On enhanced area laws of the entanglement entropy

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Zusammenfassung

In Vielteilchensystemen liefert die Reichweite der durch Verschränkung induzierten räumlichen Quantenkorrelationen eine Vielzahl von Informationen über verschiedene physikalische Eigenschaften. Eine Möglichkeit, diese Informationen zu untersuchen, ist die Betrachtung des Skalierungsverhaltens der Verschränkungsentropie des Grundzustandes in Bezug auf eine skalierte Version eines räumlichen Gebietes. In vielen Systemen wächst die Verschränkungsentropie proportional zur Oberfläche des Gebietes, was als Oberflächengesetz bezeichnet wird. In dieser Arbeit untersuchen wir den Zusammenhang zwischen dem Skalierungsverhalten der Verschränkungsentropie und Vielteilchenlokalisierung. In den letzten Jahren konnte gezeigt werden, dass eine Reihe von Systemen, von denen bekannt ist, dass sich ihr Grundzustand in der lokalisierten Phase befindet, Oberflächengesetze der Verschränkungsentropie aufweisen. Auf der anderen Seite wird allgemein angenommen, dass die Verschränkungsentropie von delokalisierten Grundzuständen nicht einem Oberflächengesetz genügt. Allerdings gibt es nur wenige Beispiele, für die ein abweichendes Skalierungsverhalten bereits gezeigt wurde. Ziel dieser Arbeit ist es, weitere Beispiele für solche Abweichungen von Oberflächengesetzen der Verschränkungsentropie im Zusammenhang mit delokalisierten Systemen zu liefern. In drei verschiedenen Modellen zeigen wir, dass die Verschränkungsentropie des Grundzustandes zumindest ein logarithmisch erweitertes Oberflächengesetz aufweist.

Der erste Teil dieser Dissertation, welcher auf einer gemeinsamen Arbeit mit L. Pastur und P. Müller [MPS20] basiert, befasst sich mit dem zufälligen Dimer-Modell. Obwohl dieses nicht-interagierende, eindimensionale Modell spektral lokalisiert ist, gibt es kritische Punkte in dem Spektrum, an denen die Lokalisierungslänge divergiert. Im Falle von geringer Unordnung wird in dieser Arbeit eine logarithmische Untergrenze für den Erwartungswert der Verschränkungsentropie gezeigt. Darüber hinaus wird für eine beliebige Unordnungsstärke eine logarithmische Untergrenze an die Verschränkungsentropie für endliche Volumen an diesen kritischen Punkten bewiesen.

Im zweiten Teil dieser Arbeit, welcher auf einer gemeinsamen Arbeit mit P. Müller [MS20] basiert, betrachten wir einen mehrdimensionalen, kontinuierlichen Schrödinger-Operator, der durch die Störung eines negativen Laplace-Operators durch ein kompakt getragenes, beschränktes Potential gegeben ist. Sowohl eine obere als auch eine untere Grenze für die Verschränkungsentropie zu einer positiven Fermi-Energie wird gezeigt. Diese Schranken beweisen, dass das Skalierungsverhalten der Verschränkungsentropie einem logarithmisch erweiterten Oberflächengesetz entspricht. Dies ist das gleiche Skalierungsverhalten, das auch bei freien Fermionen auftritt. Das Modell der freien Fermionen ist eines der wenigen delokalisierten Systeme, für die eine asymptotische Entwicklung der Verschränkungsentropie bekannt ist.

Im dritten und letzten Teil wird, basierend auf einer gemeinsamen Arbeit mit C. Fischbacher [FS20], die endliche XXZ-Spinkette mit periodischen Randbedingungen in der Ising-Phase betrachtet. Dieses Modell hat aufgrund seiner Translationsinvarianz delokalisierte Eigenzustände. Wir zeigen, dass für jeden Eigenwert im Droplet-Band mindestens ein Eigenvektor existiert, sodass die zugehörige Verschränkungsentropie mindestens logarithmisch anwächst. Für dieses Resultat setzen wir voraus, dass der Anisotropie-Parameter Δ ausreichend groß ist. Zusätzlich dazu zeigen wir eine Combes–Thomas-Abschätzung für dieses Modell, was für sich genommen ebenfalls von Interesse ist.

Summary

In many-body systems the extent and range of spatial quantum correlations induced by entanglement provide a great deal of information about several qualitative physical properties. One way of studying this information is to examine the scaling behaviour of the ground state entanglement entropy with respect to a scaled version of a distinguished spatial subregion. In various systems the entanglement entropy grows proportionally to the surface area of the subregion which is referred to as an area law. In this thesis we examine the connection between the scaling behaviour of the entanglement entropy and many-body localisation. In recent years it was show that a number of systems, which are known to be in the localised phase, exhibit area laws of the entanglement entropy. It is commonly expected that the entanglement entropies of delocalised ground states do not satisfy area laws, though not many examples of different scaling behaviours have been shown, yet. The aim of this thesis is to provide further examples of violations of area laws in the context of delocalised systems. In three different models we show that the entanglement entropy of the ground states grows at least like a logarithmically enhanced area law.

The first part of this thesis, based on joint work with P. Müller and L. Pastur [MPS20], considers the random dimer model. Even though this non-interacting, one-dimensional model is spectrally localised, there exist critical points in its spectrum at which the localisation length diverges. We consider the ground state corresponding to a Fermi energy positioned at one of these critical energies. In the case of small disorder we show a logarithmic lower bound to the expectation of the entanglement entropy. Moreover, we proof a logarithmic lower bound to the finite-volume entanglement entropy at these critical points for any disorder strength.

In the second part of this thesis, which is based on joint work with P. Müller [MS20], we consider a multi-dimensional continuum Schrödinger operator, which is given by a perturbation of a negative Laplacian by a compactly supported, bounded potential. We establish both an upper and a lower bound to the entanglement entropy corresponding to a positive Fermi energy. These bounds prove that the scaling behaviour of the entanglement entropy is a logarithmically enhanced area law. This is the same scaling behaviour as the one occurring in the case of free fermions, one of the few delocalised systems for which an asymptotic expansion of the entanglement entropy is known.

Finally, in the third and last part, based on joint work with C. Fischbacher [FS20], we consider the finite XXZ spin chain with periodic boundary conditions in the Ising phase. We show that for each eigenvalue in the droplet band there exists at least one eigenvector such that the corresponding entanglement entropy grows at least logarithmically, provided the anisotropy parameter Δ is sufficiently large. In addition, we show a Combes–Thomas estimate for this model, which may be of independent interest.

Preface

The thesis consists of an introductory chapter followed by three chapters with a detailed description of the results, including proofs. The results presented here were obtained in scientific collaboration, which resulted in the publications listed below. The relation to published material is highlighted at the beginning of each of the chapters two to four. Moreover, parts of the introduction coincide both in content and writing with material from the publications (i)-(iii) below.

Published content

(i) P. Müller, L. Pastur and R. Schulte, How much delocalisation is needed for an enhanced area law of the entanglement entropy?, *Commun. Math. Phys.* **376**, 649–679 (2020).

(ii) P. Müller and R. Schulte, Stability of the enhanced area law of the entanglement entropy, accepted by *Ann. Henri Poincaré*, (2020).

(iii) C. Fischbacher and R. Schulte, Lower bound to the entanglement entropy of the XXZ spin ring (2020), e-print arXiv:2007.00735.

We do not refer to the publications below by the numbers (i)–(iii) but by their respective numbers in the bibliography at the end of this thesis.

Chapter 1

Introduction

Quantum entanglement is an important aspect of quantum mechanics, which lies at the centre of interest of 21st century physics. First discovered by A. Einstein, B. Podolsky and N. Rosen [EPR35] it describes a type of quantum mechanical correlations without counterpart in classical physics. Such correlations impact many different aspects of quantum mechanics. They were studied extensively in the context of various branches of modern physics ranging from quantum information science over condensed matter physics to string theory [HHHH09, Laf16].

A key quantity in the context of analysing entanglement is the *bipartite entanglement entropy*, which serves as a quantifier of the entanglement between two subsystems. For a given pure state in a bipartite system it is defined as the von Neumann entropy of the corresponding reduced state [PV07]. We only consider two subsystems corresponding to a distinguished spatial subregion and its complement in this thesis. In such a situation, the entanglement entropy is sometimes also referred to as *geometric entropy*. In recent years, the scaling behaviour of the entanglement entropy has received much attention [ECP10, Laf16], with the asymptotic growth of the entanglement entropy with respect to a scaled version of a spatial region Λ, namely $\Lambda_L := L \cdot \Lambda$ for $L > 0$, being of particular interest. To study the effects of the correlations induced by entanglement in a given state, the leading asymptotic derived from this scaling proves to be rather insightful.

Analysis of the ground state entanglement entropy of various physical systems reveals a curious property – against all expeditions it is generally not extensive. Unlike the physical entropy of a thermal state, the entanglement entropy does not always satisfy a volume law, which means that it does not scale like $\sim L^d$ for a d-dimensional model. Often, the ground state entanglement entropy seems to be subject to an area law instead, which means that it is proportional to the boundary surface of the region $\sim L^{d-1}$. Other types of scaling behaviour, such as an area law with an additional logarithmic enhancement $\sim L^{d-1} \ln L$, are common, too. This is indeed an unusual observation, since generic states generally do not satisfy area laws. Page's law suggests that most of them obey a volume law instead [Pag93, FK94].

Historically, an area law of the entanglement entropy was first observed within the context of black holes. In 1973, J. Bekenstein argued that the thermodynamic entropy of a black hole, which is also called the Bekenstein–Hawking entropy, should be proportional to the horizon area [Bek73, Bek04]. Later, a connection between the Bekenstein–Hawking entropy and the ground state entanglement entropy of a free scalar bosonic field in flat space time was found [BKLS86, Sre93].

The idea of studying the entanglement between two spatial subregions was taken up soon thereafter to quantify correlations in many-body systems. Especially in one-dimensional models numerous results were found. Most notably, M. Hastings proved in his seminal work [Has07] that the ground state of a rather generic one-dimensional system with local interaction always obeys an area law, provided the ground state energy is both simple and separated by a gap from the rest of the spectrum. Area laws of the ground state entanglement entropy also occur in the context of topologically ordered two-dimensional media [KP06].

This raises the question: What does an area law of the entanglement entropy signify for a many-body state? Broadly speaking, it suggests that in this state the correlations induced by entanglement are short-ranged so that only those close to the boundary of the distinguished region yield large contributions. States with such a property may be described with relatively few parameters. This is advantageous for numerical simulations. It enables an approximation by matrix product states. The density matrix renormalisation group, a versatile algorithm often used to model one-dimensional systems, relies on this approximation [Sch05].

Apart therefrom, there is more information to be gained from the scaling behaviour of the entanglement entropy. Those cases where the entanglement entropy does not satisfy an area law are of particular interest. Such a violation of the area law may indicate a quantum critical point, a second order phase transition at zero temperature marked by the divergence of a correlation length. A logarithmic enhancement of an area law was first found in the ground states of XY and XXZ spin chains at critical points [VLRK03]. Note that the same spin models with non-critical parameters satisfy an area law. More generally, a logarithmic growth of the entanglement entropy for one-dimensional critical systems was shown by P. Calabrese and J. Cardy by using conformal field theory [CC09]. This is expected to be a purely one-dimensional phenomenon. In higher dimensions it is conjectured that the leading term of the entanglement entropy is proportional to the surface area at any point, though the criticality of the system might be encoded in a sub-leading term [MFS09, HW14]. A sub-leading term to the entanglement entropy is also of interest as a criterion for characterising topological order [HIZ05, KP06].

The scaling behaviour of the entanglement entropy has also received some attention in the context of studying the many-body localisation phase. Since this thesis concerns questions linked to this phase, localisation shall be addressed in more detail. In 1958, P. W. Anderson [And58] discovered that in some non-interacting models, used to describe disordered materials, the absence of diffusion of waves. Later on, this phenomenon was called *Anderson localisation.* The materials in question include amorphous materials and glasses, where atoms are not positioned on a periodic lattice but are rather randomly distributed, as well as alloys and materials with impurities, which are random mixtures of different atoms. The randomness included in these models causes the eigenstates of certain parts of the spectrum to be localised in space, hence the name Anderson localisation. Because of this localisation property, quantum transport is suppressed. Anderson localisation is therefore considered to be a source of an insulating behaviour other than the spectral gap responsible for the better known band insulators.

Many-body localisation requires the occurrence of a similar effect in systems with a particle-particle interaction. Contrary to the notion of Anderson localisation in a non-interacting system, the notion of localisation in a many-body system is not as clearly cut out. As we have remarked before, localisation is thought to prevent quantum transport and thus to be a source of insulating behaviour. In a non-interacting system this is equivalent to a situation in which no particle moves. However, in a many-body setting quantum transport can occur in the form of group waves, where each individual particle may move very little. Due to such complications, understanding many-body localisation in any kind of interacting system is much more challenging.

On a mathematical level, Anderson localisation has been studied extensively within the mathematical theory of random Schrödinger operators [CL90, PF92, Sto01, Kir08, AW15]. By methods such as the multi-scale analysis [FS83, FMSS85, vDK89] or the fractional moment method [AM93, Aiz94, ASFH01], localisation in some part of the spectrum was proven for a number of models. However, the same is not true for many-body localisation. Only a few rigorous results exist, mostly in specific one-dimensional systems [KP90, ARNSS17, HSS12, Mas17]. We especially point out the recent results concerning localisation in the lowest energy band of the XXZ spin chain in a disordered magnetic potential [BW17, EKS18a, EKS18b], since this thesis addresses a related topic. Even fewer attempts have been made to show many-body localisation in a more general setting [Imb16a, Imb16b]. All in all, our general understanding of many-body localisation is far from satisfactory. There still remains much to be explored.

One characteristic of a localised state are the rapidly decaying spatial correlations responsible for an area law of the entanglement entropy [BH15]. And indeed, such scaling behaviour of the engagement entropy has been proven for a number of disordered systems. First and foremost, in a system with quasi-free fermions in a disordered background potential, the many-body ground state corresponding to a Fermi energy E_F satisfies an area law of the entanglement entropy, provided E_F lies in a region of Anderson localisation [PS14, EPS17, PS18a]. For some interacting systems, which are known to be in the many-body localisation phase, area laws for the ground state entanglement entropy have been shown, too. This again includes spin-chains in a random magnetic background potential [ARS15, ARNSS17, BW18, FS18, Sto20]. Another example is the bosonic model of randomly coupled harmonic oscillators [NSS13, AR18, BSW19].

Having established so far that localisation is connected to area laws of the entanglement entropy it remains to be assessed what happens in the absence of localisation. One might expect some violation of an area law, since the correlations in delocalised states are less likely to be short-ranged. Supporting this hypothesis is the case of free fermions, for which the respective (generalised) eigenstates are clearly delocalised. In any dimension $d \in \mathbb{N}$, the entanglement entropy corresponding to a Fermi energy $E_F > 0$ satisfies a logarithmically enhanced area law, i.e. it scales like $\sim L^{d-1} \ln L$ [Wol06, HLS11, LSS14, LSS17]. Another example for such scaling behaviour of the entanglement entropy occurs in a system of quasi-free fermions in a periodic background potential in one dimension [PS18b], which is another model with delocalised eigenstates. If the scaling behaviour of the entanglement entropy is indeed different for many-body localised and delocalised states, it might serve as a localisation criterion or, at least, as an indicator for localisation. Such a criterion could benefit in further studies of the elusive many-body localisation phase. As pointed out before, there already exists a number of many-body localised systems for which an area law of the ground state entanglement entropy is confirmed. However, to the best of our knowledge there are no other results than the ones already mentioned, proving a violation of the area law in the absence of localisation. Now, the aim of this thesis is to collect further examples of logarithmically enhanced area laws in order to explore the connection between delocalisation and the scaling of the entanglement entropy. To that end three different models with eigenstates that are known to be delocalised are to be examined, two of them without and one with particle-particle interactions. These models are: the random dimer model, a system of quasi-free fermions in a compactly supported, bounded background potential and the XXZ spin chain in the Ising phase. In order to show the absence of an area law, the thesis focuses on proving lower bounds to the entanglement entropy. For the second model an upper bound is also obtained.

Before expanding on these models in more detail, the mathematical foundations of entanglement and the entanglement entropy are introduced in the next section.

1.1 Definition of the entanglement entropy

Entanglement occurs in quantum mechanical systems consisting of two or more subsystems. A state in such a system is entangled if it cannot be described in terms of separate states of each subsystem. Let us focus on a bipartite system, which is a system with two subsystems A and B. Mathematically, each subsystem is described by its own separable Hilbert space $\mathbb{H}_A$ and $\mathbb{H}_B$. The total system is described by the Hilbert space $\mathbb{H} := \mathbb{H}_A \otimes \mathbb{H}_B$, where $\otimes$ denotes the tensor product of Hilbert spaces. Here, and in the following we use the Dirac notation for vectors in a Hilbert space. A vector $|\phi\rangle \in \mathbb{H}$ is called *separable*, if there exist vectors $|\phi^A\rangle \in \mathbb{H}_A$ and $|\phi^B\rangle \in \mathbb{H}_B$ with

$$|\phi\rangle = |\phi^A\rangle \otimes |\phi^B\rangle . \tag{1.1}$$

If $|\phi\rangle \in \mathbb{H}$ is not separable, it is called *entangled*.

One method to quantifying for a given state the degree of entanglement between two subsystems is the *bipartite entanglement entropy*. This quantity depends on the reduction of a state to one of the subsystems. This is achieved by taking the *partial trace*, which we are going to introduce first.

Definition 1.1.1. *Let $\mathbb{H}_A$, $\mathbb{H}_B$ be separable Hilbert spaces. Let $\mathbb{H} := \mathbb{H}_A \otimes \mathbb{H}_B$. Let $|\psi^B\rangle \in \mathbb{H}_B$ be fixed. We define*

$$\begin{aligned} V(\psi^B) :\ \mathbb{H}_A &\to \mathbb{H}_A \otimes \mathbb{H}_B, \\ |\phi^A\rangle &\mapsto |\phi^A\rangle \otimes |\psi^B\rangle . \end{aligned} \tag{1.2}$$

Let furthermore $\{|\psi_j^B\rangle\}_j$ be an orthonormal basis of $\mathbb{H}_B$. For any trace-class operator $T : \mathbb{H} \to \mathbb{H}$ let the partial trace of T with respect to $\mathbb{H}_B$ be given by

$$\operatorname{tr}_B T := \sum_j \left[V(\psi_j^B)\right]^* T V(\psi_j^B). \tag{1.3}$$

Here, A^* denotes the Hermitian adjoint of an operator A.

Remark 1.1.2. *(i) The partial trace does not depend on the choice of the orthonormal basis $\{|\psi_j^B\rangle\}_j$.*

(ii) The partial trace $\operatorname{tr}_B T$ of a trace class operator T, is a trace class operator mapping $\mathbb{H}_A$ onto $\mathbb{H}_A$. Moreover, $\operatorname{tr}\{T\} = \operatorname{tr}\{\operatorname{tr}_B T\}$.

We are now able to reduce a state to a subsystem. Let us consider a pure state $|\phi\rangle \in \mathcal{H}$ with $\| |\phi\rangle \| = 1$, where $\| \cdot \|$ denotes the standard norm on $\mathbb{H}$. Let further $\rho(\phi) := |\phi\rangle\langle\phi|$ be the density operator associated with this state. The *reduced density with respect to the subsystem A* is given by

$$\rho_A(\phi) := \operatorname{tr}_B\{\rho(\phi)\}. \tag{1.4}$$

Let $\rho_B(\phi) := \operatorname{tr}_A\{\rho(\phi)\}$ be defined analogously. The operator $\rho_A(\phi)$ is itself a density operator defined on the Hilbert space $\mathbb{H}_A$.

Definition 1.1.3. *The bipartite entanglement entropy is defined as the von Neumann-entropy of the reduced state,*

$$S_1(A;\phi) := \operatorname{tr}\{s(\rho_A(\phi))\} \tag{1.5}$$

where tr *denotes the trace and* $s\colon [0,1] \to [0,\infty[$ *with*

$$s(\lambda) := -\lambda \log_2(\lambda) \quad \text{for all } \lambda \in [0,1]. \tag{1.6}$$

Here, $\log_2$ *denotes the binary logarithm with the convention* $0\log_2 0 := 0$.

The partial trace is identical to a projection only if $|\phi\rangle$ is separable, i.e. $\rho_A(\phi) = |\phi^A\rangle\langle\phi^A|$ if $|\phi\rangle = |\phi^A\rangle \otimes |\phi^B\rangle$ for some normalised vectors $|\phi^A\rangle \in \mathbb{H}_A$ and $|\phi^B\rangle \in \mathbb{H}_B$. Otherwise, $\rho_A(\phi)$ is a mixed state, which implies that it has eigenvalues other than zero or one. Hence, $S_1(A;\phi) = 0$ if and only if $|\phi\rangle$ is separable.

There are also other measures for the bipartite entanglement, similar to the entanglement entropy. One example for such a measure are the *Rényi entropies* [Weh78, Section II.G], which are defined as $S_\alpha(A;\phi) := \frac{1}{1-\alpha}\log_2(\operatorname{tr}\{[\rho_A(\phi)]^\alpha\})$ for $\alpha \in]0,\infty[\setminus \{1\}$. These entanglement measures are directly connected with the entanglement entropy via the relation $\lim_{\alpha\to1} S_\alpha(A;\phi) = S_1(A;\phi)$. Another important measure is the logarithmic negativity [HHHH09, VW02], which is a lower bound to the entanglement entropy. These measures are used analogously to the entanglement entropy in the context of studying the correlations between spatial regions [LSS14, BW18, AR18].

As we have mentioned before, we are interested in measuring the entanglement between a distinguished spatial subregion and its complement. Before we can determine the entanglement entropy between spatial subregions, we first have to identify the respective Hilbert spaces associated with these regions. To that effect, let us consider a d-dimensional spin system with a spin positioned on each site of the lattice $\Gamma \subseteq \mathbb{Z}^d$. The Hilbert space used to describe a single spin is $\mathbb{C}^2$. The Hilbert space for the whole system is given by $\mathbb{H}_\Gamma$, where

$$\mathbb{H}_{\mathcal{A}} := \bigotimes_{j\in\mathcal{A}} \mathbb{C}^2 \tag{1.7}$$

for any finite set $\mathcal{A}$. Of course, any subset $\varnothing \neq \Lambda \subset \Gamma$ is described by $\mathbb{H}_\Lambda$, and we have

$$\mathbb{H}_\Gamma \cong \mathbb{H}_\Lambda \otimes \mathbb{H}_{\Lambda^c}, \tag{1.8}$$

where $(\cdot)^c$ denotes the complement (with respect to Γ). We are now able to define the entanglement entropy between Λ and Λ^c for any state $|\phi\rangle \in \mathbb{H}_\Gamma$ by

$$S(\Lambda;\Gamma,\phi) := S_1(\Lambda;\phi), \tag{1.9}$$

where we substituted $\mathbb{H}_A$ in (1.5) by $\mathbb{H}_\Lambda$. Notice that we made the dependance on Γ explicit in this notation.

The model describing fermions on a finite lattice Γ is related to the one of a spin system on the same lattice. The Hilbert space corresponding to one single fermion is of course $\ell^2(\Gamma)$. Consequentially, the many-particle Hilbert space is given by the fermionic Fock space $\mathcal{F}_-(\ell^2(\Gamma))$. Applying the formalism of second quantisation enables us to identify the Fock space $\mathcal{F}_-(\ell^2(\Gamma))$ with the spin space $\mathbb{H}_\Gamma$, see Appendix A.1. We write $\mathcal{F}_-(\ell^2(\Gamma)) \cong \mathbb{H}_\Gamma$. Therefore, we are able identify the subspaces corresponding to the spatial region Λ and Λ^c as $\mathcal{F}_-(\ell^2(\Lambda)) \cong \mathbb{H}_\Lambda$ and $\mathcal{F}_-(\ell^2(\Lambda^c)) \cong \mathbb{H}_{\Lambda^c}$ respectively. Hence, the entanglement entropy for a state $|\phi\rangle \in \mathcal{F}_-(\ell^2(\Gamma))$ is defined analogously to (1.9).

An interesting simplification of the formula for the entanglement entropy exists in case that $|\phi\rangle$ is a ground state of a quasi-free fermionic system. Let $H : \ell^2(\Gamma) \to \ell^2(\Gamma)$ be a single-particle Hamiltonian and let $(|\psi_j\rangle)_{j\in\Gamma} \subset \ell^2(\Gamma)$ denote an orthonormal basis of eigenstates corresponding to the eigenvalues $(E_j)_{j\in\Gamma} \subset \sigma(H)$. Let further $E_F > \min \sigma(H)$ be a Fermi energy and let $N \equiv N(E_F) := |\{j \in \Gamma : E_j < E_F\}|$. Then the N-particle ground state of the corresponding many-body system is given by

$$|\phi\rangle := \Pi_-\Big[\bigotimes_{\substack{j\in\Gamma,\\ E_j<E_F}} |\psi_j\rangle \Big], \tag{1.10}$$

where Π_- denotes the anti-symmetrisation operator, which ensures that the state $|\phi\rangle$ is fermionic. By a straightforward calculation [Kli06] it can be seen that we can express the entanglement entropy with respect to the region $\varnothing \neq \Lambda \subseteq \Gamma$ completely in terms of the one-particle Hamiltonian,

$$S(\Lambda;\Gamma,\phi) = \operatorname{tr}\big\{h\big(1_\Lambda(X)1_{<E_F}(H)1_\Lambda(X)\big)\big\}, \tag{1.11}$$

where X denotes the position operator and $h : [0,1] \to \mathbb{R}$ with

$$h(\lambda) := -\lambda \log_2(\lambda) - (1-\lambda)\log_2(1-\lambda) \quad \text{for all } \lambda \in [0,1]. \tag{1.12}$$

We write $1_\mathcal{A}$ for the indicator function on the set $\mathcal{A}$ and, in abuse of notation, $1_{<E_F} := 1_{]-\infty,E_F[}$. Note that the Fermi projection $1_{<E_F}(H)$ takes the place of the many-body eigenstate $\rho(\phi)$.

The right-hand side of (1.11) is used to define the entanglement entropy between spatial subregions for more general quasi-free fermionic systems, both discrete and continuous.

Definition 1.1.4. *Let $\mathbb{K} \in \{\mathbb{Z},\mathbb{R}\}$ and $d \in \mathbb{N}$. Let $\Gamma \subseteq \mathbb{K}^d$ be a Borel subset, which is not a null-set. Let further H be a Hamiltonian which is densely defined on $\ell^2(\Gamma)$ (if $\mathbb{K} = \mathbb{Z}$) or $L^2(\Gamma)$ (if $\mathbb{K} = \mathbb{R}$). For any Fermi energy $E_F \in \mathbb{R}$ we define the entanglement entropy with respect to a bounded, measurable subset $\Lambda \subset \Gamma$ by*

$$S_{E_F}(\Lambda;\Gamma,H) := \operatorname{tr}\big\{h\big(1_\Lambda(X)1_{<E_F}(H)1_\Lambda(X)\big)\big\}. \tag{1.13}$$

1.2 Content of this thesis

Even without the complications added by particle-particle interactions, localisation is not always a clearly cut-out concept. There are several different definitions of localisation, which are not always equivalent. One model where this is noticeable is the random dimer model with Bernoulli disorder, which we consider in Chapter 2. This discrete model describes quasi-free fermions in a one-dimensional chain composed out of two distinctive dimer molecules, which are strung together in random order. For a definition of the corresponding random Schrödinger operator $H^\omega : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ for a certain event ω in the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, see (2.1). The dimer model exhibits *spectral localisation*, which means that for $\mathbb{P}$-almost all ω the operator H^ω has only pure-point spectrum and all eigenfunctions decay exponentially in space [DBG00]. However, there is another, stronger notion for localisation called *strong dynamical localisation*. A random Schrödinger operator $\tilde{H}^\omega$, which is densely defined on $\ell^2(\mathbb{Z}^d)$ for some $d \in \mathbb{N}$, satisfies strong dynamical localisation in the energy interval $I \subseteq \mathbb{R}$ if there exists $C, \mu \in [0, \infty[$ such that

$$\mathbb{E}\Big[\sup_{t\in\mathbb{R}} |\langle \delta_j, \mathrm{e}^{-\mathrm{i}t\tilde{H}} 1_I(\tilde{H})\delta_k\rangle|\Big] \le C\mathrm{e}^{-\mu|j-k|} \quad \text{for all } j,k \in \mathbb{Z}^d \tag{1.14}$$

where $(|\delta_k\rangle)_{k\in\mathbb{Z}^d}$ denotes the canonical basis of $\ell^2(\mathbb{Z}^d)$ and $|\cdot|$ denotes the Euclidian norm. The methods used for proving dynamical localisation, the fractional moment method and multiscale analysis, can also be used to establish exponential decay (or possibly only sub-exponential decay) of the Fermi projection [AG98, GK06a, AW15], i.e. for every $E_F \in I$, where I is an interval of strong dynamical localisation, there exists $C, \mu \in]0, \infty[$ such that

$$\mathbb{E}\big[|\langle \delta_j, 1_{<E_F}(\tilde{H})\delta_k\rangle|\big] \le C\mathrm{e}^{-\mu|k-j|} \quad \text{for all } j,k \in \mathbb{Z}^d. \tag{1.15}$$

In view of Definition 1.1.4, this is a useful insight in the context of determining an upper bound of the entanglement entropy. In [PS14], it was shown that (1.15) is a sufficient condition for ensuring an area law of the entanglement entropy in the sense that there exist constants $c, C \in]0, \infty[$ such that for $\Lambda_L := [-L, L]^d \cap \mathbb{Z}^d$ we have

$$cL^{d-1} \le \mathbb{E}\big[S_{E_F}(\Lambda_L; \mathbb{Z}^d, \tilde{H})\big] \le CL^{d-1}. \tag{1.16}$$

However, the dimer Hamiltonian H^ω neither satisfies (1.15) nor strong dynamical localisation on the whole spectrum. On the contrary, there exists superdiffusive quantum transport in this model, i.e. for every $\alpha \in]0, 1/2[$ there exists $C_\alpha > 0$ such that

$$\int_0^T \frac{\mathrm{d}t}{T} \langle \delta_0, \mathrm{e}^{\mathrm{i}tH^\omega}|X|^2\mathrm{e}^{-\mathrm{i}tH^\omega}\delta_0\rangle \ge C_\alpha T^{3/2-\alpha} \tag{1.17}$$

for all $T > 0$ and $\mathbb{P}$-almost all ω. Here and in the following, $|A|^2 := A^*A$ for any operator A. Predicted by [DWP90], this result was shown in [JSBS03]. The occurrence of transport is due to specific critical points in the spectrum of H^ω, at which the localisation length diverges. Apart from these points, the operator satisfies dynamical localisation [DBG00]. In this thesis we prove a logarithmic lower bound of the expectation of the entanglement entropy for a Fermi energy identical to one of the critical points responsible for quantum transport in this model. Hence, we prove that spectral localisation over the entire spectrum is, contrary to dynamical localisation, not sufficient to guarantee the existence of an area law of the entanglement entropy. To prove this statement, we make use of the delocalisation properties near the critical point which were studied in detail in [JSBS03].

In Chapter 3 of this thesis we consider a more general system of quasi-free fermions in $d \in \mathbb{N}$ space dimensions moving in a bounded and compactly supported background potential $V \in L_c^\infty(\mathbb{R}^d)$. The corresponding (single particle) Hamiltonian is given by $H := H_0 + V$, where the Laplacian $H_0 := -\Delta$ denotes the Hamiltonian of the free Fermi gas in d dimensions.

Let us first consider the case of free fermions without any additional potential. It was suggested in [Wol06, GK06b, Gio06, HLS11] that the entanglement entropy at any Fermi energy $E_F > 0$ of free fermions should satisfy a logarithmically enhanced area law, i.e.

$$S_{E_F}(\Lambda_L; \mathbb{R}^d, H_0) = \Sigma_0\, L^{d-1} \ln L + o\big(L^{d-1} \ln L\big) \quad \text{as } L \to \infty \tag{1.18}$$

where $\Lambda_L := L \cdot \Lambda$ is the scaled version of a bounded Lipschitz-domain $\Lambda \subset \mathbb{R}^d$ with piecewise C^1-boundary. Based on Widom's conjecture, the leading-order coefficient $\Sigma_0 \equiv \Sigma_0(d, \Lambda, E_F)$ was expected to depend only on the Fermi energy and the surface $\partial\Lambda$. Widom's conjecture was finally proven by A. Sobolev in his celebrated works [Sob13, Sob15]. This enabled H. Leschke, A. Sobolev and W. Spitzer to confirm in [LSS14] the leading asymptotic of the entanglement entropy in (1.18). For a one-dimensional system with a periodic background potential a logarithmically enhanced area law can be obtained by similar methods [PS18b].

For an arbitrary, bounded background potential V, we do not expect to encounter significantly stronger correlations induced by entanglement, as compared to the case of free fermions. We therefore predict that any such Schrödiger operator should satisfy at most a logarithmical enhancement of an area law. As a first step towards proving this conjecture, we consider compactly supported potentials only. The operator H has many similarities with H_0. Most importantly, the absolutely continuous spectrum of both operators is given by the non-negative real numbers. This implies delocalisation on the whole positive real line, which leads us to expect a logarithmically enhanced area law of the entanglement entropy. In this thesis we are able to prove for any $E_F > 0$ both an upper and a lower bound for $S_{E_F}(\Lambda_L; \mathbb{R}^d, H)$ proportional to $\sim L^{d-1} \ln L$, where $\Lambda_L := L \cdot \Lambda$ is again a scaled version of a subset $\Lambda \subset \mathbb{R}^d$ satisfying Assumption 3.1.2. We obtain this result by deriving a perturbation theory for (1.18). A limiting absorption principle for H of the form [Agm75, JM17] is required as a major technical input to our proof.

Finally, in Chapter 4, we consider the XXZ spin chain in the Ising-phase for energies in the droplet band. At an earlier point in this introduction we have already mentioned that the disordered XXZ spin chain is one of the few interacting systems, for which many-body localisation has been proven [EKS18b, EKS18a, BW17]; see [Sto20] for a survey of the most recent developments. To provide some context for the result of Chapter 4, we briefly introduce the disordered model considered in these publications. For $L \in \mathbb{N}$ let $\mathcal{V}_L := \{0, \cdots, L-1\}$. The Hamiltonian of a finite XXZ chain of length L in a disordered magnetic background field is given by $\tilde{H}_L^\omega : \ell^2(\mathcal{V}_L) \to \ell^2(\mathcal{V}_L)$,

$$\tilde{H}_L^\omega \equiv \tilde{H}_L^\omega(\Delta) := \sum_{j=1}^{L-1} \Big[\Big(\frac{1}{4} - S_j^3 S_{j-1}^3 \Big) - \frac{1}{\Delta} \big(S_j^1 S_{j-1}^1 + S_j^2 S_{j-1}^2 \big) \Big] \\ + \sum_{j=0}^{L-1} \omega_j N_j + \beta(\Delta)(N_0 + N_{L-1}) \tag{1.19}$$

for some event $\omega \in \Omega := \mathbb{R}^L$, where $S^1, S^2, S^3 \in \mathbb{C}^{2\times2}$ denote the standard spin-1/2 Pauli matrices and $N := \frac{1}{2}(1_{2\times2} - 2S^3)$ denotes the local number operator. Here, and in the following, let for any matrix $A \in \mathbb{C}^{2\times2}$ and $j \in \mathcal{V}_L$ the operator $A_j : \mathbb{H}_{\mathcal{V}_L} \to \mathbb{H}_{\mathcal{V}_L}$ denote the operator acting like A on the j-th spin. The choice of anisotropy parameter $\Delta \in \,]0,\infty[$ characterises the Ising phase. The Hamiltonian features *droplet boundary conditions*, i.e. $\beta(N_0 + N_{L-1})$ with $\beta \equiv \beta(\Delta) := \frac{1}{2}(1 - \frac{1}{\Delta})$.

The many-body localisation result most closely related to the definition of dynamical localisation, which we mentioned before in the context of non-interacting systems, is the following [EKS18b, EKS18a, BW17]: There exist an interval I at the bottom of the spectrum of $\tilde{H}_L^\omega$ such that for sufficiently large Δ there exist constants $C, \mu \in \,]0,\infty[$ such that

$$\mathbb{E}\Big[\sum_{E \in \sigma(\tilde{H}_L) \cap I} \|N_j \psi_E\| \|N_k \psi_E\| \Big] \le C \mathrm{e}^{-\mu|j-k|} \qquad \text{for all } j,k \in \mathcal{V}_L \tag{1.20}$$

where ψ_E^ω denotes the eigenstate corresponding to $E \in \sigma(\tilde{H}_L^\omega)$. The interval I is a subset of the *droplet spectrum*, which we are going to discuss in more detail in Section 4.1.

Other localisation results that have been shown as well, include dynamical exponential clustering [EKS18b, EKS18a] and zero-velocity Lieb-Robinson bounds [EKS18a]. More relevant for our purpose, an area law for the expectation of the entanglement entropy for eigenstates ψ_E corresponding to energies in the droplet spectrum has been shown in [BW18]. In addition to the area law, a logarithmic upper bound for an arbitrary deterministic magnetic field has been proven simultaneously. Such logarithmic upper bounds exist also for eigenstates corresponding to higher energies outside the droplet spectrum [ARFS20]. We now ask the following question: If many-body localisation induces in this model an area law of the entanglement entropy, is then on the other hand delocalisation accompanied by a violation of the area law? If an area law of the entanglement entropy is indeed a criterion for localisation, delocalised states must have a different scaling behaviour. To answer this question, we consider the XXZ model without a magnetic field. Furthermore, we consider cyclic boundary conditions instead of the droplet boundary conditions in (1.19). Hence, the eigenstates of the droplet spectrum are delocalised, due to the translational symmetry in this system. For a large number of eigenstates corresponding to eigenvalues in the droplet spectrum we prove a logarithmic lower bound of the entanglement entropy. As an intermediary step, we also show a Combes-Thomas estimate for this Hamiltonian, which may be of interest on its own.

Chapter 2

Logarithmic enhancement in the dimer model

Can we observe a logarithmic enhancement of the area law if we also have overall spectral localisation at the same time? To answer this question, we consider quasi-free fermions in the random dimer model with Bernoulli disorder. The dimer Hamiltonian has almost surely only pure-point spectrum with corresponding exponentially decaying eigenfunctions. However, there exist critical energies in the spectrum, where the localisation length diverges. In this chapter we show a logarithmically divergent lower bound for the entanglement entropy in the case that the Fermi energy coincides with one of these critical energies. This chapter is the result of a collaboration with P. Müller and L. Pastur. The content was already published in [MPS20].

2.1 Introduction and result

We consider a system of quasi-free fermions in the one-dimensional lattice of integers $\mathbb{Z}$. The Hamiltonian $H : \Omega \ni \omega \mapsto H^\omega$ of the random dimer model is given by the sum of the kinetic part represented by the discrete Laplacian and a random potential,

$$H^\omega := -\sum_{x\in\mathbb{Z}} \Big(|\delta_x\rangle\langle\delta_{x+1}| + |\delta_{x+1}\rangle\langle\delta_x| \Big) + v \sum_{x\in\mathbb{Z}} V^\omega(x)\, |\delta_x\rangle\langle\delta_x| . \tag{2.1}$$

Here, $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space and the realisation H^ω acts as a bounded linear operator on $\ell^2(\mathbb{Z})$ for a given disorder configuration ω. We write $\{\delta_x\}_{x\in\mathbb{Z}}$ for the canonical basis of $\ell^2(\mathbb{Z})$ and use the Dirac notation for rank-1 operators. The random potential with disorder strength $v > 0$ acts as the multiplication operator by the single-site potentials $\big(V^\omega(x)\big)_{x\in\mathbb{Z}}$, which are the realisations of a family of real-valued random variables with the

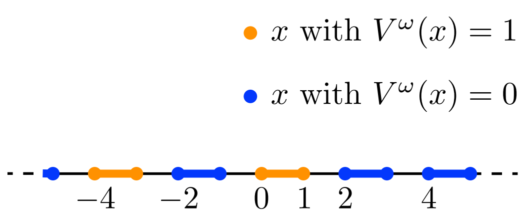

Figure 2.1: The dimer model

properties $V(2x) = V(2x+1)$ for all $x \in \mathbb{Z}$ and $\big(V(2x)\big)_{x\in\mathbb{Z}}$ are independently and identically distributed. This means that every other pair of consecutive sites shares the same value of the potential. The random variable $V(0)$ is Bernoulli distributed. It assumes one of the two different potential values $V_\pm \in \mathbb{R}$ with probability $p_\pm \in \,]0,1[$, subject to $p_+ + p_- = 1$. Without loss of generality, we set $V_- := 0$ and $V_+ := 1$. The random Schrödinger operator H describes a random infinite sequence of two kinds of homodimers linked together to an infinite chain. The random dimer model is a special case of the more general random polymer model, which was treated in [JSBS03].

The spectrum of the operator (2.1) is given by $\sigma(H^\omega) = [-2,+2] \cup [v-2, v+2]$ for $\mathbb{P}$-almost every $\omega \in \Omega$. This observation is the result of standard ergodicity argument [CL90, PF92, AW15] – here with respect to $2\mathbb{Z}$-translations. Moreover, the spectrum is almost surely pure-point [DBG00], as is common in one-dimensional random models [AW15]. For our purpose, the most interesting property of this particular model is that it exhibits characteristics of delocalisation at isolated critical energies in the sense that the localisation length diverges at these points in the spectrum [DWP90, JSBS03]. The critical energies in question occur at $\{0, v\}$, provided $v < 2$. We state and discuss the precise result in Section 2.2. Critical energies in general are isolated points in the spectrum, where the Lyapunov exponent $\mathcal{L}$ vanishes. Apart from $\{0, v\}$ there exist other critical energies in the dimer model for specific choices of the disorder strength [DBG00]. However, it is not at all clear what kind of delocalisation phenomena are to be expected at these other energies. In any case, [DBG00] proves strong dynamical localisation apart from all of these exceptional energies.

Our main result shows the existence of a logarithmic lower bound to the disorder-averaged entanglement entropy, if the following two conditions are met. First, the Fermi energy must be equal to either 0 or v. And second, the disorder strength v must be sufficiently weak. Given $L \in \mathbb{N}$, let $\Lambda_L := \{1, \dots, L\}$ be a box in $\mathbb{Z}$ consisting of $|\Lambda_L| = L$ consecutive sites.

Theorem 2.1.1. *Consider the entanglement entropy* (1.13) *for the Hamiltonian* (2.1) *of the random dimer model. Then, there exists a maximal disorder strength* $v_0 \in \,]0,2[$ *such that for every* $v \in \,]0, v_0]$ *and for a critical Fermi energy* $E_c \in \{0, v\}$*, we have*

$$\liminf_{L\to\infty} \frac{\mathbb{E}\big[S_{E_c}(\Lambda_L; \mathbb{Z}, H)\big]}{\ln L} > 0. \tag{2.2}$$

Here, $\mathbb{E}$ *denotes the expectation corresponding to the probability measure* $\mathbb{P}$.

In proving the theorem, we obtain an enhancement to the area law for a finite-volume entanglement entropy as an intermediate result. Instead of the infinite lattice $\mathbb{Z}$ we consider the finite volume $\Gamma_L := \{-L, \cdots, L-1\} \subset \mathbb{Z}$. By $H_L^\omega := 1_{\Gamma_L} H^\omega 1_{\Gamma_L}$ we denote the restriction of the infinite-volume operator H^ω to Γ_L. For a suitable choice of an L-dependent $\Lambda_L' \subset \Gamma_L$, the finite-volume entanglement entropy $S_{E_c}(\Lambda_L', \Gamma_L; H_L^\omega)$ admits a logarithmic lower bound.

Theorem 2.1.2. *Let* $v \in \,]0,2[$ *and the Fermi energy* $E_c \in \{0, v\}$ *be critical. Then there exists* $\delta' \in \,]0,1[$ *such that for all* $\delta \in \,]0, \delta']$ *the finite-volume entanglement entropy satisfies*

$$\liminf_{L\to\infty} \frac{S_{E_c}(\Lambda_L'; \Gamma_L, H_L^\omega)}{\ln L} > 0 \tag{2.3}$$

for $\mathbb{P}$-almost all $\omega \in \Omega$. Here, we have defined $\Lambda'_L := [-L, -(1-\delta)L] \cap \mathbb{Z}$.

Remark 2.1.3. *(i) The proof of Theorem 2.1.1 shows that the left-hand side of* (2.2) *is bounded from below by* 2^{-16}*, see* (2.196)*. More interestingly, the proof of Theorem 2.1.2 yields a strictly positive constant, which depends only on* v*, but not on* ω *that serves as a lower bound for the limit inferior in* (2.3)*.*

(ii) We point out that, in contrast to Theorem 2.1.1, the validity of Theorem 2.1.2 is not restricted to weak disorder. Furthermore, it provides an almost-sure bound, whereas Theorem 2.1.1 is obtained in expectation only. This is of relevance, because the entanglement entropy is known not *to be self-averaging in one dimension [PS18a]. The price we pay is that the box* Λ'_L *is attached to one boundary point of* Γ_L*. Our methods in Section 2.4 do not allow us to pass to the infinite-volume entanglement entropy in this situation.*

(iii) Finite-volume entanglement entropies with boxes attached to a boundary as in (2.3) *are often considered in physics, especially if the entanglement entropy is determined numerically, see e.g. [ISL12, PY14].*

(iv) For all energies at which the Lyapunov exponent does not vanish, the multi-scale analysis can be applied to prove strong dynamical localisation, despite the Bernoulli distribution of the random variables [CKM87, DBG00]. Some additional work then yields fast decay of the Fermi projection at all these energies. Thus, it follows from [PS14, EPS17] that the entanglement entropy exhibits an area law at all non-critical Fermi energies of the random dimer model.

2.1.1 Roadmap

In Section 2.2 we discuss in detail the delocalisation phenomena as described in [JSBS03] that occur at the critical energies. These results are the foundation for our own approach. Since the dependence on the disorder strength is crucial for our proof, we present a slightly enhanced version of the original result. The necessary additional arguments follow closely the proof presented in [JSBS03].

Next, we show an intermediate result similar to Theorem 2.1.2 for the finite-volume operator in Section 2.3. In particular, we conduct a detailed analysis of the Prüfer angles of generalised eigenfunctions to obtained a logarithmic lower bound for the finite-volume entanglement entropy.

Finally, in Section 2.4, we proof Theorem 2.1.1 by extending the finite-volume result to the infinite volume.

2.2 Delocalisation at the critical energies

The delocalisation and transport properties at critical energies of the random polymer model were studied in detail by S. Jitomirskaya, H. Schulz-Baldes and G. Stolz in [JSBS03]. They showed that in a window around the critical energies the finite-volume Hamiltonian shares many properties with the discrete Laplacian. Within this window, the eigenvalues are rigidly spaced apart from each other and the eigenfunctions are evenly spread out like plain waves. In the following, we present a short overview of these results, since they are relevant for our own. We restrict ourselves by treating the dimer model with disorder strength $v < 2$ only.

Recall that the operator H^ω has pure-point spectrum with exponentially decaying eigenfunctions. In other words, H^ω exhibits spectral localisation. However, the rate of decay of the eigenfunctions is not the same for each eigenenergy. In the dimer model, delocalisation arises from the fact that this rate of decay vanishes sufficiently fast at the critical energies.

In order to formulate this in a more rigorous way we need to consider general solutions of the eigenequation. Given $E \in \mathbb{R}$ and $\omega \in \Omega$, let $\tilde{\phi}_E^\omega : \mathbb{Z} \to \mathbb{R}$ be a non-trivial solution of the difference equation

$$-\tilde{\phi}_E^\omega(x-1) - \tilde{\phi}_E^\omega(x+1) + vV^\omega(x)\tilde{\phi}_E^\omega(x) = E\tilde{\phi}_E^\omega(x) \qquad \text{for all } x \in \mathbb{Z}. \tag{2.4}$$

This solution is an eigenfunction of H^ω if and only if it is an element of $\ell^2(\mathbb{Z})$, too. Thus, general solutions of (2.4) expand our previous notion of eigenfunctions.

Any solution of (2.4) can be constructed directly with the aid of transfer matrices. Given $V \in \{0,1\}$ and $E \in \mathbb{R}$, we define the single-step transfer matrix by

$$W_V(E) := \begin{pmatrix} vV - E & -1 \\ 1 & 0 \end{pmatrix} \in \mathbb{R}^{2\times 2}. \tag{2.5}$$

The (multi-step) transfer matrix

$$W^\omega(E; y, x) := \begin{cases} W_{V^\omega(y-1)}(E) \cdots W_{V^\omega(x)}(E) & \text{if } x < y, \\ 1_{2\times 2} & \text{if } x = y, \end{cases} \tag{2.6}$$

relates the solution of the discrete Schrödinger equation (2.4) at different sites

$$W^\omega(E; y, x) \begin{pmatrix} \tilde{\phi}_E^\omega(x) \\ \tilde{\phi}_E^\omega(x-1) \end{pmatrix} = \begin{pmatrix} \tilde{\phi}_E^\omega(y) \\ \tilde{\phi}_E^\omega(y-1) \end{pmatrix}, \tag{2.7}$$

where $x \le y$. A useful tool to study the asymptotic behaviour of the solution of (2.4) for a given energy $E \in \mathbb{R}$ is the Lyapunov-exponent

$$\mathcal{L}(E) := \lim_{L\to\infty} \frac{1}{L} \ln \| W^\omega(E; L, 0) \|, \tag{2.8}$$

where $\|\cdot\|$ denotes the operator norm. Note that this definition is well-defined. According to [PF92, Chap. V], the limit in (2.8) exists and is $\mathbb{P}$-almost-surely constant in ω. As a consequence of Oseledec's theorem, any true eigenfunction to an eigenenergy E decays exponentially with rate $\mathcal{L}(E)$ for $x \to \pm\infty$. By definition, the localisation length is given by the inverse of the Lyapunov-exponent.

At the critical energies $E \in \{0, v\}$, the Lyapunov exponent vanishes, which is synonymous to the divergence of the localisation length. To establish this result, the single-dimer transfer matrix

$$D_V(E) := \big(W_V(E)\big)^2 \tag{2.9}$$

for $V \in \{0,1\}$ is particularly helpful. Let us consider the critical energy $E = 0$ only, since $E = v$ can be treated similarly. The single-dimer transfer matrices have the properties $D_0(0) = -1_{2\times2}$, while $D_1(0)$ has the complex eigenvalues $\lambda_\pm = \left(v \pm \mathrm{i}\sqrt{4-v^2}\right)/2$. This implies $\mathcal{L}(0) = 0$ at once. Moreover, a Taylor expansion of the Lyapunov exponent is possible [JSBS03] with

$$\mathcal{L}(\varepsilon) = C\varepsilon^2 + o(\varepsilon^2) \tag{2.10}$$

for a constant $C > 0$.

We want to illustrate the consequences of (2.10) on the eigenfunctions of the finite-volume Hamiltonian. For any $L \in \mathbb{N}$ we define a particular set of solutions of the eigenequation, which we call *generalised eigenfunction*. For all $E \in \mathbb{R}$ and $\omega \in \Omega$ let $\phi^\omega_{L,E}$ be a solution to (2.4) subject to the constraints

$$\phi^\omega_{L,E}(-L-1) = 0,\ \phi^\omega_{L,E}(-L) > 0 \text{ and } \sum_{x\in\Gamma_L} \phi^\omega_{L,E}(x)^2 = 1. \tag{2.11}$$

We write

$$\psi^\omega_{L,E} := \phi^\omega_{L,E}\big|_{\Gamma_L} \tag{2.12}$$

for its restriction onto Γ_L. Only if $\phi^\omega_{L,E}(L) = 0$ is satisfied, is this restriction also an eigenfunction of the finite-volume operator H^ω_L.

The Taylor expansion of the Lyapunov exponent suggests that there exists energy windows around each of the critical energies within which the localisation length is larger than the size of Γ_L. Hence, the corresponding generalised eigenfunctions are evenly spread out over Γ_L with high probability. In this respect, they are similar to plain waves, the eigenfunctions of the discrete Laplacian. Since the leading term in (2.10) is quadratic, the width of this window is roughly $L^{-1/2}$.

Not only do the eigenfunctions of H^ω_L within the energy window resemble the ones of the discrete Laplacian, the spectral statistics of both operators also show similarities. The eigenvalues are evenly spaced, a property which is sometimes called *clock behaviour*. In order to derive this spectral property within the window we introduce Prüfer variables $r^\omega_x(E) \in [0,\infty[$ and $\theta^\omega_x(E) \in \mathbb{R}$ as the polar coordinates of the pair

$$\begin{pmatrix} \phi^\omega_{L,E}(x) \\ \phi^\omega_{L,E}(x-1) \end{pmatrix} =: r^\omega_x(E) \begin{pmatrix} \cos\left(\theta^\omega_x(E)\right) \\ \sin\left(\theta^\omega_x(E)\right) \end{pmatrix} \tag{2.13}$$

for every $x \in \mathbb{Z}$. For ease of notation, we do not keep track of the L-dependence of the Prüfer variables. The angle θ^ω_x is chosen such that it is monotonously increasing in E. Moreover, according to [LGP88, Sect. 12.2] and [JSBS03, Lemma 2], the Prüfer angle is even differentiable in E with derivative

$$\frac{\mathrm{d}}{\mathrm{d}E}\theta^\omega_\ell(E) = \left(r^\omega_\ell(E)\right)^{-2} \sum_{x=-L}^{\ell-1} \left(\phi^\omega_{L,E}(x)\right)^2 \tag{2.14}$$

for all $\ell \in \mathbb{Z}$ with $\ell \geq -L$ and $\omega \in \Omega$. Any eigenvalue E of H^ω_L must satisfy $\theta^\omega_L(E) \in \pi/2 + \pi\mathbb{Z}$ in order for $\phi^\omega_{L,E}$ to meet the boundary condition on the right border. If a generalised eigenfunction is evenly spread out, then $[\theta^\omega_L]'(E) \approx L$. Consequently, integrating 2.14 yields a distance of approximately π/L between any two consecutive eigenvalues of H_L^ω inside this window.

The precise formulation of the delocalisation properties, which we have just discussed, is contained in the next theorem.

Theorem 2.2.1 (Jitomirskaya, Schulz-Baldes, Stolz [JSBS03]). *Let* $v \in\,]0,2[$ *and* $E_c \in \{0, v\}$*. Then*

(i) For every $\alpha > 0$ *there exist a minimal length* $L_{\min} \equiv L_{\min}(\alpha, v) \in \mathbb{N}$ *and constants* $c \equiv c(\alpha, v, p_+) > 0$ *and* $C \equiv C(\alpha, v) > 1$ *with*

$$\lim_{v \downarrow 0} C = 1, \tag{2.15}$$

such that for all $L \geq L_{\min}$ *there are exceptional events* $\Omega_L(\alpha) \subseteq \Omega$ *of small probability*

$$\mathbb{P}[\Omega_L(\alpha)] \leq e^{-cL^{\alpha/2}} \tag{2.16}$$

such that for every non-exceptional $\omega \in (\Omega_L(\alpha))^c$ *the following statement is true: the eigenvalues of* H_L^ω *in the critical energy window*

$$\mathcal{W}_L \equiv \mathcal{W}_L(\alpha, E_c) := [E_c - L^{-1/2-\alpha}, E_c + L^{-1/2-\alpha}] \tag{2.17}$$

are equally spaced in the sense that any two adjacent eigenvalues E *and* E' *in* $\mathcal{W}_L$ *satisfy*

$$\frac{\pi}{C^3 L} \leq |E - E'| \leq \frac{\pi C^3}{L}. \tag{2.18}$$

Furthermore, for any $E \in \mathcal{W}_L$ *the generalised eigenfunction* $\psi_{L,E}^\omega$ *of* (2.4)*, defined as in* (2.12)*, is evenly spread out over* Γ_L *in the sense that*

$$\frac{1}{CL} \leq (r_x^\omega(E))^2 \leq \frac{C}{L} \tag{2.19}$$

for all $x \in \{-L+1, \dots, L-1\}$*.*

(ii) The density of states $\mathcal{N}'(E_c)$ *is well defined and obeys the estimate*

$$\frac{1}{2\pi C^3} \leq \mathcal{N}'(E_c) \leq \frac{C^3}{2\pi}. \tag{2.20}$$

Remark 2.2.2. *(i) Our formulation of Theorem 2.2.1(i) is a slight improvement of the original theorem in [JSBS03] concerning the value of* C*. In fact, the statement* (2.15) *on its limit for weak disorder is not provided by [JSBS03]. However, we need* C *to be sufficiently close to* 1 *for our proof of Theorem 2.1.1 to succeed. It is plausible that weak disorder should lead to a value of* C *close to* 1*. If* C *would be equal to* 1*, perfect clock behaviour of spectral statistics and perfect flatness of the eigenfunctions are the consequence. Therefore, the deviation of* C *from* 1 *encodes the aberration from these properties of the Laplacian. In order to derive* (2.15) *we repeat some arguments of [JSBS03] in Chapter 2.2.1 while carefully tracking the occurring constants. In particular, this requires additional estimates which were not needed in [JSBS03].*

(ii) The explicit two-sided bound on the density of states in Part (ii) is not contained in [JSBS03] either. Its proof is also contained in Chapter 2.2.1.

2.2.1 Proof of Theorem 2.2.1

Again, we assume $v \in\,]0,2[$ and we restrict ourselves to the case $E_c = 0$, the case of the other critical energy $E_c = v$ being analogous.

In the previous chapter it already transpired that transfer matrices are an important tool in this proof. In particular, the following similarity transform of the single-dimer transfer matrices

$$T_V(E) := M_v^{-1} D_V(E) M_v =: \begin{pmatrix} \overline{a_V(E)} & b_V(E) \\ \overline{b_V(E)} & a_V(E) \end{pmatrix} \tag{2.21}$$

with entries $a_V(E), b_V(E) \in \mathbb{C}$ is of great relevance. Here, the change of basis in $\mathbb{C}^2$ induced by

$$M_v := m_v \begin{pmatrix} \overline{p_v} & p_v \\ 1 & 1 \end{pmatrix} \qquad \text{with} \qquad p_v := \frac{1}{2}\left(v + \mathrm{i}\sqrt{4-v^2}\right) \tag{2.22}$$

simultaneously diagonalises $D_0(0)$ and $D_1(0)$, i.e. $T_0(0) = -1_{2\times 2}$ and $T_1(0)$ are both diagonal. The real parameter $m_v > 0$ is chosen such that $|\det M_v| = 1$. We remark that for every $w \in \mathbb{R}^2$ there exists $z \in \mathbb{C}$ such that

$$M_v^{-1} w = \begin{pmatrix} z \\ \overline{z} \end{pmatrix}. \tag{2.23}$$

For later usage we state the Taylor expansions of the entries of $T_V(E)$ as $E \downarrow 0$

$$\begin{aligned} a_0(E) &= -1 - E\,\frac{2\mathrm{i}}{\sqrt{4-v^2}} + \mathcal{O}(E^2), \\ a_1(E) &= -1 + \frac{v^2}{2} + \frac{v\mathrm{i}}{2}\sqrt{4-v^2} - E\left(v + \frac{(2-v^2)\mathrm{i}}{\sqrt{4-v^2}}\right) + \mathcal{O}(E^2), \\ b_0(E) &= \frac{Ev}{2}\left(-1 + \frac{v\mathrm{i}}{\sqrt{4-v^2}}\right) + \mathcal{O}(E^2), \\ b_1(E) &= -b_0(E) + \mathcal{O}(E^2). \end{aligned} \tag{2.24}$$

In analogy to (2.6), we define the modified (multi-step) dimer transfer matrix as

$$T^\omega(E; y, x) := \begin{cases} T_{V^\omega(y-1)}(E) \cdots T_{V^\omega(x)}(E) & \text{if } x < y, \\ 1_{2\times 2} & \text{if } x = y, \end{cases} \tag{2.25}$$

where $x, y \in 2\mathbb{Z}$.

