On Phase Transitions in Random Spatial Systems

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Zusammenfassung

Diese Dissertation beschäftigt sich mit der Analyse dreier Modelle, welche allesamt ihre Motivation aus dem Gebiet der statistischen Mechanik beziehen. Das erste Modell (behandelt im ersten Beitrag) ist ein System interagierender Teilchen mit drei Zuständen, während die anderen beiden Modelle (die im zweiten bis vierten Beitrag behandelt werden) Perkolationsmodelle sind.

Im ersten Modell beantworten wir die Frage nach der Existenz eines Phasenübergangs positiv. Im zweiten und dritten Modell wird das Verhalten am (und nahe am) Punkt des Phasenübergangs untersucht.

Im ersten Beitrag geht es um Teilchen, welche durch $\mathbb{Z}$ indiziert sind und sich in drei verschiedenen Zuständen befinden können, wovon wir einen als „*infiziert*“ bezeichnen. Interaktionen, für die es von einem Parameter $q \in [0, 1]$ dirigierte Regeln gibt, finden zwischen direkt benachbarten Teilchen statt. Wir untersuchen, ob die Menge der infizierten Teilchen für große Zeiten überlebt. Ist q klein genug, so zeigen wir, dass diese Menge mit positiver Wahrscheinlichkeit nie leer ist; ist q hingegen der 1 nahe genug, so stirbt die Infektion fast sicher aus. Zwischen den beiden Regimen für q findet also ein Phasenübergang statt.

Der zweite und dritte Beitrag behandeln das bekannte Knotenperkolationsmodell in hohen Dimensionen. Wir leiten die „Lace Expansion“ her, eine Identität für Zwei-Punkt-Funktion des Modells. Hieraus können Dreiecksbedingung, eine infrarote Schranke und somit Molekularfeldverhalten in hinreichend hoher Dimension abgeleitet werden. Der dritte Beitrag nutzt die Lace Expansion (und damit die Resultate des zweiten Beitrags), um die ersten drei Koeffizienten der asymptotischen Expansion des kritischen Punkts p_c explizit zu berechnen.

Der vierte Beitrag handelt vom „*Random Connection Model*“, das als Verallgemeinerung des Kontinuums-Analogs der Knotenperkolation angesehen werden kann. Durch Adaptieren der Lace Expansion wird das kritische Verhalten in hohen Dimensionen untersucht. Als Beispiel des implizierten Molekularfeldverhaltens wird außerdem die Existenz des kritischen Exponenten γ mit Wert 1 bewiesen.

Abstract

This dissertation is concerned with the analysis of three models, all of which have their motivation in the field of statistical mechanics. The first model (investigated in the first contribution) is an interacting particle system with three states, while the other two models (investigated in the second, third, and fourth contribution) are percolation models.

For the first model, we confirm the existence of a phase transition. In the second and third model, we investigate behavior of the models at (and close to) the point of the phase transition.

In the first contribution, the particles are indexed by $\mathbb{Z}$ and can be in one of three states, one of which we call "*infected*". Interactions take place with the two nearest-neighbor particles and according to some rule governed by a parameter $q \in [0, 1]$. We investigate survival (for large times) of the set of infected particles and prove that for q small enough, the infection survives with positive probability, whereas for q close enough to 1, it almost surely dies out—hence, between the different regimes of q, a phase transition occurs.

In the second and third contribution, the well-known site percolation model is considered in high dimensions. We derive the *lace expansion*, an identity for the model's two-point function, to deduce the triangle condition, the infra-red bound, and thus mean-field behavior in sufficiently high dimension. The third contribution then builds on this derived lace expansion to explicitly compute the first terms of the asymptotic expansion of the critical point p_c.

The fourth contribution investigates the random connection model, which can be viewed as a generalization of the continuum analogue of site percolation. We investigate the model's mean-field behavior in high dimension through a continuum-space adaption of the lace expansion. Moreover, as an example of the implied mean-field behavior, the critical exponent γ is proven to exist and (as on the lattice) to take value 1.

Acknowledgements

First and foremost, I would like to thank my supervisor, Markus Heydenreich, for his guidance throughout my PhD. This includes suggesting the topics treated in this thesis, but it also includes the patience with me in getting familiar with these topics. In particular in the phase where I was getting familiar with the lace expansion, his spontaneity when it came to finding time to answer questions was of immense help to me, and probably prevented many days of frustration.

His spot-on sense for always applying the right amount of micromanagement allowed me to work autonomously and develop progress on my own, while also providing the feeling of always having an option to fall back on in case a problem seemed insurmountable. In combination, this resulted in a very fulfilling experience of mathematical research, which I know not to take for granted.

I also want to thank Günter Last and Mathew Penrose for accepting to review my thesis.

My gratitude to Günter further extends to his role as a my co-author. Our numerous calls and discussions taught me (among other things) a lot on mathematical writing and presentation. A very similar description can be given about the positive influence of Remco van der Hofstad, whom I want to thank as well for his time and patience.

I want to thank Cristina Toninelli for being a co-author and for initiating the questions that lead to my first contribution of this thesis.

Marinus Gottschau, who was also part of this first contribution, is much more than a co-author, he is a very good friend. Our shared passion for mathematics is a big part of the reason I decided to pursue a PhD in the first place, while our close collaboration in the first year exemplified mathematics as a social activity in the best possible way.

A big part of my everyday life in the institute were my colleagues, and I want to thank everyone who was part of joint lunches, coffee breaks, mathematical discussions, and who made my time at the institute such an enjoyable one: Anna, Felix, Leon, Leonid, Matija, Nannan, Simon, Stein, Thomas, and Tom.

Good friends help to overcome mathematical hardships. Since I am lucky to have many of the former (too many to name), these hardships were rarely a serious problem. An equally important factor in providing an environment to focus on my studies was the continuous support of my parents, for which I am most grateful.

Contents

Chapter 1

Introduction

1.1 On a multi-type contact process

To motivate and introduce the model that is the topic of the first contribution of this thesis, we are taking two approaches. The first one is to speak of interacting particle systems very generally, the second is that of kinetically constrained spin models.

The theory of interacting particle systems is a very rich one, and a comprehensive treatment would be quite impossible here. Instead, we point to [39, 40] for two monographs on the subject. In the following, we restrict to presenting a prototypical model, which will allow us to illustrate some special features of the model investigated in [1].

Graphs. In this section, as well as in Section 1.2, we consider models on graphs. A graph is a pair $G = (V, E)$, where V is some countable set called the *vertices*. A vertex is sometimes also called *point, site*, or *individual*. The *edges* $E \subseteq \{\{x, y\} \subseteq V : x \neq y\}$ are also called *bonds*. An often-considered graph is the integer lattice $\mathbb{Z}^d$ with nearest-neighbor bonds; that is, $V = \mathbb{Z}^d$ and the edges are the pairs at l_1-distance 1. We use $\mathbb{Z}^d$ to refer to both the vertex set as well as the graph.

The contact process. We now introduce the contact process on (the graph) $\mathbb{Z}^d$ as a classical interacting particle system which generates similar questions as those investigated in [1]. It is a Feller process which can be defined via its generator. There is an alternative way of introducing it, which starts with an informal description.

At any given time $t \geq 0$, we want each vertex to be in one of the two states $\{0, 1\}$, which we call *healthy* and *infected*, respectively; hence our state space is $\{0, 1\}^{\mathbb{Z}^d}$. To each individual x, independently of the other individuals, we assign $2d + 1$ independent $\mathbb{R}_+$-valued Poisson processes ("Poisson clocks"), one of intensity 1 and the others of intensity $\lambda \geq 0$, where λ is a parameter of the model. Each of the $2d$ rate-λ-clocks is assigned to a neighbor of x. The points of these processes ("clock rings") mark the time events at which either x or one of the neighbors of x updates. In particular, if the rate-1-clock rings at time t, then x becomes healthy (no matter its state at time t^-). On the other hand, if y is a neighbor of x and the rate-λ-clock assigned to y rings at time t and x is infected at time t, then y becomes infected.

There is not much missing to turn this into a formal construction. This is called the *graphical construction*, since one can easily imagine a plot of the graph, where each vertex is assigned a time axis on which the Poisson clock rings are recorded.

A fundamental question is if the infection survives as time progresses. By this, we mean that there exist infected sites for all times $t \geq 0$. Clearly, this survival probability depends on η_0: If everyone is healthy at the start of the process, no infections will ever take place. But if we declare only the origin, $\mathbf{0} \in \mathbb{Z}^d$, infected at time 0, can the infection survive with positive probability? The answer to this question depends on λ. The clear intuition is that we have monotonicity in the sense that increasing λ increases the survival chance of the infected sites. Indeed, there is $\lambda_c \in (1, \infty)$ such that for all $\lambda < \lambda_c$, the infection dies out with probability 1, whereas it survives with positive probability for $\lambda > \lambda_c$.

The contact process is thus a simple interacting particle system that witnesses a *phase transition*, an abrupt change in the global behavior of the process.

Moreover, not only does the contact process enjoy the above monotonicity, it is also attractive and exhibits duality. Informally, the former means that adding infected sites at time 0 does not decrease the infection's survival probability, whereas the latter refers to a property of the graphical construction that also yields a contact process when considering this construction "backwards in time".

We mention these two properties to point out that the proofs for many results rely on them and without them, a lot of standard machinery is not available. And when considering a quite natural generalization to processes with three or more states per site, analogous notions are often absent or do not hold.

Maybe this makes it less surprising that results on such multi-type processes are rather scarce. One natural way of generalizing the contact process to two types of infections was considered in [44] and [15], where some monotonicity is retained. The model we now introduce falls in a similar category.

A multi-type process. We introduce a process $(\eta_t)_{t\geq 0}$ living on the line $\mathbb{Z}$. Its state space is $\{0, 1, 2\}^{\mathbb{Z}}$. We call sites in state 0 *healthy*, sites in state 1 *passive*, and sites in state 2 *infected*. We let $q \in [0, 1]$ be a model parameter. Each site is assigned one Poisson clock of rate 1. Upon a clock ring of $x \in \mathbb{Z}$,

- if at least one of x's two neighbors is healthy, then x itself becomes healthy with probability q and passive with probability $1 - q$.
- If x has no healthy neighbors, at least one infected neighbor, and x was healthy, then x becomes infected with probability $1-q$ (and remains in its state otherwise).
- If x has no healthy neighbors, at least one infected neighbor, and x was passive, then x becomes infected with probability q (and remains in its state otherwise).

From this, a formal, graphical construction can be given. We postpone the motivation for the precise dynamics for now and instead point out some properties of the model. First, note that if there is a time t such that $\eta_t(x) \neq 2$ for all $x \in \mathbb{Z}$, then the same is true for all larger times. In other words, once the infection has died out, it cannot re-emerge. Secondly, as long as a site has two passive neighbors, it will not update.

Again, a first question of interest is the one about the infection's survival. Our model differs from the contact process in that it has some "hard restrictions" that need to be satisfied in order for updates to occur (namely, at least one healthy or infected site needs to be adjacent). It retains the monotone property that adding infected sites does not decrease the survival chances. However, this does not directly translate into a monotonicity in q.

The result of our contribution [1] is to prove that for sufficiently small values of q, infection survives with positive probability (starting from only one infected site), whereas it dies out if q is sufficiently close to 1. In other words, we prove the existence of a phase transition. If monotonicity in q was true, then this would imply a critical point q_c similar to the contact process.

For models that lack the nice properties available in the classical two-type systems like the contact process, there is no canonical "toolbox" to prove the existence of phase transitions. The proof of [1] demonstrates a way how to tackle this question in a one-dimensional system which may be helpful in other systems as well.

Connection to kinetically constrained spin models. To motivate the choice of our dynamics, we point out that when starting our model with any measure supported on $\{0,1\}^{\mathbb{Z}}$, then it is the same as the Fredrickson-Anderson one-spin facilitated model (FA1f), see [18, 19]. The FA1f is a type of kinetically constrained spin model and is studied to understand "glassy" dynamics (see [46]). It is not hard to show that a product-Bernoulli-q measure is invariant for the FA1f, and it is a question of interest to show convergence to this equilibrium measure for "reasonable" initial measures. In [10], this convergence (together with a rate) is obtained for $q > 1/2$. The authors moreover conjecture that such a convergence should be true for all $q > 0$.

This is where the connection to the model considered in [1] can be made: If we couple two FA1f models, one started in equilibrium and one with another measure, and if we moreover label individuals with 0 if they are both in state 0, with 1 if they are both in state 1, and with 2 if their states disagree, we obtain a model that is dominated by the one introduced above in the sense that extinction of the infection in our model implies extinction of the disagreements. While this connection may be insightful, the result obtained in [1] about small values of q shows that the convergence to equilibrium for all $q > 0$ cannot be shown in this way.

Lastly, one may also view our multi-type process as an infection process in random dynamic environment, where the dynamic environment is precisely the FA1f model.

1.2 Percolation

1.2.1 What percolation is about

Percolation theory deals with the effects of varying the connectivity of elements (e.g., particles, sites, or bonds) in a random system. A cluster is simply a *connected* group of elements. Roughly speaking, the *percolation transition*, or *threshold*, of the system is the point at which a cluster first spans the system, i.e., the first appearance of *long-range* connectivity. In the thermodynamic limit, the percolation threshold is the point at which a cluster becomes infinite in size. The percolation transition is a wonderful example of a second-order phase transition and critical phenomenon.

> Percolation phenomena arise in a variety of applications, including transport and mechanical properties of composites and porous media, spread of diseases and fires, gelation, conductor-insulator transition in metals with disorder, fracture processes in heterogeneous rock formations, circuitry in microchips, the glass transition, sea ice, and even star formation in galaxies.
>
> (S. Torquato, *Random Heterogeneous Materials* [52])

One of the earliest contributions to percolation theory is Flory's 1941 work on gelation [17]. In 1957, Broadbent and Hammersley [11] first put percolation into a mathematical framework. The latter is what we do now.

We introduce two of the simplest and most-studied percolation models, namely *site percolation* and *bond percolation* on the hypercubic lattice $\mathbb{Z}^d$. The contributions [3, 4] work on the former, whereas we introduce the latter to draw (literature) comparisons or highlight differences.

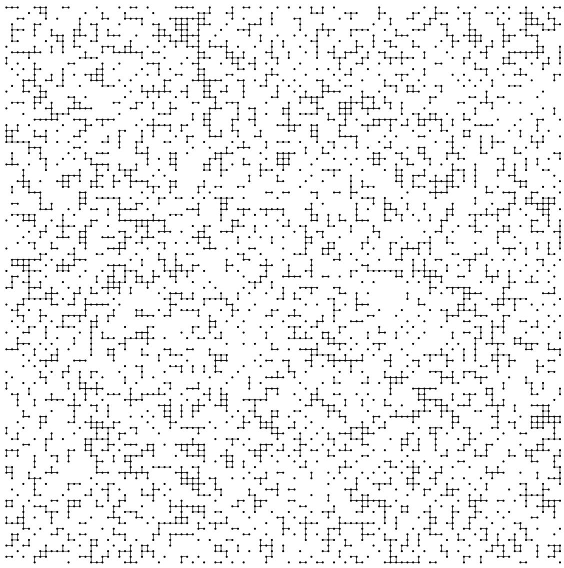
For site percolation, we take the graph $\mathbb{Z}^d$ (see Section 1.1) and declare each site, independently of all other sites, *open* (or *occupied*) with probability $p \in [0,1]$ (where p is a parameter of the model). The other sites are declared *closed* (or *vacant*). The set of open sites together with the edges both of whose endpoints are open yields a random subgraph of $\mathbb{Z}^d$. Formally, we model this with a probability space $(\Omega, \mathcal{F}, \mathbb{P}_p)$ as follows: The space Ω is $\{0,1\}^{\mathbb{Z}^d}$ (where $\omega(x) = 1$ for $\omega \in \Omega$ encodes that site x is open), the σ-algebra is the one generated by all cylinder sets, and $\mathbb{P}_p$ is a product-Bernoulli-p-measure.

For bond percolation on the other hand, we declare every bond in $E(\mathbb{Z}^d)$ open (occupied) with probability p and the other edges closed (vacant). We obtain a random subgraph with vertex set $\mathbb{Z}^d$ and the open bonds as edge set. This model can also be modeled with a product-Bernoulli measure $\mathbb{P}_p$ on the space $\{0,1\}^{E(\mathbb{Z}^d)}$ and so we write $\mathbb{P}_p^{\text{site}}$ and $\mathbb{P}_p^{\text{bond}}$ when it is necessary to avoid confusion.

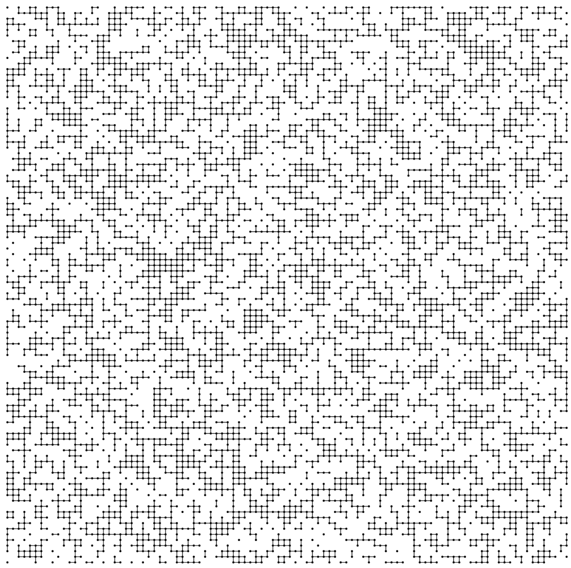
It is clear that both bond and site percolation models can be defined on any finite or locally finite graph in exactly the same way, giving rise to a zoo of different percolation models. We stress that in this section, when we do not specifically mention other models, all statements are for the bond and site percolation model on $\mathbb{Z}^d$ as introduced above.

Other percolation models. With that in mind, let us mention some of the other models that live in (or nearby) the percolation zoo. Both bond and site percolation are *spatial* in the sense that their vertex set is embedded into a (metric) space, namely, Euclidean space $\mathbb{R}^d$. In this thesis, all main models are embedded into $\mathbb{R}^d$. There are many other Euclidean lattices on which interesting percolation models are studied (the triangular lattice, the hexagonal lattice, and many more). Yet, percolation models can also be defined on graphs that are not necessarily spatially embedded, or graphs that are embedded in, say, hyperbolic space. A standard example is the d-regular tree (also called *Bethe lattice*); see Figure 1.2.

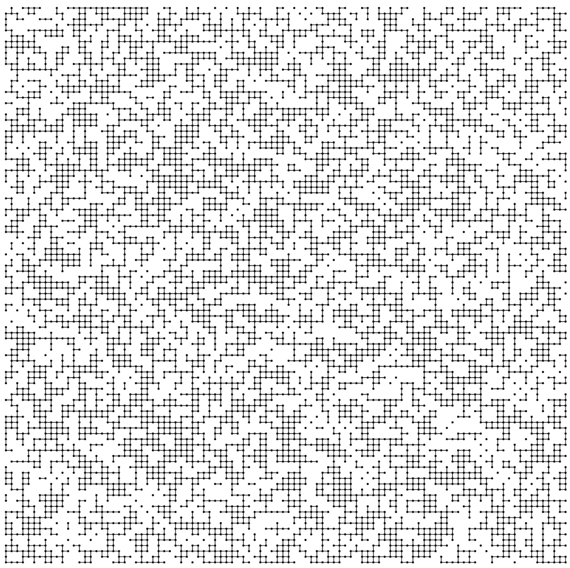
For bond percolation, it is not necessary to stick to the perspective that we take a given graph and then declare edges open and closed; instead, one could also start with a set of vertices (hence, the "empty graph") to then insert edges between nearest neighbors independently with probability p. More generally, we could allow for any pair of vertices to be joined by an edge with a probability that may now depend on factors like the spatial distance between the two vertices (given that our model is spatial), or



$p = 0.4$



$p = 0.59$



$p = 0.65$

Figure 1.1: Three realizations of the site percolation process on a subgraph of $\mathbb{Z}^2$ for different values of p.

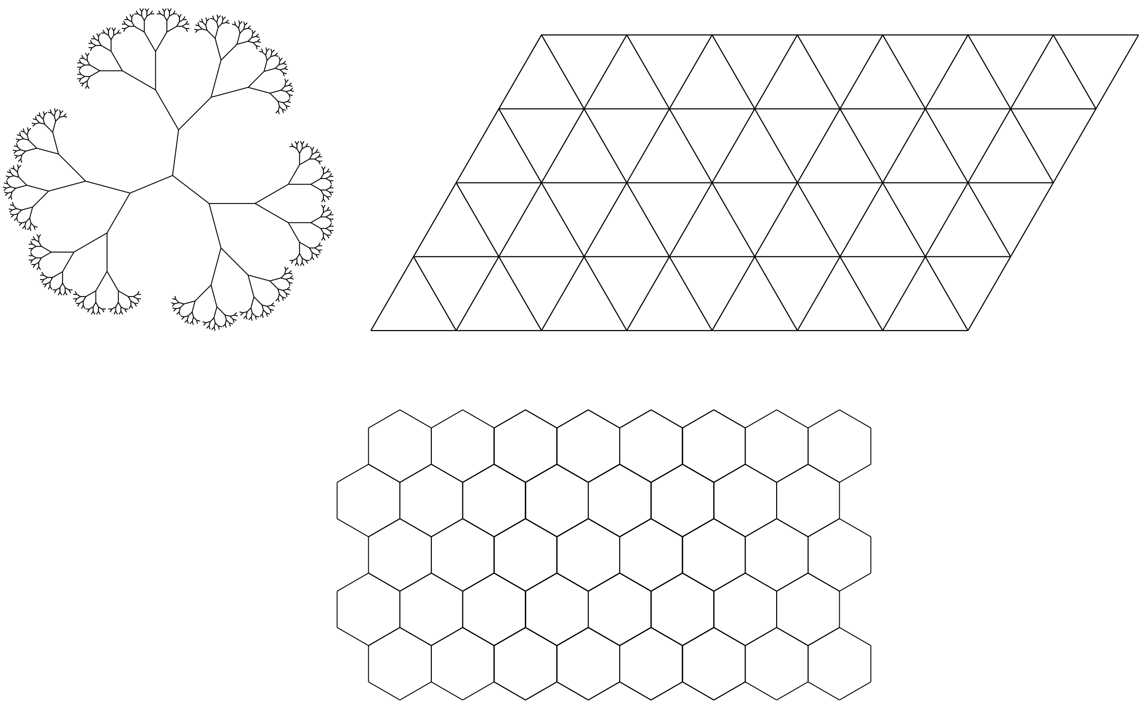


Figure 1.2: Three more lattices, or rather, parts thereof: On the upper left, the d-regular tree for $d = 3$, also called the Bethe lattice. On the upper right, the triangular lattice; and on the bottom, the hexagonal lattice. The Bethe lattice does not require a spatial embedding (but is embedded into $\mathbb{R}^2$ for display purposes), while the other two are lattices in $\mathbb{R}^2$.

a previously assigned weight of the two vertices. Models like this typically exhibit *long* edges and are therefore called *long-range percolation*.

Not all models in $\mathbb{R}^d$ stick to a prescribed lattice structure like $\mathbb{Z}^d$. Embedding the vertex set into $\mathbb{R}^d$ in a random manner as a first step and then, as a second step, devising a (possibly also random) rule by which edges are formed is the topic of continuum percolation and will be treated in more detail in Section 1.2.4.

The interesting questions. Percolation theory answers questions about the connectivity of the random subgraph sampled w.r.t. the probability measure $\mathbb{P}_p$. We make this more precise with the following definitions.

Given $\omega \in \Omega$ and $x, y \in \mathbb{Z}^d$, we say that x and y are *connected* (and we write $x \longleftrightarrow y$) if there is $k \in \mathbb{N}_0$ and a sequence $x = x_0, x_1 \dots, x_k = y$ so that $|x_i - x_{i-1}| = 1$ for all $i \in \{1, \dots, k\}$ and if, moreover, all the pairs $\{x_{i-1}, x_i\}$ are edges of ω. In site percolation, this amounts to asking that the sites $x_0, \dots, x_k$ are occupied in ω.[1]

An immediate and crucial observation is translation invariance in the sense that

$$\mathbb{P}_p(x \longleftrightarrow y) = \mathbb{P}_p(\mathbf{0} \longleftrightarrow (x-y)), \tag{1.1}$$

where $\mathbf{0}$ denotes the origin in $\mathbb{Z}^d$. Motivated by (1.1), we define the *two-point function* $\tau_p \colon \mathbb{Z}^d \to [0,1]$ (also called the *pair-connectedness function*) as

$$\tau_p(x) := \mathbb{P}_p(\mathbf{0} \longleftrightarrow x).$$

This function encodes a lot of information about our percolation model, and it is going to play a central role in [2, 3, 4]. Another central quantity is the *cluster* (or *connected component*) $\mathscr{C}(x)$ of a site $x \in \mathbb{Z}^d$, which is the set of vertices connected to x:

$$\mathscr{C}(x) := \{y \in \mathbb{Z}^d : x \longleftrightarrow y\}.$$

We say that percolation occurs if there is a site $x \in \mathbb{Z}^d$ such that $|\mathscr{C}(x)|$, the size of $\mathscr{C}(x)$, is infinite. We then say that x percolates and write $x \longleftrightarrow \infty$. Again by translation invariance, $\mathbb{P}_p(x \longleftrightarrow \infty) = \mathbb{P}_p(\mathbf{0} \longleftrightarrow \infty)$ for all sites $x \in \mathbb{Z}^d$, and we define the *percolation probability* as

$$\theta(p) := \mathbb{P}_p(\mathbf{0} \longleftrightarrow \infty).$$

Among the first results in a textbook on percolation, one will find that the function $p \mapsto \theta(p)$ is non-decreasing with $\theta(0) = 0$ and $\theta(1) = 1$, which motivates the definition of the *critical probability* (or *critical point*) as

$$p_c := \sup\{p \in [0,1] : \theta(p) = 0\}.$$

Since it is a tail event (and since we have independence in our product space),

$$\mathbb{P}_p(\exists x \in \mathbb{Z}^d : |\mathscr{C}(x)| = \infty) \in \{0,1\},$$

and it is an easy exercise to show that

$$p_c = \sup\{p \in [0,1] : \mathbb{P}_p(\exists x \in \mathbb{Z}^d : |\mathscr{C}(x)| = \infty) = 0\}.$$

[1] For technical reasons, we use a slightly altered definition of connectivity in [3, 4]. However, it is qualitatively equivalent.

Therefore, p_c is the point at which a phase transition occurs, between the almost sure non-presence and presence of an infinite cluster.

We call a model *subcritical* if $p < p_c$, *critical* for $p = p_c$ and otherwise *supercritical.* It is a natural question to ask how many infinite cluster can exist. One of the fundamental theorems in percolation theory tells us that there is at most one infinite cluster almost surely [6, 13, 20]. The argument in [13] can be extended to a broad class of graphs (so-called *amenable* graphs). It is therefore justified to speak of *the* (almost surely unique) infinite cluster, given that it exists.

But is there an infinite cluster *at* the critical point? In other words, is $\theta(p_c)$ zero or greater than zero? This question, in general, remains one of the biggest open questions in percolation theory. Since it is known that on the interval $(p_c, 1]$, the percolation function $\theta(p)$ is continuous [9], above question is equivalent to asking if θ is continuous. It is believed that $\theta(p_c) = 0$ for all dimensions $d \geq 2$. This is known to be true in dimension 2. It is also known to be true in *high dimension.* We elaborate on what high dimension means (among other things) in Section 1.2.3.

Another very natural question is to ask for the value of p_c. This value depends on the underlying graph, and for $\mathbb{Z}^d$ hence depends on the dimension d. Except for a few special cases, this is also an open problem. One such exception is bond percolation on $\mathbb{Z}^2$, where the critical point p_c^{bond} is equal to 1/2 [31, 36]. It is worth pointing out that 20 years passed between proving $p_c^{\text{bond}} \geq 1/2$ and proving $p_c^{\text{bond}} = 1/2$, and that this proof also provided the result $\theta(p_c) = 0$ stated in the previous paragraph. Some other two-dimensional lattices also belong to this set of exceptions for which p_c is known.

Another instance in which we know something about the value of p_c is when the dimension becomes large. As d tends to ∞,

$$2dp_c \to 1. \tag{1.2}$$

The intuition behind this and a finer exploration of this convergence will be discussed in Section 1.2.3

To present another fundamental result in percolation theory, let

$$\chi(p) := \mathbb{E}_p[|\mathscr{C}(\mathbf{0})|]$$

denote the expected cluster size of the origin, or *susceptibility*, and let

$$p_T := \sup\{p : \chi(p) < \infty\}$$

be a second critical point. Then if $p > p_c$, clearly $\chi(p) = \infty$ (there is a positive contribution of weight $+\infty$) and therefore $p_T \leq p_c$. The converse is also true and far from trivial [5, 43]. Hence, we have coincidence of the critical points,

$$p_c = p_T.$$

Observe that for bond percolation, by linearity of expectation,

$$\chi(p) = \sum_{x \in \mathbb{Z}^d} \tau_p^{\text{bond}}(x), \tag{1.3}$$

and so from $p_T = p_c$, we infer that τ_p is summable in the subcritical regime. The same is true for the two-point function of site percolation, and we heavily rely on this fact in [3]. We remark that it is moreover known that $\chi(p_c) = \infty$ [7, Lemma 3.1].

Various generalizations of the model have not yet been mentioned, as well as various lines of research, especially the rich theory in the sub- and supercritical regimes as well as the world of techniques that opens up when one restricts to two dimensions. Those topics lead away from the contributions of this thesis however, and so we move on to Section 1.2.2, where we sharpen our focus on the topic of critical behavior.

1.2.2 Percolation at and around criticality

The presentation in this section follows the one in [32, Chapter 1.2]. We use this section to dive deeper into the questions that arise when one wants to understand what happens at the critical point; for example, the question if there is an infinite cluster or not, raised in Section 1.2.1. To answer questions like this, it turns out that not only is it of importance to understand the model for $p = p_c$, but also when $p \nearrow p_c$ and $p \searrow p_c$, that is, when we take sequences of models whose parameter p approaches p_c out of the sub- or supercritical regime. When we speak of critical behavior of the model, we therefore also mean its near-critical behavior.

It is not at all clear how to go about systematically understanding critical behavior of a model that exhibits a phase transition in a way that percolation models do. However, many of the questions that arise can be phrased in terms of *critical exponents*, and so we start by giving an example of one such exponent. It is predicted that there exists a dimension-dependent $\beta = \beta(d) > 0$ such that

$$\theta(p) \asymp (p - p_c)^\beta \qquad \text{as } p \searrow p_c, \tag{1.4}$$

where the meaning of the '$\asymp$' symbol needs some explaining. In the literature, there are several meanings that can be assigned to this asymptotic equivalence, and for this thesis, we will only elaborate on one of them. For our purposes, (1.4) means that there are constants $0 < c_1 < c_2 < \infty$ such that

$$c_1(p - p_c)^\beta \leq \theta(p) \leq c_2(p - p_c)^\beta \tag{1.5}$$

for all $p \geq p_c$. If (1.4) indeed is true, then we say that β is the critical exponent for the percolation function and, since we defined '$\asymp$' as in (1.5), exists in the *bounded-ratio sense.*

The existence of $\beta > 0$ implies that $\theta(p_c) = 0$ and thus continuity of θ. But the predictions of physicists claim much more. Not only are both bond and site percolation models predicted to exhibit the behavior of (1.4) with the same value of β, but so is a large class of percolation models in $\mathbb{R}^d$. Roughly speaking, this is the class of models that exhibit no arbitrarily long edges and obey certain symmetries. What this prediction means is that the precise nature of the model does not matter for its critical behavior; rather, all of these "similar" models lie in the same *universality class.*

While it is very hard to give this vague physics notion of universality a precise and general mathematical definition, critical exponents like β and the ones to be defined below provide a tangible approximation in the case of percolation models. Moreover, believing in this universal behavior motivates the better understanding of the simplest models of a universality class, and therefore understanding of our bond and site percolation model.

To understand critical behavior, we therefore try to prove that certain critical exponents actually exist in the way predicted by physicists. We will now introduce some more of these exponents.

We already stated that $\chi(p)$, the expected cluster size, diverges as $p \nearrow p_c$. The critical exponent $\gamma > 0$ predicts the nature of this divergence and is defined as

$$\chi(p) \asymp (p_c - p)^{-\gamma} \qquad \text{as } p \nearrow p_c.$$

Defining the expected finite cluster size as

$$\chi^{\mathrm{f}}(p) := \mathbb{E}_p\big[|\mathscr{C}(\mathbf{0})|\mathbb{1}_{\{|\mathscr{C}(\mathbf{0})|<\infty\}}\big],$$

it is also predicted that

$$\chi^{\mathrm{f}}(p) \asymp (p - p_c)^{-\gamma} \qquad \text{as } p \searrow p_c.$$

Letting $\tau_p^{\mathrm{f}}(x) := \mathbb{P}_p(\mathbf{0} \longleftrightarrow x, |\mathscr{C}(\mathbf{0})| < \infty)$, we can define the *correlation length* as

$$\xi(p) := -\lim_{n\to\infty} \frac{n}{\log \tau_p^{\mathrm{f}}(ne_1)}, \tag{1.6}$$

where $e_1 = (1, 0, \dots, 0)$. Note that $\tau_p^{\mathrm{f}} = \tau_p$ for $p < p_c$. The existence of the limit in (1.6) (see [24, Thm. 6.44]) guarantees that $\xi(p)$ is well defined. The correlation length measures the exponential decay of the (finite) two-point function along coordinate axes. The critical exponent $\nu > 0$ is supposed to govern ξ as

$$\xi(p) \asymp (p_c - p)^{-\nu} \quad \text{as } p \nearrow p_c, \qquad \xi(p) \asymp (p - p_c)^{-\nu} \quad \text{as } p \searrow p_c.$$

The critical exponent $\delta \geq 1$ measures the decay of the cluster tail and is defined as

$$\mathbb{P}_{p_c}(|\mathscr{C}(\mathbf{0})| \geq n) \asymp n^{-1/\delta}, \qquad n \to \infty.$$

Since $\mathbb{P}_{p_c}(|\mathscr{C}(\mathbf{0})| \geq n)$ decays exponentially in the subcritical regime, δ quantifies a radically different behavior at the critical point. The occurrence of a phase transition of a system is sometimes also defined via such a change, which serves as another motivation to study critical exponents.

The exponent $\eta \geq 0$ is associated to the two-point function's critical decay and defined as

$$\tau_{p_c}(x) \asymp |x|^{-(d-2+\eta)}, \qquad |x| \to \infty.$$

Similar to the cluster size tail, $\tau_{p_c}(x)$ also decays exponentially in the subcritical regime. Lastly, we introduce two *arm exponents*. The *extrinsic arm exponent* $\rho_{\mathrm{ex}} > 0$ is defined by

$$\mathbb{P}_{p_c}(\mathbf{0} \longleftrightarrow \partial\Lambda_n) \asymp n^{-1/\rho_{\mathrm{ex}}}, \qquad n \to \infty,$$

where $\Lambda_n := \{-n, \dots, n\}^d$ and $\partial\Lambda_n = \Lambda_n \setminus \Lambda_{n-1}$. Hence, ρ_{ex} is associated to the event that the cluster of the origin extends to length at least n w.r.t. the Euclidean metric (which is seen as an extrinsic metric). Conversely, the *intrinsic arm exponent* ρ_{in} is defined as

$$\mathbb{P}_{p_c}\big(\exists x \in \mathscr{C}(\mathbf{0}) : d_{\mathscr{C}(\mathbf{0})}(\mathbf{0}, x) = n\big) \asymp n^{-1/\rho_{\mathrm{in}}}, \qquad n \to \infty,$$

where $d_{\mathscr{C}(\mathbf{0})}$ is the graph distance in the random graph induced by the vertices of $\mathscr{C}(\mathbf{0})$. The intrinsic arm exponent is therefore associated to the graph distance, which is viewed as intrinsic to the percolation model.

The existence of the critical exponents just introduced is in general not proved. After all, the existence of some of them directly implies continuity of θ, which is also open. However, there are some predictions on their values, which are summarized in Table 1.1. The values for dimension $d = 2$ are proven for site percolation on the triangular lattice [51][2]. If we believe that site and bond percolation on $\mathbb{Z}^2$ lie in the same universality class, then the critical exponents in these models should be the same.

dimension	β	γ	ν	δ	η	ρ_{ex}	ρ_{in}
2	$\frac{5}{36}$	$\frac{43}{18}$	$\frac{4}{3}$	$\frac{91}{5}$	$\frac{5}{24}$	$\frac{48}{5}$	?
3-6	?	?	?	?	?	?	?
> 6	1	1	$\frac{1}{2}$	2	0	$\frac{1}{2}$	1

Table 1.1: Some predicted (and partially proven) values for critical exponents.

For the exponents in dimensions $3, 4, 5$, and 6, only numerical approximations exist. Table 1.1 already suggests that the critical exponents become dimension-independent in dimension 7 and higher. Section 1.2.3 continues with a discussion of what is proven in this regime, where we say that the model shows *mean-field behavior.* This will finally also lead us to the contributions of [3, 4].

1.2.3 Mean-field behavior and high dimensions

If we believe the universality heuristic and the predictions in Table 1.1, then something fundamental happens around dimension 6 for our percolation models. Recall that the uniformity of the critical exponents over all the models contained in one universality class suggests that the local specifics of our model (e.g., the specific lattice structure) are so weak that they do not influence this behavior. In dimension 7 and higher, it seems that the spatial embedding of the model becomes "too weak" to influence the critical behavior in terms of critical exponents. Acknowledging this, dimension 6 is called the *upper critical dimension d_c.*

This dimension-independent behavior is called *mean-field behavior.* The critical exponents above d_c are the same as those of a much simpler model, namely, the $(2d)$-regular tree or Bethe lattice. That is why it is called a mean-field model for percolation[3]; recall that that we briefly mentioned the Bethe lattice in Section 1.2.1 (recall Figure 1.2). We denote it by $\mathbb{T}_r$ with $r = 2d$.

Before we demonstrate the simplicity of $\mathbb{T}_r$ by actually computing some of the introduced percolation quantities, let us elaborate on the relation between the two models

[2]The '$\asymp$' notion under which these exponents are proven to exist is weaker than the one we use in (1.5).

[3]The Bethe lattice is not the *only* mean-field model. For example, for purposes relating to the spatial nature of $\mathbb{Z}^d$, a spatially embedded version of the Bethe lattice may be called the mean-field model.

to strengthen the believe that above d_c, percolation on $\mathbb{Z}^d$ is "close to" percolation on $\mathbb{T}_{2d}$. We do this by arguing why the cluster of $\mathbf{0}$ is "close to" the cluster of an arbitrary designated vertex of $\mathbb{T}_{2d}$ (let us name it o and denote its cluster in $\mathbb{T}_{2d}$ by $\mathscr{C}(o;\mathbb{T}_{2d})$).

Consider percolation (bond or site) on $\mathbb{T}_r$. Thanks to the cycle-free tree structure, the subgraphs attached to different edges incident to o are disjoint and thus independent. In other words, if we explore $\mathscr{C}(o;\mathbb{T}_r)$ in a depth-first manner (or any other manner), we encounter every vertex only once. On the contrary, $\mathbb{Z}^d$ contains many cycles, so a priori, vertices can be encountered more than once. This strong independence structure in $\mathbb{T}_r$ is a big part of what makes explicit computations easy.

However, recall that we already saw in (1.2) that $2dp_c \to 1$. Hence, if $p \le p_c$ and d is not too small, we may hope that the average number of neighbors of a vertex is not much more than 1. Hence, if we explore the cluster of $\mathbf{0}$ in the percolation model on $\mathbb{Z}^d$ in a depth-first manner, then on average, we do not have much more than one new occupied bond on which we can continue our exploration. The higher the dimension, the more potential neighbors any vertex has, and the less likely it is that such an exploration path takes us back to a vertex that has already been explored, which is the only way to form a cycle. This is the intuitive argument why, as the dimension grows, $\mathscr{C}(\mathbf{0};\mathbb{Z}^d)$ looks more and more like $\mathscr{C}(o;\mathbb{T}_{2d})$.

Explicit computations on $\mathbb{T}_r$. We now compute $p_c(\mathbb{T}_r)$ as well as the critical behavior of $\chi(p)$ on $\mathbb{T}_r$. Percolation on a tree has a lot to do with branching processes, and we assume some familiarity with the latter. The offspring of the root o (that is, its potential neighborhood) is $\mathrm{Bin}(r,p)$-distributed[4], whereas the offspring of any other vertex in $\mathscr{C}(o)$ (that is, the neighbors of this vertex whose distance to o is greater) is $\mathrm{Bin}(r-1,p)$-distributed. From this, it is immediate from the extinction properties of a $\mathrm{Bin}(r-1,p)$-distributed Galton-Watson process that

$$p_c(\mathbb{T}_r) = \frac{1}{r-1},$$

and that $|\mathscr{C}(o)|$ is almost surely finite at the critical point. We now let $\mathscr{C}_{\mathrm{BP}}(x)$ denote the total offspring of a Galton-Watson process with offspring distribution $\mathrm{Bin}(r-1,p)$ and root x. Moreover, let $\chi_{\mathrm{BP}}(p) = \mathbb{E}[|\mathscr{C}_{\mathrm{BP}}(x)|]$. Then we have the recursive identity

$$\chi_{\mathrm{BP}}(p) = 1 + (r-1)p\chi_{\mathrm{BP}}(p) = \frac{1}{1-(r-1)p}.$$

From this, we can easily compute $\chi(p)$, the mean cluster size of o on $\mathbb{T}_r$, as

$$\chi(p) = 1 + rp\chi_{\mathrm{BP}}(p) = \frac{1+p}{1-(r-1)p} = \frac{1+p}{r-1}\big(p_c(\mathbb{T}_r) - p\big)^{-1}.$$

This proves that $\gamma = 1$ on the Bethe lattice.

[4]Recall that the Binomial distribution with parameters r and p, denoted by $\mathrm{Bin}(r,p)$, attains value k for $k \in \{0,\dots,r\}$ with probability $\binom{r}{k}p^k(1-p)^{r-k}$.

The random-walk picture. Note that some critical exponents are associated to a function that has no meaning without a spatial embedding; for example, η governs the spatial decay of τ_{p_c}. This is why in this context, the appropriate mean-field model for percolation is a spatial embedding of percolation on $\mathbb{T}_{2d}$. We can interpret it as a *branching random walk*. We do not give this embedding in detail. Instead point out that this branching random walk also possesses an analogue to the percolation's two-point function τ_p, and this function in turn is closely related to the Green's function of simple random walk. To introduce the latter, let

$$D(x) = \frac{1}{2d}\mathbb{1}_{\{|x|=1\}} \qquad \text{for } x \in \mathbb{Z}^d$$

be the step distribution of simple random walk. Moreover, for two functions $f, g\colon \mathbb{Z}^d \to \mathbb{R}$, let

$$(f \star g)(x) = \sum_{y \in \mathbb{Z}^d} f(y) g(x-y)$$

denote the discrete convolution and let $f^{\star n}(x) = f^{\star(n-1)} \star f$ denote the recursively defined n-fold convolution. Then

$$G_\mu(x) = \sum_{n \geq 0} \mu^n D^{\star n}(x)$$

is the Green's function of simple random walk. Setting $\mu = 1$, it counts the expected number of visits to x by a simple random walk started at the origin. The parameter μ turns this into a random walk that is killed at every step with probability $1-\mu$. To motivate μ, note that we intend to compare the Green's function to the percolation two-point function. The parameter range $p \leq p_c$ will then be analogous to $\mu \leq 1$, with $\mu = 1$ taking the role of the critical point. For all $d \geq 3$, it is known [53] that

$$G_1(x) = \frac{\text{const}}{|x|^{d-2}}\big(1 + o(1)\big) \qquad \text{as } |x| \to \infty. \tag{1.7}$$

We will come back to this fact; for now, note that if we believe the analogy between G_1 and τ_{p_c}, then (1.7) motivates the mean-field exponent $\eta = 0$.

Triangle condition and infrared bound. So far, we have established a heuristic of what is supposed to happen above the upper critical dimension, and hopefully, we have delivered some intuition along with it. Yet, the fact that d_c is supposed to be exactly 6, as opposed to any other natural number, should seem quite arbitrary up to this point. Moreover, and even worse, with the computations on $\mathbb{T}_r$ being the exception, we are missing mathematical rigor in all of the above arguments. We now take steps towards rectifying this, starting with bond percolation.

Let us start by introducing a diagrammatic convergence condition that implies some of the mean-field behavior discussed above. Introducing the (open) *triangle diagram* for bond percolation as

$$\triangle_p(x) = \sum_{y,z \in \mathbb{Z}^d} \tau_p(y)\tau_p(z-y)\tau_p(x-z) = \big(\tau_p \star \tau_p \star \tau_p\big)(x) = \tau_p^{\star 3}(x),$$

the *triangle condition* is the condition that

$$\triangle_{p_c} = \sup_{x\in\mathbb{Z}^d} \triangle_{p_c}(x) < \infty. \tag{1.8}$$

We will come back to the triangle condition for site percolation (see (1.12)). The finiteness of the double sum in $\triangle_{p_c}$ requires τ_{p_c} to vanish for $|x|\to\infty$, and so $\theta(p_c) = 0$ under the triangle condition. It was shown in [7] that under the triangle condition, $\gamma = 1$, and in [8] the mean-field values of Table 1.1 for β and δ were verified under (1.8) (both for the bond and site percolation model).

Without going into detail, most of what we introduced as mean-field behavior can be deduced from the triangle condition. And so, while there is no a-priori reason why the triangle condition should be the "right criterion", the above implications make a good case for it. Another important reason why to value (1.8) lies in the fact that on the one hand, it is so easy to state, whereas on the other, it implies most of what we can only informally group under the term of mean-field behavior. Verification of the triangle condition is the central topic of the contributions [2, 3].

Observe that the triangle diagram can be written in terms of convolutions, and so it is not a far-fetched idea to write it in terms of its Fourier transform, since it turns convolutions into products. Letting

$$\widehat{f}(k) = \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}k\cdot x} f(x)$$

for a summable function f (with '$\cdot$' the standard scalar product in $\mathbb{C}^d$) and with $k \in (-\pi,\pi]^d$ denote the discrete Fourier transform, we can use the Fourier inversion theorem to write

$$\triangle_p(x) = \int_{(-\pi,\pi]^d} \mathrm{e}^{-\mathrm{i}k\cdot x} \widehat{\tau}_p(k)^3 \frac{\mathrm{d}k}{(2\pi)^d}. \tag{1.9}$$

This rewriting can serve as motivation for a second central condition that implies mean-field behavior, which is the *infrared bound*. It states that there is an absolute constant A such that

$$|\widehat{\tau}_p(k)| \leq \frac{A}{1-\widehat{D}(k)} \tag{1.10}$$

for all $p < p_c$, where we recall that D is the step distribution of simple random walk and τ_p refers to bond percolation in this case. As a side remark, one can show that for bond percolation, $\widehat{\tau}_p(k) \geq 0$. Note that we can write the bound in (1.10) in terms of the Green's function G_1, as

$$\widehat{G}_\mu(k) = \sum_{n\geq 0} \mu^n \widehat{D}^{\star n}(k) = \frac{1}{1-\mu\widehat{D}(k)},$$

and so (1.10) establishes a relation between the percolation two-point function and the random walk Green's function in Fourier space. The infrared bound implies the triangle condition, as

$$\triangle_p(\mathbf{0}) \leq A^3 \int_{(-\pi,\pi]^d} \frac{1}{\big(1-\widehat{D}(k)\big)^3} \frac{\mathrm{d}k}{(2\pi)^d} \leq \frac{(A\pi^2 d)^3}{2^3} \int_{(-\pi,\pi]^d} \frac{1}{|k|^6} \frac{\mathrm{d}k}{(2\pi)^d}. \tag{1.11}$$

The second bound follows in few steps from the fact that $\widehat{D}(k) = d^{-1}\sum_{i=1}^d \cos(k_i)$ and from Taylor expanding the cosine function around the origin. This behavior of $\widehat{G}_1$ is the analogue to the asymptotics of (1.7) in x-space, which are much harder to prove. As a whole, it is arguably harder to avoid Fourier theory and prove closeness of τ_p and G_1 in x-space and infer the triangle condition from this.

The right-hand side of (1.11) is finite if and only if $d > 6$. Since (1.11) provides an upper bound on $\triangle_p$ that is uniform in p, the triangle condition follows by monotone convergence.

Not only have we now encountered two criteria from which we can deduce mean-field behavior, the bound in (1.11) can be taken as evidence as to why 6 should be the upper critical dimension.

The triangle condition was first identified in [7] as a criterion for mean-field behavior. It is reminiscent of the so-called *bubble condition* that plays a similar role for, among other models, *self-avoiding walk*; in this bubble condition, the quantity analogous to the two-point function is convoluted only two, not three times (this also explains the names of the conditions). Hence, the triangle condition and its implications were known before it was verified for any model.

It was Hara and Slade in their seminal paper [26] who proved the triangle condition for bond percolation in dimension $d \geq 19$, where this value for d was not verified explicitly, but rather communicated informally. Even though it was clear that it is not best possible, the value 19 stuck around as folklore for a long time. As already mentioned in Section 1.2.1, and under considerable additional effort, it was proved in [16] that the results of [26] hold for $d \geq 11$. In [26], Hara and Slade also verified the triangle condition for all $d > 6$ for a related model called *spread-out percolation*, believed to be in the same universality class. Their method of proof is called the *lace expansion*.

Lace Expansion. The lace expansion gets its name from the weakly self-avoiding walk model it was first applied to by Brydges and Spencer [12]. There, actual laces appear in the pictorial representation of some of the quantities in the proof. The method of [12] was then adapted significantly for percolation in [26] to prove the triangle condition. Modified versions of the expansion have been applied to various other models, among them self-avoiding walk [28, 29, 50], oriented percolation [34], long-range percolation [33], lattice trees and animals [27], the contact process [47], the Ising model [48], and the φ^4 model [49].

It was noted in [26] that the results for bond percolation should be adaptable to the site percolation model. Verifying this is the main content of the second contribution of this thesis [3]. We now discuss this in more detail.

But first, we stress that the notion of connectivity and the definition of the two-point function used in [3] and [4] is a slightly modified one: For x and y to be connected, the occupation status of x and y does *not* matter. In other words, the event $\{x \longleftrightarrow y\}$ is independent of the status of x and y. Otherwise the definition is as in Section 1.2.1. As a result, $\tau_p(x) = 1$ whenever x is a neighbor of the origin. We also set $\tau_p(\mathbf{0}) = 0$. Then the triangle condition for site percolation is

$$\sup_{x\in\mathbb{Z}^d} \triangle_{p_c}(x) = \sup_{x\in\mathbb{Z}^d} \tau_{p_c}^{\star 3}(x) < \infty, \tag{1.12}$$

as is the case for bond percolation. The lace expansion is an identity for the two-point function of the form

$$\tau_p(x) = C(x) + p(C \star \tau_p)(x), \tag{1.13}$$

where $C = C_p$ is called the *direct connectedness function* (we suppress the p-dependence). This convolution identity goes by the name *Ornstein-Zernike equation* (OZE) and is generally associated rather to the correlation function of particle systems. Without going into detail, it is expected to hold in most reasonable models.

Again, there is a large gap between this heuristic from physics and rigorously proven results. While many physics texts use (1.13) as a definition for C, it is not at all clear that such a C should exist. In several models, there is an informal explicit expression for C obtained through a *cluster expansion*. However, obtaining this expression rigorously has so far been only possible for small densities $p \ll p_c$, and so these approaches cannot be of help when trying to understand a model's behavior as $p \nearrow p_c$.

The lace expansion gives an explicit, albeit involved, expression for the direct connectedness function and establishes the OZE for every $p \leq p_c$. For site percolation, [3] shows that

$$C(x) = \mathbb{1}_{\{|x|=1\}} + \sum_{n \geq 0} (-1)^n \Pi_p^{(n)}(x), \tag{1.14}$$

where the functions $\Pi_p^{(n)}$, called the *lace-expansion coefficients*, have a probabilistic interpretation. The lace-expansion coefficients for bond percolation are of a very similar form. We refrain from explicitly defining either of them here, but we point out that a big part of the lace-expansion argument is to gain good control over them.

We also would like to point out that writing the OZE in terms of the Fourier transform and solving for $p\widehat{\tau}_p(k)$ (see (1.16) below) will lead to the site percolation version of the infrared bound (1.10), given that $\widehat{C}(k)$ is close to $\widehat{D}(k)$ in a certain sense. Let us elaborate on this. Multiplying (1.14) with p, we see that $pC(x)$ contains the leading term $p\mathbb{1}_{\{|x|=1\}}$, which, thanks to (1.2), approaches $D(x)$ as the dimension becomes large. The aforementioned control over the lace-expansion coefficients moreover provides us with the bounds

$$|\Pi_p^{(n)}(x)| \leq (\text{const}/d)^{n \vee 1} \tag{1.15}$$

for an absolute, p-independent constant. This justifies the notion of pC and D being "close"—at least for large values of d.

Let us now motivate the result of the third contribution, [4], which builds upon [3]. To this end, observe that in Fourier space, the OZE evaluated at zero becomes

$$\sum_{x \in \mathbb{Z}^d} \tau_p(x) = \widehat{\tau}_p(0) = \frac{\widehat{C}(0)}{1 - p\widehat{C}(0)}. \tag{1.16}$$

Recall that $p_T = p_c$, and so $\widehat{\tau}_p(0)$ diverges at p_c. As the numerator of (1.16) remains bounded, the denominator must vanish at p_c, and so

$$0 = 1 - p_c\widehat{C}(0) = 1 - 2dp_c - p_c \sum_{n \geq 0} (-1)^n \widehat{\Pi}_{p_c}^{(n)}(0).$$

We can solve this for p_c and use the estimate (1.15) to obtain

$$p_c = \frac{1}{2d} - \frac{1}{(2d)^2} \frac{\widehat{\Pi}_{p_c}^{(0)}(0) - \widehat{\Pi}_{p_c}^{(1)}(0) + \ldots \pm \widehat{\Pi}_{p_c}^{(m)}(0)}{1 + \frac{1}{2d}\sum_{n \geq 0}(-1)^n \widehat{\Pi}_{p_c}^{(n)}(0)} + \mathcal{O}(d^{-(m+2)})$$

for any $m \in \mathbb{N}$. Note that this reproves the asymptotic $2dp_c \to 1$. If one evaluates the coefficients $\Pi_{p_c}^{(n)}(0)$ carefully, much finer asymptotics for p_c can be deduced in terms of powers of $1/d$, in theory to all orders m. In [4], we derive such an expansion up to the third order and obtain

$$p_c(d) = (2d)^{-1} + \frac{5}{2}(2d)^{-2} + \frac{31}{4}(2d)^{-3} + \mathcal{O}\big((2d)^{-4}\big).$$

Results like this are not new in percolation theory. They go back to predictions made by physicists in the 70's, both for bond and for site percolation [21, 22], where even higher-order terms are predicted. For bond percolation, the prediction to order 3 was

$$p_c^{\text{bond}}(d) = (2d)^{-1} + (2d)^{-2} + \frac{7}{2}(2d)^{-3} + \mathcal{O}\big((2d)^{-4}\big). \tag{1.17}$$

It should come as no surprise that as a consequence of establishing the lace expansion for bond percolation, (1.17) was proved [30, 35]. When comparing the two critical values, we see that $p_c^{\text{site}} > p_c^{\text{bond}}$, and we can quantify the asymptotic difference. On $\mathbb{Z}^d$ for general $d \geq 2$, it is also known that $p_c^{\text{site}} > p_c^{\text{bond}}$, see [25].

1.2.4 Continuum percolation

We now move away from bond and site percolation to a percolation model that does not restrict to a fixed lattice structure for our vertex set. The general idea of *continuum percolation* can be summed up in the following way:

- First, generate a vertex set in $\mathbb{R}^d$ in a random manner.
- Second, generate an edge set; either do this in deterministic or in random fashion.

We remark that for this thesis, we restrict to $\mathbb{R}^d$ for the space in which our point set lives, even though other setups have been considered in the literature. The biggest difference to the percolation models we have seen so far lies in the first step, so we spend a few paragraphs on the very basics of point process theory (see [37] for a more comprehensive treatment). When confronted with the question on how to distribute points randomly in $\mathbb{R}^d$, one is confronted with several issues. What is a good probability space in which to model such a process? Which are good properties that characterize such a *point process*? How to parametrize the model in a way suitable to the percolation setup?

Let us make two naive requests to our process. If have have a Borel set $A \subset \mathbb{R}^d$ and $t \in \mathbb{R}^d$, then we may want the number of points in A and the one in the translated set $t + A$ to have the same distribution. Secondly, if A and B are disjoint subsets of $\mathbb{R}^d$, then the random variable counting the number of points in A should be independent of the one counting the number of points in B.

Of course, it depends very much on the motivation behind our model if these two properties are well chosen, or if they are "natural" properties; either way, we are going to adhere to them. It turns out that if we want our process to have these two properties, then only a one-parameter family of processes remains. This is the homogeneous *Poisson point process* (PPP), and its parameter, $\lambda \geq 0$, is called its *intensity*. The intensity λ is the average number of points per unit volume of the process. In other words: The larger

λ, the more points there are. We denote our homogeneous PPP by $\eta = \eta_\lambda$, and we next explain what space η lives in.

It turns out that it is most convenient to let η be a random counting measure, that is, a random variable that takes values in the set $\mathbf{N}(\mathbb{R}^d)$ of all counting measures on $\mathbb{R}^d$. For our purposes, a counting measure $\mu \in \mathbf{N}(\mathbb{R}^d)$ is a measure that assigns to each Borel set $A \in \mathcal{B}(\mathbb{R}^d)$ a number $k \in \mathbb{N}_0 \cup \{\infty\}$, and assigns a finite number if A is bounded. We denote this number by $\mu(A)$. Moreover, $\mu(\{x\}) \in \{0, 1\}$, that is, there cannot be two or more points at the same position $x \in \mathbb{R}^d$. Using the σ-algebra generated by the events $\{\mu : \mu(A) = k, A \in \mathcal{B}(\mathbb{R}^d), k \in \mathbb{N}_0 \cup \{\infty\}\}$, we have a measure space $(\mathbf{N}(\mathbb{R}^d), \mathcal{N}(\mathbb{R}^d))$. The PPP η_λ is characterized by the fact that $\eta(A)$ is Poisson-distributed with parameter $\lambda|A|$ (where $|A|$ is the Lebesgue measure of A).

A crucial property of this random counting measure η is the fact that we can a.s. write it as

$$\eta = \{X_i : i \in \mathbb{N}\}$$

for a collection of $\mathbb{R}^d$-valued random variables X_i. Hence, we can identify η with a (random) set of points in $\mathbb{R}^d$. This is important since most of the time, we want to think of η in terms of a random set rather than a random counting measure.

Now that we have introduced a way of distributing points in space, let us obtain a graph in a very simple manner: Given $\eta \subset \mathbb{R}^d$, join any two points at distance at most r by an edge. For this model, which we call the *Poisson blob model*, we can think of every point being the center of a ball of radius $\frac{r}{2}$, and any two overlapping balls are called adjacent. See Figure 1.3.

We next present two generalizations of the Poisson blob model, which represent two central models of continuum percolation. Both use additional randomness in the second step of the construction, i.e. in the formation of the edges. First, we introduce the *Boolean model*. Here, we require a (radius) distribution ρ that is supported on $\mathbb{R}_{\geq 0}$, and a sequence $(R_i)_{i \in \mathbb{N}}$ of i.i.d., ρ-distributed random variables. Now, given η, we assign to each $X_i \in \eta$ the radius R_i. Again, we can think of a ball centered at X_i, but with a random radius this time. The random graph is again obtained by joining two points if their balls overlap.

Since this model is not the topic of this thesis, we do not go into detail about what ρ should satisfy in order to produce a reasonable model (e.g., moment conditions). Instead, we introduce the second generalization, the *random connection model* (RCM for short). Here, instead of a radius distribution, we require a *connection function* $\varphi \colon \mathbb{R}^d \to [0, 1]$. Now, given two points $x, y \in \eta$, we join them by an edge with probability $\varphi(x - y)$, independently of all other edge connections. Setting $\varphi(x) = \mathbb{1}_{\{|x| \leq r\}}$, we recover the Poisson blob model.

We use ξ to denote the RCM and we denote the probability measure of the according space by $\mathbb{P}_\lambda$. Hence, λ takes the role of the model's parameter. We refrain from giving a formal construction of ξ.

A first reasonable thing to ask of φ is the symmetry

$$\varphi(x) = \varphi(-x) \qquad \text{for all } x \in \mathbb{R}^d.$$

Instead, one may also make the stronger assumption that $\varphi(x)$ depends only on the Euclidean distance $|x|$, since most reasonable connection functions satisfy this anyway.

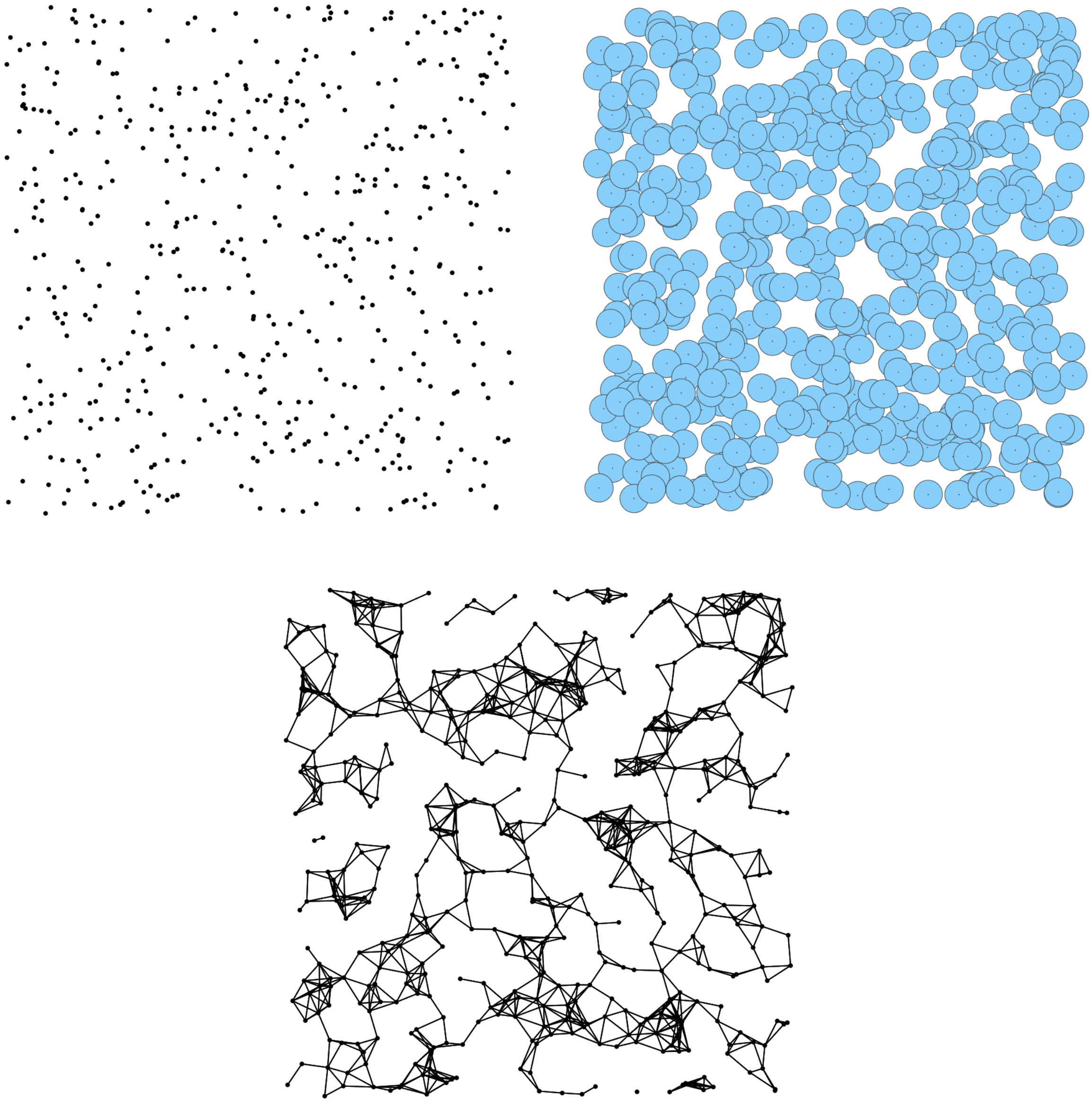


Figure 1.3: On the upper left, a realization of a Poisson point process in a box in $\mathbb{R}^2$ (conditioned to have 500 points). On the upper right, the Poisson blob model represented by balls centered at the Poisson points. Below, the underlying graph structure of this blob model. The underlying points are the same in all three figures.

A second crucial property we ask for is that

$$\int \varphi(x)\,\mathrm{d}x < \infty. \tag{1.18}$$

Here, and throughout this section, unspecified integrals are over the whole space. To motivate (1.18), we note that if this integrability condition is not satisfied, then vertices have infinite degree almost surely.

Before asking percolation questions on ξ, we need to deal with the fact that up to this point, we are not able to talk about a cluster of the origin, since any fixed point like $\mathbf{0} \in \mathbb{R}^d$ is almost surely no point of η. We are therefore also interested in $\eta^x = \eta \cup \{x\}$ for $x \in \mathbb{R}^d$, which is the PPP η augmented by x, as well as the RCM ξ^x, whose vertex set is η^x. Analogously, define $\xi^{x,y}$ for $x, y \in \mathbb{R}^d$.

We are now equipped to define the analogous quantities to those in Section 1.2.1. To highlight the difference, we refer to percolation models on lattices as discrete percolation. Two points x, y are now connected if there is a sequence of adjacent points in ξ from x to y. Our model is translation invariant, and so we define the two-point function as

$$\tau_\lambda(x) := \mathbb{P}_\lambda(\mathbf{0} \longleftrightarrow x \text{ in } \xi^{\mathbf{0},x}).$$

Note that in the above notation, it is stressed that the connection event takes place on $\xi^{\mathbf{0},x}$. The information about what points were "manually" added to the model can be essential. The cluster of the origin can be defined as

$$\mathscr{C}(\mathbf{0}, \xi^{\mathbf{0}}) := \{x \in \eta : \mathbf{0} \longleftrightarrow x \text{ in } \xi^{\mathbf{0}}\}.$$

The percolation function $\theta(\lambda)$, the expected cluster size $\chi(\lambda)$, and the critical points λ_c and λ_T are defined analogously. We point out that θ is still a non-increasing function, even though this is a little harder to show.

The Poisson blob model was introduced by Gilbert [23], which is why it is also known as the Gilbert graph. The random connection model was introduced by Penrose in [45], where, among other things, it was shown that $\lambda_c \in (0, \infty)$. Uniqueness of the infinite cluster (if it exists) was shown in [14]. In [41], Meester proved uniqueness of the critical point, i.e. $\lambda_c = \lambda_T$, and thus the continuum analogue to the $p_c = p_T$ result.

We next discuss an analogous result of the asymptotic $2dp_c \to 1$. For $\mathbb{Z}^d$, there is a natural sequence of models so that one may consider $d \to \infty$. To guarantee the same in the RCM, we need a function $\tilde{\varphi}\colon \mathbb{R} \to [0, 1]$ and can now define the RCM for every $d \in \mathbb{N}$ by using the connection function $\varphi(\cdot) = \tilde{\varphi}(|\cdot|)$. Note that (1.18) then becomes

$$\int t^d \tilde{\varphi}(t)\,\mathrm{d}t < \infty$$

for all $d \in \mathbb{N}$. The corresponding result to (1.2) was proven in [42] to be

$$\lambda_c \int \varphi(x)\,\mathrm{d}x \to 1 \qquad \text{as } d \to \infty.$$

Let us now juxtapose critical behavior of discrete and continuum percolation. The freedom of choice that we have for φ means that we can model continuum analogues of a whole variety of discrete (lattice) models. Accordingly, the critical behavior of the RCM will depend on the specific choice of φ. A large class of choices for the connection function, namely those where $\int |x|^2\varphi(x)\,\mathrm{d}x$ is finite, is expected to be in the same universality class as the discrete bond and site percolation model. In contrast, if φ has slower decay at infinity, then the according RCM is expected to be in the same universality class as certain discrete long-range percolation models (in particular, those models have a smaller upper critical dimension).

So, the range and rate of decay of the connection function supposedly dictate the critical and universal behavior. This may be interpreted in the following way: Not only is the specific structure of the lattice of discrete models non-essential for the critical behavior, the fact that the vertices are aligned in a highly regular lattice structure is not captured in the critical behavior altogether. In some sense, lattices and PPPs are extremes in the set of ways of distributing points in space: For a lattice, knowledge about very few points determines the position of all other points, whereas in a PPP we have complete spatial independence. This perspective strengthens the idea that the spatial distribution of the points (so long as it is somewhat reasonable) does not influence the critical behavior in a strong way.

However, one may also draw the conclusion that a Poisson point process generates a "special" random set of points (whatever special should mean in this context) whose critical percolation behavior resembles that of a lattice.

Both interpretations lead to the interesting task of investigating continuum percolation models based on other point processes (which are not necessarily Poisson).

Either way, it is certainly an interesting first step to strengthen arguments that continuum models based on PPPs are in the same universality class as their respective discrete cousins. The goal of [2] is exactly that.

Before laying out the contents of [2], let us argue in what way we view discrete and continuum percolation as related by comparing site percolation to the Poisson blob model. The following is also an argument for why we consider the random connection model as a continuum site (rather than bond) percolation model. Clearly, in both site percolation and the Poisson blob model, the randomness lies in the determination of the vertex set, after which the edge connections are deterministic; the parameters p and λ both serve as *point densities*. Recall that (1.3) gives an identity for $\chi(p)$ for bond percolation. The analogous identity for site percolation is

$$\chi(p) = 1 + p \sum_{x\in\mathbb{Z}^d} \tau_p^{\mathrm{site}}(x).$$

It is not hard to show that for the RCM, we have the very similarly-looking identity

$$\chi(\lambda) = 1 + \lambda \int \tau_\lambda(x)\,\mathrm{d}x.$$

Moving towards mean-field behavior, the triangle diagram for the RCM turns out to be

$$\triangle_\lambda(x) = \iint \tau_\lambda(z)\tau_\lambda(y-z)\tau_\lambda(x-y)\,\mathrm{d}z\,\mathrm{d}y = (\tau_\lambda * \tau_\lambda * \tau_\lambda)(x),$$

where '$*$' denotes the usual convolution. The triangle condition is

$$\sup_{x\in\mathbb{R}^d} \triangle_{\lambda_c}(x) < \infty, \tag{1.19}$$

strongly mirroring the one for bond and site percolation. The lace expansion for the RCM proved in [2] takes the form

$$\tau_\lambda(x) = C(x) + \lambda(C * \tau_\lambda)(x), \tag{1.20}$$

suppressing the λ-dependence for $C = C_\lambda$. We remark that the existence of such a function C was shown in [38] for $\lambda < \lambda_c$ in general dimension. Moreover, [2] shows that C is of the form

$$C(x) = \varphi(x) + \sum_{n \geq 0} (-1)^n \Pi_\lambda^{(n)}(x) \tag{1.21}$$

and so (1.20) and (1.21) are very similar to (1.13) and (1.14). To strengthen the notion that the RCM is a continuous version of a site percolation model, we remark that the OZE in bond percolation takes the slightly different form

$$\tau_p^{\text{bond}} = C + 2dp(C \star D \star \tau_p^{\text{bond}})$$

for a bond percolation's version of the direct connectedness function C. Similar to how the leading term in the direct connectedness function for site percolation related to the step distribution of simple random walk and thus the mean-field model of site percolation, the leading term in (1.21) is related to a random walk taking values in $\mathbb{R}^d$ and whose step distribution is

$$\widetilde{D}(x) = \frac{\varphi(x)}{\int \varphi(y) \, \mathrm{d}y}.$$

To no surprise, the random walk induced by $\widetilde{D}$ constitutes the (in this sense appropriate) mean-field model for the RCM. It is therefore among the topics of [2] to investigate this random walk.

The first main result of [2] is to establish the triangle condition (1.19) (and an infrared bound for $\widehat{\tau}_\lambda$) in sufficiently high dimension and to establish it for $d > 6$ in a spread-out version that we do not introduce here and that mirrors the one considered in [26]. Moreover, these results are also established for a class of long-range RCM models.

While thanks to [7, 8], the triangle condition for site percolation immediately implies that certain critical exponents attain their mean-field values and thus establishes some of the expected critical behavior, a second result of [2] is to prove that under the triangle condition for the RCM, the percolation function is continuous and the critical exponent γ exists and satisfies $\gamma = 1$.

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Chapter 2

Phase transition for a non-attractive infection process in heterogeneous environment

This chapter is the published version of

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My own contribution. This article is joint work with fellow PhD student Marinus Gottschau, my supervisor Markus Heydenreich, and Cristina Toninelli. While Cristina Toninelli and Markus Heydenreich proposed the question and gave regular input, most of the developed ideas are a collaborative effort between Marinus Gottschau and myself.

Chapter 3

Critical site percolation in high dimension

The chapter is a revised version of

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This preprint is available online at `https://arxiv.org/abs/1911.04159`.

My own contribution. This article results from numerous discussions with my supervisor, Markus Heydenreich, and is thus a collaborative effort. This actual writing was mostly done by myself.

Critical site percolation in high dimension

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Abstract

We use the lace expansion to prove an infra-red bound for site percolation on the hypercubic lattice in high dimension. This implies the triangle condition and allows us to derive several critical exponents that characterize mean-field behavior in high dimensions.

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Keywords and phrases. Site percolation, lace expansion, mean-field behavior, infra-red bound

1 Introduction

1.1 Site percolation on the hypercubic lattice

We consider site percolation on the hypercubic lattice $\mathbb{Z}^d$, where sites are independently *occupied* with probability $p \in [0,1]$, and otherwise *vacant.* More formally, for $p \in [0,1]$, we consider the probability space $(\Omega, \mathcal{F}, \mathbb{P}_p)$, where $\Omega = \{0,1\}^{\mathbb{Z}^d}$, the σ-algebra $\mathcal{F}$ is generated by the cylinder sets, and $\mathbb{P}_p = \bigotimes_{x \in \mathbb{Z}^d} \mathrm{Ber}(p)$ is a product-Bernoulli measure. We call $\omega \in \Omega$ a configuration and say that a site $x \in \mathbb{Z}^d$ is *occupied* in ω if $\omega(x) = 1$. If $\omega(x) = 0$, we say that the site x is *vacant.* For convenience, we identify ω with the set of occupied sites $\{x \in \mathbb{Z}^d : \omega(x) = 1\}$.

Given a configuration ω, we say that two points $x \neq y \in \mathbb{Z}^d$ are *connected* and write $x \longleftrightarrow y$ if there is an *occupied path* between x and y—that is, there are points $x = v_0, \dots, v_k = y$ in $\mathbb{Z}^d$ with $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ such that $|v_i - v_{i-1}| = 1$ (with $|y| = \sum_{i=1}^d |y_i|$ the 1-norm) for all $1 \leq i \leq k$, and $v_i \in \omega$ for $1 \leq i \leq k-1$ (i.e., all *inner* sites are occupied). Two neighbors are automatically connected (i.e., $\{x \longleftrightarrow y\} = \Omega$ for all x, y with $|x-y| = 1$). Many authors prefer a different definition of connectivity by requiring both endpoints to be occupied as well. These two notions are closely related and we explain our choice in Section 1.4. Moreover, we adopt the convention that $\{x \longleftrightarrow x\} = \varnothing$, that is, x is *not* connected to itself.

We define the *cluster* of x to be $\mathscr{C}(x) := \{x\} \cup \{y \in \omega : x \longleftrightarrow y\}$. Note that apart from x itself, points in $\mathscr{C}(x)$ need to be occupied. We also define the expected cluster size (or *susceptibility*) $\chi(p) = \mathbb{E}_p[|\mathscr{C}(\mathbf{0})|]$, where for a set $A \subseteq \mathbb{Z}^d$, we let $|A|$ denote the cardinality of A, and $\mathbf{0}$ the origin in $\mathbb{Z}^d$.

We define the *two-point function* $\tau_p \colon \mathbb{Z}^d \to [0,1]$ by $\tau_p(x) := \mathbb{P}_p(\mathbf{0} \longleftrightarrow x)$. The *percolation probability* is defined as $\theta(p) := \mathbb{P}_p(\mathbf{0} \longleftrightarrow \infty) = \mathbb{P}_p(|\mathscr{C}(\mathbf{0})| = \infty)$. We note that $p \mapsto \theta(p)$ is increasing and define the *critical point* for θ as

$$p_c = p_c(\mathbb{Z}^d) = \inf\{p > 0 : \theta(p) > 0\}.$$

Note that we can define a critical point $p_c(G)$ for any graph G. As we only concern ourselves with $\mathbb{Z}^d$, we write p_c or $p_c(d)$ the refer to the critical point of $\mathbb{Z}^d$.

1.2 Main result

The *triangle condition* is a versatile criterion for several critical exponents to exist and to take on their mean-field value. In order to introduce this condition, we define the *open triangle diagram* as

$$\triangle_p(x) = p^2(\tau_p * \tau_p * \tau_p)(x)$$

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and the triangle diagram as $\triangle_p = \sup_{x\in\mathbb{Z}^d} \triangle_p(x)$. In the above, the convolution '$*$' is defined as $(f*g)(x) = \sum_{y\in\mathbb{Z}^d} f(y)g(x-y)$. We also set $f^{*j} = f^{*(j-1)} * f$ and $f^{*1} \equiv f$. The triangle condition is the condition that $\triangle_{p_c} < \infty$. To state Theorem 1.1, we recall that the discrete Fourier transform of an absolutely summable function $f:\mathbb{Z}^d \to \mathbb{R}$ is defined as $\widehat{f}:(-\pi,\pi]^d \to \mathbb{C}$ with

$$\widehat{f}(k) = \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}k\cdot x} f(x),$$

where $k\cdot x = \sum_{j=1}^d k_j x_j$ denotes the scalar product. Letting $D(x) = \frac{1}{2d}\mathbb{1}_{\{|x|=1\}}$ for $x \in \mathbb{Z}^d$ be the step distribution of simple random walk, we can formulate our main theorem:

Theorem 1.1 (The triangle condition and the infra-red bound)**.** *There exist $d_0 \geq 6$ and a constant $C = C(d_0)$ such that, for all $d > d_0$,*

$$p|\widehat{\tau}_p(k)| \leq \frac{|\widehat{D}(k)| + C/d}{1-\widehat{D}(k)} \tag{1.1}$$

for all $k \in (-\pi,\pi]^d$ uniformly in $p \in [0,p_c]$ (we interpret the right-hand side of (1.1) *as ∞ for $k=0$). Additionally, $\triangle_p \leq C/d$ uniformly in $[0,p_c]$, and the triangle condition holds.*

1.3 Consequences of the infra-red bound

The triangle condition is the classical criterion for mean-field behavior in percolation models. The triangle condition implies readily that $\theta(p_c) = 0$ (since otherwise $\triangle_{p_c}$ could not be finite), a problem that is still open in smaller dimension (except $d = 2$).

Moreover, the triangle condition implies that a number of critical exponents take on their mean-field values. Indeed, using results by Aizenman and Newman [1, Section 7.7], the triangle condition implies that the critical exponent γ exists and takes its mean-field value 1, that is

$$\frac{c}{p_c - p} \leq \chi(p) \leq \frac{C}{p_c - p} \tag{1.2}$$

for $p < p_c$ and constants $0 < c < C$. We write $\chi(p) \sim (p_c - p)^{-1}$ as $p \nearrow p_c$ for the behavior of χ as in (1.2). There are several other critical exponents that are predicted to exist. For example, $\theta(p) \sim (p-p_c)^{\beta}$ as $p \searrow p_c$, and $\mathbb{P}_{p_c}(|\mathscr{C}(\mathbf{0})| \geq n) \sim n^{-1/\delta}$ as $n \to \infty$.

Barsky and Aizenman [3] show that under the triangle condition,

$$\delta = 2 \qquad \text{and} \qquad \beta = 1. \tag{1.3}$$

Their results are stated for a class of percolation models including site percolation. Hence, Theorem 1.1 implies (1.3). However, "for simplicity of presentation", the presentation of the proofs is restricted to bond percolation models.

Moreover, as shown by Nguyen [2], Theorem 1.1 implies that $\Delta = 2$, where Δ is the gap exponent.

1.4 Discussion of literature and results

Percolation theory is a fundamental part of contemporary probability theory and its foundations are generally attributed to a 1957 paper of Broadbent and Hammersley [8]. Meanwhile, a number of textbooks appeared, and we refer to Grimmett [12] for a comprehensive treatment of the subject, as well as Bollobás and Riordan [6], Werner [26] and Beffara and Duminil-Copin [4] for extra emphasis on the exciting recent progress in two-dimensional percolation.

The investigation of percolation in high dimensions was started by the seminal 1990 paper of Hara and Slade [13], who applied the lace expansion to prove the triangle condition for bond percolation in high dimension. A number of modifications and extensions of the lace expansion method for bond percolation have appeared in the meantime. The expansion itself is presented in Slade's Saint Flour notes [25]. A detailed account of the full lace expansion proof for bond percolation (including convergence of the expansion and related results) is given in a recent textbook by the first author and van der Hofstad [18].

Despite the fantastic understanding of *bond* percolation in high dimensions, *site* percolation is not yet analyzed with this method, and the present paper aims to remedy this situation. Together with van der Hofstad and Last [19], we recently applied the lace expansion to the random connection model, which can be viewed as a continuum site percolation model. The aim of this paper is to give a rigorous exhibition of the lace expansion applied to one of the simplest site percolation lattice models. We have chosen to set up the proofs in a similar fashion to corresponding work for bond percolation, making it easier to oversee by readers who are familiar with that literature. Indeed, Sections 3 and 4 follow rather closely the well-established method in [7, 18, 20], which are all based on Hara and Slade's foundational work [13]. Interestingly, there is a difference in the expansion itself, which has numerous repercussions in the diagrammatic bounds and also in the different form of the infrared bounds. We now explain these differences in more detail.

A key insight for the analysis of high-dimensional site percolation is the frequent occurrence of *pivotal points*, which are crucial for the setup of the lace expansion. Suppose two sites x and y are connected, then an intermediate vertex u is *pivotal* for this connection if every occupied path from x to y passes through u. We then break up the original connection event in two parts: a connection between x and u, and a second connection between u and y. In high dimension, we expect the two connection events to behave rather independently, except for their joint dependence on the occupation status of u. This little thought demonstrates that it is highly convenient to define connectivity events as in Section 1.1, and thus treat the occupation of the vertex u independent of the two new connection events. A more conventional choice of connectivity, where two points can only be connected if they are both occupied, is obtained a posteriori via

$$\mathbb{P}_p(\{x \longleftrightarrow y\} \cap \{x, y \text{ occupied}\}) = p^2\, \tau_p(y-x), \qquad x, y \in \mathbb{Z}^d, x \neq y. \tag{1.4}$$

As already stated in Theorem 1.1, not only are we going to show finiteness of the triangle, but we show its smallness. Therefore, having the right amount of p's in τ_p is relevant. Our definition of τ_p avoids divisions by p, not only in the triangle, but throughout Sections 2 and 3.

Paying close attention to this right amount of factors of p is a guiding thread of the technical aspects of Sections 3 and 4 that sets site percolation apart from bond percolation. Like it is done in bond percolation, the diagrammatic events by which we bound the lace-expansion coefficients depend on quite a few more points than the pivotal points. In contrast, every pair of points among these that may coincide hides a case distinction, and a coincidence case leads to a new diagram, typically with a smaller number of factors of p (see (3.2)). We handle this, for example, by encoding such coincidences in τ_p° and $\tau_p^\bullet$ (see Definition 3.3). In Section 4, the mismatching number of p's and τ_p's in $\triangle_p$ needs to be resolved.

The differing diagrams in Section 3 due to coincidences already appear in the lace-expansion coefficients of small order. This manifests itself in the answer to a classical question for high-dimensional percolation; namely, to devise an expansion of the critical threshold $p_c(d)$ when $d \to \infty$. It is known in the physics literature that

$$p_c(d) = (2d)^{-1} + \frac{5}{2}(2d)^{-2} + \frac{31}{4}(2d)^{-3} + \frac{75}{4}(2d)^{-4} + \frac{11977}{48}(2d)^{-5} + \frac{209183}{96}(2d)^{-6} + \cdots. \tag{1.5}$$

The first four terms are due to Gaunt, Ruskin and Sykes [11], the latter two were found recently by Mertens and Moore [23] by exploiting involved numerical methods.

The lace expansion devised in this paper enables us to give a rigorous proof of the first terms of (1.5). Indeed, we use the representation obtained in this paper to show that

$$p_c(d) = (2d)^{-1} + \frac{5}{2}(2d)^{-2} + \frac{31}{4}(2d)^{-3} + \mathcal{O}\left((2d)^{-4}\right) \quad \text{as } d \to \infty. \tag{1.6}$$

This is the content of a forthcoming paper [16]. Deriving p_c expansions from lace expansion coefficients has been earlier achieved for bond percolation by Hara and Slade [15] and van der Hofstad and Slade [21]. Comparing (1.6) to their expansion for bond percolation confirms that already the second coefficient is different.

Proposition 4.2 proves the convergence of the lace expansion for $p < p_c$, yielding an identity for τ_p of the form

$$\tau_p(x) = C(x) + p(C * \tau_p)(x), \tag{1.7}$$

where C is the *direct-connectedness function* and $C(\cdot) = 2dD(\cdot) + \Pi_p(\cdot)$ (for a definition of Π_p, see Definition 2.8 and Proposition 4.2). In fluid-state statistical mechanics, (1.7) is known as the *Ornstein-Zernike equation* (OZE), a classical equation that is typically associated to the total correlation function.

We can juxtapose (1.7) with the converging lace expansion for bond percolation, which yields

$$\tau_p^{\mathrm{bond}}(x) = C^{\mathrm{bond}}(x) + 2dp(C^{\mathrm{bond}} * D * \tau_p^{\mathrm{bond}})(x), \tag{1.8}$$

where $C^{\mathrm{bond}}(x) = \mathbb{1}_{\{x=\mathbf{0}\}} + \Pi_p^{\mathrm{bond}}(x)$. Thus, only for the site percolation two-point function (as defined in this paper), the lace expansion coincides with the OZE.

We want to touch on how this relates to the infra-red bound (1.1). To this end, define the random walk Green's function as $G_\lambda(x) = \sum_{m\geq 0} \lambda^m D^{*m}(x)$ for $\lambda \in (0,1]$. Consequently,

$$\widehat{G}_\lambda(k) = \frac{1}{1-\lambda\widehat{D}(k)}.$$

One of the key ideas behind the lace expansion for bond percolation is to show that the two-point function is close to G_λ in an appropriate sense (this includes an appropriate parametrization of λ). Solving the OZE in Fourier space for $\widehat{\tau}_p$ already hints at the fact that in site percolation, $p\widehat{\tau}_p$ should be close to $\widehat{G}_\lambda\widehat{D}$ and $p\tau_p$ should be close to $D * G_\lambda$. As a technical remark, we note that $\widehat{G}_\lambda$ is uniformly lower-bounded, whereas $\widehat{G}_\lambda\widehat{D}$ is not, which poses some inconvenience later on.

The complete graph may be viewed as a mean-field model for percolation, in particular when we analyze clusters on high-dimensional tori, cf. [17]. Interestingly, the distinction between bond and site percolation exhibits itself rather drastically on the complete graph: for bond percolation, we obtain the usual Erdős-Rényi random graph with its well-known phase transition, whereas for site percolation, we obtain again a complete graph with a binomial number of points.

Theorem 1.1 proves the triangle condition in dimension $d > d_0$ for sufficiently large d_0. It is folklore in the physics literature that $d_0 = 6$ suffices (6 is the "upper critical dimension") but the perturbative nature of our argument does not allow us to derive that. Instead, we only get the result for *some* $d_0 \geq 6$. For bond percolation, already the original paper by Hara and Slade [13] treated a second, spread-out version of bond percolation, and they proved that for this model, $d_0 = 6$ suffices (under suitable assumption on the spread-out nature). For ordinary bond percolation, it was announced that $d_0 = 19$ suffices for the triangle condition in [14], and the number 19 circulated for many years in the community. Finally, Fitzner and van der Hofstad [10] devised involved numerical methods to rigorously verify that an adaptation of the method is applicable for $d > d_0 = 10$. It is clear that an analogous result of Theorem 1.1 would hold for "spread-out site percolation" in suitable form (see e.g. [18, Section 5.2]).

1.5 Outline of the paper

The paper is organized as follows. The aim of Section 2 is to establish a lace-expansion identity for τ_p, which is formulated in Proposition 2.9. To this end, we use Section 2.1 to state some known results that we are going to make use of in Section 2 as well as in later sections. We then introduce a lot of the language and quantities needed to state Proposition 2.9 in Section 2.2, followed by the actual derivation of the identity in Section 2.3.

Section 3 bounds the lace-expansion coefficients derived in Section 2.3 in terms of simpler diagrams, which are large sums over products of two-point (and related) functions. Section 4 finishes the argument via the so-called bootstrap argument. First, a *bootstrap function* f is introduced in Section 4.1. Among other things, it measures how close $\widehat{\tau}_p$ is to $\widehat{G}_\lambda$ (in a fractional sense). Section 4.2 shows convergence of the lace expansion for fixed $p < p_c$. Moreover, assuming that f is bounded on $[0, p_c)$, it is shown that this convergence is uniform in p (see first and second part of Proposition 4.2). Lastly, Section 4.3 actually proves said boundedness of f.

