Current algebra, generalised geometry and integrable models in string theory

David Osten

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David Osten

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David Osten
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Erstgutachter: Prof. Dr. Dieter Lüst
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Und darum: Hoch lebe die Physik! Und höher noch das,
was uns zu ihr zwingt, – unsre Redlichkeit!

Friedrich Nietzsche,
aus: Die fröhliche Wissenschaft

Zusammenfassung

Eine Konsequenz der ausgedehnten Natur des Strings ist, dass in der Stringtheorie allgemeinere Hintergrundgeometrien als (Riemannsche) Mannigfaltigkeiten möglich sind. Insbesondere bei Kartenwechseln sind nicht nur Diffeomorphismen (oder andere Eichtransformationen) erlaubt, sondern auch (String-)Dualitätstransformationen. Solche Räume werden auch nicht-geometrische Räume genannt. Ihre mathematische Formulierung basiert auf Hitchins und Gualtieris Verallgemeinerter Geometrie.

In dieser Arbeit wird gezeigt, dass entgegen bisheriger Resulte in der Literatur die Poisson-Struktur, genauer die Stromalgebra, eines Strings nicht $\mathrm{O}(d,d)$-invariant ist und deren korrekte Beschreibung so genannte para-Hermitesche Geometrie benötigt.

Darauf aufbauend wird eine Hamiltonsche Formulierung von klassischer Stringtheorie in einem generischen, geometrischen oder nicht-geometrischen Hintergrund vorgeschlagen. Die Essenz dieser Formulierung ist eine Deformation der Stromalgebra, die durch die verallgemeinerten Flüsse, die einen solchen Hintergrund beschreiben, charakterisiert wird. Diese Formulierung ist allgemeiner als die durch eine Lagrange/-dichte, da zum Beispiel magnetisch geladene Hintegründe und solche, die die Sektionsbedingung der Verallgemeinerten Geometrie verletzen, hier auch diskutiert werden können – auf Kosten der Verletzung der Jacobi-Identität der Stromalgebra.

Zwei Anwendungen dieser Formulierung werden diskutiert: Zum einen kann man aus der deformierten Stromalgebra direkt die nicht-kommutative und nicht-assoziative Interpretation der nicht-geometrischen Hintergründe abgelesen. Zum anderen können zweierlei Verallgemeinerungen von nicht-Abelscher T-Dualität über Poisson-Lie T-Dualität hinaus abgeleitet werden. Es existiert eine nicht-Abelsche T-Dualitätsgruppe, analog zu $\mathrm{O}(d,d)$ für Abelsche T-Dualität. Außerdem existieren Versionen von Poisson-Lie-Dualität für Modelle mit generischen konstanten Verallgemeinerten Flüssen.

Eine Verallgemeinerung dieser Ergebnisse für M-branen in M-Theorie scheint möglich. Für eine M2-bran in M-Theorie in vier Dimensionen wird gezeigt, dass die Stromalgebra nicht dualitätsinvariant ist und genauso wie im Falle des Strings eine Lie-Klammer beinhaltet, die in einer para-Hermiteschen Version von exzeptioneller Verallgemeinerter Geometrie auftaucht. Im Unterschied zur Diskussion des Strings kann Kovarianz unter der Dualitätsgruppe, hier $\mathrm{SL}(5)$, nur durch die Einführung zusätzlicher Objekte, der Membranladungen, wiederhergestellt werden. Mit Hilfe der typischen doppelten dimensionalen Reduktion von M-Theorie zur Typ IIa Superstringtheorie kann man die M2-bran- und Stringströme miteinander in Beziehung setzen.

Ein anderes zentrales Thema sind integrable Modelle im Kontext von Stringtheorie. In besonders symmetrischen Hintergründen, wie der Minkowski-Raumzeit oder bestimmten Anti-de Sitter-Kompaktifizierungen, ist Stringtheorie exakt lösbar (integrabel). Deformationen dieser Hintergründe, die die Integrabilität beibehalten, wurden in den letzten Jahren ausführlich untersucht. Es stellt sich heraus, dass viele diese Deformationen klare Entsprechungen in der Verallgemeinerten Geometrie haben. In dieser Arbeit wird gezeigt, dass eine große Klasse dieser Deformationen, die homogenen Yang-Baxter-Deformationen, nichts weiter sind als die β-Transformationen der oben erwähnten nicht-Abelschen T-Dualitätsgruppe.

Abstract

One consequence of the extended nature of the string is that more general background geometries than Riemannian manifolds are possible in string theory. In particular, when gluing charts not only diffeomorphisms (or other gauge transformations of the background) but also (string) duality transformations are allowed. These geometries are called non-geometric spaces. Their mathematical formulation is based on Hitchin's and Gualtieri's *generalised* (or $\mathrm{O}(d,d)$-) *geometry*.

In this thesis it is shown that, despite previous results in the literature, the Poisson structure – to be more precise: the current algebra – of a string is not $\mathrm{O}(d,d)$-invariant. Its correct treatment requires the so-called para-Hermitian geometry.

Building on that, a Hamiltonian formulation of the classical world-sheet theory in a generic, geometric or non-geometric, background is proposed. The essence of this formulation is that the generalised fluxes, characterising such a background, describe a deformation of the current algebra. This formulation extends to backgrounds for which there is no Lagrangian description of the world-sheet theory – namely magnetically charged backgrounds and those that violate the section condition of generalised geometry, at the cost of violating the Jacobi identity of the current algebra.

Two applications of this formulation are discussed. On the one hand, one can read off the non-commutative and non-associative interpretation directly from the deformed current algebra. On the other hand, one can derive two generalisations of non-abelian T-duality that go beyond the standard factorised Poisson-Lie T-duality. There is a non-abelian T-duality group, analogous to $\mathrm{O}(d,d)$ for abelian T-duality. Moreover, there are generalisations for Poisson-Lie T-duality for models with generic constant generalised fluxes.

A generalisation of these results to M-branes in M-theory seems possible. For the membrane in M-theory compactified on a four-dimensional space, it is shown that the current algebra is not U-duality invariant. Exactly as for the string, a Lie bracket appears that is connected to para-Hermitian exceptional generalised geometry. In contrast to the string, even manifest covariance under the U-duality group, here $\mathrm{SL}(5)$, is only possible when introducing additional objects, the membrane charges. With the help of the typical double dimensional reduction from M-theory to type IIa superstring theory, one can relate the membrane and string currents.

Another central topic of this thesis is integrability in context of string theory. In particularly symmetric backgrounds, like Minkowski spacetime or certain Anti-de Sitter compactifications, string theory is exactly solvable (integrable). Deformations of these backgrounds, that preserve integrability of the world-sheet theory, have been studied extensively in the last years. It turned out that many of these deformations can be described in terms of generalised geometry. In this thesis it is shown that a big class of these deformations, the homogeneous Yang-Baxter deformations, are nothing else than the β-shifts of the non-abelian T-duality group mentioned above.

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Contents

Part I

Introduction

Space, time and gravity

Space and time have lied at the core of our understanding of nature throughout the (western) history of science, starting from the natural philosophy in ancient Greece – for example in Zeno's paradoxes or Aristotle's physics – through Galilean relativity, on which classical mechanics including Newtonian gravity is based, to special relativity and electrodynamics. This conceptual process is nicely reviewed in [1].

The advent of differential geometry in the 19th century opened the door for Einstein's theory of gravity, general relativity [2]. The identity of space and time seemed to be clarified as a curved four-dimensional manifold. The dynamics of the spacetime is described in terms of the dynamics of the metric field.

The beginning of the 20th century gave physics quantum mechanics, and finally quantum field theory. Its typical assumptions are *unitarity*, *Lorentz invariance*, *locality* and *renormalisability*, the latter ensuring the predictivity of the theory in the presence of ultraviolet (UV) divergences. The quantum theories of the other known fundamental interactions, unified in the standard model of particle physics, are very successful and fit into this framework.

General relativity, on the other hand, treated as a standard quantum field theory turns out to be non-renormalisable [3]. The appearing divergences are related to fundamental excitations being point-like and the interactions being local. So, at the scale at which quantum effects of gravity become relevant – the Planck scale –, we expect a break down of the classical notion of spacetime or violations of some of our assumptions on quantum field theories.

String theory

The resolution that string theory offers to the problem of quantum gravity is the relaxation of the assumption of *locality*. The fundamental constituents are assumed to be one-dimensional strings instead of point particles. Somewhat surprisingly, the spacetime geometry that these strings probe is, at least *a priori*, 'classical'.

Historically, string theory started as a candidate for an effective theory of the strong interaction – the dual resonance model. With the advent of quantum chromodynamics, it was discarded. It gained attention again when it turned out that it naturally contains a (massless) graviton in the spectrum and, hence, is a candidate for a quantum theory of gravity. The low-energy effective theory turns out to include Einstein gravity. Besides these excitations, string theory contains in general infinitely many more, corresponding to internal oscillation modes on the string.

In order to rid the theory of anomalies (e.g. of Lorentz symmetry) and instabilities (through the existence of tachyons), it has to be defined in a ten-dimensional spacetime and assumed to be supersymmetric. Both facts could be understood as, at first sight, fatal flaws. But, they turn out to be a reason for the conceptual richness of string theory. Compactifying the extra dimensions – assuming that the extra dimensions are small and curled up –, leaves the parameters that describe this 'internal geometry' as new fields in an effective four-dimensional theory. Also, the supersymmetry can be broken in this theory. The fascination lies in the fact, that string theory naturally includes gravity and only has *one* free parameter – the string tension. Hence, it is a potential theory of everything, in which all the parameters of the four-dimensional low-energy effective theories are fixed dynamically.

In principle, the phenomenological challenge to string theory is the following: finding a stable configuration, a vacuum, of string theory such that, in the low-energy effective limit, we obtain the standard models of particle physics and cosmology. One has to ensure that the additional (infinitely many) excitations that are inherent to string theory, should lie at mass scale beyond current experimental reach. In particular, the emergence of multiple massless scalar fields (moduli) in the low-energy effective field theories is quite generic. These have to be avoided, as they are not accounted for by the observed physics.

Unfortunately, simple solutions rarely exist and a vast and impenetrable landscape of string theory vacua was discovered. In addition, the existence of de Sitter-vacua in string theory, that would describe the proposed current cosmological state of the universe, is put in question. This leaves the question: *Why study string theory today?*

- *String theory as a quantum gravity.* As a fully-fledged and fairly well-understood theory of quantum gravity, that can also include matter and other interactions, string theory can serve as a laboratoty to test potential generic properties of quantum gravity. For example, not all potential effective field theories can be consistently coupled to quantum gravity. Recently, constraints on such effective field theories have been studied extensively in the so-called swampland program [4,5].

- *String theory as a toy model.* The non-perturbative regime of quantum field theories is generically hard to access. String theory can help to understand it better.

  One picture of string theory is as a non-trivial, interacting two-dimensional quantum field theory. There are dualities that allow to access the non-perturbative regime. Also, string theory is a conformal field theory and, in some backgrounds, an exactly solvable, 'integrable', field theory. Both frameworks best come into play in two-dimensional spacetimes, like the string world-volume.

  In the context of the holographic duality [6], string theory is a dual equivalent description of certain field theories. This duality is useful in both ways: understanding the strongly coupled regime of field theories via string theory, and understanding aspects of quantum gravity with help of standard field theory methods.

- *String theory as a source for new mathematical structures.* Physical insight from string theory has helped in the developing new approaches to several branches of mathematics, e.g. enumerative geometry [7], mirror symmetry between Calabi-Yau manifolds [8,9] or moonshine conjectures [10].

The latter two points are the ones that will be relevant in this thesis. A different kind of geometry appears, that generalises the standard Riemannian geometry and captures some stringy features. Also, we will be concerned with the notion of integrability of string theories and how far it can be stretched.

Generalised geometry

Strings are extended objects. They can probe more than one point of the spacetime at once. Hence, at length scales around the string length, one imagines a spacetime geometry in which points are meaningless. So-called *generalised geometry* [11, 12], a generalisation of Riemannian geometry, is a setting which incorporates some of this.

Some aspects of generalised geometry can be motivated purely from the point of view of the target space. Besides the metric, a 'higher' (2-form) gauge field arises in the massless spectrum of the closed string. Treating both on same footing leads to a generalisation of Riemannian geometry.

From the string perspective, generalised geometry is motivated from a symmetry of the spectrum. Along compact directions, the string possesses additional modes, that describe winding around these compact direction. When analysing the spectrum, one notices that the winding and centre of mass excitations contribute in a similar way. Exchanging the two leads to a dual, equivalent theory – this is *T-duality*. A setting, in which this duality action is a geometrical one, is generalised geometry.

Going back to how this changes our understanding of space and time – the main consequences are: the appearance of minimal length scales, more general allowed classes of spacetimes and non-commutative spacetimes. Let us briefly investigate each of these points.

Dualities. As mentioned above, generalised geometry is a geometric framework in which (string) dualities get a geometric meaning. T-duality is the duality with the most immediate consequences for the target space geometry in string theory. Consider string theory on a target space, where one dimension is a circle of radius R. Then the dual, equivalent string theory is defined on the a circle with radius

$$\tilde{R} \sim \frac{l_S^2}{R},$$

where l_S is the string length scale. This duality introduces an effective minimal length scale $R_{min} \sim l_S$ to the theory, as a theory with radius below R_{min} is dual to one with radius above.

Global non-geometry. A Riemannian manifold allows for diffeomorphisms of the metric, or generally gauge transformations of background fields, for the transition between different coordinate patches. In the presence of dualities, these can also be used to glue different patches [13]. This big class of spacetimes is called *globally non-geometric*.

Non-commutative spacetimes. A different perspective of what the microscopic nature of the spacetime is arises, if we assume that the spacetime is non-commutative: $[x^\mu, x^\nu] = \beta^{\mu\nu}$. The corresponding uncertainty relation introduces minimal areas in to theory:

$$\Delta x^\mu \cdot \Delta x^\nu \gtrsim \beta^{\mu\nu}.$$

In string theory, such β-fields arise naturally in the presence of the, above mentioned, 2-form gauge field [14]. In generalised geometry, such a β-field is a natural way to parameterise the background.

The idea, that a quantised spacetime might introduce a natural cutoff like this and, consequently, resolve divergence problems of the quantisation of general relativity, is a old as [15] or [16]. As explained for example in [17], this hope was not answered. The reason is that due to UV-IR mixing standard renormalisation theory cannot be applied.

Integrability

A theory is (classically) integrable, if it has as many symmetries as degrees of freedom. These symmetries need to be independent or, in technical terms, in involution – their generators need to have vanishing Poisson brackets with each other. With help of these symmetries, one can solve these models algebraically. The prime example from classical mechanics is the Kepler problem: due to a hidden symmetry of the $\frac{1}{r}$-potential, this model is exactly solvable.

Field theories have infinitely many degrees of freedom. So, a notion of integrability for field theories needs infinitely many symmetries in involution and a way to compare the infinities. In contrast to the point particle case, there is no general definition that achieves that. Very commonly used and also applied in this thesis is the *Lax formalism*. A two-dimensional field theory is called Lax integrable, if it is possible to rewrite the equations of motion as

$$\partial_t L_1(t, x; \lambda) - \partial_x L_0(t, x; \lambda) + [L_0(t, x; \lambda), L_1(t, x; \lambda)] = 0.$$

L_α is called the Lax connection, or Lax pair, and takes values in some auxiliary Lie algebra. λ is an auxiliary additional parameter, called *spectral parameter*. If the model possesses such a spectral-parameter dependent Lax connection, one can construct infinitely many conserved quantities. Lax integrability is also possible for field theories in higher dimensions.

String theory is classically integrable in certain very symmetric backgrounds, in particular all backgrounds that are Riemannian symmetric spaces. This includes flat Minkowski space, spheres or Anti de Sitter-spaces. One motivation for the work in this thesis is to see how far integrability stretches to models with less symmetry. This is done by deforming known integrable string backgrounds in such a way that integrability is preserved.

Main results

This thesis investigates how generalised geometry manifests itself in the (classical) world-sheet theory of a string. In particular, the connection of the Poisson structure and the generalised fluxes, a certain characterisation of backgrounds in generalised geometry, is discussed.

The starting point is the Hamiltonian formulation of a (classical) non-linear σ-model on a d-dimensional background with metric $G_{\mu\nu}$ and Kalb-Ramond field $B_{\mu\nu}$ and coordinates x^μ. The canonical Hamiltonian takes the form

$$H = \frac{1}{2}\int \mathrm{d}\sigma\, \mathcal{H}_{MN}(G,B)\mathbf{E}^M(\sigma)\mathbf{E}^N(\sigma)$$

with $\mathbf{E}_M(\sigma) = (p_\mu(\sigma), \partial x^\mu(\sigma))$, where $\partial = \partial_\sigma$, and the generalised metric

$$\mathcal{H}(G,B) = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}.$$

$x^\mu(\sigma)$, $p_\mu(\sigma)$ are the coordinate fields of the string embedding and their canonical momenta. σ is a spatial coordinate on the string worldsheet. The generalised metric is, in general, σ-dependent throught the coordinate dependence of G and B. The indices $M = 1, ...2d$ are raised and lowered by the $\mathrm{O}(d,d)$-metric

$$\eta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

Background fields and deformations of the Poisson structure. In chapter 5 a collection of theories of point particles or strings in background fields is presented. The common observation is that background fields can be introduced to a theory by a deformation of the Poisson brackets. In the Hamiltonian theory, a world-volume theory in some backgrounds can be schematically defined as

$$H = H^{\text{(free)}} \quad \text{and} \quad \Pi = \Pi^{\text{(can.)}} + \Pi^{\text{(background)}}$$

for the Hamiltonian of the free theory (without background field) $H^{\text{(free)}}$ and the canonical Poisson structure $\Pi^{\text{(can.)}}$.

This observation is by no means new for the individual examples, that are well-known in the literature. But, it is shown to be a generic feature of a σ-model description here. A similar result is derived for the Hamiltonian membrane theory in chapter 9.

Brackets on the phase space. Before one can generalise this generic result to the backgrounds of generalised geometry, one has to study the geometry of Poisson structure of the string in order to see, if and how generalised geometry is realised there.

The canonical Poisson structure of the $x^\mu(\sigma)$ and their canonical momenta, phrased in terms of the $\mathbf{E}_M$ and thus in an $\mathrm{O}(d,d)$-covariant way, is

$$\{\mathbf{E}_M(\sigma_1), \mathbf{E}_N(\sigma_2)\} = \frac{1}{2}\eta_{MN}(\partial_1 - \partial_2)\delta(\sigma_1 - \sigma_2) + \frac{1}{2}\omega_{MN}(\partial_1 + \partial_2)\delta(\sigma_1 - \sigma_2) \qquad (0.0.1)$$

without any simplifications. η is the $\mathrm{O}(d,d)$-metric and $\omega = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$.

Terms of the type $(\partial_1 + \partial_2)\delta(\sigma_1 - \sigma_2)$, like the second term in (0.0.1), have been neglected in previous literature. One should keep track of such terms as they contribute in certain situations: for open strings via the boundary and for closed strings via winding along compact directions.

The above form (0.0.1) of the current algebra is a Lie bracket, as we started with the canonical Poisson structure. As in the target space, generalised geometry and Courant algebroid properties seemed very useful, the question would be where these are hidden in the canonical current algebra (0.0.1). It turns out that, without the second term, (0.0.1) would be a Courant bracket, meaning it would be $\mathrm{O}(d,d)$-invariant and skew-symmetric but violating the Jacobi identity by a total derivative term under the σ-integral.

Moreover, the ω-term is crucial for the non-geometric interpretation of the current algebra. For example, the current algebra of the locally geometric pure **Q**-flux background is associative as expected only if it is taken in account. This is shown in section 7.2.

Strings in arbitrary σ-model backgrounds. The generalised metric, and hence the Hamiltonian, can be diagonalised by generalised vielbeins $E_A{}^M$, $\mathcal{H}_{MN}E_A{}^M E_B{}^N = \delta_{AB}$. So that, in terms of phase space variables $\mathbf{E}_A(\sigma) = E_A{}^M \mathbf{E}_M$, the Hamiltonian looks like the free one

$$H = \frac{1}{2}\int \mathrm{d}\sigma\; \delta^{AB}\mathbf{E}_A(\sigma)\mathbf{E}_B(\sigma), \tag{0.0.2}$$

whereas the background data is encoded in a deformed current algebra

$$\begin{aligned}\{\mathbf{E}_A(\sigma_1), \mathbf{E}_B(\sigma_2)\} = {}& \frac{1}{2}\eta_{AB}(\partial_1 - \partial_2)\delta(\sigma_1 - \sigma_2) - \mathbf{F}^C{}_{AB}(\sigma)\mathbf{E}_C(\sigma)\delta(\sigma_1 - \sigma_2) \\ & + \text{boundary term,}\end{aligned} \tag{0.0.3}$$

in terms of the generalised fluxes

$$\mathbf{F}_{ABC} = \left(\partial_{[A} E_B{}^M\right) E_{C]M} \tag{0.0.4}$$

with $\partial_A = E_A{}^M \partial_M$. The Jacobi identity of the current algebra (0.0.3) is equivalent to the Bianchi identity of generalised fluxes

$$\partial_{[A}\mathbf{F}_{BCD]} - \frac{3}{4}\mathbf{F}^E{}_{[AB}\mathbf{F}_{CD]E} = 0. \tag{0.0.5}$$

This formulation of the world-sheet theory works for any NSNS-background characterised by the generalised fluxes $\mathbf{F}_{ABC}$. In particular, this includes globally non-geometric backgrounds in which metric and B-field are not globally well-defined.

The Hamiltonian equations of motion of a string in a generic background take a convenient form. We recognise them as a Maurer-Cartan equation of the $\mathbf{E}_A$ treated as one-forms and pulled back to the world-sheet:

$$d\mathbf{E}^A + \frac{1}{2}\mathbf{F}^A{}_{BC}\mathbf{E}^B \wedge \mathbf{E}^C = 0 \quad \text{and} \quad \mathbf{E}_A = \delta_{AB} \star \mathbf{E}^B. \tag{0.0.6}$$

Generalisations to magnetically charged and double field theory backgrounds. The world-sheet theory as a Lagrangian σ-model is only defined in 'electric' backgrounds, i.e. those that fulfil (0.0.5) and are locally geometric. Instead, the Hamiltonian formulation in the generalised flux frame extends straightforwardly to magnetically charged and locally non-geometric backgrounds.

For a magnetically charged background, the Bianchi identity of generalised fluxes (0.0.5) is not fulfilled. This means one cannot find a generalised vielbein that will connect the deformed current algebra (0.0.3) to the canonical one (0.0.1). Analogously to the case of the point particle in an magnetic monopole background, the violation of the Bianchi identity corresponds to a violation of the Jacobi identity of the current algebra.

Double field theory considers the original target space coordinates x and their T-duals $\tilde{x}$ on the same footing, such that duality rotations become manifest symmetries. In the world-sheet theory, the dual fields $\tilde{x}_\mu(\sigma)$ are given by $p_\mu(\sigma) = \partial\tilde{x}_\mu(\sigma)$, such that one can employ the canonical Poisson structure in order to investigate of the world-sheet theory in double field theory backgrounds. There are typically two constraints on the dependence of functions on the doubled coordinates $X^M = (x^\mu, \tilde{x}_\mu)$. Violations of these constraints have consequences in the above defined Hamiltonian world-sheet theory and the canonical current algebra (0.0.1):

- weak constraint: $\partial_\mu\tilde{\partial}^\mu f(x,\tilde{x}) = \frac{1}{2}\partial_M\partial^M f(X) = 0$ for all functions f

  Allowing for a violation of the weak constraint on the generalised vielbein $E_A{}^M$ means that it might depend on original as well as dual coordinates. In that case, an additional non-local term appears in the deformed current algebra implying a modification of the algebra of world-sheet diffeomorphisms.

- strong constraint: $(\partial_M f)(\partial^M g) = 0$ for all functions f and g

  The Jacobi identity of generic functions on doubled space is only fulfilled up to strong constraint violating terms, e.g.

  $$\{\Psi,\{\phi_1,\phi_2\}\} + c.p. = \int \mathrm{d}\sigma\, \frac{1}{2}\,(\eta_{KL} + \omega_{KL})\,\phi^K_{[1}\,\partial_M\Psi\,\partial^M\phi^L_{2]} + \text{ other terms,}$$

  where $\phi_i = \int \mathrm{d}\sigma\phi_i^M\,(X(\sigma))\,\mathbf{E}_M(\sigma)$ and $\Psi, \phi_i^M(\sigma)$ are functions of the fields $X_M(\sigma) = (\tilde{x_\mu}(\sigma), x^\mu(\sigma))$. See section 7.2 for more details.

The non-geometric interpretation of the current algebra. Given such a deformed current algebra in a generalised flux frame $\mathbf{E}_A(\sigma)$, one might postulate new adapted coordinates y^a of such a non-geometric background. Generalising $\mathbf{E}_M = (p_\mu, \partial x^\mu)$, one decomposes $\mathbf{E}_A$ into $(e_{0,a}(\sigma), e_1^a(\sigma))$ and defines $\partial y^a = e_1^a$. These coordinate fields $y^a(\sigma)$ are the ones of which the zero modes potentially show the typical non-geometric, e.g. non-commutative or non-associative, behaviour. We obtain their Poisson brackets simply by integrating the deformed current algebra (0.0.1).

This is in contrast to many previous derivations of the non-geometric nature of the backgrounds which relied on solving the equations of motion first. In section 7.2 it is shown that one reproduces the known results on open strings in a constant B-field background and closed strings in a constant $\mathbf{Q}$-flux backgrounds. In principle, this approach shows that the non-commutative interpretation is an off-shell property.

Generalised T-dualities beyond Poisson-Lie T-duality. The framework easily realises abelian T-duality. For Poisson-Lie T-dualisable resp. the $\mathcal{E}$-models [18, 19], the current algebra is exactly of the kind (0.0.3) with

$$\mathbf{F}^c{}_{ab} = f^c{}_{ab}, \qquad \mathbf{F}_c{}^{ab} = \overline{f}_c{}^{ab} \qquad \text{and} \qquad \mathbf{F}_{abc} = \mathbf{F}^{abc} = 0,$$

where the constants $f^c{}_{ab}$ and $\overline{f}_c{}^{ab}$ are structure constants to a Lie bialgebra [20, 21]. The duality transformations are linearly realised in that basis. In addition to the factorised Poisson-Lie T-duality, one observes that similar to abelian T-duality there is a bigger duality group. This group, named non-abelian T-duality here, is a certain subgroup of $\mathrm{O}(d,d)$.

Of particular relevance is the so-called β-transformation subgroup of this non-abelian T-duality group. The non-abelian β-transformations are nothing else than the so-called homogeneous Yang-Baxter deformations, a particular kind of integrable deformation of string σ-models.

Moreover, it is shown that there exists an extension of Poisson-Lie T-duality for certain parameterisations of $\mathbf{F}_{ABC}$. It nevertheless relies on the same trick as Poisson-Lie T-duality, namely that a Poisson bivector on a group manifold realises a constant generalised flux background. It is also shown that these (generalised) T-dualities are canonical transformations.

Membranes in the SL(5)-theory. The connection of string current algebra and $\mathrm{O}(d,d)$ generalised geometry seems to be no coincidence. In chapter 9, the appearance of *exceptional generalised geometry* in the membrane current algebra is shown. This is done for $d = 4$, where the U-duality group is $\mathrm{SL}(5)$. Basically, all the results from the string case generalise. This includes the deformation of the membrane currents algebra by the $\mathrm{SL}(5)$ generalised fluxes and an appearance of an additional topological contribution, that break the duality invariance. An additional issue is that one needs to introduce additional objects, the membrane charges, into the theory in order to be able to write the current algebra, the generalisation of (0.0.1) in an $\mathrm{SL}(5)$-covariant way.

Organisation of the thesis

This thesis aims to be fairly self-contained and therefore contains reviews of some relevant textbook material in part II. Chapter 1 includes, besides a very basic introduction to generalities of string theory, reviews of κ-symmetry, Dirac brackets and Green-Schwarz superstring σ-models. A quite general survey of dualities in high energy physics and string theory is presented in chapter 2, including a detailed introduction to T-duality from the world-sheet perspective and its generalisations. Generalised geometry, focussing on the generalised fluxes, and integrability, with particular focus on integrable deformations of string σ-models, are introduced in chapters 3 and 4.

The main results, as introduced above, are presented in detail in part III. After a few closing remarks, the appendix contains some additional material on T-duality with respect to fermionic isometries and the action of T-duality on the RR-fields, on Lie algebra cohomology and Lie bialgebras.

Thesis publications

The material and the results presented in this thesis are based on the following publications:

- Dieter Lüst, David Osten: *Generalised fluxes, Yang-Baxter deformations and the O(d,d) structure of non-abelian T-duality*, *JHEP* **05** (2018) 165, arXiv:1803.03971
- David Osten: *On current algebras, generalised fluxes and non-geometry*, *J. Phys. A* **53** (2020) 56, arXiv:1910.00029

The material on the membrane generalisations in chapter 9 is not published yet and the author's contribution to:

Alex Arvanitakis, Chris Blair, David Osten, Daniel Thompson: *to be published*

Part II

Foundations

Chapter 1

Strings

Although many results presented in this thesis apply to generic two-dimensional σ-models, the physical motivation comes from string theory. The aim of this chapter is to introduce the basic notions necessary for the pursuit of this thesis, namely: the classical theory, Dirac brackets, the notion of κ-symmetry, string σ-models and consistent string backgrounds.

1.1 Bosonic strings in flat space

1.1.1 Actions and symmetries

String theory is the theory of one-dimensional extended objects propagating in a D-dimensional space time (also *target space* in the following). A natural action, first considered by Nambu and Goto [22, 23], is given by the area functional of the two-dimensional world-sheet Σ, generalising the action of a relativistic point particle. For a flat target space with Minkowski metric η it takes the form

$$S_{NG} = -\frac{1}{\alpha'}\int_\Sigma \mathrm{d}A = -\frac{1}{\alpha'}\int_\Sigma \mathrm{d}^2\sigma \sqrt{-\det_{\alpha\beta}\left(\partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu}\right)}. \tag{1.1.1}$$

Here, $\sigma^\alpha = (\sigma^1, \sigma^2) = (\tau, \sigma)$ are coordinates on the world-sheet and $\partial_\alpha = \frac{\partial}{\partial\sigma^\alpha}$. In comparison to standard literature, we choose their ranges to be $\tau_i < \tau < \tau_f$ and $0 \leq \sigma \leq 1$. $x^\mu(\tau, \sigma)$ are the coordinate fields of this embedding in the target space, and we write $\dot{x} = \partial_\tau x$ and $x' = \partial_\sigma x$. The only free parameter is the the Regge slope α'. Oftentimes, other choices of this parameter are used: the string tension $T = \frac{1}{2\pi\alpha'}$, the string length scale $l_s = 2\pi\sqrt{\alpha'}$ or the string mass scale $M_s = \frac{1}{\sqrt{\alpha'}}$. For the most part of the thesis α' will be set to $\alpha' = 1$.

In order to obtain the equation of motions

$$\frac{\partial}{\partial\sigma^\alpha}\frac{\partial\mathcal{L}}{\partial(\partial_\alpha x^\mu)} = 0 \tag{1.1.2}$$

by varying the action (1.1.1), we need to impose boundary conditions on the fields $x^\mu(\tau,\sigma)$:

- closed string: periodic boundary conditions $x^\mu(\tau,\sigma+1) = x^\mu(\tau,\sigma)$.
- open string: $\frac{\partial\mathcal{L}}{\partial(\partial_\sigma x^\mu)}\,\delta x^\mu(\tau,\sigma) = 0$ at $\sigma = 0,1$.

  There are typically two ways to achieve this:

  - Neumann boundary conditions: $\frac{\partial\mathcal{L}}{\partial(x'^\mu)} = 0$ at $\sigma = 0,1$.
  - Dirichlet boundary conditions: δx^μ at the $\sigma = 0,1$. This basically introduces new higher dimensional objects, the *D*-branes, into the theory.

The Nambu-Goto action is not polynomial and thus it will be difficult to quantise it. But, there is a classically equivalent action in which the square-root is removed by introducing auxiliary degrees of freedom [24, 25]

$$S_P = -\frac{1}{2\alpha'}\int_\Sigma \mathrm{d}^2\sigma\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x^\nu\eta_{\mu\nu}. \tag{1.1.3}$$

This form is named after Polyakov [26], who used it to perform the path integral quantisation of string theory. The auxiliary fields $\gamma^{\alpha\beta}$ take the form of a (non-dynamical) metric on the world-sheet. They act as Lagrangian multipliers, as their equations of motions correspond to the vanishing of the two-dimensional energy-momentum tensor

$$T_{\alpha\beta} = \frac{2}{\sqrt{-\gamma}}\frac{\delta S_P}{\delta\gamma^{\alpha\beta}} = -\frac{1}{\alpha'}\left(\partial_\alpha x^\mu\partial_\beta x^\nu - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\gamma\delta}\partial_\gamma x^\mu\partial_\delta x_\mu\right) = 0. \tag{1.1.4}$$

Global symmetries of the Polyakov action (1.1.3) correspond to isometries of the target space, in this case the Poincaré transformations $x^\mu \to a^\mu{}_\nu x^\nu + b^\nu$ with constants $a_{\mu\nu} = -a_{\nu\mu}$ and b^μ. Local symmetries consist of reparameterisations $\sigma^\alpha \to \sigma^\alpha + \delta\sigma^\alpha$, and local Weyl rescalings $\gamma_{\alpha\beta} \to \gamma_{\alpha\beta} + 2\omega\gamma_{\alpha\beta}$. The local symmetries have two important consequences: the Polyakov action exhibits full local two-dimensional conformal symmetry and moreover we can use the reparameterisation invariance to gauge fix the world-sheet metric to be conformally flat $\gamma_{\alpha\beta} = \Omega^2(\tau,\sigma)\eta_{\alpha\beta}$. With that and because of Weyl invariance, the Polyakov action becomes nothing else than the theory of D free scalar bosons on the world-sheet

$$S_P = -\frac{1}{2\alpha'}\int_\Sigma \mathrm{d}^2\sigma\;\partial_\alpha x^\mu\partial^\alpha x^\nu\eta_{\mu\nu} \tag{1.1.5}$$

with equation of motion $\partial_\alpha\partial^\alpha x^\mu = 0$. A generic solution with closed string boundary conditions takes the form:

$$x^\mu(\tau,\sigma) = x_0^\mu + \alpha' p^\mu\tau + i\sqrt{\frac{\alpha'}{4\pi}}\sum_{n\neq 0}\left(\alpha_n^\mu e^{-2\pi i n\sigma^-} + \bar{\alpha}_n^\mu e^{-2\pi i n\sigma^+}\right). \tag{1.1.6}$$

The x_0^μ and p^μ refer to the center of mass position and momentum of the string and will often be summarised as *zero modes*. The α^μ and $\bar{\alpha}^\mu$-terms describe the internal oscillations of the string, often subsumed as *oscillators*. We also employed light-cone coordinates on the world-sheet

$$\sigma^{\pm} = \tau \pm \sigma, \quad \partial_{\pm} = \frac{1}{2}(\partial_\tau \pm \partial_\sigma),$$
$$\eta_{+-} = \eta_{-+} = -\frac{1}{2}, \quad \eta_{++} = \eta_{--} = 0.$$

1.1.2 Constraints and brackets

The gauge fixing of the world-sheet metric $\gamma^{\alpha\beta}$ to (1.1.5) does not come for free. We need to impose the original equations of motions (1.1.4) as constraints now: $T_{\alpha\beta} \approx 0$ – the *Virasoro constraints*. Whereas $T_{+-} = T_{-+} = 0$ by tracelessness of the energy-moment tensor is enforced by Weyl invariance of (1.1.3), T_{++} and T_{--} are non-trivial constraints.

Algebra of constraints. The canonical Poisson brackets for the fields x^μ and their canonical conjugates $p^\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}$ are

$$\begin{aligned} \left\{x^\mu(\sigma), x^\nu(\sigma')\right\} &= \left\{p_\mu(\sigma), p_\nu(\sigma')\right\} = 0 \\ \left\{x^\mu(\sigma), p_\nu(\sigma')\right\} &= \delta^\mu_\nu \delta(\sigma - \sigma'). \end{aligned} \tag{1.1.7}$$

As all the brackets are supposed to be evaluated at equal times, the τ-argument is neglected from here on when working in the Hamiltonian formalism. By ∂ we then denote σ-derivatives, e.g. in the canonical **current algebra**

$$\begin{aligned} \left\{\partial x^\mu(\sigma), \partial x^\nu(\sigma')\right\} &= \left\{p_\mu(\sigma), p_\nu(\sigma')\right\} = 0 \\ \left\{\partial x^\mu(\sigma), p_\nu(\sigma')\right\} &= \delta^\mu_\nu \partial\delta(\sigma - \sigma'). \end{aligned} \tag{1.1.8}$$

With that, one can compute the algebra of constraints

$$\begin{aligned} \{T_{\pm\pm}(\sigma_1), T_{\pm\pm}(\sigma_2)\} &= \pm 2\left(T_{\pm\pm}(\sigma_1) + T_{\pm\pm}(\sigma_2)\right)\frac{1}{2}(\partial_1 - \partial_2)\delta(\sigma_1 - \sigma_2) \\ \{T_{\pm\pm}(\sigma_1), T_{\mp\mp}(\sigma_2)\} &= 0. \end{aligned} \tag{1.1.9}$$

In terms of their modes,

$$L_n = \frac{1}{4\pi^2}\int \mathrm{d}\sigma\, e^{-2\pi i n\sigma} T_{--}, \qquad \bar{L}_n = \frac{1}{4\pi^2}\int \mathrm{d}\sigma\, e^{-2\pi i n\sigma} T_{++}, \tag{1.1.10}$$

it takes the more familiar form of the de Witt algebra,

$$\begin{aligned} \{L_m, L_n\} &= -i(m-n)L_{m+n}, \\ \{\bar{L}_m, \bar{L}_n\} &= -i(m-n)\bar{L}_{m+n} \\ \{L_m, \bar{L}_n\} &= 0. \end{aligned} \tag{1.1.11}$$

In particular, these generate two commuting copies of the algebra of reparameterisations of S^1, here $\sigma^\pm \to f(\sigma^\pm)$. As usual, the Hamiltonian, the generator of τ-translations via $\frac{\mathrm{d}}{\mathrm{d}\tau} f = \{f, H\}$, of a time reparameterisation invariant theory is a constraint. As such, it is part of the above algebra as well and given by

$$H = \frac{1}{2} \int \mathrm{d}\sigma \left(\dot{x}^2 + x'^2 \right) = L_0 + \bar{L}_0. \tag{1.1.12}$$

in conformal gauge. Similarly, the generator of σ-translation is given by

$$P = \int \mathrm{d}\sigma \, \dot{x} \cdot x' = L_0 - \bar{L}_0. \tag{1.1.13}$$

Dirac brackets. The analysis of the algebra of constraints does in general not end here. There are many caveats in the treatment of gauge constraints, first discussed by Dirac [27, 28]. Let us discuss Dirac's procedure from a purely Hamiltonian view. We start with a canonical or 'naive' Hamiltonian $H_{can.}$ together with a bunch of constraints $\phi_n \approx 0$ (often called primary, though without a Lagrangian there is no hierarchy of constraints). The '$\approx$' indicates a *weak equality*, an equality enforced by a constraint, as opposed to the *strong* one, '$=$' describing a true identity.

The constraints should hold over time so we impose $\dot{\phi}_n = \{H, \phi_n\} \approx 0$. The naive Hamiltonian $H_{can.}$ might not satisfy this. Hence, we use the, by this point, most general choice of Hamiltonian $H = H_{can.} + c_n \phi_n \approx H_{can.}$ with constants c_n potentially to be determined. Requiring $\varphi_n = \{H, \phi_n\} \approx 0$ can have different outcomes – besides showing a fundamental inconsistency when there is no solution – there are three interesting options:

1. $\varphi_m \approx 0$ holds by use of the $\phi_n \approx 0$ alone. There is nothing more that needs to be done about these constraints.

2. φ_m is a function of the c_n, such that we make some choice of the c_n in order for $\varphi_m \approx 0$ to hold.

3. φ_m are new non-trivial functions on the phase space, but independent of the c_n. In this case, we add these φ_n to our set of constraints (often called secondary, and potentially tertiary, quartary, etc. when repeating the procedure).

The generalised Hamiltonian computed like this is enough to obtain correct equations of motion. But, also the Poisson structure has to be modified. This is done by the introduction of Dirac brackets $\{\cdot, \cdot\}_{D.B.}$ satisfying $\{\phi_m, \phi_n\}_{D.B.} = 0$, apart from requiring all other axioms of a Poisson bracket. There are two ways to understand this condition:

- When quantising, the constraints $\phi_n \approx 0$ become honest operator identities $\phi_n \approx 0 \to \hat{\phi}_n = 0$. Via canonical quantisation of the naive Poisson brackets, we would have $[\hat{\phi}_n, \hat{\phi}_m] = -\frac{1}{i\hbar} M_{nm} \neq 0$. So, unless we quantise the Dirac brackets this is inconsistent.

- The constraints should be independent, meaning their order of application should not matter. This is exactly what is measured by the Poisson bracket.

It is possible to find a basis $(\psi_\mu, \tilde{\psi}_\alpha)$ of our ideal of constraints, such that

$$\{\psi_\mu, \psi_n\} = \{\psi_\mu, \tilde{\psi}_\alpha\} = 0 \quad \text{and} \quad \{\tilde{\psi}_\alpha, \tilde{\psi}_\beta\} = C_{\alpha\beta} \tag{1.1.14}$$

with $C_{\alpha\beta}$ being invertible. The ψ_μ are called *first-class*, the $\tilde{\psi}_\alpha$ *second-class*. The Dirac bracket of two functions F, G on the phase space is then uniquely determined to be [27,28]

$$\{F, G\}_{D.B.} = \{F, G\} - \{F, \psi_\alpha\} \left(C^{-1}\right)^{\alpha\beta} \{\psi_\beta, G\} . \tag{1.1.15}$$

As a consequence of $\{\phi_m, \phi_n\}_{D.B.} = 0$, we also see that we can save ourselves from computing the generalised Hamiltonian H, as $\{\phi_n, H\}_{D.B.} = \{\phi_n, H_{can.}\}_{D.B.}$.

For the free bosonic string and the Virasoro constraints, we see that none of this is actually necessary as $\{L_m, L_n\} \sim L_{m+n} \approx 0$. Nevertheless, in the Hamiltonian treatment of the superstring and the supermembrane it becomes relevant. Also, in sections 7 and 9, a Dirac procedure is applied to obtain the canonical Poisson brackets from a higher dimensional 'generalised' phase space. Many conceptual aspects of this discussion will also appear in the study of classical integrability in section 4.

1.1.3 Quantisation

The general challenge in quantising the string is to ensure gauge invariance, in this case two-dimensional conformal symmetry, in the quantum theory and to remove the unphysical degrees of freedom. Let us outline some general ideas.

Spectrum and critical dimension. Performing the canonical quantisation $\{\, ,\, \} \to i[\, ,\,]$ of (1.1.7) on the level of the mode expansions gives

$$[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n,0}\eta^{\mu\nu}, \qquad [p_\mu, x^\nu] = i\delta^\nu_\mu. \tag{1.1.16}$$

Up to rescaling, the α- and $\bar{\alpha}$-modes correspond to right- respectively left-moving harmonic oscillator modes. A generic state is given by the creators, α^μ_{-n} for $n > 0$, of these modes acting on the vacuum state $|0; p\rangle$ characterised by the center of mass momentum p.

Due to (1.1.16), the L_0-operator is the only Virasoro operator that is subject to normal ordering ambiguities,

$$L_0 = \frac{1}{2}\sum_n : \alpha_{-n}\alpha_n : +a = \frac{\alpha'}{4}p^\mu p_\mu + N + a, \tag{1.1.17}$$

with the number operator $N = \sum_n \alpha_{-n}\alpha_n$ for the α-modes, similar expressions for the $\bar{L}_0$ and the $\bar{\alpha}$-modes and a normal ordering constant a. The mass spectrum for the closed bosonic string can be derived from the (normal ordered) Hamiltonian $H = L_0 + \bar{L}_0 + 2a$:

$$M^2 = -p^\mu p_\mu = \frac{2}{\alpha'}(N + \bar{N} + 2a). \tag{1.1.18}$$

The invariance under σ-reparameterisations is ensured by $P = L_0 - \bar{L}_0 \approx 0$ in the classical theory (1.1.13). In the quantum theory, this is called the *level matching condition*

$$(L_0 - \bar{L}_0)\,|\phi_{phys.}\rangle = 0 \quad \Rightarrow \quad N = \bar{N}. \tag{1.1.19}$$

The *Virasoro algebra* receives a modification by a central term in comparison to (1.1.11), due to the normal ordering procedure

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \tag{1.1.20}$$

for $c = D$, being the dimension of the target space. This means there is an anomaly of conformal symmetry, the so-called Weyl anomaly. The D bosons x^μ, each contributing $c = 1$, cancel this anomaly and the energy-momentum tensor remains traceless.

Due to the central term, the constraints $L_m \approx 0$ cannot be completely implemented as operator conditions on the space of physical states like $L_m\,|\phi_{phys.}\rangle = 0$ and instead

$$(L_0 - a)\,|\phi_{phys.}\rangle = 0 \quad \text{and} \quad L_m\,|\phi_{phys.}\rangle = 0, \quad \text{for } m > 0. \tag{1.1.21}$$

is the best that one can achieve. Continuing from here and identifying and removing negative norm states from the spectrum with help of (1.1.20) is called *covariant quantisation*. Closer to standard field theory methods is the quantisation based on the *Polyakov path integral*

$$Z = \int \frac{\mathcal{D}[x]\,\mathcal{D}[\gamma]}{\text{vol}(Diff) \times \text{vol}(Weyl)}\, e^{-S_P} \tag{1.1.22}$$

with help of a Faddeev-Popov procedure [26], or more modernly the *BRST quantisation* [29,30]. The *light-cone gauge quantisation* [31] offers a straightforward way to obtain the spectrum and conditions on target space dimension D and normal ordering constant a. As is typical in gauge theories, the fixing of gauge in the Polyakov action is not complete. There is a residual gauge freedom of this gauge choice. Here, these are the reparameterisations generated by the conformal Killing vectors, i.e. those that maintain the conformal gauge. A way to fix this residual gauge freedom is to set $x^+ \sim p^+\tau$ in light-cone coordinates on the target space, $x^\pm = x^0 \pm x^D$ and x^i, for $i = 1, ..., D-1$. Due to the Virasoro constraints, x^- is fixed in terms of the x^i as well. This gauge fixing breaks manifest Lorentz invariance in the target space. Imposing Lorentz invariance to hold anyway implies

$$\text{critical dimension: } D = 26, \quad \text{and} \quad a = -1. \tag{1.1.23}$$

Another consequence is that the $\alpha^\pm$-modes should be identified as the unphysical longitudinal degrees of freedom, such that a generic physical state after light-cone gauge quantisation is of the form

$$(\alpha^{i_1}_{-m_1})^{k_1} \cdot ... \cdot (\bar{\alpha}^{j_1}_{-n_1})^{l_1} \cdot ...\, |0, p\rangle \tag{1.1.24}$$

subject to the level matching condition (1.1.19). According to the mass formula (1.1.18), the lightest states in the spectrum are

$$|0;p\rangle\,, \quad \alpha^i_{-1}\bar{\alpha}^j_{-1}\,|0;p\rangle\,, \quad \dots\,. \tag{1.1.25}$$

The first one has $\alpha' M^2 = -4$ and is a *tachyon* – signalling that bosonic string theory is unstable. More interestingly, on the next level we have *massless* states that decompose under the massless little group $\mathrm{SO}(24)$ of the Lorentz group $\mathrm{SO}(1,25)$ into a symmetric traceless, a skewsymmetric and a scalar representation. These correspond to the *graviton*, *skewsymmetric* and *dilaton* excitations.

Towards string in curved space. The interaction between the string excitations are described by insertions of their *vertex operators* on the world-sheet. The one of the graviton has the form

$$V \sim \int \mathrm{d}^2\sigma \xi_{\mu\nu}(X)\sqrt{\gamma}\gamma^{\alpha\beta} : \partial_\alpha X^\mu \partial_\beta X^\nu e^{ik\cdot X} : \tag{1.1.26}$$

with a polarisation tensor $\xi_{\mu\nu}$. Introducing such an external graviton as source to the string path integral $e^{-S_P - V} = e^{-S_P}(1 - V + \dots)$ and exponentiating the graviton vertex operators, we expect to obtain the coupling to a finite metric, understood as a coherent state of gravitons,

$$S = -\frac{1}{2\alpha'}\int_\Sigma \mathrm{d}^2\sigma\; \partial_\alpha x^\mu \partial^\alpha x^\nu G_{\mu\nu}(x). \tag{1.1.27}$$

This is the natural generalisation of (1.1.5) to curved space. In comparison to flat string theory, the x-dependence of the background means that this is an interacting theory.

1.2 Non-linear σ-models and the geometric paradigm

The action (1.2.2) is part of a class of models, known as σ-models [32,33]. These are theories of maps $\phi : \Sigma_d \to \mathcal{M}$ from a d-dimensional world-volume Σ_d into a D-dimensional target space $\mathcal{M}$, such that the Lagrangian is quadratic in derivatives:

$$S = -\frac{1}{2\lambda}\int \mathrm{d}^d\sigma E_{ij}(\phi)\partial_\alpha\phi^i\partial^\alpha\phi^j. \tag{1.2.1}$$

The fields ϕ^i transform as scalars on the world-volume. In principle, the $E_{ij}(\phi)$ can include infinitely many coupling constants $E_{ij} = \sum_n E_{ijk_1\dots k_n}\phi^{k_1}\cdot\dots\cdot\phi^{k_n}$, but it is beneficial to understand it as some field dependent coupling. In particular, the infinitely many β-functions of the expansion can be resummed again to a β-functional $\beta^{(E)}_{ij}(\phi)$. The constant λ takes the role of $\hbar$ in general, or in string theory the role of α'. If $E_{ij}(\phi)$ is not constant, the σ-model is called *non-linear*.

This class of models includes quantum mechanics as a one-dimensional example. The original $\mathcal{M}$=O(3) σ-model with the physical space-time as world-volume served as a toy model for spontaneous symmetry breaking [32], the effective description of σ-mesons (giving the σ-model its name) [33] or instantons [34]. In a more string theory related setting, so-called gauged linear σ-models have been employed in the Landau-Ginzburg/Calabi-Yau correspondence [35]. The σ-models' behaviour under renormalisation group flow for $d = 2 + \epsilon$ has been described in [36–39].

1.2.1 String σ-models

We want to include finite versions of the antisymmetric tensor, the Kalb-Ramond or B-field, and the scalar excitation, the dilaton Φ, into (1.1.27) as well. The general action that describes the propagation of a string in a background generated by the *massless* states of the bosonic string is

$$S = -\frac{1}{2\alpha'} \int \left(G_{\mu\nu}(x)\, \mathrm{d}x^\mu \wedge \star \mathrm{d}x^\nu + B_{\mu\nu}(x)\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu + \frac{\alpha'}{2\pi} \Phi(x)\, \mathcal{R}^{(2)} \star 1 \right), \qquad (1.2.2)$$

the so-called string σ-model – a non-linear σ-model, with the two-dimensional world-sheet being the world-volume and the physical space-time the target space. Whereas in many of the above examples $E_{ij}(\phi)$ is the metric tensor on the target space, here, the coupling E_{ij} has a symmetric and a skewsymmetric part, corresponding to a metric and a B-field. Additionally, the dilaton term, containing the world-sheet Ricci-scalar $\mathcal{R}^{(2)}$, is not included in the generic form of a σ-model (1.2.1). It is higher order in α' and hints at the fact that string theory is not only a theory of embeddings of the world-sheet into target space but also describes two-dimensional internal gravity on that world-sheet. Consequently, there are some specialities for the string σ-model in comparison to a generic one:

Non-trivial world-sheet topology. Gravity in two dimensions is trivial, meaning that the world-sheet metric γ has no dynamics. By reason of symmetry, the two-dimensional Einstein equations vanish and hence the two-dimensional Einstein-Hilbert action describes a topological invariant: the Euler characteristic

$$\chi = 2(1-g) = \frac{1}{4\pi} \int_\Sigma \mathrm{d}^2\sigma \sqrt{-\gamma} \mathcal{R}^{(2)}.$$

Indeed, the two-dimensional string world-sheets can have different non-trivial topologies. In oriented closed string theory, the world-sheets are two-dimensional Riemann surfaces of genus g. The topology does not influence the local dynamics of the σ-model but still contributes via the dilaton term.

Double perturbative expansion. Though there is only one dimensionful constant a priori, α', there are two *dimensionless* parameters in which we can perturbatively expand the theory, each of which is relevant in a different context.

- *World-sheet loop expansion.* The first ingredient is the *string length scale* $\sqrt{\alpha'} \sim l_S$. In order to get an expansion in a dimensionless parameter, we need another characteristic scale to compare. This comes from *σ-model perturbation theory*, which is in principle simply the perturbation theory of the infinitely many couplings of (1.2.1) resp. (1.2.2). A σ-model is renormalisable in the traditional sense, if only finitely many of its coupling constants run. Oftentimes, it is instructive to parameterise the fields and couplings differently, $x^\mu(\sigma) = x_0^\mu + \sqrt{\alpha'}y^\mu(\sigma)$, as an expansion around a constant background value or a classical solution. In the first case, we have

$$G_{\mu\nu}(x)\partial^\alpha x^\mu \partial_\alpha x^\nu \sim \left(G_{\mu\nu}(x_0) + \sqrt{\alpha'}\partial_\rho G_{\mu\nu}(x_0)y^\rho + ...\right)\partial^\alpha y^\mu \partial_\alpha y^\nu. \qquad (1.2.3)$$

  The typical length scale on the target space l_T shows in the gradient of the metric

$$\sqrt{\alpha'}\partial_\rho G_{\mu\nu} \sim \frac{l_S}{l_T}, \qquad (1.2.4)$$

  giving an expansion of (1.2.2) in the dimensionless ratio l_S/l_T. This perturbation theory is applicable when the theory is weakly coupled, $l_S/l_T \ll 1$ and breaks down when the extended nature of the string becomes relevant $l_T \cong l_S \sim \sqrt{\alpha'}$.

- *Target space loop expansion.* The typical loop expansion (in target space) of a quantum field theory is governed in a very unique way in string theory. The string path integral includes a sum over different world-sheet topologies $Z = \sum_g Z_g$, in the closed string case a sum over the different genera ($\hat{=}$ target space loop numbers). Normally, this corresponds to an expansion in a dimensionless coupling constant as well. Each of the Z_g requires a separate discussion in principle, but the vacuum expectation value of the dilaton contributes in a universal factor $e^{-S_{dil.}} = e^{-\Phi_0 2(1-g)} = g_S^{2g-2}$ with $g_S = e^{\Phi_0}$. So, g_S is the 'loop counting' coupling constant in target space. This is a good example for a general feature of string model building: dimensionless parameters emerging from string theory correspond to vacuum expectation values of scalar fields.

Classical conformal invariance. We obtained the flat space σ-model (1.1.5) from (1.1.3) by gauge-fixing. In order to ensure conformal invariance on the classical level, we have to impose the Virasoro constraints (1.1.4). The dilaton term seems to break conformal invariance, but this happens at higher order in α' and thus it is a quantum correction, cancelling the $\mathcal{O}(\alpha')$ breaking of conformal invariance of the metric and B-field term.