The next Lemma corresponds to (42) in [JSBS03]].

Lemma 2.2.3 (Cf. (42) in [JSBS03]). *Given $\theta \in [0, 2\pi[$, let $e_\theta := \frac{1}{\sqrt{2}}(e^{-i\theta}, e^{i\theta})^T$. For all $v \in\,]0,2[$, $V \in \{0,1\}$ and all $E \in \mathbb{R}$ there exist maps $\Theta_V : [0,2\pi[\, \to [0,2\pi[$ and $\rho_V : [0,2\pi[\, \to\,]0,\infty[$ such that*

$$T_V(E) e_\theta = \rho_V(\theta)\, e_{\Theta_V(\theta)} \tag{2.26}$$

for all $\theta \in [0,2\pi[$. Furthermore, we have

$$\rho_V^2(\theta) = 1 + 2|b_V(E)|^2 + 2\,\mathrm{Re}\left(a_V(E) b_V(E) e^{2i\theta}\right). \tag{2.27}$$

Proof. The form of $T_V(E)$ in (2.21) implies that for every non-zero $w_z := (z, \overline{z})^T$, $z \in \mathbb{C} \setminus \{0\}$ there exists $\zeta \in \mathbb{C} \setminus \{0\}$ such that $T_V(E) w_z = w_\zeta$. Since $w_\zeta = \rho e_\Theta$ for a unique $\rho > 0$ and $\Theta \in [0, 2\pi[$, the first part of the lemma follows. The equality (2.27) is verified by a direct computation, during which the equality $1 = \det D_V(E) = |a_V(E)|^2 - |b_V(E)|^2$ is applied. □

In the following lemma, which is a modification of (49) in [JSBS03], we use the notation $|\cdot|$ for the Euclidean norm on $\mathbb{C}^2$.

Lemma 2.2.4. *Let $v \in \,]0,2[$, $L \in \mathbb{N}$, $E \in [-v, v]$, $\omega \in \Omega$ and $x, y \in \Gamma_L$ with $x \le y$.*

(i) Then there exists a constant $\tilde{C} \equiv \tilde{C}(v) \in \,]0, \infty[$ and a constant $c_v \in \,]0, \infty[$, which depends only on v and obeys

$$\lim_{v \downarrow 0} c_v = 0, \tag{2.28}$$

such that for all unit vectors $w \in \mathbb{R}^2$, $|w| = 1$, there is an angle $\xi_w \in [0, 2\pi[$ such that

$$\ln\left(|W^\omega(E; x, -L) w|^2\right) \in 2E \sum_{k=k_0}^{k_1 - 1} \mathrm{Re}\left(d_{V^\omega(2k)} e^{2i\vartheta_k}\right) + (c_v + \tilde{C} E^2 L)[-1, 1] \tag{2.29}$$

with $d_V := a_V(0) b'_V(0)$ for $V \in \{0,1\}$ and where

$$k_0 := \min\{k \in \mathbb{Z} : \; -L \le 2k\} \quad \textit{and} \quad k_1 \equiv k_1(x) := \max\{k : \; 2k \le x\}. \tag{2.30}$$

The angles $(\vartheta_k)_{k_1 > k \ge k_0} \subseteq [0, 2\pi[$ are defined recursively by $\vartheta_{k_0} := \xi_w$ and $\vartheta_{k+1} = \Theta_{V^\omega(2k)}(\vartheta_k)$ for all $k \in \{k_0, \cdots, k_1 - 2\}$.

(ii) Let $\{w_1, w_2\}$ be an orthonormal basis of $\mathbb{R}^2$. Then

$$\|W^\omega(E; y, x)\| \le 2 \max_{w \in \{w_1, w_2\}} \max_{z \in \Gamma_L} |W^\omega(E; z, -L) w|^2. \tag{2.31}$$

Proof. (i) For all $x \in \Gamma_L$ we have

$$W^\omega(E; x, -L) = W^\omega(E; x, 2k_1) M_v T^\omega(E; 2k_1, 2k_0) M_v^{-1} W^\omega(E; 2k_0, -L). \tag{2.32}$$

For $w \in \mathbb{R}^2$, $|w| = 1$, let the angle $\xi_w \in [0, 2\pi[$ be given as the unique solution of

$$e_{\xi_w} = M_v^{-1} W^\omega(E; 2k_0, -L) w / |M_v^{-1} W^\omega(E; 2k_0, -L) w|. \tag{2.33}$$

We claim that

$$\ln |W^\omega(E; x, -L) w|^2 \in \sum_{k=k_0}^{k_1 - 1} \ln\left(\rho_{V^\omega(2k)}(\vartheta_k)^2\right) + c_v[-1, 1] \tag{2.34}$$

with

$$c_v := 4 \ln\left(\|M_v\|\right) + 4 \max_{E \in [-v, v]} \max_{V \in \{0,1\}} \ln \|W_V(E)\| > 0. \tag{2.35}$$

Since $\max_{E\in[-v,v]} \|W_V(E)\|$, $\|M_v\| \to 1$ as $v \to 0$ for every $V \in \{0,1\}$, we conclude (2.28) from (2.35). To see the validity of (2.34), we iterate Lemma 2.2.3 and conclude

$$|W^\omega(E;x,-L)w| = |W^\omega(E;x,2k_1)M_v e_{\vartheta_{k_1}}| \prod_{k=k_0}^{k_1-1} \rho_{V^\omega(2k)}(\vartheta_k) \\ \times |M_v^{-1} W^\omega(E;2k_0,-L)w|. \tag{2.36}$$

Furthermore, we note that

$$\|A^{-1}\| = \|A\| \qquad \text{and} \qquad \frac{1}{\|A\|} \le |Aw| \le \|A\| \tag{2.37}$$

for any complex 2×2-matrix A with $|\det A| = 1$ and any $w \in \mathbb{C}^2$ with $|w| = 1$. Applying (2.37) to the first and last factor on the right-hand side of (2.36) yields (2.34).

Equation (2.27), together with a Taylor expansion in the energy E, using (2.24), yields the estimate

$$\ln\left(|W^\omega(E;x,-L)w|^2\right) \in 2E \sum_{k=k_0}^{k_1-1} \mathrm{Re}\left(d_{V^\omega(2k)} \mathrm{e}^{2\mathrm{i}\vartheta_k}\right) + R^\omega_{w,x}(E) + c_v[-1,1], \tag{2.38}$$

where the residual function $R^\omega_{w,x} :]-v,v[\to \mathbb{R}$ is given by

$$R^\omega_{w,x}(E) := \sum_{k=k_0}^{k_1-1} \Big[\ln(\rho_{V^\omega(2k)}(\vartheta_k)^2) - 2E\,\mathrm{Re}\left(d_{V^\omega(2k)}\mathrm{e}^{2\mathrm{i}\vartheta_k}\right)\Big] + |b_{V^\omega(2k)}(E)|^2 \Big] \tag{2.39}$$

for all $E \in]-v,v[$. We bound this function uniformly in w, x and ω

$$|R^\omega_{w,x}(E)| \le \tilde{C} L E^2, \tag{2.40}$$

where the constant $\tilde{C} \equiv \tilde{C}(v)$ is determined by yet another Taylor expansion using (2.24). This yields (2.29).

(ii) For all x, $y \in \Gamma_L$ we have

$$\|W^\omega(E;y,x)\| \le \|W^\omega(E;y,-L)\| \|W^\omega(E;x,-L)^{-1}\| \\ \le \max_{z\in\Gamma_L} \|W^\omega(E;z,-L)\|^2, \tag{2.41}$$

where we used the equality of norms in (2.37). The claim follows from the observation that for any 2×2 matrix

$$\|A\|^2 \le 2 \max_{w\in\{w_1,w_2\}} \|Aw\|^2. \tag{2.42}$$

□

The next lemma accounts for a perturbation in energy and is a variation of [DT03, Lemma 2.1] or [Sim96, Thm. 2J].

Lemma 2.2.5. *Let $E, \varepsilon \in \mathbb{R}$, $\omega \in \Omega$, $L \in \mathbb{N}$ and $G_E^\omega := \max_{x,y \in \Gamma_L, x \le y} \|W^\omega(E; y, x)\|$. Then we have for all $x \in \Gamma_L$ and all $w \in \mathbb{R}^2$ with $|w| = 1$ the estimate*

$$|W^\omega(E+\varepsilon; x, -L)w|^2 \in |W^\omega(E; x, -L)w|^2 + (G_E^\omega)^2 \big(e^{4L|\varepsilon| G_E^\omega} - 1\big) \, [-1, 1]. \tag{2.43}$$

Proof. For $V \in \{0, 1\}$ and $E, \varepsilon \in \mathbb{R}$ we observe

$$W_V(E+\varepsilon) = W_V(E) - \varepsilon \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.44}$$

and expand $W^\omega(E+\varepsilon; x, -L)$ in powers of ε. For the upper bound, this leads to the estimate

$$\begin{aligned} |W^\omega(E+\varepsilon; x, -L)w| &\le |W^\omega(E; x, -L)w| + G_E^\omega \max_{x \in \Gamma_L} \sum_{j=1}^{x+L} \tbinom{x+L}{j} \, (|\varepsilon| G_E^\omega)^j \\ &\le |W^\omega(E; x, -L)w| + G_E^\omega \sum_{j=1}^{|\Gamma_L|} \frac{\big(|\Gamma_L| |\varepsilon| G_E^\omega\big)^j}{j!} \\ &\le |W^\omega(E; x, -L)w| + G_E^\omega \big(\mathrm{e}^{2L|\varepsilon| G_E^\omega} - 1\big) \end{aligned} \tag{2.45}$$

for all $x \in \Gamma_L$ and all unit vectors $w \in \mathbb{R}^2$. To prove the lower bound, we use the inverse triangle inequality to estimate the expansion in ε according to

$$\begin{aligned} |W^\omega(E+\varepsilon; x, -L)w| &\ge |W^\omega(E; x, -L)w| - G_E^\omega \max_{x \in \Gamma_L} \sum_{j=1}^{x+L} \tbinom{x+L}{j} \, (|\varepsilon| G_E^\omega)^j \\ &\ge |W^\omega(E; x, -L)w| - G_E^\omega \big(\mathrm{e}^{2L|\varepsilon| G_E^\omega} - 1\big) \end{aligned} \tag{2.46}$$

for all $x \in \Gamma_L$ and all unit vectors $w \in \mathbb{R}^2$. We note that for any $a, b, c \ge 0$, the two-sided estimate $a \in b + c\,[-1, 1]$ implies $a^2 \in b^2 + c(2b + c)\,[-1, 1]$. In our case, we have $b := |W^\omega(E; x, -L)w| \le G_E^\omega$, which implies the claim. □

For the convenience of the reader we quote [JSBS03, Thm. 6] in our notation and note that the assumption $|\langle \mathrm{e}^{2\mathrm{i}\eta_\pm} \rangle| < 1$ is always satisfied in the dimer model.

Theorem 2.2.6 ([JSBS03, Thm. 6])**.** *Let $v \in \,]0, 2[$. For $L \in \mathbb{N}$, $\alpha > 0$, $\theta \in [0, 2\pi[$ and $E \in \mathcal{W}_L$, where $\mathcal{W}_L$ was defined in (2.17), let*

$$\Omega_L(\alpha, E, \theta) := \Big\{ \omega \in \Omega : \ \exists\, k_1 \in \big(\tfrac{1}{2}\Gamma_L\big) \cap \mathbb{Z} \text{ such that } \Big| \sum_{k=k_0}^{k_1} d_{V^\omega(2k)} e^{2i\vartheta_k} \Big| \ge L^{\alpha + \frac{1}{2}} \Big\}, \tag{2.47}$$

with d_V, k_0 and ϑ_k defined as in Lemma 2.2.4 (i) with $\vartheta_{k_0} = \theta$. Then there exist constants $C_1 \equiv C_1(\alpha, v, p_+) > 0$ and $C_2 \equiv C_2(\alpha, v, p_+) > 0$ independent of E and θ such that

$$\mathbb{P}\big(\Omega_L(\alpha, E, \theta)\big) \le C_1 e^{-C_2 L^\alpha}. \tag{2.48}$$

Lemma 2.2.7. *Let $v \in]0,2[$. For all $\alpha > 0$ there exists $L_0 \equiv L_0(\alpha, v) \in \mathbb{N}$ such that for all $L \geq L_0$ there exists a measurable subset $\Omega_L(\alpha) \subseteq \Omega$ and a constant $c \equiv c(\alpha, v, p_+) > 0$ such that*

$$\mathbb{P}\big(\Omega_L(\alpha)\big) \leq e^{-cL^{\alpha/2}} \tag{2.49}$$

and such that for all $\omega \in \big(\Omega_L(\alpha)\big)^c$, $E \in \mathcal{W}_L$ and $x \in \Gamma_L$

$$\Big|W^\omega(E; x, -L)\binom{1}{0}\Big|^2 \in [e^{-3c_v}, e^{3c_v}], \tag{2.50}$$

where the constant c_v is given in Lemma 2.2.4 (i), see (2.35).

Proof. Let $w_1 := \binom{1}{0}$ and $w_2 := \binom{0}{1}$. In view of (2.33), we define a set of modified Prüfer angles

$$\Xi := \Big\{\xi \in [0, 2\pi[\,:\, \exists W \in \{1_{2\times 2}, W_0(E), W_1(E)\}, w \in \{w_1, w_2\} \\ \text{with } e_\xi = \frac{M_v^{-1} W w}{|M_v^{-1} W w|}\Big\} \tag{2.51}$$

with cardinality $|\Xi| \leq 6$. Let

$$\Omega_L(\alpha, E) := \bigcup_{\theta \in \Xi} \Omega_L(\alpha/2, E, \theta). \tag{2.52}$$

Hence, $\mathbb{P}(\Omega_L(\alpha, E)) \leq 6C_1 \mathrm{e}^{-C_2 L^{\alpha/2}}$ by Theorem 2.2.6. We assume $L \geq v^{-2}$ so that $\mathcal{W}_L \subset [-v, v]$. Thus, for all $E \in \mathcal{W}_L$ and $\omega \in (\Omega_L(\alpha, E))^c$ the estimate (2.29) yields

$$\ln\big(|W^\omega(E; x, -L)w|^2\big) \in \big(\tilde{C} L E^2 + 2EL^{1/2+\alpha/2} + c_v\big)\, [-1, 1] \tag{2.53}$$

for all $x \in \Gamma_L$ and $w \in \{w_1, w_2\}$. Here, the constant $\tilde{C} \equiv \tilde{C}(v)$ is given in Lemma 2.2.4. Hence there exists $L_0' \equiv L_0'(\alpha, v) \geq v^{-2}$ such that for all $L \geq L_0'$, all $E \in \mathcal{W}_L$, all $\omega \in (\Omega_L(\alpha, E))^c$, all $x \in \Gamma_L$ and $w \in \{w_1, w_2\}$, we have

$$\ln\big(|W^\omega(E; x, -L)w|^2\big) \in 2c_v\, [-1, 1]. \tag{2.54}$$

The upper bound in (2.54) and the inequality in Lemma 2.2.4 (ii) imply for the quantity G_E^ω in Lemma 2.2.5

$$G_E^\omega = \max_{x,y \in \Gamma_L, x \leq y} \|W^\omega(E; y, x)\| \leq 2\mathrm{e}^{2c_v} \tag{2.55}$$

for all $\omega \in (\Omega_L(\alpha, E))^c$. We define

$$\Omega_L(\alpha) := \bigcup_{\substack{n \in \mathbb{Z}: \\ n/L^2 \in \mathcal{W}_L}} \Omega_L(\alpha, n/L^2). \tag{2.56}$$

Hence there exists $L_0'' \equiv L_0''(\alpha, v) \geq L_0'$ and $c > 0$ such that for all $L \geq L_0''$ we have

$$\mathbb{P}\big[\Omega_L(\alpha)\big] \leq 18 L^{3/2} C_1 \mathrm{e}^{-C_2 L^{\alpha/2}} \leq \mathrm{e}^{-cL^{\alpha/2}}. \tag{2.57}$$

Now, we consider a $n \in \mathbb{Z}$ such that $E_n := n/L^2 \in \mathcal{W}_L$ and an arbitrary $\omega \in (\Omega_L(\alpha))^c$. Applying Lemma 2.2.5, (2.54) and (2.55) with $w = w_1$ yields

$$\begin{aligned} |W^\omega(E;x,-L)w_1|^2 &\in |W^\omega(E_n;x,-L)w_1|^2 + 4\mathrm{e}^{4c_v}\Big(\exp(8e^{2c_v}/L) - 1\Big)\,[-1,1] \\ &\subseteq [\mathrm{e}^{-2c_v}, e^{2c_v}] + 4\mathrm{e}^{4c_v}\Big(\exp(8\mathrm{e}^{2c_v}/L) - 1\Big)\,[-1,1] \end{aligned} \tag{2.58}$$

for all $x \in \Gamma_L$ and all $E \in D_n := E_n + L^{-2}[-1,1]$. Since

$$\mathcal{W}_L \subseteq \bigcup_{\substack{n\in\mathbb{Z}:\\ E_n\in\mathcal{W}_L}} D_n \tag{2.59}$$

there exists $L_0 \equiv L_0(\alpha, v) \geq L_0''$ such that for all $L \geq L_0$, all $\omega \in (\Omega_L(\alpha))^c$, all $E \in \mathcal{W}_L$ and all $x \in \Gamma_L$ we have

$$|W^\omega(E;x,-L)w_1|^2 \in \,[\mathrm{e}^{-3c_v}, \mathrm{e}^{3c_v}]. \tag{2.60}$$

□

Proof of Theorem 2.2.1. (i) Let us first proof (2.19). For every $L \in \mathbb{N}$, $x \in \Gamma_L$, $E \in \mathbb{R}$ and $\omega \in \Omega$, we infer from (2.4) that

$$r_x^\omega(E)^2 = \phi_{L,E}^\omega(x)^2 + \phi_{L,E}^\omega(x-1)^2 = \left|W^\omega(E;x,-L)\binom{1}{0}\right|^2/(R_E^\omega)^2, \tag{2.61}$$

with the normalisation

$$(R_E^\omega)^2 := \sum_{k=0}^{L-1}\left|W^\omega(E;-L+1+2k,-L)\binom{1}{0}\right|^2. \tag{2.62}$$

Given $\alpha > 0$, Lemma 2.2.7, provides the existence of a minimal length $L_{\min} \equiv L_{\min}(\alpha, v) \geq v^{-2}$ such that for all $L \geq L_{\min}$, $\omega \in (\Omega_L(\alpha))^c$, $x \in \Gamma_L$ and $E \in \mathcal{W}_L$, the two-sided estimate

$$(R_E^\omega)^2 \in [L\mathrm{e}^{-3c_v}, L\mathrm{e}^{3c_v}] \tag{2.63}$$

is true. Thus, (2.63), another application of Lemma 2.2.7 and (2.61) yield (2.19) with the constant

$$C = \mathrm{e}^{6c_v}, \tag{2.64}$$

and (2.28) implies (2.15).

To prove the level-spacing estimate (2.18), let L_0 be as above, $L \geq L_0$, $\omega \in (\Omega_L(\alpha))^c$ and let $E, E' \in \mathcal{W}_L$ be two adjacent eigenvalues of H_L^ω with $E < E'$. Recall that for $E^{(\prime)}$ to be an eigenvalue, the boundary conditions $\phi^\omega_{L,E^{(\prime)}}(L) = 0$ have to be met on the right border of Γ_L, that is, $\theta_L^\omega(E^{(\prime)}) \in \pi/2 + \pi\mathbb{Z}$. Hence E and E' are adjacent eigenvalues if and only if the Prüfer angle difference satisfies $\theta_L^\omega(E') - \theta_L^\omega(E) = \pi$. By integrating (2.14) we obtain

$$\pi = \int_E^{E'} \mathrm{d}\varepsilon\,\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\theta_L^\omega(\varepsilon) = \int_E^{E'} \mathrm{d}\varepsilon \sum_{x=-L}^{L-1}\left(\frac{\phi_{L,\varepsilon}^\omega(x)}{r_L^\omega(\varepsilon)}\right)^2 = \int_E^{E'} \mathrm{d}\varepsilon\,\frac{1}{\big(r_L^\omega(\varepsilon)\big)^2}. \tag{2.65}$$

The eigenfunction estimate (2.19) does not apply directly to $r_L^\omega(\varepsilon)$ for $\varepsilon \in \mathcal{W}_L$, since $L \notin \Gamma_L$. We need an additional iteration with the transfer matrix

$$\big(r_L^\omega(\varepsilon)\big)^2 = \left| W_{V^\omega(L-1)}(\varepsilon) \begin{pmatrix} \cos\theta_{L-1}^\omega(\varepsilon) \\ \sin\theta_{L-1}^\omega(\varepsilon) \end{pmatrix} \right|^2 \big(r_{L-1}^\omega(\varepsilon)\big)^2. \tag{2.66}$$

We already have $\big(r_{L-1}^\omega(\varepsilon)\big)^2 \in L^{-1}[C^{-1}, C]$ for every $\omega \in (\Omega_L(\alpha))^c$ by (2.19). Since $\max_{V \in \{0,1\}} \|W_V(\varepsilon)\| \le \mathrm{e}^{c_v/4} \le C$ uniformly in $\varepsilon \in \mathcal{W}_L$ by (2.35), we deduce from (2.37) that $\big(r_L^\omega(\varepsilon)\big)^2 \in L^{-1}[C^{-3}, C^3]$. Inserting this into (2.65) yields

$$E' - E \in \frac{\pi}{L}\,[C^{-3}, C^3]. \tag{2.67}$$

(ii) The existence of the density of states $\mathcal{N}'(E_c)$ follows from [JSBS03, Thm. 3]. We do not present a proof for existence here, only for the upper and lower bound. We use Dirichlet–Neumann bracketing as well as the eigenvalue spacing within the critical energy window to show these estimates.

For $L \in \mathbb{N}$ we introduce the restricted Schrödinger operators $H_L^{\omega,\,D/N}$ with Dirichlet, respectively Neumann, boundary conditions

$$H_L^{\omega,\,D} := H_L^\omega + |\delta_{-L}\rangle\langle\delta_{-L}| + |\delta_{L-1}\rangle\langle\delta_{L-1}|\,, \tag{2.68}$$
$$H_L^{\omega,\,N} := H_L^\omega - |\delta_{-L}\rangle\langle\delta_{-L}| - |\delta_{L-1}\rangle\langle\delta_{L-1}|\,. \tag{2.69}$$

Their integrated densities of states at energy $E \in \mathbb{R}$ are given by

$$\mathcal{N}_L^{\omega,\,D/N}(E) := \operatorname{tr}\Big\{1_{\le E}\big(H_L^{\omega,\,D/N}\big)\Big\}. \tag{2.70}$$

Since $H_L^{\omega,\,D/N}$ are rank-2-perturbations of H_L^ω, the min-max-principle implies

$$\mathcal{N}_L^{\omega,\,D/N}(E) \in \operatorname{tr}\big\{1_{\le E}(H_L^\omega)\big\} + [-2, 2]. \tag{2.71}$$

According to [CL90, p. 312] Dirichlet–Neumann bracketing yields

$$\frac{1}{|\Gamma_L|}\mathbb{E}\big[\mathcal{N}_L^D(E)\big] \le \mathcal{N}(E) \le \frac{1}{|\Gamma_L|}\mathbb{E}\big[\mathcal{N}_L^N(E)\big] \tag{2.72}$$

for every $E \in \mathbb{R}$ and every $L \in \mathbb{N}$. Thus, we conclude from (2.71) and (2.72) that

$$\frac{\mathcal{N}(E+\varepsilon) - \mathcal{N}(E-\varepsilon)}{2\varepsilon} \in \frac{1}{2\varepsilon|\Gamma_L|}\mathbb{E}\Big[\operatorname{tr}\big\{1_{]E-\varepsilon, E+\varepsilon]}(H_L)\big\}\Big] + \frac{2}{\varepsilon|\Gamma_L|}\,[-1, 1] \tag{2.73}$$

for every $\varepsilon > 0$. For a fixed $\alpha \in\,]0, 1/2[$ let $\varepsilon_L := L^{-1/2-\alpha}$ be half the width of the critical energy window $\mathcal{W}_L$ around $E_c \in \{0, v\}$. This implies $\lim_{L\to\infty} \frac{2}{\varepsilon_L|\Gamma_L|} = 0$ and therefore

$$\mathcal{N}'(E_c) = \lim_{L\to\infty} \frac{1}{2\varepsilon_L|\Gamma_L|}\mathbb{E}\Big[\operatorname{tr}\big\{1_{]E_c-\varepsilon_L, E_c+\varepsilon_L]}(H_L)\big\}\Big]. \tag{2.74}$$

We estimate this limit by treating bad events in $\Omega_L(\alpha)$ separately. The bound for the probability of bad events from Theorem 2.2.1 (i) yields

$$0 \leq \lim_{L\to\infty} \frac{1}{2\varepsilon_L|\Gamma_L|}\mathbb{E}\big[1_{\Omega_L(\alpha)} \operatorname{tr}\big\{1_{]E_c-\varepsilon_L,E_c+\varepsilon_L]}(H_L)\big\}\big] \leq \lim_{L\to\infty} \frac{\mathbb{P}[\Omega_L(\alpha)]}{2\varepsilon_L} = 0. \quad (2.75)$$

Theorem 2.2.1 (i) provides the means to estimate the expectation for good events in $\big(\Omega_L(\alpha)\big)^c$. According to (i) there exists a minimal length $L_0 \in \mathbb{N}$ such that for all $L \geq L_0$ and all $\omega \in \big(\Omega_L(\alpha)\big)^c$ we have

$$\frac{2\varepsilon_L L}{\pi C^3} - 1 \leq \operatorname{tr}\big\{1_{]E_c-\varepsilon_L,E_c+\varepsilon_L]}(H_L^\omega)\big\} \leq \frac{2\varepsilon_L C^3 L}{\pi} + 1. \quad (2.76)$$

Hence, by using (2.74) and (2.75) we arrive at the following estimate

$$\begin{aligned}\mathcal{N}'(E_c) &= \lim_{L\to\infty} \frac{1}{2\varepsilon_L|\Gamma_L|}\mathbb{E}\big[1_{(\Omega_L(\alpha))^c} \operatorname{tr}\big\{1_{]E_c-\varepsilon_L,E_c+\varepsilon_L]}(H_L)\big\}\big] \\ &\in \left(\lim_{L\to\infty} \frac{L\,\mathbb{P}\big[\big(\Omega_L(\alpha)\big)^c\big]}{\pi|\Gamma_L|}\right)[C^{-3}, C^3] = \frac{1}{2\pi}[C^{-3}, C^3], \end{aligned} \quad (2.77)$$

which concludes the proof. □

2.3 Lower bound of the finite-volume entanglement entropy

2.3.1 General idea and strategy

Our aim is to construct a suitable lower bound to the finite-volume entanglement entropy that grows logarithmically in L. Apart from proving Theorem 2.1.2, this is the first step towards showing a logarithmic enhancement for the infinite-volume entanglement entropy as well, which is our main objective.

A typical first step [PS14] in obtaining a lower bound for the entanglement entropy is to replace the function h in its definition (1.13) by a parabola. The function $g : [0,1] \to \mathbb{R}_{\geq 0}$,

$$g(\lambda) := 4\lambda(1-\lambda), \quad (2.78)$$

is a lower bound to h. We further introduce the quadratic analogue to the entanglement entropy in non-interacting systems. Let $H : \ell^2(\Gamma) \to \ell^2(\Gamma)$ be a discrete Schrödinger operator defined on a lattice $\Gamma \subseteq \mathbb{Z}$. For any $\Lambda \subseteq \Gamma$ and a Fermi energy $E \in \mathbb{R}$ let

$$Q_E(\Lambda;\Gamma,H_\Gamma) := \operatorname{tr} g\big(1_\Lambda(X)1_{<E}(H_\Gamma)1_\Lambda(X)\big). \quad (2.79)$$

That $Q_E(\Lambda;\Gamma,H_\Gamma) \leq S_E(\Lambda;\Gamma,H_\Gamma)$ follows immediately from $g \leq h$.

Since the finite-volume Schrödinger operator $H^\omega_{\Gamma_L}$ has only discrete spectrum, the finite-volume quadratic entanglement entropy can be rewritten in terms of the eigenvalues $E \in \sigma(H^\omega_{\Gamma_L})$ and corresponding $\ell^2(\Gamma_L)$-normalised eigenfunctions $\psi^\omega_{L,E}$.

Lemma 2.3.1. *Let $x_1, x_2 \in \mathbb{Z}$ with $-L \le x_1 < x_2 < L-1$, $\Lambda := [x_1, x_2] \cap \mathbb{Z}$ and $E_F \in \mathbb{R}$. Then we have*

$$Q_{E_F}(\Lambda; \Gamma_L, H_L^\omega) = 4 \sum_{\substack{E,E' \in \sigma(H_L^\omega): \\ E<E_F,\, E' \ge E_F}} \frac{1}{(E'-E)^2} |\langle \psi_{L,E}^\omega, [H_L^\omega, 1_\Lambda(X)] \psi_{L,E'}^\omega \rangle|^2 \tag{2.80}$$

for all $\omega \in \Omega$, where the commutator is a boundary operator

$$[H_L^\omega, 1_\Lambda(X)] = \begin{cases} |\delta_{x_1}\rangle\langle\delta_{x_1-1}| - |\delta_{x_1-1}\rangle\langle\delta_{x_1}| + |\delta_{x_2}\rangle\langle\delta_{x_2+1}| - |\delta_{x_2+1}\rangle\langle\delta_{x_2}| & \text{if } x_1 \ne -L, \\ |\delta_{x_2}\rangle\langle\delta_{x_2+1}| - |\delta_{x_2+1}\rangle\langle\delta_{x_2}| & \text{if } x_1 = -L, \end{cases} \tag{2.81}$$

which is independent of randomness.

Proof. We introduce the abbreviations $P := 1_\Lambda(X)$ and $Q := 1_{<E_F}(H_L^\omega)$. A straightforward calculation of the trace yields

$$\begin{aligned} \frac{1}{4} \operatorname{tr} g(PQP) &= \operatorname{tr} PQP(1-Q)P \\ &= \sum_{\substack{E,E' \in \sigma(H_L^\omega): \\ E<E_F,\, E' \ge E_F}} \operatorname{tr} P |\psi_{L,E}^\omega\rangle\langle\psi_{L,E}^\omega| P |\psi_{L,E'}^\omega\rangle\langle\psi_{L,E'}^\omega| P \\ &= \sum_{\substack{E,E' \in \sigma(H_L^\omega): \\ E<E_F,\, E' \ge E_F}} |\langle \psi_{L,E}^\omega, P\psi_{L,E'}^\omega\rangle|^2. \end{aligned} \tag{2.82}$$

We write the matrix elements of P in terms of the commutator by applying the equality

$$E\langle\psi_{L,E}^\omega, P\psi_{L,E'}^\omega\rangle = \langle\psi_{L,E}^\omega, H_L^\omega P\psi_{L,E'}^\omega\rangle = E'\langle\psi_{L,E}^\omega, P\psi_{L,E'}^\omega\rangle + \langle\psi_{L,E}^\omega, [H_L^\omega, P]\psi_{L,E'}^\omega\rangle. \tag{2.83}$$

This concludes the proof. □

Remark 2.3.2. *It is clear that* (2.80) *is true for general self-adjoint operators H defined on $\ell^2(\Gamma)$ for any finite subregion $\Gamma \subseteq \mathbb{Z}^d$ with $d \in \mathbb{N}$.*

The overall idea of our argument is that the energy denominator in (2.80) provides the mechanism for a potential logarithmic enhancement to the area law. The enhancement can only occur if eigenfunctions corresponding to nearby energies have a significant overlap somewhere on the surface of Λ. For Anderson localised systems this is typically not the case, because the localisation centres of two eigenfunctions are expected to be separated by a distance that grows logarithmically with the inverse of their energy difference [Mot68, Mot70, KLP03]. Consequently, the entanglement entropy is expected to obey a strict area law for localised systems. Indeed, this was proven in [PS14, EPS17], following another line of reasoning. In the dimer model, however, localisation breaks down at the critical energies and delocalisation properties occur, as we have discussed in detail in Chapter 2.2. In particular, the eigenfunctions close to the critical energies $E_c \in \{0, v\}$ are with high probability evenly spread out over Γ_L. This provides us with sufficient overlap at the surface of Λ between any two of them. To study this effect more rigorously, we express the matrix element of the commutator in (2.81) in terms of Prüfer variables

$$\begin{aligned}\langle\psi^{\omega}_{L,E},[H^{\omega}_{L},1_{\Lambda}(X)]\psi^{\omega}_{L,E'}\rangle &= r^{\omega}_{x_2+1}(E)r^{\omega}_{x_2+1}(E')\sin\left(\theta^{\omega}_{x_2+1}(E)-\theta^{\omega}_{x_2+1}(E')\right)\\ &\quad - r^{\omega}_{x_1}(E)r^{\omega}_{x_1}(E')\sin\left(\theta^{\omega}_{x_1}(E)-\theta^{\omega}_{x_1}(E')\right),\end{aligned} \tag{2.84}$$

where $E,E'\in\sigma(H^{\omega}_{L})$. We only consider good events $\omega\in(\Omega_L(\alpha))^c$ in the following. For energies inside $\mathcal{W}_L$ the Prüfer radius at any point in Γ_L is roughly $L^{-1/2}$, see (2.19). Neglecting the possibility of cancellations between the two terms on the right-hand side of (2.84) for the moment, we thus argue that $|\langle\psi^{\omega}_{L,E},[H^{\omega}_{L},1_{\Lambda}(X)]\psi^{\omega}_{L,E'}\rangle|^2\sim L^{-2}$ for eigenvalues $E,E'\in\mathcal{W}_L$.

We restrict the double sum in (2.80) to only those eigenvalues that are inside the critical window $\mathcal{W}_L$. This still yields a lower bound to the entanglement entropy, because all summands are non-negative. Since the spacing of eigenvalues inside $\mathcal{W}_L$ is $\sim L^{-1}$ according to (2.18), we are able to approximate these sums with the double integral

$$\mathbb{E}[S_{E_c}(\Lambda;\Gamma_L,H^{\omega}_{L})]\gtrsim\int_{-L^{-\alpha-1/2}}^{-L^{-1}}\mathrm{d}E\int_{L^{-1}}^{L^{-\alpha-1/2}}\mathrm{d}E'\,\frac{1}{(E'-E)^2}\geq\frac{1-2\alpha}{4}\ln L, \tag{2.85}$$

for sufficiently large L. We note that this logarithmic divergence is derived exclusively from the artificial L-dependence of the larger box Γ_L to which the operator is restricted, rather than from Λ. Also, in the crude argument above we have neglected the possibility of cancellations of the two terms in (2.84). In fact, such cancellations do occur for a fixed bounded Λ. However, the argument can be justified for an L-dependent region Λ_L of a size proportional to L. Let us therefore set

$$\Lambda^{\gamma,\delta}_{L}:=[L_1,L_2-1]\cap\mathbb{Z}\quad\text{with}\quad L_1:=-L+\lfloor\gamma L\rfloor,\ L_2:=-L+\lfloor(\gamma+\delta)L\rfloor \tag{2.86}$$

for some $\gamma,\ \delta\in[0,1]$, where $\lfloor\cdot\rfloor$ denotes the standard floor function or Gauss bracket. If γ is chosen to be equal to zero, this region is identical to Λ'_L in Theorem 2.1.2. To prove Theorem 2.1.1 we ultimately want to replace the finite volume Γ_L with the infinite lattice $\mathbb{Z}$. To control the ensuing error, it is, however, important that Λ_L is not attached to the border of Γ_L. We therefore focus on the case $\gamma>0$ from now on.

It is altogether impossible to prevent the above-mentioned cancellations with any choice of γ and δ. However, if $\left|\langle\psi^{\omega}_{L,E},[H^{\omega}_{L},1_{\Lambda^{\gamma,\delta}_{L}}(X)]\psi^{\omega}_{L,E'}\rangle\right|\gtrsim L^{-1}$ for a sufficient number of energies $E,\ E'\in\mathcal{W}_L\cap\sigma(H^{\omega}_{L})$ we have enough good contributions in the double sum in (2.80) to ensure a logarithmic lower bound. To obtain the necessary number of good contributions, we choose

$$0<\delta\ll\gamma\ll1. \tag{2.87}$$

Their exact value will be determined by the quantity C given in Theorem 2.2.1. For technical reasons we require $C\approx1$. Recall that this is the case if the disorder strength is small as in the prerequisites of Theorem 2.1.1.

The energy dependence of the Prüfer angle may be used to determine the value of (2.84). Integrating (2.14) yields

$$\begin{aligned}\left(\theta^{\omega}_{L_1}(E)-\theta^{\omega}_{L_1}(E')\right)&\approx2^{-1}\gamma L(E-E'),\\ \left(\theta^{\omega}_{L_2}(E)-\theta^{\omega}_{L_2}(E')\right)&\approx2^{-1}(\gamma+\delta)L(E-E'),\end{aligned} \tag{2.88}$$

which suggests that the matrix elements in (2.84) are approximately of the form

$$\langle \psi^{\omega}_{L,E}, [H^{\omega}_{L}, 1_{\Lambda}(X)]\psi^{\omega}_{L,E'}\rangle \approx 2L^{-1}\sin\left(2\pi(E-E')/T^{-}\right)\cos\left(2\pi(E-E')/T^{+}\right) \quad (2.89)$$

for all eigenvalues $E,\ E' \in \mathcal{W}_L \cap \sigma(H^{\omega}_{L})$ where

$$T^{-} \equiv T^{-}(\delta) := \frac{8\pi}{L\delta} \quad \text{and} \quad T^{+} \equiv T^{+}(\gamma,\delta) := \frac{8\pi}{L(2\gamma+\delta)}. \quad (2.90)$$

Observe that the right-hand side of (2.89) is a sinusoidal function with slowly varying amplitude. The period of the fast oscillation T^{+} is much smaller than the period of the envelope T^{-}, since $\delta \ll \gamma$. However, both periods are much larger than the approximate distance between two consecutive eigenvalues $\approx \pi L^{-1}$, because $\gamma \ll 1$.

It is evident that there are eigenvalues $E,\ E' \in \mathcal{W}_L \cap \sigma(H^{\omega}_{L})$ for which the matrix element of the commutator is either very small or vanishes altogether. To identify energies that give rise to *good contributions* in the double-sum of (2.80), we proceed as follows. For a fixed $E < E_c$ we write

$$\mathcal{W}_L \cap [E_c, \infty[= \bigcup_{q=0}^{N^{-}} I_q \quad (2.91)$$

as a union over $N^{-}+1 \sim \delta\, L^{1/2-\alpha}$ intervals that lie between the nodes of the envelope

$$f_e(\cdot) := \sin(2\pi(E-\cdot)/T^{-}), \quad (2.92)$$

as shown in Figure 2.2 (a). For each interval $q \in \{0,\cdots,N^{-}\}$ we define *envelope good* interval

$$I^{eg}_q := \{\varepsilon' \in I_q : |f_e(\varepsilon')| \geq 2^{-1/2}\}, \quad (2.93)$$

see Figure 2.2 (b). Each interval I^{eg}_q, with the possible exception of $q=0$ and $q=N^{-}$, includes $\sim \delta^{-1}$ eigenvalues E'. To determine eigenvalues with good contributions, we have to consider the fast oscillation

$$f_f(\cdot) := \cos(2\pi(E-\cdot)/T^{+}). \quad (2.94)$$

Approximately half of the eigenvalues in I^{eg}_q also satisfy

$$|f_f(E')| \geq 2^{-1/2}, \quad (2.95)$$

as indicated by Figure 2.2 (c). Consequently, the absolute value of the matrix element in (2.89) for such eigenvalues is larger than L^{-1}. It is important to note that the number of good contributions in I_q for any $q \in \{1,\cdots,N^{-}-1\}$ is independent of L, since both $T^{\pm}$ and the approximate distance between consecutive eigenvalues are of order $\mathcal{O}(L^{-1})$. All in all, for each $E < E_c$ we have $\sim L^{1/2-\alpha}$ eigenvalues $E' > E_c$ that yield good contributions. This ultimately allows us to use an adaptation of the argument of (2.85) to prove a logarithmic lower bound.

The main technical difficulty of this proof arises from the eigenvalue distribution. Although the spectral statistic is close to clock behaviour, the distance between two consecutive eigenvalues is generally not exactly the same. Using the minimal and

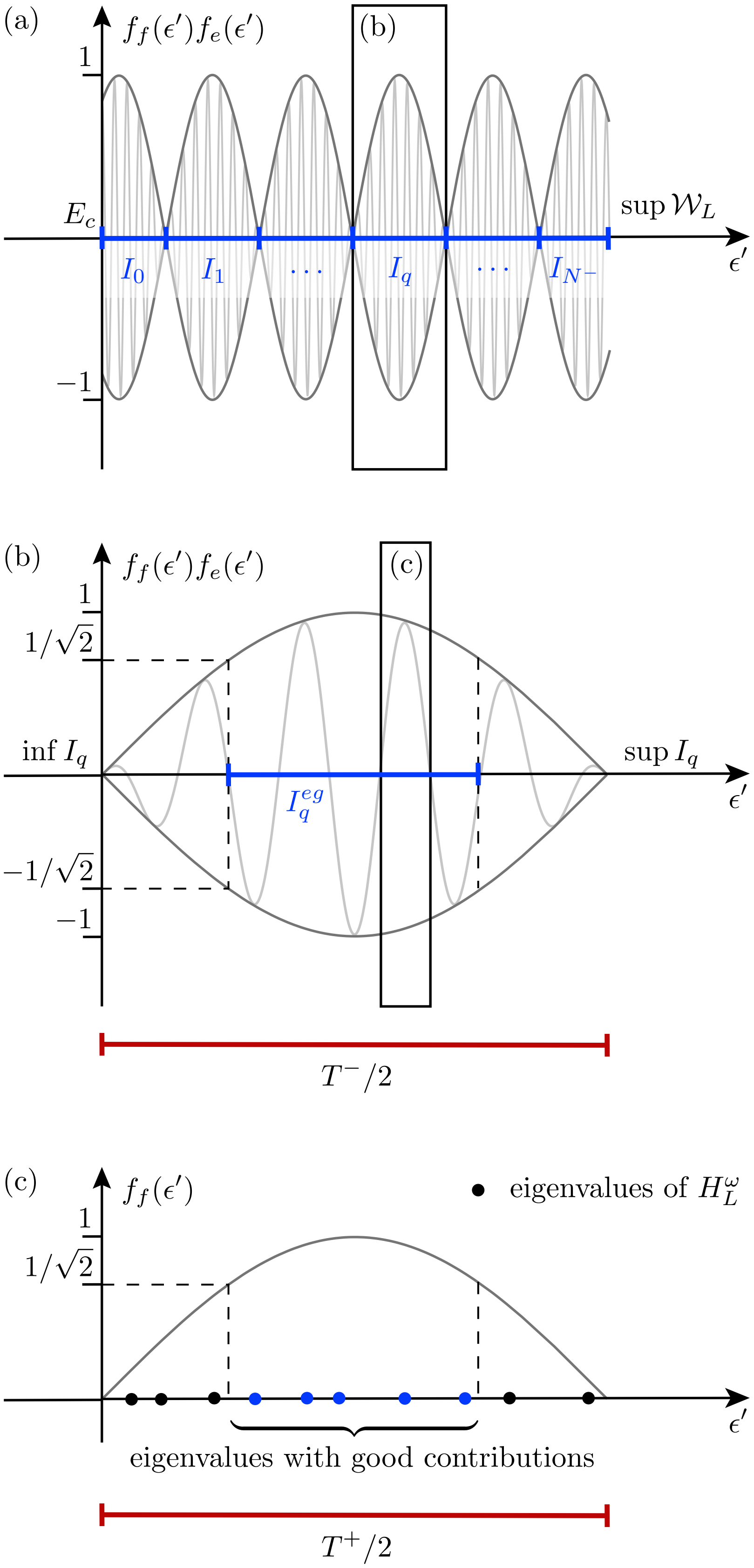

Figure 2.2: The process of identifying eigenvalues with good contributions

maximal distance between eigenvalues, we derive an upper and a lower bound for the number of good contributions in the following chapter. The somewhat artificial restriction to small disorder arises from the necessity to control the γ- and v-dependence of the logarithmic lower bound. This will be crucial for the next step towards proving Theorem 2.1.1. For further details see Chapter 2.4.1. Leaving aside this restriction, we are able to prove Theorem 2.1.2 for any disorder strength $v \in]0,2[$ by slightly modifying the arguments above.

2.3.2 Finding good contributions

Here we identify a sufficient number of good contributions for a lower bound for the modulus of (2.84) by following the general strategy outlined in the previous section. We assume without loss of generality $E_c = 0$. Throughout this subsection $L \geq L_{\min}$ and $\alpha > 0$ from Theorem 2.2.1 are fixed. We only consider elementary events $\omega \in (\Omega_L(\alpha))^c$ in this section. For the reader's convenience we drop ω in the notation of all quantities. The enumeration

$$E_{J_{\min}} < \cdots < E_{-2} < E_{-1} < E_c \leq E_0 < E_1 < \cdots < E_{J_{\max}} \tag{2.96}$$

of the $2L$ non-degenerate eigenvalues of H_L will be convenient. The labelling index runs from the negative integer $J_{\min}$ to the positive integer $J_{\max}$, which both depend on ω. Since we are only interested in eigenvalues in the critical window $\mathcal{W}_L$ below or above the critical energy, we introduce the two index sets

$$\begin{aligned} \mathcal{J}_< &:= \left\{J_{\min} < j < 0 : E_{j-1}, E_j \in \mathcal{W}_L\right\}, \\ \mathcal{J}_\geq &:= \left\{0 \leq j < J_{\max} : E_j, E_{j+1} \in \mathcal{W}_L\right\} \end{aligned} \tag{2.97}$$

and

$$J_< := \min \mathcal{J}_< - 1, \qquad J_\geq := \max \mathcal{J}_\geq + 1. \tag{2.98}$$

The next lemma analyses the step size at which the sine functions in (2.84) are sampled. This is the discrete analogue to (2.88).

Lemma 2.3.3. *Let γ, $\delta \in [0,1]$. There exists a minimal length $L_0 \equiv L_0(C,\gamma,\delta) \in \mathbb{N}$ such that for every length $L \geq L_0$ and every pair of consecutive eigenvalues $E_j, E_{j+1} \in \mathcal{W}_L$ for a $j \in \{J_<, \cdots, J_\geq\}$ we have*

$$\begin{aligned} \theta_{L_1}(E_{j+1}) - \theta_{L_1}(E_j) &\in \frac{\pi\gamma}{2C^6}\,[1, C^{12}], \\ \theta_{L_2}(E_{j+1}) - \theta_{L_2}(E_j) &\in \frac{\pi(\gamma+\delta)}{2C^6}\,[1, C^{12}]. \end{aligned} \tag{2.99}$$

Here, the quantity $C > 1$ is the one from Theorem 2.2.1, and L_1, L_2 are defined in (2.86).

Proof. To prove the first statement in (2.99) we use the explicit representation (2.14) for $(\mathrm{d}/\mathrm{d}E)\theta_{L_1}$. We see that for $E \in \mathcal{W}_L$ we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}E}\theta_{L_1}(E) &= \sum_{n=-L}^{L_1-1}\left(\frac{\psi_{L,E}(n)}{r_{L_1}(E)}\right)^2 \\ &= \frac{1}{2}\sum_{n=-L+1}^{L_1-1}\left(\frac{r_n(E)}{r_{L_1}(E)}\right)^2 + \frac{1}{2}\left(\frac{\psi_{L,E}(-L)}{r_{L_1}(E)}\right)^2 + \frac{1}{2}\left(\frac{\psi_{L,E}(L_1-1)}{r_{L_1}(E)}\right)^2. \end{aligned} \tag{2.100}$$

For $\omega \in (\Omega_L(\alpha))^c$ and all $n \in \Gamma_L \smallsetminus \{-L\}$ we conclude, by the estimate (2.19) of Theorem 2.2.1, that

$$\left(\frac{r_n(E)}{r_{L_1}(E)}\right)^2 \in \left[C^{-2}, C^2\right]. \tag{2.101}$$

Furthermore $\psi_{L,E}^2(-L) \le (r_{-L+1}(E))^2$ and $\psi_{L,E}^2(L_1-1) \le (r_{L_1-1}(E))^2$. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}E}\theta_{L_1}(E) \in \frac{\lfloor \gamma L\rfloor - 1}{2}\left[C^{-2}, C^2\right] + \frac{1}{2}\left[0, 1+C^2\right] \subseteq \frac{\gamma L}{2C^3}\,[1, C^6], \tag{2.102}$$

where there exists a $L_0 \equiv L_0(C,\gamma) \in \mathbb{N}$ such that the last inclusion is true for all $L \ge L_0$, because $C > 1$. The first statement in (2.99) now follows from integrating (2.102) over $E_{j+1} - E_j$ together with the level spacing estimate (2.18).

The verification of the second statement in (2.99) is analogous and makes the minimal length L_0 also dependent on δ. □

As in (2.89) and the following we find a condition that, if satisfied, yields a lower bound to the commutator in (2.84).

Lemma 2.3.4. *For $j \in \mathcal{J}_{\ge}$ and $k \in \mathcal{J}_{<}$ we define*

$$z_{j,k}^{\pm} := \big([\theta_{L_2}(E_j) - \theta_{L_2}(E_k)] \pm [\theta_{L_1}(E_j) - \theta_{L_1}(E_k)]\big)/2. \tag{2.103}$$

Assume that

$$|\cos z_{j,k}^{+} \sin z_{j,k}^{-}| \ge 1/2. \tag{2.104}$$

Then the estimate

$$|\langle \psi_{L,E_k}, [H_L, 1_{\Lambda_L^{\gamma,\delta}}(X)]\psi_{L,E_j}\rangle| \ge \frac{1}{CL} \tag{2.105}$$

is true for the constant $C > 1$ from Theorem 2.2.1.

Proof. Introducing $\zeta_{j,k}^{\pm} := z_{j,k}^{+} \pm z_{j,k}^{-}$, the modulus of (2.84) reads

$$\begin{aligned}&|\langle \psi_{L,E_k},[H_L, 1_{\Lambda_L^{\gamma,\delta}}(X)]\psi_{L,E_j}\rangle| \\ &\qquad = \big|r_{L_2}(E_k) r_{L_2}(E_j) \sin\zeta_{j,k}^{+} - r_{L_1}(E_k) r_{L_1}(E_j)\sin\zeta_{j,k}^{-}\big|. \end{aligned} \tag{2.106}$$

The condition (2.104) implies that

$$|\sin\zeta_{j,k}^{+} - \sin\zeta_{j,k}^{-}| = 2\,|\cos z_{j,k}^{+}\sin z_{j,k}^{-}| \ge 1, \tag{2.107}$$

and therefore that $\sin\zeta_{j,k}^{+}$ and $\sin\zeta_{j,k}^{-}$ have opposite signs. Thus, the right-hand side of (2.106) equals

$$r_{L_2}(E_k) r_{L_2}(E_j)|\sin\zeta_{j,k}^{+}| + r_{L_1}(E_k) r_{L_1}(E_j)|\sin\zeta_{j,k}^{-}|, \tag{2.108}$$

and all four Prüfer radii can be estimated from below with (2.19). This yields the lower bound

$$|\langle\psi_{L,E_k},[H_L,1_{\Lambda_L^{\gamma,\delta}}(X)]\psi_{L,E_j}\rangle| \geq \frac{1}{CL}\left(|\sin\zeta^+_{j,k}|+|\sin\zeta^-_{j,k}|\right) = \frac{1}{CL}|\sin\zeta^+_{j,k}-\sin\zeta^-_{j,k}|. \quad (2.109)$$

Now, the claim follows from again applying (2.107). □

From now on, our aim is to guarantee that condition (2.104) is satisfied for a sufficient number of indices j and k. We start with an auxiliary result.

Lemma 2.3.5. *Let $j \in \mathcal{J}_\geq$ and $k \in \mathcal{J}_<$. Then*

$$\begin{aligned} z^-_{j+1,k} - z^-_{j,k} &\in \frac{\pi}{4C^6}\left[-(C^{12}-1)\gamma+\delta\,,\,(C^{12}-1)\gamma + C^{12}\delta\right], \\ z^+_{j+1,k} - z^+_{j,k} &\in \frac{\pi(2\gamma+\delta)}{4C^6}\,[1,C^{12}] \end{aligned} \quad (2.110)$$

is true for the constant $C > 1$ in Theorem 2.2.1.

Proof. This statement is a direct consequence of (2.99), the identity

$$z^\pm_{j+1,k} - z^\pm_{j,k} = \{[\theta_{L_2}(E_{j+1}) - \theta_{L_2}(E_j)] \pm [\theta_{L_1}(E_{j+1}) - \theta_{L_1}(E_j)]\}/2 \quad (2.111)$$

and that $[a,b]+[c,d] = [a+c,b+d]$ and $[a,b]-[c,d] = [a-d,b-c]$ is true for all finite intervals $[a,b]$, $[c,d] \subseteq \mathbb{R}$. □

Now, we think of the index $k \in \mathcal{J}_<$ as being fixed, whereas the index j varies over $\mathcal{J}_\geq$ in an increasing way in steps by one. For the time being, we assume the condition

$$C^{12} - 1 < \frac{\delta}{\gamma}. \quad (2.112)$$

Its validity will be ensured later with a restriction on the disorder strength v. Thus, according to (2.110), both variables $z^\pm_{j,k}$ are strictly increasing functions in j albeit $z^+_{j,k}$ grows much faster than $z^-_{j,k}$ due to (2.87). In fact, for $C \approx 1$ we have

$$z^\pm_{j+1,k} - z^\pm_{j,k} \approx 2\pi(E_{j+1}-E_j)/T^\pm \quad (2.113)$$

for all $j \in \mathcal{J}_\geq$, where T^- is the period of the envelope and T^+ is the period of the fast oscillation of (2.89) in the last chapter. Hence, $z^-_{j,k}$ samples the envelope while $z^+_{j,k}$ samples the faster oscillation.

The condition (2.104) amounts to the requirement that sampling the oscillation produces an amplitude larger than 1/2. First, we focus on the envelope and partition $\mathcal{J}_\geq$ into smaller sets of eigenvalues between the nodes of the envelope. This partition is a discrete version of the one presented in (2.91).

Definition 2.3.6. *The set*

$$\mathcal{Z}^- := \left\{j \in \mathcal{J}_\geq :\ \sin z^-_{j,k}\sin z^-_{j-1,k} \leq 0 \text{ and } \sin z^-_{j,k} \neq 0\right\} \quad (2.114)$$

consists of those $N^- := |\mathcal{Z}^-|$ *indices where a sign change occurs in the envelope. It gives rise to a disjoint partition*

$$\mathcal{J}_\geq =: \dot{\bigcup_{q=0}^{N^-}} \mathcal{A}_q^- \tag{2.115}$$

of the index set into ranges of successive indices between the nodes of the envelope. Here, we introduced $\mathcal{A}_0^- := \{j \in \mathcal{J}_\geq : j < \min \mathcal{Z}^-\}$ *as the left-most set in the partition, which is the only one that can be empty. The requirement* $\mathcal{A}_q^- \cap \mathcal{Z}^- = \min \mathcal{A}_q^-$ *for every* $q \in \{1, \dots, N^-\}$ *renders the partition unique. The set of "envelope-good" indices in* $\mathcal{A}_q$ *is defined as*

$$\mathcal{J}_q^{\mathrm{eg}} := \{j \in \mathcal{A}_q^- : |\sin z_{j,k}^-| \geq 2^{-1/2}\}, \tag{2.116}$$

and the set of "good" indices in $\mathcal{A}_q$ *as*

$$\mathcal{J}_q^{\mathrm{g}} := \{j \in \mathcal{J}_q^{\mathrm{eg}} : |\cos z_{j,k}^+| \geq 2^{-1/2}\}, \tag{2.117}$$

where $q \in \{1, \dots, N^- - 1\}$*. For the sake of brevity, we have dropped the dependence on* $k \in \mathcal{J}_<$ *in all of the above notions.*

Remark 2.3.7. *(i) Clearly,* $j \in \mathcal{J}_q^{\mathrm{g}}$ *implies that* (2.104) *is true for this index* j *and the respective fixed index* k*.*

(ii) The set $\mathcal{A}_q$ *is the equivalence to the set of eigenvalues included in* I_q *in Figure 2.2 (a).*

Lemma 2.3.8. *Fix* $k \in \mathcal{J}_<$*. Let* $\gamma, \delta \leq 2^{-7}$ *and* $v \in \,]0, 2[$ *such that* $C \leq 2$ *and that* (2.112) *is satisfied. Then we have*

$$|\mathcal{A}_q^-| \in 2C^6 \left[\frac{1}{(C^{12}-1)\gamma + C^{12}\delta}, \frac{4}{-(C^{12}-1)\gamma + \delta} \right] \tag{2.118}$$

for every $q \in \{1, \dots, N^- - 1\}$*. The upper bound in* (2.118) *is also true for* $q = 0$ *and* $q = N^-$*. Moreover, the number of envelope-good indices is controlled by*

$$|\mathcal{J}_q^{\mathrm{eg}}| \in C^6 \left[\frac{1}{(C^{12}-1)\gamma + C^{12}\delta}, \frac{4}{-(C^{12}-1)\gamma + \delta} \right] \tag{2.119}$$

for every $q \in \{1, \dots, N^- - 1\}$*. As before,* $C > 1$ *stands for the constant from Theorem 2.2.1.*

Proof. Lemma 2.3.5 and (2.112) provide the positive bounds $a := (\pi/4C^6)[-(C^{12} - 1)\gamma + \delta]$ and $b := (\pi/4C^6)[(C^{12}-1)\gamma + C^{12}\delta]$ for the possible values of the increments $(z_{j+1,k}^- - z_{j,k}^-) \in [a, b]$. For any $q \in \{1, \cdots, N^- - 1\}$ the maximal phase difference of sample points within half of a period can be estimated as

$$\max_{j,l \in \mathcal{A}_q^-} \{z_{j,k}^- - z_{l,k}^-\} \in \,]\pi - 2b, \pi[, \tag{2.120}$$

see also Figure 2.3.2. Hence, we conclude

$$|\mathcal{A}_q^-| \in \left[\lfloor \pi - b/b \rfloor + 1, \lfloor \pi/a \rfloor + 1\right] \subseteq [\pi/(2b), 2\pi/a]. \tag{2.121}$$

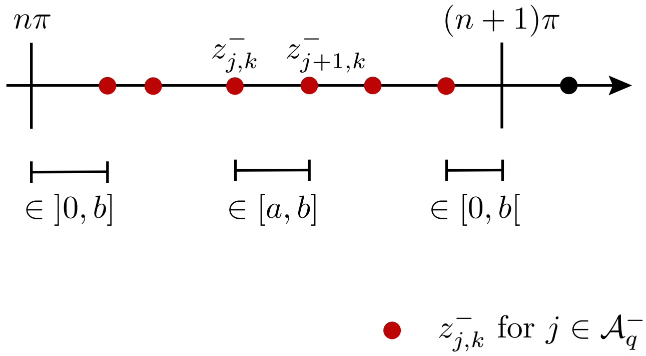

Figure 2.3: Determining the number of elements in $\mathcal{A}^-_q$.

