha(\ell_E - \ell_J)\overline{H}H + \int \mathrm{d}x^\perp \left[8\alpha(i\overline{\Phi}\partial_-\Phi - i\partial_-\overline{\Phi}\Phi) - 4(1-2\alpha)\overline{\Psi}\,\right],
\end{aligned} \tag{3.166}$$

as well as

$$\begin{aligned}
\mathcal{R}^{\text{int.}(2)}_{++} &= -8|J(\Phi)|^2 - 8|E(\Phi)|^2 + 4\sqrt{2}i\alpha(\overline{D}_+\overline{\Phi}\Psi - \overline{\Psi}D_+\Phi)|_\partial \\
&\quad - 16\int \mathrm{d}x^\perp \left[\alpha\partial_+\partial_-(\overline{\Phi}\Phi) + \tfrac{1}{2}\partial_\perp\overline{\Phi}\partial_\perp\Phi + \tfrac{1}{2}|W'(\Phi)|^2 \right.\\
&\qquad\qquad \left. - \tfrac{i}{4}\partial_-\overline{D}_+\overline{\Phi}D_+\Phi + \tfrac{i}{4}\overline{D}_+\overline{\Phi}\partial_-D_+\Phi\right], \\
\mathcal{R}^{\text{int.}(2)}_{--} &= 8i\partial_-\overline{H}H - 8i\overline{H}\partial_-H \\
&\quad - 16\int \mathrm{d}x^\perp \left[2\partial_-\overline{\Phi}\partial_-\Phi - \alpha\partial_-^2(\overline{\Phi}\Phi) - \tfrac{i}{2}\partial_-\overline{\Psi}\Psi + \tfrac{i}{2}\overline{\Psi}\partial_-\Psi\right].
\end{aligned} \tag{3.167}$$

After using equations of motion (3.111), (3.126), boundary conditions (3.131), factorization condition (3.128) and homogeneity of superpotential terms, we find that these integrated current multiplets indeed satisfy the relations

$$\begin{aligned}
\overline{D}_+\big(\mathcal{R}^{\text{int.}(2)}_{--} + 2i\partial_-\mathcal{R}^{\text{int.}(0)}_{--}\big) &= 0, \\
\overline{D}_+(\mathcal{R}^{\text{int.}(2)}_{++} + 2i\partial_-\mathcal{R}^{\text{int.}(0)}_{++}) &= 0.
\end{aligned} \tag{3.168}$$

which shows that the respective lowest components are $\overline{Q}_+$-cohomology elements (cf. section 3.3.4).

[10] See (C.26)–(C.28) for details: T_{+-} and T_{++} are contained in the (0)-pieces, while T_{--} is contained in the (2)-piece.

Appendix A

Special functions

In this appendix subsection we recall some basic functions [104, 105].
The q-Pochhammer symbol is defined by

$$(z;q)_\infty \coloneqq \begin{cases} \prod_{r=0}^{\infty}(1-q^r z) & , \quad |q|<1,\ z\in\mathbb{C}, \\ \prod_{r=1}^{\infty}\frac{1}{1-q^{-r}z} & , \quad |q|>1,\ z\in\mathbb{C}\setminus\{q^k\}_{k\in\mathbb{Z}_{>0}}. \end{cases} \tag{A.1}$$

The inversion formula

$$(z;q)_\infty = \frac{1}{(q^{-1}z;q^{-1})_\infty}, \quad |q|\gtrless 1, \tag{A.2}$$

follows directly from the definition.
The q-Pochhammer symbol for general index complex index is defined as

$$(z;q)_\alpha \coloneqq \frac{(z;q)_\infty}{(q^\alpha z;q)_\infty}, \quad |q|\gtrless 1,\ z,\alpha\in\mathbb{C}\ (\text{generic})\ , \tag{A.3}$$

from which the usual formulas follow

$$(z;q)_n = \begin{cases} \prod_{r=0}^{n-1}(1-q^r z) & , \quad n\geq 0\ , \\ \prod_{r=1}^{-n}\frac{1}{1-q^{-r}z} & , \quad n<0\ . \end{cases} \tag{A.4}$$

The q-Pochhammer symbol can also be expanded as

$$(z;q)_\infty = \exp\Big[-\sum_{r=1}^{\infty}\frac{z^r}{r(1-q^r)}\Big] = \exp\Big[\frac{1}{\log q}\sum_{n=0}^{\infty}\frac{B_n\log q^n}{n!}\,\mathrm{Li}_{2-n}(z)\Big], \tag{A.5}$$

where on the very right-hand side B_n are the Bernoulli numbers and Li_k is the k-th polylogarithm. We also define the theta function as

$$\theta_q(z) \coloneqq (z;q)_\infty(qz^{-1};q)_\infty\ . \tag{A.6}$$

Finally, we collect some useful interrelated identities to make q-gymnastics more transparent

$$(z;q)_\alpha = \frac{1}{(q^{-1}z;q^{-1})_{-\alpha}}\ , \tag{A.7a}$$

$$\frac{\theta_q(z)}{\theta_q(q^n z)} = (z;q)_n(qz^{-1};q)_{-n} = (-1)^n q^{\frac{1}{2}n(n-1)}z^n\ , \tag{A.7b}$$

$$(z;q^{-1})_n = (z^{-1};q)_n(-1)^n q^{-\frac{1}{2}n(n-1)}z^n\ . \tag{A.7c}$$

where $z,\alpha\in\mathbb{C}$ are generic and $n\in\mathbb{Z}$.

Appendix B

Cohomology and K-theory of complex Grassmannians

B.1 Cohomology

In this section, we discuss the classical cohomology of complex Grassmannians $\mathrm{Gr}(M,N)$, i.e., complex M-planes in $\mathbb{C}^N$. Results from the literature concerning different presentations elucidates the 'transition' between classical, quantum and equivariant versions of the topological rings. Throughout this section Greek letters $\mu,\nu,\rho,\ldots$ will denote Young diagrams.

B.1.1 Cell structure, Schubert calculus and Schur polynomials

Cell structure

The most 'constructive' way to present $H^*\big(\mathrm{Gr}(M,N)\big)$ is to consider its cell structure [95]. One can show that complex Grassmannians have a CW structure with cells given by the complex codimension $|\mu|$ *Schubert varieties* $C_\mu \subset \mathrm{Gr}(M,N)$

$$\{\mathrm{Gr}(M,N)=C_\varnothing,\ C_1,\ C_2,\ C_{1,1},\ \cdots, C_{(N-M)^M}\}, \tag{B.1}$$

and trivial boundary maps. The definition of C_μ depends on choice of a *complete flag*

$$0=V_0\subset V_1\subset\ldots V_N=\mathbb{C}^N, \tag{B.2}$$

and it reads

$$C_\mu \coloneqq \{p\in\mathrm{Gr}(M,N)|\dim(p\cap V_{N-M+a-\mu_a})\geq a,\ a=1,\ldots,M\}. \tag{B.3}$$

Schubert calculus

Each C_μ defines a (Schubert) cycle $[C_\mu]\in H_{2\dim-2|\mu|}\big(\mathrm{Gr}(M,N)\big)$ which in turn defines a cohomology class $\sigma_\mu\in H^{2|\mu|}\big(\mathrm{Gr}(M,N)\big)$ by Poincaré duality. The classes are independent of the choices of flags. We find that $H^*\big(\mathrm{Gr}(M,N)\big)$ is freely generated (as a vector space) by the $\binom{N}{M}$ elements

$$\{\sigma_\mu\}_{\mu\subseteq\mathcal{B}_{M\times(N-M)}}, \tag{B.4}$$

where $\mathcal{B}_{M\times(N-M)}$ is the 'full box' Young diagram $(N-M)^M$. The multiplicative structure can be similarly determined: the cup product obeys

$$\sigma_\mu \cdot \sigma_\nu = \sum_\rho C^\rho_{\mu\nu}\sigma_\rho, \tag{B.5}$$

where $C^\rho_{\mu\nu}$ are the Littlewood-Richardson coefficients, computed e.g., via Pieri and Giambelli formulas, with the added relations that $C^\rho_{\mu\nu} = 0$ whenever any of $\mu,\nu,\rho \not\subseteq \mathcal{B}_{M\times(N-M)}$.

Schur polynomials

Schur polynomials are symmetric polynomials in M variables, labeled by Young diagrams. They can be defined as [140]

$$\sigma_\mu(x_1,\ldots,x_M) \coloneqq \frac{\det(x_a^{\mu_b+M-b})_{ab}}{\prod_{a<b}(x_a-x_b)} \tag{B.6}$$

and they form a basis of the vector space of symmetric polynomials in M variables. Their multiplicative structure is

$$\sigma_\mu(x)\cdot\sigma_\nu(x) = \sum_\rho C^\rho_{\mu\nu}\sigma_\rho(x), \tag{B.7}$$

where $C^\rho_{\mu\nu}$ are (as opposed to (B.5)), unrestricted Littlewood-Richardson coefficients, computable by Pieri and Giambelli formulas. It is hence clear that Schur polynomials yield an incarnation of (duals to) Schubert cycles, and hence a presentation of Grassmannian cohomology rings in the sense that, as rings[1]

$$H^*\big(\mathrm{Gr}(M,N)\big) \cong \frac{\mathbb{C}\langle\sigma_\mu\rangle}{\big\langle \sigma_\mu\cdot\sigma_\nu - \sum_\rho C^\rho_{\mu\nu}\sigma_\rho, \{\sigma_\mu\}_{\mu\not\subseteq\mathcal{B}_{M\times(N-M)}}\big\rangle}. \tag{B.8}$$

Note that the above presentation is not a minimal one: as a ring, we in fact only need the Schur polynomials labeled by 'vertical' diagrams $\sigma_1, \sigma_{1,1},\ldots,\sigma_{1^M}$.

In the context of Schubert calculus, geometric interpretation of variables x_a. These variables, however, arise naturally when we consider different presentations of $H^*\big(\mathrm{Gr}(M,N)\big)$ based on "splitting" rings, or based on the symplectic quotient construction of $\mathrm{Gr}(M,N)$. We discuss these constructions in the next subsections.

B.1.2 Splitting ring presentation

Another presentation of $H^*\big(\mathrm{Gr}(M,N)\big)$ can be found e.g., in [120] or [141, Ex. 14.6.6]. We start with the fact that the tangent bundle satisfies $T\,\mathrm{Gr}(M,N) \cong \mathrm{Hom}(S,Q)$, where S is the tautological bundle of rank M and Q is the quotient bundle of rank $N-M$ determined by the Euler sequence

$$0 \to S \to \underline{\mathbb{C}}^N \to Q \to 0 \tag{B.9}$$

[1]The numerator denotes the free commutative polynomial ring in the symbols $\{\sigma_\mu\}$, and the denominator $\langle x,y\rangle$ denotes the ideal generated by elements x,y.

of bundles over $\mathrm{Gr}(M,N)$. The bundle $c(S^*)$ does not factorize over $H^*\big(\mathrm{Gr}(M,N)\big)$, but due to the splitting lemma there is a larger ring R with an injection $H^*\big(\mathrm{Gr}(M,N)\big) \hookrightarrow R$, such that $c(S^*)$ factorizes and we can set

$$\sum_{a=0}^{M} c_a(S^*)q^a = \prod_{a=1}^{M}(1+x_a q), \tag{B.10}$$

where q is a formal (degree 2) variable, and $x_a \in R$ are the Chern roots of S^*. From the above equation it is clear that

$$c_a(S^*) = \sigma_{1^a}(x), \tag{B.11}$$

where on the right-hand side we have the a-th elementary symmetric polynomial, written in terms of Schur polynomials [140]. One can also show that, in terms of the roots x_a, the Chern classes of Q are

$$c_i(Q) = \sigma_i(x). \tag{B.12}$$

The c_a's and c_i's generate $H^*\big(\mathrm{Gr}(M,N)\big)$ with the relation stemming from (B.9) so

$$H^*\big(\mathrm{Gr}(M,N)\big) \cong \frac{\mathbb{C}\langle(-1)^a\sigma_{1^a}(x), \sigma_i(x)\rangle_{a=1,\dots,M,i=1,\dots,N-M}}{\langle c_q(S)c_q(Q) - q^0\rangle} \tag{B.13}$$

where $c_q(E) = 1 + qc_1(E) + \ldots + q^{\mathrm{rk}}c_{\mathrm{rk}}(E)$ is the total Chern class weighted by the formal variable q.

B.1.3 Presentation via symplectic quotient construction

One can identify $\mathrm{Gr}(M,N)$ with the symplectic quotient

$$\mathrm{Gr}(M,N) \cong \mathrm{Hom}(\mathbb{C}^M, \mathbb{C}^N)//U(M) \tag{B.14}$$

where the action of $U(M)$ is acts on the right. We can then describe the cohomology of $\mathrm{Gr}(M,N)$ via the cohomology of the symplectic quotient by the maximal torus

$$(\mathbb{P}^{N-1})^M \cong \mathrm{Hom}(\mathbb{C}^M, \mathbb{C}^N)//U(1)^M. \tag{B.15}$$

The variables x_a, interpreted in the previous subsection as Chern roots of S^*, are now the H^2 generators of the a-th $\mathbb{P}^{N-1}$-factor. For symplectic quotients $A//G$ and $A//T$, where G is a non-Abelian group with maximal torus $T \subset G$ and Weyl group W, the we have from [91, 92] is

$$H^*(A//G) \cong \frac{H^*(A//T)^W}{\mathrm{ann}(e)}. \tag{B.16}$$

Here, $\mathrm{ann}(e)$ is the annihilator of e, i.e., elements of $H^*(A//T)^W$ that are annihilated upon multiplication by e, the Euler class of a vector bundle reflecting the non-Abelian nature of G. For the case of $\mathrm{Gr}(M,N)$ we may identify $A = \mathrm{Hom}(\mathbb{C}^M, \mathbb{C}^N)$, $G = U(M)$, $T = U(1)^M$, Weyl group $W = S_M$ and the result is[2]

$$H^*\big(\mathrm{Gr}(M,N)\big) \cong \frac{\mathbb{C}[x_1,\dots,x_M]^{\mathrm{Sym}}}{\langle x_1^N,\dots,x_M^N\rangle : \langle\Delta\rangle}, \qquad \Delta = \prod_{a<b}(x_a - x_b), \tag{B.17}$$

[2] In [92], $\mathrm{ann}(e)$ is computed in the case of $\mathrm{Gr}(M,N)$ to be $\langle x_1^N,\dots,x_M^N\rangle : \Delta^2$. This is an equivalent presentation to the one presented above. A careful discussion is found in [91] and a proof of this assertion in [93, Thm 1.8].

where $I : J = \{r | rJ \subset I\}$ denotes the ideal quotient. The quotient by the ideal quotient is taken inside the larger ring $\mathbb{C}[x_1, \ldots, x_M]$ (neither $\langle x_1^N, \ldots, x_M^N \rangle$ nor $\langle \Delta \rangle$ are ideals of $\mathbb{C}[x_1, \ldots, x_M]^{\mathrm{Sym}}$).

This presentation can also be interpreted as being derived from the Kirwan map [93, 142]. For symplectic quotients $A//G$ and $A//T$, where G is a non-Abelian group with maximal torus $T \subset G$, we have *surjective* maps

$$\begin{aligned} \kappa_T &: H_T^*(A) \to H^*(A//T), \\ \kappa_G &: H_G^*(A) \to H^*(A//G), \end{aligned} \tag{B.18}$$

where the left-hand sides denote equivariant cohomology. For the case of $\mathrm{Gr}(M, N)$ the equivariant cohomologies are the freely generated 'numerators'

$$H_T^*(A) = \mathbb{C}[x_1, \ldots, x_M], \quad H_G^*(A) = H_T^*(A)^W = \mathbb{C}[x_1, \ldots, x_M]^{\mathrm{Sym}}. \tag{B.19}$$

The kernel of the map κ_T is then given by $\langle x_1^N, \ldots, x_M^N \rangle$ and the kernel of the map κ_G is given precisely by taking the quotient of $H_G(A)$ by the ideal quotient $\langle x_1^N, \ldots, x_M^N \rangle : \langle \Delta \rangle$ as subrings of $H_T(A)$.

B.1.4 Deformation to quantum cohomology

There are deformations of the multiplicative structures of $H^*\big(\mathrm{Gr}(M, N)\big)$ that lead to quantum cohomology $QH^*\big(\mathrm{Gr}(M, N)\big)$. These can also be determined by deforming the presentations (B.8), (B.13) and (B.17) given above. In that order we have

1. The deformation of Littlewood-Richardson rules $C_{\mu\nu}^{\rho} \mapsto C_{\mu\nu}^{\rho}(q)$ for (B.8) can be found in [143, 144] and we do not repeat it here.

2. In [42, 43, 120], it was shown that the deformation of (B.13) is

$$QH^*\big(\mathrm{Gr}(M, N)\big) \cong \frac{\mathbb{C}\langle (-1)^a \sigma_{1^a}(x), \sigma_i(x) \rangle_{a=1,\ldots,M, i=1,\ldots,N-M}}{\langle c_q(S) c_q(Q) - q^0 - (-1)^{N-M} q^N \rangle} \tag{B.20}$$

   where the (previously dummy) variable q now counts degrees of curves.

3. Lastly, the presentation via the symplectic quotients (B.17) can be deformed by adopting the quantum relations for the Abelian quotients $\mathbb{P}^{N-1}$

$$x_a^N = q, \quad a = 1, \ldots, M \tag{B.21}$$

   and otherwise remains unchanged, so that

$$QH^*\big(\mathrm{Gr}(M, N)\big) \cong \frac{\mathbb{C}[x_1, \ldots, x_M]^{\mathrm{Sym}}}{\langle x_1^N - q, \ldots, x_M^N - q \rangle : \langle \Delta \rangle}, \qquad \Delta = \prod_{a<b} (x_a - x_b). \tag{B.22}$$

   In other words, the classical Kirwan map extends to quantum cohomology.