Quantum conformal invariance. The quantum theory should respect the conformal invariance as well. In particular, scale invariance is a strong statement as it implies that there should be no dependence on the renormalisation scale. Such a dependence is described in terms of the β-functionals. The Weyl anomaly of the theory is given by the β-functionals in a convenient renormalisation scheme in first order in α' and in first order in an expansion of G, B and Φ around flat space as

$$2\alpha' T^{\gamma}{}_{\gamma} = \beta^{(G)}_{\mu\nu}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu - i\beta^{(B)}_{\mu\nu}\epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu - \alpha' \beta^{(\Phi)}\mathcal{R}^{(26)}. \tag{1.2.5}$$

So, $\mathcal{O}(\alpha')$ Weyl invariance is indeed equivalent to scale invariance, meaning that all β-functions have to vanish

$$\beta^{(G)}_{\mu\nu} = \beta^{(B)}_{\mu\nu} = \beta^{(\Phi)} = 0. \tag{1.2.6}$$

Including higher orders in fields, the β-functionals of the (field dependent) couplings Φ, $B_{\mu\nu}$ and $G_{\mu\nu}$ up to $\mathcal{O}(\alpha')$ for critical bosonic strings read:

$$\begin{aligned}
\beta^{(G)}_{\mu\nu} &= \alpha' \mathcal{R}^{(26)}_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{\alpha'}{4}\mathbf{H}_{\mu\lambda\omega}\mathbf{H}_\nu{}^{\lambda\omega} + \mathcal{O}(\alpha'^2) \\
\beta^{(B)}_{\mu\nu} &= \frac{\alpha'}{2}\nabla^\omega \mathbf{H}_{\omega\mu\nu} + \alpha' \nabla^\omega \Phi \mathbf{H}_{\omega\mu\nu} + \mathcal{O}(\alpha'^2) \\
\beta^{(\Phi)} &= \frac{\alpha'}{2}\nabla^2\Phi + \alpha' \nabla_\omega \Phi \nabla^\omega \Phi - \frac{\alpha'}{24} H_{\mu\nu\lambda}\mathbf{H}^{\mu\nu\lambda} + \mathcal{O}(\alpha'^2).
\end{aligned} \tag{1.2.7}$$

$\mathcal{R}^{(26)}_{\mu\nu}$ denotes the target-space Ricci-tensor, ∇ is the affine connection to G and $\mathbf{H} = \mathrm{d}B$ is the three-form field strength of the antisymmetric tensor B. For consistency, it should be possible to get these equations as the equations of motion of an action. This is the case and the low-energy (meaning lowest order in α') effective spacetime action, that reproduces (1.2.7) as its equations of motion, is:

$$S = -\frac{1}{2\kappa^2}\int \mathrm{d}^{26}X \sqrt{-\det G}\, e^{-2\Phi}\left(\mathcal{R}^{(26)} - \frac{1}{12}\mathbf{H}_{\mu\nu\lambda}\mathbf{H}^{\mu\nu\lambda} + 4\nabla_\mu \Phi \nabla^\mu \Phi\right). \tag{1.2.8}$$

κ is connected to the D-dimensional Planck mass $M^{(D)}_P$ by $\kappa^2 \sim (M^{(D)}_P)^{D-2}$ resp. to the gravitational constant G by $\kappa^2 \sim G$.

1.2.2 The geometric paradigm

The σ-models that we want to consider are supposed to be subject to the geometric paradigm (see e.g. [40, 41]):

"Physical quantities of a σ-model correspond to geometric quantities of the target space."

Let us demonstrate this for the example of target space $\mathcal{M}$ being a Riemannian manifold with metric G:

- The coupling in the σ-model is given by the metric tensor G.
- The β-functional is the Ricci tensor (up to potential RG-dependent diffeomorphisms).

- The classical solutions are the minimal surfaces (resp.: volumes) of the target space.
- Global symmetries of the target space correspond to global symmetries of the σ-model.
- Assume that the target space is a Lie group G. Then the σ-model is classically integrable, meaning that it possesses infinitely many conserved charges. Almost all of these will be non-local, seemingly suggesting they have no obvious purely target space interpretation. But there is an algebraic construction, solely based on the Lie algebra $\mathfrak{g}$ of G – the so-called Yangian $Y(\mathfrak{g})$ –, that corresponds to the algebra of conserved charges in this case. See chapter 4 and references there for more details.

  In particular, this point shows that even non-local objects can have existing analogues in the pure target space geometry.

On the one hand, this paradigm seems very natural as it reflects the physicist's favourite approach of bootstrapping results with the help of symmetry and geometry. On the other hand, the previous section showed that at scales below $\sqrt{\alpha'}/l_T$ effects showing finiteness of the world-volume become important. So, we expect that ordinary Riemannian geometry, the notions of which are motivated by point particles, will not be the correct framework. One challenge, even at the classical level, will be posed in the next chapter 2 – the existence of T-duality, an equivalence of two σ-models with target spaces, that differ in terms of Riemannian geometry and even topology.

So, the question turns out to be: What is the correct kind of geometry, in which one can understand all physical quantities of a given σ-model? This is the guideline for the program of *generalised geometry*, introduced in chapter 3 and, as such, a central topic of this thesis. A main result of this thesis concerns the role that the generalised fluxes of generalised geometry play in the Hamiltonian treatment of a (not necessarily string) σ-model.

1.3 Superstrings and fluxes

The bosonic string theory sketched in the previous sections suffers from two problems: the existence of the tachyon in the spectrum and the absence of fermions. Supersymmetric string theory solves both of these issues.

1.3.1 Supersymmetric strings

Fermions and supersymmetry can be introduced in two ways in string theory – on the world-sheet and target space. Let us sketch both constructions in flat space, following in large parts the standard textbooks [42–45].

Target space supersymmetry. Starting with the bosonic Polyakov string action (1.1.3), a natural guess for the superstring action is

$$S_1 = -\frac{1}{2\alpha'}\int d^2\sigma\sqrt{\gamma}\gamma^{\alpha\beta}A_\alpha \cdot A_\beta \qquad \text{with } A^\mu_\alpha = \partial_\alpha x^\mu - i\bar{\Theta}^I\Gamma^\mu\partial_\alpha\Theta^I. \tag{1.3.1}$$

for $I = 1, 2$. Here, we introduced two Majorana-Weyl spinors Θ^I and Dirac matrices Γ on the target space. This action is invariant under the ($\mathcal{N} = 2$) supersymmetry transformations $\delta\Theta^I = \epsilon^I$ and $\delta x^\mu = i\bar{\epsilon}^I\Gamma^\mu\Theta^I$. However, this manifestly supersymmetric action is not the desired action because there are twice as many fermionic degrees of freedom as there should be – a Majorana-Weyl spinor in $D = 10$ has 16 real degrees of freedom whereas there are only eight bosonic degrees of freedom after light-cone gauge fixing. Also, the equations of motion become rather complicated and non-linear:

$$A^2 = 0 \qquad \dot{A}^\mu_\alpha = 0 \qquad \Gamma \cdot A^\alpha\partial_\alpha\Theta = 0. \tag{1.3.2}$$

Adding an extra Wess-Zumino term S_{WZ} to action S_1 resolves this problem and we obtain the Green-Schwarz (GS) action [46]

$$\begin{aligned} S_{GS} &= S_1 + S_{WZ} \\ S_{WZ} &= \frac{1}{2\alpha'}\int d^2\sigma\left(-i\epsilon^{\alpha\beta}\partial_\alpha x^\mu(\bar{\Theta}^1_\mu\Gamma_\mu\partial_\beta\Theta^1 - \bar{\Theta}^2\Gamma_\mu\partial_\beta\Theta^2)\right. \\ &\qquad \left.+\,\epsilon^{\alpha\beta}\bar{\Theta}^1\Gamma^\mu\partial_\alpha\Theta^1\bar{\Theta}^2\Gamma_\mu\partial_\beta\Theta^2\right). \end{aligned} \tag{1.3.3}$$

The result of imposing supersymmetry on S_{WZ} is that, even in the classical theory, the superstring only exists in a certain number of spacetime dimensions ($D = 3, 4, 6$ or 10), due to Fierz identities of the Γ-matrices. Similar to the bosonic string, Lorentz symmetry in the quantum theory requires $D = 3$ or $D = 10$. In the following we only work in $D = 10$.

The resolution of the problem of the superfluous degrees of freedom introduces the important concept of *κ-symmetry*. It is a local fermionic symmetry, a gauge symmetry, that allows to gauge fix half of fermionic degrees of freedom. The concrete realisation is highly technical. In other backgrounds and in other conventions it might take a very different form – in some backgrounds even the required rank of κ-symmetry might be different, e.g. in the construction of the $AdS_4 \times \mathbb{CP}^3$ superstring action [47].

Nevertheless, let us sketch the basic idea. The gauge parameters κ^I are not only Majorana-Weyl spinors but also world-sheet vectors. As the irreducible representations of the two-dimensional Lorentz group are one-dimensional, any world-sheet vector can be split into a self-dual and an anti-self-dual part. So, we impose that the gauge parameters κ^I fulfil:

$$\kappa^{1,2} = \Pi_\pm \cdot \kappa^{1,2} \tag{1.3.4}$$

with respect to projectors $\Pi^{\alpha\beta}_\pm = \frac{1}{2\sqrt{-\gamma}}\left(\sqrt{-\gamma}\gamma^{\alpha\beta} \pm \epsilon^{\alpha\beta}\right)$. This is the reason why κ-symmetry removes exactly have of the fermionic degrees of freedom. The κ-symmetry transformations then take the form

$$\delta\Theta^I \sim i\Gamma \cdot A_\alpha\kappa^I \quad \text{and} \quad \delta x^\mu \sim i\Theta^I\Gamma^\mu\delta\Theta^I. \tag{1.3.5}$$

In addition, the variation of the world-sheet metric $\gamma_{\alpha\beta}$ still has to be determined.

World-sheet supersymmetry. The Ramond-Neveu-Schwarz (RNS) formalism [48,49] introduces two world-sheet Majorana spinors that also carry a target space vector index $\Psi^\mu = (\Psi^\mu_+, \Psi^\mu_-)$. This approach is very popular in the modern textbooks, because it allows to use the powerful tools of two-dimensional superconformal field theories. After fixing superconformal gauge the Polyakov-type action describes D free bosons and D free fermions

$$S_{RNS} = -\frac{1}{2\alpha'} \int_\Sigma \mathrm{d}^2\sigma \left(\partial_\alpha x \cdot \partial^\alpha x + i\bar{\Psi} \cdot \partial\!\!\!/ \Psi \right). \tag{1.3.6}$$

In addition to the bosonic symmetries of the Polyakov action, the action is (on-shell) supersymmetric under $\partial_\pm x^\mu \leftrightarrow \bar{\epsilon}\Psi^\mu$, $\delta\Psi^\mu_\pm = -i\partial_\pm x^\mu \epsilon$. This becomes apparent in the equations of motions

$$\partial_+ \Psi^\mu_- = \partial_- \Psi^\mu_+ = 0 \quad \text{and} \quad \partial_+\partial_- x^\mu = 0. \tag{1.3.7}$$

The fermions can be expanded into (anticommuting) modes

$$\Psi^\mu_- = \sum_{n\in\mathbb{Z}+r} d^\mu_n e^{2\pi\sigma_- n}, \qquad \Psi^\mu_+ = \sum_{n\in\mathbb{Z}+r} \bar{d}^\mu_n e^{2\pi\sigma_+ n} \tag{1.3.8}$$

with the anticommutator $\{d^\mu_m, d^\nu_n\} = \eta^{\mu\nu}\delta_{m+n,0}$, and correspondingly for the left-moving modes, after canonical quantisation. They can appear in two sectors depending on their boundary conditions: the antiperiodic Neveu-Schwarz (NS) sector with $r = \frac{1}{2}$ and the periodic Ramond (R) sector with $r = 0$, meaning $\Psi(\tau, \sigma + 1) = \pm\Psi(\tau, \sigma)$ respectively.

Whereas the vacuum of the NS-sector is unique and bosonic, the R-sector contains the d^μ_0-modes which form a representation of the target-space Clifford algebra, $\Gamma^\mu \sim d^\mu_0$, as $\{d^\mu_0, d^\nu_0\} = \eta^{\mu\nu}$. Their action commutes with the fermion number operator and hence with the mass operator. So, the R-sector vacuum has to transform as a target space fermion. The full Fock space is obtained by the action of the creators α^μ_m, d^μ_m for $m < 0$ on these oscillator vacua. As these modes are target space vectors, their action preserves the statistics of the vacuum and the closed string Hilbert space decomposes into

$$\left(\mathcal{H}^{(NS)} \oplus \mathcal{H}^{(R)}\right)_{\text{right-moving}} \otimes \left(\mathcal{H}^{(NS)} \oplus \mathcal{H}^{(R)}\right)_{\text{left-moving}}.$$

After a similar procedure as for the bosonic string, e.g. via light-cone gauge quantisation, we arrive at a critical dimension of $D = 10$ and a spectrum that still contains a tachyon and does not possess target space supersymmetry. The Gliozzi-Scherk-Olive (GSO) projection [50], that arises naturally from requiring modular invariance on the torus partition function, resolves all these problems. It makes use of the so called G-parity

$$G^{(NS)} = (-1)^F \quad \text{resp.} \quad G^{(R)} = \Gamma_{11}(-1)^F \tag{1.3.9}$$

with the fermion number operator $F = \sum_{n\geq 1-r} d^i_{-n} d^i_n$ and the ten-dimensional target space Dirac matrix Γ_{11}, that characterises the chirality of a ten-dimensional spinor. The

GSO projection projects onto G-eigenspaces. On the NS-sector, one fixes $G^{(NS)} = 1$ removing the bosonic ground state, the tachyon, from the spectrum. In the R-sector G-parity fixes only a relative chirality as compared to the vacuum. Hence, states in the R-sector become (Majorana-)Weyl spinors in target space. For the open string (or the right- or left-moving sector of a closed string), we denote the R-sector ground state by $|+\rangle$ resp. $|-\rangle$, real 16-dimensional Majorana-Weyl spinors. The Dirac equation removes another half of these degrees of freedom, leaving eight.

The lightest, massless states of the spectrum after light-cone gauge quantisation and GSO-projection are

$$d^i_{-\frac{1}{2}} |0, p\rangle^{(NS)} \quad \text{and} \quad |+, p\rangle^{(R)} .$$

The massless spectrum exhibits $\mathcal{N} = 1$ supersymmetry, as expected. Bosons (NS sector states) and fermions (R-sector vacuum) both have eight degrees of freedom, as usual for massless vectors in $D = 10$.

Combining right- and left-moving sectors of a closed string, together with the level matching condition, gives two possible relative choices of a vacuum resp. GSO-projection for the RR sector. These different choices result in the type IIa and IIb superstring theories, which are discussed in the following.

Type II superstrings. The three types of possible left- or right-moving excitations can be arranged into representations of SO(8), the massless little group in $D = 10$. The NS-sector excitations correspond to the vector representation $\mathbf{8}_V$ and the two R-sector vacua of opposite chirality are Spin(8) representations, often denoted by $\mathbf{8}_S$ and $\mathbf{8}_C$. For closed string excitations, there are the two possibilities of relative chirality of the R-sector vacua, leading to the type IIa and IIb superstring:

$$\begin{array}{llcl} \text{IIa:} & (\mathbf{8}_V \oplus \mathbf{8}_S) \otimes (\mathbf{8}_V \oplus \mathbf{8}_C) & \to & G,\ B,\ \Phi,\ C_1,\ C_3,\ \text{fermions} \\ \text{IIb:} & (\mathbf{8}_V \oplus \mathbf{8}_S) \otimes (\mathbf{8}_V \oplus \mathbf{8}_S) & \to & G,\ B,\ \Phi,\ C_0\ , C_2\ ,\ C_4,\ \text{fermions.} \end{array}$$

The (bosonic) NSNS-sector is universal containing – as in the bosonic case – a metric G, a skewsymmetric tensor B and the scalar dilaton Φ. The fermionic NS/R-sector states consists of two gravitini (spin $\frac{3}{2}$) and dilatini (spin $\frac{1}{2}$). The (bosonic) RR-sector contributes p-form gauge fields C_p. Not all the field strengths F_p are defined in the usual way, for F_3, F_4, F_5 we have:

$$F_3 = \mathrm{d}C_2 - C_0\mathbf{H}, \quad F_4 = \mathrm{d}C_3 + C_1 \wedge \mathbf{H}, \quad F_5 = \mathrm{d}C_4 - \frac{1}{2}\mathrm{d}(C_2 \wedge B), \tag{1.3.10}$$

supplemented by the self-duality condition $F_5 = \star F_5$.

Potential low energy effective theories of these excitations are known: the type II supergravities,

$$S_{SUGRA} = S_{NS} + S_R + S_{CS}. \tag{1.3.11}$$

They contain the universal NSNS-sector part S_{NS} of the form (1.2.8), the kinetic terms of the RR-flux gauge fields

$$S_R^{(IIa)} = -\frac{1}{4\kappa^2}\int (F_2 \wedge \star F_2 + F_4 \wedge \star F_4)$$
$$S_R^{(IIb)} = -\frac{1}{4\kappa^2}\int \left(F_1 \wedge \star F_1 + F_3 \wedge \star F_3 + \frac{1}{2} F_5 \wedge \star F_5\right).$$

There are additional Chern-Simons (CS) terms describing interactions between the NSNS and RR gauge fields, $\mathbf{H} = \mathrm{d}B$ and $F_p = \mathrm{d}C_{p-1}$.

$$S_{CS}^{(IIa)} = -\frac{1}{4\kappa^2}\int (B \wedge \mathrm{d}C_3 \wedge \mathrm{d}C_3)$$
$$S_{CS}^{(IIb)} = -\frac{1}{4\kappa^2}\int (C_4 \wedge \mathbf{H} \wedge F_3).$$

The constant ten-dimensional gravitational constant behaves as $\kappa^2 \sim l_S^8$, due to dimensional reasons. The equations of motion of S_{SUGRA} are called *supergravity equations*. Their classical solutions are considered to be viable string backgrounds, i.e. the worldsheet theory in these backgrounds is conformally invariant at 1-loop level.

The objects that are sourcing an RR-flux F_p are the $D(p-2)$-branes as electric sources and the $D(10-p-2)$-branes as magnetic sources. Their low energy excitations are described, by a Nambu-Goto type action, namely the Dirac-Born-Infeld (DBI) action. [43]

Other superstring theories. Besides the construction sketched above, there are also type I superstrings theory (obtained by an orientifold projection from type IIb superstrings), namely the two heterotic superstring theories and the type 0 string theories (results of different GSO-projections and *not* resulting in a target space supersymmetric spectrum). Apart from their relevance in the web of string dualities leading to M-theory, they will not be relevant in the following.

Compactification. Also in the ten-dimensional superstring theories, we are still not yet in the striven for four-dimensional space-time of our physical reality. In many realistic scenarios one supposes that the background is a product manifold

$$\mathcal{M}_{1,3} \times \mathcal{M}_{int} \tag{1.3.12}$$

of our physical space-times $\mathcal{M}_{1,3}$ and an internal six-dimensional space $\mathcal{M}_6$. Phrased as a four-dimensional theory, the parameters (metric, fluxes) that describe the internal geometry, become fields of the four-dimensional theory. Such an approach first arose, independently, in work by Kaluza and Klein [51, 52], who proposed that general relativity and Maxwell theory can be unified in a five-dimensional theory of gravity.

The internal space $\mathcal{M}_{int}$ is often considered to be *'small'* and *compact*. This is because it is so far, at the energy scale of present experiments, unobserved [53]. Also, from the

Einstein-Hilbert actions in D dimensions,

$$S \sim \left(M_P^{(D)}\right)^{(D-2)} \int \mathrm{d}^D x \sqrt{-G}\mathcal{R}^{(D)},$$

we read off that

$$\left(M_P^{(4)}\right)^2 \sim \left(M_P^{(10)}\right)^8 \mathrm{Vol}(\mathcal{M}_6). \tag{1.3.13}$$

As a consequence, one has to require a finite volume of the internal space in order for the effective four-dimensional Planck constant to be finite.

The internal geometry of the compact internal manifold is the playground for phenomenological scenarios in string theory. These will not be pursued further in this thesis. Instead, the compact nature of the internal space brings two additional features into the game, *Kaluza-Klein excitations* and *winding strings*, the interplay of which leads to T-duality and generalised geometry, introduced in the following chapters. Let us assume we compactify on a circle S^1 with radius R. Any field ϕ can be decomposed into modes on the circle $\phi(x) = \sum_n \phi^{(n)} e^{2\pi i x \frac{n}{R}}$, where $0 \leq x < 1$ is the coordinate on the circle. In the lower dimensional effective theory of a compactification, these excitations $\phi^{(n)}$ appear as a tower of distinct fields, the Kaluza-Klein states, with mass $M^2 \sim \frac{n}{R}$.

The allowed momenta along the compact direction are quantised. This is also true for the centre of mass momentum p in (1.1.6), $p \sim \frac{n}{R}$ for $n \in \mathbb{Z}$ and contributing like $M^2 \sim \left(\frac{n}{R}\right)^2$ to the mass spectrum. This is true for point particles as well. A purely stringy feature is that strings can wind around such compact directions. I.e. they can be subject to the more general boundary condition $x(\tau, \sigma + 1) = x(\tau, \sigma) + Rw$, where w is the winding number of the string.

1.3.2 Superstring σ-models

The low-energy effective actions of type II superstring theories, sketched by their bosonic actions above, are known and well-versed theories in their own right as dimensional reductions of maximal supergravity. On the other hand, the superstring analogue to (1.2.2), a superstring σ-model, is not known in general. The NSNS-sector has the universal form of (1.2.2), but even the bosonic sector is not completely known, as the coupling of the world-sheet to the RR-sector is not known in general.

In case the backgrounds can be realised as semisymmetric supercosets, it is possible to construct a complete σ-model in the framework of the GS-formalism nevertheless. The most influential of these, through its relevance in the AdS/CFT-correspondence, is arguably the type IIb superstring σ-model in $\mathrm{AdS}_5 \times \mathrm{S}^5$ [54]. Starting from this supercoset construction, one can construct string σ-models in less symmetric string backgrounds via deformations. Some of these are discussed in section 4.2.

The action is of the form

$$S = -\frac{1}{2\alpha'} \int \mathrm{d}^2\sigma \left(\sqrt{\gamma}\gamma^{\alpha\beta}\mathrm{STr}(A_\alpha^{(2)} A_\beta^{(2)}) - \epsilon^{\alpha\beta}\mathrm{STr}(A_\alpha^{(1)} A_\beta^{(3)})\right) \tag{1.3.14}$$

for fields $g : \Sigma \to \mathcal{G}$, and $A = g^{-1}\mathrm{d}g$. The fact, that it is a semisymmetric supercoset means, that the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$ is $\mathbb{Z}_4$-graded, $\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)} \oplus \mathfrak{g}^{(3)}$. $\mathfrak{g}^{(1)}$ and $\mathfrak{g}^{(3)}$ denote the fermionic components, $\mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(2)}$ and $\mathfrak{g}^{(0)}$ are the bosonic Lie algebras to the isometry and isotropy groups, G and H. Hence, the bosonic part of the background is $\frac{G}{H} \subset \frac{\mathcal{G}}{H}$. The latter describes a full supersymmetric backgrounds, including fermions. Furthermore, $A^{(i)}$ denotes the projection of A onto $\mathfrak{g}^{(i)}$. Choosing a group parameterisation $g = g(x, \Theta)$ in terms of coordinates x^μ, Θ^I on the target superspace, one can deduce actions of the form (1.2.2) including fermions and couplings to the RR-fluxes.

Typical examples are:

- Type IIa or IIb strings in flat space, depending on the relative chirality of the supercharges $\mathcal{Q}^I$ [55]:

  $$\mathfrak{g} = \mathrm{span}(M_{\mu\nu}) \oplus \mathrm{span}(\mathcal{Q}^1, \bar{\mathcal{Q}}^1) \oplus \mathrm{span}(P_\mu) \oplus \mathrm{span}(\mathcal{Q}^2, \bar{\mathcal{Q}}^2).$$

  where P^μ and $M_{\mu\nu}$ are the translation and Lorentz rotations in ten-dimensional Minkowski space. In that case the second term of (1.3.14) reproduces the WZ-term in (1.3.3).

- Type IIb strings in $\mathrm{AdS}_5 \times \mathrm{S}^5$ [54]:

  $$\mathrm{AdS}_5 \times \mathrm{S}^5 = \frac{\mathrm{SU}(2,2)}{\mathrm{SO}(1,4)} \times \frac{\mathrm{SU}(4)}{\mathrm{SO}(5)} \subset \frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(1,4) \times \mathrm{SO}(5)}. \tag{1.3.15}$$

- Type IIa strings in $\mathrm{AdS}_4 \times \mathbb{CP}^3$ [47]:

  $$\mathrm{AdS}_4 \times \mathbb{CP}^3 = \frac{\mathrm{SO}(2,3)}{\mathrm{SO}(1,3)} \times \frac{\mathrm{SU}(4)}{\mathrm{U}(3) \times \mathrm{U}(1)} \subset \frac{\mathrm{OSp}(6|4)}{\mathrm{SO}(1,3) \times \mathrm{U}(3)}. \tag{1.3.16}$$

  It is assumed that the supercoset σ-model does not describe the full dynamics in that case [47].

Let us mention two important general results about these semisymmetric supercoset string σ-models:

- If the isometry group has vanishing Killing form, the background is Ricci-flat and, as a consequence, the σ-model is conformally invariant at 1-loop and, hence, a consistent string background [56]. In particular, this includes the groups to the superalgebras $\mathrm{A}(n,n) = \mathfrak{psl}(n|n;\mathbb{C})$ and its real forms (e.g. $\mathfrak{psu}(2,2|4)$), $\mathrm{D}(n+1|n)=\mathfrak{osp}(2n+2|2n)$ and the exceptional Lie superalgebra $\mathrm{D}(2|1,\alpha)$ in the classification of Lie superalgebras [57].

- The consistency of the backgrounds implies κ-symmetry of these models. The converse statement is not completely true [58]. Instead, κ-symmetry is now believed only to imply *generalised supergravity* equations of motions.

Chapter 2

Duality

2.1 Duality in high energy physics

The concept of 'duality' appears in many facets in string theory and is central in this thesis. This section aims to characterise it in the context of the related notions of 'symmetry' and 'emergence'.

2.1.1 Symmetries, emergence, duality – a disambiguation

Symmetries. Transformations of fields or of the space-time that leave certain features of the physical systems invariant are called *symmetries*. Depending on the context, one requires invariance, for example, of the equations of motions or the partition sum.

Typically, one differentiates between global and local symmetries. The key characteristic of 'global' symmetries in comparison to gauge symmetries and dualities is that they relate *different* field configurations resp. states of the *same* theory. Local or *gauge symmetries* describe *redundancies* of the mathematical description of a physical system. They map *physically equivalent* field configurations of a theory to each other.

Emergence. Effects and properties that stem from an interplay of the constituents of a (physical) system, but are not – at least superficially – described by the properties of the constituents, are called emergent. The obvious example from physics is the emergence of thermodynamics from statistical mechanics.

In natural philosophy, emergence is a very debated notion. One differentiates between a *strong* and a *weak* notion. Fundamental physics only employs the weak version, as it assumes that emergent qualities and quantities can, in principle, be derived from an underlying theory, although they might be encoded in a highly non-trivial way. The statement of strong emergence on the other hand is, that there might be emergent systems that are not even in principle understandable only in terms of their constituents or a basic underlying theory.

Emergence requires some kind of 'macroscopic' and 'microscopic' description. The transition between these two is characterised by extensive quantities, e.g. size, particle numbers or energy scales at which the physics of the systems is evaluated. The latter is typical in high energy physics, where the emergent systems at low energies are nothing else then effective field theories. One gets from the high-energy (UV) to the low-energy (IR) theory via the renormalisation group flow or by integrating out states above a certain energy scale. The other direction, finding a UV-complete, renormalisable, description of certain effective field theories (the standard model, general relativity) is the program of fundamental physics.

Other typical examples of emergent quantities in high energy physics are solitonic excitations, i.e. instantons or other states protected by topology or (super)symmetries, coherent states and other composite degrees of freedom. In string theory, D-branes and winding strings are examples for instantons, and vertex operators (1.1.26) are examples for collective degrees of freedom. Another interesting example is the emergence of the kinetic term of a gauge field, when integrating out massive fields that possess a UV description with a (non-dynamical) gauge field. This is reviewed in the case of the $\mathbb{CP}^n$ σ-model in [5].

A general expectation/conjecture/hope is that *any* dimensionless coupling in the low-energy theories is emerging (created dynamically) from a fundamental theory. In string theory this happens in the way that such couplings are obtained as vacuum expectation values of scalar fields, as in section 1 for the closed string coupling constant $g_S = \exp(\langle \Phi_0 \rangle)$.

Dualities. An (exact) equivalence of two (quantum) theories is called a duality. Interacting quantum field theories are generically hard to understand beyond the perturbative level. Hence, the maps between the states and the complete spectra of the two theories are often not known completely. Both would be necessary in order to identify a duality. *Strong-weak* (UV-IR) dualities, connecting non-perturbative to pertubative sectors of different theories are particularly interesting. Here, emergent quantities are dual to fundamental ones in the dual theory and vice versa. Such dualities are very useful for getting insights into the non-pertubative sectors and for perfoming otherwise inaccessible calculations. Sometimes, the notion of *approximate* dualities as opposed to exact dualities is used. In this case, only certain limits or subsectors of two theories are dual to each other.

Dualities, also called duality symmetries sometimes, and symmetries are very similar concepts. Both correspond to different 'parameterisations' of the path integral. Let us outline several differences between these two concepts. A symmetry maps two states to each other that have the *same* physical meaning, whereas a duality maps two states to each other that have a *different* physical interpretation. This is obvious if the duality is between two distinct theories, but it can also happen in self-dualities of a theory.

Let us try to find more rigorous characterisations. In a *Lagrangian formulation*, symmetries typically correspond to local field redefinitions, whereas dualities (if the duality map is known explicitly at all) are at best non-local ones. In the *(classical) Hamiltonian theory*, it is often the case that dualities are *canonical transformations*. In particular, a lot of them appear to be field theory generalisations of the Born duality $q \to P$, $p \to -Q$

generated for example by a generating function: $F(q, Q) = qQ$. (Continuous) global symmetries, on the other hand, act on fields or observables as $\delta F = \{F, \Phi\}$, where Φ, the generator of the symmetry, is the conserved Noether charge corresponding to that symmetry, $\{H, \Phi\} = 0$.

That *dual*ity actually refers to *two* is often the case. Then, the duality corresponds to a $\mathbb{Z}_2$ action. Sometimes, compositions of different dualities and symmetries form bigger webs of pairwise dual theories. Consequently, duality can act as bigger non-trivial groups, for example $\mathrm{SL}(2, \mathbb{R})$ in S-duality, $\mathrm{O}(d, d)$ in T-duality or the exceptional groups $\mathrm{E}_{d(d)}$ in U-duality. In such cases, there exist terms like 'plurality' or, as a mathematical example, 'triality' (for the $\mathbb{Z}_3$-symmetry of the $\mathrm{SO}(8)$ Dynkin diagram) in the literature. Following standard practice in the literature, all elements of duality groups are referred to as dualities in the following, not minding that some of these are simply (gauge) symmetries. In case the actual *dual*ities are meant, these are referred to as $\mathbb{Z}_2$- or factorised dualities.

The end of reductionism? Dualities, in particular those of the strong-weak type, challenge the paradigm of high energy physics of finding more and more 'complete' UV-theories in terms of more and more fundamental entities. The existence of these dualities implies that emergent (IR) degrees of freedom can be interpreted to be as fundamental as the microscopic (UV) degrees of freedom in a theory. In the natural philosophy literature, this has been argued to imply the end of *reductionism* and *atomism*. [59]

Instead, the proposed point of view in the following is that the existence of dualities might suggest that we did not formulate the theory in terms of the correct degrees of freedom. We strive to find a formulation of the theory that describes duality as an honest symmetry. This has a famous historic analogy: the correct formulation of quantum mechanics could only be found after dismantling the particle-wave duality.

2.1.2 Examples

Particle-wave duality. Whereas the particle-wave duality between momentum and position space representations of the free field/free particle Hilbert space is sometimes considered to be an outdated concept, it is a true duality in terms of the criteria phrased above. The duality map is exactly known, a Fourier transformation $\phi(x) \sim \int \mathrm{d}p\, e^{ipx} \hat{\phi}(p)$. It is not a symmetry transformation: it does not leave the equations of motion invariant and, moreover, it is non-local. Classically, it corresponds simply to the canonical transformation $x \to p$, $p \to -x$. One could even argue that it is a weak-strong duality – small scales in position space correspond to large scales in momentum space and vice versa.

The understanding that the particle-wave duality simply corresponds to the choice of a different basis/different coordinates of the state space will reappear in the treatment of T-duality in generalised geometry.

Bosonisation. Consider a free complex fermion Ψ and its particle-hole excitations

$$\phi(z) \sim \int_z \mathrm{d}w : \bar{\Psi}\Psi(w) : \tag{2.1.1}$$

that behave like a (free chiral) boson ϕ. Similar relations like this hold in any dimension. But, specifically in two dimensions, we have that coherent superpositions of that boson can be associated again to the elementary fermion

$$\Psi \sim : \exp(i\phi) : . \tag{2.1.2}$$

The changed statistics comes from a branch-cut in the complex logarithm. So, emergent degrees of freedom – collective resp. coherent ones – and elementary excitations change role. The duality map is exactly known.

There are versions of this as a duality of interacting theories, between the massive Thirring model and the sine-Gordon model [60] and in the Wess-Zumino-Witten model [61]. Independently, this phenomenon has been discovered in the condensed matter literature [62].

Electromagnetic duality. The (four-dimensional) vacuum Maxwell equations,

$$\mathrm{d}F = 0, \quad \mathrm{d} \star F = 0,$$

are invariant under the exchange $F \leftrightarrow \star F$, or $(\vec{E}, \vec{B}) \leftrightarrow (\vec{B}, -\vec{E})$. The Dirac quantisation condition for electric and magnetic charges $q_e \cdot q_m \in \mathbb{Z}$ shows that this is a strong-weak duality. The emergent degree of freedom here is the magnetic monopole – a solitonic excitation, corresponding to a point-like topological defect –, the fundamental one is an electrically charged probe.

As an exact quantum duality, it entered the literature as Montonen-Olive duality of supersymmetric Yang-Mills theories [63,64] and, as such, electromagnetic duality is the prototype for some dualities relevant in string theory.

Type IIb S-duality. The (non-perturbative) D-branes are no accidents of string theory, they are as fundamental as the string itself. The duality that describes this is S-duality. In type IIb it is a self-duality. Concretely, an S-duality element

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$$

acts as

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} C_2 \\ B \end{pmatrix} \to M \cdot \begin{pmatrix} C_2 \\ B \end{pmatrix} \tag{2.1.3}$$

linearly resp. fractionally linearly on the doublet of two-form NSNS- and RR-potentials resp. the axio-dilaton $\tau = C_0 + ie^{\Phi}$. It acts trivially on the metric (in Einstein frame) and the C_4 gauge potential.

There is no general proof of S-duality, not even a general map. Instead, it is used to probe the non-perturbative regime of the IIb superstring theory and demonstrated by the use of BPS-states. These are states of which certain properties are protected by supersymmetry. The half-BPS states of the type IIb theory are

	$\mathbf{H}$	F_1	F_3	F_5
electrically charged	F	$D(-1)$	$D1$	$D3$
magnetically charged	$NS5$	$D7$	$D5$	$D3$

with F being the fundamental string, $NS5$-brane the magnetic source for the $\mathbf{H}$-flux, and the $D(-1)$ the D-instanton. They are transformed into each other under the (factorised) S-duality:

$$F \leftrightarrow D1, \quad NS5 \leftrightarrow D5, \quad D3 \leftrightarrow D3. \tag{2.1.4}$$

The $D3 \leftrightarrow D3$ self-duality corresponds to the Montonen-Olive (electromagnetic) self-duality of $\mathcal{N} = 4$ super-Yang-Mills theory. It is a non-perturbative weak-strong duality in the string coupling constant g_S. For example, if we compare the tensions of the fundamental string and $D1$-brane:

$$T_F \sim \frac{g_S}{\alpha'} \leftrightarrow T_{D1} \sim \frac{1}{g_S\alpha'}. \tag{2.1.5}$$

Quantum effects break the $\mathrm{SL}(2,\mathbb{R})$ to $\mathrm{SL}(2,\mathbb{Z})$. The simplest motivation is that the fluxes are quantised, and for example the fundamental string carries integer units of B-charge. This should be preserved by the duality, allowing for $\mathrm{SL}(2,\mathbb{Z})$ as the biggest subgroup of $\mathrm{SL}(2,\mathbb{R})$ that does so.

Other non-perturbative (S-duality type) dualities of string theories include the self-duality of four-dimensional heterotic string theory, the duality between $\mathrm{SO}(32)$ heterotic string theory on T^4 and type IIa string theory on K^3-spaces and the duality of $\mathrm{SO}(32)$ heterotic theory to type I string theory.

Holographic duality. A string (or potentially more generally: quantum gravity) duality, that does not fit in the above list of examples of generalisation of the electromagnetic duality, is the holographic duality. It is most remarkable, as it makes it possible to consider gravity as an emergent phenomenon and relates gravity to a field theory (without gravity) in one dimension less:

$$\text{quantum gravity on } \mathrm{AdS}_{d+1} \quad \longleftrightarrow \quad \text{conformal field theory on } \mathbb{R}^{1,d-1}.$$

The most exact manifestation of holography is the duality of type IIb superstring theory in $\mathrm{AdS}_5\times\mathrm{S}^5$ and $\mathcal{N} = 4$ super-Yang-Mills theory with $\mathrm{SU}(N)$ gauge group [6]. It has been tested well using integrable structures on both sides of the duality [65]. Preserving some of these integrable structures and going beyond this very symmetric setting is the aim of the program of integrable deformations, introduced in chapter 4.

Two typical examples of the maps between these theories are:

- scaling dimension Δ of operators $\longleftrightarrow$ energy $E = \Delta$ of string states
- expectation values of Wilson loops, *non-perturbative* quantities, in the CFT $\longleftrightarrow$ minimal areas of string world-sheets, the Nambu-Goto action evaluated on *classical* string solution

2.2 T-duality

The duality that will play a central role in the following is (abelian) T-duality and its generalisations.

2.2.1 Strings on S^1

Closed bosonic strings. Compactifying one dimension on a circle of radius R, we have seen before that the mode expansion of a closed bosonic string takes the form

$$x^{25}(\tau, \sigma) = x_0^{25} + \alpha' p^{25} \tau + Rw\sigma + \textit{oscillators}.$$

Here, w is the winding number. The compactness of the 25th dimension leads to a quantisation of the p^{25}-momentum, the Kaluza-Klein (KK) momentum $n \in \mathbb{Z}$, $p^{25} = \frac{n}{R}$. The mass spectrum and level-matching condition change in comparison to the non-compact case:

$$\alpha' M^2 = \left[n^2 \left(\frac{\sqrt{\alpha'}}{R} \right)^2 + w^2 \left(\frac{R}{\sqrt{\alpha'}} \right)^2 \right] + 2(N + \bar{N} + 2) \tag{2.2.1}$$

$$n \cdot w = N - \bar{N}. \tag{2.2.2}$$

From here, we can read of a duality as a 'symmetry' of the spectrum. The simultaneous exchange

$$n \leftrightarrow w, \qquad R \leftrightarrow \bar{R} = \frac{\alpha'}{R} \tag{2.2.3}$$

leaves the spectrum invariant. It fits into our picture of dualities sketched above: It is a duality between two σ-models with distinct target spaces if, at the same time, the role of the *KK*-momentum (fundamental) and the winding modes (solitonic) are exchanged. In this case, it is an invariance of the full spectrum and thus we expect it to be a duality of the full quantum theory.

Under T-duality, the x^{25}-field transforms as

$$\partial_\pm x^{25} \to \pm \partial_\pm x^{25} \quad \text{resp.} \quad \mathrm{d}x^{25} \to \star \mathrm{d}x^{25}. \tag{2.2.4}$$

In this sense, T-duality could be understood as a kind of electromagnetic duality on the world-sheet. This picture will be clarified in the next section.

Open bosonic strings. As the above calculation only involved the zero modes of the closed string, the basic notion of T-duality does not change in case of the open string. The difference is that, on the one hand, open strings with free, Neumann, boundary conditions cannot wind around the the compact direction – they have the topology of a point – but they can move freely along the compactified dimension and hence possess a KK momentum. On the other hand, strings with Dirichlet boundary conditions are fixed to the D-branes and cannot have center of mass momentum orthogonal to the D-brane but they can wind around the compact direction.

Neumann string		Dirichlet string
radius of S^1 : R	$\overset{T}{\longleftrightarrow}$	radius of S^1 : α'/R
KK momentum $p = n/R$		winding number $w = n$
$Dp -$ brane		$D(p+1) -$ brane

Open strings with mixed Neumann-Dirichlet boundary conditions along the compact direction can have neither winding nor KK-momentum.

Type II superstrings. The above observation for open strings, that T-duality maps Dp- to $D(p \pm 1)$-branes, motivates that T-duality is not a self-duality of the IIa/IIb theories as in the bosonic case, but instead

$$\text{IIa on } S^1 \text{ with radius } R \overset{T}{\longleftrightarrow} \text{IIb on } S^1 \text{ with radius } \alpha'/R.$$

This is because the half-BPS Dp-branes have p being even resp. odd in IIa resp. IIb and we assume the duality to map between BPS states.

More rigorously, let us consider closed superstrings in the RNS formalism. Due to worldsheet supersymmetry, $\partial_\pm X \sim \Psi_\pm$, the T-duality map $\partial_\pm X \to \pm\partial_\pm X$ corresponds to $\Psi_\pm \to \pm\Psi_\pm$ for the world-sheet spinors $\Psi_\pm$. This implies for the zero mode in the Ramond sector, $d_0^9 \to -d_0^9$. As a consequence, $\Gamma_{11} = \Gamma_0 \cdot \Gamma_1 \cdot \ldots \cdot \Gamma_9 \to -\Gamma_{11}$ and $P_\pm = 1 \pm \Gamma_{11} \to P_\mp$, being the projector on the two chiral Weyl components. Hence, T-duality maps between the IIa and IIb theories.

The result is that the type II string theories compactified on a circle can be identified for almost all of the moduli space, i.e. except for $R \to 0$ or $R \to \infty$.

Perturbative expansions and map of couplings. T-duality is a non-perturbative (weak-strong) duality in the *dimensionless* σ-model coupling $\frac{\sqrt{\alpha'}}{R} \leftrightarrow \frac{R}{\sqrt{\alpha'}}$, describing the world-sheet loop expansion as discussed in section 1.2. At the radius $R = \sqrt{\alpha'}$ the theory is self-dual, the duality symmetry becomes a gauge symmetry. The radii $R < \sqrt{\alpha'}$ are dual to the radii $R > \sqrt{\alpha'}$. Effectively, this introduces a minimal (or maximal) length scale of the internal manifold.

In order to see what T-duality does to the g_S-expansion, let us consider the effect on the low-energy effective action. The string frame effective action scales as

$$\int \mathrm{d}^{10}x\sqrt{-G}e^{-2\Phi}\mathcal{R}^{(10)} + \ldots \sim \frac{1}{g_S^2}\int \mathrm{d}^{10}x\sqrt{-G}e^{-2\tilde{\Phi}}\mathcal{R}^{(10)} + \ldots . \tag{2.2.5}$$

Dimensional reduction on an S^1 of radius R gives

$$\frac{1}{g_S^2}\int \mathrm{d}^{10}x\sqrt{-G}e^{-2\tilde{\Phi}}\mathcal{R}^{(10)} \sim \frac{R}{g_S^2}\int \mathrm{d}^9x\sqrt{-G^{(9)}}e^{-2\tilde{\Phi}}\mathcal{R}^{(9)}. \tag{2.2.6}$$

Comparing the low-energy effective action for two T-dual theories, we see that $\frac{R}{g_S^2} \to \frac{\alpha'}{g_S^2 R}$. Hence, $g_S \to \frac{\alpha'}{R}g_S$. T-duality is a perturbative (weak-weak) duality in g_S, as it maps a perturbative expansion in g_S to another one.

2.2.2 Non-linear σ-models and abelian isometries

The product manifold setting $\mathcal{M}^9 \times S^1$ can be generalised to arbitrary circle or even toroidal fibrations: $T^d \hookrightarrow \mathcal{M}$.

The key requirement for the appearance of T-duality is that the background possesses multiple isometries k_i: $\delta x^\mu = \epsilon^i k_i^\nu \partial_\nu x^\mu$. These correspond to global symmetries of a non-linear σ-model (1.2.2) of a closed string ($\partial\Sigma = 0$), if

$$\mathcal{L}_{k_i}G = 0 \quad \text{and} \quad \mathcal{L}_{k_i}B = \mathrm{d}v_i. \tag{2.2.7}$$

This means that k_i are indeed isometries of the metric G and the Lie derivative of the B-field generates at most a gauge transformation of the **H**-flux. When the k_i commute and hence form an *abelian* algebra, the σ-model is subject to T-duality. To differentiate between this notion and the generalised versions appearing later, this is called *abelian* or *standard* T-duality in the following.

We aim to show basic features of T-duality. More details can be found in the important publications and reviews on T-duality that include [66–81]. Many basic features can be demonstrated in a slightly less general setting. Consider the non-linear σ-model

$$S = -\frac{1}{2\alpha'}\int \left(G_{\mu\nu}(x)\mathrm{d}x^\mu \wedge \star\mathrm{d}x^\nu + B_{\mu\nu}(x)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\right), \qquad \mu,\nu = 1,...,D \tag{2.2.8}$$

where we neglect the dilaton term for now and assume the target space has an isometry. Then, one can choose coordinates $x^\mu = (x^1, x^{\underline{\mu}})$ such that x^1 parameterises the isometry. G and B are then assumed to be only functions of the $x^{\underline{\mu}}$.

Buscher rules. One can rewrite the action (2.2.8) by substituting $\mathrm{d}x^1$ by gauge fields A. For this, we also have to add a Lagrange multiplier term $-\bar{x}_1\mathrm{d}A$ to the action, such that the new action is classically equivalent to (2.2.8) by enforcing that A is flat, $A = \mathrm{d}x^1$. When one integrates out the gauge field A instead of the Lagrange multiplier $\bar{x}_1$ one obtains a new σ-model of the form

$$S = -\frac{1}{2\alpha'}\int \left[\bar{G}^{\mu\nu}(\bar{x}_{\underline{\mu}})\mathrm{d}\bar{x}_\mu \wedge \star\mathrm{d}\bar{x}_\nu + \bar{B}^{\mu\nu}(\bar{x}_{\underline{\mu}})\mathrm{d}\bar{x}_\mu \wedge \mathrm{d}\bar{x}_\nu\right] \tag{2.2.9}$$

with coordinates $\tilde{x}_\mu = (\tilde{x}_1, x^{\underline{\mu}})$ where the new background $\bar{E} = \bar{G} + \bar{B}$ for the metric $\bar{G}$ and the Kalb-Ramond field $\bar{B}$ is given by the so-called *Buscher rules* [82]

$$\begin{aligned} \bar{G}^{11} &= \frac{1}{G_{11}}, \quad \bar{E}^{1\underline{\mu}} = \frac{E_{1\underline{\mu}}}{E_{11}}, \\ \bar{E}^{\underline{\mu}1} &= -\frac{E_{\underline{\mu}1}}{E_{11}}, \quad \bar{E}^{\underline{\mu\nu}} = E_{\underline{\mu\nu}} - \frac{E_{\underline{\mu}1}E_{1\underline{\nu}}}{E_{11}} \end{aligned} \tag{2.2.10}$$

in terms of the old background $E = G + B$.

Dilaton. Let us consider the dilaton term of (1.2.2), $S_{dil.} \sim \int_\Sigma \Phi \star 1$. At first glance, the dilaton is not affected by the gauging procedure described above, if it respects the isometry as well – $\frac{\partial}{\partial x^1}\Phi = 0$. But, when performing this procedure properly in the path integral, one finds that the dilaton has to transform, too [83].

A simple route to derive the T-duality transformation rule of the dilaton is to impose that the measure of the low-energy effective action $\sqrt{-G}e^{-2\Phi}$ stays invariant under T-duality. The shift

$$\Phi \quad \to \quad \Phi - \frac{1}{2}\ln G_{11} \tag{2.2.11}$$

compensates the T-duality transformation of the metric determinant.

Holonomies. Another issue in the procedure described above is related to ensure gauge invariance of the gauged σ-model (the intermediate state in the above procedure) in case of higher genus world-sheets. We would like to get rid of potential holonomies $h_{a_i} \sim \oint_{a_i} A$ of the gauge fields A along the generators a_i of the first homology group of the world-sheet. Introducing the δ-functions $\delta(h_{a_i}) \sim \sum_{n_i} e^{2\pi i n_i \oint_{a_i} A}$ in the path-integral enforces this. This is done with help of the Lagrange multipliers $\tilde{x}_1$

$$\int_\Sigma \mathrm{d}\tilde{x}_1 \wedge A \sim \oint_{a_1} \mathrm{d}\tilde{x}_1 \oint_{a_2} A - \oint_{a_1} \mathrm{d}\tilde{x}_1 \oint_{a_2} A \stackrel{!}{=} n_1 \oint_{a_2} A - n_2 \oint_{a_1} A \tag{2.2.12}$$

for example for the torus $g = 1$, where we have two cycles a_1 and a_2 as generators of the first homology group. The first step is the Riemann identity. In the second step, we recognise that we reproduce the δ-function naturally in the path-integral through the sum over all configurations of $\tilde{x}_1$ with $\oint_{a_i} \mathrm{d}\tilde{x}_1 = \pm n_i$. The consequence is, that the dual direction has to be compact as well and the dual coordinate, the Lagrangian multiplier $\tilde{x}_1$, has to be multi-valued.

So, the total derivative term $\int \mathrm{d}\tilde{x}_1 \wedge \mathrm{d}x^1$ that arose in the gauging procedure should not be neglected if we identify $A = \mathrm{d}x^1$. In general, x^1 is not single-valued. In particular,

$$\oint_{\text{spatial}} \mathrm{d}x^1 = w \in \mathbb{Z} \quad \text{resp.} \quad \oint_{\text{time-like}} \mathrm{d}x^1 = \alpha' p \in \mathbb{Z}, \tag{2.2.13}$$

for winding number w and Kaluza-Klein momentum p. In the path integral, the total derivative term corresponds to

$$\exp\left(\frac{2\pi i}{\alpha'}\int_\Sigma \mathrm{d}\tilde{x}_1\wedge \mathrm{d}x^1\right). \tag{2.2.14}$$

This exponential does not contribute to the path integral justifying that we could neglect the exponent in the action. By consulting Riemann's bilinear identity again, this relates holonomies of dual and original coordinates along non-trivial cycles on the world-sheet. In some way, this generalises the exchange of momentum and winding excitations due to T-duality and ensures that this works at higher genus world-sheets, as well. More details on this can be found in [66,73].

T-duality and topology. T-duality will, in general, map topologically distinct target spaces to each other. Circumstances in which that happens are, for example, if the toroidal bundle is degenerate at some point. As a simple example, consider $\mathbb{R}^2$ as a degenerate circle bundle in polar coordinates (R,ϕ):

$$\mathrm{d}s^2 = \mathrm{d}R^2 + R^2\mathrm{d}\phi^2 \quad \to \quad \mathrm{d}\tilde{s}^2 = \mathrm{d}R^2 + \frac{1}{R^2}\mathrm{d}\tilde{\phi}^2.$$

The dual background possesses a singularity at $R=0$, demonstrating that it is not only geometrically, but also topologically different from the original background.

Including backgrounds with $\mathbf{H}$-flux, topology change due to T-duality has been studied systematically in [84,85].

Exotic T-dualities. So far, we only considered backgrounds with isometries corresponding to compact *spatial* directions. It is possible to consider time- or light-like compact isometries, as well. These arise, for example, in AdS spaces.

Both cases have been studied with interesting results. Time-like T-duality leads to exotic string theories [86] – the 'exotic' refers to the appearence of multiple times in target space, Euclidean strings and time-like Dirichlet boundary condition. Recently, these exotic string theories attracted attention as they allow for dS solutions [87]. T-duality along null directions breaks Lorentz invariance [88]. The dual backgrounds are described by (non-relativistic) Newton-Cartan geometry [89,90].

2.2.3 The T-duality group

In case there are d such $\mathrm{U}(1)$-isometries, or in other words the target space is a toroidal fibration, the above consideration can be applied to each of the isometries. We choose coordinates such that these fibres are parameterised by the x^α in $x^\mu = (x^\alpha, x^{\underline{\mu}})$ and apply the above procedure. Combining the $\mathbb{Z}_2$-dualities discussed above with $\mathrm{GL}(d)$-transformations of the x^α gives the duality orbit

$$\bar{E} = \varphi.E = (aE+b)(cE+d)^{-1}, \tag{2.2.15}$$

where $a = \mathrm{diag}(A, \mathbb{1})$, $b = \mathrm{diag}(b, 0)$, $c = \mathrm{diag}(c, 0)$ and $d = \mathrm{diag}(D, \mathbb{1})$ are each D by D matrices and

$$\varphi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \in \mathrm{O}(d, d),$$

where $\mathrm{O}(d, d)$ is defined w.r.t. the metric $\eta = \begin{pmatrix} & \mathbb{1} \\ \mathbb{1} & \end{pmatrix}$. The duality group is $\mathrm{O}(d, d)$ and can be revealed only by considering the quadratic part in derivatives of the isometry coordinates - the transformation of non-isometry coordinates (also called spectators) is easily reproduced from (2.2.15). These are normally neglected in conceptual considerations. The action of $\mathrm{O}(d, d)$ on the RR-fluxes and a generalisation for fermionic isometries is discussed in appendix B.

Alternatively, $\mathrm{O}(d, d)$ acts linearly on the generalised metric

$$\mathcal{H}(G, B) = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}, \tag{2.2.16}$$

a $2D \times 2D$-matrix, incorporating the full information on the background. This generalised metric appears in a Hamiltonian derivation of the $\mathrm{O}(d, d)$ T-duality group. With the canonical momentum

$$p_\mu = -\left(G_{\mu\nu}\dot{x}^\nu + B_{\mu\nu}x'^\nu\right), \tag{2.2.17}$$

we can compute a first order form of the action (from now on we choose $\alpha' = 1$):

$$S = -\frac{1}{2}\int \mathrm{d}^2\sigma \left(\dot{x}^\mu p_\mu - H\right) \tag{2.2.18}$$

$$\text{with} \quad H = \frac{1}{2}\int \mathrm{d}\sigma (x', p)\mathcal{H}(G, B)\left(x', p\right)^T. \tag{2.2.19}$$

This holds generically, also in the case without isometries. If there are d commuting isometries, then we choose adapted coordinates as above and the Hamiltonian density H is invariant under linear transformations by $\varphi \in \mathrm{O}(d, d)$ of (x', p). From a similar first order form of an action we will motivate the non-abelian generalisation of the duality group $\mathrm{O}(d, d)$.

There is an $\mathrm{O}(d)\times\mathrm{O}(d)$-subgroup – or depending on the signature of the compact background $\mathrm{O}(q, d-q)\times\mathrm{O}(q, d-q)$-transformations – determined by the metric G, that leaves the generalised metric $\mathcal{H}$ invariant. As a consequence, the duality group without overcounting is

$$\frac{\mathrm{O}(d, d)}{\mathrm{O}(d) \times \mathrm{O}(d)}. \tag{2.2.20}$$

As it turns out, the generalised metric $\mathcal{H}$ itself is in $\mathrm{O}(d, d)$-element and takes values in that coset. Hence, (2.2.20) is nothing else than the moduli space of internal geometries, described by metric G and and B-field.

Any $\varphi \in$O(d,d) can be generated by elements of the following four subgroups:

- The true $\mathbb{Z}_2$-dualities, corresponding to the Buscher rules, are normally called *factorised dualities*. They fulfil $\varphi^2 = \mathbb{1}$ and 'generate' the components of O(d,d), that are not connected to the identity:

$$\varphi_{T_i} = \begin{pmatrix} \mathbb{1} - E_{ii} & E_{ii} \\ E_{ii} & \mathbb{1} - E_{ii} \end{pmatrix} \qquad \text{with} \qquad (E_{ij})_{kl} = \delta_{ik}\delta_{jl}. \tag{2.2.21}$$

- *General linear transformations* $G + B \to A^T(G+B)A$ are contained in this representation of O(d,d) as

$$\varphi_{GL} = \begin{pmatrix} A^T & \\ & A^{-1} \end{pmatrix} \qquad \text{for} \qquad A \in \mathrm{GL}(d). \tag{2.2.22}$$

- *B-shifts* by a constant skewsymmetric matrix also form a subgroup of the duality group. They correspond to gauge transformations of the **H**-flux, $\mathbf{H} = \mathrm{d}B$.

