Risk, Uncertainty and Optimisation on Infinite Dimensional Ordered Spaces

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Zusammenfassung

Die vorliegende Arbeit beleuchtet mathematische Aspekte der Theorie ökonomischer und finanzieller Risiken. In den drei versammelten Beiträgen werden das Fortsetzungsproblem für Risikomaße unter Knight'scher Unsicherheit sowie die Optimierung von Risiko- und Nutzenaufteilungen studiert.

Der erste Beitrag, *Model Spaces for Risk Measures*, widmet sich dem Fortsetzungsproblem für Risikomaße. Letztere sind als Kapitalanforderungen definiert, welche durch das Zusammenspiel einer konvexen Menge akzeptabler Positionen und eines mehrdimensionalen Security-Markts zur Absicherung nicht akzeptabler Positionen induziert werden. Die Nettoverluste, deren Risiko gemessen wird, werden durch Zufallsvariablen auf der Menge möglicher künftiger Zustände der Ökonomie modelliert. Wir betrachten allerdings Risikomaße, welche auch *Knight'sche Unsicherheit* abbilden: Die Realisierung künftiger Zustände lässt sich nur partiell oder überhaupt nicht durch einen Zufallsmechanismus mit bekannten Parametern beschreiben. In der Fortsetzungsfrage gehen wir deshalb von einer simplen Situation aus, in der das Risiko von beschränkten Nettoverlusten durch szenarioweise Überlegungen und ohne ein zugrundeliegendes Wahrscheinlichkeitsmodell bestimmt wird. Wir zeigen, dass unter milden Annahmen ein solches Risikomaß eine intrinsische probabilistische Sichtweise auf die Realisierung künftiger Zustände erlaubt. Diese kann als Ausgangspunkt einer Fortsetzung dienen, da sie im Vergleich zur modellfreien Betrachtung keinen Informationsverlust verursacht. Um das Risiko komplexerer Verluststprofile bestimmen zu können, muss ein geeigneter größerer Definitionsbereich gefunden werden. Wir konstruieren einen Banachverband von Zufallsvariablen, dessen analytische Struktur durch die ursprüngliche Risikomessung diktiert wird. Er ist invariant unter allen relevanten probabilistischen Modellen und fungiert als maximaler natürlicher Definitionsbereich des ursprünglichen Risikomaßes, da er verschiedene Fortsetzungen desselben auf unbeschränkte Zufallsvariablen erlaubt. Diese beleuchten und vergleichen wir detailliert. Die Diskussion von Bedingungen, unter welchen *reguläre Subgradienten*, d.h. sinnvolle Preisregeln, existieren, schließt sich an.

Der zweite Beitrag, *Risk Sharing for Capital Requirements with Multidimensional Security Markets*, studiert die Existenz optimaler Risikoaufteilungen. Betrachtet wird ein System von n Agenten, welche in einer Handelsperiode individuelle Nettoverluste erleiden. Deren Risiko quantifiziert jeder Agent mit einem individuellen Risikomaß, welches wie im ersten Beitrag eine aus Akzeptanzmenge und endlichdimensionalem Security-Markt resultierende Kapitalanforderung ist. Die Frage ist nun, ob die aggregierten (d.h. addierten) individuellen Risiken durch restlose Umverteilung der individuellen Verlustpositionen minimiert werden können. Nettoverluste werden als Elemente eines zugrundeliegenden Riesz-Raums modelliert. Individuell relevante und individuell akzeptable Verlustpositionen sowie individuelle Security-Märkte können sehr heterogen sein. Wir treffen einzig eine schwache Kooperationsannahme, welche den Agenten überhaupt erst ein gemeinsames Pooling der Risiken ermöglicht. Es wird gezeigt, dass nur *Pareto-optimale* Allokationen (Aufteilungen) eines gegebenen Gesamtverlusts die Aggregation der Einzelrisiken minimieren. Das minimale aggregierte Risiko eines aufgeteilten Gesamtverlusts stellt eine Kapitalanforderung für den Markt dar, die sich durch eine Markt-Akzeptanzmenge und einen globalen Security-Markt charakterisieren lässt. Das Problem lässt sich also mittels eines *repräsentativen Agenten* formulieren. Für Agentensysteme mit polyhedralen und verteilungsinvarianten Akzep-

tanzmengen wird die Existenz von Pareto-optimalen Allokationen und Equilibria gezeigt und ihre Stabilität unter Perturbation des aggregierten Verlusts analysiert. Als Anwendung werden optimale Portfolio-Aufteilungen unter Tochterfirmen in Märkten mit Transaktionskosten untersucht.

Der dritte Beitrag, *Efficient Allocations under Law-Invariance: A Unifying Approach*, wechselt die Perspektive vom Risiko von Verlusten auf den *Nutzen*, den die Aufteilung eines durch stochastische Faktoren unsicheren künftigen Guts unter n Agenten mit heterogenen Präferenzen erzeugt. Güter werden als Zufallsvariablen über einem atomlosen Wahrscheinlichkeitsraum $(\Omega, \mathcal{F}, \mathbb{P})$ modelliert. Die Präferenzen der einzelnen Agenten sind durch quasikonkave *verteilungsinvariante* Nutzenfunktionale kodiert. Alle Agenten sind also insofern kompatibel, als dass sie indifferent zwischen zwei Gütern mit derselben Verteilungsfunktion unter dem Referenzmaß $\mathbb{P}$ sind. Wir nehmen an, dass die individuellen Nutzen mit einer allgemein gehaltenen Funktion zu einem Gesamtnutzen aggregiert werden, beispielsweise durch (gewichtete) Summation. Allokationen sind *effizient*, wenn sie bei Aggregation zum optimalen, d.h. maximalen aggregierten Nutzen unter allen möglichen Allokationen führen. Für große Klassen individueller Nutzenfunktionen und Aggregationsfunktionen zeigen wir die Existenz *komonotoner* effizienter Allokationen. Hiermit verallgemeinern wir viele der bereits bekannten Resultate zu optimaler Nutzenaufteilung mit verteilungsinvarianten Nutzenkriterien und führen sie zu einem Meta-Theorem zusammen. Als Anwendungen der allgemeinen Existenztheorie zeigen wir die Existenz von Allokationen mit verschiedenen ökonomisch motivierten Optimalitätseigenschaften: (schwache, verzerrte und individuell rationale) Pareto-Optimalität, systemisch faire Allokationen und Kern-Allokationen.

Abstract

This dissertation examines mathematical aspects of the theory of economic and financial risks. The three collected contributions study the extension problem for risk measures under Knightian uncertainty as well as the optimisation of risk and utility allocations.

The first contribution, *Model Spaces for Risk Measures*, addresses the extension problem for risk measures. The latter are defined as capital requirements which are induced by the interplay of a convex set of acceptable positions and a multidimensional security market for securing non-acceptable positions. The net losses whose risk is to be measured are modelled by random variables on the set of future states of the economy. However, we consider risk measures which can also account for *Knightian uncertainty*: The realisation of future states of the economy can only partially or not at all be described with known parameters. In the extension question, we thus depart from a simple situation in which the risk of bounded net losses is determined by scenariowise considerations without an underlying probabilistic model. We show that under mild assumptions such a risk measure admits an intrinsic probabilistic perspective on the realisation of future states of the economy. It can serve as starting point for an extension as it does not lead to any loss of information compared to the model-free risk measure. In order to determine the risk of more complex loss profiles, a suitable larger domain of definition must be found. We construct a Banach lattice of random variables whose analytic structure is dictated by the initial risk measure. It is invariant under all relevant probabilistic models and plays the role of a maximal natural domain of definition of the initial risk measure, as it admits several extensions thereof to unbounded random variables. We examine and compare them in detail. A discussion of conditions under which *regular subgradients* exist, i.e. sensible pricing rules, concludes.

The second contribution, *Risk Sharing for Capital Requirements with Multidimensional Security Markets*, studies the existence of optimal risk allocations. We consider a system of n agents who incur individual net losses in a one-period market. Each agent quantifies their risk with an individual risk measure which is a capital requirement resulting from an acceptance set and a finite dimensional security market as in the first contribution. This raises the question whether the aggregated (i.e. added) individual risks can be minimised by complete redistribution of the individual losses. Net losses are modelled as elements of an ambient Riesz space. Individually relevant and individually acceptable losses as well as individual security markets may be very heterogeneous. We only make a weak cooperation assumption which allows the agents to pool their risks in the first place. It is shown that only *Pareto optimal* allocations of a given aggregated loss minimise the aggregation of the individual risks. The minimal aggregated risk of an allocated aggregated loss defines a capital requirement for the market which may be characterised by a market acceptance set and a global security market. The problem can hence be formulated with a *representative agent.* For agent systems with polyhedral and law-invariant acceptance sets, the existence of Pareto optimal allocations and equilibria is shown and their stability under perturbation of the aggregated loss is analysed. As an application, we study optimal portfolio splits among subsidiaries in markets with transaction costs.

The third contribution, *Efficient Allocations under Law-Invariance: A Unifying Approach*, shifts the perspective from the risk of losses to the *utility* produced by the allocation of a future good, uncertain for stochastic factors, among n agents with heterogeneous preferences. Goods are modelled as random variables on a non-atomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The preferences of the individual agents are encoded by quasi-concave *law-invariant* utility functions. All agents are thus compatible insofar as they are indifferent between two goods with the same distribution function under the reference measure $\mathbb{P}$. We assume that individual utilities are aggregated with a general aggregation function, for instance (weighted) summation. Allocations are *efficient* if under aggregation they lead to the optimal, i.e. maximal, aggregated utility among all attainable allocations. For wide classes of individual utility functions and aggregation functions we show the existence of *comonotone* efficient allocations. We hereby generalise many of the the known results on optimal utility allocation under law-invariant utility criteria and unify them in a meta theorem. We apply the general existence theory by showing the existence of allocations with different economically motivated optimality properties: (weak, biased, and individually rational) Pareto optimality, systemically fair allocations, and core allocations.

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Contents

Chapter 1

Introduction

> By "uncertain" knowledge ... I do not mean merely to distinguish what is known for certain from what is only probable. The game of roulette is not subject, in this sense, to uncertainty ... Even the weather is only moderately uncertain. The sense in which I am using the term is that in which the prospect of a European war is uncertain, or the price of copper and the rate of interest twenty years hence, ... or the position of private wealth-owners in the social system in 1970. About these matters there is no scientific basis on which to form any calculable probability whatever. We simply do not know. Nevertheless, the necessity for action and for decision compels us ... to do our best to overlook this awkward fact and to behave exactly as we should if we had behind us a good Benthamite calculation of a series of prospective advantages and disadvantages, each multiplied by its appropriate probability, waiting to be summed.
> (J. M. Keynes, The General Theory of Employment)

> Yet that "probability" is, in the strict sense, indefinable, need not trouble us much; it is a characteristic which it shares with many of our most necessary and fundamental ideas.
> (J. M. Keynes, The Principles of Probability)

1.1 A brief history of risk measures

The consequences of future events, as crucial as they are for making good decisions in the present, cannot be known until they have realised. Spurred by this simple truth, disciplines ranging from economics to climate sciences have over decades developed methodologies to manage the *uncertain* rationally. These all face the challenge to make conceptual and intellectual sense of uncertainty. Moreover, they have to balance sufficient caution concerning impacts which are unlikely but possible on the one hand, and pragmatism, serving the practical applicability of the reasoning in question, as well as cost efficiency on the other hand.

In the field of mathematical finance, one of the most prominent of such methodologies is the theory of risk measures. They are a mathematically rigorous way to capture the financial risk posed by a future loss — to which we will also refer as a *loss profile* or *loss*. Note that in contrast to the majority of the literature on risk measures, we define them to measure the risk of losses net of gains, not gains net of losses.[1] This is in line with the later chapters of this thesis. In the case of static risk measures, two time points are fixed, "today" and "tomorrow". The

[1] In more financial terms, they can be interpreted as liabilities net of assets or negative P&L (profits and losses).

objective is to attribute a risk level to an uncertain loss occurring tomorrow based on imperfect knowledge about the actual size of the loss available today. Ideally, this risk level should have an immediate interpretation in terms of actions which can be taken today in order to mitigate the loss tomorrow.

Risk measures were introduced in an axiomatic way in the seminal 1999 paper Artzner et al. [11]. The immediate popularity of the axiomatisation presented there is likely due to two of its most appealing features. Firstly, it offers a very flexible framework for risks which can capture "market risks as well as nonmarket risks" [11, p. 203]. Secondly, the way in which an agent measures the risk of a loss is nevertheless highly operational and dependent only on very few extrinsic *fundamentals*. Given an ambient set of possible loss profiles which may occur tomorrow and whose risk is to be measured, these fundamentals are:

(i) identifying the set of "acceptable" losses which pose a risk the agent can fathom without further mitigation.

(ii) identifying a so-called "prudent" financial reference instrument. Given a loss profile outside of the set of acceptable losses, the agent can raise extra capital and invest in this reference instrument in a way such that the future loss *combined* with the future payoff of this investment is acceptable in the sense of (i).

Bringing (i) and (ii) together, the risk of a loss profile is defined as the *minimal amount of capital* which needs to be raised and invested in the prespecified prudent financial instrument such that the combined loss profile is acceptable in the prespecified manner. Note that if the loss is acceptable already, its risk is minus the maximal amount of capital which can be obtained without losing acceptability by shorting the prudent instrument today and offsetting the liabilities tomorrow with the initial acceptable loss profile. Hereby, risk has both a monetary and an operational interpretation: What exactly needs to be done today in order to mitigate risks occurring tomorrow?

The *risk measure* associated to this procedure is eventually defined as the function on the set of potential losses mapping a particular loss to its capital requirement as computed above.

The mathematical axiomatisation of such an operational understanding of risk should ideally prescribe properties of the two fundamentals, the *acceptance set* and the *prudent instrument*, which are meaningful from an economic perspective. Without going into too much detail, two of the axioms introduced in [11] need to be singled out.

The authors model future losses whose risk is to be determined with the risk measure as a vector space of real-valued functions X over the set Ω of future states of the economy. For such a function $X : \Omega \to \mathbb{R}$, the value $X(\omega)$ is the net loss X renders provided state $\omega \in \Omega$ realises. Potential losses are thus naturally ordered by an "objective order" and can be compared. This objective order is the pointwise order $\leq$, and $X \leq Y$ holds if the net loss Y is at least as large as the net loss X in each state, i.e. $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$. However, Artzner et al. only consider finite sets of states, i.e. the cardinality of Ω is finite. Since their paper [11], considering function spaces as the domain of definition of risk measures has become standard.

The second remarkable axiom concerns the acceptance set. It is assumed to contain the negative cone, i.e. if $X : \Omega \to \mathbb{R}$ is acceptable if $X(\omega) \leq 0$ for all $\omega \in \Omega$ and no actual loss is produced. Also, the authors mainly focus on *coherent risk measures*: A risk measure is coherent if the acceptance set is a convex cone in that $\lambda X + \mu Y$ is acceptable whenever X and Y are acceptable and $\lambda, \mu \geq 0$ are non-negative scaling factors. Convexity has a widely accepted economic interpretation: If the agent diversifies between two acceptable losses, this does not increase risk to a degree at which acceptability is lost. Conicity, however, is a more questionable property. It would mean that any arbitrarily large quantity of an acceptable loss remains acceptable. Liquidity effects are thus entirely ignored in the measurement of risk. Whereas a bet on the outcome of a coin paying or costing 1€ appears to be quite acceptable for any financial agent, the same cannot be said of a similar bet paying or costing 1,000,000€. This peculiarity quickly led to the further development of the axioms and to the suggestion to replace the coherence axiom by mere convexity; cf. Föllmer & Schied [51] and Fritelli & Rosazza Gianin [55]. However, the negative cone being acceptable and the coherence of the acceptance set imply together a property which remained to be widely accepted, *monotonicity* of the acceptance set. That is, if Y is acceptable and X is a loss which is at most as large as Y ($X \leq Y$), then the agent should also accept X.

Equally important are the early contributions of Delbaen [33, 34]. Firstly, they generalise coherent risk measures decisively to situations where no assumption on the number of relevant future states of the economy is made. The set Ω may have any arbitrary cardinality. However, in certain contrast to Artzner et al. [11], losses are modelled as random variables measurable with respect to some σ-agebra $\mathcal{F} \subset 2^{\Omega}$ of events observable tomorrow, and a probability measure $\mathbb{P}$ is present assigning probabilities to these events. Losses are identified to a single equivalence class if they agree in all states outside of an "impossible" set of states of $\mathbb{P}$-probability 0. Moreover, there is a natural generalisation of the pointwise objective ordering of losses in Artzner et al. [11], given by the $\mathbb{P}$-*almost-sure* ($\mathbb{P}$-a.s.) order. More precisely, $X \leq Y$ $\mathbb{P}$-a.s. if for each choice of representatives f and g of the equivalence classes of X and Y, respectively, $f(\omega) \leq g(\omega)$ holds for all states ω outside an event of $\mathbb{P}$-probability 0.[2] *A priori*, the probability measure $\mathbb{P}$ is extrinsic. [33, 34] as well as most of the literature do neither explain the origin of $\mathbb{P}$ nor provide an interpretation of it.[3]

Another important and lasting contribution of [33, 34] concerns the reference instrument used to produce acceptability for general losses. In Artzner et al., this reference instrument is mostly illustrated as a default-free zero coupon bond in a particular currency. Its payoff is therefore deterministic and governed by the respective interest rate. Such an instrument can obviously be used as a numéraire. In Delbaen [33, 34], losses are therefore *a priori* assumed to be *discounted* by the reference instrument, i.e. each loss is expressed in units of the reference instrument:

> Here we only remark that we are working in a model without interest rate, the general case can "easily" be reduced to this case by "discounting". [34, p. 4]

An immediate consequence is *cash-additivity* of the resulting risk measure: adding or withdraw-

[2] As usual, we will in the following treat equivalence classes as random variables and make no further mention of representatives in statements which hold $\mathbb{P}$-almost surely, i.e. with full $\mathbb{P}$-probability.

[3] We will argue in Section 1.2 that the role of the reference probability measure $\mathbb{P}$ entails non-trivial interpretational queries whether a given risk measure actually accounts for Knightian uncertainty. [33, 34] mostly avoid these issues by considering risk measures on the "robust spaces" L^{∞} and L^{0} of equivalence classes of bounded and arbitrary real-valued random variables, respectively. These spaces only depend weakly on a particular probability measure $\mathbb{P}$ in that their definition merely uses the set of its null sets.

ing a constant amount m of (discounted) capital to or from a loss X results in decreasing or increasing the risk by precisely that amount. If ρ denotes the risk measure as a function of the loss,

$$\rho(X+m) = \rho(X) + m.$$

These contributions have been so style-forming that, usually, risk measures are introduced along the following lines, cf. Föllmer & Schied [52, Chapter 4]: Suppose $\mathcal{X}$ is a vector space of real-valued functions containing all constant functions and carrying a vector space order $\preceq$ which, restricted to the set of constant functions, agrees with the usual ordering on the real line. This space models (discounted) losses net of gains, the ordering admits a comparison of losses. A functional $\rho : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ is a *monetary risk measure* if is has the following two properties:

(R1) Monotonicity: Whenever $X, Y \in \mathcal{X}$ satisfy $X \preceq Y$, $\rho(X) \leq \rho(Y)$ has to hold.

(R2) Cash-additivity: $\rho(X+m) = \rho(X)+m$ for all $X \in \mathcal{X}$ and all constant functions $m \in \mathbb{R}$.

ρ is a *convex monetary risk measure* if, beside properties **(R1)** and **(R2)**, it satisfies

(R3) Quasi-convexity: For all $X, Y \in \mathcal{X}$ and all real numbers $\lambda \in [0, 1]$,

$$\rho(\lambda X + (1-\lambda)Y) \leq \max\{\rho(X), \rho(Y)\}.$$

The economic interpretation of **(R1)** is immediate: If a loss X is better in comparison to Y, its risk should be less than that of Y. The emergence of **(R2)** is explained above. Similar to convexity of the acceptance set in Artzner et al. [11], quasi-convexity reflects the generally accepted economic paradigm that diversification does not increase risk. Note that the conjunction of **(R2)** and **(R3)** is logically equivalent to the conjunction of **(R2)** and

(R4) Convexity: $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$ for all $X, Y \in \mathcal{X}$ and every choice of a parameter $\lambda \in [0, 1]$.

As the presentation here employs a very general domain of definition, i.e. the space $\mathcal{X}$, it makes both sense when $\mathcal{X}$ is a vector space of pointwisely defined functions as in Artzner et al. [11] or [52, Chapter 4.1], and when $\mathcal{X}$ is a space of equivalence classes of measurable functions over a probability space as in Delbaen [33, 34]. Nevertheless, such a definition embodies a complete shift in the *understanding* of risk measurement. A risk measure is not a functional *arising* from a particular choice of extrinsic objects such as acceptance set and reference instrument, but a functional defined *ad hoc* with particular axiomatic properties. This is important to keep in mind.

The question how in a concrete situation financial risk can actually be determined by an agent has remained untouched so far. The losses net of gains as well as the resulting risk of a financial position are often determined on the basis of historically measured or simulated payoff patterns of the same or similar classes of financial instruments. These yield a distribution of losses and gains which is *assumed* to provide relevant information for the future payoff of the financial position. Although this approach is intuitive, it is *a priori* in contrast to the framework of determining payoffs in the presence of isolated future states of the economy resolving all uncertainty as posited by Artzner et al. [11]. However, it can well be incorporated in Delbaen's setting of risk measurement, where losses are modelled by random variables in the presence of a probability measure. Each equivalence class X of real-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ induces a distribution of losses on the real line relative to the probability measure $\mathbb{P}$. This distribution can be captured, for instance, by the cumulative distribution function $F_X^{\mathbb{P}}$ defined by $F_X^{\mathbb{P}}(x) := \mathbb{P}(X \leq x)$, $x \in \mathbb{R}$. A risk measure ρ on such random variables is now called *law-invariant* if the measured risk only depends on the distribution under the reference measure $\mathbb{P}$: if X and Y have the same cumulative distribution functions, i.e. $F_X^{\mathbb{P}} = F_Y^{\mathbb{P}}$, then they pose the same risk, i.e. $\rho(X) = \rho(Y)$. Demanding such an invariance of the risk measure implies that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ in its role as collection of the future states of the economy becomes obsolete. Instead, the risk of probability distributions on the real line is measured, and the distributional patterns derived from simulations or historical data are taken at face value. Important early contributions on law-invariant risk measures are Filipović & Svindland [48], Fritelli & Rosazza Gianin [56], Jouini et al. [65], Kusuoka [74], and Svindland [93, 94]. They also demonstrated that the assumption of law-invariance has very strong and rather surprising analytic consequences. An important one which we shall get back to frequently in this introduction is that the *extension problem* can be solved for law-invariant convex monetary risk measures. Let $L^\infty := L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ denote the space of equivalence classes of bounded random variables up to agreement with $\mathbb{P}$-probability 1. Suppose $\rho : L^\infty \to \mathbb{R}$ is a law-invariant convex monetary risk measure. If the probability space is rich enough and $p \in [1, \infty)$ is an arbitrarily chosen parameter, there is a unique *lower semicontinuous* (l.s.c.), cash-additive and convex function $\rho^\sharp : L^p \to (-\infty, \infty]$ such that $\rho^\sharp(X) = \rho(X)$ for all $X \in L^\infty$; cf. [48, Theorem 2.2]. Here, $L^p := L^p(\Omega, \mathcal{F}, \mathbb{P})$ denotes the space of equivalence classes of real-valued random variables whose p-th moment with respect to the probability measure $\mathbb{P}$ is finite. Lower semicontinuity means that for each level $c \in \mathbb{R}$ the lower level set $\{X \in L^p \mid \rho^\sharp(X) \leq c\}$ is closed in the natural topology of convergence in p-th mean. $\rho^\sharp$ thus extends ρ in a consistent manner to unbounded losses. For this important property of law-invariant risk measures, we refer to Chen et al. [26], Filipović & Svindland [48], and Gao et al. [58].