Note that the presentation in terms of symplectic quotients can also be deformed to present $U(1)^N$-equivariant cohomology of $\mathrm{Gr}(M, N)$, by replacing the $\mathbb{P}^{N-1}$ relations by "equivariantized" relations

$$\prod_{i=1}^{N} (x_a - y_i) = 0, \quad a = 1, \ldots, M. \tag{B.23}$$

In the analogous presentations via Schubert cycles, Schur polynomials will be replaced by *factorial* Schur polynomials (see [145] and references therein).

B.2 K-theory

Classical K-theory for complex Grassmannians (over $\mathbb{Q}$ or $\mathbb{C}$) is isomorphic to classical cohomology, by virtue of the cell structure, via the Chern character [146]

$$\mathrm{ch} : K\big(\mathrm{Gr}(M,N)\big) \stackrel{\cong}{\to} H^*\big(\mathrm{Gr}(M,N)\big), \tag{B.24}$$

so it is in some sense redundant to discuss different presentations of K-theory. However, the isomorphism is not clear for the deformation to quantum K-theory. In the case of cohomology in the previous section in B.1.4, we saw how one can slightly deform various presentations of cohomology to obtain quantum cohomology. We want to make analogous statements for K-theory: we will recall again three related presentations of classical K-theory, and present the proposals from [1, 2] two of the three presentations are deformed to quantum K-theory. The remaining deformation is the one presented in [144].

B.2.1 K-theoretic "Schur" classes

The description of cohomology via the tautological bundle S over $\mathrm{Gr}(M,N)$ has an analog in K-theory. We may write (in some larger ring)

$$[S] = P_1 + \ldots + P_M \tag{B.25}$$

where P_a denotes the K-theoretic Chern root. It is related to the cohomological Chern root by the Chern character[3]

$$\mathrm{ch}(P_a) = e^{-x_a}. \tag{B.26}$$

We may then present K-theory using Schur polynomials in the variables $x_a^K = 1 - P_a$, so that[4]

$$K\big(\mathrm{Gr}(M,N)\big) \cong \frac{\mathbb{C}\langle \sigma_\mu(x^K)\rangle}{\left\langle \sigma_\mu(x^K)\cdot\sigma_\nu(x^K) - \sum_\rho C^\rho_{\mu\nu}\sigma_\rho(x^K), \{\sigma_\mu(x^K)\}_{\mu\not\subseteq\mathcal{B}_{M\times(N-M)}}\right\rangle}. \tag{B.27}$$

where $C^\rho_{\mu\nu}$ are the classical Littlewood-Richardson coefficients. The ch-images of the classes $\sigma_\mu(x^K) \in K\big(\mathrm{Gr}(M,N)\big)$ can be computed by expanding $x^K = 1 - e^{-x_a}$ in x_a up to order $x_a^{M(N-M)}$, and decomposing the result in terms of (cohomological) Schur polynomials $\sigma_\mu(x)$. The dimension of K-theory is of course $\binom{N}{M}$, the same as cohomology.

B.2.2 Schubert structure sheaves and Grothendieck polynomials

Another presentation in addition to (B.27) involves the Schubert varieties C_μ (B.3). In particular one can show that $K\big(\mathrm{Gr}(M,N)\big)$ is generated additively by the *structure sheaves* $\mathcal{O}_\mu$ on these subvarieties. The multiplicative structure can be computed algorithmically [115], to determine some structure constants $D^\rho_{\mu\nu}$, to give

$$K\big(\mathrm{Gr}(M,N)\big) \cong \frac{\mathbb{C}\langle \mathcal{O}_\mu\rangle_{\mu\subseteq\mathcal{B}_{M\times(N-M)}}}{\langle \mathcal{O}_\mu\cdot\mathcal{O}_\nu - \sum_\rho D^\rho_{\mu\nu}\mathcal{O}_\rho\rangle}. \tag{B.28}$$

[3] In the bulk of chapter 2, we implicitly assume that all K-classes have been evaluated via the Chern character map and we work in cohomology.

[4] See footnote 1 for notation.

The two presentations (B.27) and (B.28) can be seen to be compatible using *Grothendieck* polynomials, first introduced in [147]. These are symmetric polynomials in M variables, labeled by Young diagrams. There is a determinantal definition (see [148] and references therein)

$$\mathcal{O}_\mu(x_1,\ldots,x_M) \coloneqq \frac{\det\left(x_a^{\mu_b+M-b}(1-x_a)^{b-1}\right)_{ab}}{\prod_{a<b}(x_a-x_b)}. \tag{B.29}$$

Their structure constants match the ones of the abstract Schubert structure sheaves

$$\mathcal{O}_\mu(x)\cdot\mathcal{O}_\nu(x) = \sum_\rho D^\rho_{\mu\nu}\mathcal{O}_\rho(x), \tag{B.30}$$

and hence they form an incarnation of the Schubert structure sheaves. We find that we can present

$$K\big(\mathrm{Gr}(M,N)\big) \cong \frac{\mathbb{C}\langle\mathcal{O}_\mu(x^K)\rangle}{\langle\mathcal{O}_\mu(x^K)\cdot\mathcal{O}_\nu(x^K) - \sum_\rho C^\rho_{\mu\nu}\mathcal{O}_\rho(x^K), \{\mathcal{O}_\mu(x^K)\}_{\mu\not\subseteq\mathcal{B}_{M\times(N-M)}}\rangle}.. \tag{B.31}$$

In the above notation we have also tacitly incorporated the fact that

$$\mathrm{ch}(\mathcal{O}_\mu) = \mathcal{O}_\mu(x^K), \tag{B.32}$$

where on the left-hand side we have the (abstract) Schubert structure sheaf, and on the right-hand side we have the Grothendieck polynomial in the cohomological variable $x_a^K = 1 - e^{-x_a}$.

The Grothendieck polynomials are related to Schur polynomials by linear transformations

$$\mathcal{O}_\mu(x) = \sum_\nu U_{\mu\nu}\sigma_\nu(x), \tag{B.33}$$

that can be computed explicitly. Note that, while $\sigma_\mu(x)$ is homogeneous of degree $|\mu|$ in x, while $\mathcal{O}_\mu(x)$ is not homogenous, but its degree in x bounded below by $|\mu|$. However, the (infinite dimensional) bases of symmetric polynomials in M variables, given by Schur and Grothendieck polynomials admit compatible filtrations "by rectangular boxes"; concretely, this means that if $\mu\subset\mathcal{B}_{M\times L}$ then the coefficients $U_{\mu\nu}$ above will be non-zero only for $\nu\in\mathcal{B}_{M\times L}$. The 'compatibility' of the two presentations is reflected in the fact that the structure constants $D^\rho_{\mu\nu}$ can then be computed from the Littlewood-Richardson coefficients $C^\rho_{\mu\nu}$ and the transformation matrices $U_{\mu\nu}$.

B.2.3 Presentation via symplectic quotient

Using the symplectic quotient construction (B.14) of $\mathrm{Gr}(M,N)$ we may apply the result from [94]

$$K(A//G) \cong \frac{K(A//T)^W}{\mathrm{ann}(e)} \tag{B.34}$$

where e is the K-theoretic Euler class of a bundle over $A//T$, reflecting the non-Abelian structure of G. This is the K-theory analog of the cohomological statements (B.16) in [91–93] used for (B.17). We take $A = \mathrm{Hom}(\mathbb{C}^M,\mathbb{C}^N)$, $G = U(M)$, $T = U(1)^M$, Weyl group $W = S_M$. The K-theory of the Abelian quotients is

$$K\big((\mathbb{P}^{N-1})^M\big) \cong \frac{\mathbb{C}[P_1^{\pm1},\ldots,P_M^{\pm1}]}{\langle(1-P_1)^N,\ldots,(1-P_M)^N\rangle} \tag{B.35}$$

and the K-theory of the full symplectic quotient is

$$K\big(\operatorname{Gr}(M,N)\big) \cong \frac{\mathbb{C}[P_1^{\pm 1},\dots,P_M^{\pm 1}]^{\mathrm{Sym}}}{\langle (1-P_1)^N,\dots,(1-P_M)^N\rangle : \langle \Delta_K\rangle}, \qquad \Delta_K = \prod_{a<b}(P_a - P_b), \tag{B.36}$$

where $I : J = \{r | rJ \subset I\}$ denotes the ideal quotient. The quotient by the ideal quotient is taken inside the larger ring $\mathbb{C}[P_1^{\pm 1},\dots,P_M^{\pm 1}]$. This presentation can be mapped to the previous ones by setting $P_a = 1 - x_a^K$, i.e., we can also present[5]

$$K\big(\operatorname{Gr}(M,N)\big) \cong \frac{\mathbb{C}[x_1^K,\dots,x_M^K]}{\langle (x_1^K)^N,\dots,(x_M^K)^N\rangle : \langle \Delta_K\rangle}, \qquad \Delta_K = \prod_{a<b}(x_a^K - x_b^K). \tag{B.37}$$

Once again, this presentation arises by considering the K-theoretic Kirwan maps [94] from equivariant K-theory to K-theory of symplectic quotients

$$\kappa_G : K_G(A) \to K(A//G), \qquad \kappa_T : K_T(A) \to K(A//T). \tag{B.38}$$

The maps can be shown to be surjective, [149] and the quotient presentation above reflects the kernel of the maps.

B.2.4 Proposal for quantum deformation

In the previous section B.1, we have recalled different presentations for the cohomology ring $H^*\big(\operatorname{Gr}(M,N)\big)$, and we subsequently deformed these presentations yielding presentations for the *quantum* cohomology ring, listed in page 88. Throughout the current section, we have recalled analogous presentations for K-theory $K\big(\operatorname{Gr}(M,N)\big)$. In this subsection we argue that results from the literature and results from subsection 2.2.6 are K-theoretic analogs of the cohomological deformation presentations of B.1.4.

In particular, we want to argue that (see the list in page 88)

1. *"Schubert" presentation*:
   The quantum deformation [118] of the classical K-theory structure constants [115] is the K-theory analog of the deformation 1. of classical cohomology structure constants, where the Littlewood-Richarson coefficients in (B.8) are replaced with q-dependent coefficients.

2. *Explicit presentation á la Intrilligator, Vafa, Witten:*
   The quantum deformation of the explicit presentation (B.31) in terms of Grothendieck polynomials, proposed in [2], is the K-theory analog of the deformation 2. of classical cohomology of Grassmannians by Vafa, Intriligator and Witten.

3. *"Symplectic quotient" presentation via ideal quotient*:
   The quantum ring described in [1, 2] and discussed in 2.2.6 is a deformation of (B.36) which is the K-theory analog of the deformation 3. of classical cohomology (B.17). In other words,

$$QK\big(\operatorname{Gr}(M,N)\big) \cong \frac{\mathbb{C}[P_1^{\pm 1},\dots,P_M^{\pm 1}]^{\mathrm{Sym}}}{\mathfrak{L}_Q : \langle \Delta_K\rangle}, \qquad \Delta_K = \prod_{a<b}(P_a - P_b), \tag{B.39}$$

[5] Note that, while the K-theoretic Chern roots P_a are invertible (in the 'numerator' ring), the generators x_a^K are not.

where $\mathfrak{L}_Q$ is the ideal

$$\mathfrak{L}_Q = \langle (1-P_1)^N + Q\frac{P_1^N}{\prod_{a=1}^M P_a}, \dots, (1-P_M)^N + Q\frac{P_M^N}{\prod_{a=1}^M P_a}\rangle. \tag{B.40}$$

Once again, the quotient is taken inside the larger ring $\mathbb{C}[P_1^{\pm 1}, \dots, P_M^{\pm 1}]$. The above presentation can be again viewed as the image under a quantum Kirwan map between respecitve equivariant versions [150]. Equivalently, we may present

$$QK\big(\mathrm{Gr}(M,N)\big) \cong \frac{\mathbb{C}[D_1,\dots,D_M]^{\mathrm{Sym}}}{\mathfrak{L}_Q : \langle \Delta_K \rangle}, \quad \Delta_K = \prod_{a<b}(D_a - D_b), \tag{B.41}$$

where $\mathfrak{L}_Q$ is the ideal

$$\mathfrak{L}_Q = \langle D_1^N + Q\frac{1-D_1^N}{\prod_{a=1}^M (1-D_a)}, \dots, D_M^N + Q\frac{1-D_M^N}{\prod_{a=1}^M (1-D_a)}\rangle. \tag{B.42}$$

B.3 Quantum K-theory of low-dimensional Grassmannians

In this section we print some multiplication tables for the first few Grassmannians, computed by the proposal (B.41). The multiplication rings of their dual counterparts $\mathrm{Gr}(N-M,N) \cong \mathrm{Gr}(M,N)$ are obtained by transposing the labeling young diagrams $\mu \mapsto \mu^T$ and sending $Q \mapsto (-1)^M Q$ (which leads to a relative sign between dual pairs when N is odd). After performing this redefinition, one can check that our results agree with the ring structure from [144].

We also state the quantum q-difference equations of the corresponding I-functions as discussed in section 2.2.7, with $\hat{\delta} = 1 - q^\theta$ and $\theta = Q\frac{\partial}{\partial Q}$, and their corresponding 2D limits, matching the results from [122].

B.3.1 Quantum K theory of $\mathbf{Gr(2,3) \cong Gr(1,3)}$

The generators of $QK\big(\mathrm{Gr}(2,3)\big)$ are the structure sheaves on the Schubert varieties

$$\big\{\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_{1,1}\big\}, \tag{B.43}$$

and the multiplication table is

$$\mathcal{O}_1 * \mathcal{O}_1 = \mathcal{O}_{1,1}, \quad \mathcal{O}_1 * \mathcal{O}_{1,1} = Q\mathcal{O}_0, \quad \mathcal{O}_{1,1} * \mathcal{O}_{1,1} = Q\mathcal{O}_1. \tag{B.44}$$

The multiplication table for $\mathrm{Gr}(1,3)$ is obtained by $\mu \mapsto \mu^T$.
The quantum difference operator annihilating the I-function is

$$\mathcal{D}_{q,\mathrm{Gr}(2,3)} = \hat{\delta}^3 - Q, \tag{B.45}$$

with 2D limit

$$\mathcal{D}^{2D}_{\mathrm{Gr}(2,3)} = \theta^3 - Q^{2D}. \tag{B.46}$$

B.3.2 Quantum K theory of $\mathbf{Gr(3,4) \cong Gr(1,4)}$

The generators of $QK\big(\mathrm{Gr}(3,4)\big)$ are the structure sheaves on the Schubert varieties

$$\big\{\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_{1,1}, \mathcal{O}_{1,1,1}\big\}, \tag{B.47}$$

and the multiplication table is

$$\begin{aligned}
&\mathcal{O}_1 * \mathcal{O}_1 = \mathcal{O}_{1,1}, && \mathcal{O}_1 * \mathcal{O}_{1,1} = \mathcal{O}_{1,1,1},\\
&\mathcal{O}_1 * \mathcal{O}_{1,1,1} = -Q\mathcal{O}_0, && \mathcal{O}_{1,1} * \mathcal{O}_{1,1} = -Q\mathcal{O}_0,\\
&\mathcal{O}_{1,1} * \mathcal{O}_{1,1,1} = -Q\mathcal{O}_1, && \mathcal{O}_{1,1,1} * \mathcal{O}_{1,1,1} = -Q\mathcal{O}_{1,1}.
\end{aligned} \tag{B.48}$$

The multiplication table for $\mathrm{Gr}(1,4)$ is obtained by $Q \mapsto -Q$ and $\mu \mapsto \mu^T$.