- Performing a 'full' factorised duality $\varphi = \eta$, gives a new background $\bar{E} = g + \beta$, where β takes the role of B and is given by

$$\beta = -(G+B)^{-1}B(G-B)^{-1}. \tag{2.2.23}$$

  We will refer to β as being dual or conjugate to B. So, logically *β-shifts* by a skewsymmetric matrix form another subgroup of O(d,d). We discuss their meaning on the σ-model level in the next paragraph.

  The matrix representations of B- and β-shifts are given by

$$\varphi_B = \begin{pmatrix} \mathbb{1} & b \\ & \mathbb{1} \end{pmatrix}, \qquad \varphi_\beta = \begin{pmatrix} \mathbb{1} & \\ r & \mathbb{1} \end{pmatrix} \tag{2.2.24}$$

  for skewsymmetric b and r.

β-shifts via a generalised Buscher procedure. The σ-model interpretation of β-shifts will be of special interest in the following. We will show two points of view on β-shifts. Starting from the linear σ-model (setting $\alpha' = 1$)

$$S = -\frac{1}{2}\int \left(G_{\mu\nu}\mathrm{d}x^\mu \wedge \star \mathrm{d}x^\nu + B_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\right) \tag{2.2.25}$$

for *constant* metric G and Kalb-Ramond field B, we follow the steps

1. gauging all the U(1)-isometries: substituting $\mathrm{d}x^\mu$ by gauge fields A^μ and introducing a Lagrange multiplier term $\bar{x}_\mu \wedge A^\mu$ in the Lagrangian,

2. B-shift in the dual picture: adding a term $\bar{X}^*\beta = \beta^{\mu\nu}\mathrm{d}\bar{x}_\mu \wedge \mathrm{d}\bar{x}_\nu$ with constant and skewsymmetric coefficients $\beta^{\mu\nu}$, which is a total derivative as $\mathrm{d}\beta = 0$, to the Lagrangian,

3. step 1 for the U(1)-isometries of the dual coordinates $\bar{x}_\mu$.

We arrive at

$$S' = -\frac{1}{2}\int A^\mu \wedge \left(G_{\mu\nu} \star A^\nu + B_{\mu\nu}A^\nu\right) + \bar{A}_\mu \wedge (\beta^{\mu\nu}\bar{A}_\nu + A^\mu) + \mathrm{d}x^\mu \wedge \bar{A}_\mu. \tag{2.2.26}$$

Integrating out A^μ and $\bar{A}_\mu$ gives the standard $\mathrm{O}(d,d)$ β-shift

$$S = -\frac{1}{2}\int \mathrm{d}x^\mu \wedge \left[\tilde{G}_{\mu\nu} \star + \tilde{B}_{\mu\nu}\right] \mathrm{d}x^\nu = \int \mathrm{d}^2\sigma\, \partial_+ x^\mu \left(\frac{1}{\frac{1}{G+B} + \beta}\right)_{\mu\nu} \partial_- x^\nu. \tag{2.2.27}$$

The second perspective on a β-shift, that we will take, is the following. A β-shift cannot only be interpreted as a B-shift in the dual coordinates, but also as the introduction of a Poisson bivector $\Pi = \beta^{\mu\nu}\partial_\mu \wedge \partial_\nu$. On a symplectic leaf of Π – so choosing and restricting to coordinates such that $\left(\beta^{-1}\right)_{\mu\nu}$ exists –, we can define a two-form $\omega = \beta^{-1}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$, which is symplectic for constant $\beta^{\mu\nu}$:

$$[\Pi, \Pi]_S = 0 \quad \Leftrightarrow \quad \mathrm{d}\omega = 0, \tag{2.2.28}$$

where $[\ ,\ ]_S$ is the Schouten bracket of multivectors. On such a symplectic leaf and after integrating out the $\bar{A}_\mu$ in (2.2.26) first and redefining $A^\nu \mapsto A^\nu - \mathrm{d}x^\nu$, we get

$$S = -\frac{1}{2}\int \mathbf{D}x^\mu \wedge \left(G_{\mu\nu} \star \mathbf{D}x^\nu + B_{\mu\nu}\mathbf{D}x^\nu\right) + \beta^{-1}_{\mu\nu}A^\mu \wedge A^\nu \tag{2.2.29}$$

with $\mathbf{D}x^\mu = \mathrm{d}x^\mu - A^\mu$. With the identification

$$\mathrm{d}\bar{x}_\mu = G_{\mu\nu} \star \mathrm{d}x^\nu + B_{\mu\nu}\mathrm{d}x^\nu \tag{2.2.30}$$

between dual (winding) and the original coordinates and subtraction of a total derivative $\mathrm{d}\bar{x}_\mu \wedge \mathrm{d}x^\mu$ we obtain

$$S = -\frac{1}{2}\int A^\mu \wedge \left(G_{\mu\nu} \star A^\nu + B_{\mu\nu}A^\nu\right) + \mathrm{d}\bar{x}_\mu \wedge A^\mu + \beta^{-1}_{\mu\nu}A^\mu \wedge A^\nu. \tag{2.2.31}$$

The reason that we consider the version (2.2.31) of (2.2.26) is that it can be obtained via a different route, which was introduced in [91]:

- Given a cocycle ω (2.2.28) we can centrally extend the isometry algebra with generators $\{t_\mu\}$ (abelian in the toroidal case) in the following way (see appendix C for more details), where Z is the new central element of the algebra:

$$[t_\mu, t_\nu] = 0 \quad \mapsto \quad [t_\mu, t_\nu]' = [t_\mu, t_\nu] + \omega(t_\mu, t)Z = \beta^{-1}_{\mu\nu}Z, \quad [t_\mu, Z]' = 0. \tag{2.2.32}$$

- Starting from original action (2.2.25), we substitute $\mathrm{d}x^\mu$ by gauge fields A^μ, which we now assume to be components of a gauge field $A' = A^\mu t_\mu + CZ$ with field strength $F' = \mathrm{d}A' - [A' \overset{\wedge}{,} A']'$. Using (2.2.32) the components of F are

$$F^\mu = \mathrm{d}A^\mu \quad \text{and} \quad F^Z = \mathrm{d}C - \beta^{-1}_{\mu\nu}A^\mu \wedge A^\nu. \tag{2.2.33}$$

Instead of adding a Lagrangian multiplier term which enforces only $F^\mu = \mathrm{d}A^\mu = 0$, we also want to set $F^Z = 0$. For this we use 'extended' dual coordinates $Y_s = (\bar{x}_\mu, Y)$ and add the following term to the Lagrangian

$$\mathcal{L}_{Lag.mult.} \propto -Y_s F^s \overset{P.I.}{=} \mathrm{d}\bar{x}_\mu \wedge A^\mu + \mathrm{d}Y \wedge C + Y\beta^{-1}_{\mu\nu} A^\mu \wedge A^\nu. \qquad (2.2.34)$$

- Integrating out C leads to $Y = const.$, so that the resulting action is the same as (2.2.31). After integrating out A we are left with (2.2.27).

All these manipulations were rather trivial in the abelian case, but this analysis helps to understand the geometrical meaning and the objects to look for in the non-abelian case. We will comment on this and the connection to Lie algebra cohomology in section 8.2.2.

2.2.4 Non-abelian and Poisson-Lie T-duality

Up to now, we assumed that the isometries we consider commute. Certain aspects of abelian T-duality can be generalised to cases of non-commuting isometries and, motivated by that, even without isometries in certain algebraic circumstances. This is known as non-abelian T-duality (NATD) and Poisson-Lie T-duality.

Non-abelian T-duality. The adapted parameterisation in the case of non-commuting isometries,

$$[k_a, k_b] = f^c{}_{ab} k_c$$

uses group $\mathcal{G}$ valued fields $g\colon \Sigma \to \mathcal{G}$, with corresponding Lie algebra $\mathfrak{g}$ and structure constants $f^c{}_{ab}$. If then the 'background' field $E = G + B$ is constant, the model

$$S = -\frac{1}{2}\int \mathrm{d}^2\sigma\, (g^{-1}\partial_+ g)^a E_{ab} (g^{-1}\partial_- g)^b \qquad (2.2.35)$$

is (globally) $\mathcal{G}$ invariant under left multiplication, $g \to g_0 g$. We neglect potential additional spectator coordinates and perform a procedure, which is very similar to the abelian case: Substitute $g^{-1}\mathrm{d}g$ by $\mathfrak{g}$-valued gauge fields j, add a Lagrangian multiplier term $x_a(\mathrm{d}j + [j, j])^a$, which fixes j to be pure gauge and then integrate out j. The non-abelian T-dual model is

$$S = -\frac{1}{2}\int \mathrm{d}^2\sigma\, \partial_+ x_a \bar{E}^{ab}(x) \partial_- x_b, \qquad \bar{E}^{-1}_{ab} = E_{ab} - x_c f^c{}_{ab}. \qquad (2.2.36)$$

In contrast to the abelian case, the duality connects an isometric and a non-isometric model with each other. In general, the status of non-abelian T-duality is not as strong as the one of abelian T-duality – it is not a true duality at the quantum level, but a map between similar theories or a type of solution generating technique [92].

Also, if the trace of the structure constants of $\mathfrak{g}$ does not vanish, $f_c{}^{ac} \neq 0$, the non-abelian T-dual model (2.2.36) possesses a kind of anomaly that spoils conformal invariance in the quantum theory. For more details on non-abelian T-duality see [72, 74, 75, 91, 93–95].

Lie bialgebras. Before we discuss the further generalisation of the above, i. e. Poisson-Lie T-duality and Poisson-Lie σ-models, which will also enlighten the structure of the action (2.2.36), let us review the algebraic basics for this – Lie bialgebras – and set up our conventions.

Here and in the rest of the thesis, we consider a d-dimensional semi-simple[1] Lie algebra $\mathfrak{g}$ with corresponding Lie group $\mathcal{G}$, the Killing form κ and generators t_a fulfilling

$$[t_a, t_b] = f^c{}_{ab} t_c. \tag{2.2.37}$$

We use $\partial_a, \partial_b, \ldots$ to represent the curved derivatives corresponding to $t_a, t_b, \ldots$ treated as invariant vector fields on $\mathcal{G}$, and $\partial_\mu, \partial_\nu \ldots$ for flat derivatives.

There are two ways to define Lie bialgebras:

- *Definition via Manin triples.* We want to define a Lie bracket on the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$ in terms of the $2d$ generators $T_A = (t_a, \bar{t}^a)$ of $\mathfrak{g} \oplus \mathfrak{g}^*$, such that the canonical, non-degenerate and symmetric bilinear form, defined by

$$\langle t_a | t_b \rangle = \langle \bar{t}^a | \bar{t}^b \rangle = 0, \quad \langle t_a | \bar{t}^b \rangle = \delta_a^b \tag{2.2.38}$$

or in terms of the T_A, $\langle T_A | T_B \rangle = \eta_{AB}$ with $\eta = \begin{pmatrix} & \mathbb{1}_d \\ \mathbb{1}_d & \end{pmatrix}$, is Ad-invariant, i.e.

$$\langle T_A | [T_B, T_C] \rangle = \langle [T_C, T_A] | T_B \rangle \,. \tag{2.2.39}$$

The structure constants of a complementary pair $(\mathfrak{g}, \mathfrak{g}^\star)$ of Lagrangian (meaning maximally isotropic w.r.t. to $\langle \, | \, \rangle$) subalgebras can be constructed to be of the form

$$\begin{aligned} & [T_A, T_B] = F^C{}_{AB} T_C \\ \text{with} \quad & [t_a, t_b] = f^c{}_{ab} t_c, \quad [\bar{t}^a, \bar{t}^b] = \bar{f}_c{}^{ab} \bar{t}^c, \\ & [t_a, \bar{t}^b] = \bar{f}_a{}^{bc} t_c + f^b{}_{ca} \bar{t}^c. \end{aligned} \tag{2.2.40}$$

The Lie group to the Lie algebra $\mathfrak{d}$ is called Drinfel'd double, we denote it by $\mathcal{D}$. It contains $\mathcal{G}$ and $\bar{\mathcal{G}}$ (the Lie group to $\mathfrak{g}^\star$) as subgroups, $\mathcal{D} = G \bowtie \bar{G}$. The condition on the structure constants $f^c{}_{ab}$ and $\bar{f}_c{}^{ab}$ in order for (2.2.40) to fulfil the Jacobi identity is

$$f^c{}_{mn} \bar{f}_c{}^{ab} = f^{[a}{}_{c[m} \bar{f}_{n]}{}^{b]c}. \tag{2.2.41}$$

The triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^\star)$ is called a Manin triple. For a given $(\mathfrak{d}, \langle \, | \, \rangle)$ there can be multiple Manin triple decompositions. In the following we also use the notation

$$\mathfrak{g} \oplus_{\mathfrak{d}} \mathfrak{g}^\star \tag{2.2.42}$$

to describe a Manin triple. Consistency requires that κ^{-1} is the Killing form on $\mathfrak{g}^\star$.

[1] In principle we do not have to restrict to the semi-abelian case to perform non-abelian T-duality. This was successfully demonstrated in [96].

- *Definition via 1-cocycles.* In the maths literature bialgebras are normally defined differently but of course equivalently. A bialgebra is the pair $(\mathfrak{g}, u)$ of a Lie algebra $\mathfrak{g}$ and a $\mathfrak{g} \wedge \mathfrak{g}$-valued 1-cochain u on $\mathfrak{g}$ fulfilling

  1. 1-cocycle condition:
  $$\delta u(m,n) := \Delta(\mathrm{ad}_m)u(n) - \Delta(\mathrm{ad}_n)u(m) - u([m,n]) = 0 \qquad (2.2.43)$$
  2. 'Jacobi identity':
  $$\Delta(u) \circ u = 0, \qquad (2.2.44)$$

  where we defined the coproduct $\Delta(X) := \mathbb{1} \otimes X + X \otimes \mathbb{1}$. A $\mathfrak{g} \wedge \mathfrak{g}$-valued 1-cochain
  $$u(t_c) = u_c{}^{ab} t_a \wedge t_b \qquad (2.2.45)$$
  can be identified with a skew-symmetric bracket on $\mathfrak{g}^\star$, $[\,,\,]_{\mathfrak{g}^\star} : \mathfrak{g}^\star \wedge \mathfrak{g}^\star \to \mathfrak{g}^\star$ with structure constants $\overline{f}_c{}^{ab} \equiv u_c{}^{ab}$. Then the 1-cocycle condition (2.2.43) is equivalent to (2.2.41) and (2.2.44) corresponds to the Jacobi identity on $\mathfrak{g}^\star$. So indeed, this definition is equivalent to the Manin triple.

  If the 1-cocycle is 1-coboundary, we will call the corresponding bialgebra 1-*coboundary*. We will comment on a certain type of these 1-coboundary bialgebras in the next paragraph. But, of course, there are more possible bialgebras. In appendix D.4 we comment on this. Some relevant basics on Lie algebra cohomology are summarised in appendix C.

***R*-brackets and Yang-Baxter equations.** Given a Lie algebra $\mathfrak{g}$ with bracket $[\,,\,]$, is it possible to define another Lie bracket on $\mathfrak{g}$? A simple candidate is the so-called R-bracket
$$[m,n]_R = [m, R(n)] - [n, R(m)] \qquad \forall m,n \in \mathfrak{g} \qquad (2.2.46)$$
for some $R \in \mathrm{End}(\mathfrak{g})$. A sufficient condition on R, s.t. $[\,,\,]_R$ fulfils the Jacobi-identity, is
$$[R(m), R(n)] - R([m,n]_R) = c^2[m,n] \qquad \forall m,n \in \mathfrak{g}. \qquad (2.2.47)$$
This condition can be rewritten as
$$[(R \pm c\mathbb{1})(m), (R \pm c\mathbb{1})(n)] = (R \pm c\mathbb{1})([m,n]_R), \qquad \forall m,n \in \mathfrak{g}, \qquad (2.2.48)$$
which means that $(R \pm c\mathbb{1})$ is a Lie algebra homomorphism between $(\mathfrak{g}, [\,,\,])$ and $\mathfrak{g}_R := (\mathfrak{g}, [\,,\,]_R)$. After rescaling we can distinguish three cases of (2.2.47):

1. $c = 0$: classical Yang-Baxter equation (cYBe),
2. $c = 1$: *non-split* modified classical Yang-Baxter equation (mcYBe)
3. $c = i$: *split* modified classical Yang-Baxter equation,

each of which have distinct roles in the definition of Lie bialgebra structures of semisimple Lie algebras, which is sketched in appendix D. A more extensive review can be found for example in [97].

The connection of R-brackets of semisimple Lie algebras to their bialgebra structures is seen via the definition via 1-cocycles. The non-degenerate Killing form κ on $\mathfrak{g}$ defines a 2-vector $r = r^{ab} t_a \wedge t_b$ for each $R \in \mathrm{End}(\mathfrak{g})$, $r^{ab} = \kappa^{ac} R^b{}_c$. The $\mathfrak{g} \wedge \mathfrak{g}$-valued 1-coboundary

$$\delta r(x) = \Delta(\mathrm{ad}_x) r = [\Delta(x), r] \tag{2.2.49}$$

is trivially a 1-cocycle (2.2.43) and the condition for the Jacobi identity (2.2.44) on r can be written as

$$\Delta(\mathrm{ad}_x)\left([r,r]_S\right) = 0, \tag{2.2.50}$$

where $[\ ,\ ]_S$ is the standard Schouten bracket of multivectors.

Bivector fields on Lie groups. Typical bivectors on Lie groups are associated to a dual Lie algebra structure by the canonical relation

$$\bar{f}_c{}^{ab} = \partial_c \Pi^{ab}(e). \tag{2.2.51}$$

The *canonical Poisson vector* $\Pi = \Pi^{ab} t_a \wedge t_b$ on a Lie group $\mathcal{G}$ for a left-(right-)invariant basis $\{t_a\}$ of $T\mathcal{G}$ is given by

$$\Pi^{ab}(g) = \bar{f}_c{}^{ab} x^c - \frac{1}{2} \bar{f}_c{}^{k[a} f^{b]}{}_{dk} x^c x^d + \ldots, \quad \text{for } g = \exp(x^a t_a) \in \mathcal{G}, \tag{2.2.52}$$

when the structure constants $f, \bar{f}$ describe a Lie bialgebra.[2] The explicit form of (2.2.52) can be derived from (formally) transporting $\bar{f}_c{}^{ab} x^c$ along $\mathcal{G}$. Let us note, that Π in (2.2.52) is neither left- nor right-invariant, which allows for the fact that $\Pi(e) = 0$. For this reason we will call Π a *homogeneous* Poisson structure in the following. The Poisson bivector (2.2.52) can be constructed also in a coordinate independent manner. Given a Lie bialgebra $\mathfrak{d} = \mathfrak{g} \oplus_{\mathfrak{d}} \mathfrak{g}^\star$ and the adjoint action of a $g \in \mathcal{G}$ on the generators of $\mathfrak{d}$ we can write the homogeneous Poisson bivector as [98]

$$\Pi(g) = C(g) \cdot A^{-1}(g) \tag{2.2.53}$$

$$\text{for} \quad \mathrm{Ad}_g T_A = \begin{pmatrix} g t_a g^{-1} \\ g \bar{t}^a g^{-1} \end{pmatrix} = \begin{pmatrix} A(g) & 0 \\ C(g) & (A^T)^{-1}(g) \end{pmatrix}_A{}^B T_B.$$

From the properties of Ad_g on $\mathfrak{d}$ we can deduce the useful relation

$$\partial_a \Pi^{bc}(g) = \bar{f}_a{}^{bc} + f^{[b}{}_{ad} \Pi^{c]d}(g) \tag{2.2.54}$$

[2] E.g. $\mathcal{O}(x)$ of $\Pi^{k[a} \partial_k \Pi^{bc]}$ vanishes because of the Jacobi identity on $\mathfrak{g}^\star$, all the higher order terms because of Jacobi identity on $\mathfrak{g}^\star$ and 1-cocycle condition (8.3.7).

for $g = \exp(x^a t_a)$.

Another important class of bivectors are *invariant* bivectors. In case we have a 1-coboundary bialgebra associated to $\mathfrak{g}$ and the 0-cocycle $r = r^{ab} t_a \wedge t_b$ on $\mathfrak{g}$, we can define a non-homogeneous bivector

$$\Pi_r^{ab}(g) = r^{ab} - \bar{f}_c{}^{ab} x^c + \ldots , \quad \text{for } g = \exp(x^a t_a) \in \mathcal{G}, \tag{2.2.55}$$

which is simply r transported via left-(right) translation along $\mathcal{G}$ and the same as (2.2.52) plus a constant term.

Generically, Π_r given by (2.2.55) will not be a Poisson structure, but the bialgebra properties on the structure constants f and $\bar{f}$ mean that the Jacobiator will be constant and Ad-invariant. So for example, if r corresponds to a solution of the (modified) classical Yang-Baxter equation, then

$$\Pi_r^{k[a} \partial_k \Pi_r^{bc]}(g) = c^2 \kappa^{am} \kappa^{bn} f^c{}_{mn}, \tag{2.2.56}$$

where c^2 is the coefficient in the (modified) classical Yang-Baxter equation (2.2.47). In case r is a solution of the classical Yang-Baxter equation ($c^2 = 0$), (2.2.55) is the left(right)-invariant non-homogeneous Poisson bivector, introduced by [99].

For such 1-coboundary bialgebras we can reproduce the homogeneous Poisson bivector (2.2.52) from

$$\Pi = r - \Pi_r. \tag{2.2.57}$$

This has been noted already in [100]. But, as commented above, $\Pi_r^{ab}(e) = r^{ab}$ does not necessarily have to be a solution of a Yang-Baxter equation in order for Π to be Poisson.

Construction of Poisson-Lie σ-models from a doubled σ-model. Let us review the basic construction and geometry of Poisson-Lie σ-models. It is the natural generalisation of σ-models like (2.2.35) or (2.2.36).

A first order parent action [101–103] for Poisson-Lie σ-models is a σ-model of fields l taking value in a Drinfel'd double $\mathcal{D}$

$$S = -\frac{1}{2} \int_{\partial B} \mathrm{d}^2\sigma \, \Big(\langle l^{-1}\partial_\sigma l, l^{-1}\partial_\tau l \rangle - \langle l^{-1}\partial_\sigma l, \hat{\mathcal{H}}(l^{-1}\partial_\sigma l) \rangle \Big) + \frac{1}{12} \int_B \langle l^{-1}\mathrm{d}l \overset{\wedge}{,} [l^{-1}\mathrm{d}l \overset{\wedge}{,} l^{-1}\mathrm{d}l] \rangle \,, \tag{2.2.58}$$

where $\langle \, , \rangle$ is the canonical metric on $\mathfrak{d}$. From now on, we set $\alpha' = 1$. The operator $\hat{\mathcal{H}}$ represents a generalised metric

$$\mathcal{H}_{AB} \equiv \mathcal{H}(T_A, T_B) \equiv \langle T_A, \hat{\mathcal{H}}(T_B) \rangle = \begin{pmatrix} G_0 - B_0 G_0^{-1} B_0 & B_0 G_0^{-1} \\ -G_0^{-1} B_0 & G_0^{-1} \end{pmatrix}$$

and is defined by constant symmetric resp. skewsymmetric $d \times d$-matrices G_0 resp. B_0 given in some basis $\{T_A\} = \{t_a, \bar{t}^a\}$ of a Manin triple decomposition of $\mathfrak{d}$. As such, (2.2.58) is the natural generalisation of the toroidal first order action (2.2.18) with a few caveats:

- The polarisation term $\dot{X} \cdot P$ in the abelian case becomes a WZW-model like term.

- The non-abelian nature of $\mathfrak{d}$ means that for some choice of Manin triple decomposition the decomposition of $l = \bar{g}g^{-1} \in \mathcal{D}$ for $\bar{g} \in \bar{\mathcal{G}}$ and $g \in \mathcal{G}$ will not result into a direct decomposition of $l^{-1}\partial_\sigma l$ (which would correspond to (X'^i, P_i) in the abelian case), but instead we have:

$$\begin{aligned} l^{-1}\mathrm{d}l &= \mathrm{Ad}_g \left(-g^{-1}\mathrm{d}g + \bar{g}^{-1}\mathrm{d}\bar{g} \right) \\ &= -(g^{-1}\mathrm{d}g)^a \mathrm{Ad}_g(t_a) + (\bar{g}^{-1}\mathrm{d}\bar{g})_a \mathrm{Ad}_g(\bar{t}^a) \\ &= \left(-(g^{-1}\mathrm{d}g)^a, (\bar{g}^{-1}\mathrm{d}\bar{g})_a \right) \begin{pmatrix} \mathbb{1} & \\ \Pi(g) & \mathbb{1} \end{pmatrix} \begin{pmatrix} A(g) & \\ & (A^T)^{-1}(g) \end{pmatrix} \begin{pmatrix} t^b \\ \bar{t}_b \end{pmatrix} \end{aligned} \tag{2.2.59}$$

  where the homogeneous Poisson structure $\Pi(g)$ arises according to definition (2.2.53).

What we call a *Poisson-Lie σ-model* in this thesis is constructed as follows: Choose a Manin triple $\mathfrak{g} \oplus_\mathfrak{d} \mathfrak{g}^\star$ with a corresponding basis $\{t_a, \bar{t}^a\}$ and groups $\mathcal{G}$, $\bar{\mathcal{G}}$, and take a corresponding decomposition of l as above $l = \bar{g}g^{-1}$. We put this choice of parameterisation of $\mathcal{D}$ into (2.2.58). Then with knowledge of (2.2.59) and the help of the Polyakov-Wiegmann identity for the WZW-term, we integrate out $\bar{g}$ and obtain

$$S = \frac{1}{2} \int \mathrm{d}^2\sigma \, (g^{-1}\partial_+ g)^a \left(\frac{1}{\frac{1}{G_0+B_0} + \Pi(g)} \right)_{ab} (g^{-1}\partial_- g)^b. \tag{2.2.60}$$

This model is a σ-model for $\mathcal{G}$-valued fields g. The bialgebra structure of the original doubled σ-model is encoded in the homogeneous Poisson structure $\Pi(g)$ of the form (2.2.52), and the generalised metric finds itself in $E_0 = G_0 + B_0$.

The models discussed before belong to this class of σ-models for different choices of bialgebras:

- The *toroidal σ-model* (2.2.25) is reproduced from (2.2.60) for $\mathfrak{d}$ being abelian.

- We obtain the *$\mathcal{G}$-isometric σ-model* (2.2.35) for the so-called semi-abelian bialgebra

$$\mathfrak{d} = \mathfrak{g} \oplus_\mathfrak{d} (\mathfrak{u}(1))^d.$$

- The *non-abelian T-dual* of the above (2.2.36) corresponds to

$$\mathfrak{d} = (\mathfrak{u}(1))^d \oplus_\mathfrak{d} \mathfrak{g}.$$

  Due to the abelian structure of the target space, this Poisson structure of the form (2.2.52) is given by $\Pi_{ab}(x) = -f^c{}_{ab} x_c$.

Dirac structures. The key data of the doubled σ-model (2.2.58) is the bialgebra $\mathfrak{d}$ and the generalised metric $\mathcal{H}$ (2.2.16). This data singles out a decomposition of $\mathfrak{d}$ into so-called Dirac structures, orthogonal subspaces w.r.t. to the natural $\mathrm{O}(d,d)$-metric $\langle\, , \rangle$

$$\mathfrak{d} = \mathfrak{d}^+ \perp \mathfrak{d}^- . \tag{2.2.61}$$

Each choice of a non-degenerate $d \times d$-matrix $E_0 = G_0 + B_0$, with a metric G_0 and skewsymmetric B_0, determines such a decomposition:

$$S_a^\pm = t_a \pm E^\pm_{0,ab}\bar{t}^b, \quad \text{with } E_0^+ = E_0^T, E_0^- = E_0 \tag{2.2.62}$$

$$\mathfrak{d}^\pm = \operatorname{span}\left(S_a^\pm\right). \tag{2.2.63}$$

This basis also block-diagonalises the canonical $\mathrm{O}(d,d)$-metric

$$\langle\, , \rangle = |S_a^+\rangle\, G_0^{ab}\, \langle S_b^+| - |S_a^-\rangle\, G_0^{ab}\, \langle S_b^-| \tag{2.2.64}$$

So, the effect of the generalised metric term in (2.2.58) is to 'choose' a decomposition of $\mathfrak{d}$ into Dirac structures, which are the eigenspaces of $\hat{\mathcal{H}}$:

$$\hat{\mathcal{H}}\big|_{\mathfrak{d}^\pm} = \pm\langle\, , \rangle\big|_{\mathfrak{d}^\pm}, \quad \hat{\mathcal{H}} = |S_a^+\rangle\, G_0^{ab}\, \langle S_b^+| + |S_a^-\rangle\, G_0^{ab}\, \langle S_b^-|\,. \tag{2.2.65}$$

A crucial property of the (classical) doubled σ-model (2.2.58) is that the dynamics follows the constraint

$$\langle l^{-1}\partial_\pm l, \mathfrak{d}^\pm\rangle = 0. \tag{2.2.66}$$

This relation highlights the role of the Dirac structures and was the starting point of the investigation of Poisson-Lie T-duality in [98], even before the doubled σ-model was discovered. For a mathematical treatment of Dirac structures and Courant algebroids in the context of Poisson-Lie T-duality, see [104].

The choice of decomposition is of course independent of the basis choice of $\mathfrak{d}$, but the Dirac structure is *not manifestly* realised in the Poisson-Lie σ-model (2.2.60), the explicit form of which will depend on a choice of basis. This is the key point in our analysis of Poisson-Lie T-duality.

Poisson-Lie T-duality. Suppose we have another choice of Manin triple other than $\mathfrak{g} \oplus_{\mathfrak{d}} \mathfrak{g}^\star$ at our hand, the simplest choice of course being $\mathfrak{g}^\star \oplus_{\mathfrak{d}} \mathfrak{g}$. We choose a corresponding parametrisation of $\mathcal{D}$ in (2.2.58) by $l = g\bar{g}^{-1}$ and integrate out g (instead of $\bar{g}$ before). This gives rise to a classically equivalent σ-model

$$S = \frac{1}{2}\int \mathrm{d}^2\sigma (\bar{g}^{-1}\partial_+\bar{g})_a \bar{E}^{ab}(\bar{g})(\bar{g}^{-1}\partial_+\bar{g})_b \tag{2.2.67}$$

with $\bar{E}^{-1}(g) = G_0 + B_0 + \bar{\Pi}$, where $\bar{\Pi}$ is now the homogeneous Poisson structure on $\bar{\mathcal{G}}$, equivalent to the dual Lie algebra structure $\mathfrak{g}$. This is the Poisson-Lie T-dual of (2.2.60) and generalises the $R \leftrightarrow \frac{1}{R}$-behaviour from abelian T-duality by

$$E_0\bar{E}_0^{-1} = \mathbb{1} \quad \text{and also} \quad E(e)\bar{E}(e) = \mathbb{1}.$$

A Poisson-Lie T-duality group will consequently be the space of decompositions $l = \bar{h}h^{-1} \in \mathcal{D}$, where $h \in \mathcal{H}$, $\bar{h} \in \bar{H}$ for some $\mathcal{D} = H \bowtie \bar{H}$. At the Lie algebra level this corresponds to the set of Manin triple decompositions of the bialgebra $\mathfrak{d}$ to $\mathcal{D}$. The task to explore this space is what we set about in section 8. It will also become apparent that the construction, showing that Poisson-Lie T-duality is a canonical transformation, extends to more general setups than the Poisson-Lie σ-models.

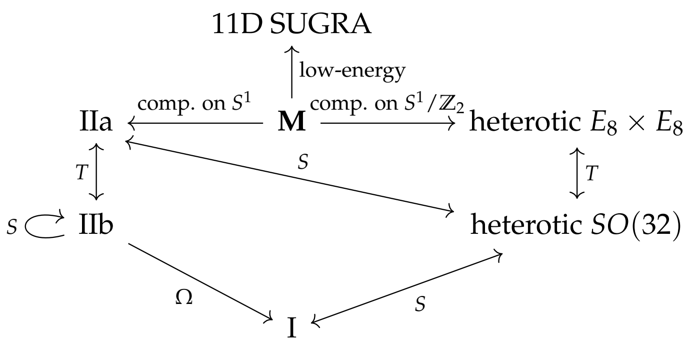
One can find generalisations of Poisson-Lie T-duality to coset spaces, open strings and supersymmetry [105–107], a canonical analysis [18, 20, 108, 109] and studies of the dual models beyond the classical level [102, 110–114] in the literature.

2.3 String dualities and M-theory

The consequence of S- and T-dualities connecting all the known superstring theories is that they should all describe the same physics. The evidence for an underlying theory that connects and unites these string theories was brought together in a collective effort in the mid 1990s, among many others in [115–117]. That theory has been called M-theory.

Eleven-dimensional supergravity. Besides the supergravity theories that arise from the superstrings theories as low-energy effective theories, there is a unique maximal supergravity in $D = 11$, first considered in [118]. The bosonic field content is very simple, consisting only of a metric G and a three-form gauge field A. From this, we can easily motivate that there are two natural objects: membranes, or M2-branes, – charged electrically under A – and the M5-branes – charged magnetically under A. These branes are BPS-states in the theory. The supposed UV-completion of eleven-dimensional supergravity is M-theory [119].

The web of dualities. The following scheme summarises the web of S- and T-dualities of the five superstring theories and, also, how these and eleven-dimensional supergravity are related to M-theory.



Ω denotes the orientifold projection.

A complete formulation of M-theory does not exist. A natural way to understand it is as a world-volume theory of the M2- and M5-branes [119, 120]. This is the route that will be taken in the rest of this thesis. The problem with this is that the quantum theory can be approached only in the perturbatively accessible string theory limits.

Besides the superstring theories, further insights into the nature of M-theory have been achieved through the matrix models in the BFSS matrix theory [121] and the AdS/CFT-correspondence, here in particular through ABJM superconformal field theory which is dual to M-theory on $AdS_4 \times S^7$ [122].

Membrane action and double dimensional reduction. The relation of the type II string theories and M-theory will be of special interest in the following.

Let us write down the bosonic part of the M2-brane action [119]

$$S = \frac{1}{2} \int d^3\sigma \left(\sqrt{-\hat{\gamma}} \hat{\gamma}^{\hat{\alpha}\hat{\beta}} \partial_{\hat{\alpha}} x^{\hat{\mu}} \partial_{\hat{\beta}} x^{\hat{\nu}} G_{\hat{\mu}\hat{\nu}} + \frac{1}{3} \epsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}} \partial_{\hat{\alpha}} x^{\hat{\mu}} \partial_{\hat{\beta}} x^{\hat{\nu}} \partial_{\hat{\gamma}} x^{\hat{\rho}} A_{\hat{\mu}\hat{\nu}\hat{\rho}} - \sqrt{-\hat{\gamma}} \right), \quad (2.3.1)$$

where $x^{\hat{\mu}}$ are coordinates on the eleven-dimensional target space and $\gamma^{\hat{\mu}\hat{\nu}}$ is the metric on the three-dimensional world-volume.

If one performs the *double dimensional reduction*, a dimensional reduction both of the world-volume and of the target space,

$$\sigma^{\hat{\alpha}} = (\sigma^{\alpha}, \rho) \qquad \hat{x}^{\hat{\mu}} = (x^{\mu}, \rho), \quad (2.3.2)$$

one arrives straightforwardly at the Polyakov action (1.1.5). In this picture, the target space is assumed to be $\mathcal{M} \times S^1$ and the membrane is wrapped around the circle. The same can be demonstrated for the full supermembrane action that reduces to the type IIa superstring action upon this double dimensional reduction [123]. In case the membrane is not wrapped around the circle, the membrane becomes the $D2$-brane of the type IIa theory. Similarly, the three-form gauge field A of eleven-dimensional supergravity decomposes into an NSNS part, $B_{\mu\nu} = A_{\mu\nu 10}$, and an RR-flux part, $C_{3,\mu\nu\rho} = A_{\mu\nu\rho}$ due to dimensional reduction to IIa supergravity.

The connection between M-theory and the heterotic string theories, on the other hand, is related to M5-brane dynamics and has first been discussed in [120].

U-duality. Toroidal compactification of eleven-dimensional supergravities on T^d possess a global symmetry group containing

$$SL(d;\mathbb{R}) \bowtie O(d-1, d-1), \quad (2.3.3)$$

describing Lorentz symmetries unbroken by the compactification on T^d and the T-duality group in one dimension less [124]. The full duality groups up to $d = 8$ are [118]:

d	U-duality group $E_{d(d)}$	H_d	$\dim(\mathcal{R}_1)$	$\dim(\mathcal{R}_2)$
1	$\mathbb{R}^+$	1	1	1
2	$SL(2;\mathbb{R}) \times \mathbb{R}^+$	$SO(2;\mathbb{R})$	3	2
3	$SL(3;\mathbb{R}) \times SL(2;\mathbb{R})$	$SO(3;\mathbb{R}) \times SO(2;\mathbb{R})$	6	3
4	$SL(5;\mathbb{R})$	$SO(5;\mathbb{R})$	10	5
5	$S0(5,5;\mathbb{R})$	$SO(5;\mathbb{R}) \times SO(5;\mathbb{R})$	16	10
6	$E_{6(6)}$	$USp(8)$	27	27
7	$E_{7(7)}$	$SU(8)$	56	133
8	$E_{8(8)}$	$SO(16)$	248	3875

Here, $E_{d(d)}$ denotes a split real form of E_d, in particular such that the T-duality group $O(d-1, d-1)$ is a subset of it. H_d is the maximal compact subgroup of $E_{d(d)}$. $E_{d(d)}/H_d$ parameterises the moduli space (the massless internal degrees of freedom) of eleven-dimensional supergravity compactified on T^d. This is similar to $\frac{O(d,d)}{O(d) \times O(d)}$, which parametrises the space of generalised metrics and, in this way, describes the moduli space of the NSNS sector in the gravity theories associated to strings. $\mathcal{R}_1$ and $\mathcal{R}_2$ are two representations of $E_{d(d)}$. The latter is the fundamental representation, whereas $\mathcal{R}_1$ is the *generalised vector* representation, also called *charge* or *flux* representation. The relevant object in the $\mathcal{R}_1$-representation is the collection of momentum and winding/'wrapping' excitations, similar to the collection of KK-momenta and winding number of a string which form an $O(d, d)$-vector. Let us demonstrate this for the $SL(5)$-theory, describing compactifications of M-theory on T^4:

$$\dim(\mathcal{R}_1) = \sharp\text{KK-momenta} + \sharp\text{wrapping configurations of an M2} = 4 + \frac{4 \cdot 3}{2} = 10.$$

A consequence of this is that a 'factorised membrane duality' cannot work in the same way as in the string case via an exchange of momentum and wrapping excitations [125]. Apart from the case $d = 3$, the number of momentum and wrapping modes do not match [126].

The full (quantum) superstring and M-theory breaks the duality group to $E_{d(d)}(\mathbb{Z})$, as expected from the arguments for T-duality and S-duality [124]. Above $d = 8$, the U-duality groups become infinite-dimensional [127–129]. In the context of matrix theory, U-duality has been explored in [130]. Exceptional field theory, a U-duality covariant setup, is explained for the case $E_{4(4)} = SL(5)$ in section 3.4.

Chapter 3

Generalised Geometry

As long as the background is globally geometric – meaning only diffeomorphisms and B-field gauge transformations are necessary for gluing coordinate patches – the metric and the B-field are globally well-defined and seem to be an appropriate description of the background. But not all backgrounds in string theory can be described as such. So called non-geometric backgrounds have been shown to arise naturally as T-duals of geometric backgrounds [131]. The ones we consider here can be understood as T-folds [13, 132, 133], meaning that we allow for patching with T-duality transformations as well. They are expected to make up a big part of the landscape of string theory [81,134–137], this includes not only duals of geometric backgrounds but also genuinely non-geometric backgrounds. These backgrounds can be described in terms of generalised geometry [11,12,138] or the *generalised fluxes*. These fluxes arise as parameters in gauged supergravities [139,140], are the basis of a formulation of double field theory [67,81,141–148] and have been shown to be related to the non-commutative and non-associative interpretations of these backgrounds [149–156].

3.1 Introduction

T-duality showed that distinct geometries can give rise to equivalent world-sheet theories. From the string theory point of view, KK-momenta p_μ and winding modes w^μ appeared on the same footing, connected by an $\mathrm{O}(d,d)$-action. But, from a point particle perspective from which most geometric notions are motivated, the two have very different meanings. This indicates that strings might require a different understanding of geometry.

A geometric object that incorporates such an $\mathrm{O}(d,d)$-action is the *generalised tangent bundle*

$$(T \oplus T^\star)M = TM \oplus T^\star M$$

which has a natural $\mathrm{O}(d,d)$ structure group, coming from the natural pairing of vectors and 1-forms. From now on, we will restrict M to be the d-dimensional internal space with coordinates x^μ. For $\phi \in (T \oplus T^\star)M$ we use indices $\phi^M = (\phi^\mu, \phi_\mu)$. From a more geometrical point of view, the elements of the generalised tangent bundle

$$v + \xi \in (T \oplus T^\star)M$$

for vector fields v, generating diffeomorphisms, and 1-form fields ξ, generating B-field gauge transformations $B \to B + \mathrm{d}\xi$, are called *generalised vector fields*. They generate the so-called *geometric subgroup* of the $\mathrm{O}(d,d)$-group. The action of such a generalised vector field $\phi = v + \xi$ is often called *generalised diffeomorphism*.

Potential algebraic structures on the generalised tangent bundle will be the topic of the next section. The geometry of this generalised tangent bundle is called *generalised geometry*, including different kinds of generalisations of notions of Riemannian geometry. A key difference is that we have two types of metrics on this generalised tangent bundle:

- the *constant* $\mathrm{O}(d,d)$-metric η
- the (non-constant) generalised metric $\mathcal{H}(G,B)$ (2.2.16)

  It defines the splitting of the generalised tangent bundle into a pair of (maximally) Lagrangian[1] subbundles L and $\bar{L}$: $(T \oplus T^\star)M = L \oplus \bar{L}$, in the way that $\mathcal{H}|_L = G$. Alternatively, the generalised metric defines the decomposition into *Dirac structure*, as already above in the discussion of Poisson-Lie T-duality.

Doubled geometry goes one step further than generalised geometry. It assumes that the generalised tangent bundle is the tangent bundle of a doubled manifold $\mathbb{M}$: $TM \oplus T^*M\ (= T\mathbb{M})$. The coordinates on this doubled manifold are $X^M = (x^\mu, \tilde{x}_\mu)$. The *dual coordinates* $\tilde{x}_\mu$ are the ones that appear in the T-dual actions. In this approach, the original target space coordinates x and their T-duals $\tilde{x}$ are treated on the same footing, such that duality rotations become manifest symmetries. In the world-sheet theory the dual fields $\tilde{x}_\mu(\sigma)$ are given by $p_\mu(\sigma) = \partial\tilde{x}_\mu(\sigma)$, in analogy to the winding which is related to ∂x^μ by $w^\mu = \oint \mathrm{d}\sigma\, \partial x^\mu$. One typically imposes two constraints on the dependence of functions of the doubled coordinates X^M:

- *strong constraint* or *section condition*: $(\partial_\mu f)(\partial^\mu g) = 0$ for all functions f and g

  Satisfying this condition is equivalent to choosing a section, a d-dimensional subspace. A section would be for example the original manifold M before doubling, parametrised by the coordinates x^μ.

  The space of inequivalent sections is again parameterised by $\frac{\mathrm{O}(d,d)}{\mathrm{O}(d)\times\mathrm{O}(d)}$. T-duality simply comes down to taking a different choice of section.

  There is no fundamental motivation for the strong constraint apart from the fact that it allows for a straightforward reduction to a d-dimensional manifold. It will be shown later that violations of strong constraint are possible – at cost of the associativity of the Poisson structure, see section 7.2.

[1] *Lagrangian* means that $\eta_{MN}\phi^M\psi^N = 0$ for all $\phi, \psi \in L$ – *maximally Lagrangian*, $\eta_{MN}\phi^M\psi^N \neq 0$ for $\phi \in L$ and $\psi \notin L$.

- *weak constraint*: $\partial_\mu \tilde{\partial}^\mu f(x, \tilde{x}) = \frac{1}{2}\partial_M \partial^M f(X) = 0$ for all functions f

  This constraint is weaker than the strong constraint, as it allows for a different choice of section for every function on the doubled space.

  The origin of the weak constraint lies in the level matching condition, coming from the invariance under σ-reparameterisations. An instance of this will be seen later in section 7.2, where it is shown that violations of the weak constraint of the background (i.e. metric and B-field) lead to violation of the Virasoro algebra.

With the help of doubled geometry, one can rephrase the target space geometry in a manifestly $O(d,d)$-covariant way. In particular, one can write down the potential low-energy effective action for the target space dynamics, such that $O(d,d)$ is a global symmetry. If one solves the section condition it reduces to the standard low-energy effective action (1.2.8). This is the approach of *double field theory*.

In this chapter, we collect and review well-known material about generalised geometry and generalised fluxes. Further details can be found in the standard reviews of double field theory [81, 144–147], generalised geometry [12, 138] and the generalised flux formulation [148].

3.2 Lie and Courant algebroids

The generalised tangent bundle is an example of an *algebroid*. In non-technical terms, an algebroid over a manifold M is a collection of algebras for each point of M. Or in other words, an algebra where the structure constants are functions on M.

In this section we collect some well-known facts about the algebroid structures relevant to us [11, 12, 81, 142, 143, 157–163].

Lie algebroid. A vector bundle $E \to M$ over a manifold M with a *Lie bracket* $[\ ,\ ]_L$, i.e. skew-symmetric and satisfying the Jacobi identity, on the space of sections $\Gamma(E)$ and an anchor, a linear map $\rho : E \to TM$, is called a Lie algebroid (over M), iff $[\ ,\ ]_L$ together with the anchor ρ satisfies the Leibniz rule

$$[\phi_1, f\phi_2]_L = (\rho(\phi_1)f)\,\phi_2 + f[\phi_1,\phi_2]_L, \qquad \text{for } \phi_1,\phi_2 \in \Gamma(E),\ f \in C^\infty(M).$$

$[\ ,\ ]$ is the Lie bracket on TM and the fact that ρ is a homomorphism of Lie brackets

$$\rho([\phi_1,\phi_2]_L) = [\rho(\phi_1),\rho(\phi_2)]\,,$$

follows from the Leibniz rule.

Courant algebroid. A *Courant algebroid* over a manifold M is a vector bundle $E \to M$, together with a bracket $[\ ,\ ]_D$ on $\Gamma(E)$, a fibre-wise non-degenerate symmetric bilinear form $\langle\ ,\ \rangle_E$ and an anchor $\rho:\ E \to TM$, satisfying the following axioms:

$$\begin{aligned}
[\phi_1,[\phi_2,\phi_3]_D]_D &= [[\phi_1,\phi_2]_D,\phi_3]_D + [\phi_2,[\phi_1,\phi_3]_D]_D \\
[\phi_1, f\phi_2]_D &= (\rho(\phi_1)f)\,\phi_2 + f[\phi_1,\phi_2]_D \\
[\phi,\phi]_D &= \frac{1}{2}\mathcal{D}\langle\phi,\phi\rangle \\
\rho(\phi_1)\langle\phi_2,\phi_3\rangle &= \langle[\phi_1,\phi_2]_D,\phi_3\rangle + \langle\phi_2,[\phi_1,\phi_3]_D\rangle
\end{aligned}$$

for $\phi_i \in \Gamma(E)$, $f \in C^\infty(M)$ and the derivation $\mathcal{D}:\ C^\infty(M) \to E$:

$$\langle\mathcal{D}f,\phi\rangle = \rho(\phi)f.$$

In the following, we call $[\ ,\ ]_D$ *Dorfman bracket*. It is also called generalised Lie derivative in the literature. From the first two axioms follows that ρ is a homomorphism of brackets. The third axiom implies that $[\ ,\ ]_D$ is not skew-symmetric, the first line describes a certain Jacobi identity for this non skew-symmetric bracket.

Skew-symmetric realisation. A Courant algebroid as defined above possesses an equivalent representation via a skew-symmetric bracket

$$[\phi_1,\phi_2]_C = \frac{1}{2}\left([\phi_1,\phi_2]_D - [\phi_2,\phi_1]_D\right) = [\phi_1,\phi_2]_D - \frac{1}{2}\mathcal{D}\langle\phi_1,\phi_2\rangle,$$

which we call *Courant bracket*. It satisfies modified axioms – in particular, the Jacobi identity only holds up to a total derivation by $\mathcal{D}$

$$[\phi_1,[\phi_2,\phi_3]_C]_C + \text{ c.p.} = \mathcal{D}\left(\frac{1}{3}\langle[\phi_1,\phi_2]_C,\phi_3\rangle + \text{ c.p.}\right). \tag{3.2.1}$$

The standard Courant algebroid on $TM \oplus T^\star M$. The Courant bracket for sections $\phi = v + \xi \in TM \oplus T^\star M$ is given by

$$[\phi_1,\phi_2]_C = [v_1,v_2] + \mathcal{L}_{v_1}\xi_2 - \mathcal{L}_{v_2}\xi_1 - \frac{1}{2}\mathrm{d}\left(\xi_2(v_1) - \xi_1(v_2)\right). \tag{3.2.2}$$

In the following we use the notation $\phi = \phi^I\partial_I$ with $\partial_I = (\partial_i, \mathrm{d}x^i)$ where the action of $\mathrm{d}x^i$ on functions is $\mathrm{d}x^i.f = 0$. Then the coordinate expression for Courant resp. Dorfman bracket is:

$$[\phi_1,\phi_2]_C^I = \phi^J_{[1}\partial_J\phi^I_{2]} - \frac{1}{2}\eta_{JK}\phi^J_{[1}\partial^I\phi^K_{2]}, \qquad [\phi_1,\phi_2]_D^I = \phi^J_{[1}\partial_J\phi^I_{2]} - \eta_{JK}\phi^J_1\partial^I\phi^K_2, \tag{3.2.3}$$

where η is the $\mathrm{O}(d,d)$ metric, which raises indices $I, J = 1, ..., 2d$. The anchor is simply projection to TM: $v + \xi \mapsto v$.

One motivation for the Courant bracket from the point of view of the study of T-dualities is, that it possesses an invariance under global $\mathrm{O}(d,d)$-transformations (manifest through the index structure in (3.2.3)) and also under the geometric subgroup of local $\mathrm{O}(d,d)$-transformations, namely diffeomorphisms and B-field gauge transformations via $\partial_i \to \partial_i + B_{ij}(x)\mathrm{d}x^j$ with $\mathrm{d}B = 0$.

3.3 Non-geometry and generalised fluxes

So far, we have parametrised the background in generalised geometry in terms of the generalised metric (2.2.16). For the most part of this thesis, we will use a different characterisation. As for metrics in Riemannian (or Lorentzian) geometry, one can write the generalised metrics in terms of generalised vielbeins.

Generalised vielbeins. A *generalised vielbein* or *frame* $E_A{}^M(x)$ is defined to be any (local) $\mathrm{O}(d,d)$-transformation in the component connected to the identity that diagonalises and trivialises the generalised metric $\mathcal{H}$, i.e.

$$E_A{}^M E_B{}^J \eta_{MN} = \eta_{AB} \quad \text{and} \quad E_A{}^M E_B{}^N \mathcal{H}_{MN} = \gamma_{AB} := \begin{pmatrix} \gamma_{ab} & 0 \\ 0 & \gamma^{ab} \end{pmatrix}, \tag{3.3.1}$$

where γ is a flat metric in the signature of the target space and is used to raise and lower indices $a, b, ... = 1, ..., d$. Indices $A, B, ... = 1, ..., 2d$ denote the 'flat' indices and are raised and lowered by η_{AB}. Unless stated otherwise, we will assume that the generalised vielbeins are (local) functions on the original target space with coordinates x^μ. With this assumption we restrict to *locally geometric* backgrounds, but below and in section 7.2 we will also discuss the generalisation to *locally non-geometric* backgrounds.

Every generalised vielbein can be generated by successively performing

$$B\text{-shifts: } E^{(B)} = \begin{pmatrix} \mathbb{1} & B \\ 0 & \mathbb{1} \end{pmatrix}, \quad GL\text{-transformations } E^{(e)} = \begin{pmatrix} e & 0 \\ 0 & (e^{-1})^T \end{pmatrix} \tag{3.3.2}$$

$$\beta\text{-shifts: } E^{(\beta)} = \begin{pmatrix} \mathbb{1} & 0 \\ \beta & \mathbb{1} \end{pmatrix}, \quad \text{factorised dualities: } E^{(T_\mu)} = \begin{pmatrix} \mathbb{1} - \delta_\mu & \delta_\mu \\ \delta_\mu & \mathbb{1} - \delta_\mu \end{pmatrix}$$

for skewsymmetric $d \times d$-matrices B and β, an invertible matrix e (a d-dimensional vielbein) and $(\delta_\mu)_{\nu\kappa} = \delta_{\mu\nu}\delta_{\mu\kappa}$.

Weitzenböck connection and generalised fluxes. The Weitzenböck connection of such a generalised flux frame is defined by

$$\mathbf{\Omega}_{C,AB} = \partial_C E_A{}^M E_{BM} \quad \text{with} \quad \partial_A := E_A{}^M \partial_M, \tag{3.3.3}$$

fulfilling $\mathbf{\Omega}^C{}_{AB} = -\mathbf{\Omega}^C{}_{BA}$ due to (3.3.1). $\partial_M = (\partial_\mu, \tilde{\partial}^\mu)$, where $\tilde{\partial}^\mu$ denotes the derivative w.r.t. to dual coordinates $\tilde{x}_\mu$. Assume that we work on a section described by the coordinates x^μ, such that $\tilde{\partial}^\mu$ vanishes.

In fact, only the totally skewsymmetric combination[2] will be relevant for us: the *generalised fluxes*:

$$\mathbf{F}_{ABC} = \mathbf{\Omega}_{[C,AB]} = \left(\partial_{[A} E_B{}^I\right) E_{C]I}. \tag{3.3.4}$$

[2]The following conventions are used:

$$v_{[a}w_{b]} = v_a w_b - v_b w_a, \qquad u_{[a}v_b w_{c]} = u_a v_b w_c + \textit{cyclic} \text{ perm.}$$

$$u_{[a}v_b w_c z_{d]} = u_a v_b w_c z_d + (-1)^{\text{sign}} \times \textit{all} \text{ permutations}$$

It includes the four fluxes – $\mathbf{H}$, $\mathbf{f}$, $\mathbf{Q}$, $\mathbf{R}$ – for different decomposition of the $\mathrm{O}(d,d)$-indices

$$\begin{aligned} \mathbf{H}_{abc} &\equiv \mathbf{F}_{abc}, \qquad & \mathbf{f}^{c}{}_{ab} &\equiv \mathbf{F}^{c}{}_{ab} = \mathbf{F}_{b}{}^{c}{}_{a} = \mathbf{F}_{ab}{}^{c} \\ \mathbf{R}^{abc} &\equiv \mathbf{F}^{abc}, \qquad & \mathbf{Q}_{c}{}^{ab} &\equiv \mathbf{F}_{c}{}^{ab} = \mathbf{F}^{b}{}_{c}{}^{a} = \mathbf{F}^{ab}{}_{c} \end{aligned}$$

In a *generalised flux frame* (3.3.1) all the information about the background is stored inside the generalised fluxes, instead of the generalised metric. The generalised metric will be trivial in that frame.