2 The expansion

2.1 The standard tools

We require two standard tools of percolation theory, namely Russo's formula and the BK inequality, both for increasing events. Recall that A is called *increasing* if $\omega \in A$ and $\omega \subseteq \omega'$ implies $\omega' \in A$. Given

ω and an increasing event A, we introduce

$$\mathsf{Piv}(A) = \{y \in \mathbb{Z}^d : \omega \cup \{y\} \in A, (\omega \setminus \{y\}) \notin A\}.$$

If A is an increasing event determined by sites in $\Lambda \subset \mathbb{Z}^d$ with $|\Lambda| < \infty$, then Russo's formula [24], proved independently by Margulis [22], tells us that

$$\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{P}_p(A) = \mathbb{E}[|\mathsf{Piv}(A)|] = \sum_{y \in \Lambda} \mathbb{P}_p(y \in \mathsf{Piv}(A)). \tag{2.1}$$

To state the BK inequality, let $\Lambda \subset \mathbb{Z}^d$ be finite and, given $\omega \in \Omega$, let

$$[\omega]_\Lambda = \{\omega' \in \Omega : \omega'(x) = \omega(x) \text{ for all } x \in \Lambda\}$$

be the cylinder event of the restriction of ω to Λ. For two events A, B, we can define the *disjoint occurrence* as

$$A \circ B = \{\omega : \exists K, L \subseteq \mathbb{Z}^d : K \cap L = \varnothing, [\omega]_K \subseteq A, [\omega]_L \subseteq B\}.$$

The BK inequality, proved by van den Berg and Kesten [5] for increasing events, states that, given two increasing events A and B,

$$\mathbb{P}_p(A \circ B) \le \mathbb{P}_p(A)\mathbb{P}_p(B). \tag{2.2}$$

The following proposition about simple random walk will be of importance later:

Proposition 2.1 (Random walk triangle, [18], Proposition 5.5)**.** *Let $m \in \mathbb{N}_0, n \ge 0$ and $\lambda \in [0,1]$. Then there exists a constant $c^{(RW)}_{2m,n}$ independent of d such that, for $d > 2n$,*

$$\int_{(-\pi,\pi]^d} \frac{\widehat{D}(k)^{2m}}{[1-\lambda\widehat{D}(k)]^n} \frac{\mathrm{d}k}{(2\pi)^d} \le c^{(RW)}_{2m,n} d^{-m}.$$

In [18], $d > 4n$ is required; however, more careful analysis shows that $d > 2n$ suffices (see [7, (2.19)]). We will also need the following related result:

Proposition 2.2 (Related random walk bounds)**.** *Let $m \in \{0,1\}$, $\lambda \in [0,1]$, and $r, n \ge 0$ such that $d > 2(n+r)$. Then, uniformly in $k \in (-\pi,\pi]^d$,*

$$\int_{(-\pi,\pi]^d} \widehat{D}(l)^{2m}\widehat{G}_\lambda(l)^n \tfrac{1}{2}\big[\widehat{G}_\lambda(l+k) + \widehat{G}_\lambda(l-k)\big]^r \frac{\mathrm{d}l}{(2\pi)^d} \le c d^{-m}, \tag{2.3}$$

$$\int_{(-\pi,\pi]^d} \widehat{D}(l)^{2m}\widehat{G}_\lambda(l)^{n-1} \big[\widehat{G}_\lambda(l+k)\widehat{G}_\lambda(l-k)\big]^{r/2} \frac{\mathrm{d}l}{(2\pi)^d} \le \tilde{c} d^{-m}, \tag{2.4}$$

where $c = c^{(RW)}_{2m,n+2r} \vee c^{(RW)}_{2m,n}$, $\tilde{c} = c^{(RW)}_{2m,n-1+2r} \vee c^{(RW)}_{2m,n-1}$, and the constants $c^{(RW)}_{\cdot,\cdot}$ are from Proposition 2.1.

Proposition 2.2 is slightly more general than [18, Exercise 5.4], so we prove it.

Proof of Proposition 2.2. Note that, letting $G_{\lambda,k}(x) := \cos(k \cdot x) G_\lambda(x)$,

$$\widehat{G}_\lambda(l \pm k) = \sum_{x \in \mathbb{Z}^d} \cos(l \cdot x) G_{\lambda,k}(x) \mp \sin(l\cdot x)\sin(k \cdot x) G_\lambda(x).$$

We can therefore rewrite the left-hand side of (2.3) as

$$\int_{(-\pi,\pi]^d} \widehat{D}(l)^{2m}\widehat{G}_\lambda(l)^n \widehat{G}_{\lambda,k}(l)^r \frac{\mathrm{d}l}{(2\pi)^d} \le \Big(\int_{(-\pi,\pi]^d} \widehat{D}(l)^{2m}\widehat{G}_\lambda(l)^n \widehat{G}_{\lambda,k}(l)^{2r} \frac{\mathrm{d}l}{(2\pi)^d}\Big)^{1/2} \times \Big(\int_{(-\pi,\pi]^d} \widehat{D}(l)^{2m}\widehat{G}_\lambda(l)^n \frac{\mathrm{d}l}{(2\pi)^d}\Big)^{1/2},$$

using Cauchy-Schwarz. The second term is at most $(c^{(\mathrm{RW})}_{2m,n} d^{-m})^{1/2}$ by Proposition 2.1, the first term is

$$\big((D^{*(2m)} * G_\lambda^{*n} * G_{\lambda,k}^{*(2r)})(\mathbf{0})\big)^{1/2} \le \big((D^{*(2m)} * G_\lambda^{*(n+2r)})(\mathbf{0})\big)^{1/2} \le (c^{(\mathrm{RW})}_{2m,n+2r} d^{-m})^{1/2}.$$

To prove (2.4), we note that

$$0 \le \widehat{G}_\lambda(l+k)\widehat{G}_\lambda(l-k) = \widehat{G}_{\lambda,k}(l)^2 - \Big(\sum_{x\in\mathbb{Z}^d} \sin(l\cdot x)\sin(k\cdot x) G_\lambda(x)\Big)^2 \le \widehat{G}_{\lambda,k}(l)^2.$$

The bound continues analogous to the one for (2.3). □

The following differential inequality is an application of Russo's formula and the BK inequality. It applies them to events which are not determined by a finite set of sites. We refer to the literature [18, Lemma 4.4] for arguments justifying this and for a more detailed proof. Observation 2.3 will be of use in Section 4.

Observation 2.3. *Let $p < p_c$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}p}\widehat{\tau}_p(0) \le \widehat{\tau}_p(0)^2, \qquad \frac{\mathrm{d}}{\mathrm{d}p}\chi(p) \le \chi(p)\widehat{\tau}_p(0).$$

As a proof sketch, note that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}\widehat{\tau}_p(0) &= \sum_{x\in\mathbb{Z}^d} \frac{\mathrm{d}}{\mathrm{d}p}\tau_p(x) = \sum_{x\in\mathbb{Z}^d}\sum_{y\in\mathbb{Z}^d} \mathbb{P}_p\big(y \in \mathsf{Piv}(\mathbf{0} \longleftrightarrow x)\big) \le \sum_{x\in\mathbb{Z}^d}\sum_{y\in\mathbb{Z}^d} \mathbb{P}_p\big(\{\mathbf{0} \longleftrightarrow y\} \circ \{y \longleftrightarrow x\}\big) \\ &\le \sum_{x\in\mathbb{Z}^d}\sum_{y\in\mathbb{Z}^d} \tau_p(y)\tau_p(x-y) = \widehat{\tau}_p(0)^2.\end{aligned}$$

The inequality for $\chi(p)$ follows from the identity $\chi(p) = 1 + p\widehat{\tau}_p(0)$.

2.2 Definitions and preparatory statements

We need the following definitions:

Definition 2.4 (Elementary definitions)**.** Let $x, u \in \mathbb{Z}^d$ and $A \subseteq \mathbb{Z}^d$.

1. We set $\omega^x := \omega \cup \{x\}$ and $\omega^{u,x} := \omega \cup \{u, x\}$.
2. We define $J(x) := \mathbb{1}_{\{|x|=1\}} = 2dD(x)$.
3. Let $\{u \longleftrightarrow x \text{ in } A\}$ be the event that $\{u \longleftrightarrow x\}$, and there is a path from u to x, all of whose inner vertices are elements of $\omega \cap A$. Moreover, write $\{u \longleftrightarrow x \text{ off } A\} := \{u \longleftrightarrow x \text{ in } \mathbb{Z}^d \setminus A\}$.
4. We define $\{u \Longleftrightarrow x\} := \{u \longleftrightarrow x\} \circ \{u \longleftrightarrow x\}$ and say that u and x are *doubly connected*.
5. We define the modified cluster of x with a designated vertex u as
$$\widetilde{\mathscr{C}}^u(x) := \{x\} \cup \{y \in \omega \setminus \{u\} : x \longleftrightarrow y \text{ in } \mathbb{Z}^d \setminus \{u\}\}.$$
6. For a set $A \subset \mathbb{Z}^d$, define $\langle A\rangle := A \cup \{y \in \mathbb{Z}^d : \exists x \in A \text{ s.t. } |x-y| = 1\}$ as the set A itself plus its external boundary.

Definition 2.4.1 allows us to speak of events like $\{a \longleftrightarrow b \text{ in } \omega^x\}$ for $a, b \in \mathbb{Z}^d$, which is the event that a is connected to b in the configuration where x is fixed to be occupied. We remark that $\{x \longleftrightarrow y \text{ in } \mathbb{Z}^d\} = \{x \longleftrightarrow y\} = \{x \longleftrightarrow y \text{ in } \omega\}$ and that $\{u \Longleftrightarrow x\} = \Omega$ for $|u - x| = 1$. Similarly, $\{u \Longleftrightarrow x\} = \varnothing$ for $u = x$. The following, more specific definitions are important for the expansion:

Definition 2.5 (Extended connection probabilities and events)**.** Let $v, u, x \in \mathbb{Z}^d$ and $A \subseteq \mathbb{Z}^d$.

1. Define
$$\{u \xleftrightarrow{\;A\;} x\} := \{u \longleftrightarrow x\} \cap \Big(\{u \not\longleftrightarrow x \text{ off } \langle A\rangle\} \cup \{x \in \langle A\rangle\}\Big).$$
In words, this is the event that u is connected to x, but either any path from u to x has an interior vertex in $\langle A\rangle$, or x itself lies in $\langle A\rangle$.

2. Define
$$\tau_p^A(u,x) := \mathbb{1}_{\{x\notin\langle A\rangle\}}\mathbb{P}_p(u \longleftrightarrow x \text{ off } \langle A\rangle).$$

3. We introduce $\mathsf{Piv}(u,x) := \mathsf{Piv}(u \longleftrightarrow x)$ as the set of pivotal points for $\{u \longleftrightarrow x\}$. That is, $v \in \mathsf{Piv}(u,x)$ if the event $\{u \longleftrightarrow x \text{ in } \omega^v\}$ holds but $\{u \longleftrightarrow x \text{ in } \omega\setminus\{v\}\}$ does not.

4. Define the events
$$E'(v,u;A) := \{v \xleftrightarrow{A} u\} \cap \{\nexists u' \in \mathsf{Piv}(v,u) : v \xleftrightarrow{A} u'\},$$
$$E(v,u,x;A) := E'(v,u;A) \cap \{u \in \omega \cap \mathsf{Piv}(v,x)\}.$$

First, we remark that $\{u \xleftrightarrow{\mathbb{Z}^d} x\} = \{u \longleftrightarrow x\}$. Secondly, note that we have the relation
$$\tau_p(x-u) = \tau_p^A(u,x) + \mathbb{P}_p(u \xleftrightarrow{A} x). \tag{2.5}$$
We next state a partitioning lemma (whose proof is left to the reader; see [19, Lemma 3.5]) relating the events E and E' to the connection event $\{u \xleftrightarrow{A} x\}$:

Lemma 2.6 (Partitioning connection events)**.** *Let $v,x \in \mathbb{Z}^d$ and $A \subseteq \mathbb{Z}^d$. Then*
$$\{v \xleftrightarrow{A} x\} = E'(v,x;A) \cup \bigcup_{u\in\mathbb{Z}^d} E(v,u,x;A),$$
and the appearing unions are disjoint.

The next lemma, titled the Cutting-point lemma, is at the heart of the expansion:

Lemma 2.7 (Cutting-point lemma)**.** *Let $v,u,x \in \mathbb{Z}^d$ and $A \subseteq \mathbb{Z}^d$. Then*
$$\mathbb{P}_p(E(v,u,x;A)) = p\mathbb{E}_p\left[\mathbb{1}_{E'(v,u;A)}\tau_p^{\widetilde{\mathscr{C}}^u(v)}(u,x)\right].$$

Proof. The proof is a special case of the general setting of [19]. Since it is essential, we present it here. We abbreviate $\widetilde{\mathscr{C}} = \widetilde{\mathscr{C}}^u(v)$ and observe that
$$\begin{aligned}\{u \in \mathsf{Piv}(v,x)\} &= \{v \longleftrightarrow u\} \cap \{u \longleftrightarrow x \text{ off } \widetilde{\mathscr{C}}\} \cap \{x \notin \langle\widetilde{\mathscr{C}}\rangle\}\\ &= \{v \longleftrightarrow u\} \cap \{u \longleftrightarrow x \text{ off } \langle\widetilde{\mathscr{C}}\rangle\} \cap \{x \notin \langle\widetilde{\mathscr{C}}\rangle\}.\end{aligned}$$
In the above, we can replace $\widetilde{\mathscr{C}}$ by $\langle\widetilde{\mathscr{C}}\rangle$ in the middle event, as, by definition, we know that, apart from u, any site in $\langle\widetilde{\mathscr{C}}\rangle \setminus \widetilde{\mathscr{C}}$ must be vacant. Now, since $E'(v,u;A) \subseteq \{v \longleftrightarrow u\}$, we get
$$E(v,u,x;A) = E'(v,u;A) \cap \{u \longleftrightarrow x \text{ off } \langle\widetilde{\mathscr{C}}\rangle\} \cap \{x \notin \langle\widetilde{\mathscr{C}}\rangle\} \cap \{u \in \omega\}.$$
Taking probabilities, conditioning on $\widetilde{\mathscr{C}}$, and observing that the status of u is independent of all other events, we see
$$\mathbb{P}_p(E(v,u,x;A)) = p\mathbb{E}_p\left[\mathbb{1}_{E'(v,u;A)}\mathbb{1}_{\{x\notin\langle\widetilde{\mathscr{C}}\rangle\}}\mathbb{E}_p\left[\mathbb{1}_{\{u\longleftrightarrow x \text{ off } \langle\widetilde{\mathscr{C}}\rangle\}}|\widetilde{\mathscr{C}}\right]\right],$$
making use of the fact that the first two events are measurable w.r.t. $\widetilde{\mathscr{C}}$. The proof is complete with the observation that under $\mathbb{E}_p$, almost surely,
$$\mathbb{1}_{\{x\notin\langle\widetilde{\mathscr{C}}\rangle\}}\mathbb{E}_p\left[\mathbb{1}_{\{u\longleftrightarrow x \text{ off } \langle\widetilde{\mathscr{C}}\rangle\}}|\widetilde{\mathscr{C}}\right] = \tau_p^{\widetilde{\mathscr{C}}}(u,x).$$
□

2.3 Derivation of the expansion

We introduce a sequence $(\omega_i)_{i\in\mathbb{N}_0}$ of independent site percolation configurations. For an event E taking place on ω_i, we highlight this by writing E_i. We also stress the dependence of random variables on the particular configuration they depend on. For example, we write $\mathscr{C}(u;\omega_i)$ to denote the cluster of u in configuration i.

Definition 2.8 (Lace-expansion coefficients)**.** Let $m \in \mathbb{N}, n \in \mathbb{N}_0$ and $x \in \mathbb{Z}^d$. We define

$$\Pi_p^{(0)}(x) := \mathbb{P}_p(\mathbf{0} \Longleftrightarrow x) - J(x),$$
$$\Pi_p^{(m)}(x) := p^m \sum_{u_0,\dots,u_{m-1}} \mathbb{P}_p\Big(\{\mathbf{0} \Longleftrightarrow u_0\}_0 \cap \bigcap_{i=1}^{m} E'(u_{i-1},u_i;\mathscr{C}_{i-1})_i\Big),$$

where we recall that $J(x) = \mathbb{1}_{\{|x|=1\}}$ and moreover $u_{-1} = \mathbf{0}, u_m = x$, and $\mathscr{C}_i = \widetilde{\mathscr{C}}^{u_i}(u_{i-1};\omega_i)$. Let

$$R_{p,n}(x) := (-p)^{n+1} \sum_{u_0,\dots,u_n} \mathbb{P}_p\Big(\{\mathbf{0} \Longleftrightarrow u_0\}_0 \cap \bigcap_{i=1}^{n} E'(u_{i-1},u_i;\mathscr{C}_{i-1})_i \cap \{u_n \xleftrightarrow{\ \mathscr{C}_n\ } x\}_{n+1}\Big).$$

Finally, set

$$\Pi_{p,n}(x) = \sum_{m=0}^{n} (-1)^m \Pi_p^{(m)}(x).$$

It should be noted that the events $E'(u_{i-1},u_i;\mathscr{C}_{i-1})_i$ appearing in Definition 2.8 take place on configuration i only if $\mathscr{C}_{i-1}$ is taken to be a fixed set—otherwise, they are events determined by configurations $i-1$ and i.

Proposition 2.9 (The lace expansion)**.** *Let $p < p_c$, $x \in \mathbb{Z}^d$, and $n \in \mathbb{N}_0$. Then*

$$\tau_p(x) = J(x) + \Pi_{p,n}(x) + p\big((J + \Pi_{p,n}) * \tau_p\big)(x) + R_{p,n}(x).$$

Proof. We have

$$\tau_p(x) = J(x) + \Pi_p^{(0)}(x) + \mathbb{P}_p(\mathbf{0} \longleftrightarrow x, \mathbf{0} \not\Longleftrightarrow x).$$

We can partition the last summand via the first pivotal point. Pointing out that $\{\mathbf{0} \Longleftrightarrow u\} = E'(0,u;\mathbb{Z}^d)$, we obtain

$$\mathbb{P}_p(\mathbf{0} \longleftrightarrow x, \mathbf{0} \not\Longleftrightarrow x) = \sum_{u\in\mathbb{Z}^d} \mathbb{P}_p(\mathbf{0} \Longleftrightarrow u, u \in \omega, u \in \mathsf{Piv}(\mathbf{0},x)) = \sum_u \mathbb{P}_p(E(\mathbf{0},u,x;\mathbb{Z}^d))$$
$$= p \sum_u \mathbb{E}_p\left[\mathbb{1}_{\{\mathbf{0}\Longleftrightarrow u\}} \cdot \tau_p^{\mathscr{C}_0}(u,x)\right]$$

via the Cutting-point lemma 2.7. Using (2.5) for $A = \mathscr{C}_0$, we have

$$\tau_p(x) = J(x) + \Pi_p^{(0)}(x) + p\sum_u \Big(J(u) + \Pi_p^{(0)}(u)\Big)\tau_p(x-u) - p\sum_u \mathbb{E}_p\Big[\mathbb{1}_{\{\mathbf{0}\Longleftrightarrow u\}} \cdot \mathbb{P}_p(u \xleftrightarrow{\ \mathscr{C}_0\ } x)\Big]. \quad (2.6)$$

This proves the expansion identity for $n = 0$. Next, Lemma 2.6 and Lemma 2.7 yield

$$\mathbb{P}_p(u \xleftrightarrow{\ A\ } x) = \mathbb{P}_p(E'(u,x;A)) + \sum_{u_1\in\mathbb{Z}^d} \mathbb{P}_p(E(u,u_1,x;A))$$
$$= \mathbb{P}_p(E'(u,x;A)) + p\sum_{u_1\in\mathbb{Z}^d} \mathbb{E}_p\Big[\mathbb{1}_{E'(u,u_1;A)} \cdot \tau_p^{\widetilde{\mathscr{C}}^{u_1}(u)}(u_1,x)\Big].$$

Plugging this into (2.6), we use (2.5) for $A = \widetilde{\mathscr{C}}^{u_1}(u)$ to extract $\Pi_p^{(1)}$ and get

$$\tau_p(x) = J(x) + \Pi_p^{(0)}(x) - \Pi_p^{(1)}(x) + p\big((J + \Pi_p^{(0)}) * \tau_p\big)(x)$$
$$+ p^2 \sum_{u_1} \tau_p(x-u_1) \sum_{u_0} \mathbb{P}_p\Big(\{\mathbf{0} \Longleftrightarrow u_0\}_0 \cap E'(u_0,u_1;\mathscr{C}_0)_1\Big) + R_{p,1}(x)$$
$$= J(x) + \Pi_p^{(0)}(x) - \Pi_p^{(1)}(x) + p\big((J + \Pi_p^{(0)} - \Pi_p^{(1)}) * \tau_p\big)(x) + R_{p,1}(x).$$

Note that all appearing sums are bounded by $\sum_y \tau_p(y)$. This sum is finite for $p < p_c$, justifying the above changes in order of summation. The expansion for general n is an induction on n where the step is analogous to the step $n = 1$ (but heavier on notation). □

3 Diagrammatic bounds

3.1 Setup, bounds for $n = 0$

We use this section to state Lemma 3.1 and state bounds on $\Pi_p^{(0)}$, which are rather simple to prove. The more involved bounds on $\Pi_p^{(n)}$ for $n \geq 1$ are dealt with in Section 3.2. Note that if $f(-x) = f(x)$, then $\widehat{f}(k) = \sum_{x \in \mathbb{Z}^d} \cos(k \cdot x) f(x)$. We furthermore have the following tool at our disposal:

Lemma 3.1 (Split of cosines, [9], Lemma 2.13)**.** *Let $t \in \mathbb{R}$ and $t_i \in \mathbb{R}$ for $i = 1, \dots, n$ such that $t = \sum_{i=1}^n t_i$. Then*

$$1 - \cos(t) \leq n \sum_{i=1}^{n} [1 - \cos(t_i)].$$

We begin by treating the coefficient for $n = 0$, giving a glimpse into the nature of the bounds to follow in Sections 3.2 and 3.3. To this end, we define the two displacement quantities

$$J_k(x) := [1 - \cos(k \cdot x)] J(x) \quad \text{and} \quad \tau_{p,k}(x) = [1 - \cos(k \cdot x)] \tau_p(x).$$

Proposition 3.2 (Bounds for $n = 0$)**.** *For $k \in (-\pi, \pi]^d$,*

$$|\widehat{\Pi}_p^{(0)}(k)| \leq p^2 (J^{*2} * \tau_p^{*2})(\mathbf{0}),$$
$$|\widehat{\Pi}_p^{(0)}(0) - \widehat{\Pi}_p^{(0)}(k)| \leq 2p^2 \Big((J_k * J * \tau_p^{*2})(\mathbf{0}) + (J^{*2} * \tau_{p,k} * \tau_p)(\mathbf{0}) \Big).$$

Proof. Note that $|x| \leq 1$ implies $\Pi_p^{(0)}(x) = 0$ by definition. For $|x| \geq 2$, we have

$$\Pi_p^{(0)}(x) \leq \mathbb{E}\Big[\sum_{y \neq z \in \omega} \mathbb{1}_{\{|y| = |z| = 1\}} \mathbb{1}_{\{y \longleftrightarrow x\} \circ \{z \longleftrightarrow x\}} \Big] \leq p^2 \Big(\sum_{y \in \mathbb{Z}^d} J(y) \tau_p(x - y) \Big)^2 = p^2 (J * \tau_p)(x)^2.$$

Summation over x gives the first bound. The last bound is obtained by applying Lemma 3.1 to the bounds derived for $\Pi_p^{(0)}(x)$:

$$\begin{aligned} |\widehat{\Pi}_p^{(0)}(0) - \widehat{\Pi}_p^{(0)}(k)| &= \sum_x [1 - \cos(k \cdot x)] \Pi_p^{(0)}(x) \\ &\leq 2p^2 \sum_x (J * \tau_p)(x) \sum_y \Big([1 - \cos(k \cdot y)] J(y) \tau_p(x - y) + J(y) [1 - \cos(k \cdot (x - y))] \tau_p(x - y) \Big). \end{aligned}$$

Resolving the sums gives the claimed convolution. □

3.2 Bounds in terms of diagrams

The main result of this section is Proposition 3.5, providing bounds on the lace-expansion coefficients in terms of so-called diagrams, which are sums over products of two-point (and related) functions. To state it, we introduce some functions related to τ_p as well as several "modified triangles" closely related to $\triangle_p$.

Definition 3.3 (Modified two-point functions)**.** Let $x \in \mathbb{Z}^d$ and define

$$\tau_p^\circ(x) := \delta_{\mathbf{0},x} + \tau_p(x), \qquad \tau_p^\bullet(x) = \delta_{\mathbf{0},x} + p\tau_p(x), \qquad \gamma_p(x) = \tau_p(x) - J(x).$$

Definition 3.4 (Modified triangles)**.** We let $\triangle_p^\circ(x) = p^2(\tau_p^\circ * \tau_p * \tau_p)(x)$, $\triangle_p^\bullet(x) = p(\tau_p^\bullet * \tau_p * \tau_p)(x)$, $\triangle_p^{\bullet\circ}(x) = p(\tau_p^\bullet * \tau_p^\circ * \tau_p)(x)$, and $\triangle_p^{\bullet\bullet\circ}(x) = (\tau_p^\bullet * \tau_p^\bullet * \tau_p^\circ)(x)$. We also set

$$\triangle_p^\circ = \sup_{x \in \mathbb{Z}^d} \triangle_p^\circ(x), \quad \triangle_p^\bullet = \sup_{\mathbf{0} \neq x \in \mathbb{Z}^d} \triangle_p^\bullet(x), \quad \triangle_p^{\bullet\circ} = \sup_{\mathbf{0} \neq x \in \mathbb{Z}^d} \triangle_p^{\bullet\circ}(x), \quad \triangle_p^{\bullet\bullet\circ} = \sup_{x \in \mathbb{Z}^d} \triangle_p^{\bullet\bullet\circ}(x),$$

and $T_p := (1 + \triangle_p) \triangle_p^{\bullet\circ} + \triangle_p \triangle_p^{\bullet\bullet\circ}$.

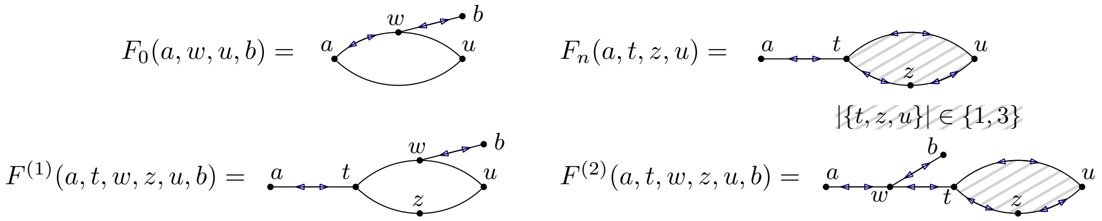


Figure 1: The F events represented graphically. For lines with double arrows, we may have coincidence of the endpoints, for lines without double arrows, we do not. The area with grey tiles indicates that its three boundary points are either all distinct or collapsed into a single point. These diagrams also serve as a pictorial representation of the function ϕ_0, ϕ, ϕ_n. There, lines with double arrows represent factors of τ_p° and lines without double arrows represent factors of τ_p.

Definitions 3.3 and 3.4 allow us to properly keep track of factors of p, which turns out to be important throughout Section 3.

Proposition 3.5 (Triangle bounds on the lace-expansion coefficients)**.** *For $n \geq 0$,*

$$p \sum_{x \in \mathbb{Z}^d} \Pi_p^{(n)}(x) \leq \triangle_p^\bullet(\mathbf{0}) \big(T_p\big)^n.$$

The proof of Proposition 3.5 relies on two intermediate steps, successively giving bounds on $\sum \Pi_p^{(n)}$. These two steps are captured in Lemmas 3.7 and 3.10, respectively. We first state the former lemma.

Recall that $\Pi_p^{(n)}$ is defined on independent percolation configurations $\omega_0, \ldots, \omega_n$. A crucial step in proving Proposition 3.5 is to group events taking place on the percolation configuration i, and then to use the independence of the different configurations. To this end, note that event $E'(u_{i-1}, u_i; \mathscr{C}_{i-1})_i$ takes place on configuration i only if $\mathscr{C}_{i-1}$ is considered to be a fixed set. Otherwise, it is a product event made up of the connection events of configuration i as well as a connection event in configuration $i-1$, preventing a direct use of the independence of the ω_i. Resolving this issue is one of the goals of Lemma 3.7; another is to give bounds in terms of the simpler events (amenable to application of the BK inequality) introduced below in Definition 3.6:

Definition 3.6 (Bounding events)**.** Let $x, y \in \mathbb{Z}^d$. We define

$$\{x \longleftrightarrow\!\!\!\!\rightsquigarrow y\} := \{x \longleftrightarrow y\} \cup \{x = y\}.$$

Let now $i \in \{1, \ldots, n\}$ and set $\vec{v}_i = (u_{i-1}, t_i, w_i, z_i, u_i, z_{i+1})$. We define

$$\begin{aligned} F_0(a, w_0, u_0, z_1) &= \Big(\{w_0 = a, |u_0 - a| = 1\} \cap \{a \leftrightsquigarrow z_1\}\Big) \\ &\quad \cup \Big(\{|u_0 - a| > 1\} \cap \big(\{a \leftrightsquigarrow w_0\} \circ \{a \longleftrightarrow u_0\} \circ \{w_0 \longleftrightarrow u_0\} \circ \{w_0 \leftrightsquigarrow z_1\}\big)\Big), \\ F_n(u_{n-1}, t_n, z_n, x) &= \{|\{t_n, z_n, x\}| \neq 2\} \cap \{u_{n-1} \leftrightsquigarrow t_n\} \\ &\quad \circ \{t_n \leftrightsquigarrow x\} \circ \{t_n \leftrightsquigarrow z_n\} \circ \{z_n \leftrightsquigarrow x\}, \\ F^{(1)}(\vec{v}_i) &= \{|\{w_i, t_i, z_i, u_i\}| = 4\} \cap \Big(\{u_{i-1} \leftrightsquigarrow t_i\} \circ \ \{t_i \longleftrightarrow w_i\} \circ \ \{t_i \longleftrightarrow z_i\} \\ &\quad \circ \ \{w_i \longleftrightarrow u_i\} \circ \ \{z_i \longleftrightarrow u_i\} \circ \ \{w_i \leftrightsquigarrow z_{i+1}\}\Big), \\ F^{(2)}(\vec{v}_i) &= \big\{w_i \notin \{z_i, u_i\}, |\{t_i, z_i, u_i\}| \neq 2\big\} \cap \Big(\{u_{i-1} \leftrightsquigarrow w_i\} \circ \ \{w_i \leftrightsquigarrow z_{i+1}\} \\ &\quad \circ \{w_i \leftrightsquigarrow t_i\} \circ \{t_i \leftrightsquigarrow u_i\} \circ \{t_i \leftrightsquigarrow z_i\} \circ \{z_i \leftrightsquigarrow u_i\}\Big). \end{aligned}$$

The coincidence requirements in $F^{(2)}$ mean that among the points t_i, w_i, z_i, u_i, the point w_i may coincide *only* with t_i; and additionally, the triple $\{t_i, z_i, u_i\}$ are either all distinct, or collapsed into a single point. The above events are depicted in Figure 1.

For intervals $[a, b]$, we use the notation $\vec{x}_{[a,b]} = (x_a, x_{a+1}, \ldots, x_b)$. This is not to be confused with the notation $\vec{v}_i$ from Definition 3.6. We use the notation $(\mathbb{Z}^d)^{(m,1)}$ to denote the set of vectors $\{\vec{y}_{[1,m]} \in (\mathbb{Z}^d)^m, y_i \neq y_{i+1} \ \forall 1 \leq i < m\}$.

Lemma 3.7 (Coefficient bounds in terms of F events)**.** *For $n \geq 1$ and $(u_0, \dots, u_{n-1}, x) \in (\mathbb{Z}^d)^{(n+1,1)}$,*

$$\{\mathbf{0} \Longleftrightarrow u_0\}_0 \cap \bigcap_{i=1}^{n} E'(u_{i-1}, u_i; \mathscr{C}_{i-1})_i$$

$$\subseteq \bigcup_{\substack{\vec{z}_{[1,n]}: z_i \in \omega_i^{u_i}, \\ w_0 \in \omega_0^{\mathbf{0}}, t_n \in \omega_n^{u_{n-1}, x}}} F_0(\mathbf{0}, w_0, u_0, z_1)_0 \cap \Big(\bigcap_{i=1}^{n-1} \Big(\bigcup_{\substack{t_i \in \omega_i^{u_{i-1}}, \\ w_i \in \omega_i}} F^{(1)}(\vec{v}_i)_i \cup \bigcup_{\substack{w_i \in \omega_i^{u_{i-1}}, \\ t_i \in \omega_i^{u_i, w_i}}} F^{(2)}(\vec{v}_i)_i \Big) \Big)$$
$$\cap F_n(u_{n-1}, t_n, z_n, x)_n.$$

The proof is analogous to the one in [19, Lemma 4.12] and we do not perform it here. The second important lemma is Lemma 3.10, and its bounds are phrased in terms of the following functions:

Definition 3.8 (The ψ and ϕ functions)**.** Let $n \geq 1$ and $a_1, a_2, b, w, t, u, z \in \mathbb{Z}^d$. We define

$$\begin{aligned}
\psi_0(b, w, u) &:= \delta_{b,w} pJ(u-b) + p\tau_p^{\bullet}(w-b)\gamma_p(u-b)\tau_p(w-u), \\
\tilde{\psi}_0(b, w, u) &:= p\tau_p^{\bullet}(w-b)\tau_p(u-b)\tau_p(w-u), \\
\psi_n(a_1, a_2, t, z, x) &:= \mathbb{1}_{\{|\{t,z,x\}| \neq 2\}} \tau_p^{\circ}(z-a_1)\tau_p^{\bullet}(t-a_2)\tau_p^{\bullet}(z-t)\tau_p^{\bullet}(x-t)\tau_p^{\circ}(x-z).
\end{aligned}$$

Moreover, we define

$$\begin{aligned}
\psi^{(1)}(a_1, a_2, t, w, z, u) &:= p^3 \mathbb{1}_{\{|\{t,w,z,u\}|=4\}} \tau_p^{\circ}(z-a_1)\tau_p^{\bullet}(t-a_2)\tau_p(w-t)\tau_p(z-t)\tau_p(u-w)\tau_p(u-z), \\
\psi^{(2)}(a_1, a_2, t, w, z, u) &:= \mathbb{1}_{\{w \notin \{z,u\}, |\{t,z,u\}| \neq 2\}} \tau_p^{\circ}(z-a_1)\tau_p^{\bullet}(w-a_2)\tau_p^{\bullet}(t-w)\tau_p^{\bullet}(z-t)\tau_p^{\bullet}(u-t)\tau_p^{\circ}(u-z),
\end{aligned}$$

and $\psi := \psi^{(1)} + \psi^{(2)}$. Furthermore, for $j \in \{1, 2\}$, let

$$\begin{aligned}
\phi_0(b, w, u, z) &:= \delta_{b,w} pJ(u-b)\tau_p^{\circ}(z-w) + p\tau_p^{\bullet}(w-b)\gamma_p(u-b)\tau_p(w-u)\tau_p^{\circ}(z-w), \\
\phi_n(a_2, t, z, x) &:= \mathbb{1}_{\{|\{t,z,x\}| \neq 2\}} \tau_p^{\bullet}(t-a_2)\tau_p^{\bullet}(z-t)\tau_p^{\bullet}(x-t)\tau_p^{\circ}(x-z), \\
\phi^{(j)}(a_2, t, w, z, u, b) &:= \frac{\tau_p^{\circ}(b-w)}{\tau_p^{\circ}(z)} \psi^{(j)}(\mathbf{0}, a_2, t, w, z, u),
\end{aligned}$$

and $\tilde{\phi}_0(b, w, u, z) := \tilde{\psi}_0(b, w, u)\tau_p^{\circ}(z-w)$ as well as $\phi := \phi^{(1)} + \phi^{(2)}$.

We remark that $\psi_0 \leq \tilde{\psi}_0$ as well as $\phi_0 \leq \tilde{\phi}_0$, and we are going to use this fact later on. In the definition of $\phi^{(j)}$, the factor $\tau_p^{\circ}(z)$ cancels out. In that sense, $\phi^{(j)}$ is obtained from $\psi^{(j)}$ by "replacing" the factor $\tau_p^{\circ}(z-a_1)$ by the factor $\tau_p^{\circ}(b-w)$, and the two functions are closely related.

We first obtain a bound on $\Pi_p^{(n)}$ in terms of the F events (this is Lemma 3.7). Bounding those with the BK inequality, we will naturally observe the ϕ functions (Lemma 3.10). To decompose them further, we would like to apply induction; for this purpose, the ψ functions are much better-suited. By introducing both the ϕ and ψ functions, we increase the readability throughout this section (and later ones).

Definition 3.9 (The Ψ function)**.** Let $w_n, u_n \in \mathbb{Z}^d$ and define

$$\Psi^{(n)}(w_n, u_n) := \sum_{\substack{\vec{t}_{[1,n]}, \vec{w}_{[0,n-1]}, \vec{z}_{[1,n]}, \vec{u}_{[0,n-1]}: \\ u_{n-1} \neq u_n}} \psi_0(\mathbf{0}, w_0, u_0) \prod_{i=1}^{n} \psi(w_{i-1}, u_{i-1}, t_i, w_i, z_i, u_i),$$

where $\vec{t}_{[1,n]}, \vec{z}_{[1,n]}, \vec{w}_{[0,n-1]} \in (\mathbb{Z}^d)^n$ and $\vec{u}_{[0,n-1]} \in (\mathbb{Z}^d)^{(n,1)}$.

We remark that $\Psi^{(n)}(x, x) = 0$, resulting from the fact that the ϕ functions in Definition 3.8 output 0 for $w = u$. We are going to make use of this later.

Lemma 3.10 (Bound in terms of ψ functions)**.** *For $n \geq 0$,*

$$p \sum_{x \in \mathbb{Z}^d} \Pi_p^{(n)}(x) \leq \sum_{w,u,t,z,x} \Psi^{(n-1)}(w, u)\psi_n(w, u, t, z, x) \leq \sum_{w,u \in \mathbb{Z}^d} \Psi^{(n)}(w, u). \tag{3.1}$$

Proof. Note first that according to Definition 2.8, the function $\Pi_p^{(n)}$ contains the event $E'(u_{i-1}, u_i; \mathscr{C}_{i-1})$, which is itself contained in $\{u_{i-1} \longleftrightarrow u_i\}$. This is why we may assume that the points $u_0, \dots, u_n$ in the definition of $\Pi_p(n)$ satisfy $u_{i-1} \neq u_i$. Together with Lemma 3.7, this yields a bound on $\Pi_p^{(n)}$ of the form

$$\begin{aligned} p\Pi_p^{(n)}(u_n) &\leq p^{n+1} \sum_{\vec{u}} \mathbb{P}_p\Big(\bigcup_{w_0, t_n, \vec{z}} F_0(\mathbf{0}, w_0, u_0, z_1)_0 \cap \bigcap_{i=1}^{n-1} \Big(\bigcup_{t_i, w_i} F^{(1)}(\vec{v}_i)_i \cup \bigcup_{t_i, w_i} F^{(2)}(\vec{v}_i)_i \Big) \\ &\qquad\qquad \cap F_n(u_{n-1}, t_n, z_n, x)_n \Big) \\ &= \sum_{\vec{t}, \vec{w}, \vec{z}, \vec{u}} p^{|\{\mathbf{0}, w_0\}|} \mathbb{P}_p\big(F_0(\mathbf{0}, w_0, u_0, z_1)\big) \\ &\qquad \times \prod_{i=1}^{n-1} \Big(p^{|\{u_{i-1}, t_i\}|+2} \mathbb{P}_p\big(F^{(1)}(\vec{v}_i)\big) + p^{|\{u_{i-1}, w_i\}|+|\{w_i, t_i\}|+|\{t_i, z_i, u_i\}|-3} \mathbb{P}_p\big(F^{(2)}(\vec{v}_i)\big) \Big) \\ &\qquad \times p^{|\{u_{n-1}, t_n\}|+|\{t_n, z_n, u_n\}|-2} \mathbb{P}_p\big(F_n(u_{n-1}, t_n, z_n, u_n)\big). \end{aligned} \tag{3.2}$$

In the first line, $\vec{t}, \vec{w}, \vec{z}$ are occupied points as in Lemma 3.7. In both lines, $\vec{u}_{[0,n]} \in (\mathbb{Z}^d)^{(n+1,1)}$, and in the second line, $\vec{t}, \vec{w}, \vec{z} \in (\mathbb{Z}^d)^n$. Crucially, the identity in (3.2) holds due to the independence of the different percolation configurations. Moreover, it is crucial here that the number of factors of p (appearing when we switch from a sum over points in ω to a sum over points in $\mathbb{Z}^d$) depends on the number of coinciding points.

We can now decompose the F events by heavy use of the BK inequality, producing bounds in terms of the ϕ functions introduced in Definition 3.8. We start by bounding

$$p^{|\{a,w\}|} \mathbb{P}_p\big(F_0(a, w, u, z)\big) \leq \phi_0(a, w, u, z),$$
$$p^{|\{a,t\}|+|\{t,z,u\}|-2} \mathbb{P}_p\big(F_n(a, t, z, x)\big) \leq \phi_n(a, t, z, x).$$

We continue to bound

$$p^{|\{a,t\}|+2} \mathbb{P}_p\big(F^{(1)}(a, t, w, z, u, b)\big) + p^{|\{a,w\}|+|\{w,t\}|+|\{t,z,u\}|-3} \mathbb{P}_p\Big(F^{(2)}(a, t, w, z, u, b)\big)\Big) \leq \phi(a, t, w, z, u, b).$$

Plugging these bounds into (3.2), we obtain the new bound

$$p\Pi_p^{(n)}(u_n) \leq \sum_{(\vec{t}, \vec{z})_{[1,n]}, (\vec{w}, \vec{u})_{[0,n-1]}} \phi_0(\mathbf{0}, w_0, u_0, z_1) \phi_n(u_{n-1}, t_n, z_n, x) \prod_{i=1}^{n-1} \phi(u_{i-1}, t_i, w_i, z_i, u_i, z_{i+1}), \tag{3.3}$$

where $\vec{t}_{[1,n]}, \vec{w}_{[0,n-1]}, \vec{z}_{[1,n]} \in (\mathbb{Z}^d)^n$, and $\vec{u}_{[0,n]} \in (\mathbb{Z}^d)^{(n+1,1)}$. We rewrite the right-hand side of (3.3) by replacing the ϕ_0, ϕ_n and ϕ functions by ψ_0, ψ_n and ψ functions. As the additional factors arising from this replacement exactly cancel out, this gives the first bound in Lemma 3.10. The observation

$$\psi_n(a_1, a_2, t, z, u) \leq \psi(a_1, a_2, t, a_2, z, u) \tag{3.4}$$

gives the second bound and finishes the proof. □

We can now prove Proposition 3.5:

Proof of Proposition 3.5. We show that

$$\sum_{w,u \in \mathbb{Z}^d} \Psi^{(n)}(w, u) \leq \triangle_p^{\bullet}(\mathbf{0}) \big(T_p\big)^n, \tag{3.5}$$

which is sufficient due to (3.1). The proof of (3.5) is by induction on n. For the base case, we bound

$$\sum_{w,u \in \mathbb{Z}^d} \Psi^{(0)}(w, u) = \sum_{w,u} \psi_0(\mathbf{0}, w, u) \leq \sum_{w,u} \tilde{\psi}_0(\mathbf{0}, w, u) = p \sum_{w,u} \tau_p^{\bullet}(w) \tau_p(u) \tau_p(u - w) = \triangle_p^{\bullet}(\mathbf{0}).$$

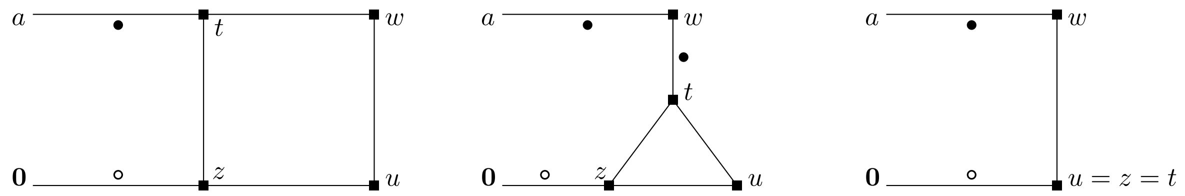


Figure 2: The pictorial representation of $\sup_{\mathbf{0}\neq a}\sum_{t,w,z,u\neq a}\psi(\mathbf{0},a,t,w,z,u)$. The last two pictures represent the case distinction captured in the indicator. Points that are summed over are marked with a square.

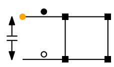
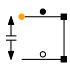
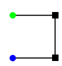
Let now $n\geq 1$. Then

$$\begin{aligned}\sum_{w,u}\Psi^{(n)}(w,u) &= \sum_{w',u'}\Psi^{(n-1)}(w',u')\sum_{z,t,w,u\neq u'}\psi(w',u',t,w,z,u)\\ &\leq \Big(\sum_{w',u'}\Psi^{(n-1)}(w',u')\Big)\Big(\sup_{w'\neq u'}\sum_{z,t,w,u\neq u'}\psi(w',u',t,w,z,u)\Big).\end{aligned}\tag{3.6}$$

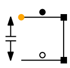
The fact that $\Psi^{(n)}(x,x)=0$ for any n and x allows us to assume $w'\neq u'$ in the supremum in the second line of (3.6). After applying the induction hypothesis, it remains to bound the second factor for $w'\neq u'$, which we rewrite as $\sup_{a\neq\mathbf{0}}\sum_{t,w,z,u\neq a}\psi(\mathbf{0},a,t,w,z,u)$ by translation invariance. As it is a sum of two terms (originating from $\psi^{(1)}$ and $\psi^{(2)}$), we start with the first one and obtain

$$\begin{aligned}&p^3\sum_{t,z}\Big(\tau_p^{\bullet}(t-a)\tau_p(z-t)\tau_p^{\circ}(z)\Big(\sum_{u,w}\tau_p(w-t)\tau_p(u-w)\tau_p(z-u)\Big)\Big)\\ &\quad\leq p\sum_{t,z}\Big(\tau_p^{\bullet}(t-a)\tau_p(z-t)\tau_p^{\circ}(z)\Big(\sup_{t,z}p^2\sum_{u,w}\tau_p(w-t)\tau_p(u-w)\tau_p(z-u)\Big)\Big)\leq\triangle_p\triangle_p^{\bullet\circ}(a).\end{aligned}\tag{3.7}$$

Before treating the second term, we show how to obtain the bound from (3.7) pictorially, using diagrams very similar to the ones introduced in Figure 2. In particular, factors of τ_p are represented by lines, factors of $\tau_p^{\bullet}$ and τ_p° by lines with an added '•' or '∘', respectively. Points summed over are represented by squares, other points (which we mostly take the supremum over, for example point a) are represented by colored disks. Hence, we interpret the factor $\tau_p(z-t)$ as a line between t and z. Since both endpoints are summed over, we display them as squares (and without labels t or z). We interpret the factor $\tau_p^{\circ}(z)$ as a (∘-decorated) line between $\mathbf{0}$ and z; the origin is represented by lack of decorating the incident line. Finally, we indicate the distinctness of a pair of points (in our case $\mathbf{0}\neq a$) by a disrupted two-headed arrow ⇕. With this notation, (3.7) becomes

$p^3\sum$  $\leq p\sum\Big($  $\Big(\sup_{\bullet,\bullet}p^2\sum$  $\Big)\Big)\leq\triangle_p^{\bullet\circ}\triangle_p.$

The second term in ψ, originating from $\psi^{(2)}$, contains an indicator. Resolving it splits this term into two further terms. We first consider the term arising from $|\{t,z,u\}|=1$, which forces $w\neq t=u=z$, and the term is of the form

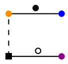
$p\sum_{u,w}\tau_p^{\circ}(u)\tau_p(w-u)\tau_p^{\bullet}(a-w)=p\sum$  $=\triangle_p^{\bullet\circ}(a).$

Turning to the term due to $|\{t,z,u\}|=3$, with a substitution of the form $y'=y-u$ for $y\in\{t,w,z\}$ in the second line, we see that

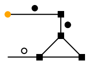
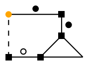
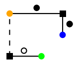
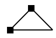
$$\begin{aligned}&p^2\sum_{t,w,z,u}\tau_p^{\bullet}(w-a)\tau_p^{\circ}(z)\tau_p^{\bullet}(t-w)\tau_p(z-t)\tau_p(u-t)\tau_p(z-u)\\ =&p^2\sum_{t',z'}\Big(\tau_p(z')\tau_p(t'-z')\tau_p(t')\Big(\sum_{u,w'}\tau_p^{\circ}(z'+u)\tau_p^{\bullet}(a-w'-u)\tau_p^{\bullet}(w-t)\Big)\Big)\end{aligned}$$

$$\leq \triangle_p \triangle_p^{\bullet\bullet\circ}. \tag{3.8}$$

This concludes the proof. However, we also want to show how to execute the bound in (3.8) using diagrams. To do so, we need to represent a substitution in pictorial form. Note that after the substitution, the sum over point u is w.r.t. two factors, namely $\tau_p^\circ(z'+u)\tau_p^\bullet(a-w'-u)$. We interpret these two factors as a line between $-u$ and z' and a line between $-(u-a)$ and w'. In this sense, the two lines do not meet in u, but they have endpoints that are a constant vector a apart. We represent this as

$$\sum_u \tau_p^\circ(z'+u)\tau_p^\bullet(a-w'-u) = \sum$$  .

The bound in (3.8) thus becomes

$p^2 \sum$  $= p^2 \sum$  $\leq p^2 \sum \Big(\Big(\sup_{\bullet,\bullet} \sum$  $\Big)$  $\Big) \leq \triangle_p^{\bullet\bullet\circ} \triangle_p,$

where we point out that we did not use $a \neq \mathbf{0}$ for the bound $\triangle_p^{\bullet\bullet\circ}$, and so it was not indicated in the diagram. □

The following corollary will be needed later to show that the limit $\Pi_{p,n}$ for $n \to \infty$ exists:

Corollary 3.11. *For $n \geq 1$,*

$$\sup_{x\in\mathbb{Z}^d} \Pi_p^{(n)}(x) \leq \triangle_p^\bullet(\mathbf{0})(1+\triangle_p^{\bullet\circ})\big(T_p\big)^{n-1}.$$

Proof. Note that

$$\Pi_p^{(n)}(x) \leq \Big(\sup_{w\neq u} \sum_{t,z} \psi_n(w,u,t,z,x)\Big) \sum_{w,u} \Psi^{(n-1)}(w,u).$$

Since we do sum over x, we bound the factors depending on x by 1, and so

$$\sum_{t,z} \psi_n(w,u,t,z,x) \leq \tau_p^\bullet(x-u)\tau_p^\circ(x-w) + \triangle_p^{\bullet\circ}(u-w) \leq 1 + \triangle_p^{\bullet\circ}(u-w)$$

implies the claim together with Proposition 3.5. □

3.3 Displacement bounds

The aim of this section is to give bounds on $p\sum_x[1-\cos(k\cdot x)]\Pi_p^{(n)}(x)$. Such bounds are important in the analysis in Section 4. We regard $[1-\cos(k\cdot x)]$ as a "displacement factor". To state the main results, Propositions 3.13 and 3.14, we introduce some displacement quantities:

Definition 3.12 (Diagrammatic displacement quantities)**.** Let $x \in \mathbb{Z}^d$ and $k \in (-\pi,\pi]^d$. Define

$$\begin{aligned}
W_p(x;k) &:= p(\tau_{p,k} * \tau_p^\circ)(x), \qquad W_p(k) = \max_{x\in\mathbb{Z}^d} W_p(x;k),\\
H_p(b_1,b_2;k) &:= p^5 \sum_{t,w,z,u,v} \tau_p(z)\tau_p(t-u)\tau_p(t-z)\tau_{p,k}(u-z)\tau_p(t-w)\tau_p(w-b_1)\tau_p(v-w)\tau_p(v+b_2-u),\\
H_p(k) &:= \max_{b_1\neq\mathbf{0}\neq b_2\in\mathbb{Z}^d} H_p(b_1,b_2;k).
\end{aligned}$$

Note that Proposition 3.2 already provides displacement bounds for $n=0$. The following two results give bounds for $n \geq 1$:

Proposition 3.13 (Displacement bounds for $n \geq 2$)**.** *For $n \geq 2$ and $x \in \mathbb{Z}^d$,*

$$p\sum_x [1-\cos(k\cdot x)]\Pi_p^{(n)}(x) \leq 11(n+1)\big(T_p\big)^{1\vee(n-2)}\big(\triangle_p^{\bullet\bullet\circ}\big)^3 W_p(k)\Big[1+\triangle_p^\circ + T_p + \frac{H_p(k)}{W_p(k)}\Big].$$

Proposition 3.14 (Displacement bounds for $n=1$)**.** *For $x \in \mathbb{Z}^d$,*

$$p\sum_x [1-\cos(k\cdot x)]\Pi_p^{(1)}(x) \le 9W_p(k)\Big[\triangle_p^\bullet(\mathbf{0})\big(\triangle_p^{\bullet\circ}+\triangle_p\big)+\triangle_p^\circ+\triangle_p\Big]+p^2(J*\tau_{p,k}*\tau_p)(\mathbf{0}).$$

In preparation for the proofs, we define a function $\bar\Psi^{(n)}$, similar to $\Psi^{(n)}$, and prove an almost identical bound to the one in Proposition 3.5. Let $\bar\Psi^{(0)}(t,z)=\phi_n(\mathbf{0},t,z,\mathbf{0})/\tau_p^\bullet(t)$. For $i\ge 1$, define

$$\bar\Psi^{(i)}(t,z)=\sum_{w,u,t',z'}\bar\Psi^{(i-1)}(t',z')\Big[\phi^{(1)}(\mathbf{0},t,w,z,u,z')+\phi^{(2)}(\mathbf{0},w,t,z,u,z')\Big]\frac{\tau_p^\bullet(t'-u)}{\tau_p^\bullet(t)}.$$

Note that in $\phi^{(2)}$, the points t and w swap roles, so that in both $\phi^{(1)}$ and $\phi^{(2)}$, u is adjacent to t' and t is the point adjacent to $\mathbf{0}$; and in particular, the factor $\tau_p^\bullet(t)$ cancels out. The following lemma, in combination with Lemma 3.10, is analogous to the bound (3.5), and so is its proof, which is omitted.

Lemma 3.15. *For $n\ge 0$,*

$$\sum_{t,z\in\mathbb{Z}^d}\bar\Psi^{(n)}(t,z)\le \triangle_p^{\bullet\bullet\circ}\big(T_p\big)^n.$$

Proof of Proposition 3.13. Setting $\vec v_i=(w_{i-1},u_{i-1},t_i,w_i,z_i,u_i)$, we use the bound

$$p\sum_x[1-\cos(k\cdot x)]\Pi_p^{(n)}(x)\le\sum_x\sum_{\vec t,\vec w,\vec z,\vec u}[1-\cos(k\cdot x)]\psi_0(\mathbf{0},w_0,u_0)\psi_n(w_{n-1},u_{n-1},t_n,z_n,x)\prod_{i=1}^{n-1}\psi(\vec v_i), \quad (3.9)$$

which is, in essence, the first bound of Lemma 3.10. The next step is to distribute the displacement factor $1-\cos(k\cdot x)$ over the $n+1$ segments. To this end, we write $x=\sum_{i=0}^n d_i$, where $d_i=w_i-u_{i-1}$ for even i and $d_i=u_i-w_{i-1}$ for odd i (with the convention $u_{-1}=\mathbf{0}$ and $w_n=u_n=x$). Over the course of this proof, we are going to drop the subscript i and are then confronted with a displacement $d=d_i$ (which is not to be confused with the dimension).

Using the Cosine-split lemma 3.1, we obtain

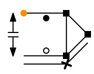
$$\begin{aligned}p\sum_x[1-\cos(k\cdot x)]\Pi_p^{(n)}(x)\le (n+1)\sum_{i=0}^n\sum_x\sum_{\vec t,\vec w,\vec z,\vec u}&[1-\cos(k\cdot d_i)]\psi_0(\mathbf{0},w_0,u_0)\\ &\times\psi_n(w_{n-1},u_{n-1},t_n,z_n,x)\prod_{j=1}^{n-1}\psi(\vec v_j),\end{aligned} \quad (3.10)$$

with d_i as introduced above. We now handle these terms for different i.

<u>Case (a): $i\in\{0,n\}$.</u> Let us start with $i=n$, so that $d_n\in\{x-u_{n-1},x-w_{n-1}\}$. The summand for $i=n$ in (3.10) is equal to

$$\begin{aligned}&\sum_{w\ne u}\Psi^{(n-1)}(w,u)\sum_{t,z,x\ne u}[1-\cos(k\cdot d')]\psi_n(w,u,t,z,x)\\ \le&\triangle_p^\bullet(\mathbf{0})\big(T_p\big)^{n-1}\max_{0\ne u}\sum_{t,z,x\ne u}[1-\cos(k\cdot d)]\psi_n(\mathbf{0},u,t,z,x),\end{aligned} \quad (3.11)$$

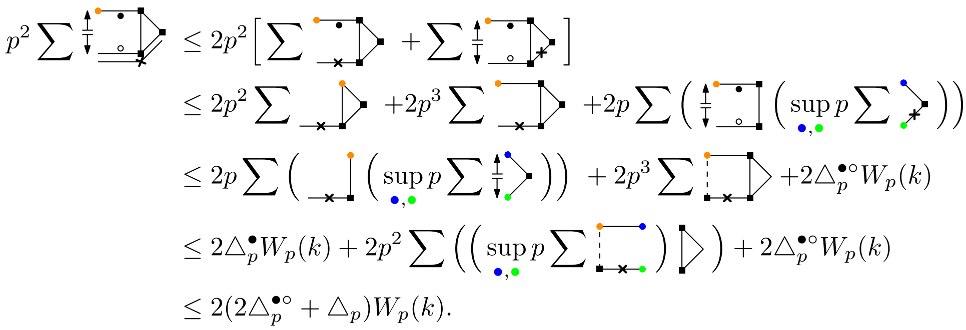
where $d'\in\{x-w,x-u\}$ and $d\in\{x,x-u\}$. We expand the indicator in ψ_n into two cases. If $t=z=x$, then we can bound the maximum in (3.11) by $p\sum_x[1-\cos(k\cdot d)]\tau_p^\circ(x)\tau_p(x-u)$, which is bounded by $W_p(k)$ for both values of d. If t,z,x are distinct points, then for $d=x$, the maximum in (3.11) becomes

$$p^2\max_{u\ne 0}\sum_{t,z,x}[1-\cos(k\cdot d)]\tau_p^\circ(z)\tau_p^\bullet(t-u)\tau_p(t-z)\tau_p(x-z)\tau_p(x-t)=p^2\sum$$  .

Note that in the pictorial representation, we represent the factor $[1-\cos(k\cdot(x-\mathbf{0}))]$ by a line from $\mathbf{0}$ to x carrying a '×' symbol. We use the Cosine-split lemma 3.1 again to bound

$$[1-\cos(k\cdot x)]\le 2\big([1-\cos(k\cdot z)]+[1-\cos(k\cdot(x-z))]\big),$$

which results in



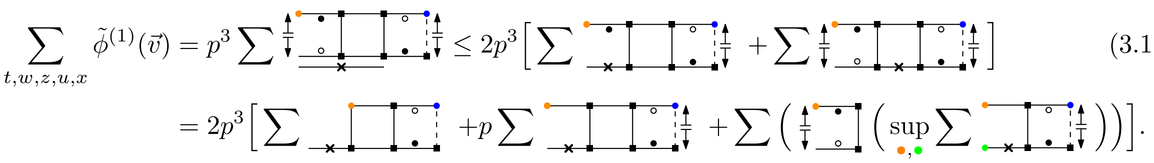
It is not hard to see that a displacement $d = x - u$ yields the same bound. Similar computations show that the case $i = 0$ yields a contribution of at most

$$\triangle_p^{\bullet\bullet\circ} (T_p)^{n-1} \triangle_p^{\bullet\circ} W_p(k).$$

Case (b): $1 \le i < n$. We want to apply both the bound (3.5) and Lemma 3.15. To this end, we rewrite the i-th summand in (3.10) as

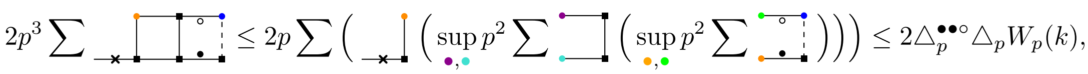
$$\begin{aligned}
&\sum_x \sum_{\vec t, \vec w, \vec z, \vec u} [1-\cos(k \cdot d_i)] \psi_0(\mathbf{0}, w_0, u_0) \psi_n(w_{n-1}, u_{n-1}, t_n, z_n, x) \prod_{j=1}^{n-1} \psi(\vec v_j) \\
&\quad = \sum_x \sum_{a_1,a_2,b_1,b_2} \Big(\Psi^{(i-1)}(a_1,a_2) \bar\Psi^{(n-i-1)}(b_1 - x, b_2 - x) \\
&\qquad\qquad \times \sum_{t,w,z,u} \underbrace{\phi(a_2,t,w,z,u,b_2)[1-\cos(k\cdot d_i)] \tau_p^\circ(z-a_1)\tau_p^\bullet(b_1-u)}_{=:\tilde\phi(a_1,a_2,t,w,z,u,b_1,b_2;k,d)} \Big) \\
&\quad \le \big(\triangle_p^\bullet(\mathbf{0})\big) \big(T_p\big)^{i-1} \sum_{b_1',b_2'} \Big(\bar\Psi^{(n-i-1)}(b_1',b_2') \max_{a_1 \ne a_2} \sum_{t,w,z,u,x} \tilde\phi(a_1,a_2,t,w,z,u,b_1'+x,b_2'+x;k,d_i) \Big) \\
&\quad \le \big(\triangle_p^\bullet(\mathbf{0}) \triangle_p^{\bullet\bullet\circ}\big) \big(T_p\big)^{n-2} \max_{a_1\ne a_2, b_1 \ne b_2} \sum_{t,w,z,u,x} \tilde\phi(a_1,a_2,t,w,z,u,b_1+x,b_2+x;k,d_i) \\
&\quad \le \big(\triangle_p^{\bullet\bullet\circ}\big)^2 \big(T_p\big)^{n-2} \max_{\mathbf{0}\ne a, \mathbf{0}\ne b} \sum_{t,w,z,u,x} \tilde\phi(\mathbf{0},a,t,w,z,u,b+x,x;k,d_i),
\end{aligned}$$

where we use the substitution $b_j' = x - b_j$ in the second line and the bound $\triangle_p^\bullet(\mathbf{0}) \le \triangle_p^{\bullet\bullet\circ}$ in the last line. It remains to bound the sum over $\tilde\phi$. We first handle the term due to $\phi^{(1)}$, and we call it $\tilde\phi^{(1)}$. Depending on the orientation of the diagram (i.e., the parity of i), the displacement $d = d_i$ is either $d = w - a = (w-t) + (t-a)$ or $d = u = (u-z) + z$. We perform the bound for $d = u$ and use the Cosine-split lemma 3.1 once, so that we now have a displacement on an actual edge. In pictorial bounds, abbreviating $\vec v = (\mathbf{0}, a, t, w, z, u, b+x, x; k, u)$, this yields

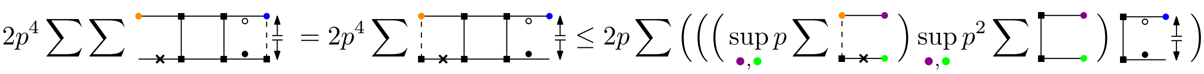


(3.12)

The bound in (3.12) consists of three summands. The first is

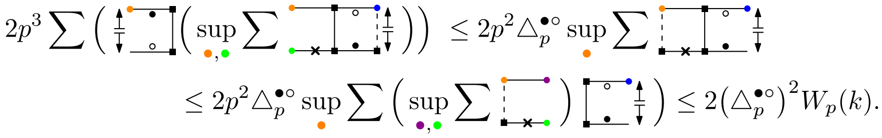


the second is



$$\leq \triangle_p^{\bullet\circ} \triangle_p W_p(k),$$

and the third is



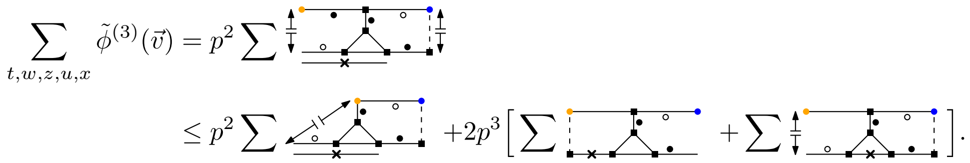
The displacement $d = w - a$ satisfies the same bound. In total, the contribution in $\tilde{\phi}$ due to $\phi^{(1)}$ is at most

$$4\big(\triangle_p^{\bullet\bullet\circ}\big)^3 \big(T_p\big)^{n-2} \big(\triangle_p^{\bullet\circ} + \triangle_p\big) W_p(k).$$

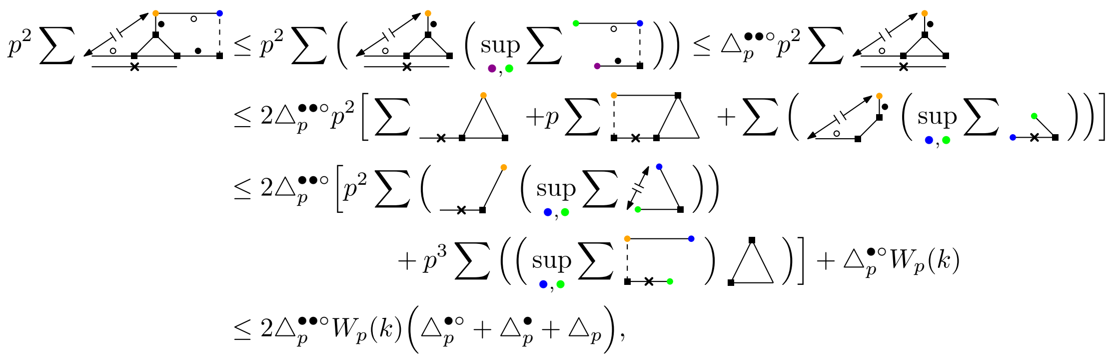
Let us now tend to $\tilde{\phi}^{(2)}$. To this end, we first write $\tilde{\phi}^{(2)} = \sum_{j=3}^{5} \tilde{\phi}^{(j)}$, where

$$\tilde{\phi}^{(3)}(\mathbf{0}, a, t, w, z, u, b + x, x; k, d) = \tilde{\phi}^{(2)}(\mathbf{0}, a, t, w, z, u, b + x, x; k, d)\mathbb{1}_{\{|\{t,z,u\}|=3\}},$$
$$\tilde{\phi}^{(4)}(\mathbf{0}, a, t, w, z, u, b + x, x; k, d) = [1 - \cos(k \cdot d)]\delta_{z,u}\delta_{t,u}\tau_p^{\circ}(u)\tau_p(w - u)\tau_p^{\bullet}(a - w)\tau_p^{\bullet}(u + x)\tau_p^{\circ}(b - w - x),$$
$$\tilde{\phi}^{(5)}(\mathbf{0}, a, t, w, z, u, b + x, x; k, d) = [1 - \cos(k \cdot d)]\delta_{z,u}\delta_{t,u}\delta_{a,w}\tau_p^{\circ}(u)\tau_p(a - u)\tau_p^{\bullet}(u + x)\tau_p^{\circ}(b - a - x).$$

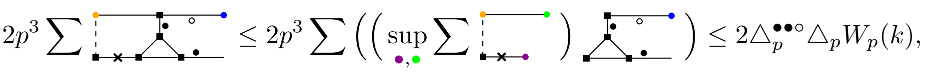
Again, we set $\vec{v} = (\mathbf{0}, a, t, w, z, u, b + x, x; k, u)$. Then


(3.13)

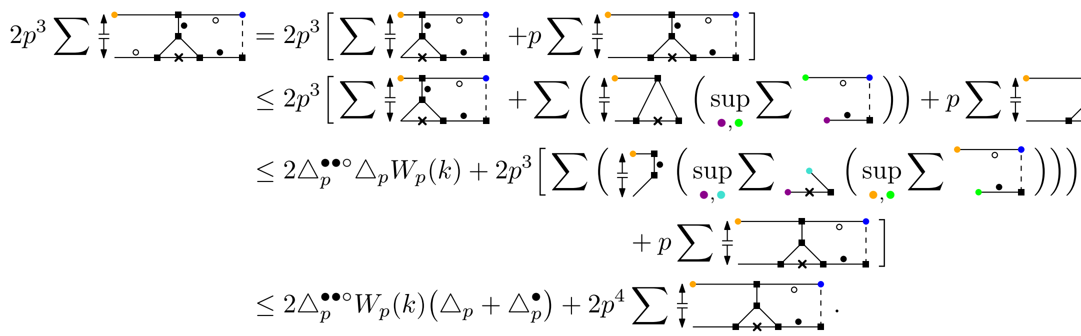
The first term in (3.13) is



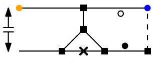
the second term is



and the third term is


(3.14)

We are left to handle the last diagram appearing in the last bound of (3.14), which contains one factor τ_p° and one factor $\tau_p^\bullet$. We distinguish the case where neither collapses (this leads to the diagram $H_p(k)$) and the case where are least one of the factors collapses. Using $\tau_p^\bullet \leq \tau_p^\circ$ and the substitution $y' = y - u$ for $y \in \{w, z, t\}$, we obtain

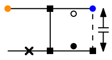
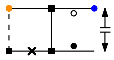
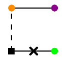
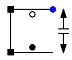
$$2p^4 \sum$$ 

$$\begin{aligned} &\leq 2H_p(k) + 4p^4 \sum_{z,t,w,u} \tau_p(t-u)\tau_p(w-t)\tau_p^\circ(u+a_2-w) \\ &\qquad \times \tau_{p,k}(z-u)\tau_p(t-z)\tau_p(z)\tau_p(w-a_1) \\ &= 2H_p(k) + 4p^4 \sum_{t',w'} \Big(\tau_p(t')\tau_p(w'-t')\tau_p^\circ(a_2-w') \sum_z \Big(\tau_{p,k}(z')\tau_p(t'-z') \\ &\qquad \times \sum_u \tau_p(z'+u)\tau_p(a_1-w'-u)\Big)\Big) \\ &\leq 2H_p(k) + 4\triangle_p^\bullet(\mathbf{0})\triangle_p^\circ W_p(k). \end{aligned}$$

In total, this yields an upper bound on (3.13) of the form

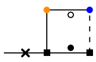
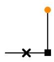
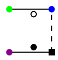
$$6(\triangle_p^{\bullet\bullet\circ})^3(T_p)^{n-2}\big[(\triangle_p^{\bullet\circ} + \triangle_p + \triangle_p^\circ)W_p(k) + H_p(k)\big].$$

The same bound is good enough for the displacement $d = w - a$. Turning to $j = 4$, we consider the displacement $d = u$ and see that

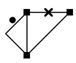
$$\sum_{t,w,z,u,x} \tilde\phi^{(4)}(\vec v) = p^2 \sum$$  $$= p^2 \sum$$  $$\leq p \sum \Big(\Big(\sup_{\bullet,\bullet} p \sum$$  $$\Big)$$  $$\Big)$$

$$\leq \triangle_p^{\bullet\circ} W_p(k),$$

which is also satisfied for $d = w - a$. Finally, $j = 5$ forces $d = u$, and we have

$$\sum_{t,w,z,u,x} \tilde\phi^{(5)}(\vec v) = p^2 \sum$$  $$\leq p \sum \Big(\Big(\sup_{\bullet,\bullet} p \sum$$  $$\Big)$$  $$\Big) \leq \triangle_p^{\bullet\bullet\circ} W_p(k),$$

and we see that this bound is not good enough for $n = 2$. To get a better bound for $n = 2$, we bound

$$p \sum_{w,u,s,t,z,x} \tilde\psi_0(\mathbf{0}, w, u)\tau_{p,k}(s-w)\tau_p(s-u)\psi_n(u,s,t,z,x) \;\leq \Big(p^2 \sum$$  $$\Big) \sup_{u \neq s} \sum_{t,z,x} \psi_n(u,s,t,z,x)$$

$$\leq \triangle_p^\bullet(\mathbf{0}) W_p(k) T_p,$$

where we recall that $\tilde\psi_0$ is an upper bound on ψ_0 (see Definition 3.8). The above bound is due to the fact that, thanks to (3.4), the supremum over the sum over ψ_n is bounded by the supremum in (3.6). □

Proof of Proposition 3.14. Let $n = 1$. Expanding the two cases in the indicator of ϕ_n gives

$$\begin{aligned} p \sum_x [1-\cos(k\cdot x)]\Pi_p^{(1)}(x) &\leq \sum_{w,u,t,z,x} [1-\cos(k\cdot x)]\phi_0(\mathbf{0},w,u,z)\phi_n(u,t,z,x) \\ &\leq p^2 \sum_{w,u,t,z,x} [1-\cos(k\cdot x)]\tilde\phi_0(\mathbf{0},w,u,z)\tau_p^\bullet(t-u)\tau_p(z-t)\tau_p(z-x)\tau_p(t-x) \\ &\quad + p \sum_{w,u,x} [1-\cos(k\cdot x)]\phi_0(\mathbf{0},w,u,z)\tau_p(x-u), \end{aligned} \tag{3.15}$$

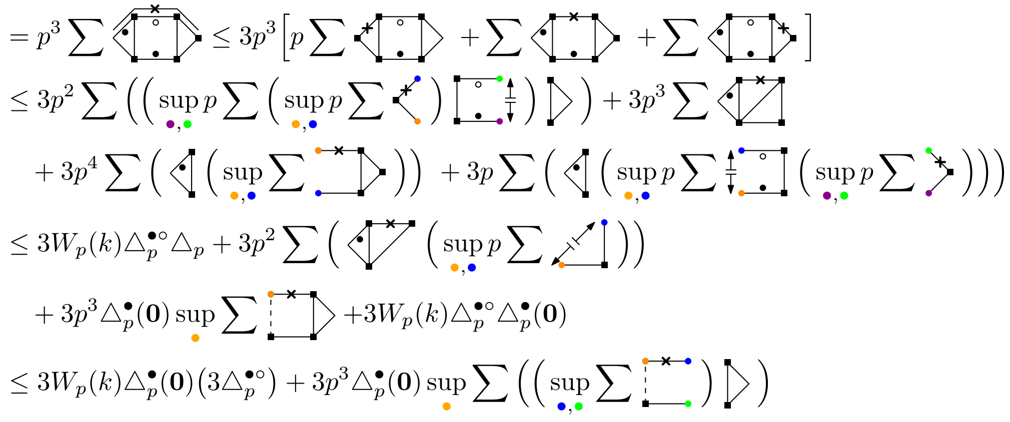
where we used the bound $\phi_0 \leq \tilde\phi_0$ (see Definition 3.8) for the first summand. Since ϕ_0 is a sum of two terms, (3.15) is equal to

$$\begin{aligned} &p^3 \sum_{w,u,t,z,x} [1-\cos(k\cdot x)]\tau_p^\bullet(w)\tau_p(u)\tau_p(u-w)\tau_p^\circ(z-w)\tau_p^\bullet(t-u)\tau_p(z-t)\tau_p(x-z)\tau_p(x-t) \\ &+ p^2 \sum_{u,x} J(u)\tau_{p,k}(x)\tau_p(x-u) \end{aligned}$$

$$+p^2 \sum_{w,u,x} [1-\cos(k\cdot x)]\tau_p^{\bullet}(w)\gamma_p(u)\tau_p(u-w)\tau_p^{\circ}(x-w)\tau_p(x-u). \tag{3.16}$$

We use the Cosine-split lemma 3.1 on the first term of (3.16) to decompose $x = u + (z-u) + (x-z)$, which gives

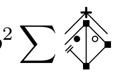
$$p^3 \sum_{w,u,t,z,x} [1-\cos(k\cdot x)]\tau_p^{\bullet}(w)\tau_p(u)\tau_p(u-w)\tau_p^{\circ}(z-w)\tau_p^{\bullet}(t-u)\tau_p(z-t)\tau_p(x-u)\tau_p(x-t)$$

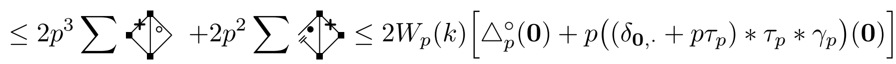


$$\le 3W_p(k)\triangle_p^{\bullet}(\mathbf{0})\big(3\triangle_p^{\bullet\circ} + \triangle_p\big).$$

The second term in (3.16) is $p^2(J * \tau_{p,k} * \tau_p)(\mathbf{0})$. Depicting the factor γ_p as a disrupted line, the third term in (3.16) is

$$p^2 \sum_{w,u,x} [1-\cos(k\cdot x)]\tau_p^{\bullet}(w)\gamma_p(u)\tau_p(u-w)\tau_p^{\circ}(x-w)\tau_p(x-u) = p^2 \sum$$





$$\le 2W_p(k)\big[\triangle_p^{\circ} + p(\tau_p * \gamma_p)(\mathbf{0}) + p^2(\tau_p^{*2} * \gamma_p)(\mathbf{0})\big] \le 2W_p(k)\big[\triangle_p^{\circ} + 2\triangle_p\big].$$

In the above, we have used that $\gamma_p(x) \le \tau_p(x)$ as well as $\gamma_p(x) \le p(J * \tau_p)(x) \le p\tau_p^{*2}(x)$. □

4 Bootstrap analysis

4.1 Introduction of the bootstrap functions

This section brings the previous results together to prove Proposition 4.2, from which Theorem 1.1 follows with little extra effort. The remaining strategy of proof is standard and described in detail in [18]. In short, it is the following: We introduce the bootstrap function f in (4.1). In Section 4.2, and in particular in Proposition 4.2, we prove several bounds in terms of f, including bounds uniform in $p \in [0, p_c)$ under the additional assumption that f is uniformly bounded.

In Section 4.3, we show that $f(0) \le 3$ and that f is continuous on $[0, p_c)$. Lastly, we show that on $[0, p_c)$, the bound $f \le 4$ implies $f \le 3$. This is called the improvement of the bounds, and it is shown by employing the implications from Section 4.2. As a consequence of this, the results from Section 4.2 indeed hold uniformly in $p \in [0, p_c)$, and we may extend them to p_c by a limiting argument.

Let us recall the notation $\tau_{p,k}(x) = [1-\cos(k\cdot x)]\tau_p(x), J_k(x) = [1-\cos(k\cdot x)]J(x)$. We extend this to $D_k(x) = [1-\cos(k\cdot x)]D(x)$. We note that $\chi(p)$ was defined as $\chi(p) = \mathbb{E}[|\mathscr{C}(\mathbf{0})|]$ and that $\chi(p) = 1 + p\sum_{x\in\mathbb{Z}^d}\tau_p(x)$. We define

$$\lambda_p = 1 - \frac{1}{\chi(p)} = 1 - \frac{1}{1 + p\widehat{\tau}_p(0)}.$$

We define the bootstrap function $f = f_1 \vee f_2 \vee f_3$ with

$$f_1(p) = 2dp, \qquad f_2(p) = \sup_{k\in(-\pi,\pi]^d} \frac{p|\widehat{\tau}_p(k)|}{\widehat{G}_{\lambda_p}(k)}, \qquad f_3(p) = \sup_{k,l\in(-\pi,\pi]^d} \frac{p|\widehat{\tau}_{p,k}(l)|}{\widehat{U}_{\lambda_p}(k,l)}, \tag{4.1}$$

where $\widehat{U}_{\lambda_p}$ is defined as

$$\widehat{U}_{\lambda_p}(k,l) := 3000[1-\widehat{D}(k)]\Big(\widehat{G}_\lambda(l-k)\widehat{G}_\lambda(l) + \widehat{G}_\lambda(l)\widehat{G}_\lambda(l+k) + \widehat{G}_\lambda(l-k)\widehat{G}_\lambda(l+k)\Big).$$

We note that $\widehat{\tau}_{p,k}$ relates to $\Delta_k \widehat{\tau}_p$, the discretized second derivative of $\widehat{\tau}_p$, as follows:

$$\Delta_k \widehat{\tau}_p(l) := \widehat{\tau}_p(l-k) + \widehat{\tau}_p(l+k) - 2\widehat{\tau}_p(l) = -2\widehat{\tau}_{p,k}(l).$$

The following result bounds the discretized second derivative of the random walk Green's function:

Lemma 4.1 (Bounds on Δ_k, [25], Lemma 5.7)**.** *Let $a(x)=a(-x)$ for all $x \in \mathbb{Z}^d$, set $\widehat{A}(k) = (1-\widehat{a}(k))^{-1}$, and let $k,l \in (-\pi,\pi]^d$. Then*

$$\begin{aligned}|\Delta_k \widehat{A}(l)| \leq \big(\widehat{|a|}(0) - \widehat{|a|}(k)\big) \times \Big(&\big[\widehat{A}(l-k) + \widehat{A}(l+k)\big]\widehat{A}(l) \\ &+ 8\widehat{A}(l-k)\widehat{A}(l+k)\widehat{A}(l)\big[\widehat{|a|}(0) - \widehat{|a|}(l)\big]\Big).\end{aligned}$$

In particular,

$$|\Delta_k \widehat{G}_\lambda(l)| \leq [1-\widehat{D}(k)]\Big(\widehat{G}_\lambda(l)\widehat{G}_\lambda(l-k) + \widehat{G}_\lambda(l)\widehat{G}_\lambda(l+k) + 8\widehat{G}_\lambda(l-k)\widehat{G}_\lambda(l+k)\Big).$$

A natural first guess for f_3 might have been $\sup p|\Delta_k \widehat{\tau}_p(l)|/|\Delta_k \widehat{G}_{\lambda_p}(l)|$. However, $\Delta_k \widehat{G}_{\lambda_p}(l)$ may have roots, which makes this guess an inconvenient choice for f_3. In contrast, $\widehat{U}_{\lambda_p}(k,l) > 0$ for $k \neq 0$. Hence, the bound in Lemma 4.1 supports the idea that f_3 is a reasonable definition.

4.2 Consequences of the diagrammatic bounds

The main result of this section, and a crucial result in this paper, is Proposition 4.2. Proposition 4.2 proves (in high dimension) the convergence of the lace expansion derived in Proposition 2.9 by giving bounds on the lace-expansion coefficients. Under the additional assumption that $f \leq 4$ on $[0,p_c)$, these bounds are shown to be uniform in $p \in [0,p_c)$.

Proposition 4.2 (Convergence of the lace expansion and Ornstein-Zernike equation)**.**

1. *Let $n \in \mathbb{N}_0$ and $p \in [0,p_c)$. Then there is $d_0 \geq 6$ and a constant $c_f = c(f(p))$ (increasing in f and independent of d) such that, for all $d > d_0$,*

$$\sum_{x\in\mathbb{Z}^d} p|\Pi_{p,n}(x)| \leq c_f/d, \qquad \sum_{x\in\mathbb{Z}^d} [1-\cos(k\cdot x)]p|\Pi_{p,n}(x)| \leq [1-\widehat{D}(k)]c_f/d, \tag{4.2}$$

$$\sup_{x\in\mathbb{Z}^d} p \sum_{m=0}^{n} |\Pi_p^{(m)}(x)| \leq c_f, \tag{4.3}$$

and

$$\sum_{x\in\mathbb{Z}^d} |R_{p,n}(x)| \leq c_f (c_f/d)^n \widehat{\tau}_p(0). \tag{4.4}$$

Consequently, $\Pi_p := \lim_{n\to\infty} \Pi_{p,n}$ is well defined and τ_p satisfies the Ornstein-Zernike equation (OZE), taking the form

$$\tau_p(x) = J(x) + \Pi_p(x) + p\big((J+\Pi_p) * \tau_p\big)(x). \tag{4.5}$$

2. *Let $f \leq 4$ on $[0,p_c)$. Then there is a constant c and $d_0 \geq 6$ such that the bounds* (4.2), (4.3), (4.4) *hold for all $d > d_0$ with c_f replaced by c for all $p \in [0,p_c)$. Moreover, the OZE* (4.5) *holds.*

The standard assumption in the lace expansion literature is a uniform bound on $f(p)$ (typically $f(p) \leq 4$ as in part *2.* of Proposition 4.2). This is part of the so-called bootstrap argument.

We first formulate part *1.* of Proposition 4.2 to demonstrate that this bootstrap argument is not necessary to obtain convergence of the lace expansion and thus establish the OZE for a *fixed* value

$p < p_c$ given that the dimension is large enough. In the context of continuum percolation, an existence result without an explicit expression is obtained in [?] in general dimension.

However, since we want to extend our results to the critical point p_c, we are interested in bounds that are uniform in p. Only here is it that the bootstrap argument (and thus Section 4.3) comes into play. In Section 4.3, we indeed prove that $f \leq 4$ and so the second part of Proposition 4.2 applies. We get the following corollary:

Corollary 4.3 (OZE at p_c). *There is d_0 such that for all $d > d_0$, the limit $\Pi_{p_c} = \lim_{p \nearrow p_c} \Pi_p$ exists and is given by $\Pi_{p_c} = \sum_{n \geq 0} (-1)^n \Pi_{p_c}^{(n)}$, where $\Pi_{p_c}^{(n)}$ is the extension of Definition 2.8 at $p = p_c$. Consequently, the bounds in Proposition 4.2 and the OZE (4.5) extend to p_c.*

Proposition 4.2 follows without too much effort as a consequence of Lemmas 4.7, 4.8, 4.9, and 4.10. Part of the lace expansion's general strategy of proof in the bootstrap analysis is to use the Inverse Fourier Theorem to write

$$\triangle_p(x) = p^2 \int_{(-\pi,\pi]^d} \mathrm{e}^{-\mathrm{i}k\cdot x} \widehat{\tau}_p(k)^3 \frac{\mathrm{d}k}{(2\pi)^d}$$

and then to use an assumed bound on f_2 to replace $\widehat{\tau}_p$ by $\widehat{G}_{\lambda_p}$. For site percolation, this poses a problem, since we are missing one factor of p. Overcoming this issue poses a novelty of Section 4. The following two observations turn out to be helpful at this:

Observation 4.4 (Convolutions of J). *Let $m \in \mathbb{N}$ and $x \in \mathbb{Z}^d$ with $m \geq |x|$. Then there is a constant $c = c(m, x)$ with $c \leq m!$ such that*

$$J^{*m}(x) = c \mathbb{1}_{\{m-|x| \text{ is even}\}} (2d)^{(m-|x|)/2}.$$

Proof. This is an elementary matter of counting the number of m-step walks from $\mathbf{0}$ to x. If $m - |x|$ is odd, then there is no way of getting from $\mathbf{0}$ to x in m steps.

So assume that $m - |x|$ is even. To get from $\mathbf{0}$ to x, $|x|$ steps must be chosen to reach x. Only taking these $|x|$ steps (in any order) would amount to a shortest $\mathbf{0}$-x-path. Out of the remaining steps, half can be chosen freely (each producing a factor of $2d$), and the other half must compensate them. In counting the different walks, we have to respect the at most $m!$ unique ways of ordering the steps. □

We remark that this also shows that the maximum is attained for $x = \mathbf{0}$ when m is even and for x being a neighbor of $\mathbf{0}$ when m is odd.

Observation 4.5 (Elementary bounds on τ_p^{*n}). *Let $n, m \in \mathbb{N}$. Then there is $c = c(m, n)$ such that, for all $p \in [0, 1]$ and $x \in \mathbb{Z}^d$,*

$$\tau_p^{*n}(x) \leq c \sum_{l=0}^{m-n} p^l J^{*(l+n)}(x) + c \sum_{j=1 \vee (n-m)}^{n} p^{m+j-n} (J^{*m} * \tau_p^{*j})(x),$$

where we use the convention that $\sum_{l=0}^{m-n}$ vanishes for $n > m$.

Proof. The observation heavily relies on the bound

$$\tau_p(x) \leq J(x) + \mathbb{E}\Big[\sum_{y \in \omega} \mathbb{1}_{\{|y|=1, y \longleftrightarrow x\}} \Big] = J(x) + p(J * \tau_p)(x). \tag{4.6}$$

Note that the left-hand side equals 0 when $x = \mathbf{0}$. We prove the statement by induction on $m - n$; for the base case, let $m \leq n$. Then we apply (4.6) to m of the n convoluted τ_p terms to obtain

$$\begin{aligned} \tau_p^{*n}(x) &\leq \big(\tau_p^{*(n-m)} * \big(J + p(J * \tau_p) \big)^{*m} \big)(x) \\ &= \sum_{l=0}^{m} \binom{m}{l} p^l \big(J^{*m} * \tau_p^{*(n-m+l)} \big)(x) = \sum_{l=n-m}^{n} \binom{m}{l+m-n} p^{l+m-n} \big(J^{*m} * \tau_p^{*l} \big)(x). \end{aligned}$$

Let now $m - n > 0$. Applying (4.6) once yields a sum of two terms, namely

$$\tau_p^{*n}(x) \leq \big(J * \tau_p^{*(n-1)} \big)(x) + p \big(J * \tau_p^{*n} \big)(x). \tag{4.7}$$

We can apply the induction hypothesis on the second term with $\tilde{m} = m-1$ and $\tilde{n} = n$, producing terms of the sought-after form. Now, observe that application of (4.6) yields

$$\big(J^{*j} * \tau_p^{*(n-j)}\big)(x) \le \big(J^{*(j+1)} * \tau_p^{*(n-j-1)}\big)(x) + p\big(J^{*(j+1)} * \tau_p^{*(n-j)}\big)(x) \tag{4.8}$$

for $1 \le j < n$. For every j, the second term can be bounded by the induction hypothesis for $\tilde{m} = m-j-1$ and $\tilde{n} = n-j$ (so that $\tilde{m}-\tilde{n} < m-n$) with suitable $c(m,n)$. Hence, we can iteratively break down (4.7); after applying (4.8) for $j = n-1$, we are left with the term $J^{*n}(x)$, finishing the proof. □

We now define

$$V_p^{(m,n)}(a) := (J^{*m} * \tau_p^{*n})(a), \quad W_p^{(m,n)}(a;k) := (\tau_{p,k} * V_p^{(m,n)})(a), \quad \widetilde{W}_p^{(m,n)}(a;k) := (J_k * V_p^{(m,n)})(a).$$

Note that W_p from Definition 3.12 relates to the above definition via $W_p = pW_p^{(0,0)} + pW_p^{(0,1)}$. Moreover, $\triangle_p(x) = p^2 V^{(0,3)}(x)$.

