Essays on Random Walks in Cones

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Dedicated to the memory of my mother.

Abstract

Consider a random walk $\{S(n), n \geq 1\}$ on $\mathbf{R}^d$, $d \geq 2$, where $S(n) = \sum_{i=1}^n X(i)$ with $\{X(n), n \geq 1\}$ is a family of i.i.d. random variables in $\mathbf{R}^d$. Let $K \subset \mathbf{R}^d$ be an open cone and $\tau_x = \inf\{n \geq 1 : x + S(n) \notin K\}$ the exit time from the cone when the random walk is started at $x \in K$. Let $Z_x = \{Z_x(n) : n \geq 0\}$ be the Markov chain defined by the random walk conditioned to stay in the cone K. That is, the distribution of $Z(n)$ is given by the distribution of $\{x + S(n), k \leq n\}$ conditional on $\tau_x > n$. The bulk of this dissertation studies different aspects of the behavior of Z_x. Classical methods typically used for the one-dimensional case (e.g. Wiener-Hopf factorization) are not applicable in the multi-dimensional setting.

In the case of zero-drift in chapter 3 we prove invariance principles and functional convergence of h-transforms and bridges to the corresponding functionals of Brownian motion. These results are used subsequently in chapter 4 to study the asymptotic behavior of the Green function of Z in the case that K is convex. The characterization of the Green function for the case of zero-drift leads to the proof of uniqueness of the harmonic function of Z in the case that K is convex. Complementary to this result, it is shown in the same chapter that whenever the random walk has non-zero drift, there are typically uncountably many harmonic functions for the case of convex cones.

Many of the results in the dissertation are proven under minimal moments assumption on the random variables $\{X(n), n \geq 1\}$. From a methodological perspective, many of the proofs use a combination of a strong approximation of random walks 'deep inside the cone' with Brownian motion, corresponding results about functionals of Brownian motion and estimates to control the behavior near the boundary. Some of the results for the continuous time limits need to be proven from scratch.

In chapter 5, we show how a similar methodology as for the case of random walks in cones can be used to study a different process, which does not exhibit the same symmetry properties as the case of random walks, even though it is a functional of $S(n)$: integrated random walks conditioned to stay positive. To be more precise, the integrated random walk is given by $\{T(n) : n \geq 0\}$ where $T(0) = x > 0$ and $T(n) = x + ny + \sum_{k=1}^n S(k)$. $T(n)$ is not a Markov chain, but if we define $S(0) = y$ then $Z_{(x,y)} = \{(T(n), S(n)) : n \geq 0\}$ is a Markov chain. The respective stopping time is $\tau_{(x,y)} = \inf\{n \geq 1 : Z(n) \notin (0, \infty) \times \mathbf{R}\}$. We prove invariance principles and functional convergence of h-transforms and bridges for $\{Z_{(x,y)} : x, y > 0\}$ conditional on $\tau_{(x,y)} > n$. The limit process now is the Kolmogorov diffusion, a two-dimensional continuous time stochastic process which consists of the integral of

a Brownian motion in its first coordinate and the Brownian motion itself in its second coordinate.

Some of the results on random walks in cones in the dissertation have proven useful for recent work that applies the probabilistic approach to enumerative combinatorics (see e.g. Kenyon et al. [2019], Bostan et al. [2018]). Moreover, they complete the study of respectively, random walks in multi-dimensional cones and integrated random walks conditioned to stay positive, started respectively in Denisov and Wachtel [2010, 2015b] and Denisov and Wachtel [2015a].

Zusammenfassung

Sei $\{S(n), n \geq 1\}$ eine Irrfahrt in $\mathbf{R}^d, d \geq 2$, mit $S(n) = \sum_{i=1}^{n} X(i)$ und $\{X(n), n \geq 1\}$ eine Familie von Zufallsvariablen mit Werten in $\mathbf{R}^d$. Sei außerdem $K \subset \mathbf{R}^d$ ein offener Kegel und $\tau_x = \inf\{n \geq 1 : x + S(n) \notin K\}$ die Austrittszeit aus dem Kegel, wenn die Irrfahrt bei $x \in K$ gestartet wird. Sei $Z_x = \{Z_x(n), n \geq 0\}$ die Markov-Kette, die durch Bedingen von $\{x + S(k), k \leq n\}$ auf $\tau_x > n$ gewonnen wird. Die Hauptergebnisse dieser Dissertation beschäftigen sich mit verschiedenen Aspekten des Verhaltens von Z, wenn $d \geq 2$ gilt. Klassische Methoden wie Wiener-Hopf-Faktoriesierungen sind im multi-dimensionalen Fall nicht anwendbar.

Im Falle von Irrfahrten mit Drift Null beweisen wir in Kapitel 3 mehrere Invarianzprinzipien und funktionale Grenzwertsätze für die h-Transformierten und Brücken der Irrfahrt. Die Limiten sind immer Funktionale der Brownschen Bewegung, gestoppt wenn sie den Kegel verlässt. Diese Resultate werden im Weiteren in Kapitel 4 benutzt um das asymptotische Verhalten der Green'schen Funktion der Irrfahrt, gestoppt wenn sie den Kegel verlässt, zu charakterisieren. Diese Charakterisierung liefert als Korollar die Eindeutigkeit der harmonischen Funktion für die gestoppte Irrfahrt. Dieses Resultat wird im selben Kapitel ergänzt, in dem gezeigt wird, dass es überabzählbar viele harmonische Funktionen für die gestoppte Irrfahrt gibt, wenn die Irrfaht einen Drift hat.

Viele der Resultate der Dissertation kommen mit minimalen Momentenannahmen an den Schritt der Irrfahrt aus. Methodologisch gesehen kombinieren viele der Beweise eine starke Approximation durch die Brownsche Bewegung der Irrfahrt 'tief im Kegel', entsprechende Resultate der Theorie der Brownschen Bewegung und Ungleichungen für das Verhalten in der Nähe des Kegelrandes. Einige der Resultate für den Limesprozess werden hier zum ersten Mal bewiesen.

Im Kapitel 5 zeigen wir, dass eine ähnliche Methodologie wie im Falle der Irrfahrt behilflich sein kann, um einen stochastischen Prozess zu studieren, der nicht die gleichen Eigenschaften wie die Irrfahrt aufweist: integrierte ein-dimensionale Irrfahrt, bedingt darauf positiv zu bleiben. Genauer formuliert, die integrierte Irrfahrt ist gegeben durch $\{T(n) : n \geq 0\}$ mit $T(0) = x > 0$ und $T(n) = x + ny + \sum_{k=1}^{n} S(k)$. $T(n)$ ist keine Markov-Kette, aber mit der Definition $S(0) = y$ wird der Prozess $Z_{(x,y)} = \{(T(n), S(n)) : n \geq 0\}$ eine Markov Kette. Die Stoppzeit von Interesse ist $\tau_{(x,y)} = \inf\{n \geq 1 : Z(n) \notin (0, \infty) \times \mathbf{R}\}$. Wir beweisen Invarianzprinzipien und funktionale Grenzwertsätze für die h-Transformierten und Brücken von $\{Z_{(x,y)} : x, y > 0\}$ bedingt darauf, dass $\tau_{(x,y)} > n$. Der Limesprozess ist nun die Kolmogorov'sche Diffusion, ein zwei-dimensionaler Prozess bestehend aus dem

Integral einer Brownschen Bewegung und der Brownschen Bewegung selbst.

Einige der Resultate in dieser Dissertation wurden vor Kurzem in Arbeiten, die die probabilistiche Perspektive in der Kombinatorik verfolgen, eingesetzt (siehe etwa Kenyon et al. [2019] oder Bostan et al. [2018]). Die Resultate in dieser Dissertation erweitern und vervollständigen die Arbeit über multi-dimensionale Irrfahrten in Kegeln und integrierte Irrfahrten bedingt darauf positiv zu bleiben, die in den Arbeiten Denisov and Wachtel [2015b, 2019] und Denisov and Wachtel [2015a] begonnen wurde.

Inhaltsverzeichnis

Chapter 1

Introduction

Conditioned random walks, and especially conditioned random walks in cones have gained considerable popularity recently. This is because they appear in some form or another in many models of probability theory, statistical mechanics, combinatorics and beyond. Sometimes they are an integral part of a probabilistic model, in other cases they appear as an auxiliary tool in proofs. This thesis brings new insights into the study of such stochastic processes.

We first give some more explicit applications of random walks in cones and of the methods used in this thesis. More explicit definitions of the mathematical objects mentioned follow within this introduction and in the next chapter.

i. Models of non-intersecting particles are sometimes used in physics (see e.g. Fisher [1984]). They can be interpreted as models of a multi-dimensional random walk in specific Weyl chambers, which are also cones.[1] Functionals of random walks in such chambers also appear in recent work: see e.g. Feierl [2012] and Feierl [2009]. Denisov and Wachtel [2015b] contains a discussion how the study of walks in Weyl chambers in Denisov and Wachtel [2010] served as a starting point for the generalization to random walks in general cones.

ii. Random walks in the quarter plane are a favorite object of study in enumerative combinatorics, not just the model per se, but also as a building block for the analysis of other models. Note that a quarter plane, or orthant more generally, is a cone. See Bostan et al. [2018] and references therein for work on counting of random walks in a quarter plane and results which use the results presented in this thesis. See Kenyon et al. [2019] for an example how results in this thesis are used as a tool in proofs for other stochastic models.

   The algebraic approach to random walks in the quarter plane is studied extensively in Fayolle et al. [1999] and applied in recent work on random walks in the quarter plane: e.g. Raschel [2012], Kurkova and Raschel [2011], Kurkova and Raschel [2012] and Raschel [2014]. The work presented here shows how the probabilistic approach is

[1] Weyl chambers of type A and B to be more precise.

poweful enough to give more general results than these works without making strong assumptions on the step of the random walk.

Random walks in the quarter plane sometimes also appear in queueing models, see e.g. Greenwood and Shaked [1977] and Cohen [1992].

iii. Doob's h-transforms as defined in this thesis are sometimes useful in the numerical simulation of physical processes, as well as in simulating rare events for certain functionals of random walks, in both one- and multi-dimensional cases.[2] See e.g. Blanchet and Glynn [2008] for examples. Moreover, because h-transforms allow to control the drift of the random walk, there may be uses of such transforms in machine learning algorithms which aim at learning outcomes which are rare under the data-generating process.

iv. The tools used in this thesis and other preceding work (Denisov and Wachtel [2015b], Denisov and Wachtel [2019]) can be fruitfully used in the study of other related Markov processes. An example of this are integrated random walks. Recent work has studied integrated random walks conditioned to stay positive, see e.g. Dembo et al. [2013] and Denisov and Wachtel [2015a]. Chapter 5 is devoted to proving invariance principles for such processes.

Grama et al. [2017] study conditioned limit theorems for functionals of products of random matrices under the condition that the functional stay away from the origin. It should not be hard to use results in this thesis to extend their work towards invariance principles for certain functionals of products of random matrices.

v. Finally, random walks conditioned to stay positive appear as a tool in the study of the asymptotic behavior of branching processes and in the study of random walks in random environments. Random walks conditioned to stay positive are the one-dimensional analogue of the processes to which most of this thesis is devoted. The results in this thesis may be potentially helpful for the study of the asymptotic behavior of branching processes in which children of each generation are tagged by discrete collection of 'features'.

We introduce the main geometric object of interest in this thesis. A cone in $\mathbf{R}^d, d \geq 1$ is an open set K such that for every $z \in K$ it holds $\lambda z \in K$ for all $\lambda > 0$. A cone is called convex if it is closed under vector addition. In $d = 1$ there are only two possible cones: $(0, \infty)$ and $(-\infty, 0)$ – we focus for simplicity and without loss of generality only on the former.

The one-dimensional case.

The study of one-dimensional random walks conditioned to stay positive is more or less complete and well-known in the classical literature. Denote for the random walk $\{S(n) := \sum_{k=1}^n X_i\}$ with $\{X(n), n \geq 1\}$ a family of i.i.d. one-dimensional random variables, the exit

[2] See Doob [1982] for a classical treatment of h-transforms.

time from the positive domain by $\tau_x = \inf\{n \geq 1 : x + S(n) \leq 0\}$, whenever the random walk is started at $x \geq 0$.

The asymptotics of the exit time probability $\mathbf{P}(\tau_x > n)$ for $x > 0$ as $n \to \infty$ have been very well studied. The same holds true for the process of the random walk conditioned to stay positive, i.e. $x + S(n) = x + \sum_{k=1}^{n} X_k$ under the condition that $\tau_x > n$. See e.g. Doney [1989], Doney [1998], Bertoin and Doney [1994] and Bolthausen [1976] as well as the references contained in these papers. We give here a brief summary of the results.

One knows that

$$\mathbf{P}(\tau_x > n) \sim V(x)\mathbf{P}(\tau_0 > n) \sim \varkappa V(x) n^{-\frac{1}{2}}, \quad n \to \infty. \tag{1.1}$$

Here, $V(x)$ is the renewal function based on ladder heights of the reflected random walk. To be more precise, define the process of ladder heights and ladder epochs $(H, T) = (H_k, T_k)_{k \geq 0}$ for the reflected random walk $\{-S(n)\}$ as follows. Let $T_0 = 0$ and define the ladder height of order k as $H_k = -S_{T_k}$ and the ladder epoch of order $k+1$ as $T_{k+1} = \inf\{j > T_k : -S_j > H_k\}$. Use the convention that $H_k = +\infty$ when $T_k = +\infty$.

Then the function V is given by

$$V(x) = \sum_{k \geq 0} \mathbf{P}(H_k \leq x), \quad x > 0. \tag{1.2}$$

Even nicer characterizations of V are known when $\mathbf{E}[|X_1|^2] < \infty$. In that case, Wald's identity leads to

$$V(x) = \frac{\mathbf{E}[-S(\tau_x)]}{\mathbf{E}[-S(\tau_0)]}.$$

Whenever $\mathbf{E}[X_i] \geq 0$, the function V is *harmonic* for the random walk killed when it leaves $(0, \infty)$. This means

$$\mathbf{E}[V(x + X_1), x + X_1 > 0] = V(x), \quad x > 0.^3 \tag{1.3}$$

In case $\mathbf{E}[X_i] < 0$, the function V is only super-harmonic and the equality in V is replaced by $\leq$. One can state this otherwise: $\{V(S(n))\mathbf{1}_{\{\tau_x > n\}}\}_{n \geq 0}$ is a martingale whenever $S(n)$ is started at x and $\mathbf{E}[X_i] \geq 0$, and a super-martingale whenever $S(n)$ is started at x and $\mathbf{E}[X_i] < 0$. Whenever harmonicity of V is ensured, one can use the theory of h- transforms due to Joseph Leo Doob, to define a Markov chain $\{\hat{S}(n)\}$ which never leaves $(0, \infty)$. This is done formally by defining

$$\mathbf{P}_x^{(V)}(\hat{S}(n) \in A) = \frac{1}{V(x)} \mathbf{E}[V(x + S(n)), A, \tau_x > n], \quad \forall A \subset (0, \infty) \text{ Borel-measurable}, x > 0.$$

Bertoin and Doney [1994] show that the measure $\mathbf{P}_x^{(V)}\left((\hat{S}(k), k \leq n) \in \cdot\right)$ is the weak limit of $\mathbf{P}(S(k), k \leq n) \in \cdot | \tau_x > n)$ as $n \to \infty$. Therefore, the two methods of defining a

[3] In the following we require any harmonic function to be non-negative.

'random walk conditioned to stay positive', via weak limit and via h-transform, coincide in the one-dimensional case.

One knows that V as defined in (1.2) is, up to multiplication with positive scalars, the only non-constant function which satisfies (1.3), whenever $\mathbf{E}[X_1] \geq 0$. When $\mathbf{E}[X_1] < 0$, only constant functions satisfy (1.3). More generally, for a given Markov chain $Z(n)$, the study of all functions V such that $V(Z(n))$ becomes a Martingale is called the study of the *Martin boundary* of $Z(n)$. The Martin boundary of a Markov chain characterizes its long-run behavior and gives insights as to how to *control* the Markov chain (see e.g. chapter 7 in Woess [2009] for a modern introduction to the Martin boundary theory for Markov chains).

Finally, we turn the attention to invariance principles. Bolthausen [1976] proves an invariance principle with the Brownian meander as a weak limit: he shows that under the assumption that $\mathbf{E}[X_i] = 0$ and $\mathbf{E}[X_i^2] = 1$, the continuous-time process $Y_n(t)$ with $Y_n(\frac{k}{n}) = \frac{S(k)}{n^{\frac{1}{2}}}$ and which is linearly interpolated otherwise, converges weakly to the Brownian meander process defined by

$$B^+(t) = \left| \frac{1}{(1-\tau)^{\frac{1}{2}}} B(\tau + (1-\tau)t) \right|, \quad t \in [0,1]$$

with B a Brownian motion and $\tau = \sup\{t \in [0,1] : B(t) = 0\}$. Bolthausen's result is limited to random walks in the domain of attraction of the standard normal distribution. Caravenna and Chaumont [2008] prove a similar result to Bolthausen's for the case of random walks in the domain of attraction of a stable law.[4] Instead of the meander of a Brownian motion conditioned to stay positive, the limit is then a 'Lévy process conditioned to stay positive'. In analogy to above, this limit can be constructed through a h-transform from the respective unconstrained Lévy process, which is the limit in the invariance principle of the unconstrained random walk.

All of the results in the one-dimensional case use, to varying extents, the Wiener-Hopf identities in the proofs, and the associated construction through ladder heights and ladder epochs (H, T). In fact, these powerful tools can be used even for random walks which are not in the domain of attraction of the standard normal distribution. E.g. a typical and powerful Wiener-Hopf identity is

$$\sum_{n\geq 0} \mathbf{E}[e^{i\lambda M_n^+}] z^n = \exp\left(\sum_{k\geq 1} \frac{\psi_k(\lambda)}{k} z^k \right), \quad |z| < 1, \lambda \in \mathbb{C}, im(\lambda) \geq 0.$$

Here $\psi_k(\lambda) = \mathbf{E}[e^{i\lambda S(k)^+}], k \geq 1$. From this identity one can extract the asymptotics of $\mathbf{P}(\tau_x > n)$ as $n \to \infty$, even when the random walk is outside of the domain of the attraction of the standard normal distribution. To exemplify, assume that $\{S(n)\}$ satisfies the condition $\mathbf{P}(S(n) > 0) \to \rho \in (0,1)$ as $n \to \infty$. Then, (1.1) holds in the form $\mathbf{P}(\tau_x > n) \to V(x) n^{-\rho} L(n)$, for a slowly varying function $n \to L(n)$.

[4] I.e. the random walk $\{S(n), n \geq 1\}$ is such that there exists a sequence of positive real numbers $\{a(n), n \geq 1\}$ such that $\frac{S(n)}{a(n)}$ converges in law to a stable law as $n \to \infty$.

The multi-dimensional case.

In dimensions $d \geq 2$, the number of possible cones K becomes uncountable. Unfortunately, there is no reasonable way of extending the Weiner-Hopf identities and the associated construction through ladder heights and epochs to the set-up with $d \geq 2$. Geometrically speaking, the problem is that the directions in which a typical random walk can leave a typical cone are highly non-unique. Therefore, a new methodology is required for the case $d \geq 2$.

Denisov and Wachtel [2015b] offers such a methodology for random walks with zero drift and finite second moment. The methodology started there and refined in Denisov and Wachtel [2019] and Duraj and Wachtel [2020] is of probabilistic nature, and doesn't rely on algebraic identities as was the case for one-dimensional random walks. It can be used to establish results for general classes of random walks in cones and related processes. In particular, it enables:

i. Establishing the asymptotics of $\mathbf{P}(\tau_x > n)$ (Denisov and Wachtel [2015b], Denisov and Wachtel [2019]), see chapter 2 here;

ii. Establishing conditional limit theorems and local limit theorems (Denisov and Wachtel [2015b, 2019]), see chapter 2 here;

iii. Establishing several invariance principles, both for random walks conditioned to stay in cones (meander and bridge), as well as for the respective h-transforms (Duraj and Wachtel [2020]), see chapter 3;

iv. Characterizing the Martin boundary for the zero-drift case (Duraj and Wachtel [2018]), see chapter 4 here;

v. Studying properties of related processes. Some examples are:

   - the non-zero drift random walk in a general convex cone (e.g. Duraj [2014a]), see subsection 4.4 of chapter 4,
   - recurrence properties for two-dimensional queuing processes, which can be studied via absorbing-reflecting random walks in $\mathbf{R}^2_+$ (e.g. the recent Peigné and Woess [2019]),
   - integrated random walks conditioned to stay positive, see chapter 5 here.

The limits that are characterized are always functionals or measure-transformations of Brownian motion killed when leaving the multi-dimensional cone. The study of Brownian motion in cones started with Burkholder [1977] to the best of knowledge. This thesis uses results from Banuelos and Smits [1997] and Garbit [2009]. Moreover, it proves some additional results regarding the convergence of h-transforms which were not present in the literature in the generality necessary for this thesis. It is well-known since Banuelos and Smits [1997] that for Brownian motions in a cone K started at $x \in K$, the exit time from the cone τ_x^{bm} has the asymptotics $\mathbf{P}(\tau_x^{bm} > t) \sim \varkappa u(x) t^{-\frac{p}{2}}$ with a $p > 0$. The parameter p is determined by the geometry of the cone K. Here, u is a harmonic function for the Brownian motion in K, i.e. it satisfies a similar relation as V in (1.3) for the Brownian motion in K. Analogous to the discrete-time process, one can define the u-transform of the Brownian motion to get a continuous process in continuous time which never leaves the cone.

The general strategy for the results in the zero-drift case always has the same broadly defined steps which we now explain.

First, one notices that under the condition $\{\tau_x > n\}$ the random walk eventually ends up 'deep inside the cone' (so-called entropic repulsion). This means that the probability of the random walk being near to the boundary of the cone, conditional on $\{\tau_x > n\}$, falls exponentially with n.

Second, one notices that the random walk deep inside the cone behaves essentially as a Brownian motion. This is due to the Donsker invariance principle and the fact that deep inside the cone there are no 'boundary effects'. Therefore, one uses a strong approximation of the random walk through Brownian motion, when the random walk reaches into the depths of the cone.

Finally, because the random walk is assumed to have finite second moments, deep inside the cone, paths with atypical $M(n) := \max_{k \le n} |S(k)|$ where n ranges up until the time of entry in the region deep inside the cone, do not contribute to the limit. To show that these large deviations don't play a role in the limit one uses versions of the powerful Fuk-Nagaev inequalities (see Nagaev [1979] and Borovkov [1972]).

The proofs of the results in this thesis build upon each other and previous technical work: e.g. the invariance principles use the local-limit theorems for random walks in cones from Denisov and Wachtel [2015b, 2019] together with extensions of the work in Garbit [2009] on Brownian motion in cones; the characterization of the Green function uses the already proven invariance principles, local-limit theorems and estimates for the Green function of the unconditioned random walk.

A slightly more formal discussion of the proof strategies follows. The rest of the thesis is devoted to formalizing the strategies and giving complete proofs for all results.

Invariance principles.

We look at a zero-drift random walk in a cone K with geometric characteristic p, i.e. so that the Brownian motion killed when leaving K satisfies $\mathbf{P}(\tau_x^{bm} > t) \sim \varkappa u(x) t^{-\frac{p}{2}}, t \to \infty$. We assume the step X_1 satisfies $\mathbf{E}[|X_1|^2] = 1$, $\mathbf{E}[|X_1|^\alpha] < \infty$ for some $\alpha > 2$ satisfying $\alpha = p$ if $p > 2$. In particular, the random walk is in the domain of attraction of the Brownian motion. More than second moments are needed for the proof of the results, because of the geometry of the cones. But the moment assumptions made are shown by examples to be minimal (for the invariance principles the examples in this spirit are in Denisov and Wachtel [2015b], whereas for the study of the Green function they are contained in the thesis). Let $K \subset \mathbf{R}^d$ be a convex cone with smooth boundary.[5] One defines for $\epsilon > 0$ small the region

$$K_{n,\epsilon} = \{x \in K : dist(x, \partial K) \geq n^{\frac{1}{2}-\epsilon}\}.$$

Note that this is the 'right' formalization of 'deep inside the cone', since the random walk is in the domain of attraction of Brownian motion. Define also the associated entry time

$$\nu_n = \min\{k \geq 1 : x + S(k) \in K_{n,\epsilon}\}.$$

Let $D[0,1]$ be the space of real-valued functions on $[0,1]$ which are right-continuous with left limits. To show an invariance principle for random walks conditioned to stay in the cone, one defines $X^{(n)}(t) = \frac{x+S([nt])}{\sqrt{n}}, t \geq 0$. One establishes the invariance principle towards the Brownian meander M_K within K, if one shows for bounded and uniformly continuous $f : D[0,1] \to \mathbf{R}_+$ that

$$\mathbf{E}[f(X^{(n)}) | \tau_x > n] \to \mathbf{E}[f(M_K)], \quad n \to \infty. \tag{1.4}$$

One knows from Denisov and Wachtel [2015b, 2019] that $\mathbf{P}(\tau_x > n) \sim \varkappa V(x) n^{-\frac{p}{2}}$ as $n \to \infty$. Chapter 4 here shows that V is unique up to scaling.

Moreover, one can split

$$\mathbf{E}[f(X^{(n)}), \tau_x > n] = \mathbf{E}[f(X^{(n)}), \tau_x > n, \nu_n \leq n^{1-\epsilon}] + \mathbf{E}[f(X^{(n)}), \tau_x > n, \nu_n > n^{1-\epsilon}].$$

Denisov and Wachtel [2015b] show that $\mathbf{P}(\tau_x > n, \nu_n > n^{1-\epsilon}) \leq e^{-O(n^\epsilon)}$, so that the second term is $o(\mathbf{P}(\tau_x > n))$. The first term can be split according to whether a big jump has occurred before entering $K_{n,\epsilon}$.

$$\begin{aligned}\mathbf{E}[f(X^{(n)}), \tau_x > n, \nu_n \leq n^{1-\epsilon}] &= \mathbf{E}[f(X^{(n)}), \tau_x > n, \nu_n \leq n^{1-\epsilon}, M(\nu_n) \leq \theta_n \sqrt{n}] \\ &\quad + \mathbf{E}[f(X^{(n)}), \tau_x > n, \nu_n \leq n^{1-\epsilon}, M(\nu_n) > \theta_n \sqrt{n}].\end{aligned}$$

Since the random walk is in the domain of attraction of the Brownian motion, the second term is also $o(\mathbf{P}(\tau_x > n))$. To show this large-deviations result one uses a version of the Fuk-Nagaev inequalities, which hold for general random walks under minimal conditions. Thus, only the first term on the right-hand side of the equality contributes to the limit. We call the paths where $\{\tau_x > n, \nu_n \leq n^{1-\epsilon}, M(\nu_n) \leq \theta_n \sqrt{n}\}$ the region of typical paths.

Once one focuses on typical paths under $\{\tau_x > n\}$, one can focus on the difference $S([nt]) - S(\nu_n)$ by defining $X_z^{(k,n)}(t) = \frac{z + (S([nt]) - S(k))\mathbf{1}_{\{t > \frac{k}{n}\}}}{\sqrt{n}}, t \geq 0, z \in K$. For typical paths of the random walk, the difference between $X^{(n)}$ and $X_{x+S(\nu_n)}^{(\nu_n,n)}$ is $o(\mathbf{P}(\tau_x > n))$ in probability. Thus, one can essentially push the random walk deep inside the cone, where it can be approximated by Brownian motion. To be more precise, one knows that one can define a probability space and a random walk and Brownian motion B, such that for $\gamma > 0$ small, the event

$$A_n = \{\sup_{k \leq n} |S([k]) - B(k)| \leq n^{\frac{1}{2}-\gamma}\},$$

[5]This is for simplicity, to cover the rest of the discussion in the introduction. Several of the results hold under weaker conditions.

is of the order n^{-r} for $r > 0$ that can be made arbitrarily small. That is, the probability that the path of the Brownian motion and of the random walk are far from each other is of order n^{-r}, for $r > 0$ that can be made arbitrarily small.

After one pushes the Brownian motion B deep inside the cone with the same steps as for the random walk $\{S(n)\}$, the fact that $\mathbf{P}(A_n) = O(n^{-r})$ and that f is uniformly continuous establishes (1.4).

For the case of the invariance principle for bridges, one looks to establish for fixed $x, y \in K$ the weak convergence of the measures $\mathbf{P}(\cdot|x + S(n) = y, \tau_x > n)$. The main idea is to split the path from x to y according to the intermediate value reached at $[(1-t)n]$ for $t \in (0,1)$ and use the invariance principle for the meander. One writes

$$\begin{aligned}&\mathbf{P}(\cdot|x + S(n) = y, \tau_x > n) \\ &\quad= \frac{\sum_{z\in K} \mathbf{P}(\cdot, x + S(tn) = z, \tau_x > tn)\mathbf{P}(z + S((1-t)n) = y, \tau_z > (1-t)n)}{\mathbf{P}(x + S(n) = y, \tau_x > n)}.\end{aligned} \tag{1.5}$$

The path until time $[(1-t)n]$ converges to a meander M_K, as established above. The path from $[(1-t)n]$ to n of $\{S(n)\}$ corresponds approximately to a path from time 0 to $[tn]$ for the random walk $\{-S(n)\}$.[6] Thus, one can write

$$\mathbf{P}(z + S((1-t)n) = y, \tau_z > (1-t)n) = \mathbf{P}(y - S((1-t)n) = z, \tilde{\tau}_y > (1-t)n),$$

for $\tilde{\tau}_y$ the respective exit time for $\{-S(n)\}$. The invariance principle for the meander therefore deals with the numerator in (1.5). The asymptotics of the denominator follow immediately from a local limit theorem in Denisov and Wachtel [2015b].

We also prove an invariance principle for the h-transforms, namely show that the V-transform of $S(n)$ converges to the u-transform of the Brownian motion started at zero.

Formally, for any bounded and uniformly continuous $f : D[0,1] \to \mathbf{R}_+$ one shows that $\mathbf{E}_x^{(V)}[f(X^{(n)})] = \frac{1}{V(x)}\mathbf{E}[f(X^{(n)})V(x+S(n)), \tau_x > n]$ converges to the expectation of f under a measure $\mathbf{P}_0^{(u)}$, which is the weak limit as $x \to 0$ of u-transforms of the Brownian motion started at $x \in K, x \to 0$.[7] The proof follows similarly to the case of the meander, except that one needs good estimates for the harmonic function V. Denisov and Wachtel [2019] shows that the difference of V and u is of the order $\max\{|x|^{p-\gamma}, \frac{|x|^p}{(dist(x,\partial K)^\gamma)}\}$. This is a tight characterization of the difference and multiple applications of this insight are used in the proof of the invariance for the h-transforms.

The Green function and Martin boundary for the case with zero drift.

To study the Green function of the random walk, one focuses on random walks on a lattice (w.l.o.g. the $\mathbb{Z}^d$ lattice).

The Green function for $\{S(n)\}$, killed when leaving K is given by

[6] This trick was first used in Caravenna and Chaumont [2013].

[7] We prove the existence of this limit.

$$G_K(x,y)=\sum_{n\geq 0}\mathbf{P}(x+S(n)=y,\tau_x>n).$$

It is well-known that all positive harmonic functions of the random walk killed when leaving K are characterized by looking at the limits $\frac{G_K(x,y)}{G_K(x',y)}$ as $|y|\to\infty$. Thus, one needs to study this limit with respect to all possible convergence directions of $\frac{y}{|y|}$ as $|y|\to\infty, y\in K$.

To study the asymptotics of the Green function for fixed $x\in K$ and $y\in K$ such that $|y|\to\infty$, one splits as follows

$$\begin{aligned}G_K(x,y)&=\sum_{n<\epsilon|y|^2}\mathbf{P}(x+S(n)=y,\tau_x>n)+\sum_{n\geq\epsilon|y|^2}\mathbf{P}(x+S(n)=y,\tau_x>n)\\&=S_1(x,y,\epsilon)+S_2(x,y,\epsilon).\end{aligned}$$

An application of the local limit theorems from Denisov and Wachtel [2015b, 2019] and results from the one-dimensional case show that

$$S_2(x,y,\epsilon)\sim c\frac{V(x)}{|y|^{2p+d-2}}u(y),\ \text{for } |y|\to\infty \text{ and } y \text{ far from } \partial K, \tag{1.6}$$

$$S_2(x,y,\epsilon)\sim c\frac{V(x)}{|y|^{p+d-1}}v_\sigma(dist(y,\partial K)),\ \text{for } |y|\to\infty \text{ and } \frac{y}{|y|}\to\sigma\in\partial K. \tag{1.7}$$

Here, v_σ is asymptotically linear, i.e. $v_\sigma(t)\sim c_\sigma t,\quad t\to\infty$. This function appears from the one-dimensional theory. For more details, note that when $y\in K$ becomes unbounded along the vector $\sigma\in\partial K$, the condition $\{\tau_x>n\}$ is satisfied with the same order in probability, as if one would require the weaker condition that the random walk stay inside the half-space whose boundary contains σ and which contains K fully. Denote in the following this half-space by H_σ. Note that this half-space exists because K is assumed convex and it is unique at the direction σ because of the smoothness assumption on the boundary of K. This construction allows to replace the condition $\{\tau_x>n\}$ with a respective one-dimensional condition by projecting the random walk along the perpendicular vector to σ. After necessary technical modifications because the half-space H_σ is usually not aligned to the lattice $\mathbb{Z}^d$, the results from the one-dimensional theory can be applied to our setting.

The restriction $n<\epsilon|y|^2$ pertaining to $S_1(x,y,\epsilon)$ is usually treated in the literature using estimates for local large deviations – one estimates the summand in $S_1(x,y,\epsilon)$. We follow a different approach – we estimate the whole sum and use estimates for Green functions together with the invariance principle. For more details, $S_1(x,y,\epsilon)$ can be first approached via a split of the path from x to y. One defines $\theta_y=\inf\{n\geq 1:|x+S(n)-y|\leq\delta|y|\}$, the entry time into the neighborhood of y with radius $\delta|y|$ and splits the path using Markov property according to the entry value in this neighborhood. In the case that y becomes unbounded in directions away from the boundary of K and $d\geq 3$, this leads to an estimate of the type

$$S_1(x,y,\epsilon)\leq\mathbf{E}[G^{(\epsilon|y|^2)}(y-x-S(\theta_y)),\tau_x>\theta_y,\theta_y\leq\epsilon|y|^2],$$

where $G^{(t)}(z) = \sum_{n<t} \mathbf{P}(S(n) = z)$ is the truncated Green function for the unconstrained random walk. Then one uses estimates of the unconstrained Green function from previous works like Uchiyama [1998], together with various versions of the invariance principles for random walks in cones. Namely, one knows from Uchiyama [1998] that for $d \geq 3$ it holds uniformly in t and z

$$G^{(t)}(z) \leq \frac{C}{1 + |z|^{d-2}}, \quad z \in \mathbb{Z}^d.$$

In the case $d = 2$ and y becoming unbounded in directions away from the boundary, one replaces the estimates of the unconstrained Green functions through the standard unconditional local limit theorem for random walk (see e.g. chapter 2 from Spitzer [1976]). Overall, with the help of the invariance principles for the V-transform of the random walk, this results in

$$y|^{2p+d-2} S_1(x, y, \epsilon) \to 0, \text{ for } |y| \to \infty \text{ and } y \text{ far from } \partial K.$$

Establishing this asymptotics requires slightly more than the moment assumption needed for the existence of the harmonic function V and of the asymptotics $\mathbf{P}(\tau_x > n) \sim \varkappa V(x) n^{-\frac{p}{2}}$, but one can show by example that the assumptions made are minimal for our proof method to achieve the asymptotics (1.6).

For the case that y moves along the boundary direction $\sigma \in \partial K$, we use the same split of the path as for the case away from the boundary, use also time reversion after the split and instead of the truncated Green function $G^{(t)}$, we make use of the truncated Green function of the half-space H_σ.

This leads to the estimate

$$S_1(x, y, \epsilon) \leq \mathbf{E}[G_{\epsilon,y}(x + S(\theta_y)), \tau_x > \theta_y, \theta_y \leq \epsilon |y|^2],$$

with

$$G_{\epsilon,y}(z) = \sum_{k<\epsilon|y|^2} \mathbf{P}(y - S(k) = z, T'_y > k),$$

where T'_y is the exit time for $\{-S(n)\}$ from the half-space H_σ. One uses results from the one-dimensional theory to bound

$$G_{\epsilon,y}(z) \leq C \min\{\frac{v'_\sigma(y)(1 + dist(z, H_y))}{|z - y|^d}, 1\}.$$

From here, the proof follows similar steps as in the case of y becoming unbounded but not approaching the boundary ∂K.

Overall, one arrives for any $d \geq 2$ to the asymptotics

$$|y|^{p+d-1} S_1(x, y, \epsilon) \to 0, \text{ for } |y| \to \infty \text{ and } \frac{y}{|y|} \to \sigma \in \partial K. \tag{1.8}$$

It follows from (1.6), (1.7) and (1.8) that for any $x, x' \in K$

$$\frac{G_K(x,y)}{G_K(x',y)} \to \frac{V(x)}{V(x')}, \qquad \text{as } |y| \to \infty.$$

This establishes uniqueness of the harmonic function (up to a positive scaling) for zero-drift random walks in cones K that are convex and have a smooth boundary. This result generalizes some recent and partial uniqueness results for random walks with small steps and specific cones (see e.g. Raschel [2014] and Bouaziz et al. [2015]).

Integrated random walks conditioned to stay positive.

The process to which chapter 5 is devoted, is not a random walk, but a derived Markov chain.

Let $\{X_i\}_{i\in\mathbb{N}}$ be i.i.d. random variables with $\mathbf{E}[X_i]=0, \mathbf{E}[X_i^2]=1$ and $\mathbf{E}[|X_i|^{2+\delta}]<\infty$ for some $\delta>0$ given and fixed throughout. For every starting point (x,y) define

$$S(n) = y + X_1 + \cdots + X_n, \quad n \in \mathbb{N}_0.$$

and

$$T(n) = x + S(1) + \cdots + S(n) = x + ny + nX_1 + (n-1)X_2 + \cdots + X_n, \quad n \in \mathbb{N}_0.$$

We define the Markov chain $Z(n) = (T(n), S(n)), n \in \mathbb{N}_0$. We study the process under the condition that $T(n)$ remains positive. We look at the exit time

$$\tau_{(x,y)} = \inf\{n \geq 0 : T(n) \leq 0\}.$$

We establish invariance principles for the Markov chain Z conditioned on $\tau_{(x,y)} > n$.

Let $X^{(n)}(t) = \left(\frac{T(tn)}{n^{\frac{3}{2}}}, \frac{S(tn)}{n^{\frac{1}{2}}}\right)$ for $t \in [0,1]$ be the scaled discrete-time process. The continuous limit process which appears in the invariance principles, is the Kolmogorov Diffusion, started at $(x,y) \in \mathbf{R}_+ \times \mathbf{R}$ given by

$$W_{(x,y)} = \left\{ \left(x + ty + \int_0^t B(s)ds, y + B(s) \right) : t \geq 0 \right\},$$

and conditioned to stay in $\mathbf{R}_+ \times \mathbf{R}$ up to time 1.

We establish precisely the same invariance principles as for the random walk in cones: convergence towards meander, bridge/excurison, and functional convergence of h-transforms. All of these are established under the assumption of zero drift. Since the statements and their proofs have an analogous strategy and implementation to the case of random walks in cones, we avoid a detailed exposition of the strategy, and just summarize its main steps.

For the case of the meander, note that, conditional on $\tau_{(x,y)} > n$, the bulk of the mass is concentrated in the region where $T(n)$ is relatively large. In this region of 'typical paths', one can use a strong approximation through the Kolmogorov diffusion. This is given by the same arguments as the case of Brownian motion in a cone, because the Kolmogorov diffusion is a continuous transformation of the paths of Brownian motion.

For the case of bridges, one uses again time reversion together with the already proven invariance principle for the meander.

Finally, for the functional convergence of h-transforms, besides the invariance principle for the meander, one needs good control for the difference between the discrete-time and continuous-time harmonic functions. This is given whenever the step of the random walk has moments above $\frac{5}{2}$.

Relation to papers of the thesis author. All new results contained in this thesis come from the following papers of the author of the thesis: Duraj [2014a], Duraj [2014b], Duraj and Wachtel [2018], Duraj and Wachtel [2020], and Bär et al. [2020]. The following chapters of the thesis are partial reformulations from sections of these papers, with some passages literally copied.

Chapter 3 of this thesis and some parts of chapter 2 are based on the paper Duraj and Wachtel [2020], which has been published in the journal *Stochastic Processes and Their Applications* this year. Theorem 1 is a result of joint discussions with Vitali Wachtel, Theorems 2 and 3 were proven by the author of the thesis.

Chapter 4, except for section 4.4, is based on the preprint Duraj and Wachtel [2018]. All the new results of this chapter (except for section 4.4) were obtained in joint discussion with Vitali Wachtel.

Chapter 5, Anhang A, section B.2 of Anhang B, and some parts of chapter 2 are based on the preprint Bär et al. [2020]. The new results on the integrated Brownian motion in Anhang A and chapter 2 were established in joint discussion with Michael Bär and Vitali Wachtel. The main results in chapter 5 (Theorems 7, 8 and 9) are due to the author of the thesis.

Sections 4.4, and B.1 of Anhang B are complementary results of lesser importance than the main results of this thesis. Section 4.4 is based on the paper Duraj [2014a] of the author, which has been published by the journal *Electronic Communications in Probability* in the year 2014. Section B.1 of Anhang B is based on the paper Duraj [2014b] of the author, which has been published by the journal *Stochastic Processes and Their Applications* in the year 2014.

Chapter 2

Technical preliminaries

This chapter summarizes several auxiliary results needed for the main body of the work of the thesis. Many of these results have been proven in preceding work and are cited accordingly whenever they are presented. Some of these results needed to be proven from scratch, because they were not found in the literature. Whenever this is the case, a proof is given in this chapter or in the appendix. The new results mentioned and proved here appear in the works Duraj and Wachtel [2018], Duraj and Wachtel [2020] and Bär et al. [2020].

2.1 Notation

In this thesis the following notation is used:

- The symbol $\sim$ means *asymptotic equivalence*. That is, for two non-negative functions f, g and a constant $(a \in \mathbf{R} \cup \{\pm\infty\})^d$ for some $d \geq 1$.

$$f(x) \sim g(x), \text{ as } x \to a \iff \lim_{x\to a} \frac{f(x)}{g(x)} = 1.$$

- For real-valued functions with real-valued arguments, the symbols o, O are used to indicate, respectively, *the function on the left has a smaller order than the function on the right*, and *the function on the right does not have a larger order*. In more detail, for a constant $a \in \mathbf{R} \cup \{\pm\infty\}$:

$$f(x) = o(g(x)), \text{ as } x \to a \iff \lim_{x\to a} \frac{f(x)}{g(x)} = 0,$$

$$f(x) = O(g(x)), \text{ as } x \to a \iff \limsup_{x\to a} \frac{f(x)}{g(x)} < \infty.$$

- For a real-valued random variable X we define

$$X^+ = \max\{X, 0\}, \qquad X^- = \min\{X, 0\}.$$

- For a random variable X and measurable event A, the expectation expression $\mathbf{E}[X, A]$ means $\mathbf{E}[X\mathbf{1}_A]$. Here, $\mathbf{1}_A$ is the indicator function of A with the properties $\mathbf{1}_A(x) = 1$ if $x \in A$ and 0 otherwise.

- For a vector $X \in \mathbf{R}^d, d \geq 1$, the notation $|X|$ denotes the euclidean norm of the vector. Recall that this coincides with the absolute value when $d = 1$.

- The short notation *i.i.d.* for a sequence of random variables means that the said sequence is *identically and independently distributed.*

- For a random walk with i.i.d. steps denoted by $\{S(n), n \geq 0\}$ ($S(0) = x$, the starting point) we denote the running maximum of the norms by $M_n := \max_{0 \leq k \leq n} |S_k|$. This encodes how far from the origin the walk has arrived within its first n steps.

- In the following, and for the rest of the thesis, we denote by $c, C, C_1, c_0, C_\epsilon, C_R$ positive constants whose value may change from line to line and whose exact value is not important for the arguments.

Additional notation is introduced as needed in the following chapters.

This thesis is about long-run behavior of random walks conditioned to stay in certain geometric domains. All of the convergence limits consist of different functionals of Brownian motion conditioned to stay in certain geometric domains. In the following subsection we gather some useful results about Brownian motion in cones and related objects.

2.2 Results about Brownian motion in cones

The new auxiliary results presented in this section are taken from Duraj and Wachtel [2020]. The rest of the results are in the literature and cited accordingly.

Denote by $\mathbb{S}^{d-1}$ the unit sphere of $\mathbf{R}^d$ and Σ an open and connected subset of $\mathbb{S}^{d-1}$. Let K be the cone generated by the rays emanating from the origin and passing through Σ, i.e. $\Sigma = K \cap \mathbb{S}^{d-1}$.

Throughout the thesis, we shall impose the following geometric conditions on the cone $K \subset \mathbf{R}^d$ for some $d \geq 2$:

- **Geometric conditions:** K is either convex or star-like. If K is star-like we assume also that its boundary is C^2. K star-like means here that there exists $x_0 \in \Sigma$ such that $x_0 + K \subset K$ and $\mathrm{dist}(x_0 + K, \partial K) > 0$.

Let $L_{\mathbb{S}^{d-1}}$ be the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$ and assume that Σ is regular with respect to $L_{\mathbb{S}^{d-1}}$. With this assumption, there exists (see, for example, Banuelos and Smits [1997]) a complete set of orthonormal eigenfunctions m_j and corresponding eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$ satisfying

$$L_{\mathbb{S}^{d-1}} m_j(x) = -\lambda_j m_j(x), \quad x \in \Sigma \qquad (2.1)$$
$$m_j(x) = 0, \quad x \in \partial\Sigma.$$

Define
$$p := \sqrt{\lambda_1 + (d/2-1)^2} - (d/2-1) > 0.$$

This is an important geometric parameter of the cone K, which appears frequently in the following.

Define the function $u(x)$ given by

$$u(x) = |x|^p m_1\left(\frac{x}{|x|}\right), \quad x \in K. \qquad (2.2)$$

Then $u(x)$ is, up to positive scaling, the unique strictly positive solution on K of the following boundary problem:

$$\Delta u(x) = 0,\ x \in K \quad \text{with } u\big|_{\partial K} = 0.$$

The function u is extended by setting $u(x)$ for $x \notin K$.

Denisov and Wachtel [2019] show the following estimates for u, which we register here in two lemmas for future use.

Lemma 1. *Assume the geometric conditions for K. Then*

$$C_1(dist(x, \partial K))^p \le u(x) \le C_2|x|^{p-1} dist(x, \partial K), \quad x \in K \qquad (2.3)$$

and

$$|\nabla u(x)| \le C_3|x|^{p-1}, \quad x \in K. \qquad (2.4)$$

Lemma 2. *Assume the geometric conditions for K. Then for $x \in K, x+y \in K$*

$$|u(x+y) - u(x)| \le C|y|(|x|^{p-1} + |y|^{p-1}), \qquad (2.5)$$

and, if in addition $|y| \le \frac{|x|}{2}$, then

$$|u(x+y) - u(x)| \le C|y||x|^{p-1}. \qquad (2.6)$$

For $p < 1$ and $x \in K$,

$$|u(x+y) - u(x)| < C|y|^p. \qquad (2.7)$$

In the following we denote

$$\tau_x^{bm} := \inf\{t > 0 : x + B(t) \notin K\},$$

the exit time from K of the Brownian motion started at $x \in K$.

Proposition 1 (Banuelos and Smits [1997], Denisov and Wachtel [2015b]). *1) There exists a finite constant C such that*

$$\mathbf{P}(\tau_x^{bm} > t) \le C\frac{|x|^p}{t^{\frac{p}{2}}}, \quad x \in K. \tag{2.8}$$

Moreover,

$$\mathbf{P}(\tau_x^{bm} > t) \sim \varkappa\frac{u(x)}{t^{\frac{p}{2}}}. \tag{2.9}$$

uniformly in $x \in K$ satisfying $|x| \le \theta_t\sqrt{t}$ with some $\theta_t \to 0$, where $\varkappa$ is a positive constant. In particular,

$$\mathbf{P}(\tau_x^{bm} > 1) \sim \varkappa u(x), \quad \text{as } x \to 0, \tag{2.10}$$

and

$$\lim_{\varepsilon\to 0}\sup_{x\in K, |x|\le\varepsilon}\left|\frac{\mathbf{P}(\tau_x^{bm} > 1)}{\varkappa u(x)} - 1\right| = 0. \tag{2.11}$$

2) The density $b_t(x,z)$ of the probability $\mathbf{P}(\tau_x^{bm} > t, x + B(t) \in dz)$ is

$$b_t(x,z) \sim \varkappa_0 t^{-\frac{d}{2}} e^{-\frac{|z|^2}{2t}} u(x)u(z)t^{-p}, \tag{2.12}$$

uniformly in $x, z \in K$ satisfying $|x| \le \theta_t\sqrt{t}$ and $|z| \le \sqrt{\frac{t}{\theta_t}}$ with some $\theta_t \to 0$.