Bianchi identities. Generalised fluxes, given as above in terms of a generalised vielbein, cannot be chosen arbitrarily but have to fulfil the (dynamical) Bianchi identity [148, 164–166]

$$\partial_{[A}\mathbf{F}_{BCD]} - \frac{3}{4}\mathbf{F}^{E}{}_{[AB}\mathbf{F}_{CD]E} = 0, \tag{3.3.5}$$

or in the decomposition into the d-dimensional fluxes

$$\begin{aligned} 0 &= \partial_{[a}\mathbf{H}_{bcd]} - \frac{3}{2}\mathbf{H}_{k[ab}\mathbf{f}^{k}{}_{cd]} = (\mathrm{d}\mathbf{H})_{abcd} \\ 0 &= \partial^{a}\mathbf{H}_{bcd} + \partial_{[b}\mathbf{f}^{a}{}_{cd]} - \mathbf{f}^{a}{}_{k[a}\mathbf{f}^{k}{}_{bc]} - \mathbf{H}_{k[bc}\mathbf{Q}_{d]}{}^{ak} \\ 0 &= \partial^{[a}\mathbf{f}^{b]}{}_{cd} + \partial_{[c}\mathbf{Q}_{d]}{}^{ab} - \mathbf{f}^{k}{}_{ab}\mathbf{Q}_{k}{}^{cd} + \mathbf{f}^{[c}{}_{m[a}\mathbf{Q}_{d]}{}^{b]k} - \mathbf{H}_{abk}\mathbf{R}^{kcd} \\ 0 &= \partial_{a}\mathbf{R}^{bcd} + \partial^{[b}\mathbf{Q}_{a}{}^{cd]} - \mathbf{Q}_{a}{}^{k[a}\mathbf{Q}_{k}{}^{bc]} - \mathbf{R}^{k[bc}\mathbf{f}^{d]}{}_{ak} \\ 0 &= \partial^{[a}\mathbf{R}^{bcd]} + \frac{3}{2}\mathbf{R}^{k[ab}\mathbf{Q}_{k}{}^{cd]}. \end{aligned} \tag{3.3.6}$$

If the fluxes violate this condition, they cannot be written in terms of a generalised vielbein via (3.3.4). In the following, we call the corresponding backgrounds *magnetically charged*.

The locally geometric T-duality chain and the non-geometric fluxes. The starting point in the T-duality chain is the flat 3-torus with h units of $\mathbf{H}$-flux, i.e.

$$\mathbf{H} = h\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3. \tag{3.3.7}$$

A choice of B-field for this $\mathbf{H}$-flux is $B = hx^3\,\mathrm{d}x^1 \wedge \mathrm{d}x^2$, such that the two commuting isometries of the background are manifest. After a T-duality along the isometry x^1 the Buscher rules [82] produce a pure metric background. This background turns out to be parallelisable, e.g. there is a globally defined frame field $e_a{}^{\mu}$. The only non-vanishing component of the generalised flux (3.3.4) is

$$\mathbf{f}^{1}{}_{23} = h \quad \text{with} \quad \mathbf{f}^{c}{}_{ab} = e_{\nu}{}^{c}e_{[a}{}^{\mu}\partial_{\mu}e_{b]}{}^{\nu}. \tag{3.3.8}$$

The interpretation of the locally geometric pure $\mathbf{f}$-flux is that it is the totally skewsymmetric combination of the spin connection of a d-dimensional vielbein.

Performing a second T-duality along x_2, we arrive at the background

$$G = \frac{1}{1+h(x^3)^2}\left((\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2\right) + (\mathrm{d}x^3)^2, \qquad (3.3.9)$$
$$H = -\frac{h}{(1+h(x^3)^2)^2}\left(1-h(x^3)^2\right)\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3$$

with identifications $x^i \sim x^i + 1$. At $x^3 + 1 \sim x^3$ it is not possible to patch geometrically. Instead, we can describe this background by the generalised vielbein

$$E_{(Q)} = \begin{pmatrix} \mathbb{1} & 0 \\ \beta & \mathbb{1} \end{pmatrix}, \; \beta = \begin{pmatrix} 0 & hx^3 & \\ -hx^3 & 0 & \\ & & 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{Q}_3{}^{12} = h. \qquad (3.3.10)$$

So, a constant β-shift by $h\,\mathrm{d}x^1\wedge\mathrm{d}x^2$ can be used to patch at $x^3+1\sim x^3$. In other words, a 3-torus solely together with a constant $\mathbf{Q}$-flux is characterised by a non-trivial monodromy of β. The background has the interpretation of a non-commutative spacetime with $\{x^1, x^2\} \sim hw^3$, where w^3 is the winding around the x^3-cycle. More details on the non-geometric interpretation of these backgrounds can be found in [81, 149–156].

Local non-geometry. Let us summarise the above in the scheme [131]

$$\mathbf{H}_{123} \overset{T_1}{\longleftrightarrow} \mathbf{f}_1{}^{23} \overset{T_2}{\longleftrightarrow} \mathbf{Q}_1{}^{23} \overset{T_3}{\longleftrightarrow} \mathbf{R}^{123}. \qquad (3.3.11)$$

The first two steps can be realised via standard abelian T-duality, whereas the last step cannot because the background (either described by a generalised metric or generalised vielbeine) does not possess a corresponding isometry for x^3. In order to allow for such T-dualities along non-isometric direction, we need to allow for the dependence on dual coordinates $\tilde{x}_\mu$. As, discussed the dependence of functions of the $2d$ coordinates $X^M = (x^\mu, \tilde{x}_\mu)$ is typically restricted by the strong or weak constraint. In the following, we implicitly work in some section, i.e. a solution to the strong constraint, unless stated otherwise. T-duality can be simply understood as transforming between different sections on this doubled space, e.g. the exchange $x^\mu \leftrightarrow \tilde{x}_\mu$.

Given a generalised flux configuration and a section, we call a background *locally* non-geometric, if one can *only* find a generalised vielbein reproducing these generalised fluxes that depends on dual coordinates as well.

One example for such a background is the missing background in (3.3.11), the pure $\mathbf{R}$-flux background. A generalised vielbein reproducing a pure $\mathbf{R}$-flux with $\mathbf{R}^{123} = h$ is

$$E_{(R)} = \begin{pmatrix} \mathbb{1} & 0 \\ \beta & \mathbb{1} \end{pmatrix}, \; \beta = \begin{pmatrix} 0 & h\tilde{x}^3 & \\ -h\tilde{x}^3 & 0 & \\ & & 0 \end{pmatrix}, \qquad (3.3.12)$$

which is obviously the formal T-dual, via $x^3 \leftrightarrow \tilde{x}_3$, of the pure $\mathbf{Q}$-flux background. It turns out that it is impossible to find a vielbein that only depends on the original coordinates x for a pure **R**-flux background. As a consequence, it turns out that one cannot write down a σ-model Lagrangian in the usual fashion, as the metric and the B-field do not depend on the original coordinates of the section alone.

This slightly unhandy definition of local non-geometry is employed because there are also generalised vielbeins, that depend on dual coordinates and describe *locally geometric* backgrounds nonetheless. This is the case for example for the pure **f**- or **Q**-flux backgrounds with possible alternative choices of generalised vielbeins:

$$
\begin{aligned}
\tilde{E}_{(f)} &= \begin{pmatrix} \mathbb{1} & B \\ 0 & \mathbb{1} \end{pmatrix}, \; B = \begin{pmatrix} 0 & h\tilde{x}^3 & \\ -h\tilde{x}^3 & 0 & \\ & & 0 \end{pmatrix}, \\
\tilde{E}_{(Q)} &= \begin{pmatrix} e & 0 \\ 0 & (e^T)^{-1} \end{pmatrix}, \; e = \begin{pmatrix} 1 & 0 & 0 \\ -h\tilde{x}^3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

Apart from section 7.2 we will only work in locally geometric backgrounds.

Examples and Lagrangians. In this paragraph, we will include some explicit examples of such generalised flux frames. Besides setting conventions for later discussion, it should be emphasised here that in our definition as components of $\mathbf{F}_{ABC}$ the physical interpretation of the fluxes **H**, **f**, **Q**, **R** is *frame dependent*. By this we mean the **Q**-flux might not correspond to a monodromy of β or closed string non-commutativity or the **R**-flux not to local non-geometry, in a generic generalised frame.

- *Geometric frame*. This is the standard frame of a Lagrangian σ-model given by a metric and a B-field. Only the **H**-flux and the geometric **f** are non-vanishing

$$
\begin{aligned}
\mathbf{H}_{abc} &= \partial_{[a} B_{bc]} + f^d{}_{[ab} B_{c]d} \\
\mathbf{f}^c{}_{ab} &= f^c{}_{ab} = e_\nu{}^c e_{[a}{}^\mu \partial_\mu e_{b]}{}^\nu \\
\mathbf{Q}_c{}^{ab} &= 0 = \mathbf{R}^{abc}
\end{aligned}
\tag{3.3.13}
$$

  The corresponding vielbein is a composition of a d-dimensional tetrad rotation and a B-shift: $E = E^{(B)} E^{(e)}$.

- *(Globally) non-geometric frame* (resp. open string variables). σ-models like

$$
S = -\frac{1}{2} \int \mathrm{d}^2\sigma \left(\frac{1}{\gamma - \Pi(x)} \right)_{ab} e_\mu{}^a e_\nu{}^b \partial_+ x^\mu \partial_- x^\nu . \tag{3.3.14}
$$

  are described by the vielbein $E = E_\Pi^{(\beta)} E^{(e)}$, with $E_\Pi^{(\beta)}$ denoting a β-shift by a bivector Π. This results in the generalised fluxes

$$
\begin{aligned}
\mathbf{H}_{abc} &= 0 \\
\mathbf{f}^c{}_{ab} &= f^c{}_{ab} \\
\mathbf{Q}_c{}^{ab} &= Q_c{}^{ab} = \partial_c \Pi^{ab} + f^{[a}{}_{dc} \Pi^{b]d} \\
\mathbf{R}^{abc} &= R^{abc} = \Pi^{d[a} \partial_d \Pi^{bc]} + f^{[a}{}_{de} \Pi^{b|d} \Pi^{|c]e}.
\end{aligned}
\tag{3.3.15}
$$

This is an important class of backgrounds as this parameterisation is relevant for open strings in NSNS-backgrounds. Also non-abelian T-duals, Poisson-Lie σ-models are of this form.

- The *e-B-Π-frame*. The next logical step is to introduce a frame in which all the fluxes are non-vanishing. The nearly exclusively used choice in the literature is the generalised flux frame for the σ-model

$$S = -\frac{1}{2}\int d^2\sigma \left(\frac{1}{\frac{1}{\gamma - B(x)} - \Pi(x)}\right)_{ab} e_\mu{}^a e_\nu{}^b \partial_+ x^\mu \partial_- x^\nu. \tag{3.3.16}$$

The corresponding generalised vielbein is of the type $E = E^{(\beta)}_{\Pi} E^{(B)} E^{(e)}$ and the resulting generalised fluxes are

$$\begin{aligned} \mathbf{H}_{abc} &= \partial_{[a}B_{bc]} + f^d{}_{[ab}B_{c]d} \\ \mathbf{f}^c{}_{ab} &= f^c{}_{ab} + \mathbf{H}_{abd}\Pi^{de} \\ \mathbf{Q}_c{}^{ab} &= Q_c{}^{ab} + \mathbf{H}_{cde}\Pi^{ad}\Pi^{be} = \partial_c\Pi^{ab} + f^{[a}{}_{dc}\Pi^{b]d} + \mathbf{H}_{cde}\Pi^{ad}\Pi^{ae} \\ \mathbf{R}^{abc} &= R^{abc} + \mathbf{H}_{def}\Pi^{ad}\Pi^{be}\Pi^{cf} = \Pi^{d[a}\partial_d\Pi^{bc]} + f^{[a}{}_{de}\Pi^{b|d}\Pi^{|c]e} + \mathbf{H}_{cde}\Pi^{ad}\Pi^{be}\Pi^{ef}. \end{aligned} \tag{3.3.17}$$

This has been derived several times in the literature [160, 166, 167].

- The *e-Π-B-frame*. The previous choice was not the only possible one. For example $E = E^{(B)}E^{(\Pi)}E^{(e)}$ is a valid parameterisation for which generically all the components of $\mathbf{F}_{ABC}$ might be non-vanishing. Here the generalised fluxes are

$$\begin{aligned} \mathbf{H}_{abc} &= \partial_{[a}B_{bc]} + f^d{}_{[ab}B_{c]d} + [B,B]^{K.S.}_{abc} + B_{[\underline{a}d}B_{\underline{b}e}Q_{\underline{c}]}{}^{de} + B_{ad}B_{be}B_{ce}\mathbf{R}^{def} \\ \mathbf{f}^c{}_{ab} &= f^c{}_{ab} + Q_{[a}{}^{dc}B_{b]c} + \Pi^{cd}\partial_d B_{ab} + B_{ab}B_{be}\mathbf{R}^{abc} \\ \mathbf{Q}_c{}^{ab} &= Q_c{}^{ab} + \mathbf{R}^{abd}B^{ac} = \partial_c\Pi^{ab} + f^{[a}{}_{dc}\Pi^{b]d} + \mathbf{R}^{abd}B^{ac} \\ \mathbf{R}^{abc} &= \Pi^{d[a}\partial_d\Pi^{bc]} + f^{[a}{}_{de}\Pi^{b|d}\Pi^{|c]e}. \end{aligned} \tag{3.3.18}$$

We recognise the (dual) Koszul derivative $\partial^c_\Pi = \Pi^{cd}\partial_d$, which is used to define the Koszul-Schouten bracket $[\ ,\ ]^{K.S}$ of forms analogously to the usual Schouten bracket of multivector fields. This vielbein corresponds to the σ-model

$$S = -\frac{1}{2}\int d^2\sigma \left(\frac{1}{\gamma^{-1} - \Pi(x)} - B(x)\right)_{ab} e_\mu{}^a e_\nu{}^b \partial_+ x^\mu \partial_- x^\nu. \tag{3.3.19}$$

- The completely general expression for $\mathbf{F}_{ABC}$ in terms of a generic generalised vielbein can be found in [81], also including a vielbein which might violate the strong constraint.

In contrast to the case in the T-duality chain (3.3.11), where only one of the fluxes **H**, **f**, **Q**, **R** was turned on, the single components have no general interpretation. E.g. here, there can be **R**-flux in a locally geometric background, if other fluxes are turned on as well.

Global non-geometry. Despite all the above examples being locally geometric, many of these backgrounds might be *globally non-geometric*. Metric and B-field, encoded in the generalised metric, are only defined locally. If the patching involves only B-field gauge transformations and d-dimensional diffeomorphisms, we call the background *globally geometric*. On the other hand, for a generic non-geometric background we can patch as

$$\mathcal{H}'_{MN}(G'(x), B'(x)) = M_M{}^K(x)(\mathcal{H}_{KL}(G(x), B(x))M_N{}^L(x) \tag{3.3.20}$$

for an $M_{KL}(x) \in$O(d,d). In a corresponding generalised flux frame (3.3.1), we have that the 'internal' generalised metric $\mathcal{H}_{AB} = E_A{}^M \mathcal{H}_{MN} E_B{}^N = \delta_{AB}$ is trivial and globally well-defined. The generalised vielbein will in general be defined only patch-wise and patched via $E'^A{}_M(x) = M_M{}^N(x) E^A{}_N(x)$. The generalised fluxes transform according to

$$\tilde{\mathbf{F}}_{ABC} = \mathbf{F}_{ABC} + E_{\underline{[A}}{}^N E_{\underline{B}}{}^K (\partial_{\underline{C}]} M^M{}_N) M_{MK}. \tag{3.3.21}$$

'O(d,d) gauge transformations' are those $M(x)$ for which the second term vanishes such that, as expected, the generalised fluxes are globally well-defined in a non-geometric background. Such O(d,d) gauge transformations include for example

- in the *geometric frame*: geometric gauge transformations, i.e. B-field gauge transformations and d-dimensional diffeomorphisms.

- in the *geometric frame with* $\mathbf{H} = 0$: certain (coordinate dependent) β-shifts in non-holonomic coordinates, s.t. both $\mathbf{Q}_c{}^{ab} = 0$ and $\mathbf{R}^{abc} = 0$. Such β-shifts exist, homogeneous Yang-Baxter deformations of group manifolds are of this kind for example [168]. It has been shown that these correspond to a *non-local* field redefinition in the Lagrangian [169].

- *frame independent*: all constant O(d,d) transformations, including factorised dualities. For example the constant $\mathbf{Q}$-flux background in the T-duality chain is of this type, where M is a constant β-shift.

As (3.3.21) shows, the allowed $M_M{}^N(x)$ depend on the generalised frame $E_A{}^M$ under investigation. Not all of these necessarily have to be interpretable as standard abelian T-duality, for example they might also correspond to non-abelian T-duality transformations [170].

Finding such a generalised flux frame for some given generalised metric $\mathcal{H}$ is non-trivial and not unique, as there is a huge gauge freedom[3]. There is in general no preferred frame, except if we can find a globally well-defined generalised vielbein (this case is called a generalised parallelisable manifold - see e.g. [138]). We are only concerned with local properties of the target space in the following, so all statements involving the generalised vielbeins $E_A{}^M$ are to be understood in a single patch.

[3]Condition (3.3.1) fixes only the gauge for the flat internal indices, the gauge freedom corresponds to the gauge freedom of the original $\mathcal{H}_{MN}$.

In section 7 we strive for a Hamiltonian formulation of classical string theory given directly in terms of these globally well-defined generalised fluxes $\mathbf{F}_{ABC}$. This formulation will only hide the fact that in principle we still need to work in the different coordinates patches in which the ${E_A}^M$ are defined. Steps towards a more rigorous discussion of global issues have been taken in [171, 172] in the present context of current algebras and loop groups as phase space.

3.4 SL(5) exceptional generalised geometry

U-duality is the duality symmetry of M-theory. The aim of exceptional generalised geometry is the same as the aim of generalised $\mathrm{O}(d,d)$-geometry in string theory: offering a geometrical framework for the target space geometry that is U-duality covariant.

As discussed in section 2.3, the U-duality groups $E_{d(d)}$ and their used representations have to be discussed for each d separately. Fairly generic for all d is the form and properties of the generalised Lie derivative[4], or Dorfman bracket, of two sections of a bundle $E \to M$

$$[\phi_1, \phi_2]^K = -\phi^L_{[1}\partial_L \phi^K_{2]} + {Y^{KL}}_{MN}\phi_1^M \partial_L \phi_2^N. \tag{3.4.1}$$

Again, we assume an extended space with coordinates $X^M = (x^\mu, \tilde{x}_{??})$, including 'dual' coordinates with different index structure for each d. Imposing a Jacobi-like identity on this bracket leads to a *section condition*

$${Y^{KL}}_{MN}\partial_K f \partial_L g = 0, \quad \text{for all functions } f, g. \tag{3.4.2}$$

Now, let us discuss the concrete realisation in the $E_{4(4)}$ =SL(5)-theory, so the theory relevant for compactifications of M-theory on a four-dimensional manifold M. The generalised tangent bundle is

$$\phi \in E = TM \oplus \Lambda^2 T^\star M. \tag{3.4.3}$$

There will be two kinds of representations of SL(5) that play a role in the following. The generalised vectors ϕ^K are in the so-called $\mathcal{R}_1$-representation, which is the (skewsymmetric) **10**-representation of SL(5). So, we can understand K as a double index $K = [kk']$. Moreover, we choose the following conventions

$$\phi^K = \frac{1}{\sqrt{2}}\phi^{kk'}, \ldots \qquad \text{e.g.} \quad \delta^K_L = \frac{1}{2}\delta^{kk'}_{ll'} = \delta^k_{[l}\delta^k_{l']}$$

The $\mathcal{R}_2$-indices are the fundamental SL(5)-indices and will not be that relevant in what follows.

The analogue of the $\mathrm{O}(d,d)$-invariant metric η in the SL(5)-theory is the epsilon tensor ϵ_{klmnp}:

$$\epsilon_{klmnp}{M_{k'}}^k {M_{l'}}^l {M_{m'}}^m {M_{n'}}^n {M_{p'}}^p = \epsilon_{k'l'm'n'p'}.$$

[4]This form include the Dorfman bracket of the Courant algebroid with ${Y^{KL}}_{MN} = \eta^{KL}\eta_{MN}$.

Some definitions regarding the ϵ- and generalised Kronecker symbols are collected in appendix A.2. The Y-tensor of the generalised Lie derivative is defined in terms of ϵ-symbols:

$$Y^{kk'll'}{}_{mm'nn'} = \epsilon^{pkk'll'}\epsilon_{pmm'nn'} \tag{3.4.4}$$

With this Y-tensor, the section condition take the form

$$\partial_{[kk'} \otimes \partial_{ll']} = 0 \tag{3.4.5}$$

In contrast to the $\mathrm{O}(d,d)$-case, there are different types of solutions to the section condition. The above decomposition into $TM \oplus \Lambda^2 T^\star M$ corresponds to the M-theory section: $k = (\kappa, 5), l = (\lambda, 5), \ldots$ and ${}_{kk'} = ({}_{\kappa}, {}^{\kappa\kappa'}) \equiv ({}_{\kappa 5}, {}^{\kappa\kappa'})$. In the M-theory section, the extended coordinates decompose as $X^{mm'} = (x^\mu, \tilde{x}_{\mu\mu'})$ into the four physical coordinates x^μ on the four-dimensional internal space and six 'dual' coordinates. The conventions have been chosen in a way that $\mathrm{d} = \mathrm{d}X^M \partial_M = \mathrm{d}x^\mu \partial_\mu + \frac{1}{2}\mathrm{d}\tilde{x}_{\mu\mu'}\tilde{\partial}^{\mu\mu'}$. The section condition is fulfilled, if $\tilde{\partial}^{\mu\mu'} f = 0$ for all functions f.

In addition to this, there are also three-dimensional sections that describe the three-dimensional internal geometry of compactifications of the ten-dimensional type II supergravities. In both cases the $\mathcal{R}_2$-indices decompose as $k = (\underline{\kappa}, 4, 5)$.

- The type IIa section follows trivially from the M-theory section by decomposing $x^\mu = (x^{\underline{\mu}}, x^4)$ and assuming that $\partial_4 = 0$, as well. This corresponds to the dimensional reduction of M-theory.
- The type IIb section is not connected that simply to the M-theory section. The physical coordinates are the three coordinates $x^{\underline{\mu\mu'}}$ in $x^\mu = (x^{\underline{\mu\mu'}}, x^{i\underline{\mu}}, x^{ij})$ with $\underline{\mu}, \ldots = 1,2,3$ and $i, .. = 4,5$.

The background, described by the metric G and the 3-form A is again encoded in a generalised metric. In the M-theory decomposition, it is given by

$$\mathcal{H}_{MN}(G) = \begin{pmatrix} G_{\mu\nu} + \frac{1}{2}A_\mu{}^{\kappa\kappa'}A_{\nu\kappa\kappa'} & \frac{1}{\sqrt{2}}A_\mu{}^{\nu\nu'} \\ \frac{1}{\sqrt{2}}A^{\mu\mu'}{}_\nu & \frac{1}{2}(G^{\mu\nu}G^{\mu'\nu'} - G^{\mu\nu'}G^{\mu'\nu}) \end{pmatrix} \tag{3.4.6}$$

Similarly to $\mathrm{O}(d,d)$ generalised geometry, one can define $\mathrm{SL}(5)$-vielbeine

$$\text{'little' vielbein:} \qquad E_{a_1}{}^{m_1} \ldots E_{a_5}{}^{m_5}\epsilon^{a_1 \ldots a_5} = \epsilon^{m_1 \ldots m_5}$$

$$\text{'big' vielbein:} \qquad E_A{}^K = \frac{1}{2}E_{aa'}{}^{kk'} = E_{[a}{}^k E_{a']}{}^{k'}$$

$$E_A{}^K E_B{}^L E_M{}^C E_N{}^D Y^{MN}{}_{KL} = Y^{CD}{}_{AB}$$

and generalised fluxes

$$\mathbf{F}^C{}_{AB} = E_N{}^C \partial_{[A} E_{B]}{}^N - Y^{CD}{}_{AE} E_N{}^E \partial E_B{}^N, \quad \text{from} \quad [E_A, E_B]_D = \mathbf{F}^C{}_{AB} E_C. \tag{3.4.7}$$

Decomposed in terms of M-theory or type II sections, they describe the collection of NSNS-, RR- and geometric (metric) fluxes and non-geometric versions of these, as in the $\mathrm{O}(d,d)$-case. In the SL(5)-theory, this has been explored in [173]. For more details on exceptional generalised geometry, also for $d \neq 4$, see [163, 174–179].

Chapter 4

Integrability

In Hamiltonian systems, (maximal) integrability means that the model allows for a maximal set of Poisson commuting (involuting) conserved quantities. This is also called integrability in the Liouville sense. The phase space is describable as a foliation and, in the finite-dimensional (and compact) case, is diffeomorphic to T^{2n} due to the Liouville-Arnold theorem. Upon finding that diffeomorphism, one has solved that system.

In quantum theories, a possible definition for quantum integrability is the factorisability of the S-matrix. This means, an S-matrix factorises into $2 \to 2$ S-matrices and, moreover, all the different possible factorisations are equivalent.

Developed to a large extend in mathematics literature, see e.g. [180], integrability experienced a renaissance in string theory as a technical tool in the prime example of the AdS/CFT correspondence. The interplay between the classical integrability of string theory in $\mathrm{AdS}_5\times\mathrm{S}^5$, the integrability of certain spin chains and the quantum integrability of $\mathcal{N} = 4$ super Yang-Mills theory offered a powerful test of the duality. The classical integrability of string theories, as a two-dimensional field theories, will be investigated in the following.

For detailed reviews of various aspects of integrability, including an introduction to quantum integrability and the inverse scattering method, see [65, 181].

4.1 Classical integrability of 2d field theories

Lax integrability. A two-dimensional field theory, for simplicity defined here on a closed string world-sheet, is called *Lax integrable*, if its equations of motion can be written as a *spectral parameter* dependent flat connection $L_\alpha(\lambda)$, the Lax connection or Lax pair. The zero-curvature condition

$$\partial_\alpha L_\beta(\lambda) - \partial_\beta L_\alpha(\lambda) - [L_\alpha(\lambda), L_\beta(\lambda)] = 0, \tag{4.1.1}$$

is called Lax connection. The appearance of the, at this stage redundant, spectral parameter $\lambda \in \mathbb{C}$ is crucial. Infinitely many conserved charges can be derived with the help of the so-called monodromy matrix

$$T(\tau,\lambda) = \mathfrak{P}\exp\left(\int_0^1 \mathrm{d}\sigma L_\sigma(\sigma,\tau;\lambda)\right). \tag{4.1.2}$$

Its time evolution is given by

$$\partial_\tau T(\tau,\lambda) = [L_\tau(0,\tau;\lambda), T(\tau,\lambda)]. \tag{4.1.3}$$

A consequence of this is that the eigenvalues $\mu_i(\lambda)$ are conserved over time. The infinitely many conserved charges arise as coefficients in a Laurent expansion of the μ in λ. As an example and basis for further investigations, let us discuss the principle chiral model.

The principal chiral model. The field theory of group G valued of a field $g(\tau,\sigma)$ with the action

$$S = -\frac{1}{2}\int \kappa_{ab} j^a \wedge \star j^b, \tag{4.1.4}$$

with $j = -g^{-1}\mathrm{d}g$ and κ being the Killing form on the Lie algebra $\mathfrak{g}$ of G, is called the principal chiral model. The equations of motion and the flatness condition for the $\mathfrak{g}$-valued currents j is:

$$\mathrm{d}\star j \quad \text{and} \quad \mathrm{d}j + \frac{1}{2}[j,j] = 0. \tag{4.1.5}$$

The Lax representation of the equations of motion of the principal chiral model can be easily computed from the ansatz

$$L = l_1 j + l_2 \star j, \tag{4.1.6}$$

where l_1 and l_2 are to be determined. Using of the equations of motion and the flatness condition for the currents j^a, one finds that the Lax connection is

$$L = \pm\frac{\lambda^2}{1-\lambda^2} j + \frac{\lambda}{1-\lambda^2}\star j \tag{4.1.7}$$

with the free 'spectral' parameter λ.

The r-s formalism. From this Lax pair one can derive the monodromy matrix and, following from this, the infinitely many conserved quantities. A necessary condition for Hamiltonian integrability (in the Liouville sense) is that the conserved quantities, that are produced in the Lax formalism, do Poisson-commute.

A sufficient condition on the Poisson bracket of the Lax pair was derived by Sklanin [182], and in the more general setting that is relevant for the principal chiral by Maillet [183]

$$\begin{aligned}\{L_\mathbf{1}(\sigma;\lambda), L_\mathbf{2}(\sigma;\mu)\} = {} & [r_\mathbf{12}(\lambda,\mu), L_\mathbf{1}(\sigma;\lambda) + L_\mathbf{2}(\sigma';\mu)]\delta(\sigma-\sigma') \\ & + [s_\mathbf{12}(\lambda,\mu), L_\mathbf{1}(\sigma;\lambda) - L_\mathbf{2}(\sigma';\mu)]\delta(\sigma-\sigma') \\ & - 2s_\mathbf{12}(\lambda,\mu)\delta'(\sigma-\sigma')\end{aligned} \tag{4.1.8}$$

This relation is formally defined on the free algebra of $\mathfrak{g}$ with $L_1 \in \mathfrak{g} \otimes \mathbb{1}$ $L_2 \in \mathbb{1} \otimes \mathfrak{g}$ and $r_{12}, s_{12} \in \mathfrak{g} \otimes \mathfrak{g}$, r being skewsymmetric and s symmetric when exchanging the two copies of $\mathfrak{g}$. The last δ'-term is called the *non-ultralocal*. It will play a central role in the geometric investigation of the Poisson structure later.

If one finds these objects r and s such that the Lax pair has the Poisson algebra (4.1.8), the conserved charges computed in the Lax formalism are in involution. Only now, the system is integrable in the Liouville sense.

Imposing the Jacobi identity on (4.1.8), one arrives at an instance of the *classical Yang-Baxter equation*:

$$[\mathcal{R}_{12}(\lambda,\mu), \mathcal{R}_{13}(\lambda,\mu)] + [\mathcal{R}_{12}(\lambda,\mu), \mathcal{R}_{23}(\lambda,\mu)] + [\mathcal{R}_{13}(\lambda,\mu), \mathcal{R}_{23}(\lambda,\mu)] = 0 \qquad (4.1.9)$$

for $\mathcal{R} = r + s$ as a sufficient condition.

Integrable string σ-models. Integrability and conformal symmetry are both powerful tools in two dimensions. But, they are independent of each other. So, a integrable σ-model does not have to be a string σ-model, meaning conformally invariant at 1-loop. In other words, the background of an integrable σ-model does not have to be a solution to the supergravity equations of motion.

The principal chiral, that served as example above, is not a string σ-model. Instead, one can generalised the integrability of the principal chiral model to Riemannian symmetric spaces (spheres, AdS, hyperbolic spaces,...), and finally semi-symmetric supercoset spaces – so, exactly to the kind of spaces for which we know the form of the GS σ-model, as shown in section 1.2.

As discussed there, these backgrounds are string backgrounds only if the isometry group has vanishing Killing form. Integrability is not connected to that property, as is κ-symmetry.

4.2 Integrable deformations of string σ-models

The criterion for integrability of a superstring σ-model is clearly very restrictive, as it only applies to very symmetric spaces. But, it is possible to deform these in a way that preserves integrability and, in some of these cases, also the property of being a string background. Let us introduce these deformations for simplicity as deformations of the principal chiral model.

4.2.1 Yang-Baxter deformations

Following the same conventions as in the principal chiral model above, the action of a Yang-Baxter deformed principal chiral model is

$$S = -\frac{1}{2} \int \mathrm{d}^2\sigma \quad (g^{-1}\partial_+ g)^a \kappa_{ac} \left(\frac{1}{\mathbb{1} - \eta R_g} \right)^c{}_b (g^{-1}\partial_- g)^b, \qquad (4.2.1)$$

where $R_g = \mathrm{Ad}_g^{-1} \circ R \circ \mathrm{Ad}_g$ (and also R) is a solution to the (modified) classical Yang-Baxter equation[1]

$$[R(t_a), R(t_b)] - R([R(t_a), t_b) + [t_a, R(t_b)]) = -c^2[t_a, t_b] \tag{4.2.2}$$

in terms of the R-operator.[2] For $c = 0$, the corresponding deformations are usually called (homogeneous) Yang-Baxter deformations and for $c = i$ η-deformations.

Their integrability was proven in [184], before they were generalised to coset and supercoset σ-models, and solutions of the classical Yang-Baxter equation as generators [185–187]. The resulting backgrounds are (super)gravity solutions if the generalising classical r-matrix is unimodular [169], meaning that the resulting dual structure constants fulfil

$$\overline{f}_b{}^{ab} = 0, \quad \text{for} \quad \overline{f}_c{}^{ab} = r^{d(a} f^{b)}{}_{dc}. \tag{4.2.3}$$

Nevertheless, starting from a κ-symmetric semi-symmetric type IIb supergravity background, all Yang-Baxter deformations preserve κ-symmetry and thus the resulting backgrounds are still solutions of so-called modified type IIb supergravity equations [58, 188–191]. The study of many examples of homogeneous Yang-Baxter deformations [187, 192–203] revealed that they seem to be related to β-shifts of abelian T-duality in case of abelian r-matrices. This was proven in the case of abelian r-matrices [204] and for general r-matrices in case of $\mathrm{AdS}_5 \times \mathrm{S}^5$ [91]. The connection of Yang-Baxter deformations and non-abelian T-duality became clearer in [169, 205, 206]. It was demonstrated, that on a purely formal level Yang-Baxter deformations are given by formal β-shifts, though there was no criterion of a connection to non-abelian T-duality (NATD) there [207]. In case of $\mathrm{AdS}_5 \times \mathrm{S}^5$-backgrounds in context of the AdS/CFT correspondence it was demonstrated that homogeneous Yang-Baxter deformations lead to Drinfel'd twists of the corresponding Hopf algebra structures on both sides of the duality [200, 208]. These aspects were investigated further in [209, 210].

The non-geometric features have been already discussed in the case of the (abelian) β-shifted S^5-background (a special case being the Lunin-Maldacena background [211]) in [212, 213], where the β-shift can be accounted for by twists of the closed string boundary conditions. The first insights into homogeneous Yang-Baxter deformations in the sense of non-geometric fluxes discussed in section 3 have been found in [214]. There the **Q**-flux of homogeneously Yang-Baxter deformed coset σ-models was studied in some examples and a T-fold interpretation of the resulting backgrounds was established.

With the help of the previously developed framework of a NATD group and generalised flux analysis of the Poisson-Lie σ-model, we will analyse Yang-Baxter deformed σ-models. In case of the homogeneous Yang-Baxter deformations this will prove the natural generalisation of [204] for non-abelian r-matrices, that the notions of homogeneous Yang-Baxter deformations and non-abelian T-duality generalisation of β-shifts of principal chiral models are exactly the same. This is proven in section 8.

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[1] Let me emphasise for clarity's sake the distinction between R-operator, related to β by $\beta^{ab} = R^a_{g\,c}\kappa^{cb}$, and the **R**-flux, defined in terms of β, as $\mathbf{R} = [\beta, \beta]_S$.

[2] This classical Yang-Baxter equation is related to (4.1.9) by $R^a{}_b = \kappa_{ac} r^{cb}$.

4.2.2 λ-deformation

A class of integrable models, that is a conformal field theory for all values of the deformation parameter, is the λ-deformation

$$S(g) = S_{WZW,k}(g) + \frac{\lambda}{\pi}\int d^2\sigma\, (g^{-1}\partial_+ g)^a \left(\frac{1}{\mathbb{1} - \lambda \mathrm{Ad}_g^{-1}}\right)_{ab} (g^{-1}\partial_+ g)^b .$$

It describes an interpolation between a WZW-model and the (factorised) non-abelian T-dual of the principal chiral model [215–219]. The deformation parameter λ can be written in terms of the WZW-level k and the coupling of a principal chiral model κ as $\lambda = \frac{k^2}{\kappa^2 + k}$. Interesting limits are

- $\lambda \to 0$: undeformed WZW model
- $k \to \infty$: non-abelian T-dual of principal chiral model on G

$$S_{NATD} = \frac{1}{\pi}\int d\sigma^2\, \partial_+ \chi_a \left(\mathbb{1} - \chi_c(f^c)\right)^{-1,ab} \partial_- \chi_b$$

- $k \ll \kappa^2$: perturbed WZW

$$S(g) = S_{WZW,k}(g) + \frac{k^2}{\pi\kappa^2}\int d\sigma^2\, (g^{-1}\partial_+ g)^a (g^{-1}\partial_+ g)^a$$

It can be completed to a supergravity solution, corresponds to certain q-deformations of the original group and has been argued to be equivalent via Poisson-Lie T-duality and analytic continuation of the deformation parameter $\eta \leftrightarrow \pm i\lambda$ to the η-deformation [19, 189, 216, 220].

Part III

Results

Chapter 5

Deformation of the current algebra – a motivation

The 'geometric paradigm' in section 1.2 emphasised the connection of target space geometry to the world-volume theories. In additional support to that statement, this chapter collects several instances of the fundamental connection of background fields and deformations of the Poisson structure. Apart from staying in the framework of Riemannian geometry, these examples employ the same conceptual step as later in chapter 7. That is finding coordinates of the phase space in which the Hamiltonian is trivial. The chapter presents material from [221].

5.1 Point particle in an electromagnetic background

As a motivational example that shares many features with the string in NSNS backgrounds, let us consider a relativistic point particle with mass m and electric charge q in an arbitrary electromagnetic background [222, 223]. At first we define it by an electric potential $\mathbf{A} = A_\mu \mathrm{d}x^\mu$ with field strength $\mathbf{F} = \mathrm{d}\mathbf{A} = F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$. A convenient choice[1] of Hamiltonian, $H = \frac{1}{2m}\left(p - qA\right)^2$, together with the canonical Poisson structure gives the equations of motion

$$\dot{x}^\mu = \frac{1}{m}\pi^\mu \equiv \frac{1}{m}(p^\mu - qA^\mu) \quad \text{and} \quad \dot{\pi}_\mu = \frac{q}{m}F_{\mu\nu}\pi^\nu. \tag{5.1.1}$$

Alternatively, this problem can be phrased in terms of new coordinates on the phase space (x^μ, π_μ) with the kinematic momentum π_μ. Let us note a few important characteristics of this formulation, which will also be key points in the string discussion:

- *Preferred non-canonical phase space coordinates*. In terms of kinematic momentum π^μ the Hamiltonian is $H = \frac{\pi_\mu \pi^\mu}{2m}$, so we have a 'free' Hamiltonian. All background

[1] The free Hamiltonian $H_{free} = \frac{e}{2m}\mathbf{p}^2$ with 4-momentum $\mathbf{p}$ is obtained via a Polyakov trick with the einbein e so that H_{free} is indeed the constraint corresponding to time reparameterisation invariance in this case. After gauge choice $e = 1$ and minimal substitution we are left with above Hamiltonian.

data – the coupling to the electromagnetic field – is encoded in the deformed Poisson brackets

$$\{x^\mu, x^\nu\} = 0, \quad \{x^\mu, \pi_\nu\} = \delta^\mu_\nu, \quad \{\pi_\mu, \pi_\nu\} = qF_{\mu\nu}, \tag{5.1.2}$$

resp. the deformed symplectic structure $\omega = \omega_0 + q\mathbf{F}$. The Jacobi identity of the Poisson bracket resp. the closedness of ω is equivalent to the Bianchi identity in the standard Maxwell equations:

$$\mathrm{d}\omega = 0 \quad \Leftrightarrow \quad \mathrm{d}\mathbf{F} = 0. \tag{5.1.3}$$

The field equations for $\mathbf{F}$ can also be phrased conveniently in terms of the symplectic structure: $\partial^\mu \omega_{\mu\nu} = 4\pi \mathbf{j}^{(e)}_\nu$.

- *Generalisation to magnetically charged backgrounds*. In this formulation there is no need to refer to the potential A, it is phrased only in terms of the field strength F. So it is well suited for generalisations to magnetically charged backgrounds with $\star\mathrm{d}\mathbf{F} = 4\pi j^{(m)}$.

  Alternatively, one could take another point of view, namely to consider this as a free particle in non-commutative or, in case $\mathrm{d}\mathbf{F} \neq 0$, even non-associative momentum space. This is why it this a typical example and toy model for the treatment of non-associative phase spaces [224–228]. Recently it has been shown that such a non-associative, or almost symplectic, phase space can be realised in a higher dimensional symplectic one [229, 230].

- *Charge algebra.* This coordinate change in phase space (a symplectomorphism in the case without magnetic sources[2]) is simply the local field redefinition from canonical to kinematic momenta

$$\begin{aligned}\omega = -\mathrm{d}\theta = \mathrm{d}(p_\mu \mathrm{d}x^\mu) &= \mathrm{d}(\pi_\mu + qA_\mu) \wedge \mathrm{d}x^\mu \\ &= \mathrm{d}\pi_\mu \wedge \mathrm{d}x^\mu + q\mathbf{F}.\end{aligned} \tag{5.1.4}$$

[2]To make this problem symmetric in electric and magnetic terms we could consider a dyon (q, g) in an electromagnetic background F, e.g. a particle with Lorentz force $\dot{\pi}_\mu = \frac{1}{m}\left(qF_{\mu\nu} + g\tilde{F}_{\mu\nu}\right)\pi^v$, thus corresponding to the deformed symplectic structure is $\omega = \omega_0 + q\mathbf{F} + g \star \mathbf{F}$, which is not symplectic anymore, as soon as we have any electric or magnetic sources for F. For the dyon then there is no (local) field redefinition anymore connecting the two formulations.

5.2 Integrable models and deformations of current algebras

The principal chiral model, the theory of the embedding of a string world-sheet into a group manifold G, is one of the most important toy models for the study of integrable σ-models. It can be defined by a Hamiltonian

$$H = \frac{1}{2}\int \mathrm{d}\sigma \left(\kappa^{ab} j_{0,a} j_{0,b} + \kappa_{ab} j_1^a j_1^b\right) \tag{5.2.1}$$

and the following Poisson structure, the current algebra,

$$\begin{aligned} \{j_{0,a}(\sigma), j_{0,b}(\sigma')\} &= -f^c{}_{ab} j_{0,c}(\sigma)\delta(\sigma-\sigma') \\ \left\{j_{0,a}(\sigma), j_1^b(\sigma')\right\} &= -f^b{}_{ca} j_1^c(\sigma)\delta(\sigma-\sigma') - \delta_b^a \partial_{\sigma'}\delta(\sigma-\sigma') \\ \left\{j_1^a(\sigma), j_1^b(\sigma')\right\} &= 0. \end{aligned} \tag{5.2.2}$$

$f^c{}_{ab}$ are structure constants to the Lie algebra $\mathfrak{g}$ of G and κ its Killing form. We take $\partial_\tau = \{\cdot, H\}$. The Hamiltonian equations of motion contain both the flatness condition and the (Euler-Lagrange) equations of motion,

$$\mathrm{d}j + \frac{1}{2}[j,j] = 0 \quad \text{and} \quad \mathrm{d}\star j = 0. \tag{5.2.3}$$

We have $j_{0,a} = (g^{-1}\partial_0 g)_a = p_a$ and $j_1^a = (g^{-1}\partial_1 g)^a = e_\mu{}^a \partial x^\mu$. This identification will be different for distinct backgrounds and is, what we later call, a (generalised) frame. We still have to define the brackets between the j_α and functions f on G:

$$\begin{aligned} \{j_{0,a}(\sigma), f(x(\sigma'))\} &= -\partial_a f(x(\sigma))\delta(\sigma-\sigma') \equiv -e_a{}^\mu \partial_\mu f(x(\sigma))\delta(\sigma-\sigma') \\ \{j_{1,a}(\sigma), f(x(\sigma'))\} &= 0, \end{aligned}$$

where we chose some coordinates x on G.

The principal chiral model possesses many deformations which preserve one its most interesting properties: its classical integrability. Interestingly, all these deformations can be understood as deformations of the current algebra (5.2.2) instead of the deformation of a Hamiltonian or Lagrangian.

- The introduction of a WZ-term in the Lagrangian can be accounted for by a change of the j_0-j_0 Poisson bracket in comparison to (5.2.2)

  $$\{j_{0,a}(\sigma), j_{0,b}(\sigma')\}_{WZW} = -\left(f^c{}_{ab} j_{0,c}(\sigma) + k f_{abc} j_1^c(\sigma)\right)\delta(\sigma-\sigma'). \tag{5.2.4}$$

  Classically, k can be considered as a deformation parameter. See for example the standard textbook [231] for more details on the Hamiltonian treatment of the WZW-model.

- The σ-model Lagrangian of the η-deformation was discovered in [184, 232] and its target space interpretation as a q-deformation of the original group manifold was given in [186]. It can also be represented by a modication of the current algebra. As such, it arose already in [233]. Compared to (5.2.2), the Poisson bracket between the j_1 is

$$\left\{ j_1^a(\sigma), j_1^b(\sigma') \right\}_\eta = \frac{\eta^2}{1-\eta^2} f^{abc} j_{0,c}(\sigma)\delta(\sigma-\sigma'). \tag{5.2.5}$$

- The λ-deformation was introduced directly in terms of a deformation of the current algebra, originally for G =SU(2) in [234] and later generalised to arbitrary groups in [215], accompanied with a Lagrangian derivation. For more details and references see section 4.2.

  Again after some rescaling of the currents compared to the original articles, the λ-deformation corresponds only to a change in the j_1-j_1-Poisson bracket:

$$\left\{ j_1^a(\sigma), j_1^b(\sigma') \right\}_\lambda = -\frac{\lambda^2}{1+\lambda^2} f^{abc} j_{0,c}(\sigma)\delta(\sigma-\sigma'). \tag{5.2.6}$$

  Phrased like this in the Hamiltonian formalism and compared to (5.2.5), we see directly that λ- and η-deformations are equivalent via analytic continuation $\eta \leftrightarrow \pm i\lambda$.

With this short survey we have motivated that in the Hamiltonian formulation deformations of the current algebra are a convenient playground. In fact, we will see that every bosonic string σ-model can be represented by the free Hamiltonian and a modified current algebra.

A related discussion of the SU(2) principal chiral model aimed on the features connected the generalised geometry can be found in [235].

5.3 String in an H-flux background

The generalisation of the point particle in an electromagnetic field (section 5.1) to strings in a geometric **H**-flux background was achieved in [236]. Consider the σ-model of a (classical) string in a geometric background, defined by metric G and Kalb-Ramond field B

$$S = -\frac{1}{2}\int \mathrm{d}x^\mu \wedge \left(G_{\mu\nu}(x) \star + B_{\mu\nu}(x)\right) \mathrm{d}x^\nu. \tag{5.3.1}$$

Twisted symplectic structure Following the same steps as before, we express the symplectic structure in terms of the kinematic momenta $\pi_\mu := p_\mu + B_{\mu\nu}(x)\partial x^\nu$

$$\begin{aligned} \omega &= \int \mathrm{d}\sigma\, \delta p_\mu(\sigma) \wedge \delta x^\mu(\sigma) \qquad (5.3.2)\\ &= \int \mathrm{d}\sigma \left(\delta\pi_\mu \wedge \delta x^\nu - \frac{1}{2}\mathbf{H}_{\mu\nu\kappa}(x)\partial x^\kappa \delta x^\mu \wedge \delta x^\nu + \frac{1}{2}\partial\left(B_{\mu\nu}(x)\delta x^\mu \wedge \delta x^\nu\right)\right). \end{aligned}$$

Up to the total derivative term, the symplectic structure is twisted in a B-field gauge independent way, by the $\mathbf{H}$-flux, similarly to the electromagnetic case (5.1.4). Imposing that the symplectic form (5.3.2) is closed,

$$\delta\omega = \frac{1}{6}\int \mathrm{d}\sigma \partial_{[i}\mathbf{H}_{jkl]}(x)\,\partial x^{\mu}\,\delta x^{\nu}\wedge\delta x^{\kappa}\wedge\delta x^{l} = 0, \tag{5.3.3}$$

requires that $\mathbf{H}$ is a closed 3-form on M. If we instead neglect the boundary contribution in the symplectic two-form (5.3.2), we get such a contribution for the closure of the symplectic form

$$\delta\omega_{bdy} = \partial\left(\mathbf{H}_{\mu\nu\kappa}(x)\delta x^{\mu}\wedge\delta x^{\nu}\wedge\delta x^{\kappa}\right) \tag{5.3.4}$$

up to a total derivative term. So together with the Hamiltonian

$$H = \frac{1}{2}\int \mathrm{d}\sigma\left(G^{ij}(x)\pi_{\mu}\pi_{\nu} + G_{\mu\nu}(x)\partial x^{\mu}\partial x^{\nu}\right) \tag{5.3.5}$$

this defines a world-sheet theory in backgrounds which are magnetically charged under the NSNS flux, e.g. the $NS5$-brane - in particular in and near these magnetic sources, but requiring that the phase space there is only almost symplectic.

The total derivative terms in (5.3.2) resp. (5.3.3), which naively vanish for closed strings, are for example relevant for

- *open strings ending on D-branes*. A contribution to the symplectic structure from a (potentially pure-gauge) B-field on the brane is the well known source for the fact, that we find non-commutative gauge theories on the brane. In the present context of deformations of the symplectic/Poisson structure this has been discussed in [236], in particular closedness of the symplectic structure requires $\mathbf{H}|_{D-\text{brane}} = 0$ if we neglect the boundary term in the current algebra. For the some of the models motivating this thesis D-branes have been discussed, i.e. Poisson-Lie σ-models [105] or λ-deformations [237].

- *winding strings*. As discussed above the winding number

$$w = \int \mathrm{d}\sigma\,\partial x(\sigma)$$

  along a compact direction is such an integral over a total derivative. In section 7.2 we show that such winding contributions need to be considered so that the current algebra still satisfies the Jacobi identity.

- *globally non-geometric backgrounds*. E.g. consider the $\mathbf{Q}$-flux background obtained from the standard T-duality chain of T^3 with q units of $\mathbf{H}$-flux, expressed in terms of a metric G and the $\mathbf{H}$-flux (3.3.9). We expect a contribution of a monodromy $\mathbf{H}(1) - \mathbf{H}(0)$. But let us note that also the Hamiltonian (5.3.5) is not well-defined at $x^3 + 1 \sim x^3$ in the geometric frame.

  Choosing the generalised flux frame instead – here in particular the one for the pure $\mathbf{Q}$-flux background – should give a globally well-defined description of the background and be used to twist the symplectic structure. This is the route we want to take in the following.

It turns out that these twists by the generalised fluxes are more conveniently defined in terms of the variables $p_\mu(\sigma)$ and $\partial x^\mu(\sigma)$ and their Poisson structure, the current algebra. The current algebra and its deformations could in principle also be phrased in terms of a symplectic structure. But in case of such a Poisson structure containing so-called *non-ultralocal* terms, the symplectic structure will be non-local:

$$\omega_{\text{current}} = \int \mathrm{d}\sigma_1 \mathrm{d}\sigma_2 \bar{\Theta}(\sigma_1 - \sigma_2)\delta p_\mu(\sigma_1) \wedge \delta(\partial x^\mu)(\sigma_2),$$

where $\bar{\Theta}$ is the step function with $\delta_\sigma \bar{\Theta} = \delta(\sigma)$. Trying to invert the **H**-twisted Poisson current algebra (5.3.2) to obtain a twisted ω_{current} we get:

$$\begin{aligned} \omega_{\mu\nu}(\sigma_1, \sigma_2) &= \begin{pmatrix} A_{\mu\nu}(\sigma_1, \sigma_2) & \delta^\nu_\mu \bar{\Theta}(\sigma_1 - \sigma_2) \\ -\delta^\nu_\mu \bar{\Theta}(\sigma_1 - \sigma_2) & 0 \end{pmatrix} \\ \text{with} \quad & \int \mathrm{d}\sigma_2 \partial_1^2 A_{\mu\nu}(\sigma_1, \sigma_2) = -H_{\mu\nu\kappa}\partial x^\kappa(\sigma_1) \end{aligned}$$

neglecting boundary terms.

Let us make a connection between what follows in the next part and the above twisting of the symplectic structure by the **H**-flux. Going to kinematic variables ∂x^ν, π_ν) the current algebra is

$$\begin{aligned} \{\partial x^\mu(\sigma_1), \partial x^\nu(\sigma_2)\} = 0, \qquad \{\partial x^\mu(\sigma_1), \pi_\nu(\sigma_2)\} &= \delta^\mu_\nu \partial_1 \delta(\sigma_1 - \sigma_2). \\ \{\pi_\mu(\sigma_1), \pi_\nu(\sigma_2)\} &= -\mathbf{H}_{\mu\nu\kappa}(\sigma_1)\partial x^\kappa(\sigma_1)\delta(\sigma_1 - \sigma_2) \\ &\quad + \int \mathrm{d}\sigma \partial \left(B_{\mu\nu}(\sigma)\delta(\sigma - \sigma_1)\delta(\sigma - \sigma_2)\right) \end{aligned} \tag{5.3.6}$$

The Jacobi identity imposes, of course equivalently to (5.3.3), $\partial_{[i}\mathbf{H}_{jkl]} = 0$ and in case we neglect the total derivative term in (5.3.6) $\mathbf{H}\big|_{\text{D-brane}} = 0$.

Chapter 6

On the geometry of the current algebra

In the previous chapter, it was shown that the Poisson structure can encode background data. The presented examples showed this for geometric backgrounds. In order to see how generalised fluxes of generalised geometry appear, we study the algebraic structure of the current algebra first.

The material presented in this chapter was part of [221].

6.1 $\mathbf{O}(d,d)$-covariant formulation of current algebra

6.1.1 Local functionals and reduction to standard algebroids

The configuration space of a closed string moving in a manifold M is the (free) loop space

$$LM = \left\{ x :\ S^1 \to M,\ \sigma \mapsto x(\sigma) \right\}.$$

We denote elements of LM by x or $x^\mu(\sigma)$, working in a coordinate patch of M. We take σ to have values between 0 and 1 and in a slight abuse of nomenclature for LM also include open strings.

The class of smooth functions on LM, that we will consider most often, are (multi-local) functionals on M

$$F :\ LM \to \mathbb{R},\ F[x] = \int \mathrm{d}\sigma_1 ... \mathrm{d}\sigma_n f(x(\sigma_1), ..., x(\sigma_n))$$

induced by *smooth* functions $f\ :\ M \times ... \times M\ \to\ \mathbb{R}$ – in particular this includes all the background fields and fluxes. We assume *no explicit* σ-dependence, as required by independence under σ-reparameterisations.

The tangent space $T(LM)$ is spanned by variational derivatives and consists of elements

$$V[x] = \int \mathrm{d}\sigma\ V^\mu[x](\sigma) \frac{\delta}{\delta x^\mu(\sigma)} \in T(LM).$$

For simplicity of the notation, we will write $V^\mu(\sigma) \equiv V^\mu[x](\sigma)$. These $V^\mu(\sigma)$ are also only implicit functions of σ, i.e. $\partial V^\mu(\sigma) \equiv \int d\sigma' \partial x^\nu(\sigma') \frac{\delta}{\delta x^\nu(\sigma')} V^\mu(\sigma)$, where $\delta = \int d\sigma\, \delta x^\mu(\sigma)\, \frac{\delta}{\delta x^\mu(\sigma)}$ is the de Rham differential on LM and we use the notation $\partial \equiv \partial_\sigma$.

Not all functions on LM are related to multilocal functionals of smooth functions on M, e.g. the winding number

$$w = \int d\sigma\, \partial x(\sigma), \tag{6.1.1}$$

where $x(\sigma) = x + w\sigma +$ *oscillators*, is a total derivative under the integral over the closed circle. The coordinate $x(\sigma)$ itself is not a *smooth* function on the circle in case of a winding string. So, not all expressions $\int d\sigma\, \partial(...)$ are expected to vanish.