Lemma 4.6 (Bounds on $V_p^{(m,n)}, W_p^{(m,n)}, \widetilde{W}_p^{(m,n)}$)**.** *Let $p \in [0, p_c)$ and $m, n \in \mathbb{N}_0$ with $d > \frac{20}{9}n$. Then there is a constant $c_f = c(m,n,f(p))$ (increasing in f) such that the following hold true:*

1. *For $m+n \ge 2$,*
$$p^{m+n-1} V_p^{(m,n)}(a) \le \begin{cases} c_f & \text{if } m+n = 2 \text{ and } a = \mathbf{0}, \\ c_f/d & \text{else.} \end{cases}$$

2. *For $m+n \ge 1$, and under the additional assumption $d > 2n+4$ for the bound on $W_p^{(m,n)}$,*
$$p^{m+n} \max\Big\{ \sup_{a \in \mathbb{Z}^d} \widetilde{W}_p^{(m,n)}(a;k), \sup_{a \in \mathbb{Z}^d} W_p^{(m,n)}(a;k) \Big\} \le [1 - \widehat{D}(k)] \times \begin{cases} c_f & \text{if } m+n \le 2, \\ c_f/d & \text{if } m+n \ge 3. \end{cases}$$

We are going to apply Lemma 4.6 for $n \le 3$, and so $d \ge 7 > 60/9$ for the dimension suffices.

Proof. Bound on V_p. We start with the case $m \ge 4$ where we can rewrite the left-hand side via Fourier transform and apply Hölder's inequality to obtain

$$\begin{aligned} p^{m+j-1}(J^{*m} * \tau_p^{*j})(a) &= p^{m+j-1} \int_{(-\pi,\pi]^d} \mathrm{e}^{-\mathrm{i}k\cdot a} \widehat{J}(k)^m \widehat{\tau}_p(k)^j \frac{\mathrm{d}k}{(2\pi)^d} \\ &\le \Big(p^{10(m-1)} \int_{(-\pi,\pi]^d} \widehat{J}(k)^{10m} \frac{\mathrm{d}k}{(2\pi)^d} \Big)^{1/10} \Big(\int_{(-\pi,\pi]^d} (p|\widehat{\tau}_p(k)|)^{10j/9} \frac{\mathrm{d}k}{(2\pi)^d} \Big)^{9/10} \\ &\le \Big(p^{10(m-1)} J^{*10m}(\mathbf{0}) \Big)^{1/10} \times f_2(p)^j \Big(\int_{(-\pi,\pi]^d} \widehat{G}_{\lambda_p}(k)^{10j/9} \frac{\mathrm{d}k}{(2\pi)^d} \Big)^{9/10}. \end{aligned} \tag{4.9}$$

We note that the number 10 in the exponent holds no special meaning other than that it is large enough to make the following arguments work. The first factor in (4.9) is handled by Observation 4.4, as

$$\begin{aligned} \Big(p^{10(m-1)} J^{*10m}(\mathbf{0}) \Big)^{1/10} &\le \Big(c p^{10(m-1)} (2d)^{5m} \Big)^{1/10} \le \Big(c (2dp)^{10(m-1)} (2d)^{-5m+10} \Big)^{1/10} \\ &\le c f_1(p)^{m-1} (2d)^{-m/2+1} \le c_f/d \end{aligned}$$

and $m \ge 4$. Regarding the second factor in (4.9), note that $10j/9 \le 10n/9 < d/2$ and so Proposition 2.1 gives a uniform upper bound. We remark that the exponent 10 in (4.9) is convenient because it is even and allows us to apply Lemma 4.6 in dimension $d \ge 7$.

If $m < 4$, then we first use Observation 4.5 with $\tilde{m} = 4-m$ to get that $p^{m+n-1} V_p^{(m,n)}$ is bounded by a sum of terms of two types, which are constant multiples of

$$p^{l+m+n-1} J^{*(l+m+n)}(a) = p^{s-1} J^{*s}(a) \qquad \text{and} \quad p^{4+j-1}(J^{*4} * \tau_p^{*j})(a), \tag{4.10}$$

where $0 \le l \le 4-m-1$ (and therefore $s \ge 2$) and $1 \le j \le n$. If s is odd, we can write $s = 2r+1$ for some $r \ge 1$, and Observation 4.4 gives

$$p^{2r} J^{*(2r+1)}(a) \le c p^{2r} (2d)^r = c(2dp)^{2r}(2d)^{-r} \le c(f_1(p))^{2r}(2d)^{-r} \le c_f/d.$$

Similarly, if s is even and $a \neq \mathbf{0}$ or $s \ge 4$, then $p^{s-1} J^{*s}(a) \le c_f/d$. Finally, if $s = 2$ and $a = \mathbf{0}$, then $p(J * J)(\mathbf{0}) \le c_f$. This shows that the terms of the first type in (4.10) are of the correct order. The second type is of the form $p^{4+j-1} V_p^{(4,j)}(a)$ and included in the previous considerations. Together this proves the claimed bound on $V_p^{(m,n)}$.

Bound on $\widetilde{W}_p$. Let first $m + n \ge 3$. Then

$$\begin{aligned} p^{m+n} \widetilde{W}_p^{(m,n)}(a;k) &= p^{m+n} \sum_{y \in \mathbb{Z}^d} J_k(y) V_p^{(m,n)}(a-y) \\ &\le p^{m+n-1} \Big(\sup_{a \in \mathbb{Z}^d} V_p^{(m,n)}(a) \Big) (2dp) \sum_{y \in \mathbb{Z}^d} [1 - \cos(k \cdot y)] D(y) \\ &\le c_f/d \times f_1(p) [1 - \widehat{D}(k)], \end{aligned}$$

applying the bound on V_p.

Consider now $m + n = 2$. Using first that $J \le \tau_p$ and then (4.6),

$$p^2 \widetilde{W}_p^{(m,n)}(a;k) \le p^2 \widetilde{W}_p^{(0,2)}(a;k) \le p^2 \widetilde{W}_p^{(2,0)}(a;k) + p^3 \widetilde{W}_p^{(2,1)}(a;k) + p^3 \widetilde{W}_p^{(1,2)}(a;k). \tag{4.11}$$

The second and third summand right-hand side of (4.11) can be dealt with as before, we only have to deal with the first summand. Indeed,

$$p^2 \widetilde{W}_p^{(2,0)}(a;k) = p^2 \sum_y J_k(y) J^{*2}(a-y) \le 2dp^2 J^{*2}(\mathbf{0}) \sum_y D_k(y) = f_1(p)^2 [1 - \widehat{D}(k)],$$

and we can choose $c_f = f_1(p)^2$.

Finally, for $m+n = 1$, we have $p\widetilde{W}_p^{(m,n)}(a;k) \le p\widetilde{W}_p^{(1,0)}(a;k) + p^2 \widetilde{W}_p^{(1,1)}(a;k)$. The second term was already bounded, the first is

$$p(J_k * J)(a) \le p \sum_y J_k(y) = f_1(p)[1 - \widehat{D}(k)].$$

Bound on W_p. We note that a combination of (4.6) and the Cosine-split lemma 3.1 yields

$$\tau_{p,k}(x) \le J_k(x) + 2p(J_k * \tau_p)(x) + 2p(J * \tau_{p,k})(x). \tag{4.12}$$

Applying this repeatedly, we can bound $p^{m+n} W_p^{(m,n)}(a;k)$ by a sum of quantities of the form $p^{s+t} \widetilde{W}_p^{(s,t)}$ (where $s + t \ge 1$) plus $c(m,n) p^{m+n} W_p^{(m,n)}$, where we can now assume $m \ge 4$ w.l.o.g. The terms of the form $\widetilde{W}_p^{(s,t)}$ were bounded above already. Similarly to how we obtained the bound (4.9), we bound the last term by applying Hölder's inequality, and so

$$\begin{aligned} p^{m+n} W_p^{(m,n)}(a;k) &\le p^{m+n} \int_{(-\pi,\pi]^d} |\widehat{J}(l)|^m |\widehat{\tau}_p(l)|^n |\widehat{\tau}_{p,k}(l)| \frac{\mathrm{d}l}{(2\pi)^d} \\ &\le \Big(p^{10(m-1)} \int_{(-\pi,\pi]^d} \widehat{J}(l)^{10m} \frac{\mathrm{d}l}{(2\pi)^d} \Big)^{1/10} \Big(\int_{(-\pi,\pi]^d} (p|\widehat{\tau}_p(l)|)^{10n/9} (p|\widehat{\tau}_{p,k}(l)|)^{10/9} \frac{\mathrm{d}l}{(2\pi)^d} \Big)^{9/10} \\ &\le c_f/d \times 3000 f(p)^{n+1} \Big(\int_{(-\pi,\pi]^d} \widehat{G}_{\lambda_p}(l)^{10n/9} \Big[\widehat{G}_{\lambda_p}(l) \big(\widehat{G}_{\lambda_p}(l-k) + \widehat{G}_{\lambda_p}(l+k) \big) \\ &\qquad + \widehat{G}_{\lambda_p}(l-k) \widehat{G}_{\lambda_p}(l+k) \Big]^{10/9} \frac{\mathrm{d}l}{(2\pi)^d} \Big)^{9/10} \\ &\le c_f/d, \end{aligned} \tag{4.13}$$

where the last bound is due to Proposition 2.2 and the value of c_f has changed in the last line. □

The proofs of the following lemmas, bounding the quantities appearing in Section 3, are direct consequences of Lemma 4.6.

Lemma 4.7 (Bounds on various triangles)**.** *Let $p \in [0, p_c)$ and $d > 6$. Then there is $c_f = c(f(p))$ (increasing in f) such that*

$$\max\{\triangle_p, \triangle_p^{\circ}, \triangle_p^{\bullet}, \triangle_p^{\bullet\circ}\} \le c_f/d, \qquad \max\{\triangle_p^{\bullet}(\mathbf{0}), \triangle_p^{\bullet\circ}(\mathbf{0}), \triangle_p^{\bullet\bullet\circ}\} \le c_f.$$

Proof. Note that

$$\triangle_p(x) = p^2 V_p^{(0,3)}(x), \qquad \triangle_p^{\circ}(x) = p^2 V_p^{(0,2)}(x) + \triangle_p(x), \qquad \triangle_p^{\bullet}(x) = pV_p^{(0,2)}(x) + \triangle_p(x),$$
$$\triangle_p^{\bullet\circ}(x) = p\tau_p(x) + \triangle_p^{\bullet}(x), \qquad \triangle_p^{\bullet\bullet\circ}(x) = \delta_{\mathbf{0},x} + \triangle_p^{\bullet\circ}(x).$$

For the bound on $p\tau_p \le p$, we use that $p \le f_1(p)/d$. For all remaining quantities, we use Lemma 4.6, which is applicable since $n \le 3$ and $\frac{20}{9} n \le \frac{60}{9} < 7 \le d$. □

Lemma 4.8 (Bound on W_p)**.** *Let $p \in [0, p_c)$ and $d > 6$. Then there is a constant $c_f = c(f(p))$ (increasing in f) such that*

$$W_p(k) \le [1 - \widehat{D}(k)] c_f.$$

Proof. By (4.12),

$$\begin{aligned} W_p(x; k) &= pW_p^{(0,1)}(x; k) + p\tau_{p,k}(x) \\ &\le pW_p^{(0,1)}(x; k) + 2p^2 \widetilde{W}_p^{(0,1)}(x; k) + 2p^2 W_p^{(1,0)}(x; k) + pJ_k(x). \end{aligned}$$

The proof follows from Lemma 4.6 together with the observation that

$$pJ_k(x) = (2dp) D_k(x) \le f_1(p) \sum_{x \in \mathbb{Z}^d} D_k(x) = f_1(p)[1 - \widehat{D}(k)]. \qquad \square$$

Lemma 4.9 (Bounds on $\Pi_p^{(0)}$ and $\Pi_p^{(1)}$)**.** *Let $p \in [0, p_c), i \in \{0, 1\}$, and $d > 6$. Then there is a constant $c_f = c(f(p))$ (increasing in f) such that*

$$p \sum_x \Pi_p^{(0)}(x) \le c_f/d, \qquad p \sum_x [1 - \cos(k \cdot x)] \Pi_p^{(i)}(x) \le [1 - \widehat{D}(k)] c_f/d.$$

Proof. We recall the two bounds obtained in Proposition 3.2. The first one yields $p|\widehat{\Pi}_p^{(0)}(k)| \le p^3 V_p^{(2,2)}(\mathbf{0})$, the second one yields

$$p\widehat{\Pi}_p^{(0)}(0) - p\widehat{\Pi}_p^{(0)}(k) \le 2p^3 \widetilde{W}_p^{(1,2)}(\mathbf{0}; k) + 2p^3 W_p^{(2,1)}(\mathbf{0}; k).$$

All of these bounds are handled directly by Lemma 4.6. Similarly, the only quantity in the bound of Proposition 3.14 that was not bounded already is $p^2 W_p^{(1,1)}(\mathbf{0}; k)$. By a combination of (4.6) and (4.12), we can bound

$$\begin{aligned} p^2 W_p^{(1,1)}(\mathbf{0}; k) &\le p^2 \Big(W_p^{(2,0)}(\mathbf{0}; k) + pW_p^{(2,1)}(\mathbf{0}; k) \Big) \\ &\le p^2 \Big(\widetilde{W}_p^{(2,0)}(\mathbf{0}; k) + 2p\widetilde{W}_p^{(2,1)}(\mathbf{0}; k) + 2pW_p^{(3,0)}(\mathbf{0}; k) + pW_p^{(2,1)}(\mathbf{0}; k) \Big). \end{aligned}$$

But $0 \le \widetilde{W}_p^{(2,0)}(\mathbf{0}; k) \le 2J^{\star 3}(\mathbf{0}) = 0$ by Observation 4.4. The other three terms are bounded by Lemma 4.6. □

Lemma 4.10 (Displacement bounds on H_p)**.** *Let $p \in [0, p_c)$ and $d > 6$. Then there is a constant $c_f = c(f(p))$ (increasing in f) such that*

$$H_p(k) \le [1 - \widehat{D}(k)] c_f/d.$$

Proof. We recall that

$$H_p(b_1, b_2; k) = p^5 \sum_{t,w,z,u,v} \tau_p(z)\tau_p(t-u)\tau_p(t-z)\tau_{p,k}(u-z)\tau_p(t-w)\tau_p(w-b_1)\tau_p(v-w)\tau_p(v+b_2-u).$$

We bound the factor $\tau_p(z) \le J(z) + p(J * \tau_p)(z)$, splitting H_p into a sum of two. The first term is easy to bound. Indeed,

$$\begin{aligned} p^5 \sum_{t,w,z} & J(z)\tau_p(t-z)\tau_p(w-t)\tau_p(b_1-w) \sum_u \tau_{p,k}(u-z)\tau_p(t-u)(\tau_p * \tau_p)(b_2-u-w) \\ &\le \triangle_p^\bullet(\mathbf{0}) p^4 \sum_{t,w,z,u} J(z)\tau_p(t-z)\tau_p(w-t)\tau_p(b_1-w)(\tau_{p,k} * \tau_p)(t-z) \\ &\le \triangle_p^\bullet(\mathbf{0}) W_p(k) p^3 V_p^{(1,3)}(b_1) \le [1-\widehat{D}(k)]c_f/d \end{aligned}$$

by the previous Lemmas 4.6-4.8. We can thus focus on bounding

$$p^6 \sum_{t,w,z,u,v} (J*\tau_p)(z)\tau_p(t-u)\tau_p(t-z)\tau_{p,k}(u-z)\tau_p(t-w)\tau_p(w-b_1)\tau_p(v-w)\tau_p(v+b_2-u). \tag{4.14}$$

To prove such a bound (and thus the lemma), we need to recycle some ideas from the proof of Lemma 4.6 in a more involved fashion. To this end, let

$$\sigma(x) := p^4(J^{*4} * \tau_p)(x) + \sum_{j=1}^4 p^{j-1} J^{*j}(x),$$

and note that $\tau_p(x) \le \sigma(x)$ by (4.6). Consequently, (4.14) is bounded by $\widetilde{H}_p(a_1, a_2; k)$, where we define

$$\widetilde{H}_p(a_1, a_2; k) = p^6 \sum_{t,w,z,u,v} (J*\sigma)(z)\sigma(t-u)\sigma(t-z)\tau_{p,k}(u-z)\sigma(t-w)\sigma(w-a_1)\sigma(v-w)\sigma(v+a_2-u).$$

By the Inverse Fourier Theorem, we can write

$$\widetilde{H}_p(a_1, a_2; k) = p^6 \int_{(-\pi,\pi]^{3d}} \mathrm{e}^{-\mathrm{i}a_1\cdot l_1 - \mathrm{i}a_2\cdot l_2}\, \widehat{J}(l_1)\widehat{\sigma}(l_1)^2\widehat{\sigma}(l_2)^2\widehat{\tau}_{p,k}(l_3)\widehat{\sigma}(l_1-l_2)\widehat{\sigma}(l_1-l_3)\widehat{\sigma}(l_2-l_3)\frac{\mathrm{d}(l_1,l_2,l_3)}{(2\pi)^{3d}}.$$

(For details on the above identity, see [19, Lemma 5.7] and the corresponding bounds on H_λ therein.) We bound

$$\begin{aligned} \widetilde{H}_p(a_1, a_2; k) \le 3000 f_3(p)[1-\widehat{D}(k)]p^5 & \int_{(-\pi,\pi]^{3d}} |\widehat{J}(l_1)|\widehat{\sigma}(l_1)^2\widehat{\sigma}(l_2)^2|\widehat{\sigma}(l_1-l_2)||\widehat{\sigma}(l_1-l_3)||\widehat{\sigma}(l_2-l_3)| \\ \times \Big(\widehat{G}_{\lambda_p}(l_3)\widehat{G}_{\lambda_p}(l_3-k) + & \widehat{G}_{\lambda_p}(l_3)\widehat{G}_{\lambda_p}(l_3+k) + \widehat{G}_{\lambda_p}(l_3-k)\widehat{G}_{\lambda_p}(l_3+k)\Big)\frac{\mathrm{d}(l_1,l_2,l_3)}{(2\pi)^{3d}}. \end{aligned} \tag{4.15}$$

Opening the brackets in (4.15) gives rise to three summands. We show how to treat the third one. Applying Cauchy-Schwarz, we obtain

$$\Big(\int_{(-\pi,\pi]^{3d}} \big[p^2|\widehat{J}(l_1)||\widehat{\sigma}(l_1)|^3\big]\big[p^2\widehat{\sigma}(l_2-l_1)^2|\widehat{\sigma}(l_2)|\big]\big[p\widehat{G}_{\lambda_p}(l_3+k)^2|\widehat{\sigma}(l_3-l_2)|\big]\frac{\mathrm{d}(l_1,l_2,l_3)}{(2\pi)^{3d}}\Big)^{1/2} \tag{4.16}$$

$$\times \Big(\int_{(-\pi,\pi]^{3d}} \big[p^2|\widehat{\sigma}(l_2)|^3\big]\big[p^2\widehat{\sigma}(l_1-l_3)^2|\widehat{J}(l_1)||\widehat{\sigma}(l_1)|\big]\big[p\widehat{G}_{\lambda_p}(l_3-k)^2|\widehat{\sigma}(l_3-l_2)|\big]\frac{\mathrm{d}(l_1,l_2,l_3)}{(2\pi)^{3d}}\Big)^{1/2} \tag{4.17}$$

The square brackets indicate how we want to decompose the integrals. We first bound (4.16), and we start with the integral over l_3. We intend to treat the five summands constituting $\widehat{\sigma}(l_3-l_2)$ simultaneously. Indeed, note that with our bound on f_2,

$$|\widehat{\sigma}(l)| \le \sum_{j=1}^4 p^{j-1}|\widehat{J}(l)|^j + p^3\widehat{J}(l)^4\widehat{G}_{\lambda_p}(l) \le 5 \max_{n\in\{0,1\}, j\in[4]} p^{(j\vee 4n)-1}|\widehat{J}(l)|^{(j\vee 4n)}\widehat{G}_{\lambda_p}(l)^n. \tag{4.18}$$

With this,

$$
\begin{aligned}
&p^{(j\vee 4n)}\int_{(-\pi,\pi]^d}|\widehat{J}(l_3-l_2)|^{(j\vee 4n)}\widehat{G}_{\lambda_p}(l_3-l_2)^n\widehat{G}_{\lambda_p}(l_3+k)^2\frac{\mathrm{d}l_3}{(2\pi)^d}\\
&\le\Big(p^{10(j\vee 4n)}\int_{(-\pi,\pi]^d}\widehat{J}(l_3)^{10(j\vee 4n)}\frac{\mathrm{d}l_3}{(2\pi)^d}\Big)^{1/10}\\
&\qquad\times\Big(\int_{(-\pi,\pi]^d}\widehat{G}_{\lambda_p}(l_3+k)^{20/9}\big[\widehat{G}_{\lambda_p}(l_3-l_2)+\widehat{G}_{\lambda_p}(l_3-2k+l_2)\big]^{10/9}\frac{\mathrm{d}l_3}{(2\pi)^d}\Big)^{9/10}\\
&\le(c_f/d)^{1/2}
\end{aligned}
$$

by the same considerations that were performed in (4.13). We use the same approach to treat the integral over l_2 in (4.16). Applying (4.18) to all three factors of $\widehat{\sigma}$ gives rise to tuples (j_i,n_i) for $i\in[3]$, and so

$$
\begin{aligned}
&p^{-1+\sum_{i=1}^3(j_i\vee 4n_i)}\int_{(-\pi,\pi]^d}|\widehat{J}(l_2-l_1)|^{\sum_{i=1}^2(j_i\vee 4n_i)}|\widehat{J}(l_2)|^{j_3\vee 4n_3}\widehat{G}_{\lambda_p}(l_2-l_1)^{n_1+n_2}\widehat{G}_{\lambda_p}(l_2)^{n_3}\frac{\mathrm{d}l_2}{(2\pi)^d}\\
&\le\Big(p^{10(-1+\sum_{i=1}^3(j_i\vee 4n_i))}\int\widehat{J}(l_2-l_1)^{10\sum_{i=1}^2(j_i\vee 4n_i)}\widehat{J}(l_2)^{10(j_3\vee 4n_3)}\frac{\mathrm{d}l_2}{(2\pi)^d}\Big)^{1/10}\\
&\qquad\times\Big(\int_{(-\pi,\pi]^d}\widehat{G}_{\lambda_p}(l_2-l_1)^{10(n_1+n_2)/9}\big[\widehat{G}_{\lambda_p}(l_2)+\widehat{G}_{\lambda_p}(l_2-2l_1)\big]^{10n_3/9}\frac{\mathrm{d}l_2}{(2\pi)^d}\Big)^{9/10}\\
&\le c_f\Big(p^{20(-1+\sum_{i=1}^2(j_i\vee 4n_i))}\int\widehat{J}(l_2-l_1)^{20\sum_{i=1}^2(j_i\vee 4n_i)}\frac{\mathrm{d}l_2}{(2\pi)^d}\Big)^{1/20}\\
&\qquad\times\Big(p^{20(j_3\vee 4n_3)}\int\widehat{J}(l_2)^{20(j_3\vee 4n_3)}\frac{\mathrm{d}l_2}{(2\pi)^d}\Big)^{1/20}\\
&\le(c_f'/d)^{1/2}.
\end{aligned}
$$

We finish by proving that the integral over l_1 in (4.16) is bounded by a constant. Indeed,

$$
\begin{aligned}
&p^{-1+\sum_{i=1}^3(j_i\vee 4n_i)}\int_{(-\pi,\pi]^d}|\widehat{J}(l_1)|^{1+\sum_{i=1}^3(j_i\vee 4n_i)}\widehat{G}_{\lambda_p}(l_1)^{n_1+n_2+n_3}\frac{\mathrm{d}l_1}{(2\pi)^d}\\
&\le\Big(p^{10(-1+\sum_{i=1}^3(j_i\vee 4n_i))}\int\widehat{J}(l_1)^{10(1+\sum_{i=1}^3 j_i\vee 4n_i)}\frac{\mathrm{d}l_1}{(2\pi)^d}\Big)^{1/10}\Big(\int_{(-\pi,\pi]^d}\widehat{G}_{\lambda_p}(l_1)^{10(n_1+n_2+n_3)/9}\frac{\mathrm{d}l_1}{(2\pi)^d}\Big)^{9/10}\\
&\le c_f\cdot p^{-1+\sum_{i=1}^3(j_i\vee 4n_i)}\Big(J^{*10(1+\sum_{i=1}^3 j_i\vee 4n_i)}(\mathbf{0})\Big)^{1/10}\\
&\le c_f'\cdot p^{-1+\sum_{i=1}^3(j_i\vee 4n_i)}(2d)^{\frac{1}{2}(1+\sum_{i=1}^3 j_i\vee 4n_i)}\le c_f''.
\end{aligned}
$$

This proves that (4.16) is bounded by $(c_f/d)^{1/2}$. Note that (4.17) is very similar to (4.16), and the same bounds can be applied to get a bound of $(c_f/d)^{1/2}$. Since the other two terms in (4.15) are handled analogously, we obtain the bound $\widetilde{H}_p(b_1,b_2;k)\le[1-\widehat{D}(k)]c_f/d$, which is what was claimed. □

Proof of Proposition 4.2. Recalling the bounds on $|\Pi_p^{(m)}(k)|$ obtained in Propositions 3.2 and 3.5, and the bounds on $|\Pi_p^{(m)}(k)-\Pi_p^{(m)}(0)|$ obtained in Propositions 3.2, 3.14, and 3.13, we can combine them with the bounds just obtained in Lemmas 4.7, 4.8, 4.9, and 4.10. This gives

$$
p|\widehat{\Pi}_p^{(m)}(k)|\le p\sum_{x\in\mathbb{Z}^d}\Pi_p^{(m)}(x)\le c_f(c_f/d)^{m\vee 1}, \tag{4.19}
$$

$$
p|\widehat{\Pi}_p^{(m)}(k)-\widehat{\Pi}_p^{(m)}(0)|=p\sum_{x\in\mathbb{Z}^d}[1-\cos(k\cdot x)]\Pi_p^{(m)}(x)\le c_f[1-\widehat{D}(k)](c_f/d)^{1\vee(m-2)}.
$$

Summing the above terms over m, we recognize the geometric series in their bounds. The series converges for sufficiently large d. If $f\le 4$ on $[0,p_c)$, we can replace c_f by $c=c_4$ in the above, so that the bounds are uniform in $p\in[0,p_c)$, which means that the value of d above which the series converges is independent of p. Hence,

$$
p|\widehat{\Pi}_n(k)|\le\sum_{m=0}^{\infty}p\Pi_p^{(m)}(x)\le c_f/d,\qquad p|\widehat{\Pi}_n(k)-\widehat{\Pi}_n(0)|\le[1-\cos(k\cdot x)]c_f/d.
$$

The bound (4.3) follows from Corollary 3.11 by analogous arguments. Recalling the definition of the remainder term yields the straight-forward bound

$$\sum_x R_{p,n}(x) \leq \sum_x \sum_u p\Pi_p^{(n)}(u)\tau_p(x-u) \leq p\widehat{\Pi}_p^{(n)}(0)\widehat{\tau}_p(0) \leq c_f(c_f/d)^n\widehat{\tau}_p(0), \tag{4.20}$$

applying (4.19). Hence, if $c_f/d < 1$ and $p < p_c$, then $\sum_x R_{p,n}(x) \to 0$ as $n \to \infty$. Again, if $f \leq 4$, we can replace c_f by $c = c_4$ in (4.20) and the smallness of (c/d) does not depend on the value of p.

The existence of the limit Π_p follows by dominated convergence with the bound (4.3). Together with (4.20), this implies that the lace expansion identity in Proposition 2.9 converges as $n \to \infty$ and satisfies the OZE. □

Corollary 4.3 as well as the main theorem now follow from Proposition 4.2 in conjunction with Proposition 4.12, which is proven in Section 4.3 below.

Proof of Corollary 4.3 and Theorem 1.1. Proposition 4.12 implies that indeed $f(p) \leq 3$ for all $p \in [0, p_c)$, and therefore, the consequences in the second part of Proposition 4.2 are valid. Lemma 4.7 together with Fatou's lemma and pointwise convergence of $\tau_p(x)$ to $\tau_{p_c}(x)$ then implies the triangle condition.

The remaining arguments are analogous to the proofs of [19, Corollary 6.1] and [19, Theorem 1.1]. The rough idea for the proof of Corollary 4.3 is to use $\theta(p_c) = 0$ (which follows from the triangle condition) to couple the model at p_c with the model at $p < p_c$, and then show that, as $p \nearrow p_c$, the (a.s.) finite cluster of the origin is eventually the same. For the full argument and the proof of the infra-red bound, we refer to [19]. □

4.3 Completing the bootstrap argument

It remains to prove that $f \leq 4$ on $[0, p_c)$ so that we can apply the second part of Proposition 4.2. This is achieved by Proposition 4.12, where three claims are made: First, $f(0) \leq 4$, secondly, f is continuous in the subcritical regime, and thirdly, f does not take values in $(3, 4]$ on $[0, p_c)$. This implies the desired boundedness of f. The following observation will be needed to prove the third part of Proposition 4.12:

Observation 4.11. *Suppose $a(x) = a(-x)$ for all $x \in \mathbb{Z}^d$. Then*

$$\tfrac{1}{2}\big|\Delta_k \widehat{a}(l)\big| \leq \widehat{|a|}(0) - \widehat{|a|}(k)$$

for all $k, l \in (-\pi, \pi]^d$ (where $\widehat{|a|}$ denotes the Fourier transform of $|a|$). As a consequence, $|\widehat{D}_k(l)| \leq 1 - \widehat{D}(k)$. Moreover, there is $d_0 \geq 6$ a constant $c_f = c(f(p))$ (increasing in f) such that, for all $d > d_0$,

$$\big|\Delta_k p\widehat{\Pi}_p(l)\big| \leq [1 - \widehat{D}(k)]c_f/d.$$

Proof. The statement for general a can be found, for example, in [18, (8.2.29)]. For convenience, we give the proof. Setting $a_k(x) = [1 - \cos(k \cdot x)]a(x)$, we have

$$\begin{aligned}\tfrac{1}{2}|\Delta_k \widehat{a}(l)| = |\widehat{a}_k(l)| &\leq \sum_{x \in \mathbb{Z}^d} \big|a(x)\cos(l \cdot x)[1 - \cos(k \cdot x)]\big| \leq \sum_{x \in \mathbb{Z}^d} |a(x)|[1 - \cos(k \cdot x)] \\ &\leq \widehat{|a|}(0) - \widehat{|a|}(k)\end{aligned}$$

The consequence about $\widehat{D}$ now follows from $D(x) \geq 0$ for all x. Moreover, the statement for $\Delta_k p\widehat{\Pi}_p(l)$ follows applying the observation to $a = \Pi_p$ together with the bounds in (4.2). □

Proposition 4.12. *The following are true:*

1. *The function f satisfies $f(0) \leq 3$.*
2. *The function f is continuous on $[0, p_c)$.*
3. *Let d be sufficiently large; then assuming $f(p) \leq 4$ implies $f(p) \leq 3$ for all $p \in [0, p_c)$.*

Consequently, there is some d_0 such that $f(p) \leq 3$ uniformly for all $p \in [0, p_c)$ and $d > d_0$.

As a remark, in the third step of Proposition 4.12, we prove the stronger statement $f_i(p) \leq 1 + \text{const}/d$ for $i \in \{1, 2\}$. In the remainder of the paper, we prove this proposition and thereby finish the proof the main theorem. We prove each of the three assertions separately.

1. Bounds on $f(0)$. This one is straightforward. As $f_1(p) = 2dp$, we have $f_1(0) = 0$.

Further, recall that $\lambda_p = 1 - 1/\chi(p)$, and so $\lambda_0 = 0$ and $\widehat{G}_{\lambda_0}(k) = \widehat{G}_0(k) = 1$ for all $k \in (-\pi, \pi]^d$. Since both $p|\widehat{\tau}_p(k)|$ and $p|\widehat{\tau}_{p,k}(l)|$ equal 0 at $p = 0$, recalling the definitions of f_2 and f_3 in (4.1), we conclude that $f_2(0) = f_3(0) = 0$. In summary, $f(0) = 0$.

2. Continuity of f. The continuity of f_1 is obvious. For the continuity of f_2 and f_3, we proceed as in [18], that is, we prove continuity on $[0, p_c - \varepsilon]$ for every $0 < \varepsilon < p_c$. This again is done by taking derivatives and bounding them uniformly in k and in $p \in [0, p_c - \varepsilon]$. To this end, we calculate

$$\frac{\mathrm{d}}{\mathrm{d}p} \frac{\widehat{\tau}_p(k)}{\widehat{G}_{\lambda_p}(k)} = \frac{1}{\widehat{G}_{\lambda_p}(k)^2} \Big[\widehat{G}_{\lambda_p}(k) \frac{\mathrm{d}\widehat{\tau}_p(k)}{\mathrm{d}p} - \widehat{\tau}_p(k) \frac{\mathrm{d}\widehat{G}_\lambda(k)}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_p} \frac{\mathrm{d}\lambda_p}{\mathrm{d}p} \Big]. \tag{4.21}$$

Since $\lambda_p = 1 - 1/\chi(p)$,

$$\frac{1}{2} \le \frac{1}{1 - \lambda_p \widehat{D}(k)} = \widehat{G}_{\lambda_p}(k) \le \widehat{G}_{\lambda_p}(0) = \chi(p) \le \chi(p_c - \varepsilon). \tag{4.22}$$

Further, since $\widehat{\tau}_p(0)$ is non-decreasing,

$$\widehat{\tau}_p(k) \le \widehat{\tau}_p(0) \le \widehat{\tau}_{p_c - \varepsilon}(0) = \frac{\chi(p_c - \varepsilon) - 1}{p_c - \varepsilon}. \tag{4.23}$$

We use Observation 2.3 to obtain

$$\Big| \frac{\mathrm{d}}{\mathrm{d}p} \widehat{\tau}_p(k) \Big| = \Big| \sum_{x \in \mathbb{Z}^d} \mathrm{e}^{\mathrm{i}k\cdot x} \frac{\mathrm{d}}{\mathrm{d}p} \tau_p(k) \Big| \le \sum_{x \in \mathbb{Z}^d} \frac{\mathrm{d}}{\mathrm{d}p} \tau_p(x) = \frac{\mathrm{d}}{\mathrm{d}p} \sum_{x \in \mathbb{Z}^d} \tau_p(x) \le \widehat{\tau}_p(0)^2,$$

and the same bound as in (4.23) applies. The interchange of sum and derivative is justified as both sums are absolutely summable. Note that $\mathrm{d}\widehat{G}_\lambda(k)/\mathrm{d}\lambda = \widehat{D}(k)\widehat{G}_\lambda(k)^2$, and this is bounded by $\chi(p_c - \varepsilon)^2$ for $\lambda = \lambda_p$ by (4.22). Finally, by Observation 2.3,

$$\frac{\mathrm{d}\lambda_p}{\mathrm{d}p} = \frac{\frac{\mathrm{d}}{\mathrm{d}p}\chi(p)}{\chi(p)^2} \le \widehat{\tau}_p(0),$$

which is bounded by (4.23) again. In conclusion, all terms in (4.21) are bounded uniformly in k and p, which proves the continuity of f_2. We can treat f_3 in the exact same manner, as it is composed of terms of the same type as the ones we just bounded.

3. The forbidden region $(3, 4]$. Note that we assume $f(p) \le 4$ in the following, and so the second part of Proposition 4.2 applies with $c = c_4$.

Improvement of f_1. Recalling the definition of $\lambda_p \in [0, 1]$, this implies

$$f_1(p) = \lambda_p - p\widehat{\Pi}_p(0) \le 1 + c_4/d.$$

Improvement of f_2. We introduce $a = p(J + \Pi_p)$, and moreover

$$\widehat{N}(k) = \frac{\widehat{a}(k)}{1 + p\widehat{\Pi}_p(0)}, \qquad \widehat{F}(k) = \frac{1 - \widehat{a}(k)}{1 + p\widehat{\Pi}_p(0)}.$$

By adapting the analogous argument from [19, proof of Theorem 1.1], we can show that $1 - \widehat{a}(k) > 0$. Therefore, under the assumption that $f(p) \le 4$, we have $p\widehat{\tau}_p(k) = \widehat{N}(k)/\widehat{F}(k) = \widehat{a}(k)/(1 - \widehat{a}(k))$, and furthermore $\lambda_p = \widehat{a}(0)$. In the following lines, M and M' denote constants (typically multiples of c_4) whose value may change from line to line. An important observation is that

$$\frac{1}{1 + p\widehat{\Pi}_p(0)} \le 1 + M/d, \qquad |\widehat{a}(k)| \le 1 + M/d, \qquad |p\widehat{\Pi}_p(k)| \le M/d.$$

We are now ready to treat f_2. Since $|\widehat{N}(k)| \le 1 + M/d$ and by the triangle inequality,

$$\left|\frac{p\widehat{\tau}_p(k)}{\widehat{G}_{\lambda_p}(k)}\right| = \left|\widehat{N}(k) + p\widehat{\tau}_p(k)[1 - \lambda_p\widehat{D}(k) - \widehat{F}(k)]\right|$$
$$\le 1 + M/d + \left|p\widehat{\tau}_p(k)\right|\left|1 - \lambda_p\widehat{D}(k) - \widehat{F}(k)\right|. \tag{4.24}$$

Also,

$$\left|1 - \lambda_p\widehat{D}(k) - \widehat{F}(k)\right| = \left|\frac{1 + p\widehat{\Pi}_p(0) - \big(2dp + p\widehat{\Pi}_p(0)\big)\big(1 + p\widehat{\Pi}_p(0)\big)\widehat{D}(k) - 1 + 2dp\widehat{D}(k) + p\widehat{\Pi}_p(k)}{1 + \widehat{\Pi}_p(0)}\right|$$
$$= \left|\frac{[1 - \widehat{D}(k)]p\widehat{\Pi}_p(0)}{1 + p\widehat{\Pi}_p(0)}\right| + \left|\frac{\widehat{a}(0)p\widehat{\Pi}_p(0)\widehat{D}(k) + p\widehat{\Pi}_p(k)}{1 + p\widehat{\Pi}_p(0)}\right|. \tag{4.25}$$

The first term in the right-hand side of (4.25) is bounded by $[1 - \widehat{D}(k)]M/d$. In the second term, recalling that $\widehat{a}(0) = \lambda_p$, we add and subtract $p\widehat{\Pi}_p(0)$ in the numerator, and so

$$\left|1 - \lambda_p\widehat{D}(k) - \widehat{F}(k)\right| \le [1 - \widehat{D}(k)]M/d + \left|\frac{[1 - \lambda_p\widehat{D}(k)]p\widehat{\Pi}_p(0) + p\big(\widehat{\Pi}_p(k) - \widehat{\Pi}_p(0)\big)}{1 + p\widehat{\Pi}_p(0)}\right|$$
$$\le [1 - \widehat{D}(k)]M'/d + [1 - \lambda_p\widehat{D}(k)]M'/d \le 3[1 - \lambda_p\widehat{D}(k)]M'/d$$

for some constant M'. In the second to last bound, we have used that $p\widehat{\Pi}_p(0) - p\widehat{\Pi}_p(k) \le [1 - \widehat{D}(k)]M/d$. For the last, we have used that $1 - \widehat{D}(k) \le 2(1 - \lambda_p\widehat{D}(k)) = 2\widehat{G}_{\lambda_p}(k)^{-1}$. Putting this back into (4.24), we obtain

$$\left|\frac{p\widehat{\tau}_p(k)}{\widehat{G}_{\lambda_p}(k)}\right| \le 1 + M/d + 3|p\widehat{\tau}_p(k)/\widehat{G}_{\lambda_p}(k)|M'/d \le 1 + 4(M \vee M')/d.$$

This concludes the improvement on f_2. Before dealing with f_3, we make an important observation:

Observation 4.13. *Given the improved bounds on f_1 and f_2,*

$$\sup_{k \in (-\pi,\pi]^d} \left|\frac{1 - \lambda_p\widehat{D}(k)}{1 - \widehat{a}(k)}\right| \le 3.$$

Proof. Consider first those p such that $2dp \le 3/7$. Then

$$\left|\frac{1 - \lambda_p\widehat{D}(k)}{1 - \widehat{a}(k)}\right| = \left|\frac{1 - 2dp\widehat{D}(k) - p\widehat{\Pi}_p(0)\widehat{D}(k)}{1 - 2dp\widehat{D}(k) - p\widehat{\Pi}_p(k)}\right| \le \frac{\frac{10}{7} + M/d}{\frac{4}{7} + M/d} \le (1 + M'/d)\tfrac{5}{2} \le 3$$

for d sufficiently large. Next, consider those $k \in (-\pi, \pi]^d$ such that $|\widehat{D}(k)| \le 7/8$. Then

$$\left|\frac{1 - \lambda_p\widehat{D}(k)}{1 - \widehat{a}(k)}\right| = \left|1 - \frac{p\widehat{\Pi}_p(0)\widehat{D}(k) - p\widehat{\Pi}_p(k)}{1 - \widehat{a}(k)}\right| \le 1 + \frac{2M/d}{1 - (1 + M/d)\frac{7}{8} - M/d} \le 1 + 16M'/d$$

for d sufficiently large. Let now p such that $2dp > 3/7$ and k such that $|\widehat{D}(k)| > 7/8$. We write $1 - \lambda_p\widehat{D}(k) = \widehat{G}_{\lambda_p}(k)^{-1}$ and $1 - \widehat{a}(k) = \widehat{\tau}_p(k)/\widehat{a}(k)$. Since $2dp|\widehat{D}(k)| \ge 3/8$, we obtain

$$\left|\frac{1 - \lambda_p\widehat{D}(k)}{1 - \widehat{a}(k)}\right| \le \frac{8}{3}\left|\frac{\widehat{\tau}_p(k)}{\widehat{G}_{\lambda_p}(k)} \cdot \frac{2dp\widehat{D}(k)}{\widehat{a}(k)}\right| \le \frac{8}{3}\big(1 + M/d\big)\left|1 - \frac{p\widehat{\Pi}_p(k)}{\widehat{a}(k)}\right|$$
$$\le \frac{8}{3}\big(1 + M/d\big)\Big(1 + \frac{M/d}{\frac{8}{3} - M/d}\Big) \le 3$$

for d sufficiently large. □

Improvement of f_3. Elementary calculations give

$$\Delta_k p\widehat{\tau}_p(l) = \underbrace{\frac{\Delta_k \widehat{a}(l)}{1-\widehat{a}(l)}}_{\text{(I)}} + \sum_{\sigma\in\pm 1} \underbrace{\frac{\big(\widehat{a}(l+\sigma k)-\widehat{a}(l)\big)^2}{(1-\widehat{a}(l))(1-\widehat{a}(l+\sigma k))}}_{\text{(II)}} + \underbrace{\widehat{a}(l)\Delta_k\Big(\frac{1}{1-\widehat{a}(l)}\Big)}_{\text{(III)}}.$$

We bound each of the three terms (I)-(III) separately. For the first term,

$$\begin{aligned}|(\mathrm{I})| &= \big|\Delta_k \widehat{a}(l)\big| \cdot \left|\frac{1-\lambda_p\widehat{D}(l)}{1-\widehat{a}(l)}\right| \cdot \widehat{G}_{\lambda_p}(l) \le 3\widehat{G}_{\lambda_p}(l)\Big|2dp\Delta_k\widehat{D}(l) + \Delta_k p\widehat{\Pi}_p(l)\Big| \\ &\le (3+M/d)[1-\widehat{D}(k)]\widehat{G}_{\lambda_p}(l)\widehat{G}_{\lambda_p}(l+k).\end{aligned}$$

In the above, we have first used Observation 4.13, then Observation 4.11, and finally the fact that $\widehat{G}_{\lambda_p}(l+k) \ge 1/2$. Note that if we obtained similar bounds on (II) and (III), we could prove a bound of the form $|\Delta_k \widehat{\tau}_p(l)| \le c\widehat{U}_{\lambda_p}(k,l)$ for the right constant c and the improvement of f_3 would be complete.

To deal with (II), we need a bound on $\widehat{a}(l+\sigma k) - \widehat{a}(l)$ for $\sigma \in \{\pm 1\}$. As in [18, (8.4.19)-(8.4.21)],

$$\begin{aligned}|\widehat{D}(l\pm k) - \widehat{D}(l)| &\le \sum_x \Big(|\sin(k\cdot x)|D(x) + [1-\cos(k\cdot x)]D(x)\Big) = 1-\widehat{D}(k) + \sum_x |\sin(k\cdot x)|D(x) \\ &\le 1-\widehat{D}(k) + \Big(\sum_x D(x)\Big)^{1/2}\Big(\sum \sin(k\cdot x)^2 D(x)\Big)^{1/2} \\ &\le 1-\widehat{D}(k) + 2\Big(\sum [1-\cos(k\cdot x)]D(x)\Big)^{1/2} \\ &\le 4[1-\widehat{D}(k)]^{1/2},\end{aligned}$$

and similarly

$$\begin{aligned}p|\widehat{\Pi}_p(l\pm k) - \widehat{\Pi}_p(l)| &\le \Big(p\sum_x |\Pi_p(x)|\Big)^{1/2}\Big(2p\sum_x [1-\cos(k\cdot x)]|\Pi_p(x)|\Big)^{1/2} + p\sum_x [1-\cos(k\cdot x)]|\Pi_p(x)| \\ &\le M[1-\widehat{D}(k)]^{1/2}/d.\end{aligned}$$

Putting this together yields

$$|\widehat{a}(l\pm k) - \widehat{a}(l)| \le 2dp|\widehat{D}(l\pm k) - \widehat{D}(l)| + p|\widehat{\Pi}_p(l\pm k) - \widehat{\Pi}_p(l)| \le (16+M/d)[1-\widehat{D}(k)]^{1/2}. \tag{4.26}$$

Combining (4.26) with Observation 4.13 yields

$$\begin{aligned}(\mathrm{II}) &\le (16+M/d)^2[1-\widehat{D}(k)]\left|\frac{1-\lambda_p\widehat{D}(l)}{1-\widehat{a}(l)}\right|\left|\frac{1-\lambda_p\widehat{D}(l+\sigma k)}{1-\widehat{a}(l+\sigma k)}\right|\widehat{G}_{\lambda_p}(l)\widehat{G}_{\lambda_p}(l+\sigma k) \\ &\le (144+M'/d)[1-\widehat{D}(k)]\widehat{G}_{\lambda_p}(l)\widehat{G}_{\lambda_p}(l+\sigma k).\end{aligned}$$

To bound (III), we want to use Lemma 4.1. We first provide bounds for the three types of quantities arising in the use of the lemma. First, note that $|\widehat{a}(l)| \le 4+M/d$. Next, we observe

$$\widehat{|a|}(0) - \widehat{|a|}(k) = \sum_x [1-\cos(k\cdot x)]\big|2dpD(x) + p\Pi_p(x)\big| \le (4+M/d)[1-\widehat{D}(k)].$$

The third ingredient we need is Observation 4.13, which produces $|1-\widehat{a}(l)|^{-1} \le 3\widehat{G}_{\lambda_p}(l)$. Putting all this together and applying Lemma 4.1 gives

$$\begin{aligned}\Delta_k\Big(\frac{1}{1-\widehat{a}(l)}\Big) &\le (4+M/d)[1-\widehat{D}(k)]\bigg(9\big[\widehat{G}_{\lambda_p}(l-k) + \widehat{G}_{\lambda_p}(l+k)\big]\widehat{G}_{\lambda_p}(l) \\ &\qquad + 216(4+M/d)\widehat{G}_{\lambda_p}(l-k)\widehat{G}_{\lambda_p}(l+k)\widehat{G}_{\lambda_p}(l)[1-\widehat{D}(l)]\bigg) \\ &\le (6912+M'/d)[1-\widehat{D}(k)]\Big(\widehat{G}_{\lambda_p}(l-k) + \widehat{G}_{\lambda_p}(l+k)]\widehat{G}_{\lambda_p}(l) + \widehat{G}_{\lambda_p}(l-k)\widehat{G}_{\lambda_p}(l+k)\Big),\end{aligned}$$

noting that $\widehat{G}_{\lambda_p}(l)[1-\widehat{D}(l)] \le 2$. In summary, $(\mathrm{I})+(\mathrm{II})+(\mathrm{III}) \le 3\widehat{U}_{\lambda_p}(k,l)$, which finishes the improvement on f_3. This finishes the proof of Proposition 4.12, and therewith also the proof of Theorem 1.1.

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Chapter 4

Expansion for the critical point of site percolation: the first three terms

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My own contribution. This article results from numerous discussions with my supervisor, Markus Heydenreich, and is thus a collaborative effort. This actual writing was mostly done by myself.

Expansion for the critical point of site percolation: the first three terms

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Abstract

We expand the critical point for site percolation on the d-dimensional hypercubic lattice in terms of inverse powers of $2d$, and we obtain the first three terms rigorously. This is achieved using the lace expansion.

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1 Introduction

We study site percolation on the hypercubic lattice $\mathbb{Z}^d$. To this end, we fix a parameter $p \in [0,1]$ and create a random subgraph of $\mathbb{Z}^d$ as follows. Each site (or vertex) $x \in \mathbb{Z}^d$, independently of all other sites, is declared *occupied* with probability p (and *vacant* otherwise). A bond (edge) between two nearest-neighbor sites in $\mathbb{Z}^d$ is an edge of the random subgraph if and only if the two sites are occupied. Denote by $\theta(p)$ the probability that there is a path starting at the origin $\mathbf{0} \in \mathbb{Z}^d$ and diverging to infinity that consists only of occupied vertices. This allows us to define the *critical point* as

$$p_c := \inf\big\{p \in [0,1] : \theta(p) > 0\big\}. \tag{1.1}$$

It is standard that $0 < p_c < 1$ in all dimensions $d \geq 2$. In general, it is not possible to write down an explicit value for $p_c = p_c(d)$ (see Table 1 for numerical values), a notable exception is site percolation on the two-dimensional triangular lattice (when $p_c = 1/2$). However, it is possible to derive an asymptotic expansion for $p_c(d)$ when $d \to \infty$. Indeed, it is known in the physics literature that

$$p_c = \sigma^{-1} + \frac{3}{2}\sigma^{-2} + \frac{15}{4}\sigma^{-3} + \frac{83}{4}\sigma^{-4} + \frac{6577}{48}\sigma^{-5} + \frac{119077}{96}\sigma^{-6} + \cdots \quad \text{for } \sigma = 2d-1 \to \infty. \tag{1.2}$$

The first four terms were found by Gaunt, Ruskin, and Sykes in 1976 [5] through exact enumeration, the final term has been obtained by Mertens and Moore [16] by exploiting involved numerical methods. When writing this in powers of $\frac{1}{2d}$, (1.2) becomes

$$p_c(d) = (2d)^{-1} + \frac{5}{2}(2d)^{-2} + \frac{31}{4}(2d)^{-3} + \frac{75}{2}(2d)^{-4} + \frac{11977}{48}(2d)^{-5} + \frac{209183}{96}(2d)^{-6} + \cdots. \tag{1.3}$$

In this paper, we extent the previously known first term by establishing the second and third term, including a rigorous bound on the error term.

Theorem 1.1 (Expansion of p_c in terms of $(2d)^{-1}$). *As $d \to \infty$,*

$$p_c(d) = (2d)^{-1} + \frac{5}{2}(2d)^{-2} + \frac{31}{4}(2d)^{-3} + \mathcal{O}\left((2d)^{-4}\right).$$

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The key technical tool for our approach is the lace expansion for site percolation. It was established in a recent paper [13], which itself draws its inspiration from Hara and Slade's seminal paper [11]. The lace expansion provides an expression for p_c in terms of *lace-expansion coefficients*, which are defined in Definition 2.5. Moreover, it provides good control over these coefficients, and the results of [13] identify already the leading order term in (1.3).

Comparison with bond percolation. It is most instructive to compare the critical thresholds for site and bond percolation. While the critical behaviour of bond- and site percolation is comparable, the actual values of the critical thresholds differ, as illustrated by the following table:

dim	2	3	4	5	6	7	8	9	10	11	12
p_c^{site}	0.5927	0.3116	0.1969	0.1408	0.1090	0.0890	0.0752	0.0652	0.0576	0.0516	0.0467
p_c^{bond}	0.5*	0.2488	0.1601	0.1182	0.0942	0.0786	0.0677	0.0595	0.0531	0.0479	0.0437

Table 1: Critical values for percolation on $\mathbb{Z}^d$, rounded to multiples of 10^{-4}. The only rigorously obtained value is for bond percolation in dimension 2 (marked with *). All other values are obtained through numerical simulation; the values for $d \geq 4$ are reported in Grassberger [8] and Mertens and Moore [16].

Grimmett and Stacey [10] prove that $p_c^{\text{site}} > p_c^{\text{bond}}$ on $\mathbb{Z}^d$ for all dimensions $d \geq 2$. This difference must be reflected in the asymptotic expansion for p_c. Indeed, Hara and Slade [12] and van der Hofstad and Slade [15] rigorously obtain a series expansion for *bond* percolation as

$$p_c^{\text{bond}}(d) = (2d)^{-1} + (2d)^{-2} + \frac{7}{2}(2d)^{-3} + \mathcal{O}\left((2d)^{-4}\right), \tag{1.4}$$

which indeed differs from the expansion of p_c^{site} in Theorem 1.1. Again, more precise estimates are known by non-rigorous methods [4, 16]:

$$p_c^{\text{bond}} = \sigma^{-1} + \frac{5}{2}\sigma^{-3} + \frac{15}{2}\sigma^{-4} + 57\sigma^{-5} + \frac{4855}{12}\sigma^{-6} + \cdots \tag{1.5}$$

for $\sigma = 2d - 1$, which is equivalent to

$$p_c^{\text{bond}}(d) = (2d)^{-1} + (2d)^{-2} + \frac{7}{2}(2d)^{-3} + 16(2d)^{-4} + 103(2d)^{-5} + \frac{9487}{12}(2d)^{-6} + \cdots.$$

We remark that (1.4) was proved in [15] also for the d-dimensional cube. More recently, an asymptotic expansion was also proven for the Hamming graph [3].

Borel summability of the coefficients. It appears that the methods devised in this paper allow to obtain an expansion as in Theorem 1.1 to all orders. Writing $s = \frac{1}{2d}$ and $\bar{p}_c(s) = p_c(d)$, this means that there is a real sequence $(\alpha_n)_{n\in\mathbb{N}}$ such that for any $M \in \mathbb{N}$,

$$\bar{p}_c(s) = \sum_{n=1}^{M-1} \alpha_n \, s^n + \mathcal{O}(s^M).$$

This was proved for bond percolation by Hofstad and Slade [14] (additionally, it was proved that the coefficients α_n are rational). However, it is expected that the radius of convergence for this series expansion is zero (even though rigorous evidence is lacking), and this non-convergence is valid in greater generality for series expansions of critical thresholds of various statistical mechanical models. The reason is that the sequence of absolute values $|\alpha_1|, |\alpha_2|, |\alpha_3|, \dots$ grows very rapidly, cf. (1.3), and therefore it is not possible to compute $\bar{p}_c(s)$ from the sequence (α_n).

Instead, we believe that the coefficients are *Borel summable*. Suppose $\bar{p}_c(s)$ has an analytic extension to the complex disc $C = \{z \in \mathbb{C} \colon \mathrm{Re}(z^{-1}) > 1\}$, and suppose there is $L > 0$ such that for all $s \in C$ and all M, we have

$$\left| \bar{p}_c(s) - \sum_{n=1}^{M-1} \alpha_n \, s^n \right| \leq L^M \, M! \, |s|^M, \tag{1.6}$$

then Sokal [17] proves that the *Borel transform* $B(t) = \sum_{n=1}^{\infty} \alpha_n t^n/n!$ exists, and $\bar{p}_c(s)$ equals the *Borel sum*

$$\bar{p}_c(s) = \frac{1}{s}\int_0^{\infty} e^{-t/s} B(t)\, dt. \tag{1.7}$$

It is, however, unclear how an analytic extension of $\bar{p}_c(s)$ could be obtained.

A rare example for which we know Borel summability is the exact solution $K_c(d)$ of the spherical model. Gerber and Fisher [6] prove that there is an expansion of $K_c(d)$ in powers of $1/d$, that the radius of convergence is zero, but that we may interpret the expansion as a Borel sum as described above. They also prove that the signs of the coefficients of K_n oscillate: the first 12 terms are positive, the next 8 are negative, the next 9 are positive, and so on. For the well-known model of self-avoiding walk, Graham [7] proves bounds for the connective constant as in (1.6).

1.1 Strategy of proof, outline of the paper

Theorem 1.1 heavily builds upon the results obtained in [13]. We use Section 2 to collect the necessary notation and results from [13] in order to prove our main result. At the heart of these results is an identity for τ_p. From this, we almost immediately get an identity for p_c in terms of so-called *lace-expansion coefficients* (see Definition 2.5). It will be clear that sufficient control over the coefficients will result in the expansion of Theorem 1.1. In fact, the results from [13] immediately give the first term of (1.3).

For the other terms in Theorem 1.1, however, we require even better control of these coefficients, which is provided by Lemma 3.1. Section 3 proves Theorem 1.1 assuming Lemma 3.1. The latter is at the heart of this paper and is proved in Section 5. As a preparation for the proof, Section 4 introduces some new notation on connection events and proves bounds on them. Those bounds are in essence an extension of some of the bounds presented in Section 2.

2 Preliminaries

2.1 Site percolation: Model and basic definitions

We introduce the model more formally. Given $p \in [0,1]$, we can choose our probability space to be $(\{0,1\}^{\mathbb{Z}^d}, \mathcal{F}, \mathbb{P}_p)$, where the σ-algebra $\mathcal{F}$ is generated by the cylinder sets, and $\mathbb{P}_p = \bigotimes_{x\in\mathbb{Z}^d} \mathrm{Ber}(p)$. We call $\omega \in \{0,1\}^{\mathbb{Z}^d}$ a configuration and say that a site $x \in \mathbb{Z}^d$ is *open* or *occupied* in ω if $\omega(x) = 1$. If $\omega(x) = 0$, we say that the site x is *closed* or *vacant*. We often identify ω with the set $\{x \in \mathbb{Z}^d : \omega(x) = 1\}$.

For two points $x \neq y \in \mathbb{Z}^d$ and a configuration ω, we write $x \longleftrightarrow y$ (and say that x is *connected* to y) if there are points $x = v_0, \dots, v_k = y$ in $\mathbb{Z}^d$ with $k \in \mathbb{N}_0$ such that $|v_i - v_{i-1}| = 1$ for all $1 \le i \le k$, and $v_i \in \omega$ for $1 \le i \le k-1$. Here, and throughout the paper, we write $|x| = \sum_{i=1^d} |x_i|$ for $x \in \mathbb{R}^d$ (which is equal to the graph distance in $\mathbb{Z}^d$). We set $\{x \longleftrightarrow x\} = \varnothing$, that is, x is *not* connected to itself. Moreover, $|x-y| = 1$ implies $\{x \longleftrightarrow y\} = \{0,1\}^{\mathbb{Z}^d}$ (neighbors are always connected).

We define the *cluster* of x to be $\mathscr{C}(x) = \{x\} \cup \{y \in \omega : x \longleftrightarrow y\}$. Note that apart form x itself, points in $\mathscr{C}(x)$ need to be occupied.

The *two-point function* $\tau_p \colon \mathbb{Z}^d \to [0,1]$ is defined as $\tau_p(x) := \mathbb{P}_p(\mathbf{0} \longleftrightarrow x)$, where $\mathbf{0}$ denotes the origin in $\mathbb{Z}^d$. The *percolation probability* is defined as $\theta(p) = \mathbb{P}_p(\mathbf{0} \longleftrightarrow \infty) = \mathbb{P}_p(|\mathscr{C}(\mathbf{0})| = \infty)$. We note that $p \mapsto \theta(p)$ is increasing and define the *critical point* for θ as in (1.1). The critical point p_c depends on the underlying graph.

For an absolutely summable function $f \colon \mathbb{Z}^d \to \mathbb{R}$, the discrete Fourier transform is defined as $\widehat{f} \colon (-\pi,\pi]^d \to \mathbb{C}$, where

$$\widehat{f}(k) = \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}k\cdot x} f(x)$$

and $k \cdot x = \sum_{j=1}^d k_j x_j$ denotes the scalar product.

2.2 The lace expansion in high dimension

We use this section to state the definitions and results from [13] needed in the proof of Theorem 1.1. We note that the below definition uses the notion of disjoint occurrence (denoted '$\circ$') related to the BK inequality (which we will use at a later stage as well). For details on both, see e.g. [2, Chapter 2] or [9, Section 2.3].

Definition 2.1 (Connection events, modified clusters)**.** Let $x, u \in \mathbb{Z}^d$ and $A \subseteq \mathbb{Z}^d$.

1. We set $\Omega := 2d$.
2. We define $J(x) := \mathbb{1}_{\{|x|=1\}} = \mathbb{1}_{\{0\sim x\}}$ and $D := J/\Omega$.
3. Let $\{u \longleftrightarrow x \text{ in } A\}$ be the event that there is a path from u to x, all of whose internal vertices are elements of $\omega \cap A$.
4. We define $\{u \Longleftrightarrow x\} := \{u \longleftrightarrow x\} \circ \{u \longleftrightarrow x\}$ and say that u and x are *doubly connected.*
5. We define the modified cluster of x with a designated vertex u as
$$\widetilde{\mathscr{C}}^u(x) := \{x\} \cup \{y \in \omega \setminus \{u\} : x \longleftrightarrow y \text{ in } \mathbb{Z}^d \setminus \{u\}\}.$$
6. Let $\langle A \rangle := A \cup \{y \in \mathbb{Z}^d : \exists x \in A : |x-y| = 1\}$.

Note that we introduce $\Omega = 2d$. For better readability, we stick to using Ω for the remainder of the paper. We also address the Landau notation $f(\Omega) \leq \mathcal{O}(g(\Omega))$ that will appear frequently throughout the paper. It is always to be understood in the sense that there exists some d_0 and a constant $C(d_0)$, such that $f(\Omega) \leq Cg(\Omega)$ for all $\Omega \geq d_0$. The constant C may depend on other appearing parameters.

We remark that $\{x \longleftrightarrow y \text{ in } \mathbb{Z}^d\} = \{x \longleftrightarrow y\} = \{x \longleftrightarrow y \text{ in } \omega\}$ and that $\{u \Longleftrightarrow x\} = \{0,1\}^{\mathbb{Z}^d}$ for $|u-x| = 1$. Similarly, $\{u \Longleftrightarrow x\} = \varnothing$ for $u = x$.

We state two elementary observations made in [13] involving J that will be important later on.

Observation 2.2 (Convolutions of J, [13, Observation 4.4])**.** *Let $m \in \mathbb{N}$ and $x \in \mathbb{Z}^d$ with $m \geq |x|$. Then there is a constant $c = c(m,x)$ with $c \leq m!$ such that*
$$J^{*m}(x) = c(m)\mathbb{1}_{\{m-|x| \text{ is even}\}}\Omega^{(m-|x|)/2}.$$

Observation 2.3 (Elementary bound on τ_p^{*n}, [13, Observation 4.5])**.** *Let $m, n \in \mathbb{N}, p \in [0,1]$ and $x \in \mathbb{Z}^d$. Then there is a constant $c = c(m,n)$ such that*
$$\tau_p^{*n}(x) \leq c \sum_{l=0}^{m-1} p^l J^{*l+n}(x) + c \sum_{j=1}^{n} p^{m+j-n} (J^{*m} * \tau_p^{*j})(x).$$

The following, more specific definitions are important to define the lace-expansion coefficients:

Definition 2.4 (Extended connection events)**.** Let $v, u, x \in \mathbb{Z}^d$ and $A \subseteq \mathbb{Z}^d$.

1. Define
$$\{u \overset{A}{\longleftrightarrow} x\} := \{u \longleftrightarrow x\} \cap \Big(\{u \not\longleftrightarrow x \text{ in } \mathbb{Z}^d \setminus \langle A \rangle\} \cup \{x \in \langle A \rangle\}\Big).$$
In words, this is the event that u is connected to x, but either any path from u to x has an interior vertex in $\langle A \rangle$, or x itself lies in $\langle A \rangle$.
2. We introduce $\mathsf{Piv}(u,x)$ as the set of pivotal points for $\{u \longleftrightarrow x\}$. That is, $v \in \mathsf{Piv}(u,x)$ if the event $\{u \longleftrightarrow x \text{ in } \omega \cup \{v\}\}$ holds but $\{u \longleftrightarrow x \text{ in } \omega \setminus \{v\}\}$ does not.
3. Define the event
$$E'(v,u;A) := \{v \overset{A}{\longleftrightarrow} u\} \cap \{\nexists u' \in \mathsf{Piv}(v,u) : v \overset{A}{\longleftrightarrow} u'\}$$

We remark that $\{u \overset{\mathbb{Z}^d}{\longleftrightarrow} x\} = \{u \longleftrightarrow x\}$. We can now define the lace-expansion coefficients. To this end, let $(\omega_i)_{i \in \mathbb{N}_0}$ be a sequence of independent site percolation configurations. For an event E taking place on ω_i, we highlight this by writing E_i. We also stress the dependence of random variables on the particular configuration they depend on. For example, we write $\mathscr{C}(u;\omega_i)$ to denote the cluster of u in configuration i.

Definition 2.5 (Lace-expansion coefficients)**.** Let $n \in \mathbb{N}_0, x \in \mathbb{Z}^d$, and $p \in [0, p_c]$. We define

$$\Pi_p^{(0)}(x) := \mathbb{P}_p(\mathbf{0} \Longleftrightarrow x) - J(x),$$
$$\Pi_p^{(n)}(x) := p^n \sum_{u_0,\dots,u_{n-1}} \mathbb{P}_p\Big(\{\mathbf{0} \Longleftrightarrow u_0\}_0 \cap \bigcap_{i=1}^n E'(u_{i-1}, u_i; \mathscr{C}_{i-1})_i\Big),$$

where $u_{-1} = \mathbf{0}, u_n = x$ and $\mathscr{C}_i = \widetilde{\mathscr{C}}^{u_i}(u_{i-1}; \omega_i)$. Let furthermore $\Pi_p(x) := \sum_{n=0}^\infty (-1)^n \Pi_p^{(n)}(x)$.

It is proved in [13] that the functions $(\Pi_p^{(n)}(x))_{n\in\mathbb{N}_0}$ are (absolutely) summable for every x and that Π_p is thus well defined. We remark that $E'(u_{i-1}, u_i; \mathscr{C}_{i-1})_i$ takes place solely on ω_i only if $\mathscr{C}_{i-1}$ is regarded as a fixed set; otherwise it takes place on ω_{i-1} as well as ω_i. Proposition 2.6 summarizes the main results of [13] (namely, Theorem 1.1 and Proposition 4.2).

Proposition 2.6 (OZE, infra-red bound and bounds on the lace-expansion coefficients)**.** *Let $p \in [0, p_c]$. Then there is $d_0 \geq 6$ such that, for all $d > d_0$, τ_p satisfies the Ornstein-Zernike equation*

$$\tau_p(x) = J(x) + \Pi_p(x) + p\big((J + \Pi_p) * \tau_p\big)(x). \tag{2.1}$$

Secondly, there is a constant $C = C(d_0)$ such that

$$p|\widehat{\tau}_p(k)| \leq \frac{|\widehat{D}(k)| + C/d}{1 - \widehat{D}(k)}, \tag{2.2}$$

where we take the right-hand side to be ∞ for $k = 0$. Thirdly, $2dp \leq 1 + C/d$, and lastly, for $n \in \mathbb{N}_0$,

$$p \sum_{x\in\mathbb{Z}^d} \Pi_p^{(n)}(x) \leq C(C/d)^{n\vee 1}. \tag{2.3}$$

As a consequence, we also have $p\sum_x \Pi_p(x) \leq C/d$.

2.3 Diagrammatic bounds

In the proofs to follow, we need another result from [13]. We formulate it in terms of a diagrammatic notation, as we are going to make use of this later as well. To this end, we introduce some quantities related to τ_p.

Definition 2.7 (Modified two-point functions and triangles)**.** Let $x \in \mathbb{Z}^d$ and define

$$\tau_p^\circ(x) := \delta_{\mathbf{0},x} + \tau_p(x), \qquad \tau_p^\bullet(x) = \delta_{\mathbf{0},x} + p\tau_p(x).$$

Moreover, let $\triangle_p^\bullet(x) = p(\tau_p^\bullet * \tau_p * \tau_p)(x), \triangle_p^{\bullet\circ}(x) = p(\tau_p^\bullet * \tau_p^\circ * \tau_p)(x)$, and $\triangle_p^{\bullet\bullet\circ}(x) = (\tau_p^\bullet * \tau_p^\bullet * \tau_p^\circ)(x)$. We also set

$$\triangle_p^\bullet = \sup_{\mathbf{0}\neq x\in\mathbb{Z}^d} \triangle_p^\bullet(x), \quad \triangle_p^{\bullet\circ} = \sup_{\mathbf{0}\neq x\in\mathbb{Z}^d} \triangle_p^{\bullet\circ}(x), \quad \triangle_p^{\bullet\bullet\circ} = \sup_{x\in\mathbb{Z}^d} \triangle_p^{\bullet\bullet\circ}(x).$$

We need the following bounds obtained in [13].

Proposition 2.8 (Triangle bounds, [13, Lemma 4.7])**.** *Let $p \in [0, p_c]$. Then there is $d_0 \geq 6$ and a constant $C = C(d_0)$ such that, for all $d > d_0$,*

$$\max\{\triangle_p, \triangle_p^\bullet, \triangle_p^{\bullet\circ}\} \leq C/d, \qquad \max\{\triangle_p^\bullet(\mathbf{0}), \triangle_p^{\bullet\circ}(\mathbf{0}), \triangle_p^{\bullet\bullet\circ}\} \leq C.$$

As part of the proof that bounds the functions $\Pi_p^{(i)}$ in [13], a first bound is formulated in terms of a long sum over products of the modified two-point functions. In a second step, those are decomposed into products of the modified triangles. We need a formulation of this intermediate bound on $\Pi_p^{(i)}$ for $i \in \{1, 2\}$ for Section 5, as well as a pictorial representation. We first state the needed bound on $\Pi_p^{(1)}$.

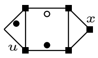
Lemma 2.9 (Diagrammatic bound on $\Pi_p^{(1)}$, [13, Lemma 3.10])**.** *Let $p \in [0, p_c]$. Then*

$$\sum_{x \in \mathbb{Z}^d} \Pi_p^{(1)}(x) \le \sum_{\substack{w,u,t,z,x \in \mathbb{Z}^d: \\ u \neq x, |\{t,z,x\}| \neq 2}} \tau_p^{\bullet}(w)\tau_p(u)\tau_p(w-u)\tau_p^{\circ}(z-w)\tau_p^{\bullet}(t-u)\tau_p^{\bullet}(z-t)\tau_p^{\bullet}(x-t)\tau_p^{\circ}(x-z). \quad (2.4)$$

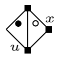
The bounds in [13] are formulated only for $p < p_c$, but as the bounds are increasing in p, a limit argument easily extends them to the critical point. We now show how we represent the bound in (2.4) in terms of pictorial diagrams. As the bound on $\Pi_p^{(2)}$ is even longer to write down, Lemma 2.10 is stated only in terms of these pictorial bounds.

The points w, u, t, z, x summed over are represented as squares, factors of τ_p are represented as lines, and lines with a '$\bullet$' ('$\circ$') symbol represent factors of $\tau_p^{\bullet}$ (τ_p°). For example, the factor $\tau_p(w-u)$ is represented as a line between two squares, which we think of as the points w and u. We interpret the factor $\tau_p(u)$ as a line between u and the origin. We indicate the position of u and x in the below diagrams. After expanding the two cases in (2.4) according to whether $|\{t,z,x\}| = 1$ or $|\{t,z,x\}| = 3$, this pictorial representation allows us to rewrite the bound in (2.4) as

$$\begin{aligned}\sum_{x \in \mathbb{Z}^d} \Pi_p^{(1)}(x) &\le p^2 \sum_{w,u,t,z,x \in \mathbb{Z}^d} \tau_p^{\bullet}(w)\tau_p(u)\tau_p(w-u)\tau_p^{\circ}(z-w)\tau_p^{\bullet}(t-u)\tau_p(z-t)\tau_p(x-t)\tau_p(x-z) \\ &\quad + p \sum_{w,u,x \in \mathbb{Z}^d} \tau_p^{\bullet}(w)\tau_p(u)\tau_p(w-u)\tau_p^{\circ}(x-w)\tau_p(x-u) \\ &\le p^2 \sum\end{aligned}$$



$+p \sum$



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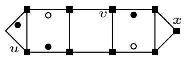
We now formulate the bound on $\Pi_p^{(2)}$; more precisely, we are going to insert a case distinguishing indicator, resulting in two bounds.

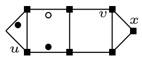
Lemma 2.10 (Diagrammatic bound on $\Pi_p^{(2)}$, [13, Lemma 3.10])**.** *Let $p \in [0, p_c]$. Then*

$$\sum_{u,v,x \in \mathbb{Z}^d} \mathbb{P}_p\Big(\{\mathbf{0} \Longleftrightarrow u\}_0 \cap E'(u,v;\mathscr{C}_0)_1 \cap E'(v,x;\mathscr{C}_1)_2 \cap \big(\{v \notin \langle \mathscr{C}_0 \rangle\}_0 \cup \{x \notin \langle \mathscr{C}_1 \rangle\}_1\big)\Big)$$

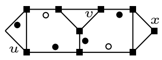
$\le p^5 \sum$



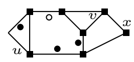
$+p^4 \sum$



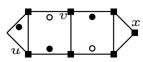
$+p^4 \sum$



$+ p^3 \sum$



$+p^3 \sum$

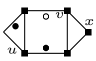


(2.5)

and

$$\sum_{u,v,x \in \mathbb{Z}^d} \mathbb{P}_p\Big(\{\mathbf{0} \Longleftrightarrow u\}_0 \cap E'(u,v;\mathscr{C}_0)_1 \cap E'(v,x;\mathscr{C}_1)_2 \cap \{v \in \langle \mathscr{C}_0 \rangle\}_0 \cap \{x \in \langle \mathscr{C}_1 \rangle\}_1\Big)$$

$\le p^2 \sum$



. (2.6)

2.4 Convolution bounds

The last result from [13] we need to state is going to be important for the proofs of Section 4.

Lemma 2.11 (Bounds on convolutions of J and τ_p, [13, Lemma 4.6])**.** *Let $m, n \in \mathbb{N}_0$ with $2m + n \ge 2$. Let moreover $p \in [0, p_c]$ and $d > 2n$. Then there is $d_0 \ge 6$ such that, for all $d > d_0$,*

$$\sup_{a \in \mathbb{Z}^d} p^{2m+n-1}\big(J^{*2m} * \tau_p^{*n}\big)(a) \le c\Omega^{1-m}$$

for some constant $c = c(m, n, d_0)$.

Again, Lemma 4.6 in [13] is stated only for $p < p_c$. A close look at the proof reveals that the bounds

$$2dp_c \le 1 + \mathcal{O}(\Omega^{-1}) \quad \text{and} \quad \sup_{k \in (-\pi,\pi]^d} \frac{p_c|\widehat{\tau}_{p_c}(k)|}{\widehat{G}_1(k)} \le 1 + \mathcal{O}(\Omega^{-1})$$

are sufficient for the statement to extend to p_c. While the former bound is part of Proposition 2.6, the latter follows from the infra-red bound (2.2) by observing that $|\widehat{D}(k)| \le 1$. The bound for $k = 0$ follows from the continuity of the Fourier transform.

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1 assuming Lemma 3.1, the latter providing an asymptotic expansion of the lace-expansion coefficients $\Pi^{(0)}, \Pi^{(1)}$, and $\Pi^{(2)}$ up to order $\mathcal{O}(\Omega^{-2})$.

Lemma 3.1 (Expansion of lace-expansion coefficients)**.** *As $d \to \infty$,*

$$\begin{aligned}
\widehat{\Pi}^{(0)}_{p_c}(0) &= \tfrac{1}{2}\Omega^2 p_c^2 + \tfrac{5}{2}\Omega^{-1} + \mathcal{O}(\Omega^{-2}), \\
\widehat{\Pi}^{(1)}_{p_c}(0) &= \Omega p_c + 2\Omega^2 p_c^2 + 4\Omega^{-1} + \mathcal{O}(\Omega^{-2}), \\
\widehat{\Pi}^{(2)}_{p_c}(0) &= 10\Omega^{-1} + \mathcal{O}(\Omega^{-2}).
\end{aligned}$$

Lemma 3.1 is the union of Lemmas 5.1, 5.2, 5.3, which are proved in Section 5. As a preparation for these proofs, we need Section 4. These proofs are lengthy considerations of numerous percolation configurations in search for contributions of the right order of magnitude (in terms of powers of Ω^{-1}). They are very mechanical in that they boil down to counting exercises and case distinctions. This also means that no new ideas are needed to extend Lemma 3.1 to higher orders of Ω^{-1} and expand the higher-order coefficients $\widehat{\Pi}^{(3)}, \widehat{\Pi}^{(4)}$, etc. The necessary effort increases exponentially however.

Proof of Theorem 1.1. Let first $p < p_c$. Taking the Fourier transform of (2.1) and solving for $\widehat{\tau}_p$ at $k = 0$ gives

$$p\widehat{\tau}_p(0) = \frac{p\Omega + p\widehat{\Pi}_p(0)}{1 - p(\Omega + \widehat{\Pi}_p(0))}. \tag{3.1}$$

A standard result is that $p\widehat{\tau}_p(0) = \mathbb{E}_p[|\mathscr{C}(\mathbf{0})|] - 1$ diverges as $p \nearrow p_c$, cf. [1]. As the enumerator of (3.1) is bounded by $1 + \mathcal{O}(\Omega^{-1})$, we conclude that p_c satisfies

$$1 - p_c(\Omega + \widehat{\Pi}_{p_c}(0)) = 0. \tag{3.2}$$

From here on out, we abbreviate $\widehat{\Pi} = \widehat{\Pi}_{p_c}(0)$ and $\widehat{\Pi}^{(m)} = \widehat{\Pi}^{(m)}_{p_c}(0)$. We know from Proposition 2.6 that $|\widehat{\Pi}/\Omega| = \mathcal{O}(\Omega^{-1})$, and so rearranging (3.2) yields

$$\Omega p_c = \frac{1}{1 + \widehat{\Pi}/\Omega} = 1 + \mathcal{O}(\Omega^{-1}). \tag{3.3}$$

Proposition 2.6 moreover provides the bound $|\widehat{\Pi}^{(m)}| = \mathcal{O}(\Omega^{1-(m \vee 1)})$ for all $m \geq 0$. We can use this to describe Ωp_c in more detail as

$$\begin{aligned}
\Omega p_c &= 1 - \frac{\widehat{\Pi}^{(0)}/\Omega - \widehat{\Pi}^{(1)}/\Omega + \widehat{\Pi}^{(2)}/\Omega + \sum_{m \geq 3} (-1)^m \widehat{\Pi}^{(m)}/\Omega}{1 + \widehat{\Pi}/\Omega} \\
&= 1 - \frac{\widehat{\Pi}^{(0)}/\Omega - \widehat{\Pi}^{(1)}/\Omega + \widehat{\Pi}^{(2)}/\Omega}{1 + \widehat{\Pi}/\Omega} + \mathcal{O}(\Omega^{-3}).
\end{aligned} \tag{3.4}$$

Simplifying (3.4) to an error term of order $\mathcal{O}(\Omega^{-2})$ gives

$$\Omega p_c = 1 - \widehat{\Pi}^{(0)}/\Omega + \widehat{\Pi}^{(1)}/\Omega + \mathcal{O}(\Omega^{-2}). \tag{3.5}$$

Plugging in the expansion for $\widehat{\Pi}^{(0)}$ and $\widehat{\Pi}^{(1)}$ from Lemma 3.1 gives $\Omega p_c = 1 + \frac{5}{2}\Omega^{-1} + \mathcal{O}(\Omega^{-2})$. Using this and the first identity of (3.3) in (3.4) gives

$$\Omega p_c = 1 - \big(\widehat{\Pi}^{(0)}/\Omega - \widehat{\Pi}^{(1)}/\Omega + \widehat{\Pi}^{(2)}/\Omega\big)\big(1 + \tfrac{5}{2}\Omega^{-1} + \mathcal{O}(\Omega^{-2})\big) + \mathcal{O}(\Omega^{-3}). \tag{3.6}$$

Applying Lemma 3.1 in (3.6) proves the theorem. □

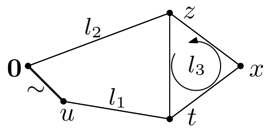


Figure 1: An illustration of the diagrammatic quantity $\Leftrightarrow^{(l)}$. The '$\sim$' symbol on the line between $\mathbf{0}$ and u means that $|u| = 1$.

4 Further bounds on connection events

This section extracts some results that are frequently used in the proofs of Section 5. We start by defining l-step connections.

Definition 4.1 (l-step connections). Let $l \in \mathbb{N}$ and $p \le p_c$.

1. We define $\{u \overset{(l)}{\longleftrightarrow} v\}$ as the event that u is connected to v via a path that contains at least l edges, and let $\tau_p^{(l)} = \mathbb{P}_p(u \overset{(l)}{\longleftrightarrow} v)$.

   We define $\{u \overset{(\ge l)}{\longleftrightarrow} v\}$ as the event that u is connected to v and the shortest path between u and v is of length at least l. Furthermore, let $\{u \overset{(\le l)}{\longleftrightarrow} v\}$ be the event that u and v are connected by a path of length at most l. Lastly, set $\{u \overset{(=l)}{\longleftrightarrow} v\} := \{u \overset{(\le l)}{\longleftrightarrow} v\} \cap \{u \overset{(\ge l)}{\longleftrightarrow} v\}$.

2. We define $\{u \overset{(l)}{\Longleftrightarrow} v\} := \cup_{j=1}^{l-1} \{u \overset{(j)}{\longleftrightarrow} v\} \circ \{u \overset{(l-j)}{\longleftrightarrow} v\}$ as the event that u and v lie in a cycle of length at least l, where all sites—except possibly u and v—are occupied.

   Let $\{u \overset{(\ge l)}{\Longleftrightarrow} v\}$ be the event that $\{u \Longleftrightarrow v\}$ and the shortest cycle containing u and v (with all other vertices occupied) is of length at least l. Similarly, let $\{u \overset{(\le l)}{\Longleftrightarrow} v\}$ be the event that $\{u \Longleftrightarrow v\}$ and the shortest cycle containing u and v is of length at most l, and let $\{u \overset{(=l)}{\Longleftrightarrow} v\} := \{u \overset{(\ge l)}{\Longleftrightarrow} v\} \cap \{u \overset{(\le l)}{\Longleftrightarrow} v\}$.

3. Also, define

$$\triangle^{(l)}(u, v, w) := \sum_{\substack{l_1, l_2, l_3 \ge 1: \\ l_1 + l_2 + l_3 = l}} \tau_p^{(l_1)}(u) \tau_p^{(l_2)}(v - u) \tau_p^{(l_3)}(w - v),$$

$$\Leftrightarrow^{(l)}(u, t, z, x) := \sum_{\substack{l_1, l_2 \ge 0, l_3 \ge 3: \\ l_1 + l_2 + l_3 = l - 1}} \big(\delta_{t,u}\delta_{\mathbf{0},l_1} + p(1 - \delta_{\mathbf{0},l_1})\tau_p^{(l_1)}(t - u)\big)\big(\delta_{\mathbf{0},z}\delta_{\mathbf{0},l_2} + (1 - \delta_{\mathbf{0},l_2})\tau_p^{(l_2)}(z)\big)$$
$$\times J(u) \triangle^{(l_3)}(t - z, x - z, \mathbf{0}).$$

See Figure 1 for an illustration of $\Leftrightarrow^{(l)}$. We remark that $\tau_p^{(1)} = \tau_p$. Moreover, note that $\mathbb{Z}^d$ is bipartite and thus contains no cycles of odd length, which is why $\{u \overset{(2l-1)}{\Longleftrightarrow} v\} = \{u \overset{(2l)}{\Longleftrightarrow} v\}$ and $\triangle^{(2l-1)}(u, v, 0) = \triangle^{(2l)}(u, v, 0)$.

The bounds stated in Lemma 4.2 provide the core tools in dealing with lower-order terms in the bounds on $\Pi^{(i)}$ in the proofs of Section 5.

Lemma 4.2 (Bounds on l-step connection probabilities). *Let $2 \le l \in \mathbb{N}, x \in \mathbb{Z}^d$ and $p \le p_c$. Then*

$$\tau_p^{(l)}(x) = \mathcal{O}\big(|x| \Omega^{1 - (l + |x|)/2}\big). \tag{4.1}$$

Moreover,

$$\sum_{x \in \mathbb{Z}^d} \mathbb{P}_p(\mathbf{0} \overset{(2l)}{\Longleftrightarrow} x) \le p \sum_{u, x \in \mathbb{Z}^d} \triangle^{(2l)}(u, x, \mathbf{0}) = \mathcal{O}\big(\Omega^{2 - l}\big) \tag{4.2}$$

and

$$p^2 \sum_{u, t, z, x \in \mathbb{Z}^d} \Leftrightarrow^{(9)}(u, t, z, x) = \mathcal{O}(\Omega^{-2}). \tag{4.3}$$

Proof. We observe that

$$\tau_p^{(l)}(x) \le \sum_{y\in\mathbb{Z}^d} J(y)\mathbb{P}_p(y \text{ occupied}, \ y \xleftrightarrow{(l-1)} x) = p(J * \tau_p^{(l-1)})(x).$$

Iterating this yields

$$\tau_p^{(l)}(x) \le p^{l-1}(J^{*(l-1)} * \tau_p)(x). \tag{4.4}$$

To prove the first part in (4.2), note that by the BK inequality,

$$\begin{aligned}\sum_{x\in\mathbb{Z}^d} \mathbb{P}_p(\mathbf{0} \overset{(2l)}{\Longleftrightarrow} x) &\le \sum_x \sum_{j=1}^{l} \tau_p^{(j)}(x)\tau_p^{(2l-j)}(x) \le \sum_x \sum_{j=1}^{l} \tau_p^{(j)}(x) p\big(J * \tau_p^{(2l-j-1)}\big)(x) \\ &\le p \sum_x \sum_{j=1}^{l} \tau_p^{(j)}(x) \sum_u \tau_p^{(1)}(u)\tau_p^{(2l-j-1)}(x-u) \le p\sum_{u,x} \triangle^{(2l)}(u,x,\mathbf{0}).\end{aligned}$$

To prove the second part of (4.2), we combine (4.4) with Observation 2.3, yielding

$$\begin{aligned} p\sum_{u,x\in\mathbb{Z}^d} \triangle^{(2l)}(u,x,\mathbf{0}) &\le \sum_{\substack{l_1,l_2,l_3:\\ l_1+l_2+l_3=2l}} \sum_{u,x\in\mathbb{Z}^d} p^{l_1-1}\big(J^{*l_1-1} * \tau_p\big)(u) \\ &\qquad \times p^{l_2-1}\big(J^{*l_2-1} * \tau_p\big)(x-u) p^{l_3-1}\big(J^{*l_3-1} * \tau_p\big)(x) \\ &= p^{2l-2} \sum_{\substack{l_1,l_2,l_3:\\ l_1+l_2+l_3=2l}} \big(J^{*2l-3} * \tau_p^{*3}\big)(\mathbf{0}) = p^{2l-2}\binom{2l-1}{2}\big(J^{*2l-3} * \tau_p^{*3}\big)(0) \\ &\le 2l^2 p^{2l-2}\Big(J^{*2l-3} * \big(J + p(J*\tau_p)\big)^{*3}\Big)(\mathbf{0}) \\ &\le 6l^2 \sum_{j=0}^{3} p^{2l-2+j}\big(J^{*2l} * \tau_p^{*j}\big)(\mathbf{0}) \le \mathcal{O}(\Omega^{2-l}), \end{aligned} \tag{4.5}$$

where the last inequality is due to Lemma 2.11. To prove the bound on $\tau_p^{(l)}$, we set $m = |x|$. note that with the bound (4.4), we can apply Observation 2.3 to obtain

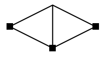
$$\begin{aligned}\tau_p^{(l)}(x) &\le p^{l-1+m}\big(J^{*l-1+m} * \tau_p\big)(x) + \sum_{j=0}^{m-1} p^{l-1+j} J^{*l+j}(x) \\ &\le \mathcal{O}(1)\Omega^{1-(l+m)/2} + \sum_{j=0}^{m-1} \mathcal{O}(1)\Omega^{1-(|x|+l+j)/2} \le |x|\mathcal{O}(1)\Omega^{1-(|x|+l)/2}.\end{aligned}$$

The first term, i.e. the term including a convolution with τ_p, is bounded using Lemma 2.11. The second term, i.e. the convolutions over J, are bounded using Observation 2.2.