The quantum difference operator annihilating the I-function is

$$\mathcal{D}_{q,\mathrm{Gr}(3,4)} = \hat{\delta}^4 + Q, \tag{B.49}$$

with 2D limit

$$\mathcal{D}^{2D}_{\mathrm{Gr}(3,4)} = \theta^4 + Q^{2D}. \tag{B.50}$$

B.3.3 Quantum K theory of $\mathbf{Gr(2,5) \cong Gr(3,5)}$

The generators of $QK\big(\mathrm{Gr}(2,5)\big)$ are the structure sheaves on the Schubert varieties

$$\big\{\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_{1,1}, \mathcal{O}_3, \mathcal{O}_{2,1}, \mathcal{O}_{3,1}, \mathcal{O}_{2,2}, \mathcal{O}_{3,2}, \mathcal{O}_{3,3}\big\}, \tag{B.51}$$

and the multiplication table is

$$\begin{aligned}
&\mathcal{O}_1 * \mathcal{O}_1 = \mathcal{O}_{1,1} - \mathcal{O}_{2,1} + \mathcal{O}_2, && \mathcal{O}_1 * \mathcal{O}_2 = \mathcal{O}_{2,1} - \mathcal{O}_{3,1} + \mathcal{O}_3,\\
&\mathcal{O}_1 * \mathcal{O}_{1,1} = \mathcal{O}_{2,1}, && \mathcal{O}_1 * \mathcal{O}_3 = \mathcal{O}_{3,1},\\
&\mathcal{O}_1 * \mathcal{O}_{2,1} = \mathcal{O}_{2,2} + \mathcal{O}_{3,1} - \mathcal{O}_{3,2}, && \mathcal{O}_1 * \mathcal{O}_{3,1} = \mathcal{O}_{3,2} + Q\left(\mathcal{O}_0 - \mathcal{O}_1\right),\\
&\mathcal{O}_1 * \mathcal{O}_{2,2} = \mathcal{O}_{3,2}, && \mathcal{O}_1 * \mathcal{O}_{3,2} = \mathcal{O}_{3,3} + Q\left(\mathcal{O}_1 - \mathcal{O}_2\right),\\
&\mathcal{O}_1 * \mathcal{O}_{3,3} = Q\mathcal{O}_2, && \mathcal{O}_2 * \mathcal{O}_2 = \mathcal{O}_{2,2} + \mathcal{O}_{3,1} - \mathcal{O}_{3,2},\\
&\mathcal{O}_2 * \mathcal{O}_{1,1} = \mathcal{O}_{3,1}, && \mathcal{O}_2 * \mathcal{O}_3 = \mathcal{O}_{3,2},\\
&\mathcal{O}_2 * \mathcal{O}_{2,1} = \mathcal{O}_{3,2} + Q\left(\mathcal{O}_0 - \mathcal{O}_1\right), && \mathcal{O}_2 * \mathcal{O}_{3,1} = \mathcal{O}_{3,3} + Q\left(\mathcal{O}_1 - \mathcal{O}_2\right),\\
&\mathcal{O}_2 * \mathcal{O}_{2,2} = Q\mathcal{O}_1, && \mathcal{O}_2 * \mathcal{O}_{3,2} = Q\left(\mathcal{O}_{1,1} - \mathcal{O}_{2,1} + \mathcal{O}_2\right),\\
&\mathcal{O}_2 * \mathcal{O}_{3,3} = Q\mathcal{O}_{2,1}, && \mathcal{O}_{1,1} * \mathcal{O}_{1,1} = \mathcal{O}_{2,2},\\
&\mathcal{O}_{1,1} * \mathcal{O}_3 = Q\mathcal{O}_0, && \mathcal{O}_{1,1} * \mathcal{O}_{2,1} = \mathcal{O}_{3,2},\\
&\mathcal{O}_{1,1} * \mathcal{O}_{3,1} = Q\mathcal{O}_1, && \mathcal{O}_{1,1} * \mathcal{O}_{2,2} = \mathcal{O}_{3,3},\\
&\mathcal{O}_{1,1} * \mathcal{O}_{3,2} = Q\mathcal{O}_2, && \mathcal{O}_{1,1} * \mathcal{O}_{3,3} = Q\mathcal{O}_3,\\
&\mathcal{O}_3 * \mathcal{O}_3 = \mathcal{O}_{3,3}, && \mathcal{O}_3 * \mathcal{O}_{2,1} = Q\mathcal{O}_1,\\
&\mathcal{O}_3 * \mathcal{O}_{3,1} = Q\mathcal{O}_2, && \mathcal{O}_3 * \mathcal{O}_{2,2} = Q\mathcal{O}_{1,1},\\
&\mathcal{O}_3 * \mathcal{O}_{3,2} = Q\mathcal{O}_{2,1}, && \mathcal{O}_3 * \mathcal{O}_{3,3} = Q\mathcal{O}_{2,2},\\
&\mathcal{O}_{2,1} * \mathcal{O}_{2,1} = \mathcal{O}_{3,3} + Q\left(\mathcal{O}_1 - \mathcal{O}_2\right), && \mathcal{O}_{2,1} * \mathcal{O}_{3,1} = Q\left(\mathcal{O}_{1,1} - \mathcal{O}_{2,1} + \mathcal{O}_2\right),\\
&\mathcal{O}_{2,1} * \mathcal{O}_{2,2} = Q\mathcal{O}_2, && \mathcal{O}_{2,1} * \mathcal{O}_{3,2} = Q\left(\mathcal{O}_{2,1} - \mathcal{O}_{3,1} + \mathcal{O}_3\right),
\end{aligned}$$

$$
\begin{aligned}
&\mathcal{O}_{2,1} * \mathcal{O}_{3,3} = Q\mathcal{O}_{3,1}, && \mathcal{O}_{3,1} * \mathcal{O}_{3,1} = Q\left(\mathcal{O}_{2,1} - \mathcal{O}_{3,1} + \mathcal{O}_{3}\right),\\
&\mathcal{O}_{3,1} * \mathcal{O}_{2,2} = Q\mathcal{O}_{2,1}, && \mathcal{O}_{3,1} * \mathcal{O}_{3,2} = Q\left(\mathcal{O}_{2,2} + \mathcal{O}_{3,1} - \mathcal{O}_{3,2}\right),\\
&\mathcal{O}_{3,1} * \mathcal{O}_{3,3} = Q\mathcal{O}_{3,2}, && \mathcal{O}_{2,2} * \mathcal{O}_{2,2} = Q\mathcal{O}_{3},\\
&\mathcal{O}_{2,2} * \mathcal{O}_{3,2} = Q\mathcal{O}_{3,1}, && \mathcal{O}_{2,2} * \mathcal{O}_{3,3} = Q^2\mathcal{O}_{0},\\
&\mathcal{O}_{3,2} * \mathcal{O}_{3,2} = Q\mathcal{O}_{3,2} + Q^2\left(\mathcal{O}_{0} - \mathcal{O}_{1}\right), && \mathcal{O}_{3,2} * \mathcal{O}_{3,3} = Q^2\mathcal{O}_{1},\\
&\mathcal{O}_{3,3} * \mathcal{O}_{3,3} = Q^2\mathcal{O}_{1,1}.
\end{aligned}
$$

The multiplication table for $\mathrm{Gr}(3,5)$ is obtained by $Q \mapsto -Q$ and $\mu \mapsto \mu^T$.
The quantum difference operator annihilating the I-function is

$$\mathcal{D}_{q,\mathrm{Gr}(2,5)} = \sum_{i=0}^{4} Q^i \mathcal{L}_i, \tag{B.52}$$

with

$$
\begin{aligned}
\mathcal{L}_0 &= -25\hat{\delta}^7(-1+\hat{\delta}+q)^3(1+3q+q^2)^3\\
\mathcal{L}_1 &= 25\hat{\delta}^{10}(q-1)q^4(1+q)(1+q+q^2)\\
&\quad + 25\hat{\delta}^3(q-1)^2q^3(1+2q)(1+3q+q^2)^3\\
&\quad - 25\hat{\delta}^4(q-1)q^4(2+9q)(1+3q+q^2)^3\\
&\quad - 5\hat{\delta}^6q^4(1+3q+q^2)^3(2+10q+63q^2)\\
&\quad + 5\hat{\delta}^5q^4(1+3q+q^2)^3(-7-16q+78q^2)\\
&\quad + 5\hat{\delta}^9(q-1)q^4(-13-13q-3q^2+2q^3+2q^4)\\
&\quad - 5\hat{\delta}^8(q-1)q^4(-12-3q+12q^2+32q^3+47q^4+24q^5)\\
&\quad + 5\hat{\delta}^7q^4(3-7q+39q^2+295q^3+815q^4+1087q^5+677q^6+191q^7+25q^8)\\
\mathcal{L}_2 &= \hat{\delta}^{10}q^9 - 5\hat{\delta}^8(q-1)q^9(1+q) + 25q^8(1+3q+q^2)^2(1+3q^2+q^4)\\
&\quad - 5\hat{\delta}^7(q-1)q^9(3+28q+51q^2+50q^3+25q^4)\\
&\quad + 5\hat{\delta}^6(q-1)q^9(4+79q+162q^2+190q^3+115q^4)\\
&\quad - 25\hat{\delta}q^8(1+3q+q^2)(5+12q+21q^2+45q^3+19q^4+18q^5+5q^6)\\
&\quad + 25\hat{\delta}^2q^8(10+45q+108q^2+240q^3+345q^4+240q^5+180q^6+72q^7+10q^8)\\
&\quad + 5\hat{\delta}^4q^8(12+104q+200q^2+525q^3+685q^4+598q^5+766q^6+210q^7+25q^8)\\
&\quad - \hat{\delta}^5q^8(25+201q-125q^2-5q^3+205q^4+875q^5+1699q^6+225q^7+25q^8)\\
&\quad - 5\hat{\delta}^3q^8(32+179q+475q^2+1190q^3+1645q^4+1193q^5+1096q^6+390q^7+50q^8)\\
\mathcal{L}_3 &= -5\hat{\delta}^7q^{15} + 5\hat{\delta}^6q^{14}(-2+5q) - 5(q-1)q^{13}(5+8q+9q^2+8q^3+5q^4)\\
&\quad + 5\hat{\delta}(q-1)q^{13}(25+38q+45q^2+42q^3+25q^4)\\
&\quad - 5\hat{\delta}^4(q-1)q^{13}(25+21q+48q^2+48q^3+25q^4)\\
&\quad - 5\hat{\delta}^2(q-1)q^{13}(50+70q+91q^2+88q^3+50q^4)\\
&\quad + 5\hat{\delta}^3(q-1)q^{13}(50+61q+93q^2+92q^3+50q^4)\\
&\quad + \hat{\delta}^5q^{13}(-25+34q-70q^2+25q^4+25q^5)\\
\mathcal{L}_4 &= (-1+\hat{\delta})^5q^{19}
\end{aligned}
\tag{B.53}
$$

with 2D limit

$$\mathcal{D}^{2D}_{\mathrm{Gr}(2,5)} = 3125\big[(\theta-1)^3\theta^7 + Q^{2D}\theta^3(3+11\theta+11\theta^2) + (Q^{2D})^2\big]. \tag{B.54}$$

B.3.4 Quantum K theory of Gr(2, 6) ≅ Gr(4, 6)

The generators of $QK\big(\mathrm{Gr}(2,6)\big)$ are the structure sheaves on the Schubert varieties

$$\big\{\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_{1,1}, \mathcal{O}_3, \mathcal{O}_{2,1}, \mathcal{O}_4, \mathcal{O}_{3,1}, \mathcal{O}_{2,2}, \mathcal{O}_{4,1}, \mathcal{O}_{3,2}, \mathcal{O}_{4,2}, \mathcal{O}_{3,3}, \mathcal{O}_{4,3}, \mathcal{O}_{4,4}\big\}, \tag{B.55}$$

and the multiplication table is

$$\begin{aligned}
&\mathcal{O}_1 * \mathcal{O}_1 = \mathcal{O}_{1,1}\mathcal{O}_{2,1} + \mathcal{O}_2, && \mathcal{O}_1 * \mathcal{O}_2 = \mathcal{O}_{2,1} - \mathcal{O}_{3,1} + \mathcal{O}_3,\\
&\mathcal{O}_1 * \mathcal{O}_{1,1} = \mathcal{O}_{2,1}, && \mathcal{O}_1 * \mathcal{O}_3 = \mathcal{O}_{3,1} - \mathcal{O}_{4,1} + \mathcal{O}_4,\\
&\mathcal{O}_1 * \mathcal{O}_{2,1} = \mathcal{O}_{2,2} + \mathcal{O}_{3,1} - \mathcal{O}_{3,2}, && \mathcal{O}_1 * \mathcal{O}_4 = \mathcal{O}_{4,1},\\
&\mathcal{O}_1 * \mathcal{O}_{3,1} = \mathcal{O}_{3,2} + \mathcal{O}_{4,1} - \mathcal{O}_{4,2}, && \mathcal{O}_1 * \mathcal{O}_{2,2} = \mathcal{O}_{3,2},\\
&\mathcal{O}_1 * \mathcal{O}_{4,1} = \mathcal{O}_{4,2} + Q\left(\mathcal{O}_0 - \mathcal{O}_1\right), && \mathcal{O}_1 * \mathcal{O}_{3,2} = \mathcal{O}_{3,3} + \mathcal{O}_{4,2} - \mathcal{O}_{4,3},\\
&\mathcal{O}_1 * \mathcal{O}_{4,2} = \mathcal{O}_{4,3} + Q\left(\mathcal{O}_1 - \mathcal{O}_2\right), && \mathcal{O}_1 * \mathcal{O}_{3,3} = \mathcal{O}_{4,3},\\
&\mathcal{O}_1 * \mathcal{O}_{4,3} = \mathcal{O}_{4,4} + Q\left(\mathcal{O}_2 - \mathcal{O}_3\right), && \mathcal{O}_1 * \mathcal{O}_{4,4} = Q\mathcal{O}_3,\\
&\mathcal{O}_2 * \mathcal{O}_2 = \mathcal{O}_{2,2} + \mathcal{O}_{3,1} - \mathcal{O}_{3,2} - \mathcal{O}_{4,1} + \mathcal{O}_4, && \mathcal{O}_2 * \mathcal{O}_{1,1} = \mathcal{O}_{3,1},\\
&\mathcal{O}_2 * \mathcal{O}_3 = \mathcal{O}_{3,2} + \mathcal{O}_{4,1} - \mathcal{O}_{4,2}, && \mathcal{O}_2 * \mathcal{O}_{2,1} = \mathcal{O}_{3,2} + \mathcal{O}_{4,1} - \mathcal{O}_{4,2},\\
&\mathcal{O}_2 * \mathcal{O}_4 = \mathcal{O}_{4,2}, && \mathcal{O}_2 * \mathcal{O}_{3,1} = \mathcal{O}_{3,3} + \mathcal{O}_{4,2} - \mathcal{O}_{4,3} + Q\left(\mathcal{O}_0 - \mathcal{O}_1\right),\\
&\mathcal{O}_2 * \mathcal{O}_{2,2} = \mathcal{O}_{4,2}, && \mathcal{O}_2 * \mathcal{O}_{4,1} = \mathcal{O}_{4,3} + Q\left(\mathcal{O}_1 - \mathcal{O}_2\right),\\
&\mathcal{O}_2 * \mathcal{O}_{3,2} = \mathcal{O}_{4,3} + Q\left(\mathcal{O}_1 - \mathcal{O}_2\right), && \mathcal{O}_2 * \mathcal{O}_{4,2} = \mathcal{O}_{4,4} + Q\left(\mathcal{O}_{1,1} - \mathcal{O}_{2,1} + \mathcal{O}_2 - \mathcal{O}_3\right),\\
&\mathcal{O}_2 * \mathcal{O}_{3,3} = Q\mathcal{O}_2, && \mathcal{O}_2 * \mathcal{O}_{4,3} = Q\left(\mathcal{O}_{2,1} - \mathcal{O}_{3,1} + \mathcal{O}_3\right),\\
&\mathcal{O}_2 * \mathcal{O}_{4,4} = Q\mathcal{O}_{3,1}, && \mathcal{O}_{1,1} * \mathcal{O}_{1,1} = \mathcal{O}_{2,2},\\
&\mathcal{O}_{1,1} * \mathcal{O}_3 = \mathcal{O}_{4,1}, && \mathcal{O}_{1,1} * \mathcal{O}_{2,1} = \mathcal{O}_{3,2},\\
&\mathcal{O}_{1,1} * \mathcal{O}_4 = Q\mathcal{O}_0, && \mathcal{O}_{1,1} * \mathcal{O}_{3,1} = \mathcal{O}_{4,2},\\
&\mathcal{O}_{1,1} * \mathcal{O}_{2,2} = \mathcal{O}_{3,3}, && \mathcal{O}_{1,1} * \mathcal{O}_{4,1} = Q\mathcal{O}_1,\\
&\mathcal{O}_{1,1} * \mathcal{O}_{3,2} = \mathcal{O}_{4,3}, && \mathcal{O}_{1,1} * \mathcal{O}_{4,2} = Q\mathcal{O}_2,\\
&\mathcal{O}_{1,1} * \mathcal{O}_{3,3} = \mathcal{O}_{4,4}, && \mathcal{O}_{1,1} * \mathcal{O}_{4,3} = Q\mathcal{O}_3,\\
&\mathcal{O}_{1,1} * \mathcal{O}_{4,4} = Q\mathcal{O}_4, && \mathcal{O}_3 * \mathcal{O}_3 = \mathcal{O}_{3,3} + \mathcal{O}_{4,2} - \mathcal{O}_{4,3},\\
&\mathcal{O}_3 * \mathcal{O}_{2,1} = \mathcal{O}_{4,2} + Q\left(\mathcal{O}_0 - \mathcal{O}_1\right), && \mathcal{O}_3 * \mathcal{O}_4 = \mathcal{O}_{4,3},\\
&\mathcal{O}_3 * \mathcal{O}_{3,1} = \mathcal{O}_{4,3} + Q\left(\mathcal{O}_1 - \mathcal{O}_2\right), && \mathcal{O}_3 * \mathcal{O}_{2,2} = Q\mathcal{O}_1,\\
&\mathcal{O}_3 * \mathcal{O}_{4,1} = \mathcal{O}_{4,4} + Q\left(\mathcal{O}_2 - \mathcal{O}_3\right), && \mathcal{O}_3 * \mathcal{O}_{3,2} = Q\left(\mathcal{O}_{1,1} - \mathcal{O}_{2,1} + \mathcal{O}_2\right),\\
&\mathcal{O}_3 * \mathcal{O}_{4,2} = Q\left(\mathcal{O}_{2,1} - \mathcal{O}_{3,1} + \mathcal{O}_3\right), && \mathcal{O}_3 * \mathcal{O}_{3,3} = Q\mathcal{O}_{2,1},\\
&\mathcal{O}_3 * \mathcal{O}_{4,3} = Q\left(\mathcal{O}_{2,2} + \mathcal{O}_{3,1} - \mathcal{O}_{3,2}\right), && \mathcal{O}_3 * \mathcal{O}_{4,4} = Q\mathcal{O}_{3,2},\\
&\mathcal{O}_{2,1} * \mathcal{O}_{2,1} = \mathcal{O}_{3,3} + \mathcal{O}_{4,2} - \mathcal{O}_{4,3}, && \mathcal{O}_{2,1} * \mathcal{O}_4 = Q\mathcal{O}_1,\\
&\mathcal{O}_{2,1} * \mathcal{O}_{3,1} = \mathcal{O}_{4,3} + Q\left(\mathcal{O}_1 - \mathcal{O}_2\right), && \mathcal{O}_{2,1} * \mathcal{O}_{2,2} = \mathcal{O}_{4,3},\\
&\mathcal{O}_{2,1} * \mathcal{O}_{4,1} = Q\left(\mathcal{O}_{1,1} - \mathcal{O}_{2,1} + \mathcal{O}_2\right), && \mathcal{O}_{2,1} * \mathcal{O}_{3,2} = \mathcal{O}_{4,4} + Q\left(\mathcal{O}_2 - \mathcal{O}_3\right),\\
&\mathcal{O}_{2,1} * \mathcal{O}_{4,2} = Q\left(\mathcal{O}_{2,1} - \mathcal{O}_{3,1} + \mathcal{O}_3\right), && \mathcal{O}_{2,1} * \mathcal{O}_{3,3} = Q\mathcal{O}_3,\\
&\mathcal{O}_{2,1} * \mathcal{O}_{4,3} = Q\left(\mathcal{O}_{3,1} - \mathcal{O}_{4,1} + \mathcal{O}_4\right), && \mathcal{O}_{2,1} * \mathcal{O}_{4,4} = Q\mathcal{O}_{4,1},
\end{aligned}$$