6.1.2 Current algebra as Lie and Courant algebroids

Algebroids over LM. Let us compute the algebra of arbitrary multilocal 'charges'. A section $\phi \in \Gamma(E)$ is given by

$$\phi = \phi[x] = \int d\sigma\, \phi^M(\sigma) \mathbf{E}_M(\sigma) \tag{6.1.2}$$

The *Poisson bracket* between these sections ϕ is

$$\begin{aligned} \{\phi_1, \phi_2\} = \int d\sigma_1 d\sigma_2\, \mathbf{E}_M(\sigma_1) \Big(-\phi^N_{[1}(\sigma_2) \frac{\delta}{\delta X^N(\sigma_2)} \phi^N_{2]}(\sigma_1) \\ + \frac{1}{2} \phi^N_{[1}(\sigma_2) \frac{\delta}{\delta X_M(\sigma_1)} \phi_{2]N}(\sigma_2) + \frac{\delta}{\delta X_M(\sigma_1)} \frac{1}{2} \left(\omega_{KL} \phi^K_1(\sigma_2) \phi^L_2(\sigma_2) \right) \Big) \end{aligned} \tag{6.1.3}$$

with $\frac{\delta}{\delta X^M(\sigma)} := \left(\frac{\delta}{\delta x^\mu(\sigma)}, 0 \right)$. Also, we have a natural anchor map $\rho : E \to T(LM)$ defined via the Poisson bracket

$$\phi \in \Gamma(E) \mapsto \rho(\phi) = \{\,\cdot\,, \phi\} = \int d\sigma \phi^\mu(\sigma) \frac{\delta}{\delta x^\mu(\sigma)} \in \Gamma\big(T(LM)\big). \tag{6.1.4}$$

The Leibniz rule follows from the properties of the fundamental Poisson brackets. Also, the Jacobi identity

$$\{\phi_1, \{\phi_2, \phi_3\}\} + \text{ c. p.} = 0 \tag{6.1.5}$$

holds *identically*, i.e. without any total derivative terms under the σ-integrals. For this, we have to use $\frac{\delta}{\delta X^M(\sigma)} F \frac{\delta}{\delta X_M(\sigma)} G$ for arbitrary functions F, G on LM, which is the *strong constraint* of double field theory on LM and follows here from our definition of $\frac{\delta}{\delta X^M(\sigma)}$. Resultantly the full (multilocal) charge algebra is a not only a Lie algebra as expected, but also a **Lie algebroid** $(E, \{\cdot,\cdot\}, \rho)$ over the free loop space LM. It is something which could be called standard Lie algebroid of the generalised tangent bundle $(T \oplus T^\star)(LM)$, for which the Lie bracket is the semi-direct product of TM, with the Lie bracket and $T^\star M$ for an arbitrary manifold M,

$$[\phi_1, \phi_2]_L = [v_1, v_2] + \mathcal{L}_{v_1} \xi_2 - \mathcal{L}_{v_2} \xi_1. \tag{6.1.6}$$

From (6.1.3), which is written in an O(d,d)-covariant way, we see that the Lie algebroid bracket is *not* invariant under O(d,d)-transformations due to the presence of the last term containing ω.

There is a natural non-degenerate inner product on $E \to LM$ induced by the $O(d,d)$-metric η on $(T \oplus T^\star)M$:

$$\langle \phi_1, \phi_2 \rangle = \int \mathrm{d}\sigma \; \eta_{IJ} \phi_1^M(\sigma) \phi_2^N(\sigma). \tag{6.1.7}$$

This product is the canonical bilinear form on $(T \oplus T^\star)LM$. Following the definition $\mathcal{D}$, $\langle \mathcal{D}F, \phi \rangle = \rho(\phi)F$, we find the derivation

$$\begin{aligned} \mathcal{D}F[x] &= \int \mathrm{d}\sigma \; \mathbf{E}_M(\sigma) \frac{\delta}{\delta X_M(\sigma)} F[x] = \int \mathrm{d}\sigma \partial x^\mu(\sigma) \frac{\delta}{\delta x^\mu(\sigma)} F[x] \\ &= \int \mathrm{d}\sigma_1 ... \mathrm{d}\sigma_n (\partial_1 + ... + \partial_n) f\left(x(\sigma_1), ..., x(\sigma_n)\right). \end{aligned} \tag{6.1.8}$$

With the help of these objects we can define the **standard Courant algebroid** on $(T \oplus T^\star)LM$, for which the Courant resp. Dorfman bracket take the form:

$$\{\phi_1, \phi_2\}_C = \int \mathrm{d}\sigma_1 \mathrm{d}\sigma_2 \mathbf{E}_M(\sigma_1) \left(-\phi^N_{[1}(\sigma_2) \frac{\delta}{\delta X^N(\sigma_2)} \phi^M_{2]}(\sigma_1) + \frac{1}{2} \phi^N_{[1}(\sigma_2) \frac{\delta}{\delta X_M(\sigma_1)} \phi_{2]N}(\sigma_2) \right) \tag{6.1.9}$$

$$\{\phi_1, \phi_2\}_D = \int \mathrm{d}\sigma_1 \mathrm{d}\sigma_2 \mathbf{E}_M(\sigma_1) \left(-\phi^N_{[1}(\sigma_2) \frac{\delta}{\delta X^N(\sigma_2)} \phi^M_{2]}(\sigma_1) + \phi^M_1(\sigma_2) \frac{\delta}{\delta X_M(\sigma_1)} \phi_2(\sigma_2) \right) \tag{6.1.10}$$

In a local form in terms of the basis $\mathbf{E}_I(\sigma)$, the relevant brackets and objects are:

- Brackets:

$$\text{Lie :} \quad \{\mathbf{E}_M(\sigma_1), \mathbf{E}_N(\sigma_2)\}[\sigma] = \frac{1}{2}\eta_{MN}(\partial_1 - \partial_2)\big(\delta(\sigma - \sigma_1)\delta(\sigma - \sigma_2)\big) + \frac{1}{2}\omega_{MN}\partial\big(\delta(\sigma - \sigma_1)\delta(\sigma - \sigma_2)\big) \tag{6.1.11}$$

$$\text{Courant :} \quad \{\mathbf{E}_M(\sigma_1), \mathbf{E}_N(\sigma_2)\}[\sigma] = \frac{1}{2}\eta_{MN}(\partial_1 - \partial_2)\big(\delta(\sigma - \sigma_1)\delta(\sigma - \sigma_2)\big) \tag{6.1.12}$$

$$\text{Dorfman :} \quad \{\mathbf{E}_M(\sigma_1), \mathbf{E}_N(\sigma_2)\}[\sigma] = \eta_{MN}\partial_1\delta(\sigma - \sigma_1)\delta(\sigma - \sigma_2) \tag{6.1.13}$$

- Non-degenerate O(d,d)-invariant inner product:

$$\langle \mathbf{E}_M(\sigma_1), \mathbf{E}_N(\sigma_2) \rangle [\sigma] = \eta_{MN}\delta(\sigma - \sigma_1)\delta(\sigma - \sigma_2) \tag{6.1.14}$$

The anchor ρ is the projection onto $T(LM)$ and we have as for any manifold that the standard Courant algebroid over LM is an *exact* Courant algebroid, as the sequence

$$T^\star(LM) \xrightarrow{\rho^T} E \xrightarrow{\rho} T(LM) \tag{6.1.15}$$

is an exact one.

But in contrast to an arbitrary manifold, we see that the derivation $\mathcal{D}$ produces total derivative terms under the σ-integral, which vanishes for multilocal functionals $F[x]$, induced by well-defined *smooth* functions $f : M \times .... \times M \to \mathbb{R}$. But for open strings, this will give a boundary contribution and even for closed strings topological quantities like winding can arise as discussed in section 5.3 – e.g. $\int \mathrm{d}\sigma\, \partial x \neq 0$ for a winding string along a compact direction parameterised by x. In particular, the last term of (6.1.3) which spoiled the $O(d,d)$-invariance should not be neglected.

In contrast to previous literature, we will keep track of the total derivative terms at times in the following. In section 7.2 it is shown a contribution from this total derivation term is indeed necessary to ensure associativity of the subalgebra of zero modes (meaning center of mass coordinate x and momentum p, and winding w) even in locally geometric backgrounds. This agrees with the discussion in this section where we expect a violation of the Jacobi identity of the Courant bracket (by a total derivation term) but, by assumption, a Lie algebroid structure of the phase space.

Let us summarise: for any manifold M, we have that of the three properties – skew-symmetry, Jacobi identity and O(d,d)-invariance – each of the three brackets – Lie, Courant or Dorfman – satisfy two identically and the third one up to a total derivation term (under $\mathcal{D}$).

Algebroids over M. In this paragraph we want to tackle two questions

- Can we find bundle maps $e_\star$, such that

$$T^\star M \xrightarrow{e_\star^T} T^\star(LM) \xrightarrow{\rho^T} E \xrightarrow{\rho} T(LM) \xrightarrow{e_\star} TM \tag{6.1.16}$$

  is a (non-exact) Courant algebroid over M with anchor $e_\star \circ \rho$?

- Does such a bundle map $e_\star$ also extend to a homomorphism of Lie resp. Courant algebroids? What happens to the total derivation terms?

In general, these questions seem to go beyond the scope of this article, both for reasons of mathematical rigor – which seems to be required if we consider bundle maps which keep track of more of the 'non-local' structure of the full current algebra – and also for physical reasons – we work in a fully generic background so far, so no mode expansion of the basis $\mathbf{E}_I(\sigma)$ is available. A stringy expansion of the full current algebra could be an interesting question for further study. A more rigorous study of the current algebra and loop space structure of the phase space can be found in previous literature [171, 172].

Nevertheless, we can find a simple example. Let us consider the bundle map

$$e^0_\star : \; v = \int \mathrm{d}\sigma_1 ... \mathrm{d}\sigma_n v^i(x(\sigma_1), ..., x(\sigma_n)) \frac{\delta}{\delta x^i(\sigma_1)} \quad \mapsto \quad v^i(x) \equiv v^i(x, ..., x)\partial_i, \tag{6.1.17}$$

which is the push-forward of the evaluation map of the loop space, $e^0 : \; LM \to M,\; x(\sigma) \mapsto x \equiv x(\sigma_0)$ for some σ_0. This bundle map is simply the projection from the loop space phase space to the phase space associated to a point of the string. It induces an anchor $e^0_\star \circ \rho$, as we can show that e_0 is an Lie algebra homomorphism. This can be used to view the current algebra as an algebroid over M, not only over LM.

It extends easily to a complete (Lie resp. Courant) algebroid homomorphism $E \to (T \oplus T^\star)M$. Consider the generic bracket on E,

$$\{\phi_1, \phi_2\}_{a,b} = \int \mathrm{d}\sigma_1 \mathrm{d}\sigma_2 \mathbf{E}_I(\sigma_1) \left(\phi^J_{[1}(\sigma_2) \frac{\delta}{\delta X^J(\sigma_2)} \phi^I_{2]}(\sigma_1) + \frac{1}{2} \phi^J_{[1}(\sigma_2) \frac{\delta}{\delta X_I(\sigma_1)} \phi_{2]J}(\sigma_2) \right.$$
$$\left. + \frac{\delta}{\delta X_I(\sigma_1)} \frac{1}{2} \left(a\, \omega_{KL} + b\, \eta_{KL}\right) \phi^K_1(\sigma_2) \phi^L_2(\sigma_2) \right) \tag{6.1.18}$$

for some $a, b \in \mathbb{R}$, which incorporates all brackets discussed in the previous section. $e^0_\star$ defines a bracket on $(T \oplus T^\star)M$

$$e^0_\star \{\phi_1, \phi_2\}^I = \{e^0_\star \phi_1, e^0_\star \phi_2\}^I = -\phi^J_{[1} \partial_J \phi^I_{2]} + \frac{1}{2} \phi^J_{[1} \partial^I \phi_{2]J} + \frac{1}{2} \left(a\, \omega_{KL} + b\, \eta_{KL}\right) \phi^K_{[1} \partial^I \phi^L_{2]} \tag{6.1.19}$$

and is a true Courant algebroid homomorphism, but the brackets (6.1.18) differ only by total derivative term under the integral, so they might be argued to be equivalent for sufficiently nice charges for closed strings – but their projections to points are truly inequivalent. This issue is quite logical because the map $e^0_\star$ does not really correspond to a point-particle limit of the string[1], but to a restriction of the total phase space (the current algebra) to a local phase space associated to one point on the string. Total derivative terms correspond to a kind of flux on the string, which adds up to zero for closed strings without winding.

The string current algebra. The current algebra derived from the canonical Poisson structure of the string is given by (1.1.8). We write it in an $\mathrm{O}(d,d)$-covariant way, defining $\mathbf{E}_M(\sigma) = (p_\mu(\sigma), \partial x^\mu(\sigma))$,

$$\{\mathbf{E}_M(\sigma_1), \mathbf{E}_N(\sigma_2)\}[\sigma] = \frac{1}{2} \eta_{MN} (\partial_1 - \partial_2) \left(\delta(\sigma - \sigma_1)\delta(\sigma - \sigma_2)\right)$$
$$+ \frac{1}{2} \omega_{MN} \partial \big(\delta(\sigma - \sigma_1)\delta(\sigma - \sigma_2)\big) \tag{6.1.22}$$

[1] An obvious candidate for this would seems to be a bundle map associated to the zero mode projection

$$\bar{e} : LM \to M,\ x(\sigma) = x_0 + \bar{x}(\sigma) \mapsto \int x_0 \equiv \mathrm{d}\sigma x(\sigma) = x_0. \tag{6.1.20}$$

But the push-forward bundle homomorphism,

$$\bar{e}_\star : \phi : \int \mathrm{d}\sigma \phi^i(\sigma) \frac{\delta}{\delta x^i(\sigma)} \mapsto \left(\mathrm{d}\sigma \phi^i(\sigma)\right), \tag{6.1.21}$$

turns out have several issues. Written as such

- It is conceptually ill-defined, because we add vectors of tangent spaces at different points. We would need to transport them back to x_0 before summing them up.
- It will not be an algebra homomorphism.

without neglecting the second total derivative term and where we employ the notation: $\{\cdot,\cdot\} = \int \mathrm{d}\sigma\,(\{\cdot,\cdot\}\,[\sigma])$. The second term containing $\omega = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$ is a total derivative under the σ-integral and not invariant under $O(d,d)$-transformations. It is the boundary term that was already discussed in section 5.3 and will be dicussed in detail in the next section.

Without the ω-term, one can easily show that (6.1.22) is a Courant bracket and in fact the bracket of canonical Courant algebroid over LM [143, 236].

6.2 The ω-term

Before discuss the modification that the ω-term brings to the standard notion of a Courant algebroid we define some notation.

Projector identities and the choice of section. Let us define the projector

$$P^{KL}{}_{MN} = \frac{1}{2}\left(Y^{KL}{}_{MN} + \Omega^{KL}{}_{MN}\right)$$

with $Y^{KL}{}_{MN} = \eta^{KL}\eta_{MN}$ and $\Omega^{KL}{}_{MN} = \eta^{KL}\omega_{MN}$ for $\mathrm{O}(d,d)$ generalised geometry. This notation is chosen in a way that is directly generalises to exceptional generalised geometry. The projector has the following properties:

- $P^{KL}{}_{MN} = P^{LK}{}_{MN}$
- $P^{KL}{}_{MN}\partial_K \otimes \partial_L = 0$ (section condition)
- Using this section condition, we have

$$P^{KL}{}_{MN}P^{NP}{}_{RS}\partial_P = P^{KL}{}_{RS}\begin{pmatrix} 0 \\ \tilde{\partial}^m \end{pmatrix} \approx 0,$$
$$P^{KL}{}_{NM}P^{NP}{}_{RS}\partial_P = P^{KL}{}_{RS}\begin{pmatrix} \partial_m \\ 0 \end{pmatrix} \approx P^{KL}{}_{RS}\partial_M.$$

In comparison to the standard Courant algebroid structure, there are two bilinear objects:

$$\begin{aligned} (\!(\phi_1,\phi_2)\!)^K &= \frac{1}{2}Y^{KL}{}_{MN}\partial_L\left(\phi_1^M\phi_2^N\right) \\ [\![\phi_1,\phi_2]\!]^K &= \frac{1}{2}\Omega^{KL}{}_{MN}\partial_L\left(\phi_1^M\phi_2^N\right) \end{aligned} \tag{6.2.1}$$

In that form, it is not yet fully apparent that they correspond to total derivative terms under the spatial world-volume integrals. But,

$$(\!(\phi_1,\phi_2)\!) = 2\eta_{MN}\int \mathrm{d}\left(\phi_1^M\phi_2^N\right) = 2\int \mathrm{d}\left(\phi_1 \bullet \phi_2\right)$$
$$[\![\phi_1,\phi_2]\!] = 2\omega_{MN}\int \mathrm{d}\left(\phi_1^M\phi_2^N\right) = 2\int \mathrm{d}\left(\phi_1 \circ \phi_2\right).$$

Lie vs. Courant brackets on the phase space. There are four natural brackets on sections of the extended tangent bundle, which differ by these total derivative terms:

$$\begin{aligned}
\text{Dorfman:}\quad & [\phi_1,\phi_2]_D^K = -2\phi^L_{[1}\partial_L\phi^K_{2]} + Y^{KL}{}_{MN}\phi_1^M\partial_L\phi_2^N \\
\text{Courant:}\quad & [\phi_1,\phi_2]_C^K = -2\phi^L_{[1}\partial_L\phi^K_{2]} + Y^{KL}{}_{MN}\phi^M_{[1}\partial_L\phi^N_{2]} \\
\text{Lie:}\quad & [\phi_1,\phi_2]_L^K = -2\phi^L_{[1}\partial_L\phi^K_{2]} + 2P^{KL}{}_{MN}\phi^M_{[1}\partial_L\phi^N_{2]} \\
\text{Dorfman-}\Omega:\quad & [\phi_1,\phi_2]_\Omega^K = -2\phi^L_{[1}\partial_L\phi^K_{2]} + 2P^{KL}{}_{MN}\phi_1^M\partial_L\phi_2^N
\end{aligned}$$

In the following table the key properties of these brackets on the extended tangent bundle are collected: besides duality invariance, skew symmetry and the Jacobiator

$$\mathcal{J}(\phi_1,\phi_2,\phi_3) = [\phi_1[\phi_2,\phi_3]] - [[\phi_1,\phi_2],\phi_3] - [\phi_2,[\phi_1,\phi_3]], \tag{6.2.2}$$

the gauge transformations generated the total derivative terms (6.2.1) via $\mathcal{L}_\phi\Psi = [\Psi,\phi]$ (w.r.t. to the corresponding bracket) are considered.

	$[\,\cdot\,,\,\cdot\,]_D$	$[\,\cdot\,,\,\cdot\,]_L$	$[\,\cdot\,,\,\cdot\,]_C$	$[\,\cdot\,,\,\cdot\,]_\Omega$
$\mathcal{J}(\phi_1,\phi_2,\phi_3)$	0	0	$\frac{1}{3}\langle\!\langle[\phi_1,\phi_2]_C,\phi_3\rangle\!\rangle + \text{c.p.}$	$\frac{1}{3}[\![[\phi_1,\phi_2]_\Omega,\phi_3]\!] + \text{c.p.}$
$[\Psi,X_1]$	0	$-\left(\langle\!\langle\Psi,X_1\rangle\!\rangle + [\![\Psi,X_1]\!]\right)$	$\langle\!\langle\Psi,X_1\rangle\!\rangle$	$[\![\Psi,X_1]\!]$
$[\Psi,X_2]$	0	$-\left(\langle\!\langle\Psi,X_2\rangle\!\rangle + [\![\Psi,X_2]\!]\right)$	$\langle\!\langle\Psi,X_2\rangle\!\rangle$	$[\![\Psi,X_2]\!]$
skew symmetry	×	✓	✓	×
duality invariance	✓	×	✓	×

for $X_1 = \langle\!\langle\phi_1,\phi_2\rangle\!\rangle$ and $X_2 = [\![\phi_1,\phi_2]\!]$.

So, to summarise the current algebra is a (non-standard) Lie algebroid over $TM \oplus T^\star M$. But, there is a bunch of associated brackets, connected to the Lie bracket by total derivative terms, with interesting properties. In particular, the Dorfman bracket of the standard Courant algebroid has

ω-term and the section. The crucial point of the 'ω-geometry' is that it, in contrast to the standard approach to generalised geometry or double field theory, allows for a reconstruction of the section from the choice of ω and vice versa.

We choose the $P^{KL}{}_{MN}$ as the fundamental object obeying the identities (9.3.1) and start with the standard section $\partial_N \approx (\partial_n, 0)$. Then, as $P^{KL}{}_{MN} = \frac{1}{2}\eta^{KL}(\eta_{MN} + \omega_{MN})$, the identities (9.3.1) imply

$$\omega = \begin{pmatrix} 2B & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{6.2.3}$$

for some skewsymmetric matrix B. If we took an arbitrary section $\partial'_M = M_M{}^N\partial_N$, ω transforms as $\omega' = M\cdot\omega\cdot M^T$, for $M \in \mathrm{O}(d,d)$. Up to a (constant) B-shift, the choice of section determines the form of ω and vice versa. This B-shift symmetry is also a well-known property of a Courant algebroid.

A conceptual consequence is that one can reconstruct the Lie algebroid structure of the current algebra from the standard Courant algebroid *plus* a choice of section.

Para-Hermitian and para-Kähler geometries. The current algebra is characterised by the pair (η, ω) and could be completed to a compatible triple (η, ω, I) by $I^M{}_N = \eta^{MK}\omega_{KN}$. If I is a real structure, $I^2 = \mathbb{1}$, the geometry is called *para-Hermitian*, if $\mathrm{d}\omega = 0$ *para-Kähler* [238–240].

In addition, a string model is defined by a generalised metric $\mathcal{H}$ in the Hamiltonian formalism. Recently, Born geometry was introduced as para-Kähler geometry of the tripel $(\eta, \omega, \mathcal{H})$, subject to the conditions [241]

$$\eta^{-1}\mathcal{H} = \mathcal{H}^{-1}\eta, \quad \omega^{-1}\mathcal{H} = -\mathcal{H}^{-1}\omega. \tag{6.2.4}$$

A central result of [239,241] was that, following from this, there exists a frame in which all the defining structures take their canonical form:

$$\mathcal{H} = \mathbb{1}, \quad \omega = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{6.2.5}$$

The input that we obtain from the Hamiltonian formulation of the string is a different though. In the generalised metric formulation, in which we worked so far, – meaning canonical coordinates on the phase space and background information encoded in the Hamiltonian via the generalised metric – we get η and ω in their canonical form and $\mathcal{H}(G, B)$ in a general background dependent form. So, unless we are in flat space, where we can choose $\mathcal{H} = \mathbb{1}$, the classical phase space geometry of the string is *not* described by Born geometry.

Chapter 7

Current algebra in the generalised flux frame

The aim of this chapter is to present a convenient formulation of the world-sheet theory which highlights the role of the generalised fluxes, making the non-geometric features more apparent than the not generally globally defined Lagrangian data G and B. The key result of this chapter is that a *Hamiltonian* description in terms of non-canonical coordinates on the string phase space achieves this objective. All the physical information about the background is encoded in a deformation of the Poisson structure

$$\Pi = \Pi^{(\eta)} + \Pi^{(\text{bdy.})} + \Pi^{(\text{flux})}. \tag{7.0.1}$$

The canonical Poisson structure consists of an $\mathrm{O}(d,d)$-invariant part $\Pi^{(\eta)}$ and a boundary contribution $\Pi^{(\text{bdy.})}$, relevant for open strings and winding along compact directions. $\Pi^{(\text{flux})}$ is characterised exactly by the generalised fluxes. Apart from $\Pi^{(\text{bdy.})}$, this perspective already appeared back in [142, 143] or in [236] for geometrical **H**-flux backgrounds. On the other hand, non-geometric fluxes were already introduced as generalised WZ-terms in first order Lagrangians [167, 242], but only for a certain choice of generalised vielbein. Other perspectives on the connection of σ-models, current algebras and generalised geometry include [162, 171, 172, 243–245]. In particular, $\mathrm{O}(d,d)$-invariant Hamiltonian setups and their non-geometric interpretation have been studied already in [246, 247].

The material in this chapter has been presented already in [221].

7.1 Hamiltonian formulation of classical string theory

In the last chapter we only had a very generic look on aspects of current algebras, valid for arbitrary backgrounds – we did not introduce any dynamics. This section aims to show how the Hamiltonian world-sheet theory in any generalised flux background can be defined by a free string Hamiltonian. All the background information is encoded in a deformation of the Poisson structure. This deformation of the current algebra will be accounted for by the generalised (geometric and non-geometric) NSNS fluxes, in perfect analogy to the point particle in an electromagnetic field. This generalises the result of [236], reviewed in section 5.3. Many aspects of this were discussed already in [167, 242] from a Lagrangian point of view and for a certain parameterisation of generalised vielbeins reviewed in section 3.

Hamilton formalism for string σ-models. Let us consider a generic string σ-model coupled to metric and B-field of a d-dimensional target space

$$S = -\frac{1}{2}\int \left(G_{\mu\nu}(x)\,\mathrm{d}x^\mu \wedge \star\mathrm{d}x^\nu + B_{\mu\nu}(x)\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\right). \tag{7.1.1}$$

Choosing conformal gauge, we find the Hamiltonian to be

$$H = \frac{1}{2}\int \mathrm{d}\sigma \mathcal{H}_{MN}(\sigma)\,\mathbf{E}^M(\sigma)\mathbf{E}^N(\sigma) \tag{7.1.2}$$

where $\mathcal{H}_{MN}(\sigma)$ is the generalised metric (2.2.16), which depends on σ via the coordinate dependence of G and B. $\mathbf{E}_M(\sigma) = (p_\mu(\sigma), \partial x^\mu(\sigma))$, where $p_\mu(\sigma)$ is the canonical momentum, fulfils the canonical current Poisson brackets (6.1.3).

Generalised fluxes in Hamiltonian formalism. Assume we have a generalised flux frame describing our background, e.g. a generalised vielbein $E_A{}^M(x)$ with

$$E_A{}^M(x)E_B{}^N(x)\mathcal{H}_{MN}\left(G(x), B(x)\right) = \gamma^{AB} = \begin{pmatrix} \gamma^{ab} & 0 \\ 0 & \gamma_{ab} \end{pmatrix}, \tag{7.1.3}$$

where γ^{ab} is some convenient flat metric in the signature of the target space. We could be tempted to phrase the Hamiltonian world-sheet theory also in terms of a new basis of the current algebra: $\mathbf{E}_A = E_A{}^M\mathbf{E}_M$. The Hamiltonian is again of the form of a 'free' Hamiltonian:

$$H = \frac{1}{2}\int \mathrm{d}\sigma\, \gamma^{AB}\mathbf{E}_A(\sigma)\mathbf{E}_B(\sigma) \tag{7.1.4}$$

Thus, all the information is expected to be encoded in the current algebra. The redefinition $\mathbf{E}_A = E_A{}^M\mathbf{E}_M$ of (6.1.22) results in the twisted current algebra

$$\begin{aligned} \left\{\mathbf{E}_A(\sigma_1), \mathbf{E}_B(\sigma_2)\right\}[\sigma] = &\frac{1}{2}\eta_{AB}(\partial_1 - \partial_2)\left(\delta(\sigma_1-\sigma)\delta(\sigma_2-\sigma)\right) \\ &+ \frac{1}{2}\partial\left(\omega_{AB}(\sigma)\delta(\sigma_1-\sigma)\delta(\sigma_2-\sigma)\right) \\ &- \mathbf{F}^C{}_{AB}(\sigma)\mathbf{E}_C(\sigma)\delta(\sigma_1-\sigma)\delta(\sigma_2-\sigma), \end{aligned} \tag{7.1.5}$$

with $\mathbf{F}_{ABC} = (\partial_{[A}E_B{}^I)E_{C]I}$. In contrast to the η-term, the total derivative term containing ω_{AB} is *not* invariant under this change of basis as

$$\omega_{AB}(\sigma) = \mathbf{E}_A{}^I(\sigma)\mathbf{E}_B{}^J(\sigma)\omega_{IJ} \neq \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}_{AB} \tag{7.1.6}$$

in general. e-transformations leave the ω_{IJ}-term invariant compared to (6.1.22), whereas for example a B- resp. a β-shift leads to

$$\omega^{(B)} = \begin{pmatrix} 2B & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad \text{resp.} \quad \omega^{(\beta)} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & -2\beta \end{pmatrix}. \tag{7.1.7}$$

Neglecting the ω-term leads to a Dorfman bracket, as discussed before,

$$\{\mathbf{E}_A(\sigma_1), \mathbf{E}_B(\sigma_2)\}_D = \eta_{AB}\partial_1\delta(\sigma_1 - \sigma_2) - \mathbf{F}^C{}_{AB}(\sigma_2)\mathbf{E}_C(\sigma_1)\delta(\sigma_1 - \sigma_2) \tag{7.1.8}$$

or decomposed into the four components **H**, **f**, **Q** and **R**:

$$\begin{aligned}
\left\{e_{0,a}(\sigma), e_{0,b}(\sigma')\right\}_D &= -\left(\mathbf{f}^c{}_{ab}(\sigma)e_{0,c}(\sigma) + \mathbf{H}_{abc}(\sigma)e_1^c(\sigma)\right)\delta(\sigma - \sigma') \\
\left\{e_{0,a}(\sigma), e_1^b(\sigma')\right\}_D &= -\left(\mathbf{f}^b{}_{ca}(\sigma)e_1^c(\sigma) + \mathbf{Q}_a{}^{bc}(\sigma)e_{0,c}(\sigma)\right)\delta(\sigma - \sigma') + \delta_a^b\partial(\sigma - \sigma') \\
\left\{e_1^a(\sigma), e_1^b(\sigma')\right\}_D &= -\left(\mathbf{Q}_c{}^{ab}(\sigma)e_1^c(\sigma) + \mathbf{R}^{abc}(\sigma)e_{0,c}(\sigma)\right)\delta(\sigma - \sigma')
\end{aligned} \tag{7.1.9}$$

with $\mathbf{E}_A(\sigma) = \left(e_{0,a}(\sigma), e_1^a(\sigma)\right)$.

Equations of motion. The Hamilton equations of motion are

$$\mathrm{d}\star\mathbf{e}_c + \frac{1}{2}\left(\mathbf{Q}_c{}^{ab} + \mathbf{H}_{cmn}\gamma^{ma}\gamma^{nb}\right)\mathbf{e}_a \wedge \mathbf{e}_b + \frac{1}{2}\mathbf{f}^{\{a}{}_{kc}\gamma^{b\}k}\mathbf{e}_a \wedge \star\mathbf{e}_b = 0 \tag{7.1.10}$$

$$\mathrm{d}\mathbf{e}^c + \frac{1}{2}\left(\mathbf{f}^c{}_{ab} + \mathbf{R}^{cmn}\gamma_{ma}\gamma_{nb}\right)\mathbf{e}^a \wedge \mathbf{e}^b + \frac{1}{2}\mathbf{Q}_{\{a}{}^{kc}\gamma_{b\}k}\mathbf{e}^a \wedge \star\mathbf{e}^b = 0 \tag{7.1.11}$$

with one-forms $\mathbf{e}^c = e_\alpha{}^c\mathrm{d}\sigma^\alpha$. In terms of a Lagrangian formulation, these correspond to an equation of motion and a world-sheet Bianchi identity. The Hamiltonian formalism does not distinguish between these two 'types' of equations of motion, showing that it is a convenient framework to study dualities.

The equations of motion of the string in an arbitrary locally geometric background can be encoded very conveniently into the O(d,d)-covariant form

$$d\mathbf{E}^A + \frac{1}{2}\mathbf{F}^A{}_{BC}\mathbf{E}^B \wedge \mathbf{E}^C = 0, \quad \text{with} \quad \mathbf{E}^A := (\mathbf{e}^a, \star\mathbf{e}_a) \quad \text{resp.} \quad \mathbf{E}^A = \gamma^{AB}\star\mathbf{E}_B. \tag{7.1.12}$$

In this form, the equations of motion are nothing else than the pullback of a structure equation for frame fields $\mathbf{E}_A$ treated as one forms together with the constraint $\star\mathbf{E}^A = \gamma^{AB}\mathbf{E}_B$.

Virasoro constraints. To complete the description of a string theory in a generalised flux background, we give the Virasoro constraints and their properties. There is, of course, nothing new to expect – they are a consequence of world-sheet reparameterisation invariance and hold identically. Similarly to the Hamiltonian, the constraints and their properties take the same form as the ones for the string in flat space. This relies solely on the fact that the $\mathbf{F}_{ABC}$ are totally skew-symmetric. The conservation of the energy-momentum tensor additionally requires the equation of motion as usual. So we can phrase the whole dynamics of a string solely in terms of the generalised fluxes without referring to the generalised vielbeins.

With the definition $T_{\alpha\beta} = \frac{2}{\sqrt{-h}}\frac{\delta S}{\delta h^{\alpha\beta}}$ and choosing a generalised flux frame $\mathbf{E}_A$ as before, these constraints take the form (in flat gauge on the world-sheet)

$$\begin{aligned} T_{00}(\sigma) = T_{11}(\sigma) &= +\frac{1}{2}\gamma^{AB}\mathbf{E}_A(\sigma)\mathbf{E}_B(\sigma) = 0, \\ T_{01}(\sigma) = T_{10}(\sigma) &= +\frac{1}{2}\eta^{AB}\mathbf{E}_A(\sigma)\mathbf{E}_B(\sigma) = 0. \end{aligned} \tag{7.1.13}$$

Moreover, their respective zero modes H and P correspond to world-sheet derivatives $\partial_\tau = \{\cdot, H\}$ and $\partial_\sigma = \{\cdot, P\}$. Even if we consider the current algebra with all boundary contributions (6.1.11), we get the standard Virasoro algebra

$$\begin{aligned} &\{T_{\pm\pm}(\sigma_1), T_{\pm\pm}(\sigma_2)\}\,[\sigma] = \pm 2\left(T_{\pm\pm}(\sigma_1) + T_{\pm\pm}(\sigma_2)\right)\frac{1}{2}(\partial_1 - \partial_2)(\delta(\sigma-\sigma_1)\delta(\sigma-\sigma_2)), \\ &\{T_{\pm\pm}(\sigma_1), T_{\mp\mp}(\sigma_2)\}\,[\sigma] = 0. \end{aligned} \tag{7.1.14}$$

Conservation of the energy momentum tensor holds on-shell (7.1.12) and for totally skew-symmetric $\mathbf{F}_{ABC}$

$$\partial_+ T_{--}(\sigma) \pm \partial_- T_{++}(\sigma) = \pm \mathbf{F}_{ABC}(\sigma)\gamma^{CD}\mathbf{E}^A(\sigma)\mathbf{E}^B(\sigma)\mathbf{E}_D(\sigma) = 0. \tag{7.1.15}$$

Let us note that in the following we continue discuss the unconstrained current algebra. In this way, the results in the next sections can be applied to generic σ-models, not only string ones. For a discussion of Dirac brackets in the current algebra in context of the generalised metric formulation, see [246, 247].

Deformation of current algebra structure and generalised fluxes. The approach taken above shows a generalisation of the previously known statement, demonstrated in chapter 5 for point particles in Maxwell background or strings in $\mathbf{H}$-flux backgrounds, that the coupling to these background fields can be encoded in a deformation of the symplectic structure of the phase space – in contrast to introducing interaction terms in the Lagrangian or the Hamiltonian. So, locally the world-sheet theory in *any* generalised flux background is characterised by

$$\{\mathbf{E}_A(\sigma_1), \mathbf{E}_B(\sigma_2)\} = \eta_{AB}\partial_1\delta(\sigma_1-\sigma_2) - \mathbf{F}^C{}_{AB}(\sigma)\mathbf{E}_C(\sigma_1)\delta(\sigma_1-\sigma_2) \tag{7.1.16}$$

in terms of the generalised fluxes $\mathbf{F}_{ABC}$, neglecting total derivative terms, together with a 'free' Hamiltonian $H = \frac{1}{2}\int \mathrm{d}\sigma \gamma^{AB}\mathbf{E}_A(\sigma)\mathbf{E}_B(\sigma)$ (and similarly the full set of Virasoro constraints).

From this point of view, we could imagine to generalise to a current algebra twisted by the Weitzenböck connection $\mathbf{\Omega}^C{}_{AB}$ (3.3.3)

$$\{\mathbf{E}_A(\sigma_1), \mathbf{E}_B(\sigma_2)\} = \eta_{AB}\partial_1\delta(\sigma_1-\sigma_2) - \mathbf{\Omega}^C_{AB}(\sigma)\mathbf{E}_C(\sigma_1)\delta(\sigma_1-\sigma_2). \tag{7.1.17}$$

This however seems to be a substantial change in the theory, as the Virasoro algebra (7.1.14) and the conservation of the energy momentum tensor (7.1.15) relies on the total skewsymmetry of $\mathbf{F}_{ABC}$.

This formulation focuses on the physical content of a background, namely the globally well-defined fluxes opposed to the potentially not globally well-defined objects in the generalised metric formulation. In the case of the point particle in an electromagnetic background or the string in **H**-flux background, this formulation also seemed to be gauge invariant under A- resp. B-field gauge transformations. Indeed, all the objects in the twisted current algebra (7.1.16) transform as a tensor under $O(d,d)$ *gauge transformations* $\mathbf{E}_{A'} \to E_{A'}{}^{A}\mathbf{E}_A$. With $\mathrm{O}(d,d)$ gauge transformation, we mean as defined in section 3 precisely those $\mathbf{E}_{A'}$, under which $\mathbf{F}_{ABC}$ transforms as a tensor. S,o all results are expected to take a gauge covariant form, as is usual in the generalised flux formulation of double field theory [148]. The Bianchi identity, which will be discussed in the next paragraph, will serve as an example for that.

If we wanted to define the Hamiltonian theory only by means of (7.1.16) and a 'free' Hamiltonian H, we need to specify the Poisson brackets between the $\mathbf{E}_A$ and functions of the phase space as well:

$$\left\{\mathbf{E}_A(\sigma), f(x(\sigma'))\right\} = -\partial_A f(x(\sigma))\delta(\sigma-\sigma') \tag{7.1.18}$$

with $\partial_A = E_A{}^M\partial_M$ and $\partial_M = (\partial_\mu, 0)$ as before.

Bianchi identities and magnetically charged backgrounds. In analogy to the examples in section 5, let us show what kind of consistency condition the Jacobi identity of the deformed Poisson brackets implies:

$$\begin{aligned} 0 &= \left\{\mathbf{E}_{[A}(\sigma_1), \left\{\mathbf{E}_B(\sigma_2), \mathbf{E}_{C]}(\sigma_3)\right\}\right\}[\sigma] \;+\; \text{c. p.} \\ &= \left(\partial_{[A}\mathbf{F}_{BCD]} - \frac{3}{4}\mathbf{F}^E{}_{[AB}\mathbf{F}_{CD]E}\right)\mathbf{E}^D(\sigma)\delta(\sigma-\sigma_1)\delta(\sigma-\sigma_2)\delta(\sigma-\sigma_3). \end{aligned} \tag{7.1.19}$$

We recognise the Bianchi identity of generalised fluxes (3.3.5) in the last line [148]

$$\partial_{[A}\mathbf{F}_{BCD]} - \frac{3}{4}\mathbf{F}^E{}_{[AB}\mathbf{F}_{CD]E} = 0. \tag{7.1.20}$$

This calculation holds exactly, meaning without neglecting total derivative terms, if we start with the full form of (7.1.5) including the total derivative term there.

Instead, we could start with (7.1.16) instead of (7.1.5), hence neglecting the total derivative term as previously done in the **H**-flux case, see section 5.3 or [236]. One reason for doing so is that the equations of motion for an open Dirichlet string for example, considering all the boundary terms coming from (6.1.22), take the inconvenient form

$$d\mathbf{E}^A(\sigma) + \frac{1}{2}\mathbf{F}^A{}_{BC}(\sigma)\mathbf{E}^B(\sigma)\wedge\mathbf{E}^C(\sigma) = \frac{1}{2}(\eta^{AB}+\omega^{AB})\gamma_{BC}\mathbf{E}^C(\sigma_1)\delta(\sigma-\sigma_1)\big|_{\sigma_1=0}^{\sigma_1=1}.$$

As a consequence, we expect an additional total derivative term in the calculation (7.1.19) – similar to the differences between Lie algebroids and Courant algebroid structures discussed in section 3. This is indeed the case, the Jacobi identity implies

$$\int \mathrm{d}\sigma\, \partial \mathbf{F}_{ABC} = 0. \tag{7.1.21}$$

In the geometric frame and for an open Dirichlet string, we reproduce $\mathbf{H}_{abc}\big|_{\text{D-brane}} = 0$ as a sufficient condition for associativity of the phase space. This reproduces the boundary contribution to open strings in an **H**-flux background in the Jacobi identity section as expected in section 5.3 resp. reference [236].

In full analogy to the point particle in magnetic monopole backgrounds, we expect violations of this Bianchi identity and thus of the Jacobi identity of our current algebra for magnetically charged backgrounds. Such backgrounds like NS5-branes and its T-duals have been studied in [148, 248, 249] in the generalised flux formulation.

In [148] also the following Bianchi identities/potential source terms have been discussed:

$$\mathcal{J} = \partial^A \mathbf{F}_A - \frac{1}{2}\mathbf{F}^A\mathbf{F}_A + \frac{1}{12}\mathbf{F}^{ABC}\mathbf{F}_{ABC},$$
$$\mathcal{J}_{AB} = \partial^C \mathbf{F}_{CAB} + 2\partial_{[A}\mathbf{F}_{B]} - \mathbf{F}^C\mathbf{F}_{CAB}$$

with $\mathbf{F}_A = \mathbf{\Omega}^B{}_{BA} + 2\partial_A d$, where d is the generalised dilaton. We do not expect an appearance of these terms in the classical world-sheet theory, as they do explicitly contain the dilaton and the Weitzenböck connection. Thus, we will not consider them in the following. From the side of gauged supergravity, both $\mathbf{F}_{ABC}$ as well as $\mathbf{F}_A$ are known to correspond to electric gauging parameters [139, 140].

Magnetic backgrounds source the Bianchi identity like

$$\partial_{[A}\mathbf{F}_{BCD]} - \frac{3}{4}\mathbf{F}^E{}_{[AB}\mathbf{F}_{CD]E} = \mathcal{J}_{ABCD}. \tag{7.1.22}$$

In the 'global' axioms, the violation of the Bianchi identity like this corresponds to a violation of the Jacobi identities of either Lie, Courant or Dorfman brackets like

$$\{\phi_1, \{\phi_2, \phi_3\}\} - \{\{\phi_1, \phi_2\}, \phi_3\} - \{\phi_2, \{\phi_1, \phi_3\}\} = \mathcal{J}(\phi_1, \phi_2, \phi_3) \tag{7.1.23}$$

where $\mathcal{J} : \Gamma(E) \wedge \Gamma(E) \wedge \Gamma(E) \to \Gamma(E)$ with $\mathcal{J}(\phi_1, \phi_2, \phi_3) = \mathcal{J}_{ABCD}\phi_1^A\phi_2^B\phi_3^C\mathbf{E}^D$ in a local basis.

In principle, this implies that inside the magnetic sources the background cannot be described anymore by a generalised vielbeins that gives the generalised flux $\mathbf{F}_{ABC}$ (3.3.4). This means that in this case we cannot untwist the current algebra and that it is not possible to find a Lagrangian description of the world-sheet theory. Phrased in other words, there are no Darboux coordinates to this problem, as the canonical Poisson bracket cannot be used to represent the then non-associative phase space. Working in the Hamiltonian formalism, we still have to specify a generalised vielbein in which all the objects are phrased, although this vielbein will not account for the full amount of $\mathbf{F}_{ABC}$.

7.2 Non-geometric interpretation

7.2.1 Weak and strong constraint from the current algebra

We saw that the action of T-duality is not well-defined without assuming some isometry. The obstruction was that the dual backgrounds would become functions of new dual coordinates $\tilde{x}_\mu(\sigma)$. These, being antiderivatives of the canonical momentum densities $p_\mu(\sigma)$, are not uniquely defined.

A natural approach to this problem is to define $X_M = (\tilde{x}_\mu, x^\mu)$ to be the fundamental fields of the phase space. Remarkably it seems, from the point of view of the Hamiltonian formalism, we would not need to 'double' phase space but instead allow a dependence of the background on the momenta in this very peculiar non-local way, namely via $\tilde{x}_\mu = \int^\sigma \mathrm{d}\sigma' p_\mu(\sigma)$.

Poisson brackets on doubled space and the strong constraint. The question is what the Poisson structure on the doubled space is, and in particular if it is a Poisson structure, i.e. if the proposed brackets fulfils the Jacobi identity. First, we look for a skewsymmetric Poisson bracket $\{X_M(\sigma), X_N(\sigma')\}$ by integrating the canonical current algebra (6.1.22). The solution is

$$\{X_M(\sigma), X_N(\sigma')\} = -\eta_{MN}\bar{\Theta}(\sigma - \sigma') \quad (+\, c\, \omega_{MN}) \tag{7.2.1}$$

with $\bar{\Theta}(\sigma) = 1/2\ \mathrm{sign}(\sigma)$, s.t. $\partial_\sigma \bar{\Theta}(\sigma) = \delta(\sigma)$ and an integration constant c. As a bracket of functionals,

$$\{\Psi_1, \Psi_2\} = \int \mathrm{d}\sigma_1 \mathrm{d}\sigma_2 \bar{\Theta}(\sigma_1 - \sigma_2)\eta^{MN} \frac{\delta\Psi_1}{\delta X^M(\sigma_1)} \frac{\delta\Psi_2}{\delta X^N(\sigma_2)}, \tag{7.2.2}$$

it will always vanish if we assume the *strong constraint* (or section condition) of double field theory. The bracket (7.2.1), without the ω-term, has been discussed already in [246, 247] from point of view of first order σ-models and the generalised metric formulation. The occurence of the constant ω-term reminds of the zero mode non-commutativity observed in [250].

It is easy to show that the above bracket satisfies the Jacobi identity on the space of functionals (with or without the ω-term). But, let us note that the bracket (7.2.1) is *not* equivalent to the canonical current algebra, when we are not neglecting $\int \mathrm{d}\sigma\, \partial(\ldots) \neq 0$ terms. For example, trying to derive (6.1.22) from (7.2.1) leads to ambiguities because $\partial_\sigma \partial_{\sigma'} \{X_M(\sigma), X_N(\sigma')\}$ and $\partial_{\sigma'} \partial_\sigma \{X_M(\sigma), X_N(\sigma')\}$ differ exactly by such a topological/total derivative term. Accepting this, we will use the fundamental brackets

$$\{X_M(\sigma), X_N(\sigma')\} = -\eta_{MN}\bar{\Theta}(\sigma - \sigma'), \qquad \{X_M(\sigma), \mathbf{E}_N(\sigma')\} = \eta_{MN}\delta(\sigma - \sigma') \tag{7.2.3}$$

supplemented by the canonical current algebra (6.1.22) for our calculations of the brackets of functionals $F[X, \mathbf{E}]$, which do not contain σ-derivatives of X or $\mathbf{E}$. In section 3 it was shown that the Jacobi identity for sections of the 'canonical Lie algebroid' holds exactly when using the canonical current algebra (and also assuming the strong constraint). Allowing for violations of the strong constraint leads to a fundamental violation of the Jacobi identity. As an example, let us compute the Jacobi identity between a functional $\Psi[X]$ and two sections of the canonical Lie algebroid $\phi_i[X, \mathbf{E}] = \int \mathrm{d}\sigma \phi_i^M[X](\sigma)\mathbf{E}_M(\sigma)$, where Ψ and the $\phi_i^M(\sigma)$ are assumed to be functional of $X(\sigma)$ only, not of its σ-derivatives. Using (7.2.3) we arrive at

$$
\begin{aligned}
\{\Psi, \{\phi_1, \phi_2\}\} + c.p. = \int \mathrm{d}\sigma_1 \mathrm{d}\sigma_2 \Bigg[& \frac{1}{2}(\eta_{JK} + \omega_{JK})\phi^J_{[1}(\sigma_1) \frac{\delta \Psi}{\delta X^I(\sigma_2)} \frac{\delta \phi^K_{2]}(\sigma_1)}{\delta X_I(\sigma_2)} \\
& + \mathbf{E}_I(\sigma_1)\mathbf{E}_J(\sigma_2) \left(\left\{ \Psi, \left\{ \phi_1^I(\sigma_1), \phi_2^J(\sigma_2) \right\} \right\} + c.p. \right) \\
& + \mathbf{E}_I(\sigma_1) \left(\tfrac{\delta \phi^I_{[1}(\sigma_1)}{\delta X^J(\sigma_2)} \left\{ \Psi, \phi^J_{2]}(\sigma_2) \right\} - \tfrac{\delta \Psi}{\delta X^J(\sigma_i)} \left\{ \phi^I_{[1}(\sigma_1), \phi^J_{2]}(\sigma_2) \right\} - \phi^J_{[1}(\sigma_2) \left\{ \Psi, \tfrac{\delta \phi^I_{2]}(\sigma_1)}{\delta X^J(\sigma_2)} \right\} \right. \\
& \left. + \ \tfrac{1}{2}(\eta_{JK} + \omega_{JK})\ \phi^J_{[1}(\sigma_2) \left\{ \Psi, \tfrac{\delta \phi^K_{2]}(\sigma_2)}{\delta X^I(\sigma_1)} \right\} - \tfrac{1}{2}(\eta_{JK} - \omega_{JK}) \tfrac{\delta \phi^J_{[1}(\sigma_2)}{\delta X^I(\sigma_1)} \left\{ \Psi, \phi^K_{2]}(\sigma_2) \right\} \right) \Bigg] .
\end{aligned}
\tag{7.2.4}
$$

We recognise the Jacobi identity of the (here unspecified) X-X Poisson bracket in the second line of (7.2.4), which we assume to vanish. Apart from that, we see other strong constraint violating terms that generically could contribute to a violation of the Jacobi identity. It might seem that taking different choices of the topological term in the canonical current algebra or a different choice X-$\mathbf{E}$-Poisson bracket than (7.2.3) could make these contributions disappear. In section 7.2 we will show that this is not the case and fundamental violation (in particular the first term) leads exactly to typical non-vanishing Jacobiators of the zero modes in generalised flux backgrounds.

The generalised flux frame and the Virasoro algebra. Let us briefly mention differences to the approach taken in before section, if we allow the generalised vielbeins itself violate the weak constraint (which is in fact not the case in the typical examples of non-geometric backgrounds – see e.g. the $\mathbf{R}$-flux backgrounds). Going to a generalised flux frame ${E_A}^I(X)$, allowing for a generic dependence on the doubled space, we get the Poisson current algebra

$$
\begin{aligned}
\left\{ \mathbf{E}_A(\sigma), \mathbf{E}_B(\sigma') \right\} &= \left\{ {E_A}^I\big(X(\sigma)\big)\mathbf{E}_I(\sigma), {E_B}^J\big(X(\sigma')\big)\mathbf{E}_J(\sigma') \right\} \\
&= \eta_{AB}\partial_\sigma \delta(\sigma - \sigma') - {\mathbf{F}^C}_{AB}(\sigma)\mathbf{E}_C(\sigma)\delta(\sigma - \sigma') - {\mathbf{G}_{AB}}^{CD}(\sigma, \sigma')\mathbf{E}_C(\sigma)\mathbf{E}_D(\sigma')\bar{\Theta}(\sigma - \sigma').
\end{aligned}
\tag{7.2.5}
$$

The last term is bilocal and vanishes if the generalised vielbein satisfies the weak constraint. It is given in terms of the Weitzenböck connection (3.3.3)

$$
{\mathbf{G}_{AB}}^{CD}(\sigma, \sigma') = \eta^{KL} \left(\partial_K {E_A}^I(\sigma) \right) \left(\partial_L {E_B}^J(\sigma') \right) {E_I}^C(\sigma) {E_J}^D(\sigma') = \eta^{KL} {\mathbf{\Omega}_{K,A}}^C(\sigma) {\mathbf{\Omega}_{L,B}}^D(\sigma').
$$

If $\mathbf{F}$ and $\mathbf{G}$ are given in terms of generalised vielbeins, then (7.2.5) is a Poisson bracket because (7.2.1) is. The equation of motion of the string in a DFT background would be also modified by the non-local and strong constraint violating $\mathbf{G}$-terms.

More crucially, this term would be responsible for a modification of the Virasoro algebra. Following the derivation of the Virasoro algebra in the generalised flux frame as before, this can be easily seen as the **G**-term is not totally antisymmetric. The connection of the weak constraint and the algebra of worldsheet diffeomorphisms was already noted from the generalised metric point of view in [246]. Nevertheless, this only occurs if the generalised vielbein or the generalised metric itself depends on coordinates and their duals at the same time.

As before, we calculate the Bianchi identity of the objects **F** and **G** by imposing the Jacobi identity on (7.2.5)

$$\partial_{[A}\mathbf{F}_{BCD]} - \frac{3}{4}\mathbf{F}^{E}{}_{[AB}\mathbf{F}_{CD]E} = \text{ strong constraint violating terms.} \tag{7.2.6}$$

Thus, one way to account for a violation of the Bianchi identities of generalised fluxes (7.1.11), e.g. in order to describe magnetically charged backgrounds, is to consider violations of the weak constraint, but only if we trade off (manifest) locality of the equations of motion for it and a modification of the Virasoro algebra for it.

7.2.2 A non-commutative and non-associative interpretation

The new observation of [221] was that the non-commutative and non-associative interpretation of non-geometric backgrounds is an off-shell (purely kinematic) property of the current algebra. One can reproduce standard results for constant a B-field in case of the open string, and constant **H**-, **f**-, **Q**- or **R**-flux for the closed string. The key characteristics are

- Leaving magnetic or locally non-geometric backgrounds aside, there should be 'Darboux coordinates' $(x^\mu(\sigma),\ p_\mu(\sigma))$ fulfilling the canonical Poisson brackets. The question is where the well-known non-geometric nature of the backgrounds is 'hidden', meaning their non-commutative and non-associative behaviour.

  Before in this chapter, we saw that beside Darboux coordinates $x^\mu(\sigma),\ p_\mu(\sigma)$, the generalised flux frame of a given background gives rise to a second preferred set of coordinates for the current algebra $\mathbf{E}_A(\sigma)$. We define 'non-geometric coordinates' y^a and 'non-geometric momenta' π_a by $\partial y^a = \mathbf{E}^a(\sigma)$ and $\pi_a(\sigma) = \mathbf{E}_a(\sigma)$. In the spirit of section 5, we dub them *'kinematic'*. Their Poisson brackets agree with the known ones usually associated to non-geometric backgrounds.

  With this we can generalise the non-geometric interpretation to more complicated generalised flux backgrounds. Also, we do not need to know the mode expansions of the fields $y^a(\sigma)$ (or impose the equations of motion) to study the non-geometric behaviour of the background.

- In the spirit of generalised geometry and double field theory, we demonstrate in the language of the current algebra, how T-dualities can be reproduced by choosing different solutions to the strong constraint.

- The significance of the non $\mathrm{O}(d,d)$-invariant boundary term in (6.1.22) or (7.1.5) lies
  - reproducing non-commutativity for the endpoints of open strings.
  - ensuring associativity for closed strings, unless we calculate the brackets of objects violating the strong constraint. In that case, the zero modes of the current algebra (and its integrated form) show that this approach reproduces the known form of non-vanishing Jacobiators in the constant **Q**- and **R**-flux backgrounds.

Open string non-commutativity. In this section we review the classic result of [14, 251] and are interested in the world-sheet dynamics of an open string in a constant B-field background. This can be expressed in terms of the open string variables resp. the non-geometric frame with flat metric and β

$$\beta^{\mu\nu} = \left(\frac{1}{G+\mathcal{F}}\right)^{\mu\rho} \mathcal{F}_{\rho\sigma} \left(\frac{1}{G-\mathcal{F}}\right)^{\sigma\nu}, \qquad \mathcal{F} = B - \mathrm{d}A = B - F, \tag{7.2.7}$$

where G is the flat Minkowski metric and F is the constant field strength of a Maxwell field. The current algebra in the generalised flux basis (the non-geometric frame) in which we have the 'free' Hamiltonian is

$$\begin{aligned} \{e_{0,\mu}(\sigma_1), e_{0,\nu}(\sigma_2)\} &= 0 \\ \{e_{0,\mu}(\sigma_1), e_1^{\nu}(\sigma_2)\} &= -\delta^{\mu}_{\nu}\partial_2\delta(\sigma_1-\sigma_2) \\ \{e_1^{\mu}(\sigma_1), e_1^{\nu}(\sigma_2)\} &= -\beta^{\mu\nu}\int \mathrm{d}\sigma\, \partial\left(\delta(\sigma-\sigma_1)\delta(\sigma-\sigma_2)\right). \end{aligned} \tag{7.2.8}$$

Now, we associate new 'non-geometric coordinates' to this new basis: meaning $e_1^a = \partial y_a(\sigma)$. Simply integrating both sides of the last line of (7.2.8) gives the result:

$$\{y^{\mu}(\sigma_1), y^{\nu}(\sigma_2)\} = \begin{cases} -\beta^{\mu\nu}, & \sigma_1=\sigma_2=1 \\ +\beta^{\mu\nu}, & \sigma_1=\sigma_2=0 \\ 0 & \text{else.} \end{cases} \tag{7.2.9}$$

This is exactly the result of [251], derived without any reference to a mode expansion. Let us note that the total derivative ω-term in the last line of (7.2.8) was crucial for this result.