(2.10) has been proven in Banuelos and Smits [1997]. (2.8) and (2.9) and 2) are proven in from Lemma 18 in Denisov and Wachtel [2015b] using results from Banuelos and Smits [1997]. The rest of 1) follows using the Brownian scaling.

Meander and bridge of Brownian motion in a cone.

Let $\{M_K(t), t \in [0,1]\}$ denote the Brownian meander in the cone K. Roughly speaking, this process is the Brownian motion conditioned to start from the vertex of the cone K and to stay in K for all $t \in [0,1]$. A rigorous construction only for the case K is a nice cone is due to Garbit [2009], see Subsection 4.3.2 in Garbit [2009] for the corresponding definition of a nice cone. According to Remark 4 in Garbit's paper, every convex cone with C^2 boundary is nice. But not every star-like cone is nice. In order to have the process M_K defined in star-like cones as well, in joint work with Vitali Wachtel, we propose in this chapter (see subsection 2.3) an alternative proof of existence of the meander M_K with starting point at the origin.

We also consider the Brownian bridge B_K^0. Roughly speaking, this is the Brownian motion conditioned to start from the vertex of the cone K, to remain in K for intermediate times and to end at the vertex at time 1. Formally, $B_K^0(0) = B_K^0(1) = 0$ and $B_K^0(t) \in K$ for all $t \in (0,1)$. The Brownian bridge in K is constructed here through a limit of random walk bridges and time reversion. For more details, we show first that the random walk bridge $\{\frac{x+S([nt])}{\sqrt{n}}, t \in [0,\sigma] | x + S(n) = y\}$ for $x, y \in K$, converges weakly for all $\sigma < 1$ to a continuous process. Moreover, the same holds for $\{\frac{x+S([nt])}{\sqrt{n}}, t \in [\sigma,1] | x + S(n) = y\}$ by time reversion. This shows tightness of the limit on $[0,1]$ and establishes existence of the continuous limit. More details are in the end of the proof of Subsection 3.2.3.[1]

h-transforms and their convergence.

For the weak convergence of random walks conditioned to stay in K at all times, one can use Doob's h-transforms. Here, we explain the h-transform of the Brownian motion.[2]

In the case of the standard d-dimensional Brownian motion $B(t)$ one can use the function u to construct the h-transform. The harmonicity property can be written in the following way:

$$u(x) = \mathbf{E}[u(x + B(t)), \tau_x^{bm} > t], \quad t > 0.$$

Then we consider the probabilistic measure $\mathbf{P}^{(u)}$ given by the following relation: For any $t > 0$ and each continuous and bounded functional $f_t : C[0,t] \mapsto \mathbb{R}$,

$$\mathbf{E}^{(u)}[f_t(B)|B(0) = x] = \mathbf{E}\left[f_t(B)\frac{u(x + B(t))}{u(x)}, \tau_x^{bm} > t\right], \quad x \in K.$$

We shall write $\mathbf{P}_x^{(u)}$ for the distribution of the process with starting point x. For our next theorem we need a convergence property of the measures $\mathbf{P}_x^{(u)}$ as $x \to 0$. With help of the convergence towards M_K we show in section 2.3 that, under our geometric assumptions,

$$\mathbf{P}_{x_n}^{(u)} \text{ converges weakly on } C[0,\infty) \text{ as } x_n \to 0, \tag{2.13}$$

under some weak restrictions on the sequence $\{x_n\}$. In the literature Chaumont [1997] has shown this relation for stable processes conditioned to stay positive. For this reason we give a proof in the next subsection.

2.3 Proofs for the Brownian meander and h-transform

Construction of Brownian meander for a large class of star-like and convex cones

In this paragraph we adapt the arguments from Garbit's paper Garbit [2009] and obtain the Brownian meander M_K for a large class of star-like and convex cones.

Let $\{x_n\}$ be a sequence of points in K such that $x_n \to 0$. We are going to show that the sequence of processes

$$\left\{x_n + B(t)|\tau_{x_n}^{bm} > 1\right\},$$

converges weakly in $C[0,1]$ under some mild restrictions on the sequence $\{x_n\}$.

We first address the convergence of finite dimensional distributions. For this we note that we have good control over the density

$$p(1,x,z) := \frac{\mathbf{P}(x + B(1) \in dz, \tau_x^{bm} > 1)}{dz}.$$

[1] The proof uses discrete time methods and therefore we leave it for the main chapters of the thesis.

[2] h-transforms for random walks will be defined and studied in chapter 3.

Namely, it follows from the proof of Lemma 5.4 in Garbit [2009] that

$$\sup_{x\in K,|x|\le\frac12}\frac{p(1,x,z)}{u(x)}\le Ce^{-\gamma|z|^2} \tag{2.14}$$

for some suitable $C,\gamma>0$. This is essentially a sharpening of (2.12). That proof doesn't use more than the analytical results from Banuelos and Smits [1997] and so is valid for all cones satisfying our geometric conditions. Using this estimate and the Brownian scaling, we obtain for every fixed $t\in(0,1]$

$$\lim_{R\to\infty}\sup_{x\in K,|x|\le\sqrt{t}/2}\frac{\mathbf{P}(|x+B(t)|>R,\tau_x^{bm}>t)}{u(x)}=0. \tag{2.15}$$

By using (2.10) and (2.11), we see that there exists $\varepsilon_0\in(0,1)$ such that

$$\mathbf{P}(\tau_x^{bm}>1)\ge\frac12\varkappa u(x)\quad\text{for all } x\in K,|x|\le\varepsilon_0. \tag{2.16}$$

Combining this relation with (2.15), we conclude that, for every fixed $t\in(0,1]$,

$$\lim_{R\to\infty}\sup_{x\in K,|x|\le\varepsilon_0\sqrt{t}/2}\mathbf{P}(|x+B(t)|>R|\tau_x^{bm}>1)=0. \tag{2.17}$$

Now one can repeat the proofs of Lemma 5.5 and Proposition 5.1 in Garbit [2009] with the following change: instead of (5.7) there one uses (2.17). This gives the desired convergence of finite dimensional distributions, since (5.7) is the only place where 'nice'-property has been used.

We now turn to the tightness. We are going to use Theorem 7.4 and its Corollary from Billingsley [1968]. Fix $\epsilon,\delta>0$. For every $t\in[\delta,1)$ we have

$$\begin{aligned}
&\mathbf{P}\left(\sup_{u\in[t,t+\delta]}|B(u)-B(t)|>\epsilon\,\middle|\,\tau_{x_n}^{bm}>1\right)\\
&\quad=\frac{\mathbf{P}\left(\sup_{u\in[t,t+\delta]}|B(u)-B(t)|>\epsilon,\tau_{x_n}^{bm}>1\right)}{\mathbf{P}(\tau_{x_n}^{bm}>1)}\\
&\quad\le\frac{\mathbf{P}\left(\sup_{u\in[t,t+\delta]}|B(u)-B(t)|>\epsilon,\tau_{x_n}^{bm}>t\right)}{\mathbf{P}(\tau_{x_n}^{bm}>1)}\\
&\quad=\frac{\mathbf{P}(\tau_{x_n}^{bm}>t)}{\mathbf{P}(\tau_{x_n}^{bm}>1)}\mathbf{P}\left(\sup_{u\in[0,\delta]}|B(u)|>\epsilon\right)\le\frac{\mathbf{P}(\tau_{x_n}^{bm}>\delta)}{\mathbf{P}(\tau_{x_n}^{bm}>1)}\mathbf{P}\left(\sup_{u\in[0,\delta]}|B(u)|>\epsilon\right)\\
&\quad\le\frac{\mathbf{P}(\tau_{x_n}^{bm}>\delta)}{\mathbf{P}(\tau_{x_n}^{bm}>1)}C_1e^{-c_0\epsilon^2/\delta}\le C_2\delta^{-p/2}e^{-c_0\epsilon^2/\delta}.
\end{aligned} \tag{2.18}$$

The second equality uses the Markov property. In the last two steps we have used the obvious bound

$$\mathbf{P}\left(\sup_{u\in[0,\delta]}|B(u)|>\epsilon\right)\le C_1e^{-c_0\epsilon^2/\delta},$$

and the relation $\mathbf{P}(\tau_{x_n}^{bm} > t) \sim \varkappa u(x_n) t^{-p/2}$ as $n \to \infty$ which follows from (2.11). Combining now (2.18) with the above-mentioned corollary to Theorem 7.4 in Billingsley [1968], we conclude tightness on $C[\delta, 1]$ for every $\delta > 0$.

In order to extend this property on the whole interval $[0,1]$ it remains to show that the probability $\mathbf{P}\left(\sup_{u\in[0,\delta]} |B(u)| > \epsilon \middle| \tau_{x_n}^{bm} > 1\right)$ can be made sufficiently small. We first note that

$$\mathbf{P}\left(\sup_{u\in[0,\delta]} |B(u)| > \epsilon \middle| \tau_{x_n}^{bm} > 1\right) \le \mathbf{P}\left(\sup_{u\in[0,\frac{\delta}{2}]} |B(u)| > \frac{3\epsilon}{4} \middle| \tau_{x_n}^{bm} > 1\right) + \mathbf{P}\left(\sup_{u\in[0,\frac{\delta}{2}]} |B(u)| \le \frac{3\epsilon}{4}, \sup_{u\in[0,\delta]} |B(u)| > \epsilon \middle| \tau_{x_n}^{bm} > 1\right).$$

For the second summand we have

$$\begin{aligned} &\mathbf{P}\left(\sup_{u\in[0,\frac{\delta}{2}]} |B(u)| \le \frac{3\epsilon}{4}, \sup_{u\in[0,\delta]} |B(u)| > \epsilon \middle| \tau_{x_n}^{bm} > 1\right) \\ &\quad \le \frac{1}{\mathbf{P}(\tau_{x_n}^{bm} > 1)} \mathbf{P}\left(\sup_{u\in[\delta/2,\delta]} |B(u) - B(\delta/2)| > \frac{\epsilon}{4}, \tau_{x_n}^{bm} > \frac{\delta}{2}\right) \\ &\quad = \frac{\mathbf{P}(\tau_{x_n}^{bm} > \frac{\delta}{2})}{\mathbf{P}(\tau_{x_n}^{bm} > 1)} \mathbf{P}\left(\sup_{u\in[0,\frac{\delta}{2}]} |B(u)| > \frac{\epsilon}{4}\right) \le \frac{\mathbf{P}(\tau_{x_n}^{bm} > \frac{\delta}{2})}{\mathbf{P}(\tau_{x_n}^{bm} > 1)} C_1 e^{-c_0 \left(\frac{\epsilon}{4}\right)^2 / \frac{\delta}{2}}. \end{aligned}$$

As a result,

$$\begin{aligned} &\mathbf{P}\left(\sup_{u\in[0,\delta]} |B(u)| > \epsilon \middle| \tau_{x_n}^{bm} > 1\right) \\ &\quad \le \mathbf{P}\left(\sup_{u\in[0,\frac{\delta}{2}]} |B(u)| > \frac{3\epsilon}{4} \middle| \tau_{x_n}^{bm} > 1\right) + \frac{\mathbf{P}(\tau_{x_n}^{bm} > \frac{\delta}{2})}{\mathbf{P}(\tau_{x_n}^{bm} > 1)} C_1 e^{-c_0 \epsilon^2 / 8\delta}. \end{aligned}$$

After N iterations we arrive at the bound

$$\begin{aligned} &\mathbf{P}\left(\sup_{u\in[0,\delta]} |B(u)| > \epsilon \middle| \tau_{x_n}^{bm} > 1\right) \\ &\quad \le \mathbf{P}\left(\sup_{u\in[0,\frac{\delta}{2^N}]} |B(u)| > \frac{3^N\epsilon}{4^N} \middle| \tau_{x_n}^{bm} > 1\right) + \sum_{k=1}^{N} \frac{\mathbf{P}(\tau_{x_n}^{bm} > \frac{\delta}{2^k})}{\mathbf{P}(\tau_{x_n}^{bm} > 1)} C_1 e^{-c_0 \frac{9^{k-1}\epsilon^2}{8^k \delta}}. \end{aligned} \tag{2.19}$$

According to (2.14) it holds

$$\sup_{|x|\le\frac{1}{2}} \frac{\mathbf{P}(\tau_x^{bm} > 1)}{u(x)} \le C_*.$$

Then by the scaling property of Brownian motion we have for $|x_n| \le \frac{\delta^{\frac{1}{2}}}{2^{\frac{k}{2}+1}}$

$$\mathbf{P}\left(\tau_{x_n}^{bm} > \frac{\delta}{2^k}\right) = \mathbf{P}\left(\tau_{x_n 2^{\frac{k}{2}}/\sqrt{\delta}} > 1\right) \le C_* \frac{2^{kp/2}}{\delta^{\frac{p}{2}}} u(x_n). \tag{2.20}$$

Set $N := \max\{k \ge 1 : 2^k \le \frac{\delta}{4|x_n|^2}\}$. Then, combining (2.16), (2.19) and (2.20), we obtain

$$\begin{aligned}\mathbf{P}\left(\sup_{u\in[0,\delta]} |B(u)| > \epsilon \middle| \tau_{x_n}^{bm} > 1\right) &\le \mathbf{P}\left(\sup_{u\in[0,\frac{\delta}{2^N}]} |B(u)| > \frac{3^N \epsilon}{4^N}\right) / \mathbf{P}(\tau_{x_n}^{bm} > 1) \\ &\quad + C\sum_{k=1}^{N} \frac{2^{kp/2}}{\delta^{p/2}} e^{-c_0 9^{k-1}\epsilon^2/8^k\delta} \\ &\le \frac{Ce^{-c_0 9^N \epsilon/8^N \delta^2}}{u(x_n)} + C\delta^{-\frac{p}{2}} e^{-c_0\epsilon^2/8\delta}.\end{aligned}$$

By the definition of N it holds $2^N \ge \frac{\delta}{8}|x_n|^{-2}$. Therefore,

$$e^{-(9/8))^N c_0\epsilon^2/\delta} \le e^{-c_1\epsilon^2/(\delta^{1-q_0}|x_n|^{2q_0})},$$

where $q_0 = \frac{\log(9/8)}{\log 2} \in (0,1)$. Consequently,

$$\mathbf{P}\left(\sup_{u\in[0,\delta]} |B(u)| > \epsilon \middle| \tau_{x_n}^{bm} > 1\right) \le C\left(\frac{e^{-c_1\epsilon^2/(\delta^{1-q_0}|x_n|^{2q_0})}}{u(x_n)} + \delta^{-\frac{p}{2}} e^{-c_0\epsilon^2/8\delta}\right).$$

For every given ϵ one can choose δ so that the second summand on the right hand side becomes as small as needed. Clearly, the first summand will also converge to zero, for an appropriate choice of δ, if we assume that the sequence x_n satisfies the condition

$$\log\left(\frac{1}{u(x_n)}\right) = O\left(|x_n|^{-2q_0}\right). \tag{2.21}$$

As a result we have tightness on $C[0,1]$ and the weak convergence on the same space for any sequence of starting points satisfying (2.21).

Let us give some sufficient conditions for the validity of (2.21). According to Lemma 1,

$$u(x) \ge C(\text{dist}(x, \partial K))^p.$$

Thus, (2.21) follows from the relation

$$\log\left(\frac{1}{\text{dist}(x_n, \partial K)}\right) = O\left(|x_n|^{-2q_0}\right). \tag{2.22}$$

In other words, in order to ensure weak convergence one has to assume that the distance $\text{dist}(x_n, \partial K)$ to the boundary is not too small when compared with the absolute value of $|x_n|$.

The restriction on the sequence of starting points is the main difference between our approach and that of Garbit. In his paper one has convergence for any sequence of starting points, but the cone should be nice.

Convergence of h-transforms for Brownian motion in cones

Proof of (2.13)**.** Let in the following $\mathbf{P}_0^{(u)}$ denote the limiting distribution in (2.13). Due to the scaling property of the Brownian motion, it suffices to prove the convergence on $C[0,1]$. Let f be a bounded and uniformly continuous function on $C[0,1]$ with values in $[0,1]$. Fix some $R>1$ and split

$$\mathbf{E}_x^{(u)}[f(B)] = \mathbf{E}_x^{(u)}[f(B), |x+B(1)| \le R] + \mathbf{E}_x^{(u)}[f(B), |x+B(1)| > R]. \tag{2.23}$$

In order to bound the second summand we use (2.14). Introducing spherical coordinates and using the estimate $u(x) \le C|x|^p$, we obtain

$$\begin{aligned}\mathbf{E}_x^{(u)}[f(B), |x+B(1)| > R] &= \frac{1}{u(x)}\mathbf{E}[f(x+B)u(x+B(1)), \tau_x^{bm} > 1, |x+B(1)| > R] \\ &\le \frac{C}{u(x)}\mathbf{E}[|x+B(1)|^p, \tau_x^{bm} > 1, |x+B(1)| > R] \\ &\le C\int_R^\infty r^{p+d-1}e^{-\gamma r^2}dr,\end{aligned}$$

uniformly in $x \in K, |x| \le \frac{1}{2}$. Therefore,

$$\sup_{x\in K, |x|\le\frac{1}{2}} \mathbf{E}_x^{(u)}[f(B), |x+B(1)| > R] \to 0 \quad \text{as } R\to\infty. \tag{2.24}$$

For the first summand in (2.23) we have the following representation

$$\begin{aligned}&\mathbf{E}_x^{(u)}[f(B), |x+B(1)| \le R] \\ &\quad = \frac{\mathbf{P}(\tau_x^{bm} > 1)}{u(x)}\mathbf{E}[f(x+B)u(x+B(1))1\{|x+B(1)| \le R\}|\tau_x^{bm} > 1].\end{aligned}$$

The functional $f(x+B)u(x+B(1))1\{|x+B(1)| \le R\}$ is bounded and the set of its discontinuities is a null set with respect to the distribution of the meander M_K. Therefore, by the convergence proven in Section 2.3,

$$\begin{aligned}&\mathbf{E}[f(x_n+B)u(x_n+B(1))1\{|x_n+B(1)| \le R\}|\tau_{x_n}^{bm} > 1] \\ &\quad \to \mathbf{E}[f(M_K)u(M_K(1))1\{M_K(1) \le R\}], \quad x_n \to 0,\end{aligned}$$

provided that x_n satisfies (2.21).

Recalling (2.10), we then have

$$\mathbf{E}_x^{(u)}[f(B), |x+B(1)| \le R] \to \varkappa\mathbf{E}[f(M_K)u(M_K(1))1\{M_K(1) \le R\}]. \tag{2.25}$$

Note that, by monotone convergence,

$$\mathbf{E}[f(M_K)u(M_K(1))1\{M_K(1) \le R\}] \to \mathbf{E}[f(M_K)u(M_K(1))] \quad \text{as } R\to\infty.$$

Combining this with (2.24) and (2.25) we finally get

$$\lim_{n\to\infty} \mathbf{E}_{x_n}^{(u)}[f(B)] = \varkappa\mathbf{E}[f(M_K)u(M_K(1))]. \tag{2.26}$$

As in the construction of the meander M_K, the restriction on the sequence of starting points is not needed if the cone is nice. In that case it suffices to use the results from Garbit [2009].

2.4 Other tools: Fuk-Nagaev-Borovkov inequalities and the KMT coupling

As mentioned in chapter 1, one step in proving convergence (after appropriate scaling) of the random walks in cones, involves studying the behavior of the (appropriately scaled) discrete-time process ‘deep’ inside the cone. The idea is that, ‘untypical’ values of $S(n)$ do not contribute in the limit, just as values near the boundary of the cone do not contribute in the limit. On the other hand, ‘typical’ values of $S(n)$ ‘deep’ in the cone contribute the bulk of the limit.

A powerful tool for estimates in formalizing the intuition above are the Fuk-Nagaev-Borovkov inequalities. These can be used to estimate the probability of large deviations uniformly on n and under minimal assumptions on the random walk step. They are therefore of interest on their own.

Proposition 2 (Fuk-Nagaev-Borovkov inequalities)**.** *Let $x, y > 0$ and $X_i, i \geq 1$ be $\mathbf{R}^d$-valued i.i.d. random variables with $\mathbf{E}[X_1] = 0, \mathbf{E}[|X_1(j)|^2] = 1$. Then it holds*

$$\mathbf{P}\left(M(n) > x, \max_{k \leq n} |X(k)| \leq y\right) \leq 2de^{x/\sqrt{d}y} \left(\frac{\sqrt{d}n}{xy}\right)^{x/\sqrt{d}y} \tag{2.27}$$

and

$$\mathbf{P}(M(n) > x) \leq 2de^{x/\sqrt{d}y} \left(\frac{\sqrt{d}n}{xy}\right)^{x/\sqrt{d}y} + n\mathbf{P}(|X(1)| > y). \tag{2.28}$$

The original form of the Fuk-Nagaev inequalities is in Lemma 22 of Denisov and Wachtel [2015b] (see also the comments therein related to the original work Fuk and Nagaev [1971]). The weaker version where $M(n)$ is replaced by $|S(n)|$ is Corollary 23 in Denisov and Wachtel [2015b]. Borovkov [1972] shows that all Fuk-Nagaev inequalities remain valid for partial maximas of sums of independent random variables.

Deep inside the cone, the appropriately scaled random walk behaves like a Brownian motion. This can be formally established through a coupling argument. The argument is old; the formulation below is from Lemma 17 in Denisov and Wachtel [2015b].

Proposition 3 (Hungarian coupling)**.** *If $\mathbf{E}[|X_1|^{2+\delta}] < \infty$ for some $\delta \in (0, 1)$, then one can define on a joint probability space a copy of $S(n) = \sum_{i=0}^{n} X_i$ and a Brownian motion $B(t)$ on the same space, such that for any γ so that $0 < \gamma < \frac{\delta}{2(2+\delta)}$ the probability of the event*

$$\mathbf{P}\left(\left\{\sup_{u \leq n} |S([u]) - B(u)| > n^{1/2-\gamma}\right\}\right) < Cn^{-r},$$

with $r = r(\delta, \gamma) = \delta/2 - 2\gamma - \gamma\delta$.

2.5 Tail asymptotics for the exit time and the local limit theorem for random walks in cones

It is instructive to summarize shortly the main results in Denisov and Wachtel [2015b] and Denisov and Wachtel [2019] with regards to random walks conditioned to stay in cones, because these results are an important building block for the work in chapters 3 and 4.

We first define the setting. We look at a random walk $\{S(n), n \geq 1\}$ on $\mathbf{R}^d, d \geq 1$, where $S(n) = \sum_{j=1}^n X(j)$ and $\{X(n), n \geq 1\}$ are i.i.d. random variables with $X \equiv X(1) = (X_1, \dots, X_2)$ a $d-$dimensional random variable.

Throughout the thesis, unless otherwise specified, we impose the following moment conditions.

- **Moment conditions:** $\mathbf{E}[|X|^\alpha]$ is finite for $\alpha = p$ if $p > 2$ or for some $\alpha > 2$ if $p \leq 2$. Moreover, $\mathbf{E}[X] = 0 \in \mathbf{R}^d$, $\mathbf{Var}(X_i) = 1$ and $\mathbf{cov}(X_i, X_j) = 0, i, j = 1, \dots, d, i \neq j$.

For $x \in K$ we let

$$\tau_x = \inf\{n \geq 1 : x + S(n) \notin K\},$$

be the exit time from the cone of the random walk started at x.

The following result characterizes a class of harmonic functions for random walks satisfying the moment conditions.

Proposition 4. *[Denisov and Wachtel [2019]] Assume that the cone K satisfies the geometric conditions and that the random walk $S(n), n \geq 1$ satisfies the moment conditions. Then the function*

$$V(x) = \lim_{n\to\infty} \mathbf{E}[u(x + S(n)), \tau_x > n)] \tag{2.29}$$

is finite and harmonic for $\{S(n)\}$ killed at leaving K, i.e.

$$V(x) = \mathbf{E}[V(x + S(n)), \tau_x > n], \quad x \in K, n \geq 1.$$

V is strictly positive on the set

$$K_+ = \{x \in K : \exists\gamma > 0 : \forall R > 0 \exists n \textit{ with } \mathbf{P}(x + S(n) \in D_{R,\gamma}, \tau_x > n) > 0\},$$

where $D_{R,\gamma} = \{x \in K : |x| \geq R, dist(x, \partial K) \geq \gamma|x|\}$.

A large part of the work in proving this result is devoted to proving several properties of the function V as defined in (2.29). We gather here the properties which are needed for chapters 3 and 4.

A. *V and u behave asymptotically the same along rays:* For any $\gamma > 0, R > 0$, uniformly in $x \in D_{R,\gamma}$ we have $V(tx) \sim u(tx)$ as $t\to\infty$.

B. *Estimates for V:* It holds for every $x \in K$

$$V(x) \leq C(1 + |x^p|). \tag{2.30}$$

Moreover, for all γ small enough

$$|V(x) - u(x)| \leq C\left(1 + |x|^{p-\gamma} + \frac{|x|^p}{(dist(x, \partial K)^\gamma)}\right). \tag{2.31}$$

C. *Monotonicity inside the cone:* If $x \in K$, then $V(x) \leq V(x + x_0)$ for all x_0 such that $x_0 + K \subset K$.

Once Theorem 4 is established, coupling arguments and Fuk-Nagaev inequalities are the main 'novel' ingredients used to prove the pendant of Proposition 1 for random walks. For this, we first formalize the expression *deep inside the cone* through the definition

$$K_{n,\epsilon} = \{x \in K : dist(x, \partial K) \geq n^{\frac{1}{2}-\epsilon}\}.$$

Proposition 5 (Cor. 1.2 in Denisov and Wachtel [2019], Lemma 20 in Denisov and Wachtel [2015b])**.** *1) For all sufficiently small $\epsilon > 0$,*

$$\mathbf{P}(\tau_y > n) = \varkappa u(y) n^{-\frac{p}{2}}(1 + o(1)), \quad n \to \infty \tag{2.32}$$

uniformly in $y \in K_{n,\epsilon}, n \geq 1$, such that $|y| \leq \theta_n \sqrt{n}$ for some $\theta_n \to 0$.

Moreover, there exists a constant C such that

$$\mathbf{P}(\tau_y > n) \leq C \frac{|y|^p}{n^{\frac{p}{2}}}, \tag{2.33}$$

uniformly in $y \in K_{n,\epsilon}, n \geq 1$. Finally, it holds for every $x \in K$

$$\mathbf{P}(\tau_x > n) \sim \varkappa V(x) n^{-\frac{p}{2}}.$$

2) For any compact set $D \subset K$ it holds true

$$\mathbf{P}(\tau_y > n, y + S(n) \in \sqrt{n} D) \sim \varkappa_0 u(y) n^{-\frac{p}{2}} \int_D e^{-\frac{|z|^2}{2}} u(z) dz, \tag{2.34}$$

uniformly in $y \in K_{n,\epsilon}$ such that $|y| \leq \theta_n \sqrt{n}$ for some $\theta_n \to 0$.

Suppose that X has support on a lattice L which is a non-degenerate linear transformation of $\mathbb{Z}^d$. Denisov and Wachtel [2015b] and Denisov and Wachtel [2019] use then the previous results, together with time reversion and standard concentration inequalities, as well as Stone's local limit theorem to prove the following local limit theorems for lattice walks in cones.

Proposition 6 (Thms. 5,6 in Denisov and Wachtel [2015b], Cor. 1.3 in Denisov and Wachtel [2019])**.** *1) Assume that X takes values on a lattice L which is a non-degenerate linear transformation of $\mathbb{Z}^d$. Then under the moment and geometric assumptions it holds*

$$\sup_{y\in D_n(x)}\left|n^{\frac{p}{2}+\frac{d}{2}}\mathbf{P}(x+S(n)=y,\tau_x>n)-C_0V(x)u\left(\frac{y}{\sqrt{n}}\right)e^{-\frac{|y|^2}{2n}}\right|\to 0,$$

where $D_n(x)=\{y\in K:\mathbf{P}(x+S(n)=y)>0\}$.

Here the constant C_0 is a product of the volume of the unit cell in L and of a factor, which depends on the periodicity of the distribution of X.

2) Under the assumptions of 1) and for every fixed $y\in K$ it holds

$$\mathbf{P}(x+S(n)=y,\tau_x>n)\sim C_1\frac{V(x)V'(y)}{n^{p+\frac{d}{2}}},$$

where V' is the harmonic function for the walk $\{-S(n)\}$ and

$$C_1=C_0^2\int_K u^2(z)e^{-|z|^2/2}dz.$$

2.6 Results about Kolmogorov diffusion conditioned to stay positive

If $\{B_t,t\geq 0\}$ is a standard Brownian motion, we define the Kolmogorov diffusion started at (x,y) to be the process given by

$$W_{(x,y)}(t)=(U_t,V_t)_{(x,y)}=\left(x+ty+\int_0^t B_s ds,\ y+B_t\right),\quad t\geq 0.$$

Kolmogorov diffusion has the following scaling property. Define the map $a_t:\mathbf{R}^2\to\mathbf{R}^2$ via $a_t(x,y)=(\frac{x}{t^{\frac{3}{2}}},\frac{y}{t^{\frac{1}{2}}})$ for $t>0$. Let $b_t=a_t^{-1}$. Then

$$a_t\left((W_s,s\leq t,W(0)=(x,y))\right)=(W_s,s\leq 1,W(0)=a_t(x,y)) \tag{2.35}$$

or equivalently

$$(W_s,s\leq t,W(0)=(x,y))=b_t\left((W_s,s\leq 1,W(0)=a_t(x,y))\right).$$

It has a long history and the reader is referred to Groeneboom et al. [1999] for more details and references on applications. The transition density of this process can be calculated via the expectations and covariances of this Gaussian process (see Denisov and Wachtel [2015a] and references therein). For $x,y,u,v\in\mathbb{R}$ and $t>0$ it is given by

$$\begin{aligned}p_t(x,y;u,v)&=\frac{\sqrt{3}}{\pi t^2}\exp\left(-\frac{6(u-x-ty)^2}{t^3}+\frac{6(v-y)(u-x-ty)}{t^2}-\frac{2(v-y)^2}{t}\right)\\&=\frac{\sqrt{3}}{\pi t^2}\exp\left(-\frac{6(u-x)^2}{t^3}+\frac{6(u-x)(v+y)}{t^2}-\frac{2(v^2+vy+y^2)}{t}\right).\end{aligned}$$

Denote the exit time from $\mathbf{R}_+ \times \mathbf{R}$ by $\tau^{bm}_{(x,y)}$. It is given as follows.

$$\tau^{bm}_{(x,y)} = \inf\{t \geq 0 : x + ty + \int_0^t B_s ds \leq 0\}.$$

Let $\bar{p}_t(x,y;u,v)$ denote the density of the Kolmogorov diffusion started at (x,y) at time t and killed if the first coordinate becomes non-positive. Let h be the function $h : \mathbf{R}_+ \times \mathbf{R} \longrightarrow \mathbf{R}$ which is harmonic for the Kolmogorov diffusion, in the sense that $\mathcal{A}h = 0$ on $\mathbf{R}_+ \times \mathbf{R}, h(0,\cdot) \equiv 0$, where $\mathcal{A} = y\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial y^2}$ is the generator of W. This can also be expressed as

$$\mathbf{E}_{(x,y)}[h(W(t)), \tau^{bm} > t] = h(x,y), \quad x > 0, y \in \mathbf{R}, t > 0.$$

In other words, $h(W(t))\mathbf{1}_{\{\tau^{bm}>t\}}$ is a nonnegative martingale. h has both an integral representation (see Appendix) and a representation in terms of confluent hypergeometric functions (see Groeneboom et al. [1999], Denisov and Wachtel [2015a], or appendix). It can be bounded by the square root of $\alpha(x,y) = \max\{|x|^{\frac{1}{3}}, |y|\}$ both from below and above (see Lemma 6 in Denisov and Wachtel [2015a]). It also has the scaling property $h(x,y) = x^{\frac{1}{6}}h(1, x^{-\frac{1}{3}}y), x > 0, y \in \mathbf{R}$.

Parts of the following result are well-known, others need a proof.

Proposition 7. *1) There exists a finite positive constant C such that*

$$\mathbf{P}_{(x,y)}(\tau^{bm} > t) \leq C\frac{h(x,y)}{t^{\frac{1}{4}}}, \quad x, y > 0, \tag{2.36}$$

Moreover,

$$\mathbf{P}_{(x,y)}(\tau^{bm} > t) \sim \varkappa\frac{h(x,y)}{t^{\frac{1}{4}}}, \tag{2.37}$$

uniformly in $x, y > 0$ satisfying $\alpha(x,y)^{\frac{1}{2}} \leq \theta_t t^{\frac{1}{4}}$ with some $\theta_t \to 0$.

In particular,

$$\lim_{\epsilon\to 0} \sup_{(x,y)\in\mathbf{R}^2_+, \alpha(x,y)\leq\epsilon^2} \left|\frac{\mathbf{P}_{(x,y)}(\tau^{bm} > 1)}{\varkappa h(x,y)} - 1\right| = 0. \tag{2.38}$$

2) For $x, y \searrow 0$ we have

$$\bar{p}_t(x,y;u,v) \sim h(x,y)\overline{h}(t,u,-v), \tag{2.39}$$

uniformly in (u,v) with $\alpha(u,v) \leq R$ ($R > 0$ arbitrary).

Moreover,

$$\sup_{x,y\in\mathbf{R}_+, \alpha(x,y)\leq 1} \frac{\bar{p}_1(x,y;u,v)}{h(x,y)} \leq \psi(u,v), \tag{2.40}$$

for an integrable function $\psi : \mathbf{R}^2 \to \mathbf{R}$ such that ψh is integrable in $\mathbf{R}^2$.

The proofs of (2.36) and (2.37) are in Denisov and Wachtel [2015a], whereas the proof of (2.38) follows from Lemma 15 there and the scaling property of Kolmogorov diffusion.[3] The proofs of (2.39) and (2.40) are in the appendix.

Kolmogorov meander, bridge conditioned to stay positive and h-transforms

We call the measure on $C[0,1]$ given by the conditional measure $W_{(x,y)}(\cdot\ |\tau^{bm} > 1)$ the *Kolmogorov meander of length one* started at $(x, y) \in \mathbf{R}_+ \times \mathbf{R}_+$.

We show the following convergence result.

$$W_{z_n}(\cdot\ |\tau^{bm} > 1) \text{ converges weakly in } C[0,\infty) \text{ as } z_n \searrow (0,0), \tag{2.41}$$

under some weak restrictions on $\{z_n : n \geq 1\}$. We denote the limit process as $W_{(0,0)}(\cdot\ |\tau^{bm} > 1)$, and call it *Kolmogorov meander started at* $(0, 0)$ *of length one*. Its respective distribution is denoted by $\mathbf{P}_{(0,0)}(W \in \cdot|\tau^{bm} > 1)$. One can characterize the density explicitly (see Appendix).

Just as for Brownian motion in cones, we construct the Kolmogorov bridge (or excursion) conditioned to stay positive, through a limit of discrete-time processes and time reversion.

Just as for random walks in cones, we can define the h-transform as follows: For any $t > 0$ and each continuous and bounded functional $f_t : C[0, t]\to\mathbf{R}$,

$$\mathbf{E}^{(h)}_{(x,y)}[f_t(W)] = \mathbf{E}_{(x,y)}\left[f_t(W)\frac{h(W(t))}{h(x,y)}, \tau^{bm} > t\right], \quad x > 0, y \in \mathbf{R}.$$

With the help of (2.41) we show the following convergence in $(C[0, 1], ||\cdot||_\infty)$.

$$\mathbf{P}^{(h)}_{z_n} \quad \text{converges weakly as } z_n \searrow (0,0), \tag{2.42}$$

under some weak restrictions on $\{z_n : n \geq 1\}$.

2.7 Proofs for the convergence of Kolmogorov diffusion

Here we prove (2.41) and (2.42), leaving the rest of the auxiliary results concering the Kolmogorov diffusion to the appendix.

Proof of (2.41)

We show that the sequence of processes with continuous paths in $\mathbf{R}^2$

$$\{W(\cdot)|W(0) = z_n, \tau^{bm} > 1\},$$

converges as $z_n \searrow (0, 0), n\to\infty$.

[3] To see this it suffices to use with $t = \frac{1}{\epsilon}$.

Convergence of finite-dimensional distributions. Note that

$$\frac{\mathbf{P}_{(x,y)}(\alpha(W(t)) > R, \tau^{bm} > t)}{h(x,y)} = \frac{\mathbf{P}_{(x,y)}(\alpha(b_t(W(1))) > R, \tau^{bm} > 1)}{t^{\frac{1}{4}} h(a_t(x,y))},$$

and from this that

$$\lim_{R\to\infty} \sup_{x,y>0, a_t(x,y)\le\frac{1}{2}} \frac{\mathbf{P}_{(x,y)}(\alpha(W(t)) > R, \tau^{bm} > t)}{h(x,y)} = 0. \tag{2.43}$$

Now we take in (2.38) $\epsilon_0 > 0$ such that

$$\mathbf{P}_{(x,y)}(\tau^{bm} > 1) \ge \frac{1}{2}\varkappa h(x,y) \text{ for all } x \in K, \alpha(x,y) \le \epsilon_0^2. \tag{2.44}$$

Combining (2.44) with (2.43) delivers for every $t \in (0,1]$ that

$$\lim_{R\to\infty} \sup_{x,y>0, a_t(x,y)\le\frac{1}{2})} \mathbf{P}_{(x,y)}(\alpha(W(t)) > R | \tau^{bm} > t) = 0. \tag{2.45}$$

We can write for the density of the meander started at (x,y)

$$m_t(x,y;u,v) = \frac{\bar{p}_t(x,y;u,v)\mathbf{P}_{(u,v)}(\tau^{bm} > 1-t)}{\mathbf{P}_{(x,y)}(\tau^{bm} > 1)} = \frac{\bar{p}_t(x,y;u,v)}{h(x,y)} \frac{h(x,y)}{\mathbf{P}_{(x,y)}(\tau^{bm} > 1)} \mathbf{P}_{(u,v)}(\tau^{bm} > 1-t).$$

(2.39) shows that the density $m_t(x,y;u,v)$ converges to $m_t(u,v)$, as $x,y \searrow 0$ uniformly for $\alpha(u,v) \le R$ and $R > 0$ arbitrary. Note that (2.40) implies that $m_t(u,v)$ is integrable. Using this fact and (2.45) it follows that

$$\lim_{x,y\searrow 0}\sup \int\int |m_t(x,y;u,v) - m_t(u,v)| \, dudv = 0.$$

It follows that $m_t(u,v)$ is also a density. Together with the Markov property, this implies the convergence of finite-dimensional distributions.

Tightness. First, we note that due to the continuous mapping theorem, it is enough to focus on showing tightness of the second coordinate of W. That is, it is sufficient to show the tightness of Brownian motion killed whenever its integral (plus a start point) becomes non-positive.

We follow a similar approach as for the construction of the meander for the Brownian motion in a cone. Namely, we use again Theorems 7.3, 7.4 in Billingsley [1968] and the Corollary of the latter result there. Since the proof is very similar to that of the meander of Brownian motion in a cone, we just highlight the differences.

Showing tightness on $C[\delta, 1]$ ($\delta \in (0,1)$ arbitrary) is the same, except the replacements of x_n by z_n, p by $\frac{1}{2}$, and we use the relation $\mathbf{P}_{z_n}(\tau^{bm} > t) \sim \varkappa h(z_n) t^{-\frac{1}{4}}, n\to\infty$ instead of $\mathbf{P}_{x_n}(\tau^{bm} > t) \sim \varkappa u(x_n) t^{-\frac{1}{4}}, n\to\infty$.

We then need to show that $\mathbf{P}_{z_n}(\sup_{u\in[0,\delta]}|B(u)|>\epsilon|\tau^{bm}>1)$ can be made sufficiently small. The proof follows exactly the same steps as for the meander of the Brownian motion in cones, but for the replacements of x_n by z_n, $u(x_n)$ by $h(z_n)$ $h(z_n)$, p by $\frac{1}{2}$. Moreover, we replace $\sup_{|x|\leq\frac{1}{2}}\frac{\mathbf{P}_x(\tau^{bm}>1)}{u(x)}\leq C^*$ by

$$\sup_{\alpha(z)\leq\frac{1}{2}}\frac{\mathbf{P}_z(\tau^{bm}>1)}{h(z)}\leq C^*,$$

for a suitable C^*. This follows from (2.38) and the fact that $\mathbf{R}_+\times\mathbf{R}_+\ni z\mapsto\mathbf{P}_z(\tau^{bm}>1)$ is continuous. We also replace $N:=\max\{k\geq 1: 2^k\leq\frac{\delta}{4|x_n|^2}\}$ by $N:=\max\{k\geq 1: 2^k\leq\frac{\delta}{4\alpha(z_n)^2}\}$. Overall, we arrive at the condition

$$\log\left(\frac{1}{\alpha(z_n)^{\frac{1}{2}}}\right)=O(\alpha(z_n)^{-2q_0}). \tag{2.46}$$

Here, $q_0=\frac{\log(9/8)}{\log 2}\in(0,1)$, just as for the case of the meander. (2.46) is easy to satisfy, e.g. take sequences $\{z_n,n\geq 1\}$ such that $\alpha(z_n)=O(n^{-\beta})$ for some $\beta>0$ as $n\to\infty$.

Proof of (2.42)

Let f be a uniformly continuous, bounded function on $C[0,1]$ with values in $[0,1]$. Fix some $S>0$. It holds

$$\mathbf{E}^{(h)}_{(x,y)}[f(W)]=\mathbf{E}^{(h)}_{(x,y)}[f(W),|W(1)|\leq S]+\mathbf{E}^{(h)}_{(x,y)}[f(W),|W(1)|>S]. \tag{2.47}$$

We have the following estimate with the help of (2.40).

$$\begin{aligned}\mathbf{E}^{(h)}_{(x,y)}[f(W),|W(1)|>S]&=\frac{1}{h(x,y)}\mathbf{E}_{(x,y)}[f(W)h(W(1)),\tau^{bm}>1,|W(1)|>S]\\&\leq\frac{C}{h(x,y)}\mathbf{E}_{(x,y)}[|W(1)|^{\frac{1}{2}},\tau^{bm}>1,|W(1)|>S]\end{aligned}$$

uniformly for $|(x,y)|\leq\frac{1}{2}, x,y>0$ and some suitable $S>0$. In particular, it follows

$$\sup_{|(x,y)|\leq\frac{1}{2},x,y>0}\mathbf{E}^{(h)}_{(x,y)}[f(W),|W(1)|>S]=o(S),\quad S\to\infty. \tag{2.48}$$

We have

$$\begin{aligned}\mathbf{E}^{(h)}_{(x,y)}[f(W),|W(1)|\leq S]&=\frac{1}{h(x,y)}\mathbf{E}_{(x,y)}[f(W)h(W(1)),\tau^{bm}_{(x,y)}>1,|W(1)|\leq S]\\&=\frac{\mathbf{P}_{(x,y)}(\tau^{bm}>1)}{h(x,y)}\mathbf{E}_{(x,y)}[f(W)h(W(1)),|W(1)|\leq S|\tau^{bm}>1].\end{aligned}$$

We recall (2.38). Moreover, the function

$$f(W)h(W(1))\mathbf{1}_{\{|W(1)|\le S\}}$$

is bounded and the set of its discontinuities is a null set w.r.t. the Lebesgue measure, i.e. also w.r.t. measure constructed by the weak convergence of $\mathbf{P}_{(x,y)}(\cdot|\tau^{bm} > 1)$ as $x, y \searrow 0$. In particular,

$$\mathbf{E}_{(x,y)}[f(W)h(W(1)), |W(1)| \le S|\tau^{bm} > 1] \to \mathbf{E}_{(0,0)}[f(W)h(W(1)), |W(1)| \le S|\tau^{bm} > 1],$$

because of the convergence of the meander of Kolmogorov diffusion as the start point converges to $(0, 0)$. Monotone convergence delivers

$$\mathbf{E}_{(0,0)}[f(W)h(W(1)), |W(1)| \le S|\tau^{bm}_{(0,0)} > 1] \to \mathbf{E}_{(0,0)}[f(W)h(W(1))|\tau^{bm}_{(0,0)} > 1], \quad S\to\infty. \tag{2.49}$$

Combining this with (2.48) shows

$$\mathbf{E}^{(h)}_{(x,y)}[f(W)] \to \varkappa \mathbf{E}_{(0,0)}[f(W)h(W(1))|\tau^{bm}_{(0,0)} > 1].$$

This finishes the proof.

2.8 Exit time and local limit theorems for integrated random walks

We summarize shortly the main results from Denisov and Wachtel [2015a] we need, leaving smaller technical results to the proofs section of chapter 5. We consider a sequence of i.i.d. real-valued random variables $\{X_i : i \ge 1\}$ with $\mathbf{E}[X_i] = 0$ and $\mathbf{E}[|X_i|^{2+\delta}] < \infty$ for some $\delta > 0$.

We define respectively, the random walk $\{S(n) : n \ge 0\}$ and the integrated random walk $\{T(n) : n \ge 0\}$ as

$$S(n) = y + X_1 + \cdots + X_n, \quad n \in \mathbb{N}_0.$$

and

$$T(n) = x + S(1) + \cdots + S(n) = x + ny + nX_1 + (n-1)X_2 + \cdots + X_n, \quad n \in \mathbb{N}_0.$$

Here $x > 0, y \in \mathbf{R}$. We look at the Markov chain $Z(n) = (T(n), S(n)), n \in \mathbb{N}_0$. We look at the following stopping time.

$$\tau_{(x,y)} = \inf\{n \ge 0 : T(n) \le 0\}.$$

Denisov and Wachtel [2015a] construct a harmonic function for $Z(n)$, killed when the first coordinate becomes non-positive and establish the asymptotic for the exit time $\tau_{(x,y)}$. They also establish local limit theorems for the same process.

Proposition 8. *[Denisov and Wachtel [2015a]] Assume the moment conditions* $\{X_i : i \geq 1\}$ *with* $\mathbf{E}[X_i] = 0$ *and* $\mathbf{E}[|X_i|^{2+\delta}] < \infty$ *for some* $\delta > 0$.

1) The function

$$V(z) := \lim_{n\to\infty} \mathbf{E}_z[h(Z(n)), \tau > n]$$

is well-defined, strictly positive on

$$K_+ = \{z \in \mathbf{R}_+ \times \mathbf{R} : z \in \mathbf{R}_+ \times \mathbf{R} : \mathbf{P}_z(Z(n) \in \mathbf{R}_+^2, \tau > n) > 0 \textit{ for some } n \geq 0\},$$

and is harmonic for the killed integrated random walk, i.e.

$$V(z) = \mathbf{E}_z[V(Z(1)), \tau > 1], \quad z \in \mathbf{R}_+ \times \mathbf{R}.$$

2) It holds

$$\mathbf{P}_z(\tau > n) \sim \varkappa \frac{V(z)}{n^{\frac{1}{4}}}, \quad n \to \infty. \tag{2.50}$$

3) The following local limit theorems hold true.

$$\sup_{\tilde{z}} \left| n^{2+\frac{1}{4}} \mathbf{P}_z(Z(n) = \tilde{z}, \tau > n) - \varkappa V(z) h\left(a_n(\tilde{z})\right) p_1(0, 0; a_n(\tilde{z})) \right| \to 0. \tag{2.51}$$

For every fixed $\tilde{z} \in K_+$

$$\lim_{n\to\infty} n^{2+\frac{1}{2}} \mathbf{P}_z(Z(n) = \tilde{z}, \tau > n) = V(z) V'(\tilde{z}), \tag{2.52}$$

for some positive function V'.

Chapter 3

Invariance principles for random walks in cones

3.1 Introduction and statement of results

The work presented in this chapter is included in Duraj and Wachtel [2020].

Invariance principles for random walks in one-dimensional cones (a half-line) are well-known. It is clear that one has only two non-trivial cones: $(0, \infty)$ and $(-\infty, 0)$ in this case. Furthermore, due to symmetry, it suffices to consider $(0, \infty)$ only. A functional central limit theorem for zero mean random walks with finite variance conditioned to stay positive has been proved by Bolthausen [1976]. Denisov et al. [2018] have generalized this result to all random walks with non-identical distributed increments and satisfying a Lindeberg-type of condition. For random walks with i.i.d. increments from the domain of attraction of a stable law, conditional invariance principles were derived by Doney [1985] and by Caravenna and Chaumont [2008].

In terms of methodology, in the case of one-dimensional walks with i.i.d. increments tools like Wiener-Hopf factorisation are feasible. These allow to obtain exact identities for distributions of conditioned walks. Unfortunately, this method does not work if the random walk is not one-dimensional. As a consequence, the case of multi-dimensional walks is much less studied in the literature. For random walks constrained to the Weyl chamber functional limit theorems have been derived in Denisov and Wachtel [2010]. A generalization to Weyl chambers of type C and D was obtained by König and Schmid [2010]. Shimura [1991] and Garbit [2011] have obtained an invariance principle for some conditioned two-dimensional random walks. Besides these special results, we are unaware of other invariance principles in the literature. Finally, a conditional limit theorem for walks starting deep in a cone was proved in Denisov and Wachtel [2015b].

Similar to Denisov and Wachtel [2015b], we consider here quite general cones and a wide class of random walks and prove various conditional functional limit theorems for walks starting at a fixed point. The main purpose of this chapter is to continue the study of random walks in cones and to derive invariance principles for various functionals of random walks constrained to stay in cones.