To prove (4.3), we split $\diamond$. First observe that when $l_1 = l_2 = 0$,

$$p^2 \sum_{u,t,z,x} J(u)\delta_{t,u}\delta_{\mathbf{0},z}\triangle^{(l_3)}(t-z, x-z, \mathbf{0}) \le p^2 \sum_{u,x} \triangle^{(l_3)}(u,x,\mathbf{0}),$$

which is in $\mathcal{O}(\Omega^{-2})$ for $l_3 = 9$. Let next $l_1 \ne 0 = l_2$. Then

$$p^3 \sum_{u,t,x} J(u)\tau_p^{(l_1)}(t-u)\triangle^{(l_3)}(t,\mathbf{0},x) \le p^3 \sum$$  $$\le \triangle_p^{\bullet}\triangle_p = \mathcal{O}(\Omega^{-2}).$$

When $l_1 = 0 \ne l_2$,

$$p^2 \sum_{u,z,x} J(u)\tau_p^{(l_2)}(z)\triangle^{(l_3)}(u-z, x-z, \mathbf{0}) = p^2 \sum_{u,z,x} \triangle^{(l_3)}(\mathbf{0},z,u)(J*\tau_p^{(l_2)})(u-z). \tag{4.6}$$

If $l_3 \geq 5$, then (4.6) is bounded by $p\triangle_p^\bullet \sum_{u,x} \triangle^{(5)}(\mathbf{0},u,x) = \mathcal{O}(\Omega^{-2})$. If $l_3 \leq 4$, then $l_2 \geq 4$. We can rewrite the left-hand side of (4.6) as

$$p^2 \sum_{u,z} \sum_{\substack{m_1,m_2,m_3: \\ m_1+m_2+m_3=l_3}} J(u)\tau_p^{(l_2)}(z)\tau_p^{(m_1)}(z-u)(\tau_p^{(m_2)} * \tau_p^{(m_3)})(z-u) \leq p\triangle_p^\bullet \sum_{u,z} \triangle^{(6)}(\mathbf{0},u,z) = \mathcal{O}(\Omega^{-2}),$$

as $l_2 + m_1 \geq 5$.

Lastly, let $l_1 \neq 0 \neq l_2$. If $l_3 \geq 5$, then

$$p^3 \sum_{u,t,z,x} J(u)\tau_p^{(l_1)}(z)\tau_p^{(l_2)}(t-u)\triangle^{(l_3)}(t-z,x-z,\mathbf{0})$$
$$= \sum_{t,z} \triangle^{(l_3)}(t,z,\mathbf{0})(\tau_p^{(l_1)} * J * \tau_p^{(l_2)})(z-t) \leq p\triangle_p \sum_{t,z} \triangle^{(6)}(t,z) = \mathcal{O}(\Omega^{-2}).$$

If $l_3 \leq 4$, then $l_1 + l_2 \geq 4$. We bound

$$p^3 \sum_{u,t,z,x} J(u)\tau_p^{(l_1)}(z)\tau_p^{(l_2)}(t-u)\triangle^{(l_3)}(t-z,x-z,0) \leq p^2\triangle_p^\bullet(J * \tau_p^{(l_2)} * \tau_p * \tau_p^{(l_1)})(\mathbf{0})$$
$$\leq \triangle_p^\bullet\big(p^4(J^{*3} * \tau_p^{*3})(\mathbf{0})\big) = \mathcal{O}(\Omega^{-2}),$$

where we used the same sequence of bounds as in (4.5). □

Lastly, we state an observation that appears enough times throughout the arguments of Section 5 for us to extract and state it here.

Observation 4.3. *Let $a \in \mathbb{Z}^d$. Let further $u \neq v$ be two neighbors of a, and set $t = v + u - a$. Then*

$$E'(u,v;\{a\}) \cap \big(\{t = a\} \cup \{t \text{ is vacant}\}\big) \subseteq \{u \xleftrightarrow{(4)} v\}.$$

Proof. Let $A = \{a\}$. We know that $E'(u,v;A) \subset \{u \longleftrightarrow v\}$. If a is vacant, then the shortest possible u-v-path that may be occupied is of length 4 and the claim holds.

On the other hand, if a is occupied, then $\{u \longleftrightarrow v\}$ holds. However, $\{u \xleftrightarrow{A} a\}$ also holds, and so for $E'(u,v;A)$ to hold, a cannot be a pivotal vertex. But in order for a not to be pivotal, there needs to be a second u-v-path, avoiding a. But either t is vacant, or $t = a$; in both cases, a second u-v-path must be of length at least 4, proving the claim. □

5 Detailed analysis of the first three lace-expansion coefficients

5.1 Analysis of $\widehat{\Pi}^{(0)}$

We recall that we write $\widehat{\Pi}^{(i)} = \widehat{\Pi}_{p_c}^{(i)}(0)$. We will also abbreviate $\mathbb{P} = \mathbb{P}_{p_c}$ and $\tau = \tau_{p_c}$ throughout Section 5. We use (3.3) a lot throughout Section 5, and we recall that it states

$$\Omega p_c = 1 + \mathcal{O}(\Omega^{-1})$$

and follows from Proposition 2.6. Moreover, we will use (4.1) of Lemma 4.2 frequently in the proofs to follow and will not mention every time we do so.

Lemma 5.1 (Finer asymptotics of $\widehat{\Pi}^{(0)}$)**.** *As $d \to \infty$,*

$$\widehat{\Pi}^{(0)} = \tfrac{1}{2}\Omega^2 p_c^2 + \tfrac{5}{2}\Omega^{-1} + \mathcal{O}(\Omega^{-2}).$$

Proof. Recall that $\widehat{\Pi}^{(0)} = \sum_x \mathbb{P}(\mathbf{0} \Longleftrightarrow x) - J(x)$. This sum only gets contributions from $|x| \geq 2$. Now,

$$\widehat{\Pi}^{(0)} = \sum_{|x|\geq 2} \mathbb{P}(\mathbf{0} \Longleftrightarrow x) = \sum_{|x|=2} \mathbb{P}\big(\mathbf{0} \overset{(\leq 4)}{\Longleftrightarrow} x\big) + \sum_{|x|\leq 3} \mathbb{P}\big(\mathbf{0} \overset{(=6)}{\Longleftrightarrow} x\big) + \sum_{|x|\geq 2} \mathbb{P}\big(\mathbf{0} \overset{(\geq 8)}{\Longleftrightarrow} x\big)$$

$$= \sum_{|x|=2} \mathbb{P}\big(\mathbf{0} \overset{(\leq 4)}{\Longleftrightarrow} x\big) + \sum_{|x|\leq 3} \mathbb{P}\big(\mathbf{0} \overset{(=6)}{\Longleftrightarrow} x\big) + \mathcal{O}(\Omega^{-2}),$$

where the last identity is due to Lemma 4.2. We first consider 4-cycles. The only points x with $|x| \geq 2$ that can form a 4-cycle with the origin are those with $|x| = 2, \|x\|_\infty = 1$. There are $\frac{1}{2}\Omega(\Omega-2)$ such points. If $x = v_1 + v_2$ (with $|v_i| = 1$) is such a point, then $\{\mathbf{0} \overset{(\leq 4)}{\Longleftrightarrow} x\}$ holds if and only if $\{v_1, v_2\} \subseteq \omega$. Therefore,

$$\sum_{|x|\geq 2} \mathbb{P}\big(\mathbf{0} \overset{(\leq 4)}{\Longleftrightarrow} x\big) = \tfrac{1}{2}\Omega(\Omega-2)p_c^2 = \tfrac{1}{2}\Omega^2 p_c^2 - \Omega^{-1} + \mathcal{O}(\Omega^{-2}). \tag{5.1}$$

We are left to consider points $|x| \geq 2$ contained in cycles of length 6 that also contain the origin. Note that this is possible for $|x| \in \{2,3\}$ and $\|x\|_\infty \in \{1,2\}$. We first claim that $\|x\|_\infty = 2$ gives a contribution of order $\mathcal{O}(\Omega^{-2})$.

Indeed, there are Ω points x with $|x| = 2$ and $\|x\|_\infty = 2$, and any such point is contained in at most $c\Omega$ many origin-including cycles of length 6 (where c is some absolute constant). Any given 6-cycle has probability p_c^4 of being present, and so the contribution is at most $c\Omega^2 p_c^4 = \mathcal{O}(\Omega^{-2})$.

Similarly, there are at most $\Omega(\Omega-2)$ points x with $|x| = 3, \|x\|_\infty = 2$, and any such point is contained in exactly one origin-including cycle of length 6. Hence, this contributes at most $\Omega^2 p_c^4 = \mathcal{O}(\Omega^{-2})$ as well.

Let now $|x| = 3, \|x\|_\infty = 1$. There are $\frac{1}{6}\Omega(\Omega-2)(\Omega-4)$ such points. Such a point spans a (3-dimensional) cube with the origin, in which two internally disjoint paths of respective length 3, making up the sought-after 6-cycle, have to be occupied. There are 9 such cycles. By inclusion-exclusion,

$$\sum_{|x|=3, \|x\|_\infty=1} \mathbb{P}\big(\mathbf{0} \overset{(=6)}{\Longleftrightarrow} x\big) \begin{cases} \leq \frac{9}{6}\Omega^3 p_c^4 = \frac{3}{2}\Omega^{-1} + \mathcal{O}(\Omega^{-2}), \\ \geq \frac{1}{6}(\Omega-4)^3\big[9p_c^4 - \binom{9}{2}p_c^5\big] = \frac{3}{2}\Omega^{-1} + \mathcal{O}(\Omega^{-2}). \end{cases} \tag{5.2}$$

Lastly, consider one of the $\frac{1}{2}\Omega(\Omega-2)$ points $x = v_1 + v_2$ with $|x| = 2, \|x\|_\infty = 1$, and $|v_i| = 1$. Note that there are precisely two paths of length 2 from $\mathbf{0}$ to x, namely the ones using v_i. To produce a relevant contribution to $\{\mathbf{0} \overset{(=6)}{\Longleftrightarrow} x\}$, we claim that exactly one of the two vertices must be vacant and the other occupied. Indeed, if both are occupied, then there is a 4-cycle containing $\mathbf{0}$ and x. If both are vacant, then the shortest possible cycle containing $\mathbf{0}$ and x is of length 8.

We assume v_1 to be occupied and v_2 to be vacant (the reverse gives the same contribution by symmetry, and we respect it with a factor of 2). It remains to count the number of paths of length 4 from $\mathbf{0}$ to x that avoid v_1 and v_2. Avoiding $\pm v_i$ gives $\Omega - 4$ options for the first step. There are two options for the second step (namely, to a neighbor of v_1 or v_2). Steps 3 and 4 are now fixed: Out of the two shortest paths to x, one is via v_i, and is not an option. In conclusion, the probability that there is a $\mathbf{0}$-x-path of length 4 traversing some fixed neighbor of $\mathbf{0}$ (which is not $\pm v_i$) first is $p_c^2(2p_c - p_c^2)$. This gives

$$\sum_{|x|=2, \|x\|_\infty=1} \mathbb{P}\big(\mathbf{0} \overset{(=6)}{\Longleftrightarrow} x\big) \begin{cases} \leq \frac{1}{2}\Omega^3 4p_c^4 = 2\Omega^{-1} + \mathcal{O}(\Omega^{-2}), \\ \geq (\Omega-4)^3 p_c^3(2p_c - p_c^2) - 4\Omega^2 p_c\binom{\Omega-4}{2}p_c^6 = 2\Omega^{-1} + \mathcal{O}(\Omega^{-2}), \end{cases} . \tag{5.3}$$

Summing up (5.1), (5.2), and (5.3) finishes the proof. □

5.2 Analysis of $\widehat{\Pi}^{(1)}$

Lemma 5.2 (Finer asymptotics of $\widehat{\Pi}^{(1)}$)**.** *As $d \to \infty$,*

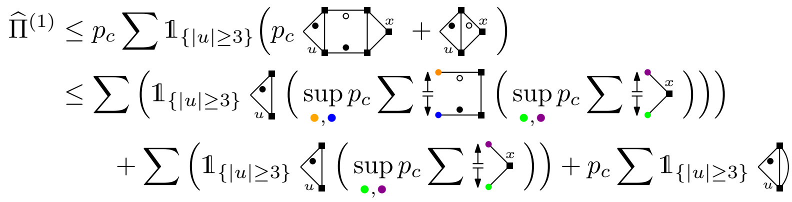
$$\widehat{\Pi}^{(1)} = \Omega p_c + 2\Omega^2 p_c^2 + 4\Omega^{-1} + \mathcal{O}(\Omega^{-2}).$$

Proof. Abbreviating $\mathscr{C}_0 = \widetilde{\mathscr{C}}^u(\mathbf{0}; \omega_0)$, we recall that

$$\widehat{\Pi}^{(1)} = p_c \sum_{u\in\mathbb{Z}^d} \sum_{x\in\mathbb{Z}^d} \mathbb{P}\big(\{\mathbf{0} \Longleftrightarrow u\}_0 \cap E'(u, x; \mathscr{C}_0)_1\big). \tag{5.4}$$

While this is a double sum over all points in $\mathbb{Z}^d$, we first prove that only small values of u give relevant contributions. To this end, assume that $|u| \geq 3$. We use the pictorial representation of the bound in

Lemma 2.9 and decompose it in terms of modified triangles introduced in Definition 2.7. In the below pictorial diagrams, points over which the supremum is taken (in particular, those points are *not* summed over) are represented by colored disks. The indicator that two such points (disks) may not coincide is represented by a disrupted two-sided arrow. Lemma 2.9 together with Proposition 2.8 then gives



$$\le \big(\triangle^{\bullet\bullet\circ}_{p_c}\triangle^{\bullet}_{p_c} + \triangle^{\bullet}_{p_c} + p_c\big) \sum \mathbb{1}_{\{|u|\ge 3\}} \cdots$$

$$\le \mathcal{O}(\Omega^{-1})\Big(\sum_u \mathbb{P}(\mathbf{0} \overset{(6)}{\Longleftrightarrow} u) + p_c \sum_{u,w} \triangle^{(6)}(u,w,\mathbf{0})\Big) = \mathcal{O}(\Omega^{-2}), \tag{5.5}$$

where the last identity is due to Lemma 4.2. When we encounter similar diagrams to the ones in (5.5) at later stages of this paper, we decompose them in the same way as performed in (5.5), but in less detail.

We consider the cases of $|u| \in \{1,2\}$ separately. For both, we make further case distinctions according to the value of $|x|$. The contributions are summarized in the following table:

$\widehat{\Pi}^{(1)}$:	$x = \mathbf{0}$	$\lvert x\rvert = 1$	$\lvert x\rvert = 2$	$\lvert x\rvert = 3$
$\lvert u\rvert = 1$	Ωp_c	$\Omega^2 p_c^2 - 2\Omega^{-1}$	$\Omega^2 p_c^2 + \Omega^{-1}$	$2\Omega^{-1}$
$\lvert u\rvert = 2$		Ω^{-1}	Ω^{-1}	Ω^{-1}

Contributions of $|u| = 1$. By rotational symmetry, we can drop the sum over u, and rewrite (5.4) as

$$p_c \sum_{|u|=1} \sum_{x\in\mathbb{Z}^d} \mathbb{P}\big(\{\mathbf{0} \Longleftrightarrow u\}_0 \cap E'(u,x;\mathscr{C}_0)_1\big)$$

$$= p_c \sum_{u,x\in\mathbb{Z}^d} J(u)\mathbb{P}\left(E'(u,x;\mathscr{C}_0)_1\right) \tag{5.6}$$

$$=\Omega p_c \sum_{x\in\mathbb{Z}^d} \mathbb{P}\left(E'(u,x;\mathscr{C}_0)_1\right). \tag{5.7}$$

In (5.7) and in the following, we take u to be an arbitrary (but fixed) neighbor of the origin. We recall that ω_i is a sequence of independent percolation configurations and an event event with subscript i takes place on ω_i. Moreover, $E'(u,x;\mathscr{C}_0)$ is indexed to take place on configuration 1, which is only accurate if $\mathscr{C}_0$ is regarded as a fixed set; otherwise the event takes place on ω_0 and ω_1.

We proceed by splitting the sum over x in (5.7) (respectively, (5.6)) into different cases.

The case of $|u| = 1, x = \mathbf{0}$ contributes Ωp_c: The event $E'(u,v;\mathscr{C}_0)_1$ in (5.7) holds, the sum collapses to 1, and the contribution is Ωp_c.

The case of $|u| = 1 = |x|$ contributes $\Omega^2 p_c^2 - 2\Omega^{-1} + \mathcal{O}(\Omega^{-2})$: There are $\Omega - 1$ choices for $x \neq u$. We exclude the special case $x = -u$ first. For other choices of x, we let $v := x + u$.

- For $x = -u$, we have $E'(u,x;\mathscr{C}_0)_1 \subseteq \{u \overset{(4)}{\longleftrightarrow} x\}_1$ by Observation 4.3. Hence, (5.7) is bounded by

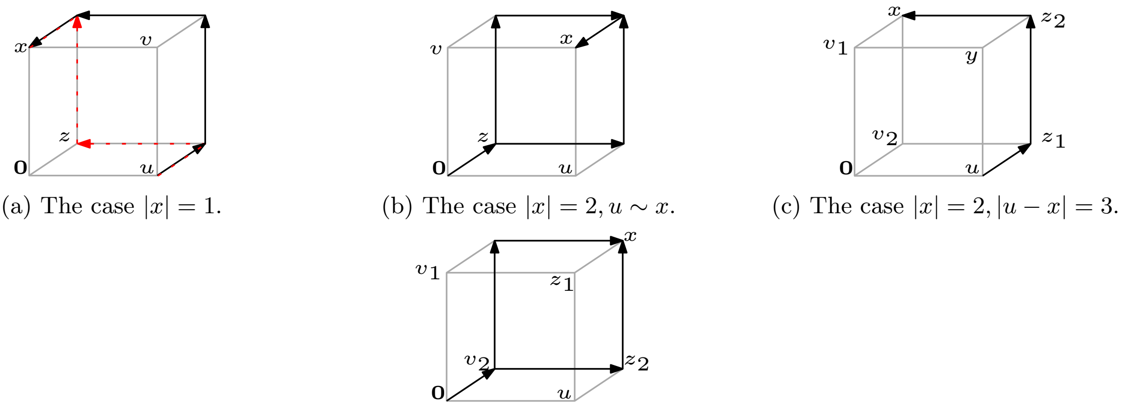
$$\Omega p_c \tau^{(4)}(u - x) = \mathcal{O}(\Omega^{-2}).$$

- Let $x \neq \pm u$ and $v \in \omega_1$. Note first that there are $\Omega - 2$ choices for x, and we can treat them equally by symmetry. Now,

$$E'(u,x;\mathscr{C}_0)_1 \cap \{v \in \omega_1\}_1 = \{v \in \omega_1\}_1 \cap \Big(\{v \notin \langle\mathscr{C}_0\rangle\}_0 \cup \{v \notin \mathsf{Piv}(u,x)\}_1\Big).$$

Note that all three appearing events on the right are independent of each other. Observing

$$\mathbb{P}(v \notin \langle\mathscr{C}_0\rangle) = 1 - \mathbb{P}\big(x \in \omega_0\big) - \mathbb{P}\big(\mathbf{0} \overset{(\ge 4)}{\longleftrightarrow} v \text{ in } \omega_0 \setminus \{u\}\big) = 1 - p_c + \mathcal{O}(\Omega^{-2}),$$



(a) The case $|x| = 1$.

(b) The case $|x| = 2, u \sim x$.

(c) The case $|x| = 2, |u - x| = 3$.

(d) The case $|x| = 3$.

Figure 2: An illustration of several appearing cases for $|u| = 1$. In the first two cases, $\mathbf{0}$ and v are vacant in ω_1. In case (a), the black path is γ_1, the red and dotted one is γ_2. In case (b), the two $\mathbf{0}$-x-paths are marked as black chains of arrows. In case (c), $\{v_1, v_2\} \cap \omega_0 = \{v_1\}$ and the only relevant u-x-path is marked in black.

$$\mathbb{P}(v \notin \mathsf{Piv}(u, x)) = \mathbb{P}(\mathbf{0} \in \omega_1) + \mathbb{P}\big(u \xleftrightarrow{(\geq 4)} x \text{ in } \omega_1 \setminus \{v\}\big) = p_c + \mathcal{O}(\Omega^{-2}),$$

we can replace the sum over x by a factor of $(\Omega - 2)$ and write (5.7) as

$$\begin{aligned}\Omega p_c(\Omega-2)\mathbb{P}\big(E'(u,x;\mathscr{C}_0)_1 \cap \{v \in \omega_1\}_1\big) &= \Omega(\Omega-2)p_c^2\Big(1 - p_c + p_c - (1-p_c)p_c\Big) + \mathcal{O}(\Omega^{-2}) \\ &= (\Omega p_c)^2(1-p_c) - 2\Omega p_c^2 + \mathcal{O}(\Omega^{-2}) \\ &= \Omega^2 p_c^2 - 3\Omega^{-1} + \mathcal{O}(\Omega^{-2}).\end{aligned}$$

- Let $x \neq \pm u, v \notin \omega_1$, and $\mathbf{0} \notin \omega_1$. For $E'(u, x; \mathscr{C}_0)_1$ to hold, there needs to be a ω_1-path between u and x. Its pivotal points cannot lie in $\langle \mathscr{C}_0 \rangle$ however. First, note that any relevant path between u and x is of length 4, as

$$\Omega p_c(\Omega-2)\mathbb{P}\big(E'(u,x;\mathscr{C}_0)_1 \cap \{u \xleftrightarrow{(\geq 6)} x\}_1\big) \leq \Omega^2 p_c \tau^{(6)}(x-u) = \mathcal{O}(\Omega^{-2}).$$

We now investigate the 4-paths from u to x that avoid $\mathbf{0}$ and v—from Lemma 5.1, we already know that there are $2(\Omega - 4)$ of them. Let z be one of the $\Omega - 4$ unit vectors satisfying $\dim\langle\langle u, x, z\rangle\rangle = 3$, where we let $\langle\langle \cdot \rangle\rangle$ denote the span. We denote by γ_1 and γ_2 the two u-x-paths of length 4 that visit $y_1 := u + z$. W.l.o.g., γ_1 visits $y_2 := y_1 + x$ second and $y_3 := y_2 - u$ third, whereas γ_2 visits z second and y_3 third. Let $\{\gamma_i \subseteq \omega_1\}$ denote the event that the 3 internal vertices of γ_i are ω_1-occupied. See Figure 2a for an illustration.

We now show that only γ_1 produces a relevant term. Assume first that $y_2 \notin \omega_1$, but $\gamma_2 \subseteq \omega_1$. For $E'(u, x; \mathscr{C}_0)_1$ to hold, $z \in \langle \mathscr{C}_0 \rangle$ must not be a pivotal point. Under $\gamma_2 \subseteq \omega_1$,

$$\{z \notin \mathsf{Piv}(u,x)\}_1 \subseteq \{\{u, y_1\} \longleftrightarrow \{y_3, x\} \text{ in } \omega_1 \setminus \{z\}\}_1. \tag{5.8}$$

Resolving the right-hand side of (5.8) by a union bound gives four connection events. The shortest ω_1-path from u to x of non-vacant vertices is of length 4. Moreover, the shortest ω_1-path from y_1 to y_3 of non-vacant vertices that avoids z is of length 4 as well, and so (5.7) is bounded by

$$\begin{aligned}&\Omega(\Omega-2)p_c \sum_z \mathbb{P}\Big(E'(u,x;\mathscr{C}_0)_1, \{\mathbf{0}, v, y_2\} \cap \omega_1 = \varnothing, \gamma_2 \subseteq \omega_1\Big) \\ &\qquad \leq \Omega^3 p_c^4 \Big(\tau^{(4)}(x-u) + \tau^{(3)}(y_3-u) + \tau^{(3)}(x-y_1) + \tau^{(4)}(y_3-y_1)\Big) = \mathcal{O}(\Omega^{-2}).\end{aligned}$$

We now show that $\gamma_1 \in \omega_1$ gives a contribution. Note that under $\{\mathbf{0}, v \notin \omega_1, \gamma_1 \subseteq \omega_1\}$,

$$E'(u,x;\mathscr{C}_0)_1 = \bigcap_{i \in \{1,2,3\}} \Big(\{y_i \notin \mathsf{Piv}(u,x)\}_1 \cup \{y_i \notin \langle \mathscr{C}_0 \rangle\}_0\Big) \tag{5.9}$$

But $\mathbb{P}(\{y_i \notin \mathsf{Piv}(u,x)\}_1 \cup \{y_i \notin \langle \mathscr{C}_0\rangle\}_0) \geq 1 - \mathbb{P}(y_i \in \langle \mathscr{C}_0\rangle) \geq 1 - \tau^{(2)}(y_i) = 1 - \mathcal{O}(\Omega^{-1})$ for all i by Lemma 4.2, and so, by inclusion-exclusion,

$$\Omega(\Omega-2)p_c \sum_z \mathbb{P}\Big(E'(u,x;\mathscr{C}_0)_1, \{\mathbf{0}, v\} \cap \omega_1 = \varnothing, \gamma_1 \subseteq \omega_1\Big)$$
$$\begin{cases} \leq \Omega(\Omega-2)(\Omega-4)(1-p_c)^2 p_c^4\big(1-\mathcal{O}(\Omega^{-1})\big) = \Omega^{-1} + \mathcal{O}(\Omega^{-2}), \\ \geq \Omega^3 p_c^4 (1-\mathcal{O}(\Omega^{-1})) - \Omega^2\binom{\Omega-4}{2} p_c^7 = \Omega^{-1} + \mathcal{O}(\Omega^{-2}). \end{cases}$$

- Let $x \neq \pm u, v \notin \omega_1$, and $\mathbf{0} \in \omega_1$. By Observation 4.3,

$$\Omega p_c (\Omega-2)\mathbb{P}\big(E'(u,x;\mathscr{C}_0)_1 \cap \{v \notin \omega_1, \mathbf{0} \in \omega_1\}\big) \leq \Omega^2 p_c^2 \tau^{(4)}(x-u) = \mathcal{O}(\Omega^{-2}).$$

<u>The case of $|u|=1, |x|=2$ contributes $\Omega^2 p_c^2 + \Omega^{-1} + \mathcal{O}(\Omega^{-2})$</u>: There are $\frac{1}{2}\Omega^2$ choices for x. We first consider the $\Omega-1$ choices neighboring u and, among those, exclude the special case $x = 2u$ first. For x a neighbor of u, we set $v := x - u$.

- Let $x = 2u$. Since $x \sim u$, we have $E'(u,x;\mathscr{C}_0)_1 = \{x \in \langle \mathscr{C}_0\rangle\}_0 \subseteq \{\mathbf{0} \xleftrightarrow{(4)} x\}_0$, and so the contribution to (5.7) is bounded by $\Omega p_c \tau^{(4)}(x) = \mathcal{O}(\Omega^{-2})$.
- Let $2u \neq x \sim u$ and $v \in \omega_0$. There are $\Omega-2$ choices for x. The event $E'(u,x;\mathscr{C}_0)_1$ holds, and so

$$\Omega p_c \sum_{2u \neq x \sim u} \mathbb{E}_0\left[\mathbb{1}_{\{v \in \omega_0\}} \mathbb{P}_1\left(E'(u,x;\mathscr{C}_0)\right)\right] = \Omega(\Omega-2)p_c^2 = \Omega^2 p_c^2 - 2\Omega^{-1} + \mathcal{O}(\Omega^{-2}).$$

- Let $2u \neq x \sim u$ and $v \notin \omega_0$. We partition

$$E'(u,x;\mathscr{C}_0)_1 = \Big(E'(u,x;\mathscr{C}_0)_1 \cap \{\mathbf{0} \xleftrightarrow{(\leq 4)} x \text{ in } \mathbb{Z}^d \setminus \{u\}\}_0\Big) \cup \Big(E'(u,x;\mathscr{C}_0)_1 \cap \{\mathbf{0} \xleftrightarrow{(\geq 6)} x \text{ in } \mathbb{Z}^d \setminus \{u\}\}_0\Big)$$

  and treat the second event by observing

$$\Omega p_c \sum_{2u \neq x \sim u} \mathbb{P}\Big(\{v \notin \omega_0\}_0 \cap \{\mathbf{0} \xleftrightarrow{(\geq 6)} x \text{ in } \mathbb{Z}^d \setminus \{u\}\}_0 \cap E'(u,x;\mathscr{C}_0)_1\Big) \leq \Omega^2 p_c \tau^{(6)}(x) = \mathcal{O}(\Omega^{-2}).$$

  As the only 2-paths from $\mathbf{0}$ to x go through u and v respectively, we can focus on paths of length 4 avoiding v and u. Hence, the status of v is independent of such paths. Let z be one of the $\Omega-4$ neighbors of $\mathbf{0}$ with $\dim\langle\langle u,v,z\rangle\rangle = 3$. For any such z, there are two $\mathbf{0}$-x-paths of length 4 that first visit z and avoid $\{v,u\}$. More precisely, these paths are $(\mathbf{0}, z, u+z, x+z, x)$ and $(\mathbf{0}, z, v+z, x+z, x)$. Let $Q_4(z)$ denote the event that at least one of these paths is in ω_0. See Figure 2b for an illustration. As the events $\{Q_4(z)\}$ are pairwise independent,

$$\{v \notin \omega_0\}_0 \cap \{\mathbf{0} \xleftrightarrow{(\leq 4)} x \text{ in } \mathbb{Z}^d \setminus \{u\}\}_0 \cap E'(u,x;\mathscr{C}_0)_1 = \{v \notin \omega_0\}_0 \cap \big(\cup_z Q_4(z)\big),$$
$$\mathbb{P}(\cup_z Q_4(z)) = (\Omega-4)\mathbb{P}(Q_4(z)) + \mathcal{O}(\Omega^{-4}) = 2(\Omega-4)p_c^3 + \mathcal{O}(\Omega^{-3}).$$

  Consequently,

$$\Omega p_c \sum_{2u \neq x \sim u} \mathbb{P}\Big(\{v \notin \omega_0\}_0 \cap \{\mathbf{0} \xleftrightarrow{(\leq 4)} x \text{ in } \mathbb{Z}^d \setminus \{u\}\}_0 \cap E'(u,x;\mathscr{C}_0)_1\Big)$$
$$= \Omega p_c (\Omega-2) 2(\Omega-4)p_c^3 + \mathcal{O}(\Omega^{-2}) = 2\Omega^{-1} + \mathcal{O}(\Omega^{-2}).$$

- Let $|u-x| = 3$ and $\|x\|_\infty = 2$. There are $\Omega-1$ choices for x. Let $2v = x$. Note first that

$$\Omega(\Omega-1)p_c \mathbb{P}\Big(\big(\{x \in \langle \mathscr{C}_0\rangle\}_0 \cup \{u \xleftrightarrow{(5)} x\}_1\big) \cap E'(u,x;\mathscr{C}_0)_1\Big)$$
$$\leq \Omega^2 p_c \Big(\tau^{(2)}(x)\tau^{(3)}(x-u) + \tau^{(5)}(x-u)\Big) = \mathcal{O}(\Omega^{-2}).$$

  The complementary event is that $x \notin \langle \mathscr{C}_0\rangle$ and the presence of a u-x-path of length 3. The former implies $v \notin \omega_0$. There are at most four potential sites that can make up internal vertices on a u-x-path of length 3, namely $\mathbf{0}, v, u+v, u+2v$. To avoid potential pivotality of $\mathbf{0}$ and v and still guarantee a path of length 3, we require $\{v+u, v+2u\} \subseteq \omega_1$. But both these vertices are of distance at least 2 from the origin, and at least one of them must be in $\langle \mathscr{C}_0 \rangle$. In conclusion,

$$\Omega(\Omega-1)p_c \mathbb{P}\Big(\{x \notin \langle \mathscr{C}_0 \rangle\}_0 \cap \{u \xleftrightarrow{(\le 3)} x\}_1 \cap E'(u,x;\mathscr{C}_0)_1\Big) \\ \le 2\Omega^2 p_c \tau^{(2)}(u+v)\tau^{(3)}(x-u) = \mathcal{O}(\Omega^{-2}).$$

- Let $|u-x|=3, \|x\|_\infty = 1$, and $x \in \langle \mathscr{C}_0 \rangle$. Write $x = v_1 + v_2$, where $|v_i|=1$. We first show that contributions arise when precisely one point in $\{v_1, v_2\}$ is ω_0-occupied. Note that when both v_1 and v_2 are vacant in ω_0, the contribution to (5.7) is bounded by $\Omega^3 p_c \tau^{(4)}(x)\tau^{(3)}(x-u) = \mathcal{O}(\Omega^{-2})$. On the other hand, if $\{v_1,v_2\} \subseteq \omega_0$, then the contribution is bounded by $\Omega^3 p_c^3 \tau^{(3)}(u-x) = \mathcal{O}(\Omega^{-2})$. Let now $v_1 \in \omega_0$ and $v_2 \notin \omega_0$ (the other case is identical and is respected by counting the contribution twice). There are $\frac{1}{2}\Omega^2(1+\mathcal{O}(\Omega^{-1}))$ choices for x. If $\{u \xleftrightarrow{(5)} x\}_1$, then the contribution to (5.7) is $\mathcal{O}(\Omega^{-2})$. Set $z_1 = u+v_2, z_2 = u+v_2+v_1$, and set $y = u+v_1$. We claim that the only u-x-path of length 3 that produces a relevant contribution is (u, z_1, z_2, x). See Figure 2c for an illustration.
  First, assume $z_1 \notin \omega_1$. Note that the only other paths of length 3 from u to x go through either $\mathbf{0}$ or y. But $\{0, y\} \subseteq \langle \mathscr{C}_0 \rangle$, and so neither $\mathbf{0}$ nor y can be a pivotal point. Hence, $E'(u,x;\mathscr{C}_0)_1 \cap \{z_1 \notin \omega_1\}$ enforces $\{\mathbf{0}, y\} \subseteq \omega_1$. To get to x and avoid pivotality of any points in $\langle \mathscr{C}_0 \rangle$, at least two points in $\{v_1, v_2, z_1\}$ must be occupied, and the contribution to (5.7) is at most

$$2\Omega p_c\big(\tfrac{1}{2}\Omega^2(1+\mathcal{O}(\Omega^{-1}))\big)p_c^2\binom{3}{2}p_c^2 = \mathcal{O}(\Omega^{-2}).$$

  If $z_1 \in \omega_1$ and $z_2 \notin \omega_1$, then the only u-x-path of length 3 through z_1 visits $v_2 \in \langle \mathscr{C}_0 \rangle$. This gives a contribution of $\mathcal{O}(\Omega^{-2})$ by the same bound as above. We may turn to the case $z_i \in \omega_1$ for $i \in \{1,2\}$. Now, under $\{v_1 \in \omega_0, \{z_1,z_2\} \subseteq \omega_1\}$, we can express $E'(u,x;\mathscr{C}_0)_1$ similarly to (5.9), replacing y_i ($i \in [3]$) by z_i ($i \in [2]$). Applying the same bounds, we obtain a contribution to (5.7) of

$$2\Omega p_c\big(\tfrac{1}{2}\Omega^2(1+\mathcal{O}(\Omega^{-1}))\big)\mathbb{P}\big(v_1 \in \omega_0, \{z_1,z_2\} \subseteq \omega_1\big)(1-\mathcal{O}(\Omega^{-1})) = \Omega^{-1} + \mathcal{O}(\Omega^{-2}).$$

- Let $|u-x|=3, \|x\|_\infty = 1$, and $x \notin \langle \mathscr{C}_0 \rangle$. Let γ be a u-x-path in ω_1. By assumption, there needs to be some $z \in \gamma$ with $z \in \langle \mathscr{C}_0 \rangle$. Consequently, z cannot be a pivotal point and so there needs to be another u-x-path $\tilde{\gamma}$ in ω_1 that contains a point $\tilde{z} \notin \gamma$ with $\tilde{z} \in \langle \mathscr{C}_0 \rangle$. Assume first that both $\gamma, \tilde{\gamma}$ are paths of length 3. If they are disjoint, then the contribution to (5.7) is at most $9\Omega^3 p_c^5 = \mathcal{O}(\Omega^{-2})$. If they share their first vertex, then, in the terminology of Figure 2c, it must be either y or z_1 (otherwise $\mathbf{0}$ is pivotal). W.l.o.g., $\tilde{\gamma}$ must then pass through z_2 and so $\tilde{z} = z_2 \in \langle \mathscr{C}_0 \rangle$ needs to hold, and the contribution to (5.7) is at most $\Omega^3 p_c^4 \tau^{(3)}(z_2) = \mathcal{O}(\Omega^{-2})$. Assume next that $\tilde{\gamma}$ is of length 5. As γ and $\tilde{\gamma}$ share at most one internal vertex (and there are two internal vertices in γ), we count a factor of p_c for the unique vertex of γ, and the contribution to (5.7) is at most $18\Omega^3 p_c^2 \tau^{(5)}(x-u) = \mathcal{O}(\Omega^{-2})$. Similarly, when both γ and $\tilde{\gamma}$ are of length at least 5, the contribution is $\mathcal{O}(\Omega^{-2})$.

The case of $|u|=1, |x|=3$ contributes $2\Omega^{-1} + \mathcal{O}(\Omega^{-2})$: Note that when $\{u \xleftrightarrow{(4)} x\}_1$, then the contribution to (5.6) is at most

$$p_c \sum_{u,x} \triangle^{(8)}(u,x,\mathbf{0}) + p_c^2 \sum_{u,t,z,x} \diamond^{(9)}(u,t,z,x) = \mathcal{O}(\Omega^{-2}) \tag{5.10}$$

by Lemma 4.2. We can therefore focus on x with $|x-u|=2$ and $\{u \xleftrightarrow{(=2)} x\}_1$. Moreover, we can assume that there is no u-x-path of length 4. Let $x = u+v_1+v_2$, where $|v_1| = 1 = |v_2|$, and assume first that $\dim\langle\langle u, v_1, v_2\rangle\rangle = 3$. There are $\frac{1}{2}(\Omega-2)(\Omega-4)$ choices for x. Let $z_i = u+v_i$ be the two internal vertices of the two shortest u-x-paths—see Figure 2d for an illustration.

We first claim that only $x \in \langle \mathscr{C}_0 \rangle$ produces a relevant contribution. Indeed, if $x \notin \langle \mathscr{C}_0 \rangle$, and as there is no u-x-path of length 4, we must have $z_i \in \omega_1 \cap \langle \mathscr{C}_0 \rangle$ for $i \in \{1,2\}$. For $\{\mathbf{0} \longleftrightarrow z_i\}_0$ to hold, either $v_i \in \omega_0$, or $\{\mathbf{0} \xleftrightarrow{(4)} z_i\}_0$, and so (5.7) is at most

$$\Omega^3 p_c \mathbb{P}\Big(\{\{z_1,z_2\} \subseteq \omega_1\} \cap \big(\{\{v_1,v_2\} \subseteq \omega_0\} \cup \{\mathbf{0} \xleftrightarrow{(4)} z_1\}_0 \cup \{\mathbf{0} \xleftrightarrow{(4)} z_2\}_0\big)\Big)$$

$$= \Omega^3 p_c^3\big(p_c^2 + \tau^{(4)}(z_1) + \tau^{(4)}(z_2)\big) = \mathcal{O}(\Omega^{-2}).$$

Turning to $x \in \langle \mathscr{C}_0 \rangle$, note that when $\{z_1, z_2\} \subseteq \omega_1$, then (5.7) is at most

$$\Omega^3 p_c \mathbb{P}\big(\{\mathbf{0} \longleftrightarrow x\}_0 \cap \{\{z_1, z_2\} \subseteq \omega_1\}\big) = \Omega^3 p_c^3 \tau^{(3)}(x) = \mathcal{O}(\Omega^{-2}).$$

W.l.o.g., we assume that $z_1 \in \omega_1$ (and $z_2 \notin \omega_1$) and (by symmetry) count the contribution twice. Now, the contribution to (5.7) is equal to

$$\Omega(\Omega-2)(\Omega-4)p_c \mathbb{P}\Big(\{x \in \langle \mathscr{C}_0 \rangle\}_0 \cap \{z_2 \notin \omega_1 \ni z_1\}_1 \cap \big(\{z_1 \notin \langle \mathscr{C}_0 \rangle_0 \cup \{z_1 \notin \mathsf{Piv}(u,x)\}_1\big)\Big). \tag{5.11}$$

If $v_1 \in \omega_0$, then $z_1 \in \langle \mathscr{C}_0 \rangle$ and so z_1 cannot be pivotal, which, in turn, forces $\{u \overset{(4)}{\longleftrightarrow} x\}_1$. But this was already shown to produce an $\mathcal{O}(\Omega^{-2})$ contribution. Further, if $\{\mathbf{0} \overset{(5)}{\longleftrightarrow} x\}_0$, then (5.11) is at most $\Omega^3 p_c^2 \tau^{(5)}(x) = \mathcal{O}(\Omega^{-2})$, and so $\mathbf{0}$ must be ω_0-connected to x by a path of length 3.

There are precisely two $\mathbf{0}$-x-paths of length 3 that use neither v_1 nor u, namely $\gamma_1 = (\mathbf{0}, v_2, v_1+v_2, x)$ and $\gamma_2 = (\mathbf{0}, v_2, z_2, x)$. If both are occupied, the contribution is $\mathcal{O}(\Omega^{-2})$. Note that

$$\mathbb{P}(z_1 \notin \langle \mathscr{C}_0 \rangle \mid \gamma_i \subseteq \omega_0) \geq 1 - 3\tau^{(2)}(z_1) = 1 - \mathcal{O}(\Omega^{-1}),$$

and so (5.11) becomes

$$\Omega^3(1-\mathcal{O}(\Omega^{-1}))p_c \mathbb{P}\Big(\big(\cup_{i=1,2}\{\gamma_i \subseteq \omega_0\}_0\big), z_1 \in \omega_1\Big) = 2\Omega^3 p_c^4(1-\mathcal{O}(\Omega^{-1})) = 2\Omega^{-1} + \mathcal{O}(\Omega^{-2}).$$

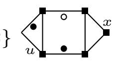
Finally, if $\dim\langle\!\langle u, v_1, v_2\rangle\!\rangle \leq 2$, then the same bounds with at least one factor of Ω in the choice of x gives a contribution of $\mathcal{O}(\Omega^{-2})$.

The case of $|u| = 1, |x| \geq 4$ contributes $\mathcal{O}(\Omega^{-2})$: The bound is the same as in (5.10).

Contributions of $|u| = 2$. If u is one of the Ω points with $|u| = 2 = \|u\|_\infty$, then $\widehat{\Pi}^{(1)}$ is bounded by $\Omega p_c \sum_x \mathbb{P}(\mathbf{0} \Longleftrightarrow u)\tau_p(u-x)$. For fixed $j = |u-x|$, this is bounded by

$$\Omega^{1+j} p_c \tau^{(2)}(u)\tau^{(4)}(u)\tau^{(j)}(x-u) = \mathcal{O}(\Omega^{-2}).$$

We now show that we can impose some further restrictions on u and x. Recall the bound in (5.5), and observe that if $x \notin \langle \mathscr{C}_0 \rangle$, then

$$p_c \sum_{|u|=2} \sum_x \mathbb{P}\big(\{\mathbf{0} \Longleftrightarrow u\}_0 \cap \{x \notin \langle \mathscr{C}_0 \rangle\}_0 \cap E'(u,x;\mathscr{C}_0)_1\big) \leq p_c^2 \sum \mathbb{1}_{\{|u|=2\}}$$  $$= \mathcal{O}(\Omega^{-2}).$$

Similar considerations enforce that $|x| \leq 3$ and $|x-u| \leq 2$ as well as $\{\mathbf{0} \overset{(\leq 4)}{\Longleftrightarrow} u\}_0$. Before going into the different cases, we note that there are $\frac{1}{2}\Omega(\Omega-2)$ choices for $u = v_1 + v_2$ (where $|v_i| = 1$), and on every choice, $\{v_1, v_2\} \subseteq \omega_0$ need to hold for a relevant contribution to arise. Taking all this into consideration, the contribution to $\widehat{\Pi}^{(1)}$ becomes

$$\tfrac{1}{2}\Omega(\Omega-2)p_c^3 \sum_{x \in \mathbb{Z}^d} \mathbb{1}_{\{|x| \leq 3, |u-x| \leq 2\}} \mathbb{P}\Big(\{x \in \langle \mathscr{C}_0 \rangle\}_0 \cap E'(u,x;\mathscr{C}_0)_1 \mid \{v_1, v_2\} \subseteq \omega_0\Big), \tag{5.12}$$

where v_1 and v_2 is a pair of arbitrary but fixed independent unit vectors (and $u = v_1 + v_2$).

The case of $|u| = 2, x = \mathbf{0}$ contributes $\mathcal{O}(\Omega^{-2})$: As $|u-x| = 2$, the contribution to (5.12) is at most $\Omega^2 p_c^3 \tau^{(2)}(x-u) = \mathcal{O}(\Omega^{-2})$.

The case of $|u| = 2, |x| = 1$ contributes $\Omega^{-1} + \mathcal{O}(\Omega^{-2})$: Note that we only need to consider $x \in \{v_1, v_2\}$ (otherwise $|u-x| = 3$). For these choices of x, both $x \in \langle \mathscr{C}_0 \rangle$ and $E'(u,x;\mathscr{C}_0)_1$ hold and the contribution to (5.12) is as claimed.

The case of $|u| = 2, |x| = 2$ contributes $\Omega^{-1} + \mathcal{O}(\Omega^{-2})$: By the indicator in (5.12), we only consider $|x-u| = 2$. Let first $\|x\|_\infty = 2$. There are only two such points at distance 2 of u, and so the contribution to (5.12) is at most $\Omega^2 p_c^3 \tau^{(2)}(x-u) = \mathcal{O}(\Omega^{-2})$.

Let thus x be one of the $2(\Omega-3)$ points with $\|x\|_\infty = 1$. W.l.o.g., we assume that $x = v_1 + v_3$, where $|v_3| = 1$. If $v_3 = -v_2$, then the contribution is bounded by $\Omega^2 p_c^3 \tau^{(2)}(x-u) = \mathcal{O}(\Omega^{-2})$. Let x be one of the remaining $2(\Omega-4)$ points with $\dim\langle\langle v_1, v_2, v_3\rangle\rangle = 3$. As $x \sim v_1$, the event $x \in \langle \mathscr{C}_0 \rangle$ holds. We partition $E'(u,x;\mathscr{C}_0)_1$ into whether $\{u \xleftrightarrow{(=2)} x\}_1$ or $\{u \xleftrightarrow{(\geq 4)} x\}_1$ and see that in the latter case, the contribution to (5.12) is at most $\Omega^3 p_c^3 \tau^{(4)}(x-u) = \mathcal{O}(\Omega^{-2})$.

For the existence of a path of length 2, either v_1 or $v_4 := x + v_2$ need to be ω_1-occupied. As $v_1 \in \mathscr{C}_0$, it cannot be a pivotal point for the ω_1-connection between u and x and there needs to be another path. The contribution to (5.12) is therefore at most $\Omega^3 p_c^4 \tau^{(2)}(x-u) = \mathcal{O}(\Omega^{-2})$. We observe that

$$E'(u,x;\mathscr{C}_0)_1 \cap \{v_4 \in \omega_1\} = \{v_4 \in \omega_1\} \cap \Big(\{v_4 \notin \mathsf{Piv}(u,x)\}_1 \cup \{\mathbf{0} \not\longleftrightarrow v_4 \text{ in } \mathbb{Z}^d \setminus \{u\}\}_0 \Big).$$

As previously, $\mathbb{P}(v_4 \notin \mathsf{Piv}(u,x)) = \mathcal{O}(\Omega^{-1})$ and $\mathbb{P}(\mathbf{0} \not\longleftrightarrow v_4 \text{ in } \mathbb{Z}^d \setminus \{u\}) = 1 - \mathcal{O}(\Omega^{-1})$, and so the contribution to (5.12) is

$$\Omega^3(1-\mathcal{O}(\Omega^{-1}))p_c^4(1+\mathcal{O}(\Omega^{-1})) = \Omega^{-1} + \mathcal{O}(\Omega^{-2}).$$

<u>The case of $|u| = 2, |x| = 3$ contributes $\Omega^{-1} + \mathcal{O}(\Omega^{-2})$:</u> We only need to consider neighbors of u, otherwise $|u - x| \geq 3$. Recall that for $|u-x| = 1$, the event $E'(u,x;\mathscr{C}_0)_1$ holds precisely when $x \in \langle \mathscr{C}_0 \rangle$. Under our conditioning, x must be connected to $\{0, v_1, v_2\}$. Note that there are two choices for x with $\|x\|_\infty = 2$. Since $\mathbb{P}(x \in \langle \mathscr{C}_0 \rangle) \leq 3 \max_{y \in \{0, v_1, v_2\}} \tau^{(2)}(x-y)) = \mathcal{O}(\Omega^{-1})$, we may focus on the $\Omega - 2$ choices of x with $\|x\|_\infty = 1$.

Let $x = u + v_3$ and set $z_1 := v_1 + v_3, z_2 := v_2 + v_3$. If $\{z_1, z_2\} \cap \omega_0 = \varnothing$, then $\{\mathbf{0} \xleftrightarrow{(5)} x\}_0$ holds, and the contribution to (5.12) is at most $\Omega^3 p_c^3 \max_{y \in \{0, v_1, v_2\}} \tau^{(3)}(x-y) = \mathcal{O}(\Omega^{-2})$. If $\{z_1, z_2\} \subset \omega_0$, then the contribution to (5.12) is at most $\Omega^3 p_c^5 = \mathcal{O}(\Omega^{-2})$.

We consider the case where $z_1 \notin \omega_0 \ni z_2$ and respect the other case with a factor of 2. The contribution to (5.12) is

$$\Omega^3(1+\mathcal{O}(\Omega^{-1})p_c^4(1+\mathcal{O}(\Omega^{-1})) = \Omega^{-1} + \mathcal{O}(\Omega^{-2}).$$

This finishes the analysis of $\widehat{\Pi}^{(1)}$. □

5.3 Analysis of $\widehat{\Pi}^{(2)}$

Lemma 5.3 (Asymptotics of $\widehat{\Pi}^{(2)}$)**.** *As $d \to \infty$,*

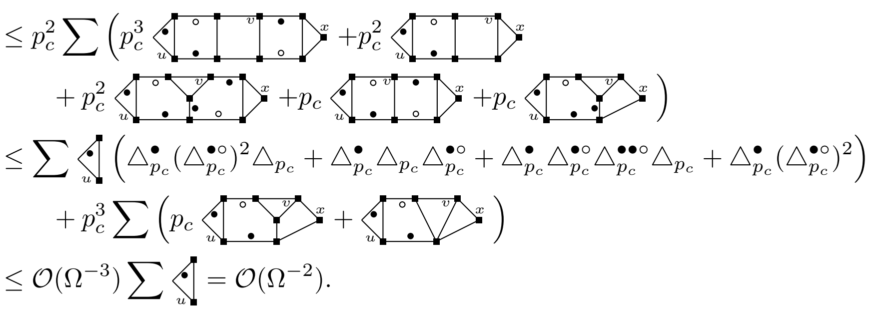
$$\widehat{\Pi}^{(2)} = 10\Omega^{-1} + \mathcal{O}(\Omega^{-2}).$$

Proof. For the proof, we recall that

$$\widehat{\Pi}^{(2)} = p_c^2 \sum_{u,v,x \in \mathbb{Z}^d} \mathbb{P}\Big(\{\mathbf{0} \Longleftrightarrow u\}_0 \cap E'(u,v;\mathscr{C}_0)_1 \cap E'(v,x;\mathscr{C}_1)_2 \Big), \tag{5.13}$$

where $\mathscr{C}_0 = \widetilde{\mathscr{C}}^u(\mathbf{0};\omega_0)$ and $\mathscr{C}_1 = \widetilde{\mathscr{C}}^v(u;\omega_1)$. We first show that when either $v \notin \langle \mathscr{C}_0 \rangle$ or $x \notin \langle \mathscr{C}_1 \rangle$, then the contribution to $\widehat{\Pi}^{(2)}$ is $\mathcal{O}(\Omega^{-2})$. Indeed, by Lemma 2.10 and Proposition 2.8,

$$p_c^2 \sum_{u,v,x \in \mathbb{Z}^d} \mathbb{P}\Big(\{\mathbf{0} \Longleftrightarrow u\}_0 \cap E'(u,v;\mathscr{C}_0)_1 \cap E'(v,x;\mathscr{C}_1)_2 \cap \big(\{v \notin \langle \mathscr{C}_0 \rangle\}_0 \cup \{x \notin \langle \mathscr{C}_1 \rangle\}_1 \big) \Big)$$

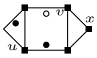


(5.14)

(5.15)

We expanded the third diagram in (5.14) to get the two diagrams of (5.15). We next show that only $|u| = 1$ gives a relevant contribution. Indeed,

$$p_c^2 \sum_{u,v,x \in \mathbb{Z}^d : |u| \geq 2} \mathbb{P}\Big(\{\mathbf{0} \Longleftrightarrow u\}_0 \cap E'(u,v;\mathscr{C}_0)_1 \cap E'(v,x;\mathscr{C}_1)_2 \cap \{v \in \langle \mathscr{C}_0 \rangle\}_0 \cap \{x \in \langle \mathscr{C}_1 \rangle\}_1\Big)$$

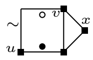
$$\leq p_c^2 \sum \mathbb{1}_{\{|u| \geq 2\}}$$

$$\leq \triangle^{\bullet}_{p_c} \triangle^{\bullet\circ}_{p_c} \sum \mathbb{1}_{\{|u| \geq 2\}}$$

$$= \mathcal{O}(\Omega^{-2}).$$

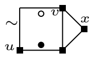
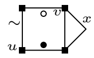
We can thus fix u to be an arbitrary neighbor of the origin and need to investigate

$$\Omega p_c^2 \sum_{v,x \in \mathbb{Z}^d} \mathbb{P}\Big(E'(u,v;\mathscr{C}_0)_1 \cap E'(v,x;\mathscr{C}_1)_2 \cap \{v \in \langle \mathscr{C}_0 \rangle\}_0 \cap \{x \in \langle \mathscr{C}_1 \rangle\}_1\Big). \tag{5.16}$$

Before going into specific cases, we exclude some of them right away: When $|x| \vee |u-x| \geq 4$, then the contribution to (5.16) is

$$p_c^2 \sum \mathbb{1}_{\{|x| \vee |u-x| \geq 4\}}$$

$$\leq \sum_{u,t,v,x} ⊲⊳^{(9)}(u,t,v,x) = \mathcal{O}(\Omega^{-2})$$

by Lemma 4.2. In the above, a line decorated with a '$\sim$' symbol denotes a direct edge. Similarly, when $|v| \geq 3$ or $|x-v| \geq 3$, the contribution to (5.16) is at most

$$p_c^2 \sum \mathbb{1}_{\{|v| \vee |x-v| \geq 3\}}$$

$$\leq p_c \triangle^{\bullet}_{p_c} \big(\tau^{(3)} * \tau * \tau^{\bullet} * J\big)(\mathbf{0}) + p_c^2 \sum \mathbb{1}_{\{|x-v| \geq 3\}}$$

$$\leq p_c \triangle^{\bullet}_{p_c} \sum_{u,v} \triangle^{(6)}(\mathbf{0},u,v) + p_c^4 \big(J^{*3} * \tau^{*3}\big)(\mathbf{0}) + p_c \triangle^{\bullet\circ}_{p_c} \sum_{t,x} \triangle^{(6)}(\mathbf{0},t,x) = \mathcal{O}(\Omega^{-2}).$$

We now investigate (5.16) by splitting the double sum over v and x. We organize this by considering the three main cases for $|v| \in \{0,1,2\}$. An overview of the contributions is given in the following table:

$\widehat{\Pi}^{(2)}$:	$x = \mathbf{0}$	$\lvert x\rvert = 1$	$\lvert x\rvert = 2$	$\lvert x\rvert = 3$
$v = \mathbf{0}$		$2\Omega^{-1}$	Ω^{-1}	
$\lvert v\rvert = 1$	Ω^{-1}		$2\Omega^{-1}$	Ω^{-1}
$\lvert v\rvert = 2$		Ω^{-1}	Ω^{-1}	Ω^{-1}

Contributions of $v = \mathbf{0}$. The events $E'(u,v;\mathscr{C}_0)_1$ and $\{v \in \langle \mathscr{C}_0 \rangle\}$ hold.

The case of $|x| = 1$ contributes $2\Omega^{-1} + \mathcal{O}(\Omega^{-2})$: First, consider the choice of $x = u$. It is easy to see that the event in (5.16) holds and the contribution is $\Omega p_c^2 = \Omega^{-1} + \mathcal{O}(\Omega^{-2})$.

Consider $0 \sim x \neq u$. As $v \sim x$, we have $E'(v,x;\mathscr{C}_1)_2 = \{x \in \langle \mathscr{C}_1 \rangle\}_1$. If $x = -u$, then $\{x \in \langle \mathscr{C}_1 \rangle\}_1 \subseteq \{u \xleftrightarrow{(4)} x\}_1$ and we receive a contribution of order $\mathcal{O}(\Omega^{-2})$. Consider now one of the $\Omega - 2$ remaining choices for x and set $z = u + x$. Then

$$\mathbb{P}(x \in \langle \mathscr{C}_1 \rangle) = \mathbb{P}(z \in \omega_1) + \mathbb{P}(z \notin \omega_1, x \in \langle \mathscr{C}_1 \rangle) = p_c + \mathcal{O}(\tau^{(4)}(x-u)) = p_c + \mathcal{O}(\Omega^{-2}),$$

yielding a contribution to (5.16) of $\Omega(\Omega-2)p_c^3 + \mathcal{O}(\Omega^{-2}) = \Omega^{-1} + \mathcal{O}(\Omega^{-2})$.

The case of $|x| = 2$ contributes $\Omega^{-1} + \mathcal{O}(\Omega^{-2})$: If $|u - x| = 3$, then the contribution to (5.16) is bounded by $\Omega^3 p_c^2 \tau^{(2)}(x-v)\tau^{(3)}(u-x) = \mathcal{O}(\Omega^{-2})$. Similarly, if $x = 2u$, we obtain a bound of $\Omega p_c^2 \tau^{(2)}(x-v) = \mathcal{O}(\Omega^{-2})$. Let therefore x be one of the $\Omega - 2$ remaining neighbors of u and note that $\{x \in \langle \mathscr{C}_1 \rangle\}$ holds.

We set $z = x - u$. If $z \notin \omega_2$, then $E'(v,x;\mathscr{C}_1)_2 \subseteq \{v \xleftrightarrow{(4)} x\}_2$ by Observation 4.3, and the contribution to (5.16) is at most $\Omega^2 p_c^2 \tau^{(4)}(x-v) = \mathcal{O}(\Omega^{-2})$. If $z \in \omega_2$, then $E'(v,x;\mathscr{C}_1)_2 = \{z \notin \mathsf{Piv}(v,x)\}_2 \cup \{z \notin \langle \mathscr{C}_1 \rangle\}_1$. By a similar argument to the one below (5.9), the contribution to (5.16) becomes

$$\Omega(\Omega-2)p_c^2 \mathbb{P}\Big(\{z \in \omega_2\} \cap \big(\{z \notin \mathsf{Piv}(v,x)\}_2 \cup \{z \notin \langle \mathscr{C}_1 \rangle\}_1\big)\Big) = \Omega^2 p_c^3 (1 - \mathcal{O}(\Omega^{-1})) = \Omega^{-1} + \mathcal{O}(\Omega^{-2}).$$

The case of $|x| = 3$ contributes $\mathcal{O}(\Omega^{-2})$: Distinguishing between $|u - x| = 4$ (at most Ω^3 choices for x) and $|u - x| = 2$ (at most Ω^2 choices), the contribution to (5.16) is at most

$$\Omega p_c^2 \tau^{(3)}(x-v)\big(\Omega^3 \tau^{(4)}(u-x) + \Omega^2 \tau^{(2)}(u-x)\big) = \mathcal{O}(\Omega^{-2}).$$

Contributions of $|v| = 1$. Let us first consider $v = -u$ and show that this case contributes $\mathcal{O}(\Omega^{-2})$. Indeed, $E'(u, v; \mathscr{C}_0)_1 \subseteq \{u \overset{(4)}{\longleftrightarrow} v\}_1$ by Observation 4.3. With the further inclusion $E'(v, x; \mathscr{C}_1)_2 \cap \{x \in \langle \mathscr{C}_1 \rangle\} \subseteq \{v \longleftrightarrow x\}_2$, we have that the contribution to (5.16) is at most

$$\begin{aligned}\Omega p_c^2 \tau^{(4)}(u-v)\Big(&\sum_{|x|=1} \tau^{(2)}(x-v) + \sum_{x: v\sim x} 1 + \sum_{|x|=2, |x-v|=3} \tau^{(3)}(x-v) \\ &+ \sum_{|x|=3, |x-v|=2} \tau^{(2)}(x-v) + \sum_{|x|=3, |x-v|=4} \tau^{(4)}(x-v)\Big) \\ &\le \mathcal{O}(\Omega^{-3})\Big(\Omega \mathcal{O}(\Omega^{-1}) + \Omega + \Omega^2 \mathcal{O}(\Omega^{-2}) + \Omega^2 \mathcal{O}(\Omega^{-1}) + \Omega^3 \mathcal{O}(\Omega^{-3})\Big) = \mathcal{O}(\Omega^{-2}).\end{aligned}$$

We may therefore take $v \neq \pm u$ to be one of the $\Omega - 2$ remaining neighbors of the origin. Set $t = v + u$. We first claim that $t \notin \omega_1$ results in an $\mathcal{O}(\Omega^{-2})$ contribution. Note that, by Observation 4.3, $E'(u, v; \mathscr{C}_0)_1 \cap \{t \notin \omega_1\} \subseteq \{u \overset{(4)}{\longleftrightarrow} v\}_1$. As there is only one choice of x such that $u \sim x \sim v$ and at most Ω choices such that $|x| = 3$ and $x \sim v$, we can bound (5.16) by

$$\begin{aligned}\Omega^2 p_c^2 \sum_{x \in \mathbb{Z}^d} \Big(&\tau^{(4)}(v-u)\big(\mathbb{1}_{\{x=\mathbf{0}\}} + \mathbb{1}_{\{|x|=1\}}\tau^{(2)}(x-v) + \mathbb{1}_{\{|x|=2, u\sim x\sim v\}} \\ &+ \mathbb{1}_{\{|x|=3, |u-x|=2=|v-x|\}}\tau^{(2)}(x-v)\big) + \mathbb{1}_{|x|=2, v\sim x}\mathbb{P}\big(E'(u,v;\mathscr{C}_0)_1 \cap \{t \notin \omega_1\} \cap \{x \in \langle \mathscr{C}_1 \rangle\}\big)\Big) \\ \le &\mathcal{O}(\Omega^{-2})\big(2 + 2\Omega\tau^{(2)}(x-v)\big) + \mathcal{O}(1) \sum_{|x|=2, x\sim v} \mathbb{P}^{(2)}\big(E'(u,v;\mathscr{C}_0)_1 \cap \{t \notin \omega_1\} \cap \{x \in \langle \mathscr{C}_1 \rangle\}\big).\end{aligned}$$

It remains to bound the last probability. There are at most Ω choices for x. If $\{u \overset{(5)}{\longleftrightarrow} x\}$, then the contribution is $\mathcal{O}(\Omega^{-2})$. Note that the u-v-path in ω_1 cannot use and is independent of the status of $\mathbf{0}$, as the origin may not be a pivotal point. Hence, if $\mathbf{0} \in \omega_1$, the contribution is at most $\Omega p_c \tau^{(4)}(v-u) = \mathcal{O}(\Omega^{-2})$. We therefore assume $\mathbf{0} \notin \omega_1$ and aim to bound

$$\Omega \mathbb{P}\big(\{u \overset{(4)}{\longleftrightarrow} v\}_1 \cap \{\mathbf{0}, t \notin \omega_1\} \cap \{u \overset{(\le 3)}{\longleftrightarrow} x\}_1\big) \tag{5.17}$$

When avoiding $\mathbf{0}$ and t, there are only two u-x-paths of length 3, namely $\gamma_1 = (u, y, z, x)$ and $\gamma_2 = (u, y, y-u, x)$, where $y := x + u - v$ and $z := y + v$. See Figure 3a for an illustration. But now, (5.17) is bounded by

$$\begin{aligned}\Omega \mathbb{P}\Big(&\{\mathbf{0}, t \notin \omega_1\} \cap \bigcup_{i=1,2} \bigcup_{s \in \gamma_i \setminus \{x\}} \{\gamma_i \subseteq \omega_1\} \circ \{s \longleftrightarrow v\}_1\Big) \\ &\le 2\Omega p_c^2 \big(\tau^{(4)}(v-u) + \tau^{(3)}(y-v) + 2\tau^{(2)}(z-v)\big) = \mathcal{O}(\Omega^{-2}).\end{aligned}$$

As a consequence, we can focus on $t \in \omega_1$, and (5.16) reduces to

$$\Omega(\Omega-2) p_c^2 \sum_{x \in \mathbb{Z}^d} \mathbb{P}\Big(E'(u,v;\mathscr{C}_0)_1 \cap E'(v,x;\mathscr{C}_1)_2 \cap \{t \in \omega_1, x \in \langle \mathscr{C}_1 \rangle\}_1\Big).$$

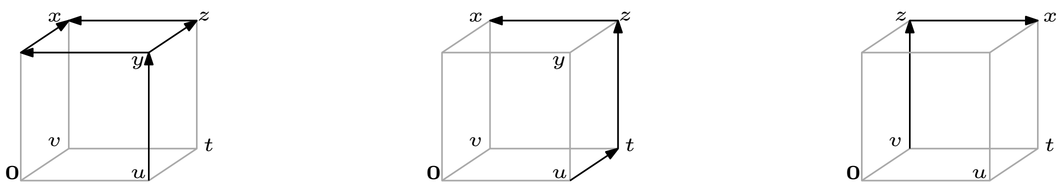
But under $t \in \omega_1$, we have $E'(u, v; \mathscr{C}_0)_1 = \{t \notin \mathsf{Piv}(u, v)\}_1 \cup \{t \notin \langle \mathscr{C}_0 \rangle\}_0$. The latter event has probability $1 - \mathcal{O}(\Omega^{-1})$, and so we can can instead investigate

$$\Omega^2 p_c^2 (1 - \mathcal{O}(\Omega^{-1})) \sum_{x \in \mathbb{Z}^d} \mathbb{P}\Big(E'(v,x;\mathscr{C}_1)_2 \cap \{t \in \omega_1, x \in \langle \mathscr{C}_1 \rangle\}_1\Big), \tag{5.18}$$

where u and v are two arbitrary (but fixed) neighbors of $\mathbf{0}$ (satisfying ($u \neq \pm v$).

The contribution of $x = \mathbf{0}$ is $\Omega^{-1} + \mathcal{O}(\Omega^{-2})$: Note that $x \in \langle \mathscr{C}_1 \rangle$ holds, and so does $E'(v, x; \mathscr{C}_1)_2$. Hence, the contribution to (5.18) is $\Omega^{-1} + \mathcal{O}(\Omega^{-2})$.

The contribution of $|x| = 1$ is $\mathcal{O}(\Omega^{-2})$: If $x \in \{\pm u, -v\}$, we can bound the contribution to (5.18) by $\Omega^2 p_c^2 \tau^{(2)}(u-v)\tau^{(2)}(x-v) = \mathcal{O}(\Omega^{-2})$ (as both $\{v \longleftrightarrow x\}_2$ and $\{u \longleftrightarrow v\}_1$ need to hold). Consider



(a) The case $|x| = 2, t \notin \omega_1$. (b) The case $|x| = 2, u \nsim x$. (c) The case $|x| = 3$.

Figure 3: An illustration of several appearing cases for $|v| = 1$. In (a), the two paths from u to x of length 3 that avoid $\mathbf{0}$ and t are drawn. In (b), the path along t, z which ensures $x \in \langle \mathscr{C}_1 \rangle$ for a contribution of Ω^{-1} is drawn. In (c), the scenario $|x - u| = 2 = |v - x|$ is shown, and the path along z ensuring $\{v \longleftrightarrow x\}_2$ is drawn in black.

thus one of the $\Omega - 4$ choices for x satisfying $\dim\langle\!\langle u, v, x \rangle\!\rangle = 3$. Conditional on $t \in \omega_1$, we have $\{x \in \langle \mathscr{C}_1 \rangle\}_1 \subseteq \{u \overset{(2)}{\longleftrightarrow} x\}_1 \cup \{t \overset{(3)}{\longleftrightarrow} x\}_1$, and so the contribution is at most

$$\Omega^3 p_c^3 \tau_p^{(2)}(x - v)\big(\tau_p^{(2)}(x - u) + \tau_p^{(3)}(x - t)\big) = \mathcal{O}(\Omega^{-2}).$$

<u>The contribution of $|x| = 2$ is $2\Omega^{-1} + \mathcal{O}(\Omega^{-2})$:</u> We can restrict to the choices of x where $v \sim x$ by the considerations made in the beginning of the proof.

- Let $x \sim u$. There is only one choice for x such that $|u - x| = |v - x| = 1$, namely $x = t$. For this choice, $E'(v, x; \mathscr{C}_1)_2$ certainly holds, and also $x \in \langle \mathscr{C}_1 \rangle$. We get a contribution of $\Omega^{-1} + \mathcal{O}(\Omega^{-2})$.
- Let $x \nsim u$. There are $\Omega - 2$ choices for x. We first exclude $x = v - u$. As $\mathbb{P}(x \in \langle \mathscr{C}_1 \rangle \mid t \in \omega_1) \leq \tau^{(4)}(x - t) + \tau^{(3)}(x - u) = \mathcal{O}(\Omega^{-2})$, the contribution in total is $\mathcal{O}(\Omega^{-2})$.
  Let now x be one of the $\Omega - 3$ remaining neighbors of v. As $v \sim x$, we have $E'(v, x; \mathscr{C}_1)_2 = \{x \in \langle \mathscr{C}_1 \rangle\}$. We set $z = x + u$ (see Figure 3b) and assume first that $z \notin \omega_1$. Then

  $$\{z \notin \omega_1 \ni t, x \in \langle \mathscr{C}_1 \rangle\} \subseteq \{z \notin \omega_1 \ni t\} \cap \big(\{u \overset{(3)}{\longleftrightarrow} x \text{ off } \{t\} \cup \{t \overset{(4)}{\longleftrightarrow} x\}\big)$$

  and the contribution to (5.18) is at most $\Omega^2 p_c^3 (1 - \mathcal{O}(\Omega^{-1}))(\tau^{(3)}(x - u) + \tau^{(4)}(x - t)) = \mathcal{O}(\Omega^{-2})$. On the other hand, if $z \in \omega_1$, then $x \in \langle \mathscr{C}_1 \rangle$ holds and (5.18) becomes

  $$\Omega^2 p_c^2 (1 - \mathcal{O}(\Omega^{-1}))(\Omega - 3)\mathbb{P}(t, z \in \omega_1) = \Omega^{-1} + \mathcal{O}(\Omega^{-2}).$$

<u>The contribution of $|x| = 3$ is $\Omega^{-1} + \mathcal{O}(\Omega^{-2})$:</u> There are at most Ω^3 choices for x such that $|u - x| = |v - x| = 4$ and there are at most $2\Omega^2$ choices where $|x - u| \neq |v - x|$. The contribution of those x to (5.18) is therefore bounded by

$$\Omega^2 p_c^2 \Big(\sum_{|x|=3} \tau^{(4)}(x - v)\tau^{(4)}(u - x) + 2 \sum_{|x-v|=2\neq|u-x|} \tau^{(2)}(x - v)\tau^{(4)}(u - x) \Big) = \mathcal{O}(\Omega^{-2}).$$

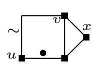
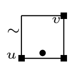
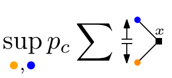
It remains to investigate those x with $|u - x| = 2 = |v - x|$. This is only possible when $x \sim t$. Let first $x = 2u + v$. By Observation 4.3, $E'(v, x; \mathscr{C}_1)_2 \cap \{t \in \omega_1\} \subseteq \{v \overset{(4)}{\longleftrightarrow} x\}$, and (5.18) is at most $\Omega^2 p_c^2 \tau^{(4)}(x - v) = \mathcal{O}(\Omega^{-2})$.

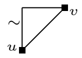
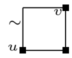
Let now x be one of the $\Omega - 3$ remaining neighbors of t (note that either $\|x\|_\infty = 1$ or $x = 2v + u$). We set $z := x - u$ and point to Figure 3c for an illustration. As t is occupied in ω_1, we have $x \in \langle \mathscr{C}_1 \rangle$. Assume now $z \notin \omega_2$. By Observation 4.3, $E'(v, x; \mathscr{C}_1)_2 \subseteq \{v \overset{(4)}{\longleftrightarrow} x\}_2$ and the contribution to (5.18) is at most $\Omega^2 p_c^3 \tau^{(4)}(x - v) = \mathcal{O}(\Omega^{-2})$. On the other hand, if $z \in \omega_2$, (5.18) becomes

$$(1 + \mathcal{O}(\Omega^{-1}))(\Omega - 3)\mathbb{P}\big(\{t \in \omega_1, z \in \omega_2\} \cap \big(\{z \notin \langle \mathscr{C}_1 \rangle\}_1 \cup \{z \notin \mathsf{Piv}(v, x)\}_2\big)\big) = \Omega^{-1} + \mathcal{O}(\Omega^{-2}).$$

Again, we have used that $\{z \notin \langle \mathscr{C}_1 \rangle\}_1$ has probability $1 - \mathcal{O}(\Omega^{-1})$ conditional on $t \in \omega_1$.

Contributions of $|v| = 2$. We first show that when $|u - v| = 3$, no relevant contributions arise. Indeed, for those v, (5.13) is at most

$p_c^2 \sum \mathbb{1}_{\{|v|=2, |u-v|=3\}}$  $\leq p_c \sum \Big(\mathbb{1}_{\{|v|=2, |u-v|=3\}}$  $\Big(\sup p_c \sum$  $\Big)\Big)$

$$\le \triangle^{\bullet}_{p_c}\Big(\sum \mathbb{1}_{\{|v|=2,|u-v|=3\}} p_c$$  $$+ p_c^2 \sum \mathbb{1}_{\{|v|=2,|u-v|=3\}}$$  $$\Big)$$

$$\le \triangle^{\bullet}_{p_c}\Big(p_c \sum_{u,v} \triangle^{(6)}(u,v,\mathbf{0}) + 2p_c^4\big(J^{*3} * \tau^{*3}\big)(\mathbf{0})\Big) = \mathcal{O}(\Omega^{-2}).$$

Moreover, $v = 2u$ implies $\{v \in \langle \mathscr{C}_0\rangle\}_0 \subseteq \{\mathbf{0} \overset{(4)}{\longleftrightarrow} v\}_0$. We can thus bound the contribution to (5.16) by $\Omega p_c \triangle^{\bullet}_p \tau^{(4)}(v) = \mathcal{O}(\Omega^{-2})$. Let v be one of the $\Omega - 2$ remaining neighbors of u, implying $E'(v,u;\mathscr{C}_0) = \{v \in \langle \mathscr{C}_0\rangle\}_0$. Let $z = v - u$. Then for $v \in \langle \mathscr{C}_0\rangle$ to hold, either $z \in \omega_0$ or there must be a path of length at least 4. In the latter case, we can bound (5.16) by $p_c^2 \sum_{u,v,t,x} ⧰^{(9)}(u,t,v,x) = \mathcal{O}(\Omega^{-2})$. We can therefore restrict to investigating

$$\Omega(\Omega-2)p_c^3 \sum_{x\in\mathbb{Z}^d} \mathbb{P}\Big(E'(v,x;\mathscr{C}_1)_2 \cap \{x \in \langle \mathscr{C}_1\rangle\}_1\Big), \tag{5.19}$$

where u is an arbitrary (but fixed) neighbor of $\mathbf{0}$ and $v \notin \{\mathbf{0}, 2u\}$ is some fixed neighbor of u.

The contribution of $x = \mathbf{0}$ is $\mathcal{O}(\Omega^{-2})$: As $\{\mathbf{0} \longleftrightarrow v\}_2$ needs to hold, we get a bound on (5.19) by $\Omega^2 p_c^3 \tau^{(2)}(v) = \mathcal{O}(\Omega^{-2})$.

The contribution of $|x| = 1$ is $\Omega^{-1} + \mathcal{O}(\Omega^{-2})$: We only need to consider $|v - x| = 1$, and there are two such choices for x. If $x = v - u$, then the contribution is bounded by $\Omega^2 p_c^3 \tau^{(2)}(u - x) = \mathcal{O}(\Omega^{-2})$.

On the other hand, if $x = u$, both $E'(v,x;\mathscr{C}_1)_2$ and $\{x \in \langle \mathscr{C}_1\rangle\}_1$ hold and the contribution to (5.19) is $\Omega^{-1} + \mathcal{O}(\Omega^{-2})$.

The contribution of $|x| = 2$ is $\Omega^{-1} + \mathcal{O}(\Omega^{-2})$: Note that only $|v - x| = 2$ may produce relevant contributions. Writing $v = u + z$, we first consider $x = u - z$. Again, $E'(v,x;\mathscr{C}_1)_2 \subseteq \{v \overset{(4)}{\longleftrightarrow} x\}_2$ by Observation 4.3, and so the contribution to (5.19) is at most $\Omega^2 p_c^2 \tau^{(4)}(v - x) = \mathcal{O}(\Omega^{-2})$. Similarly, If $|u - x| = 3$, the contribution is at most $\Omega^3 p_c^3 \tau^{(2)}(v - x)\tau^{(3)}(u - x) = \mathcal{O}(\Omega^{-2})$.

Let now y be one of the $\Omega - 4$ unit vectors satisfying $\dim\langle\!\langle u,z,y\rangle\!\rangle = 3$. Write $x = u + y$ and set $t = x + z = v + y$. We claim that we only get a relevant contribution if $t \in \omega_2$: As $\{t \notin \omega_2\} \subseteq \{v \overset{(4)}{\longleftrightarrow} x\}_2$ by Observation 4.3, this gives a bound on the contribution to (5.19) by $\Omega^3 p_c^3 \tau^{(4)}(x - v) = \mathcal{O}(\Omega^{-2})$. Under $t \in \omega_2$, (5.19) becomes

$$\begin{aligned} &\Omega^3(1 - \mathcal{O}(\Omega^{-1}))p_c^3 \mathbb{P}\Big(\{t \in \omega_2\} \cap \big(\{t \notin \mathsf{Piv}(v,x)\}_2 \cup \{t \notin \langle \mathscr{C}_1\rangle\}_1\big)\Big) \\ &\qquad = \Omega^3(1 - \mathcal{O}(\Omega^{-1}))p_c^4(1 - \mathcal{O}(\Omega^{-1})) = \Omega^{-1} + \mathcal{O}(\Omega^{-2}). \end{aligned} \tag{5.20}$$

The contribution of $|x| = 3$ is $\Omega^{-1} + \mathcal{O}(\Omega^{-2})$: We only need to consider terms where $|v - x| = 1$. Let $x = v + y$, where $|y| = 1$. If $y = z$, then $\{x \in \langle \mathscr{C}_1\rangle\} \subseteq \{v \overset{(4)}{\longleftrightarrow} x\}_1$ and the contribution to (5.19) is $\mathcal{O}(\Omega^{-3})$. For the other $\Omega - 2$ choices for x, we set $t = u + y$. When $t \notin \omega_2$, we require $\{x \in \langle \mathscr{C}_1\rangle\} \subseteq \{v \overset{(4)}{\longleftrightarrow} x\}_1$ and the contribution is $\mathcal{O}(\Omega^{-2})$. When $t \in \omega_2$, the contribution is identical to (5.20) and hence $\Omega^{-1} + \mathcal{O}(\Omega^{-2})$. □

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Chapter 5

Lace Expansion and Mean-Field Behavior for the Random Connection Model

The chapter is a revised version of

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This preprint is available online at `https://arxiv.org/abs/1908.11356`.

My own contribution. This article is joint work with my supervisor, Markus Heydenreich, as well as Remco van der Hofstad and Günter Last. Günter Last contributed Lemma 3.3, while I did most of the writing of the remainder of the paper. Numerous personal meetings and phone or video calls between the other authors and me, where ideas were obtained collaboratively, were essential to this article.

Lace Expansion and Mean-Field Behavior for the Random Connection Model

Markus Heydenreich* Remco van der Hofstad† Günter Last‡ Kilian Matzke*

August 5, 2020

Abstract

We consider the random connection model for three versions of the connection function φ: A finite-variance version (including the Boolean model), a spread-out version, and a long-range version. We adapt the lace expansion to fit the framework of the underlying continuum-space Poisson point process to derive the triangle condition in sufficiently high dimension and furthermore to establish the infra-red bound. From this, mean-field behavior of the model can be deduced. As an example, we show that the critical exponent γ takes its mean-field value $\gamma = 1$ and that the percolation function is continuous.

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Contents

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1 Introduction

1.1 The random connection model

Consider a stationary Poisson point process (PPP) η on $\mathbb{R}^d$ with intensity $\lambda \geq 0$ along with a measurable *connection function* $\varphi \colon \mathbb{R}^d \to [0,1]$. For $x \in \mathbb{R}^d$, assume that

$$\varphi(x) = \varphi(-x). \tag{1.1}$$

We assume that φ satisfies

$$0 < q_\varphi := \int \varphi(x)\,\mathrm{d}x < \infty. \tag{1.2}$$

Suppose any two distinct points $x, y \in \eta$ form an edge with probability $\varphi(y-x)$ independently of all other pairs and independently of η. This yields the random connection model (RCM), an undirected random graph denoted by ξ, whose vertex set $V(\xi)$ is η and whose edge set we denote by $E(\xi)$. We point to Section 1.4 for a brief literature overview of the RCM.

We stress the difference between η and ξ, as the former is used to denote a random set of *points*, whereas the latter is a *random graph*, which contains complete information about η as well as the additional information about edges between the points of η. It is convenient to define ξ on a probability space $(\Omega, \mathcal{F}, \mathbb{P}_\lambda)$ and to treat λ as a parameter.

For $x, y \in \mathbb{R}^d$, we use η^x and ξ^x (respectively, $\eta^{x,y}$ and $\xi^{x,y}$) to denote a PPP and an RCM augmented by x (respectively, by x and y). For ξ, this includes the random set of edges incident to x or $\{x,y\}$, with the edge probabilities governed by φ. We refer to Section 2.2 for a formal construction of the model.

We write $x \sim y$ if $\{x,y\} \in E(\xi)$ and say that x and y are *neighbors* (or *adjacent*). For $x, y \in \mathbb{R}^d$, we say that x and y are *connected* and write $x \longleftrightarrow y$ in ξ if either $x = y$ or if there is a *path* in ξ connecting x and y—that is, there are distinct $x = v_0, v_1, \dots, v_k, v_{k+1} = y \in \eta$ (with $k \in \mathbb{N}_0$) such that $v_i \sim v_{i+1}$ for $0 \leq i \leq k$. Note that for $x \neq y$ to be connected, they need to be points of η (for arbitrary $x, y \in \mathbb{R}^d$, we will often speak of the event $\{x \longleftrightarrow y \text{ in } \xi^{x,y}\}$). For $x \in \mathbb{R}^d$, we define $\mathscr{C}(x) = \mathscr{C}(x, \xi^x) = \{y \in \eta^x : x \longleftrightarrow y \text{ in } \xi^x\}$ to be the *cluster* of x.

We define the pair-connectedness (or two-point) function $\tau_\lambda \colon \mathbb{R}^d \to [0,1]$ to be

$$\tau_\lambda(x) = \mathbb{P}_\lambda(\mathbf{0} \longleftrightarrow x \text{ in } \xi^{\mathbf{0},x}), \tag{1.3}$$

where $\mathbf{0}$ denotes the origin in $\mathbb{R}^d$. The two-point function can also be defined as $\tau_\lambda \colon \mathbb{R}^d \times \mathbb{R}^d \to [0,1]$ via $\tau_\lambda(x,y) = \mathbb{P}_\lambda(x \longleftrightarrow y \text{ in } \xi^{x,y})$. The two definitions relate by translation invariance, as $\tau_\lambda(x,y) = \tau_\lambda(x-y)$, and we stick to (1.3) throughout this paper. Observe that $\lambda \mapsto \tau_\lambda(x)$ is increasing. We next define the *percolation function* $\lambda \mapsto \theta(\lambda)$ as

$$\theta(\lambda) = \mathbb{P}_\lambda(|\mathscr{C}(\mathbf{0})| = \infty),$$

where $|\mathscr{C}(x)|$ denotes the number of vertices in $\mathscr{C}(x)$. Note that $|\mathscr{C}(\mathbf{0})|$ has the same distribution as $|\mathscr{C}(x)|$ for any $x \in \mathbb{R}^d$ due to translation invariance. We next define the *critical value* for the RCM as

$$\lambda_c = \inf\{\lambda \geq 0 : \theta(\lambda) > 0\}.$$

To state our main theorem, for an (absolutely) integrable function $f \colon \mathbb{R}^d \to \mathbb{R}$, we define the Fourier transform of f to be

$$\widehat{f}(k) = \int \mathrm{e}^{\mathrm{i}k\cdot x} f(x)\,\mathrm{d}x \qquad (k \in \mathbb{R}^d),$$

where $k \cdot x = \sum_{j=1}^d k_j x_j$ denotes the scalar product. We next define the *expected cluster size* as

$$\chi(\lambda) := \mathbb{E}_\lambda[|\mathscr{C}(\mathbf{0})|] = 1 + \lambda \int \tau_\lambda(x)\,\mathrm{d}x = 1 + \lambda\widehat{\tau}_\lambda(\mathbf{0}), \tag{1.4}$$

where $\mathbf{0}$ also denotes the origin in the Fourier dual (which is also $\mathbb{R}^d$). The second, elementary but helpful, identity is proved in (2.23). This allows us to define

$$\lambda_T := \sup\{\lambda \geq 0 : \chi(\lambda) < \infty\},$$

and to point out that $\lambda_T = \lambda_c$ (proven by Meester [27]).

Let us now specify φ. In the finite-variance model defined below, our goal is to obtain a result valid for all dimensions $d \geq d_0$ for some d_0, and so we are interested in a sequence $(\varphi_d)_{d\in\mathbb{N}}$, where $\varphi_d \colon \mathbb{R}^d \to [0,1]$. Meester et al. [28] demonstrate a simple way to do this: They take a function $\tilde{\varphi} \colon \mathbb{R}_{\geq 0} \to [0,1]$ and model the RCM such that two points $x, y \in \mathbb{R}^d$ are connected with probability $\tilde{\varphi}(|x-y|)$, where $|\cdot|$ denotes Euclidean distance. The integrability condition for φ then becomes $\int t^{d-1}\tilde{\varphi}(t)\,\mathrm{d}t < \infty$.

1.2 Conditions on the connection function

Similarly to Heydenreich et al. [21], we consider three classes (or versions) for $\varphi \colon \mathbb{R}^d \to [0,1]$. Those are the "finite-variance" case, the "finite-variance spread-out" (or simply "spread-out") case, and the "long-range spread-out" (or simply "long-range") case. The second version comes with an extra spread-out parameter L, the third with L as well as another parameter α, controlling the power-law decay of φ. For each of these three cases, we make several assumptions. We give at least one example for each of the three versions.

(H1): Finite-variance model. We require φ to satisfy the following three properties:

(H1.1) The density $\varphi \equiv \varphi_d$ has a second moment, i.e. $\int |x|^2 \varphi(x)\,\mathrm{d}x < \infty$ for all d.

(H1.2) There is a function $g \colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ such that for $m \geq 3$, the m-fold convolution $\varphi^{\star m}$ of φ satisfies

$$q_\varphi^{-m} \sup_{x\in\mathbb{R}^d} \varphi^{\star m}(x) \leq g(d) = o(1) \qquad \text{as } d \to \infty$$

(we point to the notational remarks in Section 1.5 for a definition of the m-fold convolution). For $m = 2$, we make the (weaker) assumption that there exists some ε with $0 \leq \varepsilon < r_d := \pi^{-1/2}\Gamma(\frac{d}{2}+1)^{1/d}$ (i.e., r_d is the radius of the ball of volume 1) such that

$$q_\varphi^{-2} \sup_{x:|x|\geq\varepsilon} (\varphi \star \varphi)(x) \leq g(d) = o(1) \qquad \text{as } d \to \infty.$$

(H1.3) The Fourier transform $\widehat{\varphi}$ of φ satisfies

$$\liminf_{k\to\mathbf{0}} (1 - q_\varphi^{-1}\widehat{\varphi}(k))/|k|^2 > 0. \tag{1.5}$$

We now present two examples for functions that satisfy (H1). The first is $\varphi(x) = \mathbb{1}_{\{|x|\leq r\}}$ for $r > 0$, the Poisson blob model or (spherical) Boolean model. It is the most prominent example of a continuum-percolation model and in some sense the easiest of the ones commonly investigated.

Another very natural connection function is the density of a d-dimensional (independent) Gaussian, given in its standardized form by $\varphi(x) = (2\pi)^{-d/2}\exp(-|x|^2/2)$. We denote this density by $\varphi_{\mathcal{N}}$. Even though $\varphi_{\mathcal{N}}$, very much unlike the Poisson blob model, is supported on the whole space $\mathbb{R}^d$, its light tail allows us to treat it in the same way.

(H2): Spread-out model. In this version, we introduce a new parameter $L \geq 1$, upon which $\varphi = \varphi_L$ depends. It describes the range of the model and will be taken to be large. In turn, and as opposed to (H1), we can think of the dimension d as fixed. We make the following assumptions:

(H2.1) For every $L \geq 1$, the second moment exists, i.e. $\int |x|^2\varphi_L(x)\,\mathrm{d}x < \infty$.

(H2.2) There exists a constant C such that, for all $L \geq 1$, $\|\varphi_L\|_\infty \leq CL^{-d}$, where $\|\varphi_L\|_\infty = \sup_{x\in\mathbb{R}^d}\varphi_L(x)$.