Closed string non-commutativity and non-associativity. Let us demonstrate the logic explicitly for the well-known standard example of the T-duality chain of the 3-torus with constant **H**-flux.

First, let us consider the **Q**-flux background ${\mathbf{Q}_3}^{12} = h$, all other components being zero, which is described by the generalised vielbein

$$E_{(Q)} = \begin{pmatrix} \mathbb{1} & 0 \\ \beta & \mathbb{1} \end{pmatrix}, \quad \beta^{12} = hx^3. \tag{7.2.10}$$

The corresponding current algebra including boundary terms is

$$\begin{aligned}
\{e_{0,a}(\sigma_1), e_{0,b}(\sigma_2)\} &= 0 \\
\left\{e_{0,a}(\sigma_1), e_1^b(\sigma_2)\right\} &= -\delta_a^b \partial_2 \delta(\sigma_1-\sigma_2) - \mathbf{Q}_a{}^{bc} e_{0,c}(\sigma_1)\delta(\sigma_1-\sigma_2) \\
\left\{e_1^a(\sigma_1), e_1^b(\sigma_2)\right\} &= -\mathbf{Q}_c{}^{ab} e_1^c(\sigma_1)\delta(\sigma_1-\sigma_2) - \int \mathrm{d}\sigma\, \partial \left(\beta^{ab}(\sigma)\delta(\sigma-\sigma_1)\delta(\sigma-\sigma_2)\right).
\end{aligned} \tag{7.2.11}$$

Let us consider the zero modes of the 'kinematic coordinates' associated to this generalised flux frame

$$\begin{aligned}
p_a &= \int \mathrm{d}\sigma\, p_a(\sigma) = \int \mathrm{d}\sigma\, e_{0,a}(\sigma) \qquad & \tilde{y}_a &= \int \mathrm{d}\sigma' \int^{\sigma'} \mathrm{d}\sigma\, p_a(\sigma) \\
w^a &= \int \mathrm{d}\sigma\, \partial y^a(\sigma) = \int \mathrm{d}\sigma\, e_1^a(\sigma) \qquad & y^a &= \int \mathrm{d}\sigma' \int^{\sigma'} \mathrm{d}\sigma\, \partial y^a(\sigma).
\end{aligned}$$

These modes have a priori nothing to do with the original target space interpretation. This seems particular confusing in case of the winding number. But cases like this exist in the literature, there it is sometimes called 'twisted boundary conditions', see e.g. in the context of β-deformations of $AdS_5 \times S^5$ [212]. In the present case, we have

$$\partial y^a(\sigma) \equiv e_1^a(\sigma) = \delta_\mu^a \partial x^\mu(\sigma) + \beta^{ab}\delta_b^\nu p_\nu(\sigma) \tag{7.2.12}$$

and

$$w^3 = w_x^3 \qquad \text{and} \qquad w^{1/2} = w_x^{1/2} \pm h \int \mathrm{d}\sigma x^3 p_{2/1}. \tag{7.2.13}$$

The winding along the y^3 direction coincides with the actual one along the x^3 direction as also y^3 coincides with x^3 up to a constant. Now, we can integrate the current algebra (7.2.11). We use a schematic mode expansion of the kinematic coordinates

$$y^a(\sigma) = y^a + \left(w^a - \frac{1}{2}y^a\right)\sigma + y^a_{osc}(\sigma) \tag{7.2.14}$$

with $y^a_{osc}(\sigma) = y^a_{osc}(\sigma+1)$ denoting oscillator terms, of which we will not keep track explicitly as we are interested in the zero modes. Alternatively, we could approach the this calculation by inserting the most general modes expansions or $x(\sigma)$ and $p(\sigma)$, that are compatible with the boundary condition, and calculating the contributions of all the modes directly by using the field redefinition (7.2.12). This calculation also shows that all the oscillators of the y-expansion would still commute with the zero modes, such that they do not give a contribution to the Jacobi identity of the zero modes.

Integrating (7.2.11), the non-vanishing Poisson brackets of the zero modes are

$$\begin{aligned}
\left\{y^1, y^2\right\} &\sim -hw^3 + \text{ osc.}, & \left\{w^1, w^2\right\} &= -hw^3 \\
\left\{\tilde{y}_3, y^1\right\} &\sim -hp_2 + \text{ osc.}, & \left\{\tilde{y}_3, y^2\right\} &\sim hp_1 + \text{ osc.} \\
\left\{p_3, w^1\right\} &= -hp_2, & \left\{p_3, w^2\right\} &= hp_1 \qquad (7.2.15)\\
\left\{y^a, p_b\right\} &= \delta^a_b + \text{ osc.}, & \left\{\tilde{y}_a, w^b\right\} &= \delta^b_a + \text{ osc.}
\end{aligned}$$
$$\left\{y^1, w^2\right\} = \left\{y^2, w^1\right\} = -h\left(\underline{y^3 + \frac{1}{2}w^3} + \text{ osc.}\right),$$

reproducing the known non-commutative interpretation of the pure **Q**-flux background. The underlined terms only stem from the boundary term and $\sim$ denotes some neglected constant factors, including integration constants. Also, let us emphasise again that our assumptions do not imply anything about a mode expansion apart from (7.2.14) resp. the boundary conditions. So, we can discuss the non-geometric structure without solving the theory first.

Non-associativity. There are non-trivial Jacobi identities of the zero mode Poisson brackets:

$$\begin{aligned}
&\left\{\tilde{y}_3, \left\{w^1, w^2\right\}\right\} + \text{ c.p.} \sim \left\{\tilde{y}_3, \left\{y^1, y^2\right\}\right\} + \text{ c.p.} \sim h = \mathbf{Q}_3{}^{12} \qquad (7.2.16)\\
&\left\{w^1, \left\{y^2, p_3\right\}\right\} + \left\{p_3, \left\{w^1, y^2\right\}\right\} + \left\{y^2, \left\{p_3, w^1\right\}\right\} \sim 0,
\end{aligned}$$

neglecting oscillator terms. The zero mode part of the second line vanishes due to the boundary term contribution (the underlined term in (7.2.15)). The first line is a non-associativity coming from a potential violation of the strong constraint. In fact, it is exactly the expected contribution from the discussion in section 7.2.1. Specifying the general expression (7.2.4) of the violation of the Jacobi identity due to strong constraint violations to $X_I(\sigma)$ and $\mathbf{E}_A(\sigma)$ gives

$$\{X_I(\sigma_1), \{\mathbf{E}_A(\sigma_2), \mathbf{E}_B(\sigma_3)\}\} + c.p. = \frac{1}{2}\left(\eta_{MN} + \omega_{MN}\right) E_{[A}{}^M(\sigma_1)\partial_I E_{B]}{}^N(\sigma_1)\delta(\sigma_3 - \sigma_1)\delta(\sigma_2 - \sigma_1).$$

As a cross check, we obtain the same form of **Q**-flux non-associativity in the first line of (7.2.4) by inserting the generalised vielbein to the **Q**-flux background and integrating accordingly as before. All the other terms in (7.2.4) vanish in this simple example.

The other T-duality chain backgrounds. The non-associativity will not be relevant if we only 'probe' the phase space with functions $f(y^a; \mathbf{E}_A)$ resp. $f(x^a; \mathbf{E}_I)$. As $\tilde{y}_3$ is not an argument of these functions, the **Q**-flux background given by the current algebra (7.2.11) is *associative* and thus locally geometric. But, there are other different choices of solutions of the strong constraints[1] which correspond to the T-dual backgrounds of the T-duality chain (see section 3):

$f(y^1, y^2, y^3; ...)$	locally geometric **Q**-flux background,
$f(\tilde{y}_1, y^2, y^3; ...)$ or $f(y^1, \tilde{y}_2, y^3; ...)$	locally geometric **f**-flux backgrounds,
$f(\tilde{y}_1, \tilde{y}_2, y^3; ...)$	locally geometric **H**-flux background.

In addition, there are of course also the continuous $\mathrm{O}(2,2)$-transformations on the y_1, y_2.

The solutions of the strong constraint containing $\tilde{y}_3$ give non-associative phase spaces, corresponding to the locally non-geometric backgrounds:

$f(y^1, y^2, \tilde{y}_3; ...)$	locally non-geometric **R**-flux background,
$f(\tilde{y}_1, y^2, \tilde{y}_3; ...)$ or $f(y^1, \tilde{y}_2, \tilde{y}_3; ...)$	locally non-geometric **Q**-flux backgrounds,
$f(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3; ...)$	locally non-geometric **f**-flux background.

These are all locally non-geometric as the generalised vielbein depends via β on $x^3 = y^3$, which is the origin of the non-associativity.

Overall, we reproduce the well-known zero mode brackets and non-vanishing Jacobiators [81, 149–156] of the considered (non-geometric) backgrounds without imposing a mode expansion or the equations of motion.

[1] We phrase them in the phase space variables of the **Q**-flux background. To get the standard picture, e.g. of the **H**-flux we make the identifications $y^1 \leftrightarrow \tilde{y}_1$ and $y^2 \leftrightarrow \tilde{y}_2$.

Chapter 8

Generalised T-duality

Generalisations of T-duality and their understanding in the classical Hamiltonian theory have been one of the main motivations for the introduction of the generalised flux frame, as presented in the previous chapter.

The material in this chapter has been presented already in [168, 221].

8.1 T-duality as canonical transformation

The discovery and examination of (generalised) T-dualities followed the path of constructions on the Lagrangian level. A classical proof of a duality is finding that such a construction corresponds to a canonical transformation. Demanding that the equations of motions take the same form is not enough. Otherwise, for example, the principal chiral model and the WZW model would be the same. Both, equations of motion as well as Bianchi identities, can be arranged to be

$$\mathrm{d}j + \frac{1}{2} j \wedge j = 0, \quad \mathrm{d} \star j = 0 \tag{8.1.1}$$

The difference lies in the meaning of j, which in the language of the Hamiltonian formalism corresponds to different Poisson brackets of the component j, see e.g. [231] for details. So, only if the equations of motion are the same and the transformation leaves the (canonical) Poisson structure invariant, and is followingly a canonical one, we can say that the two models are dual to each other.

In this section we want to pinpoint peculiarities of T-duality from the point of view of the Hamiltonian formalism in the generalised flux frame. We reverse the logic and construct canonical transformations that can be interpreted as generalised, classical T-dualities between different σ-model Lagrangians.

8.1.1 On canonical transformations and dualities

In the classical Hamiltonian theory, there is no notion of duality, only the more general notion of canonical transformations. Let us outline some conceptual differences.

- A generalised flux $\mathbf{F}_{ABC}$ and a generalised metric $\mathcal{H}$ do *not* yet define a string σ-model Lagrangian. We need to specify a corresponding generalised vielbein ${E_A}^M$ or in other words Darboux coordinates (x, p) of our deformed current algebra.

  This choice of generalised vielbein might not be unique. Different generalised vielbeins for a given generalised flux background correspond to dual σ-models Lagrangians.

- The framework, that we choose to study dualities, are models with *constant* generalised fluxes $\mathbf{F}_{ABC}$. In slight contrast to earlier in this section, we define a generic string model in the generalised flux frame by a Hamiltonian defined by a constant generalised metric $\mathcal{H}(G_0, B_0)$.[1]

  The duality group is realised linearly. I.e. a group element ${M_{A'}}^B$ leads to a dual model defined by

$$\mathcal{H}_{A'B'}(G'_0, B'_0) = {M_{A'}}^C {M_{B'}}^D \mathcal{H}_{CD}(G_0, B_0), \quad \mathbf{F}'_{A'B'C'} = {M_{A'}}^D {M_{B'}}^E {M_{C'}}^F \mathbf{F}_{DEF} \tag{8.1.2}$$

  Given that we find generalised vielbeins, ${E_A}^M(x)$ resp. ${E'_{A'}}^I(x)$ to the original generalised fluxes $\mathbf{F}_{ABC}$ resp. the dual ones $\mathbf{F}'_{A'B'C'}$, this defines two σ-model Lagrangians with equivalent Hamiltonian dynamics.

- The ${M_{A'}}^B$ are $\mathrm{O}(d, d)$-matrices, in order to keep the current algebra (7.1.5) form-invariant.[2] We take them to be constant such that the dual generalised metric and fluxes stay constant.

- Canonical transformations are normally characterised by generating functions. Our approach instead motivates directly that

$$M'^{\ J}_{I}(x) = E'^{\ A'}_{I}(x) {M_{A'}}^B {E_B}^J(x) \tag{8.1.3}$$

  corresponds a canonical transformation, i.e. leaves the canonical Poisson brackets of the $\mathbf{E}_<$ (6.1.22) invariant. In the next section we will motivate the existence of generating functions which would generate exactly the linearly realised factorised dualities and construct closely related charges on the phase space that generate the component connected to the identity of $\mathrm{O}(d, d)$.

- From the Hamiltonian point of view, a (constant) basis change of the $\mathbf{E}_A$ does not seem to make any difference on the first sight. The point is that we keep the role of the $(e_{0,a}, e_1^a)$ resp. $(p_i, \partial x^i)$ fixed. So e.g. the $\mathbf{f}$- and $\mathbf{H}$-flux always describe the e_0-e_0 Poisson bracket and so on. Rotating the generalised fluxes around and finding

[1] If we do not relax the condition $\mathcal{H} = \mathbb{1}$ on the generalised flux frame, the component connected to the identity of $\mathrm{O}(d, d)$ will generically lead out of this condition: $M\mathcal{H}M^T \neq \mathbb{1}$.

[2] We ignore the non $\mathrm{O}(d, d)$-invariant ω-term in (6.1.22) for our considerations in this section.

new generalised vielbeins which may depend on the *same* coordinates x is, what we define to be a duality here.[3]

We could have taken the other perspective of rotating our choices of Darboux coordinates, i.e. what of the $\mathbf{E}_M$ correspond to p_μ or ∂x^μ. In the language of double field theory these would be different solutions to the strong constraint. Both perspectives are of course equivalent.

- These duality transformations resp. canonical transformations should not be realisable by *purely local* field redefinitions in the σ-model Lagrangian, otherwise we would call them symmetries.

For the remainder of this section we will discuss standard (abelian) T-duality and Poisson-Lie T-duality from this point of view. Also, we propose a generalisation, which we call *Roytenberg duality*, for the case of frames with generic constant generalised fluxes.

The general construction. Let us summarise the above considerations in the following scheme:

$$
\begin{array}{ccccc}
\begin{array}{c}\text{'Darboux}\\ \text{frame' } \mathbf{E}_I\end{array} & \begin{array}{c}\mathcal{H}\left(G(x), B(x)\right)\\ \text{can. Poisson brackets}\\ (x,p)\end{array} & \overset{\Phi}{\dashrightarrow} & \begin{array}{c}\mathcal{H}\left(\tilde{G}(\xi), \tilde{B}(\xi)\right)\\ \text{can. Poisson brackets}\\ (\xi,\pi)\end{array} \\
 & \Big\downarrow {\scriptstyle E_A{}^M(x)} & & {\scriptstyle \tilde{E}_A{}^M(\xi)} \Big\uparrow \\
\begin{array}{c}\text{generalised flux}\\ \text{frame } \mathbf{E}_A\end{array} & \begin{array}{c}\mathcal{H}\left(G_0, B_0\right)\\ \mathbf{F}_{ABC}\end{array} & \xrightarrow[\text{const. O}(d,d)]{} & \begin{array}{c}\mathcal{H}\left(\tilde{G}_0, \tilde{B}_0\right)\\ \tilde{\mathbf{F}}_{ABC}\end{array}
\end{array}
$$

The generalised fluxes $\mathbf{F}$ and $\tilde{\mathbf{F}}$ are constant. As Φ is a map between to phase spaces with canonical Poisson structure, Φ is a canonical transformation. The challenge lies in finding different parameterisations of the constant generalised fluxes in terms of vielbeins $E_A{}^M(x)$ or $\tilde{E}_A^M(\xi)$. This transformation is phrased in terms of the current algebra. It will be difficult, and probably involving non-locality, to write down the concrete transformation $(x,p) \to (\xi,\pi)$. A construction like this was used in [20] for the proof of Poisson-Lie T-duality.

Abelian T-duality. The framework for the study of abelian T-duality is a background with commuting isometries. Let us choose coordinates, such that the isometries are manifest and ignore the spectator coordinates that do not correspond to isometries.

Such a model is defined by the Hamiltonian

$$H = \frac{1}{2}\int \mathrm{d}\sigma\, \mathcal{H}^{MN}(G_0, B_0)\mathbf{E}_M(\sigma)\mathbf{E}_N(\sigma) \tag{8.1.4}$$

[3] It is here where the pure **R**-flux background fails to exist purely geometrically, as we do not find such generalised vielbein only depending on the x

with constant G_0, B_0 and – neglecting the total derivative term from (6.1.22) –

$$\{\mathbf{E}_M(\sigma_1), \mathbf{E}_N(\sigma_2)\} = \eta_{MN}\partial_1\delta(\sigma_1 - \sigma_2). \tag{8.1.5}$$

Abelian T-duality acts via $\mathrm{O}(d,d)$-matrices M as $M_M{}^N \mathbf{E}_N$, leaving the current algebra invariant, but generating new Hamiltonians. Thus, the space of dual models is given by the coset $\frac{\mathrm{O}(d,d)}{\mathrm{O}(d)\times\mathrm{O}(d)}$. This can be seen by going to the model with $\mathcal{H} = \mathbb{1}$, where $\mathrm{O}(d) \times \mathrm{O}(d)$-matrices leave the Hamiltonian as well as the canonical current algebra invariant.

Poisson-Lie T-duality. The case of *Poisson-Lie T-duality* [18, 98, 252], and included in there also non-abelian T-duality [66, 74, 253], is the one with

$$\mathbf{H} = \mathbf{R} = 0, \quad \mathbf{f}^c{}_{ab} = f^c{}_{ab}, \quad \mathbf{Q}_c{}^{ab} = \overline{f}_c{}^{ab}. \tag{8.1.6}$$

The Bianchi identities of generalised fluxes (3.3.6) reduce to Jacobi identities of the f- and $\overline{f}$-structure constants and a mixed Jacobi identity. The algebraic setting is that the generalised fluxes $\mathbf{F}^C{}_{AB}$ correspond to structure constants of a Lie bialgebra $\mathfrak{d}$. A Lie bialgebra is a $2d$-dimensional Lie algebra with a non-degenerate symmetric bilinear form $\langle\,,\,\rangle$ on $\mathfrak{d}$ given by the $\mathrm{O}(d,d)$-metric η and two (maximally) isotropic[4] subalgebras $\mathfrak{g}$ and $\mathfrak{g}^\star$, of which f resp. $\overline{f}$ are the structure constants. Together with the Hamiltonian corresponding to an arbitrary constant generalised metric $\mathcal{H}(G_0, B_0)$ this model is also known under the name $\mathcal{E}$-model[5] in the literature.

It is well-known how the corresponding generalised vielbein looks like: It is of the type $E = E^{(\beta)}E^{(e)}$ as discussed in section 3. The d-dimensional vielbein e is given by the components of the Maurer-Cartan forms to the Lie group G associated to the structure constants $f^c{}_{ab}$, $e_i{}^a = (g^{-1}\partial_i g)^a$ where g are G-valued fields. β is the homogeneous Poisson bivector Π on G defined by the dual structure constants $\overline{f}_c{}^{ab}$, fulfilling

$$\Pi(e) = 0, \qquad \partial_c \Pi^{ab}(g) = \overline{f}_c{}^{ab} + f^{[a}{}_{cd}\Pi^{b]d}. \tag{8.1.7}$$

This bivector Π is uniquely determined by such a Lie bialgebra structure. The corresponding σ-model has the form

$$S \sim \int \mathrm{d}^2\sigma \left(\frac{1}{\frac{1}{G_0+B_0} - \Pi(g)}\right)_{ab} (g^{-1}\partial_+ g)^a (g^{-1}\partial_- g)^b. \tag{8.1.8}$$

Poisson-Lie T-duality acts linearly on the deformed current algebra associated to (8.1.6). This was discovered already in [20] and discussed in present form already in [18, 19, 21]. The total factorised duality simply corresponds to $\mathbf{f} \leftrightarrow \mathbf{Q}$, respectively $\mathfrak{g} \leftrightarrow \mathfrak{g}^\star$. The full duality group, which maintains the structure of the generalised fluxes (8.1.6) of

[4] meaning $\langle \mathfrak{g}, \mathfrak{g}\rangle = 0$.

[5] named after the operator $\mathcal{E}_A{}^B = \mathcal{H}_{AC}\eta^{CB}$ fulfilling $\mathcal{E}^2 = \mathbb{1}$

the $\mathcal{E}$-model is discussed in detail in [168]. It is the group of different Manin triple decompositions of the Lie bialgebra $\mathfrak{d}$.

At the Lagrangian level, the duality can be realised by considering a 'doubled' σ-model with target being the Drinfel'd double $\mathcal{D}$, and then integrating out d.o.f.s corresponding to different (isotropic) subalgebras $\mathfrak{g}^\star$ of the Lie bialgebra $\mathfrak{d}$ [18, 102, 103]. Other approaches to Poisson-Lie T-duality via double field theory and generalised geometry include [240, 254–256].

8.1.2 Roytenberg duality - beyond the Poisson-Lie setup.

Let us consider the generic case: arbitrary *constant* generalised fluxes and a Hamiltonian corresponding to an arbitrary constant generalised metric $\mathcal{H}(G_0, B_0)$. Let us call this case *Roytenberg* model, as a configuration with a generic generic fluxes with non-vanishing **H**, **f**, **Q** and **R** was first considered in [160]. It is not clear, in contrast to the Poisson-Lie σ-model, how to find a generalised vielbein for a generic choice of constant generalised fluxes. In section (3) we introduced two choices of generalised vielbeins which generically turn on all of the four generalised fluxes. We consider choices of generalised vielbein which build upon these two and the one of the Poisson-Lie σ-model:

- $E_1 = E^{(B)}_b E^{(\beta)}_{\beta_0} E^{(\beta)}_{\Pi} E^{(e)}$
- $E_2 = E^{(\beta)}_{\beta_0} E^{(B)}_{b_0} E^{(\beta)}_{\Pi} E^{(e)}$

as before (and want to have constant generalised fluxes). We take b and β_0 to be constant, e the vielbein of a Lie group G (corresponding to Lie algebra structure constants $f^c{}_{ab}$) and $\Pi(g)$ to be again a homogeneous Poisson bivector on G, associated to dual structure constants $\overline{f}_c{}^{ab}$. This choice of $\beta = \beta_0 + \Pi(g)$ in E_1 ensures that the resulting **Q**- and **R**-flux are constant as wished. The choice of $\beta = \beta_0 + \Pi(g)$ arose as well, if we go to the complete generalised flux frame of the Poisson-Lie σ-model, i.e. $\mathcal{H} = \mathbb{1}$, see [168]. A generalised version of Poisson-Lie T-duality, called *affine* Poisson-Lie T-duality, taking into account exactly such constant β_0's and mapping between different dual choices of β_0 and $\Pi(g)$ for $B = 0$ was considered in [257]. In the language of the Poisson-Lie T-duality group studied in [168], these were 'non-abelian β-shifts'. Let us give the corresponding fluxes and σ-model Lagrangians for E_1,

$$\begin{aligned}
\mathbf{H}_{abc} &= b_{[\underline{a}d} b_{\underline{b}e} \overline{f}_{\underline{c}]}{}^{de} - b_{d[a} f^d{}_{bc]} - b_{[\underline{a}d} b_{\underline{b}e} \beta_0^{f[d} f^{e]}{}_{\underline{c}]f} + b_{ad} b_{be} b_{cf} \mathbf{R}^{def} \\
\mathbf{f}^c{}_{ab} &= f^c{}_{ab} - b_{d[a} \overline{f}_{b]}{}^{dc} + b_{e[a} \beta_0^{d[e} f^{c]}{}_{b]d} + b_{ad} b_{be} \mathbf{R}^{cde} \\
\mathbf{Q}_c{}^{ab} &= \overline{f}_c{}^{ab} - \beta_0^{d[a} f^{b]}{}_{cd} + b_{cd} \mathbf{R}^{abd} \\
\mathbf{R}^{abc} &= \beta_0^{[\underline{a}d} \beta_0^{\underline{b}e} f^{\underline{c}]}{}_{de} - \beta_0^{d[a} \overline{f}_d{}^{bc]} \\
S_1 \quad &\sim \quad \int \mathrm{d}^2\sigma \left(\frac{1}{\frac{1}{G_0+B_0} - \beta_0 - \Pi(g)} - b \right)_{ab} (g^{-1}\partial_+ g)^a (g^{-1}\partial_- g)^b,
\end{aligned} \tag{8.1.9}$$

and for E_2,

$$\begin{aligned}
\mathbf{H}_{abc} &= b_{[\underline{a}d}b_{\underline{b}e}\overline{f}_{\underline{c}]}{}^{de} - b_{d[a}f^{d}{}_{bc]} \\
\mathbf{f}^{c}{}_{ab} &= f^{c}{}_{ab} - b_{d[a}\overline{f}_{b]}{}^{cd} + \beta_0^{cd}\mathbf{H}_{abd} \\
\mathbf{Q}_{c}{}^{ab} &= \overline{f}_{c}{}^{ab} - \beta_0^{d[a}f^{b]}{}_{cd} + \beta_0^{e[a}b_{d[e}\overline{f}_{c]}{}^{b]d} + \beta_0^{ad}\beta_0^{be}\mathbf{H}^{cde} \\
\mathbf{R}^{abc} &= \beta_0^{[\underline{a}d}\beta_0^{\underline{b}e}f^{\underline{c}]}{}_{de} - \beta_0^{d[a}\overline{f}_{d}{}^{bc]} - \beta_0^{[\underline{a}d}\beta_0^{\underline{e}b}b_{f[d}\overline{f}_{e]}{}^{df} + \beta_0^{ad}\beta_0^{be}\beta_0^{df}\mathbf{H}_{def} \\
S_2 \quad &\sim \quad \int \mathrm{d}^2\sigma \left(\left(\frac{1}{\frac{1}{G_0+B_0} - \Pi(g)} - b \right)^{-1} - \beta_0 \right)^{-1}_{ab} (g^{-1}\partial_+ g)^a (g^{-1}\partial_- g)^b
\end{aligned} \tag{8.1.10}$$

So, by construction the identifications

$$b \leftrightarrow \beta_0 \quad \text{and} \quad f \leftrightarrow \overline{f} \tag{8.1.11}$$

correspond to the map between the fluxes

$$\mathbf{H} \leftrightarrow \mathbf{R} \quad \text{and} \quad \mathbf{f} \leftrightarrow \mathbf{Q}. \tag{8.1.12}$$

This would be what we call the *(factorised) Roytenberg duality* in the terminology of (8.1.2). At the Lagrangian level, the two σ-models S_1 and S_2 are (classically) dual to each other with the identifications

$$G_0^{(1)} + B_0^{(1)} = \frac{1}{G_0^{(2)} + B_0^{(2)}}, \quad \beta_0^{(1)} = b^{(1)}, \quad \beta_0^{(2)} = b^{(1)}$$
$$f^{(1)} = \overline{f}^{(2)} \quad \text{and} \quad f^{(2)} = \overline{f}^{(1)},$$

where the superscript (i) denotes the quantities in S_i and we raised and lowered the indices appropriately.

Using the two generalised vielbeins E_1 and E_2 to describe these backgrounds, the Roytenberg duality simply seems to be an extension of the Poisson-Lie T-duality group. But these vielbeins are probably not the most general description of constant generalised fluxes, so the above example might give just a vague idea, of what a Roytenberg duality is in general and what kind of σ-model Lagrangians are connected by it.

The Roytenberg duality group is the full $\frac{\mathrm{O}(d,d)}{\mathrm{O}(d)\times\mathrm{O}(d)}$ rotating the generalised fluxes and is an interesting object of further study. A Lagrangian derivation of this duality might or might not exist. But still the Hamiltonian theory is well-defined as long as the constant generalised fluxes fulfil the Bianchi identities (3.3.6).

Let us close this section with the following remark. There seems to be no difference between abelian and generalised T-dualities from the Hamiltonian point of view. We could have viewed the standard T-duality chain of section 3 in same fashion[6] – again the true problem continues to be whether we can find appropriate vielbeins to the new fluxes.

[6]The only difference is that it includes one non-isometric spectator coordinate.

8.1.3 Realisation in the Poisson algebra

In this section we want to construct the charges that generate infinitesimal $\mathrm{O}(d,d)$-transformation in different generalised flux frames. These will show the need for isometries and are closely related to generating functions of the factorised dualities, not only for abelian T-duality but also the generalised version discussed above.

Infinitesimal $\mathfrak{o}(d,d)$-transformations via charges. Let us define the *non-local* charges[7]

$$\mathfrak{Q}_{[MN]} = \frac{1}{2}\int \mathrm{d}\sigma \int^{\sigma} \mathrm{d}\sigma' \, \mathbf{E}_M(\sigma')\mathbf{E}_N(\sigma) \tag{8.1.14}$$

which generate $\mathfrak{o}(d,d)$-transformations on the $\mathbf{E}_K(\sigma)$:

$$\left\{\mathfrak{Q}_{[IJ]}, \mathbf{E}_K(\sigma)\right\} = \eta_{K[I}\mathbf{E}_{J]}(\sigma) \tag{8.1.15}$$

From this and only with help of the Jacobi identity for the $\mathbf{E}_I(\sigma)$-current algebra, it is easy to show that these charges fulfil the $\mathrm{O}(d,d)$ Lorentz algebra

$$\left\{\mathfrak{Q}_{[IJ]}, \mathfrak{Q}_{KL}\right\} = \eta_{IK}\mathfrak{Q}_{JL} \; + \text{ permutations.} \tag{8.1.16}$$

A general infinitesimal $\mathrm{O}(d,d)$-transformation

$$M_{IJ} = \mathbb{1} + m_{IJ}, \qquad m \in \mathfrak{o}(d,d), \tag{8.1.17}$$

on the phase space is generated by $m^{IJ}\mathfrak{Q}_{[IJ]}$. We have not yet made any assumptions on the background, we worked with the canonical Poisson brackets, resp. in the the generalised metric frame.

The action of these charges on functions of the original world-sheet phase space (functions of $x^\mu(\sigma)$ and $p_\mu(\sigma)$) is non-local in general. In particular, the action of the β-transformations acts non-locally on functions on the original manifold

$$\{\mathfrak{Q}_{\mu\nu}, f(x(\sigma))\} = -\tilde{x}_{[\mu}\partial_{\nu]}f(x(\sigma)), \qquad \text{with} \quad \tilde{x}_\mu(\sigma) = \int^{\sigma} \mathrm{d}\sigma' p_\mu. \tag{8.1.18}$$

So far, $\tilde{x}(\sigma)$ is a *non-local* variable on the phase space. With the definitions $\partial_M = (\partial_\mu, \tilde{\partial}^\mu)$ and $X^M(\sigma) = \left(x^\mu(\sigma), \tilde{x}_\mu(\sigma)\right)$, we have for functions in terms of this non-local variable $\tilde{x}$

$$\left\{\mathfrak{Q}_{MN}, f\big(X(\sigma)\big)\right\} = -X_{[M}(\sigma)\partial_{N]}f\big(X(\sigma)\big) \tag{8.1.19}$$

[7] We use $\int^\sigma$ as a formal expression denoting the antiderivative. More precisely we the following procedure

$$\{\mathfrak{Q}_{MN}, F(\sigma)\} = \frac{1}{4}\lim_{\sigma_0\to\sigma}\left(\int \mathrm{d}\sigma' \int_{\sigma_0}^{\sigma'} \mathrm{d}\sigma'' \left\{\mathbf{E}_M(\sigma'')\mathbf{E}_N(\sigma'), F(\sigma)\right\}\right) \tag{8.1.13}$$

where it is only important that $\sigma_0 \neq \sigma$. We will come across similar ambiguities later as well, where we will define doubled coordinates $X^M = (x^\mu, \tilde{x}_\mu)$ as fundamental fields in the phase space, with $\mathbf{E}^M = \partial X^M(\sigma)$.

in an $\mathrm{O}(d,d)$-covariant way. If we instead considered (multi-)local function(al)s on the current phase space, spanned by the $\mathbf{E}_M(\sigma)$, everything stays (multi-)local

$$\{\mathfrak{Q}_{MN}, f(\mathbf{E}_K(\sigma))\} = -\mathbf{E}_{[M}(\sigma)\frac{\partial}{\partial \mathbf{E}^{N]}} f(\sigma). \tag{8.1.20}$$

These charges are (in general) not conserved – they do not commute with the Hamiltonian. Instead, they generate infinitesimal $\mathrm{O}(d,d)$-transformations of the generalised metric as wished, *if* the generalised metric is constant (again neglecting spectator coordinates). So, the charges $\mathfrak{Q}_{MN}$ generate *abelian* T-dualities.

'Non-abelian' $\mathfrak{o}(d,d)$-transformations. $\mathfrak{Q}_{MN}$ is a tensor under constant $\mathrm{O}(d,d)$-transformations, but *not* under local ones (due to the integral). Instead, we claim that we have natural charges $\mathfrak{Q}_{AB}$ w.r.t. to some generalised vielbein $E_A{}^I(\sigma)$ by the relation

$$\left\{\mathfrak{Q}_{[AB]}, \mathbf{E}_C(\sigma)\right\} = \eta_{C[A}\mathbf{E}_{B]}(\sigma). \tag{8.1.21}$$

Such a $\mathfrak{Q}_{AB}$ exists[8]. An implicit realisation for infinitesimal fluxes would be

$$\begin{aligned} \mathfrak{Q}^{\sigma_0}_{[AB]} &= \frac{1}{2}\int \mathrm{d}\sigma \int_{\sigma_0}^{\sigma} \mathrm{d}\sigma'\ \mathbf{E}_{[A}(\sigma')\mathbf{E}_{B]}(\sigma) + \bar{\mathfrak{Q}}^{\sigma_0}_{AB} \\ \text{with}\quad \delta\bar{\mathfrak{Q}}_{AB} &= \int \mathrm{d}\sigma \left(\delta\mathbf{E}^C(\sigma)\right) M_C{}^D(\sigma) \int_{\sigma_0}^{\sigma} \mathrm{d}\sigma' (M^{-1})_D{}^E(\sigma')\ \mathbf{\Omega}_{E[A}(\sigma')\mathbf{E}_{B]}(\sigma'), \end{aligned}$$

where $M(\sigma) = \exp\left(-\int_{\sigma_0}^{\sigma} \mathrm{d}\sigma' \mathbf{\Omega}(\sigma')\right)$ and $\mathbf{\Omega}_{AB} = \mathbf{F}^C{}_{AB}\mathbf{E}_C$. Finding an integrated form of this expression for a generic background seems highly non-trivial. Nevertheless, assuming that the Poisson brackets of the $\mathbf{E}_A(\sigma)$ fulfil the Jacobi identity, the Lorentz algebra follows directly from (8.1.21).

What this means is, that for every choice of generalised vielbein $\mathbf{E}_A(\sigma)$ modulo global $\mathrm{O}(d,d)$ transformation, there is a representation of $\mathrm{O}(d,d)$ acting on the phase space.

These are simply the linearly realised (infinitesimal) $\mathrm{O}(d,d)$ transformations in the generalised flux frame $\mathbf{E}_A(\sigma)$ of the previous section. The action of constant but infinitesimal $\mathrm{O}(d,d)$-matrix $M^{AB} = \mathbb{1} + m^{AB}$, the corresponding $\mathrm{O}(d,d)$ transformation is generated by $m^{AB}\mathfrak{Q}_{AB}$ as in the abelian case and similarly the β-shifts act non-locally (in momentum) on any function $f(x)$. Again these charges are not conserved but generate $\mathrm{O}(d,d)$-transformations on a constant generalised metric defining the Hamiltonian.[9]

Generating function of factorised dualities. Factorised dualities are canonical transformations generated by generating functions of type $F[q,Q]$ [20,75]. For this type of

[8] The defining relation (8.1.21) is an ODE in σ

[9] We assumed that the algebra of the $\mathbf{E}_A(\sigma)$ fulfils the Jacobi identity. There might be problems if the background is magnetically charged or, as we will see later, violates the strong constraint.

generating function we have

$$\frac{\delta \mathcal{F}}{\delta q} = p \quad \text{and} \quad \frac{\delta \mathcal{F}}{\delta Q} = -P. \tag{8.1.22}$$

The generating function for abelian T-duality is [75]

$$\mathcal{F}[x, \tilde{x}] = -\frac{1}{2} \int \mathrm{d}\sigma \, \left(\tilde{x} \partial x - x \partial \tilde{x} \right), \tag{8.1.23}$$

leading as wished to the identifications

$$p = \partial \tilde{x} \quad \text{and} \quad \tilde{p} = \partial x. \tag{8.1.24}$$

Using the notation of the previous paragraphs, this generating function can be written as

$$\mathcal{F}_{\mathfrak{Q}_{IJ}}[x, \tilde{x}] = -\eta^{IJ} \mathfrak{Q}_{IJ} \tag{8.1.25}$$

We include the subscript $\mathfrak{Q}_{IJ}$, as we cannot treat the indices of $\mathfrak{Q}_{IJ}$ as tensor indices.

The generating function for generalised T-dualities discussed in the previous subsection are in general very difficult to construct explicitly, see e.g. the construction of the generating function of Poisson-Lie T-duality in [20]. With help of the charges $\mathfrak{Q}_{AB}$, for which we do not know the explicit form but know their algebraic properties on the phase space (8.1.21), we can easily propose a generating function for

$$\mathcal{F}_{\mathfrak{Q}_{AB}}[x, \tilde{x}] = -\eta^{AB} \mathfrak{Q}_{AB}, \tag{8.1.26}$$

which generates factorised dualities in any generalised flux frame $\mathbf{E}_A$.

The generating function and even the associated canonical transformations seem to exist for any background and for any generalised flux frame, independent of whether the background possesses (generalised) isometries or not. The problem of the canonical transformations in non-isometric backgrounds (such that generalised metric or generalised fluxes are functions of x therein) again is, that the dual fields become functions of $\tilde{x}$ non-local functions of the canonical momenta.

8.2 A non-abelian T-duality group

8.2.1 Definition

Motivated by the discussion in the last section, a candidate for a non-abelian T-duality group[10] is

$$\text{NATD group}(\mathfrak{d}) = \{\text{Manin triple decompositions of } \mathfrak{d}\}\,. \tag{8.2.1}$$

[10] The investigation of this group, although it defined on basis of, what we called here, the Poisson-Lie σ-model, will turn out to contain isometric models like the principal chiral models and their non-abelian T-duals in many cases. As it will also turn out to be a direct generalisation of the abelian T-duality group $\mathrm{O}(d,d)$, it was decided in favour of the name 'non-abelian T-duality group' against 'Poisson-Lie T-duality group' here.

The elements of the NATD group as defined above is the set vector space automorphisms φ of $\mathfrak{d}$, which

1. preserve the natural pairing $\langle\,|\,\rangle$, so $\varphi \in \mathrm{O}(d,d)$.
2. preserve the algebraic closure of $\mathfrak{g}$ and $\mathfrak{g}^\star$, i.e.

$$[\varphi(\mathfrak{g}), \varphi(\mathfrak{g})] \subset \varphi(\mathfrak{g}) \quad \text{and} \quad [\varphi(\mathfrak{g}^\star), \varphi(\mathfrak{g}^\star)] \subset \varphi(\mathfrak{g}^\star). \tag{8.2.2}$$

In the case, where $\mathfrak{d}$ is abelian, (8.2.1) naturally becomes the $\mathrm{O}(d,d)$ group of abelian T-duality.

Let us emphasise that this group (8.2.1) is the modular space of Poisson-Lie σ-models corresponding to a bialgebra $\mathfrak{d}$ (and some given G_0, B_0). This does not imply that this group contains any (non-abelian) T-duality transformation, that we can think of. As a side note, we will find that condition 2 has to be partially relaxed in order to incorporate non-abelian T-dualities of principal chiral models with respect to subgroups. Nevertheless we will be content in the study of Poisson-Lie σ-models and thus focus on the above NATD group for most of this paper.

Action on the Poisson-Lie σ-model. $\mathrm{O}(d,d)$ basis transformations on the doubled σ-model (2.2.58) result in different dual σ-models, because we project onto the same Dirac structure (which defines the model), but integrate out different d.o.f.s corresponding to $\varphi(\mathfrak{g}^\star)$. This similarity action is given by

- $(\mathfrak{d}, \mathfrak{g}', \mathfrak{g}'^\star) = (\mathfrak{d}, \varphi(\mathfrak{g}), \varphi(\mathfrak{g}^\star))$, where again φ is *not a Lie algebra*, but only a *vector space* automorphism. Generically there will be a change in algebraic structure.
- standard $\mathrm{O}(d,d)$-action on the generalised metric (2.2.16) by the inverse φ

$$\mathcal{H}(G_0' + B_0') = \varphi^{-1} \cdot \mathcal{H}(G_0 + B_0)\varphi. \tag{8.2.3}$$

So, in addition to transforming the background $G_0 + B_0$ as in abelian T-duality, we also need to account for the change in algebraic structure. The transformed σ-model looks like

$$S = -\frac{1}{2} \int \mathrm{d}^2\sigma \, (g'^{-1}\partial_+ g')^a \left(\frac{1}{\frac{1}{G_0' + B_0'} + \Pi'(g')} \right)_{ab} (g'^{-1}\partial_- g')^b \tag{8.2.4}$$

where g' takes values in $\mathcal{G}' \triangleleft \mathcal{D}$, which is the Lie group to $\varphi(\mathfrak{g})$, and Π' is the homogeneous Poisson bivector field on $\mathcal{G}'$ corresponding to the transformed dual structure $\overline{F}_c{}^{ab}$.

8.2.2 Standard subgroups

In the subsequent literature, the group (8.2.1) was studied systematically only for lower dimensional bialgebras and without physical interpretation of the transformations, following the Bianchi classification of three dimensional Lie algebras [258–262]. Now, we want to understand some concrete structure of the NATD group apart from the original 'complete' factorised non-abelian T-duality transformations (called Poisson-Lie T-dualities so far) and give an explanation on the σ-model level or, if possible, a Buscher-like procedure. As the study of a generic $\varphi \in \mathrm{O}(d,d)$ is a little unhandy, we will make use of the standard decomposition of $\mathrm{O}(d,d)$ into factories dualities and the three continuous subgroups: GL-transformations, B- and β-shifts. We look for the conditions such that these lie in the NATD group (8.2.1) and also for the meaning of these transformations on the level of the (undoubled) Poisson-Lie σ-model.

The study of the standard $\mathrm{O}(d,d)$ subgroups should help to get physical understanding of this NATD group. The definition (8.2.1) will severely restrict the allowed factorised dualities, B-shifts and β-shifts. But it is by no means to expected that these subgroups generate all elements of (8.2.1), and resultantly our investigation may only give a subgroup of (8.2.1).

Lie (bi)algebra automorphisms. For example, all the standard $\mathrm{O}(d,d)$ subgroup transformations will turn out to be generically *not* Lie algebra automorphisms. But, Lie algebra automorphisms of $\mathfrak{d}$ that also preserve the $\mathrm{O}(d,d)$-metric will also be part of the duality group, acting only on the background data G_0 and B_0. We will not consider these further.

(Non-abelian T-duality) GL-transformations. The simplest continuous subgroup of $\mathrm{O}(d,d)$, general linear transformations $\mathrm{GL}(d)$ of $\mathrm{O}(d,d)$

$$\varphi_{GL} = \begin{pmatrix} A^T & 0 \\ 0 & A^{-1} \end{pmatrix}, \quad \text{with } A \in \mathrm{GL}(d) \tag{8.2.5}$$

is clearly contained fully in the non-abelian T-duality group. It describes simultaneous basis changes of $\mathfrak{g}$ and $\mathfrak{g}^*$ preserving the $\mathrm{O}(d,d)$-metric and also the algebraic closure conditions.

Factorised non-abelian T-dualities. Factorised dualities of $\mathrm{O}(d,d)$ are the $\mathbb{Z}_2$-transformations corresponding to the maps

$$\begin{aligned} \varphi_{f.d.}:\ & t_\alpha \mapsto t'_\alpha = \bar{t}^\alpha,\ t_{\underline{\alpha}} \mapsto t'_{\underline{\alpha}} = t_{\underline{\alpha}} \\ & \bar{t}^\alpha \mapsto \bar{t}'^\alpha = t_\alpha,\ \bar{t}^{\underline{\alpha}} \mapsto \bar{t}'^{\underline{\alpha}} = \bar{t}^{\underline{\alpha}} \end{aligned} \tag{8.2.6}$$

for $\alpha = 1, ..., m$ and $\underline{\alpha} = m+1, ..., d$, for an $m \leq d$.[11] Following the definition (8.2.1) the $\varphi_{f.d.}$ are only NATD transformations, if the $\{t_a, \bar{t}^a\} = \{t_\alpha, t_{\underline{\alpha}}\,,\ \bar{t}^\alpha, \bar{t}^{\underline{\alpha}}\}$ fulfil the following conditions:

$$\bar{f}_{\underline{\gamma}}{}^{\alpha\beta} = \bar{f}_{\gamma}{}^{\underline{\alpha\beta}} = f^{\underline{\gamma}}{}_{\alpha\beta} = f^{\gamma}{}_{\underline{\alpha\beta}} = 0 \tag{8.2.7}$$

[11]This is generic as we can arrange any choice of the generators $\{t_a\}$ with help of GL-transformations.

This is equivalent to the decomposition of $\mathfrak{d}$

$$\begin{aligned}
&\mathfrak{d} = (\mathfrak{h} \oplus \mathfrak{m}) \oplus_{\mathfrak{d}} (\mathfrak{h}^\star \oplus \mathfrak{m}^\star) \\
\text{with} \qquad &[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}, \qquad [\mathfrak{h}^\star,\mathfrak{h}^\star] \subset \mathfrak{h}^\star, \\
&[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{m}, \qquad [\mathfrak{m}^\star,\mathfrak{m}^\star] \subset \mathfrak{m}^\star, \\
&[\mathfrak{h},\mathfrak{m}^\star] \subset \mathfrak{h} \oplus \mathfrak{m}^\star, \qquad [\mathfrak{h}^\star,\mathfrak{m}] \subset \mathfrak{h}^\star \oplus \mathfrak{m},
\end{aligned} \tag{8.2.8}$$

where $\mathfrak{h}$ resp. $\mathfrak{h}^\star$ is generated by $\{t_\alpha\}$ resp. $\{\bar{t}^\alpha\}$ and $\mathfrak{m}$ resp. $\mathfrak{m}^\star$ by $\{t_{\underline{\alpha}}\}$ resp. $\{\bar{t}^{\underline{\alpha}}\}$. Thus the factorised dualities act as

$$(\mathfrak{h} \oplus \mathfrak{m}) \oplus_{\mathfrak{d}} (\mathfrak{h}^\star \oplus \mathfrak{m}^\star) \quad \leftrightarrow \quad (\mathfrak{h}^\star \oplus \mathfrak{m}) \oplus_{\mathfrak{d}} (\mathfrak{h} \oplus \mathfrak{m}^\star) \tag{8.2.9}$$

on the bialgebra structure. The dual Poisson-Lie σ-model

$$L \quad \propto \quad j'^a_+ \left(\frac{1}{(E'_0)^{-1} + \Pi'} \right)_{ab} j'^b \tag{8.2.10}$$

consists of the Maurer-Cartan forms j' to the Lie group of the algebra $\mathfrak{h}^\star \oplus \mathfrak{m}$, the homogeneous Poisson structure Π' (corresponding to the new bialgebra (8.2.9)) and the transformed background E'_0, given by the standard $\mathrm{O}(d,d)$ action of $\varphi_{f,d}$ (8.2.6) on the original E_0.

The conditions (8.2.7) seem to be very restrictive - even for the semi-abelian bialgebra $\mathfrak{g} \oplus_{\mathfrak{d}} (\mathfrak{u}(1))^d$ with an abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$ these are not fulfilled in general, due to $[t_{\underline{\alpha}}, \bar{t}^\beta] = f^\beta{}_{\underline{\gamma\beta}} \bar{t}^\gamma \notin \mathfrak{h}^\star \oplus \mathfrak{m}$ in general. In the following paragraph we study exactly this scenario - the non-abelian T-duality of the principal chiral model w.r.t. to a subgroup. This will indeed demonstrate that it cannot be put into the Poisson-Lie σ-model form (2.2.60).

Subgroup non-abelian T-duality. Consider the principal chiral model on a group G, which has a subgroup H with Lie algebra $\mathfrak{h}$. We decompose the generators of G correspondingly, $\{t_a\} = \{t_\alpha, t_{\underline{\alpha}}\}$, and the model by choosing $g = hm$ with $h \in H$ and some $m \in G$, so that

$$g^{-1}\mathrm{d}g = \mathrm{Ad}_m^{-1} \left(h^{-1}\mathrm{d}h + \mathrm{d}m\, m^{-1} \right). \tag{8.2.11}$$

The action becomes

$$\begin{aligned}
S &\propto \int \mathrm{Tr} \left(\left(h^{-1}\mathrm{d}h + \mathrm{d}m\, m^{-1} \right) \wedge \star \left(h^{-1}\mathrm{d}h + \mathrm{d}m\, m^{-1} \right) \right) \\
&= \int \left\{ \mathrm{Tr} \left(\left(A + \mathrm{d}m\, m^{-1} \right) \wedge \star \left(A + \mathrm{d}m\, m^{-1} \right) \right) - \bar{x}_\alpha \left(\mathrm{d}A^\alpha + [A \overset{\wedge}{,} A]^\alpha \right) \right\}.
\end{aligned} \tag{8.2.12}$$

Integrating out the $\mathfrak{h}$-valued field A yields

$$\begin{aligned}
\tilde{S} \propto \int \mathrm{d}^2\sigma \Big\{ & \left(\partial_+ \bar{x}_\alpha + (\partial_+ m\, m^{-1})^\sigma \kappa_{\sigma\alpha} \right) \frac{1}{\kappa_{\alpha\beta} - \bar{x}_\gamma f^\gamma{}_{\alpha\beta}} \left(\partial_- \bar{x}_\beta - \kappa_{\beta\tau} (\partial_- m\, m^{-1})^\tau \right) \\
& + \mathrm{Tr} \left((\partial_+ m\, m^{-1})\, (\partial_- m\, m^{-1}) \right) \Big\}.
\end{aligned} \tag{8.2.13}$$

The equations of motion can be expressed in the following form

$$\mathrm{d}\bar{J}_a + \frac{1}{2} Q_a{}^{bc} \bar{J}_b \wedge \bar{J}_c = 0 \tag{8.2.14}$$

with the Poisson structure $\Pi_{\alpha\beta} = -\bar{x}_\gamma f^\gamma{}_{\alpha\beta}$ and the current[12]

$$\bar{J}_\pm = \pm \left(\frac{1}{\mathbb{1} \pm \Pi} \right)_{\alpha\beta} \left(\partial_\pm \bar{x}^\beta \pm (\partial_\pm m\, m^{-1}) \right) \bar{t}'^\alpha \pm \delta_{\underline{\alpha\beta}} (\partial_\pm m\, m^{-1})^{\underline{\alpha}} \bar{t}'^{\underline{\beta}} \tag{8.2.15}$$

and the new generators of the transformation (8.2.6), where the $\{t_\alpha\}$ generate a subalgebra:

$$\begin{aligned}
&[t'_\alpha, t'_\beta] = 0 \equiv F^c{}_{\alpha\beta} t'_c + H_{\alpha\beta c} \bar{t}'^c, && [\bar{t}'^\alpha, \bar{t}'^\beta] = f^\gamma{}_{\alpha\beta} \bar{t}'^\gamma \equiv Q_\gamma{}^{\alpha\beta} \bar{t}'^\gamma + R^{\alpha\beta c} t'_c \\
&[t'_\alpha, t'_{\underline{\beta}}] = f^\alpha{}_{\gamma\underline{\beta}} t'_\gamma + f^\alpha{}_{\underline{\gamma\beta}} \bar{t}'^{\underline{\gamma}} \equiv F^\gamma{}_{\alpha\underline{\beta}} t'_\gamma + H_{\alpha\underline{\beta\gamma}} \bar{t}'^{\underline{\gamma}}, && [\bar{t}'^\alpha, \bar{t}'^{\underline{\beta}}] = f^{\underline{\beta}}{}_{\underline{\gamma}\alpha} \bar{t}'^{\underline{\gamma}} \equiv Q_{\underline{\gamma}}{}^{\alpha\underline{\beta}} \bar{t}'^{\underline{\gamma}} + R^{\alpha\beta c} t'_c \\
&[t'_{\underline{\alpha}}, t'_{\underline{\beta}}] = f^{\underline{\gamma}}{}_{\underline{\alpha\beta}} t'_{\underline{\gamma}} + f^\gamma{}_{\underline{\alpha\beta}} \bar{t}'^\gamma \equiv F^{\underline{\gamma}}{}_{\underline{\alpha\beta}} t'_{\underline{\gamma}} + H_{\underline{\alpha\beta}\gamma} \bar{t}'^\gamma, && [\bar{t}'^{\underline{\alpha}}, \bar{t}'^{\underline{\beta}}] = 0 \equiv Q_c{}^{\underline{\alpha\beta}} \bar{t}'^c + R^{\underline{\alpha\beta}c} t'_c \\
&[t'_\alpha, \bar{t}'^\beta] = f^\alpha{}_{\beta\gamma} t'_\gamma + f^\alpha{}_{\beta\underline{\gamma}} \bar{t}'^{\underline{\gamma}} \equiv F^\beta{}_{\alpha\underline{\gamma}} \bar{t}'^{\underline{\gamma}} + Q_\alpha{}^{\beta\gamma} t'_\gamma, && [t'_\alpha, \bar{t}'^{\underline{\beta}}] = 0 \equiv F^{\underline{\beta}}{}_{c\alpha} \bar{t}'^c + Q_\alpha{}^{\underline{\beta}c} t'_c \\
&[t'_{\underline{\alpha}}, \bar{t}'^\beta] = f^\gamma{}_{\underline{\alpha}\beta} t'_\gamma + f^{\underline{\gamma}}{}_{\underline{\alpha}\beta} \bar{t}'^\gamma \equiv F^\beta{}_{\gamma\underline{\alpha}} \bar{t}'^\gamma + Q_{\underline{\alpha}}{}^{\beta\underline{\gamma}} t'_\gamma, && [t'_{\underline{\alpha}}, \bar{t}'^{\underline{\beta}}] = f^{\underline{\beta}}{}_{\gamma\underline{\alpha}} t'_\gamma + f^{\underline{\beta}}{}_{\underline{\gamma\alpha}} \bar{t}'^{\underline{\gamma}} = F^{\underline{\beta}}{}_{\underline{\gamma\alpha}} \bar{t}'^{\underline{\gamma}} + Q_{\underline{\alpha}}{}^{\underline{\beta}\gamma} t'_\gamma,
\end{aligned} \tag{8.2.16}$$

where we organised the resulting structure constants in the conventions of generalised fluxes.[13] So, formally (8.2.13) looks like a Poisson-Lie σ-model, but the current

$$\left(\partial_\pm \bar{x}_\alpha \pm (\partial_\pm m\, m^{-1})^\sigma \kappa_{\sigma\alpha} \right) \bar{t}^\alpha + (\partial_\pm m\, m^{-1})^{\underline{\alpha}} t_{\underline{\alpha}} \in \mathfrak{h}^\star \oplus \mathfrak{m} \tag{8.2.17}$$

is not the Maurer-Cartan form of a group, because $\mathfrak{h}^\star \oplus \mathfrak{m}$ is not closed under the Lie bracket generically. As long as $f^\alpha{}_{\underline{\beta\gamma}}$ (the only non vanishing component of H in (8.2.16)) does not vanish, it does not seem possible to arrange the Bianchi identities of (8.2.17) into a zero curvature form, which would required in order for the subgroup non-abelian T-dual model to be of the Poisson-Lie σ-model form - this agrees with (8.2.1).