The assumptions we make are the ones mentioned in chapter 2. We look at a cone K that satisfies the geometric conditions mentioned in subsection 2.2 and at a random walk that satisfies the moment conditions from subsection 2.5.

Before we state the invariance principle for the meander we collect invariance principles which are straightforward applications of the classical, unconditional invariance principle and Bayes' rule.

Remark 1. *For every sequence $\{x_n\}$ such that $\frac{x_n}{\sqrt{n}} \to x \in K$, the process $\{\frac{x_n+S([nt])}{\sqrt{n}}, t \in [0,1]\}$ conditioned on the event $\{\tau_x > n\}$ converges towards $\{x + B(t)|\tau^{bm} > 1\}_{t\in[0,1]}$ weakly in $(D[0,1], ||\cdot||_\infty)$.*

Define

$$K_+ := \{x \in K : \text{there exists } \gamma > 0 \text{ such that for every } R > 0 \\ \text{there exists } n \text{ with } \mathbf{P}(x + S(n) \in D_{R,\gamma}, \tau_x > n) > 0\},$$

where $D_{R,\gamma} = \{x \in K : |x| \geq R, dist(x, \partial K) \geq \gamma|x|\}$. Finally, let $\tilde{K}_+$ be the analogous set to K_+ where the random walk has step $-X$ instead of X. We prove the following invariance principle.

Theorem 1. *For every fixed $x \in K_+$, the process $\{\frac{x+S([nt])}{\sqrt{n}}, t \in [0,1]\}$ conditioned on $\{\tau_x > n\}$ converges towards $\{M_K(t), t \in [0,1]\}$ weakly in $(D[0,1], ||\cdot||_\infty)$.*

In Denisov and Wachtel [2015b, 2019] a positive harmonic function $V(x)$ for $S(n)$ killed at leaving K has been constructed. There it is shown that under the assumptions made here V can be expressed as the following limit

$$V(x) = \lim_{n\to\infty} \mathbf{E}[u(x + S(n)), \tau_x > n].$$

In the following we also let $\mathbf{P}^{(V)}$ denote the h-transform of $\mathbf{P}$ with the harmonic function V. It is given by the formula:

$$\mathbf{P}_x^{(V)}(S(n) \in dy) = \mathbf{P}(x + S(n) \in dy, \tau_x > n)\frac{V(y)}{V(x)}, \quad \{x \in K : V(x) > 0\}.$$

For one of the subcases of our Theorem 2 only, we expand the geometric assumptions on K by the following regularity assumption for the cone.

- (Extendability) There exists an open and connected set $\widetilde{\Sigma} \subset \mathbb{S}^{d-1}$ with $\text{dist}(\partial\Sigma, \partial\widetilde{\Sigma}) > 0$ such that $\Sigma \subset \widetilde{\Sigma}$ and the function m_1 can be extended to $\widetilde{\Sigma}$ as a solution to (2.1).

The extendability assumption is a relatively weak assumption, because holds for any two dimensional cone as well as for any cone whose $\partial\Sigma$ is real-analytic. Under this assumption one has more information on the asymptotic behaviour of the harmonic function V than in the case of our standard geometric assumptions from subsection 2.2. Therefore, for Theorem 2 below the lack of the information in the general case will be compensated by stronger moment restrictions on the walk.

Theorem 2. *Assume one of the following cases.*

A. $\mathbf{E}|X|^{\alpha+\epsilon_0} < \infty$ *for some* $\epsilon_0 > 0$*, where* α *is from the moment assumptions on the increments of the random walk.*

B. The cone satisfies the Extendability condition.

Then for every fixed $x \in K_+$*, the process* $\left\{\frac{x+S([nt])}{\sqrt{n}}, t \geq 0\right\}$ *under* $\mathbf{P}^{(V)}$ *converges weakly to* $\{B(t), t \geq 0\}$ *under* $\mathbf{P}_0^{(u)}$*.*

In the special case of the Weyl chamber of type A this result was proved by Denisov and Wachtel [2010]. In the case when $d = 1$ and $K = (0, \infty)$ Bryn-Jones and Doney [2006] proved Theorem 2 for random walks which are integer-valued, aperiodic and belong to the domain of attraction of a standard normal law. Caravenna and Chaumont [2008] have generalized this result to the whole class of asymptotically stable random walks, albeit still under $d = 1$.

It is well-known that for random walks belonging to the domain of attraction of the normal distribution one gets in the limit the three-dimensonal Bessel process for the radial part of the random walk. Grabiner [1999] has shown that the radial part of a Brownian motion conditioned to stay in Weyl chambers is also a Bessel process. It is then immediate from Theorem 2 that the rescaled radial part of a random walk converges to the corresponding Bessel process. This is actually valid in all cones satisfying our geometric assumptions.

Corollary 1. *For every fixed* $x \in K_+$*, the process* $\left\{\frac{|x+S([nt])|}{\sqrt{n}}, t \geq 0\right\}$ *under* $\mathbf{P}^{(V)}$ *converges weakly to a* $(2p+d)$*-dimensional Bessel process.*

We now turn to bridges of random walks conditioned to stay in K. Here we shall use the local limit theorems from Denisov and Wachtel [2015b], which are valid under the following assumption:

- *Lattice assumption:* X takes values on a lattice R which is a non-degenerate linear transformation of $\mathbb{Z}^d$. Furthermore, we assume that the distribution of X is strongly aperiodic, that is, for every $x \in R$, the smallest subgroup of R which contains the set

$$\{y : y = x + z \text{ with some } z \text{ such that } \mathbf{P}(X = z) > 0\}$$

  is R itself.

As above, before we state the invariance principle for the bridge we collect invariance principles which are straightforward applications of the classical, unconditional invariance principle and Bayes' rule.

Remark 2. *For every pair of lattice sequences* $\{x_n\}_n, \{y_n\}_n$ *such that* $\frac{x_n}{\sqrt{n}} \to x \in K$, $\frac{y_n}{\sqrt{n}} \to y \in K$, *the process* $\{\frac{x_n+S([nt])}{\sqrt{n}}, t \in [0,1]\}$ *conditioned on* $\{\tau_x > n, x_n + S(n) = y_n\}$ *converges towards* $\{x + B(t) | \tau_x^{bm} > 1, x + B(1) = y\}_{t\in[0,1]}$ *weakly in* $(D[0,1], ||\cdot||_\infty)$.

Theorem 3. *For all fixed* $x \in K_+$ *and* $y \in \tilde{K}_+$, *the process* $\left\{\frac{x+S([nt])}{\sqrt{n}}, t \in [0,1]\right\}$ *conditioned on* $\{\tau_x > n, x + S(n) = y\}$ *converges weakly to a continuous process* $\{B_K^0(t) : t \in [0,1]\}$, *which we shall call Brownian excursion in* K.

The proof of Theorem 3 is based on a method suggested by Caravenna and Chaumont [2013], who proved an invariance principle for bridges of one-dimensional random walks conditioned to stay positive. It is worth mentioning that their results are valid for all asymptotically stable random walks. Their method combines weak convergence towards the meander and local limit theorems for conditioned probabilities. In the case of random walks in cones we have all needed ingredients: Theorem 1 and local limit theorems in Proposition 6.

As usual, invariance principles allow to derive results about weak convergence of appropriate functionals of random processes. If one knows the limiting behaviour for one particular random walk then the same convergence is valid for all random walks satisfying the conditions in our theorems. For example, Feierl [2012, 2009] has considered functionals

$$\max_{k\le n}(x_d + S_d(k))$$

and

$$\max_{k\le n}|x_d + S_d(k) - x_1 - S_1(k)|$$

for random walk bridges in Weyl chambers $W_A := \{x \in \mathbb{R}^d : x_1 < x_2 < \ldots < x_d\}$ and $W_B := \{x \in \mathbb{R}^d : 0 < x_1 < x_2 < \ldots < x_d\}$. Assuming that every coordinate of $S(n)$ is a simple symmetric random walk, he has shown that the functionals mentioned above converge weakly after rescaling by $\sqrt{n}$. His proofs are based on very cumbersome calculations, which can be done for simple random walks only. Theorem 3 implies that one has the same limiting distributions for a much larger class of random walks.

Remark 3. *If the limiting process is a functional of the Brownian motion then one usually proves the weak convergence on the space of continuous functions. In order to do so, one considers linear interpolations of random walks. For random walks in cones this is possible only in the case of convex cones, since in a non-convex cone it can happen that a linear interpolation of points from the cone leaves the cone. This explains the choice of* D*-space with uniform metric in our theorems.* ◇

3.2 Proofs

3.2.1 Proof of convergence towards the Brownian meander

This subsection proves Theorem 1.

Let $f : D[0,1] \mapsto \mathbb{R}$ be a non-negative uniformly continuous with respect to the uniform topology function with values in $[0,1]$. Set also for brevity,

$$X^{(n)}(t) := \frac{x + S([nt])}{\sqrt{n}}, \quad t \geq 0$$

and

$$X_z^{(k,n)}(t) := \frac{z + (S([nt]) - S(k))\mathbf{1}\{t > k/n\}}{\sqrt{n}}, \quad t \geq 0.$$

It suffices to show that

$$\mathbf{E}\left[f\left(X^{(n)}\right) \middle| \tau_x > n\right] \to \mathbf{E}[f(M_K)]. \tag{3.1}$$

Similar to Denisov and Wachtel [2015b], we use the strong approximation of random walks by the Brownian motion. We shall also keep the notation from Denisov and Wachtel [2015b, 2019] for $K_{n,\epsilon}, V(x), M(n), \nu_n$.

Let $\varepsilon > 0$ be a constant and let

$$K_{n,\varepsilon} = \{x \in K : \mathrm{dist}(x, \partial K) \geq n^{1/2-\varepsilon}\}.$$

Define

$$\nu_n := \min\{k \geq 1 : x + S(k) \in K_{n,\varepsilon}\}.$$

By Theorem 1 in Denisov and Wachtel [2015b],

$$\mathbf{P}(\tau_x > n) \sim \varkappa V(x) n^{-p/2} \quad \text{as } n \to \infty, \tag{3.2}$$

where $\varkappa$ is the same constant as in (2.10). Furthermore, according to Lemma 14 from Denisov and Wachtel [2015b],

$$\mathbf{P}(\nu_n > n^{1-\varepsilon}, \tau_x > n^{1-\varepsilon}) \leq \exp\{-Cn^{\varepsilon}\}. \tag{3.3}$$

Consequently,

$$\mathbf{E}\left[f\left(X^{(n)}\right), \tau_x > n\right] = \mathbf{E}\left[f\left(X^{(n)}\right), \nu_n \leq n^{1-\varepsilon}, \tau_x > n\right] + o(\mathbf{P}(\tau_x > n)). \tag{3.4}$$

Using (2.27) and (2.28) in the proof of Lemma 24 from Denisov and Wachtel [2015b], one can easily infer that there exists a sequence θ_n which goes to 0 sufficiently slow such that

$$\lim_{n\to\infty} \mathbf{E}\left[|x + S(\nu_n)|^p; \tau_x > \nu_n, M(\nu_n) > \theta_n\sqrt{n}, \nu_n \leq n^{1-\varepsilon}\right] = 0. \tag{3.5}$$

Since f takes values in $[0,1]$,

$$\begin{aligned}\mathbf{E}\left[f\left(X^{(n)}\right), \nu_n \leq n^{1-\varepsilon}, M(\nu_n) > \theta_n\sqrt{n}, \tau_x > n\right] \\ \leq \mathbf{P}\left(\nu_n \leq n^{1-\varepsilon}, M(\nu_n) > \theta_n\sqrt{n}, \tau_x > n\right).\end{aligned}$$

By (2.33), there exists a constant C such that, uniformly in $y \in K_{n,\varepsilon}$,

$$\mathbf{P}(\tau_y > n) \le C\frac{|y|^p}{n^{p/2}}.$$

Combining this estimate with the Markov property, we obtain

$$\begin{aligned}
&\mathbf{E}\left[f\left(X^{(n)}\right), \nu_n \le n^{1-\varepsilon}, M(\nu_n) > \theta_n\sqrt{n}, \tau_x > n\right] \\
&\quad\le \mathbf{E}\left[\mathbf{P}(\tau_{x+S(\nu_n)} > n-\nu_n); \nu_n \le n^{1-\varepsilon}, M(\nu_n) > \theta_n\sqrt{n}\right] \\
&\quad\le \frac{C}{n^{p/2}}\mathbf{E}\left[|x+S(\nu_n)|^p; \tau_x > \nu_n, M(\nu_n) > \theta_n\sqrt{n}, \nu_n \le n^{1-\varepsilon}\right] \\
&\quad= o(n^{-p/2}) = o(\mathbf{P}(\tau_x > n)), \qquad (3.6)
\end{aligned}$$

where we have used (3.2) in the last step.

Let

$$C_n := \{\nu_n \le n^{1-\epsilon}, M(\nu_n) \le \theta_n\sqrt{n}\}.$$

It holds

$$\sup_{t\in[0,1]}\left|X^{(\nu_n,n)}_{x+S(\nu_n)}(t) - X^{(n)}(t)\right| = \frac{\max_{k\le\nu_n}|S(\nu_n)-S(k)|}{\sqrt{n}} \le \frac{2M(\nu_n)}{\sqrt{n}}.$$

On the set $\{M(\nu_n) \le \theta_n\sqrt{n}\}$ this is smaller than $2\theta_n$. Then, as $n \to \infty$,

$$|f(X^{(n)}) - f(X^{(\nu_n,n)}_{x+S(\nu_n)})| = o(1), \text{ uniformly on } C_n,$$

since $\theta_n \to 0, n \to \infty$. From this estimate we infer that

$$\mathbf{E}\left[f\left(X^{(n)}\right), C_n, \tau_x > n\right] = \mathbf{E}\left[f\left(X^{(\nu_n,n)}_{x+S(\nu_n)}\right), C_n, \tau_x > n\right] + o(\mathbf{P}(\tau_x > n)).$$

Then, taking into account (3.4) and (3.6), we have

$$\mathbf{E}\left[f\left(X^{(n)}\right), \tau_x > n\right] = \mathbf{E}\left[f\left(X^{(\nu_n,n)}_{x+S(\nu_n)}\right), C_n, \tau_x > n\right] + o(\mathbf{P}(\tau_x > n)). \qquad (3.7)$$

Thus, to prove the theorem, it suffices to consider the expectation on the right hand side of the above equation. Note first that, by the strong Markov property

$$\begin{aligned}
&\mathbf{E}\left[f\left(X^{(\nu_n,n)}_{x+S(\nu_n)}\right), C_n, \tau_x > n\right] \\
&= \sum_{k\le n^{1-\varepsilon}}\int_{K_{n,\varepsilon}} \mathbf{P}(\nu_n = k, \tau_x > k, M(k) \le \theta_n\sqrt{n}, x+S(k) \in dz) \\
&\qquad\qquad \times \mathbf{E}\left[f\left(X^{(k,n)}_z\right), \tau_z > n-k\right].
\end{aligned}$$

We now note that it is sufficient to show that, uniformly in $z \in K_{n,\varepsilon}$, $k \le n^{1-\varepsilon}$, such that $|z| \le \theta_n \sqrt{n}$,

$$\mathbf{E}\left[f\left(X_z^{(k,n)}\right), \tau_z > n-k\right] = (\varkappa + o(1))\mathbf{E}[f(M_K)]\frac{u(z)}{n^{p/2}}. \tag{3.8}$$

Indeed, (3.8) implies that

$$\mathbf{E}\left[f\left(X_{x+S(\nu_n)}^{(\nu_n,n)}\right), C_n, \tau_x > n\right] \sim \mathbf{E}[f(M_K)]\frac{\varkappa}{n^{p/2}}\mathbf{E}\left[u(x+S(\nu_n)), C_n, \tau_x > \nu_n\right].$$

It follows from (3.5) and Lemma 21 in Denisov and Wachtel [2015b] that

$$\mathbf{E}\left[u(x+S(\nu_n)), C_n, \tau_x > \nu_n\right] \sim \mathbf{E}\left[u(x+S(\nu_n)), \nu_n \le n^{1-\epsilon}, \tau_x > \nu_n\right] \sim V(x). \tag{3.9}$$

Now we infer from (3.7) that

$$\mathbf{E}\left[f\left(X^{(n)}\right), \tau_x > n\right] \sim \mathbf{E}[f(M_K)]\varkappa\frac{V(x)}{n^{p/2}} \sim \mathbf{E}[f(M_K)]\mathbf{P}(\tau_x > n).$$

In other words,

$$\mathbf{E}\left[f\left(X^{(n)}\right)\Big|\tau_x > n\right] \sim \mathbf{E}[f(M_K)].$$

In order to prove (3.8) we first set

$$A_n := \left\{\sup_{u \le n}|S([u]) - B(u)| \le n^{1/2-\gamma}\right\},$$

and use Proposition 3 in subsection 2.4. We have

$$\mathbf{E}\left[f\left(X_z^{(k,n)}\right), \tau_z > n-k\right] = \mathbf{E}\left[f\left(X_z^{(k,n)}\right), A_n, \tau_z > n-k\right] + O(n^{-r}).$$

One uses (2.3) from Lemma 1, to show that for all $\epsilon > 0$ sufficiently small and all $y \in K_{n,\epsilon}$ with $|y| \le \sqrt{n}$

$$n^{-r} = o(u(y)n^{-\frac{p}{2}}).$$

It follows for all $\varepsilon > 0$ sufficiently small,

$$\frac{n^{-r}}{\mathbf{P}(\tau_z > n)} \to 0,$$

uniformly in $z \in K_{n,\varepsilon}$. Consequently,

$$\begin{aligned}\mathbf{E}&\left[f\left(X_z^{(k,n)}\right), \tau_z > n-k\right] \\ &= \mathbf{E}\left[f\left(X_z^{(k,n)}\right), A_n, \tau_z > n-k\right] + o(\mathbf{P}(\tau_z > n)).\end{aligned} \tag{3.10}$$

For every $z \in K_{n,\varepsilon}$ we define

$$z^{\pm} = z \pm R_0 n^{1/2-\gamma}x_0,$$

where x_0 is such that $|x_0| = 1$, $x_0 + K \subset K$ and R_0 is such that

$$\mathrm{dist}(R_0 x_0 + K, \partial K) > 1.$$

Note also that this choice of R_0 ensures that $R_0 n^{1/2-\gamma} x_0 \subset K_{n,\gamma}$. Therefore, if we take $\varepsilon < \gamma$ then $z^{\pm} \in K_{n,\gamma}$ for all $n \geq n(\epsilon, \gamma)$ large enough.

Define

$$B_z^{(k,n)}(t) := \frac{z + (B(nt) - B(k))\mathbf{1}\{t > k/n\}}{\sqrt{n}}, \quad t \geq 0.$$

From the coupling described above we have the following relations

$$\{\tau_{z^-}^{bm} > n - k\} \cap A_n \subset \{\tau_z > n - k\} \cap A_n \subset \{\tau_{z^+}^{bm} > n - k\} \cap A_n$$

and

$$\left| f\left(X_z^{(k,n)}\right) - f\left(B_{z^\pm}^{(k,n)}\right)\right| = o(1) \quad \text{uniformly on } A_n.$$

Moreover, the latter relation is uniform in $k \leq n^{1-\varepsilon}$, $z \in K_{n,\varepsilon}$. Combining these relations with (3.10), we obtain

$$\begin{aligned} &\mathbf{E}\left[f\left(B_{z^-}^{(k,n)}\right), \tau_{z^-}^{bm} > n\right] + o(\mathbf{P}(\tau_z > n)) \\ &\quad \leq \mathbf{E}\left[f\left(X_z^{(k,n)}\right), \tau_z > n - k\right] + o(\mathbf{P}(\tau_z > n)) \\ &\qquad \leq \mathbf{E}\left[f\left(B_{z^+}^{(k,n)}\right), \tau_{z^+}^{bm} > n - n^{1-\varepsilon}\right] + o(\mathbf{P}(\tau_z > n)), \end{aligned} \tag{3.11}$$

uniformly in $k \leq n^{1-\varepsilon}$, $z \in K_{n,\varepsilon}$.

Clearly, $z^{\pm}$ satisfy (2.22) for any $z \in K_{n,\varepsilon}$. Then, applying the results from Subsection 2.2 and using the Brownian scaling, we obtain

$$\max_{k \leq n^{1-\varepsilon}} \sup_{z \in K_{n,\varepsilon}: |z| \leq \theta_n \sqrt{n}} \left| \mathbf{E}\left[f\left(B_{z^+}^{(k,n)}\right) \middle| \tau_{z^+}^{bm} > n - n^{1-\varepsilon}\right] - \mathbf{E}[f(M_K)]\right| \to 0$$

and

$$\max_{k \leq n^{1-\varepsilon}} \sup_{z \in K_{n,\varepsilon}: |z| \leq \theta_n \sqrt{n}} \left| \mathbf{E}\left[f\left(B_{z^-}^{(k,n)}\right) \middle| \tau_{z^-}^{bm} > n\right] - \mathbf{E}[f(M_K)]\right| \to 0.$$

According to Propositions 1 and 5,

$$\mathbf{P}(\tau_{z^+}^{bm} > n - n^{1-\varepsilon}) \sim \mathbf{P}(\tau_{z^-}^{bm} > n) \sim \varkappa \frac{u(z)}{n^{p/2}}$$

and

$$\mathbf{P}(\tau_z > n) \sim \varkappa \frac{u(z)}{n^{p/2}}$$

uniformly in $z \in K_{n,\varepsilon}$ with $|z| \leq \theta_n \sqrt{n}$.

Combining these relations with (3.11), we arrive at (3.8). This completes the proof of the theorem.

3.2.2 Proof of convergence of h-transforms

In this subsection we prove Theorem 2.

Most of the proof follows the same steps for both cases A and B. We note below specifically where the assumptions in A, B play a role.

Take a uniformly continuous and bounded functional in $(D[0,1], ||\cdot||_\infty)$ with $0 \le f \le 1$. We want to show that

$$\mathbf{E}_x^{(V)}[f(X^{(n)})] \to \mathbf{E}_0^{(u)}[f(B)] \quad \text{as } n \to \infty.$$

We first note that

$$\mathbf{E}_x^{(V)}[f(X^{(n)}), \nu_n > n^{1-\varepsilon}] \le \frac{1}{V(x)} \mathbf{E}[V(x+S(n)), \tau_x > n, \nu_n > n^{1-\varepsilon}].$$

By the Markov property and the harmonicity of V we get

$$\begin{aligned}
&\mathbf{E}[V(x+S(n)), \tau_x > n, \nu_n > n^{1-\epsilon}] \\
&\qquad = \mathbf{E}[\mathbf{E}[V(x+S(n))\mathbf{1}\{\tau_x > n\}|\mathcal{F}_{n^{1-\varepsilon}}], \nu_n > n^{1-\varepsilon}] \\
&\qquad = \mathbf{E}[V(x+S(n^{1-\varepsilon})), \tau_x > n^{1-\varepsilon}, \nu_n > n^{1-\varepsilon}].
\end{aligned}$$

We want to show

$$\mathbf{E}_x^{(V)}[f(X^{(n)}), \nu_n > n^{1-\varepsilon}] = o(1), \quad n \to \infty. \tag{3.12}$$

Assume first the moment condition in A. It is immediate from (2.30) that

$$V(x) \le c|x|^p, \quad x \in K, |x| \ge 1.$$

Applying the Hölder inequality and using (3.3), we obtain

$$\begin{aligned}
&\mathbf{E}[V(x+S(n^{1-\varepsilon})), \tau_x > n^{1-\varepsilon}, \nu_n > n^{1-\varepsilon}] \\
&\qquad \le \left(\mathbf{E}[|x+S(n^{1-\varepsilon})|^{\alpha+\epsilon_0}]\right)^{\frac{p}{\alpha+\epsilon_0}} \left(\mathbf{P}(\tau_x > n^{1-\varepsilon}, \nu_n > n^{1-\varepsilon})\right)^{\frac{\epsilon_0}{p+\epsilon_0}} \\
&\qquad \le C(x)e^{-cn^{\varepsilon}}.
\end{aligned}$$

This establishes (3.12) for the case of A.

Assume now the extendability condition in B. From estimates (22) and (24) found in the proof of Lemma 12 in Denisov and Wachtel [2015b] one concludes that the harmonic function $V(x)$ satisfies

$$V(x) \le u(x) + C|x|^{p-\delta}, \quad x \in K, |x| \ge 1 \tag{3.13}$$

for some $\delta > 0$ small enough. By Lemma 16 of Denisov and Wachtel [2015b] we have

$$\mathbf{E}[u(x+S(n^{1-\varepsilon})), \tau_x > n^{1-\varepsilon}, \nu_n > n^{1-\epsilon}] \le C(x)e^{-cn^{\varepsilon}}.$$

Furthermore, applying the Hölder inequality and using (3.3) once again, we have

$$\begin{aligned}
&\mathbf{E}[|x+S(n^{1-\varepsilon})|^{p-\delta}, \tau_x > n^{1-\varepsilon}, \nu_n > n^{1-\epsilon}] \\
&\quad \le C\left(\mathbf{E}[|x+S(n^{1-\varepsilon})|^{p}]\right)^{\frac{p-\delta}{p}} \left(\mathbf{P}(\tau_x > n^{1-\varepsilon}, \nu_n > n^{1-\varepsilon})\right)^{\frac{\delta}{p}} \\
&\quad \le C(x) e^{-cn^{\varepsilon}}.
\end{aligned}$$

This establishes (3.12) for the case of B.

Using the Markov property and the harmonicity of V once again, we get

$$\begin{aligned}
&\mathbf{E}_x^{(V)}[f(X^{(n)}), \nu_n \le n^{1-\varepsilon}, M(\nu_n) > \theta_n\sqrt{n}] \\
&\quad \le \frac{1}{V(x)}\mathbf{E}[V(x+S(n)), \nu_n \le n^{1-\varepsilon}, M(\nu_n) > \theta_n\sqrt{n}, \tau_x > n] \\
&\quad \le \frac{1}{V(x)}\mathbf{E}[V(x+S(\nu_n)), \nu_n \le n^{1-\varepsilon}, M(\nu_n) > \theta_n\sqrt{n}, \tau_x > \nu_n].
\end{aligned}$$

Using again $V(x) \le C|x|^p$ for $|x| \ge 1$ we have

$$\begin{aligned}
&\mathbf{E}_x^{(V)}[f(X^{(n)}), \nu_n \le n^{1-\varepsilon}, M(\nu_n) > \theta_n\sqrt{n}] \\
&\quad \le \frac{C}{V(x)}\mathbf{E}[|x+S(\nu_n)|^p, \nu_n \le n^{1-\varepsilon}, M(\nu_n) > \theta_n\sqrt{n}, \tau_x > \nu_n].
\end{aligned}$$

Then, in view of (3.5),

$$\mathbf{E}_x^{(V)}[f(X^{(n)}), \nu_n \le n^{1-\varepsilon}, M(\nu_n) > \theta_n\sqrt{n}] = o(1), \quad n \to \infty. \tag{3.14}$$

Together, (3.12) and (3.14) imply

$$\mathbf{E}_x^{(V)}[f(X^{(n)})] = \mathbf{E}_x^{(V)}[f(X^{(n)}), \nu_n \le n^{1-\varepsilon}, M(\nu_n) \le \theta_n\sqrt{n}] + o(1), \quad n \to \infty.$$

Therefore, we have to look only at the convergence of

$$\mathbf{E}_x^{(V)}[f(X^{(n)}), \nu_n \le n^{1-\varepsilon}, M(\nu_n) \le \theta_n\sqrt{n}].$$

We have already seen in the proof of Theorem 1 that

$$\left| f(X^{(n)}) - f\left(X_{x+S(\nu_n)}^{(\nu_n,n)}\right)\right| = o(1) \text{ uniformly on } \{\nu_n \le n^{1-\varepsilon}, M(\nu_n) \le \theta_n\sqrt{n}\}.$$

Therefore,

$$\begin{aligned}
&\mathbf{E}_x^{(V)}[f(X^{(n)}), \nu_n \le n^{1-\varepsilon}, M(\nu_n) \le \theta_n\sqrt{n}] \\
&\quad = \mathbf{E}_x^{(V)}\left[f\left(X_{x+S(\nu_n)}^{(\nu_n,n)}\right), \nu_n \le n^{1-\varepsilon}, M(\nu_n) \le \theta_n\sqrt{n}\right] + o(1).
\end{aligned}$$

Then, by the strong Markov property,

$$\begin{aligned}
&\mathbf{E}_x^{(V)}\left[f\left(X^{(\nu_n,n)}_{x+S(\nu_n)}\right), \nu_n \le n^{1-\varepsilon}, M(\nu_n)\le \theta_n\sqrt{n}\right] \qquad (3.15)\\
&\quad = \sum_{k=1}^{n^{1-\varepsilon}} \frac{1}{V(x)}\int_{K_{n,\varepsilon}} \mathbf{P}(x+S(k)\in dz, \tau_x>k, \nu_n=k, M(k)\le\theta_n\sqrt{n})\\
&\qquad\qquad \times \mathbf{E}[f(X^{(k,n)}_z)V(z+S(n-k)), \tau_z>n-k].
\end{aligned}$$

To estimate $\mathbf{E}[f(X^{(k,n)}_z)V(z+S(n-k)), \tau_z>n-k]$ for $k\le n^{1-\epsilon}$ we shall use once again coupling arguments. We first show that

$$\mathbf{E}[V(z+S(n-k)), \tau_z>n-k, A_n^c] = o(u(z)) \qquad (3.16)$$

and

$$\mathbf{E}[u(z+B(n-k)), \tau_z^{bm}>n-k, A_n^c] = o(u(z)) \qquad (3.17)$$

uniformly for $k\le n^{1-\varepsilon}, z\in K_{n,\varepsilon}, |z|\le\theta_n\sqrt{n}$.

Fix some $\eta>0$ and split

$$\begin{aligned}
&\mathbf{E}[V(z+S(n-k)), \tau_y>n-k, A_n^c]\\
&\quad = \mathbf{E}[V(z+S(n-k)), \tau_z>n-k, |z+S(n-k)|\le n^{\frac12+\eta}, A_n^c]\\
&\qquad\qquad + \mathbf{E}[V(z+S(n-k)), \tau_z>n-k, |z+S(n-k)|> n^{\frac12+\eta}, A_n^c]\\
&\quad =: E_{n,1}+E_{n,2}.
\end{aligned}$$

Note that

$$E_{n,1}\le Cn^{\frac{p}{2}+p\eta}\mathbf{P}(A_n^c)\le Cn^{\frac{p}{2}+p\eta-r}.$$

It follows easily from Lemma 5 in Denisov and Wachtel [2019] that

$$u(z)\ge Cn^{p/2-p\varepsilon},\ \ z\in K_{n,\varepsilon}.$$

Then, choosing ε and η sufficiently small, we get

$$E_{n,1}=o(u(z)).$$

Further,

$$\begin{aligned}
E_{n,2}&\le \mathbf{E}[V(z+S(n-k)), |z+S(n-k)|>n^{\frac12+\eta}]\\
&\le C\mathbf{E}[(M(n))^p, M(n)>n^{\frac12+\eta}].
\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}
&\mathbf{E}[(M(n))^p, M(n)>n^{\frac12+\eta}]\\
&\quad = n^{p(1/2+\eta)}\mathbf{P}(M(n)>n^{1/2+\eta}) + p\int_{n^{1/2+\eta}}^{\infty} x^{p-1}\mathbf{P}(M(n)>x)dx.
\end{aligned}$$

Using now (2.28) with $y = \frac{\varepsilon}{\sqrt{d}}x$ and taking into account the moment assumption, one easily gets

$$\mathbf{E}[(M(n))^p, M(n) > n^{\frac{1}{2}+\eta}] = o(n^{p/2-p\varepsilon}) = o(u(z)).$$

This finishes the proof of (3.16). The proof of (3.17) is even simpler.

As we have seen in the proof of Theorem 1,

$$\{\tau^{bm}_{z^-} > n\} \cap A_n \subset \{\tau_z > n\} \cap A_n \subset \{\tau^{bm}_{z^+} > n\} \cap A_n.$$

Consider the event

$$D_{n,k} := \{\mathrm{dist}(z^- + B(n-k), \partial K) \ge 2n^{(1-\gamma)/2}\}.$$

Then on the event $A_n \cap D_{n,k}$ one has also

$$\mathrm{dist}(z + S(n-k), \partial K) \ge n^{(1-\gamma)/2} \quad \text{ultimately in } n.$$

We now recall (2.31). It follows from that that for all γ small enough and n large enough0

$$|V(z+S(n-k)) - u(z+S(n-k))| \le C\frac{|z+S(n-k)|^p}{n^{\gamma(1-\gamma)/2}} \quad \text{on } A_n \cap D_{n,k}.$$

Using this estimate, we obtain for all n large enough

$$\begin{aligned}
&\mathbf{E}[f(X_z^{(k,n)})V(z+S(n-k)), \tau_z > n-k, A_n] \\
&\quad \ge \mathbf{E}[f(X_z^{(k,n)})V(z+S(n-k)), \tau_z > n-k, A_n \cap D_{n,k}] \\
&\quad \ge \mathbf{E}[f(X_z^{(k,n)})u(z+S(n-k)), \tau^{bm}_{z^-} > n-k, A_n \cap D_{n,k}] \\
&\qquad - \frac{C}{n^{\gamma(1-\gamma)/2}}\mathbf{E}[|z+S(n-k)|^p, \tau^{bm}_{z^-} > n-k, A_n].
\end{aligned}$$

Note now that

$$|(z+S(n-k)) - (z^- + B(n-k))|^p \le Cn^{(1/2-\gamma)p}$$

on the event A_n. As a result,

$$\begin{aligned}
&\mathbf{E}[f(X_z^{(k,n)})V(z+S(n-k)), \tau_z > n-k, A_n] \\
&\quad \ge \mathbf{E}[f(X_z^{(k,n)})u(z+S(n-k)), \tau^{bm}_{z^-} > n-k, A_n \cap D_{n,k}] \\
&\qquad - \frac{C}{n^{\gamma(1-\gamma)/2}}\mathbf{E}[|z^- + B(n-k)|^p, \tau^{bm}_{z^-} > n-k] \\
&\qquad - \frac{Cn^{(1/2-\gamma)p}}{n^{\gamma(1-\gamma)/2}}\mathbf{P}(\tau^{bm}_{z^-} > n-k),
\end{aligned}$$

for all n large enough.

Using the scaling property of the Brownian motion and applying (2.14) in the first inequality, we get, uniformly in $k \le n^{1-\varepsilon}$ and y such that $|y| = o(\sqrt{n})$,

$$\begin{aligned}\mathbf{E}[|y+B(n-k)|^q, \tau_y^{bm} > n-k] &= (n-k)^{q/2}\mathbf{E}\left[\left|\frac{y}{\sqrt{n-k}} + B(1)\right|^q, \tau^{bm}_{\frac{y}{\sqrt{n-k}}} > 1\right] \\ &\le C(n-k)^{q/2} u\left(\frac{y}{\sqrt{n-k}}\right) \\ &\le Cn^{(q-p)/2}u(y), \quad q > 0. \end{aligned} \tag{3.18}$$

Note that for $q = 0$ (3.18) follows immediately from (2.11).

Using this inequality with $q = p$, $y = z^-$ and recalling that $\mathbf{P}(\tau^{bm}_{z^-} > n-k) \sim Cu(z^-)n^{-p/2}$, we have for all n large enough

$$\begin{aligned}&\mathbf{E}[f(X_z^{(k,n)})V(z+S(n-k)), \tau_z > n-k, A_n] \\ &\qquad \ge \mathbf{E}[f(X_z^{(k,n)})u(z+S(n-k)), \tau^{bm}_{z^-} > n-k, A_n \cap D_{n,k}] + o(u(z^-))\end{aligned}$$

uniformly in $k \le n^{1-\varepsilon}$ and $z \in K_{n,\varepsilon}$ with $|z| \le \theta_n\sqrt{n}$.

Note that Lemma 2 implies that $u(z^{\pm}) \sim u(z)$ for all $z \in K_{n,\varepsilon}$ with $|z| \le \sqrt{n}$. We finally get for all n large enough

$$\begin{aligned}&\mathbf{E}[f(X_z^{(k,n)})V(z+S(n-k)), \tau_z > n-k, A_n] \\ &\qquad \ge \mathbf{E}[f(X_z^{(k,n)})u(z+S(n-k)), \tau^{bm}_{z^-} > n-k, A_n \cap D_{n,k}] + o(u(z)).\end{aligned}$$

We next apply the relation

$$\mathbf{E}\left[|u(z+S(n-k)) - u(z^- + B(n-k))|, \tau^{bm}_{z^-} > n-k, A_n\right] = o(u(z)), \tag{3.19}$$

which will be proved later. This estimate yields for all n large enough

$$\begin{aligned}&\mathbf{E}[f(X_z^{(k,n)})V(z+S(n-k)), \tau_z > n-k, A_n] \\ &\qquad \ge \mathbf{E}[f(X_z^{(k,n)})u(z^- + B(n-k)), \tau^{bm}_{z^-} > n-k, A_n \cap D_{n,k}] + o(u(z))\end{aligned}$$

uniformly in $k \le n^{1-\varepsilon}$ and $z \in K_{n,\varepsilon}$ with $|z| \le \theta_n\sqrt{n}$.

By the Brownian scaling, for all n large enough,

$$\begin{aligned}&\mathbf{E}[u(z^- + B(n-k)), \tau^{bm}_{z^-} > n-k, D^c_{n,k}] \\ &\quad \le (n-k)^{p/2}\mathbf{P}(\tau^{bm}_{z^-} > n-k)\left[u\left(\frac{z^-}{\sqrt{n-k}} + B(1)\right), \tilde{D}^c_{n,k}\middle| \tau^{bm}_{z^-/\sqrt{n-k}} > 1\right],\end{aligned}$$

where $\tilde{D}^c_{n,k} = \{dist(\frac{z^-}{\sqrt{n-k}} + B(1), \partial K) \le 4n^{-\frac{\gamma}{2}}\}$. The expectation on the right hand side converges to zero, due to convergence towards the meander and to (2.14). Furthermore, we already know that

$$(n-k)^{p/2}\mathbf{P}(\tau^{bm}_{z^-} > n-k) \sim \varkappa u(z^-) \sim \varkappa u(z).$$

As a result,

$$\mathbf{E}[u(z^- + B(n-k)), \tau^{bm}_{z^-} > n-k, D^c_{n,k}] = o(u(z)).$$

Consequently, for all n large enough,

$$\begin{aligned}
&\mathbf{E}[f(X^{(k,n)}_z)V(z+S(n-k)), \tau_z > n-k, A_n] \\
&\quad \geq \mathbf{E}[f(X^{(k,n)}_z)u(z^- + B(n-k)), \tau^{bm}_{z^-} > n-k, A_n] + o(u(z)).
\end{aligned}$$

As we have seen in the proof of Theorem 1,

$$f(X^{(k,n)}_z) - f\left(B^{(k,n)}_{z-}\right) = o(1)$$

uniformly in $k \leq n^{1-\varepsilon}$, $z \in K_{n,\varepsilon}$ with $|z| \leq \theta_n\sqrt{n}$ and in A_n. This implies that

$$\begin{aligned}
&\mathbf{E}[f(X^{(k,n)}_z)u(z^- + B(n-k)), \tau^{bm}_{z^-} > n-k, A_n] \\
&\qquad - \mathbf{E}[f\left(B^{(k,n)}_{z^-}\right) u(z^- + B(n-k)), \tau^{bm}_{z^-} > n-k, A_n] \\
&\quad = o\left(\mathbf{E}[u(z^- + B(n-k)), \tau^{bm}_{z^-} > n-k]\right) = o(u(z)).
\end{aligned}$$

In the last step we used the harmonicity of u and the relation $u(z^-) \sim u(z)$. From this estimate and (3.17) we infer, for all n large enough,

$$\begin{aligned}
&\mathbf{E}[f(X^{(k,n)}_z)V(z+S(n-k)), \tau_z > n-k, A_n] \\
&\quad \geq \mathbf{E}\left[f\left(B^{(k,n)}_{z^-}\right) u(z^- + B(n-k)), \tau^{bm}_{z^-} > n-k, A_n\right] + o(u(z)) \\
&\quad = \mathbf{E}\left[f\left(B^{(k,n)}_{z^-}\right) u(z^- + B(n-k)), \tau^{bm}_{z^-} > n-k\right] + o(u(z)) \\
&\quad = u(z)\mathbf{E}^{(u)}_{z^-/\sqrt{n}}\left[f\left(B^{(k,n)}_{z^-}\right)\right] + o(u(z)).
\end{aligned}$$

Then, in view of convergence (2.13), for all n large enough,

$$\mathbf{E}[f(X^{(k,n)}_z)V(z+S(n-k)), \tau_z > n-k, A_n] \geq u(z)\mathbf{E}^{(u)}_0[f(B)] + o(u(z)),$$

uniformly in $k \leq n^{1-\varepsilon}$, $z \in K_{n,\varepsilon}$ with $|z| \leq \theta_n\sqrt{n}$.

By similar arguments, for all n large enough,

$$\mathbf{E}[f(X^{(k,n)}_z)V(z+S(n-k)), \tau_z > n-k, A_n] \leq u(z)\mathbf{E}^{(u)}_0[f(B)] + o(u(z)),$$

uniformly in $k \leq n^{1-\varepsilon}$, $z \in K_{n,\varepsilon}$ with $|z| \leq \theta_n\sqrt{n}$.

Combining these inequalities and (3.15), we obtain

$$\begin{aligned}
&\mathbf{E}^{(V)}_x\left[f\left(X^{(\nu_n,n)}_{x+S(\nu_n)}\right), \nu_n \leq n^{1-\varepsilon}, M(\nu_n) \leq \theta_n\sqrt{n}\right] \\
&\quad = \frac{\mathbf{E}^{(u)}_0[f(B)] + o(1)}{V(x)}\mathbf{E}\left[u(x+S(\nu_n)), \nu_n \leq n^{1-\varepsilon}, M(\nu_n) \leq \theta_n\sqrt{n}, \tau_x > \nu_n\right].
\end{aligned}$$

Taking into account (3.9), we have

$$\mathbf{E}_x^{(V)}\left[f\left(X_{x+S(\nu_n)}^{(\nu_n,n)}\right), \nu_n \le n^{1-\varepsilon}, M(\nu_n) \le \theta_n\sqrt{n}\right] = \mathbf{E}_0^{(u)}[f(B)] + o(1).$$

Combining this with (3.12) and (3.14), we finally obtain

$$\mathbf{E}_x^{(V)}[f(X^{(n)})] \to \mathbf{E}_0^{(u)}[f(B)].$$

It remains to show (3.19). It follows from Lemma 6 in Denisov and Wachtel [2019] that

$$|u(x) - u(y)| \le C|x-y|(|x|^{p-1} + |y|^{p-1}),\ x \in K.$$

This implies

$$\begin{aligned}&|u(z+S(n-k)) - u(z^- + B(n-k))| \\ &\le C\big(|z^- - z| + |B(n-k) - S(n-k)|\big)\left(|z+S(n-k)|^{p-1} + |z^- + B(n-k)|^{p-1}\right).\end{aligned}$$

From definitions of z^- and A_n, we get for $p > 1$ the bound

$$\begin{aligned}&|u(z+S(n-k)) - u(z^- + B(n-k))| \\ &\qquad \le Cn^{1/2-\gamma}\left(|z+S(n-k)|^{p-1} + |z^- + B(n-k)|^{p-1}\right) \\ &\qquad \le Cn^{p(1/2-\gamma)} + Cn^{1/2-\gamma}|z^- + B(n-k)|^{p-1}.\end{aligned}$$

Then

$$\begin{aligned}&\mathbf{E}[|u(z+S(n-k)) - u(z^- + B(n-k))|, \tau_{z^-}^{bm} > n-k, A_n] \\ &\qquad \le Cn^{p(1/2-\gamma)} + Cn^{1/2-\gamma}\mathbf{E}[|z^- + B(n-k)|^{p-1}, \tau_{z^-}^{bm} > n-k].\end{aligned}$$

Using now (3.18) with $q = p-1$, we conclude that (3.19) is proved for $p \ge 1$.

Consider now the case $p < 1$. Lemma 2 implies that if $p < 1$ then

$$|u(z+S(n-k)) - u(z^- + B(n-k))| \le Cn^{p(1/2-\gamma)} = o(n^{p/2})$$

on the event A_n. Consequently,

$$\begin{aligned}&\mathbf{E}[|u(z+S(n-k)) - u(z^- + B(n-k))|, \tau_{z^-}^{bm} > n-k, A_n] \\ &\qquad = o(n^{p/2}\mathbf{P}(\tau_{z^-}^{bm} > n-k)) = o(u(z)).\end{aligned}$$

We conclude that (3.19) is valid also for $p < 1$.

3.2.3 Proof of convergence for bridges of random walks in a cone

For every event $B \in \sigma(\{S(k), k \le nt\})$ we have

$$\begin{aligned}
&\mathbf{P}(B|x+S(n)=y,\tau_x>n)\\
&=\frac{\sum_{z\in K}\mathbf{P}(B,x+S(nt)=z,\tau_x>nt)\mathbf{P}(z+S((1-t)n)=y,\tau_z>(1-t)n)}{\mathbf{P}(x+S(n)=y,\tau_x>n)}\\
&=\mathbf{E}\left[h^{(n)}_{x,y}\left(t,X^{(n)}(t)\right)\mathbf{1}_B|\tau_x>nt\right],
\end{aligned}\tag{3.20}$$

where

$$h^{(n)}_{x,y}(t,w)=\frac{\mathbf{P}(\tau_x>nt)\mathbf{P}\left(w\sqrt{n}+S((1-t)n)=y,\tau_{w\sqrt{n}}>(1-t)n\right)}{\mathbf{P}(x+S(n)=y,\tau_x>n)}.$$

We show now that there exists a bounded, continuous function $h(t,w)$ such that

$$\sup_{w\in K}|h^{(n)}_{x,y}(t,w)-h(t,w)|\to 0. \tag{3.21}$$

Let $\widetilde{S}(k)$ denote a random walk, whose increments are independent copies of $-X(1)$. Considering the path $w\sqrt{n}+S(k),\ k\le(1-t)\sqrt{n}$ in the reversed time, we have

$$\begin{aligned}
&\mathbf{P}\left(w\sqrt{n}+S((1-t)n)=y,\tau_{w\sqrt{n}}>(1-t)n\right)\\
&\qquad=\mathbf{P}\left(y+\widetilde{S}((1-t)n)=w\sqrt{n},\widetilde{\tau}_y>(1-t)n\right).
\end{aligned}$$

Thus, applying the local limit theorem Proposition 6, we conclude that, uniformly in $w\in K$,

$$\begin{aligned}
&n^{p/2+d/2}\mathbf{P}\left(w\sqrt{n}+S((1-t)n)=y,\tau_{w\sqrt{n}}>(1-t)n\right)\\
&\qquad\qquad-\varkappa\widetilde{V}(y)H_0\frac{u(w)}{(1-t)^{p+d/2}}e^{-|w|^2/2(1-t)}\to 0,
\end{aligned}\tag{3.22}$$

where $\widetilde{V}$ is the positive harmonic function for $\widetilde{S}$ and H_0 is a norming constant.

Furthermore, combining Propositions 5 and 6, we get

$$\frac{\mathbf{P}(\tau_x>nt)}{\mathbf{P}(x+S(n)=y,\tau_x>n)}\sim C\frac{t^{-p/2}}{\widetilde{V}(y)}n^{p/2+d/2}.$$

From this estimate and (3.22) we infer that (3.21) is valid with

$$h(t,w)=Ct^{-p/2}(1-t)^{-p/2-d/2}u(w)e^{-|w|^2/2(1-t)}.$$

Let $g_t: D[0,t]\mapsto\mathbb{R}$ be bounded and continuous. It follows from (3.20) and (3.21) that

$$\mathbf{E}[g_t(X^{(n)})|x+S(n)=y,\tau_x>n]=(1+o(1))\mathbf{E}[g_t(X^{(n)})h(t,X^{(n)}(t))|\tau_x>nt].$$

Applying now Theorem 1, we finally get

$$\mathbf{E}[g_t(X^{(n)})|x+S(n)=y,\tau_x>n]\to\mathbf{E}[g_t(t^{1/2}M_K)h(t,t^{1/2}M_K(1))]. \tag{3.23}$$

In other words, we have shown convergence in distribution on $D[0,t]$ for every fixed $t<1$. Furthermore, the right hand side in (3.23) can be used to determine all finite dimensional distributions of the suggested limiting process. Thus, it remains to show tightness on the time interval $[1-\delta,1]$.

Considering the path $x+S(k),\ k\le n$ in reversed time and using the same arguments, one obtains for every bounded and continuous $q_t: D[t,1]\mapsto\mathbb{R}$ convergence

$$\begin{aligned}&\mathbf{E}[q_t(X^{(n)})|x+S(n)=y,\tau_x>n]\\&\qquad\to\mathbf{E}[q_t((1-t)^{1/2}M_K)h(1-t,(1-t)^{1/2}M_K(1))].\end{aligned} \tag{3.24}$$

As a result, we have tightness on $[1-\delta,1]$ for every $1>\delta>0$. Therefore, the proof of the weak convergence is finished.