$\mathcal{O}_4 * \mathcal{O}_4 = \mathcal{O}_{4,4}$,
$\mathcal{O}_4 * \mathcal{O}_{3,1} = Q\mathcal{O}_2$,
$\mathcal{O}_4 * \mathcal{O}_{2,2} = Q\mathcal{O}_{1,1}$,
$\mathcal{O}_4 * \mathcal{O}_{4,1} = Q\mathcal{O}_3$,
$\mathcal{O}_4 * \mathcal{O}_{3,2} = Q\mathcal{O}_{2,1}$,
$\mathcal{O}_4 * \mathcal{O}_{4,2} = Q\mathcal{O}_{3,1}$,
$\mathcal{O}_4 * \mathcal{O}_{3,3} = Q\mathcal{O}_{2,2}$,
$\mathcal{O}_4 * \mathcal{O}_{4,3} = Q\mathcal{O}_{3,2}$,
$\mathcal{O}_4 * \mathcal{O}_{4,4} = Q\mathcal{O}_{3,3}$,
$\mathcal{O}_{3,1} * \mathcal{O}_{3,1} = \mathcal{O}_{4,4}+Q\left(\mathcal{O}_{1,1}-\mathcal{O}_{2,1}+\mathcal{O}_2-\mathcal{O}_3\right)$,
$\mathcal{O}_{3,1} * \mathcal{O}_{2,2} = Q\mathcal{O}_2$,
$\mathcal{O}_{3,1} * \mathcal{O}_{4,1} = Q\left(\mathcal{O}_{2,1}-\mathcal{O}_{3,1}+\mathcal{O}_3\right)$,
$\mathcal{O}_{3,1} * \mathcal{O}_{3,2} = Q\left(\mathcal{O}_{2,1}-\mathcal{O}_{3,1}+\mathcal{O}_3\right)$,
$\mathcal{O}_{3,1} * \mathcal{O}_{4,2} = Q\left(\mathcal{O}_{2,2}+\mathcal{O}_{3,1}-\mathcal{O}_{3,2}-\mathcal{O}_{4,1}+\mathcal{O}_4\right)$,
$\mathcal{O}_{3,1} * \mathcal{O}_{3,3} = Q\mathcal{O}_{3,1}$,
$\mathcal{O}_{3,1} * \mathcal{O}_{4,3} = Q\left(\mathcal{O}_{3,2}+\mathcal{O}_{4,1}-\mathcal{O}_{4,2}\right)$,
$\mathcal{O}_{3,1} * \mathcal{O}_{4,4} = Q\mathcal{O}_{4,2}$,
$\mathcal{O}_{2,2} * \mathcal{O}_{2,2} = \mathcal{O}_{4,4}$,
$\mathcal{O}_{2,2} * \mathcal{O}_{4,1} = Q\mathcal{O}_{2,1}$,
$\mathcal{O}_{2,2} * \mathcal{O}_{3,2} = Q\mathcal{O}_3$,
$\mathcal{O}_{2,2} * \mathcal{O}_{4,2} = Q\mathcal{O}_{3,1}$,
$\mathcal{O}_{2,2} * \mathcal{O}_{3,3} = Q\mathcal{O}_4$,
$\mathcal{O}_{2,2} * \mathcal{O}_{4,3} = Q\mathcal{O}_{4,1}$,
$\mathcal{O}_{2,2} * \mathcal{O}_{4,4} = Q^2\mathcal{O}_0$,
$\mathcal{O}_{4,1} * \mathcal{O}_{4,1} = Q\left(\mathcal{O}_{3,1}-\mathcal{O}_{4,1}+\mathcal{O}_4\right)$,
$\mathcal{O}_{4,1} * \mathcal{O}_{3,2} = Q\left(\mathcal{O}_{2,2}+\mathcal{O}_{3,1}-\mathcal{O}_{3,2}\right)$,
$\mathcal{O}_{4,1} * \mathcal{O}_{4,2} = Q\left(\mathcal{O}_{3,2}+\mathcal{O}_{4,1}-\mathcal{O}_{4,2}\right)$,
$\mathcal{O}_{4,1} * \mathcal{O}_{3,3} = Q\mathcal{O}_{3,2}$,
$\mathcal{O}_{4,1} * \mathcal{O}_{4,3} = Q\left(\mathcal{O}_{3,3}+\mathcal{O}_{4,2}-\mathcal{O}_{4,3}\right)$,
$\mathcal{O}_{4,1} * \mathcal{O}_{4,4} = Q\mathcal{O}_{4,3}$,
$\mathcal{O}_{3,2} * \mathcal{O}_{3,2} = Q\left(\mathcal{O}_{3,1}-\mathcal{O}_{4,1}+\mathcal{O}_4\right)$,
$\mathcal{O}_{3,2} * \mathcal{O}_{4,2} = Q\left(\mathcal{O}_{3,2}+\mathcal{O}_{4,1}-\mathcal{O}_{4,2}\right)$,
$\mathcal{O}_{3,2} * \mathcal{O}_{3,3} = Q\mathcal{O}_{4,1}$,
$\mathcal{O}_{3,2} * \mathcal{O}_{4,3} = Q\mathcal{O}_{4,2}+Q^2\left(\mathcal{O}_0-\mathcal{O}_1\right)$,
$\mathcal{O}_{3,2} * \mathcal{O}_{4,4} = Q^2\mathcal{O}_1$,
$\mathcal{O}_{4,2} * \mathcal{O}_{4,2} = Q\left(\mathcal{O}_{3,3}+\mathcal{O}_{4,2}-\mathcal{O}_{4,3}\right)+Q^2\left(\mathcal{O}_0-\mathcal{O}_1\right)$,
$\mathcal{O}_{4,2} * \mathcal{O}_{3,3} = Q\mathcal{O}_{4,2}$,
$\mathcal{O}_{4,2} * \mathcal{O}_{4,3} = Q\mathcal{O}_{4,3}+Q^2\left(\mathcal{O}_1-\mathcal{O}_2\right)$,
$\mathcal{O}_{4,2} * \mathcal{O}_{4,4} = Q^2\mathcal{O}_2$,
$\mathcal{O}_{3,3} * \mathcal{O}_{3,3} = Q^2\mathcal{O}_0$,
$\mathcal{O}_{3,3} * \mathcal{O}_{4,3} = Q^2\mathcal{O}_1$,
$\mathcal{O}_{3,3} * \mathcal{O}_{4,4} = Q^2\mathcal{O}_{1,1}$,
$\mathcal{O}_{4,3} * \mathcal{O}_{4,3} = Q^2\left(\mathcal{O}_{1,1}-\mathcal{O}_{2,1}+\mathcal{O}_2\right)$,
$\mathcal{O}_{4,3} * \mathcal{O}_{4,4} = Q^2\mathcal{O}_{2,1}$,
$\mathcal{O}_{4,4} * \mathcal{O}_{4,4} = Q^2\mathcal{O}_{2,2}$.

The multiplication table for $\mathrm{Gr}(4,6)$ is obtained by $\mu \mapsto \mu^T$.

The quantum difference operator annihilating the I-function is

$$\mathcal{D}_{q,\mathrm{Gr}(2,6)} = \sum_{i=0}^{8} Q^i \mathcal{L}_i, \tag{B.56}$$

where the lowest- and highest-order terms are (the most amenable to printing)

$$\begin{aligned}\mathcal{L}_0 &= 882(3+3q+18q^2+15q^3+22q^4+14q^5+6q^6+3q^7)\times \\ &\quad\times (q-1)^5(1+4q+2q^2)^4\hat{\delta}^9(\hat{\delta}+q-1)^5(\hat{\delta}+q^2-1), \\ \mathcal{L}_8 &= 729q^{57}(1-\hat{\delta})^6.\end{aligned} \tag{B.57}$$

The 2D limit of the operator is

$$\mathcal{D}^{2D}_{\mathrm{Gr}(2,6)} = 177885288(\theta-2)\big[(\theta-1)^5\theta^9-(Q^{2D})\theta^5(1+2\theta)(4+13\theta+13\theta^2)-3(Q^{2D})^2(2+3\theta)(4+3\theta)\big]. \tag{B.58}$$

B.4 Characteristic classes

As discussed in the previous section, the tangent bundle satisfies $T\,\mathrm{Gr}(M,N) \cong \mathrm{Hom}(S,Q)$, where S is the tautological bundle of rank M and Q is the quotient bundle of rank $N-M$

defined by the Euler sequence $0 \to S \to \underline{\mathbb{C}}^N \to Q \to 0$. The sequence implies the K-theory relation $[T\mathrm{Gr}(M,N)] = [\mathrm{Hom}(S,\underline{\mathbb{C}})^{\oplus N}] - [\mathrm{Hom}(S,S)] = N[S^*] - [S^* \otimes S]$ [98]. For characteristic classes G defined by multiplicative sequences $g(x)$ this means that[6] for $X = \mathrm{Gr}(M,N)$

$$G(X) = \frac{\prod_{a=1}^M g(x_a)^N}{\prod_{a<b}^M g(x_{ab})g(x_{ba})}, \quad x_{ab} = x_a - x_b \ , \tag{B.59}$$

where x_a are the Chern roots of S^*. The right-hand side can be decomposed $g(x) = \sum_\mu g_\mu \sigma_\mu$, where the *allowed* Schur polynomials σ_μ (meaning, polynomials labeled by Young diagrams fitting into an $M \times (N-M)$-box) are interpreted as the generators of $H^*\big(\mathrm{Gr}(M,N)\big)$. Thus, the Chern class with $g(x) = 1 + x$ is[7]

$$\begin{aligned} c(X) &= \frac{\prod_{a=1}^M (1+x_a)^N}{\prod_{a<b}^M (1-x_{ab}^2)} \\ &= 1 + N\sum_{a=1}^M x_a + (\frac{N^2}{2}(\sum_{a=1}^M x_a)^2 - \frac{N}{2}\sum_{a=1}^M x_a^2 + \sum_{a<b}^M x_{ab}^2) + \ldots \\ &= 1 + N\,\mathrm{tr}_R(x) + (M - \tfrac{N}{2})\,\mathrm{tr}_{SU(M)}(x^2) + \tfrac{1}{2}N(N-1)\,\mathrm{tr}_{U(1)}(x^2) + \ldots \ , \end{aligned} \tag{B.60}$$

while the Todd class with $g(x) = \frac{x}{1-e^{-x}}$ is

$$\mathrm{td}(X) = \prod_{a=1}^M \left(\frac{x_a}{(1-e^{-x_a})}\right)^N \prod_{a<b}^M \frac{(1-e^{x_{ab}})(1-e^{-x_{ab}})}{-x_{ab}^2} \ , \tag{B.61}$$

and the q-Gamma class is given by the sequence of $g(x) = \Gamma_q(1+x)$ and hence

$$\Gamma_q(X) = \frac{\prod_{a=1}^M \Gamma_q(1+x_a)^N}{\prod_{a<b}^M \Gamma_q(1+x_{ab})\Gamma_q(1-x_{ab})} \, . \tag{B.62}$$

Lastly, by mapping Chern classes to Chern characters

$$\mathrm{ch} = \mathrm{rank} + c_1 + \tfrac{1}{2}\big(c_1^2 - 2c_2\big) + \tfrac{1}{6}\big(c_1^3 - 3c_1c_2 + 3c_3\big) + \ldots \tag{B.63}$$

we find from (B.60) that the Chern character of (the tangent bundle of) X is

$$\mathrm{ch}(X) = \dim X + N\,\mathrm{tr}_R(x) + (\tfrac{N}{2} - M)\,\mathrm{tr}_{SU(M)}(x^2) + \tfrac{N}{2}\,\mathrm{tr}_{U(1)}(x^2) + \ldots \, . \tag{B.64}$$

[6] The relation can also be derived from the symplectic quotient construction $\mathrm{Gr}(M,N) \cong \mathrm{Hom}(\mathbb{C}^M, \mathbb{C}^N)//U(M)$ [92].

[7] See footnote 1 for the definitions of traces.

Appendix C

Decompositions and supercurrent multiplets in 3D

In this chapter we gather some details about the decomposition of supersymmetry algebras in our setup from chapter 3. Throughout this chapter (as in chapter 3), Greek letters $\mu, \nu, \ldots$ denote bulk spacetime indices, hatted Greek letters $\hat{\mu}, \hat{\nu}, \ldots$ denote boundary spacetime indices, while Greek letters $\alpha, \beta, \ldots$ denote spinor indices.

C.1 Decomposition of 3D $\mathcal{N} = 2$

C.1.1 Decomposition to 2D $\mathcal{N} = (0, 2)$

In this subsection we discuss decomposition of 3D $\mathcal{N} = 2$ superfields into their $(0, 2)$ subrepresentations. To do so, we use the *branching coordinate* (or *invariant coordinate*) [3, 132, 133]. It is defined such that in superspace with coordinates $(\xi^\mu, \theta^+, \theta^-)$, the representations of the preserved supercharge operators Q_+ and $\overline{Q}_+$ commute with θ^- and $\overline{\theta}^-$, i.e., contain no derivatives in these variables. Another property of the branching coordinate is that the preserved supercharge operators do not contain or generate $P_\perp$.

If we want to preserve the $(0, 2)$-subalgebra generated by $Q_+, \overline{Q}_+$, one can easily check that we need

$$\xi^\mu = x^\mu + i\delta_\perp{}^\mu \theta^2 = \big(x^+, x^-, x^\perp + i(\theta^+\overline{\theta}^- - \theta^-\overline{\theta}^+)\big). \tag{C.1}$$

Indeed, in terms of ξ^μ the operators (3.50), (3.51) are

$$\begin{aligned}
Q_+ &= \frac{\partial}{\partial\theta^+} + 2i\overline{\theta}^+\frac{\partial}{\partial\xi^+}, & Q_- &= \frac{\partial}{\partial\theta^-} + 2i\overline{\theta}^-\frac{\partial}{\partial\xi^-} - 2i\overline{\theta}^+\frac{\partial}{\partial\xi^\perp},\\
\overline{Q}_+ &= -\frac{\partial}{\partial\overline{\theta}^+} - 2i\theta^+\frac{\partial}{\partial\xi^+}, & \overline{Q}_- &= -\frac{\partial}{\partial\overline{\theta}^-} - 2i\theta^-\frac{\partial}{\partial\xi^-} + 2i\theta^+\frac{\partial}{\partial\xi^\perp},\\
D_+ &= \frac{\partial}{\partial\theta^+} - 2i\overline{\theta}^+\frac{\partial}{\partial\xi^+} + 2i\overline{\theta}^-\frac{\partial}{\partial\xi^\perp}, & D_- &= \frac{\partial}{\partial\theta^-} - 2i\overline{\theta}^-\frac{\partial}{\partial\xi^-},\\
\overline{D}_+ &= -\frac{\partial}{\partial\overline{\theta}^+} + 2i\theta^+\frac{\partial}{\partial\xi^+} - 2i\theta^-\frac{\partial}{\partial\xi^\perp}, & \overline{D}_- &= -\frac{\partial}{\partial\overline{\theta}^-} + 2i\theta^-\frac{\partial}{\partial\xi^-}.
\end{aligned} \tag{C.2}$$

In particular, the $Q_+, \overline{Q}_+$ do not contain any $\partial_\perp$ terms. The $(0, 2)$-covariant derivatives are defined by

$$\begin{aligned} D_+^{(0,2)} &= \frac{\partial}{\partial\theta^+} - 2i\overline{\theta}^+\frac{\partial}{\partial\xi^+}, \\ \overline{D}_+^{(0,2)} &= -\frac{\partial}{\partial\overline{\theta}^+} + 2i\theta^+\frac{\partial}{\partial\xi^+}, \end{aligned} \tag{C.3}$$

so we have that

$$\begin{aligned} D_+ &= D_+^{(0,2)} + 2i\overline{\theta}^-\partial_\perp, \\ \overline{D}_+ &= \overline{D}_+^{(0,2)} - 2i\theta^-\partial_\perp. \end{aligned} \tag{C.4}$$

We often drop the $(0,2)$-label when the covariant derivative type is clear from context. We may now simply perform a Taylor expansion of an $\mathcal{N}=2$ superfield $\mathcal{X}$:

$$\mathcal{X}(x,\theta,\overline{\theta}) = X^{(0)}(\xi,\theta^+,\overline{\theta}^+)+\theta^- X^{(1a)}(\xi,\theta^+,\overline{\theta}^+)+X^{(1b)}(\xi,\theta^+,\overline{\theta}^+)\,\overline{\theta}^- +\theta^-\overline{\theta}^- X^{(2)}(\xi,\theta^+,\overline{\theta}^+). \tag{C.5}$$

Since the $(0,2)$-operators $\{Q_+,\overline{Q}_+,D_+^{(0,2)},\overline{D}_+^{(0,2)}\}$ do not contain derivatives in $\theta^-,\overline{\theta}^-$, the coefficients in these variables are preserved by these operators; in other words, the coefficients are the $(0,2)$-subrepresentations of $\mathcal{X}(x,\theta,\overline{\theta})$.