This very brief summary of the evolution of static risk measures is naturally subjective in its selection and presentation of aspects. However, it presents some of the deeper tenets of and predicaments in the theory relevant for this thesis: To which degree does a theory of risk have to be operational? Can thinking about risk in terms of functionals obscure their operational interpretation? Is cash-additivity a good axiom, and what does it say about the interpretation of the objects whose loss is measured?

We close this introductory *tour de force* of mathematical risk theory with a caveat. In this thesis we will only consider losses occurring at a single particular prespecified future time point and measure the risk from the perspective of "today". Risk measurement will therefore take place in an (abstract) static market model. The rich literature on dynamic risk measurement in multi-period frameworks will not play a role. We refer the interested reader to, e.g., Acciaio & Penner [3], Detlefsen & Scandolo [38], Föllmer & Penner [50], Föllmer & Schied [52, Chapter 11], and the references therein.

Unfortunately, we will also have to ignore the literature on quasi-convex static risk measures which are *not* cash-additive. We refer to Cerreia-Vioglio et al. [23], Drapeau & Kupper [39], and El Karoui & Ravanelli [40].

The focus of this thesis lies on risk which affects the individual losses and gains of agents, not risks affecting the financial endowment or stability of a system of agents such as an external shock of the economy propagating through the banking sector. In recent years, the latter stream of mathematical risk theory, the theory of *systemic risks*, has nevertheless gained prominence, developed quickly, and produced rich amounts of literature. The type of risk measures under consideration in this thesis introduced in Section 1.4 encompass certain classes of *multivariate systemic risk measures*.[4] This link has been made explicit in the recent working paper Arduca et al. [8]. However, the theory of systemic risks extends well beyond systemic risk measures. We refer the interested reader to the monographs by Fouque & Langsam [54] and Hurd [64] as well as the references therein.

1.2 Risk, uncertainty, and the Bayesian paradigm

Before we can return to the discrepancies in the theory of risk measures unfolded in the preceding section, we need to focus on a question of epistemology: What is the "risk" measured by a risk measure?

In this question, the literature usually quotes Frank Knight's style-forming distinction of *risk* and *uncertainty*. In his seminal dissertation RISK, UNCERTAINTY, AND PROFITS [72] he argued for an understanding of risk as "a quantity susceptible of measurement", a situation with an unknown outcome whose realisation obeys parameters which can be known. Uncertainty, on the other hand, refers to situations in which the parameters determining the realisation of an outcome are ambiguous or simply impossible to be known.

> The practical difference between the two categories, risk and uncertainty, is that in the former the distribution of the outcome in a group of instances is known (either from calculation *a priori* or from statistics of past experience), while in the case of uncertainty that is not true, the reason being that it is impossible to form a group of instances, because the situation dealt with is in a high degree unique. The best example of uncertainty is in connection with the exercise of judgment or the formation of those opinions as to the future course of events, which opinions (and not scientific knowledge) actually guide most of our conduct. [72, p. 233]

Similar considerations, though of more pragmatic nature, can be found prominently in the works of Keynes [70]:

> By "uncertain" knowledge ... I do not mean merely to distinguish what is known for certain from what is only probable. The game of roulette is not subject, in this sense, to uncertainty ... The sense in which I am using the term is that in which the prospect of a European war is uncertain, or the price of copper and the rate of interest twenty years hence, ... or the position of private wealth-owners in the social system in 1970. About these matters there is no scientific basis on which to form any calculable probability whatever. We simply do not know.
> *Quoted from* [71, p. 265].

[4] More precisely, "allocate first, then aggregate"- and "first aggregate, then allocate"-type systemic risk measures; cf. [8] and the references therein.

We also refer to Keynes [69].

The devastating economic consequences of the financial crisis of 2007-2009 proved painfully that the widespread use of advanced mathematical tools had not made financial practices sustainably safe. In its aftermath, Knight's ideas enjoyed resurgent interest. The role played by quantitative methods of risk management during the lead-up to the crisis was not judged unanimously. Föllmer [49] argues that questionable and partially illegal human behaviour lay at the root of the crisis, and institutions with a strong quantitative emphasis generally fared better in the financial turmoil. The Turner Review [100] on the other hand, issued by the British Financial Services Authority (FSA), shed critical light on the very assumptions of mathematical modelling in use in a manner more than reminiscent of Knight and Keynes:

> More fundamentally, however, it is important to realize that the assumption that past distribution patterns carry robust inferences for the probability of future patterns is methodologically insecure. It involves applying to the world of social and economic relationships a technique drawn from the world of physics, in which a random sample of a definitively existing universe of possible events is used to determine the probability characteristics which govern future random samples. But it is unclear whether this analogy is valid when applied to economic and social relationships, or whether instead, we need to recognise that we are dealing not with mathematically modellable risk, but with inherent 'Knightian' uncertainty. [100, p. 45]

Such an assessment should not be ignored lightly. It is precisely this methodology of inferring the future on the basis of "past distribution patterns" which lies at the heart of law-invariant risk measurement, the both theoretically and practically most commonly used type of risk measurement.

As a rational framework for decision-making often draws its authority and influence from mathematical validation, Knight's distinction has had decisive influence on mathematical research dealing with decision-making in ambiguous situations. In particular, Knightian uncertainty (often also referred to as *ambiguity*) has over the past decade become the subject of extensive and zealous research in financial mathematics. Its intellectual impact cannot only be felt in the realm of finance or decision sciences, but also in other economic fields.[5] The distinction may sometimes be too rigid though and for instance ignore the social embeddedness of decision-making. For a critique in this respect, intellectually based on the ideas of N. Luhmann, we refer to the introduction of Drapeau & Kupper [39].

It is nowadays widely agreed that *Knightian risk* should be identified with sampling from a known distribution, whereas *Knightian uncertainty* is best understood as imperfect knowledge about the distribution the sample stems from. In line with statistical tradition, the latter is canonically modelled by a set of distributions or probability measures, although this is somewhat at odds with the original ideas of Knight and Keynes. In his review of different philosophies of uncertainty, Arrow [9] remarks:

> The statistician's problem is of the same general type as the businessman's, and even the information-getting aspects have their economic counterparts. The various

[5] Innovation economics would be a further example to mention, see Bewley [15], and Reinganum [90].

> theories which have been proposed ... as foundations for statistical inference are therefore closely related to theories of economic behavior under uncertainty. [9, pp. 409-410]

We also refer to [9, p. 418]. Therefore, Knightian uncertainty is often called *model uncertainty* in this strand of thinking. The latter refers to the impossibility to pin down the accurate or "true" probabilistic model governing the realisation of the future, provided it even exists. Instead, a potentially large set of models is taken into account which may or may not contain the true model.

This is perfectly reflected by one of the favourite mathematical toy models of the risk-uncertainty dichotomy due to Ellsberg [41, Section II]. Consider two urns containing the same known number of balls, each ball being marked with a colour from a known fixed set of colours. The composition of the first urn, i.e. the number of balls of each colour in the urn, is known. Drawing from this urn can be described with a single probability measure on a finite set of outcomes. Betting on the outcome would thus correspond to risk. The composition of the second urn, i.e. the precise number of balls of some colour in the urn, is unknown. Betting on the outcome would hence correspond to uncertainty. As the total number of balls and colours is fixed and known, drawing from that urn can nevertheless be described by taking into account a set of probability measures, each modelling a particular possible composition. This set will also contain the true model.

More fundamentally, this mathematical translation of the epistemological terms "risk" and "uncertainty" invokes the assumption that imperfect knowledge is modelled best in the language of *probabilities*. Cyert & de Groot [30, p. 524] summarise this so-called *Bayesian paradigm* poignantly:

> To the Bayesian, all uncertainty can be represented by probability distributions.

The relation between the Bayesian paradigm and uncertainty is discussed in more detail in Gilboa & Marinacci [61].

Taking the perspective that all uncertainty is best modelled in the language of probabilities, Delbaen's approach of defining risk measures on random variables relative to a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ seems to adhere to this paradigm. However, one should have a closer look at the probability measure $\mathbb{P}$. For the sake of simplicity, we consider a canonical choice of the domain of definition of a risk measure ρ: losses an agent incurs are modelled by equivalence classes of bounded random variables defined by equality outside of a $\mathbb{P}$-null set, i.e. by the space $L^\infty = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ from above. This domain of definition appears frequently in the literature. Suppose a convex monetary risk measure $\rho : L^\infty \to \mathbb{R}$ is applied which has the *Fatou property*, a mild regularity property: Every sequence $(X_n)_{n\in\mathbb{N}} \subset L^\infty$ which converges outside of a set of $\mathbb{P}$-probability 0 to some $X \in L^\infty$ pointwise[6] and is bounded in norm ($\sup_{n\in\mathbb{N}} \|X_n\|_\infty < \infty$) satisfies $\rho(X) \leq \liminf_{n\to\infty} \rho(X_n)$ — the risk of an approximated loss is essentially bounded from above by approximating risks. By [52, Theorem 4.33], ρ can then be expressed as

$$\rho(X) = \sup_{\mathbb{Q}\in\Delta_{\mathbb{P}}} \mathbb{E}_{\mathbb{Q}}[X] - \alpha(\mathbb{Q}), \quad X \in L^\infty. \tag{1.1}$$

[6] In the following, we will also use the standard terminology to refer to such sequences as *converging* $\mathbb{P}$*-almost surely* ($\mathbb{P}$-a.s.).

Here, $\Delta_{\mathbb{P}}$ denotes the set of all probability measures $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ which are absolutely continuous with respect to $\mathbb{P}$ ($\mathbb{Q} \ll \mathbb{P}$),[7] $\mathbb{E}_{\mathbb{Q}}[\cdot]$ denotes the expectation operator with respect to $\mathbb{Q}$, and $\alpha : \Delta_{\mathbb{P}} \to (-\infty, \infty]$ is a so-called *penalty function* uniformly bounded below taking values in the real numbers or $+\infty$. Probability measures with infinite penalty are clearly irrelevant for the computation of $\rho(X)$. (1.1) is a *dual representation* of the convex function ρ adapted from convex analysis. The literature often refers to it as the *robust representation* of the risk measure, in allusion to the computation of $\rho(X)$ as a "worst case" over several alternative probability models.

Justifications for convex monetary risk measures to be "a case study how to deal with model uncertainty in mathematical terms" [52, Preface to 3rd edition] and thus to account for Knightian uncertainty are usually based on such representations. In the *subjective argument* — see also Föllmer & Weber [53, p. 308] — the agent is uncertain which probability model in $\Delta_{\mathbb{P}}$ is adequate or "true". The penalty function α captures this uncertainty. The larger $\alpha(\mathbb{Q})$, the less reason the agent sees to believe in the adequacy of the probabilistic model $\mathbb{Q}$ to govern the realisation of the future. Expected losses $\mathbb{E}_{\mathbb{Q}}[X]$ are thus reduced by $\alpha(\mathbb{Q})$, and $\mathbb{Q}$ only contributes to the supremum in (1.1) in extreme cases, when caution due to the possibility of underestimating its adequacy is rational. Such an interpretation may remind the reader of Knight's "opinions as to the future course of events". Also, it is an argument essentially adapted from decision theory, more precisely from the numerical representation of preferences. Representations of a similar shape as (1.1) as worst case approaches over sets of probability measures have been obtained for multiple classes of preference structures subject to Knightian risk and Knightian uncertainty. To name a few of the most canonical examples, this applies to the multiple prior preferences due to Gilboa & Schmeidler [62] — see also [52, Chapter 2.5] for a presentation more reminiscent of risk measures —, multiplier preferences introduced by Hansen & Sargent [63] — see also [81, Section 4.2.1] —, and to variational preferences axiomatised by Maccheroni et al. [81] overarching the two aforementioned examples.[8]

One could also conceive of a *regulatory argument*. Assume a regulatory agency prescribes a set of probabilistic models $\mathrm{dom}\,(\alpha) := \{\mathbb{Q} \in \Delta_{\mathbb{P}} \mid \alpha(\mathbb{Q}) < \infty\}$ governing the realisation of states and a set of model-dependent monetary loss tolerances $\alpha : \mathrm{dom}\,(\alpha) \to \mathbb{R}$. Given a future loss X, one computes the maximal extent to which the expected loss $\mathbb{E}_{\mathbb{Q}}[X]$ exceeds the loss tolerance $\alpha(\mathbb{Q})$ over all models $\mathbb{Q}$. This procedure could be called a *stress test*.

Here and in the following, the term *model space* will refer to the domain of definition of a risk measure consisting of losses whose risk is to be measured. In the present case, the probability measure $\mathbb{P}$ defining the model space L^∞ serves as a *reference model*. We argue that it tends to play a curious double role.

- The mathematical setting of the problem does not depend too heavily on $\mathbb{P}$. In fact, it only plays the soft role of a gauge which determines which events $A \in \mathcal{F}$ the risk assessment

[7] That is, $\mathbb{Q}(N) = 0$ holds for each event $N \in \mathcal{F}$ with $\mathbb{P}(N) = 0$.

[8] It should be emphasised, however, that there is a major distinction between variational preferences and risk measures. In their numerical representation, the former penalise *expected utility*, computed with a fixed utility function over the set of consequences, but with respect to different probabilistic models. The usual argument claims that by this procedure Knightian risk — captured by expected utility under a fixed probabilistic model — is clearly separated from Knightian uncertainty — captured by taking multiple models into account and penalising them. As that utility would be linear in case of risk measures, it seems questionable if they account for risk and uncertainty simultaneously in the described manner.

inherently deems certain ($\mathbb{P}(A) = 1$) or impossible ($\mathbb{P}(A) = 0$). The model space of losses is invariant under all choices of $\mathbb{P}$ agreeing on the same notions of impossibility and certainty: If $\mathbb{P}$ and $\mathbb{P}^*$ are two *equivalent* probability measures sharing the same null sets ($\mathbb{P} \approx \mathbb{P}^*$), then $L^\infty(\Omega, \mathcal{F}, \mathbb{P}) = L^\infty(\Omega, \mathcal{F}, \mathbb{P}^*)$ and also $\Delta_\mathbb{P} = \Delta_{\mathbb{P}^*}$.

- Consider the dual representation above and assume that two equivalent probability measures $\mathbb{P} \approx \mathbb{P}^*$ satisfy $\alpha(\mathbb{P}) < \infty$ and $\alpha(\mathbb{P}^*) = \infty$. In the subjective interpretation, the agent would assign a certain degree of plausibility to $\mathbb{P}$ being the "true model", but deems $\mathbb{P}^*$ completely implausible. Therefore it should make an interpretational difference whether the underlying choice of the probability space in (1.1) is $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ and not $L^\infty(\Omega, \mathcal{F}, \mathbb{P}^*)$.

- Suppose $\mathbb{P}$ is treated as the unobservable "physical measure" which actually governs the realisation of future states and which is not known to the agent measuring risk. Does the mathematical formulation (1.1) then assume *a priori* knowledge beyond the agent's assessment of plausibility?

- Consider the classical L^p-model spaces, $p \in \{0\} \cup [1, \infty]$, which have appeared already in the discussion of the extension problem for law-invariant risk measures. (Note that L^0 is the space of equivalence classes of all real-valued random variables.) They frequently serve as domains of definition of risk measures, for instance in Filipović & Svindland [48], Frittelli & Rosazza Gianin [55], and Kaina & Rüschendorf [67]. Among them, the only ones which are invariant under equivalent choices of the probability measure are the extreme cases $p = \infty$ and $p = 0$ discussed in Delbaen [33, 34]. For unbounded random variables defined by their integrability properties, the choice of the probability model is therefore essential. If ρ accounts for Knightian uncertainty/model uncertainty, where should this meaningful probability measure $\mathbb{P}$ thus come from?

- If $\mathbb{P}$ plays a strong and not a weak role or if $\mathbb{P}$ is the unobservable physical measure, how can *a priori* assumptions like non-atomicity or standardness of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which frequently appear in the literature be justified in the presence of Knightian uncertainty?[9]

The role of the reference measure $\mathbb{P}$ becomes even more curious if one considers the case of a $\mathbb{P}$-law-invariant risk measure $\rho : L^\infty \to \mathbb{R}$ under the additional assumption that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is non-atomic. In this case, ρ automatically has the Fatou property and a robust representation in the fashion of (1.1) exists — cf. [65, 94]. Moreover, the penalisation α can always be chosen as the so-called *convex conjugate* of ρ, i.e. $\alpha = \rho^*$ for the function

$$\rho^* : \Delta_\mathbb{P} \to (-\infty, \infty], \quad \mathbb{Q} \mapsto \sup_{X \in L^\infty} \mathbb{E}_\mathbb{Q}[X] - \rho(X).$$

The set of all Radon-Nikodym derivatives

$$\mathcal{Q} := \{\tfrac{d\mathbb{Q}}{d\mathbb{P}} \mid \rho^*(\mathbb{Q}) < \infty\} \subset L^1_+$$

is a *law-invariant* or *rearrangement invariant subset* of the positive cone

[9] *Non-atomicity* of $(\Omega, \mathcal{F}, \mathbb{P})$ means that there is a random variable $U : \Omega \to \mathbb{R}$ such that the associated cumulative distribution function $F_U^\mathbb{P}$ under $\mathbb{P}$ is continuous. *Standardness* — in the terminology of [65] — is the combination of non-atomicity of $(\Omega, \mathcal{F}, \mathbb{P})$ and the separability of $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

$$L^1_+ := \{X \in L^1 \mid \mathbb{P}(X \geq 0) = 1\};$$

whenever $Z = \frac{d\mathbb{Q}}{d\mathbb{P}} \in \mathcal{Q}$ and $Z' \in L^1$ has the same cumulative distribution function as Z, i.e. $F_Z^{\mathbb{P}} = F_{Z'}^{\mathbb{P}}$, then there is a probability measure $\mathbb{Q}' \in \Delta_{\mathbb{P}}$ such that $\rho^*(\mathbb{Q}') < \infty$ and $Z' = \frac{d\mathbb{Q}'}{d\mathbb{P}}$. However, law-invariance is *not* preserved under equivalent changes of the reference measure. A functional or a set may be law-invariant with respect to $\mathbb{P}$, but they are not necessarily law-invariant with respect to an equivalent measure $\mathbb{P}^*$. In the law-invariant context, the reference measure $\mathbb{P}$ is consequently far more than a soft gauge. Specifying the correct model $\mathbb{P}$ compared to the specification of an equivalent model $\mathbb{P}^*$ *a priori* provides a plentitude of more crucial information. Hence, the two roles of the probability measure differ discernibly in the law-invariant context.

Nevertheless, the law-invariant example provides an important lesson which is the guiding thread throughout the first contribution to this thesis [76]. Suppose a risk measure $\rho : L^\infty(\Omega, \mathcal{F}, \mathbb{P}^*) \to \mathbb{R}$ with Fatou property over a non-atomic probability space $(\Omega, \mathcal{F}, \mathbb{P}^*)$ is given. We can assume without loss of generality that ρ is *normalised*, that is, $\rho(0) = 0$, and infer its dual representation (1.1) as

$$\rho(X) = \sup_{\mathbb{Q} \in \Delta_{\mathbb{P}^*}} \mathbb{E}_{\mathbb{Q}}[X] - \rho^*(\mathbb{Q}), \quad X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}^*).$$

If there is reason to believe that ρ is law-invariant with respect to a non-atomic probability measure $\mathbb{P}$ equivalent to $\mathbb{P}^*$, $\mathbb{P}$ can be found among the measures

$$(\rho^*)^{-1}(\{0\}) = \{\mathbb{Q} \in \Delta_{\mathbb{P}^*} \mid \rho^*(\mathbb{Q}) = 0 = \inf \rho^*\};$$

this follows e.g. from a slight adaptation of the arguments in [78, Remark 6.3]. In the subjective interpretation of the dual representation above, these measures would be the ones in which the agent has the highest degree of confidence. If it is identified, switching to the reference measure $\mathbb{P}$ then provides the right probabilistic lens suited to the risk measure ρ which resolves the conflict outlined above. Moreover, it facilitates further procedures such as extending the risk measure to larger domains of definition as mentioned in Section 1.1.

Note that the distinguishing feature of this approach is to assume in a first step the risk measure ρ to be given, for instance by behavioural observations of a concrete agent. Then it infers the correct probabilistic and analytic structure suited to ρ. It does not posit the probabilistic and analytic structure *a priori* and considers ρ relative to these model specifications. This should be kept in mind when risk measures are supposed to account for Knightian uncertainty.

1.3 Non-dominated models and Knightian uncertainty

As we have illustrated in the preceding section, the existence of a probabilistic reference model poses interpretational challenges when the aim is to account for Knightian uncertainty. In recent years this has led to attempts to eliminate reference probability measures from models dealing with Knightian uncertainty. These developments trade under the names of *robust* and *model-free* mathematical finance, respectively, and we would like to discuss them briefly in this short interlude.

Robust mathematical finance concerns mathematical models for situations subject to Knightian uncertainty where, in accordance with the Bayesian paradigm, the latter is captured by considering a large set $\mathcal{P}$ of probability measures potentially governing the realisation of relevant economic variables. This is akin to the set $\operatorname{dom}(\alpha)$ in the dual representation of a risk measure (1.1). The probability measures $\mathbb{P} \in \mathcal{P}$ are sometimes called *priors*.

Due to the aforementioned reasons, there has been increasing interest in recent years in *non-dominated* sets of priors. The latter means that there is no probability measure $\mathbb{P}^*$ such that $\mathcal{P} \subset \Delta_{\mathbb{P}^*}$. Exemplary publications are [16, 18, 19, 29, 60, 82, 86], to mention only a few. One of the prime examples of such sets of priors is *volatility uncertainty* in a diffusion model driven by Brownian motion. In this case, the set $\mathcal{P}$ is the set of laws of diffusions having different volatilities and is usually non-dominated. For the latter, we particularly refer to Denis & Martini [36] and Soner et al. [92].

Model-free mathematical finance studies models in which no probabilistic assumption is made on the realisation of relevant economic variables. Instead, one usually focuses on statewise evaluations. In the implicit or explicit presence of a σ-algebra $\mathcal{F}$ on the set of states Ω, this approach can in fact be seen as a limiting extreme case of robust mathematical finance in which the set $\mathcal{P}$ above is chosen to be the full set of *all* probability measures over the state space $(\Omega, \mathcal{F})$; cf. Obłój & Wiesel [87]. Such models have played a prominent role early on, for instance in the initial risk measure framework suggested by Artzner et al. [11]. We also refer to the discussion of such approaches in the monograph by Föllmer and Schied [52, Chapter 4.2] and the references therein. However, model-free finance has in recent years gained substantial complexity.

These two strands of mathematics will only play a minor or implicit role in this thesis. As promising as these attempts at a better mathematical reflection of Knightian uncertainty may seem, they pose highly non-trivial challenges three of which shall be shortly singled out here as a justification.

Firstly, if the set $\mathcal{P}$ of probability measures under consideration is not dominated, its cardinality must exceed the countable. Moreover, the priors must disperse mass among the events in such an extreme manner that no dominating measure can be found. In practical applications, however, the primitives often are empirical distributions rather than genuine random variables. Therefore one has to be very careful not to lose a clear-cut correspondence between mathematical theory and economic interpretation. The probabilities under consideration being non-dominated is furthermore not a *raison d'être* in itself. A non-dominated model should always be examined if it is supported by economic, decision-theoretic or financial reasons.

Secondly, the more Knightian uncertainty a mathematical model of an economic phenomenon "contains" interpretationally, the less tractable it tends to be mathematically. This is often mitigated either by using *ad hoc* methods, or by imposing "regularity assumptions" that make a mathematical treatment of the problem feasible, but potentially overshadow the economic content. Such assumptions should be investigated very carefully, as they may in fact reverse the endeavour to include uncertainty beyond the dominated case. An example of this effect is discussed at greater length in Section 1.6 and Liebrich & Svindland [76].