Finally, we note that the continuity of M_K implies also that B_K^0 is continuous as well. Moreover, due to $M_K(0)=0$, we have $B_K^0(0)=B_K^0(1)=0$.

3.3 Application: Convergence of the radial part of a random walk in a cone

In this subsection we prove Corollary 1.

In view of Theorem 2 it suffices to show that $|B(t)|$ under $\mathbf{P}_0^{(u)}$ is a Markov process with the desired transition kernel.

We first note that (2.26) implies that

$$\mathbf{P}_0^{(u)}(B(1)\in dx)/dx=\varkappa u(x)e(1,x)=Cu^2(x)e^{-|x|^2/2}.$$

Here, $e(t,x)$ for $t\in(0,1],x\in K$ is the density of the Meander at time 1 (see Theorem 1.1 in Garbit [2009]). Recalling that $u(x)=|x|^p m_1(x/|x|)$, we then obtain

$$\mathbf{P}_0^{(u)}(|B(1)|\in dr)/dr=Cr^{2p+d-1}e^{-r^2/2}.$$

Finally, using the scaling property of $B(t)$, we arrive at the following expression for the entrance law

$$\mathbf{P}_0^{(u)}(|B(t)|\in dr)/dr=Ct^{-p-d/2}r^{2p+d-1}e^{-r^2/2t}. \tag{3.25}$$

Under $\mathbf{P}_0^{(u)}$, $B(t)$ is a Markov process with the following transition kernel:

$$\begin{aligned}\frac{\mathbf{P}_0^{(u)}(B(t+h)\in dz|B(t)=x)}{dz}&=\frac{u(z)}{u(x)}\frac{\mathbf{P}(x+B(h)\in dz,\tau_x^{bm}>h)}{dz}\\&=\frac{u(z)}{u(x)}b_h(x,z),\end{aligned}$$

where, according Lemma 1 in Banuelos and Smits [1997],

$$b_h(x,z) = \frac{e^{-(|x|^2+|z|^2)/2h}}{h|x|^{d/2-1}|z|^{d/2-1}} \sum_{j=1}^{\infty} I_{a_j}\left(\frac{|x||z|}{h}\right) m_j\left(\frac{x}{|x|}\right) m_j\left(\frac{z}{|z|}\right),$$

with $a_j = \sqrt{\lambda_j + (d/2-1)^2}$.

Combining this with the entrance law for $B(t)$ we have

$$\frac{\mathbf{P}_0^{(u)}(B(t+h)\in dz, B(t)\in dx)}{dxdz} = Cu(z)t^{-p-d/2}u(x)e^{-|x|^2/2t}b_h(x,z)$$

Integrating over $\{x : |x| = r_1\}$ and $\{z : |z| = r_2\}$ we obtain

$$\begin{aligned}
&\frac{\mathbf{P}_0^{(u)}(|B(t)|\in dr_1, |B(t+h)|\in dr_2)}{dr_1 dr_2} \\
&= Cr_1^{d-1}r_2^{d-1}r_2^p t^{-p-d/2} r_1^p e^{-r_1^2/2t}\frac{e^{-(r_1^2+r_2^2)/2h}}{hr_1^{d/2-1}r_2^{d/2-1}} \\
&\times \sum_{j=1}^{\infty} I_{a_j}\left(\frac{r_1r_2}{h}\right)\int_{S^{d-1}}\int_{S^{d-1}} m_1(\eta)m_1(\nu)m_j(\eta)m_j(\nu)d\eta d\nu \\
&= Cr_1^{d-1}r_2^{d-1}r_2^p t^{-p-d/2} r_1^p e^{-r_1^2/2t}\frac{e^{-(r_1^2+r_2^2)/2h}}{hr_1^{d/2-1}r_2^{d/2-1}} I_{a_1}\left(\frac{r_1r_2}{h}\right).
\end{aligned}$$

Here in the last step we have used the orthogonality of eigenfunctions $m_j, j\ge 1$.

Using now the Bayes formula, we obtain

$$\begin{aligned}
&\frac{\mathbf{P}_0^{(u)}\left(|B(t+h)|\in dr_2 \big| |B(t)| = r_1\right)}{dr_2} \\
&\qquad = C\left(\frac{r_2}{r_1}\right)^{p+d/2-1} r_2 \frac{e^{-(r_1^2+r_2^2)/2h}}{h} I_{a_1}\left(\frac{r_1r_2}{h}\right).
\end{aligned}$$

Noting that $a_1 = p + (d/2-1)$ we see that the right hand side is the transition kernel of the $(2p+d)$-dimensional Bessel process.

By similar calculations one can easily show that the process $|B(t)|$ is Markovian under $\mathbf{P}_0^{(u)}$. This finishes the proof of the Corollary.

Chapter 4

The Green function of a random walk in cones

4.1 Introduction and statement of results

The work presented in this chapter is included in Duraj and Wachtel [2018].

Consider a random walk $\{S(n), n \geq 1\}$ on $\mathbf{R}^d$, $d \geq 1$, where

$$S(n) = X(1) + \cdots + X(n)$$

and $\{X(n), n \geq 1\}$ is a family of independent copies of a random variable $X = (X_1, X_2, \ldots, X_d)$. Denote by $\mathbb{S}^{d-1}$ the unit sphere of $\mathbf{R}^d$ and Σ an open and connected subset of $\mathbb{S}^{d-1}$. Let K be the cone generated by the rays emanating from the origin and passing through Σ, i.e. $\Sigma = K \cap \mathbb{S}^{d-1}$.

Let τ_x be the exit time from K of the random walk with starting point $x \in K$, that is,

$$\tau_x = \inf\{n \geq 1 : x + S(n) \notin K\}.$$

We recall the positive harmonic function $V(x)$ for the random walk $\{S(n)\}$ killed at leaving K defined in Theorem 4. That is, V satisfies

$$V(x) = \mathbf{E}[V(x + X); \tau_x > 1], \quad x \in K.$$

In this chapter we determine the asymptotic behavior of the Green function for $\{S(n)\}$ killed at leaving K and prove, by using the Martin compactification approach, the uniqueness of the positive harmonic function for such processes.

We shall strengthen the usual geometric conditions on K from chapter 2 as follows:

- **Extendability:** We assume that there exists an open and connected set $\widetilde{\Sigma} \subset \mathbb{S}^{d-1}$ with $\mathrm{dist}(\partial\Sigma, \partial\widetilde{\Sigma}) > 0$ such that $\Sigma \subset \widetilde{\Sigma}$ and the function m_1 can be extended to $\widetilde{\Sigma}$ as a solution to (2.1).

- **Convexity and regularity:** K is convex and C^2.

Besides the usual moment conditions, we impose the following assumption on the increments of the random walk:

- **Lattice assumption:** X takes values on a lattice L which is a non-degenerate linear transformation of $\mathbb{Z}^d$.

Our first result describes the asymptotic behavior of the Green function for endpoints y which lie deep inside the cone K.

Theorem 4. *Set $r_1(p) = p + d - 2 + (2-p)^+$ and assume that $\mathbf{E}|X|^{r_1(p)} < \infty$. If $|y| \to \infty$ and dist$(y, \partial K) \geq \alpha |y|$ for some positive α, then*

$$G_K(x,y) := \sum_{n=0}^{\infty} \mathbf{P}(x + S(n) = y, \tau_x > n) \sim cV(x) \frac{u(y)}{|y|^{2p+d-2}}. \tag{4.1}$$

Moreover, this relation remains valid if one replaces the moment condition $\mathbf{E}|X|^{r_1(p)} < \infty$ by the following restriction on the local structure of X_1:

$$\mathbf{P}(X = x) \leq |x|^{-p-d+1} f(|x|) \tag{4.2}$$

for some decreasing function f such that $u^{(3-p)\vee 1} f(u) \to 0$ as $u \to \infty$.

Uchiyama [1998] has shown, see Theorem 2 there, that if $d \geq 5$ and $\mathbf{E}|X|^{d-2} < \infty$ then

$$G_{\mathbb{R}^d}(0,z) \sim \frac{c}{|z|^{d-2}}, \quad |z| \to \infty.$$

If $d = 4$ or $d = 3$ then the same is valid provided that respectively $\mathbf{E}|X|^2 \log |X| < \infty$ or $\mathbf{E}|X|^2 < \infty$.

Uchiyama mentions also that this moment condition is optimal: for every $\varepsilon > 0$ there exists a random walk satisfying $\mathbf{E}|X|^{d-2-\varepsilon} < \infty$ such that

$$\limsup_{|z| \to \infty} |z|^{d-2} G_{\mathbb{R}^d}(0,z) = \infty.$$

He has considered the dimensions 4 and 5 only, but it is quite simple to show that this statement holds in every dimension $d \geq 5$. We now give an example in our setting of a random walk which shows the optimality of Uchiyama's condition and of the moment condition in Theorem 4. Our example is just a multidimensional variation of the classical Williamson example, see Williamson [1968]. Let d be greater than 4 and consider X with the following distributon. For every $n \geq 1$ and for every basis vector e_k put

$$\mathbf{P}(X = \pm 2^n e_k) = \frac{q_n}{2d},$$

where the sequence q_n is such that

$$\sum_{n=1}^{\infty} q_n = 1 \quad \text{and} \quad q_n \sim \frac{c \log n}{2^{n(d-2)}}.$$

Clearly,

$$\mathbf{E}|X|^{d-2} = \infty \quad \text{and} \quad \mathbf{E}\frac{|X|^{d-2}}{\log^{1+\varepsilon}|X|} < \infty.$$

Using now the obvious inequality $G_{\mathbb{R}^d}(0, x) \geq \mathbf{P}(X = x)$, we conclude that, for every $j = 1, 2, \dots, d$,

$$\lim_{n\to\infty} 2^{(d-2)n} G_{\mathbb{R}^d}(0, \pm 2^n e_j) = \infty.$$

If we have a cone K such that $p \geq 2$ and $e_j \in \Sigma$ for some j, then, choosing $q_n \sim \frac{c \log n}{2^{n(p+d-2)}}$, we also have

$$\lim_{n\to\infty} 2^{(p+d-2)n} G_K(e_j, (1 + 2^n)e_j) = \infty.$$

◇

But Uchiyama shows that the moment assumption $\mathbf{E}|X|^{d-2}$ is not necessary, as it can be replaced by $\mathbf{P}(X = x) = o(|x|^{-d-2})$, which implies the existence of the second moment only. In Theorem 4 we have a similar situation: the moment condition $\mathbf{E}|X|^{r_1(p)}$ is not necessary and can be replaced by the assumption (4.2), which yields the finiteness of $\mathbf{E}|X|^{p\vee 2}$ only. It has been shown in Denisov and Wachtel [2015b], if $p > 2$ then the condition $\mathbf{E}|X|^p < \infty$ is an optimal moment condition for the existence of the harmonic function $V(x)$.

We now turn to the asymptotic behavior of the Green function along the boundary of the cone.

Theorem 5. *Assume that K is convex and C^2. Assume also that $\mathbf{E}|X|^{r_1(p)+1} < \infty$. If $y/|y|$ converges to $\sigma \in \partial\Sigma$ as $|y| \to \infty$ then there exists a strictly positive function $v_\sigma(y)$ such that*

$$G_K(x, y) \sim c\frac{V(x)v_\sigma(dist(y, \partial K))}{|y|^{p+d-1}}. \tag{4.3}$$

The function v_σ is asymptotically linear, that is,

$$v_\sigma(y) \sim c_\sigma |y| \quad \text{as } y \to \infty.$$

Moreover, the same relation for G_K holds under the assumption (4.2).

Clearly, one can adapt the random walk from Example 4.1 to show that the moment assumption in Theorem 5 is minimal as well. Indeed, it suffices to take $q_n \sim \frac{c \log n}{2^{n(p+d-1)}}$ and to assume that one of the vectors $\pm e_j$ belongs to the boundary of the cone K.

Theorems 4 and 5 describe the asymptotic behavior of $G_k(x, y)$ along all possible directions inside the cone K. Combining these two results, we conclude that, for all $x, x' \in K$,

$$\frac{G(x, y)}{G(x', y)} \to \frac{V(x)}{V(x')} \quad \text{as } |y| \to \infty.$$

As a result we have the following Corollary.

Corollary 2. *Assume that the assumptions of Theorem 5 are valid. Then the function $V(x)$ is the unique, up to mutiplicative factor, positive harmonic function for $\{S(n)\}$ killed at leaving K.*

Doney [1998] has shown that the harmonic function for any one-dimensional oscillating random walk killed at leaving the positive half-axis is unique without any additional moment assumption.

For multidimensional cones much less is known. Raschel [2014] has shown the uniqueness of the positive harmonic function for random walks with small steps killed at leaving the positive quadrant $\mathbb{Z}_+^2$ and some particular cases of such random walks have been studied by the same author in Raschel [2010] and Raschel [2011]. The approach in Raschel [2014] is based on a functional equation which is satisfied by all harmonic functions. It should be also mentioned that Raschel [2010] and Raschel [2011] describe actually the asymptotic behavior of the Green function and the uniqueness of the harmonic function is just a consequence of the results on the Green function.

Uchiyama [2014] derives asymptotics for the Green function for random walks in $\mathbb{Z}^{d-1} \times \mathbb{Z}_+$, see Theorem 5. He assumes that $\mathbf{E}|X|^d$ is finite. This is slightly weaker than the moment assumption in Theorem 5, which reduces in the case of a half-space to $\mathbf{E}|X|^{d+1} < \infty$. On the other hand, Theorem 5 can be applied to any half-space, and Uchiyama's proof uses in a crucial way that the half-space is precisely $\mathbb{Z}^{d-1} \times \mathbb{Z}_+$, i.e. aligned with the gitter of the random walk.

Bouaziz et al. [2015] have shown uniqueness for a wide class of random walks with finite number of steps killed at leaving the orthant $\mathbb{Z}_+^d$, $d \geq 2$.

Ignatiouk-Robert [2020] has studied the properties of harmonic function for random walks on semigroups by introducing a special renewal structure. Applying this to a random walk in a cone, she has shown that, if the cone K satisfies the assumptions from Denisov and Wachtel [2015b] with some $p \leq 2$ and $\mathbf{E}|X|^\alpha$ is finite for some $\alpha > 2$, then the harmonic function is unique. From our results on the Green function one can deduce uniqueness if it is assumed that $\mathbf{E}|X|^{d+1} < \infty$. Therefore, for cones with $p \leq 2$ the result in Ignatiouk-Robert [2020] holds under much weaker restrictions on the random walk than Corollary 2. This does not provide any information on the asymptotic behavior of the Green function.

Raschel and Tarrago [2018] have derived an implicit asymptotic representation for the Green function of random walks in cones under the assumptions $p \geq 1$ and $\mathbf{E}|X|^{r_1(p)+p} < \infty$. They also show that the construction of the harmonic function $V(x)$ suggested in Denisov and Wachtel [2015b] works for all convex cones and all random walks satisfying $\mathbf{E}|X|^{r(p)+p} < \infty$. In other words, by increasing the number of finite moments of the random walk one can relax the geometric restrictions on the cone. This is the main difference between their assumptions and ours: we need less moments but we have to impose the condition that there exists a harmonic extension of $u(x)$ into a bigger cone and that the cone is C^2. The latter excludes, for example, Weyl chambers, which appear naturally in some models of killed random walks from statistical physics.

4.2 Proofs

4.2.1 Proof of the behavior of the Green function inside the cone

Proof of Theorem 4. Fix some $\varepsilon > 0$ and split $G_K(x,y)$ into two parts:

$$G_K(x,y) = \sum_{n<\varepsilon|y|^2} \mathbf{P}(x+S(n)=y, \tau_x > n) + \sum_{n\geq\varepsilon|y|^2} \mathbf{P}(x+S(n)=y, \tau_x > n)$$
$$=: S_1(x,y,\varepsilon) + S_2(x,y,\varepsilon).$$

By Proposition 6,

$$n^{p/2+d/2}\mathbf{P}(x+S(n)=y, \tau_x > n) = C_0 V(x) u\left(\frac{y}{\sqrt{n}}\right) e^{-|y|^2/2n} + o(1)$$

uniformly in $y \in K$. Consequently, as $|y| \to \infty$,

$$S_2(x,y,\varepsilon) = C_0 V(x) \sum_{n\geq\varepsilon|y|^2} \frac{1}{n^{p/2+d/2}} u\left(\frac{y}{\sqrt{n}}\right) e^{-|y|^2/2n} + o\left(\sum_{n\geq\varepsilon|y|^2} \frac{1}{n^{p/2+d/2}}\right)$$
$$= C_0 V(x) u(y) \sum_{n\geq\varepsilon|y|^2} \frac{1}{n^{p+d/2}} e^{-|y|^2/2n} + o\left(|y|^{-p-d+2}\right)$$
$$= C_0 V(x) u(y) |y|^{-2p-d+2} \int_\varepsilon^\infty z^{-p-d/2} e^{-1/(2z)} dz + o\left(|y|^{-p-d+2}\right).$$

Letting here $\varepsilon \to 0$ and recalling that $u(y) \geq c(\alpha)|y|^p$ for $dist(y,\partial K) \geq \alpha|y|$, we obtain

$$\lim_{\varepsilon\to 0} \lim_{|y|\to\infty} \frac{|y|^{2p+d-2}}{u(y)} S_2(x,y,\varepsilon) = C_0 V(x) \int_0^\infty z^{-p-d/2} e^{-1/(2z)} dz. \tag{4.4}$$

Therefore, it remains to show that

$$\lim_{\varepsilon\to 0} \limsup_{|y|\to\infty} \frac{|y|^{2p+d-2}}{u(y)} S_1(x,y,\varepsilon) = 0. \tag{4.5}$$

Fix additionally some small $\delta > 0$ and define

$$\theta_y := \inf\{n \geq 1 : x + S(n) \in B_{\delta,y}\},$$

where $B_{\delta,y}$ denotes the ball of radius $\delta|y|$ around point y.

Then we have

$$\begin{aligned}
S_1(x,y,\varepsilon) &= \sum_{n<\varepsilon|y^2|} \mathbf{P}(x+S(n)=y, \tau_x>n\ge\theta_y) \\
&= \sum_{n<\varepsilon|y^2|}\sum_{k=1}^{n}\sum_{z\in B_{\delta,y}} \mathbf{P}(x+S(k)=z,\tau_x>k=\theta_y)\mathbf{P}(z+S(n-k)=y,\tau_z>n-k) \\
&\le \sum_{k<\varepsilon|y|^2}\sum_{z\in B_{\delta,y}} \mathbf{P}(x+S(k)=z,\tau_x>k=\theta_y)\sum_{j<\varepsilon|y|^2-k}\mathbf{P}(z+S(j)=y) \\
&\le \mathbf{E}\left[G^{(\varepsilon|y|^2)}(y-x-S(\theta_y));\tau_x>\theta_y,\theta_y\le\varepsilon|y|^2\right], \qquad (4.6)
\end{aligned}$$

where

$$G^{(t)}(z) := \sum_{n<t}\mathbf{P}(S(n)=z).$$

We focus first on the case $d\ge 3$. Then, according to Theorem 2 in Uchiyama [1998],

$$G(z) := G^{(\infty)}(z) \le \frac{C}{1+|z|^{d-2}}, \quad z\in\mathbb{Z}^d, \qquad (4.7)$$

provided that $\mathbf{E}|X_1|^{s_d}<\infty$, where $s_d=2+\varepsilon$ for $d=3,4$ and $s_d=d-2$ for $d\ge 5$. Since $r_1(p)>s_d$, (4.7) yields

$$\begin{aligned}
&S_1(x,y,\varepsilon) \\
&\le C\mathbf{E}\left[\frac{1}{1+|y-x-S(\theta_y)|^{d-2}};\tau_x>\theta_y,\theta_y\le\varepsilon|y|^2\right] \qquad (4.8)\\
&\le C\mathbf{P}(|y-x-S(\theta_y)|\le\delta^2|y|,\tau_x>\theta_y,\theta_y\le\varepsilon|y|^2)+\frac{C(\delta)}{|y|^{d-2}}\mathbf{P}(\tau_x>\theta_y,\theta_y\le\varepsilon|y|^2).
\end{aligned}$$

Noting now that $|y-x-S(\theta_y)|\le\delta^2|y|$ yields $|X(\theta_y)|>\delta(1-\delta)|y|$ and using our moment assumption, we conclude that

$$\begin{aligned}
&\mathbf{P}(|y-x-S(\theta_y)|\le\delta^2|y|,\tau_x>\theta_y,\theta_y<\varepsilon|y|^2) \\
&\quad\le \sum_{k<\varepsilon|y|^2}\mathbf{P}(|X(k)|>\delta(1-\delta)|y|,\tau_x>k=\theta_y) \\
&\quad\le \mathbf{P}(|X|>\delta(1-\delta)|y|)\sum_{k<\varepsilon|y|^2}\mathbf{P}(\tau_x>k-1) \\
&\quad= o\left(|y|^{-r_1(p)}\mathbf{E}[\tau_x;\tau_x<|y|^2]\right) = o\left(|y|^{-d-p+2}\right). \qquad (4.9)
\end{aligned}$$

Recalling that V is harmonic for $S(n)$ killed at leaving K, we obtain

$$\begin{aligned}
&\mathbf{P}(\tau_x > \theta_y, \theta_y < \varepsilon|y|^2)\\
&\quad= \sum_{k<\varepsilon|y|^2} \sum_{z:|z-y|\le\delta|y|} \mathbf{P}(\tau_x > k, \theta_y = k, x+S(k)=z)\\
&\quad= \sum_{k<\varepsilon|y|^2} \sum_{z:|z-y|\le\delta|y|} \frac{V(x)}{V(z)}\mathbf{P}^{(V)}(\theta_y = k, x+S(k)=z)\\
&\quad\le \frac{V(x)}{\min_{\{z:|z-y|\le\delta|y|\}} V(z)}\mathbf{P}^{(V)}(\theta_y < \varepsilon|y|^2).
\end{aligned}$$

It follows from the assumption $dist(y,\partial K)\ge\alpha|y|$ and (2.31), that

$$\min_{\{z:|z-y|\le\delta|y|\}} V(z) \ge C|y|^p$$

for all δ sufficiently small. As a result,

$$|y|^p\mathbf{P}(\tau_x > \theta_y, \theta_y < \varepsilon|y|^2) \le C(x)\mathbf{P}^{(V)}\left(\max_{n<\varepsilon|y|^2}|x+S(n)| > (1-\delta)|y|\right).$$

Applying now the functional limit theorem for $S(n)$ under $\mathbf{P}^{(V)}$, see Theorem 2 and Remark 2, we conclude that

$$\lim_{\varepsilon\to 0}\limsup_{|y|\to\infty}|y|^p\mathbf{P}(\tau_x > \theta_y, \theta_y < \varepsilon|y|^2) = 0. \tag{4.10}$$

Note that the functional limit theorems from Duraj and Wachtel [2020] only require $p\vee(2+\epsilon)$-moments.

Combining (4.8)–(4.10), we infer that (4.5) is valid under the assumption $\mathbf{E}|X_1|^{r_1(p)}<\infty$ in all dimensions $d\ge 3$.

Assume now that (4.2) holds. It is clear that this restriction implies that $\mathbf{E}|X_1|^p<\infty$. Therefore, Proposition 6 is still applicable and (4.4) remains valid for all random walks satisfying (4.2). In order to show that (4.5) remains valid as well, we notice that

$$\begin{aligned}
S_1(x,y,\varepsilon) \le C\mathbf{E}&\left[\frac{1}{1+|y-x-S(\theta_y)|^{d-2}}; |y-x-S(\theta_y)|\le\delta^2|y|, \tau_x>\theta_y, \theta_y\le\varepsilon|y|^2\right]\\
&+\frac{C(\delta)}{|y|^{d-2}}\mathbf{P}(\tau_x>\theta_y, \theta_y\le\varepsilon|y|^2).
\end{aligned}$$

In view of (4.10), we have to estimate the first term on the right hand side only. For every z such that $|z-y|\le\delta^2|y|$ we have

$$\begin{aligned}
&\mathbf{P}(x+S(\theta_y)=z, \tau_x>\theta_y, \theta_y\le\varepsilon|y|^2)\\
&\quad\le \sum_{k=1}^{\varepsilon|y|^2}\sum_{z'\in K\setminus B_{\delta,y}}\mathbf{P}(x+S(k-1)=z', \tau_x>k-1)\mathbf{P}(X_k=z-z').
\end{aligned}$$

Since $|z-z'| > \delta(1-\delta)|y|$, we infer from (4.2) that

$$\begin{aligned}
&\mathbf{P}(x+S(\theta_y)=z,\tau_x>\theta_y,\theta_y\le\varepsilon|y|^2)\\
&\qquad\le C(\delta)|y|^{-p-d+1}f(\delta(1-\delta)|y|)\sum_{k=1}^{\varepsilon|y|^2}\mathbf{P}(\tau_x>k-1)\\
&\qquad\le C(\delta)|y|^{-p-d+1}f(\delta(1-\delta)|y|)\mathbf{E}[\tau_x;\tau_x<|y|^2]. \qquad (4.11)
\end{aligned}$$

Here and in the following we use that $\mathbf{E}[\tau_x;\tau_x<|y|^2]\sim C|y|^{-p+2}$ if $p\le 2$.

For every natural m there are $O(m^{d-1})$ lattice points z such that $|z-y|\in(m,m+1]$. Then, using (4.11), we obtain

$$\begin{aligned}
&\mathbf{E}\left[\frac{1}{1+|y-x-S(\theta_y)|^{d-2}};|y-x-S(\theta_y)|\le\delta^2|y|,\tau_x>\theta_y,\theta_y\le\varepsilon|y|^2\right]\\
&\qquad\le C(\delta)|y|^{-p-d+1}f(\delta(1-\delta)|y|)\mathbf{E}[\tau_x;\tau_x<|y|^2]\sum_{m=1}^{\delta^2|y|}\frac{m^{d-1}}{1+m^{d-2}}\\
&\qquad\le C(\delta)|y|^{-p-d+3}f(\delta(1-\delta)|y|)\mathbf{E}[\tau_x;\tau_x<|y|^2].
\end{aligned}$$

Recalling that $u^{(3-p)\vee 1}f(u)\to 0$, we conclude that

$$\begin{aligned}
&\mathbf{E}\left[\frac{1}{1+|y-x-S(\theta_y)|^{d-2}};|y-x-S(\theta_y)|\le\delta^2|y|,\tau_x>\theta_y,\theta_y\le\varepsilon|y|^2\right]\\
&\qquad=o(|y|^{-p-d+2}).
\end{aligned}$$

This completes the proof of the theorem for $d\ge 3$.

We now focus on $d=2$. In this case we can not use the full Green function. We will obtain bounds for $G^{(t)}(x)$ directly from the local limit theorem for unrestricted walks. More precisely, we shall use Propositions 9 and 10 from Chapter 2 in Spitzer's book Spitzer [1976], which say that

$$\mathbf{P}(S_n=z)=\frac{1}{2\pi n}e^{-|z|^2/2n}+\frac{\rho(n,z)}{|z|^2\vee n},\quad \text{as } n\to\infty \qquad (4.12)$$

where

$$\sup_{z\in\mathbb{Z}^2}\rho(n,z)\to 0,\quad \text{as } n\to\infty.$$

This asymptotic representation implies that

$$\sup_{z\in\mathbb{Z}^2}G^{(t)}(z)\le C\log t,\quad t\ge 2. \qquad (4.13)$$

Furthermore, for $|z|\to\infty$ and $t\le a|z|^2$ one has

$$G^{(t)}(z)\le\sum_{n=1}^{a|z|^2}\frac{1}{2\pi n}e^{-|z|^2/2n}+o(1)=\frac{1}{2\pi}\int_0^a\frac{1}{v}e^{-1/2v}dv+o(1).$$

As a result,

$$\sup_{z\in\mathbb{Z}^2} G^{(a|z|^2)}(z) \le C(a) < \infty. \tag{4.14}$$

Using (4.13) and (4.14), we obtain

$$S_1(x,y,\varepsilon) \le C\log|y|\mathbf{P}(|y-x-S(\theta_y)| \le \delta^2|y|, \tau_x > \theta_y, \theta_y \le \varepsilon|y|^2) \\ + C(\varepsilon)\mathbf{P}(\tau_x > \theta_y, \theta_y \le \varepsilon|y|^2)$$

According to (4.9),

$$\mathbf{P}(|y-x-S(\theta_y)| \le \delta^2|y|, \tau_x > \theta_y, \theta_y \le \varepsilon|y|^2) = o(|y|^{-r_1(p)}\mathbf{E}[\tau_x; \tau_x < |y|^2]) \\ = o(|y|^{-p}/\log|y|).$$

Combining this with (4.10), we conclude that (4.5) holds for $d = 2$. □

4.3 Random walks in a half-space

In this section we shall consider a particular cone

$$K = \left\{x \in \mathbb{R}^d : x_d > 0\right\}.$$

Since the rotations of the space do not affect our moment assumptions, the results of this section remain valid for any half-space in $\mathbb{R}^d$.

For this very particular cone we have

- $u(x) = x_d$;
- $\tau_x = \inf\{n \ge 1 : x_d + S_d(n) \le 0\}$;
- $V(x)$ depends on x_d only and is proportional to the renewal function of ladder heights of the random walk $\{S_d(n)\}$.

In other words, the exit problem from K is actually a one-dimensional problem. This allows the use of exiting results for one-dimensional walks. As a result we obtain the asymptotic behavior of the Green function.

Theorem 6. *Assume that $\mathbf{E}|X|^{d+1} < \infty$. Assume also that $x = (0,\dots,0,x_d)$ with $x_d = o(|y|)$. Then*

$$G_K(x,y) \sim c\frac{V(x)v'(y)}{|y|^d}.$$

Here, V' is the harmonic function for the killed reversed random walk $\{-S(n)\}$.

The proof of this result is based on the following simple generalization of known results for cones.

Lemma 3. *Assume that* $\mathbf{E}|X|^{2+\delta}$ *is finite. Then, uniformly in* $x_d = o(\sqrt{n})$,

(a) $\mathbf{P}(\tau_x > n) \sim V(x)n^{-1/2}$;

(b) $\{\frac{x+S([nt])}{\sqrt{n}}\}$, $t \in [0,1]$ *conditioned on* $\{\tau_x > n\}$ *converges weakly to the Brownian meander in* K;

(c) $\sup_{y\in K}\left|n^{1/2+d/2}\mathbf{P}(x+S(n)=y;\tau_x>n) - cV(x)\frac{y_d}{\sqrt{n}}e^{-|y|^2/2n}\right| \to 0.$

Proof. The first statement is the well-known result for one-dimensional random walks. The second and the third statements for fixed starting points x have been proved in Duraj and Wachtel [2020] and in Denisov and Wachtel [2015b] respectively; the former is Theorem 1 the latter is also contained in Proposition 6. To consider the case of growing x_d one has to make only one change: Lemma 24 from Denisov and Wachtel [2015b] should be replaced by the estimate

$$\lim_{n\to\infty}\frac{1}{V(x)}\mathbf{E}\left[|x+S(\nu_n)|;\tau_x>\nu_n, |x+S(\nu_n)|>\theta_n\sqrt{n}, \nu_n \le n^{1-\varepsilon}\right] = 0$$

uniformly in $x_d \le \theta_n\sqrt{n}/2$. If $x_d \ge n^{1/2-\varepsilon}$ then $\nu_n = 0$ and the expectation equals zero. If $x_d \le n^{1/2-\varepsilon}$ then one repeats the proof of Lemma 24 in Denisov and Wachtel [2015b] with p replaced by 1 and uses the part (a) of the proposition to obtain a uniform in x_d estimate for the sum $\sum_{j\le n^{1-\varepsilon}}\mathbf{P}(\tau_x > j-1)$. (In Denisov and Wachtel [2015b], the Markov inequality has been used, since one does not have the statement (a) in general cones.) $\square$

Lemma 4. *Uniformly in* y *with* $y_d = o(\sqrt{n})$,

$$\mathbf{P}(x+S(n)=y,\tau_x>n) \sim c\frac{V(x)v'(y)}{n^{1+d/2}}e^{-|y|^2/2n}.$$

Proof. Set $m = [\frac{n}{2}]$ and write

$$\begin{aligned}\mathbf{P}(x&+S(n)=y,\tau_x>n)\\ &=\sum_{z\in K}\mathbf{P}(x+S(n-m)=z,\tau_x>n-m)\mathbf{P}(z+S(m)=y,\tau_z>m).\end{aligned}$$

It holds due to part (c) of Lemma 3 that

$$\begin{aligned}\Sigma_1(A,n) :&= \sum_{z\in K:|z|>A\sqrt{n}}\mathbf{P}(x+S(n-m)=z,\tau_x>n-m)\mathbf{P}(z+S(m)=y,\tau_z>m)\\ &= Cy_d n^{-1-\frac{d}{2}}e^{-|y|^2/2n}\mathbf{E}[V(x+S(n-m)),\tau_x>n-m, |x+S(n-m)|>A\sqrt{n}]\\ &= Cy_d n^{-1-\frac{d}{2}}e^{-|y|^2/2n}V(x)\left(1-\mathbf{P}^V_x\left(\left|\frac{x+S(n-m)}{\sqrt{n}}\right|\le A\right)\right)\end{aligned}$$

It then follows from part (b) of Lemma 3 that

$$\lim_{A\to\infty}\lim_{n\to\infty}\frac{n^{1+\frac{d}{2}}}{V(x)y_d}\Sigma_1(A,n)=0.$$

It remains to show that for

$$\Sigma_2(A,n):=\sum_{z\in K:|z|\le A\sqrt{n}}\mathbf{P}(x+S(n-m)=z,\tau_x>n-m)\mathbf{P}(z+S(m)=y,\tau_y>m)$$

it holds

$$\lim_{A\to\infty}\lim_{n\to\infty}\frac{n^{1+\frac{d}{2}}}{V(x)y_d}\Sigma_2(A,n)=c>0.$$

Again with help of part (c) in Lemma 3 it follows

$$\begin{aligned}\Sigma_2(A,n)&=Cy_dn^{-1-\frac{d}{2}}e^{-|y|^2/2n}\mathbf{E}[V(x+S(n-m)),\tau_x>n-m,|x+S(n-m)|\le A\sqrt{n}]\\&=Cy_dn^{-1-\frac{d}{2}}e^{-|y|^2/2n}V(x)\mathbf{P}_x^V\left(\left|\frac{x+S(n-m)}{\sqrt{n}}\right|\le A\right)\end{aligned}$$

and the required convergence relation for $\Sigma_2(A,n)$ follows from part (b) of Lemma 3. □

Proof of Theorem 6. If y is such that $y_d\ge\alpha|y|$ for some $\alpha>0$ then it suffices to repeat the proof of Theorem 4.

We consider then the 'boundary case' $y_d=o(|y|)$.

Using Lemma 4, one obtains easily

$$\lim_{\varepsilon\to0}\lim_{|y|\to\infty}\frac{|y|^d}{V(x)v'(y)}S_2(x,y,\varepsilon)=c.$$

Namely, it follows that

$$\begin{aligned}\lim_{\varepsilon\to0}\lim_{|y|\to\infty}\frac{|y|^d}{V(x)v'(y)}S_2(x,y,\varepsilon)&=c\lim_{\varepsilon\to0}\lim_{|y|\to\infty}\sum_{n\ge\epsilon|y|^2}|y|^dn^{-1-\frac{d}{2}}e^{-\frac{|y|^2}{2n}}\\&=c\int_0^\infty\left(\frac{1}{v}\right)^{1+\frac{d}{2}}e^{-\frac{1}{2v}}dv\end{aligned}$$

and the last integral is finite. It follows that the theorem will be proven if we show that

$$\lim_{\varepsilon\to0}\lim_{|y|\to\infty}\frac{|y|^d}{V(x)V'(y)}S_1(x,y,\varepsilon)=0. \tag{4.15}$$

Using an appropriate rotation we can reduce everything to the case $y_k=o(|y|)$ for every $k=2,\dots,d-1$ and $y_1\sim|y|$. This also implies $y_d=o(|y|)$.

We first split the probability $\mathbf{P}(x+S(n)=y,\tau_x>n)$ into two parts:

$$\begin{aligned}\mathbf{P}(x+S(n)=y,\tau_x>n) =&\mathbf{P}(x+S(n)=y,\tau_x>n,\max_{k\le n}|X_1(k)|\le\gamma y_1)\\&+\mathbf{P}(x+S(n)=y,\tau_x>n,\max_{k\le n}|X_1(k)|>\gamma y_1),\end{aligned}$$

where $\gamma\in(0,1)$. Introduce the stopping time

$$\sigma_\gamma:=\inf\{k\ge 1:|X_1(k)|>\gamma y_1\}.$$

Then, by the Markov property,

$$\begin{aligned}&\mathbf{P}(x+S(n)=y,\tau_x>n,\max_{k\le n}|X_1(k)|>\gamma y_1)\\&=\sum_{k=1}^{n}\mathbf{P}(x+S(n)=y,\tau_x>n,\sigma_\gamma=k)\\&\le\sum_{k=1}^{n}\mathbf{P}(\tau_x>k-1)\mathbf{P}(|X_1|>\gamma y_1)\max_z\mathbf{P}(S(n-k)=z).\end{aligned}$$

Using now the bounds $\mathbf{P}(\tau_x>k)\le CV(x)k^{-1/2}$ and $\max_z\mathbf{P}(S(k)=z)\le Ck^{-d/2}$, we obtain

$$\begin{aligned}&\mathbf{P}(x+S(n)=y,\tau_x>n,\max_{k\le n}|X_1(k)|>\gamma y_1)\\&\le CV(x)\mathbf{P}(|X_1|>\gamma y_1)\sum_{k=1}^{n}\frac{1}{\sqrt{k}}\frac{1}{(n-k+1)^{d/2}}\\&\le CV(x)\mathbf{P}(|X_1|>\gamma y_1)\frac{(\log n)^{1\{d=2\}}}{\sqrt{n}}.\end{aligned}$$

Here, in the last step we have split the sum $\sum_{k=1}^{n}\frac{1}{\sqrt{k}}\frac{1}{(n-k+1)^{d/2}}$ into $\sum_{k=1}^{\frac{n}{2}}$ and $\sum_{k=\frac{n}{2}}^{n}$ and used elementary inequalities.

This implies that

$$\begin{aligned}&\sum_{n=1}^{\varepsilon|y|^2}\mathbf{P}(x+S(n)=y,\tau_x>n,\max_{k\le n}|X_1(k)|>\gamma y_1)\\&\qquad\le C\sqrt{\varepsilon}V(x)\mathbf{P}(|X_1|>\gamma y_1)|y|\left(\log|y|\right)^{1\{d=2\}}.\end{aligned}$$

As a result, for all random walks satisfying

$$\mathbf{E}\left[|X|^{d+1}\left(\log|X|\right)^{1\{d=2\}}\right]<\infty,$$

we have

$$\sum_{n=1}^{\varepsilon|y|^2}\mathbf{P}(x+S(n)=y,\tau_x>n,\max_{k\le n}|X_1(k)|>\gamma y_1)=o\left(\frac{V(x)}{|y|^d}\right). \tag{4.16}$$

In order to estimate $\mathbf{P}(x+S(n)=y,\tau_x>n,\max_{k\le n}|X_1(k)|\le\gamma y_1)$ we shall perform the following change of measure:

$$\overline{\mathbf{P}}(X(k)\in dz)=\frac{e^{hz_1}}{\varphi(h)}\mathbf{P}(X(k)\in dz;|X_1(k)|\le\gamma y_1),$$

where

$$\varphi(h)=\mathbf{E}\left[e^{hX_1};|X_1|\le\gamma y_1\right].$$

Therefore,

$$\begin{aligned}\mathbf{P}(x+S(n)=y,\tau_x>n,\max_{k\le n}|X_1(k)|\le\gamma y_1)\\=e^{-hy_1}\varphi^n(h)\overline{\mathbf{P}}(x+S(n)=y,\tau_x>n).\end{aligned}\tag{4.17}$$

According to (21) in Fuk and Nagaev [1971],

$$\begin{aligned}&e^{-hy_1}\varphi^n(h)\\&\le\exp\left\{-hy_1+hn\mathbf{E}[X_1;|X_1|\le\gamma y_1]+\frac{e^{h\gamma y_1}-1-h\gamma y_1}{\gamma^2y_1^2}n\mathbf{E}[X_1^2;|X_1|\le\gamma y_1]\right\}\end{aligned}$$

Choosing

$$h=\frac{1}{\gamma y_1}\log\left(1+\frac{\gamma y_1^2}{n\mathbf{E}[X_1^2;|X_1|\le\gamma y_1]}\right)\tag{4.18}$$

and noting that

$$|\mathbf{E}[X_1;|X_1|\le\gamma y_1]|=|\mathbf{E}[X_1;|X_1|>\gamma y_1]|\le\frac{1}{\gamma y_1}\mathbf{E}[X_1^2]=\frac{1}{\gamma y_1},$$

we conclude that uniformly for $n\le\gamma|y|^2$ it holds

$$e^{-hy_1}\varphi^n(h)\le\left(\frac{en}{\gamma y_1^2}\right)^{1/\gamma}.$$

Plugging this into (4.17), we obtain uniformly for $n\le\gamma|y|^2$

$$\begin{aligned}\mathbf{P}\left(x+S(n)=y,\tau_x>n,\max_{k\le n}|X_1(k)|\le\gamma y_1\right)\\\le C(\gamma)\left(\frac{n}{|y|^2}\right)^{1/\gamma}\overline{\mathbf{P}}(x+S(n)=y,\tau_x>n).\end{aligned}\tag{4.19}$$

According to Theorem 6.2 in Esseen [1968], there exists an absolute constant C such that

$$\sup_z\overline{\mathbf{P}}(S(n)=z)\le\frac{C}{n^{d/2}}\chi^{-d/2},$$

where

$$\chi := \sup_{u\ge 1}\frac{1}{u^2}\inf_{|t|=1}\overline{\mathbf{E}}\left[(t,X(1)-X(2));|X(1)-X(2)|\le u\right].$$

Since h defined in (4.18) converges to zero as $|y|\to\infty$ uniformly in $n\le\gamma|y|^2$,

$$\overline{\mathbf{E}}\left[(t,X(1)-X(2));|X(1)-X(2)|\le u\right]\to\mathbf{E}\left[(t,X(1)-X(2));|X(1)-X(2)|\le u\right]$$

for every fixed u. Since $S(n)$ is truly d-dimensional under the original measure, $\inf_{|t|=1}\mathbf{E}\left[(t,X(1)-X(2));|X(1)-X(2)|\le u\right]>0$ for all large values u.

As a result, there exists $\chi_0>0$ such that $\chi\ge\chi_0$ for all $|y|$ large enough and all $n\le\gamma|y|^2$. Consequently,

$$\sup_z\overline{\mathbf{P}}(S(n)=z)\le\frac{C\chi_0^{-d/2}}{n^{d/2}}. \tag{4.20}$$

Combining this bound with (4.19), we obtain for all $r\in(0,1)$, $\gamma<2/d$

$$\sum_{n=1}^{|y|^{2-r}}\mathbf{P}\left(x+S(n)=y,\tau_x>n,\max_{k\le n}|X_1(k)|\le\gamma y_1\right)$$
$$\le C(\gamma)\chi_0^{-d/2}|y|^{-2/\gamma}\sum_{n=1}^{|y|^{2-r}}n^{1/\gamma-d/2}\le C(\gamma)\chi_0^{-d/2}|y|^{-2/\gamma}|y|^{(2-r)(1/\gamma-d/2+1)},$$

for all $n\le\gamma|y|^2$. If we choose γ so small that $r(1/\gamma-d/2+1)>2$, then

$$\sum_{n=1}^{|y|^{2-r}}\mathbf{P}\left(x+S(n)=y,\tau_x>n,\max_{k\le n}|X_1(k)|\le\gamma y_1\right)=o\left(\frac{1}{|y|^d}\right). \tag{4.21}$$

In the case $n\ge|y|^{2-r}$ we can not ignore the condition $\tau_x>n$. By the Markov property at times $n/3$ and $2n/3$ and by (4.20),

$$\begin{aligned}
&\overline{\mathbf{P}}(x+S(n)=y,\tau_x>n)\\
&\le\sum_{z,z'}\overline{\mathbf{P}}(x+S(n/3)=z,\tau_x>n/3)\overline{\mathbf{P}}(z+S(n/3)=z')\overline{\mathbf{P}}(z'+S(n/3)=y,\tau_{z'}>n/3)\\
&=\sum_{z,z'}\overline{\mathbf{P}}(x+S(n/3)=z,\tau_x>n/3)\overline{\mathbf{P}}(z+S(n/3)=z')\overline{\mathbf{P}}(y+S'(n/3)=z',\tau_y>n/3)\\
&\le\frac{C}{n^{d/2}}\overline{\mathbf{P}}(\tau_x>n/3)\overline{\mathbf{P}}(\tau'_y>n/3).
\end{aligned}$$

Therefore, it remains to show that, uniformly in $n\in[|y|^{2-r},|y|^2]$,

$$\overline{\mathbf{P}}(\tau_x>n/3)\le C\frac{1+x_d}{\sqrt{n}}. \tag{4.22}$$

Indeed, from this estimate and from the corresponding estimate for the inversed walk we get

$$\overline{\mathbf{P}}(x+S(n)=y,\tau_x>n)\le C\frac{(x_d+1)(y_d+1)}{n^{d/2+1}}.$$

This implies with help of (4.19) that

$$\sum_{n=|y|^{2-r}}^{\varepsilon|y|^2}\mathbf{P}\left(x+S(n)=y,\tau_x>n,\max_{k\le n}|X_1(k)|\le\gamma y_1\right)$$
$$\le C\varepsilon^{1/\gamma-d/2}(x_d+1)(y_d+1)|y|^{-d}.$$

Combining this with (4.16) and with (4.21), we obtain (4.15).

To derive (4.22) we first estimate some moments of the random walk $S_d(n)$ under $\overline{\mathbf{P}}$. By the definition of this probability measure,

$$\overline{\mathbf{E}}[X_d]=\frac{1}{\varphi(h)}\mathbf{E}\left[X_de^{hX_1};|X_1|\le\gamma y_1\right].$$

For the expectation on the right hand side we have the representation

$$\begin{aligned}\mathbf{E}&\left[X_de^{hX_1};|X_1|\le\gamma y_1\right]\\&=\mathbf{E}\left[X_d;|X_1|\le\gamma y_1\right]+h\mathbf{E}\left[X_dX_1;|X_1|\le\gamma y_1\right]\\&\qquad+\mathbf{E}\left[X_d(e^{hX_1}-1-hX_1);|X_1|\le\gamma y_1\right]\\&=-\mathbf{E}\left[X_d;|X_1|>\gamma y_1\right]-h\mathbf{E}\left[X_dX_1;|X_1|>\gamma y_1\right]\\&\qquad+\mathbf{E}\left[X_d(e^{hX_1}-1-hX_1);|X_1|\le\gamma y_1\right].\end{aligned}$$

In the last step we have used the equalities $\mathbf{E}[X_d]=\mathbf{E}[X_dX_1]=0$. If

$$\mathbf{E}|X|^{3+\delta}<\infty, \tag{4.23}$$

then, by the Markov inequality,

$$\mathbf{E}\left[X_d;|X_1|>\gamma y_1\right]+h\mathbf{E}\left[X_dX_1;|X_1|>\gamma y_1\right]=o(y_1^{-2})=o(n^{-1}).$$

Therefore,

$$\mathbf{E}\left[X_de^{hX_1};|X_1|\le\gamma y_1\right]=o(n^{-1})+\mathbf{E}\left[X_d(e^{hX_1}-1-hX_1);|X_1|\le\gamma y_1\right].$$

It is obvious that $|e^x-1-x|\le\frac{x^2}{2}e^{|x|}$. Therefore,

$$\begin{aligned}\left|\mathbf{E}\left[X_d(e^{hX_1}-1-hX_1);|X_1|\le\gamma y_1\right]\right|&\le\frac{h^2}{2}\mathbf{E}\left[|X_d|X_1^2e^{h|X_1|};|X_1|\le\gamma y_1\right]\\&\le\frac{e}{2}h^2\mathbf{E}|X_d|X_1^2+h^2e^{h\gamma y_1}\mathbf{E}\left[|X_d|X_1^2;|X_1|>\frac{1}{h}\right]\\&\le\frac{e}{2}h^2\mathbf{E}|X_d|X_1^2+h^{2+\delta}e^{h\gamma y_1}\mathbf{E}|X_d||X_1|^{2+\delta}\\&\le\frac{e}{2}h^2\mathbf{E}|X|^3+h^{2+\delta}e^{h\gamma y_1}\mathbf{E}|X|^{3+\delta}.\end{aligned}$$

Here in the last step we have used Hölder's inequality. It is immediate from the definition of h that $h^2 \le cn^{-1}$. Furthermore, if $n \ge |y|^{2-r}$ with some $r < \frac{\delta}{2}$, then $h^{2+\delta}e^{h\gamma y_1} = o(n^{-1})$. From these estimates and from assumption (4.23), we obtain

$$\left|\mathbf{E}\left[X_d e^{hX_1}; |X_1| \le \gamma y_1\right]\right| \le \frac{c}{n} \tag{4.24}$$

uniformly in $n \in [|y|^{2-r}, |y|^2]$.