(H2.3) There are constants $b, c_1, c_2 > 0$ (independent of L) such that

$$1 - q_{\varphi_L}^{-1}\widehat{\varphi}_L(k) \geq \begin{cases} c_1 L^2|k|^2 & \text{for } |k| \leq bL^{-1}, \\ c_2 & \text{for } |k| > bL^{-1}. \end{cases}$$

The introduction of a spread-out parameter illustrates that $d > 6$ is the requirement that is supposedly sufficient also for (H1) (see Theorem 1.2). Let us now give an example for (H2). Let $h\colon \mathbb{R}^d \to [0,1]$ satisfy $0 < \int h(x)\,\mathrm{d}x < \infty$ and $h(x) = h(-x)$. Furthermore, assume $\int |x|^2 h(x)\,\mathrm{d}x < \infty$ and assume that $\int x_j x_l h(x)\,\mathrm{d}x \geq 0$ for all $j,l \in [d]$. Then

$$\varphi_L(x) = L^{-d} h(x/L) \qquad \text{for } x \in \mathbb{R}^d \tag{1.6}$$

satisfies (H2.1)-(H2.3). An explicit example in this class is $h(x) = \mathbb{1}_{\{|x|\leq r\}}$. Another explicit example is $h(x) = \mathbb{1}_{\{|x_i|\leq b_i \forall i \in [d]\}}$ for a collection $(b_i)_{i=1}^d$ of positive values.

(H3): Long-range spread-out model. We introduce an additional parameter $\alpha > 0$ (describing the long-range behavior of φ) so that $\varphi = \varphi_L = \varphi_{L,\alpha}$ depends on both L and α (as in (H2), d is fixed). The assumptions are now as follows:

(H3.1) For all $0 < \varepsilon \leq \alpha$, we have $\int |x|^{\alpha-\varepsilon} \varphi_L(x)\,\mathrm{d}x < \infty$.

(H3.2) =(H2.2).

(H3.3) There are constants $b, c_1, c_2 > 0$ (independent of L) such that

$$1 - q_{\varphi_L}^{-1} \widehat{\varphi}_L(k) \geq \begin{cases} c_1 (L|k|)^{\alpha\wedge 2} & \text{for } |k| \leq bL^{-1}, \\ c_2 & \text{for } |k| > bL^{-1}. \end{cases}$$

We introduce (H3) in order to observe long-range interactions, where $\varphi(x)$ decays as some inverse power of $|x|$ as $|x| \to \infty$. Even though the long-range model is defined for $\alpha > 0$, we remark that the interesting regime arises from $\alpha \leq 2$, as $\alpha > 2$ does not differ from the spread-out model (H2). As an example, set

$$h(x) = \frac{1}{(|x| \vee 1)^{d+\alpha}} \quad \text{for } x \in \mathbb{R}^d, \tag{1.7}$$

and define φ_L as in (1.6).

The following proposition verifies the required conditions for the respective examples. It is proved in the appendix A.

Proposition 1.1 (Verification of conditions for connection-function examples)**.**

(a) There exists $\rho \in (0,1)$ such that the Boolean as well as the Gaussian density satisfy (H1.1)-(H1.3) with $g(d) = \rho^d$.

(b) For non-negative, bounded, and radially symmetric h with $0 < \int |x|^2 h(x)\,\mathrm{d}x < \infty$, the density defined in (1.6) *satisfies (H2.1)-(H2.3).*

(c) The density defined in (1.6) *with h as in* (1.7) *satisfies (H3.1)-(H3.3).*

We set $\alpha = \infty$ for (H1) and (H2). This convention allows us to refrain from tedious case distinctions in later statements. In order to state the main theorem, we introduce a parameter β, dependent on the version of φ, as

$$\beta := \begin{cases} g(d)^{\frac{1}{4}} & \text{for (H1)}, \\ L^{-d} & \text{for (H2), (H3)}. \end{cases} \tag{1.8}$$

The function g in the definition of β is the same as in (H1.2). Our methods crucially depend on the fact that β can be made arbitrarily small. Not only is β important for the statement of the main theorem, it will also appear prominently throughout the paper. Whenever we speak of small β in this paper, we refer to large d for (H1) and to large L (with *fixed* d) for (H2), (H3). In particular, whenever the Landau notation $\mathcal{O}(\beta)$ appears, the asymptotics are $d \to \infty$ for (H1) and $L \to \infty$ for (H2) and (H3).

1.3 Main results

The main contribution of this paper is the establishment of the triangle condition as well as the infra-red bound for the RCM in dimension $d > 3(\alpha \wedge 2)$ and for β sufficiently small. This is achieved using the lace expansion. To formulate our main theorem, we define the *triangle* by

$$\triangle_\lambda(x) := \lambda^2 \iint \tau_\lambda(z)\tau_\lambda(y-z)\tau_\lambda(x-y)\,\mathrm{d}z\,\mathrm{d}y, \qquad \text{and } \triangle_\lambda := \sup_{x\in\mathbb{R}^d} \triangle_\lambda(x).$$

Then by the *triangle condition*, we mean that $\triangle_{\lambda_c} < \infty$. Our main result is the following theorem:

Theorem 1.2 (Infra-red bound and triangle condition)**.**

1. *If φ satisfies (H1), then there is $d^* > 12$ and a constant C such that, for all $d \geq d^*$,*

$$\lambda|\widehat{\tau}_\lambda(k)| \leq \frac{|\widehat{\varphi}(k)| + C\beta}{\widehat{\varphi}(\mathbf{0}) - \widehat{\varphi}(k)} \qquad (k \in \mathbb{R}^d) \tag{1.9}$$

   as well as $\triangle_\lambda \leq C\beta$, and both bounds are uniform in $\lambda \in [0, \lambda_c]$ (the right-hand side of (1.9) *is understood to be $+\infty$ for $k = \mathbf{0}$).*

2. *Let $d > 3(\alpha \wedge 2)$. If φ satisfies (H2) or (H3), then there is $L^* \geq 1$ and C such that* (1.9) *and $\triangle_\lambda \leq C\beta$ both hold uniformly in $\lambda \in [0, \lambda_c]$ for all $L \geq L^*$.*

Theorem 1.2 has multiple consequences (some of them are listed in Theorem 1.3), such as asymptotics of λ_c (as $d \to \infty$ or $L \to \infty$) and continuity of $\lambda \mapsto \theta(\lambda)$. Furthermore, Theorem 1.2 enables us to prove mean-field behavior in the sense that several critical exponents take their mean-field values. For example, the exponent γ is the dimension-dependent value that governs the predicted behavior

$$\chi(\lambda) \sim (\lambda_c - \lambda)^{-\gamma}, \qquad (\lambda \nearrow \lambda_c). \tag{1.10}$$

This definition of γ already assumes a certain behavior of $\chi(\lambda)$. It is also predicted that for $d > d_c$, where d_c is the *upper critical dimension* believed to be $d_c = 6$ for percolation, γ no longer depends on the dimension (it takes its *mean-field value*). We prove that the critical exponent γ exists (in a bounded-ratio sense) and takes its mean-field value 1. We point to the book [20], where other exponents (in bond percolation on the lattice) are discussed.

Theorem 1.3 (The critical point and $\gamma = 1$)**.** *Under (H1), there is $d^* > 12$ such that for all $d \geq d^*$, and under (H2) or (H3), there is $L^* \geq 1$ such that for all $L \geq L^*$*

$$\lambda(\lambda_c - \lambda)^{-1} \leq \chi(\lambda) \leq \lambda(1 + C\beta)(\lambda_c - \lambda)^{-1} \qquad \textit{for } \lambda < \lambda_c, \tag{1.11}$$

that is, the critical exponent γ takes its mean-field value 1*. Under (H1), $C = C(d^*)$ and under (H2) or (H3), $C = C(d, L^*)$. Furthermore, $\theta(\lambda_c) = 0$ and $1 \leq \lambda_c q_\varphi \leq 1 + C\beta$.*

We note that the proof of Theorem 1.3 gives the stronger result that the lower bound on $\chi(\lambda)$ in (1.11) is valid in all dimensions and does not require any set of assumptions (H1), (H2), or (H3). This implies $\chi(\lambda_c) = \infty$ for the general RCM.

As a consequence of Corollary 5.3, we get an explicit identity for λ_c. In particular, we define a function Π_λ in Proposition 5.2 that satisfies

$$\lambda_c = \frac{1}{1 + \widehat{\Pi}_{\lambda_c}(\mathbf{0})}. \tag{1.12}$$

As $\lambda \mapsto \theta(\lambda)$ is non-decreasing and the decreasing limit of continuous functions, we have that θ is continuous from the right for all $\lambda \geq 0$ (see [14, Lemma 8.9]). The fact that $\theta(\lambda_c) = 0$ implies that $\lambda \mapsto \theta(\lambda)$ is continuous on $[0, \infty)$, since the left-continuity of θ for $\lambda > \lambda_c$ can be shown by standard arguments (see [14, Lemma 8.10]). The asymptotics of λ_c for $d \to \infty$ were already shown by Meester et al. [28]. The asymptotics in the spread-out case were shown in a slightly weaker form in [32].

1.4 Literature overview and discussion

We first give some general background on percolation theory, then highlight the important literature on continuum percolation and the RCM. After this, we put the results of this paper into context.

The foundations of percolation theory are generally attributed to Broadbent and Hammersley in 1957 [7]. Several textbooks were published, we refer to Grimmett [14] as a standard reference, and Bollobás and Riordan [5], which puts an extra focus on two-dimensional percolation. More recent treatments of two-dimensional percolation are the book by Werner [39] and the survey by Beffara and Duminil-Copin [3].

A book on percolation in high dimensions was written by the first two authors [20]. It contains a self-contained proof of the lace expansion for bond percolation as well as an extensive summary of recent results on high-dimensional percolation. Another detailed description of the lace expansion is given by Slade [35], with a focus on self-avoiding walk. One of the corner stones of high-dimensional percolation is the seminal 1990 paper by Hara and Slade [18], successfully applying the lace expansion to bond percolation on $\mathbb{Z}^d$ (among other models) in sufficiently high dimension.

While this paper contains many ideas and techniques from percolation, the above references deal with discrete lattices mostly, whereas we deal with a model of continuum percolation. When highlighting the difference of the former models to continuum percolation, we refer to them as lattice percolation or discrete percolation.

Continuum percolation may be regarded as a branch of percolation theory, including some aspects of stochastic geometry, and, in particular, the theory of point processes. A textbook on the Poisson point process was written by the third author and Penrose [24]. Continuum percolation was first considered in 1961 by Gilbert [12] for the Poisson blob model. The random connection model in the way it is introduced in this paper was first introduced in 1991 by Penrose [31]. A textbook treatment of continuum percolation was given by Meester and Roy [29], also summarizing some properties of the random connection model. Among those properties is the essential result that $\lambda_c = \lambda_T$, which was first obtained in full generality in 1995 by Meester [27]. As a representative treatment of continuum percolation in the physics literature, we point to the book by Torquato [37]. More recently, the RCM was considered by the third author and Ziesche [26], and they prove that the subcritical two-point function satisfies the Ornstein-Zernike equation (OZE). We point out that (5.6) is precisely the OZE (and (1.16) is the OZE in Fourier space).

Continuum percolation has a finite-volume analogue, by restricting to a bounded domain—see the monograph by Penrose [30] about random geometric graphs. The finite-volume analogue of the RCM was investigated by Penrose [33], where it is called *soft random geometric graph*.

The RCM is related to some fundamental lattice models and has, in fact, features of both site and bond percolation. The Poisson blob model, for instance, can be considered as a continuum version of nearest-neighbor site percolation. The parameter p in the discrete setting is then analogous to the intensity λ of the PPP, as both describe the point density. In that sense, the general RCM corresponds to a discrete site percolation model with long-range connections governed by φ. The RCM can also be interpreted as bond percolation (again with long-range interactions governed by φ) on the complete graph generated by the PPP. Under this perspective the parameter p in the discrete setting can be compared with the mean degree $\lambda \int \varphi(x) \, \mathrm{d}x$ of a typical point of the PPP.

The results obtained in this paper mirror several results of lattice percolation. The treatment of nearest-neighbor models and their spread-out version, first performed by Hara and Slade [18], can be compared to our versions (H1) and (H2) for φ. For bond percolation on $\mathbb{Z}^d$, Fitzner and the second author proved that $d \geq 11$ suffices to obtain an analogue of Theorem 1.2 [10, 11]. In our corresponding regime, which is (H1), we give no quantitative bound on the dimension d. The "discrete analogue" of (H3) is long-range percolation, for which the corresponding results were obtained by the first two authors and Sakai [21].

It is worth noting that Tanemura [36] already devised a lace expansion for the Poisson blob model. For the special case of the Poisson blob model, the expansion itself is the same as the one devised in this paper. However, we were unable to give a proof of the expansion's convergence based on [36].

A possible application of the results of this paper is the deduction of the existence of several critical exponents (and their computation) other than γ. Analogous results for the lattice were proved by Aizenman and Newman [1], by Aizenman and Barsky [2], by Hara [16, 17], by Hara, the second author, and Slade [19], and furthermore by Kozma and Nachmias [22, 23] (this list is not exhaustive). These results have not yet been shown for the RCM. However, the third author together with Penrose and Zuyev [25] proved the mean-field bound on the critical exponent β for the Boolean model with random radii. It may also be possible to investigate an asymptotic expansion of the critical point λ_c (at least for specific choices of φ). We point to Torquato [38] for predictions of such results.

1.5 Overview, discussion of proof, and notation

Overview of the proof. We interpret $q_\varphi^{-1}\varphi$ as a random walk density and define its Green's function as

$$G_\mu(x) := \sum_{m\geq 0} \big(\mu/q_\varphi\big)^m \varphi^{\star m}(x), \tag{1.13}$$

where $|\mu| < 1$ and $\varphi^{\star 0}$ is a generalized (Dirac) function. In Fourier space, this gives

$$q_\varphi^{-1}\widehat{G}_\mu(k) = \frac{1}{\widehat{\varphi}(\mathbf{0}) - \mu\widehat{\varphi}(k)}, \tag{1.14}$$

noting that $\widehat{\varphi}(\mathbf{0}) = q_\varphi$. The main aim of our paper can be summarized as follows: For small β, we intend to show that $\lambda\tau_\lambda$ is close to $(\varphi \star G_\mu)$, where μ depends on λ in an appropriate way. The latter we understand much better than we understand τ_λ, in particular, we know that $(\varphi^{\star m} \star G_1^{\star 3})(x)$ is finite for $m = 1$ and small for $m = 2$. This "closeness" will allow us to transfer this result to τ_λ and prove the triangle condition.

The lace-expansion technique proceeds in three major steps, which also dictate the structure of this paper. Before the first step, we need to make sure to have the relevant tools that are analogous to those used in discrete percolation theory available to us; also, some methodology from point-process theory is introduced.This is done in Section 2.

In the first major step of the proof, which is key to proving Theorem 1.2 and is executed in Section 3, we show that the lace expansion for the two-point function τ_λ takes the form

$$\tau_\lambda = \varphi + \Pi_{\lambda,n} + \lambda\big((\varphi + \Pi_{\lambda,n}) \star \tau_\lambda\big) + R_{\lambda,n} \quad (n \in \mathbb{N}_0), \tag{1.15}$$

where the lace-expansion coefficients $\Pi_{\lambda,n}$ and $R_{\lambda,n}$ arise during the expansion and will be defined later.

Section 4 contains the second step and aims to bound $\Pi_{\lambda,n}$ and $R_{\lambda,n}$ by simpler diagrammatic functions. Those diagrams are large integrals over products of two-point functions which can then be decomposed into factors of $\triangle_\lambda$ and related quantities. We eventually want to prove $\lambda|\widehat{\Pi}_{\lambda,n}(k)| = \mathcal{O}(\beta)$ (recall that this means as $d \to \infty$ for (H1) and as $L \to \infty$ for (H2),(H3)) uniformly in $k \in \mathbb{R}^d$ and $n \in \mathbb{N}_0$, and the intermediate step of diagrammatic bounds is essential to do so.

The third step is the so-called "bootstrap argument" and it is performed in Section 5. Since the diagrammatic bounds obtained in the previous step are in terms of τ_λ itself, we need this step in order to gain meaningful bounds. More details about this are given at the beginning of Section 5. As the general strategy of proof is standard, we refer to [20] for a more detailed informal description of the methodology. However, we note that in this step, several functions are introduced, among them the fraction $|\widehat{\tau}_\lambda/\widehat{G}_{\mu_\lambda}|$, where the parametrization μ_λ satisfies $\mu_\lambda \nearrow 1$ as $\lambda \nearrow \lambda_c$. The boundedness of these functions is shown, which justifies the notion of $\widehat{\tau}_\lambda$ and $\widehat{G}_{\mu_\lambda}$ being "close".

Section 5 also contains the consequences of the completed argument. The most important of these is that if we let $n \to \infty$ in (1.15), then $R_{\lambda,n} \to 0$ and $\Pi_{\lambda,n} \to \Pi_\lambda = \mathcal{O}(\beta)$ for any $\lambda < \lambda_c$. In Fourier space, (1.15) consequently translates to

$$\widehat{\tau}_\lambda = \frac{\widehat{\varphi} + \widehat{\Pi}_\lambda}{1 - \lambda(\widehat{\varphi} + \widehat{\Pi}_\lambda)}, \tag{1.16}$$

Together with the obtained bounds on $\lambda\widehat{\Pi}_\lambda(k)$, this implies our two main results and justifies the comparison with the Green's function of the random walk with step distribution φ.

After having completed the lace expansion successfully, we prove our main theorems in Section 6.

Differences to percolation on the lattice. We informally describe the novelties that the application of the lace expansion to the RCM setting brings, in contrast to the lace expansion for, say, bond percolation on $\mathbb{Z}^d$. By virtue of the continuum space, we can use re-scaling arguments more easily (see Section 5.1), and by the underlying Poisson point process, the Mecke equation (see Section 2.3) provides an elementary but powerful tool to deal with expectations of sums over random points.

The biggest novelty in the derivation of the expansion (Section 3) is the inclusion of *thinnings* to exploit spatial independence of the RCM—see Definition 3.2 and Lemma 3.3.

The events from Section 3 now contain thinning events, which take some extra work to decompose in the fashion intended by Section 4. This is done in Definition 4.8, in Lemma 4.10 and in Lemma 4.12. We also highlight that the decomposition crucially relies on the BK inequality, which is new for the RCM (see Theorem 2.1).

While several other differences in the decomposition of Section 4 can be attributed to the site percolation nature of the RCM, it is a challenge unique to certain versions of the RCM (among them, the Poisson blob model) that $\tau_\lambda \star \tau_\lambda$ is bounded away from 0 in a neighborhood of the origin. For discrete percolation on $\mathbb{Z}^d$, the convolution $(\tau_\lambda \star \tau_\lambda)(x)$ is also bounded away from 0 when $x = \mathbf{0}$, but the continuum space forces us to deal with this issue in a different way. We point to the introduction of $\mathsf{B}^{(\varepsilon)}$ in Definition 4.13 and the discussion thereafter for more details.

Lastly, two issues arise in Section 5. The first is that τ_λ is closer to $(\varphi \star G_\mu)$ rather than G_μ, which is again a manifestation of the site-percolation nature of the RCM. The second is the fact that, unlike $\mathbb{Z}^d$, the space $\mathbb{R}^d$ has a non-compact Fourier domain (namely, $\mathbb{R}^d$ itself), which demands some extra care in the Fourier analysis of the bootstrap functions introduced in Section 5.2.

Some notation. Let us fix some helpful notation, which we will use throughout this paper:

- If not specified otherwise, ξ is used to refer to an edge-marking of a PPP (an edge-marking is the random object encoding the RCM, see Section 2.2). This PPP is the "ground process" of ξ and is always denoted by η.
- For two real numbers a, b, we use $a \wedge b := \min\{a, b\}, a \vee b := \max\{a, b\}$.
- For a vector $v \in \mathbb{R}^d$, $|v| = \|v\|_2$ denotes the Euclidean norm. Secondly, for a discrete set A, we use $|A|$ to denote the cardinality of A.
- For a natural number n, let $[n] = \{1, \dots, n\}$. Given $x_1, x_2, \dots, y_1, y_2, \dots \in \mathbb{R}^d$ as well as integers $a \le b$, we write $\vec{y}_{[a,b]} = (y_a, \dots, y_b)$, and similarly $(\vec{x}, \vec{y})_{[a,b]} = (x_a, \dots, x_b, y_a, \dots, y_b)$.
- An integral over a non-specified set is always to be understood as the integral over the whole space.
- We use $\delta_{x,y}$ to denote the distributional Dirac delta, i.e. $\int \delta_{x,y} f(x) \,\mathrm{d}x = f(y)$ for measurable $f : \mathbb{R}^d \to \mathbb{R}$. The use of $\delta_{\cdot,\cdot}$ in this paper is detailed in a remark below Definition 4.6.
- We recall that for measurable functions $f, g \colon \mathbb{R}^d \to \mathbb{R}$, we write $(f \star g)(x) = \int f(y) g(x - y) \,\mathrm{d}y$ for their convolution. Moreover, the m-fold convolution is set to be $f^{\star 0}(x) = \delta_{\mathbf{0},x}$ and $f^{\star m}(x) = (f^{\star(m-1)} \star f)(x)$.
- We recall some basic notation from graph theory. If G is a graph, then $V(G)$ is its set of vertices (points, sites), and $E(G)$ is its set of edges (bonds). Since we will be concerned with (random) graphs ξ whose vertex set is a Poisson point process η, we usually write $V(\xi) = \eta$. A subgraph G' of G is a graph where $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. For $W \subseteq V(G)$, the subgraph of G *induced* by W, denoted by $G[W]$, is the subgraph of G whose vertex set is W and where two vertices in W are adjacent in $G[W]$ if and only if they are adjacent in G.

2 Preliminaries

2.1 Point processes

We briefly recall some basic facts about point and Poisson processes, referring to [24] for a comprehensive treatment. Let $\mathbb{X}$ be a metric space with Borel σ-field $\mathcal{B}(\mathbb{X})$. Let $\mathbf{N}(\mathbb{X})$ be the set of all at most countably infinite sets $\nu \subset \mathbb{X}$. Equip $\mathbf{N}(\mathbb{X})$ with the σ-field $\mathcal{N}(\mathbb{X})$ generated by the sets $\{\nu : |\nu \cap B| = k, B \in \mathcal{B}(\mathbb{X}), k \in \mathbb{N}_0\}$, where $|\nu \cap B|$ denotes the cardinality of $\nu \cap B$. A point process on $\mathbb{X}$ is a measurable mapping $\zeta \colon \Omega \to \mathbf{N}(\mathbb{X})$ for some underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The intensity measure of a point process is the measure on $\mathbb{X}$ given by $B \mapsto \mathbb{E}[|\zeta \cap B|]$ for $B \in \mathcal{B}(\mathbb{X})$.

Let μ be a σ-finite non-atomic measure on $\mathbb{R}^d$. A Poisson point process (PPP) on $\mathbb{X}$ with intensity measure μ is a point process ζ such that the number of points $|\zeta \cap B|$ is $\mathrm{Poi}(\mu(B))$-distributed for each $B \in \mathcal{B}(\mathbb{X})$ and the random variables $|\zeta \cap B_1|, \dots, |\zeta \cap B_m|$ are independent whenever $B_1, \dots, B_m \in \mathcal{B}(\mathbb{X})$ are pairwise disjoint. We point out that in our setting, the first property implies the second (independence) property. In the case $\mathbb{X} = \mathbb{R}^d$, we call the PPP homogeneous (or stationary) with intensity λ if $\mu = \lambda \mathrm{Leb}$ with $\lambda \ge 0$ and Leb the Lebesgue measure.

Let ζ be a point process on $\mathbb{X}$ which is locally finite (the points do not accumulate in bounded sets) or has a σ-finite intensity measure. By [24, Corollary 6.5], there exist measurable mappings $\pi_i \colon \mathbf{N}(\mathbb{X}) \to \mathbb{X}$, $i \in \mathbb{N}$, such that $\zeta = \{\pi_i(\zeta) : i \le |\zeta \cap \mathbb{X}|\}$ almost surely.

In this paper we consider a homogeneous Poisson process η with intensity $\lambda \geq 0$, and we denote the underlying probability measure by $\mathbb{P}_\lambda$. We write

$$\eta = \{X_i : i \in \mathbb{N}\}, \tag{2.1}$$

where $X_i := \pi_i(\eta)$, $i \in \mathbb{N}$.

2.2 Formal construction of the RCM

As some of the later statements sensitively depend on the precise construction of the model, we give a detailed formal construction. It is convenient to construct the RCM as a deterministic functional $\Gamma_\varphi(\xi)$ of a suitable point process ξ. Following [26], we choose ξ as an *independent edge-marking* of a PPP η. We then show how to extend the construction to include deterministic points and how to extend it to include thinnings. Other (equivalent) ways to construct the RCM can be found in [8, 29]. There, the RCM is introduced as a marked PPP (hence, the information about the edges of ξ is encoded in the marks of the points). We point to the proof of Theorem 2.1, where we also require a construction in terms of a suitable marked PPP of (an approximation of) the RCM.

Construction as a point process in $(\mathbb{R}^d)^{[2]} \times [0,1]$. Recall that η denotes an $\mathbb{R}^d$-valued PPP of intensity λ, which can be written as (2.1). Let $(\mathbb{R}^d)^{[2]}$ denote the space of all sets $e \subset \mathbb{R}^d$ containing exactly two elements. Any $e \in (\mathbb{R}^d)^{[2]}$ is a potential edge of the RCM. When equipped with the Hausdorff metric, this space is a Borel subset of a complete separable metric space. Let $<$ denote the strict lexicographic ordering on $\mathbb{R}^d$. Introduce independent random variables $U_{i,j}$, $i, j \in \mathbb{N}$, uniformly distributed on the unit interval $[0,1]$ such that the double sequence $(U_{i,j})$ is independent of η. We define

$$\xi := \{(\{X_i, X_j\}, U_{i,j}) : X_i < X_j, i, j \in \mathbb{N}\}, \tag{2.2}$$

which is a point process on $(\mathbb{R}^d)^{[2]} \times [0,1]$. We interpret ξ as a random marked graph and say that ξ is an *independent edge-marking* of η. Note that η can be recovered from ξ. The definition of ξ depends on the ordering of the points of η. The distribution of ξ, however, does not.

Given an independent edge-marking ξ of η, we can define the RCM $\Gamma_\varphi(\xi)$ as a deterministic functional of ξ, given by its vertex and edge set as

$$V(\Gamma_\varphi(\xi)) = \eta = \{x \in \mathbb{R}^d : (\{x, y\}, u) \in \xi \text{ for some } y \neq x \text{ and } u \in [0,1]\}, \tag{2.3}$$

$$E(\Gamma_\varphi(\xi)) = \{\{X_i, X_j\} : X_i < X_j, U_{i,j} \leq \varphi(X_i - X_j), i, j \in \mathbb{N}\}. \tag{2.4}$$

The RCM $\Gamma_\varphi(\zeta)$ can be defined for every point process ζ on $(\mathbb{R}^d)^{[2]} \times [0,1]$ with the property that $(\{x, y\}, u) \in \zeta$ and $(\{x, y\}, u') \in \zeta$ for some $x \neq y$ and $u, u' \in [0,1]$ implies that $u = u'$. Throughout the paper, when speaking of a graph event taking place in ζ, we refer to the graph event taking place in $\Gamma_\varphi(\zeta)$.

Adding extra points. For $x_1, x_2 \in \mathbb{R}^d$, consider the random connection models driven by the point processes

$$\eta^{x_1} := \eta \cup \{x_1\}, \quad \eta^{x_1,x_2} := \eta \cup \{x_1, x_2\}.$$

To couple these models in a natural way, we extend the (double) sequence $(U_{m,n})_{m,n\geq 1}$ to a sequence $(U_{m,n})_{m,n\geq -1}$ of independent random variables uniformly distributed on $[0,1]$, independent of the Poisson process η. We then define the point process ξ^{x_1,x_2} on $(\mathbb{R}^d)^{[2]} \times [0,1]$ as

$$\xi^{x_1,x_2} := \{(\{X_i, X_j\}, U_{i,j}) : X_i < X_j, i, j \geq -1\},$$

where $(X_0, X_{-1}) := (x_1, x_2)$. We now define the point process ξ^{x_1} by removing all (marked) edges incident to x_2 from ξ^{x_1,x_2}. We define ξ^{x_2} analogously. According to our previous conventions, we can talk about events of the type $\{x_1 \longleftrightarrow x_2 \text{ in } \xi^{x_1,x_2}\}$. It is straightforward to define $\xi^{x_1,\dots,x_m}$ for arbitrary $m \geq 3$.

Including thinnings. Let $\mathbb{M} = [0,1]^{\mathbb{N}}$. It will be important to work with subgraphs of ξ that are obtained via a *thinning* of η with respect to some point process ζ. We specify this in Definition 3.2. For now, it is important that this thinning requires extra randomness, which, given η, is independent of the edge set $E(\xi)$. We model this by adding to every Poisson point in η a mark from $\mathbb{M}$.

To this end, let $(\mathbb{R}^d \times \mathbb{M})^{[2]}$ denote the space of subsets of $\mathbb{R}^d \times \mathbb{M}$ containing exactly two elements. Let $\mathbb{U}$ denote the uniform distribution on $[0,1]$ and let $Y_i = (Y_{i,k})_{k\in\mathbb{N}}$, $i \in \mathbb{N}$, be independent random elements of $\mathbb{M}$ with distribution $\mathbb{U}^{\mathbb{N}}$, independent of η. Assume that η, $(U_{i,j})_{i,j\in\mathbb{N}}$ and $(Y_i)_{i\in\mathbb{N}}$ are independent. Proceeding analogously to the definition in (2.2) we define

$$\xi := \Big\{ \big(\{(X_i, Y_i), (X_j, Y_j)\}, U_{i,j}\big) : X_i < X_j, i, j \in \mathbb{N} \Big\}, \tag{2.5}$$

which is a point process in $(\mathbb{R}^d \times \mathbb{M})^{[2]} \times [0,1]$. The RCM is constructed as before, that is, the vertex set is η and the edge set is as in (2.4), ignoring the marks (Y_i). From now on, when speaking of ξ, we assume that ξ is given by (2.5). Still, the reader might prefer to work with the simpler version (2.2). We shall clearly point out when the thinning variables are required.

We can add points $x_1, x_2 \in \mathbb{R}^d$ as follows. We let $(U_{m,n})_{m,n\geq -1}$ as above. In addition, we take independent random elements Y_0, Y_{-1} of $\mathbb{M} = [0,1]^{\mathbb{N}}$ and assume that η, $(Y_i)_{i\geq -1}$ and $(U_{m,n})_{m,n\geq -1}$ are independent. Define

$$\xi^{x_1,x_2} := \{(\{(X_i, Y_i), (X_j, Y_j)\}, U_{i,j}) : X_i < X_j, i, j \geq -1\},$$

where $(X_0, X_{-1}) := (x_1, x_2)$. The point processes ξ^{x_1}, ξ^{x_2} etc. are defined as before.

2.3 Mecke, Margulis-Russo, BK, FKG, and a differential inequality

In this section, we state some useful equalities and inequalities that are standard either in point process theory or in percolation theory.

The Mecke equation. Our first crucial tool is a version of the Mecke equation (see [24, Chapter 4]) for the independent edge-marking ξ. This fundamental equation allows us to deal with sums over points of η, which we frequently make use of. We closely follow [26]. Given $m \in \mathbb{N}$ and a measurable function $f\colon \mathbf{N}\big((\mathbb{R}^d \times \mathbb{M})^{[2]} \times [0,1]\big) \times \big(\mathbb{R}^d\big)^m \to \mathbb{R}_{\geq 0}$, the Mecke equation for ξ states that

$$\mathbb{E}_\lambda\Big[\sum_{\vec{x}\in\eta^{(m)}} f(\xi, \vec{x})\Big] = \lambda^m \int \mathbb{E}_\lambda\Big[f\left(\xi^{x_1,\ldots,x_m}, \vec{x}\right)\Big]\,\mathrm{d}\vec{x}, \tag{2.6}$$

where $\vec{x} = (x_1, \ldots, x_m)$ and $\eta^{(m)} = \{(x_1, \ldots, x_m) : x_i \in \eta, x_i \neq x_j \text{ for } i \neq j\}$. We only need the statement for $m \leq 3$, and in particular, we mostly use (2.6) for $m = 1$, yielding the univariate Mecke equation

$$\mathbb{E}_\lambda\Big[\sum_{x\in\eta} f(\xi, x)\Big] = \lambda \int \mathbb{E}_\lambda\left[f(\xi^x, x)\right]\,\mathrm{d}x. \tag{2.7}$$

Margulis-Russo formula. The Margulis-Russo formula is a well-known tool in (discrete) percolation theory and turns out to be necessary for us as well. Our version follows from a more general result (see [26, Theorem 3.2]). We write $\mathbf{N} := \mathbf{N}\big((\mathbb{R}^d \times \mathbb{M})^{[2]} \times [0,1]\big)$. Let $\Lambda \in \mathcal{B}(\mathbb{R}^d)$, $\zeta \in \mathbf{N}$, and define

$$\zeta_\Lambda := \{(\{(x,v),(y,w)\}, u) \in \zeta : \{x,y\} \subseteq \Lambda\}. \tag{2.8}$$

We call ζ_Λ the restriction of ζ to Λ. We say that $f\colon \mathbf{N} \to \mathbb{R}$ *lives* on Λ if $f(\zeta) = f(\zeta_\Lambda)$ for every $\zeta \in \mathbf{N}$. Assume that there exists a bounded set $\Lambda \in \mathcal{B}(\mathbb{R}^d)$ such that f lives on Λ. Moreover, assume that there exists $\lambda_0 > 0$ such that $\mathbb{E}_{\lambda_0}[|f(\xi)|] < \infty$. Then the Margulis-Russo formula states that, for all $\lambda \leq \lambda_0$,

$$\frac{\partial}{\partial\lambda}\mathbb{E}_\lambda[f(\xi)] = \int_\Lambda \mathbb{E}_\lambda[f(\xi^x) - f(\xi)]\,\mathrm{d}x. \tag{2.9}$$

The BK inequality. The BK inequality is a standard tool in discrete percolation and provides a counterpart to the FKG inequality, which we discuss below and which (given existing results) turns out to be much easier to prove.

Let us first informally describe what type of inequality we are aiming for, described for the type of events we need it for. Assume that we are interested in the event that there are two paths, the first between $x_1, x_2 \in \mathbb{R}^d$ and the second between $x_3, x_4 \in \mathbb{R}^d$, not sharing any vertices. This *(vertex) disjoint occurrence* is something we think of as being less likely than the probability that on two independent RCM graphs, one has an x_1-x_2-path and the second has an x_3-x_4-path. Thus, if $E \circ F$ denotes the former event, we want an inequality of the form $\mathbb{P}_\lambda(E \circ F) \leq \mathbb{P}_\lambda(E)\mathbb{P}_\lambda(F)$. We now work towards making these notions rigorous and towards proving such an inequality.

It is convenient to write $\mathbf{N} := \mathbf{N}((\mathbb{R}^d \times \mathbb{M})^{[2]} \times [0,1])$ and to denote the σ-field on $\mathbf{N}$ (as defined in Section 2.1) by $\mathcal{N}$. For $\Lambda \in \mathcal{B}(\mathbb{R}^d)$ and $\mu \in \mathbf{N}$, we define μ_Λ, the restriction of μ to all edges completely contained in $\Lambda \times \mathbb{X}$, analogously to (2.8). Define the *cylinder event* $[\![\mu]\!]_\Lambda$ as $[\![\mu]\!]_\Lambda := \{\nu \in \mathbf{N} : \nu_\Lambda = \mu_\Lambda\}$. We say that $E \in \mathcal{N}$ *lives on* Λ if $\mathbb{1}_E$ lives on Λ. We call a set $E \subset \mathbf{N}$ *increasing* if $\mu \in E$ implies $\nu \in E$ for each $\nu \in \mathbf{N}$ with $\mu \subseteq \nu$. Let $\mathcal{R}$ denote the ring of all finite unions of half-open rectangles with rational coordinates. For events $E, F \in \mathcal{N}$ we define

$$E \circ F := \{\mu \in \mathbf{N} : \exists K, L \in \mathcal{R} \text{ s.t. } K \cap L = \varnothing \text{ and } [\![\mu]\!]_K \subseteq E, [\![\mu]\!]_L \subseteq F\}. \tag{2.10}$$

If E, F are increasing events, then $E \circ F = \{\mu \in \mathbf{N} : \exists K, L \in \mathcal{R} \text{ s.t. } K \cap L = \varnothing, \mu_K \in E, \mu_L \in F\}$.

As before, we let $\eta = \{X_i : i \in \mathbb{N}\}$ denote a homogeneous Poisson process on $\mathbb{R}^d$ with intensity λ and let $\eta' := \{(X_i, Y_i) : i \in \mathbb{N}\}$ be an independent $\mathbb{U}$-marking of η (see (2.5)). By the marking theorem (see, e.g., [24, Theorem 5.6]), η' is a Poisson process with intensity measure $\lambda \text{Leb} \otimes \mathbb{U}$. Let ξ be given as in (2.5).

We will show that

$$\mathbb{P}_\lambda(\xi \in E \circ F) \leq \mathbb{P}_\lambda(\xi \in E)\mathbb{P}_\lambda(\xi \in F) \tag{2.11}$$

whenever $E, F \in \mathcal{N}$ live on a bounded set $\Lambda \in \mathcal{B}(\mathbb{R}^d)$ and are increasing.

We need a slightly more general version of this inequality involving independent random variables. Let $(\mathbb{X}_1, \mathcal{X}_1)$, $(\mathbb{X}_2, \mathcal{X}_2)$ be two measurable spaces. We say that a set $E \subset \mathbf{N} \times \mathbb{X}_i$ is *increasing* if $E^z := \{\mu \in \mathbf{N} : (\mu, z) \in E\}$ is increasing for each $z \in \mathbb{X}_i$. For increasing $E_i \in \mathcal{N} \otimes \mathcal{X}_i$, we define

$$E_1 \circ E_2 := \{(\mu, z_1, z_2) \in \mathbf{N} \times \mathbb{X}_1 \times \mathbb{X}_2 : \exists K_1, K_2 \in \mathcal{R} \text{ s.t. } K_1 \cap K_2 = \varnothing, (\mu_{K_1}, z_1) \in E_1, (\mu_{K_2}, z_2) \in E_2\}. \tag{2.12}$$

A set $E \in \mathcal{N} \otimes \mathcal{X}_i$ *lives on* Λ if $\mathbb{1}_E(\mu, z) = \mathbb{1}_E(\mu_\Lambda, z)$ for each $(\mu, z) \in \mathbf{N} \times \mathbb{X}_i$. We consider random elements W_1, W_2 of $\mathbb{X}_1$ and $\mathbb{X}_2$, respectively, and assume that ξ, W_1, W_2 are independent.

Theorem 2.1 (BK inequality)**.** *Let $E_i \in \mathcal{N} \otimes \mathcal{X}_i$, $i \in \{1,2\}$, be increasing events that live on some bounded set $\Lambda \in \mathcal{B}(\mathbb{R}^d)$. Then $\mathbb{P}_\lambda((\xi, W_1, W_2) \in E_1 \circ E_2) \leq \mathbb{P}_\lambda((\xi, W_1) \in E_1)\mathbb{P}_\lambda((\xi, W_2) \in E_2)$.*

Let us make two remarks. First, Theorem 2.1 is formulated for ξ as in (2.5), that is, for a PPP in $\mathbf{N}((\mathbb{R}^d \times \mathbb{M})^{[2]} \times [0,1])$ with a uniform measure $\mathbb{U}$ on $\mathbb{M}$. The proof is independent of the precise structure of $\mathbb{M}$ and the probability measure on it; and so Theorem 2.1 still holds true if $\mathbb{M}$ is replaced by any complete separable metric space $\mathbb{X}$ and $\mathbb{U}$ is replaced by any probability measure $\mathbb{Q}$ on $\mathbb{X}$.

Second, our proof relies on the BK inequality for marked PPPs proved by Gupta and Rao in [15] (for increasing events the BK inequality was proved by van den Berg [4]). We require a more general mark space than the one provided there. However, the result proved in [15] implies the BK inequality for general Borel spaces as mark spaces, which is sufficient for us.

Proof. We first show that it suffices to prove (2.11). Indeed, if (2.11) holds, then

$$\begin{aligned}
\mathbb{P}_\lambda((\xi, W_1, W_2) \in E_1 \circ E_2) &= \int \mathbb{P}_\lambda(\xi \in E_1^{w_1} \circ E_2^{w_2})\mathbb{P}_\lambda((W_1, W_2) \in \mathrm{d}(w_1, w_2)) \\
&\leq \iint \mathbb{P}_\lambda(\xi \in E_1^{w_1})\mathbb{P}_\lambda(\xi \in E_2^{w_2})\mathbb{P}_\lambda(W_1 \in \mathrm{d}w_1)\mathbb{P}_\lambda(W_2 \in \mathrm{d}w_2) \\
&= \mathbb{P}_\lambda((\xi, W_1) \in E_1)\mathbb{P}_\lambda((\xi, W_2) \in E_2).
\end{aligned}$$

To prove (2.11), we use the BKR inequality proven in [15]. To do so, we approximate ξ by functions of suitable independent markings of the Poisson process η.

Let $\varepsilon > 0$ and set $Q_z^\varepsilon := z + [0,\varepsilon)^d$ for $z \in \varepsilon\mathbb{Z}^d$. Define $\mathbb{M}^\varepsilon := [0,1]^{\varepsilon\mathbb{Z}^d}$ and let $\mathbb{U}^\varepsilon := \mathbb{U}^{\varepsilon\mathbb{Z}^d}$, where $\mathbb{U}$ is the uniform distribution on $[0,1]$. Let η^ε be an independent $\mathbb{U}^\varepsilon$-marking of η'. By the marking theorem [24, Theorem 5.6], η^ε is a Poisson process on $\mathbb{R}^d \times \mathbb{X} \times \mathbb{M}^\varepsilon$ with intensity measure $\lambda\mathrm{Leb}\otimes\mathbb{Q}\otimes\mathbb{U}^\varepsilon$. We write η^ε in the form

$$\eta^\varepsilon = \{(x, r, U(x, r)) : (x, r) \in \eta'\},$$

where the $U(x, r)$ are conditionally independent given η'. Moreover, given η' and $(x, r) \in \eta'$, $U(x, r) = (U_z(x, r))_{z\in\varepsilon\mathbb{Z}^d}$ is a sequence of independent uniform random variables on $[0,1]$ (for simplicity of notation, $U(x, r)$ does not reflect the dependence on ε).

We use η^ε to approximate ξ by a point process ξ^ε on $(\mathbb{R}^d \times \mathbb{X})^{[2]} \times [0,1]$ as follows. For $x \in \mathbb{R}^d$, let $q(x;\varepsilon)$ be the point $z \in \varepsilon\mathbb{Z}^d$ such that $x \in Q_z^\varepsilon$. If $(x, r), (y, s) \in \eta'$ satisfy $x < y$ (we recall that $<$ denotes strict lexicographical order) and $|\eta \cap Q^\varepsilon_{q(x;\varepsilon)}| = |\eta \cap Q^\varepsilon_{q(y;\varepsilon)}| = 1$, we let $(\{(x, r), (y, s)\}, U_{q(y;\varepsilon)}(x, r))$ be a point of ξ^ε. Let

$$R_\varepsilon := \bigcup_{z\in\Lambda^\varepsilon} \{|\eta \cap Q_z^\varepsilon| \geq 2\},$$

where $\Lambda^\varepsilon := \{z \in \varepsilon\mathbb{Z}^d : \Lambda \cap Q_z^\varepsilon \neq \varnothing\}$. A simple but crucial observation is that

$$\mathbb{P}_\lambda(\{\xi_\Lambda \in \cdot\} \cap R_\varepsilon^c) = \mathbb{P}_\lambda(\{\xi_\Lambda^\varepsilon \in \cdot\} \cap R_\varepsilon^c). \tag{2.13}$$

Next we note that, as $\varepsilon \searrow 0$,

$$\mathbb{P}_\lambda(R_\varepsilon) \leq \mathrm{diam}(\Lambda)^d \varepsilon^{-d} \left(1 - \mathrm{e}^{-\lambda\varepsilon^d} - \lambda\varepsilon^d \mathrm{e}^{-\lambda\varepsilon^d}\right) = \mathcal{O}(\varepsilon^d), \tag{2.14}$$

where diam denotes the diameter. To exploit (2.13) and (2.14), we need to recall the BKR inequality for η^ε. Set $\mathbf{N}^\varepsilon := \mathbf{N}(\mathbb{R}^d\times\mathbb{X} \times [0,1]^{\varepsilon\mathbb{Z}^d})$. For $\mu \in \mathbf{N}^\varepsilon$ and $K \in \mathcal{B}(\mathbb{R}^d)$, we set $\mu_K := \mu \cap (K \times \mathbb{X} \times [0,1]^{\varepsilon\mathbb{Z}^d})$ and $[\![\mu]\!]_K := \{\nu \in \mathbf{N}^\varepsilon : \nu_K = \mu_K\}$. Given $E', F' \in \mathbf{N}(\mathbb{R}^d \times \mathbb{X} \times [0,1]^{\varepsilon\mathbb{Z}^d})$, we define

$$E'\square F' := \{\mu \in \mathbf{N}^\varepsilon : \exists K, L \in \mathcal{R} \text{ s.t. } K \cap L = \varnothing \text{ and } [\![\mu]\!]_K \subseteq E', [\![\mu]\!]_L \subseteq F'\}. \tag{2.15}$$

If E', F' live on Λ (defined as before), then, by [15],

$$\mathbb{P}_\lambda(\eta^\varepsilon \in E'\square F') \leq \mathbb{P}_\lambda(\eta^\varepsilon \in E')\mathbb{P}_\lambda(\eta^\varepsilon \in F'). \tag{2.16}$$

By (2.13) and (2.14), we have

$$\mathbb{P}_\lambda(\xi \in E \circ F) \leq \mathbb{P}_\lambda(\{\xi \in E \circ F\} \cap R_\varepsilon^c) + \mathbb{P}_\lambda(R_\varepsilon) = \mathbb{P}_\lambda(\{\xi^\varepsilon \in E \circ F\} \cap R_\varepsilon^c) + \mathcal{O}(\varepsilon^d), \tag{2.17}$$

with E, F as in (2.11). We now use that $\xi^\varepsilon = T(\eta^\varepsilon)$ for a well-defined measurable mapping $T \colon \mathbf{N}(\mathbb{R}^d \times \mathbb{X} \times \mathbb{M}^\varepsilon) \to \mathbf{N}((\mathbb{R}^d \times \mathbb{X})^{[2]} \times [0,1])$. (Again this notation doesn't reflect the dependence on ε.) Assume that ε is rational. We assert that

$$\{\xi^\varepsilon \in E \circ F\} \cap R_\varepsilon^c \subseteq \{\eta^\varepsilon \in (T^{-1}E)\square(T^{-1}F)\} \cap R_\varepsilon^c. \tag{2.18}$$

To prove this, we assume that R_ε^c holds. Assume also that there exist disjoint $K, L \in \mathcal{R}$ (depending on η^ε) such that $T(\eta^\varepsilon)_K \in E$ and $T(\eta^\varepsilon)_L \in F$. Let

$$K' := \bigcup_{z\in\Lambda^\varepsilon : |\eta\cap Q_z^\varepsilon \cap K| = 1} Q_z^\varepsilon,$$

and define L' similarly. Since R_ε^c holds, we have that $K' \cap L' = \varnothing$ and, moreover, $T(\eta^\varepsilon)_{K'} = T(\eta^\varepsilon)_K$ as well as $T(\eta^\varepsilon)_{L'} = T(\eta^\varepsilon)_L$. By definition of T, for each $\nu \in \mathbf{N}(\mathbb{R}^d\times\mathbb{X}\times\mathbb{M}^\varepsilon)$, we have that $T(\nu)_{K'} = T(\nu_{K'})$. Let $\nu \in \mathbf{N}(\mathbb{R}^d \times \mathbb{X} \times \mathbb{M}^\varepsilon)$ be such that $\nu_{K'} = \eta^\varepsilon_{K'}$. Then $T(\nu)_{K'} = T(\eta^\varepsilon)_{K'}$. Since $T(\eta^\varepsilon)_{K'} = T(\eta^\varepsilon)_K$ and E is increasing, we obtain that $T(\nu) \in E$, that is $\nu \in T^{-1}(E)$. It follows that $[\![\eta^\varepsilon]\!]_{K'} \subset T^{-1}(E)$. In the same way, we get $[\![\eta^\varepsilon]\!]_{L'} \subset T^{-1}(F)$. This shows that (2.18) holds.

From (2.17), (2.18), and the BKR inequality (2.16), we obtain that

$$\begin{aligned}\mathbb{P}_\lambda(\xi \in E \circ F) - \mathcal{O}(\varepsilon^d) = \mathbb{P}_\lambda(\{\xi^\varepsilon \in E \circ F\} \cap R_\varepsilon^c) &\leq \mathbb{P}_\lambda(\{\eta^\varepsilon \in (T^{-1}E)\square(T^{-1}F)\} \cap R_\varepsilon^c) \\ &\leq \mathbb{P}_\lambda(\eta^\varepsilon \in (T^{-1}E)\square(T^{-1}F))\end{aligned}$$

$$
\begin{aligned}
&\le \mathbb{P}_\lambda(\eta^\varepsilon \in T^{-1}E)\mathbb{P}_\lambda(\eta^\varepsilon \in T^{-1}F) \\
&\le \mathbb{P}_\lambda(\{\xi^\varepsilon \in E\} \cap R_\varepsilon^c)\mathbb{P}_\lambda(\{\xi^\varepsilon \in F\} \cap R_\varepsilon^c) + \mathcal{O}(\varepsilon^d) \\
&= \mathbb{P}_\lambda(\xi \in E)\mathbb{P}_\lambda(\xi \in F) + \mathcal{O}(\varepsilon^d).
\end{aligned}
$$

In the second-to-last step, we have used that we can intersect events with R_ε^c at the cost of adding $\mathcal{O}(\varepsilon^d)$. In the last step, we have used (2.13). Letting $\varepsilon \to 0$, we conclude the proof. □

In the above proof of Theorem 2.1, mind that $T^{-1}(E)$ and $T^{-1}(F)$ are not increasing any more, and so we have made crucial use of the general BKR inequality (2.16).

An application of the BK inequality. We now give an example of the use of Theorem 2.1. Let ξ be given as in (2.5) and let $x_1, x_2, x_3 \in \mathbb{R}^d$. Define E as the event that there is a path between x_1 and x_2 as well as a second path between x_2 and x_3 that only shares x_2 as common vertex with the first path. More formally, let $E := \{x_1 \longleftrightarrow x_2 \text{ in } \xi^{x_1,x_2}\} \circ \{x_2 \longleftrightarrow x_3 \text{ in } \xi^{x_2,x_3}\}$. We assert that

$$
\mathbb{P}_\lambda(E) \le \tau_\lambda(x_2 - x_1)\tau_\lambda(x_3 - x_2). \tag{2.19}
$$

We want to apply Theorem 2.1 for an RCM ξ' with a slightly modified mark space; moreover, we need to identify W_1 and W_2. For each $i \in \mathbb{N}$, we let $M_i := (U_{i,0}, U_{0,i}, U_{i,-1}, U_{-1,i}, U_{i,-2}, U_{-2,i})$ and define ξ' by (2.5) with Y_i replaced by (Y_i, M_i). We also define $W_1 := (U_{0,-1}, U_{-1,0})$ and $W_2 := (U_{-1,-2}, U_{-2,-1})$. Then the RCM $\Gamma_\varphi(\xi^{x_1,x_2,x_3})$ is a (measurable) function of (ξ', W_1, W_2) and $(U_{0,-2}, U_{-2,0})$. Note that $(U_{0,-2}, U_{-2,0})$ is not required for determining the event E. Let $(\Lambda_n)_{n\in\mathbb{N}}$ with $\Lambda_n := [-n, n)^d$ and define $\tau_\lambda^n(v,u) := \mathbb{P}_\lambda(v \longleftrightarrow u \text{ in } \xi_{\Lambda_n}^{v,u})$. Then

$$
\mathbb{P}_\lambda\big(\{x_1 \longleftrightarrow x_2 \text{ in } \xi_{\Lambda_n}^{x_1,x_2}\} \circ \{x_2 \longleftrightarrow x_3 \text{ in } \xi_{\Lambda_n}^{x_2,x_3}\}\big) \le \tau_\lambda^n(x_1,x_2)\tau_\lambda^n(x_2,x_3)
$$

for every $n \in \mathbb{N}$. Monotone convergence implies (2.19).

The FKG inequality. Adapting the FKG inequality turns out to be rather straightforward. Given two increasing events E, F, we want that $\mathbb{P}_\lambda(\xi \in E \cap F) \ge \mathbb{P}_\lambda(\xi \in E)\mathbb{P}_\lambda(\xi \in F)$. Indeed, given two increasing (integrable) functions f, g, we have the more general statement

$$
\mathbb{E}_\lambda[f(\xi)g(\xi)] = \mathbb{E}_\lambda\big[\mathbb{E}_\lambda[f(\xi)g(\xi) \mid \eta]\big] \ge \mathbb{E}_\lambda\big[\mathbb{E}_\lambda[f(\xi) \mid \eta]\, \mathbb{E}_\lambda[g(\xi) \mid \eta]\big] \ge \mathbb{E}_\lambda[f(\xi)]\, \mathbb{E}_\lambda[g(\xi)]. \tag{2.20}
$$

The first inequality was obtained by applying FKG to the random graph conditioned to have η as its vertex set, the second inequality by applying FKG for point processes (see, e.g., [24]).

Truncation arguments and a differential inequality. Next, we prove elementary differentiability results as well as a differential inequality, illustrating how to put the above tools into action. Since Russo-Margulis and BK work only for events on bounded domains and we intend to use them for events of the form $\{\mathbf{0} \longleftrightarrow x \text{ in } \xi^{\mathbf{0},x}\}$, this careful treatment is necessary. We point out that this is the only instance in this paper where we carry out these finite-size approximations in such detail. We start with the differentiability of τ_λ. Recall that $\Lambda_n = [-n, n)^d$ and $\tau_\lambda^n(v, x) = \mathbb{P}_\lambda(v \longleftrightarrow x \text{ in } \xi_{\Lambda_n}^{v,x})$. We write $\tau_\lambda^n(x) := \tau_\lambda^n(\mathbf{0}, x)$.

Moreover, we want to give meaning to the event $\{x \longleftrightarrow \Lambda_n^c \text{ in } \xi_{\Lambda_n}^x\}$. To this end, we add a "ghost vertex" g in the same way we added deterministic vertices, and we add an edge between $v \in \xi_{\Lambda_n}$ and g with probability $1 - \exp(-\int_{\Lambda_n^c} \varphi(y - v)\,\mathrm{d}y)$. We now identify Λ_n^c with g.

Lemma 2.2 (Differentiability of τ_λ)**.** *Let $x \in \mathbb{R}^d$ and $\varepsilon > 0$ be arbitrary. The function $\lambda \mapsto \tau_\lambda^n(x)$ is differentiable on $[0, \lambda_c - \varepsilon]$ for any $n \in \mathbb{N}$. Furthermore, $\tau_\lambda^n(x)$ converges to $\tau_\lambda(x)$ uniformly in λ and $\frac{\mathrm{d}}{\mathrm{d}\lambda}\tau_\lambda^n(x)$ converges to a limit uniformly in λ. Consequently, $\tau_\lambda(x)$ is differentiable w.r.t. λ on $[0, \lambda_c)$ and*

$$
\lim_{n\to\infty} \frac{\mathrm{d}}{\mathrm{d}\lambda}\tau_\lambda^n(x) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\tau_\lambda(x) = \int \mathbb{P}_\lambda(\mathbf{0} \longleftrightarrow x \textit{ in } \xi^{\mathbf{0},y,x}, \mathbf{0} \not\longleftrightarrow x \textit{ in } \xi^{\mathbf{0},x})\,\mathrm{d}y. \tag{2.21}
$$

Proof. For $S \subseteq \mathbb{R}^d$, let $\{a \longleftrightarrow b \text{ in } \xi^{a,b} \text{ through } S\}$ be the event that a is connected to b in $\xi^{a,b}$, and every path uses a vertex in S. Also, let $\{x \longleftrightarrow S \text{ in } \xi^x\}$ be the event that there is $y \in \eta^x \cap S$ that is connected to x. The convergence $\tau_\lambda^n(x) \to \tau_\lambda(x)$ is uniform in $x \in \mathbb{R}^d$ and $\lambda \in [0, \lambda_c - \varepsilon]$, as

$$\begin{aligned}|\tau_\lambda(x) - \tau_\lambda^n(x)| &= \mathbb{P}_\lambda(\mathbf{0} \longleftrightarrow x \text{ in } \xi^{\mathbf{0},x} \text{ through } \Lambda_n^c) \\ &\leq \mathbb{P}_{\lambda_c-\varepsilon}(\mathbf{0} \longleftrightarrow \Lambda_n^c \text{ in } \xi^{\mathbf{0}}) \xrightarrow{n\to\infty} 0\end{aligned}$$

uniformly in $\lambda \leq \lambda_c - \varepsilon$. We further claim that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\tau_\lambda^n(x) \xrightarrow{n\to\infty} f_\lambda(x) := \int \mathbb{P}_\lambda(\mathbf{0} \longleftrightarrow x \text{ in } \xi^{\mathbf{0},y,x}, \mathbf{0} \not\longleftrightarrow x \text{ in } \xi^{\mathbf{0},x})\,\mathrm{d}y \tag{2.22}$$

uniformly in λ. A helpful identity in the proof of (2.22) is (1.4). It follows from the Mecke equation, as

$$\chi(\lambda) = 1 + \mathbb{E}_\lambda\Big[\sum_{x\in\eta} \mathbb{1}_{\{\mathbf{0}\longleftrightarrow x \text{ in } \xi^{\mathbf{0}}\}}\Big] = 1 + \lambda\int \tau_\lambda(x)\,\mathrm{d}x, \tag{2.23}$$

and it implies that $\int \tau_\lambda(x)\,\mathrm{d}x < \infty$ for $\lambda < \lambda_c = \lambda_T$. To prove (2.22), note that $\{\mathbf{0} \longleftrightarrow x \text{ in } \xi_{\Lambda_n}^{\mathbf{0},x}\}$ lives on the bounded set Λ_n, and so we can apply the Margulis-Russo formula (2.9), which gives differentiability and an explicit expression for the derivative as

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\tau_\lambda^n(x) = \int_{\Lambda_n} \mathbb{P}_\lambda\left(\mathbf{0} \longleftrightarrow x \text{ in } \xi_{\Lambda_n}^{\mathbf{0},y,x}, \mathbf{0} \not\longleftrightarrow x \text{ in } \xi_{\Lambda_n}^{\mathbf{0},x}\right)\,\mathrm{d}y. \tag{2.24}$$

As a consequence of (2.24), we can write

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}\tau_\lambda^n(x) - f_\lambda(x)\right| &= \bigg|\int_{\Lambda_n} \Big(\mathbb{P}_\lambda(\mathbf{0} \longleftrightarrow x \text{ in } \xi^{\mathbf{0},y,x}) - \mathbb{P}_\lambda(\mathbf{0} \longleftrightarrow x \text{ in } \xi^{\mathbf{0},x}) \\ &\qquad\qquad + \mathbb{P}_\lambda(\mathbf{0} \longleftrightarrow x \text{ in } \xi_{\Lambda_n}^{\mathbf{0},x}) - \mathbb{P}_\lambda(\mathbf{0} \longleftrightarrow x \text{ in } \xi_{\Lambda_n}^{\mathbf{0},y,x})\Big)\,\mathrm{d}y \\ &\quad + \int_{\Lambda_n^c} \mathbb{P}_\lambda(\mathbf{0} \longleftrightarrow x \text{ in } \xi^{\mathbf{0},y,x}, \mathbf{0} \not\longleftrightarrow x \text{ in } \xi^{\mathbf{0},x})\,\mathrm{d}y\bigg| \\ &\leq \bigg|\int_{\Lambda_n} \mathbb{P}_\lambda(\mathbf{0} \longleftrightarrow x \text{ in } \xi^{\mathbf{0},y,x} \text{ through } \Lambda_n^c) - \mathbb{P}_\lambda(\mathbf{0} \longleftrightarrow x \text{ in } \xi^{\mathbf{0},x} \text{ through } \Lambda_n^c)\,\mathrm{d}y\bigg| \\ &\quad + \int_{\Lambda_n^c} \mathbb{P}_\lambda(\mathbf{0} \longleftrightarrow y \text{ in } \xi^{\mathbf{0},y})\,\mathrm{d}y \\ &\leq \int_{\Lambda_n} \mathbb{P}_\lambda(\mathbf{0} \longleftrightarrow x \text{ in } \xi^{\mathbf{0},y,x} \text{ through } \Lambda_n^c \text{ and through } y)\,\mathrm{d}y + \int_{\Lambda_n^c} \tau_{\lambda_c-\varepsilon}(y)\,\mathrm{d}y.\end{aligned}$$

Now, observe that the event $\{\mathbf{0} \longleftrightarrow x \text{ in } \xi^{\mathbf{0},y,x} \text{ through } \Lambda_n^c \text{ and through } y\}$ is contained in

$$\begin{aligned}&\Big(\{\mathbf{0} \longleftrightarrow y \text{ in } \xi^{\mathbf{0},y}\} \circ \{y \longleftrightarrow \Lambda_n^c \text{ in } \xi^y\} \circ \{\Lambda_n^c \longleftrightarrow x \text{ in } \xi^x\}\Big) \\ \cup&\Big(\{\mathbf{0} \longleftrightarrow \Lambda_n^c \text{ in } \xi^{\mathbf{0}}\} \circ \{\Lambda_n^c \longleftrightarrow y \text{ in } \xi^y\} \circ \{y \longleftrightarrow x \text{ in } \xi^{y,x}\}\Big).\end{aligned}$$

Note that the relation '$\circ$' is associative and commutative (see also Definition 4.9). Applying the BK inequality together with $\int_{\Lambda_n^c} \tau_{\lambda_c-\varepsilon}(y)\,\mathrm{d}y = o(1)$ as $n \to \infty$ gives

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}\tau_\lambda^n(x) - f_\lambda(x)\right| &\leq \int_{\Lambda_n} \mathbb{P}_\lambda(y \longleftrightarrow \Lambda_n^c \text{ in } \xi^y) \\ &\qquad \times \Big[\tau_\lambda(y)\mathbb{P}_\lambda(x \longleftrightarrow \Lambda_n^c \text{ in } \xi^x) + \tau_\lambda(x-y)\mathbb{P}_\lambda(\mathbf{0} \longleftrightarrow \Lambda_n^c \text{ in } \xi^{\mathbf{0}})\Big]\,\mathrm{d}y + o(1) \\ &\leq 2\max_{z\in\{0,x\}} \mathbb{P}_{\lambda_c-\varepsilon}\big(z \longleftrightarrow \Lambda_n^c \text{ in } \xi^z\big)\int \tau_{\lambda_c-\varepsilon}(y)\,\mathrm{d}y + o(1) = o(1),\end{aligned}$$

as the remaining integral is again bounded and the remaining probability tends to zero uniformly in λ. The uniform convergence justifies the exchange of limit and derivative in (2.21) (see, e.g., [34, Thm. 7.17]. □

We close this section by deriving a useful differential inequality:

Lemma 2.3 (A differential inequality for $\chi(\lambda)$)**.** *Let $\lambda < \lambda_c$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\widehat{\tau}_\lambda(\mathbf{0}) \leq \widehat{\tau}_\lambda(\mathbf{0})^2.$$

Proof. First note that with (2.24) (and the BK inequality), we can bound

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\tau_\lambda^n(x) \leq \int \mathbb{P}_\lambda\left(\{\mathbf{0} \longleftrightarrow y \text{ in } \xi_{\Lambda_n}^{\mathbf{0},y}\} \circ \{y \longleftrightarrow x \text{ in } \xi_{\Lambda_n}^{y,x}\}\right) \mathrm{d}y \leq \int \tau_\lambda^n(y)\tau_\lambda^n(y,x)\,\mathrm{d}y. \tag{2.25}$$

We can now use Leibniz' integral rule in its measure-theoretic form to write $\frac{\mathrm{d}}{\mathrm{d}\lambda}\int \tau_\lambda(x)\,\mathrm{d}x = \int \frac{\mathrm{d}}{\mathrm{d}\lambda}\tau_\lambda(x)\,\mathrm{d}x$. This is justified as the integrand τ_λ is uniformly bounded by the integrable function $\tau_{\lambda_c-\varepsilon}$ for some small $\varepsilon \in (0, \lambda_c - \lambda)$. Applying Lemma 2.2 as well as (2.25), we derive

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\widehat{\tau}_\lambda(\mathbf{0}) = \int \lim_{n\to\infty} \frac{\mathrm{d}}{\mathrm{d}\lambda}\tau_\lambda^n(x)\,\mathrm{d}x \leq \int \tau_\lambda(y)\left(\int \tau_\lambda(x-y)\,\mathrm{d}x\right)\mathrm{d}y = \widehat{\tau}_\lambda(\mathbf{0})^2. \qquad \square$$

3 The expansion

3.1 Preparatory definitions

The aim of this section is to prove the expansion for τ_λ stated in (1.15). It is one of the goals of the subsequent sections to show that $R_{\lambda,n} \to 0$ as $n \to \infty$ when $\lambda < \lambda_c$. The intuitive idea behind the expansion is quite simple. Loosely speaking, $\{\mathbf{0} \longleftrightarrow x \text{ in } \xi^{\mathbf{0},x}\}$ is partitioned over the first pivotal point $u \in \eta$ for this connection (if such a point exists). That is, there is a double connection between $\mathbf{0}$ and u and we recover $\tau_\lambda(x-u)$, due to the event that $\{u \longleftrightarrow x \text{ in } \xi^{u,x}\}$. However, this is not quite true, and the overcounting error made by pretending as if the double connection event between $\mathbf{0}$ and u was *independent* of the connection event between u and x has to be subtracted. This constitutes the first step of the expansion. The second step is to further examine this error term, in which we recognize a similar structure, allowing for a similar partitioning strategy again.

Making this informal strategy of proof precise requires definitions, starting with some extended connection events:

Definition 3.1 (Connectivity terminology)**.** Let $u, v, x \in \mathbb{R}^d$.

(1.) We say u and x are 2-*connected* in ξ if $u = x \in \eta$, if $u, x \in \eta$ and $u \sim x$, or if $u, x \in \eta$ and there are (at least) two paths between u and x that are disjoint in all their interior vertices; that is, there are two paths that only share u and x as common vertices. We denote this event by $\{u \Longleftrightarrow x \text{ in } \xi\}$.

(2.) For $A \subseteq \eta$, we say that u and x are connected in ξ *off* A and write $\{u \longleftrightarrow x \text{ in } \xi \text{ off } A\}$ for the event $\{u \longleftrightarrow x \text{ in } \xi[\eta \setminus A]\}$ (where we recall that $\xi[\mu]$ is the subgraph of ξ induced by μ). In words, this is the event that $u, x \in \eta$ and there exists a path between u and x in ξ not using any vertices of A. In particular, this event fails if A contains u or x.

We remark that $\{u \Longleftrightarrow x \text{ in } \xi\} = \{u = x \in \eta\} \cup \{u \sim x \text{ in } \xi\} \cup (\{u \longleftrightarrow x \text{ in } \xi\} \circ \{u \longleftrightarrow x \text{ in } \xi\})$. Moreover, we will mostly be concerned with added deterministic points $u, x \in \mathbb{R}^d$ and hence with the event $\{u \Longleftrightarrow x \text{ in } \xi^{u,x}\}$.

The next definitions introduce thinnings, a standard concept in point-process literature. Recall from Section 2.2 that every Poisson point $X_i \in \eta$ comes with a sequence of "thinning marks" $(Y_{i,j})_{j\in\mathbb{N}}$.

Definition 3.2 (Thinning events)**.** Let $u, x \in \mathbb{R}^d$, and let $A \subset \mathbb{R}^d$ be locally finite and of cardinality $|A|$.

(1.) Set

$$\bar{\varphi}(A, x) := \prod_{y\in A}(1 - \varphi(y - x)) \tag{3.1}$$

and define $\eta_{\langle A\rangle}$ as a $\bar{\varphi}(A,\cdot)$-*thinning of* η (or simply A-*thinning of* η) as follows. We keep a point $w \in \eta$ as a point of $\eta_{\langle A\rangle}$ with probability $\bar{\varphi}(A, w)$ independently of all other points of η. To make this more explicit, we use the mappings π_i, $i \in \mathbb{N}$, introduced in Section 2.1. In particular, $(\pi_j(A))_{j\leq|A|}$ is an ordering of the points in A and $(\pi_i(\eta))_{i\in\mathbb{N}}$ is an ordering of the points in η. We keep $\pi_i(\eta) \in \eta$ as a point of $\eta_{\langle A\rangle}$ if $Y_{i,j} > \varphi(\pi_j(A) - \pi_i(\eta))$ for all $j \leq |A|$ (we say that $\pi_i(\eta)$ *survives* the A-thinning). We further define $\eta^x_{\langle A\rangle}$ as a $\bar{\varphi}(A,\cdot)$-thinning of η^x using the marks in ξ^x.

(2.) We write $u \overset{A}{\longleftrightarrow} x$ in ξ if both $\{u \longleftrightarrow x \text{ in } \xi\}$ and $\{u \not\longleftrightarrow x \text{ in } \xi[\eta_{\langle A\rangle} \cup \{u\}]\}$ take place.

In words, $\{u \overset{A}{\longleftrightarrow} x \text{ in } \xi\}$ is the event that $u, x \in \eta$ and u is connected to x in ξ, but this connection does not survive an A-thinning of $\eta \setminus \{u\}$. In particular, the connection does not survive if x is thinned out.

(3.) We define

$$\tau_\lambda^A(u,x) = \mathbb{P}_\lambda\Big(u \longleftrightarrow x \text{ in } \xi^{u,x}[\eta^x_{\langle A\rangle} \cup \{u\}]\Big). \tag{3.2}$$

In words, $\tau_\lambda^A(u,x)$ is the probability of the event that there exists an open path between u and x in an RCM driven by an A-thinning of η^x, where the point u is fixed to be present (but x is not).

Let us make some remarks about Definition 3.2. First, the definition of $\eta_{\langle A\rangle}$ is the first time we need the enriched version of ξ from Section 2.2. It is due to the independence of the sequence $(Y_{i,j})_{i,j\in\mathbb{N}}$ that the (conditional) probability of some point $y \in \eta$ being contained in $\eta_{\langle A\rangle}$ is indeed $\bar\varphi(A,y)$.

Secondly, it is due to the thinning properties of a Poisson point process that $\eta_{\langle A\rangle}$ has the distribution of an inhomogeneous PPP with intensity $\lambda\bar\varphi(A,\cdot)$ (see, e.g., [24]). Thirdly, $\{u \overset{A}{\longleftrightarrow} x \text{ in } \xi^{u,x}\}$ is *not* symmetric w.r.t. u and x, since x can be thinned out, but u can not. Lastly, note that the events considered in (2.) and (3.) of Definition 3.2 are complementary in the sense that

$$\{u \longleftrightarrow x \text{ in } \xi^{u,x}\} = \{u \longleftrightarrow x \text{ in } \xi^{u,x}[\eta^x_{\langle A\rangle} \cup \{u\}]\} \cup \{u \overset{A}{\longleftrightarrow} x \text{ in } \xi^{u,x}\}, \tag{3.3}$$

and the above union is disjoint. This observation will be an important identity in the lace expansion.

3.2 Stopping sets and cutting points

Before deriving the expansion, we state and prove the Cutting-point Lemma (see Lemma 3.6). This lemma is crucial in deriving an expansion and quite standard in the literature; we view it as an analogue of the Cutting-bond lemma (see [20, Lemma 6.4]).

One central idea in the proof of the Cutting-point Lemma 3.6 is to use the *stopping set* properties of $\mathscr{C}(v,\xi^v)$. We therefore start with Lemma 3.3, which rigorously formulates the properties we need.

To stress the dependence of ξ on η, we write $\xi(\eta) := \xi$ in the statement of Lemma 3.3 and parts of its proof, even though this notation is a bit ambiguous. First, it does not reflect the dependence of ξ on the marks U_{ij}. Secondly, the definition of ξ depends on the ordering of the points of η. The distribution of ξ, however, does not depend on this ordering.

Lemma 3.3 (Stopping-set lemma). *Let $v \in \mathbb{R}^d$. Then*

$$\mathbb{P}_\lambda\big(\xi^v[\eta^v \setminus \mathscr{C}(v,\xi^v)] \in \cdot \mid \mathscr{C}(v,\xi^v) = A\big) = \mathbb{P}_\lambda(\xi(\eta_{\langle A\rangle}) \in \cdot) \quad \textit{for } \mathbb{P}_\lambda(\mathscr{C}(v,\xi^v) \in \cdot)\textit{-a.e. } A. \tag{3.4}$$

Before giving a proof we explain the distributional identity (3.4). On the left-hand side, we have the conditional distribution of the restriction of ξ^v to the complement of the cluster $\mathscr{C}(v,\xi^v)$ given that $\mathscr{C}(v,\xi^v) = A$. On the right-hand side, we have an independent edge-marking based on the inhomogeneous Poisson process $\eta_{\langle A\rangle}$. Even though the latter is defined as an independent thinning of η (its intensity is bounded by λ), the point process $\eta \setminus \mathscr{C}(v,\xi^v)$ cannot be constructed this way. In fact, neither $\mathscr{C}(v,\xi^v)$ nor $\eta \setminus \mathscr{C}(v,\xi^v)$ is a Poisson process.

The proof is based on a recursive construction of the cluster, in ascending graph distance from the root; this is also the reason why subcriticality is not required.

Moreover, we want to point out that the following proof is for the RCM as defined in (2.2). The proof for the RCM with additional marks is essentially the same, just heavier on notation.

Proof of Lemma 3.3. The proof is similar to the one of Proposition 2 in the paper by Meester et al. [28]. Since the lemma is crucial for our paper, we give more details here. We interpret $\xi^v[\mathscr{C}(v,\xi^v)]$ as a rooted graph with root v. Let $\eta_0 := \{v\}$. For $n \in \mathbb{N}$ let η_n be the vertices of $\mathscr{C}(v,\xi^v)$ whose graph distance from the root is at most n. We assert that, for every $n \in \mathbb{N}$,

$$\mathbb{E}_\lambda[f(\xi^v[\eta \setminus \eta_n], \eta_1, \dots, \eta_n)] = \int \mathbb{E}_\lambda[f(\xi(\eta_{\langle A_{n-1}\rangle}), A_1, \dots, A_n)]\mathbb{P}_\lambda((\eta_1, \dots, \eta_n) \in \mathrm{d}(A_1, \dots, A_n)) \tag{3.5}$$

for all measurable non-negative functions f with suitable domain, where $A_0 := \{v\}$ and where we recall that the arguments of f are point processes.

We prove a slightly more general version of (3.5), which is amenable to induction. To do so, we need to introduce some further notation. Given two disjoint point processes $\eta', \eta'' \subset \eta^v$, we define $\xi^v[\eta', \eta'']$ as the random marked graph arising from ξ^v by taking all marked edges with at least one vertex in η' and no vertex outside of $\eta' \cup \eta''$. Given a point process μ on $\mathbb{R}^d$ and a locally finite set $A \subset \mathbb{R}^d$, we define a random marked graph $T(\mu, A)$ as follows. The edge set is given by $\{\{x, y\} : x \in \mu, y \in \mu \cup A\}$. The marks are given by independent random variables, uniformly distributed on $[0, 1]$ and independent of μ.

We claim that

$$\begin{aligned}&\mathbb{E}_\lambda[f(\xi^v[\eta \setminus \eta_n, \eta_n \setminus \eta_{n-1}], \eta_1, \dots, \eta_n)]\\&\quad = \int \mathbb{E}_\lambda[f(T(\eta_{\langle A_{n-1}\rangle}, A_n \setminus A_{n-1}), A_1, \dots, A_n))]\mathbb{P}_\lambda((\eta_1, \dots, \eta_n) \in \mathrm{d}(A_1, \dots, A_n))\end{aligned} \tag{3.6}$$

for all measurable non-negative functions f with suitable domain, which is clearly more general than (3.5). This can be written as

$$\mathbb{P}_\lambda(\xi^v[\eta \setminus \eta_n, \eta_n \setminus \eta_{n-1}] \in \cdot \mid (\eta_0, \dots, \eta_n) = (A_0 \dots, A_n)) = \mathbb{P}_\lambda(T(\eta_{\langle A_{n-1}\rangle}, A_n \setminus A_{n-1}) \in \cdot) \tag{3.7}$$

for $\mathbb{P}_\lambda((\eta_0, \dots, \eta_n) \in \cdot)$-a.e. $(A_0, \dots, A_n)$ and $A_0 = \{v_0\}$.

We base the proof of (3.7) on the following property. Let $h\colon \mathbb{R}^d \to [0, \infty)$ be measurable and let μ be a Poisson process with intensity function h. Further, let $A \subset \mathbb{R}^d$ be locally finite and consider the independent edge-marking $\tilde{\xi} := \xi(\mu \cup A)$ of $\mu \cup A$. Let μ^A be the set of points from μ which are directly connected to a point from A, where the connection is defined as before in terms of $\tilde{\xi}$ and the connection function φ. Then we have the distributional identity

$$(\xi[\mu \setminus \mu^A, \mu^A], \mu^A) \overset{d}{=} (T(\mu', \mu''), \mu''), \tag{3.8}$$

where μ' and μ'' are independent Poisson processes with intensity functions $h(\cdot)\bar{\varphi}(A, \cdot)$ and $h(\cdot)(1 - \bar{\varphi}(A, \cdot))$, respectively. This follows from the marking and mapping theorems for Poisson processes (see [24, Theorems 5.6 and 5.1]) applied to a suitably defined Poisson process $\tilde{\xi}$ such that $\xi(\mu \cup A)$ is (up to the marks of edges with both vertices in A) a deterministic function of $\tilde{\xi}$. The details of this construction are left to the reader.

Applying (3.8) with $A = \{v\}$ and $\mu = \eta$ gives (3.7) for $n = 1$. Suppose (3.7) is true for some $n \in \mathbb{N}$ and let $A_1, \dots, A_n$ be locally finite subsets of $\mathbb{R}^d$. Applying (3.8) with the conditional probability measure $\mathbb{P}(\cdot \mid (\eta_0, \dots, \eta_n) = (A_0, \dots, A_n))$ and with $\mu = \eta_{\langle A_{n-1}\rangle}$ as well as $A = A_n \setminus A_{n-1}$ gives (3.7) for $n + 1$.

In fact, this argument also yields that, given $(\eta_0, \dots, \eta_n)$, the point processes $\eta \setminus \eta_{n+1}$ and $\eta_{n+1} \setminus \eta_n$ are conditionally independent Poisson processes with intensity functions $\lambda\bar{\varphi}(\eta_n, \cdot)$ and $\lambda(1 - \bar{\varphi}(\eta_n \setminus \eta_{n-1}, \cdot))\bar{\varphi}(\eta_{n-1}, \cdot)$, respectively. Since

$$1 - \bar{\varphi}(\eta_n \setminus \eta_{n-1}, x) \le \sum_{w \in \eta_n \setminus \eta_{n-1}} \varphi(w - x), \quad x \in \mathbb{R}^d,$$

it follows by induction and by the integrability of φ that the point processes η_n are all finite almost surely.

Equation (3.5) shows in particular that

$$\mathbb{E}_\lambda[f(\xi^v[\eta \setminus \eta_n], \eta_n)] = \int \mathbb{E}_\lambda[f(\xi(\eta_{\langle V_{n-1}(G)\rangle}), V_n(G))]\mathbb{P}_\lambda(\xi^v[\mathscr{C}(v, \xi^v)] \in \mathrm{d}G) \tag{3.9}$$

for all measurable non-negative functions f with suitable domain, where, for a rooted graph G and $n \in \mathbb{N}_0$, $V_n(G)$ denotes the set of vertices of G whose graph distance from the root is at most n. Let $\eta_\infty = \cup_n \eta_n$ denote the vertex set $\mathscr{C}(v, \xi^v)$. Note that for a bounded Borel set, we have that $|\eta_\infty \cap B| = |\eta_n \cap B|$ for all sufficiently large n almost surely. Note also that $\xi^v[\eta \setminus \eta_n] \downarrow \xi^v[\eta \setminus \eta_\infty]$ as $n \to \infty$. Therefore, if $f(\xi^v[\eta \setminus \eta_n], \eta_n)$ is a bounded function of $|\xi^v[\eta \setminus \eta_n] \cap B_1|, \dots, |\xi^v[\eta \setminus \eta_n] \cap B_k|$ and $|\eta_n \cap B_{k+1}|, \dots, |\eta_n \cap B_m|$ for suitable measurable and bounded sets $B_1, \dots, B_m$, the left-hand side of (3.9) tends to $\mathbb{E}_\lambda[f(\xi^v[\eta \setminus \eta_\infty], \eta_\infty)]$ as $n \to \infty$.

For a similar reason, the integrand on the right-hand side converges for each fixed rooted (locally finite) graph G to $\mathbb{E}_\lambda[f(\xi(\eta_{\langle V(G)\rangle}), V(G)))]$, where $V(G)$ is the vertex set of G. Therefore, we obtain from dominated convergence that

$$\mathbb{E}_\lambda[f(\xi^v[\eta \setminus \eta_\infty], \eta_\infty)] = \int \mathbb{E}_\lambda[f(\xi(\eta_{\langle \nu \rangle}), \nu)] \mathbb{P}_\lambda(\eta_\infty \in \mathrm{d}\nu), \tag{3.10}$$

first for special non-negative f, and then, by a monotone-class argument, for general f. This implies the assertion. $\square$

Lemma 3.3 is a quite general distributional identity. We will only require the following corollary:

Corollary 3.4. *Let $v, u, x \in \mathbb{R}^d$ be distinct. Then, for $\mathbb{P}_\lambda(\mathscr{C}(v, \xi^v) \in \cdot)$-a.e. A,*

$$\mathbb{P}_\lambda\big(u \longleftrightarrow x \text{ in } \xi^{u,x} \text{ off } \mathscr{C}(v, \xi^v) \mid \mathscr{C}(v, \xi^v) = A\big) = \mathbb{P}_\lambda(u \longleftrightarrow x \text{ in } \xi^{u,x}[\eta_{\langle A \rangle} \cup \{u, x\}]).$$

We next introduce the notion of *pivotal points*. To this end, let ξ be an edge-marking of a PPP η and let $v, u, x \in \eta$. We say that $u \notin \{v, x\}$ is *pivotal* for the connection from v to x (and write $u \in \mathsf{Piv}(v, x; \xi)$) if $\{v \longleftrightarrow x \text{ in } \xi\}$ but $\{v \not\longleftrightarrow x \text{ in } \xi[\eta \setminus \{u\}]\}$. Mind that, by definition, v and x are never elements of $\mathsf{Piv}(v, x; \xi)$. We list ξ as an argument after the semicolon to indicate decorations of ξ with extra points. This way, we can speak of the event $\{u \in \mathsf{Piv}(v, x; \xi^{v,u,x})\}$ for arbitrary $v, u, x \in \mathbb{R}^d$. We use the same notation for events that are introduced later.

Note that $\mathsf{Piv}(v, x; \xi) = \mathsf{Piv}(x, v; \xi)$, but we use the notation to put emphasis on paths "from v to x". A non-symmetric property of pivotal points that we use later is the fact that $\mathsf{Piv}(v, x; \xi)$ can be ordered in the sense that there is a unique first (second, third, etc.) pivotal point that every path from v to x traverses first (second, third, etc.). Furthermore, for a locally finite set $A \subset \mathbb{R}^d$, and $v, u \in \mathbb{R}^d$, we define

$$E(v, u; A, \xi) := \{v \xleftrightarrow{\ A\ } u \text{ in } \xi\} \cap \{\nexists w \in \mathsf{Piv}(v, u; \xi) : v \xleftrightarrow{\ A\ } w \text{ in } \xi\}. \tag{3.11}$$

Let us take the time to prove an elementary partitioning identity here, which will be useful at a later stage:

Lemma 3.5 (Partition of connection events)**.** *Let $v, x \in \mathbb{R}^d$ and let $A \subset \mathbb{R}^d$ be a locally finite set. Then*

$$\mathbb{1}_{\{v \xleftrightarrow{A} x \text{ in } \xi^{v,x}\}} = \mathbb{1}_{E(v,x;A,\xi^{v,x})} + \sum_{u \in \eta} \mathbb{1}_{E(v,u;A,\xi^{v,x})} \mathbb{1}_{\{u \in \mathsf{Piv}(v,x;\xi^{v,x})\}}.$$

Proof. We prove "$\geq$" first. We first claim that the right-hand side is a sum of indicators of mutually disjoint events. Indeed, due to the ordering of pivotal points y satisfying $\{v \xleftrightarrow{\ A\ } y \text{ in } \xi^v\}$, the choice of u as the first such pivotal point is unique, making the union over first pivotal points u a disjoint one. Moreover, $E(v, x; A, \xi^{v,x})$ is the event that the set of such pivotal points is empty.

Assume now that the right-hand side takes value 1. On the one hand, if $E(v, x; A, \xi^{v,x})$ holds, then $\{v \xleftrightarrow{\ A\ } x \text{ in } \xi^{v,x}\}$ holds as well by definition. On the other hand, assume that ξ contains a point $u \in \eta = V(\xi)$ such that $\xi \in E(v, u; A, \xi^{v,x})$ and $u \in \mathsf{Piv}(v, x; \xi^{v,x})$. Due to the pivotality of u, any path γ from v to x must be the concatenation of two disjoint paths γ_1 and γ_2 (i.e., γ_1 and γ_2 share no interior vertices), where γ_1 is a path from v to u and γ_2 is a path from u to x. Since $\{v \xleftrightarrow{\ A\ } u \text{ in } \xi^{v,x}\}$ holds, there must be a vertex $\pi_i(\eta) \in \gamma_1$ that is thinned out. By definition, there is some $\pi_j(A)$ such that $Y_{i,j} \leq \varphi(\pi_j(A) - \pi_i(\eta))$. In other words, $\pi_i(\eta)$ is deleted in an A-thinning of η, and so $\{v \xleftrightarrow{\ A\ } x \text{ in } \xi^{v,x}\}$ holds. Thus, "$\geq$" holds.

To see "$\leq$", assume that $\{v \xleftrightarrow{\ A\ } x \text{ in } \xi^{v,x}\}$ holds. Then either $E(v, x; A, \xi^{v,x})$ holds, or there is at least one pivotal point y satisfying $\{v \xleftrightarrow{\ A\ } y \text{ in } \xi^v\}$. Since the pivotal points can be ordered, we can pick the first such pivotal point and call it u. This point u then satisfies $E(v, u; A, \xi^{v,u})$. $\square$

The following lemma has an analogue in discrete models, see [18, Lemma 2.1]. In bond percolation, it is called the "Cutting-bond lemma". Since Lemma 3.3 holds for arbitrary intensity, so does Lemma 3.6.

Lemma 3.6 (Cutting-point lemma)**.** *Let $\lambda \geq 0$ and let $v, u, x \in \mathbb{R}^d$ with $u \neq x$ and let $A \subset \mathbb{R}^d$ be locally finite. Then*

$$\mathbb{E}_\lambda\big[\mathbb{1}_{E(v,u;A,\xi^{v,u,x})}\mathbb{1}_{\{u\in\mathsf{Piv}(v,x;\xi^{v,u,x})\}}\big] = \mathbb{E}_\lambda\left[\mathbb{1}_{E(v,u;A,\xi^{v,u})}\cdot \tau_\lambda^{\mathscr{C}(v,\xi^v)}(u,x)\right].$$

Moreover,

$$\mathbb{P}_\lambda\left(\mathbf{0} \Longleftrightarrow u \text{ in } \xi^{\mathbf{0},u,x}, u \in \mathsf{Piv}(\mathbf{0},x;\xi^{\mathbf{0},u,x})\right) = \mathbb{E}_\lambda\left[\mathbb{1}_{\{\mathbf{0}\Longleftrightarrow u \text{ in } \xi^{\mathbf{0},u}\}}\cdot \tau_\lambda^{\mathscr{C}(\mathbf{0},\xi^{\mathbf{0}})}(u,x)\right].$$

Before proceeding with the proof, we want to stress the fact that $\tau_\lambda^{\mathscr{C}(v,\xi^v)}(u,x)$ is the random variable arising from $\tau_\lambda^A(u,x)$ by replacing the fixed set A by the random set $\mathscr{C}(v,\xi^v)$.

Proof. First, note that

$$E(v,u;A,\xi^{v,u,x}) \cap \{u \in \mathsf{Piv}(v,x;\xi^{v,u,x})\} = E(v,u;A,\xi^{v,u}) \cap \{u \in \mathsf{Piv}(v,x;\xi^{v,u,x})\}.$$

In words, we can take away vertex x from $\xi^{v,u,x}$ in the event $E(v,u;A,\xi^{v,u,x})$, since if x was necessary (or even relevant) for the connection from v to u, then u would not be pivotal. Furthermore, abbreviating $\mathscr{C} = \mathscr{C}(v,\xi^v)$,

$$\{u \in \mathsf{Piv}(v,x;\xi^{v,u,x})\} = \{v \longleftrightarrow u \text{ in } \xi^{v,u}\} \cap \{u \longleftrightarrow x \text{ in } \xi^{u,x} \text{ off } \mathscr{C}\} \cap \{x \nsim y \text{ in } \xi^{v,x}\ \forall y \in \mathscr{C}\}$$

$\mathbb{P}_\lambda$-a.s. by the following argument: If u is pivotal, then $\mathscr{C}$ contains all vertices connected to v by a path not using u, and in return any path from x to v visits u before it hits $\mathscr{C}$. Both these statements use that $u \notin \mathscr{C}$ a.s. In particular, the first two connection events on the right-hand side hold and there cannot be a direct edge from x to $\mathscr{C}$. This proves one inclusion. Conversely, if u and x are connected off $\mathscr{C}$, then x cannot lie in $\mathscr{C}$. Moreover, it cannot even lie in $\mathscr{C}(v,\xi^{v,x})$ as this would imply the existence of an edge from x to $\mathscr{C}$. Consequently, every path from v to x must pass through u. As u is connected to v, this makes u a pivotal point in $\xi^{v,u,x}$, proving the second inclusion.