Here, the last inclusion is satisfied, if $b \le \pi/2$ and $a \le \pi$. But $a \le b$ by definition, so the latter follows from the former. We point out that the assumptions of the lemma even guarantee $b \le \pi/4$, which is needed below. This establishes (2.118). Since the upper bound for the phase difference in (2.120) is trivial and also holds for $q = 0$ and $q = N^-$, we infer the validity of the upper bound in (2.118) for those two values of q, too.

Now, we turn to the proof of (2.119). Because of $\big|\{\varsigma \in [0,\pi] : |\sin\varsigma| \ge 2^{-1/2}\}\big| = \pi/2$, the maximal phase difference associated with envelope-good indices is restricted to

$$\max_{j,l\in\mathcal{J}^{\mathrm{eg}}_q}\left\{z^-_{j,k} - z^-_{l,k}\right\} \in \,](\pi/2) - 2b, \pi/2]. \tag{2.122}$$

Similarly, we conclude

$$|\mathcal{J}^{\mathrm{eg}}_q| \in \Big[\lfloor \pi/(2b) - 2\rfloor + 2, \lfloor \pi/(2a)\rfloor + 1\Big] \subseteq [\pi/(4b), \pi/a], \tag{2.123}$$

where the last inclusion follows from $a \le b \le \pi/4$. □

Next, we assert that there is a sufficient number of good indices in each $\mathcal{A}_q$ for $q \notin \{0, N^-\}$.

Lemma 2.3.9. *Fix $k \in \mathcal{J}_<$. We assume $C \le 2$, $\gamma \le 2^{-8}$, $\delta/\gamma \le 2^{-17}$ and that* (2.112) *is satisfied. Then we have*

$$|\mathcal{J}^{\mathrm{g}}_q| \ge \frac{1}{2^5 C^{18}\delta} \tag{2.124}$$

for every $q \in \{1,\dots,N^- - 1\}$. Again, $C > 1$ stands for the constant in Theorem 2.2.1.

Proof. The set

$$\mathcal{Z}^+_q := \left\{j \in \mathcal{J}^{\mathrm{eg}}_q :\ \cos z^+_{j,k}\cos z^+_{j-1,k} \le 0 \text{ and } \cos z^+_{j,k} \ne 0\right\}, \tag{2.125}$$

where $q \in \{1,\dots,N^- - 1\}$, consists of those $N^+_q := |\mathcal{Z}^+_q|$ indices where a sign change occurs in the fast oscillation within the envelope-good part of $\mathcal{A}_q$. This gives rise to a disjoint partition

$$\mathcal{J}^{\mathrm{eg}}_q =: \dot{\bigcup_{r=0}^{N^+_q}} \mathcal{A}^+_{q,r} \tag{2.126}$$

into ranges of successive indices between two nodes of the fast oscillation. Here, we introduced $\mathcal{A}^+_{q,0} := \{j \in \mathcal{J}^{\mathrm{eg}}_q : j < \min \mathcal{Z}^+_q\}$ as the left-most set in the partition, which is the only one that can be empty. The requirement $\mathcal{A}^+_{q,r} \cap \mathcal{Z}^+_q = \min \mathcal{A}^+_{q,r}$ for every $r \in \{1, \dots, N^+_q\}$ renders the partition unique.

First, we estimate the cardinality of $\mathcal{A}^+_{q,r}$ in the same way as it was done for $\mathcal{A}^-_q$ in the proof of the previous lemma. Lemma 2.3.5 provides the positive bounds $a' := \pi(2\gamma + \delta)/(4C^6)$ and $b' := \pi C^6(2\gamma + \delta)/4$ for the possible values of the increments $(z^+_{j+1,k} - z^+_{j,k}) \in [a', b']$. For any $q \in \{1, \cdots, N^- - 1\}$ and any $r \in \{1, \cdots, N^+_q - 1\}$ the maximal phase difference of sample points within half of a period of the fast oscillation can be estimated as

$$\max_{j,l \in \mathcal{A}^+_{q,r}} \left\{ z^+_{j,k} - z^+_{l,k} \right\} \in \left]\pi - 2b', \pi\right[. \tag{2.127}$$

Hence, we conclude

$$|\mathcal{A}^+_{q,r}| \in \left[\lfloor \pi/b' \rfloor, \lfloor \pi/a' \rfloor + 1\right] \subseteq [\pi/(2b'), 2\pi/a'], \tag{2.128}$$

where the last inclusion follows from $0 < a' < b' \le \pi/2$. In fact, the assumptions of the lemma even guarantee $b' \le \pi/4$, which we need below. Since the upper bound for the phase difference in (2.127) is trivial and is also true for $r = 0$ and $r = N^+_q$, we infer the validity of the upper bound in (2.128) for those two values of r, too.

In order to estimate the cardinality of $\mathcal{Z}^+_q$ for $q \in \{1, \cdots, N^- - 1\}$, we infer from (2.126) and (2.128) that

$$|\mathcal{J}^{\mathrm{eg}}_q| \le (N^+_q + 1) \frac{2\pi}{a'}. \tag{2.129}$$

The assumptions of the present lemma imply those of Lemma 2.3.8 which yields

$$|\mathcal{J}^{\mathrm{eg}}_q| \ge d := \frac{1}{2C^6 \delta}. \tag{2.130}$$

Thus, we arrive at

$$N^+_q - 1 \ge \frac{da'}{2\pi} - 2 \ge \frac{da'}{4\pi}, \tag{2.131}$$

where the second inequality holds because the assumptions of the lemma imply $da' \ge 8\pi$.

The set of "good" indices within $\mathcal{A}^+_{q,r}$ is defined as

$$\mathcal{J}^{\mathrm{g}}_{q,r} := \left\{ j \in \mathcal{A}^+_{q,r} : |\cos z^+_{j,k}| \ge 2^{-1/2} \right\} \tag{2.132}$$

so that

$$\mathcal{J}^{\mathrm{g}}_q \supseteq \biguplus_{r=1}^{N^+_q - 1} \mathcal{J}^{\mathrm{g}}_{q,r}. \tag{2.133}$$

For any $q \in \{1, \cdots, N^- - 1\}$ and any $r \in \{1, \cdots, N^+_q - 1\}$, the maximal phase difference of good sample points between two nodes of the fast oscillation can be estimated as

$$\max_{j,l \in \mathcal{J}^{\mathrm{g}}_{q,r}} \left\{ z^+_{j,k} - z^+_{l,k} \right\} \in \left](\pi/2) - 2b', \pi/2\right]. \tag{2.134}$$

Here, we used the second statement from Lemma 2.3.5, $z^+_{j+1,k} - z^+_{j,k} \in [a', b']$ with $a' := \pi(2\gamma + \delta)/(4C^6)$ and $b' := \pi C^6(2\gamma + \delta)/4$. Therefore, we conclude as in (2.123)

$$|\mathcal{J}^{\mathrm{g}}_{q,r}| \in \left[\lfloor \pi/(2b') - 2 \rfloor + 2, \lfloor \pi/(2a') \rfloor + 1\right] \subseteq [\pi/(4b'), \pi/a'], \tag{2.135}$$

where the last inclusion follows from $0 < a' < b' \le \pi/4$. Combining (2.133), (2.131) and (2.135), we obtain $|\mathcal{J}^{\mathrm{g}}_q| \ge da'/(2^4 b')$, which proves the lemma. □

2.3.3 The logarithmic lower bound

We assemble the results from the previous section and deduce a deterministic logarithmic lower bound for the quadratic analogue to the finite-volume entanglement entropy as defined in (2.79).

Theorem 2.3.10. *Let $v \in]0,2[$ and $E_c \in \{0,v\}$. We fix $\alpha \in]0,1/4[$ and $\gamma \in]0,2^{-17}[$. In addition, we assume that the quantity $C > 1$ from Theorem 2.2.1 satisfies*

$$C < 1 + \gamma^2. \tag{2.136}$$

Then there exists a minimal length $L_0 \equiv L_0(\alpha, v) > 0$ such that

$$Q_{E_c}(\Lambda_L^{\gamma,\gamma^2}; \Gamma_L, H_L^\omega) \geq 2^{-13}(1-3\alpha)\ln L \tag{2.137}$$

for all $L \geq L_0$ and all events $\omega \in (\Omega_L(\alpha))^c$.

We argue in Remark 2.2.2 (i) that the assumption (2.136) can always be satisfied by choosing the disorder strength v sufficiently small. This leads to

Corollary 2.3.11. *We fix $\gamma \in]0,2^{-17}[$. There exists a maximal disorder strength $v_0 \in]0,2[$ such that for every $v \in]0,v_0]$ and $E_c \in \{0,v\}$ there is a minimal length $L_0' \equiv L'(v) > 0$ such that for all $L \geq L_0'$*

$$\mathbb{E}\big[Q_{E_c}(\Lambda_L^{\gamma,\gamma^2}; \Gamma_L, H_L)\big] \geq 2^{-15}\ln L. \tag{2.138}$$

Proof. As $\lim_{v\downarrow 0} C = 1$ by Theorem 2.2.1, there exists a maximal disorder strength $v_0 \equiv v_0(\gamma) \in]0,2[$ such that (2.136) is true for every $v \in]0,v_0]$. We choose $\alpha = 1/6$ in Theorem 2.3.10 and infer from (2.137) that

$$\mathbb{E}\big[Q_{E_c}(\Lambda_L^{\gamma,\gamma^2}; \Gamma_L, H_L)\big] \geq 2^{-14}\ln L\, \mathbb{P}\big[(\Omega_L(\alpha))^c\big] \tag{2.139}$$

for every $L \geq L_0$. Now, the claim follows from (2.16), possibly by enlarging L_0. □

Proof of Theorem 2.3.10. Let $\omega \in (\Omega_L(\alpha))^c$ and, for the time being, $L \geq L_{\min}$, where $L_{\min}$ is the minimal length given in Theorem 2.2.1. We use the notation introduced at the beginning of Section 2.3.2 and drop ω from all quantities, as it is also done there. By restricting the double sum in Lemma 2.3.1 to energies inside the critical window, we arrive at the estimate

$$Q_{E_c}(\Lambda_L^{\gamma,\gamma^2}; \Gamma_L, H_L) \geq 4 \sum_{j\in\mathcal{J}_\geq,\, k\in\mathcal{J}_<} \frac{1}{(E_j - E_k)^2} \Big|\langle \psi_{L,E_k}, [H_L, 1_{\Lambda_L^{\gamma,\gamma^2}}(X)]\psi_{L,E_j}\rangle\Big|^2. \tag{2.140}$$

We aim to apply the lower bound for the commutator from Lemma 2.3.4. Its assumption (2.104) is satisfied for every fixed $k \in \mathcal{J}_<$ after further restricting the j-sum to good indices according to $\mathcal{J}_\geq \supseteq \bigcup_{q=1}^{N^- -1} \mathcal{J}_q^{\mathrm{g}}$, see Remark (i). This yields the lower bound

$$\frac{4}{(CL)^2} \sum_{k\in\mathcal{J}_<} \sum_{q=1}^{N^- -1} \sum_{j\in\mathcal{J}_q^{\mathrm{g}}} \frac{1}{(E_j - E_k)^2} \geq \frac{4}{(CL)^2} \sum_{k\in\mathcal{J}_<} \sum_{q=1}^{N^- -1} \frac{|\mathcal{J}_q^{\mathrm{g}}|}{(\varepsilon_q^{(k)} - E_k)^2} \tag{2.141}$$

for the right-hand side of (2.140), where we introduced

$$\varepsilon_q^{(k)} := \max_{j \in \mathcal{A}_q^-} E_j \tag{2.142}$$

for $q \in \{1, \dots, N^-\}$ and $k \in \mathcal{J}_<$. We recall that there is a suppressed k-dependence in the quantities of Definition 2.3.6 which we explicitly expressed in $\varepsilon_q^{(k)}$.

The assumptions of the theorem imply those of Lemma 2.3.9 because the elementary inequality $(1+\rho)^n \le 1 + 2^n \rho$, valid for $\rho \in [0,1]$ and $n \in \mathbb{N}$, ensures that (2.112) holds. In fact, even the stronger inequality

$$C^{12} - 1 \le 2^{12}\gamma^2 \le 2^{-5}\gamma = 2^{-5}\delta/\gamma \tag{2.143}$$

is satisfied for $\delta = \gamma^2$. Therefore, we can apply the lemma and infer that the expression

$$\frac{1/L}{2^3 C^{20}\gamma^2} \sum_{k \in \mathcal{J}_<} \sum_{q=1}^{N^- -1} \frac{\varepsilon_{q+1}^{(k)} - \varepsilon_q^{(k)}}{(\varepsilon_q^{(k)} - E_k)^2} \frac{1/L}{\varepsilon_{q+1}^{(k)} - \varepsilon_q^{(k)}} \tag{2.144}$$

is a lower bound for the right-hand side of (2.141). The energy $\varepsilon_q^{(k)}$ is the right-most next to the qth node of the envelope. Therefore we can estimate their differences as

$$0 < \varepsilon_{q+1}^{(k)} - \varepsilon_q^{(k)} \le |\mathcal{A}_{q+1}^-| \frac{\pi C^3}{L} \le \frac{\pi 2^3 C^9}{L\gamma^2} \frac{1}{1 - 2^{-5}} \le \frac{2^6 C^9}{L\gamma^2} \tag{2.145}$$

for $q \in \{1, \dots, N^- - 1\}$, independently of k. Here, the first upper bound on the difference follows from (2.18) and the second from Lemma 2.3.8 and (2.143). We note that the assumptions of Lemma 2.3.8 are weaker than those of Lemma 2.3.9. Combining (2.140), (2.141), (2.144) and (2.145), we arrive at

$$\begin{aligned} Q_{E_c}(\Lambda_L^{\gamma,\gamma^2}; \Gamma_L, H_L) &\ge \frac{1/L}{2^9 C^{29}} \sum_{k \in \mathcal{J}_<} \sum_{q=1}^{N^- -1} \frac{\varepsilon_{q+1}^{(k)} - \varepsilon_q^{(k)}}{(\varepsilon_q^{(k)} - E_k)^2} \\ &\ge \frac{1/L}{2^9 C^{29}} \sum_{k \in \mathcal{J}_<} \int_{\varepsilon_1^{(k)}}^{\varepsilon_{N^-}^{(k)}} \frac{\mathrm{d}\varepsilon}{(\varepsilon - E_k)^2}. \end{aligned} \tag{2.146}$$

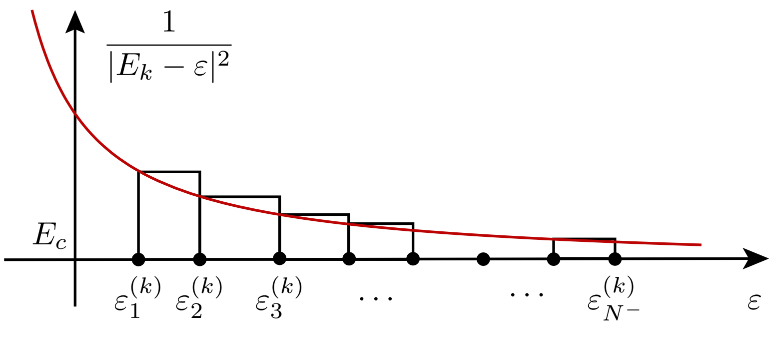

Figure 2.4: The Riemann sum of (2.146).

Now, the next step is to deduce a k-independent lower bound on the range of the ε-integration. This allows us to interchange the integral with the k-sum. Since $\varepsilon_1^{(k)}$, respectively $\varepsilon_{N^-}^{(k)}$, lies between the first, respectively last, two nodes of the envelope, we estimate as in (2.145)

$$\begin{aligned} \varepsilon_1^{(k)} &\leq E_c + \left(|\mathcal{A}_0^-| + |\mathcal{A}_1^-|\right) \frac{\pi C^3}{L} \leq E_c + \frac{2^7 C^9}{L\gamma^2}, \\ \varepsilon_{N^-}^{(k)} &\geq \max \mathcal{W}_L - \frac{\pi C^3}{L} \geq E_c + L^{-1/2-\alpha} - \frac{2^2 C^3}{L}, \end{aligned} \tag{2.147}$$

independently of k. Let $L_0 \equiv L_0(\alpha, v) \geq L_{\min}$ be large enough so that both $2^2 C^3 \leq 2^7 C^9/\gamma^2 \leq 2^{16}/\gamma^2 \leq L_0^\alpha$ and $L^\alpha - L^{-1/2+3\alpha} \geq 1$ for all $L \geq L_0$. For the rest of this proof we assume $L \geq L_0$. Then (2.147) simplifies to

$$\begin{aligned} &\varepsilon_1^{(k)} \leq E_c + L^{-1+\alpha}, \\ &\varepsilon_{N^-}^{(k)} \geq E_c + L^{-1/2-2\alpha}\left(L^\alpha - L^{-1/2+3\alpha}\right) \geq E_c + L^{-1/2-2\alpha}. \end{aligned} \tag{2.148}$$

We recall the definition of $J_<$ from (2.98) and conclude that

$$\begin{aligned} Q_{E_c}(\Lambda_L^{\gamma,\gamma^2}; \Gamma_L, H_L) &\geq \frac{1}{2^9 C^{29}} \int_{E_c+L^{-1+\alpha}}^{E_c+L^{-1/2-2\alpha}} \mathrm{d}\varepsilon \sum_{k\in\mathcal{J}_<} \frac{E_k - E_{k-1}}{(\varepsilon - E_k)^2} \frac{1/L}{E_k - E_{k-1}} \\ &\geq \frac{1}{2^{11} C^{32}} \int_{E_c+L^{-1+\alpha}}^{E_c+L^{-1/2-2\alpha}} \mathrm{d}\varepsilon \int_{E_{J_<}}^{E_{-1}} \mathrm{d}\eta \, \frac{1}{(\varepsilon-\eta)^2} \end{aligned} \tag{2.149}$$

because the level-spacing estimate (2.18) provides the bound $E_k - E_{k-1} \leq \pi C^3/L$. It also implies

$$\begin{aligned} E_{-1} &\geq E_c - \frac{\pi C^3}{L} \geq E_c - L^{-1+\alpha}, \\ E_{J_<} &\leq \min \mathcal{W}_L + \frac{\pi C^3}{L} \leq E_c - L^{-1/2-2\alpha}, \end{aligned} \tag{2.150}$$

where we argued similarly as in (2.148) for $L \geq L_0$. We thus estimate and integrate

$$\begin{aligned} Q_{E_c}(\Lambda_L^{\gamma,\gamma^2}; \Gamma_L, H_L) &\geq \frac{1}{2^{11} C^{32}} \int_{L^{-1+\alpha}}^{L^{-1/2-2\alpha}} \mathrm{d}\varepsilon \int_{-L^{-1/2-2\alpha}}^{-L^{-1+\alpha}} \mathrm{d}\eta \, \frac{1}{(\varepsilon-\eta)^2} \\ &= \frac{1}{2^{11} C^{32}} \ln\left(L^{-1+3\alpha} \frac{2}{1 + L^{-1/2+3\alpha}} \right) \geq \frac{1-3\alpha}{2^{12} C^{32}} \ln L. \end{aligned} \tag{2.151}$$

Finally, the estimate $C^{32} \leq 1 + 2^{32}\gamma^2 \leq 2$, which follows from the elementary inequality above (2.143), concludes the proof. □

2.3.4 Proof of Theorem 2.1.2

To conclude this section, we sketch the necessary modifications for the proof of Theorem 2.1.2. The goal is to obtain a similar statement to Theorem 2.3.10, which is valid for all possible coupling constants $v \in]0,2[$. We therefore cannot rely on C being arbitrarily close to one. However, the proof is much easier given that the region Λ_L' is attached to the border of Γ_L.

Proof of Theorem 2.1.2. The use of $\Lambda'_L := [-L, -(1-\delta)L] \cap \mathbb{Z} = \Lambda_L^{0,\delta}$ amounts to $\gamma = 0$ in our previous arguments. This change simplifies the matrix elements of the commutator (2.106) dramatically, since $\theta_{L_1}(E) = 0$ in this case by definition for all values $E \in \mathbb{R}$. Hence,

$$|\langle \psi_{L,E_k}, [H_L, 1_{\Lambda'_L}(X)]\psi_{L,E_j}\rangle| = \left| r_{L_2}(E_k) r_{L_2}(E_j) \sin(2z^-_{j,k}) \right| \geq \frac{1}{CL} |\sin(2z^-_{j,k})| \tag{2.152}$$

for all $k \in \mathcal{J}_<$ and $j \in \mathcal{J}_\geq$. This renders the considerations of Lemma 2.3.4 and Lemma 2.3.9 unnecessary, since only a single sine-function emerges. The overall argument, however, is similar to the one in Lemma 2.3.8 with an additional factor of 2.

We therefore redefine the set

$$\mathcal{Z}^- := \left\{ j \in \mathcal{J}_\geq : \sin(2z^-_{j,k}) \sin(2z^-_{j-1,k}) \leq 0 \text{ and } \sin(2z^-_{j,k}) \neq 0 \right\} \tag{2.153}$$

of indices where a sign change occurs in the oscillation. Again, let $N^- := |\mathcal{Z}^-|$. As before, this gives rise to a disjoint partition

$$\mathcal{J}_\geq =: \dot{\bigcup_{q=0}^{N^-}} \mathcal{A}^-_q \tag{2.154}$$

of the index set into sets of successive indices between the nodes of the oscillation. The set of good indices in $\mathcal{A}^-_q$ is defined as

$$\mathcal{J}^{\mathrm{g}}_q := \left\{ j \in \mathcal{A}^-_q : |\sin(2z^-_{j,k})| \geq 2^{-1/2} \right\}, \tag{2.155}$$

for all $q \in \{0, \cdots, N^-\}$. The proof of Lemma 2.3.5 is valid for $\gamma = 0$. It provides positive bounds $a := \delta\pi/2C^6$ and $b := \delta\pi C^6/2$ for the positive values of the increments $2(z^-_{j+1,k} - z^-_{j,k}) \in [a, b]$. From the proof of Lemma 2.3.8 we get the following estimate

$$|\mathcal{J}^{\mathrm{g}}_q| \geq \lfloor \pi/(2b) \rfloor \geq \pi/(4b), \tag{2.156}$$

where the last inclusion follows from $b \leq \pi/4$, which is true for $\delta < 2^{-1}C^{-6}$. The inequality (2.156) replaces the estimate of Lemma 2.3.9. The rest of the proof is identical to the one of Theorem 2.3.10 and Corollary 2.3.11, except that we do not take the expectation at the end but resort to the Borel–Cantelli Lemma to conclude that

$$\mathbb{P}\Big(\left\{ \omega \in \left(\Omega_L(\alpha) \right)^c \text{ for all but a finite number of } L \in \mathbb{N} \right\} \Big) = 1. \tag{2.157}$$

□

2.4 Lower bound to the infinite-volume entanglement entropy

2.4.1 General idea and strategy

In this section we deduce the main Theorem 2.1.1 from Corollary 2.3.11. The goal is to control the error arising from considering the finite-volume instead of the infinite-volume entanglement entropy. We consider a discrete interval $\Lambda \subset \Gamma_L$ and denote by

$f(H_L^\omega)$ the trivial extension of this operator from $\ell^2(\Gamma_L)$ to the space $\ell^2(\mathbb{Z})$ for any measurable function $f : \mathbb{R} \to \mathbb{R}$.

Our strategy is to apply Kreĭn's trace formula, see e.g. [Sch12, Sect. 9.7],

$$\begin{aligned}\Big|\mathrm{tr}\, g\big(1_\Lambda(X) 1_{<E_c}(H^\omega) 1_\Lambda(X)\big) - g\big(1_\Lambda(X) 1_{<E_c}(H_L^\omega) 1_\Lambda(X)\big)\Big| \\ = \Big|\int_0^1 \mathrm{d}s\, g'(s)\, \xi_L^\omega(s)\Big| \le 4\, \|\xi_L^\omega\|_{L^1}\end{aligned} \tag{2.158}$$

to the parabola g from (2.78), where

$$\xi_L^\omega : \mathbb{R} \ni s \mapsto \mathrm{tr}\, 1_{\le s}\big(1_\Lambda(X) 1_{<E_c}(H^\omega) 1_\Lambda(X)\big) - 1_{\le s}\big(1_\Lambda(X) 1_{<E_c}(H_L^\omega) 1_\Lambda(X)\big) \tag{2.159}$$

is the spectral shift function. Here, $\|\cdot\|_{L^1}$ denotes the $L^1(\mathbb{R})$-norm. It can be estimated in terms of the trace norm $\|\cdot\|_1$ of the difference

$$\|\xi_L^\omega\|_{L^1} \le \big\|1_\Lambda(X)\big(1_{<E_c}(H^\omega) - 1_{<E_c}(H_L^\omega)\big)1_\Lambda(X)\big\|_1. \tag{2.160}$$

Consequently, our aim is to find an estimate for the right-hand side of (2.160) that grows at most logarithmically in L.

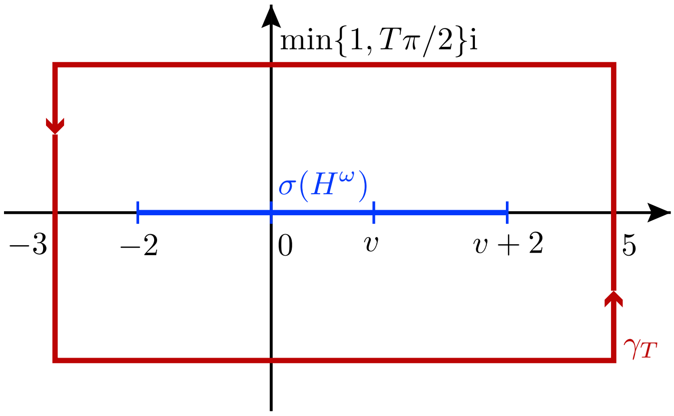

Figure 2.5: The contour γ_T

In a first step we replace $1_{<E_c}$ by $f_T(\cdot - E_c)$, where $f_T := 1/(1 + \mathrm{e}^{(\cdot)/T})$ is the Fermi–Dirac distribution for a temperature $T > 0$. For small temperatures this yields an approximation of the Fermi projection, since $\lim_{T\to 0} \|f_T - 1_{<0}\|_{L^1} = 0$. The replacement enables us to express the difference between finite- and infinite-volume operators in terms of a contour integral along the curve γ_T shown in Figure 2.5. This curve encircles both $\sigma(H_L^\omega)$ and $\sigma(H^\omega)$, but no singularities of the meromorphic function $f_T(\cdot - E_c)$, which are positioned at $(E_c + \mathrm{i}\pi T) + 2\pi\mathrm{i}T\mathbb{Z}$. Hence,

$$\begin{aligned}F_L^\omega(\Lambda, T, E_c) &:= 1_\Lambda(X)\big(f_T(H^\omega - E_c) - f_T(H_L^\omega - E_c)\big)1_\Lambda(X) \\ &= \frac{1}{2\pi\mathrm{i}} \oint_{\gamma_T} \mathrm{d}z\; f_T(z - E_c)\, 1_\Lambda(X)\Big(\frac{1}{z - H^\omega} - \frac{1}{z - H_L^\omega}\Big)1_\Lambda(X).\end{aligned} \tag{2.161}$$

This expression is advantageous, since there exists a number of well-known results on resolvents of Schrödinger operators that can be applied to estimate the contour integral.

In particular, we use the geometric resolvent equality and a Combes–Thomas estimate. For almost all $\omega \in \Omega$ we show that

$$\|F_L^\omega(\Lambda, T, E_c)\|_1 \leq 2^9|\Lambda|^2 T^{-2} \mathrm{e}^{-d_L T/6} + r_{L,1}(\Lambda, T), \tag{2.162}$$

where $r_{L,1}(\Lambda, T) > 0$ and $d_L \equiv d_L(\Lambda) := \mathrm{dist}(\Lambda, \{-L, L-1\})$ denotes the distance of the small box to the boundary of Γ_L for all $L \in \mathbb{N}$. It is noteworthy that this estimate is completely deterministic and almost surely does not depend on ω.

In a second step, we estimate the error caused by replacing the Fermi projections with the Fermi-.Dirac distribution to a temperature $T > 0$. For both H^ω and H_L^ω let us define

$$G_{(L)}^\omega(\Lambda, T, E_c) := 1_\Lambda(X)\big(f_T(H_{(L)}^\omega - E_c) - 1_{<E_c}(H_{(L)}^\omega)\big)1_\Lambda(X). \tag{2.163}$$

We estimate the expectation of these operators with

$$\mathbb{E}\big[\|G_{(L)}^\omega(\Lambda, T, E_c)\|_1\big] \leq 2C^4|\Lambda|T + r_{(L),2}(\Lambda, T), \tag{2.164}$$

where $r_{(L),2}(\Lambda, T) > 0$.

These results are summarised in the lemma below, which will be proven in Section 2.4.2.

Lemma 2.4.1. *Let $E_c \in \{0, v\}$ and $\alpha > 0$. Then there exists a minimal length $L_0 \equiv L_0(\alpha, v, p_+) \in \mathbb{N}$, such that for all $L \geq L_0$, all "temperatures" $T \in]0, \infty[$ and all discrete intervals $\Lambda \subset \Gamma_L$ we have the estimate*

$$\mathbb{E}\left[|Q_{E_c}(\Lambda; \Gamma_L, H_L) - Q_{E_c}(\Lambda; \mathbb{Z}, H)|\right] \leq 2^2 C^4|\Lambda|T + 2^9|\Lambda|^2 T^{-2} e^{-d_L T/6} + R_L(\Lambda, T) \tag{2.165}$$

with a remainder term

$$R_L(\Lambda, T) := 2^4 C + 2^9|\Lambda|^2 T e^{-d_L/6} + 2^3 C^3|\Lambda| e^{-L^{-1/2-\alpha}/T}. \tag{2.166}$$

The previous lemma finally enables us to prove Theorem 2.1.1 by perturbing the result of Corollary 2.3.11. Recall, that the spatial region $\Gamma_L^{\gamma,\gamma^2}$ considered in this corollary is of size $|\Lambda_L^{\gamma,\gamma^2}| \approx \gamma^2 L$ and at a distance $d_L(\Lambda_L^{\gamma,\gamma^2}) \approx \gamma L$ to the border of Γ_L. We choose an L-dependent temperature

$$T_L := (K \ln L)/L \tag{2.167}$$

for some constant $K \equiv K(\gamma) := \frac{25}{\gamma} > 0$.

For any $\gamma \in]0, 1/2[$ this implies $\lim_{L\to\infty} R_L(\Lambda_L^{\gamma,\gamma^2}, T_L) = 0$. The second term on the right-hand side of (2.165) also vanishes for $L \to \infty$. The remaining first term is proportional to $C^4\gamma \ln L$. Since the lower bound to the finite-volume entanglement entropy from Corollary 2.3.11 is independent of both γ and C, we are able to ascertain a logarithmic lower bound to the infinite-volume entanglement entropy by choosing γ to be sufficiently small.

2.4.2 Proof of Lemma 2.4.1

Without loss of generality we restrict ourselves in the proof of Lemma 2.4.1 to the case $E_c = 0$, the other case being analogous.

Following the strategy outlined in the previous section, we start by estimating the contour integral presented in (2.161).

Lemma 2.4.2. *The deterministic estimate*

$$\|F_L^\omega(\Lambda, T, 0)\|_1 \leq 2^9 |\Lambda|^2 \left(T e^{-d_L/6} + T^{-2} e^{-d_L T/6}\right) \tag{2.168}$$

is true for all $L \in \mathbb{N}$, $\Lambda \subset \Gamma_L$, $T > 0$ and almost all $\omega \in \Omega$.

Proof. The function f_T is meromorphic with singularities at $\mathrm{i}\pi T + 2\pi T\mathrm{i}\mathbb{Z}$. Let γ_T be a positively-oriented simple closed curve with an image that borders the rectangle

$$\left\{z \in \mathbb{C} : |\operatorname{Im}(z)| \leq \min(1, T\pi/2),\ \operatorname{Re}(z) \in [-3, 5]\right\}. \tag{2.169}$$

Note that this curve encircles the spectra of both H_L^ω and H^ω for all $L \in \mathbb{N}$ and almost all $\omega \in \Omega$, since both $\sigma(H_{(L)}^\omega) \subseteq [-2, 4]$ for all $v \in]0, 2[$. We conclude that

$$\begin{aligned} &1_\Lambda(X)\big(f_T(H^\omega) - f_T(H_L^\omega)\big)1_\Lambda(X) \\ &\qquad = \frac{1}{2\pi\mathrm{i}} \oint_{\gamma_T} \mathrm{d}z\ f_T(z)\, 1_\Lambda(X)\Big(\frac{1}{z - H^\omega} - \frac{1}{z - H_L^\omega}\Big)1_\Lambda(X). \end{aligned} \tag{2.170}$$

The geometric resolvent equation yields

$$\begin{aligned} &1_\Lambda(X)\Big(\frac{1}{z - H^\omega} - \frac{1}{z - H_L^\omega}\Big)1_\Lambda(X) \\ &\qquad = -1_\Lambda(X)\frac{1}{z - H^\omega}\left(|\delta_{-L-1}\rangle\langle\delta_{-L}| + |\delta_L\rangle\langle\delta_{L-1}|\right)\frac{1}{z - H_L^\omega}1_\Lambda(X). \end{aligned} \tag{2.171}$$

We estimate the matrix elements of the resolvent with the Combes–Thomas estimate [Kir08, Thm. 11.2]

$$\left|\Big\langle\delta_x, \Big(\frac{1}{z - H^\omega} - \frac{1}{z - H_L^\omega}\Big)\delta_y\Big\rangle\right| \leq 2\,\frac{2^2}{\operatorname{dist}(z, [-2, 4])^2}\ \mathrm{e}^{-2\operatorname{dist}(z,[-2,4])\, d_L/12} \tag{2.172}$$

for every $x, y \in \Lambda$ and every $z \notin \sigma(H_L^\omega) \cup \sigma(H^\omega)$. As to the applicability of [Kir08, Thm. 11.2], we note that based on this proof the statement can be obtained not only for $z \in \mathbb{C}$ with distance to the spectrum ≤ 1, but even if it is ≤ 12, which is satisfied in our case.

An elementary computation shows that $|f_T(z)| \leq 1$ for all z on the curve γ_T. Furthermore, for all z on the horizontal parts of γ_T where $|\operatorname{Im}(z)| = \min(1, T\pi/2)$ we find $\operatorname{dist}(z, [-2, 4]) \geq T\pi/2$. On the vertical parts where $\operatorname{Re}(z) \in \{-3, 5\}$ we have $\operatorname{dist}(z, [-2, 4]) \geq 1$. Hence,

$$
\begin{aligned}
&\big\|1_\Lambda(X)\big(f_T(H^\omega) - f_T(H^\omega_L)\big)1_\Lambda(X)\big\|_1 \\
&\qquad\qquad \le \sum_{x,y\in\Lambda} \frac{1}{2\pi}\bigg| \oint_{\gamma_T} \mathrm{d}z\, f_T(z) \Big\langle \delta_x, \Big(\frac{1}{z-H^\omega} - \frac{1}{z-H^\omega_L}\Big)\delta_y\Big\rangle \bigg| \\
&\qquad\qquad \le 2^9|\Lambda|^2 \left(T\mathrm{e}^{-d_L/6} + T^{-2}\mathrm{e}^{-d_L T/6}\right). \qquad (2.173)
\end{aligned}
$$

□

As described in the last section, we now proceed by estimating the error term (2.163) that arises from replacing the Fermi projections with $f_T(H^\omega_{(L)})$.

Lemma 2.4.3. *There exists a minimal length $\tilde{L}_0 \equiv \tilde{L}_0(v,p_+) > 1$, such that for all $L \ge \tilde{L}_0$, all $T > 0$ and all sets $\Lambda \subset \Gamma_L$ we have*

$$
\mathbb{E}\big[\|G(\Lambda,T,0)\|_1\big] \le 2|\Lambda|\left[C^3T + \big(2 + C^3L^{-1/2}\big)e^{-L^{-1/2}/T}\right]. \qquad (2.174)
$$

Proof. We recall that, given a bounded measurable function $\zeta : \mathbb{R} \to \mathbb{R}$ with decomposition $\zeta = \zeta_+ - \zeta_-$ in its positive and negative part, we have the estimate

$$
\begin{aligned}
\|1_\Lambda(X)\zeta(H)1_\Lambda(X)\|_1 &\le \|1_\Lambda(X)\zeta_+(H)1_\Lambda(X)\|_1 + \|1_\Lambda(X)\zeta_-(H)1_\Lambda(X)\|_1 \\
&= \operatorname{tr}\left\{1_\Lambda(X)|\zeta|(H)1_\Lambda(X)\right\}. \qquad (2.175)
\end{aligned}
$$

This, together with ergodicity with respect to $2\mathbb{Z}$-translations and the well-known Pastur–Shubin formula for the integrated density of states $\mathcal{N}(E) = \big(\mathbb{E}[\langle\delta_0, 1_{<E}(H)\delta_0\rangle] + \mathbb{E}[\langle\delta_1, 1_{<E}(H)\delta_1\rangle]\big)/2$ imply

$$
\begin{aligned}
&\mathbb{E}\Big[\big\|1_\Lambda(X)\big(f_T(H) - 1_{<0}(H)\big)1_\Lambda(X)\big\|_1\Big] \\
&\quad \le \mathbb{E}\Big[\operatorname{tr}\left\{1_\Lambda(X)|f_T(H) - 1_{<0}(H)|1_\Lambda(X)\right\}\Big] \le 2|\Lambda| \int_{\mathbb{R}} \mathrm{d}\mathcal{N}(E)\, |f_T(E) - 1_{<0}(E)|. \quad (2.176)
\end{aligned}
$$

We split the integral over $\mathbb{R}$ into two contributions from $\mathbb{R}_{>0}$, respectively $\mathbb{R}_{<0}$, and only discuss the one from $\mathbb{R}_{>0}$. The other one from $\mathbb{R}_{<0}$ will have the same upper bound. Thus, for every $L \in \mathbb{N}$, we infer from partial integration

$$
\begin{aligned}
\int_0^\infty \mathrm{d}\mathcal{N}(E)\, f_T(E) = &\int_0^{L^{-1/2}} \mathrm{d}E\,\big(\mathcal{N}(E) - \mathcal{N}(0)\big)\,(-f_T)'(E) \\
&+ \big(\mathcal{N}(L^{-1/2}) - \mathcal{N}(0)\big)f_T(L^{-1/2}) \\
&+ \int_{L^{-1/2}}^\infty \mathrm{d}\mathcal{N}(E)\, f_T(E). \qquad (2.177)
\end{aligned}
$$

The integral in the last line of (2.177) is bounded from above by $\mathrm{e}^{-L^{-1/2}/T}$. According to Theorem 2.2.1 (ii), there exists $\varepsilon_0 > 0$, which depends only on v and on the probabilities $p_\pm$, such that $|\mathcal{N}(E) - \mathcal{N}(0)| < 2\mathcal{N}'(0)|E|$ for all $|E| < \varepsilon_0$. From now on we assume that $L \ge \tilde{L}_0 \equiv \tilde{L}_0(v,p_+) := \varepsilon_0^{-2}$. Thus, the modulus of the term in the second line of (2.177) is bounded from above by

$$
2\mathcal{N}'(0)L^{-1/2}\,\mathrm{e}^{-L^{-1/2}/T} \le 2^{-1}C^3L^{-1/2}\,\mathrm{e}^{-L^{-1/2}/T}, \qquad (2.178)
$$

where we used $f_T \le \mathrm{e}^{-(\cdot)/T}$ and Theorem (ii). Since $(-f_T)' \ge 0$, we bound the modulus of the first integral on the right-hand side of (2.177) from above by

$$2\mathcal{N}'(0) \int_0^{L^{-1/2}} \mathrm{d}E\, E\,(-f_T)'(E) = 2\mathcal{N}'(0)\Big\{ -L^{-1/2} f_T(L^{-1/2}) + \int_0^{L^{-1/2}} \mathrm{d}E\, f_T(E)\Big\}$$
$$\le 2\mathcal{N}'(0)\,T \le 2^{-1}C^3 T. \tag{2.179}$$

Combining the three upper bounds for the contributions to (2.177), and adding the identical upper bound for the contribution from the integral over $\mathbb{R}_{<0}$ to (2.176), we obtain the claim. □

Lemma 2.4.4. *Let $\alpha > 0$. For all $L > L_{\min}$, all $T > 0$ and all discrete intervals $A \subset \Gamma_L$ we have*

$$\mathbb{E}\big[\|G_L(\Lambda, T, 0)\|_1\big] \le |\Lambda|\Big[C^4 T + \frac{2C}{L} + e^{-L^{-1/2-\alpha}/T} + e^{-cL^{\alpha/2}}\Big], \tag{2.180}$$

where $L_{\min}$, C and c originate from Theorem 2.2.1.

Proof. The principal strategy here is the same as in the proof of Lemma 2.4.3, but instead of ergodicity and regularity of the integrated density of states, we rely on the delocalisation results of Theorem 2.2.1. Thus, let $L \ge L_{\min}$ and $\omega \in (\Omega_L(\alpha))^c$. We drop ω from the notation of all quantities in this proof and infer from (2.175) that

$$\big\|1_\Lambda(X)\big(1_{<0}(H_L) - f_T(H_L)\big)1_\Lambda(X)\big\|_1 \le \sum_{x\in\Lambda} \big\langle \delta_x, \big|1_{<0}(H_L) - f_T(H_L)\big|\delta_x\big\rangle. \tag{2.181}$$

Since $|1_{<0} - f_T| \le \mathrm{e}^{-|\cdot|/T}$, we obtain for all $x \in \Lambda$

$$\big\langle \delta_x, \big|1_{<0}(H_L) - f_T(H_L)\big|\delta_x\big\rangle \le \big\langle \delta_x, 1_{\mathcal{W}_L}(H_L)\mathrm{e}^{-|H_L|/T}\delta_x\big\rangle + \mathrm{e}^{-L^{-1/2-\alpha}/T}$$
$$= \sum_{j=J_<}^{J_\ge} |\langle \delta_x, \psi_{L,E_j}\rangle|^2 \mathrm{e}^{-|E_j|/T} + \mathrm{e}^{-L^{-1/2-\alpha}/T}, \tag{2.182}$$

where $J_<$ and $J_\ge$ were defined in (2.98). Theorem 2.2.1 implies $|\psi_{L,E_j}(x)|^2 \le C/L$ for all $j \in \{J_<, \dots, J_\ge\}$ and $C/L \le (C^4/\pi)|E_j - E_{j\pm1}|$ for all $j \in \{J_<, \dots, -2\}$, respectively $j \in \{1, \dots, J_\ge\}$. This yields the following upper bound for the sum in (2.182)

$$\frac{C}{L}\sum_{j=J_<}^{J_\ge} \mathrm{e}^{-|E_j|/T} \le \frac{C^4}{\pi}\sum_{j=J_<}^{-2} |E_j - E_{j+1}|\mathrm{e}^{-|E_j|/T} + \frac{C^4}{\pi}\sum_{j=1}^{J_\ge} |E_j - E_{j-1}|\mathrm{e}^{-|E_j|/T}$$
$$+ \frac{C}{L}\big(\mathrm{e}^{-|E_{-1}|/T} + \mathrm{e}^{-|E_0|/T}\big)$$
$$\le \frac{C^4}{\pi}\int_{-L^{-1/2-\alpha}}^{L^{-1/2-\alpha}} \mathrm{d}E\,\mathrm{e}^{-|E|/T} + \frac{2C}{L} \le \frac{2C^4 T}{\pi} + \frac{2C}{L}. \tag{2.183}$$

Therefore, we conclude

$$\mathbb{E}\Big[\big\|1_\Lambda(X)\big(1_{<0}(H_L) - f_T(H_L)\big)1_\Lambda(X)\big\|_1\Big]$$
$$\le |\Lambda|\Big[C^4 T + \frac{2C}{L} + \mathrm{e}^{-L^{-1/2-\alpha}/T} + \mathbb{P}\big(\Omega_L(\alpha)\big)\Big] \tag{2.184}$$

and deduce the claim with (2.16). □

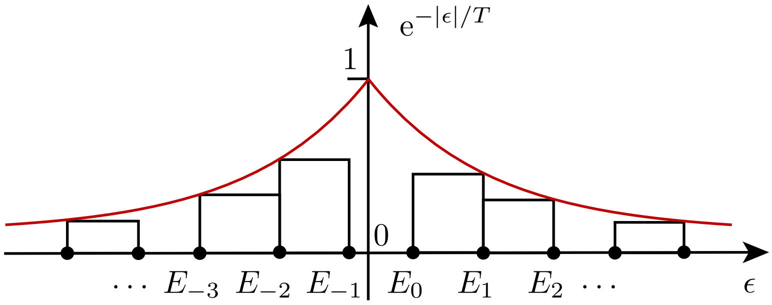

Figure 2.6: The Riemann sum of (2.183).

Proof of Lemma 2.4.1. We combine (2.158), (2.160), the triangle inequality and Lemmata 2.4.2 – 2.4.4. Furthermore, $|\Lambda| e^{-cL^{\alpha/2}} \leq 2L e^{-cL^{\alpha/2}} \leq 1 \leq C$ for all $L \geq L_1$, where the minimal length $L_1 \equiv L_1(\alpha, v, p_+)$ depends only on α and on the model parameters (but not on Λ). We set $L_0 \equiv L_0(\alpha, v, p_+) := \max\{\tilde{L}_0, L_{\min}, L_1\}$. □

2.4.3 Proof of Theorem 2.1.1

Before we turn to the proof of the main theorem, we need another perturbation result.

Lemma 2.4.5. *Let $\Lambda, \Lambda' \subseteq \Gamma_L$ and $E_F \in \mathbb{R}$. Then*

$$|Q_{E_F}(\Lambda; \Gamma_L, H_L^\omega) - Q_{E_F}(\Lambda'; \Gamma_L, H_L^\omega)| \leq 4r \quad \textit{for all } \omega \in \Omega, \tag{2.185}$$

with $r := |\Lambda \triangle \Lambda'|$, where $A \triangle B$ denotes the symmetric difference of two sets A and B.

Proof. We use the abbreviation $P := 1_{<E_F}(H_L^\omega)$. The operators $1_{\Lambda^{(\prime)}}(X) P 1_{\Lambda^{(\prime)}}(X)$ and $P 1_{\Lambda^{(\prime)}}(X) P$ share the same non-zero singular values. This and $g(0) = 0$ implies that the left-hand side of (2.185) equals

$$\begin{aligned} \left|\mathrm{tr}\left\{g\big(P1_\Lambda(X)P\big) - g\big(P1_{\Lambda'}(X)P\big)\right\}\right| &\leq 4\|P\big(1_\Lambda(X) - 1_{\Lambda'}(X)\big)P\|_1 \\ &\leq 4\|1_\Lambda(X) - 1_{\Lambda'}(X)\|_1 = 4r, \end{aligned} \tag{2.186}$$

where, in order to deduce the first inequality, we argued with Kreĭn's trace formula as in (2.158) and (2.160) but with the operators $P1_\Lambda(X)P$ and $P1_{\Lambda'}(X)P$. □

Lemma 2.4.6. *Let $\gamma \in \,]0, 1/2[$, $L' \in \mathbb{N}$ and $\Lambda_{L'} := \{1, \cdots, L'\}$. Then there exists $L \in \mathbb{N}$ such that*

$$\left|\mathbb{E}\left[Q_{E_c}(\Lambda_{L'}; \mathbb{Z}, H) - Q_{E_c}(\Lambda_L^{\gamma,\gamma^2}; \mathbb{Z}, H)\right]\right| \leq 8. \tag{2.187}$$

Proof. Let us first show, that for each $L' \in \mathbb{N}$ there exists a $L \in \mathbb{N}$ with $|\Lambda_{L'}| = |\Lambda_L^{\gamma,\gamma^2}|$. For any $\ell \in \mathbb{N}$ let

$$\gamma_\ell := |\Lambda_\ell^{\gamma,\gamma^2}| = \lfloor(\gamma + \gamma^2)\ell\rfloor - \lfloor\gamma\ell\rfloor. \tag{2.188}$$

As $\gamma < 1$ and $\gamma + \gamma^2 < 1$, we infer $\lfloor(\gamma+\gamma^2)(\ell+1)\rfloor - \lfloor(\gamma+\gamma^2)\ell\rfloor \in \{0, 1\}$ and $\lfloor\gamma(L+1)\rfloor - \lfloor\gamma L\rfloor \in \{0, 1\}$ for all $\ell \in \mathbb{N}$. Thus, we have $\gamma_{\ell+1} - \gamma_\ell \in \{-1, 0, 1\}$ for all $\ell \in \mathbb{N}$. Together with $\gamma_1 = 0$ and $\lim_{\ell\to\infty} \gamma_\ell = \infty$, this implies that there exists at least one $L \in \mathbb{N}$ with $\gamma_L = L'$.

Since H^ω is $2\mathbb{Z}$-ergodic, we have

$$\mathbb{E}\big[Q_{E_c}(A_{L'};\mathbb{Z},H)\big] = \mathbb{E}\big[Q_{E_c}(\Lambda_L^{\gamma,\gamma^2};\mathbb{Z},H)\big] \tag{2.189}$$

for either $A_{L'} = \Lambda_{L'}$ or $A_{L'} = -1+\Lambda_{L'}$, since we can shift the left border of $\Lambda_L^{\gamma,\gamma^2}$ to either 1 or 0. Since the cardinality of the symmetric difference between $\Lambda_{L'}$ and $-1+\Lambda_{L'}$ is equal to two, the claim is a consequence of Lemma 2.4.5. □

Proof of Theorem 2.1.1. We fix $\alpha := 1/6$ and $\gamma \in\,]0, 2^{-27}[$. The goal is to apply Lemma 2.4.1 with $\Lambda = \Lambda_L^{\gamma,\gamma^2}$, where $d_L \equiv d_L(\Lambda_L^{\gamma,\gamma^2}) = \lfloor \gamma L\rfloor$ and $|\Lambda_L^{\gamma,\gamma^2}| = \lfloor(\gamma+\gamma^2)L\rfloor - \lfloor\gamma L\rfloor$.

First, we have to replace the box $\Lambda_L = \{1,\dots,L\}$ by the differently positioned box $\Lambda_L^{\gamma,\gamma^2}$ according to

$$\begin{aligned}\liminf_{L\to\infty} \frac{\mathbb{E}\big[S_{E_F}(\Lambda_L;\mathbb{Z},H)\big]}{\ln L} &\geq \liminf_{L\to\infty} \frac{\mathbb{E}\big[Q_{E_F}(\Lambda_L;\mathbb{Z},H)\big]}{\ln L}\\ &= \liminf_{L\to\infty} \frac{\mathbb{E}\big[Q_{E_F}(\Lambda_L^{\gamma,\gamma^2};\mathbb{Z},H)\big]}{\ln L}.\end{aligned} \tag{2.190}$$

The equality in (2.190) follows from Lemma 2.4.6 and $\lim_{L\to\infty} \ln L/\ln|\Lambda_L^{\gamma,\gamma^2}| = 1$.

Introducing the abbreviation $\mathcal{E}_L := \mathbb{E}\big[|Q_{E_F}(\Lambda_L^{\gamma,\gamma^2};\Gamma_L,H_L) - Q_{E_F}(\Lambda_L^{\gamma,\gamma^2};\mathbb{Z},H)|\big]$, the estimate (2.190) implies

$$\begin{aligned}\liminf_{L\to\infty} \frac{\mathbb{E}\big[S_{E_F}(\Lambda_L;\mathbb{Z},H)\big]}{\ln L} &\geq \liminf_{L\to\infty} \frac{\mathbb{E}\big[Q_{E_F}(\Lambda_L^{\gamma,\gamma^2};\Gamma_L,H_L)\big]}{\ln L} - \limsup_{L\to\infty}\frac{\mathcal{E}_L}{\ln L}\\ &\geq 2^{-15} - \limsup_{L\to\infty}\frac{\mathcal{E}_L}{\ln L},\end{aligned} \tag{2.191}$$

where we used Corollary 2.3.11 in the last step, assuming that $v \in\,]0, v_0]$.

Now, we estimate the error $\mathcal{E}_L$ with Lemma 2.4.1. For that purpose we choose the temperature as $T_L := (K\ln L)/L$ with

$$K \equiv K(\gamma) := 25/\gamma. \tag{2.192}$$

We find for the residual term that

$$\lim_{L\to\infty} R_L\big(\Lambda_L^{\gamma,\gamma^2}, T_L\big) = 2^4 C. \tag{2.193}$$

Furthermore the choice of K in (2.192) implies that

$$\lim_{L\to\infty} |\Lambda_L^{\gamma,\gamma^2}|^2 T_L^{-2} \mathrm{e}^{-d_L T_L/6} = 5^{-4}\gamma^2 \lim_{L\to\infty} (\ln L)^{-2} L^{4-Kd_L/(6L)} = 0 \tag{2.194}$$

and therefore

$$\limsup_{L\to\infty} \frac{\mathcal{E}_L}{\ln L} \leq 2^7 C^4 \gamma \tag{2.195}$$

Thus, we deduce from (2.191) that

$$\liminf_{L\to\infty} \frac{\mathbb{E}\big[S_{E_F}(\Lambda_L;\mathbb{Z},H)\big]}{\ln L} \geq 2^{-15} - 2^7 C^4\gamma \geq 2^{-16}. \tag{2.196}$$

To see the validity of the last inequality, we recall from the proof of Corollary 2.3.11 that the restriction $v \leq v_0$ guarantees the bound $C < 1+\gamma^2$. Since $\gamma \in\,]0, 2^{-27}[$, we have $C^4 \leq 2$. This concludes the claim. □

Chapter 3

Stability of the enhanced area law of the entanglement entropy

One of the few models for which an enhanced area law of the entanglement entropy has already been proven is the one describing free fermions without interactions. We are now asking the question, whether this result can be extended to quasi-free fermions moving in a background potential? In this chapter, we prove an upper and a lower bound to the entanglement entropy for such a system with a compactly supported, bounded potential. The work that is presented here is the result of a collaboration with P. Müller. The content was already published in [MS20].

3.1 Introduction and Result

We consider a quasi-free Fermi gas in a background potential in $d \in \mathbb{N}$ dimensions. The model is described by the one-particle Schrödinger operator

$$H := -\Delta + V, \tag{3.1}$$

where $V \in L^\infty(\mathbb{R}^d)$ is a bounded potential and Δ denotes the Laplacian. This operator is densely defined in $L^2(\mathbb{R}^d)$.

Let us first consider the special case of free fermions described by the Hamiltonian $H_0 := -\Delta$. As we have mentioned in the introduction, it was shown in [LSS14] that free fermions satisfy a logarithmic enhancement to the area law of the entanglement entropy. Recall from (1.18) that there even exists an exact formula for the leading asymptotic growth of the entanglement entropy. Our aim is to develop a suitable perturbation theory in order to prove an enhanced area law of the entanglement entropy for certain background potentials, too.

It has been conjectured that for a general $V \in L^\infty(\mathbb{R}^d)$ any enhancement of the area law of the entanglement entropy, if it occurs at all, should not grow faster than logarithmic. As a first step towards this conjecture, we consider potentials with compact support here. The operator H is a perturbation of H_0 by the relative H_0-compact multiplication operator V [Sch12, Thm. 8.19]. It therefore has some properties in common with the unperturbed case. First and foremost, the absolutely continuous spectrum of H is identical to $\sigma_{ac}(H_0) = [0,\infty[$ [RS78, Thm. XIII.15, XIII.33 and XIII.58]. Since an absolutely continuous spectrum amounts to spectral delocalisation, we expect a logarithmically enhanced area law for the entanglement entropy. Another similarity between H and H_0 is that they both satisfy a limiting absorption principle [Agm75, JM17], which will be the main technical input in our analysis.

In our main theorem we will consider the entanglement entropy with respect to the scaled version $\Lambda_L := L\cdot\Lambda$ of a bounded subset $\Lambda\subset\mathbb{R}^d$. Before we state our main theorem let us first specify the assumptions on this subset. We require the following definition first.

Definition 3.1.1. *Let $d\in\mathbb{N}$ with $d>1$. A set $\Lambda\subset\mathbb{R}^d$ is called a* basic Lipschitz domain *if there exists a Lipschitz function $\Phi:\ \mathbb{R}^{d-1}\to\mathbb{R}$, such that with a suitable choice of Cartesian coordinates the domain Λ is represented as*

$$\Lambda = \{x = (x_1,\cdots,x_d)\in\mathbb{R}:\ x_d > \Phi(x_1,\cdots,x_{d-1})\}. \tag{3.2}$$

A set $\Lambda\subset\mathbb{R}^d$ is called a Lipschitz domain *if $\Lambda\neq\mathbb{R}^d$ and locally it can be represented by a basic Lipschitz domain, i.e. for any $z\in\Lambda$ there is a radius $r>0$ and a basic Lipschitz domain $\Lambda_0\equiv\Lambda_0(z)$ such that $B_z(r)\cap\Lambda = B_z(r)\cap\Lambda_0$.*

Assumption 3.1.2. *We consider a bounded Borel set $\Lambda\subset\mathbb{R}^d$ such that*

(i) it is a finite union of bounded intervals for $d=1$ or a Lipschitz domain with piecewise C^1-boundary for $d\geq 2$,

(ii) the origin $0\in\mathbb{R}^d$ is an interior point of Λ.