By the same arguments,

$$\begin{aligned}\varphi(h) &= \mathbf{E}\left[e^{hX_1}; |X_1| \le \gamma y_1\right]\\ &= \mathbf{P}(|X_1| \le \gamma y_1) + h\mathbf{E}\left[X_1; |X_1| \le \gamma y_1\right] + \mathbf{E}\left[e^{hX_1} - 1 - hX_1; |X_1| \le \gamma y_1\right]\\ &= 1 - \mathbf{P}(|X_1| > \gamma y_1) - h\mathbf{E}\left[X_1; |X_1| > \gamma y_1\right] + \mathbf{E}\left[e^{hX_1} - 1 - hX_1; |X_1| \le \gamma y_1\right]\\ &= 1 + o(n^{-1}). \end{aligned} \tag{4.25}$$

Combining this with (4.24), we finally obtain

$$\left|\overline{\mathbf{E}}X_d\right| \le \frac{c_1}{n}. \tag{4.26}$$

We now turn to the second and the third moments of X_d under $\overline{\mathbf{P}}$. Using (4.25) and the moment assumption we have

$$\begin{aligned}\overline{\mathbf{E}}X_d^2 &= \frac{1}{\varphi(h)}\mathbf{E}[X_d^2 e^{hX_1}; |X_1| \le \gamma y_1] = (1+o(1))\mathbf{E}[X_d^2 e^{hX_1}; |X_1| \le \gamma y_1]\\ &= \mathbf{E}[X_d^2; |X_1| \le \gamma y_1] + o(1) + O\left(\mathbf{E}\left[X_d^2(e^{hX_1} - 1); |X_1| \le \gamma y_1\right]\right)\\ &= 1 + o(1) + O\left(he^{h\gamma y_1}\right).\end{aligned}$$

Noting that $he^{h\gamma y_1} = o(1)$ for all $n \ge |y|^{2-r}$ we get

$$\overline{\mathbf{E}}X_d^2 = 1 + o(1). \tag{4.27}$$

Similarly,

$$\begin{aligned}\overline{\mathbf{E}}|X_d|^3 &= (1+o(1))\mathbf{E}[|X_d|^3 e^{hX_1}; |X_1| \le \gamma y_1]\\ &\le c\left(\mathbf{E}[|X_d|^3; |X_1| \le 1/h] + e^{h\gamma y_1}\mathbf{E}[|X_d|^3; |X_1| > 1/h]\right)\\ &\le c\left(\mathbf{E}|X_d|^3 + h^\delta e^{h\gamma y_1}\mathbf{E}|X_d|^{3+\gamma}\right)\end{aligned}$$

Using once again the fact that $h^\delta e^{h\gamma y_1} = o(1)$ for $n \ge |y|^{2-r}$, we arrive at

$$\overline{\mathbf{E}}|X_d|^3 \le c_3. \tag{4.28}$$

Now we can derive (4.22). First, it follows from (4.26) that

$$\overline{\mathbf{P}}(\tau_x > n/3) \le \overline{\mathbf{P}}(\tau^0_{x+c_1} > n/3),$$

where

$$\tau_y^0 := \inf\{k \geq 1 : y + S_d^0(k) \leq 0\} \quad \text{and} \quad S_d^0(k) = S_d(k) - k\overline{\mathbf{E}}X_d.$$

Applying Lemma 25 in Denisov et al. [2018] to the random walk S_d^0, we have

$$\overline{\mathbf{P}}(\tau_y^0 > k) \leq \frac{\overline{\mathbf{E}}[y + S_d^0(k); \tau_y^0 > k]}{\overline{\mathbf{E}}(y + S_d^0(k))^+}$$

Relations (4.27) and (4.28) allow the application of the central limit theorem to the walk $S_d^0(k)$, which gives $\overline{\mathbf{E}}(y + S_d^0(k))^+ \geq c\sqrt{k}$. Consequently,

$$\overline{\mathbf{P}}(\tau_y^0 > k) \leq \frac{C}{\sqrt{k}}\overline{\mathbf{E}}[y + S_d^0(k); \tau_y^0 > k].$$

Further, by the optional stopping theorem,

$$\begin{aligned}\overline{\mathbf{E}}[y + S_d^0(k); \tau_y^0 > k] &= y - \overline{\mathbf{E}}[y + S_d^0(\tau_y^0); \tau_y^0 \leq k] \\ &\leq y - \overline{\mathbf{E}}[y + S_d^0(\tau_y^0)].\end{aligned}$$

We now use inequality (7) in Mogul'skii [1973] which states that there exists an absolute constant A such that

$$-\overline{\mathbf{E}}[y + S_d^0(\tau_y^0)] \leq A\frac{\overline{\mathbf{E}}|X_d|^3}{\overline{\mathbf{E}}X_d^2}.$$

Combining this with (4.27) and (4.28), we finally get

$$\overline{\mathbf{P}}(\tau_y^0 > k) \leq \frac{C(y+1)}{\sqrt{k}},$$

which implies (4.22). □

4.3.1 Proof of the behavior of the Green function along the boundary of the cone

Limit theorems for random walks starting far from the origin but close to the boundary

Let $|y| \to \infty$ in such a way that $dist(y, \partial K) = o(|y|)$. Let $y_\perp \in \partial K$ be defined by the relation $dist(y, \partial K) = |y - y_\perp|$. Set $\sigma(y) := y_\perp/|y| \in \partial\Sigma$ and assume that $\sigma(\cdot)$ converges as $|y| \to \infty$ to some $\bar{\sigma} \in \partial\Sigma$. Let H_y denote a tangent hyperplane at point $y_\perp$. Let P_n be the distribution of the linear interpolation of $t \to (y + S(nt))/\sqrt{n}$ conditioned to stay in the half-space K_y containing the cone K and having boundary H_y. Then $P_n \to P$ weakly on $C[0,1]$. Denote

$$A_n := \{f \in C[0,1] : f(k/n) \in K, \forall 1 \leq k \leq n\}.$$

Then

$$\liminf A_n \supseteq \{f \in C[0,1] : f(t) \in K \text{ for all } t \in (0,1]\}$$

and

$$\limsup \overline{A_n} \subseteq \{f \in C[0,1] : f(t) \in \overline{K} \text{ for all } t \in (0,1]\},$$

where $\overline{A}$ denote the closure of A.

Denote for every fixed n by $[0,1] \ni t \mapsto S(nt)$ the linear interpolation of $S(k), k \le n$. The conditions to apply Theorem 2.3 from Durrett [1978] are given. This leads to the following invariance principle:

$$\text{as } \frac{n}{|y|^2} \to t, \quad [0,1] \ni r \to \frac{y+S(nr)}{\sqrt{n}} \text{ converges weakly}$$

$$\text{to the Brownian meander } \{B_r, r \le 1\} \text{ inside K started at } \frac{\sigma}{\sqrt{t}}$$

In particular it holds with $T_y := \inf\{n \ge 1 : y + S(n) \notin K_y\}$

$$\mathbf{P}\left(\frac{y+S(nt)}{\sqrt{n}} \in B \middle| \tau_y > n\right) \sim Q_{\sigma,t}(B) = \int_B q_{\sigma,t}(z)dz, \quad \frac{n}{|y|^2} \to t, \tag{4.29}$$

where $q_{\sigma,t}(z)$ is the density of the aforementioned Brownian meander at time t.

Theorem 2.3 in Durrett [1978] also leads to

$$\mathbf{P}(\tau_y > n | T_y > n) \to c_{\sigma,t}, \tag{4.30}$$

where

$$T_y := \inf\{n \ge 1 : y + S(n) \notin K_y\}.$$

Limiting relations (4.29) and (4.30) imply that

$$V(y) \ge c|y|^{p-1}(1 + dist(y, \partial K)). \tag{4.31}$$

Indeed, by the harmonicity of V

$$V(y) = \mathbf{E}[V(y + S(n)); \tau_y > n], \ n \ge 1.$$

Fix now some $\epsilon > 0$ and note that choosing $n = [|y|^2]$ it follows that $V(z) \sim u(z)$ uniformly as $z \to \infty$ as long as distance to ∂K of z is at least $\epsilon|z|$ (recall fact A in subsection 2.5). We obtain, as $|y| \to \infty$, and $\epsilon \to 0$

$$V(y) \ge \mathbf{P}(T_y > [|y|]^2) c_{\sigma,1} |y|^p \int_K u(z) q_{\sigma,1}(z)dz.$$

Due to results for the one-dimensional random walk we arrive at

$$\mathbf{P}(T_y > [|y|]^2) \ge c\frac{1 + dist(y, \partial K)}{|y|}.$$

This establishes (4.31).

Before continuing with the proof of Theorem 5 we record an auxiliary estimate needed in its proof.

Lemma 5. *Define*

$$\phi_\sigma(t) = c_{\sigma,t}\int_K u(z)e^{-\frac{|z|^2}{2}}q_{\sigma,t}(z)dz.$$

It holds $\phi_\sigma(t) = o(e^{-c/t})$ *as* $t \to 0$ *for some* $c > 0$.

Proof. First we record that due to the invariance principle for the halfspace it holds

$$c_{\sigma,t} = \mathbf{P}(\tau_\sigma^{bm} > t | T_\sigma^{bm} > t).$$

Here $T_y^{bm} := \inf\{t \geq 0 : y + B(t) \notin K_y\}$. Since $|\sigma| = 1$ and K is contained in $H(\sigma)$ it is clear then that $c_{\sigma,t} \to 1$ as $t \to 0$.

Write $A_K := \{z \in K : |z| > 1\}$ and $B_K := K \setminus A_K$.

It follows after a substitution for $t \leq 1$

$$\int_{A_K} u(z)e^{-\frac{|z|^2}{2}}q_{\sigma,t}(z)dz \leq t^{-\frac{p+d}{2}}e^{-\frac{1}{4t}}\int_{A_K} u(z)e^{-\frac{|z|^2}{4t}}q_{\sigma,t}(z/\sqrt{t})dz$$
$$\leq e^{-\frac{1}{4t}} \cdot_{\sigma/\sqrt{t}} [u(B(t))e^{-|B(t)|^2/2}, |B(t)| \geq \sqrt{t}|\tau^{bm} > 1] = O(e^{-\frac{1}{4t}}), \quad t \to 0.$$

For B_K we calculate for $t \leq 1$

$$\int_{B_K} u(z)e^{-\frac{|z|^2}{2}}q_{\sigma,t}(z)dz \leq t^{p/2} \cdot \mathbf{P}(|\sigma/\sqrt{t} + B(t)| \leq \sqrt{t}|\tau_{\sigma/\sqrt{t}}^{bm} > 1)$$

It is easy to see using the invariance principles for the half-space and cones as well as the limit relation $c_{\sigma,t} \to 1$ for $t \to 0$ that

$$\mathbf{P}(|\sigma/\sqrt{t} + B(t)| \leq \sqrt{t}|\tau_{\sigma/\sqrt{t}}^{bm} > 1) \leq \mathbf{P}(|\sigma/\sqrt{t} + B(t)| \leq \sqrt{t}|T_{\sigma/\sqrt{t}}^{bm} > 1).$$

Let $h \in, |h| = 1$ be so that $K_\sigma = \{y \in: h \cdot y \geq 0\}$. It holds $\{|\sigma\sqrt{t} + B(t)| \leq \sqrt{t}\} \subset \{|\sigma \cdot B(t)| \geq \frac{1}{2\sqrt{t}}\}$ for all $t > 0$ small enough. It follows then from the fact that the density of the meander of the Brownian motion in K_σ at y at time t depends only on $h \cdot y$ and t that for some $c_1 > 0$ and $c_2 > 0$ suitable

$$\mathbf{P}(|\sigma/\sqrt{t} + B(t)| \leq \sqrt{t}|\tau_{\sigma/\sqrt{t}}^{bm} > 1) \leq Ct^{-\frac{3}{2}}\sqrt{t}t^{-\frac{d-1}{2}}\int_{1/2t}^{\infty} e^{-c_1 z^2}dz = o(e^{-c_2/t}).$$

This finishes the proof. □

4.3.2 Proof of Theorem 5

To estimate the contribution coming from large values of n one does not need the limit theorems from the previous paragraph, quite rough estimates turn out to be sufficient.

Set $m=[n/2]$. Then, applying the Markov property at time m and inverting the time in the second part of the path, we obtain

$$\begin{aligned}\mathbf{P}(x+S(n)=y) &= \sum_{z\in K}\mathbf{P}(x+S(m)=z,\tau_x>m)\mathbf{P}(y+S'(n-m)=z,\tau'_y>n-m)\\ &\le \max_{z\in K}\mathbf{P}(x+S(m)=z,\tau_x>m)\mathbf{P}(\tau'_y>n-m)\end{aligned}$$

By Proposition 6,

$$\max_{z\in K}\mathbf{P}(x+S(m)=z,\tau_x>m)\le C\frac{V(x)}{m^{p/2+d/2}}.$$

Furthermore, due to results for the one-dimensional random walk

$$\mathbf{P}(\tau'_y>n-m)\le\mathbf{P}(T'_y>n-m)\le C\frac{1+dist(y,\partial K)}{\sqrt{n-m}}. \tag{4.32}$$

Combining these estimates, we obtain

$$\mathbf{P}(x+S(n)=y)\le CV(x)(1+dist(y,\partial K))n^{-(p+d+1)/2}.$$

Consequently, for $A\ge 2$ and $|y|\ge 1$,

$$\begin{aligned}\sum_{n\ge A|y|^2}\mathbf{P}(x+S(n)=y) &\le CV(x)(1+dist(y,\partial K))\sum_{n\ge A|y|^2}n^{-(p+d+1)/2}\\ &\le CV(x)A^{-(p+d-1)/2}\frac{1+dist(y,\partial K)}{|y|^{p+d-1}}. \end{aligned}\tag{4.33}$$

We turn now to the 'middle' part: $n\in(\varepsilon|y|^2,A|y|^2)$. Using again the Markov property at time $m=[n/2]$ and applying Proposition 6, we obtain

$$\begin{aligned}&\mathbf{P}(x+S(n)=y,\tau_x>n)\\ &=\sum_{z\in K}\mathbf{P}(x+S(m)=z,\tau_x>m)\mathbf{P}(y+S'(n-m)=z;\tau'_y>n-m)\\ &=\frac{C_0V(x)}{m^{p/2+d/2}}\sum_{z\in K}\left(u\left(\frac{z}{\sqrt{m}}\right)e^{-\frac{|z|^2}{2m}}+o(1)\right)\mathbf{P}(y+S'(n-m)=z;\tau'_y>n-m)\\ &=\frac{C_0V(x)}{m^{p/2+d/2}}\mathbf{E}\left[u\left(\frac{S'(n-m)}{\sqrt{m}}\right)e^{-\frac{|S'(n-m)|^2}{2m}};\tau'_y>n-m\right]+o\left(\frac{\mathbf{P}(\tau'_y>n-m)}{m^{p/2+d/2}}\right).\end{aligned}$$

Taking into account (4.32), we have

$$\begin{aligned}&\mathbf{P}(x+S(n)=y,\tau_x>n)\\ &=\frac{C_0V(x)}{m^{p/2+d/2}}\mathbf{E}\left[u\left(\frac{S'(n-m)}{\sqrt{m}}\right)e^{-\frac{|S'(n-m)|^2}{2m}};\tau'_y>n-m\right]+o\left(\frac{1+dist(y,\partial K)}{n^{(p+d+1)/2}}\right).\end{aligned}$$

Next, it follows from (4.29) and (4.30) that if $\frac{n}{|y^2|} \sim t$ then

$$\mathbf{E}\left[u\left(\frac{S'(n-m)}{\sqrt{m}}\right)e^{-\frac{|S'(n-m)|^2}{2m}}; \tau'_y > n-m\right] \sim \mathbf{P}(T'_y > n-m)\phi_\sigma(t/2).$$

Since T'_y is an exit time from a half space,

$$\mathbf{P}(T'_y > k) \sim v'(y)k^{-1/2},$$

where $v'(y)$ is the positive harmonic function for S' killed at leaving the half-space $K_{\bar{\sigma}}$. As a result,

$$\mathbf{P}(x+S(n)=y, \tau_x > n) = C_0\frac{V(x)v'(y)}{n^{(p+d+1)/2}}\phi_\sigma\left(\frac{n}{|y|^2}\right) + o\left(\frac{1+dist(y,\partial K)}{n^{(p+d+1)/2}}\right),$$

where

$$C_0 := C_0 2^{(p+d+1)/2}.$$

This representation implies that

$$\begin{aligned}
&\sum_{\varepsilon|y|^2}^{A|y|^2}\mathbf{P}(x+S(n)=y,\tau_x>n)\\
&\quad= C_0V(x)v'(y)\sum_{\varepsilon|y|^2}^{A|y|^2} n^{-(p+d+1)/2}\phi_\sigma\left(\frac{n}{2|y|^2}\right) + o\left(\frac{1+dist(y,\partial K)}{n^{(p+d-1)/2}}\right)\\
&\quad= C_0\frac{V(x)v'(y)}{|y|^{p+d-1}}\int_\varepsilon^A \phi_\sigma(t/2)t^{-(p+d+1)/2}dt + o\left(\frac{1+dist(y,\partial K)}{n^{(p+d-1)/2}}\right).
\end{aligned}$$

Combining this with (4.33) and letting $A\to\infty$, one can easily obtain

$$\lim_{|y|\to\infty}\frac{|y|^{p+d-1}}{V(x)v'(y)}S_2(x,y,\varepsilon) = C_0\int_\varepsilon^\infty \phi_\sigma(t/2)t^{-(p+d+1)/2}dt.$$

From Lemma 5 it follows

$$\lim_{\varepsilon\to0}\lim_{|y|\to\infty}\frac{|y|^{p+d-1}}{V(x)v'(y)}S_2(x,y,\varepsilon) = C_0\int_0^\infty \phi_\sigma(t/2)t^{-(p+d+1)/2}dt. \tag{4.34}$$

It remains to estimate $S_1(x,y,\varepsilon)$. We shall use the same strategy as in the proof of Theorem 4, but instead of the Green function for the whole space we shall use the Green function for the half-space K_y. More precisely,

$$\begin{aligned}
S_1(x,y,\varepsilon) &= \sum_{n<\varepsilon|y^2|} \mathbf{P}(x+S(n)=y,\tau_x>n\ge\theta_y)\\
&= \sum_{n<\varepsilon|y^2|}\sum_{k=1}^{n}\sum_{z\in B_{\delta,y}} \mathbf{P}(x+S(n)=z,\tau_x>k=\theta_y)\mathbf{P}(z+S(n-k)=y,\tau_z>n-k)\\
&= \sum_{k<\varepsilon|y|^2}\sum_{z\in B_{\delta,y}} \mathbf{P}(x+S(n)=z,\tau_x>k=\theta_y)\sum_{j<\varepsilon|y|^2-k}\mathbf{P}(z+S(j)=y,\tau_z>j)\\
&\le \sum_{k<\varepsilon|y|^2}\sum_{z\in B_{\delta,y}} \mathbf{P}(x+S(n)=z,\tau_x>k=\theta_y)\sum_{j<\varepsilon|y|^2}\mathbf{P}(y+S'(j)=z,T'_y>j)\\
&= \mathbf{E}\left[G_{\varepsilon,y}(x+S(\theta_y));\tau_x>\theta_y,\theta_y\le\varepsilon|y|^2\right],
\end{aligned}$$

where

$$G_{\varepsilon,y}(z)=\sum_{j<\varepsilon|y|^2}\mathbf{P}(y+S'(j)=z,T'_y>j).$$

Applying Theorem 6 and (4.14) to the random walk $S'(n)$, we obtain

$$G_{\varepsilon,y}(z)\le C\frac{v'(y)(1+dist(z,H_y))}{|z-y|^d}\wedge 1.$$

Therefore,

$$\begin{aligned}
S_1(x,y,\varepsilon) &\le C\mathbf{P}(|y-x-S(\theta_y)|\le\delta^2|y|,\tau_x>\theta_y,\theta_y\le\varepsilon|y|^2)\\
&\quad + C(\delta)\frac{v'(y)}{|y|^d}\mathbf{E}\left[(1+dist(x+S(\theta_y),H_y);\tau_x>\theta_y,\theta_y\le\varepsilon|y|^2\right]. \qquad (4.35)
\end{aligned}$$

The first term has been estimated in (4.9):

$$\mathbf{P}(|y-x-S(\theta_y)|\le\delta^2|y|,\tau_x>\theta_y,\theta_y\le\varepsilon|y|^2)=o(|y|^{-p-d+1}) \qquad (4.36)$$

for random walks having finite moments of order $r_2(p):=p+d-1+(2-p)^+$.

In order to estimate the second term in (4.35), we shall perform again the change of measure with the harmonic function V:

$$\begin{aligned}
&\mathbf{E}\left[(1+dist(x+S(\theta_y),H_y);\tau_x>\theta_y,\theta_y\le\varepsilon|y|^2\right]\\
&\quad = V(x)\mathbf{E}^{(V)}\left[\frac{1+dist(x+S(\theta_y),H_y)}{V(x+S(\theta_y))};\theta_y\le\varepsilon|y|^2\right].
\end{aligned}$$

Applying now (4.31), we obtain

$$\mathbf{E}\left[(1+dist(x+S(\theta_y),H_y);\tau_x>\theta_y,\theta_y\le\varepsilon|y|^2\right]\le CV(x)|y|^{-p+1}\mathbf{P}^{(V)}(\theta_y\le\varepsilon|y|^2).$$

From this estimate and (4.10) we conclude that

$$\lim_{\varepsilon\to 0}\lim_{|y|\to\infty}|y|^{p-1}\mathbf{E}\left[(1+dist(x+S(\theta_y),H_y);\tau_x>\theta_y,\theta_y\le\varepsilon|y|^2\right]=0.$$

Combining this estimate with (4.35) and (4.36) as well as (2.30) for the half-space we get

$$\lim_{\varepsilon\to 0}\lim_{|y|\to\infty}|y|^{p+d-1}S_1(x,y,\varepsilon)=0. \tag{4.37}$$

Since $v'(y)$ is bounded from below by a positive number, (4.37) and (4.34) yield the desired result for the case $\mathbf{E}[|X|^{r_2(p)}]<\infty$ due to classical results for the one-dimensional random walk.

Assume now that (4.2) holds. It is easy to see that the above proof that

$$\lim_{\varepsilon\to 0}\lim_{|y|\to\infty}\frac{|y|^{p+d-1}}{V(x)v'(y)}S_2(x,y,\varepsilon)=C_0\int_0^\infty\phi_\sigma(t)t^{-(p+d+1)/2}dt, \tag{4.38}$$

goes through again word for word. Therefore we focus on the asymptotic of $S_1(x,y,\varepsilon)$ in the following. With similar steps as above it holds

$$\begin{aligned}S_1(x,y,\varepsilon)&\le C(\delta)v'(y)\left[\frac{1+dist(x+S(\theta_y),H_y)}{|x+S(\theta_y)-y|^d},|y-x-S(\theta_y)|\le\delta^2|y|,\tau_x>\theta_y,\theta_y\le\varepsilon|y|^2\right]\\&+C(\delta)\frac{v'(y)}{|y|^d}\mathbf{E}\left[(1+dist(x+S(\theta_y),H_y);\tau_x>\theta_y,\theta_y\le\varepsilon|y|^2\right].\end{aligned}$$

The second summand can be treated just as above with help of (4.31) so that we need to show

$$\lim_{|y|\to\infty}|y|^{p+d-1}\mathbf{E}\left[\frac{1+dist(x+S(\theta_y),H_y)}{|x+S(\theta_y)-y|^d},|y-x-S(\theta_y)|\le\delta^2|y|,\tau_x>\theta_y,\theta_y\le\varepsilon|y|^2\right]=0.$$

It holds $1+dist(x+S(\theta_y),H_y)\le 1+|S(\theta_y)-y|+|y-y_\perp|=O(1)+|S(\theta_y)-y|$. To complete the proof we now show

$$\begin{aligned}S_{2,rest}(x,y,\varepsilon)&:=\mathbf{E}\left[|x+S(\theta_y)-y|^{-d+1},|y-x-S(\theta_y)|\le\delta^2|y|,\tau_x>\theta_y,\theta_y\le\varepsilon|y|^2\right]\\&=o(|y|^{-p-d+1}).\end{aligned}$$

With a similar calculation as in the proof of Theorem 4 (using (4.11)) we obtain

$$\begin{aligned}&\mathbf{E}\left[|y-x-S(\theta_y)|^{-d+1};|y-x-S(\theta_y)|\le\delta^2|y|,\tau_x>\theta_y,\theta_y\le\varepsilon|y|^2\right]\\&\le C(\delta)|y|^{-p-d+1}f(\delta(1-\delta)|y|)\mathbf{E}[\tau_x;\tau_x<|y|^2]\sum_{m=1}^{\delta^2|y|}\frac{m^{d-1}}{m^{d-1}}\\&\le C(\delta)|y|^{-p-d+2}f(\delta(1-\delta)|y|)|y|^{(2-p)^+}.\end{aligned}$$

This finishes the proof of Theorem 5.

4.4 Notes on the case of non-negative drift

The results presented in this subsection are from Duraj [2014a].

4.4.1 Introduction

The uniqueness result, up to a positive scalar, of the harmonic function for zero-drift random walks in cones opens the question as to what happens for non-zero drift random walks. We show here that for random walks of non zero drift on the euclidean space $\mathbf{R}^d, d \geq 2$, killed when leaving a convex cone with vertex in 0, there are uncountably many non-negative harmonic functions. The main assumption is finiteness of the jump generating function of the step of the random walk in a neighborhood of its preimage of 1. The proof is constructive and an adaptation of the similar proof in Ignatiouk-Robert and Loree [2010], which considers the special case of lattice random walks in the two-dimensional positive quadrant.

This establishes by example that the uniqueness result established in this chapter is special to the case of non-zero drift. We also make a conjecture about the Martin boundary of two-dimensional random walks, killed when leaving convex cones of $\mathbf{R}^2$ and comment on the difficulties in translating the Ignatiouk-Robert and Loree [2010] proof to the more general setting we are considering.

We consider a convex cone in $\mathbf{R}^d, d \geq 2$ with vertex in 0, denote by K its interior, which we assume to be nonempty throughout the paper and also a random walk on the euclidean space $\mathbf{R}^d$ with steps $X_i, i \in \mathbb{N}$ and step distribution γ. We also set $\Sigma = \overline{K} \cap \mathbb{S}^{d-1}$. We will study the random walk with the following modified moment conditions.

A1 The step distribution has

$$m := \mathbf{E}[X_1] \neq 0 \quad \text{and} \quad \mathbf{E}[|X_1|] \neq 0.$$

A2 The jump generating function $\varphi(a) := \mathbf{E}[e^{a \cdot X_1}]$ fulfills

$$D = \{a \in \mathbb{R}^2 | \varphi(a) \leq 1\} \subset int(\{a \in \mathbb{R}^2 | \varphi(a) < \infty\}).$$

A3 The random walk is lattice-valued, irreducible in $\mathbb{Z}^d$ and the killed random walk when leaving K is irreducible in K. Moreover, the angle between every two points in $\partial\Sigma$ is strictly smaller than π.

Assumption **A2** is the standard assumption made for the study of the Martin boundary of lattice random walks in the euclidean lattice (see Ney and Spitzer [1966]). It implies in particular that X_1 has all moments.

Under assumptions **A1** and **A2** it is well-known, that

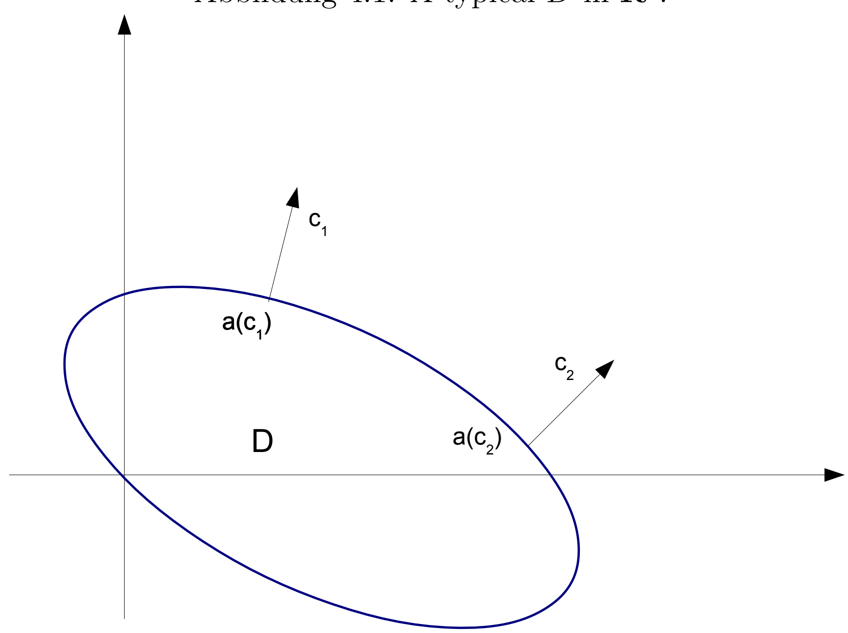
$$D = \{a \in \mathbb{R}^d : \varphi(a) \leq 1\}$$

is a strictly convex and closed set, the gradient $\nabla\varphi(a)$ exists everywhere and does not vanish on $\partial D = \{a \in \mathbb{R}^d | \varphi(a) = 1\}$. Moreover, the mapping

$$a \to q(a) = \frac{\nabla\varphi(a)}{|\nabla\varphi(a)|}$$

is a homeomorphism between ∂D and an open set of $\mathbb{S}^{d-1}$. D does not need to be bounded as the case $d = 2, \gamma = \frac{1}{3}\delta_{(1,-1)} + \frac{1}{3}\delta_{(-1,1)} + \frac{1}{3}\delta_{(-1,-1)}$ shows. If **A3** is additionally fulfilled then D is additionally compact and the image of $q(\cdot)$ is the whole sphere in d dimensions $\mathbb{S}^{d-1}$ (see Ney and Spitzer [1966] and the references therein). The inverse mapping is denoted

Abbildung 4.1: A typical D in $\mathbf{R}^2$.



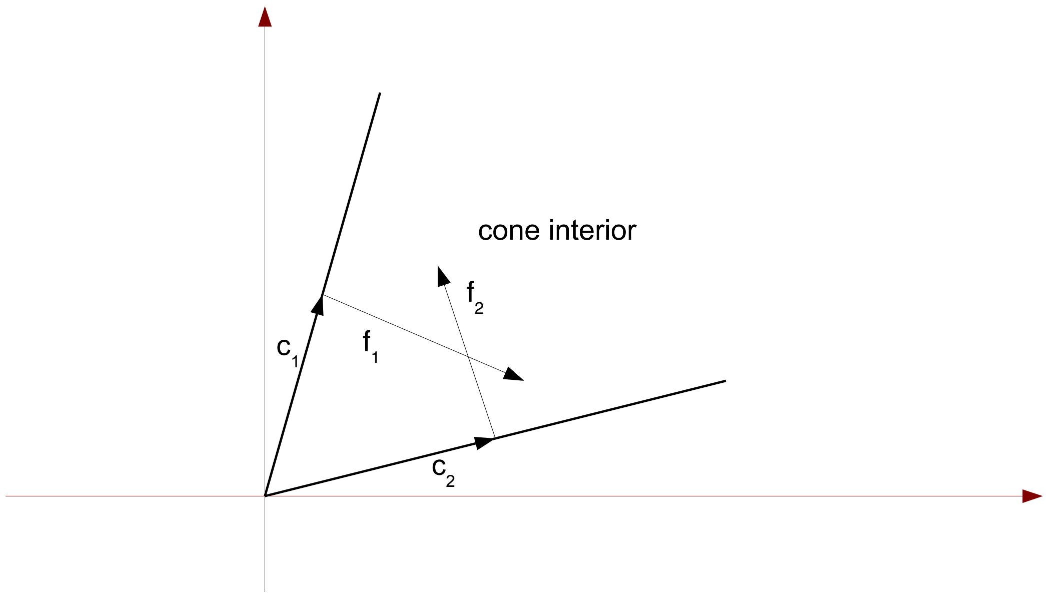
by $q \to a(q)$ and, whenever possible, we extend this map to nonzero $q \in \mathbb{R}^d$ by setting $a(q) := a\left(\frac{q}{|q|}\right)$. This definition implies that $a(q)$ is the only point in ∂D where q is normal to D. See figure 4.1 for a typical picture of D in the case $d = 2$ and that **A1**-**A3** hold.

If **A3** is not fulfilled, we make the following weaker assumption to avoid trivialities.

A4 $\Gamma = \{a \in \partial D \; |q(a) \in \Sigma = \overline{K} \cap \mathbb{S}^{d-1}\}$ is nonempty.

For the case $d = 2$ we encode the cone K as follows: take Λ to be a set of two points in the unit circle $\mathbb{S}^1$, $\Lambda = \{c_1, c_2\}$, so ordered that the angle ϕ between them is in $(0, \pi)$. The rays from (0,0) to infinity going through the $\mathbb{S}^1$-sector between the two vectors in Λ enclose a convex cone. Its interior K depends on the vectors we chose, i.e. $K = K(c_1, c_2)$. **A4** implies that at least one of the c_i is normal to ∂D if not both. We also note the unit vectors f_1 and f_2, respectively perpendicular to c_1 and c_2, pointing inwards. See figure 4.2 for a typical example. We want to prove the following.

Abbildung 4.2: A convex cone in $\mathbb{R}^2$.



Proposition 9. *(a) Under assumptions A1, A2 and A4 for every a such that* $q(a) \in int(\Sigma)$ *and* $z \in K$

$$h_a(z) = \exp(a \cdot z) - \mathbb{E}[\exp(a \cdot S(\tau_z)), \tau_z < \infty].$$

are non-negative and harmonic for the random walk, killed when leaving the cone.
(b) If in (a) $d = 2$ *and a fulfills* $q(a) = c_i,\ i \in \{1, 2\}$ *the function*

$$h_a(z) = z \cdot f_i \exp(a \cdot z) - \mathbb{E}[f_i \cdot S(\tau_z) \exp(a \cdot S(\tau_z)), \tau_z < \infty]$$

is nonnegative and harmonic for the random walk, killed when leaving the cone.
(c) The harmonic functions from (a)-(b) are strictly positive if A3 is additionally fulfilled.

These harmonic functions are just a generalization of the functions found in Ignatiouk-Robert and Loree [2010]. Intuitively, a look at figure 4.2 and at their paper suggests, that these functions must be the harmonic functions for general cones in the two-dimensional case.

For the case $q(a) \notin int(\Sigma)$ our proof method doesn't work in general for $d \geq 3$. The difficulty lies in the fact that for $d = 2$ the event $\{$*the random walk doesn't leave* K *through a specific supporting hyperplane of the cone*$\}$ can be encoded easily through the *unique* opposite supporting hyperplane. This simple characterization generalizes to $d \geq 3$ only if the cone is defined as intersection of finitely many halfspaces, which we don't pursue here since we are interested in the class of general cones. We remark also the following.

Remark 4. *In the formulation of Proposition 9 the event* $\{\tau_z < \infty\}$ *can be left out when* $m \notin K$.

Remark 5. *For $d = 1$ the only cone to consider is $(0, \infty)$. Doney [1998] fully characterizes the Martin boundary in the lattice case. His result can be used direcly in special cases even in $\mathbf{R}^d, d \geq 2$, when our assumptions are not fulfilled. For example, it shows that random walks which are cartesian products of one-dimensional lattice random walks with drift $-\infty$ and such that $\partial D = \{0\}$, killed when leaving $K = (0, \infty)^d$ have no nontrivial nonnegative harmonic function.*

Finally, one can see how the harmonicity result in Ignatiouk-Robert and Loree [2010] immediately follows from our proposition by taking $c_1 = (0, 1)$ and $c_2 = (1, 0)$ in (b).

Proposition 10 (Ignatiouk-Robert and Loree [2010]-The positive quadrant)**.** *Assume A1-A3. For every $a \in \Gamma_+ := \{a \in \partial D : q(a) \in \mathbb{R}^2_+,\ |q(a)| = 1\}$ and $z = (x_1, x_2) \in \mathbb{N}^* \times \mathbb{N}^*$*

$$h_a(z) = \begin{cases} x_1 \exp(a \cdot z) - \mathbb{E}[S_1(\tau_z) \exp(a \cdot S(\tau_z)), \tau_z < \infty], & \text{if } q(a) = (0, 1), \\ x_2 \exp(a \cdot z) - \mathbb{E}[S_2(\tau_z) \exp(a \cdot S(\tau_z)), \tau_z < \infty], & \text{if } q(a) = (1, 0) \\ \exp(a \cdot z) - \mathbb{E}[\exp(a \cdot S(\tau_z)), \tau_z < \infty], & \text{otherwise} \end{cases}$$

are strictly positive and harmonic for the random walk, killed when leaving the positive quadrant.

The next section states the natural conjecture about the Martin boundary of random walk, killed when leaving a two-dimensional convex cone, when A1-A3 are all fulfilled. We also underline where the proof in Ignatiouk-Robert and Loree [2010], which considers only the positive quadrant, breaks down for the general case. Proposition 9 is proven by adapting the proof of Proposition 10, contained in Ignatiouk-Robert and Loree [2010], to the general setting we are considering. Details of the proofs are in Duraj [2014a].

4.4.2 A Conjecture: Martin boundary for general convex cones in two dimensions

For this section only we assume that A1-A3 are fulfilled and that $d = 2$. In Ney and Spitzer [1966] the authors show that every positive harmonic function h for the random walk can be expressed as

$$h(z) = \int_C e^{c \cdot z} d\gamma(c).$$

Here γ is a positive Borel measure on some suitable set C. These types of functions are not harmonic for killed random walk on the quadrant. To makethem harmonic, one has to consider the correction term. Therefore the form of the functions in Proposition 10.

The main contribution of Ignatiouk-Robert and Loree [2010] is to show that these functions are the whole Martin boundary for the case of the positive quadrant (see Theorem 1 there).

Judging from the analogy between Proposition 9 and 10, we conjecture the following (stated analoguously to Theorem 1 in Ignatiouk-Robert and Loree [2010]).

Conjecture For the cone encoded by c_1 and c_2 as in subsection 4.4.1 and under the assumptions A1 - A4 made there, we have that :

1. A sequence of points z_n in K with $\lim_{n\to} |z_n| = +\infty$ converge to a point of the Martin boundary for the killed random walk when leaving the cone, if and only if $\frac{z_n}{|z_n|} \to q$ for some $q \in \Gamma$.

2. The full Martin Compactification of $K \cap \mathbb{Z}^2$ is homeomorphic to the closure of the set $\{w = \frac{z}{1+|z|} | z \in K \cap \mathbb{Z}^2\}$ in $\mathbb{R}^2$.

In short, the conjecture states that Proposition 9 (a)-(b) fully characterizes the Martin boundary of random walks on the two dimensional euclidean lattice, killed when leaving convex cones.

If one tries to carry over the methods of Ignatiouk-Robert and Loree [2010] to this general case, one sees that the *communication condition* contained there and the *large deviations result* can be modified to work for the more general setting as well. We will not give details how this is done, but we mention shortly that both can be proven if one augments assumption **A3** by the following.

Strong local irreducibility: There exists some uniform $R > 0$ such that for every $z \in K, e \in \mathbb{Z}^2, \quad |e| = 1$ such that $z + e \in K$ we have: there exists a path of measure non zero within $K \cap B_R(z)$ from z to $z + e$.

This assumption is neccessary, if one wants to work with the communication condition as Ignatiouk-Robert and Loree [2010] do and is fulfilled in the positive quadrant setting due to irreducibility. The obstacle for generalizing the proof in the case of the positive quadrant is the lack of Markov-additivity for local processes for the general case. We recall that a Markov Chain $\mathcal{Z}_n = (A(n), M(n))$ on a countable space $\mathbb{Z}^d \times E$ is called *Markov-additive* if for its transition matrix p it holds:

$$p((x, y), (x', y')) = p((0, y), (x' - x, y')) \text{ for all } x, x' \in \mathbb{Z}^d, y, y' \in E$$

Ignatiouk-Robert and Loree [2010] make extensive use of this property when showing the above conjecture for the case of the positive quadrant. The idea for the general case of convex cones would be to look at local processes *deep* inside the cone, where the random walk is Markov-additive in two directions. But approaching the boundary of the cone, this property disappears in general in both directions. For the positive quadrant this happens only for one direction and this is crucial for the proof in Ignatiouk-Robert and Loree [2010]. Without Markov-additivity it seems impossible to come to a usable Ratio Limit theorem as is done in Ignatiouk-Robert and Loree [2010]. On the other hand, the proof of Proposition 9 does not use Markov-additivity. This suggests the existence of more general methods than those of Ignatiouk-Robert and Loree [2010] for proving the conjecture made in this section or a similar conjecture in higher dimensions.

Chapter 5

Invariance principles for integrated random walks conditioned to stay positive

5.1 Introduction, motivation and statement of results

The work presented in this paper is included in Bär et al. [2020].

5.1.1 Motivation

Integrated random walks conditioned to stay positive have become a popular topic in probability. There are many applications of integrated random walks in models in mathematical physics and beyond, but also within probability. The reader can consult Caravenna and Deuschel [2008] for an example application or the survey articles Aurzada and Simon [2015] and Majumdar [1999]. While there are many papers that have studied integrated random walks under the condition of positivity, until now they have focused either on special cases or on characterizing asymptotic behavior of the exit time (e.g. Aurzada et al. [2014], Dembo et al. [2013], Denisov and Wachtel [2015a], Gao et al. [2014], Vysotsky [2010] and Vysotsky [2014]). In this paper we prove invariance principles for the meander and for the bridge of the integrated random walk conditioned to stay positive. Our invariance principle for the bridge establishes the weak convergence of the normalized Markov chain, whose first coordinate is the integrated random walk and the second coordinate is the random walk, started at a point and conditioned to end at another point all the while under the condition of positivity for the first component of the Markov chain.

The 'price' we have to pay for the proofs is existence of slightly more than the second moment for the step-distribution of the random walk. Just as for the case of random walks in cones, this allows the use of the normal approximation to the random walk, for the mass of probability on the random walk paths in which the positivity condition is far from being violated.

An adaptation of the strategy in Denisov and Wachtel [2015b] and Duraj and Wachtel [2020] delivers the invariance principles in our setting as well. In order to apply this strategy, we focus not on the integrated random walk (which is in general not a Markov chain), but on the *pair* integrated random walk *and* random walk. This is again a Markov chain and the condition of positivity for the integrated random walk can be formulated as a geometric condition of the new chain not leaving $\mathbf{R}_+ \times \mathbf{R}$.

Another in the proof of the invariance principles is a weak convergence result for the respective continuous limit of the above mentioned Markov chain, namely for the Kolmogorov diffusion. The Kolmogorov diffusion is the *pair* consisting of the integral of the Brownian motion and the Brownian motion itself. This is a Markov process in continuous time. The limit process conditional on posivitity of the first coordinate has been studied extensively in Groeneboom et al. [1999]. What is needed for the invariance principle for random walks is the statement that the Kolmogorov diffusion, started at some point in the non-negative quadrant, converges weakly when the starting point converges to zero within the non-negative quadrant. Groeneboom et al. [1999] proves a special case of this result but we need the full statement as above for the invariance principles. Since the proof of this result is nontrivial and lengthy, and interesting on its own, we sketch the proof and talk about other results we have proved for the Kolmogorov diffusion in a section of its own.

Once the invariance principle for the centered random walk is proven, certain invariance principles for the case with drift under modified conditioning follow in a straightforward way. We also note these down in this section.

5.1.2 Assumptions and statement of results

Let $\{X_i\}_{i\in\mathbb{N}}$ be i.i.d. random variables with $\mathbf{E}[X_i] = 0, \mathbf{E}[X_i^2] = 1$ and $\mathbf{E}[|X_i|^{2+\delta}] < \infty$ for some $\delta > 0$ given and fixed throughout. For every starting point (x, y) define

$$S(n) = y + X_1 + \cdots + X_n, \quad n \in \mathbb{N}_0.$$

and

$$T(n) = x + S(1) + \cdots + S(n) = x + ny + nX_1 + (n-1)X_2 + \cdots + X_n, \quad n \in \mathbb{N}_0.$$

We define the Markov chain $Z(n) = (T(n), S(n)), n \in \mathbb{N}_0$. We want to study the process under the condition that $T(n)$ remains positive. Therefore, the following stopping time plays a crucial role.

$$\tau_{(x,y)} = \inf\{n \geq 0 : T(n) \leq 0\}.$$

Note that we can write the law of motion of the Markov chain Z explicitly as follows.

$$Z(n+1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} Z(n) + X_{n+1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad n \geq 0, \{X_n, n \geq 1\} \text{ are i.i.d.} \tag{5.1}$$

It is easy to establish that, a two-dimensional Markov chain started at some $Z_0 \in \mathbf{R}^2$ has as components an integrated random walk and the corresponding random walk, if and only if it satisfies the law of motion as in (5.1).

The continuous counterpart of the chain Z, which appears in the limits in the invariance principles, is $W_{(x,y)} = (U,V)_{(x,y)}$, the Kolmogorov Diffusion, started at $(x,y) \in \mathbf{R}_+ \times \mathbf{R}$ and conditioned to stay in $\mathbf{R}_+ \times \mathbf{R}$ up to time 1.

Let $X^{(n)}(t) = \left(\frac{T(tn)}{n^{\frac{3}{2}}}, \frac{S(tn)}{n^{\frac{1}{2}}}\right)$ for $t \in [0,1]$ be the scaled discrete-time process. This will be considered a process in $D[0,1]$, which is equipped with the supremum norm $||\cdot||_\infty$.

Theorem 7. *The process $X^{(n)}(t)$ for $t \in [0,1]$ conditioned on $\{\tau > n\}$ and started at some fixed $(x,y) \in \mathbf{R}_+ \times \mathbf{R}_+$ converges to the Kolmogorov meander started at $(0,0)$ of length one.*

Time length of one has been chosen only due to convenience, as one can easily extend the results above to the following more general case by time scaling. In the following with $\Longrightarrow$ we denote weak convergence.

Corollary 3 (Meander of length t)**.** *For $(x,y) \in \mathbf{R}_+ \times \mathbf{R}_+$ and $t \in (0,\infty)$ we have*

$$((X^{(n)}(s))_{s\le t}|\tau > nt)_{(x,y)} \Longrightarrow (W(\cdot)_{(0,0)}|\tau^{bm} > t), \quad n\to\infty.$$

Note that this result implies in particular the limit theorem for $X^{(n)}(1)$, which is already known from Denisov and Wachtel [2015a].

For the invariance principle for bridges we look at steps of random walk defined on a lattice of $\mathbf{R}$. Without any loss of generality, we assume for the next statements that X_1 is supported on $\mathbb{Z}$ and that the resulting random walk is *aperiodic*. Aperiodicity with respect to $\mathbb{Z}$ means that for every $x \in \mathbb{Z}$, the smallest subgroup of $\mathbb{Z}$ which contains the set

$$\{y : y = x + z \text{ with some } z \text{ such that } \mathbf{P}(X_1 = z) > 0\},$$

is $\mathbb{Z}$ itself.

The necessary modifications for the general case of a lattice $a + b\mathbb{Z}$, with maximal span $b > 0$ are clear. We can then state the invariance principle for the meander as follows. We denote by a_n the scaling function $a_n(x,y) = (n^{-\frac{3}{2}}x, n^{-\frac{1}{2}}y), n \in \mathbb{N}, (x,y) \in \mathbb{Z}_+ \times \mathbb{Z}$.

Theorem 8. *For each $x,u,y,v \in \mathbb{Z}_+$ the process $(X^{(n)}(\cdot)|X^{(n)}(1) = a_n(u,v), \tau > n)$ started at $a_n(x,y)$ converges weakly to a process Y. Y is a continuous Markov process with $Y|_{D[0,t]}$ absolutely continuous w.r.t. $\mathbf{P}_0(W \in \cdot\ |\tau^{bm} > t)$. Moreover, $Y_t \searrow 0$, as $t \nearrow 1$ a.s.*

We also derive the density of Y_t w.r.t. Lebesgue measure. We call the process Y the *Kolmogorov excursion (from zero and back)*. It fulfills all obvious reasons to be called that way: it is continuous, absolutely continuous w.r.t. meander measure started at zero of length one and converges to zero as t goes to one. The proof of the invariance principle for bridges follows in broad lines the strategy already used in Caravenna and Chaumont [2013] for proving invariance principle for bridges of random walks conditioned to stay positive.

This is already used in the proof of the invariance principle for bridges in chapter 3. To recall the broad steps: first prove weak convergence up to time $t \in [0,1]$; then time-reverse the process and use convergence towards the meander to prove weak convergence for time $[t,1]$; this also incidentally proves tightness for the whole process.

Analogously to the case of meander, the restriction to time length one is artificial so that the following corollary is straightforward.

Corollary 4 (Bridge of length t)**.** *For each $x, u, y, v \in \mathbb{Z}_+$ and $t > 0$ the process $(X^{(n)(\cdot)} | X^{(n)}(1) = a_n(u,v), \tau > tn)$ started at $a_n(x,y)$ converges weakly to a process $Y^{(t)}$. $Y^{(t)}$ is a continuous Markov process with $Y_s^{(t)}$ absolutely continuous w.r.t. $\mathbf{P}_0(W_s \in \cdot \,|\tau^{bm} > s)$. Moreover, $Y_s^{(t)} \searrow 0$, as $s \nearrow t$ a.s.*

We also prove functional convergence of V-transforms for the integrated random walks conditioned to be positive. The limit process is the h-transform of the Kolmogorov diffusion introduced in 2.6.