Thirdly, intuition learned in the classical setting of a state space $(\Omega, \mathcal{F})$ endowed with a single probability measure $\mathbb{P}$ often fails in robust or model-free settings. Even more, as exemplified by the investigation of a "non-dominated version" of the Grothendieck Lemma in Maggis et al. [82], widely used analytic tools of fundamental nature do not necessarily have equivalents in the robust or model-free world. This lack of the analytic toolbox cannot always be overcome. It is still a topic of future research to systematically understand the phase transition in complexity that arises from shifting from a dominated framework to a non-dominated robust or even model-free one.

1.4 Model spaces and capital requirements

The discussion so far has shown there are good economic reasons for various choices of the model space of a risk measure. A scenariowise evaluation of incurred losses under certain information about future events may best be captured by the model space $\mathcal{L}^\infty(\Omega, \mathcal{F})$ of all bounded measurable random variables over a state space Ω endowed with a σ-algebra $\mathcal{F}$. If losses are evaluated according to their statistical and distributional properties, saying that a loss behaves according to a Gaussian distribution would require considering model spaces like $L^p(\Omega, \mathcal{F}, \mathbb{P})$, the space of equivalence classes of random variables with finite p-th moment with respect to a probability measure $\mathbb{P}$ over a measurable space $(\Omega, \mathcal{F})$, $p \in [1, \infty)$. Similar, but more exotic model spaces appearing in the literature are Orlicz hearts and Orlicz spaces; cf. [27, 58, 59]. One may even consider spaces of random variables built on the basis of a non-dominated set of probability measures $\mathcal{P}$ as in the preceding section; see, for instance, Beissner & Denis [14] and Denis et al. [35]. If a model space is furthermore infinite dimensional and considered relative to a topology, usually several choices of this topology are available and can reflect different economic phenomena. This issue is discussed in Mas Colell & Zame [83, Section 4].

All model spaces in this zoo, however, share the common trait of being *Riesz spaces*: They carry a partial order $\preceq$ which is consistent with vector space operations and with respect to which finite sets of elements have "minima" and "maxima". Having such an order at hand is a sensible minimal requirement. It allows to compare losses in an "objective" manner and to combine two losses in a best- and worst-case manner. Most of the suggested spaces, abbreviated by $\mathcal{X}$, even have the stronger property of being *Banach lattices*, i.e. they additionally carry a norm $\|\cdot\|$ such that the normed space $(\mathcal{X}, \|\cdot\|)$ is complete, and such that the norm itself is "compatible" with the order $\preceq$ on $\mathcal{X}$. For a review of the theory of Riesz spaces and Banach lattices, we refer to the classical monographs by Aliprantis & Border [5], Aliprantis & Burkinshaw [6, 7], and Meyer-Nieberg [84].

The different approaches to the model space question, supported by various different ways of economic reasoning, can hence be unified if risk measures can be defined on arbitrary ordered vector spaces or Riesz spaces. A nice side effect of such a definition would be that the risk of objects like vector-valued random variables or stochastic processes can be determined in the same setting. We therefore argue for such a flexible unifying framework throughout the thesis. It turns out that this necessitates a return to the initial *operational* definition of risk measures as in Artzner et al. [11].

Given a risk measure $\rho : \mathcal{X} \to (-\infty, \infty]$ on an arbitrary ordered vector space $(\mathcal{X}, \preceq)$, the axiom of cash-additivity **(R4)** does not have an immediate sensible counterpart. A remedial idea might be to replace cash by a reference instrument U which is positive, non-null, and available at a unit price. In terms of order, $0 \prec U$ should hold. Cash-additivity could then be replaced by the axiom

$$\forall X \in \mathcal{X}\, \forall m \in \mathbb{R}: \ \rho(X + mU) = \rho(X) + m. \tag{1.2}$$

Which implications does (1.2) have for the discounting argument from Section 1.1 though? If $\mathcal{X}$ is a function space, e.g. L^∞ over some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, its natural order is the $\mathbb{P}$-a.s. order introduced in Section 1.1: $X \leq Y$ if $\mathbb{P}(X \leq Y) = 1$. Hence, such a reference instrument U could be any random variable with the property $\mathbb{P}(U \geq 0) = 1$ and $\mathbb{P}(U > 0) > 0$. However, this could mean that U can *default* and not yield any payoff in some event, i.e. $\mathbb{P}(U = 0) > 0$. The investment in U has the role of de-risking X with a "safe" bet. Such a safe bet could be, for instance, the investment in a sovereign bond, and the possibility of a default of U cannot be ruled out *a priori* as the government-debt crisis ensuing after the last financial crisis proved. U can therefore not be assumed to be a numéraire suited for discounting. We also refer to [85, Section 1.3.1].

Even if U is a numéraire, i.e. $\mathbb{P}(U > 0) = 1$, the discounting argument leading to cash-additive risk measures cannot be performed *without changing the model space*: The space

$$\{\tfrac{X}{U} \mid X \in L^\infty\}$$

is not a subset of L^∞ unless the payoff of U is bounded below, i.e. there is some $\varepsilon > 0$ such that $\mathbb{P}(U \geq \varepsilon) = 1$. It also cannot be equipped with the usual L^∞-topology available before discounting. Note that the assumption of boundedness from below can be violated even if U will not default; the payoff can become arbitrarily small with small, but positive probability. For more information on this problem, we refer to [85, Section 1.3.2].

Moreover, as risk is usually measured *after discounting*, the discounting procedure should not add additional risk. This is questionable if the numéraire is random as above. The reader may think of the risk posed by uncertain future interest rates, an issue raised and discussed in El Karoui & Ravenelli [40]. In particular, the crucial *law-invariance property* is unstable under discounting; see also [85, Section 1.3.4]. Suppose the numéraire is not riskless and recall that $F_X^{\mathbb{P}}$ denotes the cumulative distribution function of a random variable X under $\mathbb{P}$. Then we may find losses X, Y and X', Y' such that

$$F_X^{\mathbb{P}} = F_Y^{\mathbb{P}} \text{ and } F_{X'}^{\mathbb{P}} \neq F_{Y'}^{\mathbb{P}},$$

but

$$F_{X/U}^{\mathbb{P}} \neq F_{Y/U}^{\mathbb{P}} \quad \text{and} \quad F_{X'/U}^{\mathbb{P}} = F_{Y'/U}.$$

If the suggested tradeoff is to assume U to be actually riskless, but no discounting takes place, it seems questionable financially whether such a payoff exists over long time horizons.

We refer to Munari [85, Chapter 1.3] for a thorough discussion of even more problems arising from the discounting argument and illustrating examples.

If risk measures are to be defined on general ordered vector spaces, the axiom of cash-additivity must be dropped. In light of the aforementioned conundrums, however, even replacing it by the similar-looking axiom (1.2) makes a substantial qualitative difference. Moreover, there are good financial reasons to embrace these differences.

Inspired by Farkas et al. [43, 44, 45], Frittelli & Scandolo [57], Munari [85], and Scandolo [91] we opt throughout this thesis for a generalised notion of risk measurement which returns to the initial ideas by Artzner et al. The approach makes sense on any ordered vector space. It is also interpretationally rigorous in that *risk* is identified with a *capital requirement* relative to certain *fundamentals*: a notion of acceptability of losses, portfolios of liquidly traded securities allowed for hedging, and observable prices for these securities.

Suppose an agent would like to determine the risk of loss profiles modelled by a (non-trivial) ordered vector space $(\mathcal{X}, \preceq)$. A first step would be to gain clarity about which losses pose an acceptable risk to the agent. This is reflected by choosing an *acceptance set* $\mathcal{A} \subset \mathcal{X}$ within the ambient space of losses. It should obey the following economically motivated requirements which have already made their appearance in Section 1.1:

- The set $\mathcal{A}$ is a non-empty proper subset of $\mathcal{X}$: $\emptyset \neq \mathcal{A} \subsetneq \mathcal{X}$. This excludes degenerate cases of risk measurement.

- $\mathcal{A}$ is convex: The convex combination $\lambda X + (1-\lambda)Y$ lies in $\mathcal{A}$ for all choices of $X, Y \in \mathcal{A}$ and $0 \leq \lambda \leq 1$. Economically, diversification among acceptable losses does not lead to the loss of acceptability.

- $\mathcal{A}$ is MONOTONE: Whenever a loss X is better compared to an acceptable loss Y, i.e. $X \preceq Y$ and $Y \in \mathcal{A}$, then X should be acceptable as well, i.e. $X \in \mathcal{A}$. Hence, the acceptability criterion obeys the standard financial paradigm of "more is better" (or equivalently: "less loss is better").

Having chosen such a set $\mathcal{A}$, it poses a *capital adequacy test* for an arbitrary loss $X \in \mathcal{X}$, which is deemed adequately capitalised if it belongs to the acceptance set, and inadequately capitalised otherwise.

If the loss X does not pass the capital adequacy test, the agent may in a second step want to take remedial actions. In line with the ideas of Artzner et al. [11] and the definition of a cash-additive risk measure, the agent has access to a *security market*: She can raise capital and buy a portfolio of securities in that market which, when combined with the loss profile X in question, results in the hedged position to pass the capital adequacy test.

Mathematically, a security market is a pair $(\mathcal{S}, \mathfrak{p})$ with the following properties:

- $\mathcal{S} \subset \mathcal{X}$ is a non-trivial vector space of finite dimension which contains some non-null and positive U, i.e. $0 \prec U$. We call $\mathcal{S}$ the SECURITY SPACE, its elements are called SECURITIES.

- $\mathfrak{p} : \mathcal{S} \to \mathbb{R}$ is a linear and positive functional, the latter meaning $0 \leq \mathfrak{p}(Z)$ whenever $Z \in \mathcal{S}$ satisfies $0 \preceq Z$. Moreover, we assume there is a non-null positive security $0 \prec U \in \mathcal{S}$ such that $\mathfrak{p}(U) = 1$. The functional $\mathfrak{p}$ is called the PRICING FUNCTIONAL.

Which assets qualify as securities or adequately "safe bets" for hedging may both be an individual management decision or prescribed by a regulatory oversight. The assumption that $\mathcal{S}$ is only of finite dimension is not problematic from a real-world perspective and will turn out to be mathematically instrumental to studying risk sharing problems in the second part of this thesis, Chapter 3. The definition can be immediately extended to infinite dimensional security spaces though.

The assets in the security space are liquidly traded at arbitrary quantities, which is reflected by the linearity of the pricing functional $\mathfrak{p}$. Again, the definition can be easily extended to allow for convex pricing rules which would model a non-trivial bid-ask spread in the security market, i.e. $\mathfrak{p}(Z) + \mathfrak{p}(-Z) > 0$ is possible for some $Z \in \mathcal{S}$. We will sometimes assume that the security space is a proper subspace of $\mathcal{X}$, i.e. $\mathcal{S} \subsetneq \mathcal{X}$. This merely excludes redundant situations in which only the "risk" of liquidly traded securities is determined. The assumption is met automatically if the dimension of $\mathcal{X}$ is infinite. Note that in our definition of a security market we adopt the usual market notion of order. A true gain corresponds to a positive security, a true loss or short sale to a negative security, i.e. $Z \preceq 0$. For a loss $X \in \mathcal{X}$, this requires to define the loss secured with some $Z \in \mathcal{S}$ as $X + (-Z)$. The existence of some "bond-like" security $0 \preceq U \in \mathcal{S}$ with unit price $\mathfrak{p}(U) = 1$ in the security market seems natural and well-founded in practical applications.

In contrast to the setting of Artzner et al. [11], our security spaces may be higher dimensional, and more than one asset may be available in the security market. This results in a decrease of the cost of securitisation, as the agent may invest in a portfolio of securities specifically designed to secure a particular loss rather than restricting the remedial action to investing in a single asset independent of the loss profile.

Having these fundamentals in place, we can now formulate a condition on their interplay which guarantees that they provide a rational basis to compute capital requirements and thus risks. Given an acceptance set $\mathcal{A}$ and a security market $(\mathcal{S}, \mathfrak{p})$, we say that the triplet $\mathcal{R} := (\mathcal{A}, \mathcal{S}, \mathfrak{p})$ is a RISK MEASUREMENT REGIME if the following condition holds:

$$\forall X \in \mathcal{X}: \ \sup\{\mathfrak{p}(Z) \mid Z \in \mathcal{S},\ X + Z \in \mathcal{A}\} < \infty. \tag{1.3}$$

Assume condition (1.3) is violated. Then the agent can find a financial position X and a sequence $(Z_n)_{n\in\mathbb{N}}$ of securities which become arbitrarily valuable — $0 \leq \mathfrak{p}(Z_n) \uparrow \infty$ — with the following property: The agent can sell them short in the security market, hedging with X against his short sale, without at some point failing to pass the capital adequacy test. This opens an arbitrage-like opportunity and presents a market failure of the security market $(\mathcal{S}, \mathfrak{p})$ relative to the capital adequacy test $\mathcal{A}$. It should therefore be excluded.

Now, assuming condition (1.3), we obtain an immediate definition of the risk a loss poses in the risk measurement regime $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \mathfrak{p})$ as a *capital requirement*: the infimal capital which needs to be raised and invested in a security portfolio available at market price in the security market such that the loss *hedged* with that security portfolio is acceptable and passes the capital adequacy test. The resulting risk measure associated to the risk measurement regime $\mathcal{R}$ would thus be the functional

$$\rho_{\mathcal{R}} : \mathcal{X} \to (-\infty, \infty], \quad X \mapsto \ \inf\{\mathfrak{p}(Z) \mid Z \in \mathcal{S},\ X - Z \in \mathcal{A}\} \quad (\inf \emptyset = \infty). \tag{1.4}$$

By (1.3), the risk measure $\rho_{\mathcal{R}}$ cannot produce the value $-\infty$, and it is also sensible to assume that negatively infinite risk, which would correspond to the possibility to withdraw arbitrary amounts of wealth from a loss, does not exist.

The framework of risk measures as capital requirements explained above is flexible enough to encompass both the framework of Artzner et al. [11] as well as the usual case of convex monetary risk measures with the cash-additivity property. However, it goes far beyond those two cases.

Immediate consequences of this definition are the following properties of the functional $\rho_{\mathcal{R}}$.

- $\rho_{\mathcal{R}}(Y) \leq 0$ for any choice of $Y \in \mathcal{A}$. Hence, $\rho_{\mathcal{R}}$ is a proper function, i.e. $\rho_{\mathcal{R}}^{-1}(\{-\infty\}) = \emptyset$ and $\rho_{\mathcal{R}} \not\equiv \infty$.

- $\rho_{\mathcal{R}}$ is $\preceq$-MONOTONE, i.e. it satisfies axiom **(R1)**.

- $\rho_{\mathcal{R}}$ generalises axiom **(R2)** in that it is $\mathcal{S}$-ADDITIVE: $\rho_{\mathcal{R}}(X+Z)=\rho_{\mathcal{R}}(X)+\mathfrak{p}(Z)$ for all $X\in\mathcal{X}$ and all $Z\in\mathcal{S}$.

- $\rho_{\mathcal{R}}$ satisfies axiom **(R3)** (quasi-convexity) and is even convex.

1.5 Contributions of this thesis

We have finally set the stage with some of the leitmotifs of the contributions [76, 77, 78] collected in this thesis — the risk-uncertainty problem, general model spaces, and risk measures as capital requirements. The remainder of this introduction is devoted to the discussion of the articles and the exposition of their main results. The aforementioned articles are:

(1) F.-B. Liebrich & G. Svindland (2017), MODEL SPACES FOR RISK MEASURES.
Insurance: Mathematics and Economics, Vol. 77, pp. 150–165, November 2017.

(2) F.-B. Liebrich & G. Svindland (2019), RISK SHARING FOR CAPITAL REQUIREMENTS WITH MULTIDIMENSIONAL SECURITY MARKETS.
Forthcoming in *Finance and Stochastics*.

(3) F.-B. Liebrich & G. Svindland (2019), EFFICIENT ALLOCATIONS UNDER LAW-INVARIANCE: A UNIFYING APPROACH.
Journal of Mathematical Economics, Vol. 84, pp. 28–45, October 2019.

1.6 Solving the model space problem for risk measures

In MODEL SPACES FOR RISK MEASURES [76], see also Chapter 2, we address the following question: Can the *model space problem* be solved for risk measures of type (1.4) defined on a space of measurable functions? More precisely, this problem considers a specific risk measure defined on, say, bounded measurable losses on an underlying measurable space $(\Omega,\mathcal{F})$ of states, and the corresponding risk or uncertainty preferences expressed by it. Then the following two questions arise: Is there a meaningful probabilistic structure on $(\Omega,\mathcal{F})$ *intrinsic* to the risk measure? Can a maximal domain of definition be found whose structure is *inherited* from the risk measure, which carries it as well, and which contains unbounded losses? If the answer to both questions is affirmative, such a combination of probability measure and model space would be the "right one" to harmonise the seemingly conflicting two normative paradigms of *minimal model dependence* and *maximal domain.*

The first paradigm reflects that in the presence of Knightian uncertainty, a probabilistic model $\mathbb{P}$ for the realisation of future states of the economy $\omega\in\Omega$ consistent with information given by a σ-algebra $\mathcal{F}$ on Ω may be impossible to be known or elicited, and thus a model space should not depend too heavily on some specific model. In many standard market models and financial calculations, though, unbounded distributions appear naturally as limiting objects of bounded distributions. The Central Limit Theorem, for instance, could imply that the distribution of a certain financial object is a Gaussian distribution. Similarly, log-normal distributions appear naturally in the context of the Black-Scholes-Merton model. This leads to the second paradigm of maximal domain which states that model spaces should include standard unbounded models. In this respect, [76] embeds in the literature on the extension problem for risk measures: see, for instance, Cerreia-Vioglio et al. [24] and Filipović & Svindland [48].

Our investigations begin with the completely model-free setting of a measurable space $(\Omega, \mathcal{F})$ and the space of bounded measurable real-valued functions over $(\Omega, \mathcal{F})$, $\mathcal{L}^\infty := \mathcal{L}^\infty(\Omega, \mathcal{F})$. It is a Banach lattice when equipped with the supremum norm $|X|_\infty := \sup_{\omega\in\Omega} |X(\omega)|$, $X \in \mathcal{L}^\infty$, and the pointwise order $\leq$. The elements of $\mathcal{L}^\infty$ model statewise loss estimates contingent on states $\omega \in \Omega$. We consider an agent whose risk attitudes over $\mathcal{L}^\infty$ are captured by a risk measurement regime $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \mathfrak{p})$ as described in the preceding section satisfying $\mathcal{S} \subsetneq \mathcal{X}$. At the time of writing the paper, we made the assumption that the pricing functional is in fact strictly positive, i.e. $\mathfrak{p}(Z) > 0$ for all $Z \in \mathcal{S}\backslash\{0\}$ with $Z \geq 0$. It turns out that this assumption is not needed right from the beginning and we can in fact begin with the the slightly more general definition given in the preceding section.

No structural properties have been assumed so far which would prevent the capital requirement under consideration to account both for risk or uncertainty. For practical reasons, the agent's attitude towards losses that may be large, but bounded in total and increasingly unlikely to occur, is of crucial importance. If the agent is slightly risk- or uncertainty-seeking, these losses should not really matter in the limit. Given a risk measurement regime $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \mathfrak{p})$ such that the resulting risk measure $\rho_\mathcal{R}$ is finite, we call $\rho_\mathcal{R}$ CONTINUOUS FROM ABOVE if

$$\rho_\mathcal{R}(X_n) \downarrow \rho_\mathcal{R}(X)$$

whenever $(X_n)_{n\in\mathbb{N}} \subset \mathcal{L}^\infty$ is a sequence which satisfies $X_n \downarrow X \in \mathcal{L}^\infty$ in order, i.e. $X_n(\omega) \downarrow X(\omega)$ for all $\omega \in \Omega$. Continuity from above will allow us to derive the structure which fits precisely to the agent's attitudes towards risk or uncertainty.

Let **ba** denote the space of all finitely additive set functions $\mu : \mathcal{F} \to \mathbb{R}$ with bounded total variation, i.e. the dual space of $(\mathcal{L}^\infty, |\cdot|_\infty)$ up to isometric isomorphism. Let $\mathbf{ca} \subset \mathbf{ba}$ denote the subspace of countably additive signed measures. For two non-empty sets $M, M' \subset \mathbf{ba}$, we say $M \ll M'$ if $\sup_{\nu\in M'} |\nu|(N) = 0$ implies $\sup_{\mu\in M} |\mu|(N) = 0$; here, $|\mu|$ denotes the order modulus of an element $\mu \in \mathbf{ba}$ defined by $|\mu|(A) = \sup\{\mu(B) - \mu(A\backslash B) \mid B \in \mathcal{F},\, B \subset A\}$.[10] $M \approx M'$ means that $M \ll M'$ and $M' \ll M$, and for $\nu \in \mathbf{ba}$ we set $\mathbf{ca}_\nu := \{\mu \in \mathbf{ca} \mid \{\mu\} \ll \{\nu\}\}$.

In [76, Proposition 3.1 & Corollary 3.2], we develop a thorough description of continuity from above in terms of the geometry of the fundamentals $(\mathcal{A}, \mathcal{S}, \mathfrak{p})$. If $\mathcal{R}$ is such that $\rho_\mathcal{R}$ is finite and we define the barrier cone $\mathcal{B}(\mathcal{A}) := \{\mu \in \mathbf{ba} \mid \sup_{Y\in\mathcal{A}} \int Y\, d\mu < \infty\}$, then $\rho_\mathcal{R}$ is continuous from above if and only if every lower level set $\{\mu \in \mathbf{ba} \mid \rho_\mathcal{R}^*(\mu) \leq c\}$, $c \in \mathbb{R}$, of the convex conjugate $\rho_\mathcal{R}^* : \mathbf{ba} \to (-\infty, \infty]$ defined by

$$\rho_\mathcal{R}^*(\mu) := \sup_{X\in\mathcal{L}^\infty} \int X\, d\mu - \rho_\mathcal{R}(X), \quad \mu \in \mathbf{ba},$$

[10] There is a typo in the corresponding definition in [76], cf. last paragraph on p. 153. The sentence should read: "As facilitating notation, for non-empty sets $M, M' \subseteq \mathbf{ba}$, we write $M \ll M'$ if and only if $|\nu|(A) = 0$ for all $\nu \in M'$ implies $|\mu|(A) = 0$ for all $\mu \in M$, $A \in \mathcal{F}$."

is compact in the $\sigma(\mathbf{ca}, \mathcal{L}^\infty)$-topology.[11] This is equivalent to

$$\{\mu \in \mathcal{B}(\mathcal{A}) \mid \forall\, Z \in \ker(\mathfrak{p}) :\ \int Z\, d\mu = 0\} \subset \mathbf{ca}.$$

The latter is implied by, but not equivalent to the condition $\mathcal{B}(\mathcal{A}) \subset \mathbf{ca}$. If $\mathcal{S}$ is furthermore constrained to be one-dimensional, $\rho_{\mathcal{R}}$ is continuous from above if, and only if, $\mathcal{B}(\mathcal{A}) \subset \mathbf{ca}$. Similar results have been obtained in [24].

Now, by [76, Theorem 3.3], if $\rho_{\mathcal{R}}$ is continuous from above, there exists a WEAK REFERENCE PROBABILITY MEASURE $\mathbb{P}$. This is defined as a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ such that $\rho^*_{\mathcal{R}}(\gamma\mathbb{P}) < \infty$ for a uniquely determined $\gamma > 0$, and $\{\mathbb{P}\} \approx \mathrm{dom}\,(\rho^*_{\mathcal{R}}) := \{\mu \in \mathbf{ba} \mid \rho^*_{\mathcal{R}}(\mu) < \infty\}$. Hence, $\mathrm{dom}\,(\rho^*_{\mathcal{R}}) \subset (\mathbf{ca}_{\mathbb{P}})_+$, where the latter denotes the set of all finite countably additive measures absolutely continuous with respect to $\mathbb{P}$. Note in particular the condition $\rho^*_{\mathcal{R}}(\gamma\mathbb{P}) < \infty$. In the subjective interpretation of the dual representation, this means that the agent at least assigns a certain degree of plausibility to the model $\mathbb{P}$ (or rather the appropriately scaled version of $\mathbb{P}$, where the scaling factor γ is dictated by the pricing functional $\mathfrak{p}$). Note that by means of dual representation, this implies that $\rho_{\mathcal{R}}(X) = \rho_{\mathcal{R}}(Y)$ whenever $X = Y$ $\mathbb{P}$-a.s. This result critically illustrates one of the pitfalls of the model-free approach to financial mathematics. Under a standard and conceptually not far-fetched continuity assumption, the underlying model framework is necessarily dominated.