For example, let us decompose 3D chiral field (3.107) into its $(0,2)$-submultiplets. In terms of ξ we find that the "chiral" coordinate y^μ is

$$(y^+,y^-,y^\perp) = (\xi^+ - 2i\theta^+\overline{\theta}^-, \xi^- - 2i\theta^-\overline{\theta}^-, \xi^\perp + 2i\theta^-\overline{\theta}^+). \tag{C.6}$$

The expansion of Φ_{3D} gives then

$$\Phi_{3D}(x,\theta,\overline{\theta}) = \Phi(\xi,\theta^+,\overline{\theta}^+) - 2i\theta^-\overline{\theta}^-\partial_-\Phi(\xi,\theta^+,\overline{\theta}^+) + \sqrt{2}\theta^-\Psi(\xi,\theta^+,\overline{\theta}^+), \tag{C.7}$$

where the chiral and Fermi multiplets are

$$\begin{aligned} \Phi &= \phi + \sqrt{2}\theta^+\psi_+ - 2i\theta^+\overline{\theta}^+\partial_+\phi, \\ \Psi &= \psi_- - \sqrt{2}\theta^+F - 2i\theta^+\overline{\theta}^+\partial_+\psi_- + \sqrt{2}i\overline{\theta}^+\partial_\perp\phi - 2i\theta^+\overline{\theta}^+\partial_\perp\psi_+. \end{aligned} \tag{C.8}$$

in agreement with [65]. These satisfy

$$\overline{D}_+\Phi = 0, \quad \overline{D}_+\Psi = -i\sqrt{2}\partial_\perp\Phi. \tag{C.9}$$

We can obtain the full expansion using (C.1) on the right-hand side of (C.7)

$$\Phi_{3D} = \Phi+\sqrt{2}\theta^-\Psi+i(\theta^+\overline{\theta}^- -\theta^-\overline{\theta}^+)\partial_\perp\Phi-2i\theta^-\overline{\theta}^-\partial_-\Phi-\sqrt{2}i\theta^+\theta^-\overline{\theta}^-\partial_\perp\Psi-\theta^+\overline{\theta}^+\theta^-\overline{\theta}^-\partial_\perp^2\Phi, \tag{C.10}$$

where now all (super-)functions depend on x.

C.1.2 Decomposition to 2D $\mathcal{N}=(1,1)$

In the presence of a boundary we may also preserve a 2D $\mathcal{N}=(1,1)$ subalgebra. For completeness, we also present decomposition of 3D $\mathcal{N}=2$ superfields and operators into this subalgebra.
We want to preserve the $(1,1)$-subalgebra generated by[1]

$$\begin{aligned} \mathbb{Q}_- &:= \tfrac{1}{2}(Q_- + \overline{Q}_-), \\ \mathbb{Q}_+ &:= \tfrac{i}{2}(Q_+ - \overline{Q}_+). \end{aligned} \tag{C.11}$$

[1]Note that in breaking from a 2D $(2,2)$ algebra to 2D $(1,1)$ we may additionally "pick a phase", i.e., we may choose

$$\mathbb{Q}_i := \tfrac{1}{2}(e^{iv_i}Q_i + e^{-iv_i}\overline{Q}_i),$$

These satisfy the $(1,1)$-algebra:

$$\begin{aligned}\{\mathbb{Q}_-,\mathbb{Q}_-\}&=-2i\partial_-,\\ \{\mathbb{Q}_+,\mathbb{Q}_+\}&=-2i\partial_+,\\ \{\mathbb{Q}_-,\mathbb{Q}_+\}&=0.\end{aligned} \tag{C.12}$$

Defining rotated, real Grassmann variables

$$\begin{aligned}\vartheta^-&=-i(\theta^- -\overline{\theta}^-), &\quad \tilde{\vartheta}^-&=-(\theta^-+\overline{\theta}^-),\\ \vartheta^+&=-(\theta^+ +\overline{\theta}^+), &\quad \tilde{\vartheta}^+&=-i(\theta^+-\overline{\theta}^+).\end{aligned} \tag{C.13}$$

and using the shifted "branching" coordinate

$$\begin{aligned}(\zeta^+,\zeta^-,\zeta^\perp)&=\big(x^+,x^-,x^\perp+\tfrac{i}{2}(\vartheta^-\tilde{\vartheta}^++\vartheta^+\tilde{\vartheta}^-)\big)\\ &=\big(y^+-\vartheta^+\tilde{\vartheta}^+,y^-+\vartheta^-\tilde{\vartheta}^-,y^\perp-\tfrac{1}{2}(\vartheta^-\vartheta^+-i\vartheta^-\tilde{\vartheta}^+-i\vartheta^+\tilde{\vartheta}^--\tilde{\vartheta}^-\tilde{\vartheta}^+)\big),\end{aligned} \tag{C.14}$$

the supercharges take the form

$$\begin{aligned}\mathbb{Q}_-&=-i\frac{\partial}{\partial\vartheta^-}+\vartheta^-\frac{\partial}{\partial\zeta^-},\\ \mathbb{Q}_+&=-i\frac{\partial}{\partial\vartheta^+}+\vartheta^+\frac{\partial}{\partial\zeta^+}.\end{aligned} \tag{C.15}$$

We can then define also the $(1,1)$-covariant derivatives

$$\begin{aligned}\mathbb{D}_-&=-i\frac{\partial}{\partial\vartheta^-}-\vartheta^-\frac{\partial}{\partial\zeta^-},\\ \mathbb{D}_+&=-i\frac{\partial}{\partial\vartheta^+}-\vartheta^+\frac{\partial}{\partial\zeta^+}.\end{aligned} \tag{C.16}$$

which of course satisfy

$$\begin{aligned}\{\mathbb{D}_\pm,\mathbb{D}_\pm\}&=2i\partial_\pm,\\ \{\mathbb{D}_+,\mathbb{D}_-\}&=0.\end{aligned} \tag{C.17}$$

Then, $(1,1)$-irreducible multiplets are of the form

$$\Sigma=A+i\vartheta^+B+i\vartheta^-C+i\vartheta^-\vartheta^+D. \tag{C.18}$$

Using the branching coordinate we find the decomposition

$$\Phi_{3D}(x,\theta,\overline{\theta})=\mathit{\Phi}+\tilde{\vartheta}^+\mathbb{D}_+\mathit{\Phi}-\tilde{\vartheta}^-\mathbb{D}_-\mathit{\Phi}+\tilde{\vartheta}^-\tilde{\vartheta}^+(\mathbb{D}_-\mathbb{D}_+\mathit{\Phi}-\partial_\perp\mathit{\Phi}) \tag{C.19}$$

where on the right-hand side we have super(functions) of $(\zeta,\vartheta,\tilde{\vartheta})$ and we have defined the $(1,1)$-multiplet

$$\mathit{\Phi}\coloneqq\phi-\tfrac{1}{\sqrt{2}}\vartheta^+\psi_++\tfrac{i}{\sqrt{2}}\vartheta^-\psi_-+\tfrac{1}{2}\vartheta^-\vartheta^+(\partial_\perp\phi+iF). \tag{C.20}$$

---

for $i=1,2$ [24, 32]). However, in 3D $\mathcal{N}=2$ to $(1,1)$-subalgebra breaking, we further need to impose that the charges do not generate any $\partial_\perp$-derivatives, i.e., that

$$\{\mathbb{Q}_1,\mathbb{Q}_2\}=\tfrac{i}{2}\,\mathrm{Re}(e^{i(v_1-v_2)})\partial_\perp\overset{!}{=}0,$$

which fixes the phase to $v_1-v_2=\frac{\pi}{2}+k\pi$.

Around the same point, the full branching reads

$$\begin{aligned}\Phi_{3D} &= \mathit{\Phi} + \tilde{\vartheta}^{+}\mathbb{D}_{+}\mathit{\Phi} - \tilde{\vartheta}^{-}\mathbb{D}_{-}\mathit{\Phi} + \tfrac{i}{2}(\vartheta^{-}\tilde{\vartheta}^{+} + \vartheta^{+}\tilde{\vartheta}^{-})\partial_{\perp}\mathit{\Phi} + \tilde{\vartheta}^{-}\tilde{\vartheta}^{+}(\mathbb{D}_{-}\mathbb{D}_{+}\mathit{\Phi} - \partial_{\perp}\mathit{\Phi}) \\ &\quad + \tfrac{i}{2}\tilde{\vartheta}^{-}\tilde{\vartheta}^{+}\vartheta^{+}\partial_{\perp}\mathbb{D}_{+}\mathit{\Phi} + \tfrac{i}{2}\tilde{\vartheta}^{-}\tilde{\vartheta}^{+}\vartheta^{-}\partial_{\perp}\mathbb{D}_{-}\mathit{\Phi} + \tfrac{1}{4}\vartheta^{4}\partial_{\perp}^{2}\mathit{\Phi},\end{aligned} \tag{C.21}$$

where $\vartheta^4 \coloneqq \vartheta^{+}\tilde{\vartheta}^{+}\vartheta^{-}\tilde{\vartheta}^{-}$.
The $(1,1)$-variations are induced by $\delta_{\mathrm{sym}} = i\varepsilon_{+}\mathbb{Q}_{+} + i\varepsilon_{-}\mathbb{Q}_{-}$, $\varepsilon_{\pm}$ real spinors:

$$\begin{aligned}\delta_{\mathrm{sym}}\phi &= -\tfrac{1}{\sqrt{2}}\varepsilon_{+}\psi_{+} + \tfrac{i}{\sqrt{2}}\varepsilon_{-}\psi_{-}, \\ \delta_{\mathrm{sym}}\psi_{+} &= \sqrt{2}i\varepsilon_{+}\partial_{+}\phi + \tfrac{1}{\sqrt{2}}\varepsilon_{-}(\partial_{\perp}\phi + iF), \\ \delta_{\mathrm{sym}}\psi_{-} &= -\tfrac{i}{\sqrt{2}}\varepsilon_{+}(\partial_{\perp}\phi + iF) - \sqrt{2}\varepsilon_{-}\partial_{-}\phi, \\ \delta_{\mathrm{sym}}iF &= \sqrt{2}\varepsilon_{+}(\partial_{+}\psi_{-} + \tfrac{1}{2}\partial_{\perp}\psi_{+}) - \sqrt{2}i\varepsilon_{-}(\partial_{-}\psi_{+} + \tfrac{1}{2}\partial_{\perp}\psi_{-}).\end{aligned} \tag{C.22}$$

The 3D $\mathcal{N}=2$ bulk Lagrangian of one chiral field Φ_{3D} is decomposed as follows

$$\begin{aligned}\mathcal{L} &= \int \mathrm{d}^4\theta\big(\overline{\Phi}_{3D}\Phi_{3D} - (\theta\theta)(\overline{\theta\theta})\partial_{+}\partial_{-}(\overline{\Phi}_{3D}\Phi_{3D})\big) + \int \mathrm{d}^2\theta W(\Phi_{3D}) + \int \mathrm{d}^2\overline{\theta}\overline{W}(\overline{\Phi}_{3D}) \\ &= \int \mathrm{d}^2\vartheta\Big[2(\mathbb{D}_{-}\overline{\mathit{\Phi}}\mathbb{D}_{+}\mathit{\Phi} - \mathbb{D}_{+}\overline{\mathit{\Phi}}\mathbb{D}_{-}\mathit{\Phi}) + \overline{\mathit{\Phi}}\partial_{\perp}\mathit{\Phi} - \partial_{\perp}\overline{\mathit{\Phi}}\mathit{\Phi} - 2i(W + \overline{W}) \\ &\quad + \partial_{\perp}\big[\tfrac{i}{2}\vartheta^{+}(\mathbb{D}_{+}\overline{\mathit{\Phi}}\mathit{\Phi} - \overline{\mathit{\Phi}}\mathbb{D}_{+}\mathit{\Phi}) + \tfrac{i}{2}\vartheta^{-}(\mathbb{D}_{-}\overline{\mathit{\Phi}}\mathit{\Phi} - \overline{\mathit{\Phi}}\mathbb{D}_{-}\mathit{\Phi}) \\ &\qquad + \vartheta^{-}\vartheta^{+}\big(\tfrac{1}{4}\partial_{\perp}(\overline{\mathit{\Phi}}\mathit{\Phi}) + i(W - \overline{W})\big)\big]\Big].\end{aligned} \tag{C.23}$$

The equations of motion are in superspace

$$0 = 2\mathbb{D}_{-}\mathbb{D}_{+}\mathit{\Phi} - \partial_{\perp}\mathit{\Phi} + i\overline{W}'(\overline{\mathit{\Phi}}), \tag{C.24}$$

which in components are the usual bulk equations.

C.2 Supercurrent multiplets in 3D

C.2.1 Decomposition to (0, 2)-multiplets

We decompose the bulk multiplet $\mathcal{S}_{\alpha\beta} = -2\gamma^{\mu}_{\alpha\beta}\mathcal{S}_{\mu}$ using the branching coordinate ξ according to

$$\mathcal{S}_{\alpha\beta}(x,\theta,\overline{\theta}) = \mathcal{S}^{(0)}_{\alpha\beta}(\xi,\theta^{+},\overline{\theta}^{+}) + \theta^{-}\mathcal{S}^{(1)}_{\alpha\beta}(\xi,\theta^{+},\overline{\theta}^{+}) - \overline{\theta}^{-}\overline{\mathcal{S}^{(1)}_{\alpha\beta}}(\xi,\theta^{+},\overline{\theta}^{+}) + \theta^{-}\overline{\theta}^{-}\mathcal{S}^{(2)}_{\alpha\beta}(\xi,\theta^{+},\overline{\theta}^{+}). \tag{C.25}$$

We obtain:

1. The +-direction

$$\mathcal{S}^{(0)}_{++} = 4j_{+} - 4i\theta^{+}(S_{+})_{+} - 4i\overline{\theta}^{+}(\overline{S}_{+})_{+} - 16\theta^{+}\overline{\theta}^{+}T_{++}, \tag{C.26a}$$

$$\begin{aligned}\mathcal{S}^{(1)}_{++} &= -4i(S_{+})_{-} - 2\sqrt{2}\overline{\omega}_{+} + \overline{\theta}^{+}(4i\partial_{\perp}j_{+} + 4K_{+\perp} + 4iL_{+}) - 4i\theta^{+}\overline{Y}_{+} \\ &\quad + 8\theta^{+}\overline{\theta}^{+}\partial_{+}(S_{+})_{-},\end{aligned} \tag{C.26b}$$

$$\begin{aligned}\mathcal{S}^{(2)}_{++} &= -8K_{+-} + 8\theta^{+}\partial_{\perp}(S_{+})_{-} - 8\overline{\theta}^{+}\partial_{\perp}(\overline{S}_{+})_{-} + 8\theta^{+}\partial_{-}(S_{+})_{+} - 8\overline{\theta}^{+}\partial_{-}(\overline{S}_{+})_{+} \\ &\quad - 4\sqrt{2}i\theta^{+}\partial_{+}\overline{\omega}_{-} - 4\sqrt{2}i\overline{\theta}^{+}\partial_{+}\omega_{-} - 4\sqrt{2}i\theta^{+}\partial_{\perp}\omega_{+} - 4\sqrt{2}i\overline{\theta}^{+}\partial_{\perp}\overline{\omega}_{+} \\ &\quad - 4\theta^{+}\overline{\theta}^{+}\partial_{\perp}^{2}j_{+} - 8\theta^{+}\overline{\theta}^{+}\partial_{\perp}L_{+} - 4\theta^{+}\overline{\theta}^{+}(-2\partial_{+}\partial_{\nu}j^{\nu} + \partial^{2}j_{+}).\end{aligned} \tag{C.26c}$$

2. The $-$-direction

$$\begin{aligned}\mathcal{S}^{(0)}_{--} &= 4j_- - 4i\theta^+(S_-)_+ - 4i\overline{\theta}^+(\overline{S}_-)_+ + 2\sqrt{2}\theta^+\overline{\omega}_- \\ &\quad - 2\sqrt{2}\overline{\theta}^+\omega_- - 8\theta^+\overline{\theta}^+K_{-+},\end{aligned} \tag{C.27a}$$

$$\begin{aligned}\mathcal{S}^{(1)}_{--} &= -4i(S_-)_- - 4i\theta^+\overline{Y}_- + 4\overline{\theta}^+(K_{-\perp} + i\partial_\perp j_- + iL_-) \\ &\quad + 8\theta^+\overline{\theta}^+\partial_+(S_-)_- + 4\sqrt{2}i\theta^+\overline{\theta}^+\partial_-\overline{\omega}_+,\end{aligned} \tag{C.27b}$$

$$\begin{aligned}\mathcal{S}^{(2)}_{--} &= -16T_{--} + 8\theta^+\partial_\perp(S_-)_- + 8\theta^+\partial_-(S_-)_+ - 8\overline{\theta}^+\partial_-(\overline{S}_-)_+ - 8\overline{\theta}^+\partial_\perp(\overline{S}_-)_- \\ &\quad + 4\theta^+\overline{\theta}^+\partial_\perp^2 j_- - 8\theta^+\overline{\theta}^+\partial_\perp L_- - 4\theta^+\overline{\theta}^+(-2\partial_-\partial^\nu j_\nu + \partial^2 j_-).\end{aligned} \tag{C.27c}$$

3. The $\perp$-direction

$$\begin{aligned}\mathcal{S}^{(0)}_{-+} &= -2j_\perp + 2i\theta^+(S_\perp)_+ + 2i\overline{\theta}^+(\overline{S}_\perp)_+ + \sqrt{2}\theta^+\overline{\omega}_+ \\ &\quad - \sqrt{2}\overline{\theta}^+\omega_+ + 4\theta^+\overline{\theta}^+K_{\perp+},\end{aligned} \tag{C.28a}$$

$$\begin{aligned}\mathcal{S}^{(1)}_{-+} &= +2i(S_\perp)_- - \sqrt{2}\overline{\omega}_- - 2\overline{\theta}^+(K_{\perp\perp} + i\partial_\perp j_\perp + iL_\perp) + 2i\theta^+\overline{Y}_\perp \\ &\quad - 4\theta^+\overline{\theta}^+\partial_+(S_\perp)_- - 2\sqrt{2}i\theta^+\overline{\theta}^+\partial_\perp\overline{\omega}_+ - 2\sqrt{2}i\theta^+\overline{\theta}^+\partial_+\overline{\omega}_-,\end{aligned} \tag{C.28b}$$

$$\begin{aligned}\mathcal{S}^{(2)}_{-+} &= +4K_{\perp-} - 4\theta^+\partial_\perp(S_\perp)_- + 4\overline{\theta}^+\partial_\perp(\overline{S}_\perp)_- - 4\theta^+\partial_-(S_\perp)_+ \\ &\quad + 4\overline{\theta}^+\partial_-(\overline{S}_\perp)_+ + \sqrt{2}i\theta^+\partial_-\overline{\omega}_+ + \sqrt{2}i\overline{\theta}^+\partial_-\omega_+ + 4\theta^+\overline{\theta}^+\partial_\perp L_\perp \\ &\quad + 2\theta^+\overline{\theta}^+\partial_\perp^2 j_\perp - 4\theta^+\overline{\theta}^+(\partial_\perp\partial^\nu j_\nu - \tfrac{1}{2}\partial^2 j_\perp).\end{aligned} \tag{C.28c}$$

where

$$\begin{aligned}K_{\mu\nu} &= 2T_{\nu\mu} - \eta_{\mu\nu}A - \tfrac{1}{4}\epsilon_{\mu\nu\rho}H^\rho &&= 2T_{\nu\mu} - \eta_{\mu\nu}A - \tfrac{1}{4}C_{\mu\nu}, \\ L_\mu &= \tfrac{1}{4}\epsilon_{\mu\nu\rho}F^{\nu\rho} + \epsilon_{\mu\nu\rho}\partial^\nu j^\rho &&= \tfrac{1}{4}C_\mu + \epsilon_{\mu\nu\rho}\partial^\nu j^\rho.\end{aligned} \tag{C.29}$$

where we have also defined the brane currents $C_{\mu\nu} = \epsilon_{\mu\nu\rho}H^\rho$ and $C_\mu = \epsilon_{\mu\nu\rho}F^{\nu\rho}$. The decomposition for the $\mathcal{R}$-multiplet is found simply by setting the multiplet $\mathcal{Y}_\alpha \ni (\omega_\alpha, A, Y_\mu)$ to zero.