Formally, the V-transform started at $x, y > 0$ is defined as

$$\mathbf{P}^{(V)}_{(x,y)}(Z(n) \in d(u,v)) = \mathbf{P}_{(x,y)}(Z(n) \in d(u,v), \tau > n)\frac{V(u,v)}{V(x,y)}.$$

The proof follows the same strategy as that of Theorem 2. We need an estimate for the difference between the harmonic function of the integrated random walk V and that of the harmonic function for the Kolmogorov diffusion h. This estimate is given whenever the step of the random walk has moments strictly above $\frac{5}{2}$.

Theorem 9. *Assume that $\mathbf{E}[|X|^{\frac{5}{2}+\epsilon_0}] < \infty$ for some $\epsilon_0 > 0$. Then for every fixed $(x,y) \in \mathbf{R}_+ \times \mathbf{R}_+$ the process $X^{(n)}$ under $\mathbf{P}^{(V)}$ converges weakly to the Kolmogorov diffusion under $\mathbf{P}^{(h)}_{(0,0)}$.*

Finally, we note that even though so far in the thesis, we have chosen to formulate our convergence results in $D[0,1]$ if we define $X^{(n)}(t) = \left(\frac{T(tn)}{n^{\frac{3}{2}}}, \frac{S(tn)}{n^{\frac{1}{2}}}\right)$ for $tn \in \{1, \dots, n\}$ and through linear interpolation for other $t \in [0,1]$, we can prove the invariance principles for the integrated random walk on $(C[0,1], ||\cdot||_\infty)$ as well. This is because $\mathbf{R}_+ \times \mathbf{R}$ is convex. The limit processes are the same and the proofs are similar.

5.2 Proofs

5.2.1 Proof of convergence towards Kolmogorov meander

Let $X_n(t) = (\frac{T_{tn}}{n^{\frac{3}{2}}}, \frac{S_{tn}}{n^{\frac{1}{2}}})$ for $t \in [0,1]$ and let τ be the stopping time defined above. When we don't annotate or mention it explicitly, it is assumed the processes are started at zero.

Let now $f : D[0,1] \to [0,1]$ be a bounded and uniformly continuous function. First, we introduce several helpful notations. For $k \le n, z \in \mathbf{R}_+ \times \mathbf{R}, W \in D[0,1]$ we define

$$f(z,k,W) = f(z\mathbf{1}_{\{t \le \frac{k}{n}\}} + W(t)\mathbf{1}_{\{t > \frac{k}{n}\}}).$$

The scaling function a_n is defined by $a_n(z) = (\frac{z_1}{n^{\frac{3}{2}}}, \frac{z_2}{n^{\frac{1}{2}}})$ and for $z = (x, y)$ we define $z^+ = (x + n^{\frac{3}{2}-\gamma}, y), z^- = (x - n^{\frac{3}{2}-\gamma,}, y)$. We also define

$$K_{n,\epsilon} = \{z \in \mathbf{R}_+ \times \mathbf{R} : y > 0, x \geq n^{\frac{3}{2}-3\epsilon}\},$$

for the set of values deep inside $\mathbf{R}_+ \times \mathbf{R}$. Note that for $\epsilon > 0$ small enough, $z^+, z^- \in K_{n,\epsilon'}$ if $z \in K_{n,\epsilon}$, where $\epsilon' > \epsilon$. Also let

$$\gamma_n = \inf\{k \geq 0 : Z_k \in K_{n,\epsilon}\}$$

be the corresponding entry time. This denotes the first time Z is deep inside $\mathbf{R}_+ \times \mathbf{R}$. Intuitively speaking, because of the moment assumption, under the conditioning, the process Z will spend most of the time in $K_{n,\epsilon}$ so that γ_n will be relatively small. For some $\theta_n \to 0, n \to \infty$ and so that $\theta_n \sqrt{n} \to \infty$ we use $\alpha(z) = \max\{|x|^{\frac{1}{3}}, |y|\}$ to define

$$L_{n,\epsilon} = \{z \in K_{n,\epsilon} : \alpha(z) \leq \theta_n \sqrt{n}\} = \{z : 0 < y \leq \theta_n \sqrt{n},\ n^{\frac{3}{2}-3\epsilon} \leq x \leq \theta_n^3 n^{\frac{3}{2}}\}.$$

Finally, we define for future reference the random variables $L_n := \max_{k \leq n} \alpha(Z_k)$ and $M_n := \max_{k \leq n} |S_k|$.

To show the invariance principle we have to show w.lo.g.

$$\mathbf{E}_{(x,y)}[f(X^{(n)})|\tau > n] \to \mathbf{E}_{(0,0)}[f(W)|\tau^{bm} > 1], \quad n \to \infty.$$

For the proof we will use coupling, as stated in Proposition 3. As a first step, we see easily that for all $\epsilon > 0$ small enough only the part

$$\mathbf{E}[f(X_n), \gamma_n \leq n^{1-\epsilon}|\tau > n]$$

contributes something to the limit. This is because

$$\mathbf{E}_{(x,y)}[f(X^{(n)}), \gamma_n > n^{1-\epsilon}|\tau > n] \leq \frac{\mathbf{P}_{(x,y)}(\gamma_n > n^{1-\epsilon}, \tau > n^{1-\epsilon})}{\mathbf{P}_{(x,y)}(\tau > n)} \leq C(x,y)e^{-n^{\epsilon'}}$$

with some suitable $\epsilon' > 0$ small enough. This follows from Corollary 3 and Lemma 12 in Denisov and Wachtel [2015a].
As a second step, we prove

$$\frac{\mathbf{P}_{(x,y)}(\tau > n, \gamma_n \leq n^{1-\epsilon}, L_{\gamma_n} > \theta_n \sqrt{n})}{\mathbf{P}_{(x,y)}(\tau > n)} = o(1).$$

Due to Markov property we have

$$\mathbf{P}_{(x,y)}(\tau > n, \gamma_n \leq n^{1-\epsilon}, L_{\gamma_n} > \theta_n \sqrt{n}) \leq \sum_{k \leq n^{1-\epsilon}} \int_{K_{n,\epsilon}} \mathbf{P}_z(\tau > n - n^{1-\epsilon}) \mathbf{P}(Z_{\gamma_n} \in dz, \tau > k = \gamma_n)$$

$$\leq C \frac{1}{n^{\frac{1}{4}}} \mathbf{E}_{(x,y)}[h(Z_{\gamma_n}), \gamma_n \leq n^{1-\epsilon} \tau > \gamma_n, L_{\gamma_n} > \theta_n \sqrt{n}],$$

where we have used, that

$$\mathbf{P}_z(\tau > n) \le C\frac{h(z)}{n^{\frac{1}{4}}}, \text{ uniformly in } z \in K_{n,\epsilon}, n \ge 1.$$

This is contained in Lemma 18 in Denisov and Wachtel [2015a]. We now show that

$$O\left(\mathbf{E}[h(Z_{\gamma_n}), \gamma_n < n^{1-\epsilon}\tau > \gamma_n, L_{\gamma_n} > \theta_n\sqrt{n}]\right) = o(1). \tag{5.2}$$

We use Lemma 6 in Denisov and Wachtel [2015a] to see, that on $\{\gamma_n \le n^{1-\epsilon}\}$ we have $h(Z_{\gamma_n}) \le CL_{n^{1-\epsilon}}^{\frac{1}{2}}$. Therefore,

$$\mathbf{E}_{(x,y)}[h(Z_{\gamma_n}), \gamma_n < n^{1-\epsilon}\tau > \gamma_n, L_{\gamma_n} > \theta_n\sqrt{n}] = O(\mathbf{E}_{(x,y)}[L_{n^{1-\epsilon}}^{\frac{1}{2}}, L_{n^{1-\epsilon}} > \theta_n\sqrt{n}]).$$

Since

$$L_{n^{1-\epsilon}} \le n^{\frac{1-\epsilon}{3}} M(n^{1-\epsilon})^{\frac{1}{3}} + M(n^{1-\epsilon}),$$

we have

$$\begin{aligned}
\limsup_{n\to\infty} \mathbf{E}_{(x,y)}[L_{n^{1-\epsilon}}^{\frac{1}{2}}, L_{n^{1-\epsilon}} > \theta_n\sqrt{n}] &\le \limsup_{n\to\infty} \mathbf{E}_{(x,y)}[n^{\frac{1-\epsilon}{6}} M(n^{1-\epsilon})^{\frac{1}{6}}, M(n^{1-\epsilon}) > \theta_n^3 n^{\epsilon}\sqrt{n}] \\
&\quad + \limsup_{n\to\infty} \mathbf{E}_{(x,y)}[M(n^{1-\epsilon})^{\frac{1}{2}}, M(n^{1-\epsilon}) > \theta_n\sqrt{n}] \\
&\le 2\mathbf{E}_{(x,y)}[M(n^{1-\epsilon})^{\frac{1}{2}}, M(n^{1-\epsilon}) > \theta_n\sqrt{n}],
\end{aligned}$$

for $\epsilon > 0$ small enough and θ_n going to zero slowly enough ($\theta_n^2 n^{\epsilon} \to \infty$ is sufficient).

Kolmogorov inequality now gives

$$\mathbf{E}_{(x,y)}[M(n^{1-\epsilon})^{\frac{1}{2}}, M(n^{1-\epsilon}) > \theta_n\sqrt{n}] = o(1),$$

if θ_n goes to zero slowly enough ($\theta_n^2 n^{\epsilon} \to \infty$ is again sufficient). In all, the contributing part to the limit is

$$\frac{\mathbf{E}_{(x,y)}[f(X_n), \tau > n, \gamma_n \le n^{1-\epsilon}, L_{\gamma_n} \le \theta_n\sqrt{n}]}{\mathbf{P}(\tau > n)}.$$

We note first that we have

$$\begin{aligned}
&\sup_{t\in[0,1]} \left| \left(a_n(Z(\gamma_n))\mathbf{1}_{\{t\le \frac{k}{n}\}} + X_n(t)\mathbf{1}_{\{t>\frac{k}{n}\}}\right) - X_n(t)\right| \\
&\le \max_{k\le\gamma_n} |a_n(Z(k) - Z(\gamma_n))| \le \frac{2M(\gamma_n)}{\sqrt{n}} \le 2\theta_n
\end{aligned}$$

on the set $\{L_{\gamma_n} \le \theta_n\sqrt{n}\}$. Let

$$C_n = \{\gamma_n \le n^{1-\epsilon}, L_{\gamma_n} \le \theta_n\sqrt{n}\} \in \mathcal{F}_{\gamma_n}.$$

It follows from the uniform continuity of f that

$$\mathbf{E}_{(x,y)}[f(X^{(n)}), C_n | \tau > n] = o(1) + \mathbf{E}_{(x,y)}[f(a_n(Z_{\gamma_n}), \gamma_n, X^{(n)}), C_n | \tau > n].$$

Therefore, to prove the theorem, it suffices to consider convergence of

$$\mathbf{E}_{(x,y)}[f(a_n(Z_{\gamma_n}), \gamma_n, X^{(n)}), C_n | \tau > n].$$

Note first that we can write with the Markov property

$$\mathbf{E}_{(x,y)}[f(a_n(Z_{\gamma_n}), \gamma_n, X_n), C_n, \tau > n] =$$
$$\sum_{k \le n^{1-\epsilon}} \int_{K_{n,\epsilon}} \mathbf{P}(\gamma_n = k, \tau > k, L_{\gamma_n} \le \theta_n \sqrt{n}, Z_k \in dz) \mathbf{E}_z[f(a_n(z), k, X^{(n)}) | \tau_z > n-k] \mathbf{P}_z(\tau > n-k).$$

As a next step want to show

$$\mathbf{E}_z[f(a_n(z), k, X^{(n)}) | \tau_z > n-k] = (1+o(1)) \mathbf{E}_{(0,0)}[f(B) | \tau^{bm} > 1], \tag{5.3}$$

uniformly for $z \in L_{n,\epsilon}, k \le n^{1-\epsilon}$. With (5.3), it would follow

$$\mathbf{E}_{(x,y)}[f(a_n(Z_{\gamma_n}), \gamma_n, X^{(n)}), C_n | \tau > n] = (1+o(1)) \mathbf{E}_{(0,0)}[f(B) | \tau^{bm} > 1] \frac{\mathbf{P}(\tau > n, C_n)}{\mathbf{P}(\tau > n)},$$

and we know from calculations in subsection 3.3 in Denisov and Wachtel [2015a], that

$$\frac{\mathbf{P}(\tau > n, C_n)}{\mathbf{P}(\tau > n)} = 1 + o(1).$$

This would finish the proof. Therefore, the rest of this section focuses on proving (5.3).

Define in a common probability space $S(n)$ as above and a Brownian motion $B(u)$

$$A_n = \{\sup_{u \le n} |S(u) - B(u)| \le n^{\frac{1}{2} - \gamma}\}.$$

Proposition 3 gives that

$$\mathbf{P}(A_n^c) = o(n^{-r}),$$

with $r = r(\delta, \gamma) = -2\gamma - \gamma\delta + \frac{\delta}{2}$.

First, we note, that

$$\begin{aligned}
\mathbf{E}_z[f(a_n(z), k, X^{(n)}) | \tau_z > n-k] &= \mathbf{E}_z[f(a_n(z), k, X^{(n)}), A_n | \tau_z > n-k] \\
&\quad + \mathbf{E}_z[f(a_n(z), k, X^{(n)}), A_n^c | \tau_z > n-k] \\
&= \mathbf{E}_z[f(a_n(z), k, X^{(n)}), A_n | \tau_z > n-k] + o(1),
\end{aligned}$$

for all $\epsilon > 0$ sufficiently small. This is because of $\mathbf{P}(A_n^c) = o(n^{-r})$ and the fact that uniformly for all $z \in L_{n,\epsilon}, k \le n^{1-\epsilon} : o(n^{-r}) = o(\mathbf{P}(\tau_z > n-k))$. The latter follows from the proof of Lemma 18 in Denisov and Wachtel [2015a].

Define

$$W^{(n)}(t) = \left(\frac{\int_0^{nt} B_s ds}{n^{\frac{3}{2}}}, \frac{B_{nt}}{n^{\frac{1}{2}}} \right).$$

Due to uniform continuity of f we have in A_n, uniformly for all $k \le n^{1-\epsilon}, z \in K_{n,\epsilon}$ that

$$f(a_n(z), k, X^{(n)}) = f(a_n(z), k, W^{(n)}) + o(1). \tag{5.4}$$

We know that

$$|a_n(z) - a_n(z^{\pm})| \le O(n^{-\gamma}), \tag{5.5}$$

uniformly in $z \in K_{n,\epsilon}$. One also has

$$\mathbf{P}_{z^{\pm}}(\tau^{bm} > n) = \chi h(z) n^{-\frac{1}{4}}(1 + o(1)) = \mathbf{P}_z(\tau > n), \tag{5.6}$$

uniformly in $k \le n^{1-\epsilon}, z \in K_{n,\epsilon}$. These follow from considerations in subsection 3.2 of Denisov and Wachtel [2015a].
Using (5.4),(5.5) and (5.6) and the scaling property of Brownian motion, it is easy to see that

$$\mathbf{E}_z[f(a_n(z), k, X^{(n)}), A_n | \tau_z > n - k] = (1 + o(1))\mathbf{E}_{a_n(z^{\pm})}[f(a_n(z^{\pm}), k, W)||\tau^{bm} > 1].$$

In all, we have shown that it suffices to prove

$$\mathbf{E}_{a_n(z^{\pm})}[f(a_n(z^{\pm}), k, W)|\tau^{bm} > 1] \rightarrow \mathbf{E}_{(0,0)}[f(W)|\tau^{bm} > 1], \; n \rightarrow \infty, \tag{5.7}$$

uniformly for $k \le n^{1-\epsilon}, z \in L_{n,\epsilon}$.

Notice that (2.41), together with the mild condition (2.46), implies immediately for every $\beta > 0$

$$\lim_{n\to\infty} \sup_{z \in \mathbf{R}_+ \times \mathbf{R}_+, \alpha(z) \le n^{-\beta}} \left| \mathbf{E}_z[f(W)|\tau^{bm} > 1] - \mathbf{E}_{(0,0)}[f(W)|\tau^{bm} > 1] \right| = 0. \tag{5.8}$$

Note also that for $z \in L_{n,\epsilon}$ we have $a_n(z) = O(n^{-\beta})$ uniformly for some $\beta > 0$ small, if we choose $\theta_n = n^{-p}$ for some $p > 0$ small enough.

We now show that

$$|\mathbf{E}_{a_n(z^{\pm})}[f(a_n(z^{\pm}), k, W)|\tau^{bm} > 1] - \mathbf{E}_{a_n(z^{\pm})}[f(W)|\tau^{bm} > 1]| \rightarrow 0, \tag{5.9}$$

uniformly in $k \le n^{1-\epsilon}, z \in L_{n,\epsilon}$ to complete the proof of (5.7) by the use of triangle inequality.

The difference of arguments of $f(a_n(z^{\pm}), k, W)$ and $f(W)$ is up to a constant bounded above by

$$\max_{t,s \in [0,1], |t-s| \le n^{-\epsilon}} |W_t - W_s| \text{ uniformly for all } k \le n^{1-\epsilon}.$$

Recalling that f is uniformly continuous, we can write for each $\delta' > 0$

$$\begin{aligned} &\left|\mathbf{E}_{a_n(z^{\pm})}[f(a_n(z^{\pm}), k, W)|\tau^{bm} > 1] - \mathbf{E}_{a_n(z^{\pm})}[f(W)|\tau^{bm} > 1]\right| \\ &\le C\delta' + 2\mathbf{P}_{a_n(z^{\pm})}\left(\max_{|t-s| \le n^{-\epsilon}} |W_t - W_s| \ge \delta | \tau^{bm} > 1 \right), \end{aligned}$$

for some $\delta > 0$ small enough. We now show that

$$\limsup_{n\to\infty} \mathbf{P}_{a_n(z^\pm)}(\max_{|t-s|\le n^{-\epsilon}} |W_t - W_s| \ge \delta | \tau^{bm} > 1) = 0. \tag{5.10}$$

Since the sets

$$\Gamma_n = \{\max_{|t-s|\le n^{-\epsilon}} |W_t - W_s| \ge \delta\},$$

are closed subsets of the metric space $(C[0,1], ||\cdot||_\infty)$, and moreover these sets are monotonously decreasing in n we can always write for fixed $m \in \mathbb{N}$

$$\limsup_{n\to\infty} \mathbf{P}_{a_n(z^\pm)}(\max_{|t-s|\le n^{-\epsilon}} |W_t - W_s| \ge \delta | \tau^{bm} > 1) \le \mathbf{P}_{(0,0)}(\Gamma_m | \tau^{bm} > 1),$$

uniformly for $z \in L_{n,\epsilon}$. The measure $\mathbf{P}_{(0,0)}(\cdot\ | \tau^{bm} > 1)$ is gained by convergence of densities from measures which are absolutely continuous w.r.t. to the distribution of W on $C[0,\infty)$. This implies that $\mathbf{P}_{(0,0)}(\cdot\ | \tau^{bm} > 1)$ is also absolutely continuous w.r.t. to the distribution of W on $C[0,\infty)$. Therefore, we have that $\Gamma_m \searrow \emptyset, m \to \infty$ implies $\mathbf{P}_{(0,0)}(\cdot\ | \tau^{bm} > 1)$-a.s. as well. This shows (5.10), which in turn implies (5.9). (5.8) and (5.9) imply (5.7), which in turn shows (5.3). This finishes the proof of Theorem 7.

Proof of Corollary 3 is straightforward from a time rescaling.

5.2.2 Proof of invariance principle for bridges

We now assume that the random walk moves on $\mathbb{Z}$ and is aperiodic, as described in subsection 5.1.2.

As a first step, we note the following easy property.

Lemma 6 (Time reversal). *For all $z, y \in \mathbb{Z}_+ \times \mathbb{Z}, n \in \mathbb{N}$ we have the following relation*

$$\mathbf{P}_z(Z(n) = y, \tau > n) = \mathbf{P}_{\tilde{y}}(\tilde{Z}(n) = \tilde{z}, \tilde{\tau} > n),$$

where $\tilde{Z}$ is the two-dimensional Markov chain started at $(y_1, -y_2)$, that satisfies the law of motion

$$\tilde{Z}(l+1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tilde{Z}(l) + \tilde{X}_{l+1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad l \ge 0, \{\tilde{X}_s, s \ge 1\} \text{ are i.i.d}, \tag{5.11}$$

$\tilde{X}_1$ distributed as X_1.

$\tilde{Z}$ behaves asymptotically like the Markov chain in (5.1) started at $(y_1, -y_2)$.

Proof. We have $(T_0, S_0) = (z_1, z_2)$, the recursion

$$T_l = T_{l-1} + S_l, \quad S_l = S_{l-1} + X_l, \quad 1 \le l \le n. \tag{5.12}$$

and $(T_n, S_n) = (y_1, y_2)$. Define $(\tilde{T}_0, \tilde{S}_0) = (y_1, -y_2)$ and the recursion

$$\tilde{T}_l = \tilde{T}_{l-1} + \tilde{S}_{l-1}, \quad \tilde{S}_l = \tilde{S}_{l-1} + X_{n-l+1}, \quad 1 \le l \le n. \tag{5.13}$$

Then we have $\tilde{S}_l = S_{n-l}$ and $\tilde{T}_l = T_{n-l}$. Therefore, this path transformation is such that the condition of positiveness of $\tilde{T}$ process started at $(y_1, -y_2)$ and ending in $(x_1, -x_2)$ in n-steps is preserved throughout, whenever it is preserved for the original process T started at (x_1, x_2) and ending at (y_1, y_2).

Even more is true: there is a one-to-one and measure-preserving map between paths of Z started at x and ending at y at time n, and paths of $\tilde{Z}$ started at $(y_1, -y_2)$ and ending at $(x_1, -x_2)$ at time n. In particular, one can gain results about the asymptotic behavior of $\tilde{Z}$, conditioned on not leaving $\mathbf{R}_+ \times \mathbf{R}$ until time n, from the ones about Z under the same condition. □

Remark 6. *Note that Lemma 6 clarifies that in* (2.52) *(see also equation (13) in Denisov and Wachtel [2015a]) the function V' is related to the harmonic function for the Markov chain as in* (5.1) *where the innovations of the underlying random walk are distributed as $-X_1$. Namely, if $\tilde{V}$ is the said harmonic function, then $V'(y) = \tilde{V}((y_1, -y_2))$ for $(y_1, y_2) \in \mathbf{R}_+ \times \mathbf{R}$.*

We will show: given $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{Z}_+ \times \mathbb{Z}$ and $0 \le f \le 1$ bounded and uniformly continuous and measurable w.r.t. $(D[0,t], ||\cdot||_\infty)$ for some $t \in (0,1)$ we have

$$\mathbf{E}_{a_n(x_1,x_2)}[f(X_n)|\tau > n, X_n(1) = a_n(y_1,y_2)] \to \mathbf{E}_{(0,0)}[g(W)f_t(W_t)|\tau^{bm} > 1], \quad n\to\infty,$$

with some f_t bounded and continuous. In particular, this also establishes a density for the limit process of the scaled bridges up until time t.

As a first step note that for $\mathcal{C}$ measurable w.r.t. Borel σ- algebra on $D[0,t]$ the following holds.

$$\begin{aligned}\mathbf{P}_{(x_1,x_2)}[\mathcal{C}|\tau > n, T_n = y_1, S_n = y_2] &= \frac{\mathbf{E}_x[\mathbf{1}_\mathcal{C}, \tau > nt, \mathbf{P}_{Z(nt)}(Z(n(1-t)) = y, \tau > n(1-t))]}{\mathbf{P}_x(\tau > n, T_n = y_1, S_n = y_2)} \\ &= \mathbf{E}_x[\mathbf{1}_\mathcal{C} f_t^n(X_n(t), a_n(x), a_n(y)|\tau > nt],\end{aligned}$$

with the definition

$$\begin{aligned}\mathbf{R}_+ \times \mathbf{R} \ni z \mapsto f_t^n(z,x,y) &= \frac{\mathbf{P}_z(X_n(1-t) = a_n(y), \tau > n(1-t))\mathbf{P}_x(\tau > nt)}{\mathbf{P}_x(\tau > n, X_n(1) = a_n(y))} \\ &= \frac{\mathbf{P}_{b_n(z)}(Z(n(1-t)) = y, \tau > n(1-t))\mathbf{P}_x(\tau > nt)}{\mathbf{P}_x(\tau > n, Z(n) = y)},\end{aligned}$$

where $b_n(\tilde{z}) = a_n^{-1}(\tilde{z})$ for $\tilde{z} \in \mathbb{Z}^2$. One can show the following Lemma with the help of relations in Denisov and Wachtel [2015a].

Lemma 7. *For $x, y \in \mathbb{Z}_+ \times \mathbb{Z}$ we have*

$$\lim_{n\to\infty} \sup_{z \in \mathbb{Z}_+ \times \mathbb{Z},} |f_t^n(z,x,y) - f_t(z)| = 0,$$

where $f_t(z) = \varkappa^2 t^{-\frac{1}{4}}(1-t)^{-2-\frac{1}{4}} h(z_1, -z_2) p_1(0,0; z_1, -z_2)$ is continuous and bounded.

Proof. We have

$$f_t^n(z,x,y) = \frac{\mathbf{P}_{b_n(z)}(Z(n(1-t)) = y, \tau > n(1-t))\mathbf{P}_x(\tau > nt)}{\mathbf{P}_x(\tau > n, T(n) = y_1, Z(n) = y_2)}$$
$$= \frac{\mathbf{P}_{\tilde{y}}(\tilde{Z}(n(1-t)) = b_n(\tilde{z}), \tau > n(1-t))\mathbf{P}_x(\tau > nt)}{\mathbf{P}_x(\tau > n, T(n) = y_1, S(n) = y_2)},$$

where in the last step we used Lemma 6.

First, the conditional local limit theorem from Denisov and Wachtel [2015a] (see (2.51) here or relation (12) there) yields uniformly for all $z \in \mathbb{Z}_+ \times \mathbb{Z}$ (with $\tilde{z} = (z_1, -z_2)$)

$$\mathbf{P}_{\tilde{y}}(\tilde{Z}(n(1-t)) = b_n(\tilde{z}), \tilde{\tau} > n(1-t))$$
$$\sim \varkappa n^{-2-\frac{1}{4}}(1-t)^{-2-\frac{1}{4}}V'(y)h(\tilde{z})p_1(0,0;\tilde{z}). \tag{5.14}$$

Note that we also used the scaling property of $p_t(x,y,u,v)$ here.

Second, we get by using (2.50)

$$\mathbf{P}_x(\tau > nt) \sim \varkappa V(x) n^{-\frac{1}{4}} t^{-\frac{1}{4}}. \tag{5.15}$$

Finally, we use (2.52) in Denisov and Wachtel [2015a]

$$\mathbf{P}_x(\tau > n, T(n) = y_1, S(n) = y_2) \sim n^{-2-\frac{1}{2}} V(x)V'(y). \tag{5.16}$$

We get the result about f_t by combining (5.14), (5.15) and (5.16). □

For a given functional g on $(D[0,1], ||\cdot||_\infty)$, which is uniformly continuous and bounded, we can write with the triangle inequality

$$|\mathbf{E}_x[g(X)f_t^n(X_n(t),x,y)|\tau > nt] - \mathbf{E}_{(0,0)}[g(W)f_t(W_t)|\tau^{bm} > t]|$$
$$\leq \mathbf{E}_x[g(X)|f_t^n(X_n(t),x,y) - f_t(X_n(t))||\tau > nt]$$
$$+ \left|\mathbf{E}_x[g(X)f_t(X_n(t))|\tau > nt] - \mathbf{E}_{(0,0)}[g(W)f_t(W_t)|\tau^{bm} > t]\right|$$
$$=: A_n + B_n.$$

$A_n = o(1)$ follows from Lemma 7 and the boundedness of g. $B_n = o(1)$ can be proven using Skorohod's representation theorem. For more details, take some suitable probability space and define there $X_n(\cdot)$ with distribution $\mathbf{P}(\cdot\ |\tau > nt)$ and $W.$ with distribution $\mathbf{P}(\cdot\ |\tau^{bm} > t)$ so that $X_n(\cdot) \to W.$, $\quad n \to \infty$ a.s. This is possible due to Corollary 3. It then follows due to continuity and boundedness of f_t and g with the new measure, whose expectation we denote by E, that

$$|\mathbf{E}_x[g(X)f_t(X_{nt})|\tau > nt] - \mathbf{E}_{(0,0)}[g(W)f_t(W_t)|\tau^{bm} > t]|$$
$$= |E[g(X)f_t(X_{nt})] - E[g(W)f_t(W_t)]|$$
$$\leq CE[|f_t(X_{nt}) - f_t(W_t)|] + E[f_t(W_t)|g(W) - g(X_{nt})|].$$

Dominated convergence, together with boundedness and continuity of f, g gives the result.

In all, we have proven the following.

Lemma 8. *For each $t \in (0,1)$ and $x, y \in \mathbb{Z}_+ \times \mathbb{Z}$ it holds on $D[0,t]$ that*

$$(X_n | X_n(1) = a_n(y), \tau > n)_{a_n(x)} \text{ converges weakly as } n \to \infty.$$

The limit process $Y|_{D[0,t]}$ is absolutely continuous w.r.t. Kolmogorov meander measure $\mathbf{P}_0(W \in \cdot\, | \tau^{bm} > t)$ and has continuous and bounded density f_t.

In particular, this implies tightness of the measures $(X_n | X_n(1) = a_n(y), \tau > n)_{a_n(x)}$ in $D[0,t]$. Since we know the density of $\mathbf{P}_0(W_t \in \cdot | \tau^{bm} > t)$ (see Theorem 13 in the appendix), we get a Lebesgue density for Y_t.

Corollary 5. *Y_t has Lebesgue density*

$$\begin{aligned} g_t(z) &= f_t(z) p_t^+(z_1, z_2) \\ &= \varkappa t^{-\frac{1}{4}} (1-t)^{-2-\frac{1}{4}} h(z_1, -z_2) p_1(0,0; z_1, -z_2) \mathbf{P}_{(z_1,z_2)}(\tau^{bm} > 1-t) \bar{h}(t, u, -v). \end{aligned}$$

It is obvious from the construction and the weak convergence that the laws of $(Y_s)_{s \in (0,1)}$ are consistent, in the sense that for $s < t$ we have

$$Y_t|_{C[0,s]} \stackrel{d}{=} Y_s.$$

Moreover, if we take the process $\tilde{Z}$ as defined in (5.11) from Lemma 6, and denote by $\left(\tilde{X}^{(n)}\right)_{n \in \mathbb{Z}_+}$ the process defined through $\tilde{X}^{(n)}(t) = a_n(\tilde{Z}([nt])), t \in [0,1]$ then the following result follows immediately.

Lemma 9. *For each $t \in (0,1)$ and $x, y \in \mathbf{R}_+ \times \mathbf{R}$ we have on $(D[0,t], || \cdot ||_\infty)$*

$$(\tilde{X}_n | \tilde{X}_n(1) = a_n(y), \tau > n)_{a_n(x)} \text{ converges weakly as } n \to \infty.$$

From this we get tightness of $(\tilde{X}_n | \tilde{X}_n(1) = a_n(y), \tau > n)_{a_n(x)}$ in $D[0,t]$, which amounts to tightness of the processes $(X^{(n)} | X^{(n)}(1) = a_n(y), \tau > n)_{a_n(x)}$ in $D[1-t,1]$. Using the characterizations from Theorems 15.3 and 15.5 from Billingsley [1968] we get tightness on the whole $D[0,1]$. In all, we get weak convergence to a process in $D[0,1]$, whose density at any point is characterized in Corollary 5. From Theorem 15.5 in Billingsley [1968] we get automatically that the weak limit has to be continuous, i.e. an element of $C[0,1]$.

5.2.3 Proof of convergence of h-transforms

Here we prove Theorem 9. For this, we first note some sharp estimates between V as constructed in Denisov and Wachtel [2015a] and the harmonic function h for the Kolmogorov diffusion conditioned to stay positive.

Lemma 10. *Assume that $\mathbf{E}[|X|^{\frac{5}{2}+\epsilon_0}] < \infty$ for some $\epsilon > 0$. It holds*

$$|V(z) - h(z)| \le C \min\{1, \alpha(z)^{-\frac{3}{2}-\delta}\}, \quad z \in \mathbf{R}_+ \times \mathbf{R}. \tag{5.17}$$

This is a simple implication of equation (32) in Denisov and Wachtel [2015a] (proof of Lemma 11 there) and Lemma 7 in Denisov and Wachtel [2015a]. It has the following important Corollary, which we use repeatedly throughout the proof.

Corollary 6. *Assume that* $\mathbf{E}[|X|^{\frac{5}{2}+\epsilon_0}] < \infty$. *Then the following inequalities hold true.*

$$|V(z) - h(z)| \le C\alpha(z)^{\frac{1}{2}-\Delta}, \quad \alpha(z) \ge 1, z \in \mathbf{R}_+ \times \mathbf{R}. \tag{5.18}$$

In particular, it follows

$$V(z) \le C\alpha(z)^{\frac{1}{2}}, \quad \alpha(z) \ge 1, z \in \mathbf{R}_+ \times \mathbf{R}. \tag{5.19}$$

The Corollary follows directly from Lemma 10 using Lemma 6 in Denisov and Wachtel [2015a].

Proof of Theorem 9. We want to show that given a $f \in D[0,1]$ bounded and uniformly continuous and $(x,y) \in \mathbf{R}_+ \times \mathbf{R}_+$ fixed, we have

$$\mathbf{E}^{(V)}_{(x,y)}[f(X^{(n)})] \to \mathbf{E}^{(h)}_{(0,0)}[f(W)], \quad n \to \infty.$$

Assume w.l.o.g. that values of f are in $[0,1]$. Then,

$$\mathbf{E}^{(V)}_{(x,y)}[f(X^{(n)}), \gamma_n \ge n^{1-\epsilon}] \le \frac{1}{V(x,y)}\mathbf{E}_{(x,y)}[V(Z(n)), \tau > n^{1-\epsilon}, \gamma_n \ge n^{1-\epsilon}].$$

Using Markov property and the harmonicity of V, we get

$$\begin{aligned}\mathbf{E}_{(x,y)}[V(Z(n)), \tau > n^{1-\epsilon}, \gamma_n \ge n^{1-\epsilon}] &= \mathbf{E}_{(x,y)}\left[\mathbf{E}_{(x,y)}[V(Z(n), \tau > n | \mathcal{F}_{n^{1-\epsilon}}], \tau > n^{1-\epsilon}, \gamma_n \ge n^{1-\epsilon}\right] \\ &= \mathbf{E}_{(x,y)}[V(Z(n^{1-\epsilon})), \gamma_n \ge n^{1-\epsilon}, \tau \ge n^{1-\epsilon}].\end{aligned}$$

Recalling (5.19), we use Hölder inequality to obtain

$$\begin{aligned}&\mathbf{E}_{(x,y)}[V(Z(n^{1-\epsilon})), \gamma_n \ge n^{1-\epsilon}, \tau \ge n^{1-\epsilon}] \\ &\qquad \le C\mathbf{E}_{(x,y)}[\alpha(Z(n^{1-\epsilon}))]^{\frac{1}{2}} \left(\mathbf{P}_{(x,y)}(\gamma_n \ge n^{1-\epsilon}, \tau \ge n^{1-\epsilon})\right)^{\frac{1}{2}}.\end{aligned}$$

Note that $\alpha(Z(n^{1-\epsilon})) \le 2\max\{n^{(1-\epsilon)\frac{1}{3}}M(n^{1-\epsilon})^{\frac{1}{3}}, M(n^{1-\epsilon})\}$. Together with Doob's inequality for martingales and Lemma 12 in Denisov and Wachtel [2015a] this delivers

$$\mathbf{E}^{(V)}_{(x,y)}[f(X^{(n)}), \gamma_n \ge n^{1-\epsilon}] = o(1), \quad n \to \infty.$$

We look at

$$\begin{aligned}\mathbf{E}^{(V)}_{(x,y)}[f(X^{(n)}), \gamma_n \le n^{1-\epsilon}] &= \mathbf{E}^{(V)}_{(x,y)}[f(X^{(n)}), \gamma_n \le n^{1-\epsilon}, L_{\gamma_n} > \theta_n\sqrt{n}] \\ &\quad + \mathbf{E}^{(V)}_{(x,y)}[f(X^{(n)}), \gamma_n \le n^{1-\epsilon}, L_{\gamma_n} \le \theta_n\sqrt{n}] \\ &= A^{(n)} + B^{(n)}.\end{aligned}$$

$A^{(n)}$ can be estimated as follows

$$\mathbf{E}^{(V)}_{(x,y)}[f(X^{(n)}),\gamma_n\le n^{1-\epsilon},L_{\gamma_n}>\theta_n\sqrt{n}]\le\frac{C}{V(x,y)}(\mathbf{P}(L_{n^{1-\epsilon}}>\theta_n\sqrt{n})+\mathbf{E}[L^{\frac12}_{n^{1-\epsilon}},L_{n^{1-\epsilon}}>\theta_n\sqrt{n}]),$$

for $\theta_n\to 0$ slow enough. Note that $\{L_{n^{1-\epsilon}}>\theta_n\sqrt{n}\}\subset\{M_{n^{1-\epsilon}}>\theta_n^3\sqrt{n}\}$. It holds

$$\mathbf{P}(M_{n^{1-\epsilon}}>\theta_n^3\sqrt{n})=o(1),$$

due to Kolmogorov inequality. Moreover, if θ_n goes to zero slowly enough, we have

$$\max\{((n^{1-\epsilon})M_{n^{1-\epsilon}})^{\frac13},M_{n^{1-\epsilon}}\}=M_{n^{1-\epsilon}}$$

on the set $\{M_{n^{1-\epsilon}}>\theta_n^3\sqrt{n}\}$ for all n large enough. To show that $A^{(n)}=o(1)$ it thus suffices to show

$$\mathbf{E}[M^{\frac12}_{n^{1-\epsilon}},M_{n^{1-\epsilon}}>\theta_n^3\sqrt{n}]=o(1). \tag{5.20}$$

Simple combinations of the Hölder inequality and the Kolmogorov inequality can be used to show (5.20). We now look at $B^{(n)}$. Define now for $W\in D[0,1]$

$$f(z,k,W)=f(z\mathbf{1}_{\{t\le\frac{k}{n}\}}+W(t)\mathbf{1}_{\{t>\frac{k}{n}\}}).$$

It follows from uniform continuity of f, just as for the proof of the invariance principle for the meander, that

$$\begin{aligned}&\mathbf{E}^{(V)}_{(x,y)}[f(X^{(n)}),\gamma_n\le n^{1-\epsilon},L_{\gamma_n}\le\theta_n\sqrt{n}]\\&=o(1)+\mathbf{E}^{(V)}_{(x,y)}[f(a_n(Z_{\gamma_n}),\gamma_n,X^{(n)}),\gamma_n\le n^{1-\epsilon},L_{\gamma_n}\le\theta_n\sqrt{n}].\end{aligned}$$

We can use Markov property to write

$$\begin{aligned}&\mathbf{E}^{(V)}_{(x,y)}[f(a_n(Z_{\gamma_n}),\gamma_n,X^{(n)}),\gamma_n\le n^{1-\epsilon},L_{\gamma_n}\le\theta_n\sqrt{n}]\\&=\sum_{k=0}^{n^{1-\epsilon}}\frac{1}{V(x,y)}\int_{L_{n,\epsilon}}\mathbf{P}_{(x,y)}((Z(k)\in dz,\tau>k,\gamma_n=k,L_k\le\theta_n\sqrt{n})\\&\times\mathbf{E}_z[f(a_n(z),k,X^{(n)})V(Z(n-k)),\tau>n-k]\end{aligned} \tag{5.21}$$

To continue, we first show that

$$\mathbf{E}_z[V(Z(n-k)),A_n^c,\tau>n-k]=o(h(z)), \tag{5.22}$$

$$\mathbf{E}_z[h(W(n-k)),A_n^c,\tau^{bm}>n-k]=o(h(z)) \tag{5.23}$$

uniformly for $k\le n^{1-\epsilon},z\in K_{n,\epsilon},\alpha(z)\le\theta_n\sqrt{n}$. For the first expression note

$$\begin{aligned}&\mathbf{E}_z[V(Z(n-k)),A_n^c,\tau>n-k]\\&\qquad=\mathbf{E}_z[V(z+Z(n-k)),\alpha(z+Z(n-k))\le n^{\frac12+\eta},A_n^c,\tau>n-k]\\&\qquad+\mathbf{E}_z[V(z+Z(n-k)),\alpha(z+Z(n-k))>n^{\frac12+\eta},A_n^c,\tau>n-k]\\&\qquad=A^{(n)}+B^{(n)}.\end{aligned}$$

We get with help of (5.18) and (5.19)

$$A^{(n)} \leq C(1+n^{\frac{1}{4}+\frac{\eta}{2}})\mathbf{P}(A_n^c) \leq Cn^{\frac{1}{4}+\frac{\eta}{2}-r}. \tag{5.24}$$

Here $r = \frac{\delta}{2} - 2\gamma - \gamma\delta$ from Proposition 3. Furthermore, Lemma 6 in Denisov and Wachtel [2015a] implies

$$h(z) > cn^{\frac{1}{4}-\frac{\epsilon}{2}}, \quad z \in K_{n,\epsilon}. \tag{5.25}$$

It follows that $A^{(n)} = o(h(z))$, uniformly for $z \in K_{n,\epsilon}, \alpha(z) \leq \theta_n\sqrt{n}$, whenever $\epsilon, \eta > 0$ are chosen small enough. By an analogous logic, we have

$$\begin{aligned} B^{(n)} &\leq C\mathbf{E}_z[\alpha(Z(n-k))^{\frac{1}{2}}, \alpha(Z(n-k)) > n^{\frac{1}{2}+\eta}] \\ &\leq C\mathbf{E}[M_n^{\frac{1}{2}} + \alpha(z)^{\frac{1}{2}}, M_n > n^{\frac{1}{2}+\frac{\eta}{2}}] \leq Ch(z)n^{-\frac{1}{4}+\frac{\epsilon}{2}}E[M_n^{\frac{1}{2}} + \alpha(z)^{\frac{1}{2}}, M_n > n^{\frac{1}{2}+\frac{\eta}{2}}]. \end{aligned}$$

Note that because $\alpha(z) \leq \theta_n\sqrt{n}$ with $\theta_n \to 0$, we can focus on the term $M(n)$ in the above expression.

Integration by parts gives

$$\begin{aligned} &n^{-\frac{1}{4}+\frac{\epsilon}{2}}E[M_n^{\frac{1}{2}}, M_n > n^{\frac{1}{2}+\frac{\eta}{2}}] \\ &= n^{\frac{\epsilon}{2}+\frac{\eta}{4}}\mathbf{P}(M_n > n^{\frac{1}{2}+\frac{\eta}{2}}) + \frac{1}{4}n^{-\frac{1}{4}+\frac{\epsilon}{2}}\int_{n^{\frac{1}{2}+\frac{\eta}{2}}}^{\infty} x^{-\frac{1}{2}}\mathbf{P}(M_n > x)dx \end{aligned} \tag{5.26}$$

We recall Fuk-Nagaev-Borovkov inequalities from Proposition 2:

$$\mathbf{P}(|M_n| > x) \leq 2\left(\frac{en}{xy}\right)^{\frac{x}{y}} + n\mathbf{P}(|X(1)| > y). \tag{5.27}$$

One can use this estimate with $y = xn^{-\frac{\eta}{4}}$ to show, that (5.26) is asymptotically zero, uniformly for $z \in K_{n,\epsilon}, \alpha(z) \leq \theta_n\sqrt{n}$, if we choose $\epsilon > 0$ small enough and then, given this ϵ we choose $\eta > 0$ small enough. In all, we have shown (5.22). (5.23) follows along similar lines and uses estimates for Brownian motion instead of the Fuk-Nagaev-Borovkov inequalities.

We now use coupling. Note that

$$\{\tau_{z^-}^{bm} > n\} \cap A_n \subset \{\tau_z > n\} \cap A_n \subset \{\tau_{z^+}^{bm} > n\} \cap A_n.$$

Consider the event (here we use the notation $z = (x', y')$)

$$D_{n,k} := \{x' - n^{\frac{3}{2}-\gamma} + \int_0^{n-k} B(s)ds \geq 2n^{\frac{3}{2}-\gamma}\}.$$

Then on the event $D_{n,k} \cap A_n$ it holds for $Z = (T, S)$ started at z

$$T(n-k) \geq n^{\frac{3}{2}-\gamma} \quad \text{ultimately in } n.$$

Now it follows on $D_{n,k} \cap A_n$ that

$$|V(Z(n-k)) - h(Z(n-k))| \le C\alpha(Z(n-k))^{1-\Delta}, \tag{5.28}$$

for all $\Delta > 0$ sufficiently small.

We use this and (5.18) to get for a $\Delta > 0$ small,

$$\begin{aligned}
&\mathbf{E}_z[f(a_n(z), k, X^{(n)})V(Z(n-k)), \tau > n-k, A_n] \\
&\quad \ge \mathbf{E}_z[f(a_n(z), k, X^{(n)})h(Z(n-k)), \tau_{z^-}^{bm} > n-k, A_n \cap D_{n,k}] \\
&\quad - C\mathbf{E}_z[\alpha(Z(n-k))^{\frac{1}{2}-\Delta}, \tau_{z^-}^{bm} > n-k, A_n],
\end{aligned}$$

uniformly for $k \le n^{1-\epsilon}, z \in K_{n,\epsilon}, \alpha(z) \le \theta_n \sqrt{n}$.

Note that

$$\begin{aligned}
&\mathbf{E}_z[\alpha(Z(n-k))^{\frac{1}{2}-\Delta}, \tau_{z^-}^{bm} > n-k, A_n] \\
&\quad \le C\mathbf{E}_z[|T(n-k)|^{\frac{1}{3}(\frac{1}{2}-\Delta)} + |S(n-k)|^{\frac{1}{2}-\Delta}, \tau_{z^-}^{bm} > n-k, A_n] \\
&\quad \le C\mathbf{E}_{z^-}[\alpha(W(n-k))^{\frac{1}{2}-\Delta}, \tau_{z^-}^{bm} > n-k, A_n] + C\left(n^{\frac{1}{3}(\frac{3}{2}-\gamma)(\frac{1}{2}-\Delta)} + n^{(\frac{1}{2}-\gamma)(\frac{1}{2}-\Delta)}\right)\mathbf{P}_{z^-}(\tau^{bm} > n-k) \\
&\quad =: A + B.
\end{aligned}$$

It is clear that $B = o(u(z))$ as $n \to \infty$, uniformly for all $k \le n^{1-\epsilon}, z \in K_{n,\epsilon}, \alpha(z) \le \theta_n \sqrt{n}$. Here we have used, that

$$\mathbf{P}(\tau_{z^\pm}^{bm} > n) = \varkappa h(z)(1 + o(1)), \text{ uniformly in } z \in K_{n,\epsilon},$$

which is shown in the proof of Lemma 18 of Denisov and Wachtel [2015a].

Note that

$$A \le C\mathbf{E}_z[\alpha(W(n-k))^{\frac{1}{2}-\Delta}, \tau_z^{bm} > n-k, A_n] = C(n-k)^{\frac{1}{4}(1-\Delta)}\mathbf{E}_{a_{n-k}(z)}[h(W(1))^{1-\Delta}, \tau_{a_{n-k}(z)}^{bm} > 1].$$

From here, we use the scaling property of the Kolmogorov diffusion, to get for all n large so that uniformly for all $k \le n^{1-\epsilon}, z \in K_{n,\epsilon}, \alpha(z) \le \theta_n \sqrt{n}$ it holds $\alpha(a_{n-k}(z)) \le 1$, that

$$A \le C(n-k)^{\frac{1}{4}(1-\Delta)}h(a_{n-k}(z))\int_{\mathbf{R}_+ \times \mathbf{R}} \psi(\sigma)h(\sigma)^{1-\Delta}d\sigma.$$

We use the scaling property of the function h to substitute $h(a_{n-k}(z)) = (n-k)^{\frac{1}{4}}h(z)$. This shows that $A = o(u(z))$ as $n \to \infty$, uniformly for all $k \le n^{1-\epsilon}, z \in K_{n,\epsilon}, \alpha(z) \le \theta_n \sqrt{n}$.