Consequently, under the assumption of continuity from above, we may define the same risk measure $\rho_{\mathcal{R}}$ on the model space $L^\infty_{\mathbb{P}} := L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ without any loss of information, particularly without losing continuity from above; see [76, Corollary 3.5]. (We will not suppress dependence on the reference measure $\mathbb{P}$ here.) One should also note that the translated pricing functional is necessarily strictly positive.

We also discuss a more demanding question: Given a risk measurement regime $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \mathfrak{p})$ on $L^\infty_{\mathbb{P}}$ such that the resulting risk measure is finite, normalised, i.e. $\rho_{\mathcal{R}}(0) = 0$, continuous from above, and has $\mathbb{P}$ as weak reference probability model, can we choose $\mathbb{P}$ to be a *strong reference model*? The latter means that $\rho^*_{\mathcal{R}}(\gamma\mathbb{P}) = 0$ for a suitable constant $\gamma > 0$, or equivalently

$$\forall\, X \in L^\infty_{\mathbb{P}} :\ \rho(X) \geq \gamma\mathbb{E}_{\mathbb{P}}[X].$$

The existence of strong reference models is discussed in [76, Theorems 3.9 and 3.10]. In the subjective interpretation of the dual representation, a strong reference model enjoys full confidence of the agent *and* captures the complete information given by $\mathrm{dom}\,(\rho^*_{\mathcal{R}})$, where we now consider the convex conjugate defined on $(\mathbf{ca}_{\mathbb{P}})_+$. Moreover, it may be interpreted as a generalisation of the *loadedness axiom*, cf. Wang [97] who emphasises its importance in the context of insurance premium pricing and that it unfortunately tends to be neglected as a property of risk measures and capital requirements. To give an example, [76, Proposition 3.7] shows that if the acceptance set $\mathcal{A}$ is law-invariant with respect to a probability measure $\mathbb{P}$, $\mathbb{P}$ will itself be a strong reference model under mild assumptions. This generalises the observation we made at the end of Section 1.2.

Note that the existence of weak or strong reference models does not conflict with the paradigm of minimal model dependence. They are not extrinsic, no model space dependent

[11] The latter refers to the weak topology arising from the dual pairing $\langle \mathbf{ca}, \mathcal{L}^\infty \rangle$, i.e. the weakest topology such that all linear functionals $\mathbf{ca} \ni \mu \mapsto \int X\, d\mu$, $X \in \mathcal{L}^\infty$ arbitrary, are continuous.

on a particular probability measure $\mathbb{P}$ is postulated. Instead, they are implied by or inherent to the scenariowise risk criterion $\rho_{\mathcal{R}} : \mathcal{L}^\infty \to \mathbb{R}$.

Our aim is, however, to extend the risk measure to a much larger domain of definition that also contains unbounded random variables. Assume $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \mathfrak{p})$ to be a risk measurement regime on $L^\infty_{\mathbb{P}}$ such that the resulting risk measure $\rho_{\mathcal{R}}$ is finite, normalised, continuous from above, and has $\mathbb{P}$ as weak reference model. As before we set

$$\rho^*_{\mathcal{R}}(\mu) := \sup_{X \in L^\infty_{\mathbb{P}}} \int X \, d\mu - \rho^*_{\mathcal{R}}(\mu), \quad \mu \in \mathbf{ca}_{\mathbb{P}},$$

and note that $\mathrm{dom}\,(\rho^*_{\mathcal{R}}) \subset (\mathbf{ca}_{\mathbb{P}})_+$. Define the functional

$$\rho(|X|) = \sup_{\mu \in \mathrm{dom}\,(\rho^*_{\mathcal{R}})} \int |X| \, d\mu - \rho^*_{\mathcal{R}}(\mu), \quad X \in L^0_{\mathbb{P}},$$

which will serve as a norming functional. For $c > 0$ and $X \in L^0_{\mathbb{P}}$ let

$$\|X\|_{c,\mathcal{R}} := \inf \{\lambda > 0 \mid \rho\left(\lambda^{-1}|X|\right) \leq c\} \quad (\inf \emptyset := \infty),$$

and $\|X\|_{\mathcal{R}} := \|X\|_{1,\mathcal{R}}$. The Minkowski domain for $\rho_{\mathcal{R}}$ is the set

$$L^{\mathcal{R}} := \{X \in L^0_{\mathbb{P}} \mid \|X\|_{\mathcal{R}} < \infty\}.$$

In [76, Proposition 4.2] we show that $X \in L^0_{\mathbb{P}}$ satisfies $\|X\|_{c,\mathcal{R}} < \infty$ if, and only if, $X \in L^{\mathcal{R}}$, which is the case if, and only if, $\int |X| \, d\mu < \infty$ for any choice of $\mu \in \mathrm{dom}\,(\rho^*_{\mathcal{R}})$. Moreover, if $\leq$ denotes the $\mathbb{P}$-a.s. order on $L^{\mathcal{R}}$ and $c > 0$ is given, $(L^{\mathcal{R}}, \leq, \|\cdot\|_{c,\mathcal{R}})$ is a Banach lattice, and all norms $\|\cdot\|_{c,\mathcal{R}}$ are equivalent. Furthermore $L^\infty_{\mathbb{P}} \subset L^{\mathcal{R}} \subset L^1_{\mathbb{P}}$, i.e. $L^{\mathcal{R}}$ typically extends beyond bounded random variables, and the Minkowski domain is independent of the choice of $\mathbb{P}$ among weak reference probability models. Spaces like the Minkowski domain have already been introduced in Kupper & Svindland [73], Owari [88], and Svindland [93].

Note that the structure and geometry of the Minkowski domain is completely determined by the initial risk measure defined on $L^\infty_{\mathbb{P}}$ or even $\mathcal{L}^\infty$. As a domain of definition, it thus arises from the risk measure itself and is *robust* under transformations within the relevant class of probabilistic models, the class of weak reference models. If there exists a well-defined extension of $\rho_{\mathcal{R}}$ to $L^{\mathcal{R}}$, it seems valid to call the Minkowski domain a canonical choice of the model space. One may indeed define multiple such extensions, either dually or by approximation of unbounded random variables using bounded random variables.

Consider the sets

(i) $\tilde{\mathcal{A}} := \{X \in L^{\mathcal{R}} \mid \forall \mu \in \mathrm{dom}\,(\rho^*_{\mathcal{R}}) : \int X \, d\mu \leq \rho^*_{\mathcal{R}}(\mu)\}$,

(ii) $\mathcal{A}_\xi := \{X \in L^{\mathcal{R}} \mid \sup_{m \in \mathbb{N}} \inf_{n \in \mathbb{N}} \rho_{\mathcal{R}}((-n) \vee X \wedge m) \leq 0\}$,

(iii) $\mathcal{A}_\eta := \{X \in L^{\mathcal{R}} \mid \inf_{n \in \mathbb{N}} \sup_{m \in \mathbb{N}} \rho_{\mathcal{R}}((-n) \vee X \wedge m) \leq 0\}$.

For every choice $\mathcal{A} \in \{\tilde{\mathcal{A}}, \mathcal{A}_\xi, \mathcal{A}_\eta\}$, $\mathfrak{R} := (\mathcal{A}, \mathcal{S}, \mathfrak{p})$ is a risk measurement regime on $L^{\mathcal{R}}$ such that $\rho_{\mathfrak{R}}|_{L^\infty_{\mathbb{P}}} = \rho_{\mathcal{R}}$. Moreover, as we show in [76, Theorems 4.4, 4.5, 4.7] and [76, Proposition 4.6], the following properties hold:

- $\tilde{\mathcal{A}} = \mathcal{A}_\xi$, and $\rho_{\tilde{\mathcal{R}}} = \rho_{\mathcal{R}_\xi} \leq \rho_{\mathcal{R}_\eta}$.

- $\rho_{\tilde{\mathcal{R}}}$ is order-l.s.c.[12]
- $\rho_{\tilde{\mathcal{R}}}$ preserves the dual representation of $\rho_{\mathcal{R}}$: for all $X \in L^{\mathcal{R}}$,

$$\rho_{\tilde{\mathcal{R}}}(X) = \sup_{\mu \in \mathrm{dom}\,(\rho^*_{\mathcal{R}})} \int X \, d\mu - \rho^*_{\mathcal{R}}(\mu).$$

- If for some $X \in L^{\mathcal{R}}$ we have $\rho_{\mathcal{R}_\xi}(X) = \rho_{\mathcal{R}_\eta}(X)$, then

$$\rho_{\tilde{\mathcal{R}}}(X) = \lim_{n\to\infty} \rho_{\mathcal{R}}((-n) \vee X \wedge n); \tag{1.5}$$

As a consequence, there are plenty of well-defined extensions of $\rho_{\mathcal{R}}$ to the Minkowski domain. We have demonstrated in [76, Example 5.2] that there are instances when $\rho_{\mathcal{R}_\xi} < \rho_{\mathcal{R}_\eta}$ holds locally. Hence, whether or not order lower semicontinuity is preserved by extension highly depends on the order in which the limits in the cut-off operations in (ii) and (iii) above are taken. An open question is whether or not $\rho_{\mathcal{R}_\eta}$ is l.s.c.

The paper further complements its solution of the extension problem by studying the peculiar geometry of $L^{\mathcal{R}}$, which is akin to the one of Orlicz spaces, and the dual space of $L^{\mathcal{R}}$. These findings allow us to link the validity of the local equality (1.5) to the existence of regular subgradients of $\rho_{\tilde{\mathcal{R}}}$ and $\rho_{\mathcal{R}_\eta}$, i.e. bounded linear functionals $\ell : L^{\mathcal{R}} \to \mathbb{R}$ with the property

$$\rho_{\mathcal{R}}(Y) \geq \rho_{\mathcal{R}}(X) + \ell(Y - X), \quad Y \in L^{\mathcal{R}},$$

which may be identified with integrals with respect to some countably additive measure μ on $(\Omega, \mathcal{F})$: for all $X \in \mathcal{X}$, $\ell(X) = \int X \, d\mu$. More precisely, we prove a number of sufficient conditions for the existence of such regular subgradients; cf. [76, Section 4.4]. Subgradients will also play an important role in the next contribution we discuss.

1.7 Risk sharing in agent systems

The two other papers collected in this thesis are less concerned with the conceptual challenges of Knightian uncertainty. They rather address the part of the title of this thesis so far lacking in the picture: optimisation on infinite dimensional spaces. It has been mentioned already that systemic risk — the risk a system of financial institutions faces due to a shock of the economy — will not play an explicit role in this thesis. However, we will address the question of *risk sharing* (and to a lesser extent *capital allocation*) within a system.

This problem considers an economy populated by finitely many agents. We label them with numbers $i \in \{1, ..., n\} =: [n]$. At a fixed future time point, each of them incurs a loss net of gains X_i whose risk she measures with an individual risk measure ρ_i. On a systemic level, there are two key quantities:

[12] That is whenever $(X_n)_{n\in\mathbb{N}} \subset L^{\mathcal{R}}$ is a sequence converging to $X \in L^{\mathcal{R}}$ $\mathbb{P}$-a.s. for which another element $Y \in L^{\mathcal{R}}$ can be found such that $\mathbb{P}(\sup_{n\in\mathbb{N}} |X_n| \leq |Y|) = 1$, then

$$\rho_{\tilde{\mathcal{R}}}(X) \leq \liminf_{n\to\infty} \rho_{\tilde{\mathcal{R}}}(X_n).$$

- the *aggregated loss within the system*, which should be $X := \sum_{i=1}^n X_i$ (we tacitly assume here that individual losses are vectors in an ambient vector space which enables us to add them up);
- the *aggregated risk within the system*, the quantity $\sum_{i=1}^n \rho_i(X_i)$.

The risk sharing problem now poses the question how and to which degree the aggregated risk within the system can be minimised by *redistribution.* If there are no external constraints on the nature of such a redistribution apart from the impossibility of sweeping losses under the carpet, this leads to the optimisation problem

$$\sum_{i=1}^n \rho_i(Y_i) \to \min \quad \text{subject to } \sum_{i=1}^n Y_i = X.$$

A vector $\mathbf{Y} := (Y_1, ..., Y_n)$ of portions of the aggregated loss X with the property $\sum_{i=1}^n Y_i = X$ is called an *allocation* of X. Ideally, one would hope to find an allocation $\mathbf{Y}^*$ with the property

$$\textstyle\sum_{i=1}^n \rho_i(Y_i^*) = \inf\{\sum_{i=1}^n \rho_i(Y_i) \mid \sum_{i=1}^n Y_i = X\},$$

i.e. the minimally possible level of aggregated risk is realised by redistributing according to the allocation $\mathbf{Y}^*$.

Special instances of this general problem and its twin, maximising shared utility (see also Section 1.8), have been studied extensively in the literature. Arrow [10], Borch [17] and Wilson [99] consider the problem for expected utilities and with economic applications such as medical care in mind. We will follow the path pioneered by Barrieu & El Karoui [13] and Filipović & Kupper [46], linking the problem to the *inf-convolution* of convex functions. Under the assumption that only convex monetary risk measures are involved which are law-invariant, the optimisation problem has been shown to be solvable, see Acciaio [1], Acciaio & Svindland [4], Filipović & Svindland [47], and Jouini et al. [66]. The implications of risk sharing for regulatory frameworks are discussed in, e.g., Weber [98]. For a review of the existing literature on risk sharing with monetary risk measures we refer to Embrechts et al. [42]. Another important connection can be drawn to the microeconomic theory of general equilibrium as surveyed in, e.g., Mas Colell & Zame [83] and Aliprantis & Burkinshaw [6, Chapter 8].

Our paper Risk Sharing for Capital Requirements with Multidimensional Security Markets [78], see also Chapter 3, which studies the risk sharing problem, has two major distinctive features. The first one is that, like in [76], we study the problem for *capital requirements* relative to acceptance sets and security markets as introduced in Section 1.4. To the best of our knowledge, our study is the first on risk sharing for capital requirements of this kind. We also stick to the principle of keeping the model spaces as general as possible: We consider an abstract one-period market populated by a finite number $n \geq 2$ of agents who seek to secure losses occurring at a fixed future date. The second distinctive feature is inspired by Filipović & Kupper [46]: Each agent $i \in [n]$ operates on an *(order) ideal* $\mathcal{X}_i \subset \mathcal{X}$ of an ambient Riesz space $\mathcal{X}$.[13] The latter satisfies $\mathcal{X} = \sum_{i=1}^n \mathcal{X}_i$ and collects systemic losses. The individual model spaces $\mathcal{X}_i$, $i \in [n]$, may differ however. As we keep them very general, note that they can account for Knightian uncertainty as described in the preceding sections.

[13] That is, $\mathcal{X}_i$ is a subspace of $\mathcal{X}$, and for all $X, Y \in \mathcal{X}$, $|X| \preceq |Y|$ and $Y \in \mathcal{X}_i$ imply together that $X \in \mathcal{X}_i$.

Each agent i has an acceptance set $\mathcal{A}_i \subset \mathcal{X}_i$ of acceptable loss profiles and has access to an individual security market $(\mathcal{S}_i, \mathfrak{p}_i)$ such that $\mathcal{R}_i := (\mathcal{A}_i, \mathcal{S}_i, \mathfrak{p}_i)$ is a risk measurement regime on the space $\mathcal{X}_i$. She measures risks with the resulting risk measure $\rho_i := \rho_{\mathcal{R}_i}$ defined by

$$\rho_i(X) := \inf\{\mathfrak{p}_i(Z) \mid Z \in \mathcal{S}_i,\ X - Z \in \mathcal{A}_i\}, \quad X \in \mathcal{X}_i.$$

Note that $\mathcal{R}_i$ is typically not a risk measurement regime in the larger space $\mathcal{X}$ as $\mathcal{A}_i$ may not be monotone in $\mathcal{X}$. This setting thus admits substantial heterogeneity among the agents forming the system, their risk preferences being encapsulated by the risk measurement regimes $(\mathcal{R}_1, ..., \mathcal{R}_n)$ on $(\mathcal{X}_1, ..., \mathcal{X}_n)$.

Therefore, we study a refined version of the risk sharing problem. Given an aggregated loss $X \in \mathcal{X}$ we introduce

$$\mathbb{A}_X := \{\mathbf{X} = (X_1, ..., X_n) \mid X_i \in \mathcal{X}_i,\ i \in [n],\ \text{and } X_1 + ... + X_n = X\},$$

the set of *attainable allocations* of X which respect that agent i only accepts losses in $\mathcal{X}_i$. Usually $\mathbb{A}_X \subsetneq \{\mathbf{X} \in \mathcal{X}^n \mid X_1 + ... + X_n = X\}$. The risk sharing problem we consider is

$$\sum_{i=1}^{n} \rho_i(X_i) \to \min \quad \text{subject to } \mathbf{X} \in \mathbb{A}_X \tag{1.6}$$

under an assumption of minimal cooperation among the agents:

($\star$) For all $i, j \in [n]$, the pricing functionals $\mathfrak{p}_i$ and $\mathfrak{p}_j$ agree on $\mathcal{S}_i \cap \mathcal{S}_j$. Moreover, if we set $i \sim j$ if $i \neq j$ and $\mathfrak{p}_i$ is non-trivial on $\mathcal{S}_i \cap \mathcal{S}_j$, the resulting graph

$$G = ([n], \{\{i, j\} \subset [n] \mid i \sim j\})$$

is connected in the sense of graphs.

($\star$) means that, by potentially invoking intermediaries, any two agents can exchange securities. Such a requirement is not too far-fetched, in particular if there is no heterogeneity among the model spaces. Note also that all existing results on risk sharing for convex monetary risk measures embed neatly in the framework outlined above.

Assume $X \in \mathcal{X}$ is such that there is an attainable allocation $\mathbf{X} = (X_1, \ldots, X_n) \in \mathbb{A}_X$ which solves the optimisation problem (1.6) and yields a finite optimal value. Then $\mathbf{X}$ is a *Pareto optimal* allocation of X; cf. [78, Proposition 3.6]. This economic notion of optimality means that in a redistribution the situation cannot be improved for one agent without worsening the fate of another. In the terms of the risk sharing problem, $\mathbf{X} \in \mathbb{A}_X$ is a Pareto optimal allocation of $X \in \mathcal{X}$ if for all allocations $\mathbf{Y} \in \mathbb{A}_X$ the existence of some agent $i \in [n]$ with $\rho_i(Y_i) < \rho_i(X_i)$ implies the existence of some agent $j \in [n]$ with $\rho_j(Y_j) > \rho_j(X_j)$. However, this resembles centralised redistribution which attributes to each agent a certain portion of the aggregate loss in an overall optimal way without considering individual well-being. As an "alternative", we consider redistribution by agents trading portions of losses at a certain price while adhering to individual rationality constraints. This leads to the notion of *equilibrium allocations* and *equilibrium prices*, a variant of the risk sharing problem above. In particular, each equilibrium allocation is Pareto optimal in our framework; cf. [78, Proposition 3.6].

Like Barrieu & El Karoui [13] we study the *risk sharing functional* associated to an agent system $(\mathcal{R}_1, ..., \mathcal{R}_n)$. This is the function mapping an aggregated loss to the minimal level of risk which can be obtained by redistribution:

$$\Lambda : \mathcal{X} \to [-\infty, \infty], \quad X \mapsto \inf_{\mathbf{X} \in \mathbb{A}_X} \sum_{i=1}^{n} \rho_i(X_i).$$

As a first main contribution of [78], we find that the behaviour of the interacting agents in the market is, under mild assumptions, captured by a *market capital requirement* of type (1.4):

$$\Lambda(X) = \inf\{\pi(Z) : Z \in \mathcal{M},\ X - Z \in \mathscr{A}_+\}, \quad X \in \mathcal{X},$$

where $\mathscr{A}_+ := \sum_{i=1}^{n} \mathcal{A}_i$ is a market acceptance set consisting of all losses which can be shared in a way acceptable for everyone, and $(\mathcal{M}, \pi)$ is a global security market; cf. [78, Theorem 3.1]. This so-called *representative agent formulation* allows to derive useful conditions ensuring the existence of optimal risk allocations.[14] The most useful of these criteria is the following: If Λ is a proper function (in the sense of Section 1.4) and $X \in \mathcal{X}$ admits an *optimal payoff* Z^X, i.e. $Z^X \in \mathcal{M}$ is a market security such that $X - Z^X \in \mathscr{A}_+$ is market acceptable and $\pi(Z^X) = \Lambda(X)$, then X can be allocated in a Pareto optimal way; cf. [78, Theorem 3.3].[15] In a topological setting, the most powerful criterion to allocate *any* $X \in \sum_{i=1}^{n} \mathrm{dom}\,(\rho_i) = \sum_{i=1}^{n}\{X_i \in \mathcal{X}_i \mid \rho_i(X_i) < \infty\}$ Pareto optimally is thus to check two properties: (i) Λ is a proper function, (ii) the Minkowski sum $\mathscr{A}_+ + \ker(\pi)$ is closed, where $\ker(\pi) := \pi^{-1}(\{0\})$ denotes the kernel of the global pricing functional; cf. [78, Proposition 3.4]. Considering the problem is hence also mathematically worthwhile, as usually it is difficult to decide in an infinite dimensional space if the sum of two (closed) sets is closed again. The existence of equilibria on a general Fréchet lattice is based on this theory of Pareto optima in conjunction with subdifferentiability of the risk sharing functional; cf. [78, Theorem 3.5].

As a side remark, we would like to comment here on the possible role of the risk sharing functional as a *capital allocation rule*. Suppose the market space $\mathcal{X}$ collects portfolios of one and the same firm composed of the individual contributions of subsidiaries or different business units. The risk of such a portfolio is determined with a capital requirement. A capital allocation rule then assigns portions of the required capital to the individual units. For more background on this problem, we refer to Centrone & Rosazza Gianin [22], Deprez & Gerber [37], Kalkbrener [68], Tsanakas [96], and the references therein. Now, if the risk of such a portfolio $X \in \mathcal{X}$ is determined with the risk sharing functional Λ, and if there is a Pareto optimal allocation $\mathbf{X}$ of X, a natural capital allocation rule would then be to assign capital $\rho_i(X_i)$ to business unit i.

Secondly, based on this general study of the representative agent formulation, we discuss two prominent cases of risk sharing in detail. They are characterised by the involved notions of acceptability. In these case studies, we prove that the risk sharing problem (1.6) can be solved and give mild sufficient conditions under which equilibria can be found. In the first instance

[14] By "representative agent", we mean that the behaviour of the agent system can be aggregated to be equivalent to the behaviour of a single agent of the same type. This hypothetical single agent acts on the market space $\mathcal{X}$ and is represented by the risk measurement regime $(\mathscr{A}_+, \mathcal{M}, \pi)$.

[15] The terminology of "optimal payoffs" is adopted from Baes et al. [12] who study under which conditions minimisers in the definition (1.4) of a single capital requirement can be found. In this regard [78] also complements this recent paper.