Since $E(v,u;A,\xi^{v,u}) \subseteq \{v \longleftrightarrow u \text{ in } \xi^{v,u}\}$,

$$\begin{aligned}E(v,u;A,\xi^{v,u}) &\cap \{u \in \mathsf{Piv}(v,x;\xi^{v,u,x})\} \\ &= E(v,u;A,\xi^{v,u}) \cap \{u \longleftrightarrow x \text{ in } \xi^{u,x} \text{ off } \mathscr{C}\} \cap \{x \nsim y \text{ in } \xi^{v,x}\ \forall y \in \mathscr{C}\}.\end{aligned}$$

Conditioning on $\xi' = \xi^{u,v}[\mathscr{C}(v,\xi^v) \cup \{u\}]$, we see that

$$\mathbb{E}_\lambda\big[\mathbb{1}_{E(v,u;A,\xi^{v,u,x})}\mathbb{1}_{\{u\in\mathsf{Piv}(v,x;\xi^{v,u,x})\}}\big] = \mathbb{E}_\lambda\big[\mathbb{1}_{E(v,u;A,\xi^{v,u})}\mathbb{E}_\lambda[\mathbb{1}_{\{u\longleftrightarrow x \text{ in } \xi^{u,x} \text{ off } \mathscr{C}\}}\mathbb{1}_{\{x\nsim y \text{ in } \xi^{v,x}\ \forall y\in\mathscr{C}\}} \mid \xi']\big],$$

by the fact that $E(v,u;A,\xi^{v,u})$ is measurable w.r.t. $\sigma(\xi')$. Indeed, ξ' is the graph induced by u together with all points that can be reached from v without traversing u. Now, conditionally on ξ', the last two indicators are independent: $\{x \nsim y \text{ in } \xi^{v,x}\,\forall y \in \mathscr{C}\}$ depends only on points in $\mathscr{C} \subseteq V(\xi')$ and edges between $\mathscr{C}$ and x. On the other hand, $\{u \longleftrightarrow x \text{ in } \xi^{u,x} \text{ off } \mathscr{C}\}$ depends only on points in $\eta^{u,x} \setminus \mathscr{C}$ and on edges between those points.

Together with the identities $\mathbb{P}_\lambda(x \nsim y \text{ in } \xi^{v,x}\,\forall y \in \mathscr{C} \mid \xi') = \bar\varphi(\mathscr{C},x)$ (recall (3.1)) and

$$\bar\varphi(B,x)\cdot\mathbb{P}_\lambda\big(u \longleftrightarrow x \text{ in } \xi^{u,x}[\eta_{\langle B\rangle} \cup \{u,x\}]\big) = \tau_\lambda^B(u,x)$$

for any locally finite set B (recall the definition of τ_λ^B in (3.2)), this leads to

$$\begin{aligned}\mathbb{E}_\lambda\big[\mathbb{1}_{E(v,u;A,\xi^{v,u,x})}\mathbb{1}_{\{u\in\mathsf{Piv}(v,x;\xi^{v,x})\}}\big] &= \mathbb{E}_\lambda\big[\mathbb{1}_{E(v,u;A,\xi^{v,u})}\cdot\bar\varphi(\mathscr{C},x)\cdot\mathbb{E}_\lambda[\mathbb{1}_{\{u\longleftrightarrow x \text{ in } \xi^{u,x} \text{ off } \mathscr{C}\}} \mid \xi']\big] \\ &= \mathbb{E}_\lambda\big[\mathbb{1}_{E(v,u;A,\xi^{v,u})}\cdot\bar\varphi(\mathscr{C},x)\cdot\mathbb{P}_\lambda\left(u \longleftrightarrow x \text{ in } \xi^{u,x} \text{ off } \mathscr{C} \mid \mathscr{C}\right)\big] \\ &= \mathbb{E}_\lambda\big[\mathbb{1}_{E(v,u;A,\xi^{v,u})}\cdot\bar\varphi(\mathscr{C},x)\cdot\mathbb{P}_\lambda(u \longleftrightarrow x \text{ in } \xi^{u,x}[\eta_{\langle\mathscr{C}\rangle} \cup \{u,x\}])\big] \\ &= \mathbb{E}_\lambda\big[\mathbb{1}_{E(v,u;A,\xi^{v,u})}\cdot\tau_\lambda^{\mathscr{C}}(u,x)\big].\end{aligned}$$

In the second line, we have used that $\sigma(\xi')$ and $\sigma(\mathscr{C})$ (the σ-fields generated by ξ' and $\mathscr{C}$ respectively) differ only in the information about the status of edges between points of $\mathscr{C} \cup \{u\}$. Since any connection event *off* $\mathscr{C}$ is independent of such edges, we can replace ξ' by $\mathscr{C}$ in the conditioning to use Corollary 3.4 in the third line.

The second assertion of Lemma 3.6 follows upon applying the above arguments with $E(v,u;A,\xi^{v,u})$ replaced by $\{\mathbf{0} \Longleftrightarrow u \text{ in } \xi^{\mathbf{0},u}\}$. □

3.3 The derivation of the expansion

For the following definition, we introduce a sequence of independent edge-markings $(\xi_i)_{i\in\mathbb{N}_0}$ of respective PPPs $(\eta_i)_{i\in\mathbb{N}_0}$.

Definition 3.7 (Lace-expansion coefficients)**.** For $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$, we define

$$\Pi_\lambda^{(0)}(x) := \mathbb{P}_\lambda(\mathbf{0} \Longleftrightarrow x \text{ in } \xi^{\mathbf{0},x}) - \varphi(x), \tag{3.12}$$

$$\Pi_\lambda^{(n)}(x) := \lambda^n \int \mathbb{P}_\lambda\Big(\{\mathbf{0} \Longleftrightarrow u_0 \text{ in } \xi_0^{\mathbf{0},u_0}\} \cap \bigcap_{i=1}^n E(u_{i-1}, u_i; \mathscr{C}_{i-1}, \xi_i^{u_{i-1},u_i})\Big) \,\mathrm{d}\vec{u}_{[0,n-1]}, \tag{3.13}$$

where $u_n = x$ and $\mathscr{C}_i = \mathscr{C}(u_{i-1}, \xi_i^{u_{i-1}})$ is the cluster of u_{i-1} in $\xi_i^{u_{i-1}}$. Further define

$$R_{\lambda,0}(x) := -\lambda \int \mathbb{P}_\lambda\Big(\{\mathbf{0} \Longleftrightarrow u_0 \text{ in } \xi_0^{\mathbf{0},u_0}\} \cap \{u_0 \xleftrightarrow{\;\mathscr{C}_0\;} x \text{ in } \xi_1^{u_0,x}\}\Big) \,\mathrm{d}u_0, \tag{3.14}$$

$$\begin{aligned} R_{\lambda,n}(x) := (-\lambda)^{n+1} \int \mathbb{P}_\lambda\Big(&\{\mathbf{0} \Longleftrightarrow u_0 \text{ in } \xi_0^{\mathbf{0},u_0}\} \cap \bigcap_{i=1}^n E(u_{i-1}, u_i; \mathscr{C}_{i-1}, \xi_i^{u_{i-1},u_i}) \\ &\cap \{u_n \xleftrightarrow{\;\mathscr{C}_n\;} x \text{ in } \xi_{n+1}^{u_n,x}\}\Big) \,\mathrm{d}\vec{u}_{[0,n]}. \end{aligned} \tag{3.15}$$

Additionally, define $\Pi_{\lambda,n}$ as the alternating partial sum

$$\Pi_{\lambda,n}(x) := \sum_{m=0}^n (-1)^m \Pi_\lambda^{(m)}(x). \tag{3.16}$$

We can relate $\Pi_\lambda^{(n)}$ and $R_{\lambda,n}$ in the following way. As $\mathbb{P}_\lambda(u_n \xleftrightarrow{\;A\;} x \text{ in } \xi_{n+1}^{u_n,x}) \le \tau_\lambda(x-u_n)$ for an arbitrary locally finite set A, we can bound

$$|R_{\lambda,n}(x)| \le \lambda \int \Pi_\lambda^{(n)}(u_n)\tau_\lambda(x-u_n) \,\mathrm{d}u_n \le \lambda\widehat{\tau}_\lambda(\mathbf{0})\big(\sup_{y\in\mathbb{R}^d} \Pi_\lambda^{(n)}(y)\big). \tag{3.17}$$

Our main result of this section is the following proposition:

Proposition 3.8 (Lace expansion)**.** *Let $x \in \mathbb{R}^d$ and $\lambda \in [0, \lambda_c)$. Then, for $n \ge 0$,*

$$\tau_\lambda(x) = \varphi(x) + \Pi_{\lambda,n}(x) + \lambda\big((\varphi + \Pi_{\lambda,n}) \star \tau_\lambda\big)(x) + R_{\lambda,n}(x). \tag{3.18}$$

Proof. The proof is by induction over n. After the base case (first step), we prove the case $n = 1$ (second step). The case for general n is analogous, but with heavier notation, and is only sketched (third step).

First step, $n = 0$. Using (3.12) in Definition 3.7, we observe that

$$\tau_\lambda(x) = \varphi(x) + \Pi_\lambda^{(0)}(x) + \mathbb{P}_\lambda(\mathbf{0} \longleftrightarrow x \text{ in } \xi^{\mathbf{0},x}, \mathbf{0} \not\Longleftrightarrow x \text{ in } \xi^{\mathbf{0},x}). \tag{3.19}$$

The event in the last term of the sum enforces the existence of a (first) pivotal point, and so, similar to Lemma 3.5, we can partition

$$\mathbb{1}_{\{\mathbf{0}\longleftrightarrow x \text{ in } \xi^{\mathbf{0},x}\}\cap\{\mathbf{0}\not\Longleftrightarrow x \text{ in } \xi^{\mathbf{0},x}\}} = \sum_{u\in\eta} \mathbb{1}_{\{\mathbf{0}\Longleftrightarrow u \text{ in } \xi^{\mathbf{0},x}\}\cap\{u\in\mathsf{Piv}(\mathbf{0},x;\xi^{\mathbf{0},x})\}}. \tag{3.20}$$

We set $\mathscr{C}_0 = \mathscr{C}(\mathbf{0}, \xi^{\mathbf{0}})$. Taking probabilities, we can use the Mecke formula (2.7) and then the Cutting-point Lemma 3.6 to rewrite

$$\begin{aligned} \mathbb{P}_\lambda(\mathbf{0} \longleftrightarrow x \text{ in } \xi^{\mathbf{0},x}, \mathbf{0} \not\Longleftrightarrow x \text{ in } \xi^{\mathbf{0},x}) &= \lambda \int \mathbb{P}_\lambda\left(\mathbf{0} \Longleftrightarrow u \text{ in } \xi^{\mathbf{0},u,x}, u \in \mathsf{Piv}(\mathbf{0}, x; \xi^{\mathbf{0},u,x})\right) \mathrm{d}u \\ &= \lambda \int \mathbb{E}_\lambda\Big[\mathbb{1}_{\{\mathbf{0}\Longleftrightarrow u \text{ in } \xi^{\mathbf{0},u}\}} \tau_\lambda^{\mathscr{C}_0}(u, x)\Big] \,\mathrm{d}u. \end{aligned} \tag{3.21}$$

To deal with $\tau_\lambda^{\mathscr{C}_0}(u,x)$ in (3.21), note that taking probabilities in (3.3) gives

$$\tau_\lambda^{A}(u,x) = \tau_\lambda(x-u) - \mathbb{P}_\lambda\big(u \xleftrightarrow{\ A\ } x \text{ in } \xi^{u,x}\big) \tag{3.22}$$

for a locally finite set A. We can substitute (3.22) into (3.21) with the fixed set $A = \mathscr{C}_0$. Inserting this back into (3.19) and using the independence of ξ_0 and ξ_1, we can express τ_λ as

$$\begin{aligned}
\tau_\lambda(x) &= \varphi(x) + \Pi_\lambda^{(0)}(x) + \lambda \int \mathbb{E}_\lambda\left[\mathbb{1}_{\{\mathbf{0}\Longleftrightarrow u \text{ in } \xi^{\mathbf{0},u}\}}\right]\tau_\lambda(x-u)\,\mathrm{d}u \\
&\quad - \lambda \int \mathbb{E}_\lambda\Big[\mathbb{1}_{\{\mathbf{0}\Longleftrightarrow u \text{ in } \xi_0^{\mathbf{0},u}\}}\mathbb{1}_{\{u \xleftrightarrow{\mathscr{C}_0} x \text{ in } \xi_1^{u,x}\}}\Big]\,\mathrm{d}u \qquad (3.23)\\
&= \varphi(x) + \Pi_\lambda^{(0)}(x) + \lambda\int \Big(\varphi(u) + \Pi_\lambda^{(0)}(u)\Big)\tau_\lambda(x-u)\,\mathrm{d}u + R_{\lambda,0}(x), \qquad (3.24)
\end{aligned}$$

using the definition of $R_{\lambda,0}$ in (3.14). This proves (3.18) for $n=0$. Note that all appearing integrals in (3.24) are finite (as the integrands are bounded by τ_λ), and so the rewriting via (3.22) is justified.

Second step, $n=1$. We consider the second indicator in (3.23) and its probability, regarding $\mathscr{C}_0$ as a fixed set. Thanks to Lemma 3.5, and recalling that $\mathscr{C}_1 = \mathscr{C}(u, \xi_1^u)$, we have, for any locally finite $B \subset \mathbb{R}^d$,

$$\begin{aligned}
\mathbb{P}_\lambda\Big(u \xleftrightarrow{\ B\ } x \text{ in } \xi_1^{u,x}\Big) &= \mathbb{P}_\lambda\big(E(u,x;B,\xi_1^{u,x})\big) + \mathbb{E}_\lambda\Big[\sum_{u_1\in\eta_1}\mathbb{1}_{E(u,u_1;B,\xi_1^{u,x})}\mathbb{1}_{\{u_1\in\mathsf{Piv}(u,x;\xi_1^{u,x})\}}\Big] \\
&= \mathbb{P}_\lambda\left(E(u,x;B,\xi_1^{u,x})\right) + \lambda\int \mathbb{E}_\lambda\big[\mathbb{1}_{E(u,u_1;B,\xi_1^{u,u_1,x})}\mathbb{1}_{\{u_1\in\mathsf{Piv}(u,x;\xi_1^{u,u_1,x})\}}\big]\,\mathrm{d}u_1 \\
&= \mathbb{P}_\lambda\left(E(u,x;B,\xi_1^{u,x})\right) + \lambda\int \mathbb{E}_\lambda\Big[\mathbb{1}_{E(u,u_1;B,\xi_1^{u,u_1})}\cdot\tau_\lambda^{\mathscr{C}_1}(u_1,x)\Big]\,\mathrm{d}u_1, \qquad (3.25)
\end{aligned}$$

where we have again employed Mecke's formula (2.7) and the Cutting-point Lemma 3.6. Again, we apply (3.22) with $A = \mathscr{C}_1$ to (3.25), which gives

$$\begin{aligned}
\mathbb{E}_\lambda\Big[\mathbb{1}_{E(u,u_1;B,\xi_1^{u,u_1})}\cdot\tau_\lambda^{\mathscr{C}_1}(u_1,x)\Big] &= \mathbb{P}_\lambda(E(u,u_1;B,\xi_1^{u,u_1}))\tau_\lambda(x-u_1) \\
&\quad - \mathbb{E}_\lambda\Big[\mathbb{1}_{E(u,u_1;B,\xi_1^{u,u_1})}\mathbb{1}_{\{u_1 \xleftrightarrow{\mathscr{C}_1} x \text{ in } \xi_2^{u_1,x}\}}\Big]. \qquad (3.26)
\end{aligned}$$

We now insert (3.25) with $u=u_0$ as well as the set $B = \mathscr{C}_0$ into the expansion identity (3.23). Recalling the definition of $\Pi_\lambda^{(n)}$ in (3.13), we can extract $\Pi_\lambda^{(1)}$ and apply (3.26) to perform the next step of the expansion, yielding

$$\begin{aligned}
\tau_\lambda(x) &= \varphi(x) + \Pi_\lambda^{(0)}(x) - \Pi_\lambda^{(1)}(x) + \lambda\int\Big(\varphi(u)+\Pi_\lambda^{(0)}(u)\Big)\tau_\lambda(x-u)\,\mathrm{d}u \\
&\quad - \lambda\int \tau_\lambda(x-u_1)\cdot\lambda\int\mathbb{E}_\lambda\Big[\mathbb{1}_{\{\mathbf{0}\Longleftrightarrow u_0 \text{ in } \xi_0^{\mathbf{0},u_0}\}}\mathbb{1}_{E(u_0,u_1;\mathscr{C}_0,\xi_1^{u_0,u_1})}\Big]\,\mathrm{d}u_0\,\mathrm{d}u_1 \\
&\quad + \lambda^2\int\mathbb{E}_\lambda\Big[\mathbb{1}_{\{\mathbf{0}\Longleftrightarrow u_0 \text{ in } \xi_0^{\mathbf{0},u_0}\}}\int\Big[\mathbb{1}_{E(u_0,u_1;\mathscr{C}_0,\xi_1^{u_0,u_1})}\mathbb{1}_{\{u_1\xleftrightarrow{\mathscr{C}_1} x \text{ in } \xi_2^{u_1,x}\}}\Big]\,\mathrm{d}u_1\Big]\,\mathrm{d}u_0 \\
&= \varphi(x) + \Pi_\lambda^{(0)}(x) - \Pi_\lambda^{(1)}(x) + \lambda\int\Big(\varphi(u)+\Pi_\lambda^{(0)}(u) - \Pi_\lambda^{(1)}(u)\Big)\tau_\lambda(x-u)\,\mathrm{d}u + R_{\lambda,1}(x).
\end{aligned}$$

This proves (3.18) for $n=1$. Again, we point out that the appearing integrals are finite since $\lambda<\lambda_c$.

Third step, general n. For general $n\geq 1$, we can repeat the arguments for $n=1$ and obtain

$$\begin{aligned}
\mathbb{P}_\lambda\big(u_n \xleftrightarrow{\mathscr{C}_n} x \text{ in } \xi_{n+1}^{u_n,x}\big) &= \mathbb{P}_\lambda\big(E(u_n,x;\mathscr{C}_n,\xi_{n+1}^{u_n,x})\big) \\
&\quad + \lambda\int\tau_\lambda(x-u_{n+1})\mathbb{P}_\lambda\big(E(u_n,u_{n+1};\mathscr{C}_n,\xi_{n+1}^{u_n,u_{n+1}})\big)\,\mathrm{d}u_{n+1} \\
&\quad - \lambda\int\mathbb{E}_\lambda\Big[\mathbb{1}_{E(u_n,u_{n+1};\mathscr{C}_n,\xi_{n+1}^{u_n,u_{n+1}})}\mathbb{1}_{\{u_{n+1}\xleftrightarrow{\mathscr{C}_{n+1}} x \text{ in } \xi_{n+2}^{u_{n+1},x}\}}\Big]\,\mathrm{d}u_{n+1}.
\end{aligned}$$

Plugging this into $R_{\lambda,n}(x)$, the first term yields $\Pi_\lambda^{(n+1)}(x)$, the second one yields $\lambda(\Pi_\lambda^{(n+1)}\star\tau_\lambda)(x)$, and the last one yields $R_{\lambda,n+1}(x)$. By induction, this proves the claim. □

4 Diagrammatic bounds

4.1 Warm-up: Motivation and bounds for $n = 0$

The aim of this section is to bound the lace-expansion coefficients $\Pi_\lambda^{(n)}$, which we have identified in the previous section. The bounds will be formulated in terms of somewhat simpler quantities, so-called diagrams. To this end, we first interpret the integrand in $\Pi_\lambda^{(n)}$ as the probability of an event contained in some connection event, which we can illustrate pictorially. In the next step, these connection events are decomposed by heavy use of the BK inequality into diagrams, which will turn out to be easier to analyze (this analysis is performed in Section 5). A diagram is an integral over a product of two-point and connection functions. Its diagrammatic representation is illustrated in Figure 1 and used heavily in the analysis in the later parts of this section.

To illustrate the idea of this lengthy procedure, we first illustrate it for $n = 0$. Since $\Pi_\lambda^{(0)}$ is fairly simple, this has the advantage of giving a rather compact overview of what we execute at length for general n afterwards.

The main results of this section are Propositions 4.14 and 4.19. The former gives bounds on $\widehat{\Pi}_\lambda^{(n)}(k)$, the latter gives related bounds on $\widehat{\Pi}_\lambda^{(n)}(\mathbf{0}) - \widehat{\Pi}_\lambda^{(n)}(k)$, which turn out to be important in Section 5. In preparation of the latter bounds, we state Lemma 4.1. Note that if $f(x) = f(-x)$, then

$$\widehat{f}(k) = \int f(x) \mathrm{e}^{\mathrm{i}k\cdot x} \,\mathrm{d}x = \int f(x) \cos(k \cdot x) \,\mathrm{d}x. \tag{4.1}$$

Consequently, $|\widehat{\Pi}_\lambda^{(n)}(\mathbf{0}) - \widehat{\Pi}_\lambda^{(n)}(k)| = \int [1 - \cos(k \cdot x)] \Pi_\lambda^{(n)}(x) \,\mathrm{d}x$. The following lemma is well known in the lace-expansion literature and allows to decompose factors of the form $[1 - \cos(k \cdot x)]$. It is thus titled the Cosine-split lemma:

Lemma 4.1 (Split of cosines, [10], Lemma 2.13)**.** *Let $t \in \mathbb{R}$ and $t_i \in \mathbb{R}$ for $i = 1, \dots, m$ such that $t = \sum_{i=1}^m t_i$. Then*

$$1 - \cos(t) \le m \sum_{i=1}^{m} [1 - \cos(t_i)].$$

The next definition and observation are to be seen as an intermezzo, as they are not necessary at this point. In fact, Definition 4.2 will not be of importance until Section 5. We state it here nonetheless to prove a basic relation to τ_λ, which illustrates some key ideas recurring in many of the proofs to follow:

Definition 4.2 (One-step connection probability)**.** For $x \in \mathbb{R}^d$, we define $\tilde{\tau}_\lambda(x) := \varphi(x) + \lambda(\varphi \star \tau_\lambda)(x)$.

Observation 4.3 (Relation between τ_λ and $\tilde{\tau}_\lambda$)**.** *Let $x \in \mathbb{R}^d$. Then $\tau_\lambda(x) \le \tilde{\tau}_\lambda(x)$.*

Proof. By combining Mecke's formula and the BK inequality, we obtain

$$\begin{aligned} \tau_\lambda(x) &\le \varphi(x) + \mathbb{E}_\lambda\Big[\sum_{y \in \eta} \mathbb{1}_{\{\mathbf{0} \sim y \text{ in } \xi^{\mathbf{0}}\} \cup \{y \longleftrightarrow x \text{ in } \xi^{x}\}}\Big] \\ &= \varphi(x) + \lambda \int \mathbb{P}_\lambda\big(\{\mathbf{0} \sim y \text{ in } \xi^{\mathbf{0},y}\} \cup \{y \longleftrightarrow x \text{ in } \xi^{x,y}\}\big) \,\mathrm{d}y \\ &\le \varphi(x) + \lambda \int \varphi(y) \tau_\lambda(x - y) \,\mathrm{d}y = \tilde{\tau}_\lambda(x). \end{aligned}$$

□

In the last inequality, we also used that the two intersected events are independent. This is due to the fact that $\mathbf{0} \notin \eta^{x,y}$ a.s. Whenever the first event is not a direct adjacency however (but instead also a connection event), we need to use the BK inequality instead.

We define two quantities that are of relevance for the following proposition, as well as later on in Section 4.4:

Definition 4.4 (Basic displacement functions)**.** The Fourier quantities φ_k and $\tau_{\lambda,k}$ are defined as $\varphi_k(x) = [1 - \cos(k \cdot x)]\varphi(x)$ and $\tau_{\lambda,k}(x) = [1 - \cos(k \cdot x)]\tau_\lambda(x)$.

The following proposition deals with $\Pi_\lambda^{(0)}$ and its Fourier transform:

Proposition 4.5 (Bounds for $n = 0$)**.** *Let $k \in \mathbb{R}^d$ and let $\lambda \in [0, \lambda_c)$. Then*

$$|\widehat{\Pi}_\lambda^{(0)}(k)| \le \lambda^2 (\varphi^{\star 2} \star \tau_\lambda^{\star 2})(\mathbf{0}),$$
$$|\widehat{\Pi}_\lambda^{(0)}(\mathbf{0}) - \widehat{\Pi}_\lambda^{(0)}(k)| \le \lambda^2 \Big((\varphi_k \star \varphi \star \tau_\lambda^{\star 2})(\mathbf{0}) + (\varphi^{\star 2} \star \tau_{\lambda,k} \star \tau_\lambda)(\mathbf{0}) \Big).$$

Proof. We note that for the event $\{\mathbf{0} \Longleftrightarrow x \text{ in } \xi^{\mathbf{0},x}\}$ to hold, either there is a direct edge between $\mathbf{0}$ and x, or there are vertices y, z in η that are direct neighbors of the origin and have respective disjoint paths to x that both do not contain the origin. Hence, by the multivariate Mecke equation (2.6),

$$\begin{aligned}\mathbb{P}_\lambda(\mathbf{0} \Longleftrightarrow x \text{ in } \xi^{\mathbf{0},x}) &\le \varphi(x) + \tfrac{1}{2}\mathbb{E}_\lambda\Big[\sum_{(y,z)\in\eta^{(2)}} \mathbb{1}_{(\{\mathbf{0}\sim y \text{ in } \xi^{\mathbf{0}}\}\cap\{y\longleftrightarrow x \text{ in } \xi^{x}\})\circ(\{\mathbf{0}\sim z \text{ in } \xi^{\mathbf{0}}\}\cap\{z\longleftrightarrow x \text{ in } \xi^{x}\})}\Big] \\ &= \varphi(x) + \tfrac{1}{2}\lambda^2 \iint \mathbb{P}_\lambda\big((\{\mathbf{0} \sim y \text{ in } \xi^{\mathbf{0},y}\} \cap \{y \longleftrightarrow x \text{ in } \xi^{x,y}\}) \\ &\qquad\qquad \circ (\{\mathbf{0} \sim z \text{ in } \xi^{\mathbf{0},z}\} \cap \{z \longleftrightarrow x \text{ in } \xi^{x,z}\})\big) \,\mathrm{d}y \,\mathrm{d}z.\end{aligned}$$

After applying the BK inequality to the above probability, the integral factors, and so

$$\begin{aligned}\mathbb{P}_\lambda(\mathbf{0} \Longleftrightarrow x \text{ in } \xi^{\mathbf{0},x}) &\le \varphi(x) + \tfrac{1}{2}\lambda^2 \Big(\int \mathbb{P}_\lambda(\{\mathbf{0} \sim y \text{ in } \xi^{\mathbf{0},y}\} \cap \{y \longleftrightarrow x \text{ in } \xi^{y,x}\}) \,\mathrm{d}y \Big)^2 \\ &= \varphi(x) + \tfrac{1}{2}\lambda^2 (\varphi \star \tau_\lambda)(x)^2. \end{aligned} \tag{4.2}$$

Thus, recalling that $\Pi_\lambda^{(0)}(x) = \mathbb{P}_\lambda(\mathbf{0} \Longleftrightarrow x \text{ in } \xi^{\mathbf{0},x}) - \varphi(x) \ge 0$ and dropping the factor $\frac{1}{2}$,

$$\begin{aligned}|\widehat{\Pi}_\lambda^{(0)}(k)| &= \Big| \int \cos(k \cdot x) \Pi_\lambda^{(0)}(x) \,\mathrm{d}x \Big| \le \int \Pi_\lambda^{(0)}(x) \,\mathrm{d}x \le \lambda^2 \int (\varphi \star \tau_\lambda)(x)^2 \,\mathrm{d}x \\ &= \lambda^2 \int (\varphi \star \tau_\lambda)(x)(\varphi \star \tau_\lambda)(-x) \,\mathrm{d}x = \lambda^2 (\varphi^{\star 2} \star \tau_\lambda^{\star 2})(\mathbf{0}), \end{aligned} \tag{4.3}$$

using symmetry of φ and τ_λ as well as commutativity of the convolution. For the second bound of Proposition 4.5, we apply (4.2) and obtain

$$\widehat{\Pi}_\lambda^{(0)}(\mathbf{0}) - \widehat{\Pi}_\lambda^{(0)}(k) = \int [1 - \cos(k \cdot x)] \Pi_\lambda^{(0)}(x) \,\mathrm{d}x \le \frac{\lambda^2}{2} \int (\varphi \star \tau_\lambda)(x) \int [1 - \cos(k \cdot x)] \varphi(y) \tau_\lambda(x - y) \,\mathrm{d}y \,\mathrm{d}x. \tag{4.4}$$

We call the factor $[1 - \cos(k \cdot x)]$ a displacement factor. Writing $x = y + (x - y)$, the Cosine-split Lemma 4.1 allows us to distribute it over the factors φ and τ_λ as

$$\int [1 - \cos(k \cdot x)] \varphi(y) \tau_\lambda(x - y) \,\mathrm{d}y \le 2 \Big[(\varphi_k \star \tau_\lambda)(x) + (\varphi \star \tau_{\lambda,k})(x) \Big].$$

Substituting this back into (4.4) gives the desired result. □

4.2 Bounding events for the lace-expansion coefficients

The aim of this section is to take the first step into finding simple bounds on the lace-expansion coefficients $\Pi_\lambda^{(n)}$. We start by stating the central result of this section, Proposition 4.7, while the remainder of the section is concerned with its proof.

For the proof, we first introduce the events in Definition 4.11 that allow for a simple pictorial representation and that bound the E events. As a second step, we bound these events by large products of two-point functions (through heavy use of the BK inequality), constituting the bound of Proposition 4.7. We continue to simplify this bound in Section 4.3.

Definition 4.6 introduces the quantities in terms of which the bound of Proposition 4.7 is formulated. It also introduces Dirac delta functions. We stress that in this paper, we use them primarily for convenient and more compact notation and to increase readability. In particular, they appear when applying the Mecke equation (2.7) to obtain

$$\mathbb{E}\big[\sum_{y \in \eta^u} f(y, \xi^u) \big] = \int (\lambda + \delta_{y,u}) \mathbb{E}_\lambda[f(y, \xi^y)] \,\mathrm{d}y,$$

and so the factor $\lambda + \delta_{y,u}$ encodes a case distinction of whether point y coincides with u or not.

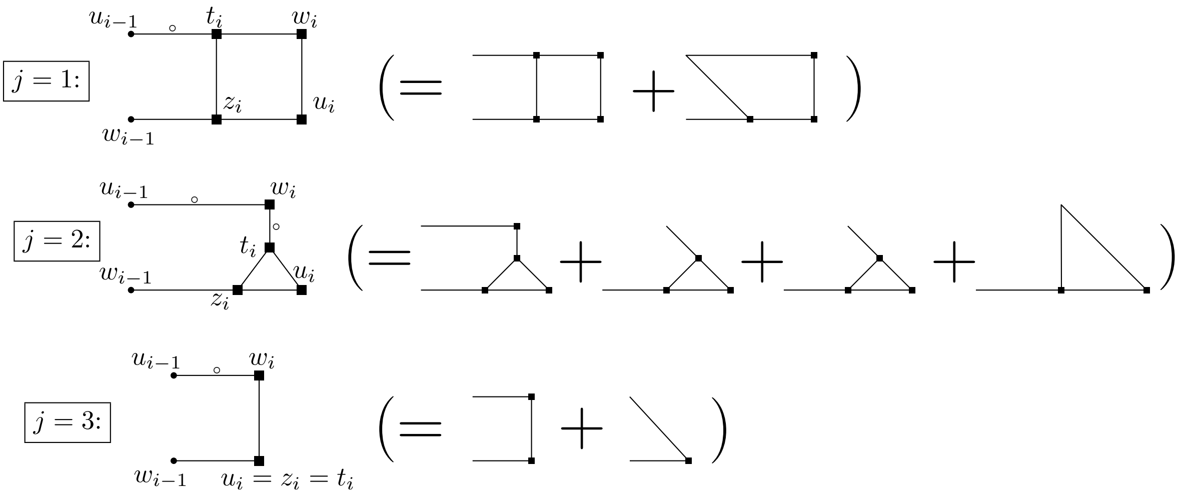


Figure 1: Diagrammatic representation of segment i, and hence of the functions $\psi^{(j)}$. Factors τ_λ are represented by lines, factors τ_λ° are represented by lines endowed with a '$\circ$'. The points t_i, w_i, z_i, u_i (the ones labeled by index i) are depicted as squares—in the later decomposition of the full diagram into segments, these are the ones integrated over when bounding segment i. The small diagrams in brackets indicate the form that the diagrams take when expanding the two terms constituting τ_λ° (i.e. when writing out all possible collapses).

Definition 4.6 (The ψ functions)**.** Let $w, x, y, z \in \mathbb{R}^d$. We set $\tau_\lambda^\circ(x) := \delta_{x,\mathbf{0}} + \lambda\tau_\lambda(x)$. Moreover, let

$$\underline{\circ}\,(x,y) = \tau_\lambda^\circ(x)\tau_\lambda(y), \quad \triangle(x,y,z) := \tau_\lambda(x-y)\tau_\lambda(y-z)\tau_\lambda(z-x),$$
$$\square(w,x,y,z) := \tau_\lambda(w-x)\tau_\lambda(x-y)\tau_\lambda(y-z)\tau_\lambda(z-w).$$

We define

$$\psi_0^{(1)}(w,u) := \lambda\triangle(\mathbf{0},w,u), \quad \psi_0^{(2)}(w,u) := \lambda\delta_{w,\mathbf{0}}\int \triangle(\mathbf{0},t,u)\,\mathrm{d}t, \quad \psi_0^{(3)}(w,u) := \varphi(u)\delta_{w,\mathbf{0}},$$
$$\psi_n^{(1)}(a,b,t,z,x) := \lambda\,\underline{\circ}\,(t-b,z-a)\triangle(t,z,x), \quad \psi_n^{(2)}(a,b,t,z,x) := \delta_{t,z}\delta_{z,x}\tau_\lambda(t-b)\tau_\lambda(z-a),$$
$$\psi^{(1)}(a,b,t,w,z,u) := \lambda^2\square(t,w,u,z)\,\underline{\circ}\,(t-b,z-a),$$
$$\psi^{(2)}(a,b,t,w,z,u) := \lambda\triangle(t,z,u)\tau_\lambda^\circ(t-w)\,\underline{\circ}\,(w-b,z-a),$$
$$\psi^{(3)}(a,b,t,w,z,u) := \delta_{z,u}\delta_{t,z}\tau_\lambda(t-w)\,\underline{\circ}\,(w-b,z-a),$$

and set $\psi_0 := \sum_{j=1}^3 \psi_0^{(j)}, \psi_n := \psi_n^{(1)} + \psi_n^{(2)}, \psi := \sum_{j=1}^3 \psi^{(j)}$.

See Figure 1 for a pictorial representation of the functions $\psi^{(j)}$.

Proposition 4.7 (Bound in terms of ψ functions)**.** *Let $n \geq 1, x \in \mathbb{R}^d$ and let $\lambda \in [0, \lambda_c)$. Then*

$$\Pi_\lambda^{(n)}(x) \leq \lambda^n \int \psi_0(w_0,u_0)\Big(\prod_{i=1}^{n-1}\psi(\vec{v}_i)\Big)\psi_n(w_{n-1},u_{n-1},t_n,z_n,x)\,\mathrm{d}\big((\vec{w},\vec{u})_{[0,n-1]},(\vec{t},\vec{z})_{[1,n]}\big),$$

where $\vec{v}_i = (w_{i-1}, u_{i-1}, t_i, w_i, z_i, u_i)$.

Throughout the paper, we use $\vec{v}_i$ as an abbreviation for various expressions.

Recall that the edge-markings in (3.13) are independent, a fact that is heavily used in the following. Unfortunately, the event taking place on graph i is not quite independent of the event taking place on graph $i-1$. However, a little restructuring together with appropriate bounding events enables us to guarantee such an independence. With the next steps, we achieve two things: On the one hand, we bound the E events by simpler ones (see Definition 4.11 and Lemma 4.12), and on the other, we exploit the independence structure.

We start by introducing a "thinning connection", defined for edge-markings of sets of points (which may not be PPPs).

Definition 4.8 (Thinning connection)**.** Let ξ_1, ξ_2 be two independent edge-markings of two locally finite sets A_1, A_2 (hence, $\xi_i = \xi_i(A_i)$). For $x, y \in \mathbb{R}^d$, define

$$\{x \leftrightsquigarrow y \text{ in } (\xi_1, \xi_2)\} := \{x \in A_1, y \in A_2\} \cap \{y \notin (A_2)_{\langle \mathscr{C}(x,\xi_1)\rangle}\}.$$

Given $\mathscr{C}(x, \xi_1)$, which is determined by ξ_1, $\{x \leftrightsquigarrow y \text{ in } (\xi_1, \xi_2)\}$ is just a thinning event in ξ_2. On the other hand, given the thinning marks of y, $\{x \leftrightsquigarrow y \text{ in } (\xi_1, \xi_2)\}$ is just a connection event in ξ_1, as x must be connected to some vertex z in ξ_1 that "thins out" y.

We will apply Definition 4.8 only for pairs (ξ_i, ξ_{i+1}) from the sequence $(\xi_i)_{i\in\mathbb{N}_0}$ used in the definition of the lace-expansion coefficients.

The next definition should be regarded as an extension of the disjoint occurrence event to multiple connection events that may overlap in their endpoints (similar to the application of the BK inequality in (2.19)), as well as to events involving '$\leftrightsquigarrow$' (living on two RCMs):

Definition 4.9 (Multiple disjoint connection events)**.** Let $m \in \mathbb{N}$ and $\vec{x}, \vec{y} \in (\mathbb{R}^d)^m$. We define $\bigcirc_m^{\leftrightarrow}((x_j, y_j)_{1\le j\le m}; \xi)$ as the event that $\{x_j \longleftrightarrow y_j \text{ in } \xi\}$ occurs for every $1 \le j \le m$ with the additional requirement that every point in η is the interior vertex of at most one of the m paths, and none of the m paths contains an interior vertex in the set $\{x_j : j \in [m]\} \cup \{y_j : j \in [m]\}$.

Moreover, let ξ_1, ξ_2 be two independent RCMs. Define $\bigcirc_m^{\leftrightsquigarrow}((x_j, y_j)_{1\le j\le m}; (\xi_1, \xi_2))$ as the event that, on the one hand, $\bigcirc_{m-1}^{\leftrightarrow}((x_j, y_j)_{1\le j\le m-1}; \xi_1)$ occurs and no path uses x_m or y_m as an interior vertex. On the other hand, $\{x_m \leftrightsquigarrow y_m \text{ in } (\xi_1[V(\xi_1) \setminus \{x_i, y_i\}_{1\le i<m}], \xi_2)\}$ occurs in such a way that at least one point z in ξ_1 that is responsible for thinning out y_m is connected to x_m by a path γ so that z as well as all interior vertices of γ are not contained in any path of the $\bigcirc_{m-1}^{\leftrightarrow}((x_j, y_j)_{1\le j\le m-1}; \xi_1)$ event.

We only require the events $\bigcirc_m^{\leftrightarrow}$ and $\bigcirc_m^{\leftrightsquigarrow}$ for distinct points $\vec{x}, \vec{y}$. We moreover remark that $\bigcirc_1^{\leftrightarrow}((x, y); \xi) = \{x \longleftrightarrow y \text{ in } \xi\}$ and $\bigcirc_1^{\leftrightsquigarrow}((x, y); (\xi_1, \xi_2)) = \{x \leftrightsquigarrow y \text{ in } (\xi_1, \xi_2)\}$. Furthermore, for distinct points u, v, x, y, almost surely,

$$\bigcirc_2^{\leftrightarrow}((u, v), (x, y); \xi^{u,v,x,y}) = \{u \longleftrightarrow v \text{ in } \xi^{u,v}\} \circ \{x \longleftrightarrow y \text{ in } \xi^{x,y}\},$$

and so on. Crucially, $\bigcirc_m^{\leftrightarrow}$ is still amenable to the use of the BK inequality (again, see the proof of (2.19)). In contrast, the thinning connection as defined in Definition 4.8 is not an increasing event. The reason for this is that adding points changes the ordering of points as given by (2.1) and thus the thinning variables. As we would like to use the BK inequality on $\bigcirc_m^{\leftrightsquigarrow}$ later on, the following observation gives an important identity for $\bigcirc_m^{\leftrightsquigarrow}$:

Lemma 4.10 (Relating $\bigcirc_m^{\leftrightsquigarrow}$ and $\bigcirc_m^{\leftrightarrow}$)**.** *Let $m \in \mathbb{N}$ and $\vec{x}, \vec{y} \in (\mathbb{R}^d)^m$. Let ξ_1, ξ_2 be two independent RCMs. Then*

$$\mathbb{P}_\lambda\big(\bigcirc_m^{\leftrightsquigarrow}\big((x_j, y_j)_{1\le j\le m}; (\xi_1^{\vec{x}_{[1,m]}, \vec{y}_{[1,m-1]}}, \xi_2^{y_m})\big)\big) = \mathbb{P}_\lambda\big(\bigcirc_m^{\leftrightarrow}\big((x_j, y_j)_{1\le j\le m}; \xi_1^{(\vec{x},\vec{y})_{[1,m]}}\big)\big).$$

Proof. Conditionally on $\xi_1^{\vec{x}_{[1,m]}, \vec{y}_{[1,m-1]}}$, the only randomness in the event $\bigcirc_m^{\leftrightsquigarrow}$ lies in $Y(y_m)$, the thinning marks of y_m in ξ_2.

Consider now $\xi_1^{(\vec{x},\vec{y})_{[1,m]}}$ and let $U(y_m)$ be the sequence of random variables determining the edges incident to y_m.

The claim follows from two facts: First, both $Y(y_m)$ and $U(y_m)$ are i.i.d. random variables distributed uniformly in $[0, 1]$, and both are independent of everything else. Secondly, given $\bar{\eta} := \eta \cup \vec{x}_{[1,m]} \cup \vec{y}_{[1,m-1]}$, both the probability of y_m having at least one neighbor in $\bar{\eta}$ as well as the probability of y_m not surviving an $\bar{\eta}$-thinning is $1 - \bar{\varphi}(\bar{\eta}, y_m)$. □

We next define the events that will be used to bound the E events:

Definition 4.11 (Bounding F events)**.** Let ξ_1, ξ_2 be two independent edge-markings, and let $n \ge 1$ and $a, b, t, w, z, u \in \mathbb{R}^d$. Define

$$F_0^{(1)}(a, w, u, b; (\xi_1, \xi_2)) := \{a \nsim u \text{ in } \xi_1\} \cap \bigcirc_4^{\leftrightsquigarrow}\big((a, u), (a, w), (u, w), (w, b); (\xi_1, \xi_2)\big),$$
$$F_0^{(2)}(a, w, u, b; (\xi_1, \xi_2)) := \{w = a\} \cap \{a \sim u \text{ in } \xi_1\} \cap \{w \leftrightsquigarrow b \text{ in } (\xi_1 \setminus \{u\}, \xi_2)\},$$
$$F_n(a, t, z, u; \xi) := \{|\{t, z, u\}| \ne 2\} \cap \bigcirc_4^{\leftrightarrow}\big((a, t), (t, z), (t, u), (z, u); \xi\big),$$

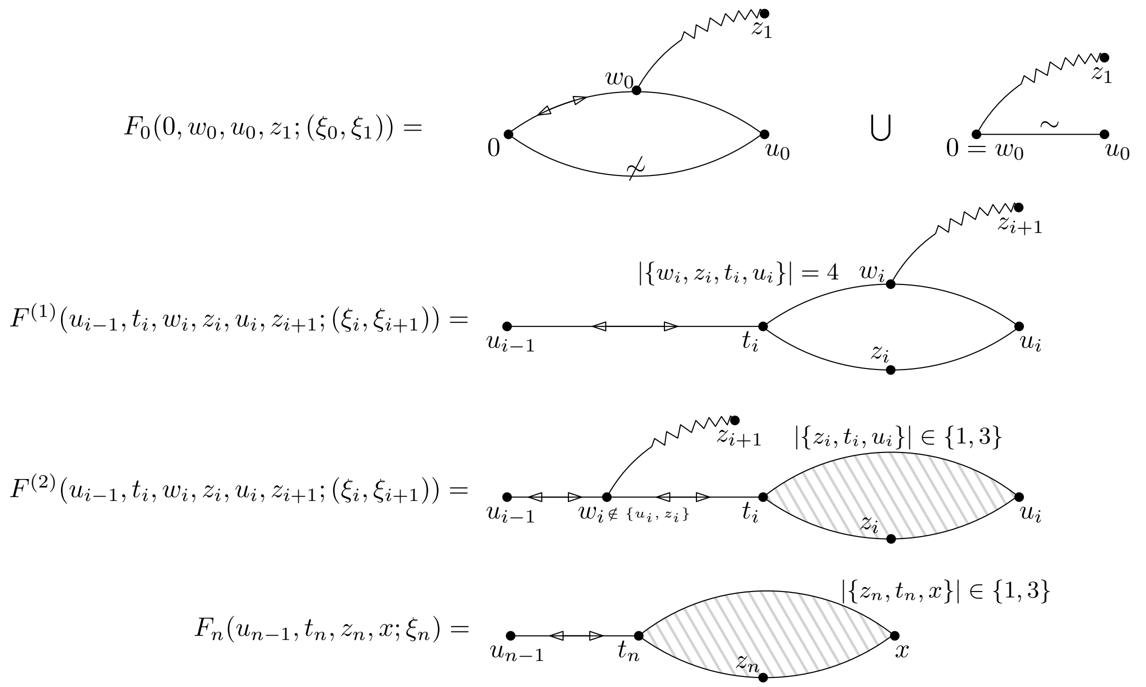


Figure 2: The full diagrammatic events. The line with a '$\sim$' symbol represents a direct edge. The line with a '$\nsim$' symbol indicates that this may *not* be a direct edge. The partially squiggly lines represent the event $\{w_i \leftrightsquigarrow z_{i+1}\}$, taking place on both ξ_i and ξ_{i+1}. Arrows on a line indicate that the two endpoints of that line may coincide. The hatched area may collapse into a single point altogether.

$$F^{(1)}(a,t,w,z,u,b;(\xi_1,\xi_2)) := \{|\{t,w,z,u\}| = 4\} \cap \bigcirc_6^{\leftrightsquigarrow}\big((a,t),(t,z),(z,u),(t,w),(w,u),(w,b);(\xi_1,\xi_2)\big),$$
$$F^{(2)}(a,t,w,z,u,b;(\xi_1,\xi_2)) := \{w \notin \{u,z\}, |\{t,z,u\}| \neq 2\} \cap \bigcirc_6^{\leftrightsquigarrow}\big((a,w),(w,t),(t,u),(t,z),(z,u),(w,b);(\xi_1,\xi_2)\big).$$

In addition, let $F_0 = F_0^{(1)} \cup F_0^{(2)}$.

Figure 2 illustrates the diagrammatic events $F_0, F^{(1)}, F^{(2)}$, and F_n. We say that a diagrammatic event *collapses* when a subset of the arguments coincides. These collapses of points turn out to be a recurring source of trouble in this section. An example of a collapse is $z = t = u$ in the event $F^{(2)}$.

The next lemma bounds the events E by the simpler F events. Recall the sequence $(\xi_i)_{i\in\mathbb{N}_0}$ from Section 3.3 and recall that it denotes a sequence of independent RCMs. We denote the respective underlying PPPs by η_i for $i \in \mathbb{N}_0$.

Lemma 4.12 (Bounds in terms of F events)**.** *Let $n \geq 1$ and let $u_0, \dots, u_n = x \in \mathbb{R}^d$ be distinct points. Write $\mathscr{C}_i = \mathscr{C}(u_{i-1}, \xi_i^{u_{i-1}}), \xi_i' = \xi_i^{u_{i-1},u_i}$, where $u_{-1} = \mathbf{0}$. Then*

$$\mathbb{1}_{\{\mathbf{0}\Longleftrightarrow u_0 \text{ in } \xi_0'\}} \prod_{i=1}^{n} \mathbb{1}_{E(u_{i-1},u_i;\mathscr{C}_{i-1},\xi_i')} \leq \sum_{\vec{z}_{[1,n]}: z_i \in \eta_i^{u_i}} \Big(\sum_{w_0 \in \eta_0^{\mathbf{0}}} \mathbb{1}_{F_0(\mathbf{0},w_0,u_0,z_1;(\xi_0',\xi_1'))} \Big) \Big(\sum_{t_n \in \eta_n^{u_{n-1},x}} \mathbb{1}_{F_n(u_{n-1},t_n,z_n,x;\xi_n')} \Big)$$
$$\times \prod_{i=1}^{n-1} \Big(\sum_{\substack{t_i \in \eta_i^{u_{i-1}}, \\ w_i \in \eta_i}} \mathbb{1}_{F^{(1)}(u_{i-1},t_i,w_i,z_i,u_i,z_{i+1};(\xi_i',\xi_{i+1}'))} + \sum_{\substack{w_i \in \eta_i^{u_{i-1}}, \\ t_i \in \eta_i^{w_i,u_i}}} \mathbb{1}_{F^{(2)}(u_{i-1},t_i,w_i,z_i,u_i,z_{i+1};(\xi_i',\xi_{i+1}'))} \Big).$$

Proof. We first prove the following assertion:

$$\mathbb{1}_{E(u_{n-1},x;\mathscr{C}_{n-1},\xi_n')} \leq \sum_{z_n \in \eta_n^x} \sum_{t_n \in \eta_n^{u_{n-1},x}} \mathbb{1}_{F_n(u_{n-1},t_n,z_n,x;\xi_n')} \mathbb{1}_{\{u_{n-2} \leftrightsquigarrow z_n \text{ in } (\xi_{n-1}^{u_{n-2}},\xi_n')\}}. \tag{4.5}$$

Recall the definition of $E(u_{n-1}, x; \mathscr{C}_{n-1}, \xi'_n)$ in (3.11) and, specifically, recall Definition 3.1(2). The event in the left-hand side of (4.5) is contained in the event that u_{n-1} is connected to x, but this connection breaks down after a $\mathscr{C}_{n-1}$-thinning of η_n^x. We distinguish two cases under which this can happen:

Case (a): In this case, x is thinned out and $E(u_{n-1}, x; \mathscr{C}_{n-1}, \xi'_n)$ is contained in

$$\{u_{n-1} \longleftrightarrow x \text{ in } \xi'_n\} \cap \{u_{n-2} \leftrightsquigarrow x \text{ in } (\xi_{n-1}^{u_{n-2}}, \xi'_n)\} = F_n(u_{n-1}, x, x, x; \xi'_n) \cap \{u_{n-2} \leftrightsquigarrow x \text{ in } (\xi_{n-1}^{u_{n-2}}, \xi'_n)\}.$$

Case (b): In this case, x is not thinned out. Now, the occurrence of E implies that there is at least one interior point on the path between u_{n-1} and x that is thinned out in a $\mathscr{C}_{n-1}$-thinning. We claim that we can pick one such point to be z_n and satisfy the bound in (4.5).

Let t_n be the last pivotal point in $\mathsf{Piv}(u_{n-1}, x; \xi'_n)$ (again, we use that $\mathsf{Piv}(u_{n-1}, x; \xi'_n)$ can be ordered in the direction from u_{n-1} to x) and set $t_n = u_{n-1}$ when $\mathsf{Piv}(u_{n-1}, x; \xi'_n) = \varnothing$. By definition of t_n as last pivotal point, we have $\{t_n \Longleftrightarrow x \text{ in } \xi_n^x\}$.

Moreover, the second part in the definition of the event $E(u_{n-1}, x; \mathscr{C}_{n-1}, \xi'_n)$ (recall (3.11)) forces all of these paths from t_n to x to break down after a $\mathscr{C}_{n-1}$-thinning, while t_n itself cannot be thinned out. Hence, there is a thinned-out point on each path between t_n and x. We can pick any of them to be the point z_n in the F_n event in (4.5). This proves (4.5).

Abbreviating $\vec{v}_i = (u_{i-1}, t_i, w_i, z_i, u_i, z_{i+1})$ for $i \in [n-1]$, we further assert that, for $z_{i+1} \in \eta_{i+1}^{u_{i+1}}$,

$$\begin{aligned}&\mathbb{1}_{E(u_{i-1}, u_i; \mathscr{C}_{i-1}, \xi'_i)} \mathbb{1}_{\{u_{i-1} \leftrightsquigarrow z_{i+1} \text{ in } (\xi_i^{u_{i-1}}, \xi'_{i+1})\}} \\ &\quad \le \sum_{z_i \in \eta_i^{u_i}} \mathbb{1}_{\{u_{i-2} \leftrightsquigarrow z_i \text{ in } (\xi_{i-1}^{u_{i-2}}, \xi'_i)\}} \Big(\sum_{\substack{t_i \in \eta_i^{u_{i-1}}, \\ w_i \in \eta_i}} \mathbb{1}_{F^{(1)}(\vec{v}_i; (\xi'_i, \xi'_{i+1}))} + \sum_{\substack{w_i \in \eta_i^{u_{i-1}}, \\ t_i \in \eta_i^{w_i, u_i}}} \mathbb{1}_{F^{(2)}(\vec{v}_i; (\xi'_i, \xi'_{i+1}))} \Big). \end{aligned} \tag{4.6}$$

Recall how the thinning events in Definition 3.2 were introduced via the mappings (π_j) from (2.1). If $z_{i+1} = \pi_j(\eta_{i+1}^{u_{i+1}})$ and the second event on the left-hand side of (2.1) occurs, then there must be a point $\pi_l(\eta_i^{u_{i-1}}) \in \mathscr{C}(u_{i-1}, \xi_i^{u_{i-1}})$ such that $Y_{j,l} \le \varphi(\pi_l(\eta_i^{u_{i-1}}) - \pi_j(\eta_{i+1}^{u_{i+1}}))$, where $(Y_{j,l})_{l \in \mathbb{N}}$ are the thinning variables associated to z_{i+1}. Informally speaking, $\pi_l(\eta_i^{u_{i-1}})$ is responsible for thinning out z_{i+1}. Let γ denote a path in $\xi_i^{u_{i-1}}$ from u_{i-1} to $\pi_l(\eta_i^{u_{i-1}})$.

Turning to the event E, again we start by considering the case $u_i \notin (\eta_i)_{\langle \mathscr{C}_{i-1} \rangle}$, i.e. the case where u_i is thinned out. In this case, the event E implies

$$\{u_{i-1} \longleftrightarrow u_i \text{ in } \xi'_i\} \cap \{u_{i-2} \leftrightsquigarrow u_i \text{ in } (\xi_{i-1}^{u_{i-2}}, \xi'_i)\}.$$

Letting w_i be the last point γ shares with the path from u_{i-1} to u_i (where $w_i = u_{i-1}$ is possible), we obtain $F^{(2)}(\vec{v}_i; (\xi'_i, \xi'_{i+1}))$ for $t_i = z_i = u_i$.

In the case where $u_i \in (\eta_i)_{\langle \mathscr{C}_{i-1} \rangle}$, we set t_i to be last pivotal point for the connection between u_{i-1} and u_i (if there is no pivotal point, set $t_i = u_{i-1}$). By definition of E, there is a path $\tilde{\gamma}$ between u_{i-1} and t_i (possibly of length 0) and there must be two disjoint t_i-u_i-paths in ξ^{u_i} (call them γ' and γ''), both of length at least two and both containing an interior point that is thinned out.

Let w_i be the last point γ shares with $\tilde{\gamma} \cup \gamma' \cup \gamma''$. If w_i lies on $\tilde{\gamma}$, we pick a thinned-out point on γ' and call it z_i to obtain the event $F^{(2)}$. If w_i lies on γ' or γ'', we pick a thinned-out point from the respective other path (γ'' or γ'), call it z_i, and obtain the event $F^{(1)}$. This proves (4.6).

We can now recursively bound the events in Lemma 4.12 (from n to 1), and, setting $\vec{v}_0 = (\mathbf{0}, w_0, u_0, z_1; (\xi'_0, \xi'_1))$, it remains to prove that

$$\mathbb{1}_{\{\mathbf{0} \Longleftrightarrow u_0 \text{ in } \xi'_0\}} \mathbb{1}_{\{\mathbf{0} \leftrightsquigarrow z_1 \text{ in } (\xi_0^{\mathbf{0}}, \xi'_1)\}} \le \sum_{w_0 \in \eta_0^{\mathbf{0}}} \mathbb{1}_{F_0(\vec{v}_0)}. \tag{4.7}$$

Again, there must be a point $\pi_l(\eta_0^{\mathbf{0}}) \in \mathscr{C}_0$ that is responsible for thinning z_1 out. Let γ be a path in $\xi_0^{\mathbf{0}}$ from $\mathbf{0}$ to $\pi_l(\eta_0^{\mathbf{0}})$.

Moreover, we can partition the event $\{\mathbf{0} \Longleftrightarrow u_0 \text{ in } \xi^{\mathbf{0}, u_0}\}_0$ as follows: When $\mathbf{0} \nsim u_0$, there are two disjoint paths (γ' and γ'' say) from $\mathbf{0}$ to u_0, both of length at least 2. On the other hand, when $\mathbf{0} \sim u_0$, we consider γ' and γ'' to be the degenerate paths containing only the origin $\mathbf{0}$.

Let w_0 be the last vertex γ shares with γ' or γ'' (thus, $w_0 = \mathbf{0}$ is possible). Requiring the three paths $\gamma', \gamma'', \gamma$ to be present, and respecting the two cases of the double connection between $\mathbf{0}$ and u_0, results precisely in $F_0(\vec{v}_0)$, proving (4.7) and therefore Lemma 4.12. □

Proof of Proposition 4.7. We use Lemma 4.12 to give a bound on $\Pi_\lambda^{(n)}(x)$. It involves sums over random points on each of the $n+1$ configurations $\xi_0, \dots, \xi_n$. In the following, we intend to apply the Mecke formula to deal with these sums. In particular, we use the Mecke formula (2.7) on ξ_0 and the multivariate Mecke formula (2.6) on ξ_i for $1 \le i \le n$. The F events in the indicators imply that some of the "extra point" coincidences vanish (for example, under $z_n = x$, application of the Mecke formula for the sum over t_n produces a term where $t_n \ne x = z_n$ a.s., but this term vanishes due to the restrictions in F_n). Taking this into consideration, recalling the definition of $\Pi_\lambda^{(n)}$ in (3.16), and abbreviating $\vec{v}_i = (u_{i-1}, t_i, w_i, z_i, u_i, z_{i+1})$, gives

$$\begin{aligned}\Pi_\lambda^{(n)}(x) \le \lambda^n \int \mathbb{E}_\lambda\Big[&\sum_{z_n \in \eta_n^x} \sum_{t_n \in \eta_n^{u_{n-1},x}} \mathbb{1}_{F_n(u_{n-1},t_n,z_n,x;\xi_n')} \\ &\times \prod_{i=1}^{n-1} \sum_{z_i \in \eta_i^{u_i}} \Big(\sum_{t_i \in \eta_i^{u_{i-1}}} \sum_{w_i \in \eta_i} \mathbb{1}_{F^{(1)}(\vec{v}_i;(\xi_i',\xi_{i+1}'))} + \sum_{w_i \in \eta_i^{u_{i-1}}} \sum_{t_i \in \eta_i^{w_i,u_i}} \mathbb{1}_{F^{(2)}(\vec{v}_i;(\xi_i',\xi_{i+1}'))}\Big) \\ &\times \sum_{w_0 \in \eta_0^{\mathbf{0}}} \mathbb{1}_{F_0(\mathbf{0},w_0,u_0,z_1;(\xi_0';\xi_1'))}\Big] \,\mathrm{d}\vec{u}_{[0,n-1]} \\ = \lambda^n \int \mathbb{E}_\lambda\Big[&(\lambda + \delta_{w_0,\mathbf{0}}) \mathbb{1}_{F_0(\mathbf{0},w_0,u_0,z_1;(\xi_0'';\xi_1''))} \prod_{i=1}^{n-1} \Big(\lambda^2(\lambda+\delta_{t_i,u_{i-1}}) \mathbb{1}_{F^{(1)}(\vec{v}_i;(\xi_i'',\xi_{i+1}''))} \\ &+ (\lambda(\lambda+\delta_{t_i,w_i}) + \delta_{z_i,u_i}\delta_{t_i,u_i})(\lambda+\delta_{w_i,u_{i-1}}) \mathbb{1}_{F^{(2)}(\vec{v}_i;(\xi_i'',\xi_{i+1}''))}\Big) \\ &\times (\lambda(\lambda+\delta_{t_n,u_{n-1}}) + \delta_{z_n,x}\delta_{t_n,x}) \mathbb{1}_{F_n(u_{n-1},t_n,z_n,x;\xi_n'')}\Big] \,\mathrm{d}\big((\vec{u},\vec{w})_{[0,n-1]}, (\vec{z},\vec{t})_{[1,n]}\big),\end{aligned} \tag{4.8}$$

where we set $\xi_0'' := \xi_0^{\mathbf{0},w_0,u_0}$, $\xi_i'' := \xi_i^{u_{i-1},t_i,w_i,z_i,u_i}$, and $\xi_n'' := \xi_n^{u_{n-1},t_n,z_n,x}$.

To simplify (4.8), we exploit the independence of the ξ_i. Note that for every i, there are four events that depend on ξ_i'', namely $F^{(j)}(\vec{v}_i; (\xi_i'', \xi_{i+1}''))$ and $F^{(j)}(\vec{v}_{i-1}; (\xi_{i-1}'', \xi_i''))$ (for $j = 1,2$, respectively). The latter two are thinning events that depend on ξ_i'' only through $Y(z_i)$, the thinning mark associated to the deterministic point z_i (see the remark after the definition of the thinning connection in Definition 4.8). The former two are connection events in ξ_i'', and they are independent of $Y(z_i)$. As a consequence, the expectation on the right-hand side of (4.8) factorizes, and so

$$\begin{aligned}\Pi_\lambda^{(n)}(x) \le \lambda^n \int &\mathbb{E}_\lambda\Big[(\lambda+\delta_{w_0,\mathbf{0}}) \mathbb{1}_{F_0(\mathbf{0},w_0,u_0,z_1;(\xi_0'';\xi_1''))}\Big] \prod_{i=1}^{n-1} \mathbb{E}_\lambda\Big[\lambda^2(\lambda+\delta_{t_i,u_{i-1}}) \mathbb{1}_{F^{(1)}(\vec{v}_i;(\xi_i'',\xi_{i+1}''))} \\ &+ \big(\lambda(\lambda+\delta_{w_i,t_i}) + \delta_{z_i,u_i}\delta_{t_i,u_i}\big)(\lambda+\delta_{w_i,u_{i-1}}) \mathbb{1}_{F^{(2)}(\vec{v}_i;(\xi_i'',\xi_{i+1}''))}\Big] \\ &\times \mathbb{E}_\lambda\Big[\big(\lambda(\lambda+\delta_{t_n,u_{n-1}}) + \delta_{z_n,x}\delta_{t_n,x}\big) \mathbb{1}_{F_n(u_{n-1},t_n,z_n,x;\xi_n'')}\Big] \,\mathrm{d}\big((\vec{u},\vec{w})_{[0,n-1]}, (\vec{z},\vec{t})_{[1,n]}\big) \\ = \lambda^n \int &(\lambda+\delta_{w_0,\mathbf{0}}) \mathbb{P}_\lambda\big(F_0(\mathbf{0},w_0,u_0,z_1;(\xi_0'';\xi_1''))\big) \prod_{i=1}^{n-1} \Big[\lambda^2(\lambda+\delta_{t_i,u_{i-1}}) \mathbb{P}_\lambda\big(F^{(1)}(\vec{v}_i;(\xi_i'',\xi_{i+1}''))\big) \\ &+ \big(\lambda(\lambda+\delta_{w_i,t_i}) + \delta_{z_i,u_i}\delta_{t_i,u_i}\big)(\lambda+\delta_{w_i,u_{i-1}}) \mathbb{P}_\lambda\big(F^{(2)}(\vec{v}_i;(\xi_i'',\xi_{i+1}''))\big)\Big] \\ &\times \big(\lambda(\lambda+\delta_{t_n,u_{n-1}}) + \delta_{z_n,x}\delta_{t_n,x}\big) \mathbb{P}_\lambda\big(F_n(u_{n-1},t_n,z_n,x;\xi_n'')\big) \,\mathrm{d}\big((\vec{u},\vec{w})_{[0,n-1]}, (\vec{z},\vec{t})_{[1,n]}\big).\end{aligned} \tag{4.9}$$

The goal is to bound the appearing events using the BK inequality, and thus bound $\Pi_\lambda^{(n)}(x)$ in terms of so-called diagrams (integrals of large products of two-point functions that are conveniently organized). Figure 2 already suggests how to decompose the probability of the respective events via the BK inequality. Note that, abbreviating $\vec{v}_n = (u_{n-1}, t_n, z_n, x)$, we can directly use the BK inequality to obtain

$$\begin{aligned}&\int (\lambda(\lambda+\delta_{t_n,u_{n-1}}) + \delta_{z_n,x}\delta_{t_n,x}) \mathbb{P}_\lambda(F_n(\vec{v}_n;\xi_n'')) \,\mathrm{d}t_n \\ &\le \int \underbrace{\lambda\tau_\lambda^\circ(t_n-u_{n-1})\triangle(t_n,z_n,x)}_{=:\phi_n^{(1)}(\vec{v}_n)} + \underbrace{\delta_{t_n,z_n}\delta_{z_n,x}\tau_\lambda(x-u_{n-1})}_{=:\phi_n^{(2)}(\vec{v}_n)} \,\mathrm{d}t_n.\end{aligned} \tag{4.10}$$

To decompose the other factors, we first make use of Lemma 4.10 to deal with the $\bigcirc_m^{\leftrightsquigarrow}$ event, and then proceed as in (4.10) by applying the BK inequality. For $1 \leq i < n$, we recall that $\vec{v}_i = (u_{i-1}, t_i, w_i, z_i, u_i, z_{i+1})$ and bound, using Lemma 4.10

$$\begin{aligned}
&\int \lambda^2(\lambda + \delta_{t_i,u_{i-1}})\mathbb{P}_\lambda\big(F^{(1)}(\vec{v}_i; (\xi_i'', \xi_{i+1}''))\big)\,\mathrm{d}(t_i, w_i) \\
&= \int \lambda^2(\lambda + \delta_{t_i,u_{i-1}})\mathbb{P}_\lambda\big(\bigcirc_6^{\leftrightarrow}\big((u_{i-1}, t_i), (t_i, z_i), (z_i, u_i), (t_i, w_i), (w_i, u_i), (w_i, z_{i+1}); \xi_i''\big)\big)\,\mathrm{d}(t_i, w_i) \\
&\leq \int \underbrace{\lambda^2 \,\underline{\circ}\,(t_i - u_{i-1}, z_{i+1} - w_i)\Box(t_i, w_i, u_i, z_i)}_{=:\phi^{(1)}(\vec{v}_i)}\,\mathrm{d}(t_i, w_i),
\end{aligned} \tag{4.11}$$

as well as

$$\begin{aligned}
&\int (\lambda(\lambda + \delta_{w_i,t_i}) + \delta_{z_i,u_i}\delta_{t_i,u_i})(\lambda + \delta_{w_i,u_{i-1}})\mathbb{P}_\lambda\big(F^{(2)}(\vec{v}_i; (\xi_i'', \xi_{i+1}''))\big)\,\mathrm{d}(t_i, w_i) \\
&= \int \Big[\lambda(\lambda + \delta_{w_i,t_i})(\lambda + \delta_{w_i,u_{i-1}})\mathbb{P}_\lambda\big(\bigcirc_6^{\leftrightarrow}\big((u_{i-1}, w_i), (w_i, t_i), (t_i, u_i), (t_i, z_i), (z_i, u_i), (w_i, z_{i+1}); \xi_i''\big) \\
&\qquad + \delta_{z_i,u_i}\delta_{t_i,u_i}(\lambda + \delta_{w_i,u_{i-1}})\mathbb{P}_\lambda\big(\bigcirc_3^{\leftrightarrow}\big((u_{i-1}, w_i), (w_i, u_i), (w_i, z_{i+1}); \xi_i''\big)\big)\Big]\,\mathrm{d}(t_i, w_i) \\
&\leq \int \Big[\underbrace{\lambda\triangle(t_i, z_i, u_i)\tau_\lambda^\circ(t_i - w_i)\,\underline{\circ}\,(w_i - u_{i-1}, z_{i+1} - w_i)}_{=:\phi^{(2)}(\vec{v}_i)} \\
&\qquad + \underbrace{\delta_{z_i,u_i}\delta_{t_i,u_i}\tau_\lambda(t_i - w_i)\,\underline{\circ}\,(w_i - u_{i-1}, z_{i+1} - w_i)}_{=:\phi^{(3)}(\vec{v}_i)}\Big]\,\mathrm{d}(t_i, w_i).
\end{aligned} \tag{4.12}$$

Analogously, with $\vec{v}_0 = (\mathbf{0}, w_0, u_0, z_1)$,

$$\begin{aligned}
&\int (\lambda + \delta_{w_0,\mathbf{0}})\mathbb{P}_\lambda\big(F_0(\mathbf{0}, w_0, u_0, z_1; (\xi_0'', \xi_1''))\big)\,\mathrm{d}w_0 \\
&= \int (\lambda + \delta_{w_0,\mathbf{0}})\mathbb{P}_\lambda\big(\{\mathbf{0} \nsim u_0 \text{ in } \xi_0''\} \cap \bigcirc_4^{\leftrightarrow}\big((\mathbf{0}, u_0), (\mathbf{0}, w_0), (u_0, w_0), (w_0, z_1); \xi_0''\big)\big) + \delta_{w_0,\mathbf{0}}\varphi(u_0)\tau_\lambda(z_1)\,\mathrm{d}w_0 \\
&\leq \int \underbrace{\lambda\triangle(\mathbf{0}, w_0, u_0)\tau_\lambda(z_1 - w_0)}_{=:\phi_0^{(1)}(\vec{v}_0)} + \delta_{w_0,\mathbf{0}}(\tau_\lambda(u_0) - \varphi(u_0))\tau_\lambda(u_0)\tau_\lambda(z_1) + \underbrace{\delta_{w_0,\mathbf{0}}\varphi(u_0)\tau_\lambda(z_1)}_{=:\phi_0^{(3)}(\vec{v}_0)}\,\mathrm{d}w_0 \\
&\leq \int \Big[\phi_0^{(1)}(\vec{v}_0) + \underbrace{\delta_{w_0,\mathbf{0}}\int \triangle(\mathbf{0}, t_0, u_0)\,\mathrm{d}t_0\tau_\lambda(z_1)}_{=:\phi_0^{(2)}(\vec{v}_0)} + \phi_0^{(3)}(\vec{v}_0)\Big]\,\mathrm{d}w_0.
\end{aligned} \tag{4.13}$$

Writing $\phi_0 := \sum_{j=1}^3 \phi_0^{(j)}, \phi := \sum_{j=1}^3 \phi^{(j)}$, and $\phi_n := \sum_{j=1}^2 \phi_n^{(j)}$, we can substitute these new bounds into (4.9) and obtain

$$\Pi_\lambda^{(n)}(x) \leq \lambda^n \int \phi_0(\vec{v}_0)\Big(\prod_{i=1}^{n-1}\phi(\vec{v}_i)\Big)\phi_n(\vec{v}_n)\,\mathrm{d}\big((\vec{u}, \vec{w})_{[0,n-1]}, (\vec{z}, \vec{t})_{[1,n]}\big). \tag{4.14}$$

The proof is completed with the observation that

$$\tau_\lambda(z_i - w_{i-1})\phi(u_{i-1}, t_i, w_i, z_i, u_i, z_{i+1}) = \tau_\lambda(z_{i+1} - w_i)\psi(w_{i-1}, u_{i-1}, t_i, w_i, z_i, u_i),$$

as well as $\phi_0(\mathbf{0}, w_0, u_0, z_1) = \tau_\lambda(z_1 - w_0)\psi_0(w_0, u_0)$ and $\tau_\lambda(z_n - w_{n-1})\phi_n(u_{n-1}, t_n, z_n, x) = \psi_n(w_{n-1}, u_{n-1}, t_n, z_n, x)$ and a telescoping product identity. □

Proposition 4.7 gives a bound on $\Pi_\lambda^{(n)}$ in terms of a diagram, which is, to be more accurate, itself a sum of $2 \cdot 3^n$ diagrams, i.e.,

$$\Pi_\lambda^{(n)}(x) \leq \lambda^n \sum_{\vec{j}_{[0,n]}} \int \psi_0^{(j_0)}\left(\prod_{i=1}^{n-1}\psi^{(j_i)}\right)\psi_n^{(j_n)}\,\mathrm{d}\big((\vec{w}, \vec{u})_{[0,n-1]}, (\vec{t}, \vec{z})_{[1,n]}\big), \tag{4.15}$$

where the sum is over all vectors $\vec{j}$ with $1 \leq j_i \leq 3$ for $0 \leq i < n$ and $j_n \in \{1,2\}$, and where the arguments of the ψ functions were omitted. If we were to expand every factor of τ_λ° (regarding it as a sum of two terms), then $\psi^{(1)}$ and $\psi^{(3)}$ turn into a sum of two terms each, whereas $\psi^{(2)}$ turns into a sum of four terms (similarly, $\psi_n^{(1)}$ turns into a sum of two terms). In that sense, there are eight types of interior segments. We point to Figure 1 for an illustration of the ψ functions and these eight types.

In Section 4.3, we want to give an inductive bound on $\int \Pi_\lambda^{(n)}(x)\,\mathrm{d}x$. To this end, define the function $\Psi^{(n)}$, which is almost identical to the bound obtained by Proposition 4.7, but better suited for the induction performed in Section 4.3. Define

$$\Psi^{(n)}(w_n,u_n) := \int \psi_0(w_0,u_0)\prod_{i=1}^{n}\psi(w_{i-1},u_{i-1},t_i,w_i,z_i,u_i)\,\mathrm{d}\big((\vec{w},\vec{u})_{[0,n-1]},(\vec{t},\vec{z})_{[1,n]}\big). \tag{4.16}$$

Note that $\Psi^{(n)}$ is similar to the bound in Proposition 4.7, but with ψ_n replaced by ψ. Since

$$\psi_n(w_{n-1},u_{n-1},t_n,z_n,x) \leq \sum_{j\in\{2,3\}}\int \psi^{(j)}(w_{n-1},u_{n-1},t_n,w_n,z_n,x)\,\mathrm{d}w_n,$$

we arrive at the new bound

$$\int \Pi_\lambda^{(n)}(x)\,\mathrm{d}x \leq \lambda^n \int \Psi^{(n-1)}(w,u)\psi_n(w,u,t,z,x)\,\mathrm{d}(w,u,t,z,x) \leq \lambda^n \iint \Psi^{(n)}(w,u)\,\mathrm{d}w\,\mathrm{d}u. \tag{4.17}$$

In the following two sections, we heavily rely on the bound obtained in (4.17).

4.3 Diagrammatic bounds on the lace-expansion coefficients

Having obtained the bound (4.17) is a good start, but this bound is still a highly involved integral. The aim of this section is to decompose $\Psi^{(n)}$ into much simpler objects, namely triangles $\triangle_\lambda$ and similar quantities (recall that $\triangle_\lambda(x) = \lambda^2\tau_\lambda^{\star 3}(x)$ and $\triangle_\lambda = \sup_x \triangle_\lambda(x)$). The latter are introduced in Definition 4.13; the central result of this section is Proposition 4.14.

Definition 4.13 (Modified triangles)**.** Define $\triangle_\lambda^\circ(x) = \lambda(\tau_\lambda^\circ \star \tau_\lambda \star \tau_\lambda)(x)$ and $\triangle_\lambda^{\circ\circ}(x) = (\tau_\lambda^\circ \star \tau_\lambda^\circ \star \tau_\lambda)(x)$. Define

$$\triangle_\lambda^\circ = \sup_{x\in\mathbb{R}^d}\triangle_\lambda^\circ(x), \qquad \triangle_\lambda^{\circ\circ} = \sup_{x\in\mathbb{R}^d}\triangle_\lambda^{\circ\circ}(x), \qquad \triangle_\lambda^{(\varepsilon)} = \sup_{x\in\mathbb{R}^d:|x|\geq\varepsilon}\triangle_\lambda^\circ(x).$$

Furthermore, set

$$\mathsf{B}^{(\varepsilon)} = \Big(\lambda\int \mathbb{1}_{\{|z|<\varepsilon\}}\,\mathrm{d}z\Big)^{1/2},$$

and

$$U_\lambda = 3\triangle_\lambda^{\circ\circ}\triangle_\lambda^\circ, \qquad U_\lambda^{(\varepsilon)} = 5\triangle_\lambda^{\circ\circ}\big(\triangle_\lambda + \triangle_\lambda^{(\varepsilon)} + \mathsf{B}^{(\varepsilon)} + (\mathsf{B}^{(\varepsilon)})^2\big), \qquad \bar{U}_\lambda = 2(1+U_\lambda)U_\lambda^{(\varepsilon)}.$$

We remark that $\triangle_\lambda \leq \triangle_\lambda^\circ \leq \triangle_\lambda^{\circ\circ}$. Moreover, as $\tau_\lambda(\mathbf{0}) = 1$, we have $\triangle_\lambda^{\circ\circ} \geq 1$.

Proposition 4.14 (Bounds for general n)**.** *Let $n \geq 0$. Then*

$$\lambda^{n+1}\iint \Psi^{(n)}(w,u)\,\mathrm{d}w\,\mathrm{d}u \leq 2(2\triangle_\lambda^\circ + \lambda + 1)\big(U_\lambda \wedge \bar{U}_\lambda\big)^n.$$

Before working towards the proof of Proposition 4.14, let us motivate it. Similarly to discrete percolation, we want to bound $\int \Pi_\lambda^{(n)}(x)\,\mathrm{d}x$ in terms of $\triangle_\lambda$ and $\triangle_\lambda^\circ$. In turn, we hope to prove that the latter two quantities become small as β becomes small. To see our motivation for introducing the ε-triangle, consider for a moment the Poisson blob model, $\varphi = \mathbb{1}_{\mathbb{B}^d}$, as a representative of the finite-variance model (H1). We have no hope here of $\triangle_\lambda^\circ$ becoming small, as

$$\lambda^{-1}\triangle_\lambda^\circ(\mathbf{0}) \geq (\tau_\lambda \star \tau_\lambda)(\mathbf{0}) \geq (\varphi\star\varphi)(\mathbf{0}) = 1.$$

This issue arises *only* for the finite-variance model, and most prominently for the Poisson blob model—under (H2) and (H3), we later prove that $\triangle_\lambda^\circ$ is small whenever β is small. However, as it turns out, we are able to prove that $\triangle_\lambda^{(\varepsilon)}$ becomes small (as β becomes small) for some ε (namely, the one assumed to exist under assumption (H1.2)). On the other hand, it is clear that for any ε smaller than the radius of the unit volume ball (i.e., $\varepsilon < r_d = \pi^{-1/2}\Gamma(\frac{d}{2}+1)^{1/d}$), we have $\mathsf{B}^{(\varepsilon)} \xrightarrow{d\to\infty} 0$.

Proposition 4.14 implies a bound which avoids ε completely, and which is substantially easier to prove. This bound suffices for the connection functions of (H2) and (H3). Additionally, we have a bound containing $\triangle_\lambda^{(\varepsilon)}$ and $\mathsf{B}^{(\varepsilon)}$, which is necessary for (H1). We prove Proposition 4.14 without specifying ε (we do this later). However, ε should be thought of as an arbitrary, but small enough, value (smaller than r_d suffices).

As a first step, we introduce some related quantities, which will be of help not only in the proof of Proposition 4.14, but also in Section 4.4 below. We define

$$\breve\Psi^{(n)}(w_0,u_0,w_n,u_n) = \sum_{\vec{j}_{[1,n]}\in[3]^n} \int \prod_{i=1}^n \psi^{(j_i)}(w_{i-1},u_{i-1},t_i,w_i,z_i,u_i)\,\mathrm{d}\big((\vec w,\vec u)_{[1,n-1]},(\vec t,\vec z)_{[1,n]}\big),$$

$$\breve\Psi^{(n,\geq\varepsilon)}(w_0,u_0,w_n,u_n) = \mathbb{1}_{\{|w_0-u_0|\geq\varepsilon\}}\breve\Psi^{(n)}(w_0,u_0,w_n,u_n),$$

$$\breve\Psi^{(n,<\varepsilon)}(w_0,u_0,w_n,u_n) = \mathbb{1}_{\{|w_n-u_n|<\varepsilon\}}\breve\Psi^{(n)}(w_0,u_0,w_n,u_n).$$

The following lemma, providing some bounds on the quantities just introduced, will be at the heart of the proof of Proposition 4.14:

Lemma 4.15 (Bound on $\breve\Psi^{(n)}$ diagrams)**.** *Let $n \geq 1$ and $\varepsilon > 0$. Then*

$$\sup_{a,b\in\mathbb{R}^d} \lambda^n \iint \breve\Psi^{(n)}(a,b,x,y)\,\mathrm{d}x\,\mathrm{d}y \leq \min\Big\{\big(U_\lambda\big)^n, U_\lambda\big(\bar U_\lambda\big)^{n-1}\Big\},$$

$$\max_{\bullet\in\{<\varepsilon,\geq\varepsilon\}} \sup_{a,b\in\mathbb{R}^d} \lambda^n \iint \breve\Psi^{(n,\bullet)}(a,b,x,y)\,\mathrm{d}x\,\mathrm{d}y \leq \big(\bar U_\lambda\big)^{n-1} U_\lambda^{(\varepsilon)},$$

$$\sup_{a,b\in\mathbb{R}^d} \lambda^n \iint \mathbb{1}_{\{|a-b|\geq\varepsilon\}}\breve\Psi^{(n,<\varepsilon)}(a,b,x,y)\,\mathrm{d}x\,\mathrm{d}y \leq \big(\bar U_\lambda\big)^{n-1}\big(U_\lambda^{(\varepsilon)}\big)^2.$$

Proof. The proof is by induction on n. The induction hypothesis is that the three inequalities in Lemma 4.15 hold for $n-1$.

Base case, bound on $\breve\Psi^{(1)}$. Let $n=1$. By translation invariance,

$$\sup_{a,b}\lambda\int\psi^{(j)}(a,b,t,w,z,u)\,\mathrm{d}(t,w,z,u) = \sup_a \lambda\int\psi^{(j)}(\mathbf{0},a,t,w,z,u)\,\mathrm{d}(t,w,z,u) \tag{4.18}$$

for $j=1,2,3$. Starting with $j=1$, the integral on the right-hand side of (4.18) is equal to

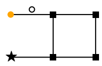
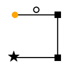
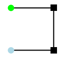
$$\begin{aligned}\lambda^3 \iint \tau_\lambda(z)\tau_\lambda(t-z)\tau_\lambda^\circ(a-t)\Big(\iint \tau_\lambda(u-z)\tau_\lambda(w-u)\tau_\lambda(t-w)\,\mathrm{d}u\,\mathrm{d}w\Big)\,\mathrm{d}z\,\mathrm{d}t&\\ = \lambda\iint \tau_\lambda(z)\tau_\lambda(t-z)\tau_\lambda^\circ(a-t)\triangle_\lambda(t-z)\,\mathrm{d}z\,\mathrm{d}t&\\ \leq \triangle_\lambda\triangle_\lambda^\circ(a) \leq \triangle_\lambda^{\circ\circ}\triangle_\lambda^\circ,&\end{aligned}$$

as $\triangle_\lambda \leq \triangle_\lambda^{\circ\circ}$. For $j=2$, we substitute $y'=y-u$ for $y\in\{t,w,z\}$, and we can bound (4.18) by

$$\begin{aligned}&\lambda^2\int \tau_\lambda(z)\tau_\lambda(u-z)\tau_\lambda(t-u)\tau_\lambda(t-z)\tau_\lambda^\circ(w-t)\tau_\lambda^\circ(a-w)\,\mathrm{d}(t,w,z,u)\\ &= \lambda^2\iint \tau_\lambda(z')\tau_\lambda(t')\tau_\lambda(t'-z')\Big(\iint \tau_\lambda(z'+u)\tau_\lambda^\circ(a-w'-u)\tau_\lambda^\circ(w'-t')\,\mathrm{d}w'\,\mathrm{d}u\Big)\,\mathrm{d}z'\,\mathrm{d}t'\\ &= \lambda^2\iint \tau_\lambda(z')\tau_\lambda(t')\tau_\lambda(t'-z')\triangle_\lambda^{\circ\circ}(a+z'-t')\,\mathrm{d}z'\,\mathrm{d}t'\\ &\leq \triangle_\lambda^{\circ\circ}\triangle_\lambda(\mathbf{0}) \leq \triangle_\lambda^{\circ\circ}\triangle_\lambda^\circ.\end{aligned}$$

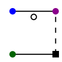
For $j=3$, the integral in (4.18) is $\triangle_\lambda^\circ(a) \leq \triangle_\lambda^\circ \leq \triangle_\lambda^{\circ\circ}\triangle_\lambda^\circ$. In total, this gives the claimed bound for $n=1$.

We show how we represent bounds of the above form in pictorial format by repeating the above bounds for $j = 1, 2$. This is redundant at this point, but as the pictorial bounds are more accessible as well as more efficient, this will make later bounds easier to read. For $j = 1$, letting $\vec{v} = (t, w, z, u)$, the above bound is executed pictorially as

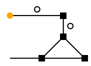
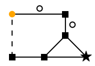
$$\lambda \sup_a \int \psi^{(1)}(\mathbf{0}, a, \vec{v})\, \mathrm{d}\vec{v} = \lambda^3 \sup \int$$  $$\leq \lambda^3 \sup \Big(\int$$  $$\Big(\sup \int$$  $$\Big)\Big) \leq \triangle_\lambda^\circ \triangle_\lambda.$$

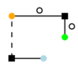
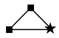
Let us explain the above line in more detail. As was the case in Figure 1, factors of τ_λ become lines, factors of τ_λ° become lines with a '$\circ$', points integrated over become black squares, and points over which we take the supremum become colored disks. The color indicates the location of the point in the diagram (however, using different colors is not essential). We note that the factor $\tau_\lambda(z)$ is interpreted as a line between z and the origin $\mathbf{0}$. We denote the origin either by '★' or by putting no symbol at all. To avoid cluttering the diagrams, we only use the '★' symbol for the origin to highlight changes in position due to substitutions.

To give the pictorial bound for $j = 2$, we have to represent the performed substitution. Note that after the substitution, the variable u appears in the two factors $\tau_\lambda^\circ(w' + u - a)$ and $\tau_\lambda(z' + u)$. We interpret this as a line between $-u$ and z' as well as a line between $a - u$ and w'. In this sense, the two lines do not meet in u, but they have endpoints that are a constant vector a apart. We represent this as

$$\int \tau_\lambda^\circ(w' + u - a)\tau_\lambda(z' + u)\, \mathrm{d}u = \int$$ $$(w' + u - a, z' + u)\, \mathrm{d}u = \int$$  $$.$$

In other words, we represent the pair of points u and $u - a$ with a dashed line. One endpoint of this dashed line will always be a square (representing u in our case), the other a colored disk. With this notation, the pictorial bound for $j = 2$ is

$$\lambda \sup_a \int \psi^{(2)}(\mathbf{0}, a, \vec{v})\, \mathrm{d}\vec{v} = \lambda^3 \sup \int$$  $$= \lambda^3 \sup \int$$ 

$$\leq \lambda^3 \int \Big(\Big(\sup \int$$  $$\Big)$$  $$\Big) \leq \triangle_\lambda^{\circ\circ} \triangle_\lambda.$$

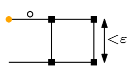
Note that the origin moved (from lower left to lower right) after the substitution.

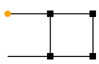
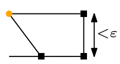
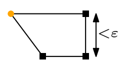
Base case, bound on $\breve{\Psi}^{(1,\geq\varepsilon)}$. To deal with $\breve{\Psi}^{(1,\geq\varepsilon)}$, we again have to bound the three types. For $j = 1, 2$, we can drop the indicator and recycle the bounds obtained on (4.18). For $j = 3$, we observe that

$$\lambda \int \mathbb{1}_{\{|a| \geq \varepsilon\}} \psi^{(3)}(\mathbf{0}, a, t, w, z, u)\, \mathrm{d}(t, w, z, u) = \mathbb{1}_{\{|a| \geq \varepsilon\}} \triangle_\lambda^\circ(a) \leq \triangle_\lambda^{(\varepsilon)},$$

which gives the desired bound.