Next we prove, that

$$\begin{aligned}
&\mathbf{E}_z[h(Z(n-k)), \tau_{z^-}^{bm} > n-k, A_n] \\
&\qquad = \mathbf{E}_{z^-}[h(W(n-k)), \tau^{bm} > n-k, A_n] + o(h(z)),
\end{aligned} \tag{5.29}$$

uniformly for $k \leq n^{1-\epsilon}, z \in K_{n,\epsilon}, \alpha(z) \leq \theta_n \sqrt{n}$.
Lemma 6 in Denisov and Wachtel [2015a] shows

$$\left| \frac{\partial^{i+j} h}{\partial x^i \partial y^j}(x,y) \right| \leq C\alpha(x,y)^{\frac{1}{2}-3i-j}. \tag{5.30}$$

We have for some $\theta_n \to 0$ with help of Lemma 6 and Lemma 18 in Denisov and Wachtel [2015a]

$$\begin{aligned} &\mathbf{E}_{z^-}[h(W(n-k)), \alpha(W(n-k)) \leq \theta_n \sqrt{n}, \tau^{bm} > n-k, A_n] \\ &\leq C\theta_n^{\frac{1}{2}} n^{\frac{1}{4}} \mathbf{P}_{z^-}(\tau^{bm} > n-k) = o(h(z)). \end{aligned} \tag{5.31}$$

Here, we have used

$$h(z^{\pm}) = (1+o(1))h(z), \tag{5.32}$$

uniformly for $z \in K_{n,\epsilon}$, which is shown in the proof of Lemma 18 in Denisov and Wachtel [2015a].

Moreover, for all large enough n and if θ_n goes to zero slowly enough, it holds

$$\alpha(Z(n-k)) \leq \frac{1}{2}\theta_n \sqrt{n}, \quad \text{on } A_n, \text{ whenever } \alpha(W(n-k)) \leq \theta_n \sqrt{n}.$$

Here Z is started at z. We estimate

$$\begin{aligned} &\mathbf{E}_{z^-}[h(Z(n-k)), \alpha(W(n-k)) \leq \theta_n \sqrt{n}, \tau^{bm} > n-k, A_n] \\ &\leq \mathbf{E}_z[h(Z(n-k)), \alpha(Z(n-k)) \leq \theta_n \sqrt{n}, \tau > n-k, A_n]. \end{aligned}$$

We use this last inequality and an estimate analogous to (5.31) to get

$$\mathbf{E}_{z^-}[h(Z(n-k)), \alpha(W(n-k)) \leq \theta_n \sqrt{n}, \tau^{bm} > n-k, A_n] = o(h(z)). \tag{5.33}$$

For $\alpha(W(n-k)) > \theta_n \sqrt{n}$ it holds again for n large enough and θ_n going slowly enough to zero, that

$$\alpha(Z(n-k)) > \frac{1}{2}\theta_n \sqrt{n}.$$

We now use Taylor formula together with (5.30) to show that on $A_n \cap \{\alpha(W(n-k)) > \theta_n \sqrt{n}\}$ for n large enough and θ_n it holds

$$|h(Z(n-k)) - h(W(n-k))| = o(n^{\frac{1}{4}}). \tag{5.34}$$

Here Z is started at z and W at z^-.

To see (5.34) note that in general we have

$$|h(z+w) - h(z)| \leq C\left(|w_1|\alpha(z+t_0 w)^{-\frac{5}{2}} + |w_2|\alpha(z+t_0 w)^{-\frac{1}{2}} \right), \tag{5.35}$$

for some suitable $t_0 \in [0,1]$. Replace here z by $W(n-k)$ and w by $Z(n-k) - W(n-k)$. It holds

$$\alpha(w) \leq Cn^{\frac{1}{2}-\frac{1}{3}\gamma},$$

and

$$\alpha(z+t_0 w) \geq C\alpha(z) - \alpha(w) \geq C\theta_n\sqrt{n},$$

for all n large enough, whenever θ_n goes to zero slowly enough. Here we have used the inequality $\alpha(z_1+z_2) \leq C\left(\alpha(z_1)+\alpha(z_2)\right)$ for $z_1, z_2 \in \mathbf{R}^2$.

We combine this with (5.35) to arrive at

$$|h(Z(n-k)) - h(W(n-k))| \leq Cn^{\frac{1}{4}-\gamma}\left(\theta_n^{-\frac{5}{2}} + \theta_n^{-\frac{1}{2}}\right) = o(n^{\frac{1}{4}}), \quad n\to\infty,$$

whenever θ_n goes to zero slowly enough.

It follows that

$$\begin{aligned}&\mathbf{E}_z[|h(Z(n-k)) - h(W(n-k)), \alpha(W(n-k)) > \theta_n\sqrt{n}, \tau^{bm}_{z^-} > n-k, A_n] \\ &\quad = o(n^{\frac{1}{4}})\mathbf{P}_{z^-}(\tau^{bm} > n-k) = o(h(z)),\end{aligned}$$

uniformly for $k \leq n^{1-\epsilon}, z \in K_{n,\epsilon}, \alpha(z) \leq \theta_n\sqrt{n}$. This finishes the proof of (5.29).

We return to the main proof. So far we have shown

$$\begin{aligned}&\mathbf{E}_z[f(a_n(z), k, X^{(n)})V(Z(n-k)), \tau > n-k, A_n] \\ &\quad \geq \mathbf{E}_z[f(a_n(z), k, X^{(n)})h(W(n-k)), \tau^{bm}_{z^-} > n-k, A_n \cap D_{n,k}] + o(h(z)),\end{aligned}$$

uniformly for $k \leq n^{1-\epsilon}, z \in K_{n,\epsilon}, \alpha(z) \leq \theta_n\sqrt{n}$.

It holds

$$\begin{aligned}&\mathbf{E}_{z^-}[h(W(n-k)), \tau^{bm}_{z^-} > n-k, D^c_{n,k}] \\ &\leq (n-k)^{\frac{1}{4}}\mathbf{P}_{z^-}(\tau^{bm} > n-k)\mathbf{E}_{a_{n-k}(z^-)}[h(a_{n-k}(W(n-k))), \tilde{D}_{n,k} | \tau^{bm} > 1],\end{aligned}$$

with $\tilde{D}_{n,k} = \{\frac{x'}{n^{\frac{3}{2}}} - n^{-\gamma} + \int_0^1 B(s)ds \leq 4n^{-\gamma}\}$. Due to convergence towards the meander, as well as the properties of the function ψ, the expectation converges to zero as $n\to\infty$. Using Lemma 15 in Denisov and Wachtel [2015a] together with (5.32) it follows for all n large enough that

$$\begin{aligned}&\mathbf{E}_z[f(a_n(z), k, X^{(n)})V(Z(n-k)), \tau > n-k, A_n] \\ &\quad \geq \mathbf{E}_z[f(a_n(z), k, X^{(n)})h(W(n-k)), \tau^{bm}_{z^-} > n-k, A_n] + o(h(z)),\end{aligned} \tag{5.36}$$

uniformly for $k \leq n^{1-\epsilon}, z \in K_{n,\epsilon}, \alpha(z) \leq \theta_n\sqrt{n}$.

Analogous to above, we can prove

$$\begin{aligned}&\mathbf{E}_z[f(a_n(z), k, X^{(n)})V(Z(n-k)), \tau > n-k, A_n] \\ &\quad \leq \mathbf{E}_z[f(a_n(z), k, X^{(n)})h(W(n-k)), \tau^{bm}_{z^+} > n-k, A_n] + o(h(z)),\end{aligned} \tag{5.37}$$

uniformly for $k \le n^{1-\epsilon}, z \in K_{n,\epsilon}, |z_2| \le \theta_n\sqrt{n}$.

In the proof of Theorem 7 we have shown that

$$f(a_n(z), k, X^{(n)}) = f(a_n(z), k, W^{(n)}) + o(1), \tag{5.38}$$

uniformly for $k \le n^{1-\epsilon}, z \in K_{n,\epsilon}, |z_2| \le \theta_n\sqrt{n}$. Here $W^{(n)} = a_n(W)$ is the rescaled Kolmogorov diffusion. (5.38) implies that

$$\begin{aligned}&\mathbf{E}_z[|f(a_n(z), k, X^{(n)}) - f(a_n(z), k, W^{(n)})|h(W(n-k)), \tau^{bm}_{z^\pm} > n-k, A_n] \qquad (5.39)\\ &\quad= o\left(\mathbf{E}_{z^\pm}[h(W(n-k)), \tau^{bm} > n-k]\right) = o(h(z)),\end{aligned}$$

uniformly for $k \le n^{1-\epsilon}, z \in K_{n,\epsilon}, |z_2| \le \theta_n\sqrt{n}$. Here we have used the harmonicity of h as well as (5.32). From this we infer, after performing a scaling, that

$$\begin{aligned}&h(z^-)\mathbf{E}^{(h)}_{a_n(z^-)}[f(a_n(z), k, W^{(n)})] + o(h(z))\\ &\le \mathbf{E}_z[f(a_n(z), k, X^{(n)})V(Z(n-k)), \tau > n-k, A_n]\\ &\le h(z^+)\mathbf{E}^{(h)}_{a_n(z^+)}[f(a_n(z), k, W^{(n)})] + o(h(z)),\end{aligned}$$

uniformly for $k \le n^{1-\epsilon}, z \in K_{n,\epsilon}, |z_2| \le \theta_n\sqrt{n}$.

Given the convergence of the h-transforms for the Kolmogorov diffusion, we arrive at

$$\mathbf{E}_z[f(a_n(z), k, X^{(n)})V(Z(n-k)), \tau > n-k, A_n] = h(z)\mathbf{E}^{(h)}_0[f(W)] + o(h(z)), \tag{5.40}$$

uniformly for $k \le n^{1-\epsilon}, z \in K_{n,\epsilon}, |z_2| \le \theta_n\sqrt{n}$. We return to (5.21), which implies with (5.40) that

$$\begin{aligned}&\mathbf{E}^{(V)}_{(x,y)}[f(a_n(Z(\gamma_n)), \gamma_n, X^{(n)}), \gamma_n \le n^{1-\epsilon}, L_{\gamma_n} \le \theta_n\sqrt{n}] \qquad (5.41)\\ &\quad= \frac{\mathbf{E}^{(h)}_0[f(W)] + o(1)}{V(x,y)}\mathbf{E}_{(x,y)}[h(Z(\gamma_n)), \gamma_n \le n^{1-\epsilon}, \tau > \gamma_n, L_{\gamma_n} \le \theta_n\sqrt{n}]. \qquad (5.42)\end{aligned}$$

A combination of Lemmas 20 and 21 from Denisov and Wachtel [2015a] shows

$$\lim_{n\to\infty} \mathbf{E}_{(x,y)}[h(Z(\gamma_n)), \gamma_n \le n^{1-\epsilon}, \tau > \gamma_n, L_{\gamma_n} \le \theta_n\sqrt{n}] = V(x,y).$$

This finishes the proof.

□

Appendices

Anhang A

On Kolmogorov diffusion conditioned to be positive

A.1 Introduction

We recall the definition of the Kolmogorov diffusion. Given $\{B_t, t \geq 0\}$ a standard Brownian motion, the Kolmogorov diffusion started at (x, y) is the process given by

$$W_{(x,y)}(t) = (U_t, V_t)_{(x,y)} = \left(x + ty + \int_0^t B_s ds,\ y + B_t \right), \quad t \geq 0.$$

The Kolmogorov diffusion has been studied since 1934, when it first occurred in Kolmogorov [1934]. The Kolmogorov diffusion with the first coordinate conditioned to be positive and started at the origin is used in various contexts (see Lachal [2003, p.111] for a list of applications). We recall the transition density of the Kolmogorov diffusion, which can be calculated via expectations and covariances of this Gaussian process (see McKean [1962]). For $x, y, u, v \in \mathbb{R}$ and $t > 0$ it is given by

$$\begin{aligned} p_t(x, y; u, v) &= \frac{\sqrt{3}}{\pi t^2} \exp\left(-\frac{6(u - x - ty)^2}{t^3} + \frac{6(v - y)(u - x - ty)}{t^2} - \frac{2(v - y)^2}{t} \right) \qquad \text{(A.1)} \\ &= \frac{\sqrt{3}}{\pi t^2} \exp\left(-\frac{6(u - x)^2}{t^3} + \frac{6(u - x)(v + y)}{t^2} - \frac{2(v^2 + vy + y^2)}{t} \right). \end{aligned}$$

In McKean [1962] the behavior of a Kolmogorov diffusion started in $(0, -|z|)$ that is stopped, when it hits zero is studied. Lachal [1991] has done a more general analysis, where the process can start in any (x, y) and is stopped when its first coordinate hits a given level.

Lemma 11 (Joint distribution of (τ_a, V_{τ_a})-Lachal [1991])**.** *The joint distribution of (τ_a, V_{τ_a}), where τ_a is the stopping time $\tau_a := \inf_{t>0}\{U_t = a\}$, is determined by the density f_t, which* *is recursively defined by*

1. $f_s(0,-|z|;0,w) := \mathbb{P}_{(0,-|z|)}\big(\tau_0 \in ds, V_{\tau_0} \in dw\big)/ds\,dw =$
$$= \frac{3|w|}{\pi\sqrt{2\pi}s^2} e^{-(2/s)(z^2-|zw|+w^2)} \int_0^{4|zw|/s} \theta^{-1/2} e^{-(3\theta/2)}\,d\theta$$

and

2. $f_t(x,y;a,z) := \mathbb{P}_{(x,y)}\big(\tau_a \in dt, V_{\tau_a} \in dz\big)/dt\,dz =$
$$= |z|\Big[p_t(x,y,a,z) - \int_0^t \int_0^\infty f_s(0,-|z|;0,w)p_{t-s}(x,y,a,-\varsigma w)\,dw\,ds\Big]1_A(z),$$

where $A = [0,\infty)$ *if* $x < a$, $A = (-\infty,0]$ *if* $x > a$ *and* ς *is the sign of* $(a-x)$.

Using this, Lachal (see Lachal [2003, p.128]) described the transition density of (U,V) killed when U hits $a \in \mathbb{R}$.

Lemma 12 (Formula for $\overline{p}$)**.** *For* $t,x,u > 0$ *and* $y,v \in \mathbb{R}$ *and where* A *is again* $(-\infty,0]$ *if* $x > a$ *and* $[0,\infty)$ *if* $x < a$ *we have*

$$\overline{p}_t^a(x,y;u,v) := \mathbb{P}_{(x,y)}\big((U_t,V_t) \in du\,dv, \tau_a > t\big)/du\,dv =$$
$$= p_t(x,y;u,v) - \int_0^t \int_A f_s(x,y;a,z)p_{t-s}(a,z;u,v)\,dz\,ds.$$

As the probability that a Kolmogorov diffusion started in (x,y) for $x > 0, y \in \mathbf{R}$ has a positive first coordinate for a given time $t > 0$ is positive, the corresponding conditioned process can be defined through Bayes' rule and is described entirely by $\overline{p}$ and the known probability $\mathbb{P}_{(x,y)}(\tau^{bm} > T)$. The existence of (U,V) conditioned to be positive forever and started in $(0,0)$ can be proven as a weak limit by showing the convergence of transition density and tightness of the corresponding family of probability measures. In Groeneboom et al. [1999] the authors have analyzed the convergence behavior of $\mathbb{P}_{(x,y)}(\tau_0 > T)$ for $T \to \infty$ and thus deduced that the transition density of $(\tilde{U},\tilde{V})^{(x,y)}$, the Kolmogorov diffusion with the first coordinate conditioned to be positive forever started in (x,y), has the form

$$h(x,y)^{-1}\overline{p}_t(x,y;u,v)h(u,v) \text{ for } x > 0 \text{ and } y \in \mathbb{R}, \tag{A.2}$$

In Groeneboom et al. [1999] the case of sending $(0,y) \downarrow 0$ for the conditioned process started in $(0,y)$ is also considered. Our aim is the density of the conditioned process started at the origin and running up until time 1.

Results of Groeneboom, Jongbloed and Wellner

In this subsection a part of the work done in Groeneboom et al. [1999] is outlined, which is needed for our own calculations. We begin with the definition of the auxiliary function h which is used to describe the behavior of $\mathbb{P}_{(x,y)}(\tau^{bm} > T)$ for $T \to \infty$. The function h can be expressed by another auxiliary function g, which is a function of one argument and satisfies the scaling property $h(x,y) = x^{1/6} g(x^{-1/3} y)$. As g can be expressed by confluent hypergeometric functions and their convergence behavior is known, we can calculate the convergence behavior of g and thus that of h. This subsection concludes with the introduction of the function $\overline{h}$, which appears in the asymptotics of $\bar{p}$.

Harmonic function h and the behavior of $\mathbb{P}_{(x,y)}(\tau_0 > T)$. We use the shorthand

$$q_t(x,y;u,v) := p_t(x,y;u,v) - p_t(x,y;u,-v).$$

The function $h : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ is defined by

$$h(x,y) := \int_{s=0}^{\infty} \int_{w=0}^{\infty} w^{3/2} q_s(x,y;0,-w) \mathrm{d}s \mathrm{d}w = \tag{A.3}$$
$$\frac{2\sqrt{3}}{\pi} \int_{s=0}^{\infty} \int_{w=0}^{\infty} w^{3/2} \exp(-6x^2 s^3 - 6xys^2 - 2(y^2 + w^2)s) \sinh(6xws^2 + 2yws) \mathrm{d}s \mathrm{d}w.$$

Groeneboom et al. [1999] characterize $\mathbb{P}_{(x,y)}(\tau_0 > T)$ as $T \to \infty$.

Lemma 13 (Groeneboom et al. [1999]). *If (U,V) starts at (x,y) with $x > 0, y \in \mathbb{R}$ we have*

$$\mathbb{P}_{(x,y)}(\tau_0 > T) \sim \frac{3\Gamma(1/4)}{2^{3/4}\pi^{3/2}} \frac{h(x,y)}{T^{1/4}} \text{ as } T \to \infty.$$

The asymptotic behavior of h. We analyze the convergence behavior of $\overline{p}$ and h. We start with h and in order to simplify the discussion we introduce from Groeneboom et al. [1999] the function $g : \mathbb{R} \to (0,\infty)$ defined by

$$g(y) := h(1,y) = \int_{s=0}^{\infty} \int_{w=0}^{\infty} w^{3/2} q_s(1,y;0,-w) \mathrm{d}s \mathrm{d}w. \tag{A.4}$$

We recall the scaling property of g, which can be easily shown from (A.3) by a change of variables $s \to x^{-2/3} s$ and $w \to x^{1/3} w$.

Lemma 14 (Representation of h via g). *The functions g and h are related by*

$$h(x,y) = x^{1/6} g(x^{-1/3} y).$$

Using results concerning confluent hypergeometric functions in Groeneboom et al. [1999] it is shown how the function g can be expressed in terms of two such functions, the Tricomi function U and the function V.

Tricomi's function U is defined as the unique solution of the confluent hypergeometric equation with the convergence behavior

$$U(a,c,z) \sim z^{-a} \sum_{s=0}^{\infty} (-1)^s \frac{(a)_s (1+a-c)_s}{s! z^s} \text{ for } z \to \infty \text{ in } |\mathrm{ph}(z)| < \frac{3}{2}\pi,$$

where ph denotes the phase of a complex number and $()_s$ is the so called Pochhammer's notation defined as

$$(a)_0 := 1 \text{ and } (a)_s := a(a+1)(a+2)\dots(a+s-1) \text{ for } s \in \mathbb{N}.$$

The confluent hypergeometric function V is defined as

$$V(a,c,z) := e^z U(c-a,c,-z) \text{ for } |\mathrm{ph}(z)| < \frac{1}{2}\pi, \mathrm{Re}(a) > 0.$$

See Olver [1974][p.256] for more on these definitions and further theory concerning confluent hypergeometric equations and functions. Now we can state the representation of g (see Groeneboom et al. [1999] for a proof).

Lemma 15. *The function g has the representation*

$$\begin{aligned} g(y) &= \ (\frac{2}{9})^{1/6} y \ U\left(\frac{1}{6}, \frac{4}{3}, \frac{2}{9}y^3\right) && \text{for } y > 0, \\ g(y) &= -(\frac{2}{9})^{1/6}\frac{1}{6} y \ V\left(\frac{1}{6}, \frac{4}{3}, \frac{2}{9}y^3\right) && \text{for } y < 0, \\ g(0) &= \ (\frac{2}{9})^{-1/6}\frac{\Gamma(1/3)}{\Gamma(1/6)}. \end{aligned}$$

Results about the density $\bar{p}$

The results proven in the remainder of Appendix A are included in Bär et al. [2020].

The main results of this section of the appendix are the following two Propositions.

Proposition 11. *There exists $R > 0, g : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}_+$ with the property*

$$\int_{\mathbf{R}_+}\int_{\mathbf{R}} g(u,v)h(u,v)dvdu < \infty,$$

so that,

$$\sup_{(x,y)\in\mathbf{R}_+\times\mathbf{R}_+, \alpha(x,y)\le\frac{1}{2}} \bar{p}_1(x,y;u,v) \le g(u,v)h(x,y), \quad \text{for } (u,v) \in \mathbf{R}_+\times\mathbf{R}; u \ge R \text{ or } |v| \ge R.$$

The auxiliary function $\overline{h}$. We finally introduce another auxiliary function, which is also used in Groeneboom et al. [1999].

The function $\overline{h}$ is defined by

$$\overline{h}(t,x,y) := \frac{4\sqrt{3}}{\sqrt{2\pi}} \int_{s=0}^{t}\int_{w=0}^{\infty} w^{3/2}s^{-1/2}p_s(0,w;0,0)q_{t-s}(x,y;0,-w)\mathrm{d}s\mathrm{d}w. \tag{A.5}$$

With this definition we can state the asymptotic of the density $\bar{p}_t(x,y;u,v)$ as $x,y \searrow 0$.

Proposition 12. *For $x, y \searrow 0$ we have*

$$\overline{p}_t(x, y; u, v) \sim h(x, y)\overline{h}(t, u, -v),$$

uniformly in (u, v) with $\alpha(u, v) \le R$ ($R > 0$ arbitrary).

Note that the proof of Proposition 12 given here establishes formally also the result stated in Lemma 16 of Denisov and Wachtel [2015a].

Before giving the proofs of the Propositions 11 and 12, we give an application. We use Proposition 12 to calculate the density of the meander $W_{(0,0)}(\cdot|\tau^{bm} > 1)$.

Proposition 13. *The transition density of the process $W_{(0,0)}(\cdot\ |\tau^{bm} > 1)$ is proportional to $\mathbf{P}_{(u,v)}(\tau^{bm} > 1 - t)\bar{h}(t, u, -v)$.*

Proof. We look at the convergence of densities, i.e. of

$$\frac{\bar{p}_t(x, y; u, v)\mathbf{P}_{(u,v)}(\tau^{bm} > 1 - t)}{\mathbf{P}_{(x,y)}(\tau^{bm} > 1)}$$

as $x, y \searrow 0$. We use Proposition 12 $\overline{p}_t(x, y, u, v) \sim h(x, y)\overline{h}(t, u, -v)$. The scaling property of Brownian motion, implies

$$\mathbf{P}_{(x,y)}(\tau^{bm} > 1) = \mathbf{P}_{(\lambda^{\frac{3}{2}}x, \lambda^{\frac{1}{2}}y)}(\tau^{bm} > \lambda).$$

This implies in turn, that

$$\mathbf{P}_{(x,y)}(\tau^{bm} > 1) = \mathbf{P}_{(1, yx^{-\frac{1}{3}})}(\tau^{bm} > x^{-\frac{2}{3}}) \quad \text{for } x^{\frac{1}{3}} \ge y,$$
$$\mathbf{P}_{(x,y)}(\tau^{bm} > 1) = \mathbf{P}_{(xy^{-3},1)}(\tau^{bm} > y^{-2}) \quad \text{for } x^{\frac{1}{3}} < y.$$

Using uniform continuity of $h(1, t), h(t, 1)$ as functions in t, Lemma 15 from Denisov and Wachtel [2015a] with $\theta_t = \max\{x^{\frac{1}{6}}, y^{\frac{1}{2}}\}, t = x^{-\frac{2}{3}}$ and the scaling properties of h we get in all

$$\mathbf{P}_{(x,y)}(\tau^{bm} > 1) \sim \varkappa h(x, y), \text{ as } x, y \searrow 0$$

The limit density is $C\mathbf{P}_{(u,v)}(\tau^{bm} > 1)\bar{h}(t, u, -v)$, where C is a suitable normalization constant. □

We calculate here also the density of the meander of length $t \ne 1$. For this we note that if the meander is started at some point $(x, y) \in \mathbf{R}_+ \times \mathbf{R}_+$ then the density at some time $s \le t$ is

$$p_t^+(s; x, y, ; u, v) = \frac{\bar{p}_s(x, y; u, v)\mathbf{P}_{(u,v)}(\tau^{bm} > t - s)}{\mathbf{P}_{(x,y)}(\tau^{bm} > t)}.$$

Here we can write

$$\mathbf{P}_{(x,y)}(\tau^{bm} > t) = \mathbf{P}_{(xt^{-\frac{3}{2}}, yt^{-\frac{1}{2}})}(\tau^{bm} > 1) \sim h(xt^{-\frac{3}{2}}, yt^{-\frac{1}{2}}) = t^{-\frac{1}{4}}h(x, y), \text{ as } x, y \searrow 0$$

Using then again $\overline{p}_s(x, y; u, v) \sim h(x, y)\overline{h}(s, u, -v)$ from above we get

$$p_t^+(s; x, y, ; u, v) \sim t^{\frac{1}{4}}\bar{h}(s, u, -v)\mathbf{P}_{(u,v)}(\tau^{bm} > t - s) \text{ as } x, y \searrow 0.$$

In all, we have proven the following.

Corollary 7. *The density of meander of length $t \in (0,\infty)$ with start point in 0 is*

$$p_t^+(s;u,v) = Ct^{\frac{1}{4}}\bar{h}(s,u,-v)\mathbf{P}_{(u,v)}(\tau^{bm} > t-s), \quad s \le t, (u,v) \in \mathbf{R}_+ \times \mathbf{R}.$$

Auxiliary results for the proofs

Using the Lemmas above we can analyze the convergence behavior of the function h.

Lemma 16 (Convergence behavior of h)**.** *For $x \downarrow 0$ and $y \to 0$ we have*

A. $h(x,y) \sim x^{1/6}g(c)$ as $x \downarrow 0$ and $y \to 0$ such that $x^{-1/3}y \to c$ for $c \in \mathbb{R}$.

B. $h(x,y) \sim y^{1/2}$ as $x \downarrow 0$ and $y \to 0$ such that $x^{-1/3}y \to +\infty$.

C. $h(x,y) \sim \frac{3}{4}x(-y)^{-5/2}e^{\frac{2}{9}\frac{y^3}{x}}$ as $x \downarrow 0$ and $y \to 0$ such that $x^{-1/3}y \to -\infty$.

Proof. A. This is a simple consequence of the representation of h and g.

B. For $y \to +\infty$ we have

$$g(y) = \left(\frac{2}{9}\right)^{\frac{1}{6}} yU\left(\frac{1}{6},\frac{4}{3},\frac{2}{9}y^3\right) \sim \left(\frac{2}{9}\right)^{\frac{1}{6}} y\ \frac{2}{9}y^3\Big)^{-\frac{1}{6}} = y^{\frac{1}{2}},$$

where we have used the definition of U. Thus, we have for h

$$h(x,y) = x^{\frac{1}{6}}g(x^{-\frac{1}{3}}y) \sim x^{\frac{1}{6}}(x^{-\frac{1}{3}}y)^{\frac{1}{2}} = y^{\frac{1}{2}}.$$

C. For $y \to -\infty$ we have

$$\begin{aligned} g(y) &= -\left(\frac{2}{9}\right)^{\frac{1}{6}}\frac{1}{6}yV\left(\frac{1}{6},\frac{4}{3},\frac{2}{9}y^3\right) = -\left(\frac{2}{9}\right)^{\frac{1}{6}}\frac{1}{6}ye^{\frac{2}{9}y^3}U\left(\frac{7}{6},\frac{4}{3},-\frac{2}{9}y^3\right) \\ &\sim -\left(\frac{2}{9}\right)^{\frac{1}{6}}\frac{1}{6}ye^{\frac{2}{9}y^3}\left(-\frac{2}{9}y^3\right)^{-\frac{7}{6}} = \frac{3}{4}(-y)^{-\frac{5}{2}}e^{\frac{2}{9}y^3}. \end{aligned}$$

Here we have used the properties of U. Thus we have for h

$$h(x,y) = x^{\frac{1}{6}}g(x^{-\frac{1}{3}}y) \sim x^{\frac{1}{6}}\frac{3}{4}(-x^{-\frac{1}{3}}y)^{-\frac{5}{2}}e^{\frac{2}{9}\frac{y^3}{x}} = \frac{3}{4}x(-y)^{-\frac{5}{2}}e^{\frac{2}{9}\frac{y^3}{x}}.$$

□

Before giving our proof of Propositions 11 and 12 we establish some useful estimates in the following Lemma.

Lemma 17. *(a) Define the quadratic forms $Q_1(u,v,z) = 6u^2 - 6uv + 2v^2 + 2z^2 - 6uz + 2vz$ and $Q_2(u,v,z) = 6u^2 - 6uv + 2v^2 + 2z^2 + 6uz - 2vz$. Then it follows*

$$Q_1(u,v,z) \ge c[(z+u)^2 + (v-u)^2],$$

$$Q_2(u,v,z) \geq c[(z-u)^2 + (v-u)^2].$$

(b) Define

$$\tilde{\rho}(u,v,z) := (u+v)\exp(-(u-v)^2)\left[\exp(-(z-u)^2) + \exp(-(z+u)^2)\right].$$

Then the function $\mathbf{R}_+ \times \mathbf{R} \ni (u,v) \mapsto h(u,v)\sup_{|z|\leq\epsilon}\tilde{\rho}(u,v,z)$ *is integrable for any* $\epsilon > 0$.

(c) For all $u, z > 0$ *and* v

$$p_1(u,-v;0,\pm z) \leq \frac{\sqrt{3}}{\pi}\exp\left(-6u^2 - v^2 + 9 - z^2\right).$$

(d) For all $u, z > 0, v$ *and* $s \in (0,1)$

$$|q_{1-s}(u,-v;0,\pm z)| \leq \frac{C}{(1-s)^2}\exp\left(-\frac{6u^2}{(1-s)^3} - \frac{v^2+z^2}{1-s}\right).$$

Proof. (a) This can be proved through lengthy algebra. Another way forward is to note that $Q_1(\tilde{z}) = \tilde{z}^t A\tilde{z}, \tilde{z} \in \mathbf{R}^3$ with the positive semidefinite matrix

$$A = \begin{pmatrix} 6 & -3 & -3 \\ -3 & 2 & 1 \\ -3 & 1 & 2 \end{pmatrix}.$$

A has eigenvalues $0, 1, 9$ and diagonalization given by

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}.$$

From here, simple algebra delivers the estimate for Q_1.

Similarly, $Q_2(\tilde{z}) = \tilde{z}^t B\tilde{z}, \tilde{z} \in \mathbf{R}^3$ with the positive semidefinite matrix

$$B = \begin{pmatrix} 6 & -3 & 3 \\ -3 & 2 & -1 \\ 3 & -1 & 2 \end{pmatrix}.$$

B has eigenvalues $0, 1, 9$ and diagonalization given by

$$B = \begin{pmatrix} -1 & 0 & 2 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{6} & \frac{1}{6} \end{pmatrix}.$$

From here, simple algebra delivers the estimate for Q_2.

(b) Recall that $h(u,v) \leq c\alpha(u,v)^{\frac{1}{2}}$. Note also that $u+v \leq c(\alpha(u,v)^3 + \alpha(u,v))$. Hence we concentrate on the exponential part

$$\iota : (u,v,z) \mapsto \exp(-(u-v)^2)\left[\exp(-(z-u)^2) + \exp(-(z+u)^2)\right].$$

Suppose that $|u-v| \leq 1$. Then trivially $u \geq R \implies v \geq R-1$ and $v \geq R \implies u \geq R-1$. Hence this region does not cause problems for integrability.

Suppose that $u \geq v+1$. In particular, $u \geq v$. Then for every $z \leq \epsilon$ it holds $\iota(u,v,z) \leq C_\epsilon \exp(-\frac{1}{2}(u^2+v^2))$. This holds true both when $v \geq 0$ and when $v < 0$.

Finally, suppose that $v \geq u+1$. In particular, $v > 0$. Then $\exp(-(u-v)^2) \leq \exp(-v)\exp(u)$, so that altogether, for every $|z| \leq \epsilon$, it holds $\iota(u,v,z) \leq C_\epsilon \exp(-v)\exp(c_\epsilon(u-u^2))$. Overall, it follows that the function is integrable over the region $|u-v| \geq 1$ also. This finishes the proof of part (b).

(c) Recall that

$$\begin{aligned} p_1(u,-v;0,z) &= \frac{\sqrt{3}}{\pi}\exp\left(-6u^2+6(v-z)-2(z^2-vz+v^2)\right) \\ &= \frac{\sqrt{3}}{\pi}\exp\left(-6u^2-2v^2+(v-z)(6+2z)\right). \end{aligned}$$

Maximize the parabola $-v^2+(v-z)(6+2z)$ w.r.t v and substitute the value of the maximized function in the exponent. This delivers the statement.

(d) This is straightforward from (c) and the scaling property of $\bar{p}$. □

Next, we register a helpful result about densities (see Lachal [2003, p.128]).

Lemma 18 (Equation for the densities)**.** *For $x,y,u,v \in \mathbb{R}$ we have*

$$\begin{aligned} p_t(x,y;u,v) &= p_t(u,-v;x,-y), \\ \overline{p}_t(x,y;u,v) &= \overline{p}_t(u,-v;x,-y). \end{aligned}$$

As a next step we prove a decomposition for $\bar{p}_1(x,y;u,v)$, which is needed in the proof of the Propositions 11 and 12.

Lemma 19. *It holds true for x,$u > 0$ and y,$v \in \mathbb{R}$*

$$\begin{aligned} \overline{p}_t(x,y;u,v) &= p_t(x,y;u,v) - p_t(x,y;-u,-v) \\ &\quad - \int_0^t \int_0^\infty q_{t-s}(u,-v;0,z)|z|q_s(x,y;0,-z)\,dw\,dr\,dz\,ds \\ &\quad + \int_0^1 \int_0^\infty \int_0^s \int_0^\infty zq_{1-s}(u,-v;0,z)f_r(0,-z;0,w)q_{s-r}(x,y;0,-w)dwdrdzds. \end{aligned}$$

Proof. Recalling Lemma 12, we have for x,$u > 0$ and y,$v \in \mathbb{R}$ that

$$\overline{p}_t(x,y;u,v) = p_t(x,y;u,v) - \int_0^t \int_{-\infty}^0 f_s(x,y;0,z)p_{t-s}(0,z;u,v)\mathrm{d}z\mathrm{d}s. \tag{A.6}$$

By $p_t(x,y;u,v) = p_t(u,-v,x,-y)$ (see Lemma 18) and the substitution $z \to -z$ equals

$$p_t(x,y;u,v) - \int_0^t \int_0^\infty f_s(x,y;0,-z)p_{t-s}(u,-v;0,z)\mathrm{d}z\mathrm{d}s. \tag{A.7}$$

As $q_t(x,y;u,v) := p_t(x,y,u,v) - p_t(x,y;u,-v)$, (A.7) equals

$$p_t(x,y;u,v) - \int_0^t \int_0^\infty f_s(x,y;0,-z)(q_{t-s}(u,-v;0,z) + p_{t-s}(u,-v;0,-z))\mathrm{d}z\mathrm{d}s. \tag{A.8}$$

Because every path which goes from (x,y) to $(-u,-v)$ in time t has to pass through $\{0\} \times (-\infty, 0]$ at least once, we have using the symmetry of density, that

$$\begin{aligned} &p_t(x,y;-u-v) \\ =&\int_0^t \int_0^\infty f_s(x,y;0,-z)p_{t-s}(0,-z;-u,-v))\mathrm{d}z\mathrm{d}s \\ =&\int_0^t \int_0^\infty f_s(x,y;0,-z)p_{t-s}(0,z;u,v))\mathrm{d}z\mathrm{d}s \\ =&\int_0^t \int_0^\infty f_s(x,y;0,-z)p_{t-s}(u,-v;0,-z))\mathrm{d}z\mathrm{d}s. \end{aligned} \tag{A.9}$$

Applying (A.9) to (A.8), we get that $\overline{p}_t(x,y;u,v)$ equals

$$p_t(x,y;u,v) - p_t(x,y;-u-v) - \int_0^t \int_0^\infty f_s(x,y;0,-z)q_{t-s}(u,-v;0,z)\mathrm{d}z\mathrm{d}s. \tag{A.10}$$

Now we look closer at the double integral in (A.10) and apply Lemma 11:

$$\begin{aligned} &\int_0^t \int_0^\infty f_s(x,y;0,-z)q_{t-s}(u,-v;0,z)\mathrm{d}z\mathrm{d}s = \\ &\int_0^t \int_0^\infty q_{t-s}(u,-v;0,z)|z|\Big[p_s(x,y;0,-z) - \\ &\qquad \int_0^s \int_0^\infty f_r(0,-|z|;0,w)p_{s-r}(x,y;0,-w)\mathrm{d}w\mathrm{d}r\Big]\mathrm{d}z\mathrm{d}s. \end{aligned} \tag{A.11}$$

Since every path from $(0,-z)$ to $(x,-y)$ hits $\{0\} \times [0,\infty)$ at least once, we have

$$\begin{aligned} &\int_0^s \int_0^\infty f_r(0,-z;0,w)p_{s-r}(x,y;0,-w))\mathrm{d}w\mathrm{d}r \\ =&\int_0^s \int_0^\infty f_r(0,-z;0,w)p_{s-r}(0,w;x,-y))\mathrm{d}w\mathrm{d}r \\ =&p_s(0,-z,x,-y) = p_s(x,y;0,z), \end{aligned}$$

where we have used Markov property. By this calculation and the definition of q the integrand of (A.11) is

$$q_{t-s}(u,-v;0,z)|z| \left[q_s(x,y;0,-z) - \int_0^s \int_0^\infty f_r(0,-|z|;0,w)q_{s-r}(x,y;0,w)dwdr\right].$$

This gives the result. □

A.2 Proofs of Propositions 11 and 12

Proof of Proposition 11. We recall the decomposition of $\bar{p}$ from Lemma 19.

$$\begin{aligned}\overline{p}_t(x,y;u,v) &= p_t(x,y;u,v) - p_t(x,y;-u,-v)\\ &- \int_0^t \int_0^\infty q_{t-s}(u,-v;0,z)|z|q_s(x,y;0,-z)\mathrm{d}w\mathrm{d}r\mathrm{d}z\mathrm{d}s\\ &+ \int_0^1 \int_0^\infty \int_0^s \int_0^\infty zq_{1-s}(u,-v;0,z)f_r(0,-z;0,w)q_{s-r}(x,y;0,-w)dwdrdzds.\end{aligned}$$

Step 1. First we deal with the part

$$p_1(x,y;u,v) - p_1(x,y;-u,-v).$$

Using the definition of p_1 this is equal to

$$\begin{aligned}&p_1(x,y;u,v) - p_1(x,y;-u,-v)\\ &= \frac{2\sqrt{3}}{\pi}\exp\left(-6(u^2+x^2)+6(uv-xy)-2(v^2+y^2)\right)\sinh\left(12ux+6(uy-vx)-2vy\right).\end{aligned}$$

Now recall, that there exists $c>0$ with

$$-6u^2 \pm 6uv - 2v^2 \le -c(u^2+v^2).$$

It follows

$$\exp\left(-6(u^2+x^2)+6(uv-xy)-2(v^2+y^2)\right) \le \exp\left(-c(u^2+v^2)\right)\exp\left(-c(x^2+y^2)\right).$$

Here, we have used $x,y>0$.

Using the estimate $\sinh(x) \le x\cosh(x) \le xe^x$ for $x \ge 0$ it follows that there exists some $c'>0$ with

$$\sinh\left(12ux+6(uy-vx)-2vy\right) \le (12|u|+6|v|)(x+y)\exp(c'(|u|+|v|)).$$

Here we have used that (x,y) is bounded. Recall from Lemma 6 in Denisov and Wachtel [2015a]

$$h(x,y) \ge c\max\{x^{\frac{1}{6}}, y^{\frac{1}{2}}\},$$

with $c>0$ suitable. In all, we have for suitable $\gamma>0$

$$|p_1(x,y;u,v) - p_1(x,y;-u,-v)| = o(1)h(x,y)\exp\left(-\gamma(u^2+v^2)\right), \text{ as } x,y \searrow 0.$$

Step 2. Next we look at

$$\frac{1}{h(x,y)}\int_0^1\int_0^\infty q_{1-s}(u,-v;0,z)|z|q_s(x,y;0,-z)dzds.$$

We first consider the integration domain $0<\epsilon<s<1$ for an $\epsilon\in(0,1)$ fixed throughout. Using the estimates $\sinh(x)\le|x|\cosh(x)$, Lemma 6 in Denisov and Wachtel [2015a], $\epsilon<s$ and the definition for p_s we have

$$\begin{aligned}
q_s(x,y;0,-z) &= \frac{2\sqrt{3}}{\pi s^2}\exp\left(-\frac{6x^2}{s^3}-\frac{6xy}{s^2}-\frac{2y^2+2z^2}{s}\right)\sinh\left(\frac{6xz}{s^2}+\frac{2yz}{s}\right)\\
&\le \left(\frac{6xz}{s^2}+\frac{2yz}{s}\right)(p_s(x,y;0,z)+p_s(x,y;0,-z))\\
&\le C_\epsilon z(x+y)=C_\epsilon zh(x,y)o(1),\ \text{ as } x,y\searrow 0.
\end{aligned}$$

In the last inequality we use extensively (A.1) and the fact that $\alpha(x,y)\le\frac{1}{2}$. For more details, note first that

$$\begin{aligned}
p_s(x,y;0,z)+p_s(x,y;0,-z) &= \frac{\sqrt{3}}{\pi s^2}\exp\left(-\frac{6x^2}{s^3}-\frac{6x(z+y)}{s^2}-\frac{2(z^2+zy+y^2)}{s}\right)\\
&\quad+\frac{\sqrt{3}}{\pi s^2}\exp\left(-\frac{6x^2}{s^3}-\frac{6x(y-z)}{s^2}-\frac{2(z^2-zy+y^2)}{s}\right).
\end{aligned}$$

Note also that the quadratic forms $(z,y)\mapsto z^2\pm zy+y^2$ are positive definite. This delivers easily that for $s\in[\epsilon,1],(x,y)\in\mathbf{R}_+\times\mathbf{R}_+$ such that $\alpha(x,y)\le\frac{1}{2}$, the sum $p_s(x,y;0,z)+p_s(x,y;0,-z)$ is bounded by a positive constant that depends on ϵ.

Now we look to control the integral

$$\int_\epsilon^1\int_0^\infty z^2q_{1-s}(u,-v;0,z)dzds.$$

Use (d) from Lemma 17 to get

$$\begin{aligned}
\int_\epsilon^1\int_0^\infty z^2q_{1-s}(u,-v;0,z)dzds &\le C\int_\epsilon^1\int_0^\infty z^2(1-s)^{-2}\exp\left(-\frac{6u^2+v^2+z^2}{1-s}\right)dzds\\
&= C\int_0^\infty t^2e^{-t^2}dt\cdot\int_0^{1-\epsilon}\frac{1}{\sqrt{s}}ds\cdot\exp\left(-\frac{6u^2+v^2}{1-\epsilon}\right)\\
&\le C\exp(-\frac{C'}{1-\epsilon}(u^2+v^2)).
\end{aligned}$$

Next we consider the integration domain $0\le s\le\epsilon<1$. We look at

$$\frac{1}{h(x,y)}\int_0^\epsilon\int_0^\infty q_{1-s}(u,-v;0,z)|z|q_s(x,y;0,-z)dzds.$$

We perform the change of variables $s \to x^{\frac{2}{3}} s$, $z \to x^{\frac{1}{3}} z$ to get after the scaling property of q_s

$$\begin{aligned}
&\int_0^{\epsilon}\int_0^{\infty} q_{1-s}(u,-v;0,z)|z|q_s(x,y;0,-z)dzds \\
&= \int_0^{x^{-\frac{2}{3}}\epsilon}\int_0^{\infty} x^{\frac{4}{3}} q_{1-x^{\frac{2}{3}}s}(u,-v;0,x^{\frac{1}{3}}z)|z|q_{x^{\frac{2}{3}}s}(x,y;0,-x^{\frac{1}{3}}z)dzds \\
&= \int_0^{x^{-\frac{2}{3}}\epsilon}\int_0^{\infty} q_{1-x^{\frac{2}{3}}s}(u,-v;0,x^{\frac{1}{3}}z)|z|q_s(1,x^{-\frac{1}{3}}y;0,-z)dzds.
\end{aligned}$$

Now recall that $1-x^{\frac{2}{3}}s \in [1-\epsilon,1]$ for $s \in [0,x^{-\frac{2}{3}}\epsilon]$ and use (d) from Lemma 17 to conclude that

$$q_{1-x^{\frac{2}{3}}s}(u,-v;0,x^{\frac{1}{3}}z) \le C_\epsilon x^{\frac{1}{3}} z(u+v)\exp(-C(6u^2+v^2+x^{\frac{2}{3}}z^2)).$$

Furthermore, $h(x,y) \ge c|x|^{\frac{1}{6}}$ by Lemma 6 in Denisov and Wachtel [2015a], i.e. one has $x^{\frac{2}{3}} = h(x,y)o(1)$ as $x,y \searrow 0$. Therefore, it suffices to show that the following expression is finite

$$\int_0^{\infty}\int_0^{\infty} zq_s(1,x^{-\frac{1}{3}}y;0,-z)dzds.$$

We use (d) from Lemma 17 to see that it suffices

$$\int_0^{\infty}\int_0^{\infty} zs^{-2}e^{-\frac{1}{s^3}-\frac{z^2}{s}}dzds < \infty.$$

This is true.

These two steps establish the result.

Step 3. Finally, we have to look at the following four-fold integral.

$$I(x,y,u,v) = \int_0^1\int_0^{\infty}\int_0^s\int_0^{\infty} zq_{1-s}(u,-v;0,z)f_r(0,-z;0,w)q_{s-r}(x,y;0,w)dwdrdzds.$$

Since the argument is particularly lengthy, we split this step into several intermediate steps.

Step 3-a: preparations.

We first use the identity

$$q_s(x,y;u,-v) = -q_s(x,y;u,v)$$

twice and then substitute $r \to s-r$ to get

$$I(x,y,u,v) = \int_0^1\int_0^{\infty}\int_0^s\int_0^{\infty} zq_{1-s}(u,-v;0,-z)f_{s-r}(0,-z;0,w)q_r(x,y;0,-w)dwdrdzds.$$

According to the formula for f_r from Lemma 11 we have

$$f_{s-r}(0,-z;0,w) = \frac{3we^{-\frac{2(z^2-zw+w^2)}{s-r}}}{\pi\sqrt{2\pi}(s-r)^2}\int_0^{\frac{4zw}{s-r}} \theta^{-\frac{1}{2}}e^{-\frac{3\theta}{2}}d\theta.$$

Note that $\int_0^x \theta^{-\frac{1}{2}}e^{-\frac{3\theta}{2}}d\theta \le 2\sqrt{x}$ for all $x>0$. Note also that the quadratic form $(z,w)\mapsto 2(z^2-zw+w^2)$ is positive definite. These two facts deliver the estimate

$$zf_{s-r}(0,-z;0,w) \le C\frac{(zw)^{\frac{3}{2}}}{(s-r)^{\frac{5}{2}}}\exp(-c\frac{z^2+w^2}{s-r}). \tag{A.12}$$

Replacing (A.12) into I, recalling that $q_r(x,y;0-w)>0$ for $x,y,w>0$ and interchanging the integration sequence (the integrand is non-negative) delivers

$$I(x,y,u,v) \le C\int_0^1\int_0^\infty w^{\frac{3}{2}}q_r(x,y;0,-w)\left(\int_0^\infty\int_r^1 \frac{z^{\frac{3}{2}}\exp(-c\frac{z^2}{s-r})}{(s-r)^{\frac{5}{2}}}q_{1-s}(u,-v;0,-z)dsdz\right)dwdr.$$

Recalling (A.3), we focus the analysis in the following on the function

$$\psi(u,v,r,t,a,b) := \int_a^b\int_r^t \frac{z^{\frac{3}{2}}\exp(-c\frac{z^2}{s-r})}{(s-r)^{\frac{5}{2}}}q_{1-s}(u,-v;0,-z)dsdz,$$
$$\text{for } 0<r<t\le 1, u,v>0, 0\le a<b\le\infty.$$

Step 3-b: integrability of $\mathbf{R}_+\times\mathbf{R}\ni(u,v)\mapsto h(u,v)\sup_{r\in[0,1]}\psi(u,v,r,1,\epsilon,\infty)$ *for arbitrary* $\epsilon>0$.

Note that $t^{-\frac{5}{2}}e^{-\frac{c}{2}t^2}=O(1), t\to\infty$. We use this fact, part (d) of Lemma 17 together with the substitution $z\mapsto\frac{z}{\sqrt{s-r}}$ and the fact that $s-r\in(0,1)$ to estimate

$$\psi(u,v,r,\epsilon,\infty) \le C_\epsilon\left(\int_0^\infty z^4e^{-\frac{c}{2}z^2}dz\right)\cdot\int_0^1\frac{1}{(1-s)^2}\exp\left(-\frac{6u^2}{(1-s)^3}-\frac{v^2}{1-s}\right)ds.$$

$z\mapsto z^4e^{-\frac{c}{2}z^2}$ is integrable. Suppose that $u\ge R$. Then it follows

$$\begin{aligned}\int_0^1\frac{1}{(1-s)^2}\exp\left(-\frac{6u^2}{(1-s)^3}-\frac{v^2}{1-s}\right)ds &\le C\exp(-c(u^2+v^2))\int_0^1\frac{1}{(1-s)^2}\exp\left(-\frac{R^2}{(1-s)^3}\right)ds\\ &\le C_R\exp(-c(u^2+v^2)).\end{aligned} \tag{A.13}$$

Suppose that $u<R, |v|\ge R$. Then a similar estimate to (A.13) holds true. This finishes the integrability required.