Remark 3.1.3. *Assumption 3.1.2(i) is taken from [LSS14] and guarantees the validity of the enhanced area law* (1.18) *for the free Fermi gas which is proven there. Assumption 3.1.2(ii) does not impose any restriction because it can always be achieved by a translation of the potential V in Theorem 3.1.4.*

The main result of this section is summarised in the following theorem.

Theorem 3.1.4. *Let $\Lambda\subset\mathbb{R}^d$ be as in Assumption 3.1.2 and let $V\in L^\infty(\mathbb{R}^d)$ have compact support. Then, for every Fermi energy $E>0$ there exist constants $\Sigma_l\equiv\Sigma_l(\Lambda,E)\in\,]0,\infty[$ and $\Sigma_u\equiv\Sigma_u(\Lambda,E,V)\in\,]0,\infty[$ such that*

$$\Sigma_l \leq \liminf_{L\to\infty}\frac{S_E(\Lambda_L;\mathbb{R}^d,H)}{L^{d-1}\ln L} \leq \limsup_{L\to\infty}\frac{S_E(\Lambda_L;\mathbb{R}^d,H)}{L^{d-1}\ln L} \leq \Sigma_u. \tag{3.3}$$

Remark 3.1.5. *(i) The constant Σ_l can be expressed in terms of the coefficient $\Sigma_0\equiv\Sigma_0(d,\Lambda,E)$ given in* (3.9) *in the leading term of the unperturbed entanglement entropy $S_E(\Lambda_L;\mathbb{R}^d,H_0)$ for large L, cf.* (1.18)*. The explicit form*

$$\Sigma_l = \frac{3\Sigma_0}{2\pi^2}, \tag{3.4}$$

is derived in (3.86).

(ii) *If $d > 1$, the constant Σ_u can also be expressed in terms of Σ_0. According to* (3.80) *and* (3.84)*, we have*

$$\Sigma_u = 2508\Sigma_0. \tag{3.5}$$

In particular, this constant is independent of V. The numerical prefactor in (3.5) *can be improved by using the alternative approach described in Remark 3.2.8. In $d = 1$ dimension, however, we only obtain a constant Σ_u which also depends on V, because there is an additional contribution from* (3.84)*.*

(iii) *Since both Σ_l and Σ_u do not depend on V in case that $d > 1$, we conjecture that the result can be improved to $\Sigma_l = \Sigma_u = \Sigma_0$.*

(iv) *B. Pfirsch and A. Sobolev [PS18b] proved that the coefficient of the leading-order term of the enhanced area law is not altered by adding a periodic potential in $d = 1$. Therefore, we expect the V-dependence of Σ_u in $d = 1$ to be an artefact of our method.*

(v) *At negative energies there is at most discrete spectrum of H. Thus, if $E < 0$ the Fermi function can be smoothed out without changing the operator $1_{<E}(H)$. Therefore, the operator kernel of $1_{<E}(H)$ has fast polynomial decay [GK03] and $S_E(\Lambda_L; \mathbb{R}^d, H) = \mathcal{O}(L^{d-1})$ follows as in [PS14, EPS17]. In other words, the growth of the entanglement entropy is at most an area law. The same is true for $E = 0$ because eigenvalues cannot accumulate from below at* 0 *due to the boundedness of V and its compact support.*

(vi) *The stability analysis we perform in this paper requires only that the spatial domain Λ is a bounded measurable subset of $\mathbb{R}^d$ which has an interior point. The stronger assumptions we make are to ensure the validity of Widom's formula for the unperturbed system as proven in [LSS14].*

3.2 Proof of Theorem 3.1.4

3.2.1 General idea and strategy

Our strategy is a perturbational approach, which bounds the entanglement entropy of H in terms of the one of H_0 for large volumes. Before we sketch our strategy for our proof, we revisit the main result by H. Leschke, A. Sobolev and W. Spitzer concerning the entanglement entropy of free fermions.

Theorem 3.2.1 (Leschke, Sobolev, Spitzer [LSS14])**.** *Let $E > 0$ and $\Lambda \subset \mathbb{R}^d$ be a set satisfying Assumption 3.1.2 (i). Let $f : [0,1] \to \mathbb{R}$ with $f|_{]0,1[} \in C^\infty(]0,1[, \mathbb{C})$ such that there exists $\beta \in]0,1[$ with $|f(x)| \leq (x(1-x))^\beta$ for all $x \in [0,1]$. Then*

$$\operatorname{tr}\{f(1_{\Lambda_L} 1_{<E}(H_0) 1_{\Lambda_L})\} = I(f) J(\partial\Lambda, E) L^{d-1} \ln L + o(L^{d-1} \ln L) \tag{3.6}$$

as $L \to \infty$. Here,

$$I(f) := \frac{1}{4\pi^2} \int_0^1 dt \, \frac{f(t)}{t(1-t)} \tag{3.7}$$

and

$$J(\partial\Lambda, E) := \frac{2}{\Gamma[(d+1)/2]}\left(\frac{E}{4\pi}\right)^{\frac{d-1}{2}} |\partial\Lambda|, \tag{3.8}$$

where $|\partial\Lambda|$ *denotes the surface area of the boundary of* Λ *and* Γ *denotes the gamma function.*

Remark 3.2.2. *We note that by this formula we can easily deduce the value of* Σ_0 *from* (1.18) *as*

$$\Sigma_0 \equiv \Sigma_0(d, \Lambda, E) := \frac{E^{(d-1)/2}}{2^d 3 \ln 2\, \Gamma[(d+1)/2]\pi^{(d-1)/2}}|\partial\Lambda|. \tag{3.9}$$

Our first step towards proving Theorem 3.1.4 is an estimate of the function h in (1.12) of the form

$$g \le h \le -3g\log_2 g, \tag{3.10}$$

where

$$g : [0,1] \to [0,1],\ \lambda \mapsto \lambda(1-\lambda). \tag{3.11}$$

See Lemma B.2.1 for a proof of the lower bound in (3.10) and Lemma B.2.2 for a proof of the upper bound. Note that both g and $g\log_2 g$ satisfy the conditions of the function f in Theorem 3.2.1. Therefore, these functions give rise to bounds for the entanglement entropy of order $\mathcal{O}(L^{d-1}\ln L)$ in the case of free fermions.

The estimate (3.10) leads us to consider the operators $g\big(1_{\Lambda_L}1_{<E}(H_{(0)})1_{\Lambda_L}\big)$. By a straight forward calculation we get

$$g\big(1_{\Lambda_L}1_{<E}(H_{(0)})1_{\Lambda_L}\big) = \big|1_{\Lambda_L^c}1_{<E}(H_{(0)})1_{\Lambda_L}\big|^2, \tag{3.12}$$

where $|A|^2 := A^*A$ for any bounded operator A, and the superscript ^{c} indicates the complement of a set. Recall that $\Lambda_L = L\cdot\Lambda$ for a set $\Lambda \subseteq \mathbb{R}^d$ satisfying Assumption 3.1.2 and $L \in \mathbb{R}$. Now, the key to our perturbative approach is an estimate of the Hilbert–Schmidt norm of the operator difference $1_{\Lambda_L^c}[1_{<E}(H_0) - 1_{<E}(H)]1_{\Lambda_L}$. This result is summarised in the following lemma.

Lemma 3.2.3. *Let* $\Lambda \subset \mathbb{R}^d$ *satisfy Assumption 3.1.2 (ii) and let* $V \in L^\infty(\mathbb{R}^d)$ *have compact support in* $[-R_V, R_V]^d$ *for some* $R_V > 0$*. Then for every Fermi energy* $E > 0$ *there exists a constant* $C_2 \equiv C_2(d, \Lambda, V, E) > 0$ *such that for all* $L > 0$ *we have the bound*

$$\big\|1_{\Lambda_L^c}\big[1_{<E}(H_0) - 1_{<E}(H)\big]1_{\Lambda_L}\big\|_2 \le C_2. \tag{3.13}$$

Here, $\|\cdot\|_p$ *denotes the von Neumann–Schatten norm for* $p \in [1,\infty[$.

Before we outline the proof of this lemma, let us first illustrate on the example of the lower bound in Theorem 3.1.4 why such an estimate is useful. The lower bound in (3.10) and (3.12) imply that

$$\big\|1_{\Lambda_L^c}1_{<E}(H_{(0)})1_{\Lambda_L}\big\|_2^2 = \operatorname{tr}\big\{g\big(1_{\Lambda_L}1_{<E}(H_{(0)})1_{\Lambda_L}\big)\big\} \le S_E(\Lambda_L;\mathbb{R}^d, H_{(0)}). \tag{3.14}$$

We have established already that the left-hand side of (3.14) in the unperturbed case of free fermions grows like $L^{d-1}\ln L$ as $L \to \infty$. Since the Hilbert-Schmidt norm of the operator difference in Lemma 3.2.3 is of order $\mathcal{O}(1)$, it follows immediately that the lower bound to the entanglement entropy in the perturbed case has the same growth. For the upper bound in Theorem 3.1.4 some further analysis is necessary, since we have to account for the additional factor of $\log_2 g$ in the upper bound in (3.10). Especially in the case of $d = 1$ this leads to further complications, which we will discuss in Remark 3.2.8. However, the upper bound is ultimately derived from Lemma 3.2.3, too.

We conclude by sketching the main idea for the proof of Lemma 3.2.3, which is presented in Section 3.2.2. An important technical input to our approach is that both H_0 and H satisfy a limiting absorption principle, in the sense that for any $E > 0$ there exists a constant $C_{LA} \equiv C_{LA}(d, V, E) > 0$ such that

$$\sup_{\substack{z \in \mathbb{C}:\, \mathrm{Re}\, z = E, \\ \mathrm{Im}\, z \neq 0}} \left\| \langle X \rangle^{-1} \frac{1}{H_{(0)} - z} \Pi_c(H_{(0)}) \langle X \rangle^{-1} \right\| \leq C_{LA}, \tag{3.15}$$

where X denotes the position operator, $\langle \cdot \rangle := \sqrt{1 + |\cdot|^2}$ the Japanese bracket and $\Pi_c(H_{(0)})$ the projection onto the continuous spectral subspace of $H_{(0)}$. Such a limiting absorption principle exists for any Schrödinger operator with compactly supported, bounded potentials [Agm75, Thm. 4.2], see also e.g. [JM17].

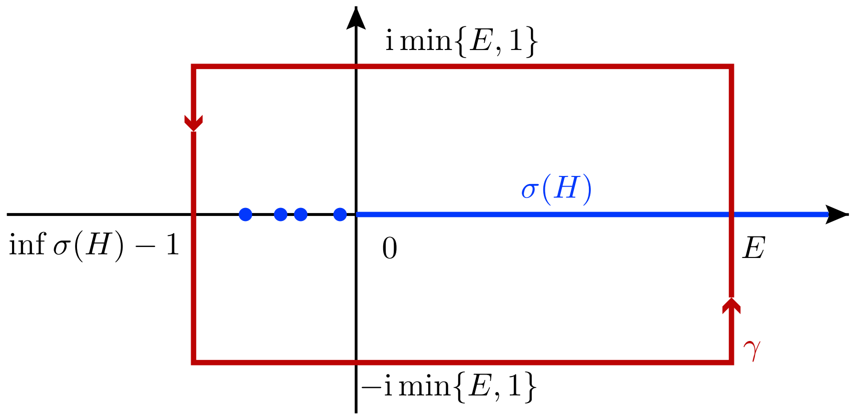

Figure 3.1: The contour γ

Let us now introduce the notation $\Gamma_n := n + [0,1]^d$ for the closed unit cube translated by $n \in \mathbb{Z}^d$. For this proof we need an estimate of decay in space of the Fermi projection difference $1_{<E}(H_0) - 1_{<E}(H)$. We begin by representing the Fermi projection in terms of a contour integral for a contour γ as in Figure 3.1, namely

$$1_{\Gamma_n}\big(1_{<E}(H_0) - 1_{<E}(H)\big)1_{\Gamma_m} = -\frac{1}{2\pi \mathrm{i}} \oint_\gamma \mathrm{d}z \; 1_{\Gamma_n} \left(\frac{1}{H_0 - z} - \frac{1}{H - z} \right) 1_{\Gamma_m} \tag{3.16}$$

for any $m, n \in \mathbb{Z}^d$. Here, the right-hand side of (3.16) exists as a Bochner integral with respect to the operator norm. Note that the limiting absorption principle of $H_{(0)}$ ensures the integrability of the integrand as well as the equality in (3.16). A more general version of this equality is proven in the Appendix B.1. The limiting absorption principle of H together with the resolvent equality also enable us to estimate the Hilbert–Schmidt norm of the integrand. We show that

$$\left\| 1_{\Gamma_n}\big(1_{<E}(H_0) - 1_{<E}(H)\big)1_{\Gamma_m}\right\|_2^2 \lesssim \frac{1}{(|n||m|)^{(d-1)}(|n|+|m|)^2} \tag{3.17}$$

for any $n, m \in \mathbb{Z}^d$ at a sufficiently large distance to $\operatorname{supp} V$. The only thing that remains to be done is to sum over all $n, m \in \mathbb{Z}^d$ with $\Lambda_L^c \cap \Gamma_n \neq \varnothing$ and $\Lambda_L \cap \Gamma_m \neq \varnothing$ for any $L > 0$. Note that the decay in (3.17) is sufficient to guarantee an upper bound independent of L.

3.2.2 Proof of Lemma 3.2.3

In order to show this crucial lemma, we need another preparatory result, regarding the decay in space of the resolvent of H_0. For $z \in \mathbb{C}\backslash\mathbb{R}$ let $G_0(\,\cdot\,,\,\cdot\,;z) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be the kernel of the resolvent $\frac{1}{H_0 - z}$. The explicit formula for $G_0(\,\cdot\,,\,\cdot\,;z)$ is well known. Likewise, there exists an estimate for $G_0(\cdot,\cdot;z)$ evaluated for large arguments, i.e. there exists $R \equiv R(d) > 0$ and $C \equiv C(d) > 0$ such that for all $x, y \in \mathbb{R}^d$ with Euclidean distance $|x-y| \geq R/|z|^{1/2}$ we have

$$|G_0(x,y;z)| \leq C|z|^{(d-3)/4}\,\frac{\mathrm{e}^{-|\operatorname{Im}\sqrt{z}||x-y|}}{|x-y|^{(d-1)/2}}. \tag{3.18}$$

For a reference, see [ST70] and [AS64, Chap. 9.2] for $d \geq 2$ and [AGHKH88, Chap. I.3.1] for $d = 1$. Here, $\sqrt{\cdot}$ denotes the principal branch of the square root.

Recall that $\Gamma_l = l + [0,1]^d$ for any $l \in \mathbb{Z}^d$.

Lemma 3.2.4. *Let $d \in \mathbb{N}$ and $V \in L_c^\infty(\mathbb{R}^d)$ with compact support in $[-R_V, R_V]^d$ for some $R_V > 0$. Given $z \in \mathbb{C} \smallsetminus \mathbb{R}$, let $\ell_0 \equiv \ell_0(d,V,z) := 2\sqrt{d}(R_V + 1) + R(d)/|z|^{1/2}$.*

Then, there exists a constant $C_1 \equiv C_1(d,V) > 0$ such that for any $z \in \mathbb{C}\smallsetminus\mathbb{R}$ and any $n \in \mathbb{Z}^d \smallsetminus\,]-\ell_0, \ell_0[^{\,d}$ we have

$$\left\| |V|^{1/2}\frac{1}{H_0 - z}1_{\Gamma_n}\right\|_4 \leq C_1|z|^{(d-3)/4}\,\frac{e^{-|\operatorname{Im}\sqrt{z}||n|/2}}{|n|^{(d-1)/2}}. \tag{3.19}$$

Proof. Let $z \in \mathbb{C}\smallsetminus\mathbb{R}$. Since the Hilbert–Schmidt norm of an operator can be computed in terms of the integral kernel, we get

$$\begin{aligned}\left\| |V|^{1/2}\frac{1}{H_0 - z}1_{\Gamma_n}\right\|_4^4 &= \left\| 1_{\Gamma_n}\frac{1}{H_0 - \overline{z}}\,|V|\,\frac{1}{H_0 - z}1_{\Gamma_n}\right\|_2^2 \\ &= \int_{\Gamma_n}\mathrm{d}x\int_{\Gamma_n}\mathrm{d}y\left|\int_{\mathbb{R}^d}\mathrm{d}\xi\; G_0(x,\xi;\overline{z})\,|V(\xi)|\,G_0(\xi,y;z)\right|^2. \end{aligned} \tag{3.20}$$

For every $n \in \mathbb{Z}^d \smallsetminus\,]-\ell_0, \ell_0[^{\,d}$, every $x \in \Gamma_n$ and every $\xi \in \operatorname{supp}V$, we infer that $|x-\xi| \geq R(d)/|z|^{1/2}$. Therefore the Green's-function estimate (3.18) yields

$$|G_0(x,\xi;z)| \leq 2^{(d-1)/2}C(d)|z|^{(d-3)/4}\,\frac{\mathrm{e}^{-|\operatorname{Im}\sqrt{z}||n|/2}}{|n|^{(d-1)/2}} \tag{3.21}$$

because

$$|x-\xi| \geq |x| - \sqrt{d}R_V \geq |n| - \sqrt{d}(R_V+1) \geq \frac{|n|}{2}. \tag{3.22}$$

This implies the lemma. □

Next, we show that the Fermi projection has a representation as a Riesz projection, which is a consequence of the limiting absorption principle. In the Appendix B.1, we show a more general version of this result, since it may be of independent interest.

Lemma 3.2.5. *Let $V \in L_c^\infty(\mathbb{R}^d)$. We fix an energy $E > 0$ and consider two compact subsets $\Gamma, \Gamma' \subset \mathbb{R}^d$. Then we have the representation*

$$1_\Gamma 1_{<E}\big(H_{(0)}\big) 1_{\Gamma'} = -\frac{1}{2\pi i} \oint_\gamma dz\, 1_\Gamma \frac{1}{H_{(0)} - z} 1_{\Gamma'}. \tag{3.23}$$

The right-hand side of (3.23) *exists as a Bochner integral with respect to the operator norm, and the integration contour γ is a closed curve in the complex plane $\mathbb{C}$ which traces the boundary of the rectangle $\big\{z \in \mathbb{C} : |\operatorname{Im} z| \leq \min\{E, 1\},\ \operatorname{Re} z \in [-1 + \inf \sigma(H), E]\big\}$ once in counter-clockwise direction, see Figure 3.1.*

Proof. The lemma follows from the corresponding abstract result in Theorem B.1.1 in the appendix. Indeed, according to [Agm75, Thm. 4.2], see also e.g. [JM17], both H and H_0 satisfy a limiting absorption principle at any $E > 0$,

$$\sup_{z \in \mathbb{C}:\, \operatorname{Re} z = E,\, \operatorname{Im} z \neq 0} \Big\| \langle X \rangle^{-1} \frac{1}{H_{(0)} - z} \Pi_c\big(H_{(0)}\big) \langle X \rangle^{-1} \Big\| < \infty \tag{3.24}$$

with X being the position operator, $\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}$ the Japanese bracket and $\Pi_c\big(H_{(0)}\big)$ the projection onto the continuous spectral subspace of $H_{(0)}$. Also, $\sigma_{pp}(H) \subset\,]-\infty, 0]$ because the potential V is bounded and compactly supported [RS78, Cor. on p. 230]. □

Next, we prove the estimate (3.17).

Lemma 3.2.6. *Let $V \in L_c^\infty(\mathbb{R}^d)$ have compact support in $[-R_V, R_V]^d$ for some $R_V > 0$. Then for every Fermi energy $E > 0$ there exists $\tilde{c} \equiv \tilde{c}(d, V, E) > 0$ and $\ell_1 \equiv \ell_1(d, V, E) > 2\sqrt{d}$ such that for all $n, m \in \mathbb{Z}^d \smallsetminus\,]-\ell_1, \ell_1[^d$ we have the bound*

$$\big\| 1_{\Gamma_n} \big(1_{<E}(H_0) - 1_{<E}(H)\big) 1_{\Gamma_m} \big\|_2 \leq \frac{\tilde{c}}{(|n||m|)^{(d-1)/2}(|n| + |m|)} \tag{3.25}$$

Proof. We fix $E > 0$. To estimate the difference between the perturbed and the unperturbed Fermi projections we express them in terms of a contour integral as stated in Lemma 3.2.5. We set

$$\ell_1 \equiv \ell_1(d, V, E) := \max_{z \in \operatorname{img}(\gamma)} \big\{\ell_0(d, V, z)\big\} \in\,]2\sqrt{d}, \infty[, \tag{3.26}$$

where ℓ_0 is defined in Lemma 3.2.4 and $\operatorname{img}(\gamma)$ denotes the image of the curve γ in Lemma 3.2.5. We obtain for all $m, n \in \mathbb{Z}^d \setminus]-\ell_1, \ell_1[^{\,d}$

$$1_{\Gamma_n}\big(1_{<E}(H_0) - 1_{<E}(H)\big)1_{\Gamma_m} = -\frac{1}{2\pi\mathrm{i}} \oint_\gamma \mathrm{d}z\, 1_{\Gamma_n}\Big(\frac{1}{H_0 - z} - \frac{1}{H - z}\Big)1_{\Gamma_m}. \tag{3.27}$$

The Bochner integral exists even with respect to the Hilbert–Schmidt norm, as will follow from the estimates (3.31) and (3.37) below. We point out that (3.37) relies again on the limiting absorption principle (3.24).

In order to estimate the integral in (3.27) we apply the resolvent identity twice to the integrand. The integrand then reads

$$1_{\Gamma_n}\Big(\frac{1}{H_0 - z}V\frac{1}{H_0 - z} - \frac{1}{H_0 - z}V\frac{1}{H - z}V\frac{1}{H_0 - z}\Big)1_{\Gamma_m}. \tag{3.28}$$

This yields the Hilbert–Schmidt norm estimate

$$\begin{aligned}\Big\|1_{\Gamma_n}\Big(&\frac{1}{H_0 - z} - \frac{1}{H - z}\Big)1_{\Gamma_m}\Big\|_2 \\ &\le \Big\|1_{\Gamma_n}\frac{1}{H_0 - z}|V|^{1/2}\Big\|_4\Big(1 + \Big\||V|^{1/2}\frac{1}{H - z}|V|^{1/2}\Big\|\Big)\Big\||V|^{1/2}\frac{1}{H_0 - z}1_{\Gamma_m}\Big\|_4.\end{aligned} \tag{3.29}$$

Lemma 3.2.4 already provides bounds for the first and third factor on the right-hand side of (3.29). To estimate the second factor, we employ two different methods, depending on the location of z on the contour. To that end we split the curve γ into two parts. We denote by γ_1 the right vertical part of γ with image $\operatorname{img}(\gamma_1) = \big\{z \in \mathbb{C} : \operatorname{Re} z = E, |\operatorname{Im} z| \le \min\{E, 1\}\big\}$. The remaining part of the curve γ is denoted by γ_2.

Let us first consider the curve γ_2. We observe

$$\operatorname{dist}\big(z, \sigma(H_{(0)})\big) \ge \min\{1, E\} \quad \text{for all } z \in \operatorname{img}(\gamma_2). \tag{3.30}$$

Therefore, the middle factor in the second line of (3.29) is bounded from above by $(1 + \|V\|_\infty/\min\{1, E\})$. Since the curve γ_2 does not intersect $[0, \infty[$, there exists $\zeta_2 \equiv \zeta_2(V, E) > 0$ such that $|\operatorname{Im}\sqrt{z}|/2 \ge \zeta_2$ for all $z \in \operatorname{img}(\gamma_2) \setminus \mathbb{R}$. Hence, according to Lemma 3.2.4 we estimate (3.29) by

$$\Big\|1_{\Gamma_n}\Big(\frac{1}{H_0 - z} - \frac{1}{H - z}\Big)1_{\Gamma_m}\Big\|_2 \le \frac{c_2\, \mathrm{e}^{-\zeta_2(|n|+|m|)}}{(|n||m|)^{(d-1)/2}} \le \frac{c_2/\zeta_2}{(|n||m|)^{(d-1)/2}(|n| + |m|)} \tag{3.31}$$

for all $z \in \operatorname{img}(\gamma_2) \setminus \mathbb{R}$ with

$$c_2 \equiv c_2(d, V, E) := C_1^2\Big(\max_{z \in \operatorname{img}(\gamma_2)} |z|^{(d-3)/2}\Big)\Big(1 + \frac{\|V\|_\infty}{\min\{1, E\}}\Big) < \infty. \tag{3.32}$$

We now turn our attention to γ_1, the part of the contour that intersects the continuous spectrum of H. Writing $1 = \Pi_{pp}(H) + \Pi_c(H)$ and recalling $\sigma_{pp}(H) \subset]-\infty, 0]$, see the end of the proof of Lemma 3.2.5, we infer

$$\Big\||V|^{1/2}\frac{1}{H - z}|V|^{1/2}\Big\| \le \frac{\|V\|_\infty}{E} + \Big\||V|^{1/2}\frac{1}{H - z}\Pi_c(H)|V|^{1/2}\Big\| \tag{3.33}$$

for every $z \in \operatorname{img}(\gamma_1) \setminus \mathbb{R}$. The second term on the right-hand side admits the uniform upper bound

$$\|\langle X\rangle |V|^{1/2}\|^2 \sup_{\substack{z\in\mathbb{C}:\,\operatorname{Re} z=E,\\ \operatorname{Im} z\neq 0}} \Big\| \langle X\rangle^{-1} \frac{1}{H-z} \Pi_c(H) \langle X\rangle^{-1} \Big\| \leq (1+dR_V^2)\|V\|_\infty C_{LA}. \tag{3.34}$$

Here, the constant $C_{LA} \equiv C_{LA}(d,V,E) < \infty$ was given in (3.15) and is derived from the limiting absorption principle.

In addition, we need a lower bound for the decay rate of the exponential in (3.19) along the curve γ_1. We write $\operatorname{img}(\gamma_1) \ni z = E + \mathrm{i}\eta$ with $|\eta| \leq \min\{1, E\}$. Then

$$|\operatorname{Im}\sqrt{z}| = \sqrt[4]{E^2+\eta^2}\, \alpha(|\eta|/E) \geq \sqrt{E}\, \alpha(|\eta|/E), \tag{3.35}$$

with $\alpha : [0,\infty[\, \to [0,1]$, $x \mapsto \sin\big(\frac{1}{2}\arctan x\big)$. We note that $\sin y \geq y(1-y^2/6)$ for all $y \geq 0$, $\arctan x \leq \pi/2$ and $\arctan x \geq x/2$ for all $x \in [0,1]$. Therefore, we infer the existence of a constant $\zeta_1 \equiv \zeta_1(E) > 0$ such that

$$|\operatorname{Im}\sqrt{z}|/2 \geq \zeta_1|\eta| \qquad \text{for all } z = E+\mathrm{i}\eta \in \operatorname{img}(\gamma_1). \tag{3.36}$$

By applying Lemma 3.2.4 together with (3.36), as well as (3.33) and (3.34), we get the estimate

$$\Big\| 1_{\Gamma_n} \Big(\frac{1}{H_0 - z} - \frac{1}{H-z} \Big) 1_{\Gamma_m} \Big\|_2 \leq \frac{c_1 \mathrm{e}^{-\zeta_1|\eta|(|n|+|m|)}}{(|n||m|)^{(d-1)/2}} \tag{3.37}$$

from (3.29) and any $\operatorname{img}(\gamma_1) \ni z = E+\mathrm{i}\eta$ with $|\eta| \leq \min\{1,E\}$. Here, we introduced the constant

$$c_1 \equiv c_1(d,V,E) := C_1^2\Big(\max_{z\in\operatorname{img}(\gamma_1)} |z|^{(d-3)/2}\Big)\Big[1 + \big(E^{-1} + (1+dR_V^2)C_{LA}\big)\|V\|_\infty\Big]. \tag{3.38}$$

We are now able to estimate the contour integral in (3.27) with the help of the bounds (3.31) and (3.37)

$$\begin{aligned} \big\| 1_{\Gamma_n} \big(1_{<E}(H_0) - 1_{<E}(H) \big) 1_{\Gamma_m} \big\|_2 &\leq \frac{\tilde{c}_2}{(|n||m|)^{(d-1)/2}(|n|+|m|)} \\ &\quad + \int_{-1}^{1} \mathrm{d}\eta\, \frac{c_1 \mathrm{e}^{-\zeta_1|\eta|(|n|+|m|)}}{2\pi(|n||m|)^{(d-1)/2}} \\ &= \frac{\tilde{c}}{(|n||m|)^{(d-1)/2}(|n|+|m|)} \end{aligned} \tag{3.39}$$

for all $m,n \in \mathbb{Z}^d \setminus\,]-\ell_1,\ell_1[^{\,d}$, where

$$\tilde{c}_2 \equiv \tilde{c}_2(d,V,E) := \frac{c_2(E+\|V\|_\infty+2)}{\pi\zeta_2} \quad \text{and} \quad \tilde{c} \equiv \tilde{c}(d,V,E) := \frac{c_1}{\pi\zeta_1} + \tilde{c}_2. \tag{3.40}$$

□

Proof of Lemma 3.2.3. In order to prove the lemma for any $L > 0$, we introduce a length $L_0 > 0$, which will be determined below, and first consider the case of $L \in]0, L_0]$. In this case we have

$$\left\| 1_{\Lambda_L^c} \big(1_{<E}(H_0) - 1_{<E}(H) \big) 1_{\Lambda_L} \right\|_2^2 \leq \left\| \big(1_{<E}(H_0) - 1_{<E}(H) \big) 1_{\Lambda_{L_0}} \right\|_2^2. \tag{3.41}$$

Following [Sim82, Thm. B.9.2 and its proof], we infer the existence of a constant $C_S \equiv C_S(d, V, E)$ such that

$$\left\| 1_{<E}\big(H_{(0)}\big) 1_{\Gamma_m} \right\|_1 \leq C_S \tag{3.42}$$

uniformly in $m \in \mathbb{Z}^d$. By applying the binomial inequality $(a+b)^2 \leq 2a^2 + 2b^2$ for $a, b \in \mathbb{R}$ and the inequality $\|A\|_2^2 \leq \|A\|_1$ for any trace-class operator A with $\|A\| \leq 1$, we estimate the right-hand side of (3.41) by

$$\begin{aligned} 2\Big(& \left\| 1_{<E}(H_0) 1_{\Lambda_{L_0}} \right\|_2^2 + \left\| 1_{<E}(H) 1_{\Lambda_{L_0}} \right\|_2^2 \Big) \\ & \leq \sum_{m \in \Xi_{L_0}} 2\Big(\left\| 1_{<E}(H_0) 1_{\Gamma_m} \right\|_1 + \left\| 1_{<E}(H) 1_{\Gamma_m} \right\|_1 \Big) \leq 4 C_S |\tilde{\Lambda}_{L_0}| < \infty, \end{aligned} \tag{3.43}$$

where we introduce the *coarse-grained box volumes*

$$\tilde{\Lambda}_\ell^{(\mathrm{ext})} := \bigcup_{m \in \Xi_\ell^{(\mathrm{ext})}} \Gamma_m \quad \text{with} \quad \Xi_\ell^{(\mathrm{ext})} := \left\{ m \in \mathbb{Z}^d : \Gamma_m \cap \Lambda_\ell^{(c)} \neq \varnothing \right\} \tag{3.44}$$

and $\ell > 0$. The sets $\Xi_\ell^{(\mathrm{ext})}$ are illustrated by Figure 3.2. We note that $\tilde{\Lambda}_\ell^{\mathrm{ext}}$ is not the complement of $\tilde{\Lambda}_\ell$. It will be needed below.

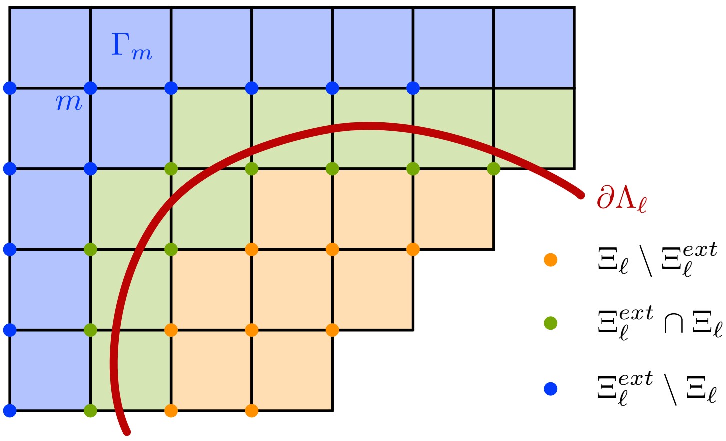

Figure 3.2: Examples for elements of $\Xi_\ell^{(ext)}$.

In order to tackle the other case of $L > L_0$ we first determine a suitable value for L_0 as follows: we recall that the origin is an interior point of the bounded domain Λ. Hence, there exists a length $L_0 \equiv L_0(d, \Lambda, V, E) > 0$ such that for all $L \geq L_0$

$$\tilde{\Lambda}_L^{\mathrm{ext}} \subset \mathbb{R}^d \setminus]-\ell_1, \ell_1[^{\,d}, \tag{3.45}$$

where ℓ_1 was given in Lemma 3.2.6. Now, we cover Λ_L^c and $\Lambda_L \smallsetminus \Lambda_{L_0}$ by unit cubes. Hence, we have

$$\begin{aligned}\left\|1_{\Lambda_L^c}\big(1_{<E}(H_0) - 1_{<E}(H)\big)1_{\Lambda_L}\right\|_2^2 &\le \left\|1_{\Lambda_L^c}\big(1_{<E}(H_0) - 1_{<E}(H)\big)1_{\Lambda_{L_0}}\right\|_2^2 \\ &\quad + \sum_{\substack{n\in\Xi_L^{\mathrm{ext}} \\ m\in\Xi_L\cap\Xi_{L_0}^{\mathrm{ext}}}} \left\|1_{\Gamma_n}\big(1_{<E}(H_0) - 1_{<E}(H)\big)1_{\Gamma_m}\right\|_2^2. \end{aligned} \tag{3.46}$$

The first term on the right-hand side of (3.46) is estimated by (3.41) and (3.43). To bound the double sum in (3.46) from above, we use Lemma 3.2.6, which is applicable due to the definition (3.45) of L_0, and obtain

$$\left\|1_{\Lambda_L^c}\big(1_{<E}(H_0) - 1_{<E}(H)\big)1_{\Lambda_L}\right\|_2^2 \le 4C_S|\tilde\Lambda_{L_0}| + \sum_{\substack{n\in\Xi_L^{\mathrm{ext}} \\ m\in\Xi_L\cap\Xi_{L_0}^{\mathrm{ext}}}} \frac{\tilde c^2}{(|n||m|)^{d-1}|n|^2}, \tag{3.47}$$

where $\tilde c$ is as in Lemma 3.2.6. We conclude from the definition of ℓ_1 that $|l| \ge |u| - \sqrt{d} \ge |u|/2$ for every $l \in \Xi_L^{\mathrm{ext}} \cup (\Xi_L \cap \Xi_{L_0}^{\mathrm{ext}})$ and every $u \in \Gamma_l \subseteq \mathbb{R}^d \smallsetminus\,]-\ell_1, \ell_1[\,^d$. Therefore, we infer that the double sum in (3.47) is upper bounded by the double integral

$$\int_{\tilde\Lambda_L} \mathrm{d}x \int_{\tilde\Lambda_L^{\mathrm{ext}}} \mathrm{d}y\, \frac{(2^d\tilde c)^2}{(|x||y|)^{d-1}|y|^2} = (2^d\tilde c)^2 \int_{\frac{L_0}{L}\tilde\Lambda_L} \frac{\mathrm{d}x}{|x|^{d-1}} \int_{\frac{L_0}{L}\tilde\Lambda_L^{\mathrm{ext}}} \frac{\mathrm{d}y}{|y|^{d+1}}. \tag{3.48}$$

But $\tilde\Lambda_L^{(\mathrm{ext})} \subseteq \bigcup_{x\in\Lambda_L^{(c)}}(x + [-1,1]^d)$ so that the scaled domains satisfy

$$\frac{L_0}{L}\,\tilde\Lambda_L^{(\mathrm{ext})} \subseteq \bigcup_{x\in\Lambda_{L_0}^{(c)}} \Big(x + \frac{L_0}{L}[-1,1]^d\Big) \subseteq \bigcup_{x\in\Lambda_{L_0}^{(c)}} \big(x + [-1,1]^d\big) =: K_{L_0}^{(\mathrm{ext})} \tag{3.49}$$

for any $L \ge L_0$. Clearly, K_{L_0} is bounded. Furthermore, we ensure that $K_{L_0}^{\mathrm{ext}}$ has a positive distance to the origin. This is always the case, since $\ell_1 > 2\sqrt{d}$ and (3.45) imply

$$K_{L_0}^{ext} \subset \mathbb{R}^d \smallsetminus\,]-\ell_1+1, \ell_1-1[^d. \tag{3.50}$$

It follows that the right-hand side of (3.48) is bounded from above by some constant $c_3 \equiv c_3(d,\Lambda,V,E) < \infty$, uniformly in $L \ge L_0$. Combining this with (3.41), (3.43), (3.47) and (3.48), we arrive at the final estimate

$$\sup_{L>0}\left\|1_{\Lambda_L^c}\big(1_{<E}(H_0) - 1_{<E}(H)\big)1_{\Lambda_L}\right\|_2^2 \le 4C_S|\tilde\Lambda_{L_0}| + c_3 =: C_2^2. \tag{3.51}$$

□

3.2.3 Proof of the upper bound

We begin with an interpolation result.

Lemma 3.2.7. *Let $\Lambda \subset \mathbb{R}^d$ be as in Assumption 3.1.2 (ii), let $V \in L^\infty_c(\mathbb{R}^d)$ and fix $E > 0$. Then there exists a constant $C_3 \equiv C_3(d, \Lambda, V, E) > 0$ such that for all $s \in]1/2, 1[$ and all $L \geq 1$ we have*

$$\big\| 1_{\Lambda_L^c} \big(1_{<E}(H) - 1_{<E}(H_0)\big) 1_{\Lambda_L} \big\|_{2s}^{2s} \leq C_3 L^{2d(1-s)}. \tag{3.52}$$

Proof. Given a trace-class operator A and $s \in]1/2, 1[$, we conclude from the interpolation inequality, see e.g. [Tao10, Lemma 1.11.5],

$$\|A\|_{2s}^{2s} \leq \|A\|_1^{2(1-s)} \|A\|_2^{2(2s-1)}. \tag{3.53}$$

Let us consider the operator

$$A_L := 1_{\Lambda_L^c} \big(1_{<E}(H) - 1_{<E}(H_0)\big) 1_{\Lambda_L}. \tag{3.54}$$

We do already know $\|A_L\|_2^2 \leq C_2^2$ for all $L \geq 1$, as was shown in Lemma 3.2.3. Since Λ is bounded, there exists $r \equiv r(\Lambda) \in [1, \infty[$ such that $\Lambda \subset [-r, r]^d$. The estimate (3.42) now implies $\|A_L\|_1 \leq 2(2\lceil rL \rceil)^d C_S \leq 2(4rL)^d C_S$, where we used $\lceil a \rceil \leq 2a$ for all $a \geq 1$. This proves the claim with

$$(2^{2d+1} r^d C_S)^{2(1-s)} C_2^{2(2s-1)} \leq 2^{2d+1} r^d (1 + C_S)(C_2^2 + 1) =: C_3 \equiv C_3(d, \Lambda, V, E). \tag{3.55}$$

$\square$

Remark 3.2.8. *Lemma 3.2.7 allows for a quick proof of the upper bound in Theorem 2.1.1, if we restrict ourselves to the case $d \geq 2$. First, we notice that we can estimate*

$$h(\lambda) \leq \frac{6}{1-s} \big(g(\lambda)\big)^s \tag{3.56}$$

for any $\lambda \in [0, 1]$ and $s \in [1/2, 1]$, as we show Lemma B.2.1. By applying this estimate to the entanglement entropy and rewrite it with (3.12) *we obtain*

$$\begin{aligned} S_E(\Lambda_L; \mathbb{R}^d, H) &\leq \frac{6}{1-s} \big\| 1_{\Lambda_L^c} 1_{<E}(H) 1_{\Lambda_L} \big\|_{2s}^{2s} \\ &\leq \frac{12}{1-s} \Big(\big\| 1_{\Lambda_L^c} 1_{<E}(H_0) 1_{\Lambda_L} \big\|_{2s}^{2s} + \|A_L\|_{2s}^{2s} \Big). \end{aligned} \tag{3.57}$$

Here, A_L is defined in (3.54)*. According to the lemma and subsequent remarks in [LSS14], the first term on the right-hand side scales like $\mathcal{O}(L^{d-1} \ln L)$. The second term is of order $\mathcal{O}\big(L^{2d(1-s)}\big)$ according to Lemma 3.2.7. If we choose $s \equiv s(d, \varepsilon) := 1 - \varepsilon(2d)^{-1}$ for any $\varepsilon \in [0, 1]$ the second term is of the order $\mathcal{O}(L^\varepsilon)$, and thus subleading as compared to the first term in all but one dimensions.*

Unfortunately, there is no choice for s which yields only a logarithmic growth in $d = 1$. To appropriately bound the term $(1-s)^{-1}\mathcal{O}\big(L^{2d(1-s)}\big)$ in (3.57) *requires an L-dependent choice of s with $s \equiv s(L) \to 1$ as $L \to \infty$. However, such a choice of s leads to an additional diverging prefactor $(1-s)^{-1}$ multiplying the asymptotic $\mathcal{O}(L^{d-1} \ln L)$ from the first term.*

We now present an approach, which yields the optimal upper bound of order $\mathcal{O}(L^{d-1} \ln L)$ for all dimensions.

Lemma 3.2.9. *Let A and B be two compact operators with $\|A\|, \|B\| \le e^{-1/2}/3$ and consider the function*

$$f : [0, \infty[\to [0,1], \ x \mapsto -1_{[0,1]}(x)\, x^2 \log_2(x^2). \tag{3.58}$$

Then we have

$$\operatorname{tr}\{f(|A|)\} \le 4 \operatorname{tr}\{f(|B|)\} + 4 \operatorname{tr}\{f(|A-B|)\}. \tag{3.59}$$

For any compact operator A let $\big(a_n(A)\big)_{n\in\mathbb{N}} \subseteq [0,\infty[$ denote the non-increasing sequence of its singular values. They coincide with the eigenvalues of the self-adjoint operator $|A|$.

Proof of Lemma 3.2.9. By assumption, we have $0 \le a_{2n}(A) \le a_{2n-1}(A) \le \mathrm{e}^{-1/2}/3$ for all $n \in \mathbb{N}$. Since the function f is monotonously increasing on $[0, \mathrm{e}^{-1/2}]$, we deduce

$$\operatorname{tr}\{f(|A|)\} = \sum_{n\in\mathbb{N}} f\big(a_n(A)\big) \le 2 \sum_{n\in\mathbb{N}} f\big(a_{2n-1}(A)\big). \tag{3.60}$$

The singular values of any compact operators A and B satisfy the inequality

$$a_{n+m-1}(A) \le a_n(B) + a_m(A-B) \tag{3.61}$$

for all $n, m \in \mathbb{N}$ [Woj91, Prop. 2 in Sect. III.G]. We point out that the right-hand side of (3.61) does not exceed the upper bound $\mathrm{e}^{-1/2}$, because of $\|A-B\| \le \|A\| + \|B\| \le (2/3)\mathrm{e}^{-1/2}$. Together with the monotonicity of f, we conclude from (3.60) that

$$\operatorname{tr}\{f(|A|)\} \le 2 \sum_{n\in\mathbb{N}} f\big(a_n(B) + a_n(A-B)\big). \tag{3.62}$$

Next, we claim that

$$f(x+y) \le -2(x^2+y^2)\log_2\big[(x+y)^2\big] \le 2f(x) + 2f(y) \tag{3.63}$$

for all $x, y \ge 0$ with $x + y < 1$. The first estimate follows from the binomial inequality together with $-\log_2[(x+y)^2] \ge 0$ for $x+y<1$, the second estimate from $(x+y)^2 \ge x^2$, respectively $(x+y)^2 \ge y^2$, and the fact that $-\log_2$ is monotonously decreasing. Combining (3.62) and (3.63), we arrive at

$$\operatorname{tr}\{f(|A|)\} \le 4 \sum_{n\in\mathbb{N}} \Big[f\big(a_n(B)\big) + f\big(a_n(A-B)\big)\Big]. \tag{3.64}$$

This concludes the proof. □

Proof of the upper bound in Theorem 2.1.1. Let $L \ge 1$ and $E > 0$. Lemma B.2.2 and (3.12) yield

$$S_E(\Lambda_L; \mathbb{R}^d, H) \le 3 \sum_{n=1}^{\infty} f\big(a_n(1_{\Lambda_L^c} 1_{<E}(H) 1_{\Lambda_L})\big), \tag{3.65}$$

where f was defined in Lemma 3.2.9. In order to apply Lemma 3.2.9, we will decompose the compact operator $1_{\Lambda_L^c} 1_{<E}\big(H_{(0)}\big) 1_{\Lambda_L}$ into a part bounded by $\mathrm{e}^{-1/2}/3$ in norm and a finite-rank operator. To this end, we introduce

$$N_{(0)} \equiv N_{(0)}(\Lambda, V, E, L) := \min\Big\{n \in \mathbb{N} :\ a_n\big(1_{\Lambda_L^c} 1_{<E}(H_{(0)}) 1_{\Lambda_L}\big) \le \mathrm{e}^{-1/2}/3\Big\} - 1, \tag{3.66}$$

the number of singular values of $1_{\Lambda_L^c} 1_{<E}\big(H_{(0)}\big) 1_{\Lambda_L}$ which are larger than $\mathrm{e}^{-1/2}/3$. We define $F_{(0)}$ as the contribution from the first $N_{(0)}$ singular values in the singular-value decomposition of $1_{\Lambda_L^c} 1_{<E}\big(H_{(0)}\big) 1_{\Lambda_L}$. Hence, $\mathrm{rank}(F_{(0)}) = N_{(0)}$ and $\|F_{(0)}\| \le 1$. The remainder

$$Q_{(0)} := 1_{\Lambda_L^c} 1_{<E}(H_{(0)}) 1_{\Lambda_L} - F_{(0)} \tag{3.67}$$

satisfies $\|Q_{(0)}\| \le \mathrm{e}^{-1/2}/3$ by definition of $N_{(0)}$. We note the upper bound

$$N_{(0)} \le 9\mathrm{e} \sum_{n=1}^{N_{(0)}} \Big(a_n\big(1_{\Lambda_L^c} 1_{<E}(H_{(0)}) 1_{\Lambda_L}\big)\Big)^2 \le 9\mathrm{e} \big\|1_{\Lambda_L^c} 1_{<E}(H_{(0)}) 1_{\Lambda_L}\big\|_2^2. \tag{3.68}$$

Using Lemma 3.2.3, we further estimate N in terms of unperturbed quantities

$$N \le 18\mathrm{e} \big\|1_{\Lambda_L^c} 1_{<E}(H_0) 1_{\Lambda_L}\big\|_2^2 + 18\mathrm{e} C_2^2. \tag{3.69}$$

The identity (3.12) and the lower bound in (B.10) imply immediately the inequality $\|1_{\Lambda_L^c} 1_{<E}(H_0) 1_{\Lambda_L}\|_2^2 \le S_E(\Lambda_L; \mathbb{R}^d, H_0)$ so that we obtain

$$N_0 \le 9\mathrm{e} S_E(\Lambda_L; \mathbb{R}^d, H_0) \quad \text{and} \quad N \le 18\mathrm{e} S_E(\Lambda_L; \mathbb{R}^d, H_0) + 18\mathrm{e} C_2^2 \tag{3.70}$$

for later usage.

We deduce from (3.61) and $\mathrm{rank}(F) = N$ that for all $n \in \mathbb{N}$

$$a_{n+N}(Q+F) \le a_n(Q) + a_{N+1}(F) = a_n(Q) \le \mathrm{e}^{-1/2}/3. \tag{3.71}$$

Hence, (3.65) implies that

$$S_E(\Lambda_L; \mathbb{R}^d, H) \le 3 \sum_{n=1}^{N} f\big(a_n(Q+F)\big) + 3 \sum_{n=1}^{\infty} f\big(a_n(Q)\big) \le 3N + 3\,\mathrm{tr}\{f(|Q|)\}, \tag{3.72}$$

where we used the monotonicity of f on $[0, \mathrm{e}^{-1/2}]$ and $f \le 1$. Now, Lemma 3.2.9 allows to estimate (3.72) so that

$$S_E(\Lambda_L; \mathbb{R}^d, H) \le 3N + 12\,\mathrm{tr}\{f(|Q_0|)\} + 12\,\mathrm{tr}\{f(|\delta Q|)\}, \tag{3.73}$$

where $\delta Q := Q - Q_0$. The rank of $\delta F := F - F_0$ obeys

$$\delta N \equiv \delta N(\Lambda, V, E, L) := \mathrm{rank}(\delta F) \le N + N_0. \tag{3.74}$$

We deduce again from (3.61) and from the definition of δN that for all $n \in \mathbb{N}$

$$a_{n+2\delta N}(\delta Q) = a_{(n+\delta N)+(\delta N+1)-1}(\delta Q) \le a_{n+\delta N}(\delta Q + \delta F). \tag{3.75}$$

Yet another application of (3.61) and the definition of δN yield for all $n \in \mathbb{N}$

$$a_{n+\delta N}(\delta Q + \delta F) \le a_n(\delta Q) \le \|\delta Q\| \le 2\mathrm{e}^{-1/2}/3. \tag{3.76}$$

Therefore the singular values in (3.75) lie in the range where the function f is monotonously increasing. Hence, we obtain

$$\begin{aligned} \mathrm{tr}\{f(|\delta Q|)\} &\le \sum_{n=1}^{2\delta N} f\big(a_n(\delta Q)\big) + \sum_{n \in \mathbb{N}} f\big(a_{\delta N+n}(\delta Q + \delta F)\big) \\ &\le 2\,\delta N + \sum_{n \in \mathbb{N}} f\big(a_n(\delta Q + \delta F)\big), \end{aligned} \tag{3.77}$$

where the second line follows from $0 \le f \le 1$.

Now, we repeat the arguments from (3.75) to (3.77) for Q_0 instead of δQ, F_0 instead of δF and N_0 instead of δN. This implies

$$\operatorname{tr}\{f(|Q_0|)\} \le 2N_0 + \sum_{n\in\mathbb{N}} f\big(a_n(Q_0 + F_0)\big). \tag{3.78}$$

The sum in (3.78) is bounded from above by the unperturbed entanglement entropy, which follows from (3.67), the definition of f, (3.12) and the lower bound in Lemma B.2.2, whence

$$\operatorname{tr}\{f(|Q_0|)\} \le 2N_0 + S_E(\Lambda_L; \mathbb{R}^d, H_0). \tag{3.79}$$

Next, we combine (3.73), (3.70), (3.77), (3.74) and (3.79) to obtain

$$S_E(\Lambda_L; \mathbb{R}^d, H) \le 2508\, S_E(\Lambda_L; \mathbb{R}^d, H_0) + 1322\, C_2^2 + 12 \sum_{n\in\mathbb{N}} f\big(a_n(\delta Q + \delta F)\big). \tag{3.80}$$

In order to estimate the sum in (3.80), we appeal to the definitions of δQ and δF, (3.67), the definition of f and (B.9) to deduce

$$\sum_{n\in\mathbb{N}} f\big(a_n(\delta Q + \delta F)\big) \le \frac{1}{1-s} \left\| 1_{\Lambda_L^c}\big(1_{<E}(H_0) - 1_{<E}(H)\big)1_{\Lambda_L} \right\|_{2s}^{2s} \tag{3.81}$$

for any $s \in\,]0,1[$. Restricting ourselves to $s \in\,]1/2,1[$ allows us to apply Lemma 3.2.7 so that

$$\sum_{n\in\mathbb{N}} f\big(a_n(\delta Q + \delta F)\big) \le \frac{C_3}{1-s}\, L^{2d(1-s)}, \tag{3.82}$$

where $C_3 = C_3(d, \Lambda, V, E) > 0$ is given in Lemma 3.2.7 and independent of s. Assuming $L \ge 8$, we choose the L-dependent exponent

$$s \equiv s(L) := 1 - \frac{1}{\ln L} \in\,]1/2, 1[\tag{3.83}$$

which implies

$$\sum_{n\in\mathbb{N}} f\big(a_n(\delta Q + \delta F)\big) \le C_3 \mathrm{e}^{2d} \ln L. \tag{3.84}$$

The entanglement entropy of a free Fermi gas exhibits an enhanced area law, i.e. $S_E(\Lambda_L; \mathbb{R}^d, H_0) = \mathcal{O}(L^{d-1}\ln L)$ according to Theorem 3.2.1, so that the claim follows from (3.80) together with (3.84). □

3.2.4 Proof of the lower bound

Proof of the lower bound in Theorem 2.1.1. We fix $L > 0$ and $E > 0$. The lower bound in (B.10), the identity (3.12) and the elementary inequality $(a-b)^2 \ge a^2/2 - b^2$ for $a, b \in \mathbb{R}$ imply

$$\begin{aligned} S_E(\Lambda_L; \mathbb{R}^d, H) &\ge \operatorname{tr}\left\{g\big(1_{\Lambda_L} 1_{<E}(H) 1_{\Lambda_L}\big)\right\} = \|1_{\Lambda_L^c} 1_{<E}(H) 1_{\Lambda_L}\|_2^2 \\ &\ge \frac{1}{2}\left\|1_{\Lambda_L^c} 1_{<E}(H_0) 1_{\Lambda_L}\right\|_2^2 - \left\|1_{\Lambda_L^c}\big(1_{<E}(H_0) - 1_{<E}(H)\big)1_{\Lambda_L}\right\|_2^2. \end{aligned} \tag{3.85}$$

The second term on the right-hand side is uniformly bounded in L according to Lemma 3.2.3. The first term is identical to $\operatorname{tr}\left\{g\big(1_{\Lambda_L}1_{<E}(H_0)1_{\Lambda_L}\big)\right\}$. According to Theorem 3.2.1, the leading behaviour of its asymptotic expansion in L is of order $L^{d-1}\ln L$. Hence,

$$\liminf_{L\to\infty}\frac{S_E(\Lambda_L;\mathbb{R}^d,H)}{L^{d-1}\ln L}\geq\frac{1}{2}\lim_{L\to\infty}\frac{\operatorname{tr}\left\{g\big(1_{\Lambda_L}1_{<E}(H_0)1_{\Lambda_L}\big)\right\}}{L^{d-1}\ln L}=\frac{3}{2\pi^2}\,\Sigma_0=:\Sigma_l. \tag{3.86}$$

□

Chapter 4

Logarithmic Enhancement in the droplet band of the XXZ spin ring

Recently, it was shown that the disordered XXZ spin chain exhibits many-body localisation phenomena in the droplet band. It is one of the few interacting systems, for which an area law of the entanglement entropy has been proven. On the other hand, the XXZ spin chain without disorder is clearly not localised. For the area law to be indeed a criterion for localisation, the entanglement entropy must have a different scaling behaviour in this case.

In the following chapter we show a logarithmically divergent lower bound to the finite-volume entanglement entropy for eigenstates in the droplet band. The work that is presented here is the result of a collaboration with C. Fischbacher. The content was already published in [FS20].

4.1 Introduction and Result

We consider the XXZ model on a discrete ring of finite size $L \in \mathbb{N}$. We describe the ring using the graph $\mathcal{G}_L := (\mathcal{V}_L, \mathcal{E}_L)$ with vertex set $\mathcal{V}_L := \{0, 1, \dots, L-1\}$ and edge set

$$\mathcal{E}_L := \big\{\{j, (j+1) \bmod L\} : \ j \in \mathcal{V}_L\big\}. \tag{4.1}$$

On each vertex we imagine a spin-1/2 particle, represented by the two dimensional vector space $\mathbb{C}^2$. Let $|\uparrow\rangle := \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $|\downarrow\rangle := \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ denote the canonical basis of the single site vector space, which we interpret as "up-spin" and "down-spin". The Hilbert space of the whole system is given by the tensor product $\mathbb{H}_L := \mathbb{H}_{\mathcal{V}_L}$. Recall that $\mathbb{H}_{\mathcal{A}} = \bigotimes_{j\in\mathcal{A}} \mathbb{C}^2$ for any set $\mathcal{A}$.

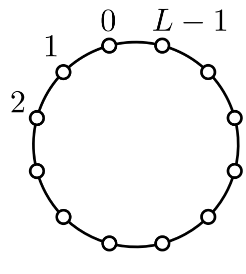

Figure 4.1: The ring described by the graph $\mathcal{G}_L$.

The XXZ Hamiltonian with cyclic boundary conditions $H_L : \mathbb{H}_L \to \mathbb{H}_L$ only contains interaction terms of sites that are connected by an edge in $\mathcal{E}_L$. It is given by

$$H_L \equiv H_L(\Delta) := \sum_{\{j,k\}\in\mathcal{E}_L} h_{jk}(\Delta), \tag{4.2}$$

where $\Delta > 1$ is the *anisotropy parameter* and the two-site operator h_{jk} describes an interaction between two spins located at the two sites $\{j,k\} \in \mathcal{E}_L$. It is defined as

$$h_{jk} \equiv h_{jk}(\Delta) := \left(\frac{1}{4} - S_j^3 S_k^3\right) - \frac{1}{\Delta}\left(S_j^1 S_k^1 + S_j^2 S_k^2\right), \tag{4.3}$$

with S^1, S^2 and S^3 being the standard spin–1/2 matrices

$$S^1 := \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad S^2 := \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix} \quad \text{and} \quad S^3 := \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}. \tag{4.4}$$

Here and in the following, for any $A \in \mathbb{C}^{2\times 2}$ the notation A_j refers to a spin operator acting as A on the site $j \in \mathcal{V}_L$ and as the identity anywhere else. The ground state energy of this Hamiltonian is given by 0 with a corresponding two-dimensional eigenspace spanned by $|\uparrow\rangle^{\otimes L}$ ("all spins-up") and $|\downarrow\rangle^{\otimes L}$ ("all spins-down").