A modified definition of a NATD group. Motivated by the above considerations, let us give a refined version for the definition of the NATD group

$$\begin{aligned}
\text{mod. NATD group } (\mathfrak{d}) &= \{\text{Manin pair decompositions of } \mathfrak{d}\} \\
&\simeq \{ (\varphi : \ \mathfrak{d} \to \mathfrak{d}) \in \mathrm{O}(d,d) : \ [\varphi(\mathfrak{g}^\star), \varphi(\mathfrak{g}^\star)] \subset \varphi(\mathfrak{g}^\star) \} \,,
\end{aligned} \tag{8.2.18}$$

[12] Superficially, it looks as if this would have increased the degrees of freedom, but the variation of (8.2.13) with respect to $(\partial m\, m^{-1})^\alpha$ vanishes by the equations of motion for $\bar{x}_\alpha$.

[13] Because we start with a semi-abelian bialgebra and the $f^c{}_{ab}$ fulfil the Jacobi identity, the coefficients H, F, Q and R fulfil the standard Bianchi identities of non-geometric fluxes [131]. We will comment further on this topic in section 3.

which goes beyond the notion of Poisson-Lie σ-models, but includes the previous case.[14] From the perspective of the construction from a doubled σ-model (2.2.58), the above scenario is plausible, because for integrating out the degrees of freedom consistency requires only, that $\mathfrak{g}^\star$ is a subalgebra (resp. $\tilde{\mathcal{G}}$ a subgroup), but not $\mathfrak{g}$. The refined conditions in comparison to (8.2.7) for factorised dualities were already stated in [100] with slight differences[15]:

$$\begin{aligned} &(\mathfrak{h} \oplus \mathfrak{m}) \oplus_{\mathfrak{d}} (\mathfrak{h}^\star \oplus \mathfrak{m}^\star) \\ \text{with} \quad &[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{m}^\star,\mathfrak{m}^\star] \subset \mathfrak{m}^\star \quad \text{and} \quad [\mathfrak{h},\mathfrak{m}^\star] \subset \mathfrak{h} \oplus \mathfrak{m}^\star, \end{aligned} \tag{8.2.20}$$

Of course, the setting of bialgebras resp. Drinfel'd doubles, Poisson-Lie σ-models and Poisson-Lie T-duality is very narrow. A consistent treatment of the modified definition (8.2.18) would require a different setting, i.e. one, which is *not* based on a bialgebra $\mathfrak{d}$, but on any even dimensional Lie algebra, which admits an $\mathrm{O}(d,d)$-invariant metric and has one maximally isotropic subalgebra. This would include very different settings, of course bialgebras but e.g. also the setup discussed in [263], with a symmetric space decomposition $\mathfrak{d} = \mathfrak{m}^{(0)} \oplus \mathfrak{m}^{(1)}$, where $\mathfrak{m}^{(0)}$ is an isotropic subspace and -algebra w.r.t. to the $\mathrm{O}(d,d)$ metric and $\mathfrak{m}^{(1)}$ is complementary isotropic subspace, but fulfils $[\mathfrak{m}^{(1)},\mathfrak{m}^{(1)}] = \mathfrak{m}^{(0)}$ and thus does not close.

Nevertheless, we will continue to work with the more restrictive Manin triple definition (8.2.1), as it gives already some interesting insights in the subgroups in the component connected to identity of the duality group (8.2.1), which we will study in the following.

Non-abelian T-duality B-shifts. Let us come to the B-shift and β-shift subgroups of the NATD group, that have been considered in the literature first in [168]. In context of abelian T-duality B-shifts correspond to gauge transformations of the Kalb-Ramond field B, leaving the $\mathbf{H}$-flux, $\mathbf{H} = \mathrm{d}B$, invariant. The expectation is that this behaviour generalises to the $\mathbf{H}$-flux of the Poisson-Lie σ-model (2.2.60) and the B-shifts of the NATD group (8.2.1) (from now on NATD B-shifts).

In this paragraph we are going to discuss how the NATD B-shift looks and how it acts on the Poisson-Lie σ-model. B-shifts in $\mathrm{O}(d,d)$ are of the form

$$\varphi_B = \begin{pmatrix} \mathbb{1} & \sigma_{ab} \\ 0 & \mathbb{1} \end{pmatrix} \tag{8.2.21}$$

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[14] A Manin pair $(\mathfrak{d},\mathfrak{g}^\star)$ is a pair consisting of a $2d$-dimensional Lie algebra $\mathfrak{d}$ admitting an $\mathrm{O}(d,d)$-metric and a Lagrangian subalgebra of $\mathfrak{d}$, here denoted by $\mathfrak{g}^\star$.

[15] In [100] the authors required that for subgroup Poisson-Lie T-duality the dual flatness condition should decompose fully

$$\mathrm{d}\bar{j}_\alpha + \frac{1}{2}\bar{f}_\alpha{}^{bc}\bar{j}_b\bar{j}_c \equiv \mathrm{d}\bar{j}_\alpha + \frac{1}{2}\bar{f}_\alpha{}^{\beta\gamma}\bar{j}_\beta \wedge \bar{j}_\gamma = 0. \tag{8.2.19}$$

But, the factorised Poisson-Lie T-duality map φ, acting on the currents $j = g^{-1}\mathrm{d}g$ and $\bar{j}_{\pm,a} = \pm\left(\frac{1}{E_0^{-1} \pm \Pi}\right)_{ab} j^b_\pm$, is more complicated than $j^\alpha \leftrightarrow \bar{j}_\alpha$ and $E_0 \leftrightarrow E_0^{-1}$ in the abelian case – it consists also of exchanging $\Pi \to \Pi'$. Our analysis shows, that condition (8.2.19) is not required for consistency of Poisson-Lie T-duality with a skewsymmetric $d \times d$-matrix σ_{ab}. The transformed algebra relations are

$$[t'_a, \bar{t}'_b] = F^c{}_{ab} t'_c + H_{abc} \bar{t}'^c \tag{8.2.22}$$

$$\text{with} \quad F^c{}_{ab} = f^c{}_{ab} + \sigma_{k[a} \bar{f}_{b]}{}^{kc} \quad \text{and} \quad H_{abc} = \sigma_{[a|d} \sigma_{|b|e} \bar{f}_{|c]}{}^{de} - \sigma_{k[a} f^k{}_{bc]} \stackrel{!}{=} 0, \tag{8.2.23}$$

by imposing algebraic closure on $\varphi_B(\mathfrak{g})$. The only relevant Jacobi identities, we need to check for these new algebra relations (8.2.22) are:

$$\begin{aligned} F^k{}_{[ab} F^d{}_{c]k} &= f^k{}_{ab} f^d{}_{ck} - \sigma_{ma} \sigma_{nb} \bar{f}_k{}^{[mn} \bar{f}_c{}^{d]k} + \bar{f}_c{}^{dk} H_{abk} \\ &\quad + \sigma_{cm} \left(f^k{}_{ab} \bar{f}_k{}^{dm} - f^{[d}{}_{k[a} \bar{f}_{b]}{}^{m]k} \right) + (\text{c. p. of } (abc)) \qquad (8.2.24) \\ F^k{}_{ab} \bar{f}_k{}^{mn} - F^{[m}{}_{k[a} \bar{f}_{b]}{}^{n)k} &= f^k{}_{ab} \bar{f}_k{}^{mn} - f^{(d}{}_{k[a} \bar{f}_{b]}{}^{n]k} + \sigma_{l[a} \bar{f}_{b]}{}^{l[k} \bar{f}_k{}^{mn]} \end{aligned}$$

which vanish due to Jacobi identities of the original bialgebra and the condition (8.2.23), $H \equiv 0$. This condition is the crucial requirement of a NATD B-shift in comparison to the abelian case, where it is trivially fulfilled. Let us distinguish three cases to understand it better:

1. $\mathfrak{g}^\star$ is abelian: $\bar{f}_c{}^{ab} \equiv 0$

   Then $0 \equiv H_{abc} = -\sigma_{k[a} f^k{}_{bc]}$, means that $\sigma_{ab} (g^{-1}\mathrm{d}g)^a \wedge (g^{-1}\mathrm{d}g)^b$ is a closed 2-form on $\mathcal{G}$. In this case the NATD B-shift simply adds of a 2-cocycle term to the Lagrangian:

   $$S \propto \int \mathrm{d}^2\sigma (g^{-1}\partial_+ g)^a \left[G_0 + B_0 + \sigma\right]_{ab} (g^{-1}\partial_- g)^b, \tag{8.2.25}$$

   which is a gauge transformation of the **H**-flux. Later we will argue that this is a generic feature also in the generic case.

2. $\mathfrak{g}$ is abelian: $f^c{}_{ab} \equiv 0$

   $\sigma_{ab} \bar{t}^a \wedge \bar{t}^b$ is a solution of the classical Yang-Baxter equation on $\mathfrak{g}^\star$. In this case a σ-model interpretation is possible in the dual picture - there B-shifts will be β-shifts of an isometric model. We will show in the next paragraph on NATD β-shifts, that these are indeed easier to understand in this specific case and that we can employ a generalised Buscher procedure there.

3. generic case:

   Generically σ_{ab} will be neither a 2-cocycle on $\mathfrak{g}$, nor a solution to the classical Yang-Baxter equation on $\mathfrak{g}^\star$. (8.2.23) says that the failures for both cancel each other out. We can also view (8.2.23) as 2-cocycle condition of σ_{ab} w.r.t. the *new* structure constants in (8.2.23)

   $$H_{abc} = -\sigma_{k[a} F^k{}_{bc]} = 0. \tag{8.2.26}$$

If we restrict to start with $f^c{}_{ab}$ being a 1-coboundary algebra to $\overline{f}_c{}^{ab}$ with $f^c{}_{ab} = -\overline{f}^{bc}_{(a}\tau_{b)d}$ for some $\tau = \tau_{ab}\overline{t}^a \wedge \overline{t}^b$ and make the ansatz $\sigma_{ab} = \tau'_{ab} - \tau_{ab}$, condition (8.2.23) becomes

$$\overline{f}_e{}^{cd}\left[\tau'_{ac}\tau'_{bd} - \tau_{ac}\tau_{bd}\right] + \text{c.p. of } (abe) = 0 \tag{8.2.27}$$

which is satisfied if τ' and τ fulfil *the same* Yang-Baxter *like* equation. In this case we can understand a NATD B-shift as

- exchanging the 1-coboundary bialgebra structures $f^c{}_{ab} \leftrightarrow F^c{}_{ab}$ that fit to $\mathfrak{g}^\star$, which is unaffected by the B-shift. This will also change $\Pi \rightarrow \tilde{\Pi}$ with $\tilde{\Pi}$ being of the standard form (2.2.52) corresponding to the dual structure constants $\overline{f}_c{}^{ab}$ but now on a new group with structure constants $F^c{}_{ab}$.
- standard action of φ_B on $E_0 = G_0 + B_0$.

A generic Buscher-like procedure or some other action on the (non-doubled) Lagrangian level, which reproduces the NATD B-shift action, has not been found yet, but it is not necessarily expected to exist, as there is also none for the factorised dualities. The justification for these transformation thus lies in the common origin in the same doubled σ-model (2.2.58).

Non-abelian T-duality β-shifts. From the perspective of the definition of the NATD group (8.2.1), β-shifts are exactly conjugate to the previously encountered NATD B-shifts. On the other hand, the formulation of the Poisson-Lie σ-model is not duality symmetric, so NATD β-shifts and their action on Poisson-Lie σ-model deserve some attention on their own. β-shifts in $\mathrm{O}(d,d)$ are of the form

$$\varphi_\beta = \begin{pmatrix} \mathbb{1} & 0 \\ r^{ab} & \mathbb{1} \end{pmatrix} \tag{8.2.28}$$

with a skewsymmetric $d \times d$-matrix r^{ab}. The transformed algebra relations are

$$[\overline{t}'^a, \overline{t}'^b] = \overline{F}_c{}^{ab}\overline{t}'^c + R^{abc}t'_c \tag{8.2.29}$$

$$\text{with} \quad \overline{F}_c{}^{ab} = \overline{f}_c{}^{ab} + r^{k[a}f^{b]}{}_{kc} \quad \text{and} \quad R^{abc} = r^{[a|d}r^{|b|e}f^{|c]}{}_{de} - r^{k[a}\overline{f}_k{}^{bc]} \stackrel{!}{=} 0, \tag{8.2.30}$$

Also, the solutions to the closure condition are basically the same as those in the B-shift case. In the dual picture (understand a NATD β-transformation as the sequence *NATD factorised duality - NATD B-shift - NATD factorised duality*) the interpretation is exactly the same as NATD B-shifts.

1. <u>$\mathfrak{g}$ is abelian: $f^c{}_{ab} \equiv 0$</u>

   $r = r^{ab}t_a \wedge t_b$ is a symplectic 2-form on $\mathfrak{g}^\star$. In this case the NATD β-shift is indeed easiest understood in the dual picture, where it is simply a NATD B-shift with the intuitive σ-model interpretation as discussed in the previous paragraph.

2. $\mathfrak{g}^\star$ is abelian: $\overline{f}_c{}^{ab} \equiv 0$

   Then $r = r^{ab} t_a \wedge t_b$ is a solution of the classical Yang-Baxter equation on $\mathfrak{g} \otimes \mathfrak{g}$. The resulting σ-model will be discuss below.

3. generic case:

   If both $\mathfrak{g}$ and $\mathfrak{g}^\star$ are non-trivial, then we use (8.2.30). Let us restrict to 1-coboundary bialgebras with $\overline{f}_c{}^{ab} = -f^{[a}{}_{bc} s^{b]d}$ for some $s = s^{ab} t_a \wedge t_b$. With the ansatz $r^{ab} = s'^{ab} - s^{ab}$ condition (8.2.30) becomes

$$f^e{}_{cd} \left[s'^{ac} s'^{bd} - s^{ac} s^{bd} \right] + \text{c.p. of } (abe) = 0, \tag{8.2.31}$$

   which is satisfied if s' and s fulfil *the same* Yang-Baxter *like* equation. NATD β-shifts switch between different choices of dual Lie algebra for a given $f^c{}_{ab}$, in a way that r corresponds to a 2-cocycle on the new dual algebra

$$r^{k[a} \overline{F}_k{}^{bc]} = 0. \tag{8.2.32}$$

   This incorporates the cases, where we can 'β-untwist' to standard $\mathcal{G}$-isometric σ-models.

   Exactly dual to the generic NATD B-shift case, these kinds of NATD β-transformations can be thought of as connecting Poisson-Lie σ-models for the same quasi-isometry algebra $\mathfrak{g}$ but different dual structure $\mathfrak{g}^\star$ connected by (8.2.31) fulfilling condition (8.2.30). This condition appeared in [257] as a condition for a correspondence between equivalent Poisson bivectors on a group.

   In general, the NATD β-shifted Poisson-Lie σ-model is given by

$$S = -\frac{1}{2} \int \mathrm{d}^2\sigma \, (g^{-1}\partial_+ g)^a \left(\frac{1}{\frac{1}{G_0 + B_0} - r + \Pi'(g)} \right)_{ab} (g^{-1}\partial_- g)^b. \tag{8.2.33}$$

σ-model interpretation via generalised Buscher procedure. Generically, exactly as for the NATD B-shifts or factorised dualities, a derivation of NATD β-shifts only on the (non-doubled) Lagrangian level is not available. But in the semi-abelian case, the $\mathcal{G}$-isometric model (2.2.35), it is possible and mediated by the non-abelian generalisation of the 'generalised Buscher procedure' (see section 2). It was introduced already in [91] to show for certain examples on AdS_5, that homogeneous Yang-Baxter deformations are non-abelian T-duality transformations, and explained further in [91]. We explain this generalised Buscher procedure in generality here

1. Start with a Lie group $\mathcal{G}$ and consider the following σ-model for group $\mathcal{G}$

$$S = -\frac{1}{2} \int \left(g^{-1} \mathrm{d} g \right)^a \wedge (G_0 \star + B_0)_{ab} \left(g^{-1} \mathrm{d} g \right)^b, \tag{8.2.34}$$

   with constant metric G_0 and B_0.

2. Given a 2-cocycle ω on $\mathfrak{g}$ we define a central extension $\mathfrak{e}$ of $\mathfrak{g}$ by a central element Z with the new bracket $[\,,\,]'$ by

$$[t_a, t_b] = f^c{}_{ab} t_c \quad \rightarrow \quad [t_a, t_b]' = [t_a, t_b] + \omega_{ab} Z \tag{8.2.35}$$

and the field strength F' of an $\mathfrak{e}$-valued gauge field $A' = A^a t_a + CZ$ by

$$\begin{aligned} F' &= \mathrm{d}A' - [A' \overset{\wedge}{,} A']' \\ F'^{,a} &= F^a = \mathrm{d}A^a - [A \overset{\wedge}{,} A]^a \quad \text{and} \quad F'^{,Z} = \mathrm{d}C - \omega_{ab} A^a \wedge A^b. \end{aligned} \tag{8.2.36}$$

3. Again on a symplectic leaf of the 2-cocycle ω, which actually defines a subalgebra[16], this defines a Poisson structure $\Pi(g)$

$$\omega = \omega_{ab} \bar{t}^a \wedge \bar{t}^b = \Pi^{-1}_{\alpha\beta} \bar{t}^\alpha \wedge \bar{t}^\beta. \tag{8.2.37}$$

Gauging $\left(g^{-1}\mathrm{d}g\right)^a \mapsto A^a$ but fixing the field strength F' to be zero, instead of F, via adding the Lagrangian multiplier term

$$\mathcal{L}_{Lag.mult.} \propto -Y_s F^s = \bar{X}_a \wedge A^a + Y \tag{8.2.38}$$

and integrating out C and the Lagrangian multipliers $Y_s = (\bar{X}_a, Y)$ leaves, similarly to earlier calculations, the σ-model

$$S \sim \int \mathrm{d}^2\sigma \, \left(g^{-1}\partial_+ g\right)^a \left(g^{-1}\partial_- g\right)^b \left(\frac{1}{\frac{1}{G_0 + B_0} + \Pi}\right)_{ab}. \tag{8.2.39}$$

As the Poisson bivector Π here should be invertible on a symplectic leaf $T\mathcal{G}$ and thus non-vanishing everywhere on $\mathcal{G}$, it can be only of the forms

$$\Pi_R^{ab}(g) = r^{ab} \quad \text{or} \quad \Pi_L^{ab}\left(g = \exp(x^a t_a)\right) = r^{ab} - r^{k[a} f^{b]}{}_{kc} x^c + \ldots \tag{8.2.40}$$

as discussed in section 2. In the sense of our definition of the action on (8.2.34) of the NATD group in section 8.2.1 only the latter has the striven for form, which agrees with the one of a NATD β-shifts by $-r$. In fact both versions make sense, as

$$(g^{-1}\partial_+ g)^a \left(\frac{1}{E_0^{-1} + \Pi_L}\right)_{ab} (g^{-1}\partial_- g)^b = (\partial_+ g\, g^{-1})^a \left(\frac{1}{A^T E_0^{-1} A + \Pi_R}\right)_{ab} (\partial_- g\, g^{-1})^b \tag{8.2.41}$$

The constant Π_R corresponds to the β-shift (and a GL-transformation by $A(g)$ of the inner automorphism corresponding to the adjoint action on g acting on E_0) corresponding to the right isometries G_R of the principal chiral model. This is well known in contexts of Yang-Baxter deformations, where E_0 is ad-invariant. We will come back to this later.

Our conventions on Lie algebra cohomology and the connection of 2-cocycles to central extensions are very briefly reviewed in appendix C.

[16] A closed Chevalley-Eilenburg 2-cocycle defines a so-called quasi-Frobenius subalgebra, which is exactly the space, where the 2-cocycle is non-degenerate [169].

8.2.3 Yang-Baxter deformations as β-shifts.

The role of abelian and non-abelian resp. Poisson-Lie T-duality in the study of integrable deformations of string σ-models has been widely discussed. I. e. the λ-deformations [216–219] were constructed as interpolations between a WZW-model and the (factorised) non-abelian T-dual of principal chiral model, as discussed in section 4.2. Yang-Baxter deformations were introduced based on Poisson-Lie T-duality and are generated by solutions of the modified classical Yang-Baxter equation [232]. A connection to β-shifts seems plausible.

We see that the Yang-Baxter deformed Lagrangian

$$S = -\frac{1}{2}\int \mathrm{d}^2\sigma \quad (g^{-1}\partial_+ g)^a \kappa_{ac} \left(\frac{1}{\mathbb{1} - \eta R_g}\right)^c{}_b (g^{-1}\partial_- g)^b,$$

is of the form of our above definition of a NATD β-shift starting from a principal chiral model. For this, we can express R_g conveniently in the language used in this section

$$(R_g)^a{}_c \kappa^{cb} = r^{ab} - \Pi^{ab}(g), \tag{8.2.42}$$

where $\Pi(g)$ is the homogeneous Poisson structure corresponding to the R-bracket of R. In order for the Yang-Baxter deformation to be a NATD β-shift, r^{ab} has to fulfil (8.2.30), which for $\overline{f}_c{}^{ab} = 0$ in case of the principal chiral model is

$$r^{[a|m} r^{|b|n} f^{|c]}{}_{mn} = 0, \tag{8.2.43}$$

which is exactly the (homogeneous) classical Yang-Baxter equation for a bivector r on $\mathfrak{g}$. This proofs the conjecture, that homogeneous Yang-Baxter deformations are exactly the same as NATD β-shifts of principal chiral models.

Let us also discuss the Yang-Baxter deformed model at the level of generalised fluxes.

$$\begin{aligned} \mathbf{Q}_c{}^{ab} &= 0 \\ \mathbf{R}^{abc} &= -\eta^2 c^2 \kappa^{ak} f_k{}^{bc}, \end{aligned} \tag{8.2.44}$$

The $\mathbf{Q}$-flux clearly vanishes because the Yang-Baxter deformation is a formal β-shift. The $\mathbf{R}$-flux of the deformed model vanishes as expected for $c = 0$. For $c = i$, the η-deformation, we see that (4.2.1) is a realisation of a *geometric* $\mathbf{R}$-flux background and, of course, the η-deformation is not a NATD β-shift.

Let us compare these results with the ones in [214], where the authors studied examples of deformed coset σ-models. The key results there was, that the deformed backgrounds should be interpreted as T-folds, because going around closed cycles we pick up a monodromy in β, which is described by a non-vanishing $\mathbf{Q}$-flux. So our result (8.2.44) of vanishing $\mathbf{Q}$-flux might seem overly simplistic, but in contrast to Yang-Baxter deformed principal chiral models we have $\beta^{ab} = \left(R_g \circ P\right)^a{}_c \kappa^{cb}$ for Yang-Baxter deformed coset σ-models, where P is the projector on the coset algebra. This projector makes the algebraic situation much more diverse, which apparently also leads to a non-vanishing $\mathbf{Q}$-flux.

Bi-Yang-Baxter deformations. A very natural generalisation of the Yang-Baxter deformation (4.2.1), which turns out to be still integrable [264, 265], is

$$S = -\frac{1}{2}\int d^2\sigma (g^{-1}\partial_+ g)^a \kappa_{ac} \left(\frac{1}{\mathbb{1} - \xi R - \eta R_g}\right)^c{}_b (g^{-1}\partial_- g)^b. \tag{8.2.45}$$

Originally it was introduced for R being a solution mcYBe(i), but is also integrable for R solution of classical Yang-Baxter deformation. Formally, it corresponds then to separate β-shifts on the isometries $G_R \times G_L$ for the principal chiral model with separate scales ξ and η. This is becomes clear as

$$(g^{-1}\partial_+ g)^a \kappa_{ac} \left(\frac{1}{\mathbb{1} - \eta R_g}\right)^c{}_b (g^{-1}\partial_- g)^b = (\partial_+ g\, g^{-1})^a \kappa_{ac} \left(\frac{1}{\mathbb{1} - \eta R}\right)^c{}_b (\partial_- g\, g^{-1})^b.$$

For our purpose we generalise to the case, where the R-bracket fulfils the Jacobi identity, and which is not generically integrable.

Let us rewrite

$$\beta^{ab} = (\xi R + \eta R_g)^a{}_c \kappa^{cb} = (\xi + \eta) r^{ab} - \eta \Pi^{ab}(g). \tag{8.2.46}$$

The generalised fluxes for this model take the form

$$\begin{aligned} \mathbf{Q}_c{}^{ab} &= -\xi f^{[a}{}_{cd} r^{b]d} \\ \mathbf{R}^{abc} &= -(\xi+\eta)^2 r^{[a|m} r^{|b|n} f^{|c]}{}_{mn}. \end{aligned} \tag{8.2.47}$$

For $\xi = -\eta$ they become especially simple

$$\begin{aligned} \mathbf{H}^{(0)}_{abc} &= \eta \kappa_{am}\kappa_{bn} f^{[m}{}_{cd} r^{n]d} = \eta \bar{f}_{abc} \\ \mathbf{f}^c{}_{ab} &= f^c{}_{ab} \\ \mathbf{Q}_c{}^{ab} &= \eta f^{[a}{}_{cd} r^{b]d} = \eta \bar{f}_c{}^{ab} \\ \mathbf{R}^{abc} &= 0. \end{aligned} \tag{8.2.48}$$

Then, (8.2.45) describes an **R**-flux free model which is (in case R is not a solution to the classical Yang-Baxter equation) not related to the principal chiral model via a NATD β-shift, because the **Q**-flux is changed.

8.3 Drinfel'd doubles and generalised double field theory

The formulation of double field theory doubles coordinates in order to make T-duality manifest – the physical space is then reached after applying a constraint, the section condition (see [67, 68, 142, 145–147, 254]). There have been approaches to incorporate non-abelian T-duality resp. Poisson Lie T-duality in double field theory [144, 255, 256, 263, 266].

If we would like to make the NATD group (8.2.1) a manifest symmetry of a theory on a doubled space, a natural candidate for this doubled space is the Drinfel'd double $\mathcal{D}$ and a (strong) section condition then is mediated by the projection onto the (local) Dirac structures $\mathfrak{d} = \mathfrak{d}^+ \perp \mathfrak{d}^-$. There are multiple possible candidates for mathematical structures describing this 'splitting' of a Drinfel'd double, some of which could be used for constructions proposed in [238,239,267,268].

We will briefly introduce some natural candidates for such a splitting structure and argue that a para-complex structure is the most natural of these and allows us to view double field theory on Drinfel'd doubles in the framework of double field theory on para-Hermitian manifolds [238,239,268].

Canonical para-complex structure. Given a bialgebra $\mathfrak{d}$ and a basis of a Manin triple decomposition $\{t_a, \bar{t}^a\}$, a canonical object describing the splitting is the linear operator

$$J(t_a) = t_a, \qquad J(\bar{t}^a) = -\bar{t}^a. \tag{8.3.1}$$

This is an almost para-complex structure because $J^2 = \mathbb{1}$ and it has d-dimensional ± 1-eigenbundles. J is chosen in a way, that these eigenbundles are also maximally isotropic subspaces w.r.t. to $\langle\,|\,\rangle$. J is integrable as its Nijenhuis-tensor

$$N_J(X,Y) = -J^2([X,Y]) + J([J(X),Y] + [X,J(Y)]) - [J(X),J(Y)] \tag{8.3.2}$$

vanishes for $X, Y \in \mathfrak{d}$. More precisely

$$\begin{aligned} N_J(t_a,t_b) = 0 \quad &\Leftrightarrow \quad [\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g} \\ N_J(\bar{t}^a,\bar{t}^b) = 0 \quad &\Leftrightarrow \quad [\mathfrak{g}^\star,\mathfrak{g}^\star] \subset \mathfrak{g}^\star \\ N_J(t_a,\bar{t}^b) \equiv 0 &\equiv N_J(\bar{t}^a,t_b). \end{aligned} \tag{8.3.3}$$

This opens a new perspective on J: Given a $2d$-dimensional Lie algebra with an Ad-invariant $\mathrm{O}(d,d)$-metric, then the choice of a complementary pair of maximally isotropic subspaces w.r.t. to the $\mathrm{O}(d,d)$-metric defines an almost (para-)complex structure J. These subspaces are closed subalgebras, iff the almost (para-)complex structure is integrable. Thus a Manin triple decomposition $(\mathfrak{d},\mathfrak{g},\mathfrak{g}^\star)$ can equivalently be described by the pair $(\mathfrak{d}, J)$ with an integrable para-complex structure J.

The invariance group of the integrability of J is exactly the NATD group (8.2.1).

Non-degenerate 2-form. Given a metric $\langle\,|\,\rangle$ and a (para)-complex structure J it is possible to complete a compatible triple $(\eta,\, J,\, \omega_J)$ with a non-degenerate two-form ω_J via

$$\omega_J(X,Y) = \langle J(X)|Y\rangle\,. \tag{8.3.4}$$

Considering the $\mathrm{O}(d,d)$-metric and the para-complex structure J (8.3.1) above we get

$$\omega_J = t_a \wedge \bar{t}^a. \tag{8.3.5}$$

With help of the Maurer-Cartan structure equation we compute

$$\mathrm{d}\omega = -\frac{1}{2}\left(f^a{}_{bc} t_a \wedge \bar{t}^b \wedge \bar{t}^c + \bar{f}_a{}^{bc} \bar{t}^a \wedge t_b \wedge t_c\right), \tag{8.3.6}$$

so the 2-form ω_J is symplectic, resp. $\mathcal{D}$ is a para-Kähler manifold, iff $\mathfrak{d}$ is abelian. For the generic case in which we are interested in here, the apparatus for para-Kähler manifolds as mentioned in [238, 268] and, thus, the straightforward interpretation of the doubled space as some 'phase space' is not applicable.

Other almost (para)-complex structures. Two other (families of) candidates for a splitting structure have been recently discussed in detail in [269] in a slightly different setting. The consideration in [269] are more general than the one we need. They consider a Lie group, which is a semidirect product of two Lie groups of equal dimension d, $\mathcal{Q} = H \ltimes K$, so the corresponding Lie algebra is $\mathfrak{q} = \mathfrak{h} \oplus \mathfrak{k}$, where $\mathfrak{h}$ is a subalgebra and $\mathfrak{k}$ is an ideal. Following this definition, they have to consider general representations $\pi : \mathfrak{h} \to \mathrm{End}(\mathfrak{k})$ describing the action $[\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}$. The study of Drinfel'd doubles fixes the choice of representation such that it is compatible with the Ad-invariant $\mathrm{O}(d,d)$-metric.

In the framework of Drinfel'd doubles, in which we are interested, they are only applicable for the semi-abelian Drinfel'd double $\mathcal{D} = T^\star G$ with $\mathfrak{d} = \mathfrak{g} \oplus_{\mathfrak{d}} (\mathfrak{u}(1))^d$, where we can define two almost (para)-complex structures,[17] given a (vector space) isomorphism $\theta :\ \mathfrak{g} \to \mathfrak{g}^* = (\mathfrak{u}(1))^d$

- almost para-complex structure $\mathbb{I} : \mathfrak{d} \to \mathfrak{d},\ (m,n) \mapsto (\theta^{-1}(n), \theta(m))$
- almost complex structure $\mathbb{J} : \mathfrak{d} \to \mathfrak{d},\ (m,n) \mapsto (-\theta^{-1}(n), \theta(m))$

Theorem 3.2. of [269], adjusted to our case, states that these structures are integrable, iff the isomorphism θ is an 1-cocycle of $(\mathfrak{g}, \mathrm{ad}(\mathfrak{g})\big|_{\mathfrak{g}^*})$, meaning that

$$[m, \theta(n)] - [n, \theta(m)] - \theta([m,n]) = 0, \quad \forall m, n \in \mathfrak{g}. \tag{8.3.7}$$

There are two simple possibilities to fulfil this condition:

- $\mathfrak{g}$ is abelian $\Rightarrow$ $\mathfrak{d}$ is abelian. In this case any isomorphism θ will do and we could for example choose the canonical harmonic isomorphism w.r.t. to the $\mathrm{O}(d,d)$-metric: $\sharp_\eta :\ \mathfrak{g} \to \mathfrak{g}^*,\ t_a \mapsto \bar{t}^a$, such that the integrable (para)-complex structures become
$$(m,n) \mapsto (\pm\flat_\eta(n), \sharp_\eta(m)). \tag{8.3.8}$$
- $\mathfrak{g}$ is a quasi-Frobenius algebra. We can define a non-degenerate 2-form $\omega = \theta_{ab}\bar{t}^a \wedge \bar{t}^b$ on G, where $\theta :\ t_a \mapsto \theta_{ab}\bar{t}^b$. The 1-cocycle condition means, that ω is symplectic ($\mathrm{d}\omega = 0$) resp. that $(\theta^{-1})^{ab} t_a \wedge t_b$ is a solution of the classical Yang-Baxter equation.

[17] In principle we could define the same structures for a generic Drinfel'd double, but only in the case of the semi-abelian double we can solve the integrability condition in a straightforward way and apply the results of [269].

The canonical para-complex structure J (8.3.1) can be obtained via $J = \mathbb{I} \circ \mathbb{J}$. The integrability of J does not depend on the integrability of $\mathbb{I}$ or $\mathbb{J}$ but only on the algebraic decomposition of $\mathfrak{d}$.

These (para)-complex structures $\mathbb{I}$ and $\mathbb{J}$ have not been applied yet to the geometric study of ordinary DFT (abelian bialgebra) or integrable deformations (quasi-Frobenius semi-abelian bialgebra case), where they might be useful. Many more details can be found in [269].

Chapter 9

An outlook to M-theory: M2-branes in the SL(5)-theory

Generalised (O(d,d)-) geometry emerged from the string current algebra. An interesting question is whether there is also a connection between the membrane current algebra and exceptional generalised geometry – interesting, because para-Hermitian versions of exceptional generalised geometry have only been discussed recently [270].

As discussed in chapter 3, the treatment of U-duality depend very much on the dimension of the (non necessarily compactified) target. Here, we work in $d = 4$ in the SL(5)-theory.

9.1 M2 current algebra

9.1.1 SL(5) generalised Lie derivative

The aim is to see how the generalised Lie derivative

$$[\phi_1, \phi_2]_D^K = 2\phi_{[1}^L \partial_L \phi_{2]}^K - Y^{KL}{}_{MN}\phi_1^M \partial_L \phi_2^N \tag{9.1.1}$$

is encoded in the current algebra of an M2 brane. We start by arranging the current algebra in the **10**-representation of SL(5)

$$\mathbf{Z}_K = \frac{1}{\sqrt{2}}\mathbf{Z}_{kk'} = \frac{1}{\sqrt{2}}\left(p_\kappa, \mathrm{d}x^\kappa \wedge \mathrm{d}x^{\kappa'}\right). \tag{9.1.2}$$

From the canonical Poisson brackets of the $p(\sigma)$ and $x(\sigma)$, one can derive, with the help of the identity $\epsilon^{\alpha\beta}\partial_\alpha x^\mu(\sigma)\partial_\beta\delta(\sigma-\sigma') = \epsilon^{\alpha\beta}\partial_\alpha x^\mu(\sigma')\partial_\beta\delta(\sigma-\sigma')$, the current algebra

$$\{\mathbf{Z}_K(\sigma), \mathbf{Z}_L(\sigma')\} = \mathrm{d}x^\mu(\sigma) \wedge \left(\frac{1}{2}(\mathrm{d}-\mathrm{d}')\delta(\sigma-\sigma')\epsilon_{\mu KL} + \frac{1}{2}(\mathrm{d}+\mathrm{d}')\delta(\sigma-\sigma')\omega_{\mu,KL}\right). \tag{9.1.3}$$

As in the string case, we can consider the 'Dorfman current algebra'

$$\left\{\mathbf{Z}_K(\sigma), \mathbf{Z}_L(\sigma')\right\}_D = \epsilon_{\mu KL} \mathrm{d}x^\mu(\sigma) \wedge \mathrm{d}\delta(\sigma - \sigma'), \tag{9.1.4}$$

an SL(5)-invariant contribution of (9.1.3), leading to the generalised Lie derivative:

$$\{\phi_1, \phi_2\}_D = -2 \int \phi_{[1}^L \partial_L \phi_{2]}^K \mathbf{Z}_K \epsilon_{\mu KL} + \int \phi_1^K \partial_{\mu'} \phi_2^L \mathrm{d}x^\mu \wedge \mathrm{d}x^{\mu'} \tag{9.1.5}$$

$$\delta_\nu^\mu \delta_{\nu'}^{\mu'} \mathrm{d}x^\nu \wedge \mathrm{d}x^{\nu'} \partial_{\mu'} = \frac{1}{2} \epsilon^{\mu\mu'}{}_{\nu\nu'} \mathrm{d}x^\nu \wedge \mathrm{d}x^{\nu'} \partial_{\mu'}$$

$$= \frac{1}{4} \epsilon^{\mu m m' n n'} \mathbf{Z}_{nn'} \partial_{mm'} = \epsilon^{\mu MN} \mathbf{Z}_N \partial_M$$

$$\Rightarrow \quad \{\phi_1, \phi_2\}_D^N(\sigma) = \phi_{[1}^M \partial_M \phi_{2]}^N - Y^{MN}{}_{KL} \phi_1^K \partial_M \phi_2^L \tag{9.1.6}$$

for local functionals $\phi = -\int \phi^K(\sigma) \mathbf{Z}_K(\sigma)$ and by use of the section condition $\tilde{\partial}^{\mu\mu'}\phi(X) = 0$ and the canonical Poisson brackets in $\mathcal{R}_1$-indices, $\left\{\mathbf{Z}_K(\sigma), X^L(\sigma')\right\} = -\delta_K^L \delta(\sigma - \sigma')$.

The second term in (9.1.3), including

$$\omega_{\mu,KL} = \frac{1}{2} \begin{pmatrix} 0 & -\epsilon_{\mu\kappa}{}^{\lambda\lambda'} \\ \epsilon_{\mu\lambda}{}^{\kappa\kappa'} & 0 \end{pmatrix}, \tag{9.1.7}$$

breaks the SL(5)-invariance of (9.1.3), but makes it a Lie bracket. It corresponds to a boundary/topological contribution, i.e. treated as a distribution

$$\omega_{\mu,KL} \int \mathrm{d}x^\mu \wedge \mathrm{d}\varphi, \tag{9.1.8}$$

giving wrapping contributions $\sim \int \mathrm{d}x^\mu \wedge \mathrm{d}x^{\mu'}$. Also the difference between Courant and Dorfman bracket is of this form of a total differential under a spatial world volume integral. This result is completely analogous to the string resp. $\mathrm{O}(d,d)$ case [221]. There, the winding contribution was equivalent to a topological term in the action, necessary for its $\mathrm{O}(d,d)$-invariance.

In principle, this is the approach in [271], but there the authors did not phrase the current algebra in an SL(5)-covariant way. Also, their approach to unveil the Courant algebroid nature was to substitute a 'Lie bracket part' of their result by the generalised Lie derivative. Here, the SL(5)-invariance and Courant bracket nature of the current algebra up to the ω-/boundary terms is obvious.

9.1.2 Twist by generalised vielbein and the embedding tensor

In analogy to the string case [221], we aim to diagonalise the Hamiltonian and all the constraints when going to the generalised flux frame

$$\mathbf{Z}_A(\sigma) = E_A{}^K(\sigma) \mathbf{Z}_K(\sigma). \tag{9.1.9}$$

in order to characterise the model via a twist of the current algebra. For the canonical (Lie) Poisson bracket (9.1.3), we have

$$\{\mathbf{Z}_A(\sigma), \mathbf{Z}_B(\sigma')\} = j^c(\sigma) \wedge \left(\frac{1}{2}(\mathrm{d} - \mathrm{d}')\delta(\sigma - \sigma')\epsilon_{cAB} + \frac{1}{2}(\mathrm{d} + \mathrm{d}')\delta(\sigma - \sigma')\omega_{c,AB}(\sigma, \sigma')\right) - \mathbf{F}^C{}_{[AB]}(\sigma)\mathbf{Z}_C(\sigma)\delta(\sigma - \sigma') \tag{9.1.10}$$

with $j^c = E_\mu{}^c \mathrm{d}x^\mu$. The ϵ-symbol is $\mathrm{SL}(5)$-invariant, whereas

$$\omega_{c,AB}(\sigma, \sigma') = E_A{}^K(\sigma) E_B{}^L(\sigma') E_c{}^\mu(\sigma) \omega_{\mu,KL} \tag{9.1.11}$$

is not $\mathrm{SL}(5)$-invariant, but is again necessary for (9.1.10) to be a Lie bracket. The twist is characterised by the skewsymmetric component of the $\mathrm{SL}(5)$ generalised fluxes:

$$[E_A, E_B]_D = \mathbf{F}^C{}_{AB} E_C \tag{9.1.12}$$

or

$$\mathbf{F}^C{}_{AB} = 2E_N{}^C \partial_{[A} E_{B]}{}^N - Y^{CD}{}_{AE} E_N{}^E \partial_D E_B{}^N \tag{9.1.13}$$

From this definition, it is quite obvious to see, why they should appear in the current algebra that reproduces the generalised Lie derivative. The 'full' $\mathrm{SL}(5)$ generalised fluxes appear as the twist for the corresponding Dorfman bracket

$$\{\mathbf{Z}_A(\sigma), \mathbf{Z}_B(\sigma')\}_D = j^c(\sigma) \wedge \mathrm{d}\delta(\sigma - \sigma')\epsilon_{cAB} - \mathbf{F}^C{}_{AB}(\sigma)\mathbf{Z}_C(\sigma)\delta(\sigma - \sigma') \tag{9.1.14}$$

Some kind of Courant algebroid conditions will put conditions on the $\mathbf{F}^C{}_{AB}$, corresponding to a (dynamical) Bianchi identity of these fluxes.

9.2 charges and SL(5)-covariance

A difference so far is, that the membrane currents (9.1.2) are not manifestly $\mathrm{SL}(5)$-covariant. This can be dealt with by the introduction of additional objects, the membrane charges, as for example in [272].

9.2.1 Dorfman bracket

One can write the current algebra in a manifestly $\mathrm{SL}(5)$-invariant way with the use of a charge $\mathbf{q}_m$. E.g. the 'Dorfman' current bracket can be written as

$$\{\mathbf{Z}_K(\sigma), \mathbf{Z}_L(\sigma')\}_D = \epsilon_{mKL} \mathbf{q}_{m'} \mathrm{d}X^{mm'}(\sigma) \wedge \mathrm{d}\delta(\sigma - \sigma') \tag{9.2.1}$$

or with help of a 1-form valued ($\mathrm{SL}(5)$-invariant) 'metric', that can be used to lower the indices,

$$\boldsymbol{\eta}^{M2}_{KL} = \eta_{KL,M} \mathrm{d}X^M \equiv \frac{1}{2}\epsilon_{mKL} \mathbf{q}_{m'} \mathrm{d}X^{mm'}, \qquad \text{s.t.} \quad \mathrm{d}\boldsymbol{\eta}^{M2} = 0, \tag{9.2.2}$$

with $\eta_{kk'll',mm'} = \epsilon_{kk'll'[m}\mathbf{q}_{m']}$, as

$$\left\{\mathbf{Z}_K(\sigma), \mathbf{Z}_L(\sigma')\right\}_D = 2\boldsymbol{\eta}^{M2}_{KL} \wedge \mathrm{d}\delta(\sigma - \sigma'). \tag{9.2.3}$$

Acting on local functionals $\phi = -\int \phi^K \mathbf{Z}_K$ this gives

$$\{\phi_1, \phi_2\}_D = -2\int \phi^J_{[1}\partial_J\phi^I_{2]}(\sigma)\, \mathbf{Z}_I(\sigma) + \int \epsilon_{mKL}\mathbf{q}_{m'}\phi^K_1\partial_N\phi^L_2(\sigma)\, \mathrm{d}X^{mm'} \wedge \mathrm{d}X^N(\sigma). \tag{9.2.4}$$

The generalised Lie derivative is reproduced if we identify

$$\mathrm{d}X^{mm'} \wedge \mathrm{d}X^{nn'}\mathbf{q}_{m'}\partial_{nn'} = \frac{3}{2}\mathrm{d}X^{[m|m'} \wedge \mathrm{d}X^{|nn']}\mathbf{q}_{m'}\partial_{nn'} = \epsilon^{mnn'K}\mathbf{Z}_K\partial_{nn'}, \tag{9.2.5}$$

with the consistency condition $\mathbf{q}_{[m}\partial_{nn']} = 0$, was used in the first step.

In the string case, we had that the currents $\mathbf{E}_I = (p_i, \partial x^i)$ are related to the doubled coordinates like $\mathbf{E}_I = \eta_{IJ}\partial X^J$. This allowed to derive the (Dorfman, Courant, Lie) Poisson brackets of the $X_I(\sigma)$. In a similar fashion the 'extended coordinate fields' X^M are related to the membrane currents $\mathbf{Z}_M$ by lowering the index with $\boldsymbol{\eta}_{MN}$ and taking the spatial world-sheet differential, leading to the following objects and their M-theory decompositions

$$\begin{aligned}
\text{0-forms (coordinates):}\qquad & X^M = \frac{1}{\sqrt{2}}X^{mm'} = \frac{1}{\sqrt{2}}\left(x^\mu, \tilde{x}_{\mu\mu'}\right) \\
\text{1-forms:}\qquad & X_M = \boldsymbol{\eta}_{MN}X^N = \frac{1}{2}\epsilon_{lMN}\mathbf{q}_{l'}X^{llprime}\mathrm{d}X^N, \qquad \mathbf{Z}^M \equiv \mathrm{d}X^M \\
\text{2-forms (currents):}\qquad & \mathbf{Z}_M = \mathrm{d}X_M = \boldsymbol{\eta}_{MN} \wedge \mathrm{d}X^N = \frac{1}{2}\epsilon_{MLk}\mathbf{q}_{k'}\mathrm{d}X^{kk'} \wedge \mathrm{d}X^L \\
& = \frac{1}{\sqrt{2}}Z_{mm'} = \frac{1}{\sqrt{2}}\left(\frac{1}{2}\mathrm{d}\tilde{x}_{\mu\nu} \wedge \mathrm{d}x^\nu, \mathrm{d}x^\mu \wedge \mathrm{d}x^{\mu'}\right).
\end{aligned} \tag{9.2.6}$$

The last identification of $\mathbf{Z}_M$ with $\mathrm{d}X^K \wedge \mathrm{d}X^L$ is equivalent to (9.2.5):

$$\begin{aligned}
\epsilon^{nrr'kk'}\mathbf{Z}_{kk'} &= \frac{1}{4}\epsilon^{nrr'kk'}\epsilon_{mll'kk'}\mathbf{q}_{m'}\mathrm{d}X^{mm'} \wedge \mathrm{d}X^{ll'} = \frac{1}{4}\; 2!\; 3!\; \delta^n_{[m}\delta^r_l\delta^{r'}_{l']}\; \mathbf{q}_{m'}\mathrm{d}X^{mm'} \wedge \mathrm{d}X^{ll'} \\
&= 3\; \mathbf{q}_{m'}\mathrm{d}X^{[n|m'} \wedge \mathrm{d}X^{|rr']} \left(= 2\; \epsilon^{nrr'K}\mathbf{Z}_K\right)
\end{aligned}$$

and then as above in (9.2.5).

9.2.2 Dirac bracket approach

We assume to start with an extended phase space associated to extended coordinate fields $Y^K(\sigma)$ and canonical momenta $P_L(\sigma)$ with

$$\left\{Y^K(\sigma), P_L(\sigma')\right\} = \delta^K_L\delta(\sigma - \sigma'), \qquad \left\{Y^K(\sigma), Y^L(\sigma')\right\} = \left\{P_K(\sigma), P_L(\sigma')\right\} = 0. \tag{9.2.7}$$

We treat the identification (9.2.5) as constraints

$$\Phi_K = P_K + \boldsymbol{\eta}_{KLM} \mathrm{d}Y^L \wedge \mathrm{d}Y^M \approx 0 \tag{9.2.8}$$

on the extended phase space, such that we could consider the extended coordinate fields as 'fundamental' fields, similar to the string case. Here, we define $\boldsymbol{\eta}_{KLM}$ as

$$\begin{aligned} &\boldsymbol{\eta}_{kk'll'mm'} = \epsilon_{kk'll'm}\mathbf{q}_{m'} \\ \text{with} \qquad &\boldsymbol{\eta}_{KLM} = \boldsymbol{\eta}_{LKM} \qquad \text{and} \qquad \boldsymbol{\eta}_{KLM} + \boldsymbol{\eta}_{LMK} + \boldsymbol{\eta}_{MKL} = 0. \end{aligned}$$

With that the algebra of constraints is

$$\begin{aligned} C_{KL}(\sigma,\sigma') = \{\Phi_K(\sigma), \Phi_L(\sigma')\} = &-3\, \boldsymbol{\eta}_{KLM} \mathrm{d}X^M(\sigma) \wedge \frac{1}{2}(\mathrm{d}-\mathrm{d}')\delta(\sigma-\sigma') \\ &+ 2\, \boldsymbol{\eta}_{M[KL]} \mathrm{d}X^M(\sigma) \wedge \frac{1}{2}(\mathrm{d}+\mathrm{d}')\delta(\sigma-\sigma'), \end{aligned} \tag{9.2.9}$$

if we do not ignore the boundary contributions $\sim (\mathrm{d}+\mathrm{d}')\delta(\sigma-\sigma')$. The hope would be that the current algebra and the brackets of the extended coordinates follow as Dirac brackets of the above constrained system, e.g.

$$\left\{Y^K(\sigma), Y^L(\sigma')\right\}_{D.B.} = (C^{-1})^{KL}(\sigma,\sigma'). \tag{9.2.10}$$

It seems illusive to calculate this. But it seems difficult to reproduce the current algebra as well via

$$\{\mathbf{Z}_K(\sigma), \mathbf{Z}_L(\sigma')\} = \{P_K(\sigma), P_L(\sigma')\}_{D.B.}.$$

Besides this, there are two reasons to be sceptical about the Dirac bracket approach for the extended membrane phase space:

- It should be possible to phrase the result of the Dirac bracket approach in a manifestly SL(5) invariant manner (meaning all expressions are only phrased in terms of the ϵ-symbol and the membrane charge $\mathbf{q}$). This is because the only two ingredients are the naive Poisson brackets on the extended phase space and the constraints, both of which are SL(5) invariant. The current algebra (9.1.3) on the other hand is not SL(5) invariant, due to the ω-term, see below.

- As usual, there is a mismatch (in contrast to the O(d,d) case) between the number of extended coordinates (10) and the number of fields in the M2 phase space (8). This could be resolved, if there are any secondary constraints or non-vanishing Poisson brackets with the diffeomorphism constraints.

Alternative approach to an X-X bracket. An alternative way to obtain the X-X brackets and check self-consistency of the approach via differentiating and integrating the canonical Poisson brackets could be the following. We start with the original (not independent) $\mathbf{Z}$-X phase space and impose vanishing Poisson brackets on the constraints

$$\mathbf{\Phi}_K = \mathbf{Z}_K(\sigma) + \boldsymbol{\eta}_{KLM}\mathrm{d}X^L \wedge \mathrm{d}X^M. \tag{9.2.11}$$

Ignoring the topological terms and using the membrane current algebra (9.1.4), we would for example arrive at the condition

$$4\boldsymbol{\eta}_{K[MN]}\boldsymbol{\eta}_{L[ST]}\mathrm{d}X^M(\sigma)\mathrm{d}'X^S(\sigma') \wedge \mathrm{d}\mathrm{d}' \left\{X^N(\sigma), X^T(\sigma')\right\} = \boldsymbol{\eta}_{KLM}\mathrm{d}X^M \wedge \mathrm{d}\delta(\sigma - \sigma') \tag{9.2.12}$$

on the X-X Poisson bracket in order for $\{\mathbf{\Phi}_K(\sigma), \mathbf{\Phi}_L(\sigma')\} = 0$ to hold. This expression is different, but a bit similar to the expression (9.2.10) and it seems similarly illusive to compute anything from here.

9.3 Towards a para-Hermitian exceptional geometry

In total analogy to the string case in chapter 6, we study the role that the ω-term plays in exceptional generalised geometry.

ω-term and the section. We define the projector

$$P^{KL}{}_{MN} = \frac{1}{2}\left(Y^{KL}{}_{MN} + \Omega^{KL}{}_{MN}\right)$$

with$\Omega^{KL}{}_{MN} = \epsilon^{pKL}\omega_{pMN}$

As the $P^{KL}{}_{MN}$ resp. $\Omega^{KL}{}_{MN}$ always appears contracted with a derivative ∂_L, the so far undetermined component ω_{5MN} does not appear if we work on a section. This projector has the following properties:

- $P^{KL}{}_{MN} = P^{LK}{}_{MN}$
- $P^{KL}{}_{MN}\partial_K \otimes \partial_L = 0$ (section condition)
- Using this section condition, one derives

$$P^{KL}{}_{MN}P^{NP}{}_{RS}\partial_L \otimes \partial_P \approx 0$$
$$P^{KL}{}_{NM}P^{NP}{}_{RS}\partial_{(L} \otimes \partial_{P)} \approx P^{KL}{}_{RS}\partial_{(L} \otimes \partial_{M)} \tag{9.3.1}$$
$$\text{e.g.}\quad P^{KL}{}_{NM}P^{NP}{}_{RS}\left(\partial_L\phi^M_{[1}\right)\left(\partial_P\phi^S_{2]}\right) \approx P^{KL}{}_{RS}\left(\partial_L\phi^P_{[1}\right)\left(\partial_P\phi^S_{2]}\right) + \underbrace{\frac{1}{2}\epsilon^{\lambda\pi\kappa\kappa'}\epsilon_{\mu\sigma\rho\rho'}\left(\partial_\lambda\phi^\mu_{[1}\right)\left(\partial_\pi\phi^\sigma_{2]}\right)}_{=0}. \tag{9.3.2}$$

As also in the usual exceptional generalised geometry [176], the identities are a weaker than in the string case.

Again, one can define two total derivative objects

$$\begin{aligned}
(\!(\phi_1, \phi_2)\!)^K &= \frac{1}{2} Y^{KL}{}_{MN} \partial_L \left(\phi_1^M \phi_2^N \right) \\
[\![\phi_1, \phi_2]\!]^K &= \frac{1}{2} \Omega^{KL}{}_{MN} \partial_L \left(\phi_1^M \phi_2^N \right)
\end{aligned} \tag{9.3.3}$$

In the (world-volume) SL(5)-theory, they are given explicitly as

$$\begin{aligned}
(\!(\phi_1, \phi_2)\!) &= 2\epsilon_{\mu MN} \int \mathrm{d}x^\mu \wedge \mathrm{d} \left(\phi_1^M \phi_2^N \right) = 2 \int \mathrm{d} \left(\phi_1 \bullet \phi_2 \right) \\
[\![\phi_1, \phi_2]\!] &= 2\omega_{\mu MN} \int \mathrm{d}x^\mu \wedge \mathrm{d} \left(\phi_1^M \phi_2^N \right) = 2 \int \mathrm{d} \left(\phi_1 \circ \phi_2 \right),
\end{aligned}$$

where we defined the bullet products $\bullet, \circ : \Gamma(E) \wedge \Gamma(E) \to T^\star M$, for the ϵ- and the ω-symbol respectively. The appearing brackets have the same properties as the ones discussed in chapter 7. The true canonical current algebra (9.1.3) is a non-standard Lie algebroid on the extended tangent bundle $TM \oplus \Lambda^2 M$.

Taking the opposite route and choose the $P^{KL}{}_{MN}$ as the fundamental object obeying the identities (9.3.1), one sees that a choice of ω is equivalent to a choice of (M-theory) section up to a gauge-transformation of the three-form gauge fields. This is, again, in full analogy to the string case in chapter 6.

Non SL(5)-invariance of ω-term. In the string case, a ω-term as in (9.1.3) broke the O(d, d)-invariance of the current algebra. We would expect the same for the SL(5)-invariance of (9.1.3).