(see [78, Section 4]), we abstract the notion that individual losses are contingent on the future state of the economy. A loss is deemed acceptable if certain capital thresholds are not exceeded under a fixed finite set of linear aggregation rules which may vary from agent to agent. Also, we fully account for the possibility that model spaces may differ in the sense outlined above. This reflects that the scenarios or states relevant to one agent may not be the ones relevant to another. Hence, each agent may have a set of relevant scenarios and one of irrelevant ones in which she demands neutrality of her stake in the loss. The latter condition could also be interpreted as *uncertainty aversion* which demands loss neutrality in states whose potential realisation is subject to extreme Knightian uncertainty. The resulting agent systems on Fréchet lattices will be called *polyhedral*, and they admit optimal payoffs, Pareto optimal allocations, and equilibria under suitable mild conditions; see [78, Theorem 4.3 & Corollary 4.5].

In [78, Section 5], we analyse systems of agents operating on one and the same space of random variables. For each agent, whether or not a given loss is deemed adequately capitalised only depends on the distributional properties under a fixed reference probability measure $\mathbb{P}$ for which $(\Omega, \mathcal{F}, \mathbb{P})$ is non-atomic. Hence, the acceptance sets $\mathcal{A}_i$, $i \in [n]$, will be *law-invariant* with respect to $\mathbb{P}$. However, the security spaces $\mathcal{S}_i$ are of finite dimension and usually *not* law-invariant. Securitisation depends on the potentially varying joint distribution of the loss and the security and is thus statewise rather than distributional. This results in a lack of law-invariance of the individual risk measures ρ_i, which is particularly interesting against the backdrop of the existing literature. So far, a powerful solution theory for the risk sharing problem has only been available in case that the individual risk measures are convex $\mathbb{P}$-law-invariant monetary risk measures, cf. [47, 66]. We will utilise these findings, but our results on the existence of optimal payoffs, Pareto optima, and equilibria go far beyond this case; cf. [78, Theorems 5.3, 5.7, Corollary 5.5].

We study the risk sharing problem for law-invariant acceptance sets on general rearrangement invariant Banach lattices of $\mathbb{P}$-integrable random variables, see [78, Theorem 5.8].[16] We will do so, however, by first considering the maximal case in which all agents operate on the model space L^1 of all $\mathbb{P}$-integrable random variables. We later localise this procedure to general model spaces. This technique also features prominently in [77], see the following section.

Thirdly, in case of polyhedral agent systems and agent systems with law-invariant acceptance sets, we carefully study continuity properties of the set-valued map assigning to an aggregated loss its optimal risk allocations. They reflect an important question in optimisation, namely how sensitive the set of optimisers is under slight misspecifications of the input of the optimisation problem. In the context of polyhedral agent systems, the map assigning *all* Pareto optima to a certain loss is *lower hemicontinuous*; cf. [78, Theorem 4.9]. This property means that a given optimal risk allocation stays close to optimal under a slight perturbation of the underlying aggregated loss. However, the heterogeneity of the involved model spaces and their very general nature of being Fréchet lattices necessitate further assumptions for this theorem to hold outside of a finite dimensional framework, see [78, Assumptions 4.6–4.8]. In [78, Theorems 5.7 & 5.8], we prove for law-invariant acceptance sets that the set of Pareto optimal allocations (of a specific shape) is *upper hemicontinuous*. Approximating a complex loss with simpler losses and

[16] That is, $L^\infty \subset \mathcal{X} \subset L^1$ is rearrangement invariant or law-invariant as a subset of L^1, and it carries a law-invariant and complete lattice norm $\|\cdot\|$. In contrast to [77], we assume in [78] the norm $\|\cdot\|$ to be order continuous: Whenever $(X_n)_{n\in\mathbb{N}} \subset \mathcal{X}$ is a sequence satisfying $X_n \downarrow 0$ $\mathbb{P}$-a.s., we have $\|X_n\| \downarrow 0$.

calculating optimal risk allocations for these will yield an optimal risk allocation for the complex loss as a limit point. It is therefore a useful property from a numerical point of view. Upper hemicontinuity transfers to the map assigning equilibria to a certain extent, see [78, Theorem 5.9].

Fourthly, providing a more financial application, we study optimal portfolio splitting problems in the spirit of Tsanakas [96] and Wang [97]. Assuming the presence of transaction costs, we study how a financial institution can split an aggregated loss produced by a portfolio optimally among subsidiaries. These are assumed to be subject to potentially varying regulatory regimes and to have access to potentially varying security markets. Building on previous results, we show that this problem admits solutions in the two frameworks considered above, cf. [78, Corollary 6.2].

1.8 Maximising aggregated utility under law-invariance

The third contribution in this thesis, EFFICIENT ALLOCATIONS UNDER LAW-INVARIANCE: A UNIFYING APPROACH [77], see also Chapter 4, diverges from the theme of risk measures predominant in [76, 78] and the rather financial language these papers employ. It studies the twin problem of the risk sharing problem in [78]: How can aggregated *utility* within a system of agents be maximised?

This requires to clarify what we mean by "aggregation". If — as in [78] — each agent in a group of agents perceives a risk which has a direct monetary interpretation as a capital requirement, the total capital required in the system can only be the sum of all individual risks. Hence, the formulation of the risk sharing problem in (1.6) seems to be the only sensible one if the objective is to minimise aggregated risk. In contrast, if the individual contributions are *utilities* expressed in *utils*, aggregation may, with good reasons, look very differently.

Suppose a *commodity* X is being shared among $n \geq 2$ agents who may be identified with the numbers $\{1, ..., n\} =: [n]$. Each one obtains a share X_i, $i \in [n]$. Assume furthermore that the utility the share X_i presents for agent i is given by a real number of utils $\mathfrak{U}_i(X_i) \in \mathbb{R}$. Then aggregating the utility within the group of agents by simply adding individual utilities,

$$\textstyle\sum_{i=1}^n \mathfrak{U}_i(X_i),$$

is very different from ignoring suffering agents with negative utility, which would correspond to the aggregation

$$\textstyle\sum_{i=1}^n \mathfrak{U}_i(X_i)^+.$$

This point of view is again the opposite of another way of aggregation ignoring agents with non-negative utility who fare well:

$$\textstyle\sum_{i=1}^n -\mathfrak{U}_i(X_i)^-.^{17}$$

Each of these choices of aggregation has its own economic motivation and justification. Consequently, our endeavour in [77] is to harness the full economic spectrum of aggregation of utility.

[17] For a real number $x \in \mathbb{R}$, $x^+ := \max\{x, 0\}$ denotes the positive part, whereas $x^- := \max\{0, -x\}$ denotes the negative part.

We focus on a *commodity space* $\mathcal{X}$ consisting of *commodities* or *goods* $X \in \mathcal{X}$, and we assume $\mathcal{X}$ to be a vector space. Each individual agent has preferences over the commodities in $\mathcal{X}$ expressed by a convex preference relation $\preceq_i$, $i \in [n]$. These preferences have a numerical representation by a vector $\mathfrak{U} := (\mathfrak{U}_1, ..., \mathfrak{U}_n)$ of *utility functions* $\mathfrak{U}_i : \mathcal{X} \to [-\infty, \infty)$ such that agent i weakly prefers Y to X ($X \preceq_i Y$) if, and only if $\mathfrak{U}_i(X) \leq \mathfrak{U}_i(Y)$. Convexity of the preference relation is translated as quasi-concavity of the utility function: For all $X, Y \in \mathcal{X}$ and all $\lambda \in [0,1]$, we have $\mathfrak{U}_i(\lambda X + (1-\lambda)Y) \geq \min\{\mathfrak{U}_i(X), \mathfrak{U}_i(Y)\}$. Note that we exclude the case of infinite utility, but not the case of infinite disutility.

Throughout [77] we assume that goods are modelled as *Savage acts*, i.e. real-valued random variables on a measurable space $(\Omega, \mathcal{F})$ of future states of the world. Riskiness in the realisation of the states $\omega \in \Omega$ is assumed to be modelled by a reference probability measure $\mathbb{P}$ such that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is non-atomic. As such, the acts are risky future quantities, but may also be interpreted as consumption patterns. We will identify two Savage acts if they lead to the same consequences $\mathbb{P}$-a.s., i.e. $\mathbb{P}(X = Y) = 1$. Moreover, we will assume that the first moment of all $X \in \mathcal{X}$ exists, i.e. $\mathcal{X} \subset L^1$.

As before, potential individual preference relations expressed by the collection of utility functions $\mathfrak{U}$ are complemented by an objective ordering of the random variables in terms of the $\mathbb{P}$-a.s. order. In a financial context, the utility functions would widely be assumed to reflect the "more is better"-paradigm of *monotonicity in the* $\mathbb{P}$*-a.s. order*, meaning that $\mathfrak{U}_i(X) \leq \mathfrak{U}_i(Y)$ whenever Y is better than X in the objective order, $\mathbb{P}(X \leq Y) = 1$. The self-evidence of this assumption outside of finance appears problematic in light of finiteness of resources as well as the adverse social or ecological effects of economic activities. Our analysis will therefore not rely on monotonicity whatsoever. When studying the efficiency of *allocations*, we will therefore distinguish between sharing *with* and *without* free disposal. In the first case, the aggregated good $X \in \mathcal{X}$ has to be shared without any remainder, e.g. because of external constraints. Hence, one considers the *consumption set* Γ_X consisting of all vectors $\mathbf{X} = (X_1, ..., X_n) \in \mathcal{X}^n$ with the property $X_1 + ... + X_n = X$. In the second case, a unanimously rejected remainder (in the objective order) may be left aside in the sharing scheme. That is, one considers the consumption set $\widehat{\Gamma}_X$ consisting of all $\mathbf{X} \in \mathcal{X}^n$ with the property $X_1 + ... + X_n \leq X$ with $\mathbb{P}$-probability 1. Note that in contrast to, e.g., [78], each vector $\mathbf{X} := (X_1, ..., X_n)$ with $X_1 + ... + X_n = X$ or $X_1 + ... + X_n \leq X$, respectively, is at least hypothetically feasible, and there are no constraints on how X can be shared.

Economic efficiency of an allocation means that it has certain economic optimality properties. The notion of Pareto optimality, here *Pareto efficiency*, has already been introduced above. It means that no agent can improve her share (within a certain fixed consumption set of allocations) without worsening the fate of another agent. The following observation is immediate: If $X \in \mathcal{X}$ is a good and $\mathbf{X}^*$ is an allocation in Γ_X or $\widehat{\Gamma}_X$ such that for appropriate positive weights $w_1, ..., w_n > 0$

$$\sum_{i=1}^n w_i \mathfrak{U}_i(X_i^*) = \sup_{\mathbf{X} \in \Gamma_X} \sum_{i=1}^n w_i \mathfrak{U}_i(X_i), \quad \text{or} \quad \sum_{i=1}^n w_i \mathfrak{U}_i(X_i^*) = \sup_{\mathbf{X} \in \widehat{\Gamma}_X} \sum_{i=1}^n w_i \mathfrak{U}_i(X_i), \text{ respectively,}$$

and if the optimal value $\sup\{\sum_{i=1}^n w_i \mathfrak{U}_i(X_i) \mid \mathbf{X} \in \Gamma_X\}$ or $\sup\{\sum_{i=1}^n w_i \mathfrak{U}_i(X_i) \mid \mathbf{X} \in \widehat{\Gamma}_X\}$, respectively, is a real number, then $\mathbf{X}^*$ is Pareto optimal within the consumption set Γ_X or $\widehat{\Gamma}_X$, respectively.

The quantity $\sum_{i=1}^n w_i \mathfrak{U}_i(X_i)$ can be seen as aggregated utility. More precisely, the function $\Lambda_{\mathbf{w}}(\mathbf{y}) := \sum_{i=1}^n w_i y_i$, $\mathbf{y} \in [-\infty, \infty)^n$, which is non-decreasing in the coordinatewise order on $[-\infty, \infty)^n$, is used to aggregate the vector

$$\mathfrak{U}(\mathbf{X}) := (\mathfrak{U}_1(X_1), ..., \mathfrak{U}_n(X_n))$$

of individual utilities arising from allocation $\mathbf{X}$. The core idea of [77] is thus to study efficiency properties of allocations of $X \in \mathcal{X}$ by considering an optimisation problem

$$\Lambda(\mathfrak{U}(\mathbf{X})) \to \max \quad \text{subject to } \mathbf{X} \in \Gamma, \tag{1.7}$$

which is designed in a way such that the associated maximisers automatically have the desired efficiency property. Here, $\Gamma \in \{\Gamma_X, \widehat{\Gamma}_X\}$ is a consumption set corresponding to sharing with and without free disposal, and the function $\Lambda : [-\infty, \infty)^n \to [-\infty, \infty)$ denotes an arbitrary *aggregation function* being non-decreasing in the coordinatewise order. We will see that considering general aggregation functions Λ introduces a substantial degree of freedom which allows to study numerous efficiency properties of allocations. Also, the problem is embedded in the classical mathematical literature on general equilibrium, cf. Debreu [32] and Mas-Colell & Zame [83].

(1.7) is an optimisation problem on an infinite dimensional space and therefore hard to solve. However, special instances of it have been shown to be solvable — see, for instance, [31, 66, 89] — under the assumption of *law-invariance* of the individual utility functions: For all $X, Y \in \mathcal{X}$ which agree in distribution under $\mathbb{P}$ ($F_X^{\mathbb{P}} = F_Y^{\mathbb{P}}$), $\mathfrak{U}_i(X) = \mathfrak{U}_i(Y)$ holds for all $i \in [n]$. Economically, one might say $\mathfrak{U}_i(X) = \mathfrak{U}_i(Y)$ whenever X and Y induce the same *lottery* over the real line under $\mathbb{P}$, and agents with law-invariant preferences display *consequentialism* in that they are indifferent between two Savage acts yielding the same lottery consequences.

In many special instances, (1.7) has *comonotone* maximisers if each $\mathfrak{U}_i$, $i \in [n]$, is law-invariant. An allocation $\mathbf{X}$ of $X \in \mathcal{X}$ is comonotone if there are n non-decreasing functions $f_i : \mathbb{R} \to \mathbb{R}$ summing up to the identity — $f_1 + ... + f_n = id_{\mathbb{R}}$ — such that $X_i = f_i(X)$, $i \in [n]$. The key are so-called *comonotone improvement* results as given by [21, 47, 75, 80], and [77] aims at exploring their full mathematical power.

Its main contribution is to prove the existence of comonotone maximisers in (1.7), and thus of economically desirable allocations, in a wide range of situations by laying the groundwork in clear-cut meta results. The range of applications is rich, the assumptions on the individual utility functions very general (essentially only quasi-concavity and law-invariance), and the commodity space may be any rearrangement invariant Banach lattice of $\mathbb{P}$-integrable random variables. In this respect, we contribute to unifying the existing literature on the problem.

Before we outline our main contributions, we emphasise that the question is neatly embedded in a rich literature on sharing problems as described above. We refer to Acciaio [1, 2], Carlier & Dana [20], Chateauneuf et al. [25], Dana [31], Jouini et al. [66], Liu et al. [79], and Ravanelli & Svindland [89], beside the references on the twin problem of risk sharing cited above. Moreover, the technical parts of the paper contribute to the general theory of law-invariant functions with convex level sets, a prominent line of research; cf. [23, 26, 28, 48, 58, 65, 94, 95].

Two of the main results of [77] which essentially lie at the heart of the solution theory are [77, Theorem 18 & Proposition 20]. Suppose that on a general rearrangement invariant

Banach lattice $L^\infty \subset \mathcal{X} \subset L^1$ with law-invariant lattice norm $\|\cdot\|$ a quasi-concave function $f : \mathcal{X} \to [-\infty, \infty)$ is given. Then under the mild assumption of order upper semicontinuity, in case of $\mathcal{X} \subsetneq L^1$ in terms of the strong Fatou property,[18] law-invariance is equivalent to two standard notions of risk aversion: (i) monotonicity in the concave order, i.e. $f(X) \le f(Y)$ holds whenever every risk averse expected utility agent weakly prefers Y to X; (ii) dilatation monotonicity, i.e. for every finite measurable partition Π of the state space, f ranks the act associated to more information encoded by Π, i.e. the conditional expectation $\mathbb{E}[X|\sigma(\Pi)]$, higher than X itself: $f(X) \le f(\mathbb{E}[X|\sigma(\Pi)])$.[19] Moreover, in any of these cases, one may define a unique consistent extension $f^\sharp : L^1 \to [-\infty, \infty)$ of f which is upper semicontinuous, quasi-concave, and proper whenever f is proper.[20] This extension result is key to solving the problem (1.7) on general rearrangement invariant Banach lattices as commodity spaces by means of a localisation procedure, provided one has developed a solution theory on the "canonical" commodity space L^1; cf. [77, Theorem 23].

As we reduced the general case of (1.7) to the problem on the commodity space $\mathcal{X} = L^1$, we only consider agents $i \in [n]$ endowed with utility functions $\mathfrak{U}_i : L^1 \to [-\infty, \infty)$ which are proper, quasi-concave, upper semicontinuous, and law-invariant. As they are non-decreasing in the concave order, we may further reduce the problem for $X \in L^1$ as

$$\sup_{\mathbf{X}\in\Gamma_X} \Lambda(\mathfrak{U}(\mathbf{X})) = \sup_{\mathbf{f}\in\mathbb{C}(n)} \Lambda\left(\mathfrak{U}(\mathbf{f}(X))\right)$$

and

$$\sup_{\mathbf{X}\in\widehat{\Gamma}_X} \Lambda(\mathfrak{U}(\mathbf{X})) = \sup\left\{\Lambda\left(\mathfrak{U}(f_1(X), ..., f_n(X))\right) \mid \mathbf{f} \in \mathbb{C}(n+1),\ f_{n+1}(X) \ge 0\right\},$$

provided free disposal is allowed. Here, for $k \in \mathbb{N}$, $\mathbb{C}(k)$ denotes the set of *comonotone k-partitions of the identity*, i.e. $\mathbf{f} : \mathbb{R} \to \mathbb{R}^k$, each coordinate function f_i is non-decreasing, and $f_1 + ... + f_k = id_{\mathbb{R}}$; cf. [77, Proposition 7]. In order for this supremum to be a maximum and the optimal value to be a real number, the most natural approach would be to select a maximising sequence $(\mathbf{f}^k)_{k\in\mathbb{N}} \subset \mathbb{C}(n)$ of $\mathbb{C}(n+1)$ and show that the allocations $\mathbf{f}^k(X)$ converges to a maximiser of (1.7). It turns out, however, that this requires to find a constant $\gamma(X) > 0$ such that a maximising sequence satisfies $\sum_i |f_i(0)| \le \gamma(X)$; cf. [77, Proposition 8]. To this end, we introduce the property of an aggregation function Λ to be COERCIVE for the individual utilities restricted to riskless commodities, i.e. $\mathfrak{U}_i|_{\mathbb{R}}$. This is a rather mild property — cf. [77, Appendix A] — which allows us to prove a general existence result on maximisers of (1.7) — both with and without free disposal, [77, Theorem 12].

We thereafter exploit the degree of freedom the aggregation function Λ provides and prove the existence of allocations with different efficiency properties. All results are derived from suitable conditions on the behaviour of the utility functions on riskless commodities. Given a

[18] That is, if $(X_n)_{n\in\mathbb{N}} \subset \mathcal{X}$ is a sequence converging to $X \in \mathcal{X}$ $\mathbb{P}$-a.s. which satisfies $\sup_{n\in\mathbb{N}} \|X_n\| < \infty$, the estimate $\limsup_{n\to\infty} f(X_n) \le f(X)$ holds; cf. [26, 58].

[19] This result complements and generalises previous similar results, see Cerreia-Vioglio et al. [23, Theorem 5.1], Cherny & Grigoriev [28, Corollary 1.3], and Svindland [95, Theorem 2.7].

[20] In contrast to the preceding sections, we say here that a function $f : \mathcal{X} \to [-\infty, \infty]$ is PROPER if $f^{-1}(\{\infty\}) = \emptyset$ and $f \not\equiv -\infty$.

parameter $0 < \alpha \leq 1$, we use the aggregation function

$$\Lambda_\alpha(\mathbf{y}) := \alpha \min_{1\leq i\leq n} y_i + (1-\alpha) \max_{1\leq i\leq n} y_i, \quad \mathbf{y} \in [-\infty, \infty)^n,$$

to obtain *(biased) weakly Pareto efficient* allocations as maximisers for problem (1.7); cf. [77, Theorem 29]. If we choose the aggregation function

$$\Xi_\alpha(\mathbf{y}) := \sum_{\emptyset \neq S \subset [n]} \alpha \min_{1\leq i\leq n} y_i + (1-\alpha) \max_{i\in S} y_i, \quad \mathbf{y} \in [-\infty, \infty)^n,$$

maximisers will be efficient in the sense of *core allocations*, which reflect a game-theoretic notion of fairness; cf. [77, Theorem 31].
We have already seen that for a vector of positive weights $\mathbf{w} = (w_1, ..., w_n) \in (0, \infty)^n$, the aggregation function

$$\Lambda_{\mathbf{w}}(\mathbf{y}) := \sum_{i=1}^n w_i y_i, \quad \mathbf{y} \in [-\infty, \infty)^n,$$

leads to Pareto efficient allocations; cf. [77, Theorem 33]. With a symbolic modification of the utility functions, one can also show the existence of Pareto efficient allocations respecting certain individual rationality constraints prescribing minimal utility which must not be undercut in a redistribution of an allocation; cf. [77, Theorem 35]. Finally, in [77, Theorem 36], we use the aggregation function

$$\Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}(\mathbf{y}) := \sum_{i=1}^n -p_i y_i^- + q_i(y_i - r_i)^+, \quad \mathbf{y} \in [-\infty, \infty)^n,$$

where $(\mathbf{p}, \mathbf{q}, \mathbf{r}) \in (\mathbb{R}^n_+)^3$ is a collection of parameters, to find allocations which are *systemically fair*. This application is motivated by the theory of systemic risk measures.

Bibliography

[1] Acciaio, B. (2007), Optimal Risk Sharing with Non-Monotone Monetary Functionals. *Finance and Stochastics*, 11:267-289.

[2] Acciaio, B. (2009), Short Note on Inf-Convolution Preserving the Fatou Property. *Annals of Finance*, 5:281-287.

[3] Acciaio, B., and I. Penner (2011), Dynamic Convex Risk Measures. In: Di Nunno, G., and B. Øksendal (eds.), *Advanced Mathematical Methods for Finance*, pp. 1-34. Springer.

[4] Acciaio, B., and G. Svindland (2009), Optimal Risk Sharing with Different Reference Probabilities. *Insurance: Mathematics and Economics*, 44(3):426-433.

[5] Aliprantis, C. and K. Border (1999), *Infinite Dimensional Analysis – A Hitchhiker's Guide*. 3rd edition, Springer.

[6] Aliprantis, C., and O. Burkinshaw (2003), *Locally Solid Riesz Spaces with Applications to Economics*. American Mathematical Society.

[7] Aliprantis, C., and O. Burkinshaw (2006), *Positive Operators*. Springer.

[8] Arduca, M., P. Koch-Medina, and C. Munari (2019), A Simple Approach to Dual Representations of Systemic Risk Measures. Preprint, arXiv:1906.10933.

[9] Arrow, K. J. (1951), Alternative Approaches to the Theory of Choice in Risk-Taking Situations. *Econometrica*, 19:404-437.

[10] Arrow, K. J. (1963), Uncertainty and the Welfare Economics of Medical Care. *American Economic Review*, 5:941-973.

[11] Artzner, P., F. Delbaen, J.-M. Eber, and D. Heath (1999), Coherent Measures of Risk. *Mathematical Finance*, 9(3):203-228.

[12] Baes, M., P. Koch-Medina, and C. Munari (2019), Existence, Uniqueness, and Stability of Optimal Portfolios of Eligible Assets. Forthcoming in *Mathematical Finance*. Available online at https://doi.org/10.1111/mafi.12205

[13] Barrieu, P., and N. El Karoui (2005), Inf-Convolution of Risk Measures and Optimal Risk Transfer. *Finance and Stochastics*, 9(2):269-298.

[14] Beissner, P., and L. Denis (2018), Duality and General Equilibrium Theory under Knightian Uncertainty. *SIAM Journal on Financial Mathematics*, 9(1):381-400.