An important property of the XXZ Hamiltonian is that it preserves the magnetisation in 3-direction. In other words, the magnetisation operator $M_L^3 := \sum_{j\in\mathcal{V}_L} S_j^3$ satisfies

$$[H_L, M_L^3] = 0. \tag{4.5}$$

This implies in particular that every eigenspace of M_L^3 is also a reducing subspace of H_L. In a way, the operator M_L^3 quantifies the number of down-spins in a given state. Its spectrum is given by $\sigma(M_L^3) = \{L/2 - n : n \in \{0,\cdots,L\}\}$ – for all $n \in \{0,\cdots,L\}$ the eigenspace corresponding to the eigenvalue $L/2-n$ contains vectors with n down-spins. We therefore treat each down-spin as a particle. A natural basis of eigenvectors can be constructed by means of the spin lowering operator

$$S^- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{4.6}$$

which is the operator satisfying $S^-|\uparrow\rangle = |\downarrow\rangle$ and $S^-|\downarrow\rangle = 0$. For any finite set $\mathcal{A}$, we now introduce the canonical basis $\{|\delta_x^{\mathcal{A}}\rangle\}_{x\in\mathcal{P}(\mathcal{A})}$ of $\mathbb{H}_{\mathcal{A}}$, where $|\delta_\varnothing^{\mathcal{A}}\rangle := |\uparrow\rangle^{\otimes|\mathcal{A}|}$ and

$$|\delta_x^{\mathcal{A}}\rangle := \prod_{j\in x} S_j^- |\delta_\varnothing^{\mathcal{A}}\rangle \qquad \text{for all } \varnothing \neq x \in \mathcal{P}(\mathcal{A}), \tag{4.7}$$

where $\mathcal{P}(\mathcal{A})$ denotes the power set for any set $\mathcal{A}$. The vector $|\delta_x^{\mathcal{A}}\rangle$ represents a state with a down-spin particle at each position in $x \subseteq \mathcal{A}$. The N-particle subspaces of $\mathbb{H}_{\mathcal{A}}$ are now defined as

$$\mathbb{H}_{\mathcal{A}}^N := \operatorname{span}\{|\delta_x^{\mathcal{A}}\rangle : x \subseteq \mathcal{A} \text{ with } |x| = N\}. \tag{4.8}$$

For any $N \in \{0,\cdots,L\}$ is $\mathbb{H}_L^N := \mathbb{H}_{\mathcal{V}_L}^N = \ker\left(M_L^3 - (L/2 - N)\right)$. Since each $\mathbb{H}_L^N$ reduces the operator H_L, we express it as the direct sum

$$H_L = \bigoplus_{N=0}^{L} H_L^N, \tag{4.9}$$

where H_L^N is the restriction of H_L to $\mathbb{H}_L^N$ for all $N \in \{0,1,\ldots,L\}$. By a short calculation we see that the operators H_L^L and H_L^0 are identical to the zero operator on $\mathbb{H}_L^L =$

$\mathrm{span}\{|\delta^L_{\mathcal{V}_L}\rangle\}$ and $\mathbb{H}^0_L = \mathrm{span}\{|\delta^L_\varnothing\rangle\}$ respectively. Here and in the following, we use the short-hand notation $|\delta^L_x\rangle := |\delta^{\mathcal{V}_L}_x\rangle$ for any $x \subseteq \mathcal{V}_L$.

In this work we are not interested in the entanglement entropy of the ground state of H_L itself. For any region $\Lambda \subseteq \mathcal{V}_L$ both $|\delta^L_\varnothing\rangle$ and $|\delta^L_{\mathcal{V}_L}\rangle$ are separable with respect to the spatial decomposition $\mathbb{H}_L = \mathbb{H}_\Lambda \otimes \mathbb{H}_{\Lambda^c}$. Consequently, the entanglement entropy of both of these ground states vanishes for any Λ_L, which is an extreme version of an area law. Instead, we consider eigenstates of the N-particle Hamiltonian H^N_L, where the particle number $N \equiv N(\varepsilon) := \lfloor \varepsilon L \rfloor$ is determined by a constant particle density $\varepsilon \in \,]0,1[$. Note that the ground state energy of H^N_L is bounded from below by $1 - \frac{1}{\Delta}$, see [NSS06].

The anisotropy parameter Δ is chosen to be positive in our model. This choice is sometimes referred to as the "ferromagnetic" case, since in such a model it is energetically favourable for all the spins to be parallel. Hence, the set of ground states of H_L include $|\delta^L_\varnothing\rangle$ and $|\delta^L_{\mathcal{V}_L}\rangle$. Special cases of the ferromagnetic model include the Heisenberg model for $\Delta = 1$ and the Ising model for $1/\Delta = 0$ or "$\Delta = \infty$". In this thesis we are interested in the case $\Delta \in \,]1,\infty[$, which lies somewhere in between. This choice for the anisotropy parameter is called the *Ising phase*.

The N-particle Hamiltonian in the Ising phase has a property that is crucial for our approach. The eigenvectors corresponding to eigenvalues included in the *droplet spectrum*

$$I_1 \equiv I_1(\Delta) := \Big[1 - \frac{1}{\Delta}, 2\Big(1 - \frac{1}{\Delta}\Big)\Big[\tag{4.10}$$

at the bottom of the spectrum favour configurations where all down-spin particles are clustered together. Such clusters are also referred to as droplets, which is why these low-energy eigenstates are also called *droplet states*. It is important to note that these droplet states are not in any sense localised. Since the Hamiltonian is translational invariant, all possible droplet configurations in its eigenvectors have the same weight. The eigenvectors to larger energies may also contain significant contributions of configurations with an increasing number of clusters. For a more detailed description we refer to Section 4.2. Seen from a physical point of view, this means that breaking up clusters costs energy, while clustering together saves energy.

It can also be observed in the infinite-volume model that droplet configurations are energetically favoured for low energies. The infinite-volume N-particle Hamiltonian H^N_∞ is defined analogous to (4.2) and (4.9) with the only difference that the underlying graph is given by $\mathcal{G} := \{\mathbb{Z}, \mathcal{E}\}$ with edge sets $\mathcal{E} := \big\{\{j, j+1\} : \; j \in \mathbb{Z}\big\}$ instead of $\mathcal{G}_L$. In a way, H^N_∞ can be thought of as the limit of the finite-volume operator for $L \to \infty$. At least the ground state energy of the finite system is known to converge towards $\inf \sigma(H^N_\infty)$ in the thermodynamical limit [NSS06]. This Hamiltonian has a distinctive lowest band, which is separated from the rest of the spectrum by a spectral gap for sufficiently large Δ [NSS06]. Moreover, the infinite-volume model is also translational invariant. As a consequence, for large Δ the spectrum of H^N_∞ in the lowest band is absolutely continuous, which can be proven with the help of the Bethe ansatz [NSS06]. The Bethe ansatz also yields a set of generalised eigenstates corresponding to energies in the lowest energy band, where droplet configurations maintain the largest contributions [NSS06, NS01, FS14, FS18].

The structure of the eigenvectors of H^N_L for low energies and even more so the absolutely continuous spectrum in the lowest energy band of H_∞^N hint at delocalisation phenomena. We therefore expect a scaling behaviour from the entanglement entropy that is not an area law. This situation changes entirely if we add a random magnetic field to the XXZ Hamiltonian, since such a disordered model does exhibit many-body localisation phenomena in the lowest energy band. Recently, this model has been rigorously studied in [EKS18b, EKS18a, BW17]; see also [Sto20] for a survey of the newest developments. Even more of interest to us is the fact, that V. Beaud and S. Warzel proved an area law for the entanglement entropy for this model in [BW18]. A difference in the scaling behaviour of the entanglement entropy between the XXZ chain with and without disordered magnetic field would indicate that area laws do indeed distinguish between localised and delocalised states.

Our main result is a logarithmic lower bound to the entanglement entropy of low-energy states whose eigenenergy belongs to the interval I_1.

Theorem 4.1.1. *Let $\varepsilon \in]0, 1/16[$ and $\theta \in]\varepsilon, 1/16[$. For $L \in \mathbb{N}$, let $N \equiv N(L) := \lfloor \varepsilon L \rfloor$ and $\Lambda_L := \{0, \cdots, 2\lfloor \theta L \rfloor\} \subset \mathcal{V}_L$. Then there exists $\Delta_0 \equiv \Delta_0(\varepsilon) > 3$ such that for all $\Delta \geq \Delta_0$, $L \in \mathbb{N}$ there exists an orthonormal system of eigenstates $\{|\varphi_L^N(\Delta, E)\rangle\}_{E \in \sigma(H_L^N) \cap I_1} \subseteq \mathbb{H}_L^N$, where $|\varphi_L^N(\Delta, E)\rangle$ is an eigenstate corresponding to $E \in \sigma(H_L^N) \cap I_1$, such that*

$$\liminf_{L \to \infty} \inf_{E \in \sigma(H_L^N) \cap I_1} \frac{S(\Lambda_L; \mathcal{V}_L, \varphi_L^N(\Delta, E))}{\ln L} \geq \frac{\varepsilon}{2 \ln 2}. \tag{4.11}$$

Remark 4.1.2. *(i) While we have made the particular choice for Λ_L to scale proportionally to the ring size L and not independently of it, our result nevertheless shows that an area law could not possibly exist in the generic case. We considered a similar choice in Theorem 2.1.2.*

(ii) We only prove a lower bound to the entanglement entropy and no upper bound. In [BW18], a logarithmic upper bound was shown for the XXZ model with droplet boundary conditions and with any additional bounded magnetic field. The proof of this statement is based on a Combes–Thomas estimate. We expect a logarithmic estimate to be true for our case of a finite XXZ spin chain with cyclic boundary conditions, since we showed a similar Combes–Thomas estimate.

(iii) Not every eigenvector to an eigenenergy $E \in I_1$ is covered by Theorem 4.1.1. We need them to reflect the translational symmetry of the system. However, since H_L is translational invariant there exists at least one eigenvector with these properties for each eigenvalue. To the best of our knowledge, it has not yet been generally established, whether the eigenvalues in the lowest energy band are simple or not. We do know though that the ground state energy is simple, see Lemma C.1.2. This implies that the ground state of H_L^N always satisfies (4.11).

4.1.1 Roadmap

Section 4.2 is dedicated to obtaining estimates on low-energy eigenfunctions of the XXZ Hamiltonian. Our focus lies on determining to which extent the weight of the eigenfunctions is concentrated in the droplet configurations. To this end, we firstly exploit the ring's translational symmetry and define a suitable Fourier transform. We then introduce an equivalent formulation of the XXZ Hamiltonian using Schrödinger operators, which we will apply to show an appropriate Combes–Thomas estimate in Theorem 4.2.8. The constructions presented in this section follow the one of [EKS18b, ARFS20] closely. The main new feature of our estimate here is that the ring's symmetry is taken into account, which allows us to obtain an additional factor of $L^{-1/2}$.

In Section 4.3 we show that in the Ising limit there exist low-energy eigenstates exhibiting the desired logarithmic lower bound. These eigenstates are entirely composed out of droplet configurations. Therefore, we regard the low-energy eigenstates of the Ising phase as a perturbation of the Ising limit. In the remainder of this section we sketch our strategy on deriving a logarithmic lower bound to the entanglement entropy for the Ising phase from the Ising limit.

For our perturbative approach it is crucial to bound the difference between low-energy eigenstates in the Ising-phase and a matching eigenstate in the Ising limit. In particular, we need to estimate the difference of the reduced densities of the aforementioned states. In Section 4.4 we derive an estimate for the entries of this operator difference. Some rather technical auxiliary results are necessary in order to deal with the underlying geometry of the ring.

After this, in Section 4.5, we derive an upper bound to control the difference between the two reduced density operators with respect to a von Neumann-Schatten quasi-norm. We show that this estimate converges to zero for $\Delta \to \infty$, which enables us to use it as the base of our perturbation theory. We derive this bound by estimating the singular values of the reduced density difference first. This finally enables us to derive a logarithmic lower bound for the Ising phase from the bound of the Ising limit, by using Kreĭns trace formula.

4.2 Estimating low energy eigenfunctions

4.2.1 Estimates for low energy eigenstates

In the last chapter we claimed that the mass of the low-energy eigenfunctions of H_L^N is mainly concentrated in the droplet configurations where all spin-down particles are clustered together. The aim of this section is to quantify this statement.

To describe the N-particle subspace $\mathbb{H}_L^N$, we introduce the graph of N-particle configurations $\mathcal{G}_L^N$ first. We are only interested in the non-trivial cases $N \notin \{0, L\}$. Therefore, let us assume $L, N \in \mathbb{N}$ with $N < L$ for the whole section below. Recall that the graph $\mathcal{G}_L := (\mathcal{V}_L, \mathcal{E}_L)$ describes the spin ring. The corresponding graph distance between two sites $j, k \in \mathcal{V}_L$ is given by

$$d_L(j,k) = L/2 - \big||j-k| - L/2\big|. \tag{4.12}$$

Following [FS18], we define $\mathcal{G}_L^N := (\mathcal{V}_L^N, \mathcal{E}_L^N)$ as the N-th symmetric product of $\mathcal{G}_L$, where

$$\mathcal{V}_L^N := \{x \subseteq \mathcal{V}_L : |x| = N\} \quad \text{and} \quad \mathcal{E}_L^N := \big\{\{x,y\} : x, y \in \mathcal{V}_L^N,\ x \triangle y \in \mathcal{E}_L\big\}. \tag{4.13}$$

Here, $x \triangle y$ denotes the symmetric difference between the two subsets $x, y \subseteq \mathcal{V}_L$. A vertex of $\mathcal{G}_L^N$ represents the positions of the N spin-down particles. As in [FS18], we now identify $\mathbb{H}_L \cong \ell^2(\mathcal{P}(\mathcal{V}_L))$ and $\mathbb{H}_L^N \cong \ell^2(\mathcal{V}_L^N)$.

As we have remarked before, configurations where all spin-down particles are clustered together are energetically favoured in the XXZ model, if the anisotropy parameter Δ is larger than one. These droplet configurations are given by

$$\mathcal{D}_L^N := \big\{\{(j+1) \bmod L, \cdots, (j+N) \bmod L\} : j \in \mathcal{V}_L\big\}. \tag{4.14}$$

How close a given configuration is to a droplet configuration is of particular interest to us. To quantify the distance between two given configurations we therefore introduce the graph distance $d_L^N(\cdot,\cdot)$ on $\mathcal{G}_L^N$. The graph distance counts how many times individual particles have to be moved from one site to a neighbouring site to transform one configuration into the other. For an example of the graph distance see Figure 4.2.

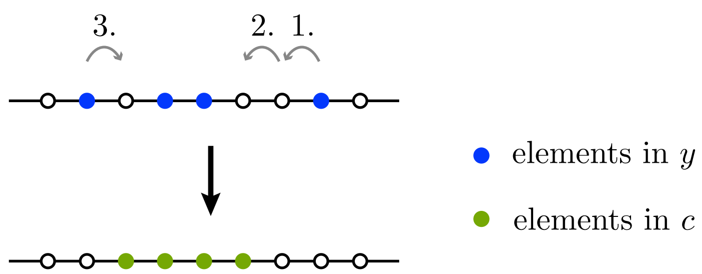

Figure 4.2: Configurations $y \in \mathcal{V}_L^4$ and $c \in \mathcal{D}_L^4$ with distance $d_L^4(y,c) = 3$.

The following theorem estimates the eigenfunctions of the XXZ Hamiltonian in the lowest energy band. Recall that the lowest energy band is included in $I_1 \equiv I_1(\Delta) = [1 - \frac{1}{\Delta}, 2(1-\frac{1}{\Delta})[$. The proof of this theorem is based on the approach in [EKS18b, ARFS20].

Theorem 4.2.1. *Let $L, N \in \mathbb{N}$ with $N < L$ and $\Delta > 3$. For any $E \in \sigma(H_L^N) \cap I_1$ there exists a corresponding eigenstate $|\varphi_L^N\rangle \equiv |\varphi_L^N(\Delta, E)\rangle \in \mathbb{H}_L^N$ such that*

$$\left|\left\langle \delta_x^L, \varphi_L^N \right\rangle\right| \leq \frac{2^4}{\sqrt{L}} \cdot e^{-\mu_1 d_L^N(x, \mathcal{D}_L^N)}, \tag{4.15}$$

for all $x \in \mathcal{V}_L^N$, where $d_L^N(x, \mathcal{A}) := \min_{y \in \mathcal{A}} d_L^N(x,y)$ for all $\mathcal{A} \subseteq \mathcal{V}_L^N$ and

$$\mu_1 \equiv \mu_1(\Delta) := \ln\left(1 + \frac{\Delta - 1}{8}\right). \tag{4.16}$$

Remark 4.2.2. *(i) The translational symmetry of the ring is reflected in the factor $L^{-1/2}$ in (4.15). It indicates that the eigenstates are indeed delocalised in the sense that the mass of a low energy eigenstate is not concentrated in one droplet but rather distributed evenly over all droplet configurations. This factor constitutes the main difference between this result and the results of [EKS18b, ARFS20]. The presence of this additional factor is crucial for proving the logarithmic lower bound to the entanglement entropy.*

(ii) The rate of decay μ_1 diverges for $\Delta \to \infty$. This indicates that in the Ising limit the eigenfunctions are concentrated on the droplet configurations alone. For further information, see Section 4.3.

4.2.2 Fourier transform

Let $L, N \in \mathbb{N}$ with $N < L$ be fixed. In order to exploit the ring's translational symmetry, we define a suitable Fourier transform. To this end, for any $\gamma \in \mathbb{Z}$ we define the translations $T_L^\gamma : \mathcal{P}(\mathcal{V}_L) \to \mathcal{P}(\mathcal{V}_L)$ by

$$T_L^\gamma x = \{(j+\gamma) \bmod L : j \in x\} \quad \text{for all } x \subseteq \mathcal{V}_L. \tag{4.17}$$

For every $\gamma \in \mathbb{Z}$, the unitary translation operator $\tilde{T}_L^\gamma : \ell^2(\mathcal{P}(\mathcal{V}_L)) \to \ell^2(\mathcal{P}(\mathcal{V}_L))$ is given by

$$(\tilde{T}_L^\gamma \psi)(x) = \psi(T_L^\gamma x) \quad \text{for all } \psi \in \ell^2(\mathcal{P}(\mathcal{V}_L)),\ x \in \mathcal{P}(\mathcal{V}_L). \tag{4.18}$$

Due to translation symmetry of H_L, we obtain $[\tilde{T}_L^\gamma, H_L] = 0$ for any $\gamma \in \mathbb{Z}$.

Now, let "$\approx$" denote the equivalence relation on $\mathcal{V}_L^N$ defined as

$$x \approx y :\Leftrightarrow \exists \gamma \in \{0, 1, \ldots, L-1\} \text{ such that } \tilde{T}_L^\gamma x = y. \tag{4.19}$$

Moreover, let $\widehat{\mathcal{V}}_L^N \subset \mathcal{V}_L^N$ be a fixed set of representatives for each equivalence class induced by "$\approx$". For an element $\hat{x} \in \widehat{\mathcal{V}}_L^N$ we denote the corresponding equivalence class by $[\hat{x}]$. We define $\hat{d}_L^N : \widehat{\mathcal{V}}_L^N \times \widehat{\mathcal{V}}_L^N \to \mathbb{N}_0$ by

$$\hat{d}_L^N(\hat{x}, \hat{y}) := \min_{\gamma \in \mathbb{Z}} d_L^N(\hat{x}, T_L^\gamma \hat{y}) \text{ for all } \hat{x}, \hat{y} \in \widehat{\mathcal{V}}_L^N. \tag{4.20}$$

Lemma 4.2.3. *$\hat{d}_L^N$ is a metric on $\widehat{\mathcal{V}}_L^N$.*

Proof. Since d_L^N is a metric, we conclude $\hat{d}_L^N(\hat{x}, \hat{y}) = 0$ if and only if there exists a $\gamma \in \mathcal{V}_L$ such that $\hat{x} = T_L^\gamma \hat{y}$. By definition of $\widehat{\mathcal{V}}_L^N$, this implies that $\hat{x} = \hat{y}$.

Now, for all $\hat{x}, \hat{y} \in \widehat{\mathcal{V}}_L^N$ let us consider

$$\hat{d}_L^N(\hat{x}, \hat{y}) = \min_{\gamma \in \mathbb{Z}} d_L^N(\hat{x}, T_L^\gamma \hat{y}) = \min_{\gamma \in \mathbb{Z}} d_L^N(T_L^{-\gamma} \hat{x}, \hat{y}) = \min_{\gamma \in \mathbb{Z}} d_L^N(\hat{y}, T_L^{-\gamma} \hat{x}) = \hat{d}_L^N(\hat{y}, \hat{x}). \tag{4.21}$$

Finally, for any $\hat{x}, \hat{y}, \hat{z} \in \widehat{\mathcal{V}}_L^N$ and any $\sigma \in \mathcal{V}_L$ consider

$$\begin{aligned} \hat{d}_L^N(\hat{x}, \hat{z}) &= \min_{\gamma \in \mathbb{Z}} d_L^N(\hat{x}, T_L^\gamma \hat{z}) \le \min_{\gamma \in \mathbb{Z}} \left(d_L^N(\hat{x}, T_L^\sigma \hat{y}) + d_L^N(T_L^\sigma \hat{y}, T_L^\gamma \hat{z}) \right) \\ &= d_L^N(\hat{x}, T_L^\sigma \hat{y}) + \min_{\gamma \in \mathbb{Z}} d_L^N(\hat{y}, T_L^{\gamma-\sigma} \hat{z}) = d_L^N(\hat{x}, T_L^\sigma \hat{y}) + \hat{d}_L^N(\hat{y}, \hat{z}). \end{aligned} \tag{4.22}$$

Minimizing over $\sigma \in \mathbb{Z}$ now yields the desired triangle inequality $\hat{d}_L^N(\hat{x}, \hat{z}) \le \hat{d}_L^N(\hat{x}, \hat{y}) + \hat{d}_L^N(\hat{y}, \hat{z})$ and thus the lemma. □

We note that not all equivalence classes have the same cardinality. In fact, for any $\hat{x} \in \widehat{\mathcal{V}}_L^N$ the number of elements in $[\hat{x}]$ is given by

$$n_{\hat{x}} := |[\hat{x}]| = \min\{\gamma \in \mathbb{N} : T_L^\gamma \hat{x} = \hat{x}\}. \tag{4.23}$$

Moreover, for any $\hat{x} \in \widehat{\mathcal{V}}_L^N$ the number L is divisible by $n_{\hat{x}}$. Let us now define the unitary Fourier transform. To this end, let

$$\mathbb{S}_L^N := \left\{\phi \in \ell^2\big(\mathcal{V}_L \times \widehat{\mathcal{V}}_L^N\big) : \ \forall \hat{x} \in \widehat{\mathcal{V}}_L^N \ \forall \gamma \notin \frac{L}{n_{\hat{x}}}\{0, \cdots, n_{\hat{x}} - 1\} \text{ we have } \phi(\gamma, \hat{x}) = 0\right\}. \tag{4.24}$$

The scalar product $\langle \cdot, \cdot \rangle_{\mathbb{S}_L^N}$ on this space is defined in the following way:

$$\langle \phi_1, \phi_2 \rangle_{\mathbb{S}_L^N} := \sum_{\gamma=0}^{L-1} \sum_{\hat{x} \in \widehat{\mathcal{V}}_L^N} \frac{1}{L/n_{\hat{x}}} \overline{\phi_1(\gamma, \hat{x})} \phi_2(\gamma, \hat{x}), \tag{4.25}$$

for any ϕ_1, $\phi_2 \in \mathbb{S}_L^N$. Since the factors $\frac{1}{L/n_{\hat{x}}}$ are positive for all $\hat{x} \in \widehat{\mathcal{V}}_L^N$ this map does indeed define s positive definite, symmetric sesquilinear form. Moreover, for any $f \in \mathbb{S}_L^N$, we define $\|f\|_{\mathbb{S}_L^N} := \sqrt{\langle f, f \rangle_{\mathbb{S}_L^N}}$. The Fourier transform $\mathfrak{F}_L^N$ is given by

$$\begin{aligned} \mathfrak{F}_L^N : \qquad & \ell^2(\mathcal{V}_L^N) \to \mathbb{S}_L^N \\ (\mathfrak{F}_L^N \psi)(\gamma, \hat{x}) &:= \frac{1}{\sqrt{L}} \sum_{z=0}^{L-1} e^{-\frac{2\pi \mathrm{i}}{L} \gamma z} \psi(T_L^z \hat{x}) \quad \text{ for all } (\gamma, \hat{x}) \in \mathcal{V}_L \times \widehat{\mathcal{V}}_L^N. \end{aligned} \tag{4.26}$$

Lemma 4.2.4. *The Fourier transform is well-defined. Furthermore, it is unitary and its adjoint is given by*

$$\begin{aligned} (\mathfrak{F}_L^N)^* : \qquad & \mathbb{S}_L^N \to \ell^2(\mathcal{V}_L^N) \\ ((\mathfrak{F}_L^N)^* \phi)(x) &:= \frac{1}{\sqrt{L}} \sum_{\gamma=0}^{L-1} e^{\frac{2\pi i}{L} \gamma z} \phi(\gamma, \hat{x}) , \end{aligned} \tag{4.27}$$

where $\hat{x} \in \widehat{\mathcal{V}}_L^N$ *and* $z \in \{0, \cdots, n_{\hat{x}} - 1\}$ *are uniquely determined by* $x = T_L^z \hat{x}$.

Proof. Firstly, let us prove that $\mathfrak{F}_L^N$ is well-defined by showing that it indeed maps into $\mathbb{S}_L^N$. For $\psi \in \ell^2(\mathcal{V}_L^N)$, $\hat{x} \in \widehat{\mathcal{V}}_L^N$ and $\gamma \notin (L/n_{\hat{x}})\{0, \cdots, n_{\hat{x}} - 1\}$ we obtain

$$\begin{aligned} (\mathfrak{F}_L^N \psi)(\gamma, \hat{x}) &= \frac{1}{\sqrt{L}} \sum_{\zeta=0}^{n_{\hat{x}}-1} \sum_{k=0}^{L/n_{\hat{x}}-1} \mathrm{e}^{-\frac{2\pi \mathrm{i}}{L}(\zeta + k n_{\hat{x}})\gamma} \psi(T_L^{\zeta + k n_{\hat{x}}} \hat{x}) \\ &= \frac{1}{\sqrt{L}} \sum_{\zeta=0}^{n_{\hat{x}}-1} \mathrm{e}^{-\frac{2\pi \mathrm{i}}{L} \zeta \gamma} \psi(T_L^\zeta \hat{x}) \left[\sum_{k=0}^{L/n_{\hat{x}}-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{L/n_{\hat{x}}} k \gamma} \right] = 0. \end{aligned} \tag{4.28}$$

In the first step of (4.28) we used that for every $z \in \mathcal{V}_L$ there exist a unique $\zeta \in \{0, \cdots, n_{\hat{x}} - 1\}$ and a $k \in \{0, \cdots, L/n_{\hat{x}} - 1\}$ such that $z = \zeta + k n_{\hat{x}}$. The last equality is due to the fact that the sum over all the $L/n_{\hat{x}}$-th roots of unity is equal to zero.

Let us now show that the adjoint of $\mathfrak{F}_L^N$ is indeed given by (4.27). To this end, let $\psi \in \ell^2(\mathcal{V}_L^N)$ and $\phi \in \mathbb{S}_L^N$. Then

$$\begin{aligned}\langle \phi, \mathfrak{F}_L^N \psi \rangle_{\mathbb{S}_L^N} &= \frac{1}{\sqrt{L}} \sum_{\gamma=0}^{L-1} \sum_{\hat{x} \in \widehat{\mathcal{V}}_L^N} \sum_{\zeta=0}^{n_{\hat{x}}-1} \sum_{k=0}^{L/n_{\hat{x}}-1} \frac{1}{L/n_{\hat{x}}} \overline{\phi(\gamma,\hat{x})} \mathrm{e}^{-\frac{2\pi \mathrm{i}}{L}\gamma(\zeta+kn_{\hat{x}})} \psi(T_L^\zeta \hat{x}) \\ &= \frac{1}{\sqrt{L}} \sum_{\hat{x} \in \widehat{\mathcal{V}}_L^N} \sum_{\zeta=0}^{n_{\hat{x}}-1} \overline{\left[\sum_{\gamma=0}^{L-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{L}\gamma\zeta} \phi(\gamma,\hat{x}) \left[\frac{1}{L/n_{\hat{x}}} \sum_{k=0}^{L/n_{\hat{x}}-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{L/n_{\hat{x}}}\gamma k} \right] \right]} \psi(T_L^\zeta \hat{x}) \end{aligned} \tag{4.29}$$

We note for any $\gamma \in (L/n_{\hat{x}})\mathbb{Z}$ that $\frac{1}{L/n_{\hat{x}}} \sum_{k=0}^{L/n_{\hat{x}}-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{L/n_{\hat{x}}}\gamma k} = 1$. Hence (4.29) is equal to

$$\sum_{x \in \mathcal{V}_L^N} \overline{((\mathfrak{F}_L^N)^* \phi)(x)} \psi(x) = \langle (\mathfrak{F}_L^N)^* \phi, \psi \rangle . \tag{4.30}$$

To show that indeed $(\mathfrak{F}_L^N)^* = (\mathfrak{F}_L^N)^{-1}$ take any $\psi \in \ell^2(\mathcal{V}_L^N)$ and $x \in \mathcal{V}_L^N$. There exist unique $\hat{x} \in \widehat{\mathcal{V}}_L^N$ and $z \in \{0, \cdots, n_{\hat{x}} - 1\}$ such that $x = T_L^z \hat{x}$. We obtain

$$((\mathfrak{F}_L^N)^* \mathfrak{F}_L^N \psi)(T_L^z \hat{x}) = \frac{1}{L} \sum_{\gamma \in \mathcal{V}_L \cap (L/n_{\hat{x}})\mathbb{Z}} \sum_{\zeta=0}^{L-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{L}\gamma z} \mathrm{e}^{-\frac{2\pi \mathrm{i}}{L}\gamma\zeta} \psi(T_L^\zeta \hat{x}), \tag{4.31}$$

where we used that $\mathfrak{F}_L^N \psi \in \mathbb{S}_L^N$. By applying the coordinate shift $\sigma = \frac{\gamma}{L/n_{\hat{x}}}$ we see that (4.31) is equal to

$$\begin{aligned}\frac{1}{L} \sum_{\sigma=0}^{n_{\hat{x}}-1} \sum_{\xi=0}^{n_{\hat{x}}-1} \sum_{k=0}^{L/n_{\hat{x}}-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{n_{\hat{x}}}(z-(\xi+kn_{\hat{x}}))\sigma} \psi(T_L^\xi \hat{x}) \\ = \frac{1}{n_{\hat{x}}} \sum_{\xi=0}^{n_{\hat{x}}-1} \left[\sum_{\sigma=0}^{n_{\hat{x}}-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{n_{\hat{x}}}(z-\xi)\sigma} \right] \psi(T_L^\xi \hat{x}) = \psi(T_L^z \hat{x}). \end{aligned} \tag{4.32}$$

It can be shown analogously that $\mathfrak{F}_L^N (\mathfrak{F}_L^N)^* = 1$. □

4.2.3 The Schrödinger operator formulation

Let again $L, N \in \mathbb{N}$ with $N < L$ be fixed. In [FS18] it was shown that the N-particle Hamiltonian is unitarily equivalent to a discrete Schrödinger operator H_L^N acting on $\mathbb{H}_L^N \cong \ell^2(\mathcal{V}_L^N)$ with

$$H_L^N \cong -\frac{1}{2\Delta} A_L^N + W_L^N . \tag{4.33}$$

Here, A_L^N denotes the adjacency operator on $\mathcal{G}_L^N$

$$(A_L^N \psi)(x) := \sum_{y:\, x \sim y} \psi(y) \quad \text{for all } x \in \mathcal{V}_L^N, \tag{4.34}$$

where we use the notation $x \sim y$ for $\{x, y\} \in \mathcal{E}_L^N$. Furthermore, the operator W_L^N is a multiplication by the function $W : \mathcal{P}(\mathcal{V}_L) \to \mathbb{N}_0$ restricted to $\mathbb{H}_L^N$ which counts the number of connected components of a configuration $x \in \mathcal{P}(\mathcal{V}_L)$

$$W(x) := \frac{1}{2} \left| \{ \{\alpha, \beta\} \in \mathcal{E}_L : \alpha \in x, \beta \notin x \} \right| . \tag{4.35}$$

Let us now consider the Fourier transform of the Hamiltonian $\hat{H}_L^N := \mathfrak{F}_L^N H_L^N (\mathfrak{F}_L^N)^*$, with $\hat{A}_L^N$ and $\hat{W}_L^N$ being defined analogously.

Lemma 4.2.5. *For any $\phi \in \mathbb{S}_L^N$, $\hat{x} \in \widehat{\mathcal{V}}_L^N$ and $\gamma \in \mathcal{V}_L$ we have*

$$(\hat{H}_L^N \phi)(\gamma, \hat{x}) = -\frac{1}{2\Delta} \sum_{\hat{y}} a_{L,\gamma}^N(\hat{x}, \hat{y}) \phi(\gamma, \hat{y}) + W(\hat{x}) \phi(\gamma, \hat{x}), \tag{4.36}$$

where the matrix elements of $a_{L,\gamma}^N$ are given by

$$a_{L,\gamma}^N(\hat{x}, \hat{y}) = \sum_{\substack{z \in \{0,\cdots,n_{\hat{y}}-1\}: \\ T_L^z \hat{y} \sim \hat{x}}} e^{\frac{2\pi i}{L} \gamma z}. \tag{4.37}$$

Proof. Firstly, concerning the potential W_L^N, we observe that for any $\phi \in \mathbb{S}_N^L$ we get

$$(\mathfrak{F}_L^N W_L^N (\mathfrak{F}_L^N)^* \phi)(\gamma, \hat{x}) = W(\hat{x}) \phi(\gamma, \hat{x}). \tag{4.38}$$

Let us now consider the adjacency operator A_L^N.

$$\begin{aligned}(\mathfrak{F}_L^N A_L^N (\mathfrak{F}_L^N)^* \phi)(\gamma, \hat{x}) &= \frac{1}{\sqrt{L}} \sum_{z=0}^{L-1} \mathrm{e}^{-\frac{2\pi \mathrm{i}}{L} \gamma z} (A_L^N (\mathfrak{F}_L^N)^* \phi)(T_L^z \hat{x}) \\ &= \frac{1}{\sqrt{L}} \sum_{z=0}^{L-1} \sum_{y: y \sim T_L^z \hat{x}} \mathrm{e}^{-\frac{2\pi \mathrm{i}}{L} \gamma z} ((\mathfrak{F}_L^N)^* \phi)(y)\end{aligned} \tag{4.39}$$

For any $y \in \mathcal{V}_L^N$ there exists a unique $\hat{y} \in \widehat{\mathcal{V}}_L^N$ and $\sigma \in \{0, \cdots, n_{\hat{y}} - 1\}$ such that $y = \tilde{T}_L^\sigma \hat{y}$. Hence,

$$(\mathfrak{F}_L^N A_L^N (\mathfrak{F}_L^N)^* \phi)(\gamma, \hat{x}) = \frac{1}{L} \sum_{z=0}^{L-1} \sum_{\hat{y}} \sum_{\substack{\sigma \in \{0,\cdots,n_{\hat{y}}-1\}: \\ T_L^{\sigma - z} \hat{y} \sim \hat{x}}} \sum_{\xi=0}^{L-1} \mathrm{e}^{-\frac{2\pi \mathrm{i}}{L} z(\gamma - \xi)} \mathrm{e}^{\frac{2\pi \mathrm{i}}{L} \xi(\sigma - z)} \phi(\xi, \hat{y}). \tag{4.40}$$

For any $\hat{y} \in \widehat{\mathcal{V}}_L^N$ and $\xi \notin (L/n_{\hat{y}})\{0, \cdots, n_{\hat{y}} - 1\}$ we have $\phi(\xi, \hat{y}) = 0$, since $\phi \in \mathbb{S}_L^N$. We therefore consider only $\xi \in (L/n_{\hat{y}})\{0, \cdots, n_{\hat{y}} - 1\}$. The second factor in (4.40) is subsequently given by

$$\mathrm{e}^{\frac{2\pi \mathrm{i}}{L} \xi(\sigma - z)} = \mathrm{e}^{\frac{2\pi \mathrm{i}}{n_{\hat{y}}} \frac{\xi}{L/n_{\hat{y}}} (\sigma - z) \bmod n_{\hat{y}}}. \tag{4.41}$$

By changing the summation index in (4.40) from σ to $\zeta := (\sigma - z) \bmod n_{\hat{y}}$ we conclude that (4.40) is equal to

$$\frac{1}{L} \sum_{\hat{y}} \sum_{\substack{\zeta \in \{0,\cdots,n_{\hat{y}}-1\}: \\ T_L^\zeta \hat{y} \sim \hat{x}}} \sum_{\xi=0}^{L-1} \left[\sum_{z=0}^{L-1} \mathrm{e}^{-\frac{2\pi \mathrm{i}}{L} z(\gamma - \xi)} \right] \mathrm{e}^{\frac{2\pi \mathrm{i}}{L} \xi \zeta} \phi(\xi, \hat{y}) = \sum_{\hat{y}} \sum_{\substack{\zeta \in \{0,\cdots,n_{\hat{y}}-1\}: \\ T_L^\zeta \hat{y} \sim \hat{x}}} \mathrm{e}^{\frac{2\pi \mathrm{i}}{L} \gamma \zeta} \phi(\gamma, \hat{y}). \tag{4.42}$$

This concludes the proof. □

Remark 4.2.6. *The operator $\hat{A}_L^N$ is selfadjoint on $\mathbb{S}_L^N$, since it is unitarily equivalent to the selfadjoint operator A_L^N. This implies in particular that for all $\gamma \in \mathcal{V}_L$ and $\hat{x}, \hat{y} \in \widehat{\mathcal{V}}_L^N$ we obtain*

$$\frac{1}{L/n_{\hat{x}}} a_{L,\gamma}^N(\hat{x}, \hat{y}) = \frac{1}{L/n_{\hat{y}}} a_{L,\gamma}^N(\hat{y}, \hat{x}). \tag{4.43}$$

Now, we decompose $\mathbb{S}_L^N$ into fibre spaces corresponding to the fibre index $\gamma \in \mathcal{V}_L$. We obtain

$$\mathbb{S}_L^N = \bigoplus_{\gamma=0}^{L-1} \mathbb{S}_{L,\gamma}^N , \tag{4.44}$$

where

$$\mathbb{S}_{L,\gamma}^N := \{\phi \in \mathbb{S}_L^N : \ \forall \hat{x} \in \widehat{\mathcal{V}}_L^N, \, \forall \sigma \in \mathcal{V}_L, \, \sigma \neq \gamma, \text{ we have } \phi(\sigma, \hat{x}) = 0\}. \tag{4.45}$$

In Lemma 4.2.5 it is shown that for each $\gamma \in \mathcal{V}_L$ the subspace $\mathbb{S}_{L,\gamma}^N$ reduces $\hat{H}_L^N$. Consequently, we decompose

$$\hat{H}_L^N = \bigoplus_{\gamma=0}^{L-1} \hat{H}_{L,\gamma}^N, \tag{4.46}$$

where $\hat{H}_{L,\gamma}^N := \hat{H}_L^N|_{\mathbb{S}_{L,\gamma}^N}$. Analogously, we set $\hat{A}_{L,\gamma}^N := \hat{A}_L^N|_{\mathbb{S}_{L,\gamma}^N}$ and $\hat{W}_{L,\gamma}^N := \hat{W}_L^N|_{\mathbb{S}_{L,\gamma}^N}$ and thus obtain

$$\hat{H}_L^N = \bigoplus_{\gamma=0}^{L-1} \hat{H}_{L,\gamma}^N = \bigoplus_{\gamma=0}^{L-1} \Big(-\frac{1}{2\Delta} \hat{A}_{L,\gamma}^N + \hat{W}_{L,\gamma}^N \Big) \tag{4.47}$$

and

$$\sigma(H_L^N) = \sigma(\hat{H}_L^N) = \bigcup_{\gamma=0}^{L-1} \sigma(\hat{H}_{L,\gamma}^N). \tag{4.48}$$

4.2.4 Combes–Thomas estimate on fibre operators and proof of Theorem 4.2.1

Let again $L, N \in \mathbb{N}$ with $N < L$ be fixed. For the reader's convenience we will omit the indices N and L in the following proofs. However, every quantity may depend on N and L unless stated otherwise.

Lemma 4.2.7. *For all $\gamma \in \mathcal{V}_L$ the operator $\hat{A}_{L,\gamma}^N$ satisfies*

$$-2\hat{W}_{L,\gamma}^N \leq \hat{A}_{L,\gamma}^N \leq 2\hat{W}_{L,\gamma}^N. \tag{4.49}$$

Proof. It is sufficient to prove only the upper bound. The lower bound follows analogously by considering $-\hat{A}_\gamma$.
Let $\hat{x} \in \widehat{\mathcal{V}}$. Equation (4.37) implies

$$\sum_{\hat{y} \in \widehat{\mathcal{V}}} |a_\gamma(\hat{x}, \hat{y})| \leq \sum_{\hat{y}} \sum_{\substack{z \in \{0, \cdots, n_{\hat{y}}\}: \\ T^z \hat{y} \sim \hat{x}}} \left| e^{\frac{2\pi i}{L} \gamma z} \right| = \sum_{\substack{y \in \mathcal{V}: \\ y \sim \hat{x}}} 1. \tag{4.50}$$

According to (4.35) we get

$$\sum_{\hat{y} \in \widehat{\mathcal{V}}} |a_\gamma(\hat{x}, \hat{y})| \leq 2W(\hat{x}). \tag{4.51}$$

Now, consider an arbitrary $\phi \in \mathbb{S}_\gamma$. Then

$$\begin{aligned} \langle \phi, \hat{A}_\gamma \phi \rangle_{\mathbb{S}} &= \sum_{\hat{x}, \hat{y} \in \widehat{\mathcal{V}}} \overline{\phi(\gamma, \hat{x})} \frac{1}{L/n_{\hat{x}}} a_\gamma(\hat{x}, \hat{y}) \phi(\gamma, \hat{y}) \\ &\leq \Big[\sum_{\hat{x}, \hat{y} \in \widehat{\mathcal{V}}} |\phi(\gamma, \hat{x})|^2 \frac{1}{L/n_{\hat{x}}} |a_\gamma(\hat{x}, \hat{y})| \Big]^{1/2} \Big[\sum_{\hat{x}, \hat{y} \in \widehat{\mathcal{V}}} |\phi(\gamma, \hat{y})|^2 \frac{1}{L/n_{\hat{x}}} |a_\gamma(\hat{x}, \hat{y})| \Big]^{1/2}. \end{aligned} \tag{4.52}$$

By the identity (4.43), we obtain

$$\langle \phi, \hat{A}_\gamma \phi \rangle_{\mathbb{S}} \leq \sum_{\hat{x},\,\hat{y}\in\widehat{\mathcal{V}}} |\phi(\gamma,\hat{x})|^2 \frac{1}{L/n_{\hat{x}}} |a_\gamma(\hat{x},\hat{y})|. \tag{4.53}$$

Hence, by applying (4.51) we arrive at

$$\langle \phi, \hat{A}_\gamma \phi \rangle_{\mathbb{S}} \leq 2 \sum_{\hat{x}} |\phi(\gamma,\hat{x})|^2 \frac{1}{L/n_{\hat{x}}} W(\hat{x}) = 2 \langle \phi, \hat{W}_\gamma \phi \rangle_{\mathbb{S}}. \tag{4.54}$$

□

We are now able to prove a Combes–Thomas estimate on a fibre. The following is an adaptation of the proof of a similar result on the infinite-volume XXZ chain [ARFS20, EKS18b, EKS18a].

Theorem 4.2.8. *For any $\gamma \in \mathcal{V}_L$ and any multiplication operator $\hat{Y}^N_{L,\gamma} : \mathbb{S}^N_{L,\gamma} \to \mathbb{S}^N_{L,\gamma}$, consider the Hamiltonian $\hat{O}^N_{L,\gamma} = -\frac{1}{2\Delta}\hat{A}^N_{L,\gamma} + \hat{W}^N_{L,\gamma} + \hat{Y}^N_{L,\gamma}$. Moreover, let $z \notin \sigma(\hat{O}^N_{L,\gamma})$ be such that*

$$\left\| (\hat{W}^N_{L,\gamma})^{1/2} (\hat{O}^N_{L,\gamma} - z)^{-1} (\hat{W}^N_{L,\gamma})^{1/2} \right\| \leq \frac{1}{\kappa^N_L(z)} < \infty, \tag{4.55}$$

for some $\kappa^N_L(z) > 0$. Then for all $\mathcal{A}, \mathcal{B} \subset \widehat{\mathcal{V}}^N_L$ we have

$$\left\| 1_{\mathcal{A}} \big(\hat{O}^N_{L,\gamma} - z \big)^{-1} 1_{\mathcal{B}} \right\| \leq \frac{2}{\kappa^N_L(z)}\, e^{-\eta^N_L(z) \hat{d}^N_L(\mathcal{A},\mathcal{B})}, \tag{4.56}$$

where $\hat{d}^N_L(\mathcal{A},\mathcal{B}) := \inf\{\hat{d}^N_L(\hat{x},\hat{y}) : \hat{x} \in \mathcal{A},\ \hat{y} \in \mathcal{B}\}$ for all $\mathcal{A}, \mathcal{B} \in \mathcal{P}(\widehat{\mathcal{V}}^N_L)$ and

$$\eta^N_L(z) = \ln\left(1 + \frac{\kappa^N_L(z)\Delta}{2} \right). \tag{4.57}$$

Proof. Let us first observe that (4.49) implies that for any $\gamma \in \{0, \cdots, L-1\}$

$$-2 \leq (\hat{W}_\gamma)^{-1/2} \hat{A}_\gamma (\hat{W}_\gamma)^{-1/2} \leq 2. \tag{4.58}$$

Now, for any $\mathcal{A} \subseteq \widehat{\mathcal{V}}$, let $\rho_{\mathcal{A},\gamma} : \mathbb{S}_\gamma \to \mathbb{S}_\gamma$ be the operator of multiplication by $\hat{d}(\mathcal{A},\cdot)$, i.e. $(\rho_{\mathcal{A},\gamma}\phi)(\gamma,\hat{x}) := \hat{d}(\mathcal{A},\hat{x})\phi(\gamma,\hat{x})$ for any $\phi \in \mathbb{S}_\gamma$. For any $\eta > 0$ let us define

$$\hat{O}_{\eta,\gamma} := \mathrm{e}^{-\eta\rho_{\mathcal{A},\gamma}} \hat{O}_\gamma \mathrm{e}^{\eta\rho_{\mathcal{A},\gamma}} \tag{4.59}$$

and $\hat{B}_{\eta,\gamma} := \hat{O}_{\eta,\gamma} - \hat{O}_\gamma$. Observe that

$$\hat{B}_{\eta,\gamma} = -\frac{1}{2\Delta}\big(\mathrm{e}^{-\eta\rho_{\mathcal{A},\gamma}} \hat{A}_\gamma \mathrm{e}^{\eta\rho_{\mathcal{A},\gamma}} - \hat{A}_\gamma \big). \tag{4.60}$$

Now, for any $\phi \in \mathbb{S}_\gamma$, consider

$$\begin{aligned} &\left\| (\hat{W}_\gamma)^{-1/2} \hat{B}_{\eta,\gamma} (\hat{W}_\gamma)^{-1/2} \phi \right\|^2_{\mathbb{S}} \\ &= \frac{1}{4\Delta^2} \sum_{\hat{x}} \frac{1}{L/n_{\hat{x}}} \left| \sum_{\hat{y}} W^{-1/2}(\hat{x}) W^{-1/2}(\hat{y}) \big(\mathrm{e}^{\eta[\rho_{\mathcal{A},\gamma}(\hat{y}) - \rho_{\mathcal{A},\gamma}(\hat{x})]} - 1 \big) a_\gamma(\hat{x},\hat{y}) \phi(\gamma,\hat{y}) \right|^2 \end{aligned} \tag{4.61}$$

We note that for all $\gamma \in \{0, \cdots, L-1\}$ and all $\hat{x}, \hat{y} \in \widehat{\mathcal{V}}$ we have $|a_\gamma(\hat{x}, \hat{y})| \le a_0(\hat{x}, \hat{y})$ which follows from (4.37). Furthermore, we have $|\mathrm{e}^{\eta[\hat{d}(\hat{x},\mathcal{A}) - \hat{d}(\hat{y},\mathcal{A})]} - 1| \le (\mathrm{e}^\eta - 1)$ for all $\hat{x}, \hat{y} \in \widehat{\mathcal{V}}$ with $\hat{d}(\hat{x}, \hat{y}) = 1$. Hence, (4.61) is bounded by

$$\begin{aligned}
&\frac{1}{4\Delta^2}\big(\mathrm{e}^\eta - 1\big)^2 \sum_{\hat{x}} \frac{1}{L/n_{\hat{x}}}\bigg[\sum_{\hat{y}} W^{-1/2}(\hat{x}) W^{-1/2}(\hat{y}) a_0(\hat{x}, \hat{y}) |\phi(\gamma, \hat{y})|\bigg]^2 \\
&\le \frac{1}{4\Delta^2}\big(\mathrm{e}^\eta - 1\big)^2 \big\|(\hat{W}_0)^{-1/2} \hat{A}_0 (\hat{W}_0)^{-1/2} \tilde{\phi}\big\|^2, \qquad (4.62)
\end{aligned}$$

where $\tilde{\phi} \in \mathbb{S}_0$ is defined by $\tilde{\phi}(\tilde{\gamma}, \hat{x}) := \delta_{\gamma,0} |\phi(\tilde{\gamma}, \hat{x})|$ for all $\hat{x} \in \widehat{\mathcal{V}}$ and $\tilde{\gamma} \in \{0, \cdots, L-1\}$. The function $\tilde{\phi}$ is indeed an element of $\mathbb{S}_0$, since for all $\hat{x} \in \widehat{\mathcal{V}}$ we have $0 \in L/n_{\hat{x}}\{0, \cdots, n_{\hat{x}} - 1\}$. Clearly, $\|\tilde{\phi}\|_{\mathbb{S}} = \|\phi\|_{\mathbb{S}}$. By using (4.58) and (4.62) we further estimate the left-hand side of (4.61) and eventually get

$$\big\|(\hat{W}_\gamma)^{-1/2} \hat{B}_{\eta,\gamma} (\hat{W}_\gamma)^{-1/2}\big\| \le \frac{1}{\Delta}(e^\eta - 1). \qquad (4.63)$$

For $\eta \equiv \eta(z)$ as in (4.57) it now follows that

$$\begin{aligned}
&\big\|(\hat{W}_\gamma)^{-1/2} \hat{B}_{\eta,\gamma} (\hat{O}_\gamma - z)^{-1} (\hat{W}_\gamma)^{1/2}\big\| \\
&= \big\|(\hat{W}_\gamma)^{-1/2} \hat{B}_{\eta,\gamma} (\hat{W}_\gamma)^{-1/2} (\hat{W}_\gamma)^{1/2} (\hat{O}_\gamma - z)^{-1} (\hat{W}_\gamma)^{1/2}\big\| \le \frac{(\mathrm{e}^\eta - 1)}{\Delta\kappa(z)} = \frac{1}{2}. \qquad (4.64)
\end{aligned}$$

Using the resolvent identity we get

$$\begin{aligned}
&(\hat{W}_\gamma)^{1/2} (\hat{O}_{\eta,\gamma} - z)^{-1} (\hat{W}_\gamma)^{1/2} \big[I + (\hat{W}_\gamma)^{-1/2} \hat{B}_{\eta,\gamma} (\hat{O}_\gamma - z)^{-1} (\hat{W}_\gamma)^{1/2}\big] \\
&= (\hat{W}_\gamma)^{1/2} (\hat{O}_\gamma - z)^{-1} (\hat{W}_\gamma)^{1/2}. \qquad (4.65)
\end{aligned}$$

By further applying the elementary inequality $\|(I+C)^{-1}\| \le (1 - \|C\|)^{-1}$ for any operator $C : \mathbb{S}_\gamma \to \mathbb{S}_\gamma$, $\|C\| < 1$, we obtain from (4.55) and (4.64) that

$$\begin{aligned}
\big\|(\hat{W}_\gamma)^{1/2} (\hat{O}_{\eta,\gamma} - z)^{-1} (\hat{W}_\gamma)^{1/2}\big\| &\le \big\|(\hat{W}_\gamma)^{1/2} (\hat{O}_\gamma - z)^{-1} (\hat{W}_\gamma)^{1/2}\big\| \\
&\times \big\|\big[I + (\hat{W}_\gamma)^{-1/2} \hat{B}_{\eta,\gamma} (\hat{O}_\gamma - z)^{-1} (\hat{W}_\gamma)^{1/2}\big]^{-1}\big\| \le \frac{2}{\kappa(z)}. \qquad (4.66)
\end{aligned}$$

We conclude

$$\begin{aligned}
&\big\|1_{\mathcal{A}} (\hat{W}_\gamma)^{1/2} (\hat{O}_\gamma - z)^{-1} (\hat{W}_\gamma)^{1/2} 1_{\mathcal{B}}\big\| = \big\|1_{\mathcal{A}} \mathrm{e}^{\eta\rho_{\mathcal{A}}} (\hat{W}_\gamma)^{1/2} (O_{\eta,\gamma} - z)^{-1} (\hat{W}_\gamma)^{1/2} \mathrm{e}^{-\eta\rho_{\mathcal{A}}} 1_{\mathcal{B}}\big\| \\
&\le \big\|(\hat{W}_\gamma)^{1/2} (\hat{O}_{\eta,\gamma} - z)^{-1} (\hat{W}_\gamma)^{1/2}\big\| \big\|\mathrm{e}^{-\eta\rho_{\mathcal{A}}} 1_{\mathcal{B}}\big\| \le \frac{2}{\kappa(z)} \mathrm{e}^{-\eta \hat{d}(\mathcal{A},\mathcal{B})}, \qquad (4.67)
\end{aligned}$$

which is the desired result. □

We use this Combes–Thomas estimate to deduce pointwise upper bounds to eigenfunctions of the fibre operators. These estimates apply uniformly to all eigenstates corresponding to eigenvalues in a certain energy range. These energy ranges are associated with configurations of K or less clusters $\widehat{\mathcal{V}}^N_{L,K} := \{\hat{x} \in \widehat{\mathcal{V}}^N_L : W(\hat{x}) \le K\}$ and are given by

$$\tilde{I}_{K,\delta} := \Big[1 - \frac{1}{\Delta}, (K + 1 - \delta)\Big(1 - \frac{1}{\Delta}\Big)\Big], \qquad (4.68)$$

where $\delta \in]0,1[$ and $K \in \mathbb{N}$.

For $K \in \mathbb{N}$ with $K \le \|\hat{W}_L^N\|$ and $\gamma \in \mathcal{V}_L$, let $\hat{P}^N_{L,K,\gamma} : \mathbb{S}^N_{L,\gamma} \to \mathbb{S}^N_{L,\gamma}$ be the orthogonal projection given by

$$\hat{P}^N_{L,K,\gamma} := 1_{\le K}(\hat{W}^N_{L,\gamma}). \tag{4.69}$$

Let further the projection $\hat{P}^N_{L,K} : \mathbb{S}^N_L \to \mathbb{S}^N_L$ be defined by

$$\hat{P}^N_{L,K} := \bigoplus_{\gamma \in \mathcal{V}_L} \hat{P}^N_{L,K,\gamma}. \tag{4.70}$$

Theorem 4.2.9. *Let $K \in \mathbb{N}$ with $K \le \|\hat{W}_L^N\|$, $\delta \in]0,1[$ and $\gamma \in \mathcal{V}_L$. For any $E \in \sigma(\hat{H}^N_{L,\gamma}) \cap \tilde{I}_{K,\delta}$ let $\phi^N_{L,\gamma} \equiv \phi^N_{L,\gamma}(\Delta, E) \in \mathbb{S}^N_{L,\gamma}$ be a corresponding eigenstate. Then, for any $\mathcal{A} \subseteq \widehat{\mathcal{V}}^N_L$ we obtain*

$$\|1_{\mathcal{A}}\phi^N_{L,\gamma}\| \le \frac{2(K+1)^2}{\delta} \cdot e^{-\tilde{\mu}_K \hat{d}^N_L(\mathcal{A}, \widehat{\mathcal{V}}^N_{L,K})} \|\hat{P}^N_{L,K,\gamma}\phi^N_{L,\gamma}\|, \tag{4.71}$$

where

$$\tilde{\mu}_K \equiv \tilde{\mu}_K(\delta, \Delta) := \ln\Big(1 + \frac{\delta(\Delta-1)}{2(K+1)}\Big). \tag{4.72}$$

Proof. Let us define the multiplication operator $\hat{Y}_{K,\gamma} := (K+1)(1-1/\Delta)\hat{P}_{K,\gamma}$. Then

$$\begin{aligned}(\hat{W}_\gamma)^{-1/2}\big(\hat{H}_\gamma + \hat{Y}_{K,\gamma} - E\big)(\hat{W}_\gamma)^{-1/2} &= -\frac{1}{2\Delta}(\hat{W}_\gamma)^{-1/2}\hat{A}_\gamma(\hat{W}_\gamma)^{-1/2} + 1 \\ &\quad + (K+1)\Big(1-\frac{1}{\Delta}\Big)\hat{P}_{K,\gamma}(\hat{W}_\gamma)^{-1} - E(\hat{W}_\gamma)^{-1}\end{aligned} \tag{4.73}$$

By using the result of Lemma 4.2.7 as well as $E \in \tilde{I}_{K,\delta}$ we estimate

$$-\frac{1}{2\Delta}(\hat{W}_\gamma)^{-1/2}\hat{A}_\gamma(\hat{W}_\gamma)^{-1/2} + 1 \ge \Big(1 - \frac{1}{\Delta}\Big). \tag{4.74}$$

Moreover,

$$(K+1)\Big(1-\frac{1}{\Delta}\Big)\hat{P}_{K,\gamma}(\hat{W}_\gamma)^{-1} - E\hat{P}_{K,\gamma}(\hat{W}_\gamma)^{-1} \ge \delta\Big(1-\frac{1}{\Delta}\Big)\hat{P}_{K,\gamma} \tag{4.75}$$

and

$$-E(1-\hat{P}_{K,\gamma})(\hat{W}_\gamma^{-1}) \ge -\frac{E}{K+1}(1-\hat{P}_{K,\gamma}) \ge \Big(1-\frac{1}{\Delta}\Big)\Big(-1+\frac{\delta}{K+1}\Big)(1-\hat{P}_{K,\gamma}). \tag{4.76}$$

Hence, (4.73) is estimated from below by

$$(\hat{W}_\gamma)^{-1/2}\big(\hat{H}_\gamma + \hat{Y}_{K,\gamma} - E\big)(\hat{W}_\gamma)^{-1/2} \ge \frac{\delta}{K+1}\Big(1-\frac{1}{\Delta}\Big). \tag{4.77}$$

This implies that $E \notin \sigma(\hat{H}_\gamma + \hat{Y}_{K,\gamma})$ and in particular, we get

$$\big\|(\hat{W}_\gamma)^{1/2}\big(\hat{H}_\gamma + \hat{Y}_{K,\gamma} - E\big)^{-1}(\hat{W}_\gamma)^{1/2}\big\| \le \frac{(K+1)\Delta}{\delta(\Delta-1)}. \tag{4.78}$$

By Theorem 4.2.8, this implies that

$$\|1_{\mathcal{A}}(\hat{H}_\gamma + \hat{Y}_{K,\gamma} - E)^{-1} 1_{\mathcal{B}}\| \le \frac{2\Delta(K+1)}{\delta(\Delta-1)} \cdot \Big(1 + \frac{\delta(\Delta-1)}{2(K+1)}\Big)^{-\hat{d}(\mathcal{A},\mathcal{B})} \tag{4.79}$$

for any $\mathcal{A}, \mathcal{B} \in \widehat{\mathcal{V}}$. Now, consider

$$\|1_{\mathcal{A}}\phi_\gamma\|_{\mathbb{S}} = \Big\|1_{\mathcal{A}}\big(\hat{H}_\gamma + \hat{Y}_{K,\gamma} - E\big)^{-1}\big(\hat{H}_\gamma + \hat{Y}_{K,\gamma} - E\big)\phi_\gamma\Big\|_{\mathbb{S}}. \tag{4.80}$$

We note that since ϕ_γ is an eigenfunction of $\hat{H}_\gamma$ we have $(\hat{H}_\gamma - E)\phi_\gamma = 0$. Hence, (4.80) is equal to

$$\begin{aligned} &(K+1)\Big(1 - \frac{1}{\Delta}\Big)\Big\|1_{\mathcal{A}}\big(\hat{H}_\gamma + \hat{Y}_{K,\gamma} - E\big)^{-1}\hat{P}_{K,\gamma}\phi_\gamma\Big\|_{\mathbb{S}} \\ &\le \frac{2(K+1)^2}{\delta} \cdot \Big(1 + \frac{\delta(\Delta-1)}{2(K+1)}\Big)^{-\hat{d}(\mathcal{A},\widehat{\mathcal{V}}_K)} \|\hat{P}_{K,\gamma}\phi_\gamma\|_{\mathbb{S}}, \end{aligned} \tag{4.81}$$

which is the desired result. □

Applying this result regarding fibre operators to the full N-particle Hamiltonian yields Theorem 4.2.1. In fact, our result can be applied to obtain estimates for eigenfunctions with eigenenergy in the K-cluster band $\tilde{I}_{K,\delta}$ for any K and not just $K = 1$.