Base case, bound on $\breve{\Psi}^{(1,<\varepsilon)}$. We show that we can bound $\breve{\Psi}^{(1,<\varepsilon)}$ by either $U_\lambda^{(\varepsilon)}$ or $(U_\lambda^{(\varepsilon)})^2$, giving the desired second and third inequality of Lemma 4.15 and thus concluding the base case. The first bound on $\breve{\Psi}^{(1,<\varepsilon)}$ (the one for $j = 1$) is given pictorially as

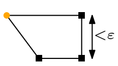
$$\lambda \sup_a \int \mathbb{1}_{\{|w-u| < \varepsilon\}} \psi^{(1)}(\mathbf{0}, a, t, w, z, u)\, \mathrm{d}(\vec{v}) = \lambda^3 \sup \int$$ 

$$\leq \lambda^4 \sup \int$$  $$+ \lambda^3 \sup \int$$  $$\leq \triangle_\lambda^2 + \lambda^3 \sup \int$$  $$. \tag{4.19}$$

An arrow with a '$< \varepsilon$' denotes the indicator of the two endpoints being less than ε apart. The disappearing line in the last bound means that we have applied the (rough) bound $\tau_\lambda \leq 1$. We investigate the second term in the bound of (4.19) and write

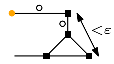
$$\tau_\lambda^{(\varepsilon)}(x) := \mathbb{1}_{\{|x| < \varepsilon\}} \tau_\lambda(x).$$

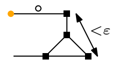
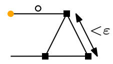
In the subsequent lines, the substitutions $w' = w - u$ and $z' = z - u$ give

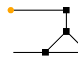
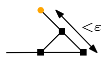
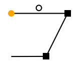
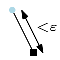
$$\lambda^3 \sup \int$$  $$= \lambda^3 \sup_a \int \tau_\lambda(a - z)\tau_\lambda(z - u)\tau_\lambda^{(\varepsilon)}(w - u)\tau_\lambda(w - a)\, \mathrm{d}(w, z, u)$$

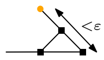
$$= \lambda^3 \sup_a \int \tau_\lambda^{(\varepsilon)}(w') \Big(\iint \tau_\lambda(u+w'-a)\tau_\lambda(a-u-z')\tau_\lambda(z') \,\mathrm{d}u\,\mathrm{d}z' \Big) \,\mathrm{d}w'$$
$$= \lambda \int \tau_\lambda^{(\varepsilon)}(w') \triangle_\lambda(w') \,\mathrm{d}w' \le \big(\mathsf{B}^{(\varepsilon)}\big)^2 \triangle_\lambda.$$

It is here that we see why $\mathsf{B}^{(\varepsilon)}$ was defined with a square root: It allows us to extract two factors of $\mathsf{B}^{(\varepsilon)}$ in the above. For $j=2$, the bounds are

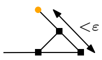
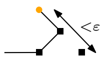
$$\lambda \sup_a \int \mathbb{1}_{\{|w-u|<\varepsilon\}} \psi^{(2)}(\mathbf{0},a,t,w,z,u) \,\mathrm{d}(\vec{v}) = \lambda^2 \sup \int$$ 

$$= \lambda^3 \sup \int$$  $$+\lambda^2 \sup \int$$ 

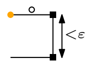
$$\le \lambda^4 \sup \int$$  $$+\lambda^3 \sup \int$$  $$+\lambda \sup \Big(\int$$  $$\Big(\lambda \sup \int$$  $$\Big)\Big)$$

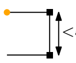
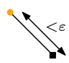
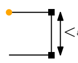
$$\le \triangle_\lambda^2 + \lambda^3 \sup \int$$  $$+\triangle_\lambda^\circ \big(\mathsf{B}^{(\varepsilon)}\big)^2.$$

To deal with the middle term, we see that, uniformly in $a \in \mathbb{R}^d$,

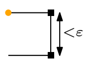
$$\lambda^3 \int$$  $$\le \lambda^3 \int$$  $$= \lambda^3 \int \tau_\lambda(z)\tau_\lambda(t-z)\tau_\lambda(a-t) \Big(\int \mathbb{1}_{\{|a-u|<\varepsilon\}} \,\mathrm{d}u \Big) \,\mathrm{d}z\,\mathrm{d}t \le \triangle_\lambda \big(\mathsf{B}^{(\varepsilon)}\big)^2.$$

Finally, the contribution of $j=3$ is bounded by

$$\lambda \sup_a \int \mathbb{1}_{\{|w-u|<\varepsilon\}} \psi^{(3)}(\mathbf{0},a,t,w,z,u) \,\mathrm{d}\vec{v} = \lambda \sup \int$$ 

$$\le \lambda^2 \sup \int$$  $$+\lambda \sup \int$$  $$\le \lambda^2 \sup \int$$  $$+\big(\mathsf{B}^{(\varepsilon)}\big)^2.$$

We next bound the first term in the above. With the change of variables $z' = z - w$,

$$\lambda^2 \int$$  $$= \lambda^2 \iint \tau_\lambda(z)\tau_\lambda^{(\varepsilon)}(w-z)\tau_\lambda(a-w) \,\mathrm{d}z\,\mathrm{d}w$$
$$= \lambda \int \tau_\lambda^{(\varepsilon)}(z') \Big(\lambda \int \tau_\lambda(z'+w)\tau_\lambda(a-w) \,\mathrm{d}w \Big) \,\mathrm{d}z' \le \triangle_\lambda^\circ \big(\mathsf{B}^{(\varepsilon)}\big)^2, \tag{4.20}$$

as $\lambda(\tau_\lambda \star \tau_\lambda)(x) \le \triangle_\lambda^\circ(x)$. Summing these contributions, $\lambda \iint \breve{\Psi}^{(1,<\varepsilon)}(a,b,x,y)\,\mathrm{d}x\,\mathrm{d}y$ is bounded by

$$2\triangle_\lambda^2 + 2\triangle_\lambda \big(\mathsf{B}^{(\varepsilon)}\big)^2 + 2\triangle_\lambda^\circ \big(\mathsf{B}^{(\varepsilon)}\big)^2 + \big(\mathsf{B}^{(\varepsilon)}\big)^2 \le 2\triangle_\lambda^2 + (4\triangle_\lambda^\circ + 1)\big(\mathsf{B}^{(\varepsilon)}\big)^2 \le U_\lambda^{(\varepsilon)} \wedge \big(U_\lambda^{(\varepsilon)}\big)^2,$$

as required. This concludes the base case.

Inductive step, $n>1$. Let now $n>1$ and assume that the lemma is true for $n-1$. Then

$$\lambda^n \iint \breve{\Psi}^{(n)}(a,b,x,y)\,\mathrm{d}x\,\mathrm{d}y = \lambda \iint \breve{\Psi}^{(1)}(a,b,s,t) \Big(\lambda^{n-1} \iint \breve{\Psi}^{(n-1)}(s,t,x,y)\,\mathrm{d}x\,\mathrm{d}y \Big) \,\mathrm{d}s\,\mathrm{d}t \le \big(U_\lambda\big)^n.$$

For the second bound, a case distinction between $|s-t| \ge \varepsilon$ and $|s-t|<\varepsilon$ gives

$$\lambda^n \iint \breve{\Psi}^{(n)}(a,b,x,y)\,\mathrm{d}x\,\mathrm{d}y = \lambda \iint \breve{\Psi}^{(1)}(a,b,s,t) \Big(\lambda^{n-1} \iint \breve{\Psi}^{(n-1,\ge\varepsilon)}(s,t,x,y)\,\mathrm{d}x\,\mathrm{d}y \Big) \,\mathrm{d}s\,\mathrm{d}t$$
$$+ \lambda \iint \breve{\Psi}^{(1,<\varepsilon)}(a,b,s,t) \Big(\lambda^{n-1} \iint \breve{\Psi}^{(n-1)}(s,t,x,y)\,\mathrm{d}x\,\mathrm{d}y \Big) \,\mathrm{d}s\,\mathrm{d}t$$
$$\le U_\lambda(2(1+U_\lambda))^{n-2} \big(U_\lambda^{(\varepsilon)}\big)^{n-1} + U_\lambda^{(\varepsilon)} U_\lambda \big((2(1+U_\lambda))U_\lambda^{(\varepsilon)}\big)^{n-2}$$
$$= 2U_\lambda(2(1+U_\lambda))^{n-2} \big(U_\lambda^{(\varepsilon)}\big)^{n-1}.$$

The same case distinction for $\breve{\Psi}^{(n,\ge\varepsilon)}$ yields

$$\lambda^n \iint \breve{\Psi}^{(n,\ge\varepsilon)}(a,b,x,y)\,\mathrm{d}x\,\mathrm{d}y = \lambda \iint \breve{\Psi}^{(1,\ge\varepsilon)}(a,b,s,t) \Big(\lambda^{n-1} \iint \breve{\Psi}^{(n-1,\ge\varepsilon)}(s,t,x,y)\,\mathrm{d}x\,\mathrm{d}y \Big) \,\mathrm{d}s\,\mathrm{d}t$$

$$
\begin{aligned}
&+ \lambda \iint \mathbb{1}_{\{|a-b|\geq \varepsilon\}} \breve{\Psi}^{(1,<\varepsilon)}(a,b,s,t)\Big(\lambda^{n-1} \iint \breve{\Psi}^{(n-1)}(s,t,x,y)\,\mathrm{d}x\,\mathrm{d}y\Big)\,\mathrm{d}s\,\mathrm{d}t \\
&\leq U_\lambda^{(\varepsilon)}(2(1+U_\lambda))^{n-2}\big(U_\lambda^{(\varepsilon)}\big)^{n-1} + \big(U_\lambda^{(\varepsilon)}\big)^2 U_\lambda\big((2(1+U_\lambda))U_\lambda^{(\varepsilon)}\big)^{n-2},
\end{aligned}
$$

which is at most $(2(1+U_\lambda))^{n-1}\big(U_\lambda^{(\varepsilon)}\big)^n$. The bounds for $\breve{\Psi}^{(n,<\varepsilon)}$ follow similarly. Having initiated and advanced the induction hypothesis, the claim follows by induction. □

Proof of Proposition 4.14. For $n = 0$,

$$
\lambda \iint \Psi^{(0)}(w,u)\,\mathrm{d}w\,\mathrm{d}u = \sum_{j=1}^{3} \lambda \iint \psi_0^{(j)}(\mathbf{0},w,u)\,\mathrm{d}w\,\mathrm{d}u = 2\triangle_\lambda(\mathbf{0}) + \lambda \int \varphi(u)\,\mathrm{d}u \leq 2\triangle_\lambda + \lambda, \qquad (4.21)
$$

which is certainly bounded by $2\triangle_\lambda^\circ + \lambda + 1$. Let now $n \geq 1$ and note that

$$
\lambda^{n+1} \iint \Psi^{(n)}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \lambda^{n+1} \iint \Psi^{(0)}(w,u)\Big(\iint \breve{\Psi}^{(n)}(w,u,x,y)\,\mathrm{d}x\,\mathrm{d}y\Big)\,\mathrm{d}w\,\mathrm{d}u.
$$

At this stage, we can employ the bound on $\breve{\Psi}^{(n)}$ from Lemma 4.15 to obtain the bound in terms of U_λ (without $U_\lambda^{(\varepsilon)}$). To obtain the second bound, we continue and observe that

$$
\begin{aligned}
\lambda^{n+1} \iint \Psi^{(n)}(x,y)\,\mathrm{d}x\,\mathrm{d}y &\leq \Big(\lambda \iint \Psi^{(0)}(w,u)\,\mathrm{d}w\,\mathrm{d}u\Big)\Big(\sup_{w,u} \lambda^n \iint \breve{\Psi}^{(n,\geq\varepsilon)}(w,u,x,y)\,\mathrm{d}x\,\mathrm{d}y\Big) \\
&\quad + \Big(\lambda \iint \mathbb{1}_{\{|w-u|<\varepsilon\}} \Psi^{(0)}(w,u)\,\mathrm{d}w\,\mathrm{d}u\Big)\Big(\sup_{w,u} \lambda^n \iint \breve{\Psi}^{(n)}(w,u,x,y)\,\mathrm{d}x\,\mathrm{d}y\Big) \\
&\leq (2\triangle_\lambda + \lambda)\big(\bar{U}_\lambda\big)^n \\
&\quad + \Big(\lambda \iint \mathbb{1}_{\{|w-u|<\varepsilon\}} \Psi^{(0)}(w,u)\,\mathrm{d}w\,\mathrm{d}u\Big) U_\lambda \big(\bar{U}_\lambda\big)^{n-1}.
\end{aligned}
$$

To finish the proof, we need a bound similar to (4.21) with the extra indicator $\mathbb{1}_{\{|w-u|<\varepsilon\}}$, i.e. we still are confronted with a sum of three terms. The one for $j = 3$ is directly bounded by $(\mathsf{B}^{(\varepsilon)})^2$. For the two terms $j = 1, 2$, we proceed similarly to (4.20) (setting $a = \mathbf{0}$). This results in the bound

$$
\lambda^2 \iint \tau_\lambda^{(\varepsilon)}(z)\tau_\lambda(z-y)\tau_\lambda(y)\,\mathrm{d}z\,\mathrm{d}y \leq \triangle_\lambda^\circ\big(\mathsf{B}^{(\varepsilon)}\big)^2.
$$

Thus, using that $\mathsf{B}^{(\varepsilon)} \leq 1$,

$$
\begin{aligned}
\lambda^{n+1} \iint \Psi^{(n)}(x,y)\,\mathrm{d}x\,\mathrm{d}y &\leq \big(\bar{U}_\lambda\big)^{n-1}\Big((2\triangle_\lambda + \lambda)\bar{U}_\lambda + U_\lambda(2\triangle_\lambda^\circ + 1)\big(\mathsf{B}^{(\varepsilon)}\big)^2\Big) \\
&\leq 2(2\triangle_\lambda^\circ + \lambda + 1)\big(\bar{U}_\lambda\big)^n.
\end{aligned}
$$

□

The following bounds on $\Pi_\lambda^{(n)}(x)$ are going to be important to define Π_{λ_c} later in Section 5:

Corollary 4.16. *Let $n \geq 1$. Then*

$$
\sup_{x \in \mathbb{R}^d} \Pi_\lambda^{(n)}(x) \leq 2(2\triangle_\lambda^\circ + \lambda + 1)^2\big(U_\lambda \wedge \bar{U}_\lambda\big)^{n-1}.
$$

Proof. Let $x \in \mathbb{R}^d$. Note that writing the statement of Proposition 4.7 in terms of $\Psi^{(n)}$ (defined in (4.16)) gives

$$
\begin{aligned}
\Pi_\lambda^{(n)}(x) &\leq \lambda^n \iint \Psi^{(n-1)}(w,u)\Big(\int \psi_n(w,u,t,z,x)\,\mathrm{d}(t,z)\Big)\,\mathrm{d}(w,u) \\
&\leq (\triangle_\lambda^\circ + 1)\lambda^n \iint \Psi^{(n-1)}(w,u)\,\mathrm{d}(w,u).
\end{aligned}
$$

Applying Proposition 4.14 implies the statement. □

4.4 Diagrammatic bounds with displacement

The main results of this section are Propositions 4.19 and 4.18. To state them, we need another definition:

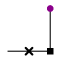
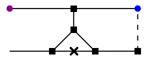
Definition 4.17 (Further displacement bound quantities)**.** Let $x, k \in \mathbb{R}^d$. Recalling that $\tau_{\lambda,k}(x) = [1-\cos(k\cdot x)]\tau_\lambda(x)$, we define the displacement bubble as

$$W_\lambda(x;k) = \lambda(\tau_{\lambda,k} \star \tau_\lambda)(x), \qquad W_\lambda(k) = \sup_{x\in\mathbb{R}^d} W_\lambda(x;k).$$

Furthermore, let $H_\lambda(k) := \sup_{a,b} H_\lambda(a,b;k)$, where

$$H_\lambda(a,b;k) := \lambda^5 \int \tau_\lambda(z)\tau_{\lambda,k}(u-z)\tau_\lambda(t-u)\tau_\lambda(t-z)\tau_\lambda(w-t)\tau_\lambda(a-w)\tau_\lambda(x+b-w)\tau_\lambda(x-u)\,\mathrm{d}(t,w,z,u,x).$$

In terms of pictorial diagrams, we can represent $W_\lambda(a;k)$ and $H_\lambda(a,b;k)$ as

$W_\lambda(a;k) = \lambda \int$  and $H_\lambda(a,b;k) = \lambda^5 \int$  .

Since $W_\lambda(a;k) = \lambda \int \tau_{\lambda,k}(y)\tau_\lambda(a-y)\,\mathrm{d}y$, the line carrying the factor $[1-\cos(k\cdot y)]$, which is the one representing $\tau_{\lambda,k}(y)$, is marked with a '×'.

Propositions 4.18 and 4.19 provide bounds on $\lambda|\widehat\Pi_\lambda^{(n)}(\mathbf{0}) - \widehat\Pi_\lambda^{(n)}(k)| = \lambda\int[1-\cos(k\cdot x)]\Pi_\lambda^{(n)}(x)\,\mathrm{d}x$:

Proposition 4.18 (Displacement bound for $n=1$)**.** *For* $k\in\mathbb{R}^d$,

$$\lambda\int[1-\cos(k\cdot x)]\Pi_\lambda^{(1)}(x)\,\mathrm{d}x \le 31(1+\lambda)(U_\lambda \wedge \bar U_\lambda)W_\lambda(k) + 2\lambda^3\big[(\tau_{\lambda,k}\star\tau_\lambda\star\varphi^{\star 2})(\mathbf{0}) + (\varphi_k \star \tau_\lambda^{\star 2}\star\varphi)(\mathbf{0})\big]$$
$$+ \min\Big\{\lambda[1-\widehat\varphi(k)]\triangle_\lambda^{\circ}, \big(\lambda[1-\widehat\varphi(k)]\triangle_\lambda^{(\varepsilon)} + 4W_\lambda(k)\big(\mathsf{B}^{(\varepsilon)}\big)^2\big)\Big\}.$$

Proposition 4.19 (Displacement bounds for $n\ge 2$)**.** *For* $n\ge 2$ *and* $k\in\mathbb{R}^d$,

$$\lambda\int[1-\cos(k\cdot x)]\Pi_\lambda^{(n)}(x)\,\mathrm{d}x$$
$$\le 60(n+1)\big(\lambda + (\triangle_\lambda^{\circ\circ})^2\big)^2\Big[W_\lambda(k)\big(U_\lambda\big)^{1\vee(n-2)}\triangle_\lambda^{\circ\circ}(1+U_\lambda) + \big(U_\lambda\big)^{n-2}H_\lambda(k)\Big]. \tag{4.22}$$

Moreover,

$$\lambda\int[1-\cos(k\cdot x)]\Pi_\lambda^{(n)}(x)\,\mathrm{d}x$$
$$\le 60(n+1)\big(\lambda + (\triangle_\lambda^{\circ\circ})^2\big)^2\Big[W_\lambda(k)\big(\bar U_\lambda\big)^{1\vee(n-2)}\Big(\triangle_\lambda^{\circ\circ} + U_\lambda + \bar U_\lambda\Big) + \big(\bar U_\lambda\big)^{n-2}H_\lambda(k)\Big]. \tag{4.23}$$

The proof of Proposition 4.19 needs another preparatory lemma (and definition), which we now state. We introduce $\bar\Psi^{(0)}$ and $\bar\Psi^{(n)}$ for $n\ge 1$, which are similar to $\Psi^{(n)}$, as

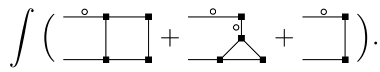
$$\begin{aligned}
\bar\Psi^{(0)}(w_0,z_0) &:= \lambda\triangle(w_0,z_0,\mathbf{0}) + \delta_{\mathbf{0},w_0}\delta_{\mathbf{0},z_0},\\
\bar\Psi^{(n)}(w_n,z_n) &:= \int \phi_n(u_1,w_0,z_0,\mathbf{0})\\
&\quad\times\prod_{i=1}^{n-1}\Big(\phi^{(1)}(u_{i+1},w_i,t_i,z_i,u_i,z_{i-1}) + \sum_{j=2}^{3}\phi^{(j)}(u_{i+1},t_i,w_i,z_i,u_i,z_{i-1})\Big)\\
&\quad\times\Big(\lambda^2\Box(w_n,t_n,u_n,z_n)\tau_\lambda(z_{n-1}-t_n) + \lambda\triangle(t_n,z_n,u_n)\,\underline{\overline{\circ}}\,(t_n-w_n,z_{n-1}-w_n)\\
&\qquad + \delta_{z_n,u_n}\delta_{t_n,z_n}\tau_\lambda(t_n-w_n)\tau_\lambda(z_{n-1}-w_n)\Big)\,\mathrm{d}\big((\vec w,\vec z)_{[0,n-1]},(\vec t,\vec u)_{[1,n]}\big).
\end{aligned}$$

Much like $\Psi^{(n)}$, $\bar\Psi^{(n)}$ is a product over "segments", and these segments (mostly) are a sum of three terms each. The three terms of the $\Psi^{(n)}$ segments and the $\bar\Psi^{(n)}$ segments are quite similar in nature—see the proof sketch of Lemma 4.20. We want to stress the fact that in $\phi^{(1)}$, the labels of the points w_i and t_i are swapped. We also define $\bar\Psi^{(n,<\varepsilon)}(w,z) := \bar\Psi^{(n)}(w,z)\mathbb{1}_{\{|w-z|<\varepsilon\}}$. The following lemma is very much in the spirit of Lemma 4.15 and Proposition 4.14:

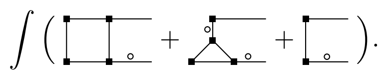
Lemma 4.20. *For $n \geq 0$,*

$$\lambda^{n+1} \iint \bar{\Psi}^{(n)}(w,z)\,\mathrm{d}w\,\mathrm{d}z \leq (\triangle_\lambda + \lambda)\big(U_\lambda \wedge \bar{U}_\lambda\big)^n,$$
$$\lambda^{n+1} \iint \bar{\Psi}^{(n,<\varepsilon)}(w,z)\,\mathrm{d}w\,\mathrm{d}z \leq (\triangle_\lambda + \lambda)\big(\bar{U}_\lambda\big)^{n+1}.$$

The proof for the statement is not performed, as it is analogous to the one of Proposition 4.14. The factor $(\triangle_\lambda + \lambda)$ stems from the base case and is not identical to the one in Proposition 4.14. We give a pictorial sketch of why we obtain the same bounds in the inductive step. We note that $\Psi^{(n)}$ consists of $\Psi^{(0)}$ times n factors of ψ, which is a sum of three terms represented pictorially as



On the other hand, $\bar{\Psi}^{(n)}$ consists of $\bar{\Psi}^{(0)}$ times n factors of the form



The pictures are identical for $j = 1, 3$ and almost identical for $j = 2$, and, most importantly, they can be bounded in the exact same way.

Proof of Proposition 4.19. What makes this proof more cumbersome is the displacement factor $[1-\cos(k \cdot x)]$. In making use of Lemma 4.1, we would like to "split it up" and distribute it over the single segments of the diagram. The $(n+1)$st segment diagram $\phi_0(\vec{v}_0)\big(\prod_{i=1}^{n-1}\phi(\vec{v}_i)\big)\phi_n(\vec{v}_n)$ that we have derived in (4.14) contains a product of factors of τ_λ, which sit along the "top" of the diagram and whose arguments sum up to x. Hence, we can rewrite $x = \sum_{i=0}^n d_i$, where the term d_i is the displacement which falls on the i-th segment along the top. We note that we can replace "top" by "bottom" in the previous sentences, and that in our later depictions of diagrams, we often use the bottom to carry the displacement.

Since the even-indexed segments appear in the diagram in the way displayed in Figure 1 and the odd-indexed ones appear upside down (of course, this is just a matter of perspective), the displacement d_i depends on the parity of i and might be of the form $d_i = w_i - u_{i-1}$ or $d_i = u_i - w_{i-1}$ for $i \in \{1, \dots, n\}$ (where $u_n = w_n = x$), whereas $d_0 = w_0$, since we can fix the orientation of the first segment. Depending on the particular type of segment i, other forms are possible (for degenerate segments, d_i might collapse to $\mathbf{0}$ altogether). A key step is the inequality

$$1 - \cos(k \cdot x) \leq (n+1)\sum_{i=0}^{n}[1-\cos(k\cdot d_i)]$$

due to the Cosine-split Lemma 4.1, which allows us to split the diagram into a sum of $(n+1)$ diagrams, each of which only contains a local displacement d_i. We can thus hope to use the bounds on Ψ and $\bar{\Psi}$ provided by Proposition 4.14 and Lemma 4.20 for all but one segment.

Before we get to actual bounds, we may have to split d_i once more into a sum of two terms for $i > 0$, so that each of these terms appears as an argument of some factor τ_λ. This strengthens the hope of obtaining a bound on $\widehat{\Pi}_\lambda^{(n)}(\mathbf{0}) - \widehat{\Pi}_\lambda^{(n)}(k)$ in terms of a sum of $(n+1)$ terms, each looking rather similar to the bound we have obtained for $\widehat{\Pi}_\lambda^{(n)}(\mathbf{0})$—that is, n out of the $(n+1)$ segments are bounded by known quantities and one designated factor contains $W_\lambda(k)$ (or the related $H_\lambda(k)$ diagram). For $\vec{v}_i = (w_{i-1}, u_{i-1}, t_i, w_i, z_i, u_i)$ to be specified below, we define

$$\psi^{(4)}(\vec{v}_i) = \tau_\lambda(w_i - u_{i-1})\tau_\lambda(u_i - w_i)\tau_\lambda(w_{i-1}-u_i)\delta_{z_i,u_i}\delta_{t_i,u_i},$$
$$\psi^{(5)}(\vec{v}_i) = \tau_\lambda(u_i - u_{i-1})\tau_\lambda(w_{i-1}-u_i)\delta_{w_i,u_{i-1}}\delta_{z_i,u_i}\delta_{t_i,u_i},$$

so that $\psi^{(3)} = \psi^{(4)} + \psi^{(5)}$. The reason to split $\psi^{(3)}$ up further is to single out $\psi^{(5)}$, which will need some special treatment at a later stage of the proof. For $j \in \{1,2,4,5\}$ and $i \notin \{0,n\}$, we aim to bound

$$\lambda^{n+1}\int \psi_0(\vec{v}_0)\Big(\prod_{l=1}^{i-1}\psi(\vec{v}_l)\Big)[1-\cos(k\cdot d_i)]\psi^{(j)}(\vec{v}_i)\prod_{l=i+1}^{n}\psi(\vec{v}_l)\,\mathrm{d}\big((\vec{t},\vec{z})_{[1,n]},(\vec{w},\vec{u})_{[0,n-1]},x\big), \tag{4.24}$$

where we write $\vec{v}_0 = (w_0, u_0)$ and $\vec{v}_l = (w_{l-1}, u_{l-1}, t_l, w_l, z_l, u_l)$ (with $u_n = x$) for $l \in \{1, \ldots, n\}$. For $i = 0$ and $i = n$, the quantities analogous to (4.24) that we need to bound are

$$\lambda^{n+1} \int [1 - \cos(k \cdot d_0)] \psi_0(\vec{v}_0) \prod_{l=1}^{n} \psi(\vec{v}_l) \, \mathrm{d}\big((\vec{t}, \vec{z})_{[1,n]}, (\vec{w}, \vec{u})_{[0,n-1]}, x\big), \tag{4.25}$$

$$\lambda^{n+1} \int \psi_0(\vec{v}_0) \Big(\prod_{l=1}^{n-1} \psi(\vec{v}_l) \Big) [1 - \cos(k \cdot d_n)] \psi_n(\vec{v}_n) \, \mathrm{d}\big((\vec{t}, \vec{z})_{[1,n]}, (\vec{w}, \vec{u})_{[0,n-1]}, x\big). \tag{4.26}$$

Our proof proceeds as follows. We devise a strategy of proof which gives good enough bounds for all $n \geq 2$, all j and all displacements d_i—except when $n = 2$ and $j = 5$. In a second step, we consider this special scenario separately. We divide the general proof into three cases:

(a) The displacement is on the first segment, i.e. $i = 0$.
(b) The displacement is on the last segment, i.e. $i = n$.
(c) The displacement is on an interior segment, i.e. $0 < i < n$.

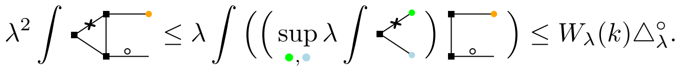
Case (a): The only option for d_0 is $d_0 = w_0$, and so $j = 1$ is the only contributing case (otherwise $d_0 = \mathbf{0}$ and thus $[1 - \cos(k \cdot d_0)] = 0$). Next, note that we can rewrite (4.25) as

$$\begin{aligned} &\lambda^{n+1} \int [1 - \cos(k \cdot w)] \psi_0^{(1)}(\mathbf{0}, w, u) \,\bar{\Psi}^{(n-1)}(t + x - u, z + x - w) \bar{\Psi}^{(n-1)}(t, z) \, \mathrm{d}(w, u, t, z, x) \\ &\leq (\lambda + \triangle_\lambda) \big(U_\lambda \wedge \bar{U}_\lambda\big)^{n-1} \times \sup_{a \in \mathbb{R}^d} \int \lambda [1 - \cos(k \cdot (w - x))] \psi_0^{(1)}(x, w, u) \,\bar{\Psi}(u, w - a) \, \mathrm{d}(w, u, x), \end{aligned} \tag{4.27}$$

where the bound is by virtue of Lemma 4.20. Here we also see the need to introduce $\bar{\Psi}^{(n)}$. The integral in the right-hand side of (4.27) is

$$\lambda^2 \int \Big(\int \tau_{\lambda,k}(w - x) \tau_\lambda(u - x) \, \mathrm{d}x \Big) \tau_\lambda(w - u) \tau_\lambda^\circ(u) \tau_\lambda(a - w) \, \mathrm{d}(w, u) \leq W_\lambda(k) \triangle_\lambda^\circ(a). \tag{4.28}$$

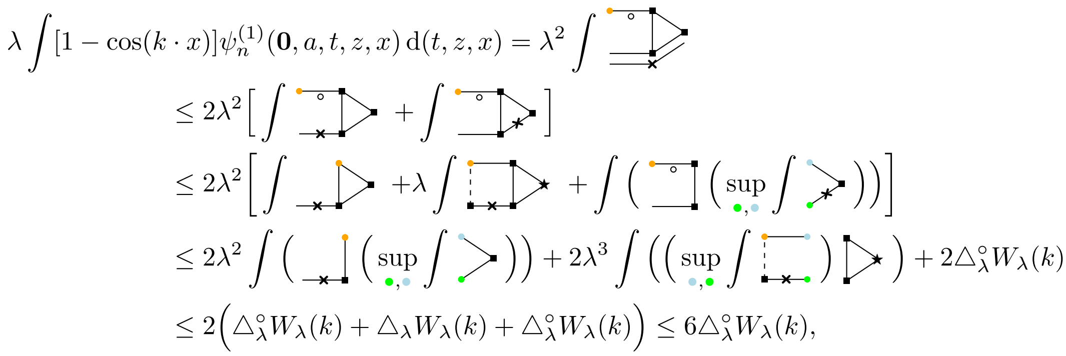
As previously, we show how we represent this bound pictorially. We can bound (4.28) as



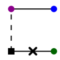
Case (b): We turn to $i = n$ and note that, depending on the parity of n, either $d_n = x - w_{n-1}$ or $d_n = x - u_{n-1}$. Suppose first that $d_n = x - w_{n-1}$. We can write (4.26) as

$$\begin{aligned} &\lambda^{n+1} \int [1 - \cos(k \cdot (x - w))] \Psi^{(n-1)}(w, u) \psi_n(w, u, t, z, x) \, \mathrm{d}(w, u, t, z, x) \\ &\qquad \leq 2(2\triangle_\lambda^\circ + \lambda + 1) \big(U_\lambda \wedge \bar{U}_\lambda\big)^{n-1} \times \lambda \sup_{a \in \mathbb{R}^d} \int [1 - \cos(k \cdot x)] \big(\psi_n^{(1)} + \psi_n^{(2)}\big)(\mathbf{0}, a, t, z, x) \, \mathrm{d}(t, z, x), \end{aligned}$$

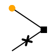
where the bound is by Proposition 4.14. Tending to $\psi_n^{(1)}$ first, we use the Cosine-split Lemma 4.1 to write $x = (x - z) + z$:



where we recall the usage of the '$\star$' symbol for the origin after substitution. In the above, note that

$$\int \tau_{\lambda,k}(b_1+x)\tau_\lambda(b_2+x-a)\,\mathrm{d}x = \int$$  $$= W_\lambda(b_2-b_1-a;k).$$

For $\psi_n^{(2)}$,

$$\lambda\int[1-\cos(k\cdot x)]\psi_n^{(2)}(\mathbf{0},a,t,z,x)\,\mathrm{d}(t,z,x) = \lambda\int$$  $$= W_\lambda(a;k) \le W_\lambda(k).$$

The bounds for $d_n = x-u_{n-1}$ are the same due to symmetry. In total, noting that $\triangle_\lambda \le 2\triangle_\lambda^\circ + 1 \le (\triangle_\lambda^{\circ\circ})^2$, cases (a) and (b) contribute at most

$$2\big(\lambda+(\triangle_\lambda^{\circ\circ})^2\big)\big(U_\lambda\wedge\bar{U}_\lambda\big)^{n-1}W_\lambda(k)\big(\triangle_\lambda^\circ+6\triangle_\lambda^\circ+1\big) \le 16\big(\lambda+(\triangle_\lambda^{\circ\circ})^2\big)\big(U_\lambda\wedge\bar{U}_\lambda\big)^{n-1}W_\lambda(k)\triangle_\lambda^{\circ\circ}.$$

Case (c): Let $i\in\{1,\dots,n-1\}$. We rewrite (4.24) as

$$\lambda^{n+1}\int\Psi^{(i-1)}(w_{i-1},u_{i-1})[1-\cos(k\cdot d_i)]\psi^{(j)}(\vec{v}_i)\,\overline{\underline{\circ}}\,(b_1+x-u_i,b_2+x-w_i) \\ \times\bar{\Psi}^{(n-i-1)}(b_1,b_2)\,\mathrm{d}(\vec{v}_i,\vec{b}_{[1,2]},x), \tag{4.29}$$

where $\vec{v}_i=(w_{i-1},u_{i-1},t_i,w_i,z_i,u_i)$ and $d_i\in\{w_i-u_{i-1},u_i-w_{i-1}\}$. In the next lines, we drop the subscript i, set $\vec{v}=(\mathbf{0},a,t,w,z,u,b_1,b_2,x;d,k)$, $\vec{y}=(t,w,z,u,x)$, $d\in\{w-a,u\}$, and write

$$\tilde{\psi}_k^{(j)}(\vec{v}) = [1-\cos(k\cdot d)]\psi^{(j)}(\mathbf{0},a,t,w,z,u)\,\overline{\underline{\circ}}\,(b_1+x-u,b_2+x-w).$$

Employing our bounds on $\Psi^{(i-1)}$, $\bar{\Psi}^{(n-i)}$, and $\bar{\Psi}^{(n-i,<\varepsilon)}$, we see that (4.29) is bounded by

$$\lambda^{n+1}\int\Psi^{(i-1)}(a_1,a_2)\Big[\sup_a\int\tilde{\psi}_k^{(j)}(\vec{v})\Big(\mathbb{1}_{\{|b_1-b_2|\ge\varepsilon\}}\bar{\Psi}^{(n-i-1)}(b_1,b_2) \\ +\bar{\Psi}^{(n-i-1,<\varepsilon)}(b_1,b_2)\Big)\,\mathrm{d}(\vec{y},\vec{b}_{[1,2]})\Big]\,\mathrm{d}\vec{a}_{[1,2]} \\ \le 2\big(\lambda+(\triangle_\lambda^{\circ\circ})^2\big)^2\big(U_\lambda\wedge\bar{U}_\lambda\big)^{n-2}\sup_{a,b_1,b_2}\int\lambda\tilde{\psi}_k^{(j)}(\vec{v})\Big(1\wedge\big(\mathbb{1}_{\{|b_1-b_2|\ge\varepsilon\}}+\bar{U}_\lambda\big)\Big)\,\mathrm{d}\vec{y},$$

using Proposition 4.14 and Lemma 4.20. To obtain (4.22), we are only interested in a bound in terms of U_λ, which is

$$2\big(\lambda+(\triangle_\lambda^{\circ\circ})^2\big)^2\big(U_\lambda\big)^{n-2}\sup_{a,b_1,b_2}\int\lambda\tilde{\psi}_k^{(j)}(\vec{v})\,\mathrm{d}\vec{y}.$$

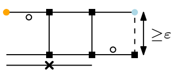
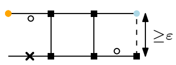
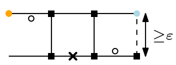
We next examine the integral of $\tilde{\psi}_k^{(j)}$. Depending on whether we aim to give a bound in terms of U_λ or $\bar{U}_\lambda$, we have the indicator $\mathbb{1}_{\{|b_1-b_2|\ge\varepsilon\}}$ present in the integral. We show that in most cases, our bound will contain a factor of either $\triangle_\lambda^\circ$ or $\triangle_\lambda^{(\varepsilon)}$. The latter is relevant for the bound (4.23), the former for the bound (4.22). We perform the bounds in terms of $\bar{U}_\lambda$, as it will be easy to see how to obtain the bounds in terms of U_λ from them (by dropping the indicator $\mathbb{1}_{\{|b_1-b_2|\ge\varepsilon\}}$). First, we note that

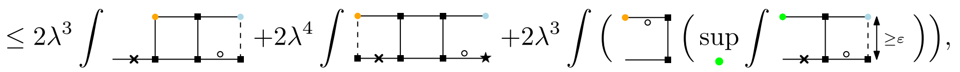
$$\lambda^2\sup_{a,b_1,b_2}\int\tilde{\psi}_k^{(j)}(\vec{v})\,\mathrm{d}\vec{y} = \lambda^2\sup_{a,b}\int[1-\cos(k\cdot d)]\psi^{(j)}(\mathbf{0},a,t,w,z,u)\,\overline{\underline{\circ}}\,(x-u,b+x-w)\,\mathrm{d}\vec{y}.$$

We now turn to the particular values for j and d. Starting with $j=1$, recall that $d\in\{u,w-a\}$. Again, we turn to pictorial bounds. In the following lines, we use an arrow together with '$\ge\varepsilon$' to represent indicators of the form $\mathbb{1}_{\{|\cdot|\ge\varepsilon\}}$.

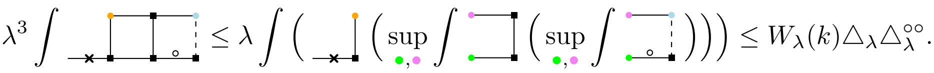
Setting $\vec{v}=(\mathbf{0},a,t,w,z,u,b,\mathbf{0},x;u,k)$ and $\vec{y}=(t,w,z,u,x)$, the bound for $d=u$ can be obtained as

$$\lambda\int\tilde{\psi}_k^{(1)}(\vec{v})\mathbb{1}_{\{|b|\ge\varepsilon\}}\,\mathrm{d}\vec{y} = \lambda^3\int[1-\cos(k\cdot u)]\tau_\lambda(z)\tau_\lambda^\circ(a-t)\square(z,t,w,u) \\ \times\tau_\lambda^\circ(x-u)\tau_\lambda(b+x-w)\mathbb{1}_{\{|b|\ge\varepsilon\}}\,\mathrm{d}\vec{y}$$

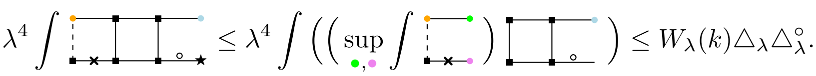
$$= \lambda^3\int$$  $$\le 2\lambda^3\int$$  $$+2\lambda^3\int$$ 

 (4.30)

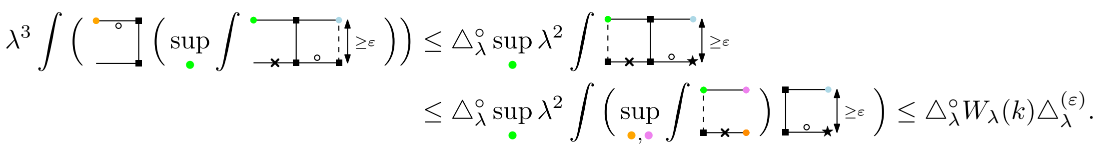
where we have used the Cosine-split Lemma 4.1. The first summand in the right-hand side of (4.30) is bounded via



The second summand in the r.h.s. of (4.30) is bounded by



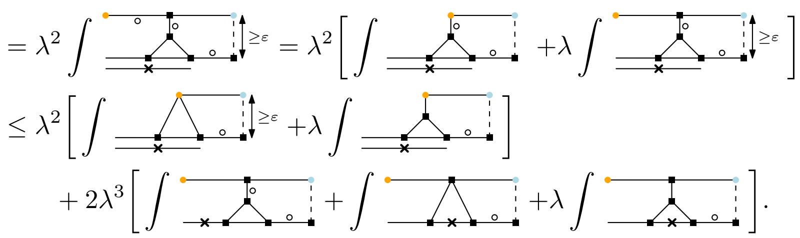
In the third summand in the r.h.s. of (4.30), we obtain the bound



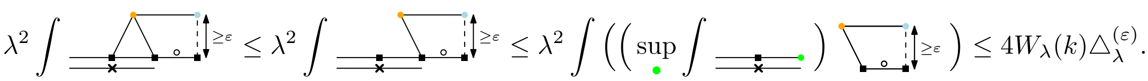
Note that this third summand is the only one where we obtain a bound in terms of $\triangle_\lambda^{(\varepsilon)}$. Not having the indicator present, we easily see that we get another factor of $\triangle_\lambda^\circ$ instead. Again, the displacement for $d = w - a$ yields the same upper bound due to symmetry. The contribution of (4.30) is therefore bounded by $6W_\lambda(k)(U_\lambda \wedge \bar{U}_\lambda)$.

We turn to $j = 2$ and see that, similarly to (4.30),

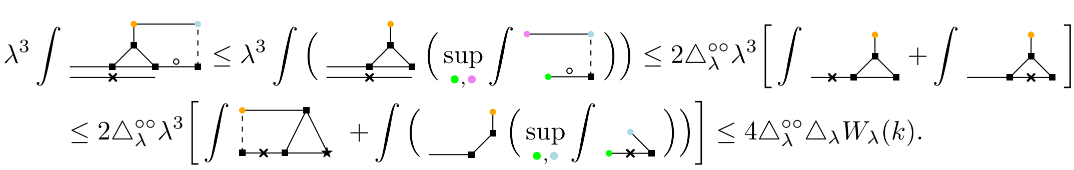
$$\lambda \int \tilde{\psi}_k^{(2)}(\vec{v}) \mathbb{1}_{\{|b|\geq\varepsilon\}} \,\mathrm{d}\vec{y} = \lambda^2 \int [1-\cos(k\cdot u)]\tau_\lambda(z)\tau_\lambda^\circ(a-w)\tau_\lambda^\circ(t-w)\triangle(z,t,u) \times \tau_\lambda^\circ(x-u)\tau_\lambda(b+x-w)\mathbb{1}_{\{|b|\geq\varepsilon\}} \,\mathrm{d}\vec{y}$$

 (4.31) (4.32)

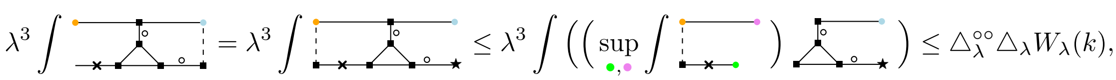
We investigate the five bounding diagrams separately. The first summand in (4.31) is



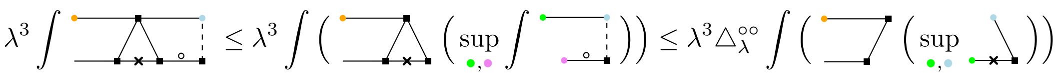
Without the indicator $\mathbb{1}_{\{|b|\geq\varepsilon\}}$ present, this bound becomes $4W_\lambda(k)\triangle_\lambda^\circ$. The second summand in (4.31) is bounded by



The first summand in (4.32) is

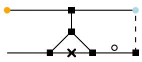
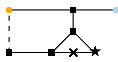
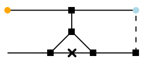


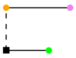
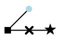
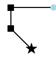
the second is



$$\leq \triangle_\lambda^{\circ\circ} \triangle_\lambda W_\lambda(k),$$

and the third is

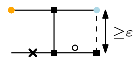
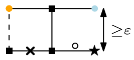
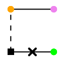
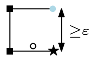
$\lambda^4 \int$  $= \lambda^4 \int$  $+\lambda^5 \int$ 

$\leq \lambda^4 \int \Big(\Big(\sup_{\bullet} \int \Big(\sup_{\bullet,\bullet} \int$  $\Big)$  $\Big)$  $\Big) + H_\lambda(k) \leq \triangle_\lambda^{\circ} \triangle_\lambda W_\lambda(k) + H_\lambda(k).$

The displacement $d = w - a$ is handled like the first term of (4.32) by symmetry, and thus the bound for $d = u$ suffices. We conclude that the joint contribution of (4.31) and (4.32) is bounded by $14 W_\lambda(k)(U_\lambda \wedge \bar{U}_\lambda(k)) + 2H_\lambda(k)$.

We turn to $j = 4$, for which we get

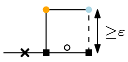
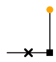
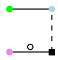
$$\lambda \int \tilde{\psi}_k^{(4)}(\vec{v}) \mathbb{1}_{\{|b| \geq \varepsilon\}} \, \mathrm{d}\vec{y} = \lambda^2 \int [1 - \cos(k \cdot u)] \tau_\lambda(u) \tau_\lambda^\circ(a - w) \tau_\lambda(u - w) \times \tau_\lambda^\circ(x - u) \tau_\lambda(b + x - w) \mathbb{1}_{\{|b| \geq \varepsilon\}} \, \mathrm{d}(w, u, x)$$

$= \lambda^2 \int$  $= \lambda^2 \int$  $\leq \lambda \int \Big(\Big(\sup_{\bullet,\bullet} \int$  $\Big)$  $\Big)$

$$\leq \triangle_\lambda^{(\varepsilon)} W_\lambda(k).$$

Again, without the indicator, we get a bound of $\triangle_\lambda^\circ W_\lambda(k)$ instead, and again, the bound for the displacement $d = w - a$ is the same by symmetry. The contribution is therefore at most $W_\lambda(k)(U_\lambda \wedge \bar{U}_\lambda)$. Finally, $j = 5$ (where $d = u$ is the only option) yields

$$\lambda \int \tilde{\psi}_k^{(5)}(\vec{v}) \mathbb{1}_{\{|b| \geq \varepsilon\}} \, \mathrm{d}\vec{y} = \lambda \int [1 - \cos(k \cdot u)] \tau_\lambda(u) \tau_\lambda(z - a) \tau_\lambda^\circ(x - u) \tau_\lambda(b + x - a) \mathbb{1}_{\{|b| \geq \varepsilon\}} \, \mathrm{d}(u, x)$$

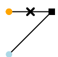
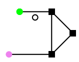
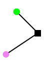
$= \lambda \int$  $\leq \lambda \int \Big($  $\Big(\sup_{\bullet,\bullet} \int$  $\Big)\Big) \leq \triangle_\lambda^{\circ\circ} W_\lambda(k).$

We see that for $n = 2$ and $j = 5$, the obtained bound is not quite what is claimed in Proposition 4.19. Hence, we still need to control

$$\lambda^3 \int \psi_0^{(j)}(\mathbf{0}, w, u) \tau_{\lambda,k}(z - w) \tau_\lambda(z - u) \psi_2(u, z, y_1, y_2, x) \, \mathrm{d}(w, u, z, y_1, y_2, x) \tag{4.33}$$

for $j \in [3]$. For $j \in [2]$, we can bound (4.33) by

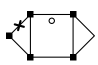
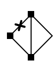
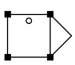
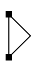
$$\lambda^3 \int \psi_0^{(j)}(\mathbf{0}, w, u) \Big(\sup_{w,u} \int \tau_{\lambda,k}(z - w) \tau_\lambda(z - u) \Big(\sup_{z,u} \int \psi_2(u, z, y_1, y_2, x) \, \mathrm{d}(y_1, y_2, x) \Big) \mathrm{d}u \Big) \mathrm{d}(w, u)$$

$= \lambda^3 \int \psi_0^{(j)}(\mathbf{0}, w, u) \Big(\sup_{\bullet,\bullet} \int$  $\Big(\sup_{\bullet,\bullet} \lambda \int$  $+ \sup_{\bullet,\bullet} \int$  $\Big)\Big) \mathrm{d}(w, u)$

$$\leq \lambda W_\lambda(k) \triangle_\lambda^\circ (1 + \triangle_\lambda^\circ) \int \psi_0^{(j)}(\mathbf{0}, w, u) \, \mathrm{d}(w, u) \leq W_\lambda(k) \triangle_\lambda \triangle_\lambda^\circ (1 + \triangle_\lambda^\circ) \leq 2 W_\lambda(k)(U_\lambda \wedge \bar{U}_\lambda),$$

as is easily checked for both $j = 1, 2$. For $j = 3$, we shift (4.33) by $-x$, whereupon (4.33) is equal to

$$\lambda^3 \int \psi_0^{(j)}(-x, w, u) \tau_{\lambda,k}(z - w) \tau_\lambda(z - u) \psi_2(u, z, y_1, y_2, \mathbf{0}) \, \mathrm{d}(w, u, z, y_1, y_2, x)$$

$\leq \lambda^4 \int$  $+ \lambda^3 \int$  $\leq \lambda^4 \int \Big(\Big(\sup_{\bullet,\bullet} \int$  $\Big)$  $\Big) + \lambda^3 \int \Big(\Big(\sup_{\bullet,\bullet} \int$  $\Big)$  $\Big)$

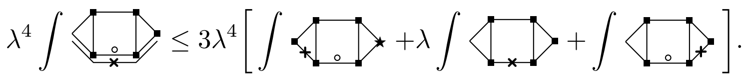
$$\leq W_\lambda(k) \triangle_\lambda (\triangle_\lambda^\circ + 1) \leq W_\lambda(k)(U_\lambda \wedge \bar{U}_\lambda).$$

Carefully putting together all these bounds finishes the proof. □

Proof of Proposition 4.18. Let now $n = 1$. By (4.14), we get a bound on $\lambda \int [1 - \cos(k \cdot x)] \Pi_\lambda^{(1)}(x) \, \mathrm{d}x$ of the form

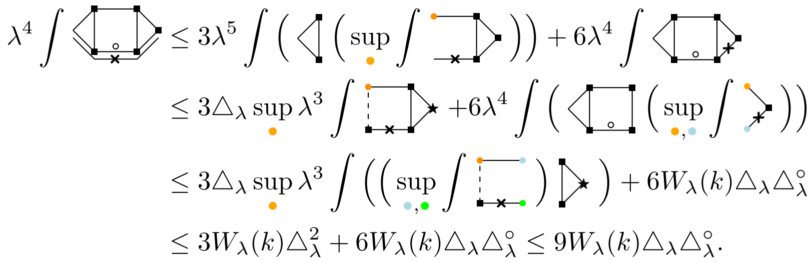
$$\lambda^2 \int [1 - \cos(k \cdot x)] \sum_{j_0=1}^{3} \sum_{j_1=1}^{2} \psi_0^{(j_0)}(w, u) \psi_1^{(j_1)}(w, u, t, z, x) \, \mathrm{d}(w, u, t, z, x). \tag{4.34}$$

This results in a sum of six diagrams, which we bound one by one. Again, we want to use the Cosine-split Lemma 4.1 in order to break up the displacement factor $[1-\cos(k\cdot x)]$ and distribute it over edges of the diagrams. Most terms follow analogously to $n \geq 2$, and so we only do the pictorial representations thereof. For $(j_0, j_1) = (1,1)$, we get the bound

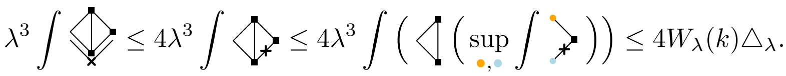


(4.35)

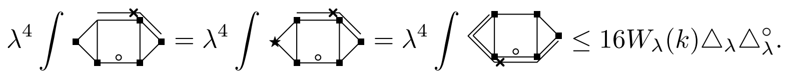
The first and third diagram on the r.h.s. of (4.35) are the same by symmetry, and so



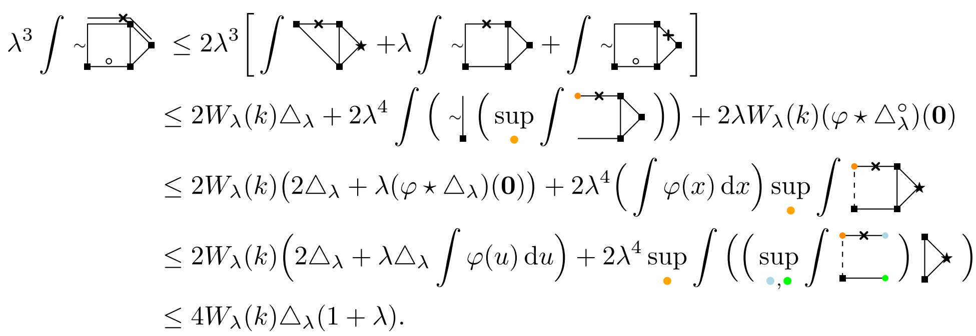
Similarly, symmetry for $(j_0, j_1) = (1,2)$ gives



By substitution, we can reduce the diagrams $(j_0, j_1) = (2,1)$ to the one from $(j_0, j_1) = (1,1)$, as



The diagram $(j_0, j_1) = (2,2)$, which is $\lambda^2 \int \triangle(\mathbf{0}, w, u) W_\lambda(u;k)\,\mathrm{d}u\,\mathrm{d}w$, can be bounded directly by $W_\lambda(k)\triangle_\lambda$. When $(j_0, j_1) = (3,1)$, we see a direct edge, which is pictorially represented with an extra '$\sim$'. The diagram therefore is



When $(j_0, j_1) = (3,2)$, we apply Observation 4.3 to get a bound of the form

$$\begin{aligned} \lambda^2(\tau_{\lambda,k} \star \tau_\lambda \star \varphi)(\mathbf{0}) &= \lambda^2(\varphi_k \star \tau_\lambda \star \varphi)(\mathbf{0}) + \lambda^3 \int [1-\cos(k\cdot x)](\varphi \star \tau_\lambda)^2(x)\,\mathrm{d}x \\ &\leq \lambda^2(\varphi_k \star \tau_\lambda \star \varphi)(\mathbf{0}) + 2\lambda^3\Big((\tau_{\lambda,k} \star \tau_\lambda \star \varphi^{\star 2})(\mathbf{0}) + (\varphi_k \star \tau_\lambda^{\star 2} \star \varphi)(\mathbf{0})\Big), \end{aligned}$$

where the Cosine-split Lemma 4.1 was used in the second line to distribute the factor $[1-\cos(k\cdot x)]$ over $(\varphi \star \tau_\lambda)$. We still have to handle the first summand. To this end, we use $1 = \mathbb{1}_{\{|x|\geq\varepsilon\}} + \mathbb{1}_{\{|x|<\varepsilon\}}$ and obtain

$$\begin{aligned} \lambda^2(\varphi_k \star \tau_\lambda \star \varphi)(\mathbf{0}) &\leq \lambda^2 \int \varphi_k(x)\Big(\mathbb{1}_{\{|x|\geq\varepsilon\}}\triangle^\circ_\lambda(x)\Big)\,\mathrm{d}x + \lambda^2 \int \tau^{(\varepsilon)}_\lambda(x)[1-\cos(k\cdot x)](\tau_\lambda \star \tau_\lambda)(x)\,\mathrm{d}x \\ &\leq \lambda \triangle^{(\varepsilon)}_\lambda \int \varphi_k(x)\,\mathrm{d}x + 4\lambda^2 \int \tau^{(\varepsilon)}_\lambda(x)(\tau_{\lambda,k} \star \tau_\lambda)(x)\,\mathrm{d}x \end{aligned}$$

$$\leq \lambda \triangle_\lambda^{(\varepsilon)} \int \varphi_k(x)\,\mathrm{d}x + 4\big(\mathsf{B}^{(\varepsilon)}\big)^2 W_\lambda(k),$$

where, again, we have used the Cosine-split Lemma 4.1 in the second line. By just the bound from the first line, we get $\lambda^2(\varphi_k \star \tau_\lambda \star \varphi)(\mathbf{0}) \leq \lambda \triangle_\lambda^\circ \int \varphi_k(x)\,\mathrm{d}x$. To finish the proof, note that $\int \varphi_k(x)\,\mathrm{d}x = 1 - \widehat{\varphi}(k)$. Carefully putting together all six bounds gives the statement. □

5 Bootstrap analysis

5.1 Re-scaling of the intensity measure

To increase the readability and avoid cluttering the technical proofs, we are going to assume throughout Section 5 (as well as in the appendix) that

$$q_\varphi = \int \varphi(x)\,\mathrm{d}x = 1. \tag{5.1}$$

A re-scaling argument similar to [29, Section 2.2] justifies this without losing generality. Let us briefly elaborate on this: We can scale $\mathbb{R}^d$ by a factor of $q_\varphi^{-1/d}$ (that is, the new unit radius ball is the previous ball of radius $q_\varphi^{1/d}$) and what we obtain has the distribution of an RCM model with parameters

$$\lambda^* = \mathbb{E}_\lambda\big[\big|\eta \cap [0, q_\varphi^{1/d}]^d\big|\big] = \lambda q_\varphi, \qquad \varphi^*(x) = \varphi\left(x q_\varphi^{1/d}\right).$$

For this new model,

$$\int \varphi^*(x)\,\mathrm{d}x = q_\varphi^{-1} \int \varphi(y)\,\mathrm{d}y = 1.$$

By the re-scaling, $\{\mathbf{0} \longleftrightarrow x \text{ in } \xi^{\mathbf{0},x}\}$ becomes $\{\mathbf{0} \longleftrightarrow x/q_\varphi^{1/d} \text{ in } \xi^{\mathbf{0},x/q_\varphi^{1/d}}\}$, we also have

$$\tau_\lambda(x) = \tau^*_{\lambda^*}\left(x/q_\varphi^{1/d}\right),$$

where τ^*_μ is the two-point function in the RCM governed by the connection function φ^*. Clearly, $\lambda_c q_\varphi = \lambda_c^*$. Another short computation shows that $\triangle_\lambda(x) = (\lambda^*)^2\big(\tau^*_{\lambda^*} \star \tau^*_{\lambda^*} \star \tau^*_{\lambda^*}\big)\big(x/q_\varphi^{1/d}\big)$, so the triangle condition holds in the original model precisely when it holds in the re-scaled model.

As an example, under the normalization assumption, $\widehat{G}_{\mu_\lambda}(k) = (1 - \mu\widehat{\varphi}(k))^{-1}$.

5.2 Introduction of the bootstrap functions

The analysis of Section 5 follows the arguments in the paper by Heydenreich et al. [21], adapting them to the continuum setting. Some parts follow the presentation given there almost verbatim. We define

$$\mu_\lambda := 1 - \frac{1}{\widehat{\tau}_\lambda(\mathbf{0})}$$

for $\lambda \geq 0$. Note that $\widehat{\tau}_\lambda(\mathbf{0})$ is increasing in λ and $\widehat{\tau}_0(\mathbf{0}) = 1$. Furthermore, as $\lambda \nearrow \lambda_c$, we have $\widehat{\tau}_\lambda(\mathbf{0}) \nearrow \infty$ and so $\mu_\lambda \nearrow 1$. In summary, $\mu_\lambda \in [0,1]$. Setting

$$\Delta_k a(l) := a(l-k) + a(l+k) - 2a(l)$$

to be the discretized second derivative of a function $a\colon \mathbb{R}^d \to \mathbb{C}$, we are in a position to define $f := f_1 \vee f_2 \vee f_3$ with

$$f_1(\lambda) := \lambda, \qquad f_2(\lambda) := \sup_{k \in \mathbb{R}^d} \frac{|\widehat{\tau}_\lambda(k)|}{\widehat{G}_{\mu_\lambda}(k)}, \qquad f_3(\lambda) := \sup_{k,l \in \mathbb{R}^d} \frac{|\Delta_k \widehat{\tau}_\lambda(l)|}{\widehat{U}_{\mu_\lambda}(k,l)}, \tag{5.2}$$

where we recall (1.13) and (1.14) for the Green's function G_μ. Moreover, $\widehat{U}_{\mu_\lambda}$ is defined as

$$\widehat{U}_{\mu_\lambda}(k,l) := 84[1 - \widehat{\varphi}(k)]\Big(\widehat{G}_{\mu_\lambda}(l-k)\widehat{G}_{\mu_\lambda}(l) + \widehat{G}_{\mu_\lambda}(l)\widehat{G}_{\mu_\lambda}(l+k) + \widehat{G}_{\mu_\lambda}(l-k)\widehat{G}_{\mu_\lambda}(l+k)\Big)$$

and will serve as an upper bound on $\Delta_k \widehat{G}_{\mu_\lambda}(l)$ by Lemma 5.1 below. This function $\widehat{U}_{\mu_\lambda}$ has nothing to do with the functions $U_\lambda, \bar{U}_\lambda$ from Section 4.

Let us point out that we show $f(\lambda) < \infty$ for all $\lambda \in [0, \lambda_c)$ as part of the proof of Proposition 5.9. In particular, we show that $f(0) \leq 2$ and that for $\lambda < \lambda_c$ and $i \in [3]$, f_i is differentiable on $[0, \lambda]$ with uniformly bounded derivative.

Let us now explain the introduction of Δ_k. The crucial observation (see Lemma 5.1 (i)) is that

$$\widehat{\tau}_{\lambda,k}(l) = -\tfrac{1}{2}\Delta_k \widehat{\tau}_\lambda(l),$$

where we recall that $\tau_{\lambda,k}(x) = [1-\cos(k\cdot x)]\tau_\lambda(x)$ is defined in Definition 4.17 and appears in $W_\lambda(k)$. This is why we are interested in a bound on f_3. We next state a lemma that collects some simple facts about the discretized second derivative, relates it to quantities of interest and states an important bound. The proof of Lemma 5.1 can be found in the literature [6, 35] for the discrete setting and carries over verbatim.

Lemma 5.1 (Bounds on the discretized second derivative, [35], Lemma 5.7)**.** *Let $a\colon \mathbb{R}^d \to \mathbb{R}$ be a measurable, symmetric and integrable function. Let $a_k(x) := [1-\cos(k\cdot x)]a(x)$. Then*

(i) $\Delta_k \widehat{a}(l) = -2\widehat{a}_k(l)$ and

(ii) $|\Delta_k \widehat{a}(l)| \leq 2(\widehat{|a|}(\mathbf{0}) - \widehat{|a|}(k))$ for all $k, l \in \mathbb{R}^d$. In particular, using (i), $\widehat{\varphi}_k(l) \leq 1 - \widehat{\varphi}(k)$.

(iii) For $k, l \in \mathbb{R}^d$ and $\widehat{A}(k) = (1-\widehat{a}(k))^{-1}$,

$$|\Delta_k \widehat{A}(l)| \leq \big[\widehat{|a|}(\mathbf{0}) - \widehat{|a|}(k)\big]\Big(\big(\widehat{A}(l-k)+\widehat{A}(l+k)\big)\widehat{A}(l) + 8\widehat{A}(l-k)\widehat{A}(l)\widehat{A}(l+k)\big(\widehat{|a|}(\mathbf{0}) - \widehat{|a|}(l)\big)\Big).$$

In particular,

$$|\Delta_k \widehat{G}_\mu(l)| \leq [1-\widehat{\varphi}(k)]\Big(\widehat{G}_\mu(l)\widehat{G}_\mu(l+k) + \widehat{G}_\mu(l)\widehat{G}_\mu(l-k) + 8\widehat{G}_\mu(l-k)\widehat{G}_\mu(l+k)\Big) \leq \widehat{U}_{\mu_\lambda}(k,l).$$

The remainder of Section 5 is organized as follows. In Section 5.3, we prove among other things that the lace expansion converges for each fixed $\lambda \in [0, \lambda_c)$, provided that β is sufficiently small (recall that this means d large for (H1) and L large for (H2), (H3)). Moreover, we prove that under the additional assumption $f \leq 3$ on $[0, \lambda_c)$, the smallness of β required for the convergence of the lace expansion does not depend on λ. To do so, we derive several bounds on triangles and related quantities in terms of the function f. In Section 5.4, we prove that $f(0) \leq 2$ and that f is continuous on $[0, \lambda_c)$. We then use the results obtained in Section 5.3 to show that in fact $f \leq 2$ on $[0, \lambda_c)$ whenever β is sufficiently small. This in turn implies that the bounds obtained in Section 5.3 under the additional assumption $f \leq 3$ are true. In percolation theory, this (at first glance circular) argument is known as the bootstrap argument. From there, the main theorems follow with only little extra work.

5.3 Consequences of the bootstrap bounds

We state Proposition 5.2, which proves bounds on the lace-expansion coefficients for fixed λ and consequently shows that the lace-expansion identity (3.18) becomes the Ornstein-Zernike equation in the limit $n \to \infty$. While Sections 3 and 4 were valid for general φ, the proofs in Section 5.3 rely heavily on Propositions A.2 and A.3, and therefore on the assumptions made on φ in (H1), (H2), or (H3).

Proposition 5.2 (Convergence of the lace expansion and OZE)**.** *Let $\lambda \in [0, \lambda_c)$. Let $d > 12$ be sufficiently large under (H1) and let $d > 3(\alpha \wedge 2)$ and L be sufficiently large under (H2) and (H3). Then there is $c_f = c(f(\lambda))$ (which is increasing in f and independent of d for (H1)) such that*

$$\int \sum_{n\geq 0} \Pi_\lambda^{(n)}(x)\,\mathrm{d}x \leq c_f \beta, \qquad \int [1-\cos(k\cdot x)] \sum_{n\geq 0} \Pi_\lambda^{(n)}(x)\,\mathrm{d}x \leq c_f [1-\widehat{\varphi}(k)]\beta, \tag{5.3}$$

$$\sup_{x\in\mathbb{R}^d} \sum_{n\geq 0} \Pi_\lambda^{(n)}(x) \leq c_f, \tag{5.4}$$

and

$$\sup_{x\in\mathbb{R}^d} |R_{\lambda,n}(x)| \leq \lambda\widehat{\tau}_\lambda(\mathbf{0})(c_f\beta)^n. \tag{5.5}$$

Furthermore, the limit $\Pi_\lambda := \lim_{n\to\infty}\Pi_{\lambda,n}$ *exists and is an integrable function with Fourier transform* $\widehat{\Pi}_\lambda(k) = \lim_{n\to\infty}\widehat{\Pi}_{\lambda,n}(k)$ *for* $k\in\mathbb{R}^d$. *Lastly,* τ_λ *satisfies the Ornstein-Zernike equation, taking the form*

$$\tau_\lambda = \varphi + \Pi_\lambda + \lambda\big((\varphi+\Pi_\lambda)\star\tau_\lambda\big). \tag{5.6}$$

Corollary 5.3 (Uniform convergence of the lace expansion)**.** *Assume that* $f\leq 3$ *on* $[0,\lambda_c)$. *Let* $d>12$ *be sufficiently large under (H1) and let* $d>3(\alpha\wedge 2)$ *and* L *be sufficiently large under (H2) and (H3). Then there is* c *(independent of* λ *and, for (H1), also independent of* d*) such that*

- *the bounds* (5.3), (5.4), (5.5) *hold with* c_f *replaced by* c,
- *the OZE* (5.6) *holds.*

We now state and prove Lemmas 5.5-5.8, which (partially) rely on Lemma 5.4. All of these lemmas will also deal with constants c_f and c. As in the statement of Proposition 5.2, they are independent of d for (H1). Proposition 5.2 will be a rather direct consequence of these lemmas.

We recall that $\tilde{\tau}_\lambda(x)=\varphi(x)+\lambda(\varphi\star\tau_\lambda)(x)$, $\tau_{\lambda,k}(x)=[1-\cos(k\cdot x)]\tau_\lambda(x)$, $\varphi_k(x)=[1-\cos(k\cdot x)]\varphi(x)$, and we furthermore set $\tilde{\tau}_{\lambda,k}(x):=[1-\cos(k\cdot x)]\tilde{\tau}_\lambda(x)$. With this, for $a,k\in\mathbb{R}^d, m,n\in\mathbb{N}_0$ and $\lambda\geq 0$, we define

$$V_\lambda^{(m,n)}(a) := \lambda^{m+n-1}(\varphi^{\star m}\star\tau_\lambda^{\star n})(a), \qquad W_\lambda^{(m,n)}(a;k) := \lambda^{m+n}(\tau_{\lambda,k}\star\varphi^{\star m}\star\tau_\lambda^{\star n})(a),$$
$$\widetilde{W}_\lambda^{(m,n)}(a;k) := \lambda^{m+n}(\varphi_k\star\varphi^{\star m}\star\tau_\lambda^{\star n})(a).$$

Note that $W_\lambda^{(0,1)}(a;k)=W_\lambda(a;k)$, with W_λ from Definition 4.17.

Lemma 5.4 (Bounds on $V_\lambda, W_\lambda, \widetilde{W}_\lambda$)**.** *Let* $\lambda\in[0,\lambda_c)$ *and let* $m,n\in\mathbb{N}_0$ *be such that* $n\leq 3$ *and* $m+n\geq 2$. *Let* $d>4n$ *under (H1) and* $d>(\alpha\wedge 2)n$ *under (H2) and (H3). Then there is* $c_f = c(f(\lambda),m,n)$ *(increasing in* f*) such that*

$$\sup_{a\in\mathbb{R}^d} V_\lambda^{(m,n)}(a) \leq \begin{cases} c_f\beta^{((m+n)\wedge 3)-2} & \text{under (H1)},\\ c_f\beta & \text{under (H2), (H3)},\end{cases}$$
$$\sup_{a\in\mathbb{R}^d} \widetilde{W}_\lambda^{(m,n)}(a;k) \leq c_f[1-\widehat{\varphi}(k)]\beta^{((m+n)\wedge 3)-2}.$$

If furtermore $n\leq 1$, *then*

$$\sup_{a\in\mathbb{R}^d} W_\lambda^{(m,n)}(a;k) \leq c_f[1-\widehat{\varphi}(k)]\beta^{((m+n)\wedge 3)-2}.$$

We remark that Lemma 5.4 produces a bound in terms of β as soon as $m+n\geq 3$.

Proof. We start by observing that, for all $m\in\mathbb{N}_0$ and $n\in\mathbb{N}$,

$$V_\lambda^{(m,n)} \leq V_\lambda^{(m+1,n-1)} + V_\lambda^{(m+1,n)} \leq 2\max_{l\in\{n-1,n\}} V_\lambda^{(m+1,l)}, \tag{5.7}$$

which is a direct consequence of $\tau_\lambda\leq\varphi+\lambda(\varphi\star\tau_\lambda)$ (i.e., Observation 4.3). Analogous bounds hold for $W_\lambda^{(m,n)}$ and $\widetilde{W}_\lambda^{(m,n)}$. What (5.7) means is that we can "increase m by possibly decreasing n". We can therefore assume $m\geq 2$ without loss of generality (i.e., replace m by $m+n$). The bound for V_λ follows from the Fourier inverse formula, as

$$V_\lambda^{(m,n)}(a) = \lambda^{m+n-1}\int \mathrm{e}^{-\mathrm{i}a\cdot l}\widehat{\varphi}(l)^m\widehat{\tau}_\lambda(l)^n\frac{\mathrm{d}l}{(2\pi)^d} \leq f(\lambda)^{m+n-1}\int|\widehat{\varphi}(l)|^m|\widehat{\tau}_\lambda(l)|^n\frac{\mathrm{d}l}{(2\pi)^d},$$

where we used that $\lambda\leq f(\lambda)$. We next apply the bound $|\widehat{\tau}_\lambda(l)|\leq f_2(\lambda)\widehat{G}_{\mu_\lambda}(l)$, yielding

$$V_\lambda^{(m,n)}(a) \leq f(\lambda)^{m+2n-1}\int|\widehat{\varphi}(l)|^m\widehat{G}_{\mu_\lambda}(l)^n\frac{\mathrm{d}l}{(2\pi)^d} = f(\lambda)^{m+2n-1}\int\frac{|\widehat{\varphi}(l)|^m}{[1-\mu_\lambda\widehat{\varphi}(l)]^n}\frac{\mathrm{d}l}{(2\pi)^d}. \tag{5.8}$$

Applying Proposition A.2 gives the claim. Next, note that

$$\widetilde{W}_\lambda^{(m,n)}(a;k) \le \lambda^{m+n} \int |\widehat{\varphi}_k(l)||\widehat{\varphi}(l)|^m |\widehat{\tau}_\lambda(l)|^n \frac{\mathrm{d}l}{(2\pi)^d} \le f(\lambda)^{m+2n}[1-\widehat{\varphi}(k)] \int |\widehat{\varphi}(l)|^m \widehat{G}_{\mu_\lambda}(l)^n \frac{\mathrm{d}l}{(2\pi)^d}$$

due to Lemma 5.1(ii). Since this is the same bound as in (5.8), we can apply Proposition A.2 again. Lastly,

$$\begin{aligned} W_\lambda^{(m,n)}(a;k) &\le 84 f(\lambda)^{m+2n}[1-\widehat{\varphi}(k)] \int |\widehat{\varphi}(l)|^m \widehat{G}_{\mu_\lambda}(l)^{n+1} \Big(\widehat{G}_{\mu_\lambda}(l-k) + \widehat{G}_{\mu_\lambda}(l+k)\Big) \frac{\mathrm{d}l}{(2\pi)^d} \\ &\quad + 84 f(\lambda)^{m+2n}[1-\widehat{\varphi}(k)] \int |\widehat{\varphi}(l)|^m \widehat{G}_{\mu_\lambda}(l)^n \widehat{G}_{\mu_\lambda}(l-k)\widehat{G}_{\mu_\lambda}(l+k) \frac{\mathrm{d}l}{(2\pi)^d}. \end{aligned}$$

Both of these terms can be bounded using Proposition A.3 (if $n=0$ in the second line, we can multiply the integrand with $\widehat{G}_{\mu_\lambda}(l)$ at the cost of a factor of 2). Recall that we have replaced m by $m+n$, which yields the exponents in the statement of Lemma 5.4. □

The following lemma applies Lemma 5.4 to deduce bounds on several triangle quantities. It is crucial in the sense that it gives a bound on $\triangle_\lambda$. We later prove that this bound is uniform in λ, implying the triangle condition.

Lemma 5.5 (Bounds on triangles)**.** *Let $\lambda \in [0, \lambda_c)$ and let $d > 12$ under (H1) and $d > 3(\alpha \wedge 2)$ under (H2) and (H3). Let further ε be given as in (H1.2). Then there is $c_f = c(f(\lambda))$ (increasing in f) such that*

$$\triangle_\lambda \le c_f \beta, \qquad \triangle_\lambda^{(\varepsilon)} \le c_f\beta, \qquad \triangle_\lambda^{\circ\circ} \le c_f, \qquad \triangle_\lambda^{\circ} \le \begin{cases} c_f & \text{under (H1)}, \\ c_f\beta & \text{under (H2), (H3)}. \end{cases}$$

Proof. Observe that

$$\begin{aligned} \triangle_\lambda(x) &\le \lambda^2(\tau_\lambda \star \tilde{\tau}_\lambda \star \tilde{\tau}_\lambda)(x) = \lambda^2\Big(\tau_\lambda \star \big(\varphi + \lambda(\varphi\star\tau_\lambda)\big) \star \big(\varphi + \lambda(\varphi\star\tau_\lambda)\big)\Big)(x) \\ &= V_\lambda^{(2,1)}(x) + 2V_\lambda^{(2,2)}(x) + V_\lambda^{(2,3)}(x) \le c_f\beta \end{aligned}$$

by Lemma 5.4. Similarly, we get $\triangle_\lambda^\circ(x) = \triangle_\lambda + \lambda\tau_\lambda^{\star 2}(x) \le \triangle_\lambda + V_\lambda^{(2,0)}(x) + 2V_\lambda^{(2,1)}(x) + V_\lambda^{(2,2)}(x)$. Applying Lemma 5.4 again, this is bounded by c_f under (H1), and by $c_f\beta$ under (H2), (H3). The bound $\triangle_\lambda^{\circ\circ} \le 1 + \triangle_\lambda^\circ$ gives $\triangle_\lambda^{\circ\circ} \le c_f$. We recall that by Observation 4.3, we have $\triangle_\lambda^\circ(x) \le \lambda(\varphi\star\varphi)(x) + \triangle_\lambda(x)$, and thus

$$\triangle_\lambda^{(\varepsilon)} \le c_f\beta + \sup_{x:|x|\ge\varepsilon} \lambda(\varphi\star\varphi)(x) \le c_f'\beta,$$

since φ satisfies (H1.2) under (H1). □

Lemma 5.6 (Bound on W_λ)**.** *Let $\lambda \in [0,\lambda_c)$ and let $d > 12$ under (H1) and $d > 3(\alpha\wedge 2)$ under (H2) and (H3). Then there is $c_f = c(f(\lambda))$ (increasing in f) such that*

$$W_\lambda(k) \le c_f[1-\widehat{\varphi}(k)].$$

Proof. Recall the definition of $W_\lambda(k)$ in Definition 4.17. By Observation 4.3 and Lemma 4.1, we obtain

$$\begin{aligned} W_\lambda(k) &\le \sup_x \lambda \int [1-\cos(k\cdot y)]\big(\varphi(y) + \lambda(\varphi\star\tau_\lambda)(y)\big)\tilde{\tau}_\lambda(x-y)\,\mathrm{d}y \\ &\le \sup_x \lambda\Big(\big[\varphi_k + 2\lambda\varphi_k\star\tau_\lambda + 2\lambda\varphi\star\tau_{\lambda,k}\big] \star \big[\varphi + \lambda(\varphi\star\tau_\lambda)\big]\Big)(x) \\ &= \sup_x \Big(\widetilde{W}_\lambda^{(1,0)}(x;k) + 3\widetilde{W}_\lambda^{(1,1)}(x;k) + 2\widetilde{W}_\lambda^{(1,2)}(x;k) + 2W_\lambda^{(2,0)}(x;k) + 2W_\lambda^{(2,1)}(x;k)\Big). \end{aligned}$$

We note that all summands except $\widetilde{W}_\lambda^{(1,0)}(x;k)$ are bounded by $c_f[1-\widehat{\varphi}(k)]$ by Lemma 5.4. The statement now follows from Lemma 5.4 together with

$$\widetilde{W}_\lambda^{(1,0)}(x;k) = \lambda\int \varphi_k(y)\varphi(x-y)\,\mathrm{d}y \le \lambda\int\varphi_k(y)\,\mathrm{d}y = f_1(\lambda)[1-\widehat{\varphi}(k)]. \qquad \square$$

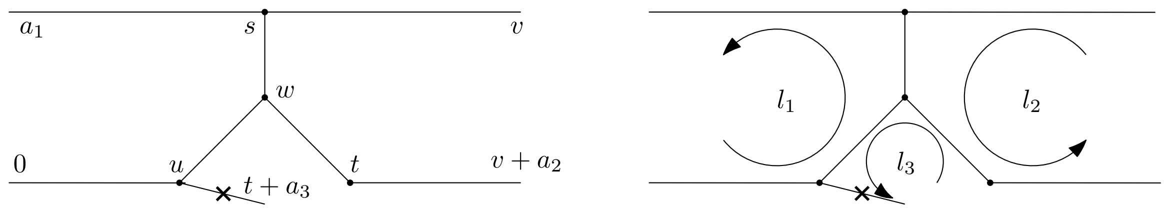


Figure 3: The diagram $H'_\lambda(a_1,a_2,a_3;k)$ and the schematic Fourier diagram $\widehat{H}'_\lambda(l_1,l_2,l_3;k)$.

The next lemma deals with the required extra treatment of the diagrams $\Pi_\lambda^{(0)}$ and $\Pi_\lambda^{(1)}$ with an added displacement:

Lemma 5.7 (Displacement bounds on $\Pi_\lambda^{(0)}$ and $\Pi_\lambda^{(1)}$)**.** *Let $\lambda \in [0,\lambda_c)$ and let $d>12$ under (H1) and $d>3(\alpha\wedge 2)$ under (H2) and (H3). Let further $i\in\{0,1\}$. Then there is $c_f=c(f(\lambda))$ (increasing in f) such that*

$$\lambda\int[1-\cos(k\cdot x)]\Pi_\lambda^{(i)}(x)\,\mathrm{d}x \le c_f\beta[1-\widehat{\varphi}(k)].$$

Proof. Let first $i=0$. With the bound (4.3) and a split of the cosine via Lemma 4.1, we get

$$\begin{aligned}\lambda\int[1-\cos(k\cdot x)]\Pi_\lambda^{(0)}(x)\,\mathrm{d}x &\le \lambda^3\int[1-\cos(k\cdot x)](\varphi\star\tau_\lambda)(x)^2\,\mathrm{d}x\\ &\le 2\widetilde{W}_\lambda^{(1,2)}(\mathbf{0};k)+2W_\lambda^{(2,1)}(\mathbf{0};k)\le c_f\beta[1-\widehat{\varphi}(k)]\end{aligned}$$

by Lemma 5.4. Let now $i=1$. Recalling the bound in Proposition 4.18, the claimed result follows from noting that the two appearing convolutions are $W_\lambda^{(2,1)}$ and $\widetilde{W}_\lambda^{(1,2)}$. □

Lemma 5.8 (Bound on H_λ)**.** *Let $\lambda\in[0,\lambda_c)$ and let $d>12$ under (H1) and $d>3(\alpha\wedge 2)$ under (H2) and (H3). Then there is $c_f=c(f(\lambda))$ (increasing in f) such that*

$$H_\lambda(k)\le c_f\beta[1-\widehat{\varphi}(k)].$$

Proof. We define $\tilde{\tilde{\tau}}_{\lambda,k}(x):=\varphi_k(x)+2\lambda[(\varphi_k\star\tau_\lambda)(x)+(\varphi\star\tilde{\tau}_{\lambda,k})(x)]$ and note that $\tilde{\tau}_{\lambda,k}\le\tilde{\tilde{\tau}}_{\lambda,k}$ by the Cosine-split Lemma 4.1. With this, by setting

$$\begin{aligned}H'_\lambda(a_1,a_2,a_3;k):=\int&\tilde{\tau}_\lambda(s-a_1)\tilde{\tau}_\lambda(v-s)\tilde{\tau}_\lambda(s-w)\tilde{\tau}_\lambda(u)\tilde{\tau}_\lambda(w-u)\\ &\times\tilde{\tau}_\lambda(t-w)\tilde{\tau}_\lambda(v+a_2-t)\tilde{\tilde{\tau}}_{\lambda,k}(t+a_3-u)\,\mathrm{d}(s,t,u,v,w),\end{aligned}$$

we have the bound $H_\lambda(k)\le f(\lambda)^5\sup_{a_1,a_2}H'_\lambda(a_1,a_2,\mathbf{0};k)$. The Fourier inversion formula yields

$$H'_\lambda(a_1,a_2,\mathbf{0};k)=\int \mathrm{e}^{-\mathrm{i}l_1\cdot a_1}\mathrm{e}^{-\mathrm{i}l_2\cdot a_2}\widehat{H}'_\lambda(l_1,l_2,l_3;k)\frac{\mathrm{d}(l_1,l_2,l_3)}{(2\pi)^{3d}}. \tag{5.9}$$

Using that $b_1=(b_1-s)+(s-w)+(w-u)+u$, $b_2=(v+b_2-t)+(t-w)+(w-s)+(s-v)$ and $b_3=(t+b_3-u)+(u-w)+(w-t)$, with appropriate substitution, this leads to

$$\begin{aligned}\widehat{H}'_\lambda(l_1,l_2,l_3;k)&=\int \mathrm{e}^{\mathrm{i}l_1\cdot b_1}\mathrm{e}^{\mathrm{i}l_2\cdot b_2}\mathrm{e}^{\mathrm{i}l_3\cdot b_3}H'_\lambda(b_1,b_2,b_3;k)\,\mathrm{d}(b_1,b_2,b_3)\\ &=\int \mathrm{e}^{\mathrm{i}l_1\cdot u}\tilde{\tau}_\lambda(u)\mathrm{e}^{\mathrm{i}l_1\cdot(b_1-s)}\tilde{\tau}_\lambda(b_1-s)\mathrm{e}^{\mathrm{i}l_2\cdot(s-v)}\tilde{\tau}_\lambda(s-v)\mathrm{e}^{\mathrm{i}l_2\cdot(v+b_2-t)}\tilde{\tau}_\lambda(v+b_2-t)\\ &\qquad\times\mathrm{e}^{\mathrm{i}l_3\cdot(t+b_3-u)}\tilde{\tilde{\tau}}_{\lambda,k}(t+b_3-u)\mathrm{e}^{\mathrm{i}(l_1-l_2)\cdot(s-w)}\tilde{\tau}_\lambda(s-w)\mathrm{e}^{\mathrm{i}(l_1-l_3)\cdot(w-u)}\tilde{\tau}_\lambda(w-u)\\ &\qquad\times\mathrm{e}^{\mathrm{i}(l_2-l_3)\cdot(t-w)}\tilde{\tau}_\lambda(t-w)\,\mathrm{d}(b_1,b_2,b_3,s,t,u,v,w)\\ &=\widehat{\tilde{\tau}}_\lambda(l_1)^2\widehat{\tilde{\tau}}_\lambda(l_2)^2\widehat{\tilde{\tilde{\tau}}}_{\lambda,k}(l_3)\widehat{\tilde{\tau}}_\lambda(l_1-l_2)\widehat{\tilde{\tau}}_\lambda(l_1-l_3)\widehat{\tilde{\tau}}_\lambda(l_2-l_3),\end{aligned}$$

where we point to Figure 3 for an interpretation of the variables l_i as cycles. Since $2\widehat{G}_\mu\ge 1$,

$$|\widehat{\tilde{\tau}}_\lambda(l)|\le|\widehat{\varphi}(l)|+\lambda|\widehat{\varphi}(l)||\widehat{\tau}_\lambda(l)|\le 3f(\lambda)^2|\widehat{\varphi}(l)|\widehat{G}_{\mu_\lambda}(l).$$

Similarly, with Lemma 5.1(ii),

$$|\widehat{\tilde{\tilde{\tau}}}_{\lambda,k}(l_3)| \leq 4f(\lambda)^2[1-\widehat{\varphi}(k)]\widehat{G}_{\mu_\lambda}(l_3) + 2f(\lambda)^2|\widehat{\varphi}(l_3)|\,\widehat{U}_{\mu_\lambda}(k,l_3).$$

We can now go back and plug the above bounds into (5.9) to obtain

$$\begin{aligned} H_\lambda(k) \leq\ & 4\times 3^7 f(\lambda)^{16}[1-\widehat{\varphi}(k)]\int \widehat{\varphi}(l_1)^2\widehat{G}_{\mu_\lambda}(l_1)^2\widehat{\varphi}(l_2)^2\widehat{G}_{\mu_\lambda}(l_2)^2\widehat{G}_{\mu_\lambda}(l_3) \\ & \times (|\widehat{\varphi}|\cdot\widehat{G}_{\mu_\lambda})(l_1-l_2)(|\widehat{\varphi}|\cdot\widehat{G}_{\mu_\lambda})(l_3-l_1)(|\widehat{\varphi}|\cdot\widehat{G}_{\mu_\lambda})(l_3-l_2)\frac{\mathrm{d}(l_1,l_2,l_3)}{(2\pi)^{3d}} \qquad (5.10) \\ & + 336\times 3^7 f(\lambda)^{14}[1-\widehat{\varphi}(k)]\int \widehat{\varphi}(l_1)^2\widehat{G}_{\mu_\lambda}(l_1)^2\widehat{\varphi}(l_2)^2\widehat{G}_{\mu_\lambda}(l_2)^2|\widehat{\varphi}(l_3)| \\ & \times \Big[\widehat{G}_{\mu_\lambda}(l_3)\big(\widehat{G}_{\mu_\lambda}(l_3+k)+\widehat{G}_{\mu_\lambda}(l_3-k)\big)+\widehat{G}_{\mu_\lambda}(l_3+k)\widehat{G}_{\mu_\lambda}(l_3-k)\Big] \\ & \times (|\widehat{\varphi}|\cdot\widehat{G}_{\mu_\lambda})(l_1-l_2)(|\widehat{\varphi}|\cdot\widehat{G}_{\mu_\lambda})(l_1-l_3)(|\widehat{\varphi}|\cdot\widehat{G}_{\mu_\lambda})(l_2-l_3)\frac{\mathrm{d}(l_1,l_2,l_3)}{(2\pi)^{3d}}. \qquad (5.11) \end{aligned}$$

Hence, $H_\lambda(k)$ is bounded by the sum of the terms (5.10) and (5.11). The latter is itself a sum of two terms: By (5.11)(i) we refer to the term in (5.11) containing $\widehat{G}_{\mu_\lambda}(l_3+k)+\widehat{G}_{\mu_\lambda}(l_3-k)$, and by (5.11)(ii) we refer to the one containing $\widehat{G}_{\mu_\lambda}(l_3+k)\widehat{G}_{\mu_\lambda}(l_3-k)$. Hence, we have to bound the three terms (5.10), (5.11)(i), (5.11)(ii). We start with the term (5.10), apply Hölder's inequality and bound the integral by

$$\begin{aligned} &\Big(\int (\widehat{\varphi}(l_1)^2\widehat{G}_{\mu_\lambda}(l_1)^3(|\widehat{\varphi}|\cdot\widehat{G}_{\mu_\lambda})(l_2)^3\widehat{G}_{\mu_\lambda}(l_3)(|\widehat{\varphi}|\cdot\widehat{G}_{\mu_\lambda})(l_3-l_1)^{1/2}(|\widehat{\varphi}|\cdot\widehat{G}_{\mu_\lambda})(l_3-l_2)^{3/2}\ \frac{\mathrm{d}(l_1,l_2,l_3)}{(2\pi)^{3d}}\Big)^{2/3} \\ &\quad\times\Big(\int \widehat{\varphi}(l_1)^2\widehat{G}_{\mu_\lambda}(l_3)(|\widehat{\varphi}|\cdot\widehat{G}_{\mu_\lambda})(l_1-l_2)^3(\widehat{\varphi}\cdot\widehat{G}_{\mu_\lambda})(l_3-l_1)^2\ \frac{\mathrm{d}(l_1,l_2,l_3)}{(2\pi)^{3d}}\Big)^{1/3}. \qquad (5.12) \end{aligned}$$

The second integral in (5.12) is simpler to deal with: We substitute $l_3' = l_3-l_1$ to then bound $\widehat{G}_{\mu_\lambda}(l_3'+l_1)\leq\widehat{G}_{\mu_\lambda}(l_3'+l_1)+\widehat{G}_{\mu_\lambda}(l_3'-l_1)$ and use Proposition A.3 to resolve the integral over l_3'. We then resolve the integral over l_2 to obtain a factor $C\beta$ and note that the remaining integral over l_1 is bounded by 1.