[15] Bewley, T. F. (2001), Market Innovation and Entrepreneurship: A Knightian View. In: Debreu, G., W. Neuefeind, and W. Trockel (eds.), *Economics Essays: A Festschrift for Werner Hildenbrand*, pp. 41-58. Springer.

[16] Bion-Nadal, J., and M. Kervarec (2012), Risk Measuring under Model Uncertainty. *Annals of Applied Probability*, 22(1):213-238.

[17] Borch, K. (1962), Equilibrium in a Reinsurance Market. *Econometrica*, 30:424-444.

[18] Bouchard, B., and M. Nutz (2015), Arbitrage and Duality in Nondominated Discrete-Time Models. *Annals of Applied Probability*, 25(2):823-859.

[19] Burzoni, M., M. Frittelli, and M. Maggis (2017), Model-Free Super Hedging Duality. *Annals of Applied Probability*, 27(3):1452-1477.

[20] Carlier, G., and R.-A. Dana (2008), Two-Persons Efficient Risk-Sharing and Equilibria for Concave Law-Invariant Utilities. *Economic Theory*, 36(2):189-223.

[21] Carlier, G., R.-A. Dana, and A. Galichon (2012), Pareto Efficiency for the Concave Order and Multivariate Comonotonicity. *Journal of Economic Theory*, 147(1):207-229.

[22] Centrone, F., and E. Rosazza Gianin (2018), Capital Allocation à la Aumann-Shapley for Non-Differentiable Risk Measures. *European Journal of Operational Research*, 267(2):667-675.

[23] Cerreia-Vioglio, S., F. Maccheroni, M. Marinacci, and L. Montrucchio (2011), Risk Measures: Rationality and Diversification. *Mathematical Finance*, 21:743-774.

[24] Cerreia-Vioglio, S., F. Maccheroni, M. Marinacci, and A. Rustichini (2014), Niveloids and their Extensions: Risk Measures on Small Domains. *Journal of Mathematical Analysis and Applications*, 413:343-360.

[25] Chateauneuf, A., R.-A. Dana, and J.-M. Tallon (2000), Optimal Risk-Sharing Rules and Equilibria with Choquet-Expected-Utility. *Journal of Mathematical Economics*, 34(2):191-214.

[26] Chen, S., N. Gao, and F. Xanthos (2018), The Strong Fatou Property of Risk Measures. *Dependence Modeling*, 6(1):183-196.

[27] Cheridito, P., and T. Li (2009), Risk Measures on Orlicz Hearts. *Mathematical Finance*, 19(2):189-214.

[28] Cherny, A. S., and P. G. Grigoriev (2007), Dilatation Monotone Risk Measures are Law Invariant. *Finance and Stochastics*, 11(2):291-298.

[29] Cohen, S. N. (2012), Quasi-Sure Analysis, Aggregation and Dual Representations of Sublinear Expectations in General Spaces. *Electronic Journal of Probability*, 17(62):1-15.

[30] Cyert, R. M., and M. H. DeGroot (1974), Rational Expectations and Bayesian Analysis. *Journal of Political Economy*, 82:521-536.

[31] Dana, R.-A. (2011), Comonotonicity, Efficient Risk-Sharing and Equilibria in Markets with Short-Selling for Concave Law-Invariant Utilities. *Journal of Mathematical Economics*, 47:328-335.

[32] Debreu, G. (1982), Existence of Competitive Equilibrium. In: Arrow, K. J., and M. D. Intriligator (eds.), *Handbook of Mathematical Economics*, Vol. 2, pp. 697-743. Elsevier.

[33] Delbaen, F. (2000), *Coherent Risk Measures*. Lecture notes, Cattedra Galileiana, Scuola Normale Superiore di Pisa.

[34] Delbaen, F. (2002), Coherent Risk Measures on General Probability Spaces. In: Sandmann, K., and P. J. Schönbucher (eds.), *Advances in Finance and Stochastics: Essays in Honor of Dieter Sondermann*, pp. 1-37. Springer.

[35] Denis, L., Y. Hu, and S. Peng (2011), Function Spaces and Capacity Related to a Sublinear Expectation: Application to G-Brownian Motion Paths. *Potential Analysis*, 34(2):139-161.

[36] Denis, L., and C. Martini (2006), A Theorectical Framework for the Pricing of Contingent Claims in the Presence of Model Uncertainty. *Annals of Applied Probability*, 16(2):827-852.

[37] Deprez, O., and H. U. Gerber (1985), On Convex Principles of Premium Calculation. *Insurance: Mathematics and Economics*, 4:79-189.

[38] Detlefsen, K., and G. Scandolo (2005), Conditional and Dynamic Convex Risk Measures. *Finance and Stochastics*, 9:539-561.

[39] Drapeau, S., and M. Kupper (2013), Risk Preferences and their Robust Representations. *Mathematics of Operations Research*, 38(1):28-62.

[40] El Karoui, N., and C. Ravanelli (2009), Cash Subadditive Risk Measures under Interest Rate Ambiguity. *Mathematical Finance*, 19:561-590.

[41] Ellsberg, D. (1961), Risk, Ambiguity and the Savage Axioms. *Quarterly Journal of Economics*, 75(4):643-669.

[42] Embrechts, P., H. Liu, and R. Wang (2017), Quantile-Based Risk Sharing. *Operations Research*, 66(4):936-949.

[43] Farkas, E. W., P. Koch Medina, and C. Munari (2013), Beyond Cash-Additive Risk Measures: When Changing the Numéraire Fails. *Finance and Stochastics*, 18(1):145-173.

[44] Farkas, E. W., P. Koch Medina, and C. Munari (2014), Capital Requirements with Defaultable Securities. *Insurance: Mathematics and Economics*, 55:58-67.

[45] Farkas, E. W., P. Koch Medina, and C. Munari (2015), Measuring Risk with Multiple Eligible Assets. *Mathematics and Financial Economics*, 9(1):3-27.

[46] Filipović, D., and M. Kupper (2008), Equilibrium Prices for Monetary Utility Functions. *International Journal of Theoretical and Applied Finance*, 11:325-343.

[47] Filipović, D., and G. Svindland (2008), Optimal Capital and Risk Allocations for Law- and Cash-Invariant Convex Functions. *Finance and Stochastics*, 12(3):423-439.

[48] Filipović, D., and G. Svindland (2012), The Canonical Model Space for Law-Invariant Convex Risk Measures is L^1. *Mathematical Finance*, 22(3):585-589.

[49] Föllmer, H. (2009), Alles richtig und trotzdem falsch? Anmerkungen zur Finanzkrise und zur Finanzmathematik. *Mitteilungen der DMV*, 17:148-154.

[50] Föllmer, H., and I. Penner (2006), Convex Risk Measures and the Dynamics of their Penalty Functions. *Statistics & Decisions*, 24(1):61-96.

[51] Föllmer, H., and A. Schied (2002), Convex Measures of Risk and Trading Constraints. *Finance and Stochastics*, 6(4):429-447.

[52] Föllmer, H., and A. Schied (2016), *Stochastic Finance: An Introduction in Discrete Time*. 4th edition, De Gruyter.

[53] Föllmer, H., and S. Weber (2015), The Axiomatic Approach to Risk Measures for Capital Determination. *Annual Review of Financial Economics*, 7:301-337.

[54] Fouque, J.-P., and J. Langsam (eds.) (2013), *Handbook on Systemic Risk*. Cambridge University Press.

[55] Frittelli, M., and E. Rosazza Gianin (2002), Putting Order in Risk Measures. *Journal of Banking and Finance*, 26(7):1473-1486.

[56] Frittelli, M., and E. Rosazza Gianin (2005), Law Invariant Convex Risk Measures. *Advances in Mathematical Economics*, 7:33-46.

[57] Frittelli, M., and G. Scandolo (2006), Risk Measures and Capital Requirements for Processes. *Mathematical Finance*, 16(4):589-612.

[58] Gao, N., D. Leung, C. Munari, and F. Xanthos (2018), Fatou Property, Representations, and Extensions of Law-Invariant Risk Measures on General Orlicz spaces. *Finance and Stochastics*, 22(2):395-415.

[59] Gao, N., D. Leung, and F. Xanthos (2017), Closedness of Convex Sets in Orlicz Spaces with Applications to Dual Representation of Risk Measures. Forthcoming in *Studia Mathematica*. Available online at https://doi.org/10.4064/sm180404-3-1.

[60] Gao, N., and C. Munari (2018), Surplus-Invariant Risk Measures. Preprint, arXiv:1707.04949.

[61] Gilboa, I., and M. Marinacci (2013), Ambiguity and the Bayesian Paradigm. In: Acemoglu, D., M. Arellano, and E. Dekel (eds.), *Advances in Economics and Econometrics. Tenth World Congress, Volume I: Economic Theory*, pp. 179-242. Cambridge University Press.

[62] Gilboa, I., and D. Schmeidler (1989), Maxmin Expected Utility with Non-Unique Prior. *Journal of Mathematical Economics*, 18(2):141-153.

[63] Hansen, L. P., and T. J. Sargent (2001), Robust Control and Model Uncertainty. *American Economic Review*, 91(2):60-66.

[64] Hurd, T. R. (2016). *Contagion! Systemic Risk in Financial Networks*. Springer.

[65] Jouini, E., W. Schachermayer, and N. Touzi (2006), Law Invariant Risk Measures have the Fatou Property. *Advances in Mathematical Economics*, 9:49-71.

[66] Jouini, E., W. Schachermayer, and N. Touzi (2008), Optimal Risk Sharing For Law Invariant Monetary Utility Functions. *Mathematical Finance*, 18(2):269-292.

[67] Kaina, M., and L. Rüschendorf (2009), On Convex Risk Measures on L^p-Spaces. *Mathematical Methods of Operations Research*, 69(3):475-495.

[68] Kalkbrener, M. (2005), An Axiomatic Approach to Capital Allocation. *Mathematical Finance*, 15(3):425-437.

[69] Keynes, J. M. (1921), The Principles of Probability. In: Johnson, E., and D. Moggridge (eds.) (1978), *Collected Writings of John Maynard Keynes, Vol. 8*. Royal Economic Society.

[70] Keynes, J. M. (1937), The General Theory of Employment. *The Quarterly Journal of Economics*, 51(2):209-223.

[71] Keynes, J. M., R. Skidelsky (ed.) (2015), *The Essential Keynes*. Penguin, London.

[72] Knight, F. (1921/1964), *Risk, Uncertainty and Profit*. Reprints of Economic Classics, Augustus M. Kelley Pubs., New York.

[73] Kupper, M., and G. Svindland (2011), Dual Representation of Monotone Convex Functions on L^0. *Proceedings of the AMS*, 139(11):4073-4086.

[74] Kusuoka, S. (2001), On Law-Invariant Coherent Risk Measures. *Advances in Mathematical Economics*, 3:83-95.

[75] Landsberger, M., and I. Meilijson (1994), Comonotone Allocations, Bickel-Lehmann Dispersion and the Arrow-Pratt Measure of Risk Aversion. *Annals of Operations Research*, 52(2):97-106.

[76] Liebrich, F.-B., and G. Svindland (2017), Model Spaces for Risk Measures. *Insurance: Mathematics and Economics*, 77:150-165.

[77] Liebrich, F.-B., and G. Svindland (2019), Efficient Allocations under Law-Invariance: A Unifying Approach. *Journal of Mathematical Economics*, 84:28-45.

[78] Liebrich, F.-B., and G. Svindland (2019), Risk Sharing for Capital Requirements with Multidimensional Security Markets. Forthcoming in *Finance and Stochastics*.

[79] Liu, P., R. Wang, and L. Wei (2019), Is the Inf-convolution of Law-invariant Preferences Law-invariant?. Preprint.

[80] Ludkovski, M., and V. Rüschendorf (2008), On Comonotonicity of Pareto Optimal Risk Sharing. *Statistics & Probability Letters*, 78(10):1181-1188.

[81] Maccheroni, F., M. Marinacci, and A. Rustichini (2006), Ambiguity Aversion, Robustness, and the Variational Representation of Preferences. *Econometrica*, 74(6):1447-1498.

[82] Maggis, M., T. Meyer-Brandis, and G. Svindland (2018), Fatou Closedness under Model Uncertainty. *Positivity*, 22(5):1325-1343.

[83] Mas-Colell, A., and W. Zame (1991), Equilibrium Theory in Infinite Dimensional Spaces. In: *Handbook of Mathematical Economics*, Vol. 4, pp. 1835-1898. Elsevier.

[84] Meyer-Nieberg, P. (1991), *Banach Lattices*. Springer.

[85] Munari, C. (2015), *Measuring Risk Beyond the Cash-Additive Paradigm*. PhD thesis, ETH Zürich. Available online at https://doi.org/10.3929/ethz-a-010475509.

[86] Nutz, M. (2014), Superreplication under Model Uncertainty in Discrete Time. *Finance and Stochastics*, 18(4):791-803.

[87] Obłój, J., and J. Wiesel (2018), A Unified Framework to Modelling Financial Markets in Discrete Time. Preprint, arXiv:1808.06430.

[88] Owari, K. (2014), Maximum Lebesgue Extensions of Monotone Convex Functions. *Journal of Functional Analysis*, 266(6):3572-3611.

[89] Ravanelli, C., and G. Svindland (2014), Comonotone Pareto Optimal Allocations for Law Invariant Robust Utilities on L^1. *Finance and Stochastics*, 18:249-269.

[90] Reinganum, J. F. (1984), Practical Implications of Game Theoretic Models of R&D. *American Economic Review*, 74(2):61-66.

[91] Scandolo, G. (2004), Models of Capital Requirements in Static and Dynamic Settings. *Economic Notes by Banca Monte dei Paschi di Siena SpA*, 33(3):415-435.

[92] Soner, H. M., N. Touzi, J. Zang (2011), Quasi-Sure Stochastic Analysis Through Aggregation. *Electronic Journal of Probability*, 16:1844-1879.

[93] Svindland, G. (2010), Subgradients of Law-Invariant Convex Risk measures on L^1. *Statistics & Decisions*, 27(2):169-199.

[94] Svindland, G. (2010), Continuity Properties of Law-Invariant (Quasi-)Convex Risk Functions on L^∞. *Mathematics and Financial Economics*, 3(1):39-43.

[95] Svindland, G. (2014), Dilatation Monotonicity and Convex Order. *Mathematics and Financial Economics*, 8:241-247.

[96] Tsanakas, A. (2009), To Split or Not to Split: Capital Allocation with Convex Risk Measures. *Insurance: Mathematics and Economics*, 44(2):268-277.

[97] Wang, R. (2016), Regulatory Arbitrage of Risk Measures. *Quantitative Finance*, 16(3):337-347.

[98] Weber, S. (2018), Solvency II, or How to Sweep the Downside Risk Under the Carpet. *Insurance: Mathematics and Economics*, 82:191-200.

[99] Wilson, R. (1968), The Theory of Syndicates. *Econometrica*, 36:119-132.

[100] *The Turner Review: A Regulatory Response to the Global Banking Crisis*. Published by the British Financial Services Authority (2009). Available online at `www.fsa.gov.uk/pubs/other/turner_review.pdf`.

Chapter 2

Model Spaces for Risk Measures

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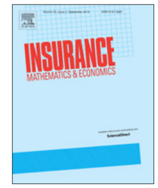

Model spaces for risk measures

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ABSTRACT

We show how risk measures originally defined in a model free framework in terms of acceptance sets and reference assets imply a meaningful underlying probability structure. Hereafter we construct a maximal domain of definition of the risk measure respecting the underlying ambiguity profile. We particularly emphasise liquidity effects and discuss the correspondence between properties of the risk measure and the structure of this domain as well as subdifferentiability properties.

1. Introduction

There is an ongoing debate on the *right* model space for financial risk measures, i.e. about what an ideal domain of definition for risk measures would be. Typically – as risk occurs in face of randomness – the risks which are to be measured are identified with real-valued random variables on some measurable space $(\Omega, \mathcal{F})$. The question which causes debate, however, is which space of random variables one should use as model space.

Since *risk* is often understood as Knightian uncertainty (Knight, 1921) about the underlying probabilistic mechanism, many scholars argue that model spaces should be robust in the sense of not depending too heavily on some specific probabilistic model. We refer to this normative viewpoint as *paradigm of minimal model dependence*. The literature usually suggests one of the following model spaces:

(i) $\mathcal{L}^0$ or $L^0_{\mathbb{P}}$, the spaces of all random variables or $\mathbb{P}$-almost sure ($\mathbb{P}$-a.s.) equivalence classes of random variables for some probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$, respectively, see Delbaen (2000, 2002);

(ii) $\mathcal{L}^\infty$ or $L^\infty_{\mathbb{P}}$, the spaces of all bounded random variables or $\mathbb{P}$-a.s. equivalence classes of bounded random variables, respectively, see Delbaen (2000, 2002), Föllmer and Schied (2002, 2011), Kusuoka (2001), Laeven and Stadje (2013) and the references therein;

(iii) $L^p_{\mathbb{P}}$, $p \in [1, \infty)$, the space of $\mathbb{P}$-a.s. equivalence classes of random variables with finite pth moment, or more generally Orlicz hearts, see e.g. Bellini et al. (2017), Cheridito and Li (2009), Frittelli and Rosazza Gianin (2002) and Rockafellar et al. (2006).

The spaces in (i) and (ii) satisfy minimal model dependence in that $\mathcal{L}^0$ and $\mathcal{L}^\infty$ are completely model free, whereas $L^0_{\mathbb{P}}$ and $L^\infty_{\mathbb{P}}$ in fact only depend on the null sets of the probability measure $\mathbb{P}$. The problem with choosing $\mathcal{L}^0$ or $L^0_{\mathbb{P}}$, however, is that these spaces are in general too large to reasonably define aggregation based risk measures on them. The latter would require some kind of integral to be well-defined. Moreover, if $(\Omega, \mathcal{F})$ is not finite, $\mathcal{L}^0$ or $L^0_{\mathbb{P}}$ do not allow for a locally convex topology which make them unapt for optimisation. Important applications of risk measures, however, use them as objective functions or constraints in optimisation problems. Since $\mathcal{L}^\infty$ and $L^\infty_{\mathbb{P}}$ are Banach spaces – so in particular locally convex spaces – and satisfy minimal model dependence, these model spaces have become most popular in the literature, and amongst them in particular $L^\infty_{\mathbb{P}}$ due to nicer analytic properties; see Delbaen (2000, 2002); Föllmer and Schied (2002, 2011), Kusuoka (2001), Laeven and Stadje (2013) and the references therein. In applications, however, unbounded models for risks are standard, like the log-normal distribution in Black–Scholes market models, etc. Assuming frictionless markets, there is no upper bound on the volumes and thus value of financial positions. Hence

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unbounded distributions appear quite naturally as limiting objects of bounded distributions, and in statistical modelling of random payoffs, where no upper bound can be assumed *a priori*. Also, risks with unbounded support and potentially heavy-tailed distributions are commonly employed in the insurance business. From this point of view model spaces should satisfy the *paradigm of maximal domain* in that they should at least be sufficiently large to include these standard unbounded models, and the model spaces in (iii) have been proposed to resolve this issue. Problematic though is the strong dependence of $L^p_{\mathbb{P}}$, $p \in [1, \infty)$, (or in general Orlicz hearts) on the probability measure $\mathbb{P}$ in that they are not invariant under equivalent changes of measure anymore. Consequently, maximal domain and minimal model dependence seem to be conflicting paradigms.

In the special case of law-invariant risk measures the measured risk is fully determined by the distribution of the risk under a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. Thus law-invariance already entails the existence of a *meaningful* reference probability model $\mathbb{P}$, and the risk measurement is fully depending on $\mathbb{P}$. Hence, the ambiguity structure is such that it is no conceptual problem to define these risk measures on, for instance, $L^1_{\mathbb{P}}$ (see Filipovic and Svindland, 2012).[1] The latter observation shows that the paradigms of minimal model dependence and maximal domain may not be as conflicting as they seem, as long as the underlying probability structure is determined by the considered risk measure. Clearly, a model space like $L^\infty_{\mathbb{P}}$ is sufficiently robust to carry any kind of risk measure. But given a specific risk measure – say defined on $L^\infty_{\mathbb{P}}$ – and the corresponding ambiguity attitude reflected by it, a model space which respects this ambiguity attitude, which also carries the risk measure, and which is probably larger than $L^\infty_{\mathbb{P}}$, is also a reasonable model space for that particular risk measure—like in the (unambiguous) case of a law-invariant risk measure and the model space $L^1_{\mathbb{P}}$.

Our starting point is an *a priori* completely model free setting on the model space $\mathcal{L}^\infty$ and a generalised notion of risk measurement adopted from Farkas, Koch Medina and Munari in, e.g., Farkas et al. (2015) and Munari in Munari (2015): all it requires is a notion of acceptability of losses (encoded by an acceptance set $\mathcal{A} \subseteq \mathcal{L}^\infty$), a portfolio of liquidly traded securities allowed for hedging (represented by a subspace $\mathcal{S} \subseteq \mathcal{L}^\infty$), and a set of observable prices for these securities (a linear functional $\mathfrak{p}$ on $\mathcal{S}$). Using such a risk measurement regime $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \mathfrak{p})$, we can define the risk $\rho_{\mathcal{R}}(X)$ to be the minimal price one has to pay in order to secure the loss $X \in \mathcal{L}^\infty$ with a portfolio in $\mathcal{S}$. This approach has the indisputable advantage of a clear operational interpretation. Section 2 introduces this kind of risk measurement in a unifyingly general framework. In Section 3, we observe that under a standard approximation property of finite risk measures – continuity from above – they automatically imply a reference probability measure $\mathbb{P}$ which allows us to view the risk measure as defined on $L^\infty_{\mathbb{P}}$ without any loss of information. The observation that this often assumed property necessarily implies that the framework is dominated sheds new and critical light on the current discussion on model free and robust finance. Next, we demonstrate that under some further conditions, e.g., sensitivity and strict monotonicity, we can even find a *strong* reference probability measure $\mathbb{P}^* \approx \mathbb{P}$ such that additionally

$$\forall X \in L^\infty_{\mathbb{P}}: \quad \rho_{\mathcal{R}}(X) \geq c\mathbb{E}_{\mathbb{P}^*}[X]$$

holds for a suitable constant $c > 0$. These strong reference probability measures serve as a class of benchmark models in that risk estimation with $\rho_{\mathcal{R}}$ is uniformly more conservative than using the linear risk estimation rules $X \mapsto c\mathbb{E}_{\mathbb{P}^*}[X]$.

In Section 4.1, we discuss how these considerations lead to a Banach space $L^{\mathcal{R}}$ typically much larger than $L^\infty_{\mathbb{P}}$, which has a multitude of desirable properties, such as

- invariance under all strong and weak reference probability models;
- a geometry completely determined by the risk measure $\rho_{\mathcal{R}}$;
- robustness in that it carries an extension of the initial risk criterion $\rho_{\mathcal{R}}$, denoted by $\rho_{\tilde{\mathcal{R}}}$, which preserves the functional form of $\rho_{\mathcal{R}}$, the dual representation, and thus convexity and lower semicontinuity. Moreover, this extension $\rho_{\tilde{\mathcal{R}}}$ is a capital requirement in terms of unchanged hedging securities and pricing functionals, but with a notion of acceptability obtained by consistently extending constraints defining $\mathcal{A}$ to $L^{\mathcal{R}}$.

In the latter sense, $L^{\mathcal{R}}$ can be seen as a natural maximal domain of definition of the initial risk criterion on which the ambiguity attitude is preserved.