Step 3-c: integrability of $\mathbf{R}_+\times\mathbf{R}\ni(u,v)\mapsto h(u,v)\sup_{r,\epsilon:r+\epsilon<1}\psi(u,v,r+\epsilon,1,0,\epsilon)$.

We use first a similar estimate as in Step 2 of the proof based on the fact that $\sinh(x) \leq |x|\cosh(x)$. Namely, it holds

$$\begin{aligned} q_{1-s}(u,-v;0,-z) \leq & \frac{C}{(1-s)^2}\exp\left(-\frac{6u^2}{(1-s)^3}+\frac{6uv}{(1-s)^2}-\frac{2v^2+2z^2}{1-s}\right) \\ & \times \sinh\left(z(\frac{6u}{(1-s)^2}-\frac{2v}{1-s})\right) \\ & \leq Cz\left(\frac{6u}{(1-s)^2}+\frac{2v}{1-s}\right)\exp\left(-\frac{6u^2}{(1-s)^3}+\frac{6uv}{(1-s)^2}-\frac{2v^2+2z^2}{1-s}\right) \\ & \times \frac{1}{(1-s)^2}\cosh\left(z(\frac{6u}{(1-s)^2}-\frac{2v}{1-s})\right). \\ = & Cz\rho(u,v,s,z), \end{aligned}$$

with

$$\begin{aligned} \rho(u,v,s,z) := & \left(\frac{6u}{(1-s)^2}+\frac{2v}{1-s}\right)\exp\left(-\frac{6u^2}{(1-s)^3}+\frac{6uv}{(1-s)^2}-\frac{2v^2+2z^2}{1-s}\right) \\ & \times \frac{1}{(1-s)^2}\cosh\left(z(\frac{6u}{(1-s)^2}-\frac{2v}{1-s})\right). \end{aligned}$$

After straightforward algebra and use of part (a) of Lemma 17 it follows that

$$\begin{aligned} \rho(u,v,s,z) \leq \bar\rho(u,v,s,z) := & \frac{1}{(1-s)^{\frac{5}{2}}}\left(\frac{6u}{(1-s)^{\frac{3}{2}}}+\frac{2v}{\sqrt{1-s}}\right)\times \\ & \left[\exp\left(-cQ_1\left(\frac{u}{(1-s)^{\frac{3}{2}}},\frac{v}{\sqrt{1-s}},\frac{z}{\sqrt{1-s}}\right)\right)+\exp\left(-cQ_2\left(\frac{u}{(1-s)^{\frac{3}{2}}},\frac{v}{\sqrt{1-s}},\frac{z}{\sqrt{1-s}}\right)\right)\right]. \end{aligned}$$

To show integrability of integrability of $\mathbf{R}_+ \times \mathbf{R} \ni (u,v) \mapsto h(u,v)\psi(u,v,r+\epsilon,1,0,\epsilon)$ for $\epsilon > 0$ small so that $\epsilon + r < 1$, we note first that it is without loss of generality to show it for $r \leq \frac{1}{2}$ and $\epsilon \leq \frac{1}{4}$. This is because the integrand is non-negative for all $r \in (0,1), \epsilon, u, v > 0$. We focus on such values of r, ϵ in the following.

We use the substitution $z \mapsto \frac{z}{\sqrt{s-r}}$ to arrive at

$$\psi(u,v,r+\epsilon,1,0,\epsilon) \leq \int_{r+\epsilon}^1 \int_0^{\frac{\epsilon}{\sqrt{s-r}}} z^{\frac{5}{2}} e^{-cz^2} \frac{1}{(s-r)^{\frac{3}{4}}}\bar\rho(u,v,s,z\sqrt{s-r})dzds.$$

Note first, that $z \mapsto z^{\frac{5}{2}}e^{-cz^2}, z \geq 0$ is bounded, and that $\frac{1}{(s-r)^{\frac{3}{4}}} \leq C_\epsilon$.

We make use of the substitutions $u \mapsto \frac{u}{(1-s)^{\frac{3}{2}}}, (v,z) \mapsto \frac{(v,z)}{\sqrt{1-s}}$ together with the fact that $z\sqrt{s-r} \leq \epsilon$ to see, that to complete Step 3-c, it is sufficient to show that

$$\int_0^\infty \int_0^\infty h(u,v) \sup_{z\leq\epsilon} \bar\rho(u,v,0,z)dudv$$

is finite. But this follows from easy substitutions and part (b) of Lemma 17.

Step 3-d: integrability of $\mathbf{R}_+ \times \mathbf{R} \times \ni (u,v) \mapsto h(u,v) \sup_{r,\epsilon: r+\epsilon<1} \psi(u,v,r,r+\epsilon,0,\epsilon)$.

Suppose first that $r > \frac{1}{2}$. Then we are integrating s over a range $[r, r+\epsilon]$, which can be expanded to $[\frac{1}{2}, \frac{3}{4}) \cup [\frac{3}{4}, 1]$ (recall that the integrand is non-negative). Step 3-c shows the integrability needed in the range $[\frac{3}{4}, 1]$. Hence, in the following we work on the integrability of $\mathbf{R}_+ \times \mathbf{R}_+ \times \ni (u,v) \mapsto h(u,v)\psi(u,v,r,\frac{3}{4},0,\epsilon)$ for $\epsilon > 0$ for some $r \le \frac{1}{2}$. Note that $1 - s \in [\frac{1}{4}, 1)$ under these conditions.

Analogous to the Step 3-c, we use the substitution $z \mapsto \frac{z}{\sqrt{s-r}}$ to arrive at

$$\psi(u,v,r,\frac{3}{4},0,\epsilon) \le \int_r^{\frac{3}{4}} \int_0^{\frac{\epsilon}{\sqrt{s-r}}} z^{\frac{5}{2}} e^{-cz^2} \frac{1}{(s-r)^{\frac{3}{4}}} \bar{\rho}(u,v,s,z\sqrt{s-r}) dz ds.$$

Note that $z\sqrt{s-r} \le \epsilon$. We estimate for $z' \le \epsilon$

$$\bar{\rho}(u,v,s,z') \le C(u+v) \exp\left(-c(\frac{u}{1-s} - v)^2\right) \times \left[\exp\left(-c(z' - \frac{u}{1-s})^2\right) + \exp\left(-c(z' - \frac{u}{1-s})^2\right)\right].$$

By steps analogous to the proof of part (b) of Lemma 17, one arrives at the integrability of

$$\mathbf{R}_+ \times \mathbf{R} \ni (u,v) \mapsto h(u,v) \cdot \sup_{s \in [\frac{3}{4},0], z \le \epsilon} \bar{\rho}(u,v,s,z). \tag{A.14}$$

Finally, recall that $(0,\infty) \ni z \mapsto z^{\frac{5}{2}} e^{-cz^2}$ is integrable, as is $(0,1) \ni t \mapsto \frac{1}{t^{\frac{3}{4}}}$.[1] This, together with (A.14) finishes the proof of Step 3-d.

Step 4. Step 3 has shown that the four-fold integral from the decomposition in Lemma 19 leads to a bound of the type

$$I(x,y,u,v) \le C \int_0^1 \int_0^\infty w^{\frac{3}{2}} q_r(x,y;0,-w) \psi(u,v,r,1,0,\infty) dw dr,$$

with a function $\mathbf{R}_+ \times \mathbf{R} \times (0,1) \ni (u,v,r) \mapsto \psi(u,v,r,1,0,\infty)$ so that

$$\int_0^\infty \int_0^\infty h(u,v) \sup_{r \in (0,1)} \psi(u,v,r,1,0,\infty) du dv < \infty. \tag{A.15}$$

Making use of the steps 1 and 2, of (A.15), and of the integral representation of h in (A.3) finishes the proof of the proposition.

□

Proof of Proposition 12. Due to the scaling properties of the functions $\bar{h}, h$ and of the density p it is w.l.o.g. to take $t = 1$.

[1] We make the substition $t \mapsto s - r$ in $\int_r^{\frac{3}{4}} \frac{1}{(s-r)^{\frac{3}{4}}} ds$.

We use the equations from Lemma 18. Due to the proof of Lemma 11 it is enough to look at

$$\lim_{x,y\searrow 0}\frac{1}{h(x,y)}\int_0^1\int_0^\infty\int_0^s\int_0^\infty zq_{1-s}(u,-v;0,z)f_r(0,-z;0,w)q_{s-r}(x,y;0,-w)dwdrdzds. \tag{A.16}$$

Lemma 11 and Lemma 17 ensures a dominating function for the integral whenever $|(u,v)|$ is large enough. Otherwise, for $|(u,v)|$ bounded points (a), (b) from Lemma 17 ensure that there is a dominating function and we can interchange integration and limit in (A.16). It is easy to see the convergence is also uniform for $\alpha(u,v)\leq R$.

After transformations similar to the ones in the proof of Lemma 11 we need to verify the limit (uniformly for $\alpha(u,v)\leq R$)

$$\int_0^1\int_0^\infty\int_0^\infty\int_0^\infty\int_0^4\lim_{x\downarrow 0,y\downarrow 0}\frac{1}{h(x,y)}|z|^{3/2}|w|^{3/2}q_{1-s}(u,-v;0,-z)x^{1/6}q_r(1,x^{-1/3}y;0,-w)$$
$$\times\frac{3}{\pi\sqrt{2\pi}(s-x^{2/3}r)^{5/2}}e^{-\frac{2}{(s-x^{2/3}r)}(z^2-|x^{1/3}zw|+x^{2/3}w^2)}\theta^{-1/2}e^{-\frac{3|x^{1/3}zw|\theta}{2(s-x^{2/3}r)}}\mathrm{d}\theta\mathrm{d}w\mathrm{d}r\mathrm{d}z\mathrm{d}s=\overline{h}(1,u,-v).$$

First we look at the case $x^{-\frac{1}{3}}y\to c$. Using $h(x,y)\sim x^{1/6}g(c)$ the integrand equals after substitutions $r\mapsto x^{\frac{2}{3}}r, z\mapsto x^{\frac{1}{3}z}$

$$\lim_{x\downarrow 0,y\downarrow 0}\frac{1}{g(c)}|z|^{3/2}|w|^{3/2}q_{1-s}(u,-v;0,-z)q_r(1,x^{-1/3}y;0,-w)$$
$$\times\frac{3}{\pi\sqrt{2\pi}(s-x^{2/3}r)^{5/2}}e^{-\frac{2}{(s-x^{2/3}r)}(z^2-|x^{1/3}zw|+x^{2/3}w^2)}\theta^{-1/2}e^{-\frac{3|x^{1/3}zw|\theta}{2(s-x^{2/3}r)}}.$$

Solving the limit, this is equal to

$$\frac{1}{g(c)}|z|^{3/2}|w|^{3/2}q_{1-s}(u,-v;0,-z)q_r(1,c;0,-w)\frac{3}{\pi\sqrt{2\pi}s^{5/2}}e^{-\frac{2}{s}z^2}\theta^{-1/2},$$

uniformly for $\alpha(u,v)\leq R$.

After calculating the θ integral and sorting the terms, we have the integral

$$\begin{aligned}&\frac{1}{g(c)}\int_0^1\int_0^\infty\frac{4\sqrt{3}}{\sqrt{2\pi}}|z|^{3/2}s^{-1/2}q_{1-s}(u,-v,0,-z)\frac{\sqrt{3}}{\pi s^2}e^{-\frac{2}{s}z^2}\\&\qquad\times\int_0^\infty\int_0^\infty|w|^{3/2}q_r(1,c;0,-w)\mathrm{d}w\mathrm{d}r\mathrm{d}z\mathrm{d}s\\&=\int_0^1\int_0^\infty\frac{4\sqrt{3}}{\sqrt{2\pi}}|z|^{3/2}s^{-1/2}q_{1-s}(u,-v,0,-z)p_s(0,z;0,0)\\&=\overline{h}(1,u,-v),\end{aligned}$$

where in the last steps we used the definitions of h,g,p and $\overline{h}$.
Finally, we look at the case $x^{-1/3}y\to\infty$.

Note that $x^{-1/3}y \to \infty$ for $x, y \searrow 0$ is equivalent to $x = 0, y \downarrow 0$. To see this, one can perform a translation of the start points (x, y). Using (4.5) of Groeneboom et al. [1999] we have

$$\overline{p}_t(0, y; u, v) \sim \sqrt{y}\overline{h}(t, u, -v) \text{ for } y \downarrow 0.$$

It is easy to see from the proof in Groeneboom et al. [1999] that the convergence is uniform in (u, v) as long as they remain bounded.

With Lemma 16 we know that $\sqrt{y} \sim h(x, y)$ as $x, y \searrow 0$ and $x^{-1/3}y \to \infty$. This concludes the proof. □

Anhang B

Results on the case of non-zero drift

B.1 The case of non-zero drift with exponential moments

The results presented in this section are from Duraj [2014b].

Formally, we study the multidimensional counterpart of the following one dimensional problem:

Let $S(n)$ be a real valued random walk with negative drift started at some $x \in (0, +\infty)$. Find the asymptotics of the exit time $\tau_x = \inf\{n \geq 0 : S(n) \leq 0\}$. This is by now a classical result. For example, in Doney [1989] the asymptotics is found if the jump of the random walk fulfills the following Cramér-type condition :
$R(h) = \mathbb{E}[e^{hX}]$ is finite in some $[0, B]$ for $B \leq \infty$, and $0 < \lim_{h\uparrow B} \frac{R'(h)}{R(h)} \leq \infty$. The asymptotics is then

$$\mathbb{P}(\tau_x > n) \sim V(x)\mu^{-n}n^{-\frac{3}{2}} \text{ as } n \to \infty. \tag{B.1}$$

Here, $\mu = \frac{1}{\mathbb{E}[e^{h_0 X}]}$ and h_0 is the unique solution of $R'(h) = 0$ and the assumption on the jump implies in particular that it has a finite second moment. For two real-valed sequences, by the notation $a(n) \sim b(n)$, as $n \to \infty$, we mean the property $\lim_{n\to\infty} \frac{a(n)}{b(n)} = 1$.

Note that the positive open real halfline is a cone. Our main aim is to find an analogue to (B.1) for random walks in dimensions greater than one, killed when leaving cones in the respective euclidean space and to derive some weak convergence results for the conditioned process.

Even though the idea of a Cramér condition and use of an exponential change of measure will be helpful for the multidimensional case as well, methodologically the multidimensional problem is different from the one dimensional case. Therefore we first briefly recall the main idea of the study of latter problem, as it is done in Doney [1989].

Its study is facilitated by the obvious relation $\mathbb{P}(\tau_x > n) = \mathbb{P}(L_n \geq -x)$ with $L_n = \min_{i=1..n} S(i)$. This makes possible the use of the following classical relations (see also the similar equations (2.5) and (2.6) in Doney [1989])

$$\sum_{n\geq 0} s^n \mathbb{E}[e^{sL_n}] = \exp\{\sum_{n\geq 1} s^n a_n(s)\} \tag{B.2}$$

and

$$\sum_{n\geq 0} s^n \mathbb{P}(\tau_0 > n) = \exp\{\sum_{n\geq 1} s^n a_n\}, \tag{B.3}$$

where $a_n(t) = \frac{1}{n}\{\mathbb{E}[e^{tS(n)}, S(n) < 0] + \mathbb{P}(S_n \geq 0)\}$ and $a_n = \frac{1}{n}\mathbb{P}(S_n \geq 0)$. Then getting the asymptotics of $a_n(t)$ and a_n yields the asymptotics of the exit time. To this aim one makes use of a change of measure, for example for a_n we have

$$\begin{aligned} a_n &= \frac{1}{n}\int_0^\infty \mathbb{P}(S_n \in dx) \\ &= \frac{1}{n}\int_0^\infty e^{-hx}(\mathbb{E}[e^{hx}])^n \mathbb{P}(\hat{S}_n \in dx) \\ &= \frac{(\mathbb{E}[e^{hx}])^n}{n}\mathbb{E}[e^{-\hat{S}_n}, \hat{S}_n \geq 0]. \end{aligned}$$

Here $\hat{S}_n$ is the driftless random walk gained from an exponential change of measure of X through the density $\frac{e^{hx}}{\mathbb{E}[e^{hx}]}$. It is then easy to see, for example if the random walk is discrete by expanding and using a local limit theorem, that the expectation in the last line is asymptotically $\frac{const}{\sqrt{n}}$.

As already mentioned, there is no hope of some similarly helpful relation as (B.2) and (B.3) for the multidimensional case. A way of attacking the multidimensional case is supplied by the work Denisov and Wachtel [2015b]. One could use some of their results after reducing the case of a nonzero drift to that of the zero drift. For this we impose a Cramér condition and use an exponential change of measure to turn the nonzero drift random walk into a driftless one already at the beginning. Nevertheless, it turns out that one has to refine and specialize some crucial estimate from the driftless case to accomplish the proof for the nonzero drift case. This refinement is the hardest part of the proof.

B.1.1 Assumptions and Statement of Results

Let $S(n)$ be a d-dimensional random walk, $d \geq 2$. Its jumps are i.i.d. copies of some random variable
$X = (X_1, X_2, ...X_d)$ which takes value in the euclidian lattice $\mathbb{Z}^d$.
We assume

Assumption 1 (Cramér condition) : The set
$\Omega = \{h \in \mathbb{R}^d : R(h) = \mathbb{E}[e^{h\cdot X}] < \infty\}$ has a nonempty interior containing 0 and there exists some nonzero h in the interior such that $\nabla R(h) = \mathbb{E}[Xe^{h\cdot X}] = 0$.

We also assume that the random walk is truly d-dimensional, i.e. it does not live on a hyperplane of the euclidian space. Thus we impose
Assumption 2 (Non Collinearity) : For every nonzero $c \in \mathbb{R}^d$ we have
$\mathbb{P}(c \cdot X = 0) < 1$.

We also assume that the random walk is strongly aperiodic.
Assumption 3 (Strong aperiodicity): X fulfills the following condition:

For every $x \in \mathbb{Z}^d$ the smallest subgroup of $\mathbb{Z}^d$ containing

$$\{y : y = x + z \text{ with some } z \text{ s.t. } \mathbb{P}(X = z) > 0\}$$

is the whole group.

Note that Assumption 1 implies that $c := R(h)$ is smaller than 1 and that $\mathbb{E}[X] = \nabla R(0)$ is nonzero, since the function $R(h)$ is strictly convex and C^∞ in the interior of Ω. Thus we are in the nonzero drift case. Also, due to convexity of Ω and Assumption 1, it follows by a Taylor expansion of $R(h)$ around 0 that $\mathbb{E}[X] \cdot h < 0$.

Define $c = \mathbb{E}[e^{h\cdot X}]$ and let $\tilde{X}$ be a random variable with density

$$\mathbb{P}(\tilde{X} \in dz) = \frac{1}{c} e^{h\cdot z} \mathbb{P}(X \in dz), \tag{B.4}$$

defined on the same probability space as $S(n)$ is. As a consequence of the Cramér assumption we know its associated random walk $\tilde{S}(n) = \sum_{i=1}^n \tilde{X}(i)$ is driftless. Clearly, the non-collinearity assumption holds for $\tilde{X}$ again, since we are dealing with an equivalent change of measure. This implies that $\mathbb{E}\left[\tilde{X} \cdot \tilde{X}^t\right]$ is a positive definite matrix. This ensures the existence of an invertible $d \times d$-matrix M such that $\hat{X} = M\tilde{X}$ has $\mathbb{E}\left[\hat{X} \cdot \hat{X}^t\right] = \mathbb{I}_{d\times d}$, where $\mathbb{I}_{d\times d}$ is the Identity matrix of dimension d. We denote by $\hat{S}(n)$ its corresponding random walk. It has uncorrelated components with zero drift, since also $\mathbb{E}[\hat{X}] = M\mathbb{E}[\tilde{X}] = 0$.

Due to the Cramér condition, we know that $\tilde{X}$ and $\hat{X}$ have all moments. The state space of $\hat{S}(n)$ is $M\mathbb{Z}^d$. It is again strongly aperiodic in its state space. We denote from now on by $\hat{y}$ the vector My for $y \in \mathbb{Z}^d$. For the original random walk we recall the stopping time

$$\tau_x = \inf\{n \geq 0 : x + S(n) \notin K\} \tag{B.5}$$

and introduce by $\hat{\tau}_{\hat{x}}$ the corresponding stopping time for $\hat{S}(n)$,

$$\hat{\tau}_{\hat{x}} = \inf\{n \geq 0 : \hat{x} + \hat{S}(n) \notin \hat{K} = MK\}. \tag{B.6}$$

Here K is an open cone containing h of Assumption 1 and $x \in \mathbb{Z}^d$ always.

Assumption 4 (Convexity): K is convex.

Moreover, our method works only for cones where the series $\sum_{y\in K\cap\mathbb{Z}^d} e^{-h\cdot y}$ is convergent, so that letting $\Sigma := K \cap \mathbb{S}^{d-1}$ we have to impose

Assumption 5: For every $x \in \partial\Sigma$ we have $|\angle(x, \frac{h}{|h|})| < \frac{\pi}{2}$.

This assumption is not fulfilled for two-dimensional cones of opening bigger or equal to π, which contain h in its interior. For such cones $h \cdot y \leq 0$ for some $y \in K \cap \mathbb{Z}^d$ can happen so that $\sum_{y\in K\cap\mathbb{Z}^d} e^{-h\cdot y}$ diverges.

For cones fulfilling Assumption 5, it follows that $h\cdot y$ can be bounded from below by a constant times the norms of h and y for $y\in K$. Thus this assumption can be considered as minimal for our method. This assumption and the fact $\mathbb{E}[X]h<0$ imply that the random walks in consideration have a drift which points outside of the cone.

One can not use the method presented here to get the exact asymptotic of the probabilities of a two dimensional random walks conditioned to stay in a half-plane. Nevertheless, under some conditions, we can use (B.1) to get the tail asymptotics of the exit time.

If the cone has the form $K=\{x\in\mathbb{R}^d|a\cdot x>0\}$ for some nonzero $a\in\mathbb{R}^d$, the jump X has $a\cdot\mathbb{E}[X]<0$, there exist some $B>0$ such that $\mathbb{E}[e^{ah\cdot X}]<\infty$ for $h\in[0,B]$, and finally $\lim_{h\uparrow B}\frac{\mathbb{E}[a\cdot Xe^{ha\cdot X}]}{\mathbb{E}[e^{ha\cdot X}]}>0$, then we are precisely in the conditions of Doney [1989] for the random walk with jump $a\cdot X$ and can use (B.1) to get the asymptotics of the exit time.

As a last restriction, we have to impose some additional regularity on ∂K.

Assumption 6 (Regularity): $\partial\Sigma$ is C^2 and the solution of

$$\begin{cases}\Delta u=0, & \text{if } x\in MK,\\ u(x)=0, & \text{if } x\in\partial MK\end{cases}$$

is extendable to the respective solution on a bigger cone, which strictly contains MK, i.e. there exists some cone $\tilde{K}$ which strictly contains MK and for which u is extendable to a solution of the Dirichlet problem for the cone $\tilde{K}$. It is also clear, M being invertible i.e. a C^∞-diffeomorphism, that $\partial(MK\cap S^{d-1})$ is C^2 if $\partial\Sigma$ is C^2. We note here, that if the original random walk has indipendent components, then M is a diagonal positive definite matrix so that $MK=K$ and the Assumption 6 is then made on the original cone.

With an eye on applications, our assumptions are not as restrictive as they seem. For example, Assumption 6 is always fulfilled for cones K with real-analytic $\partial\Sigma$ (see references in Banuelos and Smits [1997]). In particular, every two dimensional cone of opening of less or equal to $\frac{\pi}{2}$ fulfills all of Assumptions 4, 5 and 6, since $\partial\Sigma$ contains just two points. Moreover, depending on the specific setting of the original problem one wants to study, it may be possible to make a linear transformation of the cone and the jump and reduce the setting to a random walk living in some cone of the form $\tilde{K}\times\mathbb{R}^{d-r}$ (for some $r\le d$ and $\tilde{K}$ suitable), whose projection on $\mathbb{R}^r$ fulfills our assumptions.

We also note, that for our proof to go through we could weaken Assumption 1 as our proof only needs the existence of a nonzero h with $R(h)<1$, $\mathbb{E}[Xe^{h\cdot X}]=0$ and $\mathbb{E}[|X|^\alpha e^{h\cdot X}]<\infty$, where α is some suitable real number depending on the transformation M. Here, α has to be greater than 2 and equal to p if p itself is greater than 2. We have stated Assumption 1 to show the analogy with the one dimensional problem.

In this setting we are able to prove the following Proposition.

Proposition 14. *Let V be the harmonic function from Proposition 4 for $\{\hat{S}(n)\}$ and V' the respective harmonic function for $\{-\hat{S}(n)\}$. It holds for $n\to\infty$, for $A\subset K\cap\mathbb{Z}^d$*

$$\mathbb{P}(x+S(n)\in A,\tau_x>n)\sim\rho c^n n^{-(p+\frac{d}{2})}e^{h\cdot x}V(x)\sum_{y\in A}e^{-h\cdot y}V'(y).$$

In particular,

$$\mathbb{P}(\tau_x > n) \sim \varrho c^n n^{-(p+\frac{d}{2})} e^{h\cdot x} V(x). \tag{B.7}$$

Relation (B.7) is the multidimensional counterpart of (B.1) and U, U' are suitable functions defined in the next section. We note that one does not need Assumption 3 for (B.7). Indeed, given the original random walk is not strongly aperiodic, we could redefine the probabilities on a suitable subset grid of the euclidian grid and get again (B.7) through the same calculations. Assumption 3 has been stated for expository reasons. The proof of Proposition 14 is based on the similar results for the driftless random walk $\hat{S}(n)$, but is not a straightforward application, since the estimates on the tail probability for the exit time in the driftless case are not sharp enough to justify an interchange of limit and sum, which is crucial for the proof of the nonzero drift case. Sharpening this estimate for our setting is the essential in our method.

Having Proposition 14, we derive from it the asymptotic behavior of the conditioned process. As a first simple Corollary it is immediate to derive

Corollary 8.

$$\frac{\mathbb{P}_x(\tau = n)}{\mathbb{P}_y(\tau = n)} \longrightarrow e^{h\cdot(x-y)} \frac{V(x)}{V(y)} \quad , \qquad n \to \infty.$$

The one dimensional version of this limit is found in Doney [1989].
In Daley [1969] and Iglehart [1974b] the authors find the limit process for the one-dimensional conditioned random walks to be quasistationary. In our setting, as a simple corollary from Proposition 14, one can check that the respective Yaglom limit exists for the multidimensional case.

Corollary 9 (Yaglom Limit)**.** *We have for $x \in K \cap \mathbb{Z}^d$, $A \subset K$ and $n \to \infty$*

$$\mathbb{P}(x + S(n) \in A | \tau_x > n) \longrightarrow \mu(A) = \kappa \sum_{y \in A \cap \mathbb{Z}^d} e^{-h\cdot y} V'(y).$$

Here, $\kappa > 0$ is the norming constant so that μ is a probability distribution on K. Furthermore, μ is quasistationary for the conditioned process.

Existence of the Yaglom limit is not always given. For example, one sees directly that for driftless random walks $S(n)$ due to Proposition 6 we have

$$\mathbb{P}(x + S(n) = y | \tau_x > n) \longrightarrow 0, \quad n \to \infty.$$

I.e. the Yaglom Limit doesn't exist in this case (it needs to be a genuine probability distribution). For some examples on the fact that even for extensively studied processes like birth-death processes both existence and non-existence can happen, see Van Doorn [1991]. Duraj [2014b] also show with a simple example that one cannot expect uniqueness of quasistationary distributions in our setting.

The previous results can be used to prove some simple weak convergence results for the conditioned process. For the one dimensional case, the analysis of such limits in Iglehart

[1974b], Iglehart [1974a] and Daley [1969] is complete. They use the methodology of the one-dimensional problem, which as previously noted, has no bite in the multidimensional setting. Using the results above Duraj [2014b] proves the following for the exit distribution of the random walk, conditioned on exiting at a specific time: For $x \in K \cap \mathbb{Z}^d$ and $y \in \mathbb{Z}^d \setminus K$ and $n \to \infty$

$$\mathbb{P}(x + S(\tau_x) = y | \tau_x = n) \longrightarrow \frac{\chi}{1-c} \sum_{z \in K} e^{-h \cdot z} V'(z) \mathbb{P}(z + S(1) = y),$$

where χ is a positive constant. For bridges for random walks in our setting it holds

Proposition 15. *For $A \subset K \cap \mathbb{Z}^d$ finite, $z \in K$ and $t \in (0,1)$*

$$n^{p+\frac{d}{2}} \mathbb{P}(x + S([tn]) \in A | \tau_x > n, x + S(n) = z) \sim \frac{\rho}{(t(1-t))^{p+\frac{d}{2}}} \sum_{y \in A} V(y) V'(y), \quad n \to \infty.$$

Finally, Proposition 14 makes it possible to construct a Markov chain on $K \cap \mathbb{Z}^d$ conditioned to never leave the cone. Namely, we get the markov chain Z with transition matrix

$$p(x,y) = \frac{1}{c} \mathbb{P}(x + S(1) = y) e^{h \cdot (x-y)} \frac{V(x)}{V(y)}, \quad \text{for } x, y \in K \cap \mathbb{Z}^d$$

by looking at the weak limit of

$$\mathbb{P}(x + S(1) = y_1, x + S(2) = y_2, \ldots, x + S(k) = y_k, \tau_x > n | \tau_x > n),$$

as $n \to \infty$. It is then easy to prove the following: Z is a strictly stochastic and transient Markov chain on $K \cap \mathbb{Z}^d$. Z is probably *the physically right* process to take as a random walk conditioned to never leave the cone and it is important to note that it is not constructed through a *Doob-h*-transform, but involves instead a c-harmonic function, for some suitable $c \in (0,1)$. In some of the cases considered in applications, like two-dimensional random walks in two-dimensional cones, there are uncountably many positive harmonic functions (see section 4.4 in chapter 4). So there are uncountably many possible definitions through *Doob-h*-transforms of a random walk conditioned to stay in a cone in this case. Proposition B.1.1 tells us that none of them is equal to the process gained through the weak limit procedure.

To conclude this section, we comment on a special case which is ubiquitous in applications.

Example. There has been extensive research on random walks in the quarter plane. In the monograph Fayolle et al. [1999], the authors develop analytical and algebraic methods to study such random walks under strong assumptions on the distribution of the jump. These methods have been successfully used, for example in Kurkova and Raschel [2011] and Raschel [2012] to study random walks of non-zero drift with small steps. In comparison, the assumptions here are less stringent and we follow a pure probabilistic approach.

In our setting, assume that $d = 2$ and that the jump of the random walk has negative drift $m = \mathbb{E}[X] = (m_1, m_2)$ with $m_1, m_2 < 0$. Also assume that there exists some $h_0 > 0$ such that for $\tilde{h}_i \leq h_0, \quad i = 1, 2$ we have

$$\mathbb{E}[e^{\sum_{i=1}^{2} \tilde{h}_i X_i}] < \infty.$$

Assume also the existence of some $h = (h_1, h_2)$ with $h_0 > h_i > 0, \quad i = 1, \dots, d$ such that

$$\mathbb{E}\left[X_j e^{\sum_{i=1}^{2} h_i X_i}\right] = 0 \quad j = 1, 2.$$

Last, assume that the random walk is strongly aperiodic and that X has a strictly two-dimensional distribution. Under these conditions, if we take for K the interior of the positive quadrant, all of the Assumptions 1-6 are fulfilled. In particular, Theorem 1 holds and yields the asymptotics for $\mathbb{P}(\tau_x > n)$ for $x \in \mathbb{N}^2$ and all its immediate corollaries apply. In this setting we can give an explicit form for M. Namely, using Example 2 from Denisov and Wachtel [2015b] one can calculate M this way: define first

$$c_i = \mathbb{E}[(\tilde{X}_i)^2].$$

Then

$$\alpha = \mathbf{Cov}\left(\frac{\tilde{X}_1}{\sqrt{c_1}}, \frac{\tilde{X}_2}{\sqrt{c_2}}\right) \in (-1, +1),$$

due to non collinearity. With this, we have then

$$M = \frac{1}{\sqrt{1-\alpha^2}} \begin{pmatrix} \frac{\cos\varphi}{\sqrt{c_1}} & \frac{-\sin\varphi}{\sqrt{c_2}} \\ \frac{-\sin\varphi}{\sqrt{c_1}} & \frac{\cos\varphi}{\sqrt{c_2}} \end{pmatrix},$$

where φ is such that $\sin(2\varphi) = \alpha$. The terms $\frac{1}{\sqrt{c_i}}$ norm the variables $\tilde{X}_i$ into having variance 1, while the cos and sin-functions cause a rotation, which makes the components uncorrelated, as can be seen by straightforward computation. In the case of two-dimensional cones the value of p is known and equal to $\frac{\pi}{arccos(-\alpha)}$.

B.2 On integrated random walks conditioned to stay positive: Applications to the case with drift

The results presented in this subsection of the Appendix are contained in Bär et al. [2020].

The case where $\mathbf{E}[X] \neq 0$ can be considered under two different settings. We develop the exponential case and just remark shortly on the case without exponential moments.

Consider a random variable which has finite exponential moments in a right neighborhood of zero:

$$M(t) = \mathbf{E}[e^{tX_1}] < \infty, \quad 0 \leq t \leq t_0, \quad t_0 > 0.$$

Denote by μ the law of X_1 and define the laws

$$\mu_t(dz) = \frac{e^{tX_1}}{M(t)} d\mu(z).$$

Let $X^{(t)}$ be a random variable with law μ_t. Then we have for the laws μ_t^n of the respective vector $(X_1^{(t)}, \ldots, X_n^{(t)})$ under the change of exponential measure

$$\mu_t^n(dz) = \frac{e^{tS^{(t)}(n)}}{M(t)^n} d\mu^n(dz),$$

where $S^{(t)}(n)$ is the respective random walk. Note that $f(t) = (\log M(t))' = \frac{\mathbf{E}[X_1 e^{tX_1}]}{\mathbf{E}[e^{tX_1}]}$ for $t \in (0, t_0)$ is a bijective function in t and increasing. Take some c from its image, i.e. assume there exists t_c with $f(t_c) = c$. Then it is easy to show that $\mathbf{E}[X^{(t_c)}] = c$. Note also that $\sigma^2 = Var(X^{(t_c)}) = (\log M(t))''$ and that we can get an invariance principle for

$$X^{(n,t_c)}(s) = \left(\frac{T(ns) - \frac{ns(ns+1)}{2}c}{\sigma^3 n^{\frac{3}{2}}}, \frac{S_{ns} - nsc}{\sigma n^{\frac{1}{2}}} \right), \quad s \in [0,1]$$

under the measure μ_t. The condition now is for the integrated random walk to stay strictly above a quadratic function. Define thus

$$\tau = \inf\{k \geq 0 : T(k) \leq \frac{k(k+1)c}{2}\}.$$

Then we have the following invariance principle for the meander.[1]

Theorem 10. *Let $X^{(n,t_c)}(s)$ for $s \in [0,1]$ as above under the measure μ_t. Then these processes, started in $(x,y) \in \mathbf{R}_+ \times \mathbf{R}$ and conditioned on $\tau > n$, converge to the meander of the Kolmogorov Diffusion, conditioned on not leaving $\mathbf{R}_+ \times \mathbf{R}$ before time 1. In formulas*

$$(X^{(n,t_c)}(\cdot) | \tau > n)_{(x,y)} \Longrightarrow (W(\cdot) | \tau^{bm} > 1)_{(0,0)}, \quad n \to \infty$$

in $(D[0,1], || \cdot ||_\infty)$.

Similarly, we get invariance principles for the case of bridges.

Theorem 11. *For $x, u \in \mathbf{R}_+, y, v \in \mathbf{R}$ we have*

$$(X^{(n,t_c)}(\cdot) | X^{(n,t_c)} = a_n(u,v), \tau > n)_{a_n(x,y)}$$

converges weakly to the Kolmogorov excursion of length 1 w.r.t. uniform topology in $D[0,1]$.

Extensions to arbitrary time length $t > 0$ are again straigthforward. We close by remarking the case with $\mathbf{E}[X_1] \neq 0$, where no exponential assumption is assumed.

[1] Note that what we get is a whole family of invariance principles, indexed by c in the image of $f(\cdot)$.

Remark 7. *Note that the above invariance principles also hold for random walks which instead: fulfill a second moment condition of* $\mathbf{E}[|X_1|^{2+\delta}] < \infty$ *for some* $\delta > 0$ *and have drift* $\mathbf{E}[X_1] = c \neq 0$. *The steps are the same as for the exponential case above and we don't need to make a change of measure. Instead we look at*

$$X^{(n,c)}(s) = \left(\frac{T_{ns} - \frac{ns(ns+1)}{2}c}{Var[X_1]^{\frac{3}{2}} n^{\frac{3}{2}}}, \frac{S_{ns} - nsc}{Var[X_1]^{\frac{1}{2}} n^{\frac{1}{2}}} \right), \quad s \in [0,1]$$

in place of $X^{(n,t_c)}(s)$.

Literaturverzeichnis

Aurzada, F., Dereich, S., and Lifshits, M. (2014). Persistence probabilities for an integrated random walk bridge. *Probab. Math. Statist.*, 34:1–22.

Aurzada, F. and Simon, T. (2015). Persistence probabilities and exponents. In *Lévy Matters, Lecture Notes in Math.*, volume 2149. Springer, Cham.

Banuelos, R. and Smits, R. (1997). Brownian motion in cones. *Probab. Theory Related Fields*, 108:299–319.

Bär, M., Duraj, J., and Wachtel, V. (2020). Invariance principles for integrated random walks conditioned to stay positive. *arXiv preprint 2020*.

Bertoin, J. and Doney, R. (1994). On conditioning a random walk to stay non-negative. *Ann. Probab.*, 22(4):2152–2167.

Billingsley, P. (1968). *Convergence of Probability Measures*. John Wiley and Sons.

Blanchet, J. and Glynn, P. (2008). Efficient rare-event simulation for the maximum of heavy-tailed random walks. *Ann. Appl. Probab.*, 18(4):1351–1378.

Bolthausen, E. (1976). On a functional central limit theorem for random walks conditioned to stay positive. *Ann. Probab.*, 4(3):480–485.

Borovkov, A. (1972). Notes on inequalities for sums of independent random variables. *Theory Probab. App.*, 17(3):556–557.

Bostan, A., Bousquet-Mélou, M., and Melczer, S. (2018). Counting walks with large steps in an orthant. *arxiv preprint*.

Bouaziz, A., Mustapha, S., and Sifi, M. (2015). Discrete harmonic functions on an orthant in $\mathbb{Z}^d$. *Electron. Commun. Probab.*, 20:paper no. 52.

Bryn-Jones, A. and Doney, R. (2006). A functional limit theorem for random walks conditioned to stay positive. *J. London. Math. Soc.*, 74(1):244–258.

Burkholder, D. L. (1977). Exit times of brownian motion, harmonic majorization, and hardy spaces. *Adv. Math.*, 26:182–205.

Caravenna, F. and Chaumont, L. (2008). Invariance principles for random walks conditioned to stay positive. *Ann. Inst. H. Poincaré Probab. Statist.*, 44:170–190.

Caravenna, F. and Chaumont, L. (2013). An invariance principle for random walk bridges conditioned to stay positive. *Electron. J. Probab.*, 18(60):1–32.

Caravenna, F. and Deuschel, J.-D. (2008). Pinning and wetting transition for $(1+1)-$dimensional fields with Laplacian interaction. *Ann. Probab.*, 36(6):2388–2433.

Chaumont, L. (1997). Excursion normalisée, méandre et pont pour les processus de Lévy stables. *Bull. Sci. Math.*, 121:377–403.

Cohen, J. W. (1992). *Analysis of Random Walks.* Studies in Probability, Optimization and Statistics. IOS Press, Amsterdam.

Daley, D. (1969). Quasistationary behavior of a left-continuous random walk. *Ann. Math. Stat.*, 40(2):532–539.

Dembo, A., Ding, J., and Gao, F. (2013). Persistence probabilities of iterated partial sums. *Ann. Inst. H. Poincaré Probab. Stat.*, 49(3):873–884.

Denisov, D., Sakhanenko, A., and Wachtel, V. (2018). First-passage times for random walks with non-identically distributed increments. *Ann. Probab.*, 46(6):3313–3350.

Denisov, D. and Wachtel, V. (2010). Conditional limit theorems for ordered random walks. *Electron J. Probab.*, 15(11):292–322.

Denisov, D. and Wachtel, V. (2015a). Exit times for integrated random walks. *Ann. Inst. H. Poincaré Probab. Stat.*, 51(1):167–193.

Denisov, D. and Wachtel, V. (2015b). Random walks in cones. *Ann. Probab.*, 43(3):992–1044.

Denisov, D. and Wachtel, V. (2019). Alternative constructions of a harmonic function for a random walk in a cone. *Electron. J. Probab.*, 24:paper no. 92.

Doney, R. (1985). Conditional limit theorems for asymptotically stable random walks. *Z. für Wahrsc. und Verw. Geb.*, 70:351–360.

Doney, R. (1989). On the asymptotic behaviour of first passage times for transient random walk. *Probab. Theor. Rel. Fields.*, 81:239–246.

Doney, R. (1998). The Martin boundary and ratio limit theorems for killed random walks. *J. London Math. Soc.*, 58:761–768.

Doob, J. L. (1982). *Classical potential theory and its probabilistic counterpart: advanced problems.* Springer-Verlag, New York.

Duraj, J. (2014a). On harmonic functions of killed random walks in convex cones. *Electron. Commun. Probab.*, 80:1–10.

Duraj, J. (2014b). Random walks in cones: the case of non-zero drift. *Stoch. Proc. Appl.*, 124(4):1503–1518.

Duraj, J. and Wachtel, V. (2018). Green function of a random walk in a cone. *arxiv preprint.*

Duraj, J. and Wachtel, V. (2020). Invariance principles for random walks in cones. *Stoch. Proc. and Their Appl.*, 130(7):3920–3942.

Durrett, R. (1978). Conditioned limit theorems for some null recurrent markov processes. *Ann. Probab.*, 6(5):798–828.

Esseen, C. (1968). On the concentration function of a sum of independent random variables. *Z. Wahrsch. Verw. Gebiete*, 9:290–308.

Fayolle, G., Iasnogorodski, R., and Malyshev, V. (1999). *Random Walks in the Quarter Plane: Algebraic Methods, Boundary Value Problems, Applications to Queueing Systems and Analytic Combinatorics.* Springer Verlag.

Feierl, T. (2009). The height and range of watermelons without wall. *Eur. J. Combin.*, 34(1):138–154.

Feierl, T. (2012). The height of watermelons with wall. *J. Phys. A.*, 45:Paper No. 9.

Fisher, M. E. (1984). Walks, walls, wetting and melting. *J. Stat. Phys.*, 34:667–729.

Fuk, D. and Nagaev, S. (1971). Probability inequalities for sums of independent random variables. *Theory Probab. Appl.*, 16(4):643–660.

Gao, F., Liu, Z., and Yang, X. (2014). Conditional persistence of gaussian random walks. *Electron. Commun. Probab.*, 19(70):1–9.

Garbit, R. (2009). Brownian motion conditioned to stay in a cone. *J. Math. Kyoto Univ.*, 49(3):573–592.

Garbit, R. (2011). A central-limit theorem for two-dimensional random walks in a cone. *Bulletin de la SMF*, 139(2):271–286.

Grabiner, D. (1999). Browian motion in a weyl chamber, non-colliding particles, and random matrices. *Ann. Inst. H. Poincaré Probab. Statist.*, 35(2):177–204.

Grama, I., Le Page, E., and Peigné, M. (2017). Conditioned limit theorems for products of random matrices. *Probab. Theory Rel. Fields*, 168:601–639.

Greenwood, P. and Shaked, M. (1977). Fluctuations of random walk in R^d and storage systems. *Adv. in Appl. Probab.*, 9(3):566–587.

Groeneboom, P., Jongbloed, G., and Wellner, J. (1999). Integrated brownian motion conditioned to be positive. *Ann. Probab.*, 27(3):12831303.

Iglehart, D. (1974a). Functional central limit theorems for random walks conditioned to stay positive. *Ann. Probab.*, 2(4):608–619.

Iglehart, D. (1974b). Random walks with negative drift conditioned to stay positive. *J. Appl. Prob.*, 11(4):742–751.

Ignatiouk-Robert, I. (2020). Harmonic functions of random walks in a semi-group via ladder heights. *forthcoming J. Theor. Probab.*

Ignatiouk-Robert, I. and Loree, C. (2010). Martin boundary of a killed random walk on a quadrant. *Ann. Probab.*, 38(3):1106–1142.

Kenyon, R., Miller, J., Sheffield, S., and Wilson, D. (2019). Bipolar orientations on planar maps and SLE_{12}. *Ann. Probab.*, 47(3):1240–1269.

Kolmogorov, A. (1934). Zufällige bewegungen. *Annals of Mathematics, Second Series*, 35(1):116–117.

König, W. and Schmid, P. (2010). Random walks conditioned to stay in weyl chambers of type c and d. *Electron. Comm. Probab.*, 15:286–296.

Kurkova, I. and Raschel, K. (2011). Random walks in $\mathbb{Z}_+^2$ with non-zero drift absorbed at the axes. *Bull. Soc. Math. Fra.*, 139(3):341–387.

Kurkova, I. and Raschel, K. (2012). On the functions counting walks with small steps in the quarter plane. *Publ. Math. Inst. Hautes Études Sci.*, 116:69–114.

Lachal, A. (1991). Sur le premier instant de passage de l'intégrale du mouvement brownien. *Ann. Inst. Henri Poincaré*, 27(3):385–405.

Lachal, A. (2003). Application de la théorie des excursions à l'intégrale du mouvement brownien. In Azéma, J., Émery, M., Ledoux, M., and Yor, M., editors, *Séminaire de Probabilités XXXVII. Lecture Notes in Mathematics*, volume 1832. Springer, Berlin, Heidelberg.

Majumdar, S. (1999). Persistence in non-equilibrium systems. *Current Science*, 77(3):370–375.

McKean, H. (1962). A winding problem for a resonator driven by a white noise. *J. Math. Kyoto Univ.*, 2(2):227–235.

Mogul'skii, A. (1973). Absolute estimates for moments of certain boundary functionals. *Theory Probab. Appl.*, 18(2):350–357.

Nagaev, S. V. (1979). Large deviations of sums of independent random variables. *Ann. Probab.*, 7(5):745–789.

Ney, P. and Spitzer, F. (1966). The martin boundary for random walk. *Trans. Amer. Math. Soc.*, 121(1):116–132.

Olver, F. (1974). *Asymptotics and Special Functions.* New York: Academic Press.

Peigné, M. and Woess, W. (2019). Recurrence of 2-dimensional queueing processes, and random walk exit times from the quadrant. *arXiv preprint.*

Raschel, K. (2010). Green functions and martin compactification for killed random walks related to su(3). *Elect. Comm. in Probab.*, 15:176–190.

Raschel, K. (2011). Green functions for killed random walks in the Weyl chamber of Sp(4). *Ann. Inst. H. Poincaré Probab. Stat.*, 47(4):1001–1019.

Raschel, K. (2012). Counting walks in a quadrant: A unified approach via boundary value problems. *J. Eur. Math. Soc.*, 14:749–777.

Raschel, K. (2014). Random walks in the quarter plane, discrete harmonic functions and conformal mappings. *Stoch. Proc. Appl.*, 124(10):3147–3178.

Raschel, K. and Tarrago, P. (2018). Martin boundary or random walks in convex cones. *arXiv preprint.*

Shimura, M. (1991). A limit theorem for two-dimensional random walk conditioned to stay in a cone. *Yokohama Math. J.*, 39:21–36.

Spitzer, F. (1976). *Principles of random walk.* Graduate Texts in Mathematics. Springer, Berlin, 2 edition.

Uchiyama, K. (1998). Green's functions for random walks on $\mathbb{Z}^n$. *Proc. London Math. Soc.*, 77(1):215–240.

Uchiyama, K. (2014). Green's functions of random walks on the upper half plane. *Tohoku Math. J.*, 66(2):289–307.

Van Doorn, E. (1991). Quasistationary distributions and convergence to quasi-stationarity of birth-death processes. *Adv. Appl. Probab.*, 23(4):683–700.

Vysotsky, V. (2010). On the probability that integrated random walk stay positive. *Stoch. Proc. Appl.*, 120(7):1178–1193.

Vysotsky, V. (2014). Positivity of integrated random walks. *Ann. Inst. H. Poincaré Probab. Stat.*, 50(1):195–213.

Williamson, J. (1968). Random walks and Riesz kernels. *Pacific J. Math.*, 25(2):393–415.

Woess, W. (2009). *Denumerable Markov Chains: Generating Functions, Boundary Theory, Random Walks on Trees*. European Mathematical Society.