To deal with the first integral in (5.12), we first consider the integral over l_3 and use Hölder's inequality to get

$$\begin{aligned} &\sup_{l_1,l_2}\int (|\widehat{\varphi}|\cdot\widehat{G}_{\mu_\lambda})(l_3-l_2)^{3/2}\widehat{G}_{\mu_\lambda}(l_3)(|\widehat{\varphi}|\cdot\widehat{G}_{\mu_\lambda})(l_1-l_3)^{1/2}\ \frac{\mathrm{d}l_3}{(2\pi)^d} \\ &\quad\leq \sup_{l_1,l_2}\Big(\int \widehat{\varphi}(l_3')^2\widehat{G}_{\mu_\lambda}(l_3')^2\widehat{G}_{\mu_\lambda}(l_3'+l_2)\ \frac{\mathrm{d}l_3'}{(2\pi)^d}\Big)^{3/4}\Big(\int \widehat{\varphi}(l_3'')^2\widehat{G}_{\mu_\lambda}(l_3'')^2\widehat{G}_{\mu_\lambda}(l_3''+l_1)\ \frac{\mathrm{d}l_3''}{(2\pi)^d}\Big)^{1/4}, \end{aligned}$$

where we have substituted $l_3' = l_3-l_2$ and $l_3'' = l_3-l_1$. Again, we use that $\widehat{G}_{\mu_\lambda}$ is nonnegative and bound $\widehat{G}_{\mu_\lambda}(l_3'+l_2)\leq\widehat{G}_{\mu_\lambda}(l_3'+l_2)+\widehat{G}_{\mu_\lambda}(l_3'-l_2)$ in the first integral and $\widehat{G}_{\mu_\lambda}(l_3''+l_1)\leq\widehat{G}_{\mu_\lambda}(l_3''+l_1)+\widehat{G}_{\mu_\lambda}(l_3''-l_1)$ in the second. Proposition A.3 then completes the bounds. The remaining integral over l_1 and l_2 is handled straightforwardly, the latter yielding a factor of $C\beta$.

To bound the integral in (5.11)(i), let $\widehat{D}_{\mu_\lambda,k}(l)=\widehat{G}_{\mu_\lambda}(l-k)+\widehat{G}_{\mu_\lambda}(l+k)$. We apply the Cauchy-Schwarz inequality to bound the term from above by

$$\begin{aligned} &\Big(\int \big[\widehat{\varphi}(l_1)^4\widehat{G}_{\mu_\lambda}(l_1)^3\big]\big[(\widehat{\varphi}\cdot\widehat{G}_{\mu_\lambda})(l_2-l_1)^2\widehat{G}_{\mu_\lambda}(l_2)\big]\big[\widehat{\varphi}(l_3)^2\widehat{G}_{\mu_\lambda}(l_3)\widehat{D}_{\mu_\lambda,k}(l_3)^2\big]\ \frac{\mathrm{d}(l_1,l_2,l_3)}{(2\pi)^{3d}}\Big)^{1/2} \\ &\quad\times\Big(\int \big[(\widehat{\varphi}\cdot\widehat{G}_{\mu_\lambda})(l_1-l_3)^2\widehat{G}_{\mu_\lambda}(l_1)\big]\big[\widehat{\varphi}(l_2)^4\widehat{G}_{\mu_\lambda}(l_2)^3\big]\big[(\widehat{\varphi}\cdot\widehat{G}_{\mu_\lambda})(l_3-l_2)^2\widehat{G}_{\mu_\lambda}(l_3)\big]\ \frac{\mathrm{d}(l_1,l_2,l_3)}{(2\pi)^{3d}}\Big)^{1/2}, \end{aligned}$$

which is easily decomposed as indicated by the square brackets. For the integral in (5.11)(ii), we use Cauchy-Schwarz to obtain a bound of the form

$$\Big(\int \big[\widehat{\varphi}(l_1)^4\widehat{G}_{\mu_\lambda}(l_1)^3\big]\big[(\widehat{\varphi}\cdot\widehat{G}_{\mu_\lambda})(l_2-l_3)^2\widehat{G}_{\mu_\lambda}(l_2)\big]\big[\widehat{\varphi}(l_3)^2\widehat{G}_{\mu_\lambda}(l_3-k)^2\widehat{G}_{\mu_\lambda}(l_3+k)\big]\ \frac{\mathrm{d}(l_1,l_2,l_3)}{(2\pi)^{3d}}\Big)^{1/2}$$

$$\times \left(\int \left[(\widehat{\varphi}\cdot \widehat{G}_{\mu_\lambda})(l_1-l_2)^2 \widehat{G}_{\mu_\lambda}(l_1)\right] \left[\widehat{\varphi}(l_2)^4 \widehat{G}_{\mu_\lambda}(l_2)^3\right] \left[(\widehat{\varphi}\cdot \widehat{G}_{\mu_\lambda})(l_3-l_1)^2 \widehat{G}_{\mu_\lambda}(l_3+k)\right] \frac{\mathrm{d}(l_1,l_2,l_3)}{(2\pi)^{3d}} \right)^{1/2}.$$

To resolve the integral over l_3 in the first factor, we use $\widehat{G}_{\mu_\lambda} \geq 0$ to bound

$$\int \widehat{\varphi}(l_3)^2 \widehat{G}_{\mu_\lambda}(l_3-k)^2 \widehat{G}_{\mu_\lambda}(l_3+k) \frac{\mathrm{d}l_3}{(2\pi)^d} \leq \int \widehat{\varphi}(l_3)^2 \widehat{D}_{\mu_\lambda,k}(l_3)^3 \frac{\mathrm{d}l_3}{(2\pi)^d}, \tag{5.13}$$

which is bounded by Proposition A.3. The integral over l_3 in the second factor is handled similarly; the integrals over l_1 and l_2 can be handled in exactly the same way as in the bound on (5.11)(i). □

Proof of Proposition 5.2. We first recall from Section 4 and Propositions 4.14 and 4.19 as well as Corollary 4.16 therein that $\Pi_\lambda^{(n)}$, $\int \Pi_\lambda^{(n)}(x)\,\mathrm{d}x$, and $\int[1-\cos(k\cdot x)]\Pi_\lambda^{(n)}(x)\,\mathrm{d}x$ are bounded in terms of $\triangle_\lambda, \triangle_\lambda^\circ, \triangle_\lambda^{\circ\circ}, W_\lambda(k)$, and $H_\lambda(k)$ for $n \geq 0$ (with an extra term for $\int[1-\cos(k\cdot x)]\Pi_\lambda^{(1)}(x)\,\mathrm{d}x$ not of this form but addressed in Lemma 5.7). Recalling these bounds and combining them with the four lemmas just proved gives

$$\lambda \int \Pi_\lambda^{(n)}(x)\,\mathrm{d}x \leq (c'_f\beta)^{n\vee 1}, \qquad \lambda \int [1-\cos(k\cdot x)]\Pi_\lambda^{(n)}(x)\,\mathrm{d}x \leq [1-\widehat{\varphi}(k)](c'_f\beta)^{(n-1)\vee 1},$$
$$\Pi_\lambda^{(n)}(x) \leq c'_f (c'_f\beta)^{(n-1)},$$

for some $c'_f = c'(f(\lambda))$ increasing in f. Now, if $c'_f\beta < 1$,

$$\lambda \int \sum_{m=0}^n \Pi_\lambda^{(m)}(x)\,\mathrm{d}x = \sum_{m=0}^n \lambda \int \Pi_\lambda^{(m)}(x)\,\mathrm{d}x \leq \sum_{m=0}^\infty (c'_f\beta)^{m\vee 1} = c'_f\beta(1+(1-c'_f\beta)^{-1}).$$

If $c'_f\beta < 1/2$, then we can choose $c_f = 4c'_f$. The other two bounds follow similarly.

Note that, by dominated convergence, this implies that the limit Π_λ is well defined, and so is its Fourier transform. We are left to deal with $R_{\lambda,n}(x)$. Recalling the bound (3.17) and combining it with the bound on $\Pi_\lambda^{(n)}$ from Corollary 4.16 implies

$$\sup_{x\in\mathbb{R}^d} |R_{\lambda,n}(x)| \leq \lambda \widehat{\tau}_\lambda(\mathbf{0}) \sup_{x\in\mathbb{R}^d} |\Pi_\lambda^{(n)}(x)| \leq f(\lambda)\widehat{\tau}_\lambda(\mathbf{0})(c_f\beta)^n.$$

As $\widehat{\tau}_\lambda(\mathbf{0})$ is finite for $\lambda < \lambda_c$, the right-hand side vanishes as $n \to \infty$ for sufficiently small β. As a consequence, $R_{\lambda,n} \to 0$ uniformly in x, which proves (5.6) for $\lambda < \lambda_c$. □

5.4 The bootstrap argument

The missing piece to prove our main theorems is Proposition 5.9, proving that on $[0,\lambda_c)$, the function $f = f_1 \vee f_2 \vee f_3$ defined in (5.2) is continuous, bounded by 2 at 0 and that $f \leq 3$ implies $f \leq 2$.

Proposition 5.9 (The forbidden-region argument)**.** *The following three statements are true:*

1. *The function f satisfies $f(0) \leq 2$.*
2. *The function f is continuous on $[0,\lambda_c)$.*
3. *Moreover, $f(\lambda) \notin (2,3]$ for all $\lambda \in [0,\lambda_c)$ provided that $d > 3(\alpha\wedge 2)$ and $\beta \ll 1$ (i.e., d sufficiently large for (H1) and L sufficiently large for (H2), (H3)).*

Consequently, $f(\lambda) \leq 2$ holds uniformly in $\lambda < \lambda_c$ for $d > 3(\alpha\wedge 2)$ and $\beta \ll 1$.

Proof. We show that (1.)-(3.) hold for the functions f_1, f_2, f_3 separately. The result then follows for f itself.

(1.) Bound for $\lambda = 0$. Trivially, $f_1(0) = 0$. Note that $\mu_0 = 0$, and so $\widehat{G}_{\mu_0} \equiv 1$. Also, $\tau_0 = \varphi$ and so $\widehat{\tau}_0 = \widehat{\varphi}$. From this we infer $f_2(\lambda) \leq 1$. Lastly, by Lemma 5.1(ii),

$$f_3(0) = \sup_{k,l} \frac{|\Delta_k \widehat{\varphi}(l)|}{252[1-\widehat{\varphi}(k)]} \leq \frac{1}{126}.$$

(2.) Continuity of the bootstrap function. The continuity of f_1 is obvious. The same idea used in the discrete $\mathbb{Z}^d$ case is used to handle the other two bootstrap functions. More precisely, we make use of a known result, formulated by Slade [35, Lemma 5.13]. The idea is summarized as follows:

- We want to prove the continuity of the supremum of a family $(h_\alpha)_{\alpha \in B}$ of functions (α is either k or the tuple (k, l), and B is either $\mathbb{R}^d$ or $(\mathbb{R}^d)^2$).
- To this end, we fix an arbitrary $\rho > 0$ and show that (h_α) is equicontinuous on $[0, \lambda_c - \rho]$, i.e. that for $\varepsilon > 0$ there exists $\delta > 0$ such that $|s - t| < \delta$ implies $|h_\alpha(s) - h_\alpha(t)| \le \varepsilon$ uniformly in $\alpha \in B$.
- We prove this equicontinuity by taking derivatives with respect to λ and bounding this derivative uniformly in α on $[0, \lambda_c - \rho]$.
- Furthermore, we prove that $(h_\alpha)_{\alpha \in B}$ is uniformly bounded on $[0, \lambda_c - \rho]$.
- This implies that $t \mapsto \sup_{\alpha \in B} h_\alpha(t)$ is continuous on $[0, \lambda_c - \rho]$. As ρ was arbitrary, we get the desired continuity on $[0, \lambda_c)$.

A first important observation is that we actually have to deal with the supremum of a family $(|h_\alpha|)_{\alpha \in B}$ of functions, which might cause headaches when taking derivatives. However, as an immediate consequence of the reverse triangle inequality, the equicontinuity of $(h_\alpha)_{\alpha \in B}$ implies the equicontinuity of $(|h_\alpha|)_{\alpha \in B}$, as $||h_\alpha(x + t)| - |h_\alpha(x)|| \le |h_\alpha(x + t) - h_\alpha(x)|$.

We start with f_2 and consider

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{\widehat{\tau}_\lambda(k)}{\widehat{G}_{\mu_\lambda}(k)} = \frac{1}{\widehat{G}_{\mu_\lambda}(k)^2} \left[\widehat{G}_{\mu_\lambda}(k) \frac{\mathrm{d}\widehat{\tau}_\lambda(k)}{\mathrm{d}\lambda} - \widehat{\tau}_\lambda(k) \left. \frac{\mathrm{d}\widehat{G}_\mu(k)}{\mathrm{d}\mu} \right|_{\mu = \mu_\lambda} \times \frac{\mathrm{d}\mu_\lambda}{\mathrm{d}\lambda} \right],$$

which we treat by bounding every appearing term. With $2\widehat{G}_{\mu_\lambda}(k) \ge 1$, we start by noting

$$\frac{1}{2} \le \frac{1}{1 - \mu_\lambda \widehat{\varphi}(k)} = \widehat{G}_{\mu_\lambda}(k) \le \widehat{G}_{\mu_\lambda}(\mathbf{0}) = \widehat{\tau}_\lambda(\mathbf{0}) \le \widehat{\tau}_{\lambda_c - \rho}(\mathbf{0}) = \frac{\chi(\lambda_c - \rho) - 1}{\lambda_c - \rho},$$

where the last term is finite. The finiteness of $\widehat{\tau}_\lambda(\mathbf{0})$ turns out to be helpful several times, as it also bounds $|\widehat{\tau}_\lambda(k)| \le \widehat{\tau}_\lambda(\mathbf{0})$ uniformly in k. The derivative of the two-point function satisfies

$$\left| \frac{\mathrm{d}}{\mathrm{d}\lambda} \widehat{\tau}_\lambda(k) \right| = \left| \int \mathrm{e}^{\mathrm{i}k \cdot x} \frac{\mathrm{d}}{\mathrm{d}\lambda} \tau_\lambda(x) \, \mathrm{d}x \right| \le \int \frac{\mathrm{d}}{\mathrm{d}\lambda} \tau_\lambda(x) \, \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}\lambda} \int \tau_\lambda(x) \, \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}\lambda} \widehat{\tau}_\lambda(\mathbf{0}) \le \widehat{\tau}_\lambda(\mathbf{0})^2.$$

The exchange of derivative and integral is justified as the integrand τ_λ is bounded uniformly in λ by the integrable function $\tau_{\lambda_c - \rho}$. The last bound is Lema 2.3, and so we make use of $\widehat{\tau}_\lambda(\mathbf{0})$ being finite again.

By definition of $\widehat{G}_\mu$ (recall (1.14)), $|\frac{\mathrm{d}}{\mathrm{d}\mu} \widehat{G}_\mu(k)| \le \widehat{G}_\mu(k)^2 \le \widehat{G}_\mu(\mathbf{0})^2$ which, for $\mu = \mu_\lambda$, equals $\widehat{\tau}_\lambda(\mathbf{0})^2$. Lastly, $\frac{\mathrm{d}}{\mathrm{d}\lambda} \mu_\lambda = \frac{\mathrm{d}}{\mathrm{d}\lambda} \widehat{\tau}_\lambda(\mathbf{0}) / \widehat{\tau}_\lambda(\mathbf{0})^2 \le 1$ by Lemma 2.3.

This proves the continuity of f_2. It is not hard to see that f_3 can be treated in a similar way:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{\Delta_k \widehat{\tau}_\lambda(l)}{\widehat{U}_{\mu_\lambda}(k, l)} = \frac{1}{\widehat{U}_{\mu_\lambda}(k, l)^2} \left[\widehat{U}_{\mu_\lambda}(k, l) \frac{\mathrm{d}\Delta_k \widehat{\tau}_\lambda(l)}{\mathrm{d}\lambda} - \Delta_k \widehat{\tau}_\lambda(l) \left. \frac{\mathrm{d}\widehat{U}_\mu(k, l)}{\mathrm{d}\mu} \right|_{\mu = \mu_\lambda} \times \frac{\mathrm{d}\mu_\lambda}{\mathrm{d}\lambda} \right].$$

Recalling the definitions of $\Delta_k \widehat{\tau}_\lambda(l)$ and $\widehat{U}_{\mu_\lambda}(k, l)$, similar bounds as used for f_2 can be applied.

(3.) The forbidden region. To show the claim, we assume that $f(\lambda) \le 3$ for $\lambda \in [0, \lambda_c)$ and then show that this implies $f(\lambda) \le 2$. The assumption $f(\lambda) \le 3$ allows us to apply Corollary 5.3.

Throughout this whole part, we use M and $\tilde{M}$ to denote constants whose exact value may change from line to line. We stress that they are independent of d for (H1) and independent of L for (H2), (H3). We start by setting $a := \varphi + \Pi_\lambda$, and hence

$$\widehat{a}(k) = \widehat{\varphi}(k) + \widehat{\Pi}_\lambda(k).$$

By Proposition 5.2, $\widehat{\tau}_\lambda$ takes the form $\widehat{\tau}_\lambda(k) = \widehat{a}(k)/(1 - \lambda \widehat{a}(k))$, and thus

$$\mu_\lambda = \lambda + \frac{\widehat{\Pi}_\lambda(\mathbf{0})}{\widehat{a}(\mathbf{0})}. \tag{5.14}$$

Also, we will frequently use that $|\widehat{\Pi}_\lambda(k)| \le M\beta$ uniformly in k.

Consider first f_1. Applying Proposition 5.2 to (1.4) gives $\chi(\lambda) = (1-\lambda\widehat{a}(\mathbf{0}))^{-1}$, and so

$$\lambda = \frac{1-\chi(\lambda)^{-1}}{1+\widehat{\Pi}_\lambda(\mathbf{0})} \le (1-\chi(\lambda)^{-1})(1+M\beta) \le 1+M\beta. \tag{5.15}$$

Using $1 \ge \chi(\lambda)^{-1} \searrow 0$ for $\lambda \nearrow \lambda_c$ implies the required bound and, moreover, $\lambda = 1+\mathcal{O}(\beta)$ for $\lambda \nearrow \lambda_c$. What we have used here, and will frequently use in this section, is that when we are confronted with an expression of the form $(1-\widehat{g}(k))^{-1}$, where $|\widehat{g}(k)| \le M\beta$, we can choose β small enough so that $M\beta < 1$ and there exists a constant $\tilde{M}$ such that

$$0 \le \frac{1}{1-\widehat{g}(k)} \le \frac{1}{1-M\beta} = \sum_{l\ge 0}(M\beta)^l \le 1+\tilde{M}\beta.$$

To deal with f_2, we use the "split" $f_2 = f_4 \vee f_5$, where we introduce $A_\rho := \{k \in \mathbb{R}^d : |\widehat{\varphi}(k)| \le \rho\}$ and set

$$f_4(\lambda) := \sup_{k\in A_\rho} \frac{|\widehat{\tau}_\lambda(k)|}{\widehat{G}_{\mu_\lambda}(k)}, \qquad f_5(\lambda) := \sup_{k\in A_\rho^c} \frac{|\widehat{\tau}_\lambda(k)|}{\widehat{G}_{\mu_\lambda}(k)}.$$

The precise value of ρ does not matter, but for practical purposes, we set it to be $\rho = 1/4$. Now, for $k \in A_\rho$, we see that

$$\frac{|\widehat{\tau}_\lambda(k)|}{\widehat{G}_{\mu_\lambda}(k)} = |\widehat{a}(k)| \cdot \frac{(1-\mu_\lambda\widehat{\varphi}(k))}{1-\lambda\widehat{a}(k)} \le (\rho+M\beta)\cdot\frac{1+\rho}{1-(1+M\beta)(\rho+M\beta)} \le 1+\tilde{M}\beta.$$

Hence, $f_4(\lambda) \le 1+M\beta$. We turn to f_5 and consequently to $k \in A_\rho^c$. We define

$$\widehat{N}(k) = \widehat{a}(k)/\widehat{a}(\mathbf{0}), \qquad \widehat{F}(k) = (1-\lambda\widehat{a}(k))/\widehat{a}(\mathbf{0}), \qquad \widehat{Q}(k) = (1+\widehat{\Pi}_\lambda(k))/\widehat{a}(\mathbf{0}),$$

so that $\widehat{\tau}_\lambda(k) = \widehat{N}(k)/\widehat{F}(k)$. Rearranging gets us to

$$\begin{aligned}\frac{\widehat{\tau}_\lambda(k)}{\widehat{\varphi}(k)\widehat{G}_{\mu_\lambda}(k)} &= \widehat{N}(k)\frac{1-\mu_\lambda\widehat{\varphi}(k)}{\widehat{\varphi}(k)\widehat{F}(k)} = \widehat{Q}(k) + \widehat{N}(k)\frac{1-\mu_\lambda\widehat{\varphi}(k) - \frac{\widehat{Q}(k)}{\widehat{N}(k)}\widehat{\varphi}(k)\widehat{F}(k)}{\widehat{\varphi}(k)\widehat{F}(k)} \\ &= \widehat{Q}(k) + \tfrac{\widehat{N}(k)}{\widehat{F}(k)}\widehat{\varphi}(k)^{-1}\left[1-\mu_\lambda\widehat{\varphi}(k) - \tfrac{\widehat{\varphi}(k)\widehat{Q}(k)}{\widehat{N}(k)}\widehat{F}(k)\right].\end{aligned} \tag{5.16}$$

The extracted term $\widehat{Q}(k)$ satisfies $|\widehat{Q}(k)| \le 1+M\beta$. We further observe that

$$\frac{\widehat{\varphi}(k)\widehat{Q}(k)}{\widehat{N}(k)} = \frac{\widehat{\varphi}(k)(1+\widehat{\Pi}_\lambda(k))}{\widehat{\varphi}(k)+\widehat{\Pi}_\lambda(k)} = 1 - \frac{[1-\widehat{\varphi}(k)]\widehat{\Pi}_\lambda(k)}{\widehat{\varphi}(k)+\widehat{\Pi}_\lambda(k)} =: 1-\widehat{b}(k).$$

Recalling identity (5.14) for μ_λ, we can rewrite the quantity $[1-\mu_\lambda\widehat{\varphi}(k) - (1-\widehat{b}(k))\widehat{F}(k)]$, appearing in (5.16), as

$$\begin{aligned}&\frac{1+\widehat{\Pi}_\lambda(\mathbf{0}) - \big[\lambda+\widehat{\Pi}_\lambda(\mathbf{0})+\lambda\widehat{\Pi}_\lambda(\mathbf{0})\big]\widehat{\varphi}(k) - 1 + \lambda(\widehat{\varphi}(k)+\widehat{\Pi}_\lambda(k)) + \widehat{b}(k)(1-\lambda\widehat{a}(k))}{1+\widehat{\Pi}_\lambda(\mathbf{0})} \\ &\qquad = \frac{[1-\widehat{\varphi}(k)]\left(\widehat{\Pi}_\lambda(\mathbf{0})+\lambda\widehat{\Pi}_\lambda(\mathbf{0})\right) + \lambda[\widehat{\Pi}_\lambda(k)-\widehat{\Pi}_\lambda(\mathbf{0})] + \widehat{b}(k)(1-\lambda\widehat{a}(k))}{1+\widehat{\Pi}_\lambda(\mathbf{0})}.\end{aligned}$$

Noting that $|\widehat{\Pi}_\lambda(\mathbf{0}) - \widehat{\Pi}_\lambda(k)| \le M[1-\widehat{\varphi}(k)]\beta$ by Corollary 5.3, the first three terms are bounded by $M[1-\widehat{\varphi}(k)]\beta$ for some constant M. Using (5.15), the last term is

$$\left|\frac{\widehat{b}(k)(1-\lambda\widehat{a}(k))}{1+\widehat{\Pi}_\lambda(\mathbf{0})}\right| = \left|\frac{1-\widehat{\varphi}(k)}{1+\widehat{\Pi}_\lambda(\mathbf{0})}\cdot\frac{\widehat{\Pi}_\lambda(k)}{\widehat{\varphi}(k)+\widehat{\Pi}_\lambda(k)} - \frac{\lambda[1-\widehat{\varphi}(k)]\widehat{\Pi}_\lambda(k)}{1+\widehat{\Pi}_\lambda(\mathbf{0})}\right|$$

$$
\begin{aligned}
&= [1-\widehat{\varphi}(k)]\frac{|\widehat{\Pi}_\lambda(k)|}{|1+\widehat{\Pi}_\lambda(\mathbf{0})|}\left|\frac{1}{\widehat{\varphi}(k)+\widehat{\Pi}_\lambda(k)}-\lambda\right| \\
&\leq [1-\widehat{\varphi}(k)](1+M\beta)|\widehat{\Pi}_\lambda(k)|\Big(2\rho^{-1}+\lambda(1+M\beta)\Big) \\
&\leq \tilde{M}[1-\widehat{\varphi}(k)]\beta.
\end{aligned}
$$

Again, we require β to be small; in this case we want that $|\widehat{\Pi}_\lambda(k)| \leq M\beta < \rho/2$. Putting these just acquired bounds back into (5.16), we can find constants $M, \tilde{M}$ such that

$$
\begin{aligned}
\left|\frac{\widehat{\tau}_\lambda(k)}{\widehat{\varphi}(k)\widehat{G}_{\mu_\lambda}(k)}\right| &\leq |\widehat{Q}(k)| + M\beta\left|\frac{\widehat{N}(k)}{\widehat{F}(k)}\right|\left|\frac{1-\widehat{\varphi}(k)}{\widehat{\varphi}(k)}\right| \\
&\leq 1+M\beta+2M\beta(1+\tilde{M}\beta)\frac{1}{|\widehat{\varphi}(k)|}\left|\frac{\widehat{\tau}_\lambda(k)}{\widehat{G}_{\mu_\lambda}(k)}\right| \leq 1+M\beta+2\cdot 3\cdot M\rho^{-1}\beta(1+\tilde{M}\beta)) \\
&\leq 1+\bar{M}\beta, \qquad (5.17)
\end{aligned}
$$

for some constant $\bar{M}$. Note that we have used the bound $[1-\widehat{\varphi}(k)] \leq 2[1-\mu_\lambda\widehat{\varphi}(k)]$ to get from $\widehat{G}_1(k)^{-1}$ to $\widehat{G}_{\mu_\lambda}(k)^{-1}$. As $|\widehat{\tau}_\lambda(k)|/\widehat{G}_{\mu_\lambda}(k) \leq |\widehat{\tau}_\lambda(k)/(\widehat{\varphi}(k)\widehat{G}_{\mu_\lambda}(k))|$, this concludes the improvement of f_5.

Before we treat f_3, we introduce f_6 given by

$$
f_6(\lambda) := \sup_{k\in\mathbb{R}^d}\frac{1-\mu_\lambda\widehat{\varphi}(k)}{|1-\lambda(\widehat{\varphi}(k)+\widehat{\Pi}_\lambda(k))|}.
$$

We show that $f(\lambda)\leq 3$ implies $f_6(\lambda)\leq 2$. To do so, consider first $k\in A_\rho$ and choose β small enough so that $1-\lambda(\rho+|\widehat{\Pi}_\lambda(k)|)>0$ (note that $\lambda\vee(1+|\widehat{\Pi}_\lambda(k)|)\leq 1+M\beta$). We then have

$$
\frac{1-\mu_\lambda\widehat{\varphi}(k)}{1-\lambda\widehat{a}(k)} \leq \frac{1+\rho}{1-(1+M\beta)(\rho+M\beta)} \leq 2, \qquad (5.18)
$$

for $\rho = 1/4$ and β sufficiently small. Now, when $k\in A_\rho^c$, we have

$$
\frac{1-\mu_\lambda\widehat{\varphi}(k)}{1-\lambda\widehat{a}(k)} = \left|\frac{\widehat{\tau}_\lambda(k)}{\widehat{\varphi}(k)\widehat{G}_{\mu_\lambda}(k)}\right|\cdot\left|\frac{\widehat{\varphi}(k)}{\widehat{a}(k)}\right| \leq (1+M\beta)\left|1-\frac{\widehat{\Pi}_\lambda(k)}{\widehat{a}(k)}\right| \leq (1+M\beta)\left(1+\frac{M\beta}{\rho-M\beta}\right), \qquad (5.19)
$$

which is bounded by $1+\tilde{M}\beta$. Note that for the first bound in (5.19), we used the estimate established in (5.17), which is stronger than a bound on f_5. In conclusion, (5.19) together with (5.18) shows $f_6(\lambda)\leq 2$. We are now equipped to improve the bound on f_3. As a first step, elementary calculations give the identity

$$
\Delta_k\widehat{\tau}_\lambda(l) = \underbrace{\frac{\Delta_k\widehat{a}(l)}{1-\lambda\widehat{a}(l)}}_{\text{(I)}} + \sum_{\sigma=\pm1}\underbrace{\frac{\lambda\big(\widehat{a}(l+\sigma k)-\widehat{a}(l)\big)^2}{\big(1-\lambda\widehat{a}(l)\big)\big(1-\lambda\widehat{a}(l+\sigma k)\big)}}_{\text{(II)}} + \underbrace{\widehat{a}(l)\Delta_k\left(\frac{1}{1-\lambda\widehat{a}(l)}\right)}_{\text{(III)}}.
$$

We bound each of the three terms separately. We note that by Lemma 5.1 and Corollary 5.3, we have $|\Delta_k\widehat{\Pi}_\lambda(l)| \leq |\widehat{\Pi}_\lambda(\mathbf{0})-\widehat{\Pi}_\lambda(k)| \leq [1-\widehat{\varphi}(k)]M\beta$. With this in mind,

$$
\begin{aligned}
|\text{(I)}| = |\Delta_k\widehat{a}(l)|\cdot\left|\frac{1-\mu_\lambda\widehat{\varphi}(l)}{1-\lambda\widehat{a}(l)}\right|\cdot\widehat{G}_{\mu_\lambda}(l) &\leq 2\widehat{G}_{\mu_\lambda}(l)\Big|\Delta_k\widehat{\varphi}(l)+\Delta_k\widehat{\Pi}_\lambda(l)\Big| \\
&\leq 2\widehat{G}_{\mu_\lambda}(l)\Big|1-\widehat{\varphi}(k)+[1-\widehat{\varphi}(k)]M\beta\Big| = 2(1+M\beta)[1-\widehat{\varphi}(k)]\widehat{G}_{\mu_\lambda}(l) \\
&\leq 4(1+\tilde{M}\beta)[1-\widehat{\varphi}(k)]\widehat{G}_{\mu_\lambda}(l)\widehat{G}_{\mu_\lambda}(l+k),
\end{aligned}
$$

where we have used the improved bound on f_6 and $2\widehat{G}_{\mu_\lambda}(l+k)\geq 1$. This type of bound will be sufficient for our purposes, and we will aim for similar bounds on (II) and (III). Recalling that $\partial_k^{\pm}g(l) = g(l\pm k) -$

$g(l)$, we are interested in $\partial_k^{\pm}\widehat{\varphi}(l)$ and $\partial_k^{\pm}\widehat{\Pi}_\lambda(l)$ to deal with (II). Note that for g with $g(x) = g(-x)$, we have

$$\begin{aligned} |\partial_k^{\pm}\widehat{g}(l)| &= \left|\int \mathrm{e}^{\mathrm{i}l\cdot x}(\mathrm{e}^{\pm \mathrm{i}k\cdot x} - 1)g(x)\,\mathrm{d}x\right| \leq \int \left|\mathrm{e}^{\pm\mathrm{i}k\cdot x} - 1\right| \cdot |g(x)|\,\mathrm{d}x \\ &\leq \int \Big([1-\cos(k\cdot x)] + |\sin(k\cdot x)|\Big)|g(x)|\,\mathrm{d}x. \end{aligned} \tag{5.20}$$

Now, with the help of the Cauchy-Schwarz inequality and (5.20),

$$\begin{aligned} |\partial_k^{\pm}\widehat{\varphi}(l)| &\leq \left(\int \varphi(x)\,\mathrm{d}x\right)^{1/2}\left(\int \sin(k\cdot x)^2\varphi(x)\,\mathrm{d}x\right)^{1/2} + \int [1-\cos(k\cdot x)]\varphi(x)\,\mathrm{d}x \\ &= 1\cdot\left(\int [1-\cos(k\cdot x)^2]\varphi(x)\,\mathrm{d}x\right)^{1/2} + [1-\widehat{\varphi}(k)] \\ &\leq 2\left(\int [1-\cos(k\cdot x)]\varphi(x)\,\mathrm{d}x\right)^{1/2} + [1-\widehat{\varphi}(k)] \\ &= 2[1-\widehat{\varphi}(k)]^{1/2} + [1-\widehat{\varphi}(k)] \leq 4[1-\widehat{\varphi}(k)]^{1/2}. \end{aligned}$$

Similarly,

$$\begin{aligned} |\partial_k^{\pm}\widehat{\Pi}_\lambda(l)| &\leq \left(\int |\Pi_\lambda(x)|\,\mathrm{d}x\right)^{1/2}\left(2\int [1-\cos(k\cdot x)]|\Pi_\lambda(x)|\,\mathrm{d}x\right)^{1/2} + \int [1-\cos(k\cdot x)]|\Pi_\lambda(x)|\,\mathrm{d}x \\ &\leq (M\beta)^{1/2}(2M[1-\widehat{\varphi}(k)]\beta)^{1/2} + M[1-\widehat{\varphi}(k)]\beta \\ &\leq \tilde{M}[1-\widehat{\varphi}(k)]^{1/2}\beta. \end{aligned}$$

We deal with the denominator in (II) by noting that, for $\sigma \in \{-1,0,1\}$,

$$\frac{1}{1-\lambda\widehat{a}(l+\sigma k)} = \frac{1-\mu_\lambda\widehat{\varphi}(l+\sigma k)}{1-\lambda\widehat{a}(l+\sigma k)}\widehat{G}_{\mu_\lambda}(l+\sigma k) \leq 2\widehat{G}_{\mu_\lambda}(l+\sigma k), \tag{5.21}$$

employing the improved bound on f_6 again. In summary,

$$\begin{aligned} \text{(II)} &\leq (1+M\beta)\Big((4+M\beta)[1-\widehat{\varphi}(k)]^{1/2}\Big)^2 4\widehat{G}_{\mu_\lambda}(l)\widehat{G}_{\mu_\lambda}(l\pm k) \\ &\leq 64(1+\tilde{M}\beta)[1-\widehat{\varphi}(k)]\widehat{G}_{\mu_\lambda}(l)\widehat{G}_{\mu_\lambda}(l\pm k). \end{aligned}$$

Turning to (III), we note that $|\widehat{a}(l)| \leq 1+M\beta$. We treat the second factor with Lemma 5.1, and so we recall that we can recycle the bound observed in (5.21) to get $(1-\lambda\widehat{a}(l))^{-1} \leq (1+M\beta)\widehat{G}_{\mu_\lambda}(l)$. We furthermore obtain the bound

$$\begin{aligned} \lambda\big(\widehat{|a|}(\mathbf{0}) - \widehat{|a|}(k)\big) &= \lambda\int [1-\cos(k\cdot x)]|\varphi(x)+\Pi_\lambda(x)|\,\mathrm{d}x \\ &\leq (1+M\beta)[1-\widehat{\varphi}(k)]. \end{aligned}$$

Substituting this into Lemma 5.1, we obtain

$$\begin{aligned} \Delta_k\frac{1}{1-\lambda\widehat{a}(l)} &\leq (1+M\beta)^3\Big(\widehat{G}_{\mu_\lambda}(l-k) + \widehat{G}_{\mu_\lambda}(l+k)\Big)\widehat{G}_{\mu_\lambda}(l)[1-\widehat{\varphi}(k)] \\ &\quad + 8(1+M\beta)^5\widehat{G}_{\mu_\lambda}(l-k)\widehat{G}_{\mu_\lambda}(l)\widehat{G}_{\mu_\lambda}(l+k)[1-\widehat{\varphi}(l)]\cdot[1-\widehat{\varphi}(k)] \\ &\leq 16(1+M\beta)[1-\widehat{\varphi}(k)]\times\Big(\widehat{G}_{\mu_\lambda}(l-k)\widehat{G}_{\mu_\lambda}(l) + \widehat{G}_{\mu_\lambda}(l)\widehat{G}_{\mu_\lambda}(l+k) + \widehat{G}_{\mu_\lambda}(l-k)\widehat{G}_{\mu_\lambda}(l+k)\Big). \end{aligned}$$

Putting everything together, we are done, as $|\Delta_k\widehat{\tau}_\lambda(l)| \leq (1+M\beta)\widehat{U}_{\mu_\lambda}(k,l)$. This finishes the proof of Proposition 5.9. □

6 Proof of the main theorems

Proposition 5.2 and Corollary 5.3 give rise to a corollary, extending the Ornstein-Zernike equation to λ_c:

Corollary 6.1 (The OZE at the critical point)**.** *The Ornstein-Zernike equation* (5.6) *extends to* λ_c*. In particular, the limit* $\Pi_{\lambda_c}^{(n)} = \lim_{\lambda \nearrow \lambda_c} \Pi_\lambda^{(n)}$ *exists for every* $n \in \mathbb{N}_0$*, and so does* $\Pi_{\lambda_c} = \sum_{n \geq 0} (-1)^n \Pi_{\lambda_c}^{(n)}$*.*

Proof. We can define $\Pi_{\lambda_c}^{(n)}$ for $n \in \mathbb{N}_0$ by extending the definitions in (3.12) and (3.13) to λ_c. We claim that $\Pi_\lambda^{(n)} \to \Pi_{\lambda_c}^{(n)}$ for every $n \in \mathbb{N}_0$ as $\lambda \nearrow \lambda_c$.

In order to prove this, let $(\lambda_m)_{m\in\mathbb{N}}$ be in increasing sequence with $\lambda_m \nearrow \lambda_c$. We write (3.12) and (3.13) as $\Pi_\lambda^{(n)}(x) = \lambda^n \int \mathbb{P}_\lambda(A^{(n)}) \,\mathrm{d}\vec{u}$. Define

$$h_\lambda^{(n)}(x, \vec{u}) := \lambda^n \int \psi_0(w_0, u_0) \Big(\prod_{i=1}^{n-1} \psi(\vec{v}_i) \Big) \psi_n(w_{n-1}, u_{n-1}, t_n, z_n, x) \,\mathrm{d}\big(\vec{w}_{[0,n-1]}, (\vec{t}, \vec{z})_{[1,n]}\big)$$

with $\vec{v}_i = (w_{i-1}, u_{i-1}, t_i, w_i, z_i, u_i)$. By Proposition 4.7, we have $\Pi_\lambda^{(n)}(x) \leq \int h_\lambda^{(n)}(x, \vec{u}) \,\mathrm{d}\vec{u}$. Moreover, Corollary 4.16 and Corollary 5.2 show that

$$\int h_\lambda^{(n)}(x, \vec{u}) \,\mathrm{d}\vec{u} \leq C (C\beta)^{n-1} \tag{6.1}$$

for all $\lambda < \lambda_c$, and C is independent of λ. As $\lambda \mapsto h_\lambda$ is increasing, h_{λ_c} is also integrable. A close inspection of the proof of Proposition 4.7 shows that we may follow the steps in the proof with x and $\vec{u}$ as fixed arguments (i.e., we do not integrate over the points $\vec{u}$) to get $\lambda^n \mathbb{P}_\lambda(A^{(n)}) \leq h_{\lambda_c}$ for all $\lambda \leq \lambda_c$. Hence, by dominated convergence, it suffices to show $|\mathbb{P}_{\lambda_c}(A^{(n)}) - \mathbb{P}_{\lambda_m}(A^{(n)})| \to 0$ as $m \to \infty$.

Recall that the event $A^{(n)}$ takes place on $n+1$ independent RCMs. For $0 \leq i \leq n$, let $(\eta_{i,m})_{m\in\mathbb{N}}$ and $(\tilde{\eta}_{i,m})_{m\in\mathbb{N}}$ be sequences of PPPs of intensities $\lambda_m, \tilde{\lambda}_m := \lambda_c - \lambda_m$; and for fixed m, let those $2n+2$ PPPs be independent. Hence, the superposition of the two PPPs $\eta_{i,m}$ and $\tilde{\eta}_{i,m}$ forms a PPP of intensity λ_c. We can moreover couple the marks determining edge connections, so that $\tilde{\xi}'_{i,m} = \xi_i^{u_{i-1},u_i}(\eta'_{i,m})$ (where $\eta'_{i,m} = \eta_{i,m}^{u_{i-1},u_i}$) forms an RCM of intensity λ_m and $\xi'_{i,m} = \xi_i^{u_{i-1},u_i}(\eta'_{i,m} + \tilde{\eta}_{i,m})$ forms an RCM of intensity λ_c with the further property that $\tilde{\xi}'_{i,m} = \xi'_{i,m}[\eta'_{i,m}]$ (recall that $\xi(\eta)$ is the RCM with underlying PPP η). We now write

$$\begin{aligned} \big|\mathbb{P}_{\lambda_c}(A^{(n)}) - \mathbb{P}_{\lambda_m}(A^{(n)})\big| = \Big|\mathbb{E}\Big[&\mathbb{1}_{\{\mathbf{0} \Longleftrightarrow u_0 \text{ in } \xi'_{0,m}\}} \prod_{i=1}^n \mathbb{1}_{E(u_{i-1}, u_i; \xi'_{i,m})} \\ &- \mathbb{1}_{\{\mathbf{0} \Longleftrightarrow u_0 \text{ in } \tilde{\xi}'_{0,m}\}} \prod_{i=1}^n \mathbb{1}_{E(u_{i-1}, u_i; \tilde{\xi}'_{i,m})}\Big]\Big|, \end{aligned} \tag{6.2}$$

where $\mathbb{E}$ denotes the joint probability measure. In order to bound (6.2), we define $\mathscr{C}_{i,m} = \mathscr{C}(u_{i-1}, \xi_i^{u_{i-1}}(\eta_{i,m}^{u_{i-1}} + \tilde{\eta}_{i,m}))$ as well as $\tilde{\mathscr{C}}_{i,m} = \mathscr{C}(u_{i-1}, \xi_i^{u_{i-1}}(\eta_{i,m}^{u_{i-1}}))$ for $0 \leq i \leq n$ and claim that

$$\mathbb{P}(\mathscr{C}_{i,m} \neq \tilde{\mathscr{C}}_{i,m}) \xrightarrow{m\to\infty} 0. \tag{6.3}$$

Note that $\theta(\lambda_c) = 0$ is proven in Theorem 1.3 without using Corollary 6.1, so we may use that $\mathscr{C}_{i,m}$ is finite almost surely. Next, let $(\tilde{\Lambda}_m)_{m\in\mathbb{N}}$ be an increasing sequence of subsets of $\mathbb{R}^d$ exhausting $\mathbb{R}^d$ and satisfying $|\tilde{\Lambda}_m| \tilde{\lambda}_m \to 0$ as $m \to \infty$. Then

$$\begin{aligned} \mathbb{P}(\mathscr{C}_{i,m} \neq \tilde{\mathscr{C}}_{i,m}) &\leq \mathbb{P}\big(\tilde{\eta}_{i,m} \cap \tilde{\Lambda}_m \neq \varnothing\big) + \mathbb{P}_{\lambda_c}(\mathbf{0} \longleftrightarrow \tilde{\Lambda}_m^c \text{ in } \xi_{i,m}^{\mathbf{0}}) \\ &= 1 - \mathrm{e}^{-|\tilde{\Lambda}_m| \tilde{\lambda}_m} + \mathbb{P}_{\lambda_c}(\mathbf{0} \longleftrightarrow \tilde{\Lambda}_m^c \text{ in } \xi_{i,m}^{\mathbf{0}}) \xrightarrow{m\to\infty} 0, \end{aligned}$$

proving (6.3). We now proceed to bound (6.2) by observing that conditional on $\mathscr{C}_{i,m} = \tilde{\mathscr{C}}_{i,m}$ for all $0 \leq i \leq n$, the event $A^{(n)}$ occurs in $(\xi'_{i,m})_{i=0}^n$ if and only if it occurs in $(\tilde{\xi}'_{i,m})_{i=0}^n$. Consequently,

$$\big|\mathbb{P}_{\lambda_c}(A^{(n)}) - \mathbb{P}_{\lambda_m}(A^{(n)})\big| \leq \mathbb{P}\big(\exists\, i \in \{0, \dots, n\} : \mathscr{C}_{i,m} \neq \tilde{\mathscr{C}}_{i,m}\big) \leq (n+1)\mathbb{P}(\mathscr{C}_{0,m} \neq \tilde{\mathscr{C}}_{0,m}) \xrightarrow{m\to\infty} 0.$$

This proves that $|\mathbb{P}_{\lambda_c}(A^{(n)}) - \mathbb{P}_{\lambda_m}(A^{(n)})| \to 0$ as $m \to \infty$.

The functions $\int h_\lambda(n)(\cdot, \vec{u})\,\mathrm{d}\vec{u}$ serve as dominating functions for $\Pi_\lambda^{(n)}(\cdot)$, and they are summable by (6.1) for sufficiently small $C\beta$. Hence, $\sum_{n\geq 0}(-1)^n \Pi_{\lambda_c}^{(n)} =: \Pi_{\lambda_c}$ converges absolutely and satisfies $\Pi_{\lambda_c} = \lim_{\lambda\nearrow\lambda_c}\Pi_\lambda$ by dominated convergence.

Moreover, Π_{λ_c} is integrable by the uniform bounds in (5.3) of Corollary 5.3. Hence, we can take the limit $\lambda \nearrow \lambda_c$ in (5.6), extending it to λ_c with Π_{λ_c} as defined above. □

Before proving Theorem 1.2, let us note that when applying Proposition 5.2 in the model where q_φ is not normalized to be 1, the only difference in the statement of results is that

$$\widehat{\Pi}_\lambda(\mathbf{0}) - \widehat{\Pi}_\lambda(k) \leq c[\widehat{\varphi}(\mathbf{0}) - \widehat{\varphi}(k)]\beta. \tag{6.4}$$

Proof of Theorem 1.2. Let $\lambda < \lambda_c$ first. We reuse the notation $a = \varphi + \Pi_\lambda$. In Fourier space, Proposition 5.2 gives

$$\widehat{\tau}_\lambda(k)\big(1 - \lambda(\widehat{\varphi}(k) + \widehat{\Pi}_\lambda(k))\big) = \widehat{\varphi}(k) + \widehat{\Pi}_\lambda(k). \tag{6.5}$$

We claim that $1 - \lambda(\widehat{\varphi}(k) + \widehat{\Pi}_\lambda(k)) > 0$ for all k. To this end, assume first that there exists $k \in \mathbb{R}^d$ with $\widehat{\varphi}(k) + \widehat{\Pi}_\lambda(k) = \lambda^{-1}$. As

$$|\widehat{\tau}_\lambda(k)| \leq \widehat{\tau}_\lambda(\mathbf{0}) < \infty,$$

the left-hand side of (6.5) vanishes, and so also the right-hand side must satisfy $\widehat{a}(k) = 0$, which directly contradicts the assumption $\widehat{a}(k) = \lambda^{-1}$.

But for $k = \mathbf{0}$, the right-hand side of (6.5) is positive for sufficiently small β (it is at least $q_\varphi - C\beta$); since also $\widehat{\tau}_\lambda(\mathbf{0}) > 0$, we must have $1 - \lambda(\widehat{\varphi}(\mathbf{0}) + \widehat{\Pi}_\lambda(\mathbf{0})) > 0$. From this, the continuity of the Fourier transform implies that there cannot be a k with $1 - \lambda(\widehat{\varphi}(k) + \widehat{\Pi}_\lambda(k)) < 0$, proving the claim.

We now divide by $(1 - \lambda\widehat{a}(k))$ in (6.5) to obtain

$$\begin{aligned}\lambda|\widehat{\tau}_\lambda(k)| = \frac{\lambda|\widehat{a}(k)|}{1 - \lambda\widehat{a}(k)} &= \frac{\lambda|\widehat{\varphi}(k) + \widehat{\Pi}_\lambda(k)|}{\underbrace{1 - \lambda(\widehat{\varphi}(\mathbf{0}) + \widehat{\Pi}_\lambda(\mathbf{0}))}_{>0} + \lambda[\widehat{\varphi}(\mathbf{0}) - \widehat{\varphi}(k)] + \lambda[\widehat{\Pi}_\lambda(\mathbf{0}) - \widehat{\Pi}_\lambda(k)]} \\ &\leq \frac{|\widehat{\varphi}(k) + \widehat{\Pi}_\lambda(k)|}{[\widehat{\varphi}(\mathbf{0}) - \widehat{\varphi}(k)] - |\widehat{\Pi}_\lambda(\mathbf{0}) - \widehat{\Pi}_\lambda(k)|} \leq \frac{|\widehat{\varphi}(k)| + \mathcal{O}(\beta)}{[\widehat{\varphi}(\mathbf{0}) - \widehat{\varphi}(k)](1 + \mathcal{O}(\beta))} = \frac{|\widehat{\varphi}(k)| + \mathcal{O}(\beta)}{[\widehat{\varphi}(\mathbf{0}) - \widehat{\varphi}(k)]},\end{aligned} \tag{6.6}$$

using the bound (5.3) for $\widehat{\Pi}_\lambda(\mathbf{0}) - \widehat{\Pi}_\lambda(k)$. This proves the infra-red bound for $\lambda < \lambda_c$.

Let now $\lambda = \lambda_c$ and $k \neq \mathbf{0}$. Note that $\widehat{a}(\mathbf{0}) = \lambda_c^{-1}$. For contradiction, assume that $\widehat{a}(k) = \lambda_c^{-1}$ as well. We can write

$$\begin{aligned}0 = 1 - \lambda_c(\varphi(k) + \widehat{\Pi}_{\lambda_c}(k)) &= \underbrace{1 - \lambda_c(\widehat{\varphi}(\mathbf{0}) + \widehat{\Pi}_{\lambda_c}(\mathbf{0}))}_{=0} + \lambda_c[\widehat{\varphi}(\mathbf{0}) - \widehat{\varphi}(k)] + \lambda_c[\widehat{\Pi}_{\lambda_c}(\mathbf{0}) - \widehat{\Pi}_{\lambda_c}(k)] \\ &= \lambda_c[\widehat{\varphi}(\mathbf{0}) - \widehat{\varphi}(k)](1 + \mathcal{O}(\beta)),\end{aligned}$$

using the bound (5.3) for $\widehat{\Pi}_{\lambda_c}(\mathbf{0}) - \widehat{\Pi}_{\lambda_c}(k)$. But as $\widehat{\varphi}(\mathbf{0}) - \widehat{\varphi}(k) > 0$ for $k \neq \mathbf{0}$, this yields a contradiction.

With Corollary 6.1, we get identity (6.5) for λ_c, and by the above argument, we can again divide by $(1 - \lambda_c\widehat{a}(k))$ and obtain the same bound as in (6.6).

The bound $\triangle_\lambda \leq C\beta$ is obtained in Lemma 5.5. The uniformity in λ together with monotone convergence implies the triangle condition. □

Proof of Theorem 1.3. Mind that the asymptotic behavior of λ_c was noted in (5.15) and the line below. Identity (1.12) follows from the lace-expansion identity (6.5) for $k = \mathbf{0}$, keeping in mind that $\widehat{\tau}_\lambda(\mathbf{0})$ diverges for $\lambda \nearrow \lambda_c$ (this was already used in the proof for Theorem 1.2).

For continuity of θ, assume that both $\mathbf{0}$ and x are in the (a.s. unique) infinite component. This implies $\mathbf{0} \longleftrightarrow x$, and so $0 \leq \theta(\lambda_c)^2 \leq \tau_{\lambda_c}(x)$ for all $x \in \mathbb{R}^d$ via the FKG inequality (2.20). But $\tau_{\lambda_c}(x) \to 0$ for almost all $|x| \to \infty$ due to the triangle condition (a little extra effort shows that this holds for *all* x), which implies that $\theta(\lambda_c) = 0$.

To prove $\gamma \geq 1$, we rely on the work done in Lemmas 2.2 and 2.3. We start by proving $\widehat{\tau}_{\lambda_c}(\mathbf{0}) = \infty$. Recall that $\tau_\lambda^n(x,y) = \mathbb{P}_\lambda(x \longleftrightarrow y \text{ in } \xi_{\Lambda_n}^{x,y})$ and define

$$\chi^n(\lambda) := \sup_{x \in \Lambda_n} \int \tau_\lambda^n(x,y)\,\mathrm{d}y \qquad \text{for } \lambda \geq 0.$$

Mind that this is *not* the expected size of the largest cluster in Λ_n. We claim that $1/\chi^n(\lambda)$ is an equicontinuous sequence with $\chi^n(\lambda) \to \widehat{\tau}_\lambda(\mathbf{0})$ for every $\lambda \geq 0$. From this, we get continuity of $1/\widehat{\tau}_\lambda(\mathbf{0})$.

As $\tau_\lambda^n(x,y) \nearrow \tau_\lambda(x-y)$, we get $\chi^n(\lambda) \nearrow \widehat{\tau}_\lambda(\mathbf{0})$ by monotone convergence and thus $1/\chi^n(\lambda) \to 1/\widehat{\tau}_\lambda(\mathbf{0})$. For the equicontinuity, we show that $1/\chi^n(\lambda)$ has uniformly bounded derivative. First, note that the same arguments used in Lemma 2.3 show that, uniformly in x,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \int \tau_\lambda^n(x,y)\,\mathrm{d}y \leq \chi^n(\lambda)^2. \tag{6.7}$$

We want to relate (6.7) to $\frac{\mathrm{d}}{\mathrm{d}\lambda}\chi^n(\lambda)$. Given λ and ε, let $v_{\lambda,\varepsilon} \in \Lambda_n$ be a point such that

$$\int \tau_\lambda^n(v_{\lambda,\varepsilon},y)\,\mathrm{d}y \geq \chi^n(\lambda) - \varepsilon^2.$$

The exact choice of $v_{\lambda,\varepsilon}$ (as it is not unique) does not matter. This gives

$$\begin{aligned}\chi^n(\lambda+\varepsilon) - \chi^n(\lambda) &\leq \int \big(\tau_{\lambda+\varepsilon}^n(v_{\lambda+\varepsilon,\varepsilon},y) - \tau_\lambda^n(v_{\lambda+\varepsilon,\varepsilon},y)\big)\,\mathrm{d}y + \varepsilon^2 \\ &\leq \sup_{v \in \Lambda_n} \int \big(\tau_{\lambda+\varepsilon}^n(v,y) - \tau_\lambda^n(v,y)\big)\,\mathrm{d}y + \varepsilon^2. \end{aligned} \tag{6.8}$$

Similarly to Lemma 2.2, we can show that $\lambda \mapsto \chi^n(\lambda)$ is continuous and almost everywhere differentiable. We divide (6.8) by ε and let $\varepsilon \to 0$. We claim that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\chi^n(\lambda) \leq \sup_{v\in\Lambda_n} \int \frac{\mathrm{d}}{\mathrm{d}\lambda}\tau_\lambda^n(v,y)\,\mathrm{d}y \leq \chi^n(\lambda)^2. \tag{6.9}$$

The exchange of limit and supremum is justified since the integral in (6.8) converges to $\frac{\mathrm{d}}{\mathrm{d}\lambda}\int \tau_\lambda^n(v,y)\,\mathrm{d}y$ uniformly in v, and then (6.9) follows by employing the Leibniz integral rule and (6.7) for the second bound. Rearranging, this yields

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{1}{\chi^n(\lambda)} \geq -1.$$

In summary, the functions $\lambda \mapsto 1/\chi^n(\lambda)$ form a non-increasing family with uniformly bounded derivative, which gives us equicontinuity. Since also $1/\chi^n(\lambda) \to 1/\widehat{\tau}_\lambda(\mathbf{0})$ pointwise, this limit is also continuous and, since it attains zero for every $\lambda > \lambda_c$, we have $1/\widehat{\tau}_{\lambda_c}(\mathbf{0}) = 0$.

We can now integrate the inequality $\frac{\mathrm{d}}{\mathrm{d}\lambda}\widehat{\tau}_\lambda(\mathbf{0})^{-1} \geq -1$ between λ and λ_c, so that

$$\frac{1}{\widehat{\tau}_{\lambda_c}(\mathbf{0})} - \frac{1}{\widehat{\tau}_\lambda(\mathbf{0})} = -\frac{1}{\widehat{\tau}_\lambda(\mathbf{0})} \geq -(\lambda_c - \lambda). \tag{6.10}$$

Hence, $\widehat{\tau}_\lambda(\mathbf{0}) \geq (\lambda_c-\lambda)^{-1}$ and, with (1.4), we obtain $\chi(\lambda) \geq \lambda(\lambda_c - \lambda)^{-1}$. This shows $\gamma \geq 1$.

To prove $\gamma \leq 1$, let $\lambda \in (0,\lambda_c)$. We have to repeat some of the calculations from the diagrammatic bounds. Note that

$$\mathbb{P}_\lambda(u \in \mathsf{Piv}(\mathbf{0},x;\xi^{\mathbf{0},x})) = \mathbb{E}_\lambda\big[\mathbb{1}_{\{\mathbf{0}\longleftrightarrow u \text{ in } \xi^{\mathbf{0},u}\}}\tau_\lambda^{\mathscr{C}(\mathbf{0},\xi^{\mathbf{0}})}(u,x)\big]$$

follows in the same manner from Lemma 3.3 as Lemma 3.6. Applying Lemma 2.2 and then (3.3) gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}\widehat{\tau}_\lambda(\mathbf{0}) &= \int \frac{\mathrm{d}}{\mathrm{d}\lambda}\tau_\lambda(x)\,\mathrm{d}x = \iint \mathbb{P}_\lambda\big(u \in \mathsf{Piv}(\mathbf{0},x;\xi^{\mathbf{0},u,x})\big)\,\mathrm{d}u\,\mathrm{d}x \\ &= \iint \mathbb{E}_\lambda[\mathbb{1}_{\{\mathbf{0}\longleftrightarrow u \text{ in } \xi^{\mathbf{0},u}\}}\tau_\lambda(x-u)]\,\mathrm{d}u\,\mathrm{d}x\end{aligned}$$

$$- \iint \mathbb{E}_\lambda \left[\mathbb{1}_{\{\mathbf{0} \longleftrightarrow u \text{ in } \xi_0^{\mathbf{0},u}\}} \mathbb{1}_{\{u \xleftrightarrow{\mathscr{C}(\mathbf{0},\xi_0^{\mathbf{0}})} x \text{ in } \xi_1^{u,x}\}} \right] \mathrm{d}u \, \mathrm{d}x. \tag{6.11}$$

The first integral on the right-hand side of (6.11) is $\widehat{\tau}_\lambda(\mathbf{0})^2$. For the second integral, we recall the '$\leftrightsquigarrow$' notation from Definition 4.8. With this, we can bound the second integrand on the r.h.s. of (6.11) by

$$\begin{aligned}
&\mathbb{E}_\lambda \Big[\mathbb{1}_{\{\mathbf{0} \longleftrightarrow u \text{ in } \xi_0^{\mathbf{0},u}\}} \sum_{y \in \eta_1^x} \mathbb{1}_{\{\mathbf{0} \leftrightsquigarrow y \text{ in } (\xi_0^{\mathbf{0}}, \xi_1^x)\}} \mathbb{1}_{\{u \longleftrightarrow y \text{ in } \xi_1^u\} \circ \{y \longleftrightarrow x \text{ in } \xi_1^x\}} \Big] \\
&\quad = \mathbb{E}_\lambda \big[\mathbb{1}_{\{\mathbf{0} \longleftrightarrow u \text{ in } \xi_0^{\mathbf{0},u}\}} \mathbb{1}_{\{\mathbf{0} \leftrightsquigarrow x \text{ in } (\xi_0^{\mathbf{0}}, \xi_1^x)\}} \big] \tau_\lambda(x-u) \\
&\qquad + \lambda \int \mathbb{E}_\lambda \big[\mathbb{1}_{\{\mathbf{0} \longleftrightarrow u \text{ in } \xi_0^{\mathbf{0},u}\}} \mathbb{1}_{\{\mathbf{0} \leftrightsquigarrow y \text{ in } (\xi_0^{\mathbf{0}}, \xi_1^{x,y})\}} \big] \mathbb{P}_\lambda \big(\{u \longleftrightarrow y \text{ in } \xi^{u,y}\} \circ \{y \longleftrightarrow x \text{ in } \xi^{y,x}\} \big) \, \mathrm{d}y \\
&\quad \leq \int \mathbb{E}_\lambda \big[\mathbb{1}_{\{\mathbf{0} \longleftrightarrow u \text{ in } \xi_0^{\mathbf{0},u}\}} \mathbb{1}_{\{\mathbf{0} \leftrightsquigarrow y \text{ in } (\xi_0^{\mathbf{0}}, \xi_1^y)\}} \big] \tau_\lambda(y-u) \tau_\lambda^\circ(x-y) \, \mathrm{d}y.
\end{aligned} \tag{6.12}$$

Note that

$$\mathbb{1}_{\{\mathbf{0} \longleftrightarrow u \text{ in } \xi_0^{\mathbf{0},u}\}} \mathbb{1}_{\{\mathbf{0} \leftrightsquigarrow y \text{ in } (\xi_0^{\mathbf{0}}, \xi_1^y)\}} \leq \sum_{a \in \eta_0^{\mathbf{0}}} \mathbb{1}_{\bigcirc_3^{\leftrightsquigarrow}((\mathbf{0},a),(a,u),(a,y);(\xi_0^{\mathbf{0},u}, \xi_1^y))}, \tag{6.13}$$

where we recall $\bigcirc_3^{\leftrightsquigarrow}$ from Definition 4.9. We now plug (6.12) back into (6.11) and apply (6.13), with the intent to use Lemma 4.10. The second integral on the r.h.s. of (6.11) is hence bounded by

$$\begin{aligned}
&\int \Big(\delta_{a,\mathbf{0}} \mathbb{P}_\lambda \big(\bigcirc_2^{\leftrightsquigarrow} ((\mathbf{0},u),(\mathbf{0},y);(\xi_0^{\mathbf{0},u}, \xi_1^y)) \big) + \lambda \mathbb{P}_\lambda \big(\bigcirc_3^{\leftrightsquigarrow} ((\mathbf{0},a),(a,u),(a,y);(\xi_0^{\mathbf{0},a,u}, \xi_1^y)) \big) \Big) \\
&\qquad \qquad \times \tau_\lambda(y-u) \tau_\lambda^\circ(x-y) \, \mathrm{d}(a,u,x,y) \\
&\qquad \leq \int \tau_\lambda^\circ(a) \tau_\lambda(u-a) \tau_\lambda(y-a) \tau_\lambda(y-u) \tau_\lambda^\circ(x-y) \, \mathrm{d}(a,y,u,x) \\
&\qquad = \lambda^{-2} \triangle_\lambda(\mathbf{0}) \chi(\lambda)^2.
\end{aligned}$$

The above estimate is achieved by first applying Lemma 4.10 and then the BK inequality. In summary, $\frac{\mathrm{d}}{\mathrm{d}\lambda} \widehat{\tau}_\lambda(\mathbf{0}) \geq \widehat{\tau}_\lambda(\mathbf{0})^2 - \lambda^{-2} \triangle_\lambda \chi(\lambda)^2$. Rearranging yields

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{1}{\widehat{\tau}_\lambda(\mathbf{0})} \leq -1 + \triangle_\lambda \frac{\lambda^{-2} \chi(\lambda)^2}{\widehat{\tau}_\lambda(\mathbf{0})^2} \leq -1 + \triangle_\lambda (1 + \lambda^{-1} + \lambda^{-2}) \leq -1/2.$$

In this, the last inequality holds when the triangle $\triangle_\lambda$ is small enough (guaranteed by Theorem 1.2) and $\lambda > 0$. We hence get a lower bound counterpart to (6.9) and can integrate as in (6.10) to get $\gamma \leq 1$. $\square$

A Random walk properties

We first prove Proposition 1.1. For simplicity of presentation, we are going to assume throughout the appendix that $q_\varphi = 1$ (see Section 5.1).

Proof of Proposition 1.1 (b). By definition, (H2.1) and (H2.2) hold. It remains to prove (H2.3). Note that the constants b, c_1, c_2 do not depend on L, but on the function h and hence implicitly also on the dimension d. (The function h is fixed throughout the proof.)

Note first that $\widehat{\varphi}_L(k) = \widehat{h}(Lk)$. Hence, without loss of generality, we prove (H2.3) for $\varphi = h$ and $L = 1$.

For each $k \in \mathbb{R}^d$, the first and second partial derivatives of $\widehat{h}$ are given by

$$\frac{\partial}{\partial k_j} \widehat{h}(k) = \mathrm{i} \int x_j h(x) \mathrm{e}^{\mathrm{i}k \cdot x} \, \mathrm{d}x, \qquad \frac{\partial^2}{\partial k_j \partial k_l} \widehat{h}(k) = - \int x_j x_l h(x) \mathrm{e}^{\mathrm{i}k \cdot x} \, \mathrm{d}x.$$

Thus, we obtain from the multivariate Taylor theorem (applied to the origin) with the Peano form of the remainder term that

$$\widehat{h}(k) \leq 1 - \frac{1}{2} \sum_{j=1}^d k_j^2 \int x_j^2 h(x) \, \mathrm{d}x - \frac{1}{2} \sum_{j,l=1}^d k_j k_l \int x_j x_l h(x) \big(\mathrm{e}^{\mathrm{i}sk \cdot x} - 1 \big) \, \mathrm{d}x,$$

where $s = s(k) \in (0,1)$, and where we have used that $\int x_j h(x)\,\mathrm{d}x = 0$ and that $\int x_j x_l h(x)\,\mathrm{d}x \geq 0$ for all $j,l \in \{1,\dots,d\}$. By dominated convergence, $\int x_j x_l h(x)(\mathrm{e}^{\mathrm{i}k'\cdot x} - 1)\,\mathrm{d}x \to 0$ as $|k'| \to 0$ for all $j,l \in \{1,\dots,d\}$, so that the first inequality in (H2.3) holds for a suitable c_1 and all sufficiently small $b > 0$. The second inequality in (H2.3) then follows from the fact that $\widehat{h}$ is bounded away from 1 outside any compact neighborhood of the origin. (Otherwise $h(x) = 0$ for almost every $x \in \mathbb{R}^d$, a contradiction.) □

The next proofs will use the following classical fact on the Fourier transform of the indicator function of a ball; see e.g. [13, Section B.5]. Let $r > 0$ and g_r be the indicator function of the ball in $\mathbb{R}^d$ with radius r centered at the origin. Then

$$\widehat{g}_r(k) = \Big(\frac{2\pi r}{|k|}\Big)^{d/2} J_{d/2}(|k|r), \quad k \in \mathbb{R}^d, \tag{A.1}$$

where, for $a > -1/2$, the *Bessel function* $J_a \colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ is given by

$$J_a(x) := \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+a+1)} \Big(\frac{x}{2}\Big)^{2m+a}, \quad x \geq 0.$$

It is helpful to note here that $b_d := \pi^{d/2}\Gamma(d/2+1)^{-1}$ is the volume of a ball in $\mathbb{R}^d$ with radius 1 and $r_d := \pi^{-1/2}\Gamma(d/2+1)^{1/d}$ is the radius of the unit volume ball (we denote the latter by $\mathbb{B}^d$).

Proof of Proposition 1.1(c). Since we assume that $q_{\varphi_L} = 1$, we assume h and φ_L to be given as

$$h(x) = (|x| \vee r_d)^{-d-\alpha} \qquad \text{and} \qquad \varphi_L(x) = \frac{h(x/L)}{\int h(x/L)\,\mathrm{d}x}.$$

Note that we only need to consider $\alpha < 2$, as the other case is covered in the spread-out (finite-variance) model. We only consider the first bound in (H3.3), as the other properties follow similarly to (H2).

Given $L \geq 1$ and $k \in \mathbb{R}^d$, we again use that $\widehat{\varphi}_L(k) = \widehat{\varphi}_1(Lk) = c_h\widehat{h}(Lk)$, where

$$c_h^{-1} = \int h(x)\,\mathrm{d}x = \int_{|x|\leq r_d} (r_d)^{-d-\alpha}\,\mathrm{d}x + \int_{|x|>r_d} |x|^{-d-\alpha}\,\mathrm{d}x = (r_d)^{-d-\alpha} + \frac{db_d}{\alpha(r_d)^\alpha},$$

and where we have used polar coordinates. Therefore it is no loss of generality to assume $L = 1$. By (A.1),

$$\int_{|x|\leq r_d} \big(1 - \mathrm{e}^{\mathrm{i}k\cdot x}\big)\,\mathrm{d}x = \bar{b}_d - \sum_{m=0}^{\infty} \frac{(-1)^m \pi^{d/2}(r_d)^d}{m!\Gamma(m+\frac{d}{2}+1)}\Big(\frac{|k|r_d}{2}\Big)^{2m} = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}\pi^{d/2}(r_d)^d}{m!\Gamma(m+\frac{d}{2}+1)}\Big(\frac{|k|r_d}{2}\Big)^{2m},$$

which is $\mathcal{O}(|k|^2) = o(|k|^\alpha)$. By the polar representation of the Lebesgue measure, we further have

$$\int_{|x|>r_d} |x|^{-d-\alpha}\big(1 - \mathrm{e}^{\mathrm{i}k\cdot x}\big)\,\mathrm{d}x = \int_{\mathbb{S}^{d-1}} \int_{r_d}^{\infty} t^{-1-\alpha}\big(1 - \mathrm{e}^{\mathrm{i}tk\cdot u}\big)\,\mathrm{d}t\,\nu_{d-1}(\mathrm{d}u),$$

where the outer integration is with respect to ν_{d-1}, the Hausdorff measure on the unit sphere $\mathbb{S}^{d-1} = \{u \in \mathbb{R}^d : |u| = 1\}$. By the symmetry property of ν_{d-1} and a change of variables this equals

$$\int_{\mathbb{S}^{d-1}} |k\cdot u|^\alpha \int_{|k\cdot u|r_d}^{\infty} s^{-1-\alpha}(1-\cos s)\,\mathrm{d}s\,\nu_{d-1}(\mathrm{d}u) \tag{A.2}$$

$$= \int_{\mathbb{S}^{d-1}} |k\cdot u|^\alpha \int_0^{\infty} s^{-1-\alpha}(1-\cos s)\,\mathrm{d}s\,\nu_{d-1}(\mathrm{d}u) - \int_{\mathbb{S}^{d-1}} |k\cdot u|^\alpha \int_0^{|k\cdot u|r_d} s^{-1-\alpha}(1-\cos s)\,\mathrm{d}s\,\nu_{d-1}(\mathrm{d}u).$$

Since $1-\cos s \leq s^2/2$, the second integral on the right-hand side of (A.2) is bounded by

$$\frac{1}{2}\int_{\mathbb{S}^{d-1}} |k\cdot u|^\alpha \int_0^{|k\cdot u|r_d} s^{1-\alpha}\,\mathrm{d}s\,\nu_{d-1}(\mathrm{d}u) = \frac{(r_d)^{2-\alpha}}{2(2-\alpha)} \int_{\mathbb{S}^{d-1}} |k\cdot u|^2 \nu_{d-1}(\mathrm{d}u),$$

while the first integral equals $c_2 \int_{\mathbb{S}^{d-1}} |k\cdot u|^\alpha \nu_{d-1}(\mathrm{d}u)$, where $c_2 := \int_0^\infty s^{-1-\alpha}(1-\cos s)\,\mathrm{d}s < \infty$. By [13, Section D.3], we have for each $a > 0$ that

$$\int_{\mathbb{S}^{d-1}} |k\cdot u|^a \nu_{d-1}(\mathrm{d}u) = 2(d-1)b_{d-1}|k|^a \int_0^1 s^a(1-s^2)^{(d-3)/2}\,\mathrm{d}s.$$

Summarizing, we see that

$$1 - c_h\widehat{h}(k) = c_h\int_{|x|\le r_d}(1-\mathrm{e}^{\mathrm{i}k\cdot x})\,\mathrm{d}x + c_h\int_{|x|>r_d}|x|^{-d-\alpha}(1-\mathrm{e}^{\mathrm{i}k\cdot x})\,\mathrm{d}x = c_3|k|^\alpha + o(|k|^\alpha),$$

where $c_3 := 2c_hc_2(d-1)b_{d-1}\int_0^1 s^\alpha(1-s^2)^{(d-3)/2}\,\mathrm{d}s$. This implies the result. □

Before giving the proof of Proposition 1.1(a), we make the following observation about $\varphi = \mathbb{1}_{\mathbb{B}^d}$:

Observation A.1. *Let $\varepsilon > 0$, $m \ge 3$ and $\varphi = \mathbb{1}_{\mathbb{B}^d}$. Then there exists $\rho \in (0,1)$ and $C > 0$ such that*

$$(i)\ \sup_{x\in\mathbb{R}^d}\varphi^{\star m}(x) \le C\rho^d, \qquad (ii)\ \sup_{x\in\mathbb{R}^d:|x|\ge\varepsilon}\varphi^{\star 2}(x) \le C\rho^d.$$

Proof. Throughout this proof, $|\cdot|$ refers to the Lebesgue measure for sets as well as to the Euclidean norm for vectors. To prove (i), we note that $\sup_x \varphi^{\star(m+1)}(x) \le \sup_x \varphi^{\star m}(x)$, so we only consider $m = 3$. Next, note that the supremum is in fact a maximum, attaining its maximal value at $x = \mathbf{0}$. This follows, for example, from the logconcavity of φ (see, e.g.,[9, Theorem 2.18])

Recall that $r_d = \pi^{-1/2}\Gamma(\frac{d}{2}+1)^{1/d}$. We use $B(x,r)$ to denote a ball around x of radius r, so that $\mathbb{B}^d(x) = B(x, r_d)$. Fix $\delta \in (0,1)$. We see that

$$\begin{aligned}\varphi^{\star 3}(\mathbf{0}) &= \int \mathbb{1}_{\mathbb{B}^d}(y)\int \mathbb{1}_{\mathbb{B}^d}(y-z)\mathbb{1}_{\mathbb{B}^d}(-z)\,\mathrm{d}z\,\mathrm{d}y = \int_{\mathbb{B}^d}\int_{\mathbb{B}^d}\mathbb{1}_{\mathbb{B}^d}(y-z)\,\mathrm{d}z\,\mathrm{d}y\\ &= \int_{\mathbb{B}^d}|\mathbb{B}^d(\mathbf{0})\cap\mathbb{B}^d(y)|\,\mathrm{d}y \le \int_{\delta\mathbb{B}^d}1\,\mathrm{d}z + |\mathbb{B}^d\setminus\delta\mathbb{B}^d|\sup_{y\in\mathbb{B}^d\setminus\delta\mathbb{B}^d}|\mathbb{B}^d(\mathbf{0})\cap\mathbb{B}^d(y)|.\end{aligned}$$

The first term is simply δ^d. We estimate $|\mathbb{B}^d\setminus\delta\mathbb{B}^d| \le 1$ and are left to treat $|\mathbb{B}^d(\mathbf{0})\cap\mathbb{B}^d(y)| =: f(|y|)$. Note that $f(t)$ is non-increasing in t, and so the supremum is attained for $t = \delta r_d$. Let y be a point with $|y| = \delta r_d$, for example $y = (\delta r_d, 0,\dots,0)$. We claim that

$$\mathbb{B}^d\cap\mathbb{B}^d(y) \subseteq B\Big(y/2, \sqrt{r_d^2 - |y|^2/4}\Big) = B\Big(y/2, r_d\sqrt{1-\delta^2/4}\Big) = y/2 + \sqrt{1-\delta^2/4}\,\mathbb{B}^d. \tag{A.3}$$

Assuming this claim, (i) follows directly from $|\sqrt{1-\delta^2/4}\,\mathbb{B}^d| = (1-\delta^2/4)^{d/2}$. It remains to prove (A.3).

Let $x \in \mathbb{B}^d(\mathbf{0})\cap\mathbb{B}^d(y)$. Due to symmetry, we assume w.l.o.g. that $x_1 \ge t/2$. Now, recalling $y = (t, 0\dots,0)$ (where $t = |y| = \delta r_d$),

$$|x - y/2|^2 = (x_1 - t/2)^2 + \sum_{i=2}^d x_i^2 = |x|^2 - x_1t + \tfrac14 t^2 \le r_d^2 - t^2/4,$$

which proves $x \in B(y/2, \sqrt{r_d^2 - t^2/4})$.

To prove (ii), note that, by (A.3), we have $\varphi^{\star 2}(x) = |\mathbb{B}^d(\mathbf{0})\cap\mathbb{B}^d(x)| = f(|x|) \le (1-\varepsilon^2/4)^{d/2}$ for $|x| \ge \varepsilon$. □

Proof of Proposition 1.1(a). It is clear that both densities have all moments. The Gaussian distribution is easy to handle, as $\widehat{\varphi}_{\mathcal{N}}(k) = \exp(-\frac12|k|^2)$, and so $1 - \widehat{\varphi}_{\mathcal{N}}(k) = \frac12|k|^2 + o(|k|^2)$. Moreover, $\varphi_{\mathcal{N}}^{\star m}(\mathbf{0}) \le (\varphi_{\mathcal{N}}\star\varphi_{\mathcal{N}})(\mathbf{0}) = (2\sqrt{\pi})^{-d}$.

The convolution statements for $\varphi = \mathbb{1}_{\mathbb{B}^d}$ are shown in Observation A.1. It remains to prove (H1.3). Taking $k \in \mathbb{R}^d$ and choosing $r = r_d = \pi^{-1/2}\Gamma(d/2+1)^{1/d}$ in (A.1) gives

$$1 - \widehat{\varphi}(k) = (2\pi)^{d/2}\sum_{m=1}^\infty \frac{(-1)^{m+1}}{m!\Gamma(m+d/2+1)}\frac{|k|^{2m}r_d^{2m+d}}{2^{2m+d/2}} = a_d|k|^2 + |k|^4R_d(k), \tag{A.4}$$

where $a_d := (2\pi)^{d/2} r_d^{2+d} \Gamma(d/2+2)^{-1} 2^{-2-d/2}$ and

$$R_d(k) := (2\pi)^{d/2} \sum_{m=2}^{\infty} \frac{(-1)^{m+1}}{m!\Gamma(m+d/2+1)} \frac{|k|^{2m-4} r_d^{2m+d}}{2^{2m+d/2}}.$$

Since $\Gamma(d/2+2) = (d/2+1)\Gamma(d/2+1)$,

$$a_d = \frac{r_d^2}{4(d/2+1)} = \frac{\Gamma(d/2+1)^{2/d}}{4\pi(d/2+1)}.$$

At this stage, we recall the well-known bounds $(2\pi x)^{1/2}(x/\mathrm{e})^x < \Gamma(x+1) < (2\pi x)^{1/2}(x/\mathrm{e})^x \mathrm{e}$, valid for each $x > 0$. Using the first inequality with $x = d/2$ gives

$$a_d > \frac{(\pi d)^{1/d} d}{8\pi \mathrm{e}(d/2+1)} \geq c > 0, \tag{A.5}$$

where $c > 0$ does not depend on d. Further, for $|k| \leq 1$,

$$|R_d(k)| \leq \sum_{m=2}^{\infty} \frac{r_d^{2m+d}}{m!\Gamma(m+d/2+1)2^{2m+d/2}} = \sum_{m=2}^{\infty} \frac{\Gamma(d/2+1)^{(2m+d)/d}}{m!\Gamma(m+d/2+1)2^{2m+d/2}\pi^{(2m+d)/2}}.$$

Using the above bounds for the Gamma function, we obtain that

$$|R_d(k)| \leq \sum_{m=2}^{\infty} \frac{(\pi d)^{(2m+d)/(2d)} (d/(2\mathrm{e}))^{(2m+d)/2} \mathrm{e}^{(2m+d)/d}}{m!(2\pi(m+d/2))^{1/2}((m+d/2)/\mathrm{e})^{(m+d/2)} 2^{2m+d/2} \pi^{(2m+d)/2}}.$$

Since, trivially, $d/(2\mathrm{e}) \leq (m+d/2)/\mathrm{e}$, we see that

$$\begin{aligned}|R_d(k)| &\leq \sum_{m=2}^{\infty} \frac{(\pi d)^{(2m+d)/(2d)} \mathrm{e}^{(2m+d)/d}}{m!(2\pi(m+d/2))^{1/2} 2^{2m+d/2} \pi^{(2m+d)/2}} \leq \sum_{m=2}^{\infty} \frac{(\pi d)^{m/d} (\pi d)^{1/2} \mathrm{e}^{2m/d} \mathrm{e}}{m!(2\pi)^{1/2}(d/2)^{1/2} 2^{2m} \pi^m} \\ &= \sum_{m=2}^{\infty} \frac{(\pi d)^{m/d} \mathrm{e}^{2m/d} \mathrm{e}}{m! 2^{2m} \pi^m}.\end{aligned}$$

The latter series is bounded uniformly in d, so that the assertion follows from (A.1) and (A.5). □

The following two proposition are used in a crucial way in Section 5.

Proposition A.2 (Random walk s-condition)**.**

1. *Let φ satisfy (H1). Let $s \in \mathbb{N}_0$, $2 \leq m \in \mathbb{N}$, and $d > 4s$. Then there is a constant c_s^{RW} (independent of d) such that, for all $\mu \in [0,1]$ and with $\beta = g(d)^{1/4}$ as in* (1.8)*,*

$$\int \frac{|\widehat{\varphi}(k)|^m}{[1-\mu\widehat{\varphi}(k)]^s} \frac{\mathrm{d}k}{(2\pi)^d} \leq c_s^{RW} \beta^{2((m\wedge 3)-2)}.$$

2. *Let φ satisfy (H2) or (H3). Let $s \in \mathbb{N}_0$, $2 \leq m \in \mathbb{N}$, and $d > (\alpha \wedge 2)s$. Then there is a constant c_s^{RW} (independent of L) such that, for all $\mu \in [0,1]$ and with $\beta = L^{-d}$ as in* (1.8)*,*

$$\int \frac{|\widehat{\varphi}(k)|^m}{[1-\mu\widehat{\varphi}(k)]^s} \frac{\mathrm{d}k}{(2\pi)^d} \leq c_s^{RW} \beta.$$

Proposition A.3 (Related Fourier integrals)**.** *Let $d > 12$ for (H1) and $d > 3(\alpha \wedge 2)$ for (H2) and (H3). Then for $n \in \{1,2,3\}$ and $m \in \{2,3\}$, uniformly in $\mu \leq 1$ and $k \in \mathbb{R}^d$, and with β as in* (1.8)*,*

$$\int |\widehat{\varphi}(l)|^m \widehat{G}_\mu(l)^{3-n} [\widehat{G}_\mu(l+k) + \widehat{G}_\mu(l-k)]^n \frac{\mathrm{d}l}{(2\pi)^d} \leq 2^n c_3^{RW} \beta^{m-2}, \tag{A.6}$$

$$\int |\widehat{\varphi}(l)|^m \widehat{G}_\mu(l) \widehat{G}_\mu(l+k) \widehat{G}_\mu(l-k) \frac{\mathrm{d}l}{(2\pi)^d} \leq c_3^{RW} \beta^{m-2}, \tag{A.7}$$

where c_3^{RW} is as in Proposition A.2.

Proof of Proposition A.2 for the finite-variance model (H1). We first note that w.l.o.g. we can restrict to considering $m \in \{2,3\}$. By Cauchy-Schwarz,

$$\int \frac{|\widehat{\varphi}(k)|^m}{[1-\mu\widehat{\varphi}(k)]^s}\frac{\mathrm{d}k}{(2\pi)^d} \le \left(\int \widehat{\varphi}(k)^{2m-2}\frac{\mathrm{d}k}{(2\pi)^d}\right)^{1/2}\left(\int \frac{\widehat{\varphi}(k)^2}{[1-\mu\widehat{\varphi}(k)]^{2s}}\frac{\mathrm{d}k}{(2\pi)^d}\right)^{1/2}.$$

The integral in the first factor is just $(\varphi\star\varphi)(\mathbf{0}) \le 1$ for $m=2$, and $\varphi^{\star 2m-2}(\mathbf{0}) = \varphi^{\star 4}(\mathbf{0}) = \mathcal{O}(\beta^4)$ for $m=3$. Hence, the first factor is $\mathcal{O}(\beta^{2(m-2)})$ and it remains to prove the boundedness of the second integral.

Note that as soon as $\mu<1$, the denominator is bounded away from zero and the boundedness follows from the integrability of the enumerator. To get a bound that is uniform in μ, we set $\mu=1$.

We split the area of integration and first consider $\{k : |k| \le \varepsilon\}$, where we choose $\varepsilon>0$ small enough such that $1-\widehat{\varphi}(k) \ge c|k|^2$ for all $|k|\le\varepsilon$. Applying (1.5),

$$\int_{|k|\le\varepsilon} \frac{\widehat{\varphi}(k)^2}{[1-\widehat{\varphi}(k)]^{2s}}\frac{\mathrm{d}k}{(2\pi)^d} \le c^{-2s}\int_{|k|\le\varepsilon} |k|^{-4s}\frac{\mathrm{d}k}{(2\pi)^d},$$

which is finite for $d>4s$. For $|k|>\varepsilon$, we have $1-\widehat{\varphi}(k)>c>0$ as in the proof of Proposition 1.1(b). Consequently,

$$\int_{|k|>\varepsilon} \frac{\widehat{\varphi}(k)^2}{[1-\widehat{\varphi}(k)]^{2s}}\frac{\mathrm{d}k}{(2\pi)^d} \le c^{-2s}\int \widehat{\varphi}(k)^2\frac{\mathrm{d}k}{(2\pi)^d} = c^{-2s}(\varphi\star\varphi)(\mathbf{0}).$$ □

Proof of Proposition A.2 for the spread-out models (H2), (H3). Again, we set $\mu=1$, since otherwise the statement is clear. We consider the regions $|k|\le b/L$ and $|k|>b/L$. Applying (H2.3) (resp., (H3.3)),

$$\int_{|k|\le bL^{-1}} \frac{|\widehat{\varphi}_L(k)|^m}{[1-\widehat{\varphi}_L(k)]^s}\frac{\mathrm{d}k}{(2\pi)^d} \le \frac{1}{c_1^s L^{(\alpha\wedge 2)s}}\int_{|k|\le bL^{-1}} \frac{1}{|k|^{(\alpha\wedge 2)s}}\frac{\mathrm{d}k}{(2\pi)^d} \le CL^{-d}$$

for $d>(\alpha\wedge 2)s$. Note that C depends on b, c_1, α, d, but not on L. For the second region,

$$\begin{aligned}\int_{|k|>bL^{-1}} \frac{|\widehat{\varphi}_L(k)|^m}{[1-\widehat{\varphi}_L(k)]^s}\frac{\mathrm{d}k}{(2\pi)^d} &\le c_2^{-s}\int |\widehat{\varphi}_L(k)|^m\frac{\mathrm{d}k}{(2\pi)^d} \le c_2^{-s}(\varphi_L\star\varphi_L)(\mathbf{0})\\ &\le c_2^{-s}\|\varphi_L\|_\infty\int \varphi_L(x)\,\mathrm{d}x \le CL^{-d},\end{aligned}$$

using (H2.2) in the last bound. □

Proof of Proposition A.3. We show the proof for (H1); the spread-out models work similarly. To prove (A.7), we point out that

$$\widehat{G}_\mu(l\pm k) = \int [\cos(l\cdot x)\cos(k\cdot x) \mp \sin(l\cdot x)\sin(k\cdot x)]G_\mu(x)\,\mathrm{d}x. \tag{A.8}$$

Setting $G_{\mu,k}(x) = \cos(k\cdot x)G_\mu(x)$, we thus have

$$0 \le \widehat{G}_\mu(l-k)\widehat{G}_\mu(l+k) = \widehat{G}_{\mu,k}(l)^2 - \left(\int \sin(l\cdot x)\sin(k\cdot x)G_\mu(x)\,\mathrm{d}x\right)^2 \le \widehat{G}_{\mu,k}(l)^2, \tag{A.9}$$

and so, using the Cauchy-Schwarz inequality, (A.7) is bounded by

$$\begin{aligned}&\left(\int \widehat{\varphi}(l)^{2m-2}\widehat{G}_\mu(l)\widehat{G}_{\mu,k}(l)^2\frac{\mathrm{d}l}{(2\pi)^d}\right)^{1/2}\left(\int \widehat{\varphi}(l)^2\widehat{G}_\mu(l)\widehat{G}_{\mu,k}(l)^2\frac{\mathrm{d}l}{(2\pi)^d}\right)^{1/2}\\ &\quad= \Big(\big(\varphi^{\star 2m-2}\star G_\mu\star G_{\mu,k}^{\star 2}\big)(\mathbf{0})\big(\varphi^{\star 2}\star G_\mu\star G_{\mu,k}^{\star 2}\big)(\mathbf{0})\Big)^{1/2}.\end{aligned} \tag{A.10}$$

Using that $\big(\varphi^{\star 2m-2}\star G_\mu\star G_{\mu,k}^{\star 2}\big)(\mathbf{0}) \le \big(\varphi^{\star 2m-2}\star G_\mu^{\star 3}\big)(\mathbf{0})$, we continue as in the proof of Proposition A.2. By (A.8), $\widehat{G}_{\mu_\lambda}(l+k)+\widehat{G}_{\mu_\lambda}(l-k) = 2\widehat{G}_{\mu,k}(l)$, and so we can write (A.6) as

$$\int |\widehat{\varphi}(l)|^m\widehat{G}_\mu(l)^{3-n}2^n\widehat{G}_{\mu,k}(l)^n\frac{\mathrm{d}l}{(2\pi)^d},$$

which is bounded analogously to above. □

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