We also consider the following monotone extensions of $\rho_{\mathcal{R}}$ to unbounded loss profiles in $L^{\mathcal{R}}$ given by

$$\begin{aligned}\xi(X) &= \lim_{m\to\infty}\lim_{n\to\infty}\rho_{\mathcal{R}}((-n)\vee X\wedge m)\\ &= \sup_{m\in\mathbb{N}}\inf_{n\in\mathbb{N}}\rho_{\mathcal{R}}((-n)\vee X\wedge m)\end{aligned}$$

and

$$\begin{aligned}\eta(X) &= \lim_{n\to\infty}\lim_{m\to\infty}\rho_{\mathcal{R}}((-n)\vee X\wedge m)\\ &= \inf_{n\in\mathbb{N}}\sup_{m\in\mathbb{N}}\rho_{\mathcal{R}}((-n)\vee X\wedge m)\end{aligned}$$

which have been studied in for instance Delbaen (2002) and Svindland (2010b). One would maybe expect that always $\rho_{\tilde{\mathcal{R}}} = \xi = \eta$, but it turns out that $\rho_{\tilde{\mathcal{R}}} = \xi$ always holds, whereas $\rho_{\tilde{\mathcal{R}}} \neq \eta$ is possible, see Example 5.2. We characterise the often desirable regular situation when monotone approximation of risks in the following sense

$$\rho_{\tilde{\mathcal{R}}}(X) = \eta(X) = \lim_{n\to\infty}\rho_{\mathcal{R}}((-n)\vee X\wedge n) \tag{1.1}$$

is possible, see Theorem 4.7, and show that (1.1) holds if $\rho_{\tilde{\mathcal{R}}}$ shows sufficient continuity in the tail of the risk X. For instance, any risk measure to which some kind of monotone or dominated convergence rule can be applied will satisfy (1.1). In Section 4.2, we decompose $L^{\mathcal{R}}$ into subsets with a clear interpretation in terms of liquidity risk and show how $L^{\mathcal{R}}$ allows to view properties of the risk measure $\rho_{\tilde{\mathcal{R}}}$ through a topological lens. Finally, in Section 4.4, we address the issue of subdifferentiability of $\rho_{\tilde{\mathcal{R}}}$ on $L^{\mathcal{R}}$ based on a brief treatment of the dual of $L^{\mathcal{R}}$ in Section 4.3. Subgradients play an important role in risk optimisation and appear as pricing rules in optimal risk sharing schemes, see e.g. Jouini et al. (2008) and Svindland (2010b). We shall see that the topology on $L^{\mathcal{R}}$ being determined by $\rho_{\mathcal{R}}$ is fine enough to guarantee a rich class of points where $\rho_{\tilde{\mathcal{R}}}$ is subdifferentiable, thereby further illustrating how suited the model space $L^{\mathcal{R}}$ is to $\rho_{\mathcal{R}}$. Beside their mere existence, we also aim for reasonable conditions guaranteeing that subgradients correspond to measures on $(\Omega, \mathcal{F})$—which means ruling out singular elements that may exist in the dual space of $L^{\mathcal{R}}$. The motivation for this is the same as in case of $L^\infty_{\mathbb{P}}$ which in general also admits singular elements in its dual space. It is questionable whether such singular dual elements are reasonable as, for instance, pricing rules, because their effect lies mostly in the tails of the distribution, and the lack of countable additivity contradicts the paradigm of diminishing marginal risk. Also, measures show a by far better analytic behaviour which may prove to be crucial when solving optimisation problems. Our findings suggest that singular elements do not really matter in a wide range of instances. In particular, we will also see that the local equality (1.1) characterised in Theorem 4.7 is closely related to regular subgradients of $\rho_{\tilde{\mathcal{R}}}$ and η. In Section 5 we collect illustrating examples. Some cumbersome proofs are outsourced to the Appendices A and B.

[1] In fact, law-invariant risk measures are completely unambiguous.

2. Some preliminaries

Notation and terminology: Given a set $M \neq \emptyset$ and a function $f : M \to [-\infty, \infty]$, we define the DOMAIN of f to be the set $\mathrm{dom}(f) = \{m \in M \mid f(m) < \infty\}$. f is called PROPER if it does not attain the value $-\infty$ and $\mathrm{dom}(f) \neq \emptyset$.

For a subset A of a topological space $(\mathcal{X}, \tau)$, we denote by $\mathrm{cl}_\tau(A)$ and $\mathrm{int}_\tau(A)$ the closure and interior of A, respectively, with respect to the topology τ. If $(\mathcal{X}, \tau)$ is a topological vector space and τ is generated by a norm $\|\cdot\|$ on $\mathcal{X}$, we will replace the subscript τ by $\|\cdot\|$.

A triple $(\mathcal{X}, \tau, \preceq)$ is called ORDERED TOPOLOGICAL VECTOR SPACE if (X, τ) is a topological vector space and $\preceq$ is a partial vector space order compatible with the topology in that the positive cone of $\mathcal{X}$, denoted by $\mathcal{X}_+ := \{X \in \mathcal{X} \mid 0 \preceq X\}$, is τ-closed. We define $\mathcal{X}_{++} := \mathcal{X}_+ \setminus \{0\}$, and $\mathcal{X}_-$ and $\mathcal{X}_{--}$ analogously. If $(\mathcal{X}, \tau, \preceq)$ is a Riesz space and $X, Y \in \mathcal{X}$, we set $X \vee Y := \sup\{X, Y\}$, $X \wedge Y := \inf\{X, Y\}$, $X^+ := X \vee 0$, and $X^- := (-X) \vee 0$.[2]
In this section we define risk measurement regimes and risk measures, discuss some properties a risk measure may enjoy, and introduce the building blocks for a duality theory.

Definition 2.1. Let $(\mathcal{X}, \tau, \preceq)$ be an ordered topological vector space. An ACCEPTANCE SET is a non-empty proper and convex subset $\mathcal{A}$ of $\mathcal{X}$ which is monotone, i.e. $\mathcal{A} - \mathcal{X}_+ \subseteq \mathcal{A}$. A SECURITY SPACE is a finite-dimensional linear subspace $\mathcal{S} \subsetneq \mathcal{X}$ containing a non-null positive element $U \in \mathcal{S} \cap \mathcal{X}_{++}$. We refer to the elements $Z \in \mathcal{S}$ as security portfolios, or simply securities. A PRICING FUNCTIONAL on $\mathcal{S}$ is a positive linear functional $\mathfrak{p} : \mathcal{S} \to \mathbb{R}$ such that $\mathfrak{p}(Z) > 0$ for all $Z \in \mathcal{S} \cap \mathcal{X}_{++}$.

A triple $\mathcal{R} := (\mathcal{A}, \mathcal{S}, \mathfrak{p})$ is a RISK MEASUREMENT REGIME if $\mathcal{A}$ is an acceptance set, $\mathcal{S}$ is a security space and $\mathfrak{p}$ is a pricing functional on $\mathcal{S}$ such that

$$\forall X \in \mathcal{X} : \quad \sup\{\mathfrak{p}(Z) \mid Z \in \mathcal{S}, X + Z \in \mathcal{A}\} < \infty. \tag{2.1}$$

The RISK MEASURE associated to a risk measurement regime $\mathcal{R}$ is the functional

$$\rho_\mathcal{R} : \mathcal{X} \to (-\infty, \infty], \quad X \mapsto \inf\{\mathfrak{p}(Z) \mid Z \in \mathcal{S}, X - Z \in \mathcal{A}\}. \tag{2.2}$$

Our definition of risk measures is inspired by Farkas et al. (2015) and Munari (2015). Note that:

(a) The elements $X \in \mathcal{X}$ model *losses*, not gains. Thus $\rho_\mathcal{R}(X)$ is the minimal amount which has to be invested today in some security portfolio $Z \in \mathcal{S}$ with payoff $-Z$ in order to reduce the loss X tomorrow to an acceptable level.
(b) We prescribe convexity of the acceptance set $\mathcal{A}$ which means that diversification is not penalised: if X and Y are acceptable so is the diversified $\lambda X + (1 - \lambda)Y$ for any $\lambda \in (0, 1)$.
(c) The notion of a risk measurement regime depends on the interplay of $\mathcal{A}$, $\mathcal{S}$ and $\mathfrak{p}$ by means of (2.1); this condition guarantees that $\rho_\mathcal{R}$ is a proper function. Farkas et al. (2015, Propositions 1 and 2) yield criteria for $\mathcal{R}$ to be a risk measurement regime in our sense.

If $\mathcal{S} = \mathbb{R} \cdot U$ for some $U \in \mathcal{X}_{++}$ and $\mathfrak{p}(mU) = m$, $m \in \mathbb{R}$, the setting of Farkas et al. (2013, 2014) with a single liquid eligible asset can be recovered from Definition 2.1. If $\mathcal{X}$ is a Riesz space with weak unit $\mathbf{1}$, $\mathcal{S} = \mathbb{R} \cdot \mathbf{1}$ and $\mathfrak{p}(m\mathbf{1}) = m$, $m \in \mathbb{R}$, the definition covers *convex monetary risk measures* as comprehensively discussed in Föllmer and Schied (2011).[3] The following is easily verified:

Lemma 2.2. *Let $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \mathfrak{p})$ be a risk measurement regime on an ordered topological vector space $\mathcal{X}$. Then $\rho_\mathcal{R}$ is convex,* MONOTONE, *i.e. $X \preceq Y$ implies $\rho_\mathcal{R}(X) \leq \rho_\mathcal{R}(Y)$, and $\mathcal{S}$-* ADDITIVE, *i.e. $\rho_\mathcal{R}(X + Z) = \rho_\mathcal{R}(X) + \mathfrak{p}(Z)$ holds for all $X \in \mathcal{X}$ and all $Z \in \mathcal{S}$.*

In the same abstract setting we introduce further properties a risk measure can enjoy.

Definition 2.3. Let $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \mathfrak{p})$ be a risk measurement regime on an ordered topological vector space $(\mathcal{X}, \tau, \preceq)$ and let $\rho_\mathcal{R}$ be the associated risk measure.

- $\rho_\mathcal{R}$ is called FINITE if it only takes finite values, or equivalently $\mathcal{A} + \mathcal{S} = \mathcal{X}$.[4]
- $\rho_\mathcal{R}$ is NORMALISED if $\rho_\mathcal{R}(0) = 0$, or equivalently $\sup_{Z \in \mathcal{A} \cap \mathcal{S}} \mathfrak{p}(Z) = 0$.
- $\rho_\mathcal{R}$ is COHERENT if for any $X \in \mathcal{X}$ and for any $t > 0$ $\rho_\mathcal{R}(tX) = t\rho_\mathcal{R}(X)$ holds.
- $\rho_\mathcal{R}$ is SENSITIVE if it satisfies $\rho_\mathcal{R}(X) > \rho_\mathcal{R}(0)$ for all $X \in \mathcal{X}_{++}$.
- $\rho_\mathcal{R}$ is LOWER SEMICONTINUOUS (l.s.c.) if every lower level set $\{X \in \mathcal{X} \mid \rho_\mathcal{R}(X) \leq c\}$, $c \in \mathbb{R}$, is τ-closed.
- $\rho_\mathcal{R}$ is CONTINUOUS FROM ABOVE if it is finite and for any $(X_n)_{n \in \mathbb{N}} \subseteq \mathcal{X}$ with $X_n \downarrow X$ in order $\rho_\mathcal{R}(X) = \lim_n \rho_\mathcal{R}(X_n)$ holds.

Remark 2.4.

(i) Normalisation implies that the negative cone $\mathcal{X}_-$ (no losses) will be acceptable, which is economically sound. Every risk measure satisfying $\rho_\mathcal{R}(0) \in \mathbb{R}$ can be normalised by translating the acceptance set. Indeed, let $U \in \mathcal{S} \cap \mathcal{X}_{++}$ and define $r := \frac{\rho_\mathcal{R}(0)}{\mathfrak{p}(U)}$ and $\check{\mathcal{A}} := \{X + rU \mid X \in \mathcal{A}\}$. If $\mathcal{R}$ is a risk measurement regime, then so is $\check{\mathcal{R}} := (\check{\mathcal{A}}, \mathcal{S}, \mathfrak{p})$. Moreover,

$$\begin{aligned} -\rho_{\check{\mathcal{R}}}(0) &= \sup\{\mathfrak{p}(Z) \mid Z \in \mathcal{S}, Z - rU \in \mathcal{A}\} \\ &= \sup\{\mathfrak{p}(W) + \mathfrak{p}(rU) \mid W \in \mathcal{S} \cap \mathcal{A}\}. \end{aligned}$$

This implies that $-\rho_{\check{\mathcal{R}}}(0) = -\rho_\mathcal{R}(0) + \rho_\mathcal{R}(0) = 0$ holds.
(ii) Recall that, in contrast to a large share of the literature on risk measures, random variables model losses, not gains, in our setting. Consequently, our notion of continuity from above is not the same as continuity from above in the sense of Föllmer and Schied (cf. Föllmer and Schied, 2011, Lemma 4.21). The equivalent notion in the aforementioned monograph would be *continuity from below* (cf. Föllmer and Schied, 2011, Theorem 4.22), which together with lower semicontinuity of a risk measure implies the *Lebesgue property*—see Föllmer and Schied (2011).
(iii) Our notion of continuity from above means that approximating the risk of complex payoffs by the one of potentially easier but *worse* financial instruments is meaningful as long as the payoffs range in a bounded regime.
(iv) Lower semicontinuity of $\rho_\mathcal{R}$ implies that $\{X \in \mathcal{X} \mid \rho_\mathcal{R}(X) \leq 0\} = \mathrm{cl}_\tau(\mathcal{A} + \ker(\mathfrak{p}))$. In particular, it is implied by $\mathcal{A} + \ker(\mathfrak{p})$ being closed (see Farkas et al., 2015, Proposition 4) and invariant under translations of the acceptance set along $\mathcal{S}$. From an economic perspective this property is not too demanding: security spaces are always finite-dimensional in our setting, hence lower semicontinuity is, e.g., implied

[2] For details concerning ordered vector spaces, we refer to Chapters 5 and 7 of Aliprantis and Border (1999). Since risk measures will appear in this treatment on different domains of definition – in all cases spaces of random variables endowed with a pointwise or almost sure order and with varying topologies – we define them as functionals on ordered topological vector spaces. However, the reader may think of $\mathcal{X}$ as a space of random variables and of $\preceq$ as a pointwise or almost sure order on the latter.

[3] In the following, we will refer to this particular case with the term *monetary risk measures*.

[4] Farkas et al. (2015, Propositions 1–3) give further criteria to decide whether $\rho_\mathcal{R}$ is finite or not.

by the condition $\mathcal{A} \cap \ker(\mathfrak{p}) = \{0\}$ (cf. Farkas et al., 2015, Proposition 5). The latter is sometimes referred to as *absence of good deals of the first kind* (cf. Jaschke and Küchler, 2001).

(v) Note that in the case of $\mathcal{X}$ being a Banach lattice with norm $\|\cdot\|$, every finite risk measure is norm-continuous and therefore also norm-l.s.c. This follows from Ruszczyński and Shapiro (2006, Proposition 1): Suppose $\mathcal{X}$ is a Banach lattice and $f : \mathcal{X} \to (-\infty, \infty]$ is a proper convex and monotone function. Then f is continuous on $\operatorname{int}\operatorname{dom}(f)$. We will make frequent use of this fact throughout the paper.

For many questions a dual point of view on risk measures is crucial. In our case, its formulation requires the following concepts:

Definition 2.5. Assume $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \mathfrak{p})$ is a risk measurement regime on an ordered topological vector space $(\mathcal{X}, \tau, \preceq)$ with topological dual $\mathcal{X}^*$. We define the SUPPORT FUNCTION of $\mathcal{A}$ by

$$\sigma_{\mathcal{A}} : \mathcal{X}^* \to (-\infty, \infty], \quad \ell \mapsto \sup_{Y \in \mathcal{A}} \ell(Y), \tag{2.3}$$

and $\mathcal{B}(\mathcal{A}) := \operatorname{dom}(\sigma_{\mathcal{A}})$. Moreover, the EXTENSION SET will refer to the set of positive, continuous extensions of $\mathfrak{p}$ to $\mathcal{X}$, namely $\mathcal{E}_{\mathfrak{p}} := \{\ell \in \mathcal{X}^*_+ \mid \ell|_{\mathcal{S}} = \mathfrak{p}\}$.

3. Model spaces of bounded random variables, and weak and strong reference probability models

3.1. The model space $\mathcal{L}^\infty$ and weak reference probability measures

Fix a measurable space $(\Omega, \mathcal{F})$ and let $\mathcal{L}^\infty := \mathcal{L}^\infty(\Omega, \mathcal{F})$ be the set of bounded measurable real-valued functions. We recall that $\mathcal{L}^\infty$ is a Banach lattice with norm $|X|_\infty := \sup_{\omega\in\Omega}|X(\omega)|$ when equipped with the pointwise order $\leq$, so in particular an ordered topological vector space. On the level of Riesz spaces, $\Omega \ni \omega \mapsto 1$ is an order unit of $\mathcal{L}^\infty$.[5] The dual space of $\mathcal{L}^\infty$ may be identified with **ba**, the space of all finitely additive set functions $\mu : \mathcal{F} \to \mathbb{R}$. As usual **ca** denotes the countably additive set functions in **ba**, and $\mathbf{ca}_+$ is the set of finite measures. In the following, the notation will not distinguish between $m \in \mathbb{R}$ and the function $\Omega \ni \omega \mapsto m$.

In this section we study risk measures on $\mathcal{L}^\infty$. First of all, note that in the $\langle \mathcal{L}^\infty, \mathbf{ba}\rangle$-duality monotonicity of $\mathcal{A}$ implies that $\mathcal{B}(\mathcal{A}) \subseteq \mathbf{ba}_+$ has to hold, and an application of the Hahn–Banach Separation Theorem shows

$$\operatorname{cl}_{|\cdot|_\infty}(\mathcal{A}) = \{X \in \mathcal{L}^\infty \mid \forall \mu \in \mathcal{B}(\mathcal{A}) : \int X\,d\mu \leq \sigma_{\mathcal{A}}(\mu)\}. \tag{3.1}$$

We will mostly assume finiteness of $\rho_{\mathcal{R}}$, which is justified by the domain of definition $\mathcal{L}^\infty$—that is *bounded* losses which typically should be hedgeable at potentially large, but finite cost. $\rho_{\mathcal{R}}$ is for instance finite whenever the security space $\mathcal{S}$ contains some $U \in \mathcal{L}^\infty_{++}$ being uniformly bounded away from 0, i.e. $U \geq \delta$ for some constant $\delta > 0$. In Farkas et al. (2013, 2014), such securities are called NON-DEFAULTABLE. We will show that if the acceptance set is "nice enough", then any finite risk measure arising from it in an *a priori* model-free framework like $\mathcal{L}^\infty$ indeed implies a probabilistic model, a so-called weak reference model; see Theorem 3.3. As a first step towards this result, we show now that continuity from above mainly depends on the geometry of the acceptance set $\mathcal{A}$. To this end, let us recall the notion of the DUAL CONJUGATE of $\rho_{\mathcal{R}}$ being defined as

$$\rho^*_{\mathcal{R}} : \mathbf{ba} \to (-\infty, \infty], \quad \mu \mapsto \sup_{X\in\mathcal{L}^\infty} \int X\,d\mu - \rho_{\mathcal{R}}(X). \tag{3.2}$$

[5] Recall that $e \in \mathcal{X}_+$ is an order unit of a Riesz space $(\mathcal{X}, \preceq)$ if $\{X \in \mathcal{X} \mid \exists \lambda > 0 : |X| \preceq \lambda e\} = \mathcal{X}$.

Proposition 3.1. *Assume $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \mathfrak{p})$ is a risk measurement regime such that $\rho_{\mathcal{R}}(0) \in \mathbb{R}$.*

(i) *If $\rho_{\mathcal{R}}$ is l.s.c., $\mathcal{B}(\mathcal{A}) \cap \mathcal{E}_{\mathfrak{p}}$ is non-empty and for all $\mu \in$ **ba** it holds that*

$$\rho^*_{\mathcal{R}}(\mu) = \begin{cases} \sigma_{\mathcal{A}}(\mu) & \text{if } \mu \in \mathcal{B}(\mathcal{A}) \cap \mathcal{E}_{\mathfrak{p}}, \\ \infty & \text{otherwise}. \end{cases} \tag{3.3}$$

For all $X \in \mathcal{L}^\infty$ we have

$$\rho_{\mathcal{R}}(X) = \sup_{\mu\in\operatorname{dom}(\rho^*_{\mathcal{R}})} \int X\,d\mu - \rho^*_{\mathcal{R}}(\mu). \tag{3.4}$$

Moreover, if $\rho_{\mathcal{R}}$ is coherent, then

$$\rho^*_{\mathcal{R}}(\mu) = \begin{cases} 0 & \text{if } \mu \in \mathcal{B}(\mathcal{A}) \cap \mathcal{E}_{\mathfrak{p}}, \\ \infty & \text{otherwise}, \end{cases} \tag{3.5}$$

and

$$\rho_{\mathcal{R}}(X) = \sup_{\mu\in\operatorname{dom}(\rho^*_{\mathcal{R}})} \int X\,d\mu, \quad X \in \mathcal{L}^\infty. \tag{3.6}$$

(ii) *Assume $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \mathfrak{p})$ is a risk measurement regime such that $\rho_{\mathcal{R}}$ is finite, then for every $c \in \mathbb{R}$ the lower level set $E_c := \{\mu \in \mathbf{ba} \mid \rho^*_{\mathcal{R}}(\mu) \leq c\}$ of $\rho^*_{\mathcal{R}}$ is $\sigma(\mathbf{ba}, \mathcal{L}^\infty)$-compact.*

(iii) *Suppose the risk measure $\rho_{\mathcal{R}}$ associated to the risk measurement regime $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \mathfrak{p})$ is finite. Then $\rho_{\mathcal{R}}$ is continuous from above if and only if every lower level set E_c, $c \in \mathbb{R}$, of $\rho^*_{\mathcal{R}}$ is $\sigma(\mathbf{ca}, \mathcal{L}^\infty)$-compact. Hence, if $\mathcal{B}(\mathcal{A}) \subseteq \mathbf{ca}$, then $\rho_{\mathcal{R}}$ is continuous from above.*

(iv) *In the situation of (iii), if $\mathcal{S}$ is constrained to be one-dimensional, then $\rho_{\mathcal{R}}$ is continuous from above if and only if $\mathcal{B}(\mathcal{A}) \subseteq \mathbf{ca}$.*

Corollary 3.2. *Assume $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \mathfrak{p})$ is a risk measurement regime such that $\rho_{\mathcal{R}}$ is finite. Then $\rho_{\mathcal{R}}$ is continuous from above if and only if $\mathcal{B}(\mathcal{A}) \cap \ker(\mathfrak{p})^\perp \subseteq \mathbf{ca}$. Here,*

$$\ker(\mathfrak{p})^\perp := \left\{\mu \in \mathbf{ba} \,\middle|\, \forall Z \in \ker(\mathfrak{p}) : \int Z\,d\mu = 0\right\}$$

denotes the annihilator of $\ker(\mathfrak{p})$.

For the special case of a monetary risk measure, parts of Proposition 3.1 are well-known, see e.g. Föllmer and Schied (2011, Theorem 4.22 and Corollary 4.35). However, to our knowledge, so far there is no proof of Proposition 3.1 and Corollary 3.2 in this general form in the literature. As the proofs of these results are quite technical and thus lengthy we provide them in Appendix A. Note that the representation (3.4) is in terms of pricing rules consistent with $(\mathcal{S}, \mathfrak{p})$ in that $\mu \in \operatorname{dom}(\rho^*_{\mathcal{R}})$ only if $\mu|_{\mathcal{S}} = \mathfrak{p}$. If $S = \mathbb{R}$ and $\mathfrak{p} = id_{\mathbb{R}}$, these pricing rules can be identified with probability measures. Finally, we remark that continuity from above is indeed mainly a property of the acceptance set as Proposition 3.1(iii) and (iv) and Corollary 3.2 show: If $\mathcal{A}$ is regular in the sense that $\mathcal{B}(\mathcal{A}) \subseteq \mathbf{ca}$, then every finite risk measure is continuous from above. In particular, taking a single hedging asset or multiple ones will have no effect on continuity from above provided $\mathcal{A}$ is properly chosen. Non-regularity of the acceptance set in that $\mathcal{B}(\mathcal{A}) \setminus \mathbf{ca} \neq \emptyset$, however, is equivalent to the fact that no finite risk measure with a single security is continuous from above; higher-dimensional security spaces may smooth out the irregularity of $\mathcal{A}$, as illustrated in Example 5.1.