C.2.2 3D $\mathcal{N}=2$ supersymmetry variations of components

Under a variation induced by $\delta_{\text{sym}} = \xi Q - \overline{\xi Q}$, we may compute from (3.54)

$$\delta_{\text{sym}}\lambda_\alpha = i\xi_\alpha D - i(\gamma^\mu\xi)_\alpha(H_\mu + \tfrac{i}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho}) + \tfrac{i}{2}\overline{\xi}_\beta C, \tag{C.30a}$$
$$\delta_{\text{sym}}D = \xi\gamma^\mu\partial_\mu\overline{\lambda} - \overline{\xi}\gamma^\mu\partial_\mu\lambda, \tag{C.30b}$$
$$\delta_{\text{sym}}H_\mu = -\xi\partial_\mu\overline{\lambda} + \overline{\xi}\partial_\mu\lambda, \tag{C.30c}$$
$$\epsilon_{\mu\nu\rho}\delta_{\text{sym}}F^{\nu\rho} = -2i\epsilon_{\mu\nu\rho}(\xi\gamma^\rho\partial^\nu\overline{\lambda} + \overline{\xi}\gamma^\rho\partial^\nu\lambda), \tag{C.30d}$$
$$\delta_{\text{sym}}C = 0, \tag{C.30e}$$
$$\delta_{\text{sym}}\omega_\alpha = \sqrt{2}\xi_\alpha B - \tfrac{1}{2\sqrt{2}}\overline{\xi}_\alpha C - \sqrt{2}i(\gamma^\mu\overline{\xi})_\alpha Y_\mu, \tag{C.30f}$$
$$\delta_{\text{sym}}B = -\sqrt{2}i(\overline{\xi}\gamma^\mu\partial_\mu\omega), \tag{C.30g}$$
$$\delta_{\text{sym}}Y_\mu = \sqrt{2}(\xi\partial_\mu\omega), \tag{C.30h}$$
$$\delta_{\text{sym}}j_\mu = \tfrac{1}{\sqrt{2}}(\xi\gamma_\mu\overline{\omega}) - \tfrac{1}{\sqrt{2}}(\overline{\xi}\gamma_\mu\omega) - i(\xi S_\mu) - i(\overline{\xi S}_\mu), \tag{C.30i}$$
$$\begin{aligned}\delta_{\text{sym}}(S_\mu)_\alpha = {}& \tfrac{i}{4}(\gamma_\mu\xi)_\alpha\overline{C} + \epsilon_{\mu\rho\nu}(\gamma^\nu\xi)_\alpha\overline{Y}^\rho + \overline{\xi}_\alpha\big(\tfrac{1}{4}\epsilon_{\mu\nu\rho}F^{\nu\rho} + \epsilon_{\mu\nu\rho}\partial^\nu j^\rho\big) \\ & - i(\gamma^\nu\overline{\xi})_\alpha(2T_{\nu\mu} - \tfrac{1}{4}\epsilon_{\mu\nu\rho}H^\rho + i\partial_\nu j_\mu - i\partial_\rho j^\rho\eta_{\mu\nu}),\end{aligned} \tag{C.30j}$$
$$\begin{aligned}\delta_{\text{sym}}K_{\mu\nu} = -\tfrac{1}{2}\big[& 2\epsilon_{\nu\rho\lambda}\xi\gamma^\lambda\partial^\rho S_\mu - \sqrt{2}i\epsilon_{\nu\mu\rho}\xi\partial^\rho\overline{\omega} - \tfrac{i}{\sqrt{2}}\eta_{\mu\nu}\xi\gamma_\rho\partial^\rho\overline{\omega} \\ & - 2\epsilon_{\nu\rho\lambda}\overline{\xi}\gamma^\lambda\partial^\rho\overline{S}_\mu - \sqrt{2}i\epsilon_{\nu\mu\rho}\overline{\xi}\partial^\rho\omega - \tfrac{i}{\sqrt{2}}\eta_{\mu\nu}\overline{\xi}\gamma_\rho\partial^\rho\omega\big],\end{aligned} \tag{C.30k}$$
$$\begin{aligned}\delta_{\text{sym}}T_{\mu\nu} = {}& -\tfrac{i}{4\sqrt{2}}\eta_{\mu\nu}(\xi\gamma_\rho\partial^\rho\overline{\omega}) - \tfrac{1}{2}\xi\gamma_{(\nu}\gamma_\rho\partial^\rho S_{\mu)} + \tfrac{1}{2}\xi\partial_{(\nu}S_{\mu)} \\ & - \tfrac{i}{4\sqrt{2}}\eta_{\mu\nu}(\overline{\xi}\gamma_\rho\partial^\rho\omega) - \tfrac{1}{2}\overline{\xi}\gamma_{(\nu}\gamma_\rho\partial^\rho\overline{S}_{\mu)} + \tfrac{1}{2}\overline{\xi}\partial_{(\nu}\overline{S}_{\mu)}.\end{aligned} \tag{C.30l}$$

C.2.3 Decomposition of bulk constraints

Using the branching coordinate ξ and the expansions

$$\begin{aligned}\chi^B_\alpha &= \chi^{B(0)}_\alpha + \theta^-\chi^{B(1a)}_\alpha + \overline{\theta}^-\chi^{B(1b)}_\alpha + \theta^-\overline{\theta}^-\chi^{B(2)}_\alpha, \\ \mathcal{Y}^B_\alpha &= \mathcal{Y}^{B(0)}_\alpha + \theta^-\mathcal{Y}^{B(1a)}_\alpha + \overline{\theta}^-\mathcal{Y}^{B(1b)}_\alpha + \theta^-\overline{\theta}^-\mathcal{Y}^{B(2)}_\alpha.\end{aligned} \tag{C.31}$$

we may rewrite the constraints (3.53) for the bulk $\mathcal{S}$-multiplet to the following collection of equations:
From $\overline{D}_-\chi_\alpha = \frac{1}{2}\epsilon_{-\alpha}C$ (recall $\epsilon_{-+} = -1$ otherwise zero):

$$\chi^{B(1b)}_\alpha = \tfrac{1}{2}\epsilon_{-\alpha}C, \tag{C.32a}$$
$$\chi^{B(2)}_\alpha = -2i\partial_-\chi^{B(0)}_\alpha. \tag{C.32b}$$

From $\overline{D}_+\chi_\alpha = \frac{1}{2}\epsilon_{+\alpha}C$:

$$\overline{D}_+\chi^{B(0)}_\alpha = \tfrac{1}{2}\epsilon_{+\alpha}C, \tag{C.33a}$$
$$\overline{D}_+\chi^{B(1a)}_\alpha = 2i\partial_\perp\chi^{B(0)}_\alpha, \tag{C.33b}$$
$$\overline{D}_+\chi^{B(2)}_\alpha = 0. \tag{C.33c}$$

From $\operatorname{Im} D^\alpha\chi_\alpha = 0$:

$$\operatorname{Im}(D_+\chi^{B(0)}_- - \chi^{B(1a)}_+) = 0, \tag{C.34a}$$
$$\overline{D}_+\overline{\chi^{B(1a)}_-} + \chi^{B(2)}_+ - 2i\partial_-\chi^{B(0)}_+ - 2i\partial_\perp\chi^{B(0)}_- = 0, \tag{C.34b}$$
$$\operatorname{Im}(D_+\chi^{B(2)}_- - 2i\partial_-\chi^{B(1a)}_+ - 2i\partial_\perp\chi^{B(1a)}_-) = 0. \tag{C.34c}$$

From $D_\alpha \mathcal{Y}_\beta + D_\beta \mathcal{Y}_\alpha = 0$ we obtain

$$D_+\mathcal{Y}_-^{B(0)} + \mathcal{Y}_+^{B(1a)} = 0, \tag{C.35a}$$
$$D_+\mathcal{Y}_-^{B(1a)} = 0, \tag{C.35b}$$
$$D_+\mathcal{Y}_-^{B(1b)} = 2i\partial_\perp\mathcal{Y}_-^{B(0)} + \mathcal{Y}_+^{B(2)} - 2i\partial_-\mathcal{Y}_+^{B(0)}, \tag{C.35c}$$
$$D_+\mathcal{Y}_+^{B(0)} = 0, \tag{C.35d}$$
$$D_+\mathcal{Y}_+^{B(1a)} = 0, \tag{C.35e}$$
$$D_+\mathcal{Y}_+^{B(1b)} = 2i\partial_\perp\mathcal{Y}_+^{B(0)}, \tag{C.35f}$$
$$D_+\mathcal{Y}_+^{B(2)} = -2i\partial_\perp\mathcal{Y}_+^{B(1b)}, \tag{C.35g}$$
$$\mathcal{Y}_-^{B(1a)} = 0, \tag{C.35h}$$
$$\mathcal{Y}_-^{B(2)} = 2i\partial_-\mathcal{Y}_-^{B(0)}. \tag{C.35i}$$

From $\overline{D}^\alpha \mathcal{Y}_\alpha + C = 0$ we obtain

$$\overline{D}_+\mathcal{Y}_-^{B(0)} + \mathcal{Y}_+^{B(1b)} + C = 0, \tag{C.36a}$$
$$\overline{D}_+\mathcal{Y}_-^{B(1a)} + 2i\partial_\perp\mathcal{Y}_-^{B(0)} + \mathcal{Y}_+^{B(2)} + 2i\partial_-\mathcal{Y}_+^{B(0)} = 0, \tag{C.36b}$$
$$\overline{D}_+\mathcal{Y}_-^{B(1b)} = 0, \tag{C.36c}$$
$$\overline{D}_+\mathcal{Y}_-^{B(2)} - 2i\partial_\perp\mathcal{Y}_-^{B(1b)} - 2i\partial_-\mathcal{Y}_+^{B(1b)} = 0. \tag{C.36d}$$

And the relation $\overline{D}^\beta \mathcal{S}_{\alpha\beta} = \chi_\alpha + \mathcal{Y}_\alpha$:

$$\chi_\alpha^{B(0)} + \mathcal{Y}_\alpha^{B(0)} = \overline{D}_+\mathcal{S}_{\alpha-}^{B(0)} - \overline{\mathcal{S}}_{\alpha+}^{(1)}, \tag{C.37a}$$
$$\chi_\alpha^{B(1a)} + \mathcal{Y}_\alpha^{B(1a)} = -\overline{D}_+\mathcal{S}_{\alpha-}^{(1)} - \mathcal{S}_{\alpha+}^{B(2)} - 2i\partial_\perp\mathcal{S}_{\alpha-}^{B(0)} - 2i\partial_-\mathcal{S}_{\alpha+}^{B(0)}, \tag{C.37b}$$
$$\tfrac{1}{2}\epsilon_{-\alpha}C + \mathcal{Y}_\alpha^{B(1b)} = \overline{D}_+\overline{\mathcal{S}}_{\alpha-}^{(1)}, \tag{C.37c}$$
$$\chi_\alpha^{B(2)} + \mathcal{Y}_\alpha^{B(2)} = \overline{D}_+\mathcal{S}_{\alpha-}^{B(2)} + 2i\partial_\perp\overline{\mathcal{S}}_{\alpha-}^{(1)} + 2i\partial_-\overline{\mathcal{S}}_{\alpha+}^{(1)}. \tag{C.37d}$$

C.2.4 Decompositions of bulk improvements

Using the branching coordinate and the expansions (C.25) and (C.31) we may compute the improvements of decomposed multiplets:

$$\mathcal{S}_{++}^{B(0)} \mapsto \mathcal{S}_{++}^{B(0)} + [D_+, \overline{D}_+]U^{B(0)}, \tag{C.38a}$$
$$\mathcal{S}_{--}^{B(0)} \mapsto \mathcal{S}_{--}^{B(0)} - 2U^{B(2)}, \tag{C.38b}$$
$$\mathcal{S}_{+-}^{B(0)} \mapsto \mathcal{S}_{+-}^{B(0)} + D_+\overline{U^{B(1)}} - \overline{D}_+U^{B(1)}. \tag{C.38c}$$

$$\mathcal{S}_{++}^{B(1)} \mapsto \mathcal{S}_{++}^{B(1)} + [D_+, \overline{D}_+]U^{B(1)} + 4i\partial_\perp D_+U^{B(0)}, \tag{C.39a}$$
$$\mathcal{S}_{--}^{B(1)} \mapsto \mathcal{S}_{--}^{B(1)} - 4i\partial_-U^{B(1)}, \tag{C.39b}$$
$$\mathcal{S}_{+-}^{B(1)} \mapsto \mathcal{S}_{+-}^{B(1)} - D_+U^{B(2)} + 2i\partial_\perp U^{B(1)} - 2i\partial_-D_+U^{B(0)}. \tag{C.39c}$$

$$\mathcal{S}^{B(2)}_{++} \mapsto \mathcal{S}^{B(2)}_{++} + [D_+, \overline{D}_+]U^{B(2)} + 4i\partial_\perp D_+ \overline{U^{B(1)}} + 4i\partial_\perp \overline{D}_+ U^{B(1)} - 8\partial_\perp^2 U^{B(0)}, \tag{C.40a}$$

$$\mathcal{S}^{B(2)}_{--} \mapsto \mathcal{S}^{B(2)}_{--} - 8\partial_-^2 U^{B(0)}, \tag{C.40b}$$

$$\mathcal{S}^{B(2)}_{+-} \mapsto \mathcal{S}^{B(2)}_{+-} - 2i\partial_- D_+ \overline{U^{B(1)}} - 2i\partial_- \overline{D}_+ U^{B(1)} + 8\partial_-\partial_\perp U^{B(0)}. \tag{C.40c}$$

$$\chi^{B(0)}_{+} \mapsto \chi^{B(0)}_{+} + 2\overline{D}_+ D_+ \overline{U^{B(1)}}, +4i\partial_\perp \overline{D}_+ U^{B(0)}, \tag{C.41a}$$

$$\chi^{B(0)}_{-} \mapsto \chi^{B(0)}_{-} + 2\overline{D}_+ U^{B(2)} - 4i\partial_- \overline{D}_+ U^{B(0)}, \tag{C.41b}$$

$$\chi^{B(1a)}_{+} \mapsto \chi^{B(1a)}_{+} + 2\overline{D}_+ D_+ U^{B(2)} - 4i\partial_\perp \overline{D}_+ U^{B(1)} - 4i\partial_\perp D_+ \overline{U^{B(1)}} + 8\partial_\perp^2 U^{B(0)} + 4i\partial_- \overline{D}_+ D_+ U^{B(0)}, \tag{C.41c}$$

$$\chi^{B(1a)}_{-} \mapsto \chi^{B(1a)}_{-} + 4i(2\partial_- \overline{D}_+ U^{B(1)} - \partial_\perp U^{B(2)} + 2i\partial_\perp \partial_- U^{B(0)}), \tag{C.41d}$$

$$\chi^{B(1b)}_{\alpha} \mapsto \chi^{B(1b)}_{\alpha}, \tag{C.41e}$$

$$\chi^{B(2)}_{+} \mapsto \chi^{B(2)}_{+} - 4i\partial_-(\overline{D}_+ D_+ \overline{U^{B(1)}} + 2i\partial_\perp \overline{D}_+ U^{B(0)}), \tag{C.41f}$$

$$\chi^{B(2)}_{-} \mapsto \chi^{B(2)}_{-} - 4i(\partial_- \overline{D}_+ U^{B(2)} - 2i\partial_-^2 \overline{D}_+ U^{B(0)}). \tag{C.41g}$$

$$\mathcal{Y}^{B(0)}_{+} \mapsto \mathcal{Y}^{B(0)}_{+} - D_+ \overline{D}_+ \overline{U^{B(1)}}, \tag{C.42a}$$

$$\mathcal{Y}^{B(0)}_{-} \mapsto \mathcal{Y}^{B(0)}_{-} + \overline{D}_+ U^{B(2)} + 2i\partial_\perp \overline{U^{B(1)}} + 2i\partial_- \overline{D}_+ U^{B(0)}, \tag{C.42b}$$

$$\mathcal{Y}^{B(1a)}_{+} \mapsto \mathcal{Y}^{B(1a)}_{+} - D_+ \overline{D}_+ U^{B(2)} - 2i\partial_\perp D_+ \overline{U^{B(1)}} - 2i\partial_- D_+ \overline{D}_+ U^{B(0)}, \tag{C.42c}$$

$$\mathcal{Y}^{B(1a)}_{-} \mapsto \mathcal{Y}^{B(1a)}_{-}, \tag{C.42d}$$

$$\mathcal{Y}^{B(1b)}_{+} \mapsto \mathcal{Y}^{B(1b)}_{+} - 2i\partial_\perp \overline{D}_+ \overline{U^{B(1)}}, \tag{C.42e}$$

$$\mathcal{Y}^{B(1b)}_{-} \mapsto \mathcal{Y}^{B(1b)}_{-} + 4i\partial_- \overline{D}_+ \overline{U^{B(1)}}, \tag{C.42f}$$

$$\mathcal{Y}^{B(2)}_{+} \mapsto \mathcal{Y}^{B(2)}_{+} + 2i\partial_- D_+ \overline{D}_+ \overline{U^{B(1)}} - 2i\partial_\perp \overline{D}_+ U^{B(2)} + 4\partial_\perp^2 \overline{U^{B(1)}} + 4\partial_\perp \partial_- \overline{D}_+ U^{B(0)}, \tag{C.42g}$$

$$\mathcal{Y}^{B(2)}_{-} \mapsto \mathcal{Y}^{B(2)}_{-} + 2i\partial_- \overline{D}_+ U^{B(2)} - 4\partial_-\partial_\perp \overline{U^{B(1)}} - 4\partial_-^2 \overline{D}_+ U^{B(0)}. \tag{C.42h}$$