Corollary 4.2.10. *Let $K \in \mathbb{N}$. For every $E \in \tilde{I}_{K,\delta} \cap \sigma(H_L^N)$ there exists an eigenstate $|\psi_L^N\rangle \equiv |\psi_L^N(\Delta, E)\rangle \in \mathbb{H}_L^N$ such that for all $x \in \mathcal{V}_L^N$ we obtain*

$$|\langle\delta_x^L, \psi_L^N\rangle| \le \frac{2(K+1)^2}{\delta\sqrt{L}} \cdot e^{-\tilde{\mu}_K d_L^N(x, \mathcal{V}_{L,K}^N)}, \tag{4.82}$$

where $\tilde{\mu}_K(\delta, \Delta)$ was defined in (4.72)*.*

Proof. According to (4.48), for every $E \in \tilde{I}_{K,\delta} \cap \sigma(H)$ there exists a fibre index $\gamma \in \{0, \cdots, L\}$ such that $E \in \sigma(\hat{H}_\gamma)$. Let $\phi \equiv \phi(\Delta, E) \in \mathbb{S}_\gamma$ be a normalized eigenvector of $\hat{H}_\gamma$ for the eigenvalue E. Let $|\psi\rangle \equiv |\psi(\Delta, E)\rangle \cong (\mathfrak{F})^*\phi$ be the corresponding eigenstate of H.

Since $\phi \in \mathbb{S}_\gamma$ and by (4.27) we have

$$\psi(T^z\hat{x}) = \frac{1}{\sqrt{L}} \mathrm{e}^{\frac{2\pi \mathrm{i}}{L} z\gamma} \phi(\gamma, \hat{x}) \quad \text{for all } z \in \mathbb{Z} \text{ and } \hat{x} \in \widehat{\mathcal{V}}. \tag{4.83}$$

The result now follows from Theorem 4.2.9, since

$$|\langle\delta_x, \psi\rangle| = \frac{1}{\sqrt{L}}|\phi(\gamma, \hat{x})| = \frac{1}{\sqrt{L}}\|1_{\{\hat{x}\}}\phi\| \le \frac{2(K+1)^2}{\delta\sqrt{L}} \cdot \mathrm{e}^{-\tilde{\mu}_K \hat{d}(\hat{x}, \widehat{\mathcal{V}}_K)} \tag{4.84}$$

and

$$\hat{d}(\hat{x}, \widehat{\mathcal{V}}_K) = \min_\gamma d(T^\gamma\hat{x}, \widehat{\mathcal{V}}_K) = \min_\gamma d(x, T^\gamma\widehat{\mathcal{V}}_K) = d(x, \mathcal{V}_K), \tag{4.85}$$

where we used that $\bigcup_\gamma T^\gamma\widehat{\mathcal{V}}_K = \mathcal{V}_K$. □

Proof of Theorem 4.2.1. Recall that $I_1 = [1 - \frac{1}{\Delta}, 2(1 - \frac{1}{\Delta})[$ and $\mathcal{D}_L^N = \mathcal{V}_{L,1}^N$. According to Lemma C.1.1 we have $\sigma(H_L^N) \cap]1, 2(1-1/\Delta)[= \varnothing$, since $\Delta > 3$. Hence $I_1 \cap \sigma(H_L^N) = \tilde{I}_{1,1/2} \cap \sigma(H_L^N)$. The claim follows immediately from Corollary 4.2.10 with $\delta = 1/2$, $K = 1$ and $\mu_1(\Delta) = \tilde{\mu}_1(\Delta, 1/2)$. □

Remark 4.2.11. *(i) In Lemma C.1.1, it was shown that for $\Delta > 2$ and all $\gamma \in \mathcal{V}_L$, each fibre operator $\hat{H}_{L,\gamma}^N$ has exactly one eigenvalue $E_\gamma \in [(1-1/\Delta), 2(1-1/\Delta)[$. Let $\left\{ |\varphi_{L,\gamma}^N(\Delta)\rangle \right\}_{\gamma \in \mathcal{V}_L} \subset \mathbb{H}_L^N$ be the orthonormal set of corresponding eigenstates, which is unique up to phase factors.*

(ii) From Lemma C.1.2 it follows that $E_0 < E_\gamma$ for any $\gamma \neq 0$. This implies in particular that $|\varphi_{L,0}^N(\Delta)\rangle$ is the unique ground state of H_L^N.

4.3 Perturbing the Ising limit

Let again $N, L \in \mathbb{N}$ with $N < L$. The main idea for Theorem 4.1.1 is to view it as a perturbative result of the Ising limit "$\Delta = \infty$". From Theorem 4.2.1 it readily follows that for all $\gamma \in \mathcal{V}_L$ the density operator $\rho(\varphi_{L,\gamma}^N(\Delta))$ converges weakly for $\Delta \to \infty$ towards

$$\rho_{L,\gamma}^N := \rho(\varphi_{L,\gamma}^N(\infty)) = \sum_{\zeta,\xi \in \mathcal{V}_L} \frac{1}{\sqrt{L}} \mathrm{e}^{\frac{2\pi \mathrm{i}}{L}(\zeta - \xi)\gamma} \left| \delta_{T_L^\zeta \hat{x}_0}^L \right\rangle\!\left\langle \delta_{T_L^\xi \hat{x}_0}^L \right|, \tag{4.86}$$

with

$$|\varphi_{L,\gamma}^N(\infty)\rangle := \sum_{\zeta \in \mathcal{V}_L} \frac{1}{\sqrt{L}} \mathrm{e}^{\frac{2\pi \mathrm{i}}{L}\zeta\gamma} \left| \delta_{T_L^\zeta \hat{x}_0}^L \right\rangle \in \mathbb{H}_L^N \tag{4.87}$$

where $\hat{x}_0 \in \widehat{\mathcal{V}}_L^N \cap \mathcal{D}_L^N$ is the unique representative of all droplets in $\mathcal{V}_L^N$. The state $|\varphi_{L,\gamma}^N(\infty)\rangle$ itself is an eigenstate of the Ising Hamiltonian

$$H_L(\infty) := \sum_{\{j,k\} \in \mathcal{E}_L} \left(\frac{1}{4} - S_j^3 S_k^3 \right), \tag{4.88}$$

since $\operatorname{span}\left\{ |\delta_x^L\rangle : x \in \mathcal{D}_L^N \right\}$ is the eigenspace to the groundstate energy 1. This fact follows immediately from the construction presented in the last chapter, since the N-particle Hamiltonian $H_L^N(\infty)$ can be identified with the multiplication operator W_L^N.

As we will see in the following, the entanglement entropy of $\rho_{L,\gamma}^N$ satisfies the desired logarithmic correction to the area law. In order to calculate the entanglement entropy of a given pure state $|\psi\rangle \in \mathbb{H}_L^N$ recall that it is necessary to determine its partial trace first. We are interested in the partial trace with respect to the decomposition $\mathbb{H}_L = \mathbb{H}_\Lambda \otimes \mathbb{H}_{\Lambda^c}$ for a region $\Lambda \subseteq \mathcal{V}_L$. Due to the constant particle number of $|\psi\rangle$, the partial trace can be decomposed into a direct sum of operators acting on the n-particle subspaces $\mathbb{H}_\Lambda^n$ for $n \in \{0, \cdots, N\}$.

Lemma 4.3.1. *Let $\Lambda \subset \mathcal{V}_L$. For any state $|\psi\rangle \in \mathbb{H}_L^N$ and for all $n \in \left\{0, \cdots, \min\{|\Lambda|, N\}\right\}$ there exists a $\rho_\Lambda^n(\psi) \in L(\mathbb{H}_\Lambda^n)$ such that*

$$\operatorname{tr}_{\Lambda^c}\left\{\rho(\psi)\right\} = \bigoplus_{n=0}^{\min\{|\Lambda|,N\}} \rho_\Lambda^n(\psi). \tag{4.89}$$

Furthermore for any $n \in \{0, \cdots, \min\{|\Lambda|, N\}\}$ *and any* $y, y' \in \mathbb{H}_L^n$ *we have*

$$\langle \delta_y^\Lambda, \rho_\Lambda^n(\psi)\delta_{y'}^\Lambda \rangle = \sum_{\substack{z \in \mathcal{P}(\Lambda^c), \\ |z| = N-n}} \langle \delta_{y \cup z}^L, \rho(\psi)\delta_{y' \cup z}^L \rangle \,. \tag{4.90}$$

Proof. First, we remark that the N-particle space $\mathbb{H}_L^N$ is a subspace of $\mathbb{H}_L = \mathbb{H}_\Lambda \otimes \mathbb{H}_{\Lambda^c}$. The elements of the canonical bases of these vector spaces are related in the following way: For $x \subseteq \mathcal{V}_L$ with $y := x \cap \Lambda \subseteq \Lambda$ and $z := x \cap \Lambda^c \subseteq \Lambda^c$ we have

$$|\delta_x^{\mathcal{V}_L}\rangle = |\delta_y^\Lambda\rangle \otimes |\delta_z^{\Lambda^c}\rangle \,. \tag{4.91}$$

Notice that $\mathcal{P}(\mathcal{V}_L) = \{y \cup z : \; y \in \mathcal{P}(\Lambda) \text{ and } z \in \mathcal{P}(\Lambda^c)\}$. The statement is shown by applying the definition of the partial trace together with (4.91). Hence,

$$\operatorname{tr}_{\Lambda^c}\{\rho(\psi)\} = \sum_{y,y' \in \mathcal{P}(\Lambda)} \left[\sum_{z \in \mathcal{P}(\Lambda^c)} \langle \delta_{y \cup z}^L, \rho(\psi)\delta_{y' \cup z}^L \rangle \right] |\delta_y^\Lambda\rangle\langle\delta_{y'}^\Lambda| \,. \tag{4.92}$$

□

Readily, we now apply this lemma to the N-particle eigenstate $|\varphi_{L,\gamma}^N(\infty)\rangle \in \mathbb{H}_L^N$ of the Ising limit. For $n \in \{0, \cdots, N\}$ let therefore

$$\rho_{L,\Lambda,\gamma}^n := \rho_\Lambda^n(\varphi_{L,\gamma}^N(\infty)). \tag{4.93}$$

To simplify calculations, we impose some further restrictions on the region and the particle number. We consider a connected region

$$\Lambda_L := \{\lambda_-, \cdots, \lambda_+\} \subset \mathcal{V}_L, \tag{4.94}$$

where the boundary points $\lambda_-, \lambda_+ \in \mathcal{V}_L$ satisfy $\lambda_+ - \lambda_- = 2\lfloor \theta L \rfloor$ for some $\theta \in]0, 1/2[$. Recall that $N = \lfloor \varepsilon N \rfloor$ with a constant particle density ε. The particle density is chosen to be small, while Λ_L is chosen to be smaller than half of the ring but at the same time also significantly larger than the particle number. The exact conditions are given in Assumption 4.4.15.

It is easy to see that under these assumptions we have

$$\rho_{L,\Lambda_L,\gamma}^n = \frac{1}{L}(|\delta_{y_+^n}^{\Lambda_L}\rangle\langle\delta_{y_+^n}^{\Lambda_L}| + |\delta_{y_-^n}^{\Lambda_L}\rangle\langle\delta_{y_-^n}^{\Lambda_L}|) \tag{4.95}$$

for all $n \in \{1, \cdots, N-1\}$, where

$$y_\pm^n \equiv y_\pm^n(\Lambda_L) := \lambda_\pm \mp \{0, \cdots, n-1\}. \tag{4.96}$$

This is a consequence of the fact that only two droplet configurations in $\mathcal{D}_L^N$ contain exactly n-particles inside Λ_L and $N-n$ inside Λ_L^c, see Figure 4.3. For any $n \in \{1, \cdots, N-1\}$ the operator in (4.95) has two non-trivial eigenvalues, both of size $1/L$. Hence,

$$\operatorname{tr}\{s(\rho_{L,\Lambda_L,\gamma}^n)\} = \frac{2}{L}\log_2 L. \tag{4.97}$$

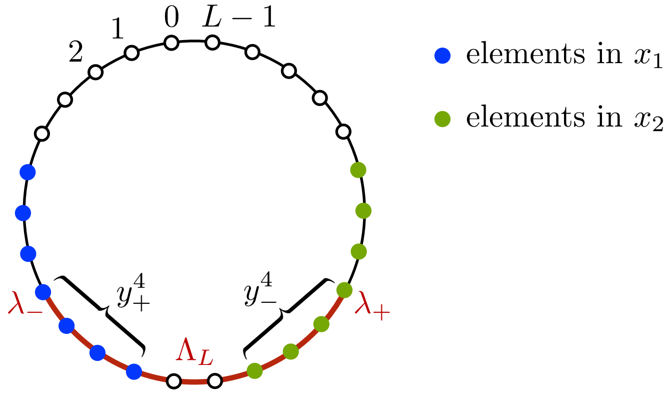

Figure 4.3: The only two droplets $x_1, x_2 \in \mathcal{D}_L^7$ with exactly four particles inside Λ_L.

Recall that the entropy function s was given by $s(x) = -x \log_2 x$ for all $x \in]0,1[$. The entanglement entropy of $|\varphi_{L,\gamma}^N(\infty)\rangle$ in total can be easily estimated from below by

$$S(\Lambda_L; \mathcal{V}_L, \varphi_{L,\gamma}^N(\infty)) = \sum_{n=0}^{N} \operatorname{tr}\left\{s\big(\rho_{L,\Lambda_L,\gamma}^n\big)\right\} \geq \frac{2(N-1)}{L} \log_2 L \geq \varepsilon \log_2 L \tag{4.98}$$

for all $L > 3\varepsilon^{-1}$. Since $|\Lambda_L| = 2\lfloor \theta L \rfloor + 1$, this constitutes a logarithmically enhanced area law.

Our strategy for proving Theorem 4.1.1 is to extend the result of the Ising limit to low energy eigenstates of the Ising phase. We therefore view $\rho(\varphi_{L,\gamma}^N)$ as a perturbation of $\rho_{L,\gamma}^N$. Our aim is to show the inequality

$$\left|\operatorname{tr}\left\{s\big(\rho_{L,\Lambda_L,\gamma}^n\big)\right\} - \operatorname{tr}\left\{s\big(\rho_{\Lambda_L}^n[\varphi_{L,\gamma}^N(\Delta)]\big)\right\}\right| \leq \frac{1}{L} \log_2 L. \tag{4.99}$$

for sufficiently many $n \in \{0, \cdots, N\}$. For proving a lower bound to the entanglement entropy it suffices to prove this inequality for $n \in \{\lceil N/2 \rceil, \cdots, N-1\}$. We are able to throw away all contributions of $s(\rho_{\Lambda_L}^n(\varphi_{L,\gamma}^N))$ for all $n < \lceil N/2 \rceil$, since s is a non-negative function. In total we obtain

$$\operatorname{tr}\left\{s\big(\rho_{\Lambda_L}[\varphi_{L,\gamma}^N(\Delta)]\big)\right\} \geq \sum_{n=\lceil N/2 \rceil}^{N-1} \operatorname{tr}\left\{s\big(\rho_{\Lambda_L}^n[\varphi_{L,\gamma}^N(\Delta)]\big)\right\} \geq \frac{\varepsilon}{4} \log_2 L \tag{4.100}$$

for all $L > 4\varepsilon^{-1}$, which constitutes a logarithmically enhanced area law of the entanglement entropy.

The only thing that remains to be done is showing the estimate (4.99). Like in Section 2.4.1, our main tool to this effect is Kreĭns trace formula, see e.g. [Sch12, Sect. 9.7]. We get

$$\left|\operatorname{tr}\left\{s\big(\rho_{\Lambda_L}^n[\varphi_{L,\gamma}^N(\Delta)]\big)\right\} - \operatorname{tr}\left\{s\big(\rho_{L,\Lambda_L,\gamma}^n\big)\right\}\right| = \left|\int_0^1 \mathrm{d}t \; s'(t) \xi_{L,\Lambda_L,\gamma}^n(t)\right|, \tag{4.101}$$

where

$$\xi_{L,\Lambda_L,\gamma}^n : \mathbb{R} \ni t \mapsto \operatorname{tr}\left\{1_{\leq t}\big(\rho_{\Lambda_L}^n[\varphi_{L,\gamma}^N(\Delta)]\big) - 1_{\leq t}\big(\rho_{L,\Lambda_L,\gamma}^n\big)\right\} \tag{4.102}$$

denotes the spectral shift function. We remark here that (4.101) does not follow directly from Kreĭns trace formula, since s cannot be extended to $\mathbb{R}$ to be a $C^1(\mathbb{R})$-function. However, we achieve (4.101) by approximating s with compactly supported C^∞-functions.

Both $s'1_{[0,1]}$ and $\xi^n_{L,\Lambda_L,\gamma}$ are L^p-integrable for any $p \in [1,\infty[$. According to [CHN01, Thm. 2.1] the L^p-norm of the spectral shift function is bounded by

$$\|\xi^n_{L,\Lambda_L,\gamma}\|_p \le \|\rho^n_{\Lambda_L}[\varphi^N_{L,\gamma}(\Delta)] - \rho^n_{L,\Lambda_L,\gamma}\|^{1/p}_{1/p}, \tag{4.103}$$

where $\|\cdot\|_{1/p}$ denotes the $1/p$-th von Neumann-Schatten quasinorm. Let us introduce the notation

$$D^{N,n}_{L,\Lambda_L,\gamma} \equiv D^n_{L,\Lambda_L,\gamma}(\Delta) := \rho^n_{\Lambda_L}[\varphi^N_{L,\gamma}(\Delta)] - \rho^n_{L,\Lambda_L,\gamma}. \tag{4.104}$$

We claim that for all $n \in \{\lceil N/2 \rceil, \cdots, N-1\}$ we have

$$\|D^{N,n}_{L,\Lambda_L,\gamma}\|^{1/p}_{1/p} \lesssim L^{-1/p} \mathrm{e}^{-\mu_1/p}, \tag{4.105}$$

where $\mu_1 \equiv \mu_1(\Delta)$ was defined in (4.16). It is reasonable to expect such an estimate because of Theorem 4.2.1. On the one hand, this theorem implies that the density operator $\rho^n_{\Lambda_L}[\varphi^N_{L,\gamma}]$ contains a factor L^{-1}. We recall from (4.95) that this is true for the density $\rho^n_{L,\Lambda_L,\gamma}$, too. The eigenvalues of $D^{N,n}_{L,\Lambda_L,\gamma}$ therefore also contain a factor L^{-1}, which then leads to the factor $L^{-1/p}$ in (4.105). On the other hand, all contributions to the state $|\varphi^N_{L,\gamma}\rangle$ of configurations other than droplets are exponentially small with respect to μ_1. Broadly speaking, to subtract $\rho^n_{L,\Lambda_L,\gamma}$ is to remove the large contributions of droplet configurations altogether. We therefore predict that the bound in 4.105 contains a factor $\mathrm{e}^{-\mu_1/p}$ as well.

In our final step we estimate the right-hand side of (4.101) by using Hölder's inequality. Note that the L^q-norm of the derivative of s satisfies

$$\|s'1_{[0,1]}\|_q \lesssim q \tag{4.106}$$

for all $q \in [1,\infty[$. To attain the right scaling behaviour in L we choose the Hölder coefficients to be L-dependent, namely $q \equiv q(L) := \ln L$ and $p \equiv p(L) := (1-[q(L)]^{-1})^{-1}$. By applying (4.106) and (4.105) we get

$$\left|\operatorname{tr}\left\{s\big(\rho^n_{\Lambda_L}[\varphi^N_{L,\gamma}(\Delta)]\big)\right\} - \operatorname{tr}\left\{s\big(\rho^n_{L,\Lambda_L,\gamma}\big)\right\}\right| \le \|s'1_{[0,1]}\|_q \|\xi^n_{L,\Lambda_L,\gamma}\|_p \lesssim \frac{\ln L}{L}\mathrm{e}^{-\mu_1(\Delta)/2}, \tag{4.107}$$

for all $L \ge 8$. Recall that according to Theorem 4.2.1 the decay rate $\mu_1(\Delta)$ diverges for $\Delta \to \infty$. It is therefore possible to determine $\Delta_0 > 3$ such that for all $\Delta \ge \Delta_0$ the inequality (4.99) is satisfied.

4.4 Estimating the entries of $D^{N,n}_{L,\Lambda_L,\gamma}$

4.4.1 General idea and strategy

Let us again consider $n, N, L \in \mathbb{N}$ with $N = \lfloor \varepsilon L \rfloor$ for an $\varepsilon \in]0,1[$ and $n < N < L$, as well as the region Λ_L given in (4.94). The purpose of this chapter is to provide bounds for

$$\langle \delta^{\Lambda_L}_y, D^{N,n}_{L,\Lambda_L,\gamma}(\Delta)\delta^{\Lambda_L}_{y'}\rangle \tag{4.108}$$

for any two configurations $y, y' \subseteq \Lambda_L$ with $n = |y'| = |y|$. Let us introduce the set

$$\mathcal{V}^m(\mathcal{A}) := \{y \subseteq \mathcal{A} : |y| = m\} \tag{4.109}$$

for any countable set $\mathcal{A}$ and $m \in \mathbb{N}_0$ with $m \le |\mathcal{A}|$.

Our main result of this section is given by the following lemma. It is an intermediate step towards estimating the eigenvalues of $D^{N,n}_{L,\Lambda_L,\gamma}(\Delta)$.

Lemma 4.4.1. *Let $\gamma \in \mathcal{V}_L$ and $\Delta > 3$ such that $\mu_1(\Delta) \ge \ln 2$, where $\mu_1 \equiv \mu_1(\Delta)$ was defined in (4.16). Let us assume $\varepsilon \in]0,1[$, $N \equiv N(\varepsilon) = \lfloor \varepsilon L \rfloor$ and Λ_L satisfy Assumption 4.4.15. Then there exists $L_0 \equiv L_0(\varepsilon) > 0$ such that for all $L \ge L_0$, $n \in \mathbb{N}$ with $N/2 < n < N$ and $y, y' \in \mathcal{V}^n(\Lambda_L)$ we have*

$$|\langle \delta_y^{\Lambda_L}, D^{N,n}_{L,\Lambda_L,\gamma}(\Delta)\delta_{y'}^{\Lambda_L}\rangle| \le \frac{2^{34}}{L} e^{-\mu_1} e^{-\mu_1(h_L^n(y)+h_L^n(y'))}. \tag{4.110}$$

Here the function $h_L^n : \mathcal{V}^n(\Lambda_L) \to [0,\infty[$ is defined by

$$h_L^n(y) := \begin{cases} \min\left\{d_L^{n+1}(y \cup \{a_{\pm,1}(\Lambda_L)\}, \mathcal{D}_L^{n+1}), L^{5/4}\right\} - 1 & \text{for } y \notin \{y_+^n, y_-^n\}, \\ 0 & \text{for } y \in \{y_+^n, y_-^n\}, \end{cases} \tag{4.111}$$

where $a_{\pm,1}(\Lambda_L) := \lambda_\pm \pm 1$ are the boundary points of Λ_L^c and $y_\pm^n$ was defined in (4.96).

Remark 4.4.2. *The function h_L^n defined in (4.111) naturally combines two measurements. On the one hand, it measures at what distance a given configuration $y \in \mathcal{V}^n(\Lambda_L)$ is to the closest droplet configuration. On the other hand, it quantifies how far away the aforementioned droplet configuration is from the boundary. The anchor particles at $a_{\pm,1}(\Lambda_L)$ are included in the definition of h_L^n to cover the second aspect. For an illustration of this function see Figure 4.4.*

For technical reasons, the function also contains a cut-off at $L^{5/4}$.

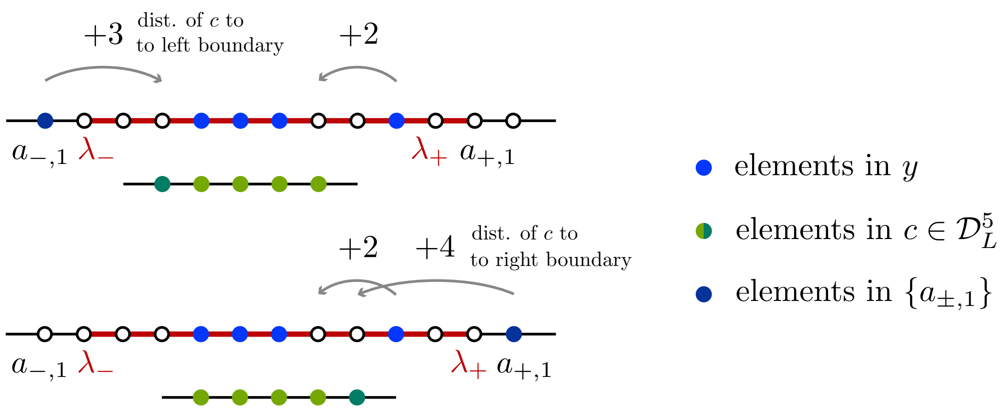

Figure 4.4: A configuration $y \in \mathcal{V}^4(\Lambda_L)$ with $h_L^4(y) = 5$

To prove Lemma 4.4.1 it is necessary to consider a sum of the form

$$F^N_{L,\Lambda_L}(y,\mu) := \sum_{\substack{z \in \mathcal{V}^{N-n}(\Lambda_L), \\ y \cup z \notin \mathcal{D}_L^N}} \mathrm{e}^{-\mu d_L^N(z \cup y, \mathcal{D}_L^N)}, \tag{4.112}$$

where $y \in \mathcal{V}^n(\Lambda_L)$ and $\mu > 0$. For $\Delta > 3$ and any two configurations $y, y' \in \mathcal{V}^n(\Lambda_L) \setminus \{y^n_\pm\}$ we have

$$\begin{aligned}\left|\left\langle \delta_y^{\Lambda_L}, D^{N,n}_{L,\Lambda_L,\gamma}\delta_{y'}^{\Lambda_L}\right\rangle\right| &= \left|\left\langle \delta_y^{\Lambda_L}, \rho^n_{\Lambda_L}[\varphi^N_{L,\gamma}(\Delta)]\delta_{y'}^{\Lambda_L}\right\rangle\right| \\ &\leq \frac{2^8}{L}\sqrt{F^N_{L,\Lambda_L}(y, 2\mu_1)F^N_{L,\Lambda_L}(y', 2\mu_1)},\end{aligned} \tag{4.113}$$

where we used (4.92) and Theorem 4.2.1 to derive this bound. Note that the first equality in (4.113) is only true, because we have excluded the configurations $y^n_\pm$ that were defined in (4.96). These configurations yield the main contributions to $\rho^n_{\Lambda_L}(\varphi^N_{L,\gamma}(\Delta))$ and the only contributions to $\rho^n_{L,\Lambda_L,\gamma}$, since they are derived from the droplet configurations in $|\varphi^N_{L,\gamma}(\Delta)\rangle$ and $|\varphi^N_{L,\gamma}(\infty)\rangle$ respectively. These configurations will be treated separately in Section 4.4.3, by determining upper and lower bounds to $|\langle \delta_c^{\mathcal{V}_L}, \varphi^N_{L,\gamma}(\Delta)\rangle|$ for any droplet $c \in \mathcal{D}^N_L$.

The sum in (4.112) is reminiscent of the geometric sum. This leads us to expect for any $y \in \mathcal{V}^n(\Lambda_L)$ and $\mu > 0$ an estimate of (4.112) that decays exponentially in y with respect to the function h^n_L, i.e.

$$F^N_{L,\Lambda_L}(y, \mu) \lesssim \mathrm{e}^{-\mu}\mathrm{e}^{-\mu h^n_L(y)}. \tag{4.114}$$

We remark that this result together with (4.113) proves Lemma 4.4.1 for most configurations y, y'.

The proof of an estimate of the form (4.114) is rather technically involved. For our approach it is necessary to restrict ourselves to a region Λ_L and a particle density ε that satisfy Assumption 4.4.15 as well as $\mu > \ln 2$. Furthermore, we only consider such configurations where more particles are inside Λ_L than outside, i.e. $N/2 < n < N$.

The main technical difficulty arises from the peculiar geometry of the ring, since it is not at all obvious what the graph distance between two given configurations is. It is similarly difficult to determine the distance to $\mathcal{D}^N_L$ for a given configuration. The situation on the infinite line stands in contrast to the one on the ring. Recall that the graph of the line was given by $\mathcal{G} = \{\mathbb{Z}, \mathcal{E}\}$ with edge-sets $\mathcal{E} = \big\{\{j, j+1\} : j \in \mathbb{N}\big\}$. The graph distance of the corresponding N-particle graph $\mathcal{G}^N = \{\mathcal{V}^N(\mathbb{Z}), \mathcal{E}^N_L\}$ can be calculated with the help of the formula [ARFS20]

$$d^N(x, x') = \sum_{j=1}^{N} |x_i - x'_i| \tag{4.115}$$

for any configurations $x^{(\prime)} = \{x_1^{(\prime)}, \cdots, x_N^{(\prime)}\} \in \mathcal{V}^N(\mathbb{Z})$ with $x_1^{(\prime)} < \cdots < x_N^{(\prime)}$. Moreover, it is known which droplet configurations are closest to x. Let

$$c^N_m := m + \left\{-\left\lfloor \frac{N-1}{2}\right\rfloor, \cdots, \left\lceil \frac{N-1}{2}\right\rceil\right\} \tag{4.116}$$

be a droplet centred around the site $m \in \mathbb{Z}$ and let $\mathcal{D}^N := \{c^N_m : m \in \mathbb{Z}\}$ denote the set of all droplets. The droplets closest to $x \in \mathcal{V}^N(\mathbb{Z})$ are centred around a site in

$$\mathcal{W}(x) := \{m \in \mathbb{Z} : d^N(x, c^N_m) = d^N(x, \mathcal{D}^N)\} = \left\{x_{\lfloor (N+1)/2\rfloor}, \cdots, x_{\lceil (N+1)/2\rceil} - 1\right\}, \tag{4.117}$$

as has been shown in [ARFS20, Lemma A.1]. This can be used to calculate $d^N(x, \mathcal{D}^N)$ and therefore also to estimate sums of the form (4.112).

Our main strategy now is to cut the ring open along one edge $e \in \mathcal{E}_L$ so that we can treat it analogous to the line. To understand why this is even possible, we have to turn our attention back to the N-particle graph distance.

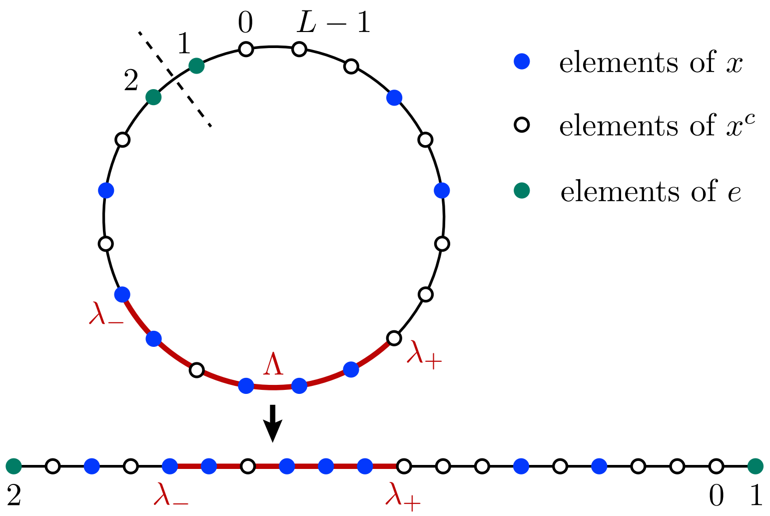

Figure 4.5: Cutting the ring open alongside the edge e.

Recall that the graph distance between two configurations $x, x' \in \mathcal{V}_L^N$ is the length of a shortest path from x to x'. For the exact definition of a path we refer to Definition 4.4.3. If we think of a configuration as the position of N individual hard-core particles, a path is a sequence of such configurations where in each step one particle hops to a neighbouring site. We claim that for each shortest path from a droplet to an arbitrary configuration there exists at least one edge $e \in \mathcal{E}_L$, which is not crossed by any particle at any time. This implies that we can remove the edge e from the graph altogether without changing the graph distance – we can cut the ring open alongside e, see Figure 4.5.

In general we cannot immediately determine for any given configuration $x \in \mathcal{V}_L^N$ which droplet $c \in \mathcal{D}_L^N$ satisfies $d_L^N(x, c) = d_L^N(x, \mathcal{D}_L^N)$. Consequently, we do not know at which edge we can cut the ring open in order to calculate this distance. The situation is different, if most particles of x are concentrated in a small sector of the ring such as Λ_L. In this case we prove that the closest droplet configurations must be centred around a site in the same region. Moreover, we show that we can cut the ring open alongside an edge outside of this sector. In view of these restrictions we only consider $n \in \mathbb{N}$ with $N/2 < n < N$, since this property ensures that all configurations in the summand of (4.112) are concentrated in Λ_L.

Let us now consider two configurations $y \in \mathcal{V}^n(\Lambda_L)$ and $z \in \mathcal{V}_L^{N-n}(\Lambda_L^c)$ for $N/2 < n < N$. Let $x := y \cup z \in \mathcal{V}_L^N$ and let $c \in \mathcal{D}_L^N$ be a droplet with $d_L^N(x, c) = d_L^N(x, \mathcal{D}_L^N)$. We have already established that c is centred around a site in Λ_L. To calculate the graph distance between x and c we now cut the ring open alongside a suitable edge $e \in \mathcal{E}_L$ and transform the ring into a line as described above. We only know that e is an edge somewhere in Λ_L^c, but we do not know its exact position. Let us assume that the cut divides the configuration z into ℓ particles to the left and r particles to the right of Λ_L for $\ell, r \in \mathbb{N}_0$ with $\ell + r = N - n$. Since we do not know the exact position of e we also do not now the value of the parameters ℓ, r, which may depend on y and z alike.

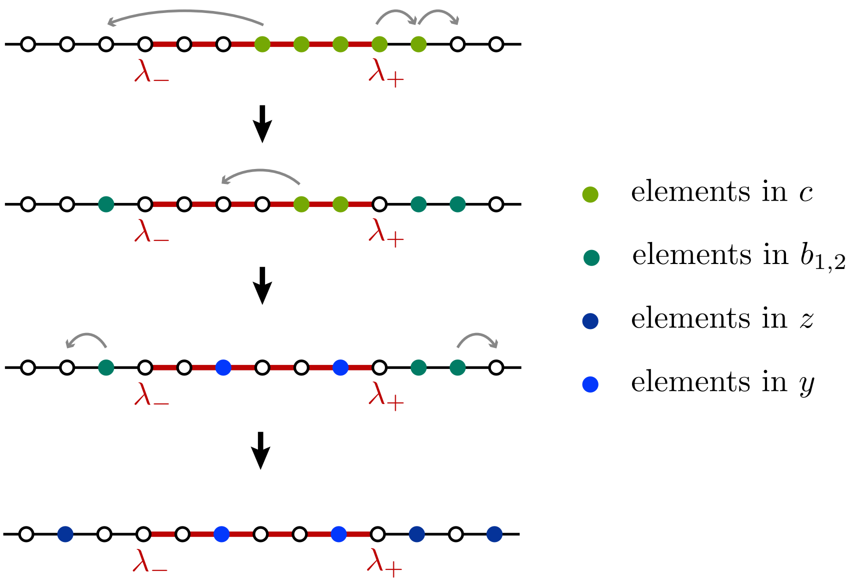

Figure 4.6: The construction of a path from c to $x = y \cup z$ via the configuration $y \cup b_{1,2}$, which constitutes a shortest path

In the next step, we construct a particular shortest path from c to x. First, we move the first ℓ particles on the left side of c into the positions $\lambda_- - \{1, \cdots, \ell\}$ outside of Λ_L and the last r particles on the right side of c into the positions $\lambda_+ + \{1, \cdots, r\}$. Then we move the remaining particles inside of Λ_L into the configuration y. We denote this intermediate configuration by $y \cup b_{\ell,r}$ where $b_{\ell,r} := [\lambda_- - \{1, \cdots, \ell\}] \cup [\lambda_+ + \{1, \cdots, r\}]$. In the last stage of this construction we move all particles outside of Λ_L into their final positions z. For an illustration of this path see Figure 4.6.

We now have

$$d_L^N(y \cup z, \mathcal{D}_L^N) = d^N(x, c) = d^N(y \cup z, y \cup b_{\ell,r}) + d^N(y \cup b_{\ell,r}, c), \tag{4.118}$$

where the second equality is due to the fact that there exists a shortest path from c to x passing through $y \cup b_{\ell,r}$. Here, in a slight abuse of notation, we denote by d^N the graph norm of the N-particle graph corresponding to the ring cut open alongside the edge e.

With the help of (4.118) the estimate (4.114) can be shown by executing the sum over all $z \in \mathcal{V}^{N-n}(\Lambda_L)$. This mainly concerns the first term on the right-hand side in (4.118), since the second term does not depend on z directly but only on ℓ and r. Note that we can immediately calculate this term by using (4.115). The function h_L^n emerges from the second term on the right-hand side of (4.118). To understand why, we point out that $b_{\ell,r}$ includes at least one of the boundary sites $a_{\pm,1}(\Lambda_L^c)$ for any ℓ and r. The only remaining obstacle is our lack of knowledge of the exact values of ℓ and r for any given configurations z and y. This difficulty is overcome by summing over all possible combinations $\ell, r \in \mathbb{N}_0$ with $\ell + r = N - n$.

4.4.2 Some technical preliminaries

For the entirety of this section, let $L, N \in \mathbb{N}$ with $N < L$ be fixed. Let $I(N) := \{1, \cdots, N\}$. We first concern ourselves with the peculiar geometry of the graph $\mathcal{V}_L^N$ and its graph norm.

Definition 4.4.3. *For any two points $x, y \in \mathcal{V}_L^N$ let*

$$\mathbf{u} := (u^{(0)}, \cdots, u^{(k)}) \in \bigtimes_{l=0}^{k} \mathcal{V}_L^N = (\mathcal{V}_L^N)^{k+1} \tag{4.119}$$

be called a path from x to y of length $k \in \mathbb{N}_0$, if and only if $u^{(0)} = x$, $u^{(k)} = y$ and $d_L^N(u^{(l-1)}, u^{(l)}) = 1$ for all $l \in \{1, \cdots, k\}$. If $k = d_L^N(x, y)$, then we call $\mathbf{u}$ a shortest path from x to y.

We first prove an auxiliary lemma necessary for showing the second equality (4.118).

Lemma 4.4.4. *Let $x, y \in \mathcal{V}_L^N$ and let $\mathbf{u}$ be a shortest path from x to y of length $k = d_L^N(x, y)$. Let $k_0 \in \{0, \cdots, k\}$,*

$$\mathbf{v} := (u^{(0)}, \cdots, u^{(k_0)}) \quad \text{and} \quad \mathbf{w} := (u^{(k_0)}, \cdots, u^{(k)}). \tag{4.120}$$

Then $\mathbf{v}$ is a shortest path from x to $u^{(k_0)}$ and $\mathbf{w}$ is a shortest path from $u^{(k_0)}$ to y. Moreover,

$$d_L^N(x, y) = d_L^N(x, u^{(k_0)}) + d_L^N(u^{(k_0)}, y). \tag{4.121}$$

Proof. The path $\mathbf{v}$ is a path from x to $u^{(k_0)}$ of length k_0. Therefore $d_L^N(x, u^{(k_0)}) \le k_0$. Analogously, $\mathbf{w}$ is a path from $u^{(k_0)}$ to y of length $k - k_0$ with $d_L^N(u^{(k_0)}, y) \le k - k_0$. Hence,

$$k = d_L^N(x, y) \le d_L^N(x, u^{(k_0)}) + d_L^N(u^{(k_0)}, y) \le k_0 + (k - k_0) = k. \tag{4.122}$$

This implies equality in (4.122) and consequently $d_L^N(x, u^{(k_0)}) = k_0$ and $d_L^N(u^{(k_0)}, y) = k - k_0$. □

As we have emphasised before, it will be useful to consider each element $z \in \mathcal{V}_L^N$ as a set of N distinguishable, hard-core particles. We use the following convention to label each individual particle: For each $z \in \mathcal{V}_L^N$ there exists a unique $(z_1, \cdots, z_N) \in (\mathcal{V}_L)^N$ with $z_1 < z_2 < \cdots < z_N$ such that $z = \{z_j : j \in I(N)\}$. We want to track each particle along the path $\mathbf{u}$ from x to y. To this end, we now construct a sequence $(\tilde{u}^{(l)})_{l \le k} \subseteq (\mathcal{V}_L)^N$ with the property that $u^{(l)} = \{\tilde{u}_j^{(l)} : j \in I(N)\}$ for all $0 \le l \le k$. Firstly, we set $\tilde{u}^{(0)} := (z_1, \cdots, z_N)$. For all $1 \le l \le k$, we then define

$$\begin{cases} \tilde{u}_j^{(l)} := \tilde{u}_j^{(l-1)} & \text{for all } j \in I(N) \text{ with } \tilde{u}_j^{(l-1)} \in u^{(l)}, \\ \tilde{u}_j^{(l)} \in u^{(l)} \backslash u^{(l-1)} & \text{else.} \end{cases} \tag{4.123}$$

Note that $\tilde{u}^{(l)}$ is well-defined for all $1 \le l \le k$, since the configuration $u^{(l)}$ is obtained by moving exactly one particle in $u^{(l-1)}$ to an unoccupied neighbouring site in $\mathcal{V}_L$. Hence, there is always exactly one $j_0 \in I(N)$ such that $\tilde{u}_{j_0}^{(l+1)} \neq \tilde{u}_{j_0}^{(l)}$.

For the next lemma we require the following definition. For any $j \in I(N)$ we denote by

$$L_j^{\mathbf{u}} := \sum_{l=1}^{k} d_L(\tilde{u}_j^{(l-1)}, \tilde{u}_j^{(l)}) \tag{4.124}$$

the distance which the j-th particle has travelled along the path $\mathbf{u}$.

Lemma 4.4.5. *For any $x, y \in \mathcal{V}_L^N$ the graph distance is given by*

$$d_L^N(x,y) = \min_{\sigma \in \mathfrak{S}_N^{cyc}} \sum_{j=1}^{N} d_L(x_j, y_{\sigma(j)}) = \min_{\sigma \in \mathfrak{S}_N} \sum_{j=1}^{N} d_L(x_j, y_{\sigma(j)}). \tag{4.125}$$

Here, $\mathfrak{S}_N^{(cyc)}$ denotes the set of (cyclic) permutations of the set $I(N)$.

Proof. Let $\mathbf{u}$ be an arbitrary shortest path from x to y of length $k := d_L^N(x,y)$. For any $l \in \{0, \cdots, k\}$ let $\tau_l \equiv \tau_l(\mathbf{u}) \in \mathfrak{S}_N$ be the uniquely defined permutation such that

$$0 \le \tilde{u}_{\tau_l(1)}^{(l)} < \tilde{u}_{\tau_l(2)}^{(l)} < \cdots < \tilde{u}_{\tau_l(N)}^{(l)} \le L-1. \tag{4.126}$$

We now claim that $\tau_l \in \mathfrak{S}_N^{cyc}$ for all $0 \le l \le k$, which we prove by induction. For the base case $l = 0$, the statement is true since $\tau_0 = \mathrm{id} \in \mathfrak{S}_N^{cyc}$. Now assume that for $l < k$, there exists a $\tau_l \in \mathfrak{S}_N^{cyc}$ such that (4.126) is satisfied. To show that the statement is then also true for $l+1$, we distinguish three cases:

- First case: $\tilde{u}_{\tau_l(1)}^{(l)} = 0$ and $\tilde{u}_{\tau_l(1)}^{(l+1)} = L-1$. This implies $\tilde{u}_{\tau_l(N)}^{(l+1)} < L-1$. According to the induction hypothesis we have $0 = \tilde{u}_{\tau_l(1)}^{(l)} < \tilde{u}_{\tau_l(2)}^{(l)} < \cdots < \tilde{u}_{\tau_l(N)}^{(l)} < L-1$, therefore we conclude
$$0 < \tilde{u}_{\tau_l(2)}^{(l+1)} < \tilde{u}_{\tau_l(3)}^{(l+1)} < \cdots < \tilde{u}_{\tau_l(N)}^{(l+1)} < \tilde{u}_{\tau_l(1)}^{(l+1)} = L-1. \tag{4.127}$$
The permutation $\tau_{l+1} = \tau_l \circ \sigma$ satisfies (4.126) at the position $l+1$, where $\sigma \in \mathfrak{S}_N^{cyc}$ is the uniquely defined cyclic permutation with $\sigma(1) = 2$. This implies clearly that $\tau_{l+1} \in \mathfrak{S}_N^{cyc}$, since the composition of two cyclic permutations is cyclic.

- Second case: $\tilde{u}_{\tau(N)}^{(l)} = L-1$ and $\tilde{u}_{\tau_l(N)}^{(l+1)} = 0$. By a completely analogous argument as for the first case, we get $\tau_{l+1} = \tau_l \circ \sigma^{-1} \in \mathfrak{S}_N^{cyc}$.

- The third case covers any other situation. Let $j_0 \le N$ be the unique index for which $\{\tilde{u}_{\tau_l(j_0)}^{(l)}, \tilde{u}_{\tau(j_0)}^{(l+1)}\} \in \mathcal{E}_L$. Since the previous two cases have been excluded, observe that the only two possibilities are $\tilde{u}_{\tau(j_0)}^{(l+1)} = \tilde{u}_{\tau_l(j_0)}^{(l)} \pm 1 \neq \tilde{u}_{\tau_l(j_0 \pm 1)}^{(l)}$. In either case, it is important to note that this implies
$$\tilde{u}_{\tau_l(j_0-1)}^{(l)} = \tilde{u}_{\tau_l(j_0-1)}^{(l+1)} < \tilde{u}_{\tau_l(j_0)}^{(l+1)} < \tilde{u}_{\tau_l(j_0+1)}^{(j+1)} = \tilde{u}_{\tau_l(j_0+1)}^{(l)}. \tag{4.128}$$
Hence, $\tau_{l+1} = \tau_l \in \mathfrak{S}_N^{cyc}$.

Since each step on the path moves exactly one particle to a neighbouring position we have

$$k = \sum_{l=1}^{k} \sum_{j=1}^{N} d_L(\tilde{u}_j^{(l)}, \tilde{u}_j^{(l-1)}) = \sum_{j=1}^{N} L_j^{\mathbf{u}}. \tag{4.129}$$

Moreover, for any $j \in I(N)$ we have

$$d_L(x_j, y_{\tau_k^{-1}(j)}) = d_L(\tilde{u}_j^{(0)}, \tilde{u}_j^{(k)}) \leq \sum_{l=1}^{k} d_L(\tilde{u}_j^{(l-1)}, \tilde{u}_j^{(l)}) = L_j^{\mathbf{u}}. \tag{4.130}$$

Since $\tau_k^{-1} \in \mathfrak{S}_N^{cyc}$ and since (4.129) is true we obtained:

$$\min_{\tau \in \mathfrak{S}_N^{cyc}} \sum_{j=1}^{N} d_L(x_j, y_{\tau(j)}) \leq \sum_{j=1}^{N} L_j^{\mathbf{u}} = k. \tag{4.131}$$

On the other hand, it was shown in detail in [FS18, Appendix A] that $d_L^N(x,y) = \min_{\sigma \in \mathfrak{S}_N} \sum_{j=1}^{N} d_L(x_j, y_{\sigma(j)})$. Since $\mathfrak{S}_N^{cyc} \subseteq \mathfrak{S}_N$, we immediately obtain

$$k = d_L^N(x,y) = \min_{\sigma \in \mathfrak{S}_N} \sum_{j=1}^{N} d_L(x_j, y_{\sigma(j)}) \leq \min_{\sigma \in \mathfrak{S}_N^{cyc}} \sum_{j=1}^{N} d_L(x_j, y_{\sigma(j)}), \tag{4.132}$$

which concludes the proof. □

Corollary 4.4.6. *Let $x, y \in \mathcal{V}_L^N$, $k := d_L^N(x,y)$, and let $\mathbf{u}$ be a shortest path between x and y. Then*

$$L_j^{\mathbf{u}} = d_L(\tilde{u}_j^{(0)}, \tilde{u}_j^{(k)}) \leq L/2 \tag{4.133}$$

and $\tilde{u}_j^{(k)} \in \{(x_j \pm L_j^{\mathbf{u}}) \bmod L\}$ for all $j \in I(N)$.

Proof. Equation (4.125) immediately implies equality in (4.130) and (4.131). Due to the definition of d_L we have

$$L_j^{\mathbf{u}} = d_L(\tilde{u}_j^{(0)}, \tilde{u}_j^{(k)}) = d_L(x_j, y_{\tau_k^{-1}(j)}) \leq L/2. \tag{4.134}$$

This already yields $\tilde{u}_j^{(k)} \in \{(x_j \pm L_j^{\mathbf{u}}) \bmod L\}$ for all $j \in I(N)$. □

Corollary 4.4.6 implies that along any shortest path $\mathbf{u}$ from x to y of length k, each individual particle moves, if at all, either clockwise or counter-clockwise. We therefore define

$$I_{\pm}^{\mathbf{u}} := \{j \in I(N) : \; L_j^{\mathbf{u}} \neq 0 \text{ and } \exists l \in \{0, \cdots, k\} \text{ with } \tilde{u}_j^{(l)} = (\tilde{u}_j^{(0)} \pm 1) \bmod L\} \quad \text{and} \tag{4.135}$$

$$I_0^{\mathbf{u}} := \{j \in I(N) : \; L_j^{\mathbf{u}} = 0\}. \tag{4.136}$$

Note that $I(N) = I_+^{\mathbf{u}} \cup I_-^{\mathbf{u}} \cup I_0^{\mathbf{u}}$ for any shortest path $\mathbf{u}$. The definition of $L_j^{\mathbf{u}}$ as well as (4.133) imply that

$$\{u_j^{(l)} : \; 0 \leq l \leq k\} = \{(x_j \pm \xi) \bmod L : \; 0 \leq \xi \leq L_j^{\mathbf{u}}\} \quad \text{for all } j \in I_{\pm}^{\mathbf{u}} \cup I_0^{\mathbf{u}}. \tag{4.137}$$

Let $\kappa_j^{(l)} \equiv \kappa_j^{(l)}(\mathbf{u}) \in \{0,\cdots,L_j^{\mathbf{u}}\}$ for any $j \in I(N)$ and any $l \in \{0,\cdots,k\}$ such that

$$\tilde{u}_j^{(l)} = (\tilde{u}_j^{(0)} \pm \kappa_j^{(l)}) \bmod L \quad \text{for all } j \in I_\pm^{\mathbf{u}} \cup I_0^{\mathbf{u}}. \tag{4.138}$$

It follows from the previous observations that the quantity $\kappa_j^{(l)}$ is well-defined.

The next lemma further establishes that we can indeed consider $\mathbf{u}$ as a path of hard-core particles that cannot move past each other.

Lemma 4.4.7 (hard-core particle property)**.** *Let* $e_0 := \{0, L-1\} \in \mathcal{E}_L$*,* $c \in \mathcal{D}_L^N$ *with* $e_0 \notin c$ *and* $x \in \mathcal{V}_L^N$*. Moreover, let* $\mathbf{u}$ *be a shortest path from* c *to* x *of length* $k := d_L^N(c,x)$*. Then:*

(i) If $i,j \in I_-^{\mathbf{u}}$ *with* $i<j$ *then* $\kappa_i^{(l)} \geq \kappa_j^{(l)}$ *for all* $l \in \{0,\cdots,k\}$*.*

(ii) If $i,j \in I_+^{\mathbf{u}}$ *with* $i<j$ *then* $\kappa_i^{(l)} \leq \kappa_j^{(l)}$ *for all* $l \in \{0,\cdots,k\}$*.*

(iii) We have the following inequalities:

$$\sup I_-^{\mathbf{u}} \leq \inf(I_0^{\mathbf{u}} \cup I_+^{\mathbf{u}}) \qquad \textit{and} \qquad \sup(I_-^{\mathbf{u}} \cup I_0^{\mathbf{u}}) \leq \inf I_+^{\mathbf{u}}, \tag{4.139}$$

where we use the convention that $\inf\varnothing = \infty$ *and* $\sup\varnothing = -\infty$*.*

(iv) If $1 \in I_-^{\mathbf{u}} \cup I_0^{\mathbf{u}}$ *and* $N \in I_+^{\mathbf{u}} \cup I_0^{\mathbf{u}}$ *then*

$$L_1^{\mathbf{u}} + L_N^{\mathbf{u}} \leq L - N. \tag{4.140}$$

Proof. Since $c \in \mathcal{D}_L^N$ with $e_0 \notin c$ we have $c_j = c_1 + j - 1$ for all $j \in I(N)$. Let $J \equiv J(k) := \{0,\cdots,k\}$.

(i) It suffices to show that if $j, j+1 \in I_-^{\mathbf{u}}$, then $\kappa_j^{(l)} \geq \kappa_{j+1}^{(l)}$ for all $l \in J$. Let us define $f:\ J \to \mathbb{Z},\ l \mapsto \kappa_j^{(l)} - \kappa_{j+1}^{(l)}$. Suppose there exists a $k_0 \in J$ such that $f(k_0) < 0$. Due to the definition of the path we have $f(0) = 0$ and $|f(l) - f(l-1)| \in \{0,1\}$ for all $l \in J \setminus \{0\}$. Now, suppose there exists a $k_0 \in J$ such that $f(k_0) < 0$. This would imply that there exist a $k_1 \in J$, $k_1 \leq k_0$, with $f(k_1) = -1$. Hence

$$u_j^{(k_1)} = (c_j - \kappa_j^{(k_1)}) \bmod L = (c_j - f(k_1) - \kappa_{j+1}^{(k_1)}) \bmod L = u_{j+1}^{(k_1)}, \tag{4.141}$$

since $c_j - f(k_1) = c_{j+1}$, which is a contradiction.

(ii) Analogous to (i).

(iii) We only prove the first inequality in (4.139). The right-hand side follows analogously. If $I_-^{\mathbf{u}} = \varnothing$, then the result is trivial. So, from now on, we assume that $I_-^{\mathbf{u}} \neq \varnothing$. Suppose $\inf(I_0^{\mathbf{u}} \cup I_+^{\mathbf{u}}) < \sup I_-^{\mathbf{u}}$. This implies that there exists a $j \in I_-^{\mathbf{u}}$ such that $j-1 \in I_0^{\mathbf{u}} \cup I_+^{\mathbf{u}}$. Consider the function $g:\ J \to \mathbb{Z},\ l \mapsto \kappa_j^{(l)} + \kappa_{j-1}^{(l)}$. We again have $g(0) = 0$, $g(k) \geq L_j^{\mathbf{u}} \geq 1$ and $|g(l) - g(l-1)| \in \{0,1\}$ for all $l \in J \setminus \{0\}$. Hence, there exists $k_0 \in J$ with $g(k_0) = 1$. This implies

$$\tilde{u}_{j-1}^{(k_0)} = (c_{j-1} + \kappa_{j-1}^{(k_0)}) \bmod L = (c_{j-1} + g(k_0) - \kappa_j^{(k_0)}) \bmod L = \tilde{u}_j^{(k_0)}, \tag{4.142}$$

which is a contradiction.

(iv) Let us consider the function $p: J \to \mathbb{N}_0$, $l \mapsto \kappa_1^{(l)} + \kappa_N^{(l)}$. Suppose $L_1^{\mathbf{u}} + L_N^{\mathbf{u}} > L - N$. Since $p(k) = L_1^{\mathbf{u}} + L_N^{\mathbf{u}}$ and $p(l) - p(l-1) \in \{0,1\}$ for all $l \in J \setminus \{0\}$, we conclude that there exists a $k_0 \in J$ with $p(k_0) = L - N + 1$. Hence, by using $c_N = c_1 + N - 1 = c_1 - p(k_0) + L$, we obtain

$$u_1^{(k_0)} = (c_1 - \kappa_1^{(k_0)}) \bmod L = (c_1 - p(k_0) + \kappa_N^{(k_0)}) \bmod L = u_N^{(k_0)}, \tag{4.143}$$

which is a contradiction.

□

Recall from Section 4.4.1 that our strategy to calculate the graph distance of two configurations relies on cutting the ring open alongside a suitable edge e. Given a path $\mathbf{u}$ of length k and an edge $e \in \mathcal{E}_L$, we say that $\mathbf{u}$ does not cross e, if $u^{(l-1)} \triangle u^{(l)} \neq e$ for all $l \in \{1, \cdots, k\}$.