Motivated by (9.2.9), a candidate formed only of the ϵ-symbol and the membrane charge $\mathbf{q}_m$ for such an ω is

$$\tilde{\omega}_{KL} = 2\boldsymbol{\eta}_{M[KL]} \mathrm{d}X^M. \tag{9.3.4}$$

Let us compare this to (9.1.3) for $\mathbf{q}_5 = 1$ and $\mathbf{q}_\mu = 0$

$$\tilde{\omega}_{\kappa\kappa',\lambda\lambda'} = 0, \qquad \tilde{\omega}_{\kappa\kappa',\lambda 5} = -\tilde{\omega}_{\lambda 5,\kappa\kappa'} = \epsilon_{\kappa\kappa'\lambda\lambda'} \mathrm{d}x^{\lambda'}, \qquad \tilde{\omega}_{\kappa 5,\lambda 5} = -2\mathrm{d}\tilde{x}_{\kappa\lambda}. \tag{9.3.5}$$

Apart from the $\tilde{\omega}_{\kappa 5,\lambda 5}$-component, this is the ω-term in (9.1.3). In $\tilde{\omega}_{KL} \wedge \frac{1}{2}(\mathrm{d} + \mathrm{d}')\delta(\sigma - \sigma')$ this $\tilde{\omega}_{\kappa 5,\lambda 5}$-component would corresponds to terms like (treated as a distribution)

$$\sim \int \mathrm{d}\tilde{x}_{\mu\mu'} \wedge \mathrm{d}x^{\mu'} = \int p_\mu. \tag{9.3.6}$$

In a way, this makes sense as $\tilde{\omega}$ is SL(5)-invariant and the other components of $\tilde{\omega}$, which were also present in the ω-term in the M2 current algebra (9.1.3), for example corresponded to wrapping contributions $\sim \int \mathbf{Z}^{\mu\mu'} = \int \mathrm{d}x^\mu \wedge \mathrm{d}x^{\mu'}$.

9.4 String currents

In the end, M-theory and string theory are deeply connected. This connection shows also in the current algebra. One can straightforwardly relate the membrane currents to the currents of the type II string theories.

9.4.1 Double reduction of membrane current algebra

In order to reduce from M-theory section (9.1.3) to the IIA section in the current algebra, we perform the usual double dimensional reduction:

$$x^4(\sigma^1,\sigma^2)=\sigma^2,\quad x^{\underline{\mu}}(\sigma^1,\sigma^2)=x^{\underline{\mu}}(\sigma^1)\equiv x^{\underline{\mu}}(\sigma). \tag{9.4.1}$$

The M2 current (9.1.2) becomes

$$\mathbf{Z}_{mm'}=\Big(\mathrm{d}x^{\underline{\mu}}\wedge \mathrm{d}x^{\underline{\mu}'}\ ,\ \mathrm{d}x^{\underline{\mu}}\wedge \mathrm{d}x^4\ ,\ p_{\underline{\mu}}\ ,\ p_4\Big)=\Big(0\ ,\ \mathrm{d}x^{\underline{\mu}}\wedge \mathrm{d}\sigma^2\ ,\ p_{\underline{\mu}}\ ,\ 0\Big) \tag{9.4.2}$$

such that with $\mathbf{Z}_{mm'}\to \mathbf{z}_{mm'}(\sigma^1)\wedge \mathrm{d}\sigma^2$

$$\mathbf{z}_{mm'}=(\underbrace{\epsilon^{\underline{\mu}\underline{\mu}'\underline{\nu}}\mathbf{z}_{\underline{\mu}\underline{\mu}'}\ ,\ \mathbf{z}_{\underline{\mu}5}}_{\mathbf{z}_{\underline{\mathcal{M}}}}\ ,\ \underbrace{\mathbf{z}_{\underline{\mu}4}\ ,\ \mathbf{z}_{45}}_{\mathbf{z}_{\tilde{\mathcal{M}}}})=\Big(\mathrm{d}x^{\underline{\mu}}\ ,\ p_{\underline{\mu}}\ ,\ 0\ ,\ ?\Big). \tag{9.4.3}$$

At the same time, the M2 current algebra (9.1.3) reduces to

$$\{\mathbf{z}_K(\sigma),\mathbf{z}_L(\sigma')\}=\eta_{KL}\frac{1}{2}(\mathrm{d}-\mathrm{d}')\delta(\sigma-\sigma')+\frac{1}{2}\omega_{KL}(\mathrm{d}+\mathrm{d}')\delta(\sigma-\sigma') \tag{9.4.4}$$

$$\{\mathbf{z}_K(\sigma),\mathbf{z}_L(\sigma')\}_D=\eta_{KL}\mathrm{d}\delta(\sigma-\sigma'), \tag{9.4.5}$$

with $\eta_{KL}=\epsilon_{4KL}$ and $\omega_{KL}=\omega_{4,KL}$. The only non-vanishing components are:

$$\eta_{\underline{\mu}\underline{\mu}'\underline{\nu}5}=\eta_{\underline{\nu}5\underline{\mu}\underline{\mu}'}=\omega_{\underline{\mu}\underline{\mu}'\underline{\nu}5}=-\omega_{\underline{\nu}5\underline{\mu}\underline{\mu}'}=\epsilon_{\underline{\mu}\underline{\mu}'\underline{\nu}} \tag{9.4.6}$$

or in the conventions of (9.4.3)

$$\eta_{KL}=\begin{pmatrix}\eta_{\underline{\mathcal{K}}\underline{\mathcal{L}}} & 0\\ 0 & 0\end{pmatrix},\quad \underline{\eta}=\begin{pmatrix}0 & \mathbb{1}_3\\ \mathbb{1}_3 & 0\end{pmatrix},$$

$$\omega_{KL}=\begin{pmatrix}\omega_{\underline{\mathcal{K}}\underline{\mathcal{L}}} & 0\\ 0 & 0\end{pmatrix},\quad \underline{\omega}=\begin{pmatrix}0 & -\mathbb{1}_3\\ \mathbb{1}_3 & 0\end{pmatrix}.$$

Restricted to the $\underline{\mathcal{K}},\underline{\mathcal{L}},...$ indices, these are the canonical $\mathrm{O}(3,3)$ metric and the components of the canonical symplectic form ω as expected from the string discussion. T-duality transformations are defined by $M_M{}^N\in\mathrm{GL}(5)$ with

$$M_M{}^K M_N{}^L\eta_{KL}=\eta_{MN} \tag{9.4.7}$$

With that, the SL(5) type IIb section can be approached as well.

Let us do a schematic discussion of the IIa and IIb section decompositions of the currents $\mathbf{z}_M(\sigma)$ *without* the double reduction

- type IIa section:

$$\mathbf{z}_{mm'} = \left(\mathbf{z}_{\underline{\mu\mu'}}\ ,\ \mathbf{z}_{\underline{\mu}4}\ ,\ \mathbf{z}_{\underline{\mu}5}\ ,\ \mathbf{z}_{45}\right) = \left(j^{(F1)}_{\underline{\mu}}\ ,\ j^{\underline{\mu\mu'}}_{(D2)}\ ,\ p_{\underline{\mu}}\ ,\ j_{(D0)}\right) \tag{9.4.8}$$

The $D0$- and $D2$ currents vanish in the double dimensional reduction.

- type IIb section:

$$\mathbf{z}_{mm'} = \left(\mathbf{z}_{\underline{\mu\mu'}}\ ,\ \mathbf{z}_{\underline{\mu}4}\ ,\ \mathbf{z}_{\underline{\mu}5}\ ,\ \mathbf{z}_{45}\right) = \left(p_{\underline{\mu\mu'}}\ \ j^{(D1)}_{\underline{\mu}}\ ,\ j^{(F1)}_{\underline{\mu}}\ ,\ z\right) \tag{9.4.9}$$

$p_{\underline{\mu\mu'}}$ is the canonical momentum to the type IIb section coordinate fields $x^{\underline{\mu\mu'}}$, to which $j^{(F1)}_{\underline{\mu}}(\sigma) = \epsilon_{\underline{\mu\nu\nu'}}\mathrm{d}x^{\underline{\nu\nu'}}(\sigma)$ is the (dualised) typical string current. These two are the T-dual variables to type IIa momentum and $F1$ current via

$$\begin{aligned} p_{\underline{\mu\mu'}} &= \epsilon_{\underline{\mu\mu'\nu}}\tilde{p}^{\underline{\nu}} \quad \xrightarrow{T\text{-duality}} \quad \mathrm{d}x^{\underline{\nu}}_{(IIa)} \\ j^{(F1)}_{\underline{\mu}} &= \epsilon_{\underline{\mu\nu\nu'}}\mathrm{d}x^{\underline{\nu\nu'}} = \mathrm{d}\tilde{x}_{\mu} \quad \xrightarrow{T\text{-duality}} \quad p^{(IIa)}_{\mu}. \end{aligned}$$

The $D1$-current is $j^{(D1)}_{\underline{\mu}} = \epsilon_{\underline{\mu\nu\nu'}}\mathrm{d}y^{\underline{\nu\nu'}}$, for a $D1$ coordinate field y, and is related to the type IIa $D2$-current via T-duality.

S-duality obviously acts as an SL(2) on the $k = 4,5$ indices, and rotates the $F1$ and $D1$ currents into each other and leaves the $\mathbf{z}_{45} = z\,\epsilon_{45}$ invariant.

9.4.2 Charges and SL(5)-covariance

The type IIa section above motivates

$$\{\mathbf{z}_K(\sigma), \mathbf{z}_L(\sigma')\} = \mathbf{q}^m \left(\epsilon_{mKL}\frac{1}{2}(\mathrm{d} - \mathrm{d}')\delta(\sigma - \sigma') + \frac{1}{2}\omega_{m,KL}(\mathrm{d} + \mathrm{d}')\delta(\sigma - \sigma')\right) \tag{9.4.10}$$

$$\{\mathbf{z}_K(\sigma), \mathbf{z}_L(\sigma')\}_D = \mathbf{q}^m\epsilon_{mKL}\mathrm{d}\delta(\sigma - \sigma'), \tag{9.4.11}$$

as the SL(5) string current algebra using the string charge $\mathbf{q}^m$, fulfilling $\mathbf{q}^m\partial_{mm'} = 0$. But it seems to hold again for both the IIa and IIb section. The SL(5)-invariant $M \to IIa$ reduction condition would be something like:

$$\mathbf{q}^m\sigma^2 = \mathbf{q}_{m'}X^{mm'}. \tag{9.4.12}$$

Let us proceed in the same way as for the membrane current algebra, and try to reproduce the generalised Lie derivative form the Dorfman current algebra (9.4.11)

$$\{\phi_1, \phi_2\} = -2\int \phi^K_{[1}\partial_K\phi^L_{2]}\mathbf{z}_L(\sigma) + \mathbf{q}^p\epsilon_{pKL}\int \phi^K_1\partial_M\phi^L_2\mathrm{d}X^M. \tag{9.4.13}$$

We need to make a similar identification as in the M2 brane case. Let us define $\eta_{KL} = \frac{1}{2}\mathbf{q}^p\epsilon_{pKL}$ in an SL(5)-invariant way. Then

$$\mathbf{z}_K = \eta_{KL}\mathrm{d}X^L \tag{9.4.14}$$

respectively

$$\epsilon^{pmm'nn'}\mathbf{z}_{nn'} = \frac{1}{4}\epsilon^{pmm'nn'}\mathbf{q}^k\epsilon_{kll'nn'}\mathrm{d}X^{ll'} = \frac{1}{4}3!\,2!\;\mathbf{q}^{[p}\mathrm{d}X^{mm']} = 3\mathbf{q}^{[p}\mathrm{d}X^{mm']} \tag{9.4.15}$$

$$3\;\mathbf{q}^{[p}\mathrm{d}X^{mm']}\partial_{mm'} = \mathbf{q}^p\mathrm{d}X^{mm'}\partial_{mm'}. \tag{9.4.16}$$

The last step made use of $\mathbf{q}^m\partial_{mm'} = 0$. With that (9.4.13) becomes

$$\{\phi_1, \phi_2\} = -2\int \phi^K_{[1}\partial_K\phi^L_{2]}\mathbf{z}_L(\sigma) + Y^{MN}{}_{KL}\int \phi^K_1\partial_M\phi^L_2\mathbf{z}_N \tag{9.4.17}$$

as wished.

As the η_{KL} here is defined only in terms of the ϵ-symbol and the charge $\mathbf{q}^m$, it seems possible to describe T-duality in an SL(5)-invariant way as well. As in the previous section, O$(3,3)$ T-duality transformations seem to be generated by $M_M{}^N \in$GL(5) with

$$M_M{}^K M_N{}^L\eta_{KL} = \eta_{MN} \tag{9.4.18}$$

Part IV

Summary and Outlook

Summary

The central result of this thesis was introduced in section 7. The world-sheet theory in a generic NSNS background, including non-geometric ones, can be defined as follows. In terms of some phase space variables $\mathbf{E}_A(\sigma)$, there is a Hamiltonian in a background independent form $H \sim \int \mathrm{d}\sigma\, \delta^{AB}\mathbf{E}_A(\sigma)\mathbf{E}_B(\sigma)$ and similarly for the Virasoro constraints. Instead, the information about the background is encoded in the Poisson structure. This is most conveniently formulated in terms of the current algebra (the algebra of the $\mathbf{E}_A(\sigma)$)

$$\{\mathbf{E}_A(\sigma_1), \mathbf{E}_B(\sigma_2)\} = \Pi^{(\eta)}_{AB}(\sigma_1,\sigma_2) + \Pi^{(\mathrm{bdy.})}_{AB}(\sigma_1,\sigma_2) + \Pi^{(\mathrm{flux})}_{AB}(\sigma_1,\sigma_2). \tag{9.4.19}$$

Π^{η} is the $\mathrm{O}(d,d)$-invariant part of the canonical current algebra (6.1.22), whereas

$$\Pi^{\mathrm{flux}}_{AB}(\sigma_1,\sigma_2) = -\mathbf{F}_{ABC}(\sigma_1)\mathbf{E}^C(\sigma_1)\delta(\sigma_2-\sigma_1)$$

is characterised solely by the generalised flux $\mathbf{F}_{ABC}$, building on known results in the literature [167, 236, 242]. This formulation seems to be the world-sheet version of the generalised flux formulation of generalised geometry resp. double field theory [138, 148].

In case of an electric and locally geometric background, meaning the Bianchi identity (3.3.5) is fulfilled, there is a connection to Darboux coordinates (x^μ, p_μ) on the phase space resp. a Lagrangian formulation. This connection is given by a choice of generalised vielbein $E_A{}^M(c)$, s.t. $\mathbf{E}_A(\sigma) = E_A{}^M(x(\sigma))(p_\mu(\sigma), \partial x^\mu(s\sigma))$ and $\mathbf{F}_{ABC} = (\partial_{[A}E_B{}^M)E_{C]M}$.

In the cases of a magnetically charged NSNS background (like an $NS5$-brane) or a locally non-geometric background the Hamiltonian world-sheet theory as define above is still defined. But there are some obstructions in either case. In former, the Bianchi identity of generalised fluxes is sourced. Resultantly, the associated current algebra violates the Jacobi identity and thus there cannot be Darboux coordinates on the phase space associated to the sourcing world-volume. In the non-geometric case, see section 7.2 for more details, there will be a violation of the Jacobi identity if we consider certain functions of the 'doubled' string phase space. E.g. for the Jacobi identity of a functional Ψ and two sections $\phi_i = \int \mathrm{d}\sigma \phi_i^M(\sigma)\mathbf{E}_M(\sigma)$, we obtained

$$\{\Psi,\{\phi_1,\phi_2\}\} + c.p. = \int \mathrm{d}\sigma_1\mathrm{d}\sigma_2 \frac{1}{2}(\eta_{NK}+\omega_{NK})\phi^N_{[1}(\sigma_1)\frac{\delta\Psi}{\delta X^M(\sigma_2)}\frac{\delta\phi^K_{2]}(\sigma_1)}{\delta X_M(\sigma_2)} + \text{ others.}$$

In case the generalised vielbein itself depends an original coordinate and its dual at the same time, an additional term in (9.4.19) appears that potentially leads to a non-local contribution to the equations of motion and a modification of the Virasoro algebra.

One difference to previous discussions in the literature is the consideration of the total derivative term,

$$\Pi^{\mathrm{bdy.}}_{AB}(\sigma_1,\sigma_2) = \int \mathrm{d}\sigma\partial\left(\omega_{AB}(\sigma)\delta(\sigma-\sigma_1)\delta(\sigma-\sigma_2)\right).$$

This occurs in this form as a *non* $\mathrm{O}(d,d)$-invariant boundary contribution from the canonical current algebra (6.1.22). Terms like this in the current algebra itself or its Jacobi identity make the difference between a Lie or a Courant algebroid structure of the phase space $(T \oplus T^{\star})LM$. This was discussed in detail in section 3. For open strings, they lead to the known constraint of $\mathbf{H}\big|_{\text{D-brane}} = 0$ [236] and the non-commutativity at the ends of the open string [251]. For closed strings, a winding contribution from this term is necessary such that the standard $\mathbf{Q}$-flux background is an associative background.

A first application of this formulation is the observation that (generalised) T-dualities act linearly on the variables in the generalised flux frame. A typical non-trivial example of a constant generalised flux background is the Poisson-Lie σ-model. So far in the literature, the $\mathbb{Z}_2$- or factorised Poisson-Lie duality has been studied. But, the full duality group, dubbed non-abelian T-duality (NATD) group, is bigger in general:

$$\text{NATD group}(\mathfrak{d}) = \{\text{Manin triple decompositions of } \mathfrak{d}\}$$

for a Poisson-Lie σ-model corresponding to the Lie bialgebra $\mathfrak{d}$. A method was proposed that allowed to get some insights into this group, which is a subgroup of $\mathrm{O}(d,d)$, consisting of more than only factorised dualities. Conditions of the typical $\mathrm{O}(d,d)$-transformations, i.e. factorised dualities, GL-transformations, B-shifts and β-shifts such that they lie in this non-abelian T-duality group, were derived. In some simple cases, an explicit interpretation of these transformation on the (non-doubled) Lagrangian level was found. A result of the analysis this non-abelian T-duality group is that homogeneous Yang-Baxter deformations are nothing else than non-abelian T-duality β-transformations following our definition.

This analysis revealed some interesting structures, but also that the above definition does not give the whole (classical) duality structure of a Poisson-Lie σ-model. Relaxing the closure conditions of the definition of the Manin triple leads to the proposal of a generalisation of Poisson-Lie T-duality to *Roytenberg duality*, applicable to models with constant generalised fluxes. The duality group is again the full $\mathrm{O}(d,d)$ group. This was shown using a certain parameterisation of the constant generalised flux based on the ones of Poisson-Lie σ-models in chapter 8.

A second application is a direct derivation of the well-known non-commutative and non-associative behaviour of some generalised flux backgrounds from the deformed current algebra in section 7.2. This interpretation does not rely on a mode expansion or even on imposing the equations of motion, it is purely kinematic. Also, it extends straightforwardly to any generalised flux background.

The generalisation to the Hamiltonian treatment of *membrane σ-models* was successful as well, shown for the $\mathrm{SL}(5)$ theory. One can follow the same steps as in the string case: phrasing the Hamiltonian, the constraints and the current algebra in terms of exceptional generalised geometry, understanding the additional appearance of non duality invariant topological contribution. As in the string case, both closed and open membranes could be treated like this. A caveat in comparison to the string case is that the membrane charges have to be introduced in order for the formulation to be manifestly duality covariant. A similar treatment for the higher $E_{d(d)}$ groups should be possible as well.

Potential applications and open problems

Part of the original motivation was the study of integrable deformation, as these can be conveniently represented as deformations of the current algebra – see section 5. The discussion in this thesis connecting the possible deformations of the current algebra for string σ-models to generalised fluxes hints at a connection of generalised geometry to the Hamilton formulation of integrable σ-models. From a purely technical side, there is also an argument to maybe expect a connection to integrability. The currents $\mathbf{e}_a$, used here to write down the equations of motion (7.1.10), are the ones which are used to calculate the Lax pair in all the examples – principal chiral model, η-deformation, λ-deformation, Yang-Baxter deformation.

Potentially, one could generalise this to the *Green-Schwarz superstring*, whereas an RNS formulation was already given in [246] in context of the generalised metric formulation. In particular, introducing RR-fluxes into the deformation of the current algebra could be interesting to obtain a direct understand the world-sheet dynamics in an RR-flux backgrounds – as generically the RR-flux terms in the non-linear σ-model are not known explicitly. For the Green-Schwarz superstring, a completely kinematic description includes κ-symmetry, which on the other hand is also closely connected to the supergravity equations [58] and thus dynamics of the background. The fact that this formalism relies on a flat internal space might be useful to define spacetime fermions in a background independent way and a formulation of the Green-Schwarz superstring, that is not only valid in very symmetric spacetimes. In principle, the generalised flux formulation of the current algebra in the Green-Schwarz approach was given already in [142], but without the topological term and without a non-geometric interpretation of the supersymmetric version of the generalised fluxes $\mathbf{F}_{ABC}$ occurring there.

Recently a generalisation of Poisson-Lie T-duality to higher gauge theories was proposed [273], it would be interesting to investigate whether such dualities are realised in a membrane current algebra in a similarly simple fashion as (generalised) T-dualities here.

As demonstrated in [242] it is not advantageous to parameterise the background by the generalised fluxes in order to calculate the 1-loop β-function and check the quantum conformality like this. But this formulation might be potentially a good framework to quantise the string canonically. In particular for constant generalised fluxes the equations of motion (7.1.12) take the form of a (constrained) Maurer-Cartan structure equations of a $2d$-dimensional (non-compact) Lie group. If it would be possible to construct a mode expansion, it seems possible to quantise the bosonic theory directly as also the Virasoro constraint take a simple form in the generalised flux frame.

An open technical problem is the relation of the canonical (deformed) current algebra (9.4.19) including topological/total derivative terms and the 'double field theory' algebra $\{X^M(\sigma), X^N(\sigma')\} = -\eta^{MN}\bar{\Theta}(\sigma-\sigma')$, as they are not equivalent. It would be very useful to understand this better as the source of the non-associativity associated to strong constrain violations in section 7.2 seems to lie in this relation. Also, it was mentioned before that apart from the fact that we assumed our generalised fluxes to be globally well-defined tensors we only discussed local properties of our globally non-geometric backgrounds. Previous work discussing current algebras, loop algebras and their global properties is [171, 172]. Connecting these approaches and the generalised flux formulation of non-geometric background seems to be an important step for future work.

Appendix A

Important identities

A.1 δ-distribution

The standard δ-distribution $f(\sigma) = \int d\sigma' f(\sigma')\delta(\sigma - \sigma')$ behaves in a maybe unexpected way, when applied to functions on compact spaces. In particular $(\partial_1 + \partial_2)\delta(\sigma_2 - \sigma_1) \neq 0$, but instead

$$\int d\sigma_1 d\sigma_2\, \phi(\sigma_1, \sigma_2)(\partial_1 + \partial_2)\delta(\sigma_2 - \sigma_1) = \int d\sigma\, \partial\, (\phi(\sigma, \sigma))\,. \tag{A.1.1}$$

As discussed in the main text, this term is not vanishing in general. Strings can have non-trivial winding around a non-trivial cycles in target space. In that case, the coordinate fields $x(\sigma)$ are not smooth, such that in particular $\int d\sigma\, \partial x(\sigma)$. Open string world-sheets have boundaries which would contribute to the above expression as well.

In many calculations it is helpful to write

$$\delta(\sigma_2 - \sigma_1) = \int d\sigma \delta(\sigma - \sigma_1)\delta(\sigma - \sigma_2), \tag{A.1.2}$$

to see for example that $(\partial_1 + \partial_2)\delta(\sigma_2 - \sigma_1) = \int d\sigma \partial\delta(\sigma - \sigma_1)\delta(\sigma - \sigma_2)$. In a similar fashion, the following distributional identities can be derived:

$$\begin{aligned}
&\int d\sigma f(\sigma)(\partial_1 + \dots + \partial_n)\big(\delta(\sigma - \sigma_1) \cdot \dots \cdot \delta(\sigma - \sigma_n)\big) \\
&\qquad = \int d\sigma\, \big((\partial f(\sigma)) + (n-1) f(\sigma)\partial\big)\big(\delta(\sigma - \sigma_1) \cdot \dots \cdot \delta(\sigma - \sigma_n)\big) \\
&\frac{1}{2} e(\sigma_1) \cdot e^{-1}(\sigma_2)(\partial_1 - \partial_2)\big(\delta(\sigma - \sigma_1)\delta(\sigma - \sigma_2)\big) \\
&\qquad = \frac{1}{2}(\partial_1 - \partial_2)\big(\delta(\sigma - \sigma_1)\delta(\sigma - \sigma_2)\big)\mathbb{1} - \big((\partial e) \cdot e^{-1}\big)(\sigma)\delta(\sigma - \sigma_1)\delta(\sigma - \sigma_2) \\
&\big(f(\sigma_2)\partial_1 + f(\sigma_1)\partial_2\big)\delta(\sigma - \sigma_1)\delta(\sigma - \sigma_2) \\
&\qquad = \big(\partial f(\sigma)\big)\delta(\sigma - \sigma_1)\delta(\sigma - \sigma_2) + \partial\big(f(\sigma)\delta(\sigma - \sigma_1)\delta(\sigma - \sigma_2)\big)
\end{aligned} \tag{A.1.3}$$

for arbitrary (matrix-valued) functions e and f which hold without any additional boundary terms.

A.2 ϵ-symbols

The invariant of the SL(5)-group is the ϵ-symbol:

$$\epsilon^{m_1 \dots m_5}, \epsilon_{m_1 \dots m_5}.$$

There is no object, that allows to raise or lower the indices. The **10**-indices are related to the fundamental ones by:

$$\phi^K = \frac{1}{\sqrt{2}} \phi^{kk'}.$$

With that the important Y-tensor in the theory is

$$Y^{MN}{}_{KL} = \epsilon^{rMN} \epsilon_{rKL} = \frac{1}{4} \epsilon^{rmm'nn'} \epsilon_{rkk'll'}. \tag{A.2.1}$$

Contractions of the ϵ-symbol are related to the generalised Kronecker symbol:

$$\epsilon^{i_1 \dots i_k j_1 \dots j_{n-k}} \epsilon_{i_1 \dots i_k k_1 \dots k_{n-k}} = k!\, \delta^{j_1 \dots j_{n-k}}_{k_1 \dots k_{n-k}} \tag{A.2.2}$$

$$= k!\, (n-k)!\, \delta^{j_1}_{[k_1} \dots \delta^{j_{n-k}}_{k_{n-k}]} \tag{A.2.3}$$

In the M-theory section $\frac{1}{2}\epsilon^{\kappa\kappa'\lambda\lambda'}$ resp. $\frac{1}{2}\epsilon_{\kappa\kappa'\lambda\lambda'}$ can be used to raise and lower the $\kappa\kappa'$-indices and translate between SL(5) indices and 'generalised tangent' bundle indices, i.e. ${}_{kk'} = ({}_{\kappa}, {}_{\kappa\kappa'})$ resp. ${}_{kk'} = ({}_{\kappa}, {}^{\kappa\kappa'})$. With these conventions, the SL(5) 'η-symbol' [274] is nothing else than a component of the ϵ-tensor with 'raised' indices in the M2 section decomposition

$$\eta_{\mu,KL} = \begin{pmatrix} 0 & \delta^{[\lambda}_{\mu} \delta^{\lambda']}_{\kappa} \\ \delta^{[\kappa}_{\mu} \delta^{\kappa']}_{\lambda} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2}\epsilon_{\mu\kappa}{}^{\lambda\lambda'} \\ \frac{1}{2}\epsilon_{\mu\lambda}{}^{\kappa\kappa'} & 0 \end{pmatrix} \equiv \frac{1}{2} \epsilon_{\mu kk'll'} = \epsilon_{\mu KL} \tag{A.2.4}$$

$$\eta_{5,KL} = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & \epsilon^{\kappa\kappa'\lambda\lambda'} \end{pmatrix} = \frac{1}{2} \epsilon_{5kk'll'} = \epsilon_{5KL} \tag{A.2.5}$$

Appendix B

T-duality and fermions

B.1 OSp$(d_b, d_b|2d_f)$ as superduality group

Consider a background with d_b bosonic and d_f fermionic isometries and $d = d_b + d_f$. Let us write the coordinates as

$$Z^M = (Z^a, Z^{\underline{a}}) = (x^\mu, \theta^\alpha, Z^{\underline{a}}), \qquad \text{with } \mu = 1, ..., d_b \text{ and } \alpha = 1, ..., d_f. \tag{B.1.1}$$

These d_f fermionic isometries should, of course, be understood as being generated by d_f anticommuting supercharges $\{\mathcal{Q}^\alpha, \mathcal{Q}^\beta$. This has been explored in [204].

Introducing $s(a)$, $s(a) = 0$ for a bosonic and $s(a) = 1$ for a fermionic index, the matrix representation of a factorised T-duality along the isometry coordinate Z^a is

$$\phi_{T_a} = \begin{pmatrix} \mathbb{1}_d - E_a & -E_a \\ -(-1)^{s(a)} E_a & \mathbb{1}_d - E_a \end{pmatrix}. \tag{B.1.2}$$

We can further consider GL$(d_b|d_f)$ coordinate transformations $Z^a \to \bar{Z}^a = A^a{}_b Z^b$ with a supermatrix $A = \begin{pmatrix} m & \eta \\ \vartheta & n \end{pmatrix} \in \mathrm{GL}(d_b|d_f)$. Supertransposition is defined as

$$A^{ST} = \begin{pmatrix} m & \eta \\ \vartheta & n \end{pmatrix}^{ST} = \begin{pmatrix} m^T & \vartheta^T \\ -\eta^T & n^T \end{pmatrix}.$$

The group element of such a GL$(d_b|d_f)$-transformation with the action (2.2.15) on the background components E, containing fermions now, is given similarly to GL(d) action in bosonic T-duality by

$$\phi_{GL} = \begin{pmatrix} (A^{ST})^{-1} & \\ & A \end{pmatrix} \qquad \text{for } A \in \mathrm{GL}(d_b|d_f). \tag{B.1.3}$$

It is easy to show that both (B.1.2) and (B.1.3) are elements of a group with elements

$$\phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \text{with } A, B, C, D \in \mathbb{R}^{(d_b|d_f)\times(d_b|d_f)}$$

fulfilling a pseudoorthogonality relation $\phi J \phi^{ST} = J$ with

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{ST} := \begin{pmatrix} A^{ST} & C^{ST} \\ B^{ST} & D^{ST} \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} & & \mathbb{1}_{d_b} & \\ & & & \mathbb{1}_{d_f} \\ \mathbb{1}_{d_b} & & & \\ & -\mathbb{1}_{d_f} & & \end{pmatrix}.$$

This is a representation of the orthosymplectic group $\mathrm{OSp}(d_b, d_b|2d_f)$ and nicely generalises the $\mathrm{O}(d_b, d_b)$-group of bosonic T-duality. This group was introduced in [142]. Up to some subtleties, B- and β-shifts can be defined similarly, see [204] for more details.

B.2 Spinor Representation of $\mathrm{O}(d,d)$

Additionally to the background field of the bosonic sector $E = G + B$, for which $\mathrm{O}(d,d)$-transformations act naturally on the generalised metric, we need to know how the RR-fields F in type II supergravities transform under T-duality. These transform in a spinor representation [275, 276]. Let us briefly sketch the construction of the spinor representation. We start with an auxiliary fermionic Fock space:

- Let $\Gamma_\mu = (\Gamma_\mu)_{\alpha\beta}$ be $\mathrm{O}(d,d)$ Dirac-matrices, meaning $2d$ $2^d \times 2^d$-matrices fulfilling $\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu}$ with $\eta = \begin{pmatrix} 0 & \mathbb{1}_d \\ \mathbb{1}_d & 0 \end{pmatrix}$.
- Due to the structure of η, the operators

  $$(\Psi^\dagger)^\mu = \frac{1}{\sqrt{2}}\Gamma_\mu \qquad \Psi_\mu = \frac{1}{\sqrt{2}}\Gamma_{d+\mu} \quad \text{for } \mu = 1, ..., d$$

  behave like fermionic creation and annihilation operators:

  $$\{\Psi_\mu, (\Psi^\dagger)^\nu\} = \delta^\nu_\mu \mathbb{1} \qquad \{\Psi_\mu, \Psi_\nu\} = \{(\Psi^\dagger)^\mu, (\Psi^\dagger)^\nu\} = 0. \tag{B.2.1}$$

- We define our spinor representation $S(\phi)$ of a $\phi \in \mathrm{O}(d,d)$ as usual

  $$S(\phi)^{-1} \cdot \Gamma_\mu \cdot S(\phi) = \Gamma_\nu \phi^\nu{}_\mu. \tag{B.2.2}$$

We use the (isomorphic) correspondence of differential forms and fermionic Fock operators:

$$\begin{aligned} \text{generic differential form} \quad &\leftrightarrow \quad \text{fermionic Fock operator acting on the vacuum} \\ \mathrm{d}x^\mu \wedge ... \quad &\leftrightarrow \quad (\Psi^\dagger)^\mu \\ \iota_\mu \quad &\leftrightarrow \quad \Psi_\mu \\ \Omega = \sum_{n=1}^{d} \Omega_{\mu_1\mu_2...\mu_n} \mathrm{d}x^{\mu_1} \wedge \; ... \; \wedge \mathrm{d}x^{\mu_n} \quad &\leftrightarrow \quad \hat{\Omega} \equiv \sum_{n=1}^{d} \Omega_{\mu_1\mu_2...\mu_n} (\Psi^\dagger)^{\mu_1} ... (\Psi^\dagger)^{\mu_n}. \end{aligned}$$

ι_μ is the contraction with ∂_μ. Spinor representatives $\hat{S}(\phi)$ of all the subgroups of T-duality, B-shifts, β-shifts, GL-transformations and factorised T-dualities (for the $\mathrm{O}(d,d)$-representations ϕ as in section) are

$$\begin{aligned} S(\varphi_\beta) &= \exp\left(\frac{1}{2} r^{\mu\nu}\Psi_\mu\Psi_\nu\right), \\ S(\varphi_b) &= \exp\left(-b_{\mu\nu}(\Psi^\dagger)^\mu(\Psi^\dagger)^\nu\right) \\ S(\varphi_{GL}) &= \det(A)\exp\left((\Psi^\dagger)^\mu \ln(A)_\mu{}^\nu\Psi_\nu\right) \\ S(\varphi_{T_\mu}) &= (\Psi^\dagger)_\mu + \Psi_\mu \end{aligned} \tag{B.2.3}$$

for skewsymmetric matrices b, r, and $A \in \mathrm{GL}(d)$. We define

$$\begin{aligned} \Gamma_{2d+1} &= \frac{1}{2^d}\Pi_{\mu=1}^d(\Gamma_\mu+\Gamma_{\mu+d})(\Gamma_\mu-\Gamma_{\mu+d}) \sim \Pi_{\mu=1}^d((\Psi^\dagger)_\mu+\Psi_\mu)((\Psi^\dagger)_\mu-\Psi_\mu) \\ &= \Pi_{\mu=1}^d(\Psi_\mu(\Psi^\dagger)_\mu-(\Psi^\dagger)_\mu\Psi_\mu) = \Pi_{\mu=1}^d(N_\mu^\dagger-N_\mu) = (-1)^{\Sigma_{\mu=1}^d N_\mu} \equiv (-1)^{N_F} \end{aligned}$$

in order to discuss Weyl representations, satisfying $\{\Gamma_{2d+1},\Gamma_r\}=0$. The N_F eigenvalue determines the chirality: the projections to the different eigenspaces defines the distinct Majorana-Weyl representations. Only $S(\varphi_{T_\mu})$ in (B.2.3) changes $(-1)^{N_F}$. This is another way of seeing that (factorised) T-duality transformations change the type of the type II supergravity, whereas the component connected to the identity preserves it.

These quite simple manifestations of generic $\mathrm{O}(d,d)$-transformations should not hide the fact that this is not the whole story: Consider for example the coupling of the Green-Schwarz string on $\mathrm{AdS}_5 \times S^5$ to the Ramond-Ramond 5-form F_5 (see [54])

$$\mathcal{L}_{RR5} \sim \partial_\alpha x^\lambda \partial^\alpha x^\nu \bar{\Theta}\Gamma_\lambda\Gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5}\Gamma_\nu\Theta F_{\mu_1\mu_2\mu_3\mu_4\mu_5}. \tag{B.2.4}$$

The spinors Θ are not affected by the generalised T-duality transformations directly (similarly to the non-isometry bosonic coordinates), but the transformed metric $G_{\mu\nu}$ 'creates' new ten-dimensional Dirac matrices (by means of the spacetime Clifford algebra) and thus the above term $\mathcal{L}_{RR5}$ will still change significantly in general. We also have to take into account that the Hodge duality conditions on the differential forms F_p in type II supergravities theory change in a background with a different metric.

Appendix C

Lie algebra cohomology

For convenience of the reader we state the basic notions and the results used in this thesis of Chevalley-Eilenburg Lie algebra cohomology here (see e.g. [277,278]).

C.1 Definition

A M-valued *k-cochain* u is a k-linear skewsymmetric map $u: \mathfrak{g} \wedge ... \wedge \mathfrak{g} \to M$, being an element of $\Lambda^k \mathfrak{g}^\star \otimes M$. In general we take M to be a vector space of a representation ρ of $\mathfrak{g}$ and as such a $\mathfrak{g}$-module.

We can define a coboundary $\delta u \in \Lambda^{k+1}\mathfrak{g}^\star \otimes M$ of such a k-cochain via a *coboundary operator* δ, defined as

$$\begin{aligned} \delta: \ & \Lambda^k \mathfrak{g}^\star \otimes M \to \Lambda^{k+1}\mathfrak{g}^\star \otimes M, \ u \mapsto \delta u \\ \delta u(x_0, x_1, ..., x_k) &= \sum_{i=0}^{k} (-1)^i \rho(x_i). \left(u(x_0, ..., \hat{x}_i, ..., x_n)\right) \\ &\quad + \sum_{i<j} (-1)^{i+j} u([x_i, x_j], x_0, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_k). \end{aligned} \tag{C.1.1}$$

As $\delta^2 = 0$, we can define the *k-th cohomology vector space* of $\mathfrak{g}$ w.r.t. the representation ρ as usual as

$$H^k(\mathfrak{g}, M) = \frac{k\text{-cocycles}}{k\text{-coboundaries}}. \tag{C.1.2}$$

C.2 Classic results

C.2.1 Compact Lie algebras and lower cohomology groups

Let $\mathfrak{g}$ be a compact semi-simple (finite-dimensional) Lie algebra, then the Chevalley-Eilenburg cohomologies $H^1(\mathfrak{g}, M)$ and $H^2(\mathfrak{g}, M)$ are trivial [279]. From this two statements follow:

- There are no non-abelian compact quasi-Frobenius algebras.

- There are no non-trivial solutions of the classical Yang-Baxter equation (D.1.3) on compact Lie algebras, except on their abelian subalgebras.

On the other hand in the *non-compact* case we can hope to find non-trivial elements of H^1 and H^2. For a nice discussion of the corresponding (M-valued) group cohomology see appendix of [156].

C.2.2 $H^2(\mathfrak{g}, M)$ and central extensions

Given a Lie algebra $\mathfrak{g}$ we consider the exact sequence

$$0 \hookrightarrow \mathfrak{h} \overset{i}{\hookrightarrow} \mathfrak{e} \overset{s}{\twoheadrightarrow} \mathfrak{g} \twoheadrightarrow 0, \tag{C.2.1}$$

so $\mathrm{Ker}(s) = \mathrm{Im}(i)$ is an ideal of the extended algebra $\mathfrak{e}$ and $\mathfrak{g}$ can be reproduced by $\mathfrak{g} \simeq \frac{\mathfrak{e}}{\mathrm{Im}(i)}$. If $\mathfrak{h}$ is abelian, $\mathfrak{e}$ is called *central extension*.

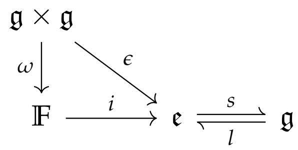
$H^2(\mathfrak{g}, M)$ has the nice interpretation as central extensions of $\mathfrak{g}$ by the $\mathfrak{g}$-module M, then considered to be an abelian algebra. Let us demonstrate the isomorphism explicitly for the trivial representation[1] $M = \mathbb{F}$, which is the case of interest in section 8.2.2.

- Given a 2-cocycle ω on $\mathfrak{g}$, let us define a bracket $[\ ,\ ]'$ on the extended algebra which viewed as a $\mathbb{F}$-vector space is $\mathfrak{e} = \mathfrak{g} \oplus \mathbb{F}Z$, where Z is the generator of $\mathfrak{h}$

$$[m_1 + r_1 Z, m_2 + r_2 Z]' = [m_1, m_2] + \omega(m_1, m_2) Z, \qquad \forall m, n \in \mathfrak{g},\ r_1, r_2 \in \mathbb{F}. \tag{C.2.2}$$

  The 2-cocycle condition on ω is equivalent to $[\ ,\ ]'$ fulfilling the Jacobi identity. Also, if we take $\mathfrak{s}$ in (C.2.1) to be the canonical projection of $[\ ,\ ]'$ back on $\mathfrak{g}$, this naturally reproduces the original Lie bracket $[\ ,\ ]$ on $\mathfrak{g}$. So $\mathfrak{s}$ is a Lie algebra homomorphism.

- Given a central extension $\mathbb{F} \overset{i}{\hookrightarrow} \mathfrak{e} \overset{s}{\twoheadrightarrow} \mathfrak{g}$, consider the diagram:



  where l is a section of $s:\ \mathfrak{e} \to \mathfrak{g}$, i.e. $s \circ l = \mathrm{id}_{\mathfrak{g}}$. We use l to define the map

$$\epsilon : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{e}, \quad \epsilon(m_1, m_2) = l([m_1, m_2]) - [l(m_1), l(m_2)] \text{ for } m_1, m_2 \in \mathfrak{g}, \tag{C.2.3}$$

  for which holds $\epsilon(m_1, [m_2, m_3]) + \text{c.p} = 0$ due to Jacobi identities in $\mathfrak{e}$ and $\mathfrak{g}$. Thus ϵ and $\omega = i^{-1} \circ \epsilon$ are $\mathfrak{e}$- resp. $\mathbb{F}$-valued 2-cocyles on $\mathfrak{g}$. This proves that for any central extension we can find a 2-cocycle.

[1]The extension to higher dimensional modules M, in case they are vector spaces themselves, corresponds basically to independent central extensions of the form for each generator of M as discussed below.

Appendix D

Yang-Baxter equations and bialgebras

D.1 Classical Yang-Baxter equation and Poisson-Lie groups

Given a Lie group $\mathcal{G}$, we can define a Poisson bracket compatible with group multiplication [99] on $C^\infty(\mathcal{G})$ as

$$\{f, g\} \equiv \Pi^{ab} X_a[f] X_b[g], \tag{D.1.1}$$

where $X_a \in \chi(\mathcal{G})$ are the left (right) invariant vector fields associated with the $t_a \in \mathfrak{g}$. Evaluated at the identity ($\Pi^{ab}(e) = r^{ab}$) skew-symmetry and Jacobi-identity of $\{\ ,\ \}$ are equivalent to

$$r = r^{ab} t_a \otimes t_b \ \in \ \mathfrak{g} \otimes \mathfrak{g} \tag{D.1.2}$$

being skew-symmetric ($r^{ab} = -r^{ba}$) and fulfilling the classical Yang-Baxter equation (cYBe)

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \equiv [r, r]_S = 0, \tag{D.1.3}$$

where the indices are tensor space indices.

Such a Poisson structure on $\mathcal{G}$ induces a Lie algebra structure on $\mathfrak{g}^*$, the dual vector space of the Lie algebra $\mathfrak{g}$, generated by d dual generators $\bar{t}^a$:

$$[\bar{t}^a, \bar{t}^b] \equiv \left[\mathrm{d}X^a\big|_e, \mathrm{d}X^b\big|_e\right] := \mathrm{d}\{X^a, X^b\}\big|_e = \left(\partial_c \Pi^{ab}\right) \mathrm{d}X^c\big|_e \equiv \bar{f}_c{}^{ab} \bar{t}^c. \tag{D.1.4}$$

The Jacobi identity for a Lie algebra with structure constants $\bar{f}_c{}^{ab}$ is fulfilled due to r fulfilling the cYBe, if $\bar{f}_c{}^{ab}$ is given in terms of r^{ab} and the original structure constants by

$$\bar{f}_c{}^{ab} = r^{ad} f^b{}_{dc} - r^{bd} f^a{}_{dc},$$

which can be shown by expanding the Poisson bivector

$$\Pi^{ab} = r^{ab} - r^{[a|d} f^{|b]}{}_{dc} X^c + \ldots\,, \tag{D.1.5}$$

for coordinates X^a associated to Lie algebra generators t_a. Viewed from the Lie group of the dual Lie algebra $\mathfrak{g}^\star$, Π^{ab} is a closed two-form.

With such a pair of structure constants $f^c{}_{ab}$, $\overline{f}_c{}^{ab}$ on $\mathfrak{g}$ resp. $\mathfrak{g}^\star$, we can define a Lie bialgebra via (2.2.40), where the Jacobi identity (2.2.41) is fulfilled, if r is a solution of the classical Yang-Baxter equation.

D.2 Complex double

Consider the complexification of a (real) simple Lie algebra $\mathfrak{g}$

$$\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \tag{D.2.1}$$

with the corresponding involution $\tau(m \otimes u) = m \otimes \bar{u}$, $\forall m \in \mathfrak{g}$, $u \in \mathbb{C}$ fixing $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$. Suppose we have a solution of the *non-split* mcYBe R, so that $\mathfrak{g}$ with the corresponding R-bracket $[\ ,\ ]_R$, denoted by $\mathfrak{g}_R$, is a Lie algebra and thus $R^- = R - i :\ \mathfrak{g}_R \hookrightarrow \mathfrak{g}^{\mathbb{C}}$ is injective and Lie algebra homomorphism due to (2.2.48). The sequence[1]

$$\mathfrak{g}_R \overset{R^+}{\hookrightarrow} \mathfrak{g}^{\mathbb{C}} \overset{\mathcal{I}}{\twoheadrightarrow} \mathfrak{g} \tag{D.2.2}$$

with $\mathcal{I}$ being the projection on the imaginary part w.r.t. τ and $\mathcal{I}\big|_{\mathrm{Im}(R^-)}$ being bijective describes the 'splitting' of $\mathfrak{g}^{\mathbb{C}}$ into a Drinfel'd double

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus_{\mathfrak{d}} \mathfrak{g}_R \equiv \mathrm{Ker}(\mathcal{I}) \oplus_{\mathfrak{d}} \mathrm{Im}(R^-) \tag{D.2.3}$$

w.r.t. to the bilinear form

$$\langle m + in | m' + in' \rangle := \gamma\left(m, n'\right) + \gamma\left(n, m'\right), \quad \text{for } m, m', n, n' \in \mathfrak{g}, \tag{D.2.4}$$

where γ is the (non-degenerate) Killing form of $\mathfrak{g}$ and R has to be skewsymmetric for $\langle\ |\ \rangle$ to be non-degenerate and g_R to be Lagrangian.

Example. Take the Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$ with standard generators $\{h, e, f\}$ fulfilling the commutation relations

$$[h, e] = e, \quad [h, f] = -f, \quad [e, f] = 2h$$

and with Killing form $\gamma(h, h) = \frac{1}{2}\gamma(e, f) = 1$. Then

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C}) = \mathrm{span}_{\mathbb{R}}\{ih, i(e+f), e-f\} \oplus \mathrm{span}_{\mathbb{R}}\{h, e, ie\} \tag{D.2.5}$$

is the decomposition into the compact real form $\mathfrak{su}(2)$ and the borel algebra $\mathfrak{b}$ of positive roots and is of the form (D.2.3) for the (canonical) Drinfel'd-Jimbo R-operator

$$R:\ h \mapsto 0, \quad e \mapsto ie, \quad f \mapsto -if. \tag{D.2.6}$$

[1] This is *not* an exact sequence.

The above structure w.r.t. to the Cartan-Weyl basis can be generalised in a straightforward manner. For reviews see [97, 100].

Group decomposition. In contrast to the Poisson-Lie case the decomposition can be defined also at the level of the corresponding Lie group $G^{\mathbb{C}}$:

$$G^{\mathbb{C}} = GG_R = G_R G, \tag{D.2.7}$$

where G resp. G_R is the Lie group to $\mathfrak{g}$ resp. $\mathfrak{g}_R$, meaning that we can write each $g \in G^{\mathbb{C}}$ as $g = h_1 h_2$ for $h_1 \in G$ and $h_2 \in G_R$ and vice versa. For the canonical Drinfel'd-Jimbo R-operator on a compact Lie algebra as above this is equivalent to the Iwasawa decomposition. In the case where the real form is non-compact we can find such a decomposition for each non-connected component.

D.3 Real double

Consider a (real) simple Lie algebra $\mathfrak{g}$, admitting a solution R of the split mcYBe and the direct (Lie algebra) sum of $\mathfrak{g}$ with itself:

$$\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g} \tag{D.3.1}$$

with projectors $p_{1,2} : \mathfrak{d} \to \mathfrak{g}$ on the first/second copy and the diagonal subalgebra

$$\mathfrak{g}^{\delta} := \{(m, m) | m \in \mathfrak{g}\} \subset \mathfrak{d}. \tag{D.3.2}$$

$R^{\pm} = (R \pm 1)$ are Lie algebra homomorphisms between $\mathfrak{g}_R$ and $\mathfrak{g}$ similarly to the complex double case. Consider the maps

$$\iota : \mathfrak{g}_R \hookrightarrow \mathfrak{d}, \; m \mapsto (R^+(m), R^-(m))$$
$$\nu : \mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g} \twoheadrightarrow \mathfrak{g}, \; (m, n) \mapsto \frac{1}{2}(m - n)$$

which are injective respectively surjective Lie algebra homomorphisms, such that the sequence

$$\mathfrak{g}_R \overset{\iota}{\hookrightarrow} \mathfrak{d} \overset{\nu}{\twoheadrightarrow} \mathfrak{g} \tag{D.3.3}$$

describes a splitting of $\mathfrak{d}$, as $\nu|_{\mathrm{Im}(\iota)}$ is bijective. The decomposition

$$\mathfrak{d} = g^{\delta} \oplus_{\mathfrak{d}} g_R \equiv \mathrm{Ker}(\nu) \oplus_{\mathfrak{d}} \mathrm{Im}(\iota) \tag{D.3.4}$$

is a Drinfel'd double w.r.t. to the bilinear form on $\mathfrak{g} \oplus \mathfrak{g}$

$$\langle (m, n) | (m', n') \rangle := \gamma(m, n') - \gamma(n, m'), \quad \text{for } m, m', n, n' \in \mathfrak{g}. \tag{D.3.5}$$

Again the non-degeneracy of $\langle \; | \; \rangle$ and also Lagrangian property of $\mathfrak{g}_R$ depends on the skew-symmetry of R.

As in the complex case, there is also a decomposition (for each connection component) $G \otimes G \simeq G^{\delta}G = GG^{\delta}$.

Example. Consider the 'analytic continuation' $R \to -iR$ of the Drinfel'd-Jimbo R-operator of the non-split mcYBe

$$R: \; h \mapsto 0, \quad e \mapsto e, \quad f \mapsto -f. \tag{D.3.6}$$

for the above generators of $\mathfrak{sl}(2,\mathbb{C})$. Of course it is now not an endomorphism on the compact real form $\mathfrak{su}(2)$ anymore, but on the split real form $\mathfrak{sl}(2,\mathbb{R})$. The R-bracket then is

$$[h,e]_R = e, \quad [h,f]_R = 0, \quad [e,f]_R = 0, \tag{D.3.7}$$

so, similar to the complex double case $\mathfrak{g}_R \simeq \mathfrak{b} = \text{span}_{\mathbb{R}}(h,e,ie)$. We can decompose the as

$$\begin{aligned} \mathfrak{d} &= \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) = (\mathfrak{sl}(2,\mathbb{R}))^{\delta} \oplus_{\mathfrak{d}} \mathfrak{g}_R \\ &= \mathfrak{span}_{\mathbb{R}} (h \oplus h, e \oplus e, f \oplus f) \oplus_{\mathfrak{d}} \mathfrak{span}_{\mathbb{R}} (h \oplus (-h), e \oplus 0, 0 \oplus f) \, . \end{aligned} \tag{D.3.8}$$

This again directly generalises to general split real forms and their doubles.

D.4 Lie bialgebras without Yang-Baxter equations?

In the physics literature, mostly bialgebras corresponding to solutions of Yang-Baxter equations were considered. These are bialgebras, where the defining 1-cocycle is an 1-coboundary of a 0-cocycle $r = r^{ab} t_a \wedge t_b$, so (2.2.43) is satisfied automatically, and where the (modified) classical Yang-Baxter equation (2.2.47) for the operator $R^a{}_b = r^{ac}\kappa_{cb}$, with the Killing form κ on $\mathfrak{g}$ holds, which is a sufficient condition for (2.2.46) to fulfil (2.2.44).

So, these span only a subspace of 1-coboundary bialgebras - in general R does not have to be a solution of the (modified) classical Yang-Baxter equation. Also there are non-coboundary 1-cocycles fulfilling (2.2.44). Let us demonstrate now that these more general bialgebra structures are not at all exotic.

D.4.1 bialgebras on the torus.

The trivial examples are possible bialgebras to an abelian Lie algebra. There are no non-trivial Chevalley-Eilenburg coboundaries on an abelian algebra, so any structure constants $\overline{f}_a{}^{bc}$ correspond to a 1-cocycle on an abelian algebra, fulfil the Jacobi identity and resultantly define a possible bialgebra, the semi-abelian bialgebra $(\mathfrak{u}(1))^d \oplus_{\mathfrak{d}} \mathfrak{g}^{\star}$.

D.4.2 bialgebras for $\mathfrak{sl}(2,\mathbb{R})$

Consider the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ with generators (h,e,f) and the commutation relations

$$[h,e] = e, \quad [h,f] = -f \quad \text{and} \quad [e,f] = 2h. \tag{D.4.1}$$

There are well-known r-matrices on $\mathfrak{sl}(2,\mathbb{R})$:

- the jordanian r-matrix $r = h \wedge e$, solving the classical Yang-Baxter equation.
- the Drinfel'd-Jimbo r-matrix $r = c\, e \wedge f$ being a solution to the modified classical Yang-Baxter equation.

As such they correspond to bialgebra structures. But simply solving conditions (2.2.43) and (2.2.44) shows, that also a generic skewsymmetric r-matrix $r = r^{ab} t_a \wedge t_b$ generates coboundary bialgebras. It seems to be a three parameter space of bialgebras, but we use the fact that the r-matrix

$$r = e \wedge f + \frac{1}{2}\left(A^2 h \wedge e + \frac{1}{A^2} h \wedge f\right) \tag{D.4.2}$$

is equivalent to the jordanian r-matrix $r = h \wedge a$ via conjugation of the $\mathfrak{sl}(2)$-generators

$$t' = S t S^{-1} \quad \text{with} \quad S = \begin{pmatrix} -\frac{\sqrt{2}}{A} & 0 \\ -\frac{1}{\sqrt{2}A} & -\frac{A}{\sqrt{2}} \end{pmatrix}. \tag{D.4.3}$$

Up to a total scale which can be absorbed into the definition of the dual generators the most general r-matrix leading to an 1-coboundary satisfying (2.2.44) is

$$r = Ae \wedge f + Bh \wedge e, \tag{D.4.4}$$

which is simply the sum of the jordanian and the Drinfel'd-Jimbo r-matrix. There seem to be no non-coboundary bialgebra structures on $\mathfrak{sl}(2, \mathbb{R})$.

D.4.3 bialgebras for AdS$_3$

AdS$_3$ has the isometry algebra $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$. There is a big amount of solutions already from Yang-Baxter type solution (e.g. abelian [204], jordanian [201], Drinfel'd-Jimbo,...). but other 1-coboundary and 1-cocycle Lie algebras exist. Let us give a non-Yang-Baxter 1-coboundary, which is closely related to the Drinfel'd-Jimbo solution of the mcYBe:

$$r = A\left(e_1 \wedge f_1 + \frac{1}{2} h_1 \wedge h_2\right) + B\left(e_2 \wedge f_2 - \frac{1}{2} h_1 \wedge h_2\right). \tag{D.4.5}$$

Let us note, that there are only unimodular dual Lie algebras for AdS$_3$, which are 1-coboundaries of solutions to the classical Yang-Baxter equation.

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