C.2.5 Explicit bulk components of $\mathcal{S}_\mu$ for LG model

We compute the components of the supercurrent multiplet for the Landau-Ginzburg model:

$$\begin{aligned} \mathcal{S}_{\alpha\beta} &= D_\alpha \Phi_{3D} \overline{D}_\beta \overline{\Phi}_{3D} + D_\beta \Phi_{3D} \overline{D}_\alpha \overline{\Phi}_{3D}, \\ \chi_\alpha &= -\tfrac{1}{2} \overline{D}^2 \mathcal{D}_\beta (\overline{\Phi}_{3D} \Phi_{3D}), \\ \mathcal{Y}_\alpha &= -\overline{D}^2 \overline{\Phi}_{3D} D_\alpha \Phi_{3D}. \end{aligned} \tag{C.43}$$

according to the expansions (3.54). In other words, we are in the $\mathcal{S}$-frame (cf. page 56) and we obtain:

$$
\begin{aligned}
&R\text{-"current"}: && j^\mu = (\overline{\psi}\gamma^\mu\psi), && \text{(C.44a)}\\
&\text{supercurrent}: && S_{\mu\alpha} = \sqrt{2}(\gamma^\nu\gamma_\mu\psi)_\alpha\partial_\nu\overline{\phi} - \sqrt{2}i(\gamma_\mu\overline{\psi})_\alpha\overline{W}', && \text{(C.44b)}\\
&\text{lowest in } \chi_\alpha: && \lambda_\alpha = 2\sqrt{2}(\gamma^\mu\overline{\psi})_\alpha\partial_\mu\phi + 2\sqrt{2}iW'\psi_\alpha, && \text{(C.44c)}\\
&\text{lowest in } \mathcal{Y}_\alpha: && \omega_\alpha = 4W'\psi_\alpha, && \text{(C.44d)}\\
&\text{EM tensor}: && T_{\nu\rho} = (\partial_\rho\phi\partial_\nu\overline{\phi} + \partial_\nu\phi\partial_\rho\overline{\phi}) - \eta_{\nu\rho}(|\partial\phi|^2 + |W'|^2) && \\
& && \qquad + \tfrac{i}{2}(\partial_{(\rho}\overline{\psi}\gamma_{\nu)}\psi) - \tfrac{i}{2}(\overline{\psi}\gamma_{(\nu}\partial_{\rho)}\psi), && \text{(C.44e)}\\
&\text{irrelevant auxilliary}: && A = -4|W'|^2 + i\partial_\mu\overline{\psi}\gamma^\mu\psi - i\overline{\psi}\gamma^\mu\partial_\mu\psi, && \text{(C.44f)}\\
&\{Q,S\}\ \text{1-brane charge}: && Y_\mu = 4\partial_\mu W, && \text{(C.44g)}\\
&\{\overline{Q},S\}\ \text{1-brane charge}: && H^\mu = -2i\partial^\mu(\overline{\psi}\psi), && \text{(C.44h)}\\
&\{\overline{Q},S\}\ \text{0-brane charge}: && \epsilon_{\rho\mu\nu}F^{\mu\nu} = -4\epsilon_{\rho\mu\nu}\partial^\mu j^\nu - 8i\epsilon_{\rho\mu\nu}\partial^\mu\phi\partial^\nu\overline{\phi}. && \text{(C.44i)}
\end{aligned}
$$

Note that all (Hodge duals to) brane currents are *exact* forms. This is to be expected, since we are working on a trivial space, and it only shows local triviality in general backgrounds. For example, if W is not a properly defined function, then Y^μ is not identically exact.

Bibliography

[1] H. Jockers, P. Mayr, U. Ninad, and A. Tabler. “Wilson Loop Algebras and Quantum K-Theory for Grassmannians”. *J. High Energ. Phys.* (2020). DOI: 10.1007/JHEP10(2020)036. arXiv: 1911.13286.

[2] H. Jockers, P. Mayr, U. Ninad, and A. Tabler. “Wilson Loop Algebras and Quantum K-Theory for Grassmannians II”. *To Appear* (2020). arXiv: 201x.xxxxx.

[3] I. Brunner, J. Schulz, and A. Tabler. “Boundaries and Supercurrent Multiplets in 3D Landau-Ginzburg Models”. *J. High Energ. Phys.* (2019). DOI: 10.1007/JHEP06(2019)046. arXiv: 1904.07258.

[4] N. Seiberg. “Naturalness Versus Supersymmetric Non-Renormalization Theorems”. *Physics Letters B* (1993). DOI: 10.1016/0370-2693(93)91541-T. arXiv: hep-ph/9309335.

[5] R. Blumenhagen, D. Lüst, and S. Theisen. *Basic Concepts of String Theory*. Springer-Verlag, 2013. DOI: 10.1007/978-3-642-29497-6.

[6] N. Beisert et al. “Review of AdS/CFT Integrability: An Overview”. *Lett. Math. Phys.* (2012). DOI: 10.1007/s11005-011-0529-2. arXiv: 1012.3982.

[7] N. A. Nekrasov. “Seiberg-Witten Prepotential From Instanton Counting”. *Adv. Theor. Math. Phys.* (2002). arXiv: hep-th/0206161.

[8] V. Pestun. “Localization of Gauge Theory on a Four-Sphere and Supersymmetric Wilson Loops”. *Commun. Math. Phys.* (2012). DOI: 10.1007/s00220-012-1485-0. arXiv: 0712.2824.

[9] V. Pestun and M. Zabzine. “Introduction to Localization in Quantum Field Theory”. *J. Phys. A: Math. Theor.* (2017). DOI: 10.1088/1751-8121/aa5704. arXiv: 1608.02953.

[10] C. Beem, T. Dimofte, and S. Pasquetti. “Holomorphic Blocks in Three Dimensions”. *J. High Energ. Phys.* (2014). DOI: 10.1007/JHEP12(2014)177.

[11] F. Benini and S. Cremonesi. “Partition Functions of N=(2,2) Gauge Theories on S$\hat{2}$ and Vortices”. *Commun. Math. Phys.* (2015). DOI: 10.1007/s00220-014-2112-z. arXiv: 1206.2356.

[12] F. Benini and W. Peelaers. “Higgs Branch Localization in Three Dimensions”. *J. High Energ. Phys.* (2014). DOI: 10.1007/JHEP05(2014)030. arXiv: 1312.6078.

[13] F. Benini and A. Zaffaroni. “A Topologically Twisted Index for Three-Dimensional Supersymmetric Theories”. *J. High Energ. Phys.* (2015). DOI: 10.1007/JHEP07(2015)127. arXiv: 1504.03698.

[14] F. Benini and B. L. Floch. "Supersymmetric Localization in Two Dimensions". *J. Phys. A: Math. Theor.* (2017). DOI: 10.1088/1751-8121/aa77bb. arXiv: 1608.02955.

[15] C. Closset, S. Cremonesi, and D. S. Park. "The Equivariant A-Twist and Gauged Linear Sigma Models on the Two-Sphere". *J. High Energ. Phys.* (2015). DOI: 10.1007/JHEP06(2015)076. arXiv: 1504.06308.

[16] N. Doroud, J. Gomis, B. le Floch, and S. Lee. "Exact Results in D=2 Supersymmetric Gauge Theories". *J. High Energ. Phys.* (2013). DOI: 10.1007/JHEP05(2013)093. arXiv: 1206.2606.

[17] K. Hori and M. Romo. "Exact Results In Two-Dimensional (2,2) Supersymmetric Gauge Theories With Boundary" (2013). arXiv: 1308.2438.

[18] A. Kapustin, B. Willett, and I. Yaakov. "Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter". *J. High Energ. Phys.* (2010). DOI: 10.1007/JHEP03(2010)089. arXiv: 0909.4559.

[19] A. Kapustin and B. Willett. "Generalized Superconformal Index for Three Dimensional Field Theories" (2011). arXiv: 1106.2484.

[20] A. Kapustin and B. Willett. "Wilson Loops in Supersymmetric Chern-Simons-Matter Theories and Duality" (2013). arXiv: 1302.2164.

[21] S. Sugishita and S. Terashima. "Exact Results in Supersymmetric Field Theories on Manifolds with Boundaries". *J. High Energy Phys.* (2013). DOI: 10.1007/JHEP11(2013)021. arXiv: 1308.1973.

[22] Y. Yoshida and K. Sugiyama. "Localization of 3d $\mathcal{N} = 2$ Supersymmetric Theories on $S^1 \times D^2$". *Prog Theor Exp Phys* (2020). DOI: 10.1093/ptep/ptaa136. arXiv: 1409.6713.

[23] P. Candelas, X. C. De La Ossa, P. S. Green, and L. Parkes. "A Pair of Calabi-Yau Manifolds as an Exactly Soluble Superconformal Theory". *Nuclear Physics B* (1991). DOI: 10.1016/0550-3213(91)90292-6.

[24] K. Hori et al. *Mirror Symmetry*. American Mathematical Society, 2003.

[25] D. Cox and S. Katz. *Mirror Symmetry and Algebraic Geometry*. Mathematical Surveys and Monographs. American Mathematical Society, 1999. DOI: http://dx.doi.org/10.1090/surv/068.

[26] P. S. Aspinwall et al. *Dirichlet Branes and Mirror Symmetry*. Providence, RI: American Mathematical Society, Clay Mathematics Institute, 2009.

[27] E. Witten. "Topological Quantum Field Theory". *Commun.Math. Phys.* (1988). DOI: 10.1007/BF01223371.

[28] M. Atiyah. "Topological Quantum Field Theories". *Publications Mathématiques de l'Institut des Hautes Scientifiques* (1988). DOI: 10.1007/BF02698547.

[29] J. Lurie. *On the Classification of Topological Field Theories*. International Press of Boston, 2009. arXiv: 0905.0465.

[30] N. Carqueville, C. Meusburger, and G. Schaumann. "3-Dimensional Defect TQFTs and Their Tricategories". *Advances in Mathematics* (2020). DOI: 10.1016/j.aim.2020.107024. arXiv: 1603.01171.

[31] E. Witten. "Topological Sigma Models". *Commun.Math. Phys.* (1988). DOI: 10.1007/BF01466725.

[32] K. Hori. "Linear Models of Supersymmetric D-Branes". *Symplectic Geometry and Mirror Symmetry*. World Scientific, 2001. DOI: 10.1142/9789812799821_0005. arXiv: hep-th/0012179.

[33] H. Ooguri, Y. Oz, and Z. Yin. "D-Branes on Calabi-Yau Spaces and Their Mirrors". *Nuclear Physics B* (1996). DOI: 10.1016/0550-3213(96)00379-3. arXiv: hep-th/9606112.

[34] K. Hori, A. Iqbal, and C. Vafa. "D-Branes And Mirror Symmetry" (2000). arXiv: hep-th/0005247.

[35] T. Okazaki and S. Yamaguchi. "Supersymmetric Boundary Conditions in Three Dimensional N = 2 Theories". *Phys. Rev. D* (2013). DOI: 10.1103/PhysRevD.87.125005. arXiv: 1302.6593.

[36] A. Gadde, S. Gukov, and P. Putrov. "Fivebranes and 4-Manifolds". *In Memory of Friedrich Hirzebruch*. Progress in Mathematics. 2016. DOI: 10.1007/978-3-319-43648-7_7. arXiv: 1306.4320.

[37] F. Aprile and V. Niarchos. "$\mathcal{N} = 2$ Supersymmetric Field Theories on 3-Manifolds with A-Type Boundaries". *J. High Energ. Phys.* (2016). DOI: 10.1007/JHEP07(2016)126. arXiv: 1604.01561.

[38] A. Gadde, S. Gukov, and P. Putrov. "Walls, Lines, and Spectral Dualities in 3d Gauge Theories". *J. High Energ. Phys.* (2014). DOI: 10.1007/JHEP05(2014)047.

[39] T. Dimofte, D. Gaiotto, and N. M. Paquette. "Dual Boundary Conditions in 3d SCFT's". *J. High Energ. Phys.* (2018). DOI: 10.1007/JHEP05(2018)060. arXiv: 1712.07654.

[40] G. Festuccia and N. Seiberg. "Rigid Supersymmetric Theories in Curved Superspace". *J. High Energ. Phys.* (2011). DOI: 10.1007/JHEP06(2011)114. arXiv: 1105.0689.

[41] W. Lerche, C. Vafa, and N. P. Warner. "Chiral Rings in N = 2 Superconformal Theories". *Nuclear Physics B* (1989). DOI: 10.1016/0550-3213(89)90474-4.

[42] K. Intriligator. "Fusion Residues". *Mod. Phys. Lett. A* (1991). DOI: 10.1142/S0217732391004097.

[43] C. Vafa. "Topological Mirrors and Quantum Rings" (1991). arXiv: hep-th/9111017.

[44] E. Witten. "Two-Dimensional Gravity and Intersection Theory on Moduli Space". *Surv. Differ. Geom.* (1990). DOI: 10.4310/SDG.1990.v1.n1.a5.

[45] M. Kontsevich and Y. Manin. "Gromov-Witten Classes, Quantum Cohomology, and Enumerative Geometry". *Commun.Math. Phys.* (1994). DOI: 10.1007/BF02101490. arXiv: hep-th/9402147.

[46] M. Kontsevich, Y. Manin, and R. Kaufmann. "Quantum Cohomology of a Product". *Invent math* (1996). DOI: 10.1007/s002220050055.

[47] Y. Ruan and G. Tian. "A Mathematical Theory of Quantum Cohomology". *J. Differential Geom.* (1995). DOI: 10.4310/jdg/1214457234.

[48] W. Fulton and R. Pandharipande. "Notes on Stable Maps and Quantum Cohomology" (1997). arXiv: alg-geom/9608011.

[49] D. McDuff and D. Salamon. *J-Holomorphic Curves and Symplectic Topology.* 2nd ed. American Mathematical Society Colloquium Publications. Providence, R.I: American Mathematical Society, 2012.

[50] A. B. Givental. "Equivariant Gromov-Witten Invariants". *Int Math Res Notices* (1996). DOI: 10.1155/S1073792896000414. arXiv: alg-geom/9603021.

[51] E. Witten. "On the Structure of the Topological Phase of Two-Dimensional Gravity". *Nuclear Physics B* (1990). DOI: 10.1016/0550-3213(90)90449-N.

[52] R. Dijkgraaf, H. Verlinde, and E. Verlinde. "Topological Strings in d<1". *Nuclear Physics B* (1991). DOI: 10.1016/0550-3213(91)90129-L.

[53] A. B. Givental. "On the WDVV-Equation in Quantum K-Theory". *Michigan Math. J.* (2000). DOI: 10.1307/mmj/1030132720. arXiv: math/0003158.

[54] Y.-P. Lee. "Quantum K-Theory I: Foundations" (2001). arXiv: math/0105014.

[55] A. Givental and V. Tonita. "The Hirzebruch–Riemann–Roch Theorem in True Genus-0 Quantum K-Theory". *Math. Sci. Res. Inst. Publ* (2011). arXiv: 1106.3136.

[56] H. Iritani, T. Milanov, and V. Tonita. "Reconstruction and Convergence in Quantum K-Theory via Difference Equations". *Int. Math. Res. Not.* (2015). DOI: 10.1093/imrn/rnu026. arXiv: 1309.3750.

[57] A. B. Givental. "Symplectic Geometry of Frobenius Structures". *Frobenius Manifolds: Quantum Cohomology and Singularities.* Aspects of Mathematics. 2004. DOI: 10.1007/978-3-322-80236-1_4. arXiv: math/0305409.

[58] H. Iritani. "Quantum D-Modules and Generalized Mirror Transformations". *Topology* (2008). DOI: 10.1016/j.top.2007.07.001. arXiv: math/0411111.

[59] A. Givental. "Permutation-Equivariant Quantum K-Theory I-IX" (2015–2017). URL: https://math.berkeley.edu/~giventh/perm/perm.html.

[60] K. Intriligator and N. Seiberg. "Mirror Symmetry in Three Dimensional Gauge Theories". *Phys. Lett. B* (1996). DOI: 10.1016/0370-2693(96)01088-X. arXiv: hep-th/9607207.

[61] O. Aharony et al. "Aspects of N=2 Supersymmetric Gauge Theories in Three Dimensions". *Nucl. Phys. B* (1997). DOI: 10.1016/S0550-3213(97)00323-4. arXiv: hep-th/9703110.

[62] N. Dorey and D. Tong. "Mirror Symmetry and Toric Geometry in Three-Dimensional Gauge Theories". *J. High Energy Phys.* (2000). DOI: 10.1088/1126-6708/2000/05/018.

[63] D. Tong. "Dynamics of N=2 Supersymmetric Chern-Simons Theories". *J. High Energy Phys.* (2000). DOI: 10.1088/1126-6708/2000/07/019. arXiv: hep-th/0005186.