Lemma 4.4.8 (cutting lemma)**.** *Let $N < L/2$. Let $c \in \mathcal{D}_L^N$ and $x \in \mathcal{V}_L^N$. Then there exists a shortest path $\mathbf{u}$ from c to x and an edge $e \in \mathcal{E}_L$ with $\min_{m \in c} d_L(m, e) \geq \lceil L/2 \rceil - N$ such that $\mathbf{u}$ does not cross e.*

Proof. Due to the translational symmetry of the system we may assume w.l.o.g. that $c = \{0, \cdots, N-1\}$. Let $\mathbf{u}$ be a shortest path of length $k := d_L^N(c, x)$ connecting c and x. To prove the lemma, we have to show that there exists an $e \in \mathcal{E}_L$ such that for all $l \in \{1, \cdots, k\}$ we have $u^{(l-1)} \triangle u^{(l)} \in \mathcal{E}_L \setminus \{e\}$. However, it suffices to show that for all $j \in I(N)$ we have $e \notin \{u_j^{(l)} : 0 \leq l \leq k\}$. We distinguish between three cases.

- First case: $I_-^{\mathbf{u}} = \varnothing$, which means that no particle moves clockwise. Let $e := \{N - 1 + \lfloor L/2 \rfloor, N + \lfloor L/2 \rfloor\}$. For all $l \in \{0, \cdots, k\}$ and for all $j \in I(N)$ we have

$$\{u_j^{(l)} : 0 \leq l \leq k\} = (j-1) + \{0, \cdots, L_j^{\mathbf{u}}\} \subseteq \{0, \cdots, N - 1 + \lfloor L/2 \rfloor\}, \tag{4.144}$$

where we used Corollary 4.4.6 for the second inclusion. Since e is not a subset of the right-hand side of (4.144), this proves the claim for this case.

- Second case: $I_+^{\mathbf{u}} = \varnothing$, which means no particle moves anti-clockwise. Analogously to the first case, we see that the edge $e := \{\lceil L/2 \rceil - 1, \lceil L/2 \rceil\}$ satisfies the claim.

- Third case: both $I_-^{\mathbf{u}}$ and $I_+^{\mathbf{u}}$ are non-empty. Let us first note that due to Lemma 4.4.7 (iii) $\max I_-^{\mathbf{u}} \leq \min(I_0^{\mathbf{u}} \cup I_+^{\mathbf{u}})$ and $\max(I_0^{\mathbf{u}} \cup I_-^{\mathbf{u}}) \leq \min I_+^{\mathbf{u}}$. This implies that $N \in I_+^{\mathbf{u}}$ and $1 \in I_-^{\mathbf{u}}$. According to Lemma 4.4.7 (iv) we have $L_1^{\mathbf{u}} + L_N^{\mathbf{u}} \leq L - N$ and thus

$$\tilde{u}_1^{(k)} = L - L_1^{\mathbf{u}} > N + L_N^{\mathbf{u}} - 1 = \tilde{u}_N^{(k)}. \tag{4.145}$$

Let $e \in \mathcal{E}_L$ with $e \subset \{\tilde{u}_N^{(k)}, \cdots, \tilde{u}_1^{(k)}\}$. For any $j \in I_+^{\mathbf{u}} \cup I_0^{\mathbf{u}}$ and any $l \in \{0, \cdots, k\}$ we have

$$\tilde{u}_j^{(l)} \in (j-1) + \{0, \cdots, \kappa_j^{(l)}\} \subseteq \{0, \cdots, \tilde{u}_N^{(k)}\} =: \mathcal{A}_+, \tag{4.146}$$

where we used Lemma 4.4.7 (ii) for the second inclusion. Analogously, for any $j \in I_-^{\mathbf{u}} \cup I_0^{\mathbf{u}}$ and any $l \in \{0, \cdots, k\}$ we have

$$u_j^{(l)} \in \{0, \cdots, N\} \cup \{L - L_1^{\mathbf{u}}, \cdots, L - 1\} =: \mathcal{A}_-. \tag{4.147}$$

Since $e \notin \mathcal{A}_{\pm}$, this shows that $\mathbf{u}$ does not cross e.

Moreover, since $\tilde{u}_N^{(k)} \leq N - 1 + \lfloor L/2 \rfloor$ and $\tilde{u}_1^{(k)} \geq \lceil L/2 \rceil$ there exists at least one edge $e \in \mathcal{E}_L$ with $e \subseteq \{\lceil L/2 \rceil, \cdots, N - 1 + \lfloor L/2 \rfloor\} \subseteq \{\tilde{u}_1^{(k)}, \cdots, \tilde{u}_N^{(k)}\}$ such that e has a distance of at least $\lceil L/2 \rceil - N$ to any $m \in c$.

□

Recalling the definition of a droplet on the infinite line (4.116), we now introduce a notation for a droplet in $\mathcal{V}_L^N$ centred around a site $m \in \mathcal{V}_L$ on the ring

$$c_{L,m}^N := \{j \bmod L : \ j \in c_m^N\}. \tag{4.148}$$

Analogously to (4.117), for any $x \subseteq \mathcal{V}_L$, $|x| > 0$, we define the set of centres of droplets that are closest to this configuration

$$\mathcal{W}_L(x) := \left\{m \in \mathcal{V}_L : \ d_L^{|x|}(x, c_{L,m}^{|x|}) = d_L^{|x|}(x, \mathcal{D}_L^{|x|})\right\}. \tag{4.149}$$

Recall from (4.117) that on the line, the set $\mathcal{W}(x)$ is explicitly known and contains the middle element x_κ for

$$\kappa \equiv \kappa(N) := \left\lfloor \frac{N+1}{2} \right\rfloor. \tag{4.150}$$

By applying the cutting lemma we want to apply this result to the ring.

Lemma 4.4.9. *Let $N < L/2$ and $x \in \mathcal{V}_L^N$. Then*

$$\mathcal{W}_L(x) \cap x \neq \varnothing. \tag{4.151}$$

Furthermore, let $m \in \mathcal{W}_L(x) \cap x$. If there exists a shortest path $\mathbf{u}$ from $c_{L,m}^N$ to x that does not cross $e_0 := \{0, L-1\}$ and $e_0 \notin c_{L,m}^N$, then $m = x_\kappa$.

Proof. Let $\nu \in \mathcal{W}_L(x)$ and define $c := c_{L,\nu}^N$. According to Lemma 4.4.8 there exists a shortest path $\mathbf{u}$ from c to x and an edge $e \in \mathcal{E}_L$ with $\min_{j \in c} d_L(j, e) > \lceil L/2 \rceil - L/2 \geq 0$ such that $\mathbf{u}$ does not cross e. This implies $e \notin c$. Let us assume w.l.o.g. that $e = e_0$. In any other case we can choose a suitable rotation by $\gamma \in \mathbb{Z}$ such that $T_L^\gamma e = e_0$ and consider $T_L^\gamma x$, $T_L^\gamma c$ and the path $\mathbf{u}_\gamma := (T_L^\gamma u^{(0)}, \cdots, T_L^\gamma u^{(k)})$ instead.

Since $\mathbf{u}$ does not cross e_0, it can also be viewed as a path on the infinite line $\mathcal{G}^N$. Therefore, we have $d_L^N(x, c) = d^N(x, c)$. Let us consider the droplet $c' := c_{x_\kappa}^N = c_{L,x_\kappa}^N$. In this case, (4.117) states that $\{x_\kappa\} = \mathcal{W}(x) \cap x$. Hence,

$$d_L^N(x, \mathcal{D}_L^N) = d^N(x, c) \geq d^N(x, c') = \sum_{j=1}^N |c'_j - x_j| \geq \sum_{j=1}^N d_L(c'_j, x_j) \geq d_L^N(c', x), \tag{4.152}$$

where we applied Lemma 4.4.5 to achieve the final estimate. Recall from (4.115) that d^N denotes the graph distance on the infinite line. Since $c' \in \mathcal{D}_L^N$ we have equality in (4.152) and therefore $x_\kappa \in \mathcal{W}_L(x) \cap x$. Moreover, if $\nu \in x$ it follows immediately that $\nu = x_\kappa$, since $d^N(x, c) = d^N(x, c')$ and the set $\mathcal{W}(x) \cap x$ contains only one element. □

By making further assumptions on the configuration x, we are able to determine $\mathcal{W}_L(x) \cap x$ precisely. For now, let us only consider configurations that are contained in a sufficiently small sector of the ring.

Here, a sector of size $\theta \in]0, 1/2[$ around a site $m \in \mathcal{V}_L^N$ is given by

$$\mathcal{S}_{L,m}(\theta) := \{k \in \mathcal{V}_L : \ d_L(k,m) < \theta L\}. \tag{4.153}$$

Lemma 4.4.10. *Let $M := \lfloor (L-1)/2 \rfloor$, $\beta \in]0, 1/4[$, $N < \beta L$ and $x \in \mathcal{V}^N\big(\mathcal{S}_{L,M}(1/4 - \beta/2)\big)$. Then*

$$\{x_\kappa\} = \mathcal{W}_L(x) \cap x, \tag{4.154}$$

where κ was defined in (4.150).

Furthermore, no shortest path $\mathbf{u}$ *from c^N_{L,x_κ} to x crosses $e_0 := \{0, L-1\}$. Moreover, we have $\{1, \cdots, \kappa\} \subseteq I^{\mathbf{u}}_0 \cup I^{\mathbf{u}}_-$ and $\{\kappa, \cdots, N\} \subseteq I^{\mathbf{u}}_0 \cup I^{\mathbf{u}}_+$.*

Proof. Let $\nu \in \mathcal{S}_{L,M}(1/4 - \beta/2)$. We claim that no shortest path $\mathbf{u}$ from $c^N_{L,\nu}$ to x crosses the edge e_0. Let $c := c^N_{L,\nu}$ and $k := d^N_L(x, c)$. As in the proof of Lemma 4.4.8, it is sufficient to show that for any $j \in I(N)$ we have $e_0 \not\subseteq \{\tilde{u}^{(l)}_j : \ 0 \le l \le k\}$.

Suppose there exists a $j \in I(N)$ such that $e_0 \subseteq \mathcal{Z}_j := \{\tilde{u}^{(l)}_j : \ 0 \le l \le k\}$, which readily implies $j \notin I^{\mathbf{u}}_0$. But according to (4.137), we know that for any $j \in I^{\mathbf{u}}_\pm$, we have

$$|\mathcal{Z}_j| = |\{c_j, \cdots, (c_j \pm L^{\mathbf{u}}_j) \bmod L\}| = L^{\mathbf{u}}_j + 1 \le \lfloor L/2 \rfloor + 1, \tag{4.155}$$

where we used Corollary 4.4.6 to estimate $L^{\mathbf{u}}_j$. W.l.o.g., let us assume $j \in I^{\mathbf{u}}_-$. Since we assumed $e_0 \subseteq \mathcal{Z}_j$, we deduce from (4.137) that

$$\{c_j, (c_j - L^{\mathbf{u}}_j) \bmod L\} \cup (\mathcal{S}_{L,M}(1/4))^c \subseteq \mathcal{Z}_j. \tag{4.156}$$

Notice that both $c_j \in \mathcal{S}_{L,M}(1/4)$ and $c_j - L^{\mathbf{u}}_j \in x \subseteq \mathcal{S}_{L,M}(1/4)$. Hence,

$$|\mathcal{Z}_j| \ge 2 + |(\mathcal{S}_{L,M}(1/4))^c| \ge 2 + \lfloor L/2 \rfloor, \tag{4.157}$$

which is a contradiction.

Since no shortest path $\mathbf{u}$ from $c^N_{L,\nu}$ to x crosses e_0, observe that $\mathbf{u}$ can also be viewed as a path on the graph induced by N particles on the infinite line. Hence,

$$d^N_L(x, c^N_{L,x_j}) = d^N(x, c^N_{L,x_j}) \tag{4.158}$$

and $\mathcal{W}_L(x) \cap x = \mathcal{W}(x) \cap x = \{x_\kappa\}$ according to (4.117).

Let us now consider a shortest path $\mathbf{v}$ from $c' := c^N_{L,x_\kappa}$ to x. Lemma 4.4.7 implies that $\tilde{v}^{(k)}_j = x_j$ and $L^{\mathbf{v}}_j = d_L(x_j, c'_j)$ for all $j \in I(N)$. Hence, $L^{\mathbf{v}}_\kappa = 0$ and therefore $\kappa \in I^{\mathbf{z}}_0$. The rest of the statement follows from Lemma 4.4.7 (iii). $\square$

4.4.3 The mass of the droplet configurations

Let $\gamma \in \mathcal{V}_L$. We are now able to estimate the contribution of droplet configurations to the eigenstate $|\varphi^N_{L,\gamma}(\Delta)\rangle$.

First, we need two auxiliary results to estimate the sum over all non-droplet contributions. The first of these lemmata is an adaptation of a similar result in [ARFS20, Proof of Theorem 6.1].

Lemma 4.4.11. *Let* $N \in \mathbb{N}$ *and*

$$\mathcal{X}^N := \{\chi = (\chi_1, \cdots, \chi_N) \subseteq \mathbb{N}_0^N : \ \chi_1 \leq \cdots \leq \chi_N\}. \tag{4.159}$$

Then for all $\mu \geq \ln 2$ *we have*

$$\sum_{\chi \in \mathcal{X}^N \setminus \{0\}} e^{-\mu|\chi|_1} \leq 30 e^{-\mu}, \tag{4.160}$$

where $|\cdot|_1$*-denotes the* ℓ^1*-norm of* $\mathbb{Z}^N$*.*

Proof. Let $\Psi : \mathbb{N}_0^N \to \mathcal{X}^N$ with

$$x \mapsto \Psi(x) := (\Psi_1(x), \cdots, \Psi_N(x)), \tag{4.161}$$

where for each $j \in I(N)$ we have defined $\Psi_j(x) := \sum_{i=1}^j x_i$. Note that Ψ is a bijection.
For any $x \in \mathbb{N}_0^N$ we therefore get

$$|\Psi(x)|_1 = \sum_{j=1}^N \Psi_j(x) = \sum_{k=1}^N (N-k+1)x_k \tag{4.162}$$

and since Ψ is a bijection, we have

$$\sum_{\chi \in \mathcal{X}^N} \mathrm{e}^{-\mu|\chi|_1} = \sum_{x \in \mathbb{N}_0^N} \mathrm{e}^{-\mu|\Psi(x)|_1} = \sum_{x \in \mathbb{N}_0^N} \prod_{k=1}^N \mathrm{e}^{-\mu(N-k+1)x_k}, \tag{4.163}$$

which yields

$$\sum_{\chi \in \mathcal{X}^N} \mathrm{e}^{-\mu|\chi|_1} = \prod_{k=1}^N \sum_{y=0}^\infty \mathrm{e}^{-\mu y(N-k+1)} = \prod_{k=1}^N \frac{1}{1-\mathrm{e}^{-\mu(N-k+1)}}. \tag{4.164}$$

We obtain the following estimate, which is uniform in N:

$$\sum_{\chi \in \mathcal{X}^N} \mathrm{e}^{-\mu|\chi|_1} \leq \prod_{m=1}^\infty \frac{1}{1-\mathrm{e}^{-\mu m}} \leq \exp\left(\frac{2\mathrm{e}^{-\mu}}{1-\mathrm{e}^{-\mu}}\right), \tag{4.165}$$

where we have used that $\ln(1-\lambda)^{-1} \leq 2\lambda$, whenever $\lambda \in]0, 1/2]$ and $\mathrm{e}^{-\mu} \leq 2^{-1}$ for $\mu \geq \ln 2$. Hence,

$$\sum_{\chi \in \mathcal{X}^N \setminus \{0\}} \mathrm{e}^{-\mu|\chi|_1} \leq \exp\left(\frac{2\mathrm{e}^{-\mu}}{1-\mathrm{e}^{-\mu}}\right) - 1 \leq 4\mathrm{e}^2\mathrm{e}^{-\mu}, \tag{4.166}$$

since $\mathrm{e}^\lambda - 1 \leq \lambda \mathrm{e}^\lambda$ for all $\lambda \geq 0$. □

Lemma 4.4.12. *Let* $N < L/2$ *and* $\mu \geq \ln 2$*. Then*

$$\sum_{x \in \mathcal{V}_L^N} e^{-\mu d_L^N(x, \mathcal{D}_L^N)} \leq L(1 + 2^9 e^{-\mu}). \tag{4.167}$$

Proof. For any $m \in \mathcal{V}_L$ let

$$\mathcal{B}^N_{L,m} := \{x \in \mathcal{V}^N_L : \ m \in \mathcal{W}_L(x) \cap x\}. \tag{4.168}$$

By Lemma 4.4.9 we have

$$\bigcup_{m \in \mathcal{V}_L} \mathcal{B}^N_{L,m} = \mathcal{V}^N_L. \tag{4.169}$$

Let $x \in \mathcal{B}^N_{L,m}$. According to Lemma 4.4.8, there exists an edge e with $\max_{j\in c} d_L(j,e) > 0$ and a shortest path $\mathbf{u}$ from $c := c^N_{L,m}$ to x that does not cross e. Pick $\gamma \in \mathbb{Z}$ such that $T^\gamma_L e = e_0 := \{0, L-1\}$. Let $x' := T^\gamma_L x$, $c' := T^\gamma_L c$ and $\mathbf{v} := \{T^\gamma_L u^{(0)}, \cdots, T^\gamma_L u^{(k)}\}$, where $k := d^N_L(x,c)$. Let us define $\chi_- \equiv \chi_-(x) \in \mathbb{N}_0^{\kappa-1}$, $\chi_+ \equiv \chi_+(x) \in \mathbb{N}_0^{N-\kappa}$ by

$$\chi_{-,j} := d_L(x'_{\kappa-j}, c'_{\kappa-j}) \qquad \text{for } j \le \kappa - 1, \tag{4.170}$$

$$\chi_{+,j} := d_L(x'_{\kappa+j}, c'_{\kappa+j}) \qquad \text{for } j \le N - \kappa. \tag{4.171}$$

We want to show that $\chi_+ \in \mathcal{X}^{N-\kappa}$ and $\chi_- \in \mathcal{X}^{\kappa-1}$. Since $\mathbf{v}$ does not cross e_0 we have $v^{(k)}_j = x'_j$ for all $j \in I(N)$. By Lemma 4.4.9 we have $x'_\kappa = c'_\kappa$. Hence $\kappa \in I^{\mathbf{v}}_0$, since $L^{\mathbf{v}}_\kappa = d_L(x'_\kappa, c'_\kappa) = 0$. As a consequence of Lemma 4.4.7 (iii) we conclude $\{1, \cdots, \kappa\} \subseteq I^{\mathbf{v}}_0 \cup I^{\mathbf{v}}_-$ and $\{\kappa, \cdots, N\} \subseteq I^{\mathbf{v}}_0 \cup I^{\mathbf{v}}_+$

By Lemma 4.4.7 (i), we get

$$\chi_{-,j} = L^{\mathbf{v}}_{\kappa-j} \le L^{\mathbf{v}}_{\kappa-j-1} = \chi_{-,j+1} \quad \text{for all } 1 \le j < \kappa - 1, \tag{4.172}$$

and therefore $\chi_- \in \mathcal{X}^{\kappa-1}$. Analogously $\chi_+ \in \mathcal{X}^{N-\kappa}$. Furthermore,

$$d^N_L(x,c) = d^N_L(x',c') = \sum_{j=1}^N L^{\mathbf{v}}_j = |\chi_-|_1 + |\chi_+|_1. \tag{4.173}$$

Note that each pair $(\chi_-, \chi_+) \in \mathcal{X}^{\kappa-1} \times \mathcal{X}^{N-\kappa}$ corresponds to one $x \in \mathcal{B}^N_{L,m}$ only, since

$$x = \{(m + j + \chi_{+,j}) \bmod L : \ j \le N - \kappa\} \cup \{m\} \cup \{(m - j - \chi_{-,j}) \bmod L : \ j \le \kappa - 1\}. \tag{4.174}$$

We therefore obtain

$$\sum_{x \in \mathcal{B}^N_{L,m}} \mathrm{e}^{-\mu d^N_L(x,c)} \le \sum_{\chi_- \in \mathcal{X}^{\kappa-1}} \sum_{\chi_+ \in \mathcal{X}^{N-\kappa}} \mathrm{e}^{-\mu(|\chi_-|_1 + |\chi_+|_1)} \le (1 + 30\mathrm{e}^{-\mu})^2 \le 1 + 2^9 \mathrm{e}^{-\mu} \tag{4.175}$$

where we applied Lemma 4.4.11 and $\mu \ge \ln 2$. Together with (4.169), this concludes the proof. □

We now turn our attention to the low-energy eigenfunctions of $H^N_L(\Delta)$. Let us examine the contribution of the droplet configurations.

Lemma 4.4.13. *Let $L, N \in \mathbb{N}$ with $N < L/2$. Let $\gamma \in \mathcal{V}_L$ and $\Delta > 3$ such that $\mu_1(\Delta) \ge \ln 2$, where μ_1 was defined in* (4.16). *Then*

$$\frac{1}{L}\Big(1 - 2^{17} e^{-2\mu_1}\Big) \le |\langle \delta^L_x, \varphi^N_{L,\gamma}\rangle|^2 \le \frac{1}{L} \tag{4.176}$$

for all $x \in \mathcal{D}^N_L$, where $|\varphi^N_{L,\gamma}(\Delta)\rangle$ was defined in Remark 4.2.11 (i).

Proof. Analogously to (4.83), the definition of $|\varphi_{L,\gamma}^N\rangle$ implies that

$$|\langle\delta_x^L,\varphi_{L,\gamma}^N\rangle| = |\langle\delta_{x_0}^L,\varphi_{L,\gamma}^N\rangle| \tag{4.177}$$

for all droplets $x \in [x_0]$, where $x_0 \in \widehat{\mathcal{V}}_L^N \cap \mathcal{D}_L^N$ is the unique representative in $\widehat{\mathcal{V}}_L^N$ of a droplet. Hence, by the results of Theorem 4.2.1, we have

$$\begin{aligned} 1 &= L|\langle\delta_{x_0}^L,\varphi_{L,\gamma}^N\rangle|^2 + \sum_{x\in\mathcal{V}_L^N\smallsetminus\mathcal{D}_L^N} |\langle\delta_x^L,\varphi_{L,\gamma}^N\rangle|^2 \\ &\le L|\langle\delta_{x_0}^L,\varphi_{L,\gamma}^N\rangle|^2 + \sum_{x\in\mathcal{V}_L^N\smallsetminus\mathcal{D}_L^N} \frac{2^8}{L}\mathrm{e}^{-2\mu_1 d(x,\mathcal{D}_L^N)}. \end{aligned} \tag{4.178}$$

The first equality already yields the upper bound in (4.176). For the lower bound, we use Lemma 4.4.12 to estimate the last term on the right-hand side of (4.178) in the following way

$$1 \le L|\langle\delta_{x_0}^L,\varphi_{L,\gamma}^N\rangle|^2 + 2^{17}\mathrm{e}^{-2\mu_1}. \tag{4.179}$$

This concludes the proof. □

Lemma 4.4.14. *Let $L, N \in \mathbb{N}$ with $N < L/2$. Let $\gamma \in \mathcal{V}_L$ and $\Delta > 3$ such that $\mu_1(\Delta) \ge \ln 2$. Then for all $x, x' \in \mathcal{V}_L^N$ we have*

$$|\langle\delta_x^L,(\rho(\varphi_{L,\gamma}^N) - \rho_{L,\gamma}^N)\delta_{x'}^L\rangle| \le \frac{2^{17}}{L}\begin{cases} e^{-2\mu_1} & \text{if } x, x' \in \mathcal{D}_L^N, \\ e^{-\mu_1[d_L^N(x,\mathcal{D}_L^N)+d_L^N(x',\mathcal{D}_L^N)]} & \text{else.} \end{cases} \tag{4.180}$$

Proof. Let again $x_0 \in \widehat{\mathcal{V}}_L^N \cap \mathcal{D}_L^N$. We only need to discuss the case $x, x' \in [x_0] = \mathcal{D}_L^N$. All other cases follow immediately from Theorem 4.2.1, since $\langle\delta_x^L,\rho_{L,\gamma}^N\delta_{x'}^L\rangle = 0$ if either x or x' are not an element of $\mathcal{D}_L^N$. Let $x = T_L^\zeta\hat{x}_0$ and $x' = T_L^\xi x_0$ for some $\xi, \zeta \in \mathcal{V}_L$. Remark 4.2.11 implies that

$$\langle\delta_x^L,\rho(\varphi_{L,\gamma}^N)\delta_{x'}^L\rangle = \mathrm{e}^{\frac{2\pi\mathrm{i}}{L}(\zeta-\xi)\gamma}\langle\delta_{x_0}^L,\rho(\varphi_{L,\gamma}^N)\delta_{x_0}^L\rangle = \mathrm{e}^{\frac{2\pi\mathrm{i}}{L}(\zeta-\xi)\gamma}|\langle\delta_{x_0}^L,\varphi_{L,\gamma}^N\rangle|^2, \tag{4.181}$$

while Definition (4.86) implies

$$\langle\delta_x^L,\rho_{L,\gamma}^N\delta_{x'}^L\rangle = \mathrm{e}^{\frac{2\pi\mathrm{i}}{L}(\zeta-\xi)\gamma}\langle\delta_{x_0}^L,\rho_{L,\gamma}^N\delta_{x_0}^L\rangle = \frac{1}{L}\mathrm{e}^{\frac{2\pi\mathrm{i}}{L}(\zeta-\xi)\gamma}. \tag{4.182}$$

By applying Lemma 4.4.13 we obtain

$$|\langle\delta_x^L,(\rho(\varphi_{L,\gamma}^N) - \rho_{L,\gamma}^N)\delta_{x'}^L\rangle| \le \frac{2^{17}}{L}\mathrm{e}^{-2\mu_1}. \tag{4.183}$$

□

4.4.4 Moving particles to the boundary of Λ_L

In this section, we construct the shortest path from a given configuration x to the closest droplet configuration, which was depicted in Figure 4.6. In the following, we will require some further assumptions on Λ_L and N.

Assumption 4.4.15. *Let $\varepsilon \in]0,1/16[$ and $\theta \in]\varepsilon,1/16[$ be fixed. For $L \in \mathbb{N}$ let $N \equiv N(\varepsilon,L) := \lfloor \varepsilon L \rfloor$. Let*

$$\Lambda_L \equiv \Lambda_L(\theta) := \mathcal{S}_{L,M}(\theta) = \{\lambda_-,\cdots,\lambda_+\} \subseteq \mathcal{V}_L, \tag{4.184}$$

where $M := \lfloor (L-1)/2 \rfloor$, $\lambda_- \equiv \lambda_-(\theta,L) := \min \Lambda_L$ and $\lambda_+ \equiv \lambda_+(\theta,L) := \max \Lambda_L$.

Let us now introduce some additional notation to classify configurations. Let $\Gamma = \{\gamma_-,\cdots,\gamma_+\} \subseteq \mathcal{S}_{L,M}(1/4)$ with $\gamma_- < \gamma_+$ be a connected subset.

For $x \subseteq \mathcal{V}_L$ let

$$x^{in} \equiv x^{in}(\Gamma) := x \cap \Gamma, \tag{4.185}$$
$$x^{out} \equiv x^{out}(\Gamma) := x \setminus \Gamma. \tag{4.186}$$

If $|x^{in}|, |x^{out}| > 0$, let $\ell, r \in \mathbb{N}_0$ be arbitrary such that $\ell + r = |x^{out}|$. The idea is to further split x^{out} into a configuration of ℓ particles which are thought of as being close to γ_- and a configuration of r particles close to γ_+. Note that for every such ℓ and r, there exists a unique permutation $\sigma_{\ell,r} \equiv \sigma_{\ell,r}(x,\Gamma) \in \mathfrak{S}_N^{cyc}$ with the property that

$$x^{in} = \{x_{\sigma_{\ell,r}(j)} : \ \ell < j \leq N - r\}. \tag{4.187}$$

We then define

$$\begin{aligned} x_-^{\ell,out} &\equiv x_-^{\ell,out}(x,\Gamma) := \{x_{\sigma_{\ell,r}(j)} : \ j \leq \ell\} \qquad \text{and} \\ x_+^{r,out} &\equiv x_+^{r,out}(x,\Gamma) := \{x_{\sigma_{\ell,r}(j)} : \ j > N - r\}. \end{aligned} \tag{4.188}$$

We have $x^{out} = x_-^{\ell,out} \cup x_+^{r,out}$. For an example see Figure 4.7.

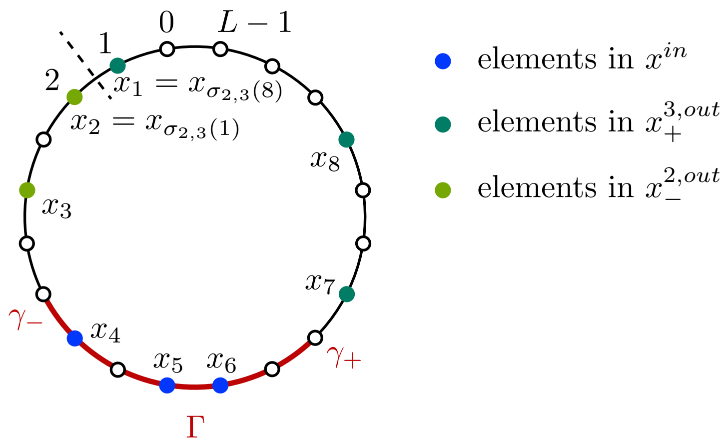

Figure 4.7: Example for the separation of a configuration x in x^{in}, $x_+^{3,out}$ and $x_-^{2,out}$

For any $j \in I(N)$ let

$$a_{\pm,j}(\Gamma) := (\gamma_\pm \pm j) \bmod L. \tag{4.189}$$

Then there exist unique $\chi_-^\ell \equiv \chi_-^\ell(x,\Gamma) = (\chi_{-,1}^\ell, \cdots, \chi_{-,\ell}^\ell) \in \mathcal{X}^\ell$ and $\chi_+^r \equiv \chi_+^r(x,\Gamma) = (\chi_{+,1}^r, \cdots, \chi_{+,r}^r) \in \mathcal{X}^r$ with $\chi_{-,\ell}^\ell, \chi_{+,r}^r < L$ such that

$$x_{\sigma_{\ell,r}(\zeta)} = (a_{-,\ell-\zeta+1} - \chi_{-,\ell-\zeta+1}^\ell) \bmod L \quad \text{for all } 1 \le \zeta \le \ell, \tag{4.190}$$

$$x_{\sigma_{\ell,r}(N+1-\xi)} = (a_{+,r-\xi+1} + \chi_{+,r-\xi+1}^r) \bmod L \quad \text{for all } 1 \le \xi \le r. \tag{4.191}$$

Finally, let us denote the special configuration in $\mathcal{V}_L^{\ell+r}(\Gamma^c)$ that consists of two clusters of size ℓ and r at the boundary of Γ^c by

$$b_{\ell,r} \equiv b_{\ell,r}(\Gamma) := \{a_{-,j} : j \le \ell\} \cup \{a_{+,j} : j \le r\}. \tag{4.192}$$

Lemma 4.4.16. *Let ε, θ and N satisfy Assumption 4.4.15. Let $\Gamma = \{\gamma_-, \cdots, \gamma_+\} \subseteq \mathcal{S}_{L,M}(\theta + 2\varepsilon)$. Moreover, let $n \in \mathbb{N}$ with $n < N$ and $x \in \mathcal{V}_L^N$ with $|x^{in}(\Gamma)| = n$. Let $r, \ell \in \mathbb{N}_0$ such that $\ell + r = N - n$ and $c \in \mathcal{D}_L^N$ with $\{c_j : \ell < j \le N - r\} \subseteq \Gamma$. Assume that $\mathbf{u}$ is a shortest path from c to x of length $k := d_L^N(c,x)$ such that*

$$\tilde{u}_j^{(k)} = x_{\sigma_{\ell,r}(j)} \quad \textit{for all } j \in I(N) \tag{4.193}$$

and in addition that

$$\{1, \cdots, \ell\} \subseteq I_0^{\mathbf{u}} \cup I_-^{\mathbf{u}} \qquad \textit{and} \qquad c_\zeta \ge a_{-,\ell-\zeta+1} \quad \textit{for } \zeta \le \ell, \tag{4.194}$$

$$\{N - r + 1, \cdots, N\} \subseteq I_0^{\mathbf{u}} \cup I_+^{\mathbf{u}} \qquad \textit{and} \qquad c_{N+1-\xi} \le a_{+,r-\xi+1} \quad \textit{for } \xi \le r. \tag{4.195}$$

Then there exists a shortest path $\mathbf{v}$ from c to x and $k_0 \in \{0, \cdots, k\}$ with

$$v^{(k_0)} = x^{in} \cup b_{\ell,r}(\Gamma) \subseteq \mathcal{S}_{L,M}(1/4 - \varepsilon/2). \tag{4.196}$$

Furthermore,

$$k - k_0 = \sum_{\zeta=1}^{\ell} \chi_{-,\zeta}^\ell + \sum_{\xi=1}^{r} \chi_{+,\xi}^r = |\chi_-^\ell(x,\Gamma)|_1 + |\chi_+^r(x,\Gamma)|_1. \tag{4.197}$$

Proof. For all $j \in I(N)$ let

$$\mathcal{Z}_j := \{\tilde{u}_j^{(l)} : 0 \le l \le k\}. \tag{4.198}$$

We claim that for all $j \in \{\ell + 1, \cdots, N - r\} \cap (I_0^{\mathbf{u}} \cup I_\pm^{\mathbf{u}})$ one has

$$\mathcal{Z}_j = \{c_j, \cdots, (c_j \pm L_j^{\mathbf{u}}) \bmod L\} \subseteq \Gamma. \tag{4.199}$$

Suppose this is not true. This would imply that there exists a $j \in I(N)$ with $\Gamma^c \cup \{\tilde{u}_j^{(0)}, \tilde{u}_j^{(k)}\} \subseteq \mathcal{Z}_j$. By Assumption (4.193) we have $\tilde{u}_j^{(k)} = x_{\sigma_{\ell,r}(j)} \in \Gamma$ and $\tilde{u}_j^{(0)} = c_j \in \Gamma$ for all $\ell < j \le N - r$. Hence,

$$|\mathcal{Z}_j| = |\Gamma^c| + 2 > L/2 + 1 \ge L_j^{\mathbf{u}} + 1 = |\mathcal{Z}_j|, \tag{4.200}$$

where we used $|\Gamma^c| \ge |(\mathcal{S}_{L,M}(1/4))^c| > L/2 - 1$. This is a contradiction.

We claim that for all $\zeta \le \ell$ and for all $\xi \le r$ we have

$$a_{-,\ell-\zeta+1} \in \mathcal{Z}_\zeta = \{c_\zeta, \cdots, (c_\zeta - L_\zeta^{\mathbf{u}}) \bmod L\}, \tag{4.201}$$

$$a_{-,r-\xi+1} \in \mathcal{Z}_{N+1-\xi} = \{c_{N+1-\xi}, \cdots, (c_{N+1-\xi} + L_{N+1-\xi}^{\mathbf{u}}) \bmod L\}. \tag{4.202}$$

We only present a proof for (4.201), since (4.202) follows analogously. For $\zeta = \ell$ we have $\tilde{u}_\ell^{(k)} \in x^{out}(\Gamma) \subseteq \Gamma^c$ according to Assumption (4.193). Together with Assumption (4.194) this implies $\gamma_- - 1 = a_{-,1} \in \mathcal{Z}_\ell$. The claim now follows from an inductive argument as well as from Lemma 4.4.7 (i).

Let us define

$$K_{-,\ell} := \begin{cases} d_L(c_\ell, a_{-,1}) & \text{for } \ell > 0, \\ 0 & \text{for } \ell = 0, \end{cases} \quad \text{and} \quad K_{+,r} := \begin{cases} d_L(c_{N+1-r}, a_{+,1}) & \text{for } r > 0, \\ 0 & \text{for } r = 0. \end{cases} \tag{4.203}$$

Then (4.201) and (4.202) imply that for all $\zeta \le \ell$ and for all $\xi \le r$

$$K_{-,\ell} = d_L(c_\zeta, a_{-,\ell-\zeta+1}) \le L_\zeta^{\mathbf{u}} \quad \text{and} \quad K_{+,r} = d_L(c_{N+1-\xi}, a_{-,r-\xi+1}) \le L_{N+1-\xi}^{\mathbf{u}}. \tag{4.204}$$

Let us now give an iterative construction of a path $\mathbf{v} = (v^{(0)}, \cdots, v^{(k)})$ starting from c. To this end, set $\tilde{v}^{(0)} := (c_1, \cdots, c_N)$. For $\zeta \in \{1, \cdots, \ell\}$ and $l \in (\zeta - 1)K_{-,\ell} + \{1, \cdots, K_{-,\ell}\}$, let

$$\tilde{v}^{(l)} := (\tilde{v}_1^{(l-1)}, \cdots, \tilde{v}_\zeta^{(l-1)} - 1, \cdots, \tilde{v}_N^{(l-1)}). \tag{4.205}$$

Let $k_1 := \ell K_{-,\ell}$. For $\xi \in \{1, \cdots, r\}$ and $l \in k_1 + (\xi - 1)K_{+,r} + \{1, \cdots, K_{+,r}\}$ let

$$\tilde{v}^{(l)} := (\tilde{v}_1^{(l-1)}, \cdots, \tilde{v}_{N+1-\xi}^{(l-1)} + 1, \cdots, \tilde{v}_N^{(l-1)}). \tag{4.206}$$

The path $\mathbf{v}$ has the property that it moves all particles of the configuration $\{c_j : j \le \ell \text{ or } j > N - r\}$ into the configuration $b_{\ell,r}(\Gamma)$ outside the boundary of Γ.

In the next step, we move the particles that are still remaining inside of Γ into the configuration $x^{in} = x^{in}(\Gamma)$. Let $k_2 := k_1 + rK_{+,r}$. Let $\ell' := |\{j \in I_-^{\mathbf{u}} : j > \ell\}|$, $r' := |\{j \in I_+^{\mathbf{u}} : j \le N - r\}|$. For $\zeta \in \{1, \cdots, \ell'\}$ and $l \in k_2 + \sum_{j=1}^{\zeta-1} L_{\ell+j}^{\mathbf{u}} + \{1, \cdots, L_\zeta^{\mathbf{u}}\}$ we set

$$\tilde{v}^{(l)} := (\tilde{v}_1^{(l-1)}, \cdots, \tilde{v}_{\ell+\zeta}^{(l-1)} - 1, \cdots, \tilde{v}_N^{(l-1)}). \tag{4.207}$$

Let $k_3 := k_2 + \sum_{j=1}^{\ell'} L_{\ell+j}^{\mathbf{u}}$. For $\xi \in \{1, \cdots, r'\}$ and $l \in k_3 + \sum_{j=1}^{\xi-1} L_{N-r-j}^{\mathbf{y}} + \{1, \cdots, L^{N-r-\xi}\}$ we set

$$\tilde{v}^{(l)} := (\tilde{v}_1^{(l-1)}, \cdots, \tilde{v}_{N-r-\xi}^{(l-1)} + 1, \cdots, \tilde{v}_N^{(l-1)}). \tag{4.208}$$

The fact that this construction is well-defined follows from the statement in (4.199), since no particle with index $j \in \{\ell+1, \cdots, N-r\}$ leaves Γ and therefore does not intersect the configuration $b_{\ell,r}$. In the last step, we move the configuration $b_{\ell,r}$ into $x^{out}(\Gamma)$. Let $k_0 := k_3 + \sum_{j=1}^{r'} L_{\ell+j}^{\mathbf{u}}$. By construction we have $v^{(k_0)} = b_{\ell,r} \cup x^{in}(\Gamma)$. Notice that by definition of $\chi_-^\ell \equiv \chi_-^\ell(x, \Gamma)$ and $\chi_+^r \equiv \chi_+^r(x, \Gamma)$, we obtain for all $\zeta \le \ell$ and $\xi \le r$ that

$$\chi_{-,\ell-\zeta+1}^\ell = d_L(\tilde{u}_\zeta^{(k)}, a_{-,\ell-\zeta+1}) = L_\zeta^{\mathbf{u}} - K_{-,\ell}, \tag{4.209}$$

$$\chi_{+,r-\xi+1}^r = d_L(\tilde{u}_{N+1-\xi}^{(k)}, a_{+,r-\xi+1}) = L_{N+1-\xi}^{\mathbf{u}} - K_{+,r}. \tag{4.210}$$

For all $\zeta \in \{1, \cdots, \ell\}$ and $l \in k_0 + \sum_{j=1}^{\zeta-1} \chi_{-,\ell-j+1}^\ell + \{1, \cdots, \chi_{-,\ell-\zeta+1}^\ell\}$ we set

$$\tilde{v}^{(l)} := (\tilde{v}_1^{(l-1)}, \cdots, (\tilde{v}_\zeta^{(l-1)} - 1) \bmod L, \cdots, \tilde{v}_N^{(l-1)}). \tag{4.211}$$

Let $k_4 := k_0 + \sum_{\zeta=1}^{\ell} \chi_{-,\zeta}^{\ell}$. For $\xi \in \{1,\cdots,r\}$ and $l \in k_4 + \sum_{j=1}^{\xi-1} \chi_{+,r-j+1}^{r} + \{1,\cdots,\chi_{+,r-\xi+1}^{r}\}$ we set

$$\tilde{v}^{(l)} := (\tilde{v}_1^{(l-1)},\cdots,(\tilde{v}_{N+1-\xi}^{(l-1)}+1) \bmod L,\cdots,\tilde{v}_N^{(l-1)}). \tag{4.212}$$

By construction, $\mathbf{v}$ is a shortest path from c to x, since it has a length of $k = \sum_{j=1}^{N} L_j^{\mathbf{u}}$ and $\tilde{v}^{(k)} = x$. Moreover, by construction as well as (4.209) and (4.210), we get

$$k - k_0 = \sum_{\zeta=1}^{\ell} \chi_{-,\zeta}^{\ell} + \sum_{\xi=1}^{r} \chi_{+,\xi}^{r}. \tag{4.213}$$

Finally, we note that $v^{(k_0)} \subseteq \mathcal{S}_{L,M}(1/4 - \varepsilon/2)$. This follows from the fact that for all $m \in b_{\ell,r}$, one has $d_L(m,M) < (\theta + 2\varepsilon)L + (\ell + r) \le (1/4 - \varepsilon/2)L$, where we used $\ell + r \le N/2 < \varepsilon L/2$ and $\varepsilon < \theta < 1/16$. □

To apply Lemma 4.4.16 to any configurations x and a set Γ we have to make sure that the rather technical conditions (4.193)-(4.195) are met. A sufficient condition for these assumptions to be satisfied is $c \subseteq \Gamma$.

Lemma 4.4.17. *Let $L > 8$. Let ε, θ and N satisfy Assumption 4.4.15. Let $\Lambda_L' := \mathcal{S}_{L,M}(\theta + 2\varepsilon)$. Fix $n \in \mathbb{N}$ such that $N/2 < n < N$ and $x \in \mathcal{V}_L^N$ with $x^{in}(\Lambda_L') \in \mathcal{V}^n(\Lambda_L')$. Let $c \in \mathcal{D}_L^N$, with $c \subseteq \Lambda_L'$ and set $k := d_L^N(x,c)$. Then there exists $r, \ell \in \mathbb{N}_0$ with $r + \ell = N - n$ and a shortest path $\mathbf{v}$ from c to x such that $\tilde{v}_j^{(k)} = x_{\sigma_{\ell,r}(j)}$ for all $j \in I(N)$. Moreover, there exists a $k_0 \in \{0,\cdots,k\}$ such that*

$$v^{(k_0)} = x^{in}(\Lambda_L') \cup b_{\ell,r}(\Lambda_L') \subseteq \mathcal{S}_{L,M}(1/4 - \varepsilon/2). \tag{4.214}$$

Proof. Let $\lambda_-' := \min \Lambda_L'$ and $\lambda_+' := \max \Lambda_L'$. Then $\Lambda_L' = \{\lambda_-',\cdots,\lambda_+'\}$.

Let $\mathbf{u}$ be a shortest path from c to x. According to Lemma 4.4.8 there exists an edge $e \in \mathcal{E}_L$ with $\max_{j \in I(N)} d_L(c_j, e) \ge (1/2 - \varepsilon)L$, which is not crossed by the path $\mathbf{u}$. Notice that this implies $e \subseteq (\Lambda_L')^c$, since $L > 8$. We define the index sets

$$J_\pm^{\mathbf{u},in} := \{j \in I_\pm^{\mathbf{u}} : \tilde{u}_j^{(k)} \in x^{in}(\Lambda_L')\} \quad \text{and} \quad J_\pm^{\mathbf{u},out} := \{j \in I_\pm^{\mathbf{u}} : \tilde{u}_j^{(k)} \in x^{out}(\Lambda_L')\}. \tag{4.215}$$

We claim that

$$\max J_+^{\mathbf{u},in} \le \min J_+^{\mathbf{u},out} \quad \text{and} \quad \max J_-^{\mathbf{u},out} \le \min J_-^{\mathbf{u},in}. \tag{4.216}$$

These statements are a consequence of Lemma 4.4.7. Here, we only prove the first inequality, the other one follows analogously. Suppose that $\max J_x^{\mathbf{u},in} > \min J_+^{\mathbf{u},out}$. This implies that there exists a $j \in J_+^{\mathbf{u},out}$ such that $j+1 \in J_+^{\mathbf{u},in}$. According to Lemma 4.4.7 (i) we have $L_j^{\mathbf{u}} \le L_{j+1}^{\mathbf{u}}$ and therefore also

$$\lambda_+' \ge \tilde{u}_{j+1}^{(k)} = c_{j+1} + L_{j+1}^{\mathbf{u}} > c_j + L_j^{\mathbf{u}} = \tilde{u}_j^{(k)} \ge \lambda_-', \tag{4.217}$$

which is a contradiction to $j \in J_+^{\mathbf{u},out}$.

Let $\ell := |J_-^{\mathbf{u},out}|$ and $r := |J_+^{\mathbf{u},out}|$. Together with (4.216) this implies $\{j : j \le \ell\} = J_-^{\mathbf{u},out} \subseteq I_-^{\mathbf{u}}$ and $\{j : j > N - r\} = J_+^{\mathbf{u},out} \subseteq I_+^{\mathbf{u}}$. Furthermore, Lemma 4.4.7 yields $\tilde{u}_j^{(k)} = x_{\sigma_{\ell,r}(j)}$ for all $j \in I(N)$. Moreover, for all $\zeta \in \{1,\cdots,\ell\}$ we have $c_\zeta \ge \lambda_- > a_{-,\ell-\zeta+1}(\Lambda_L')$ and for all $\xi \in \{1,\cdots,r\}$ we have $c_{N+1-\xi} \le \lambda_+ < a_{+,r-\xi+1}(\Lambda_L')$, since $c \subseteq \Lambda_L'$.

Lemma 4.4.16 now yields the proposition. □

Let us now define the set of negligible configurations that are in a large enough distance to the set of cluster configurations. This set is given by

$$\mathcal{C}_L^N := \{x \in \mathcal{V}_L^N : d_L^N(x, \mathcal{D}_L^N) \geq L^{3/2}\}. \tag{4.218}$$

Lemma 4.4.18. *Let ε, θ, N and Λ_L satisfy Assumption 4.4.15. There exists a $L_0 \equiv L_0(\varepsilon) > 0$ such that for all $L \geq L_0$, $n \in \mathbb{N}$ with $N/2 < n < N$ and $x \in \mathcal{V}_L^N \smallsetminus \mathcal{C}_L^N$ with $x \cap \Lambda_L \in \mathcal{V}^n(\Lambda_L)$ one has*

$$\mathcal{W}_L(x) \subseteq \mathcal{S}_{L,M}(\theta + \varepsilon)\,. \tag{4.219}$$

Moreover, for all $m \in \mathcal{W}_L(x)$ we have

$$c_{L,m}^N \subseteq \mathcal{S}_{L,M}(\theta + 2\varepsilon). \tag{4.220}$$

Proof. Let us first show (4.219). Let $\nu \in (\mathcal{S}_{L,M}(\theta+\varepsilon))^c$. Then for all $\xi \in c_{L,\nu}^N$ one has

$$d_L(\xi, \Lambda_L) \geq d_L(\nu, \Lambda_L) - \lceil (N+1)/2 \rceil \geq \varepsilon L - 2\varepsilon L/3 \tag{4.221}$$

for all $L \geq L_1$ with $L_1 \equiv L_1(\varepsilon) := 9/\varepsilon$. Since n particles of x are located inside of Λ_L we have

$$d_L^N(x, c_{L,\nu}^N) \geq n\varepsilon L/3 \geq \varepsilon^2 L^2/6, \tag{4.222}$$

where we applied Lemma 4.4.5. Let $L_0 \equiv L_0(\varepsilon) > L_1$, such that $\varepsilon^2 L_0^{1/2}/6 > 1$. For any $L \geq L_0$ this implies $d_L^N(x, c_{L,\nu}^N) \geq L^{3/2}$. Since $x \notin \mathcal{C}_L^N$ by assumption, we conclude that $\nu \notin \mathcal{W}_L(x)$.

For all $L \geq L_0$ and $m \in \mathcal{W}_L(x) \subseteq \mathcal{S}_{L,M}(\theta+\varepsilon)$, observe that for all $\xi \in c_{L,m}^N$, we have

$$d_L(\xi, M) \leq d_L(\xi, m) + d_L(m, M) < \lceil (N+1)/2 \rceil + (\theta+\varepsilon)L \leq (\theta + 2\varepsilon)L\,, \tag{4.223}$$

from which we conclude $c_{L,m}^N \subseteq \mathcal{S}_{L,M}(\theta + 2\varepsilon)$. This concludes the lemma. □

Combining Lemma 4.4.17 and Lemma 4.4.18 enables us to construct a path satisfying Assumptions (4.193)-(4.195) for an arbitrary configuration $x \in \mathcal{V}_L^N \smallsetminus \mathcal{C}_L^N$.

Lemma 4.4.19. *Let ε, θ, N and Λ_L satisfy Assumption 4.4.15. Let $L \geq L_0$, with L_0 as in Lemma 4.4.18. Let $n \in \mathbb{N}$ with $N/2 < n < N$, $y \in \mathcal{V}^n(\Lambda_L)$, $x \in \mathcal{V}_L^N \smallsetminus \mathcal{C}_L^N$ with $x^{in}(\Lambda_L) = y$. Then there exist $\ell, r \in \mathbb{N}_0$ with $r + \ell = N - n$ such that $y_{\kappa-\ell} \in \mathcal{W}_L(x)$ for $\kappa := \lfloor (N+1)/2 \rfloor$ and*

$$c_{L,y_{\kappa-\ell}}^N \subseteq \mathcal{S}_{L,M}(1/4 - \varepsilon/2). \tag{4.224}$$

Furthermore, there exists a shortest path $\mathbf{v}$ from $c_{L,y_{\kappa-\ell}}^N$ to x and a $k_0 \in \{1, \cdots, k\}$ with $k := d_L^N(x, \mathcal{D}_L^N)$ such that

$$v^{(k_0)} = y \cup b_{\ell,r}(\Lambda_L) \subseteq \mathcal{S}_{L,M}(1/4 - \varepsilon/2), \tag{4.225}$$

and

$$k - k_0 = \sum_{\zeta=1}^{\ell} \chi_{-,\zeta}^{\ell} + \sum_{\xi=1}^{r} \chi_{+,\xi}^{r} = |\chi_-^{\ell}(x, \Lambda_L)|_1 + |\chi_+^{r}(x, \Lambda_L)|_1. \tag{4.226}$$

Proof. By Lemma 4.4.9 there exists $m \in \mathcal{W}_L(x) \cap x$. From Lemma 4.4.18 we know

$$c := c^N_{L,m} \subseteq \Lambda'_L := \mathcal{S}_{L,M}(\theta + 2\varepsilon) \subseteq \mathcal{S}_{L,M}(1/4 - \varepsilon/2). \tag{4.227}$$

According to Lemma 4.4.17 there exists a shortest path $\mathbf{u}$ from c to x and $\ell', r' \in \mathbb{N}_0$ with $\ell' + r' = N - |x^{in}(\Lambda'_L)|$ and $k_1 \in \{0, \cdots, k\}$ such that

$$z := u^{(k_1)} = b_{\ell',r'}(\Lambda'_L) \cup x^{in}(\Lambda'_L) \subseteq \mathcal{S}_{L,M}(1/4 - \varepsilon/2), \tag{4.228}$$

and $\tilde{u}^{(k)}_j = x_{\sigma_{\ell',r'}(j)}$ for all $j \in I(N)$.

Let us define

$$\ell := \ell' + |\{\nu \in x^{in}(\Lambda'_L) : \nu < \lambda_-\}| = |\{j \in I(N) : z_j < \lambda_-\}| \quad \text{and} \tag{4.229}$$

$$r := r' + |\{\nu \in x^{in}(\Lambda'_L) : \nu > \lambda_+\}| = |\{j \in I(N) : z_j > \lambda_+\}|. \tag{4.230}$$

These quantities satisfy $\ell + r = N - |\{j \in I(N) : z_j \in \Lambda_L\}| = N - n$. Note that by this definition $\sigma_{\ell',r'}(x, \Lambda'_L) = \sigma_{\ell,r}(x, \Lambda_L)$. Hence, $\tilde{u}^{(k)}_j = x_{\sigma_{\ell,r}(j)}$ for all $j \in I(N)$ and $y_j = z_{j+\ell}$ for all $j \in \{1, \cdots, n\}$.

Next, we show that $m = z_\kappa$. First, we claim that

$$m \in \mathcal{W}_L(z). \tag{4.231}$$

To see this, take any $\nu \in \mathcal{W}_L(z)$ and any shortest path $\mathbf{v}$ from $c' := c^N_{L,\nu}$ to z. Lemma 4.4.4 indicates that

$$k_2 := d^N_L(z, c') = d^N_L(z, \mathcal{D}^N_L) \le d^N_L(z, c) = k_1. \tag{4.232}$$

The path $\{v^{(0)}, \cdots, v^{(k_2)}, u^{(k_1+1)}, \cdots, u^{(k)}\}$ is therefore a path from c' to x of length $k + (k_2 - k_1)$ and therefore – using (4.232) – we get

$$k = d^N_L(x, \mathcal{D}^N_L) \le k + (k_2 - k_1) \le k. \tag{4.233}$$

Hence $k_1 = k_2$. Equality in (4.232) implies $m \in \mathcal{W}_L(z)$. According to Lemma 4.4.18 we have $\mathcal{W}_L(z) \subseteq \mathcal{S}_{L,M}(\theta + \varepsilon) \subseteq \Lambda'_L$, since $d^N_L(z, \mathcal{D}^N_L) \le d^N_L(x, \mathcal{D}^N_L) < L^{3/2}$, because $x \in \mathcal{V}^N_L \setminus \mathcal{C}^N_L$. Hence

$$m \in x \cap \mathcal{W}_L(z) = x \cap \mathcal{W}_L(z) \cap \Lambda'_L = z \cap \mathcal{W}_L(z) = \{z_\kappa\}, \tag{4.234}$$

where we used $z \subseteq \mathcal{S}^N_{L,M}(1/4 - \varepsilon/2)$ together with Lemma 4.4.10. Therefore $m = z_\kappa = y_{\kappa-\ell}$.

Lemma 4.4.4 states that $\mathbf{w} := (u^{(0)}, \cdots, u^{(k_1)})$ is a shortest path from c to z. According to Lemma 4.4.10 we have $\kappa \in I^{\mathbf{w}}_0$, since $m = z_\kappa$. This also implies $\kappa \in I^{\mathbf{u}}_0$, since by construction of the path $\mathbf{u}$, for all $l \in \{k_1, \cdots, k\}$ one has $\tilde{u}^{(l)}_\kappa = \tilde{u}^{(k_1)}_\kappa$. In this case, it follows from Lemma 4.4.7 (iii) that $\{j : j \le \ell\} \subseteq \{0, \cdots, \kappa\} \subseteq I^{\mathbf{u}}_0 \cup I^{\mathbf{u}}_-$ and for all $\zeta \le \ell$ holds

$$a_{-,\ell-\zeta+1} \le z_{\ell+1} - (\ell - \zeta + 1) \le c_{\ell+1} - (\ell - \zeta + 1) = c_\zeta. \tag{4.235}$$

Analogously, we have $\{j : j > N - r\} \subseteq \{\kappa, \cdots, N\} \subseteq I^{\mathbf{u}}_0 \cup I^{\mathbf{u}}_+$ and for all $\xi \le r$ one therefore gets

$$a_{+,r-\xi+1} \ge c_{N+1-\xi}. \tag{4.236}$$

Thus, according to Lemma 4.4.16 there exists a path $\mathbf{v}$ with all the properties stated in the proposition. □

4.4.5 Proof of Lemma 4.4.1

Before we prove Lemma 4.4.1 we show the estimate (4.114).

Lemma 4.4.20. *Let ε, θ, N and Λ_L satisfy Assumption 4.4.15. There exists $L_0 \equiv L_0(\varepsilon) > 0$ such that for all $L \geq L_0$, $n \in \mathbb{N}$ with $N/2 < n < N$, $y \in \mathcal{V}^n(\Lambda_L)$ and $\mu \geq \ln 2$ one has*

$$\sum_{\substack{z \in \mathcal{V}^{N-n}(\Lambda_L^c), \\ z \cup y \notin \mathcal{D}_L^N}} e^{-\mu d_L^N(y \cup z, \mathcal{D}_L^N)} \leq 333 e^{-\mu} e^{-\mu h_L^n(y)}, \tag{4.237}$$

where h_L^n was defined in (4.111).

Proof. Let

$$\mathcal{A}'(y) := \left\{ z \in \mathcal{V}^{N-n}(\Lambda_L^c) : \ y \cup z \notin \mathcal{C}_L^N \right\} \subseteq \mathcal{V}^{N-n}(\Lambda_L^c), \tag{4.238}$$

$$\mathcal{A}(y) := \left\{ z \in \mathcal{A}'(y) : \ y \cup z \notin \mathcal{D}_L^N \right\} \subseteq \mathcal{A}'(y). \tag{4.239}$$

There exists $L_1 \in \mathbb{N}$ such that $L^{1/2} - \ln L / \ln 2 \geq L^{1/4}$ for all $L \geq L_1$. Hence for all $L \geq L_1$ we get

$$\sum_{z \in (\mathcal{A}(y))^c} \mathrm{e}^{-\mu d_L^N(y \cup z, \mathcal{D}_L^N)} \leq |(\mathcal{A}(y))^c| \mathrm{e}^{-\mu L^{3/2}} \leq \mathrm{e}^{-\mu L^{5/4}} \leq \mathrm{e}^{-\mu} \mathrm{e}^{-\mu h_L^n(y)}, \tag{4.240}$$

where we used $|(\mathcal{A}(y))^c| \leq |\Lambda_L^c|^{N-n} \leq L^L$ as well as $\mu \geq \ln 2$. We partition $\mathcal{A}^{(\prime)}(y)$ into smaller subsets. For any $\ell, r \in \mathbb{N}_0$ with $\ell + r = N - n$ let $c^\ell := c_{L, y_{\kappa-\ell}}^N$ with $\kappa := \lfloor (N+1)/2 \rfloor$. Let us further define

$$\begin{aligned} \mathcal{A}_{\ell,r}^{(\prime)}(y) := \Big\{ z \in \mathcal{A}^{(\prime)}(y) : d_L^N(y \cup z, \mathcal{D}_L^N) = d_L^N(c^\ell, y \cup b_{\ell,r}) + d_L^N(y \cup b_{\ell,r}, y \cup z), \\ d_L^N(y \cup b_{\ell,r}, y \cup z) = \sum_{\zeta=1}^{\ell} \chi_{-,\zeta}^\ell + \sum_{\xi=1}^{r} \chi_{+,\xi}^r \Big\}, \end{aligned} \tag{4.241}$$

where $\chi_-^\ell(x, \Lambda_L) \in \mathcal{X}^\ell$ and $\chi_+^r(x, \Lambda_L) \in \mathcal{X}^r$ were defined in (4.190) and (4.191).