[64] K. Intriligator and N. Seiberg. "Aspects of 3d $\mathcal{N}=2$ Chern-Simons-Matter Theories". *J. High Energ. Phys.* (2013). DOI: 10.1007/JHEP07(2013)079.

[65] E. Witten. "Phases of $\mathcal{N} = 2$ Theories In Two Dimensions". *Nuclear Physics B* (1993). DOI: 10.1016/0550-3213(93)90033-L. arXiv: hep-th/9301042.

[66] K. Hori and D. Tong. "Aspects of Non-Abelian Gauge Dynamics in Two-Dimensional N=(2,2) Theories". *J. High Energy Phys.* (2007). DOI: 10.1088/1126-6708/2007/05/079. arXiv: hep-th/0609032.

[67] H. Jockers and P. Mayr. "A 3d Gauge Theory/Quantum K-Theory Correspondence". *Adv. Theor. Math. Phys.* (2020). DOI: 10.4310/ATMP.2020.v24.n2.a4. arXiv: 1808.02040.

[68] H. Jockers and P. Mayr. "Quantum K-Theory of Calabi-Yau Manifolds". *J. High Energ. Phys.* (2019). DOI: 10.1007/JHEP11(2019)011. arXiv: 1905.03548.

[69] T. T. Dumitrescu and N. Seiberg. "Supercurrents and Brane Currents in Diverse Dimensions". *J. High Energy Phys.* (2011). DOI: 10.1007/JHEP07(2011)095. arXiv: 1106.0031.

[70] N. P. Warner. "Supersymmetry in Boundary Integrable Models". *Nuclear Physics B* (1995). DOI: 10.1016/0550-3213(95)00402-E. arXiv: hep-th/9506064.

[71] N. A. Nekrasov and S. L. Shatashvili. "Quantization of Integrable Systems and Four Dimensional Gauge Theories" (2009). DOI: 10.1142/9789814304634_0015. arXiv: 0908.4052.

[72] N. A. Nekrasov and S. L. Shatashvili. "Supersymmetric Vacua and Bethe Ansatz". *Nucl. Phys. B - Proc. Suppl.* (2009). DOI: 10.1016/j.nuclphysbps.2009.07.047. arXiv: 0901.4744.

[73] N. Nekrasov. "Four Dimensional Holomorphic Theories". Ph.D. thesis. SCGP Stony Brook University, 1996. URL: http://scgp.stonybrook.edu/people/faculty/bios/nikita-nekrasov.

[74] D. Maulik and A. Okounkov. "Quantum Groups and Quantum Cohomology" (2012). arXiv: 1211.1287.

[75] A. Okounkov. "Lectures on K-Theoretic Computations in Enumerative Geometry" (2015). arXiv: 1512.07363.

[76] M. Aganagic and A. Okounkov. "Elliptic Stable Envelopes" (2016). arXiv: 1604.00423.

[77] M. Aganagic and A. Okounkov. "Quasimap Counts and Bethe Eigenfunctions". *Mosc. Math. J.* (2017). DOI: 10.17323/1609-4514-2017-17-4-565-600. arXiv: 1704.08746.

[78] V. Gorbounov and C. Korff. "Quantum Integrability and Generalised Quantum Schubert Calculus". *Advances in Mathematics* (2017). DOI: 10.1016/j.aim.2017.03.030. arXiv: 1408.4718.

[79] V. Gorbounov and C. Korff. "Equivariant Quantum Cohomology and Yang-Baxter Algebras" (2014). arXiv: 1402.2907.

[80] J. Lurie. "A Survey of Elliptic Cohomology". *The Abel Symposium*. Springer, 2009. DOI: 10.1007/978-3-642-01200-6_9.

[81] T. Coates and A. Givental. "Quantum Cobordisms and Formal Group Laws". *The Unity of Mathematics*. Progress in Mathematics. Birkhäuser Boston, 2007. DOI: 10.1007/0-8176-4467-9_4.

[82] K. Ueda and Y. Yoshida. "3d N=2 Chern-Simons-Matter Theory, Bethe Ansatz, and Quantum K -Theory of Grassmannians". *J. High Energ. Phys.* (2020). DOI: 10.1007/JHEP08(2020)157. arXiv: 1912.03792.

[83] A. N. Redlich. "Parity Violation and Gauge Noninvariance of the Effective Gauge Field Action in Three Dimensions". *Phys. Rev. D* (1984). DOI: 10.1103/PhysRevD.29.2366.

[84] A. N. Redlich. "Gauge Noninvariance and Parity Nonconservation of Three-Dimensional Fermions". *Phys. Rev. Lett.* (1984). DOI: 10.1103/PhysRevLett.52.18.

[85] L. Alvarez-Gaumé and E. Witten. "Gravitational Anomalies". *Nuclear Physics B* (1984). DOI: 10.1016/0550-3213(84)90066-X.

[86] Y. Wen. "K-Theoretic I-Function of $V//_\theta \mathbf{G}$ and Application" (2019). arXiv: 1906.00775.

[87] Y. Ruan and M. Zhang. "The Level Structure in Quantum K-Theory and Mock Theta Functions" (2018). arXiv: 1804.06552.

[88] H. Dong and Y. Wen. "Level Correspondence of K-Theoretic I-Function in Grassmann Duality" (2020). arXiv: 2004.10661.

[89] A. Givental and X. Yan. "Quantum K-Theory of Grassmannians and Non-Abelian Localization" (2020). arXiv: 2008.08182.

[90] Y. Ruan and M. Zhang. "Verlinde/Grassmannian Correspondence and Rank 2 δ-Wall-Crossing" (2019). arXiv: 1811.01377.

[91] S. Tolman and J. Weitsman. "The Cohomology Rings of Symplectic Quotients". *Commun. Anal. Geom.* (2003). DOI: 10.4310/CAG.2003.v11.n4.a6.

[92] S. Martin. "Symplectic Quotients by a Nonabelian Group and by Its Maximal Torus" (2000). arXiv: math/0001002.

[93] L. C. Jeffrey, A.-L. Mare, and J. M. Woolf. "The Kirwan Map, Equivariant Kirwan Maps, and Their Kernels". *J. Für Reine Angew. Math.* (2005). DOI: 10.1515/crll.2005.2005.589.105. arXiv: math/0211297.

[94] M. Harada and G. D. Landweber. "The K-Theory of Abelian Symplectic Quotients". *Math. Res. Lett.* (2008). DOI: 10.4310/MRL.2008.v15.n1.a6. arXiv: math/0601294.

[95] P. Griffiths and J. Harris. *Principles of Algebraic Geometry*. 1st Edition. Wiley-Interscience, 1994.

[96] A. Libgober. "Chern Classes and the Periods of Mirrors". *Math. Res. Lett.* (1999). DOI: 10.4310/MRL.1999.v6.n2.a2. arXiv: math/9803119.

[97] L. Katzarkov, M. Kontsevich, and T. Pantev. "Hodge Theoretic Aspects of Mirror Symmetry" (2008). DOI: 10.1090/pspum/078. arXiv: 0806.0107.

[98] S. Galkin, V. Golyshev, and H. Iritani. "Gamma Classes and Quantum Cohomology of Fano Manifolds: Gamma Conjectures". *Duke Math. J.* (2016). DOI: 10.1215/00127094-3476593. arXiv: 1404.6407.

[99] H. Iritani. "An Integral Structure in Quantum Cohomology and Mirror Symmetry for Toric Orbifolds" (2009). DOI: 10.1016/j.aim.2009.05.016. arXiv: 0903.1463.

[100] M. Herbst, K. Hori, and D. Page. "Phases Of N=2 Theories In 1+1 Dimensions With Boundary" (2008). arXiv: 0803.2045.

[101] J. Knapp, M. Romo, and E. Scheidegger. "Hemisphere Partition Function and Analytic Continuation to the Conifold Point". *Commun. Number Theory Phys.* (2017). DOI: 10.4310/CNTP.2017.v11.n1.a3. arXiv: 1602.01382.

[102] S. Cecotti and C. Vafa. "Topological—Anti-Topological Fusion". *Nuclear Physics B* (1991). DOI: 10.1016/0550-3213(91)90021-O.

[103] S. Cecotti, D. Gaiotto, and C. Vafa. "tt* Geometry in 3 and 4 Dimensions". *J. High Energ. Phys.* (2014). DOI: 10.1007/JHEP05(2014)055. arXiv: 1312.1008.

[104] G. Gasper and M. Rahman. *Basic Hypergeometric Series*. Cambridge University Press, 2004.

[105] L. J. Slater. *Generalized Hypergeometric Functions*. Reissue. Cambridge: Cambridge University Press, 2009.

[106] K. Hori and C. Vafa. "Mirror Symmetry" (2000). arXiv: hep-th/0002222.

[107] A. Bertram, I. Ciocan-Fontanine, and B. Kim. "Two Proofs of a Conjecture of Hori and Vafa". *Duke Math. J.* (2005). DOI: 10.1215/S0012-7094-04-12613-2. arXiv: math/0304403.

[108] I. Ciocan-Fontanine and B. Kim. "Moduli Stacks of Stable Toric Quasimaps". *Advances in Mathematics* (2010). DOI: 10.1016/j.aim.2010.05.023. arXiv: 0908.4446.

[109] I. Ciocan-Fontanine, B. Kim, and D. Maulik. "Stable Quasimaps to GIT Quotients". *J. Geom. Phys.* (2011). DOI: 10.1016/j.geomphys.2013.08.019. arXiv: 1106.3724.

[110] I. Ciocan-Fontanine and B. Kim. "Wall-Crossing in Genus Zero Quasimap Theory and Mirror Maps". *Alg. Geom.* (2014). DOI: 10.14231/AG-2014-019.

[111] I. Ciocan-Fontanine and B. Kim. "Big I-Functions". *Adv. Stud. Pure Math.* (2016). DOI: 10.2969/aspm/06910323. arXiv: 1401.7417.

[112] S. Hosono, A. Klemm, S. Theisen, and S.-T. Yau. "Mirror Symmetry, Mirror Map and Applications to Calabi-Yau Hypersurfaces". *Commun.Math. Phys.* (1995). DOI: 10.1007/BF02100589. arXiv: hep-th/9308122.

[113] N. Nekrasov. "BPS/CFT Correspondence IV: Sigma Models and Defects in Gauge Theory". *Lett Math Phys* (2019). DOI: 10.1007/s11005-018-1115-7. arXiv: 1711.11011.

[114] W. Gu, L. Mihalcea, E. Sharpe, and H. Zou. "Quantum K Theory of Symplectic Grassmannians" (2020). arXiv: 2008.04909.

[115] A. S. Buch. "A Littlewood-Richardson Rule for the K-Theory of Grassmannians". *Acta Math.* (2002). DOI: 10.1007/BF02392644. arXiv: math/0004137.

[116] F. Benini and A. Zaffaroni. "Supersymmetric Partition Functions on Riemann Surfaces" (2017). DOI: 10.1090/pspum/096. arXiv: 1605.06120.

[117] C. Closset and H. Kim. “Comments on Twisted Indices in 3d Supersymmetric Gauge Theories”. *J. High Energ. Phys.* (2016). DOI: 10.1007/JHEP08(2016)059. arXiv: 1605.06531.

[118] A. S. Buch and L. C. Mihalcea. “Quantum K-Theory of Grassmannians”. *Duke Math. J.* (2011). DOI: 10.1215/00127094-2010-218. arXiv: 0810.0981.

[119] C. Vafa. “Topological Landau-Ginzburg Models”. *Mod. Phys. Lett. A* (1991). DOI: 10.1142/S0217732391000324.

[120] E. Witten. “The Verlinde Algebra And The Cohomology Of The Grassmannian” (1993). arXiv: hep-th/9312104.

[121] E. Verlinde. “Fusion Rules and Modular Transformations in 2D Conformal Field Theory”. *Nuclear Physics B* (1988). DOI: 10.1016/0550-3213(88)90603-7.

[122] V. V. Batyrev, I. Ciocan-Fontanine, B. Kim, and D. van Straten. “Conifold Transitions and Mirror Symmetry for Calabi-Yau Complete Intersections in Grassmannians”. *Nuclear Physics B* (1998). DOI: 10.1016/S0550-3213(98)00020-0. arXiv: alg-geom/9710022.

[123] Y.-P. Lee and R. Pandharipande. “A Reconstruction Theorem in Quantum Cohomology and Quantum K-Theory”. *Am. J. Math.* (2004). JSTOR: 40067902.

[124] S. Pasquetti. “Factorisation of N = 2 Theories on the Squashed 3-Sphere”. *J. High Energ. Phys.* (2012). DOI: 10.1007/JHEP04(2012)120. arXiv: 1111.6905.

[125] A. Bilal. “Supersymmetric Boundaries and Junctions in Four Dimensions”. *J. High Energy Phys.* (2011). DOI: 10.1007/JHEP11(2011)046. arXiv: 1103.2280.

[126] L. Di Pietro, N. Klinghoffer, and I. Shamir. “On Supersymmetry, Boundary Actions and Brane Charges”. *J. High Energy Phys.* (2016). DOI: 10.1007/JHEP02(2016)163. arXiv: 1502.05976.

[127] S. Ferrara and B. Zumino. “Transformation Properties of the Supercurrent”. *Nucl.Phys.B* (1974). DOI: 10.1016/0550-3213(75)90063-2.

[128] V. Ogievetsky and E. Sokatchev. “On a Vector Superfield Generated by the Supercurrent”. *Nuclear Physics B* (1977). DOI: 10.1016/0550-3213(77)90318-2.

[129] S. J. Gates Jr, M. T. Grisaru, M. Rocek, and W. Siegel. “Superspace, or One Thousand and One Lessons in Supersymmetry” (2001). arXiv: hep-th/0108200.

[130] M. Magro, I. Sachs, and S. Wolf. “Superfield Noether Procedure”. *Ann. Phys.* (2002). DOI: 10.1006/aphy.2002.6239. arXiv: hep-th/0110131.

[131] Z. Komargodski and N. Seiberg. “Comments on Supercurrent Multiplets, Supersymmetric Field Theories and Supergravity”. *J. High Energy Phys.* (2010). DOI: 10.1007/JHEP07(2010)017. arXiv: 1002.2228.

[132] N. Drukker, D. Martelli, and I. Shamir. “The Energy-Momentum Multiplet of Supersymmetric Defect Field Theories”. *J. High Energy Phys.* (2017). DOI: 10.1007/JHEP08(2017)010. arXiv: 1701.04323.

[133] N. Drukker, I. Shamir, and C. Vergu. “Defect Multiplets of N=1 Supersymmetry in 4d”. *J. High Energy Phys.* (2018). DOI: 10.1007/JHEP01(2018)034.

[134] M. Dedushenko. “Chiral Algebras in Landau-Ginzburg Models”. *J. High Energ. Phys.* (2018). DOI: 10.1007/JHEP03(2018)079. arXiv: 1511.04372.

[135] I. V. Melnikov. "An Introduction to Two-Dimensional Quantum Field Theory with (0,2) Supersymmetry". *LectNotes Phys* (2019). DOI: 10.1007/978-3-030-05085-6.

[136] M. Bullimore, T. Dimofte, D. Gaiotto, and J. Hilburn. "Boundaries, Mirror Symmetry, and Symplectic Duality in 3d $\mathcal{N} = 4$ Gauge Theory". *J. High Energy Phys.* (2016). DOI: 10.1007/JHEP10(2016)108. arXiv: 1603.08382.

[137] I. Brunner, M. Herbst, W. Lerche, and B. Scheuner. "Landau-Ginzburg Realization of Open String TFT". *J. High Energy Phys.* (2006). DOI: 10.1088/1126-6708/2006/11/043. arXiv: hep-th/0305133.

[138] A. Kapustin and Y. Li. "Topological Correlators in Landau-Ginzburg Models with Boundaries". *Adv. Theor. Math. Phys.* (2003). arXiv: hep-th/0305136.

[139] C. I. Lazaroiu. "On the Boundary Coupling of Topological Landau-Ginzburg Models" (2003). DOI: 10.1088/1126-6708/2005/05/037.

[140] I. G. Macdonald. *Symmetric Functions and Hall Polynomials*. Oxford Mathematical Monographs. Oxford University Press, 1995.

[141] W. Fulton. *Intersection Theory*. Springer-Verlag, 1998. DOI: 10.1007/978-1-4612-1700-8.

[142] F. C. Kirwan. *Cohomology of Quotients in Symplectic and Algebraic Geometry*. Princeton University Press, 1984. DOI: 10.2307/j.ctv10vm2m8. JSTOR: j.ctv10vm2m8.

[143] A. Bertram. "Quantum Schubert Calculus". *Advances in Mathematics* (1997). DOI: 10.1006/aima.1997.1627. arXiv: alg-geom/9410024.

[144] A. S. Buch. "Quantum Cohomology of Grassmannians". *Compos. Math.* (2003). DOI: 10.1023/A:1023908007545. arXiv: math/0106268.

[145] D. Laksov. "A Formalism for Equivariant Schubert Calculus". *Algebra Number Theory* (2009). DOI: 10.2140/ant.2009.3.711.

[146] A. Hatcher. *Vector Bundles & K-Theory*. 2017. URL: https://pi.math.cornell.edu/~hatcher/VBKT/VBpage.html.

[147] A. Lascoux and M.-P. Schützenberger. "Structure de Hopf de l'anneau de Cohomologie et de l'anneau de Grothendieck d'une Variété de Drapeaux". *C. R. Acad. Sci. Paris Sér. I Math.* (1982).

[148] K. Motegi and K. Sakai. "Vertex Models, TASEP and Grothendieck Polynomials". *J. Phys. A: Math. Theor.* (2013). DOI: 10.1088/1751-8113/46/35/355201.

[149] M. Harada and G. Landweber. "Surjectivity for Hamiltonian G-Spaces in K-Theory". *Trans. Amer. Math. Soc.* (2007). DOI: 10.1090/S0002-9947-07-04164-5. arXiv: math/0503609.

[150] E. González and C. Woodward. "Quantum Kirwan for Quantum K-Theory" (2020). arXiv: 1911.03520.