Lemma 4.4.4 and Lemma 4.4.19 imply immediately that there exists a $L_2 \equiv L_2(\varepsilon) > L_1$ such that for all $L \geq L_2$ we get the equality

$$\mathcal{A}(y) = \bigcup_{\substack{\ell, r \in \mathbb{N}_0: \\ \ell + r = N - n}} \mathcal{A}_{\ell,r}(y) = \mathcal{A}_{N-n,0}(y) \cup \mathcal{A}_{0,N-n}(y) \cup \bigcup_{\substack{\ell, r \in \mathbb{N}: \\ \ell + r = N - n}} \mathcal{A}_{\ell,r}'(y). \tag{4.242}$$

Let us first consider the case $\ell, r \in \mathbb{N}$. Definition (4.241) implies, together with Lemma 4.4.11 for $\mu \geq \ln 2$ that

$$\begin{aligned} \sum_{z \in \mathcal{A}_{\ell,r}'(y)} \mathrm{e}^{-\mu d_L^N(y \cup z, \mathcal{D}_L^N)} &\leq \mathrm{e}^{-\mu d_L^N(c^\ell, y \cup b_{\ell,r})} \Big(\sum_{\chi^\ell \in \mathcal{X}^\ell} \mathrm{e}^{-\mu |\chi^\ell|_1} \Big) \Big(\sum_{\chi^r \in \mathcal{X}^r} \mathrm{e}^{-\mu |\chi^r|_1} \Big) \\ &\leq \mathrm{e}^{-\mu d_L^N(c^\ell, y \cup b_{\ell,r})} (1 + 30 \mathrm{e}^{-\mu})^2. \end{aligned} \tag{4.243}$$

Now we estimate the first factor on the right-hand side of (4.243) uniformly in ℓ, r. Both c^ℓ and $y \cup b_{\ell,r}$ are subsets of $\mathcal{S}_{L,M}(1/4 - \varepsilon/2)$. By Lemma 4.4.10 we have $y_{\kappa-\ell} \in \mathcal{W}_L(y \cup b_{\ell,r})$ with

$$d_L^N(y \cup b_{\ell,r}, c^\ell) = d^N(y \cup b_{\ell,r}, c^\ell) \geq \sum_{j=1}^n |y_j - c_{j+\ell}^\ell| + |a_{-,1} - c_\ell^\ell| + |a_{+,1} - c_{N-r+1}^\ell|, \tag{4.244}$$

where we applied (4.158) and (4.115). For all $\ell \in \{1, \cdots, N-n-1\}$ we have

$$|a_{-,1} - c_\ell^\ell| + |a_{+,1} - c_{N-r+1}^\ell| \geq |a_{+,1} - a_{-,1}| - |c_{N-r+1}^\ell - c_\ell^\ell| \geq 2\theta L - (n+1) \geq \varepsilon L, \tag{4.245}$$

where we used that $a_{+,1} - a_{-,1} \geq d_L(a_{+,1}, M) + d_L(a_{-,1}, M) \geq 2\theta L$, as well as $n+1 \leq N < \varepsilon L$ and $\theta > \varepsilon$. Hence, for all $y \in \mathcal{V}^n(\Lambda_L)$ we have either $|a_{-,1} - c_\ell^\ell| \geq \varepsilon L/2 \geq \varepsilon L/4 + 1$ or $|a_{+,1} - c_{r+\ell+1}^\ell| \geq \varepsilon L/2 \geq \varepsilon L/4 + 1$ for all $L \geq L_0 \equiv L_0(\varepsilon) := \max\{L_2, 4/\varepsilon\}$. This implies, together with (4.244) that

$$d_L^N(y \cup b_{\ell,r}, c^\ell) - 1 \geq h_L^n(y) + \varepsilon L/4. \tag{4.246}$$

Hence, by combining (4.243) and (4.246) we find

$$\sum_{\substack{\ell,r\in\mathbb{N},\\ \ell+r=N-n}} \sum_{z\in\mathcal{A}'_{\ell,r}(y)} \mathrm{e}^{-\mu d_L^N(y\cup z, \mathcal{D}_L^N)} \leq 2\mathrm{e}^{-1}/\ln(2)(1+30\mathrm{e}^{-\mu})^2 \mathrm{e}^{-\mu} e^{-\mu h_L^n(y)} \leq 272\mathrm{e}^{-\mu} e^{-\mu h_L^n(y)}, \tag{4.247}$$

where we used that $(N-n)\mathrm{e}^{-\mu\varepsilon L/4} \leq (\varepsilon L/2)2^{-\varepsilon L/4} \leq 2\mathrm{e}^{-1}/\ln(2)$ for all $\mu \geq \ln 2$.

Let us now consider the case $\ell = 0$ or $r = 0$. There are only two configurations $y \in \mathcal{V}^n(\Lambda_L)$ such that there exists a $z \in \mathcal{V}^{N-n}(\Lambda_L^c)$ with $y \cup z \in \mathcal{D}_L^N$, namely y_+^n and y_-^n. The configurations $z_-^n \equiv z_-^n(\Lambda_L) := b_{N-n,0}$ and $z_+^n \equiv z_+^n(\Lambda_L) := b_{0,N-n}$ satisfy $y_\pm^n \cup z_\pm^n \in \mathcal{D}_L^N$. There are no other configurations in $\mathcal{V}^{N-n}(\Lambda_L^c)$ with this property. We further restrict ourselves to the case $\ell = N-n$ and $r = 0$. The other case can be treated analogously. Now our approach depends on whether $y = y_-^n$ or not. We have

$$\mathcal{A}_{N-n,0}(y) \subseteq \begin{cases} \mathcal{A}'_{N-n,0}(y) & \text{for } y \neq y_-^n, \\ \mathcal{A}'_{N-n,0}(y) \setminus \{z_-^n\} & \text{for } y = y_-^n. \end{cases} \tag{4.248}$$

Analogous to (4.243) we obtain

$$\sum_{z\in\mathcal{A}_{N-n,0}(y)} \mathrm{e}^{-\mu d_L^N(y\cup z, \mathcal{D}_L^N)} \leq \mathrm{e}^{-\mu d_L^N(y\cup b_{N-n,0}, c^{N-n})} \begin{cases} \sum_{\chi\in\mathcal{X}^{N-n}} \mathrm{e}^{-\mu|\chi|_1} & \text{for } y \neq y_-^n, \\ \sum_{\chi\in\mathcal{X}^{N-n}\setminus\{0\}} \mathrm{e}^{-\mu|\chi|_1} & \text{for } y = y_-^n, \end{cases} \tag{4.249}$$

and for all $y \in \mathcal{V}_L^n(\Lambda_L)$

$$d_L^N(y \cup b_{N-n,0}, c^{N-n}) \geq d_L^{n+1}(y \cup \{a_{-,1}\}, \mathcal{D}_L^{n+1}) \geq \begin{cases} h_L^n(y) + 1 & \text{for } y \neq y_-^n, \\ h_L^n(y) & \text{for } y = y_-^n. \end{cases} \tag{4.250}$$

Lemma 4.4.11 then implies

$$\sum_{z\in\mathcal{A}_{N-n,0}(y)} \mathrm{e}^{-\mu d_L^N(y\cup z, \mathcal{D}_L^N)} \leq \begin{cases} (1+30\mathrm{e}^{-\mu})\mathrm{e}^{-\mu}\mathrm{e}^{-\mu h_L^n(y)} & \text{for } y \neq y_-^n, \\ 30\mathrm{e}^{-\mu}\mathrm{e}^{-\mu h_L^n(y)} & \text{for } y = y_-^n. \end{cases} \tag{4.251}$$

Hence, by (4.242), (4.247) and (4.251), as well as the definition of h_L^n

$$\sum_{z\in\mathcal{A}(y)} \mathrm{e}^{-\mu d_L^N(y\cup z, \mathcal{D}_L^N)} \leq 332\mathrm{e}^{-\mu}\mathrm{e}^{-\mu h_L^n(y)} \tag{4.252}$$

where we used $\mu \geq \ln 2$. Together with (4.240), this concludes the proof. □

We are now able to prove Lemma 4.4.1 by combining the results of Section 4.4.3 and Lemma 4.4.20.

Proof of Lemma 4.4.1. Let $z_-^n \equiv z_-^n(\Lambda_L) := b_{N-n,0}$ and $z_+^n \equiv z_+^n(\Lambda_L) := b_{0,N-n}$. Then for all $y \in \mathcal{V}^n(\Lambda_L)$ and all $z \in \mathcal{V}^{N-n}(\Lambda_L^c) \setminus \{z_\pm^n\}$ we have that $y \cup z \notin \mathcal{D}_L^N$ is not a droplet configuration. By Lemma 4.3.1 we obtain for all $y, y' \in \mathcal{V}^n(\Lambda_L)$ that

$$\begin{aligned} |\langle \delta_y^{\Lambda_L}, D_{L,\Lambda_L,\gamma}^{N,n}(\Delta)\delta_{y'}^{\Lambda_L}\rangle| \le \sum_{\eta\in\{\pm\}} |\langle \delta_{y\cup z_\eta^n}^L, (\rho[\varphi_{L,\gamma}^N(\Delta)] - \rho_{L,\gamma}^N)\delta_{y'\cup z_\eta^n}^L\rangle| \\ + \sum_{z\in\mathcal{V}_{\Lambda_L}^{N-n}\setminus\{z_\pm^n\}} |\langle \delta_{y\cup z}^L, \rho[\varphi_{L,\gamma}^N(\Delta)]\delta_{y'\cup z}^L\rangle|. \end{aligned} \tag{4.253}$$

By Theorem 4.2.1 and Lemma 4.4.20, using Cauchy–Schwarz, we further estimate

$$\sum_{z\in\mathcal{V}_{\Lambda_L}^{N-n}\setminus\{z_\pm^n\}} |\langle \delta_{y\cup z}^L, \rho(\varphi_{L,\gamma}^N)\delta_{y'\cup z}^L\rangle| \le \frac{2^8}{L} 333\mathrm{e}^{-2\mu_1}\mathrm{e}^{-\mu_1(h_L^n(y)+h_L^n(y'))}, \tag{4.254}$$

for all $L \ge L_0$, where $L_0 \equiv L_0(\varepsilon)$ was given in Lemma 4.4.20. Moreover, according to Lemma 4.4.14 we derive the estimate

$$\begin{aligned} &|\langle \delta_{y\cup z_\eta^n}^L, [\rho(\varphi_{L,\gamma}^N) - \rho_{L,\gamma}^N]\delta_{y'\cup z_\eta^n}^L\rangle| \\ &\qquad \le \frac{2^{17}}{L}\begin{cases} \mathrm{e}^{-2\mu_1} & \text{if } y = y' = y_\eta^n, \\ \mathrm{e}^{-\mu_1(d_L^N(y\cup z_\eta^n, \mathcal{D}_L^N) + d_L^N(y'\cup z_\eta^n, \mathcal{D}_L^N))} & \text{else} \end{cases} \end{aligned} \tag{4.255}$$

for all $\eta \in \{\pm\}$. Lemma 4.4.20 implies for all $L \ge L_0$, $\eta \in \{\pm\}$ and all $y \in \mathcal{V}^n(\Lambda_L) \setminus \{y_\eta^n\}$ that

$$\mathrm{e}^{-\mu_1 d_L^N(y\cup z_\eta^n, \mathcal{D}_L^N)} \le 333\mathrm{e}^{-\mu_1}\mathrm{e}^{-\mu_1 h_L^n(y)}, \tag{4.256}$$

since in this case $y \cup z_\eta^n \notin \mathcal{D}_L^n$. On the other hand, if $y = y_\eta^n$ we have

$$\mathrm{e}^{-\mu_1 d_L^N(y_\eta^n\cup z_\eta^n, \mathcal{D}_L^N)} = 1 = \mathrm{e}^{-\mu_1 h_L^n(y_\eta^n)}. \tag{4.257}$$

Finally, if $y, y' \in \{y_\pm^n\}$ with $y' \ne y$ we have

$$\mathrm{e}^{-\mu_1[d_L^N(y\cup z_\eta^n, \mathcal{D}_L^N) + d_L^N(y'\cup z_\eta^n, \mathcal{D}_L^N)]} \le \mathrm{e}^{-\mu_1}. \tag{4.258}$$

By combining (4.253), (4.254), (4.255), (4.256), (4.257) and (4.258), we conclude the proof. □

4.5 The logarithmic lower bound

4.5.1 General idea and strategy

The last remaining step to prove Theorem 4.1.1 is to show the estimate (4.105). We achieve this by estimating the singular values of $D_{L,\Lambda_L,\gamma}^{N,n}(\Delta)$ in turn.

Bounds for the entries of $D_{L,\Lambda_L,\gamma}^{N,n}(\Delta)$ were already given in Lemma 4.4.1. We note that most entries are exponentially small with respect to the distance function h_L^n. The large entries are concentrated in configurations that are both close to being a droplet configuration and near the boundary of Λ_L. Let us order the contributions in $\mathcal{V}^n(\Lambda_L)$ with respect to h^n_L. We introduce a bijective map

$$\Xi^n_L : \left\{1, \cdots, m\right\} \to \mathcal{V}^n(\Lambda_L) \tag{4.259}$$

where $m := |\mathcal{V}^n(\Lambda_L)|$ with the property that $h^n_L \circ \Xi^n_L$ is monotonously increasing. If we take a look at a matrix representation of $D^{N,n}_{L,\Lambda_L,\gamma}(\Delta)$ with respect to the basis $\left\{ |\delta^{\Lambda_L}_{\Xi^n_L(j)}\rangle : j \in \{1, \cdots, m\}\right\}$ we see that the large entries are found in the upper left corner of the matrix.

We exploit the structure of $D^{N,n}_{L,\Lambda_L,\gamma}(\Delta)$ to estimate its eigenvalues. To this end, we split up $D^{N,n}_{L,\Lambda_L,\gamma}(\Delta)$ into a sum of operators of lower rank. For all $j \in \{0, \cdots, m\}$ we define

$$R^{N,n}_{L,\Lambda_L,\gamma,j} \equiv R^{N,n}_{L,\Lambda_L,\gamma,j}(\Delta) := \sum_{\substack{r,s\in\mathbb{N}:\\ j<r,s\le m}} \left\langle \delta^{\Lambda_L}_{\Xi^n_L(r)}, D^{N,n}_{L,\Lambda_L,\gamma}(\Delta)\delta^{\Lambda_L}_{\Xi^n_L(s)} \right\rangle \left|\delta^{\Lambda_L}_{\Xi^n_L(r)}\right\rangle\!\left\langle\delta^{\Lambda_L}_{\Xi^n_L(s)}\right| \tag{4.260}$$

and

$$S^{N,n}_{L,\Lambda_L,\gamma,j} \equiv S^{N,n}_{L,\Lambda_L,\gamma,j}(\Delta) := D^{N,n}_{L,\Lambda_L,\gamma}(\Delta) - R^{N,n}_{L,\Lambda_L,\gamma,j}(\Delta). \tag{4.261}$$

For an illustration of this partition see Figure 4.8.

$$D^{N,n}_{L,\Lambda_L,\gamma} = \begin{array}{c} \begin{array}{cc} \Xi^n_L(1) & \Xi^n_L(j+1) \\ \downarrow & \downarrow \end{array} \\ \underbrace{\left(\begin{array}{ccc|ccc} * & \cdots & * & * & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ * & \cdots & * & * & \cdots & * \\ \hline * & \cdots & * & & & \\ \vdots & \ddots & \vdots & & 0 & \\ * & \cdots & * & & & \end{array}\right)}_{= S^{N,n}_{L,\Lambda_L,\gamma,j}} \end{array} + \underbrace{\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{array}{ccc} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{array} \end{array}\right)}_{= R^{N,n}_{L,\Lambda_L,\gamma,j}} \leftarrow \Xi^n_L(j+1)$$

Figure 4.8: Partition of $D^{N,n}_{L,\Lambda_L,\gamma}(\Delta)$

We can estimate maximal singular values of the smaller operator $R^{N,n}_{L,\Lambda_L,\gamma,j}$ by calculating the maximum absolute row sum norm of the operators. We show for all $j \in \{1, \cdots, m\}$ that

$$\lambda_1(R^{N,n}_{L,\Lambda_L,\gamma,j}) \lesssim L^{-1} \mathrm{e}^{-\mu_1} \mathrm{e}^{-\mu_1 h^n_L \circ \Xi^n_L(j)}, \tag{4.262}$$

where $(\lambda_s)_{s\le m}$ denotes the non-increasing set of singular values. The rank of the operator $S^{N,n}_{L,\Lambda_L,\gamma,j}$ on the other hand is by definition lower or equal than $2j$. Hence, by using Inequality (3.61) for singular values, we get for any $j \in \mathbb{N}$ with $j \le |\mathcal{V}^n(\Lambda_L)|$ the following bound

$$\lambda_{2j+1}(D^{N,n}_{L,\Lambda_L,\gamma}) \le \lambda_{2j+1}(S^{N,n}_{L,\Lambda_L,\gamma,j}) + \lambda_1(R^{N,n}_{L,\Lambda_L,\gamma,j}) = \lambda_1(S^{N,n}_{L,\Lambda_L,\gamma,j}) \lesssim L^{-1} \mathrm{e}^{-\mu_1} \mathrm{e}^{-\mu_1 h^n_L \circ \Xi^n_L(j)}. \tag{4.263}$$

Since the singular values in (4.263) are exponentially decreasing with respect to $h_L^n \circ \Xi_L^n$, we are able to derive (4.105) with a similar method as presented in the previous section.

4.5.2 Estimating the von Neumann–Schatten quasinorm of $D_{L,\Lambda_L,\gamma}^{N,n}$

The objective here is to prove the convergence of $\rho_{\Lambda_L}^n[\varphi_{L,\gamma}^N(\Delta)]$ to $\rho_{L,\Lambda_L,\gamma}^n$ in the von Neumann–Schatten quasinorm $\|\cdot\|_{1/p}$ for a any $p \in]1,\infty[$.

Before we estimate the von Neumann–Schatten quasinorm of $D_{L,\Lambda_L,\gamma}^{N,n}$, we show the following auxiliary result.

Lemma 4.5.1. *Let ε, θ, N and Λ_L satisfy Assumption 4.4.15. There exists $L_0 \equiv L_0(\varepsilon) > 0$ such that for all $L \geq L_0$, $n \in \mathbb{N}$ with $N/2 < n < N$ and $\mu \geq \ln 2$ holds*

$$\sum_{y \in \mathcal{V}^n(\Lambda_L) \setminus \{y_\pm^n\}} e^{-\mu h_L^n(y)} \leq 2^{11} e^{-\mu}. \tag{4.264}$$

Proof. For any $y \in \mathcal{V}^n(\Lambda_L)$, we have $y' := y \cup \{a_{+,1}\} \subseteq \mathcal{S}_{L,M}(1/4 - \varepsilon/2)$. Hence, according to Lemma 4.4.10, we have $y_\kappa \in \mathcal{W}_L(y')$ with $\kappa := \lfloor (n+2)/2 \rfloor$. For any $t \in \{0, \cdots, |\Lambda_L| - n\}$ let

$$\mathcal{B}_t^n := \{y \in \mathcal{V}^n(\Lambda_L) : \, y_\kappa = \lambda_+ - (n - \kappa + t)\}. \tag{4.265}$$

Hence,

$$\mathcal{V}^n(\Lambda_L) = \bigcup_{t=0}^{|\Lambda_L|-n} \mathcal{B}_j^n. \tag{4.266}$$

Let us now consider a $y \in \mathcal{B}_t^n$ for a $t \in \{0, \cdots, |\Lambda_L| - n\}$. Let $c := c_{L,y_\kappa}^{n+1}$. We define $\chi'_- \equiv \chi'_-(y) \in \mathcal{X}^{\kappa-1}$ and $\chi'_+ \equiv \chi'_+(y) \in \mathcal{X}^{n-\kappa}$ such that

$$\chi'_{-,j}(y) := |(y_\kappa - j) - c_{\kappa-j}| \quad \text{for } 1 \leq j \leq \kappa - 1, \tag{4.267}$$

$$\chi'_{+,j}(y) := |(y_\kappa + j) - c_{\kappa+j}| \quad \text{for } 1 \leq j \leq n - \kappa. \tag{4.268}$$

Then, according to Lemma 4.4.10,

$$d_L^{n+1}(y \cup \{a_{+,1}\}, \mathcal{D}_L^{n+1}) = d^{n+1}(y \cup \{a_{+,1}\}, \mathcal{D}_L^{n+1}) = \sum_{j=1}^{\kappa-1} \chi'_{-,j} + \sum_{j=1}^{n-\kappa} \chi'_{+,j} + t. \tag{4.269}$$

Hence, according to Lemma 4.4.11, we obtain for all $t \in \{0, \cdots, |\Lambda_L| - n\}$ and all $\mu \geq \ln 2$,

$$\sum_{y \in \mathcal{B}_t^n} \mathrm{e}^{-\mu d_L^{n+1}(y \cup \{a_{+,1}\}, \mathcal{D}_L^{n+1})} \leq \mathrm{e}^{-\mu t}(1 + 30\mathrm{e}^{-\mu})^2. \tag{4.270}$$

Therefore, by (4.266) we have

$$\sum_{y \in \mathcal{V}^n(\Lambda_L) \setminus \{y_+^n\}} \mathrm{e}^{-\mu d_L^{n+1}(y \cup \{a_{+,1}\}, \mathcal{D}_L^{n+1})} \leq \Big(1 + \frac{\mathrm{e}^{-\mu}}{1 - \mathrm{e}^{-\mu}}\Big)(1 + 30\mathrm{e}^{-\mu})^2 - 1 \leq 1022\mathrm{e}^{-\mu} \tag{4.271}$$

where we used that $\mu \geq \ln 2$. By an analogous method we obtain the same bound for the sum over $\exp(d_L^{n+1}(y \cup \{a_{-,1}\}, \mathcal{D}_L^{n+1}))$.

Let $L_0 \in \mathbb{N}$ such that $L_0^{1/4} - \ln L_0 > 2$. By using $n \le N < \mu L$, we get for all $L \ge L_0$ that

$$\sum_{y \in \mathcal{V}^n(\Lambda_L) \setminus \{y^n_\pm\}} \mathrm{e}^{-\mu(L^{5/4}-1)} \le L^n \mathrm{e}^{-\mu(L^{5/4}-1)} \le \mathrm{e}^{-\mu(L(L^{1/4}-\ln L)-1)} \le \mathrm{e}^{-\mu} \tag{4.272}$$

By the definition of h^n_L in (4.111) we obtain

$$\sum_{y \in \mathcal{V}^n(\Lambda_L) \setminus \{y^n_\pm\}} \mathrm{e}^{-\mu h^n_L(y)} \le \sum_{\eta \in \{\pm\}} \sum_{y \in \mathcal{V}^n(\Lambda_L) \setminus \{y^n_\eta\}} \mathrm{e}^{\mu} \mathrm{e}^{-\mu d^{n+1}_L(y \cup \{a_{\eta,1}\}, \mathcal{D}^{n+1}_L)} + \mathrm{e}^{-\mu} \le 2^{11} \mathrm{e}^{-\mu}, \tag{4.273}$$

where we used (4.271). □

Now we prove Estimate (4.263) with the method which we presented in the previous section.

Lemma 4.5.2. *Let $\gamma \in \mathcal{V}_L$. Let ε, θ, N and Λ_L satisfy Assumption 4.4.15 and let $n \in \mathbb{N}$ with $N/2 < n < N$. Let $\Delta > 3$ such that $\mu_1 \ge \ln 2$. Then, there exists a $L_0 \equiv L_0(\varepsilon) > 0$ such that for all $L \ge L_0$ and all $j \in \mathbb{N}$ with $j \le \dim \mathbb{H}^n_{\Lambda_L}$ we have*

$$\lambda_j(D^{N,n}_{L,\Lambda_L,\gamma}) \le \frac{2^{45}\, e^{-\mu_1}}{L} e^{-\mu_1 h^n_L \circ \Xi^n_L(\lceil j/2 \rceil)}. \tag{4.274}$$

Proof. Let $m := \dim \mathbb{H}^n_\Lambda = |\mathcal{V}^n(\Lambda)|$. Recall from (4.260) and (4.261) that for any $j \in \{0, \cdots, m\}$ we have

$$D^{N,n}_{L,\Lambda_L,\gamma} = R^{N,n}_{L,\Lambda_L,\gamma,j} + S^{N,n}_{L,\Lambda_L,\gamma,j}. \tag{4.275}$$

First, we estimate the largest singular value of R^n_j for all $j \in \{0, \cdots, m\}$. We obtain

$$\lambda_1(R^{N,n}_{L,\Lambda_L,\gamma,j}) \le \sup_{\psi \ne 0} \frac{\| R^{N,n}_{L,\Lambda_L,\gamma,j} \psi \|_\infty}{\|\psi\|_\infty} = \max_{k>j} \sum_{l>j} |\langle \delta^{\Lambda_L}_{\Xi^n(k)}, D^{N,n}_{L,\Lambda_L,\gamma} \delta^{\Lambda_L}_{\Xi^n(l)} \rangle|, \tag{4.276}$$

where $\|\cdot\|_\infty$ denotes the supremum norm on $\mathbb{H}^n_\Lambda \cong \mathbb{C}^m$. The right-hand side of (4.276) is sometimes referred to as the maximum absolute row sum norm. According to Lemma 4.5.1 and Lemma 4.4.1 there exists a $L_0 \equiv L_0(\varepsilon)$ such that

$$\lambda_1(R^{N,n}_{L,\Lambda_L,\gamma,j}) \le \frac{2^{34}}{L} \mathrm{e}^{-\mu_1} \mathrm{e}^{-\mu_1 h^n_L \circ \Xi^n_L(j+1)} \sum_{l>j} \mathrm{e}^{-\mu_1 h^n_L \circ \Xi^n_L(l)} \le \frac{2^{45}}{L} \mathrm{e}^{-\mu_1} \mathrm{e}^{-\mu_1 h^n_L \circ \Xi^n_L(j+1)} \tag{4.277}$$

for all $L \ge L_0$, where we used the monotonicity of $h^n_L \circ \Xi^n_L$ and $\mu_1 \ge \ln 2$. Since $R^{N,n}_{L,\Lambda_L,\gamma,0} = D^{N,n}_{L,\Lambda_L,\gamma}$ this also implies

$$\lambda_2(D^{N,n}_{L,\Lambda_L,\gamma}) \le \lambda_1(D^{N,n}_{L,\Lambda_L,\gamma}) \le \frac{2^{45} \mathrm{e}^{-\mu_1}}{L} = \frac{2^{45}}{L} \mathrm{e}^{-\mu_1} \mathrm{e}^{-\mu_1 h^n_L \circ \Xi^n_L(1)}, \tag{4.278}$$

where we used that $h^n_L(\Xi^n_L(1)) = h^n_L(y^n_\pm) = 0$. By definition of $R^{N,n}_{L,\Lambda_L,\gamma,j}$ we have $\operatorname{rank}(S^{N,n}_{L,\Lambda_L,\gamma,j}) \le 2j$. Hence,

$$\lambda_{2j+1}(S^{N,n}_{L,\Lambda_L,\gamma,j}) = 0. \tag{4.279}$$

By the well-known Inequality (3.61), we deduce that for all $j \in \mathbb{N}$ with $2j+1 \leq m$ or $2j+2 \leq m$ we have

$$\lambda_{2j+2}(D^{N,n}_{L,\Lambda_L,\gamma}) \leq \lambda_{2j+1}(D^{N,n}_{L,\Lambda_L,\gamma}) \leq \lambda_{2j+1}(S^{N,n}_{L,\Lambda_L,\gamma,j}) + \lambda_1(R^{N,n}_{L,\Lambda_L,\gamma,j}) \\ \leq \frac{2^{45}\mathrm{e}^{-\mu_1}}{L}\mathrm{e}^{-\mu_1 h^n_L \circ \Xi^n_L(j+1)}, \tag{4.280}$$

where we used (4.277) and (4.279). This concludes the proof. □

We are now finally able to show the estimate anticipated in (4.105).

Lemma 4.5.3. *Let $\gamma \in \mathcal{V}_L$. Let ε, θ, N and Λ_L satisfy Assumption 4.4.15 and let $n \in \mathbb{N}$ with $N/2 < n < N$. Let $p \in]1,\infty[$ and $\Delta > 3$ such that $\mu_1/p \geq \ln 2$. Then there exists a $L_0 \equiv L_0(\varepsilon) > 0$ such that for all $L \geq L_0$ we obtain*

$$\|D^{N,n}_{L,\Lambda_L,\gamma}\|^{1/p}_{1/p} \leq \frac{2^{56}}{L^{1/p}}e^{-\mu_1/p}. \tag{4.281}$$

Proof. Again, let $m := \dim \mathbb{H}^n_{\Lambda_L}$. Then

$$\|D^{N,n}_{L,\Lambda_L,\gamma}\|^{1/p}_{1/p} = \sum_{j=1}^{m} \lambda_j^{1/p}(D^{N,n}_{L,\Lambda_L,\gamma}). \tag{4.282}$$

We remark that $\{y^n_\pm\} \subseteq \{y : h^n_L(y) = 0\}$. Therefore, by Lemma 4.5.1 and Lemma 4.5.2, there exists a $L_0 \equiv L_0(\varepsilon) > 0$ such that for all $L \geq L_0$ holds

$$\|D^{N,n}_{L,\Lambda_L,\gamma}\|^{1/p}_{1/p} \leq \frac{2^{45/p}\mathrm{e}^{-\mu_1/p}}{L^{1/p}}\left(2 + 2^{11}\mathrm{e}^{-\mu_1/p}\right) \leq \frac{2^{56}}{L^{1/p}}\mathrm{e}^{-\mu_1/p}, \tag{4.283}$$

where we used that $\mu_1/p \geq \ln 2$. □

4.5.3 Proof of Theorem 4.1.1

We are now prepared to prove the logarithmically enhanced area law as stated in Theorem 4.1.1 by implementing the strategy that was outlined in Section 4.3.

Lemma 4.5.4. *Let $\mathbb{H}$ be a finite dimensional Hilbert space, $A, T \in L(\mathbb{H})$ be self adjoint operators such that $\sigma(A), \sigma(A+T) \subseteq [0,1]$. Let $p, q \in]1,\infty[$ such that $1/p + 1/q = 1$. Then*

$$\left|\operatorname{tr}\{s(A+T) - s(A)\}\right| \leq \|T\|^{1/p}_{1/p}(1 + \|\ln(\cdot)1_{]0,1[}(\cdot)\|_q). \tag{4.284}$$

Proof. Kreĭn's theorem for the spectral shift function [Sch12] states that

$$\operatorname{tr}\{f(A+T) - f(A)\} = \int_{\mathbb{R}} f'(t)\xi(t)\,\mathrm{d}t \tag{4.285}$$

for any compactly supported and smooth function $f \in C_c^\infty(\mathbb{R})$, where

$$\xi : \mathbb{R} \ni t \mapsto \operatorname{tr}\{1_{\leq t}(A+T) - 1_{\leq t}(A)\} \tag{4.286}$$

denotes the spectral shift function. Since s is not differentiable at 0, we cannot apply this result directly. We therefore define a family of suitable auxiliary functions $(s_\eta)_{\eta\in\mathbb{N}} \subseteq C_0^\infty(\mathbb{R})$ such that $\lim_{\eta\to\infty} s_\eta(t) = s(t)$ for all $t \in [0,1]$. Let $\chi \in C^\infty(\mathbb{R})$ be a function such that $\chi(\mathbb{R}) = [0,1]$, $\chi(t) = 0$ for $t \le 1/2$ and $\chi(t) = 1$ for $t \ge 1$. For $\eta \in \mathbb{N}$ and $\tau \in \mathbb{R}$ let

$$s_\eta(\tau) := \chi(2-\tau) \int_0^\tau s'(t)\chi(\eta t)\, \mathrm{d}t. \tag{4.287}$$

Since both $s_\eta(0) = s(0) = 0$ and $\lim_{\eta\to 0} \|(s'_\eta - s')1_{]0,1[}\|_p = 0$ for all $p \in [1,\infty[$ we conclude that $\lim_{\eta\to\infty} s_\eta(t) = s(t)$ for all $t \in [0,1]$. Both A and $A+T$ have a finite number of eigenvalues. Hence,

$$\lim_{\eta\to\infty} \left|\operatorname{tr}\{s_\eta(A+T) - s_\eta(A)\} - \operatorname{tr}\{s(A+T) - s(A)\}\right| = 0. \tag{4.288}$$

For any $p,q \in]1,\infty[$, $1/p + 1/q = 1$, and $\eta \in \mathbb{N}$ we obtain by applying (4.285) to s_η that

$$\left|\operatorname{tr}\{s_\eta(A+T) - s_\eta(A)\}\right| \le \|\xi\|_p \|s'_\eta 1_{]0,1[}\|_q, \tag{4.289}$$

where we used that $\xi(s) = 0$ for $s > 1$. According to [CHN01, Thm. 2.1] the first term of the right-hand side is bounded by $\|T\|_{1/p}^{1/p}$. We estimate the second term by

$$\|s'_\eta 1_{[0,1[}\|_q = \|s'(\cdot)\chi(\eta\cdot)1_{]0,1[}(\cdot)\|_q \le \|s' 1_{]0,1[}\|_q \le 1 + \|\ln(\cdot)1_{]0,1[}(\cdot)\|_q. \tag{4.290}$$

Together with (4.288), this yields (4.284). □

Remark 4.5.5. *Observe that with the simple substitution $y = \ln(1/x)$ we obtain for all $q \in]1,\infty[$ the elementary estimate*

$$\|\ln(\cdot)1_{]0,1[}(\cdot)\|_q = \left(\int_0^1 dx\ \ln(1/x)^q\right)^{1/q} = \left(\int_0^1 dy\ y^q e^{-y}\right)^{1/q} = \Gamma(q+1)^{1/q} \le \lceil q \rceil \le 2q, \tag{4.291}$$

where Γ denotes the Gamma function. Here we used, $\Gamma(q+1) \le \lceil q\rceil! \le \lceil q\rceil^{\lceil q\rceil -1}$.

Proof of Theorem 4.1.1. By (4.48), for every $E \in \sigma(H_L^N) \cap I_1$ there exists at least one $\gamma \in \mathcal{V}_L$ such that $E = \inf\sigma(\hat{H}_{L,\gamma}^N)$. Let $|\varphi_{L,\gamma}^N\rangle$ be the corresponding eigenvector, which was defined in Remark 4.2.11 (i).

The state $|\varphi_{L,\gamma}^N\rangle$ is an eigenstate of the unitary translation operator $\tilde{T}_L^\sigma$ for all $\sigma \in \mathbb{Z}$, too. Therefore, for any two sets $\Gamma^{(\prime)} \subseteq \mathcal{V}_L$ with $\Gamma' = T_L^\sigma \Gamma$ for some $\sigma \in \mathbb{Z}$ we have

$$\operatorname{tr}_{\Gamma'}\left\{\rho(\varphi_{L,\gamma}^N)\right\} = \operatorname{tr}_{\Gamma}\left\{\rho(\varphi_{L,\gamma}^N)\right\}. \tag{4.292}$$

We therefore assume w.l.o.g. that Λ_L is defined as in Assumption 4.4.15.

Let $n \in \mathbb{N}$ with $N/2 < n < N$ and $p,q > 1$ such that $1 = 1/p + 1/q$. Let $\Delta > 3$ such that $\mu_1(\Delta)/p \ge \ln 2$. According to Lemma 4.5.3, Lemma 4.5.4 and Remark 4.5.5, there exists a $L'_0 \equiv L'_0(\varepsilon) > \mathrm{e}^2$ such that

$$\left|\operatorname{tr}\left\{s\big(\rho_{\Lambda_L}^n(\varphi_{L,\gamma}^N)\big) - s\big(\rho_{L,\Lambda_L,\gamma}^n\big)\right\}\right| \le \frac{2^{56}}{L^{1/p}} \mathrm{e}^{-\mu_1/p} \frac{1+2q}{\ln 2} \tag{4.293}$$

for all $L \ge L'_0$.

We choose $p, q > 1$ to depend on L as follows: Let

$$q \equiv q(L) := \ln(L) \quad \text{and} \quad p \equiv p(L) := (1 - 1/\ln(L))^{-1}. \tag{4.294}$$

This implies $L^{1/p} = \mathrm{e}^{-1} L$. For all $L \geq \mathrm{e}^2$ and $\Delta > 25$ we have $1/p \geq 1/2$ and $\mu_1(\Delta)/p > \ln 2$. For any $L \geq L_0'$ we bound (4.293) by

$$\left| \operatorname{tr}\left\{ s\big(\rho^n_{\Lambda_L}\big(\varphi^N_{L,\gamma}\big)\big) - s\big(\rho^n_{L,\Lambda_L,\gamma}\big) \right\} \right| \leq \frac{2^{57}}{L} \mathrm{e}^{-\mu_1/2} (2 + 2\log_2(L)). \tag{4.295}$$

Corollary 4.2.10 implies that $\mu_1(\Delta)$ diverges for $\Delta \to \infty$. Therefore, there exists a $\Delta_0 > 25$ such that

$$2^{57} \mathrm{e}^{-\mu_1/2} \leq \frac{1}{2} \tag{4.296}$$

for all $\Delta \geq \Delta_0$. Hence, by applying (4.97) as well as (4.296), we obtain

$$\begin{aligned} \operatorname{tr}\left\{ s\big(\rho^n_{\Lambda_L}(\varphi^N_{L,\gamma})\big) \right\} &\geq \operatorname{tr}\left\{ s\big(\rho^n_{L,\Lambda_L,\gamma}\big) \right\} - \left| \operatorname{tr}\left\{ s\big(\rho^n_{\Lambda_L}(\varphi^N_{L,\gamma})\big) - s\big(\rho^n_{L,\Lambda_L,\gamma}\big) \right\} \right| \\ &\geq \frac{\log_2 L - 1}{L}. \end{aligned} \tag{4.297}$$

For the entanglement entropy this implies

$$S(\Lambda_L; \mathcal{V}_L, \varphi^N_{L,\gamma}(\Delta)) \geq \sum_{\substack{n \in \mathbb{N}: \\ N/2 < n < N}} \operatorname{tr}\left\{ s\big(\rho^n_{\Lambda_L}(\varphi^N_{L,\gamma})\big) \right\} \geq (N/2 - 1) \frac{\log_2 L - 1}{L}. \tag{4.298}$$

We notice that $\lim_{L \to \infty} \frac{N/2 - 1}{L} = \varepsilon/2$. Hence, for all $\Delta \geq \Delta_0$ we have

$$\liminf_{L \to \infty} \frac{S(\Lambda_L; \mathcal{V}_L, \varphi^N_{L,\gamma}(\Delta))}{\ln L} \geq \frac{\varepsilon}{2 \ln 2}. \tag{4.299}$$

$\square$

Appendix A

Auxiliary results concerning the definition of the entanglement entropy

A.1 Connection between spin system and fermionic Fock space

Let $d \in \mathbb{N}$ and $\Gamma \subseteq \mathbb{Z}^d$ be a finite set. We want to show that we can identify the Fock space $\mathcal{F}_-(\ell^2(\Gamma))$ with $\mathbb{H}_\Gamma$.

Let us first construct a basis to $\mathbb{H}_\Gamma$. Recall that the canonical basis to the single spin space $\mathbb{C}^2$ is given by $\{|\uparrow\rangle, |\downarrow\rangle\}$ with $|\uparrow\rangle = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $|\downarrow\rangle = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$. A natural basis of eigenvectors for $\mathbb{H}_\Gamma$ is given by $\left\{|\delta_x^\Gamma\rangle\right\}_{x \in \mathcal{P}(\Gamma)}$, where $|\delta_\varnothing^\Gamma\rangle = |\uparrow\rangle^{\otimes|\Gamma|}$ and

$$|\delta_x^\Gamma\rangle = \Big(\prod_{j\in x} S_j^-\Big)|\delta_\varnothing^\Gamma\rangle \qquad \text{for all } \varnothing \neq x \in \mathcal{P}(\Gamma). \tag{A.1}$$

Recall that S_j^- denotes the spin lowering operator acting on the site $j \in \Gamma$, where $S^- = \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$.

On the other hand, we can construct a basis for $\mathcal{F}_-(\ell^2(\Gamma))$ with the help of the formalism of second quantisation. By a slight abuse of notation, we denote by $\{|\delta_j\rangle\}_{j\in\Gamma}$ the canonical basis of the single-particle space $\ell^2(\Gamma)$. Furthermore, for any $j \in \Gamma$ we denote by $a_j^* : \mathcal{F}_-(\ell^2(\Gamma)) \to \mathcal{F}_-(\ell^2(\Gamma))$ the creation operator, which creates a particle in the state $|\delta_j\rangle$. Let $|0\rangle$ denote the vacuum state. The vectors

$$\Big(\prod_{j\in x} a_j^*\Big)|0\rangle \qquad \text{for all } \varnothing \neq x \in \mathcal{P}(\Gamma) \tag{A.2}$$

form an orthonormal basis of $\mathcal{F}_-(\ell^2(\Gamma))$.

We point out that the spin lowering operator S_j^- in (A.1) has the same function as the creation operator in a_j^* in (A.2). Therefore, we define the isometry $\Phi : \mathbb{H}_\Gamma \to \mathcal{F}_-(\ell^2(\Gamma))$ by $\Phi(|\delta_x^\Gamma\rangle) := \Big(\prod_{j\in x} a_j^*\Big)|0\rangle$ for all $x \in \mathcal{P}(\Gamma)$. This identifies $\mathcal{F}_-(\ell^2(\Gamma))$ with $\mathbb{H}_\Gamma$.

Appendix B

Auxiliary results concerning the stability of enhanced area laws

B.1 Contour integral representation of the Fermi projection

The following representation (B.2) of the Fermi projection in terms of a Riesz projection with the integration contour cutting through the continuous spectrum is a key ingredient to our proof of Lemma 3.2.3. However, it may be of independent interest.

Theorem B.1.1. *Let K be a densely defined self-adjoint operator in a Hilbert space $\mathcal{H}$, which is bounded from below and satisfies a limiting absorption principle at $E \in \mathbb{R}$ in the sense that there exists a bounded operator B on $\mathcal{H}$ with inverse B^{-1} which is possibly only densely defined and unbounded, such that*

$$\mathcal{S}_E := \sup_{z\in\mathbb{C}:\, \mathrm{Re}\, z=E,\, \mathrm{Im}\, z\neq 0} \left\| B \frac{1}{K-z} \Pi_c(K) B \right\| < \infty. \tag{B.1}$$

Here, $\Pi_c(K)$ denotes the projection onto the continuous spectral subspace of K. Let A_1, A_2 be two bounded operators on $\mathcal{H}$ such that $\|A_1 B^{-1}\| < \infty$ and $\|B^{-1} A_2\| < \infty$. Finally, we assume that there are no eigenvalues of K near E, i.e. $\mathrm{dist}\big(\sigma_{pp}(K), E\big) > 0$. *Then we have the representation*

$$A_1 1_{<E}(K) A_2 = -\frac{1}{2\pi i} \oint_\gamma dz\, A_1 \frac{1}{K-z} A_2. \tag{B.2}$$

The right-hand side of (B.2) *exists as a Bochner integral with respect to the operator norm $\|\cdot\|$, and the integration contour γ is a closed curve in in the complex plane $\mathbb{C}$ which, for $s>0$, traces the boundary of the rectangle $\big\{z \in \mathbb{C} : \, |\mathrm{Im}\, z| \le s, \, \mathrm{Re}\, z \in [-1 + \inf \sigma(K), E]\big\}$ once in the counter-clockwise direction.*

Proof. Let $\varepsilon > 0$ and let γ_ε be the curve γ without the vertical line segment from $E - \mathrm{i}\varepsilon$ to $E + \mathrm{i}\varepsilon$. Since $\|(K-z)^{-1}\|$ is uniformly bounded for z in the image of γ_ε, it suffices to verify that

$$\int_{-\varepsilon}^{\varepsilon} \mathrm{d}\eta \left\| A_1 \frac{1}{K-E-\mathrm{i}\eta} A_2 \right\| < \infty \tag{B.3}$$

in order to show the existence of the right-hand side of (B.2) as a Bochner integral with respect to the operator norm. But

$$\begin{aligned}\Big\| A_1 \frac{1}{K-E-\mathrm{i}\eta} A_2 \Big\| &\le \Big\| A_1 \frac{1}{K-E-\mathrm{i}\eta} \Pi_{pp}(K)\, A_2 \Big\| \\ &\quad + \|A_1 B^{-1}\| \|B^{-1} A_2\| \Big\| B \frac{1}{K-E-\mathrm{i}\eta} \Pi_c(K) B \Big\| \\ &\le \frac{\|A_1\|\|A_2\|}{\operatorname{dist}\big(\sigma_{pp}(K), E\big)} + \|A_1 B^{-1}\| \|B^{-1} A_2\| \, \mathcal{S}_E \end{aligned} \tag{B.4}$$

uniformly in $\eta \in [-\varepsilon, \varepsilon]$, and the estimate (B.3) holds.

It remains to prove the equality in (B.2). Let $\varphi, \psi \in \mathcal{H}$. Since the contour integral along γ exists in the Bochner sense with respect to the operator norm, we equate

$$\begin{aligned}\Big\langle \varphi, \Big(\oint_\gamma \mathrm{d}z\, A_1 \frac{1}{K-z} A_2 \Big) \psi \Big\rangle &= \lim_{\varepsilon \searrow 0} \int_{\gamma_\varepsilon} \mathrm{d}z \, \langle \varphi, A_1 \frac{1}{K-z} A_2 \psi \rangle \\ &= \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \mathrm{d}\mu_{(A_1^*\varphi),(A_2\psi)}(\lambda) \int_{\gamma_\varepsilon} \mathrm{d}z \, \frac{1}{\lambda - z}, \end{aligned} \tag{B.5}$$

where we introduced the complex spectral measure $\mu_{\varphi,\psi} := \langle \varphi, 1_\bullet(K)\psi \rangle$ of K and used Fubini in the last step. On the other hand, we apply the residue theorem to conclude

$$-2\pi\mathrm{i} \, \langle \varphi, A_1 1_{<E}(K) A_2 \psi \rangle = \int_{\mathbb{R}} \mathrm{d}\mu_{(A_1^*\varphi),(A_2\psi)}(\lambda) \int_\gamma \mathrm{d}z \, \frac{1}{\lambda - z}, \tag{B.6}$$

which is justified because E is not an eigenvalue of K. The right-hand side of (B.6) equals

$$\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \mathrm{d}\mu_{(A_1^*\varphi),(A_2\psi)}(\lambda) \int_{\gamma_\varepsilon} \mathrm{d}z \, \frac{1}{\lambda - z} + \mathrm{i} \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \mathrm{d}\mu_{(A_1^*\varphi),(A_2\psi)}(\lambda) \int_{-\varepsilon}^{\varepsilon} \mathrm{d}\eta \, \frac{1}{\lambda - E - \mathrm{i}\eta}. \tag{B.7}$$

The explicit computation, using symmetry,

$$\int_{-\varepsilon}^{\varepsilon} \mathrm{d}\eta \, \frac{1}{\lambda - E - \mathrm{i}\eta} = \int_{-\varepsilon}^{\varepsilon} \mathrm{d}\eta \, \frac{\lambda - E}{(\lambda - E)^2 + \eta^2} = 2 \arctan\Big(\frac{\varepsilon}{\lambda - E} \Big) \tag{B.8}$$

holds for every real $\lambda \neq E$. Therefore, dominated convergence implies that the second limit in (B.7) vanishes. Here, we used again that E is not an eigenvalue of K. Since φ and ψ are arbitrary, the theorem follows from (B.5) to (B.7). □

Remark B.1.2. *Theorem B.1.1 readily generalises from Fermi projections to spectral projections of more general intervals.*

B.2 Estimates of h

Lemma B.2.1. *For all $s \in]0,1[$ and all $x \in [0,1]$ we have*

$$-x \log_2 x \le \frac{x^s}{1-s}. \tag{B.9}$$

and

$$g(x) \le h(x) \le \frac{6}{1-s}\big(g(x)\big)^s, \tag{B.10}$$

where g was defined in (3.11).

Proof. We introduce the continuous function $\varphi : [0,1] \to [0,\infty[,\ x \mapsto -x^{1-s}\log_2 x$. The first claim follows from the observation

$$0 \le \varphi \le \frac{1}{1-s}, \tag{B.11}$$

which is true because $\varphi(1) = \varphi(0) = 0$ and φ has a unique maximum at $\mathrm{e}^{-1/(1-s)}$.

Due to the symmetry $h(x) = h(1-x)$ and $g(x) = g(1-x)$ for all $x \in [0,1]$ it is sufficient to prove (B.10) for all $x \in [0,1/2]$ only. As for the upper bound in (B.10), we note that with $\psi : [0,1/2] \to [0,\infty[,\ x \mapsto -(1-x)\log_2(1-x)$, we have

$$\psi(x) \le \frac{x}{\ln 2} \le \frac{x^s}{\ln 2} \quad \text{for all } x \in [0,1/2], \tag{B.12}$$

because $\psi(0) = 0$ and $\psi' \le 1/\ln 2$. This and (B.11) imply

$$h(x) = x^s\varphi(x) + \psi(x) \le x^s\Big(\frac{1}{\ln 2} + \frac{1}{1-s}\Big) \le \frac{6}{1-s}\big(x(1-x)\big)^s \tag{B.13}$$

for all $x \in [0,1/2]$.

The argument for the lower bound is similar to the above. Since $h(0) = g(0) = 0$ it suffices to show $h' \ge g'$ on $]0,1/2]$. We observe $h'(1/2) = g'(1/2) = 0$, introduce $\gamma(y) := g'(-y+1/2) = 2y$, $\eta(y) := h'(-y+1/2) = \log_2((1+2y)/(1-2y))$ for $y \in [0,1/2[$ and verify $\eta' \ge 2 = \gamma'$. This yields the claim. □

Lemma B.2.2. *For every $x \in [0,1]$ we have*

$$-g(x)\log_2 g(x) \le h(x) \le -3g(x)\log_2 g(x). \tag{B.14}$$

Proof. Since $g(x) \le \min\{x, 1-x\}$ for all $x \in [0,1]$, the left inequality of the claim follows from

$$-g(x)\log_2 g(x) = -g(x)\big(\log_2 x + \log_2(1-x)\big) \le h(x). \tag{B.15}$$

For the right inequality we consider only $x \in [0,1/2]$, which suffices by symmetry. We rewrite

$$-3g(x)\log_2 g(x) - h(x) = -xp(x)\log_2 x - q(x)\log_2(1-x) \tag{B.16}$$

with $p(x) := 2 - 3x$ and $q(x) := -1 + 4x - 3x^2$. The polynomial q is negative on the interval $[0,1/3[$ and positive on $]1/3,1/2]$ while p is positive everywhere on $[0,1/2]$. Therefore, for all $x \in [1/3,1/2]$ we have

$$-3g(x)\log_2 g(x) - h(x) \ge 0. \tag{B.17}$$

On the other hand, we claim that

$$\log_2(1-x) \ge 2x\log_2 x \tag{B.18}$$

for all $x \in [0, 1/3]$ because the function $[0, 1/2] \ni x \mapsto -2x \log_2 x + \log_2(1 - x)$ vanishes at $x = 0$ and at $x = 1/2$ and is concave. Therefore, it must be non-negative. Inserting (B.18) into (B.16), we obtain

$$-3g(x) \log_2 g(x) - h(x) \geq -x(\log_2 x)\big(p(x) + 2q(x)\big) \geq 0 \tag{B.19}$$

because $p(x) + 2q(x) = 5x - 6x^2 \geq 0$ for all $x \in [0, 1/3]$. $\square$

Appendix C

Auxiliary results concerning the XXZ spin ring

C.1 Uniqueness of fibre operator ground states

Lemma C.1.1. *Let $\Delta > 2$, let $L, N \in \mathbb{N}$ with $1 < N < L$. Then for any $\gamma \in \mathcal{V}_L$, the operator $\hat{H}^N_{L,\gamma}$ has exactly one eigenvalue in $[1 - \frac{1}{\Delta}, 1]$ and no eigenvalues in $]1, 2 - \frac{2}{\Delta}[$.*

Proof. By Lemma 4.2.7, we get

$$\hat{H}^N_{L,\gamma} = -\frac{1}{2\Delta}\hat{A}^N_{L,\gamma} + \hat{W}^N_{L,\gamma} \geq \left(1 - \frac{1}{\Delta}\right)\hat{W}^N_{L,\gamma} \geq \left(1 - \frac{1}{\Delta}\right) . \tag{C.1}$$

There exists exactly one element $\hat{x}_0 \in \widehat{\mathcal{V}}^N_L \cap \mathcal{D}^N_L$. Clearly, it satisfies $W(\hat{x}_0) = 1$. For any other $\hat{x} \in \widehat{\mathcal{V}}^N_L \setminus \{\hat{x}_0\}$ we have $W(\hat{x}) \geq 2$. Let $\tilde{\phi}^N_{L,\gamma} \in \mathbb{S}^N_{L,\gamma}$ be defined by $\tilde{\phi}^N_{L,\gamma}(\sigma, \hat{x}) := \delta_{\gamma,\sigma}\delta_{\hat{x}_0,\hat{x}}$.

Hence, the operator

$$\hat{H}^N_{L,\gamma} + \left(1 - \frac{1}{\Delta}\right)|\tilde{\phi}^N_{L,\gamma}\rangle\langle\tilde{\phi}^N_{L,\gamma}| \geq \left(1 - \frac{1}{\Delta}\right)\left(\hat{W}^N_{L,\gamma} + |\tilde{\phi}^N_{L,\gamma}\rangle\langle\tilde{\phi}^N_{L,\gamma}|\right) \geq \left(2 - \frac{2}{\Delta}\right) \tag{C.2}$$

is a rank-one perturbation of $\hat{H}^N_{L,\gamma}$. Therefore, according to the min-max-principle, the unperturbed operator $\hat{H}^N_{L,\gamma}$ has at most one eigenvalue below $(2 - \frac{2}{\Delta})$. On the other hand, since $\langle \phi^N_{L,\gamma,0}, \hat{H}^N_{L,\gamma}\phi^N_{L,\gamma,0}\rangle = 1$, there exists at least one eigenvalue ≤ 1. By assuming $\Delta > 2$ we get $1 < 2 - \frac{2}{\Delta}$. This concludes the proof. □

For $\gamma = 0$, it follows from the explicit structure of the fibre operator $\hat{H}^N_{L,0}$ that it has a unique ground state $\hat{\varphi}^N_{L,0}$ which can be chosen to be strictly positive. The same is true for the original operator H^N_L. This will allow us to conclude that $\varphi^N_{L,0} := (\mathfrak{F}^N_L)^*\hat{\varphi}^N_{L,0}$ is the ground state of H^N_L. The main tool for our result will be an idea presented in [YY66], where the existence of a strictly positive ground state for the XXZ model on the ring was established. However, let us also point out that this piece of the proof follows from the Allegretto–Piepenbrink theorem shown in [HK11].

Lemma C.1.2. *Let $N, L \in \mathbb{N}$, $0 < N < L$. Moreover, let $E_0 \equiv E_0(L, N, \Delta) = \inf \sigma(\hat{H}^N_{L,0})$. Then E_0 is non-degenerate and the corresponding eigenvector $\hat{\varphi}^N_{L,0} \in \mathbb{S}^N_{L,0}$ can be chosen such that $\|\hat{\varphi}^N_{L,0}\| = 1$ and $\hat{\varphi}^N_{L,0}(0, \hat{x}) > 0$ for all $\hat{x} \in \widehat{\mathcal{V}}^N_L$. In addition, $\varphi^N_{L,0} := (\mathfrak{F}^N_L)^* \hat{\varphi}^N_{L,0}$ is the unique ground state of H^N_L.*

Proof. Firstly, note that if we choose the constant $C > 2N \geq \|W^N_L\| = \|\hat{W}^N_{L,0}\|$, we get that the matrix representations with respect to the canonical basis of both operators $A_1 := (C1_{\mathbb{H}^N_L} - H^N_L)$ and $A_2 := (C1_{\mathbb{S}^N_{L,0}} - \hat{H}^N_{L,0})$ have only non–negative entries. Moreover, note that since A_1 and A_2 are irreducible, we can choose $D \geq \dim(\mathbb{H}^N_L)$ large enough, such that the matrix entries of A^D_1 and A^D_2 will all be strictly positive. Hence, by the Perron-Frobenius Theorem, the largest eigenvalue of each of these operators A^D_1 and A^D_2 is positive, non-degenerate and the corresponding eigenfunctions can be chosen to be strictly positive. Let $\varphi^N_{L,0}$ and $\hat{\varphi}^N_{L,0}$ denote the eigenfunctions for A^D_1 and A^D_2 respectively, that satisfy these properties. Clearly, $\varphi^N_{L,0}$ and $\hat{\varphi}^N_{L,0}$ will then be the eigenfunctions of H^N_L and $\hat{H}^N_{L,0}$ corresponding to the respective minima E_0 and $\hat{E}_0$ of the spectra. Now, since H^N_L and $\hat{H}^N_L$ are unitarily equivalent via the Fourier transform $\mathfrak{F}^N_L$, the function $(\mathfrak{F}^N_L)^* \hat{\varphi}^N_{L,0}$ is also an eigenfunction of H^N_L and thus $\hat{E}_0 \in \sigma(H^N_L)$. However, from the explicit form of $(\mathfrak{F}^N_L)^*$ as given in (4.27), one sees that since $\hat{\varphi}^N_{L,0} \in \mathbb{S}^N_{L,0}$ we get that $(\mathfrak{F}^N_L)^* \hat{\varphi}^N_{L,0}$ is a strictly positive eigenfunction of H^N_L. Thus, we conclude that $(\mathfrak{F}^N_L)^* \hat{\varphi}^N_{L,0} = \varphi^N_{L,0}$ and consequently $E_0 = \hat{E}_0$. □

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Eidesstattliche Versicherung

(Siehe Promotionsordnung vom 12.07.11, § 8 Abs. 2 Pkt. 5.)

Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

Ruth Schulte

Meersburg, 30.09.2020	Ruth Schulte
Ort, Datum	Unterschrift Doktorand/in