The following theorem is the already advertised main result of this section. As facilitating notation, for non-empty sets of set functions $M, M' \subseteq \mathbf{ba}$, we write $M \ll M'$ if and only if $\nu(A) = 0$ for all $\nu \in M'$ implies $\mu(A) = 0$ for all $\mu \in M$, $A \in \mathcal{F}$. We set $M \approx M'$ to mean that both $M \ll M'$ and $M' \ll M$. Instead of $\{\mu\} \ll \{\nu\}$ or $\{\mu\} \approx \{\nu\}$, we shall write $\mu \ll \nu$ and $\mu \approx \nu$. Finally, we define $\mathbf{ba}_\nu := \{\mu \in \mathbf{ba} \mid \mu \ll \nu\}$, and $\mathbf{ca}_\nu$ analogously.

Theorem 3.3. *Let $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \mathfrak{p})$ be a risk measurement regime such that $\rho_\mathcal{R}$ is finite and continuous from above.*

(i) *There exists a* WEAK REFERENCE PROBABILITY MEASURE $\mathbb{P}$, *that is a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ such that $\rho^*_\mathcal{R}(c\mathbb{P}) < \infty$ for a suitable $c > 0$ and*

$$\mathbb{P} \approx \mathrm{dom}(\rho^*_\mathcal{R}). \tag{3.7}$$

(ii) *For $\mathbb{P}$ as in* (i) *we have that* $\mathrm{dom}(\rho^*_\mathcal{R}) \subseteq (\mathbf{ca}_\mathbb{P})_+$.

(iii) *If $\rho_\mathcal{R}$ is normalised, then $E_0 = \{\mu \in \mathbf{ca} \mid \rho^*_\mathcal{R}(\mu) = 0\} \neq \emptyset$.*

Proof. For (i), recall from Proposition 3.1 that the assumption on $\rho_\mathcal{R}$ implies that any lower level set $E_c := \{\mu \in \mathbf{ca}_+ \mid \rho^*_\mathcal{R}(\mu) \le c\}$, $c \in \mathbb{R}$, is $\sigma(\mathbf{ca}, \mathcal{L}^\infty)$-compact. Together with convexity, this implies *countable convexity*, i.e.

$$(\lambda_k)_{k\in\mathbb{N}} \subseteq [0,1], \quad \sum_{k=1}^\infty \lambda_k = 1,$$
$$(\mu_k)_{k\in\mathbb{N}} \subseteq E_c \implies \sum_{k=1}^\infty \lambda_k\mu_k \in E_c. \tag{3.8}$$

By Bogachev (2007, Theorem 4.7.25, (iv) $\Rightarrow$ (i)), E_n, $n \in \mathbb{N}$, also has compact closure in the weak topology $\sigma(\mathbf{ca}, \mathbf{ca}^*)$. As E_n is already closed in the weaker topology $\sigma(\mathbf{ca}, \mathcal{L}^\infty)$, E_n has to be weakly compact. The proof of Bogachev (2007, Theorem 4.7.25, (i) $\Rightarrow$ (ii)) shows the existence of a sequence $(\mu^n_l)_{l\in\mathbb{N}} \subseteq E_n$ such that $E_n \approx \{\mu^n_l \mid l \in \mathbb{N}\}$. We set $\nu_n := \sum_{l=1}^\infty 2^{-l}\mu^n_l$, which lies in E_n by (3.8), and satisfies $\nu_n \approx E_n$. By (3.4), the sequence $(\nu_n)_{n\in\mathbb{N}}$ satisfies $\nu_n(\Omega) \le \nu_n(\Omega) - \rho^*_\mathcal{R}(\nu_n) + n \le \rho_\mathcal{R}(1) + n$. Define

$$\nu := \sum_{n\in\mathbb{N}} 2^{-n}\nu_n, \quad c_N := \sum_{n=1}^N 2^{-n},$$
$$\zeta_N := c_N^{-1}\sum_{n=1}^N 2^{-n}\nu_n, \quad N \in \mathbb{N}.$$

$\nu \in \mathbf{ca}_+$ follows from the estimate $\nu(\Omega) \le \sum_{n=1}^\infty 2^{-n}(\rho_\mathcal{R}(1) + n) = \rho_\mathcal{R}(1) + 2$. Every non-trivial scalar multiple of ν satisfies (3.7), and moreover, $\nu = \lim_N \zeta_N$ with respect to $\sigma(\mathbf{ca}, \mathcal{L}^\infty)$. Lower semicontinuity and convexity of $\rho^*_\mathcal{R}$ and $\rho^*_\mathcal{R}(\nu_n) \le n$ imply

$$\rho^*_\mathcal{R}(\nu) \le \liminf_{N\to\infty} \rho^*_\mathcal{R}(\zeta_N) \le \lim_{N\to\infty} c_N^{-1}\sum_{n=1}^N 2^{-n}n = 2.$$

Choosing $c := \nu(\Omega)$, the probability measure $\mathbb{P} := \frac{1}{c}\nu$ is a weak reference probability model.
(ii) is an immediate consequence of (3.7). In order to prove (iii) note that normalisation implies $0 = \rho_\mathcal{R}(0) = -\inf\{\rho^*_\mathcal{R}(\mu) \mid \mu \in \mathrm{dom}(\rho^*_\mathcal{R})\}$. Hence, $\rho^*_\mathcal{R} \ge 0$ and the family of subsets $(E_k)_{k\in(0,1]}$ of the compact set E_1 has the finite intersection property. Therefore $E_0 = \bigcap_{k\in(0,1]} E_k \neq \emptyset$. $\square$

Remark 3.4. Continuity from above is sufficient but not necessary for the existence of weak reference probability models. However, without continuity from above anything can happen. For example, let $(\Omega, \mathcal{F})$ be the open unit interval $(0,1)$ endowed with its Borel sets $\mathbb{B}((0,1))$, and let $\mathbb{P}$ be the Lebesgue measure on $(0,1)$. Consider

$$\mathrm{ess\,sup}(X) := \sup\{m \in \mathbb{R} \mid \mathbb{P}(X \le m) = 1\},$$

and set

$$\mathcal{A}_1 := \{X \in \mathcal{L}^\infty \mid \mathrm{ess\,sup}(X) \le 0\},$$
$$\mathcal{A}_2 := \{X \in \mathcal{L}^\infty \mid \sup_{\omega\in\Omega} X(\omega) \le 0\}.$$

The triples $\mathcal{R}_i = (\mathcal{A}_i, \mathbb{R}, id_\mathbb{R})$, $i = 1, 2$, are risk measurement regimes. In the first case, $\mathrm{dom}(\rho^*_{\mathcal{R}_1}) = (\mathbf{ba}_\mathbb{P})_+$, and in the second $\mathrm{dom}(\rho^*_{\mathcal{R}_2}) = \mathbf{ba}_+$. Thus neither $\rho_{\mathcal{R}_1}$ nor $\rho_{\mathcal{R}_2}$ is continuous from above. $\mathbb{P}$, however, is a weak reference probability model for $\rho_{\mathcal{R}_1}$, whereas in the case of $\rho_{\mathcal{R}_2}$ there is no weak reference probability model as Ω is uncountable.

Whenever a probability measure $\mathbb{P}$ satisfies (3.7) and $X, Y \in \mathcal{L}^\infty$ are equal $\mathbb{P}$-almost surely ($\mathbb{P}$-a.s.), (3.4) shows that $\rho_\mathcal{R}(X) = \rho_\mathcal{R}(Y)$. Hence, we may view $\rho_\mathcal{R}$ as a function on the space of equivalence classes $L^\infty_\mathbb{P} := L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ with the corresponding properties. We recall that the least upper bound

$$\|X\|_\infty := \inf\{m \in \mathbb{R} \mid \mathbb{P}(|X| \le m) = 1\}, \quad X \in L^\infty_\mathbb{P},$$

is a norm on $L^\infty_\mathbb{P}$, making it into a Banach lattice together with the $\mathbb{P}$-almost sure order, and the equivalence class generated by $\Omega \ni \omega \mapsto 1$ is an order unit of $L^\infty_\mathbb{P}$. Its dual may be identified with $\mathbf{ba}_\mathbb{P}$. Let $\iota : \mathcal{L}^\infty \to L^\infty_\mathbb{P}$ be the canonical embedding, then it is straightforward to prove the following result.

Corollary 3.5. *In the situation of Theorem 3.3 define $\rho : L^\infty_\mathbb{P} \to \mathbb{R}$ by $\rho(\tilde{X}) = \rho_\mathcal{R}(X)$, where $X \in \mathcal{L}^\infty$ satisfies $\iota(X) = \tilde{X}$. Then ρ is well-defined and agrees with the risk measure $\rho_{(\iota(\mathcal{A}),\iota(\mathcal{S}),\tilde{\mathfrak{p}})}$ on $L^\infty_\mathbb{P}$, where $\tilde{\mathfrak{p}}(\tilde{Z}) = \mathfrak{p}(Z)$ whenever $\tilde{Z} = \iota(Z)$. It is norm-continuous and continuous from above. The dual function*

$$\rho^*(\mu) := \sup_{\tilde{X}\in L^\infty_\mathbb{P}} \int X\,d\mu - \rho(\tilde{X}), \quad \mu \in \mathbf{ba}_\mathbb{P}, \tag{3.9}$$

where X denotes an arbitrary representative of $\tilde{X}$, agrees with $\rho^*_\mathcal{R}|_{\mathbf{ba}_\mathbb{P}}$. Also

$$\rho(\tilde{X}) = \sup_{\mu\in\mathrm{dom}(\rho^*_\mathcal{R})} \int X\,d\mu - \rho^*_\mathcal{R}(\mu), \quad \tilde{X} \in L^\infty_\mathbb{P},$$

where X and $\tilde{X}$ are related as before.

3.2. The model space $L^\infty_\mathbb{P}$ and strong reference probability measures

Supported by our results in Theorem 3.3 and Corollary 3.5 we will from now on consider the model space $L^\infty_\mathbb{P}$, acceptance sets $\mathcal{A} \subseteq L^\infty_\mathbb{P}$, security spaces $\mathcal{S} \subseteq L^\infty_\mathbb{P}$, pricing functionals $\mathfrak{p} : \mathcal{S} \to \mathbb{R}$ and resulting finite risk measures $\rho_\mathcal{R}$ directly defined on $L^\infty_\mathbb{P}$, where $\mathbb{P}$ is a weak reference probability model for $\rho_\mathcal{R}$. Moreover, in the following we will stick to the usual convention of identifying an equivalence class of random variables in $L^\infty_\mathbb{P}$ with an arbitrary representative of that class.

By similar reasoning as in Proposition 3.1 and Theorem 3.3 we have the following result.

Lemma 3.6. *Let $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \mathfrak{p})$ be a risk measurement regime on $L^\infty_\mathbb{P}$ such that $\rho_\mathcal{R}$ is finite and normalised. Define*

$$\rho^*_\mathcal{R}(\mu) := \sup_{X\in L^\infty_\mathbb{P}} \int X\,d\mu - \rho_\mathcal{R}(X), \quad \mu \in \mathbf{ba}_\mathbb{P}. \tag{3.10}$$

Then $\rho_\mathcal{R}$ is continuous from above if and only if all lower level sets $E_c := \{\mu \in \mathbf{ba}_\mathbb{P} \mid \rho^*_\mathcal{R}(\mu) \le c\}$, $c \in \mathbb{R}$, of $\rho^*_\mathcal{R}$ are $\sigma(\mathbf{ca}_\mathbb{P}, L^\infty_\mathbb{P})$-compact, which is in particular implied by

$$\mathcal{B}(\mathcal{A}) := \left\{\mu \in \mathbf{ba}_\mathbb{P} \,\middle|\, \sup_{Y\in\mathcal{A}} \int Y\,d\mu < \infty\right\} \subseteq \mathbf{ca}_\mathbb{P}.$$

In that case

$$\rho_\mathcal{R}(X) = \sup_{\mu\in(\mathbf{ca}_\mathbb{P})_+} \int X\,d\mu - \rho^*_\mathcal{R}(\mu), \quad X \in L^\infty_\mathbb{P}. \tag{3.11}$$

In particular, $\mathrm{dom}(\rho^*_\mathcal{R}) = \mathcal{B}(\mathcal{A}) \cap \{\nu \in (\mathbf{ca}_\mathbb{P})_+ \mid \forall Z \in \mathcal{S} : \int Z\,d\nu = \mathfrak{p}(Z)\}$, and $E_0 \neq \emptyset$. If $\rho_\mathcal{R}$ is positively homogeneous, then the analogues of (3.5) and (3.6) hold as well.

We devote the remainder of this section to the question whether there is a STRONG REFERENCE MODEL, i.e. whether there an element in

$\mathcal{P} := \{\mu \in E_0 \mid \mu \approx \mathbb{P}\}$.

This notion is well-known in the case of law-invariant monetary risk measures, and the result can be generalised in our setting:

Proposition 3.7. *Let $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \mathfrak{p})$ be a risk measurement regime on $L^\infty_\mathbb{P}$ such that $\rho_\mathcal{R}$ is normalised, and assume*

(i) *the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless;*
(ii) *$\mathcal{A}$ is the acceptance set $\{X \in L^\infty_\mathbb{P} \mid \mathfrak{r}(X) \le 0\}$ of a normalised, $\mathbb{P}$-law-invariant monetary risk measure $\mathfrak{r}$ which is continuous from above;*
(iii) *$\mathfrak{p} = c\mathbb{E}_\mathbb{P}[\cdot]$ for a suitable constant $c > 0$.*

Then $\mathbb{P} \in \mathcal{P}$.

Proof. By Svindland (2010a, Proposition 1.1) and Föllmer and Schied (2011, Corollary 4.65), $\mathfrak{r}$ is dilatation monotone: for every sub-σ-algebra $\mathcal{G} \subseteq \mathcal{F}$ and every $X \in L^\infty_\mathbb{P}$, the estimate $\mathfrak{r}(\mathbb{E}_\mathbb{P}[X|\mathcal{G}]) \le \mathfrak{r}(X)$ holds. Thus, for every $X \in L^\infty_\mathbb{P}$, $\mathbb{E}_\mathbb{P}[X] = \mathfrak{r}(\mathbb{E}_\mathbb{P}[X|\{\emptyset, \Omega\}]) \le \mathfrak{r}(X)$. We conclude $\sigma_\mathcal{A}(\mathbb{P}) = \sup_{Y \in L^\infty_\mathbb{P}:\, \mathfrak{r}(Y) \le 0} \mathbb{E}_\mathbb{P}[Y] = 0$. □

Clearly, sensitivity (cf. Definition 2.3) is necessary to have $\mathcal{P} \neq \emptyset$, but apart from the coherent case it is not sufficient. As an example consider two probability measures $\mathbb{Q} \ll \mathbb{P}$ such that $\mathbb{Q} \not\approx \mathbb{P}$. Define

$$\mathbb{P}_\beta := \beta\mathbb{Q} + (1-\beta)\mathbb{P},$$
$$\rho : L^\infty_\mathbb{P} \ni X \mapsto \sup_{\beta \in [0,1]} \mathbb{E}_{\mathbb{P}_\beta}[X] - (1-\beta)^2.$$

For $\mathcal{R} = (\{X \mid \rho(X) \le 0\}, \mathbb{R}, id_\mathbb{R})$, we have that $\rho = \rho_\mathcal{R}$ is a sensitive risk measure with $E_0 = \{\mathbb{Q}\}$ and $\mathcal{P} = \emptyset$.

In the following we will use the notation $\mathcal{F}_+ := \{A \in \mathcal{F} \mid \mathbb{P}(A) > 0\}$.

Lemma 3.8. *Let $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \mathfrak{p})$ be a risk measurement regime such that $\rho_\mathcal{R}$ is finite, continuous from above, and coherent. Then $\mathcal{P} \neq \emptyset$ if and only if $\rho_\mathcal{R}$ is sensitive.*

Proof. We only prove sufficiency. If $\rho_\mathcal{R}$ is coherent, then $\mathrm{dom}(\rho^*_\mathcal{R}) = E_0$; see (3.5). Moreover, continuity from above implies $E_0 \subseteq (\mathbf{ca}_\mathbb{P})_+$. As $\rho_\mathcal{R}$ is sensitive, we have that $0 < \rho_\mathcal{R}(\mathbf{1}_A) = \sup_{\mu \in E_0} \mu(A)$ for all $A \in \mathcal{F}_+$. Consequently, there is $\mu_A \in E_0$ such that $\mu_A(A) > 0$. In other words, $E_0 \approx \mathbb{P}$. (Föllmer and Schied, 2011, Theorem 1.61) shows that there is a countable family $(\mu_n)_{n\in\mathbb{N}} \subseteq E_0$ such that $\{\mu_n \mid n \in \mathbb{N}\} \approx \mathbb{P}$. (3.8) ensures that also $\nu := \sum_{n\in\mathbb{N}} \frac{1}{2^n}\mu_n \in E_0$, i.e. $\mathcal{P} \neq \emptyset$. □

The following theorems state sufficient conditions under which $\mathcal{P} \neq \emptyset$ without requiring coherence of $\rho_\mathcal{R}$. First, we characterise the strong condition $E_0 = \mathcal{P}$ with the ability of $\rho_\mathcal{R}$ to identify arbitrage.

Theorem 3.9. *Let $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \mathfrak{p})$ be a risk measurement regime, and suppose that $\rho_\mathcal{R}$ is finite, normalised, continuous from above, and sensitive. The following are equivalent:*

(i) *$E_0 = \mathcal{P}$;*
(ii) *For all $A \in \mathcal{F}_+$ we have $\rho_\mathcal{R}(-k\mathbf{1}_A) < 0$ for $k > 0$ sufficiently large;*
(iii) *For all $X \in (L^\infty_\mathbb{P})_{++}$ we have $\rho_\mathcal{R}(-X) < 0$.*

Moreover, $E_0 = \mathcal{P}$ if $\rho_\mathcal{R}$ is strictly monotone, i.e. $\rho_\mathcal{R}(X) < \rho_\mathcal{R}(Y)$ whenever $Y - X \in (L^\infty_\mathbb{P})_{++}$.

Proof. (iii) trivially implies (ii). Now assume (i) does not hold, i.e. there is some $\mu \in E_0 \backslash \mathcal{P}$, hence $\mu(A) = 0$ for some $A \in \mathcal{F}_+$. From $0 = \rho_\mathcal{R}(0) \ge \rho_\mathcal{R}(-k\mathbf{1}_A) \ge -k\mu(A) = 0$ we infer $\rho_\mathcal{R}(-k\mathbf{1}_A) = 0$ for all $k > 0$, contradicting (ii). This shows that (ii) implies (i). In order to show that (iii) is implied by (i), assume we can find a $X \neq 0$ in the negative cone with $\rho_\mathcal{R}(X) = 0$. As the level sets of $\rho^*_\mathcal{R}$ are $\sigma(\mathbf{ca}_\mathbb{P}, L^\infty_\mathbb{P})$-compact, we can find a $\mu \in \mathrm{dom}(\rho^*_\mathcal{R})$ such that $0 = \rho_\mathcal{R}(X) = \int X\,d\mu - \rho^*_\mathcal{R}(\mu)$. This implies $\rho^*_\mathcal{R}(\mu) = 0 = \int X\,d\mu$, a CONTRADICTION to $E_0 = \mathcal{P}$. Finally, strict monotonicity clearly implies (iii) by normalisation. □

The next aim is a characterisation of $\mathcal{P} \neq \emptyset$ in terms of the components of the risk measurement regime $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \mathfrak{p})$.

Theorem 3.10. *Suppose that $\rho_\mathcal{R}$ is finite, continuous from above, normalised, and sensitive. Let $\mathbf{C} \subseteq L^\infty_\mathbb{P}$ be the smallest weakly* closed convex cone containing $\mathcal{A} + \ker(\mathfrak{p})$.*

(i) *$\mathcal{P} \neq \emptyset$ if and only if $\mathbf{C} \cap (L^\infty_\mathbb{P})_{++} = \emptyset$.*
(ii) *$\mathcal{P} \neq \emptyset$ if $\mathcal{A} + \ker(\mathfrak{p})$ satisfies the RULE OF EQUAL SPEED OF CONVERGENCE: Let $(X_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ and $(Z_n)_{n\in\mathbb{N}} \subseteq \ker(\mathfrak{p})$ be sequences such that $\|X_n + Z_n\|_\infty \le 1$ for all n. Suppose $(t_n)_{n\in\mathbb{N}}$ is such that $t_n \uparrow \infty$. If the rescaled vectors*

$$V_n := t_n(X_n + Z_n)$$

satisfy $V_n^- \to 0$ in probability, then for all sets $B \in \mathcal{F}_+$, it holds that

$$\limsup_{n\to\infty} \mathbb{P}(B \cap \{V_n^+ \ge \varepsilon\}) < \mathbb{P}(B).$$

(iii) *$\mathcal{P} = \emptyset$ if there are sequences $(X_n)_{n\in\mathbb{N}}$, $(Z_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ such that*

$$\sup_{n\in\mathbb{N}} \|t_n(X_n + Z_n)\|_\infty < \infty$$

violating the rule of equal speed of convergence

A proof is given in Appendix B.

4. The Minkowski domain of a risk measure

4.1. Construction of the Minkowski domain and extension results

Throughout Section 4 fix an acceptance set $\mathcal{A} \subseteq L^\infty_\mathbb{P}$, a security space $\mathcal{S} \subseteq L^\infty_\mathbb{P}$, and let $\mathfrak{p} : \mathcal{S} \to \mathbb{R}$ be a pricing functional such that $\rho_\mathcal{R} : L^\infty_\mathbb{P} \to \mathbb{R}$ is a normalised finite risk measure which is continuous from above. Based on the results in Section 3, we assume that $\mathbb{P}$ is a weak reference probability model, i.e. $\gamma\mathbb{P} \in \mathrm{dom}(\rho^*_\mathcal{R})$ for a suitable constant $\gamma > 0$. The aim of this section is to lift $\rho_\mathcal{R}$ to a domain of definition denoted by $L^\mathcal{R}$ whose structure is completely characterised by $\rho_\mathcal{R}$ and thus consistent with the initial risk measurement regime, although it is in general strictly bigger than $L^\infty_\mathbb{P}$. The typical argument for restricting risk measures to bounded random variables—namely, that this space is robust and thus not conflicting with the ambiguity expressed by $\rho_\mathcal{R}$—is not valid in this case, since $L^\mathcal{R}$ will completely reflect the ambiguity as perceived by $\rho_\mathcal{R}$. To this end, we remark that

$$\rho(|X|) := \sup_{\mu \in \mathrm{dom}(\rho^*_\mathcal{R})} \int |X|\,d\mu - \rho^*_\mathcal{R}(\mu), \tag{4.1}$$

where $\rho^*_\mathcal{R}$ is given in (3.10), is well-defined for all $X \in L^0_\mathbb{P} := L^0(\Omega, \mathcal{F}, \mathbb{P})$, possibly taking the value ∞. In this sense the objects appearing in the following definition are well-defined.

Definition 4.1. For $c > 0$ and $X \in L^0_\mathbb{P}$ let

$$\|X\|_{c,\mathcal{R}} := \inf\left\{\lambda > 0 \,\middle|\, \rho\left(\frac{|X|}{\lambda}\right) \le c\right\} \quad (\inf\emptyset := \infty),$$

and $\|X\|_{\mathcal{R}} := \|X\|_{1,\mathcal{R}}$. The MINKOWSKI DOMAIN for $\rho_{\mathcal{R}}$ is the set

$L^{\mathcal{R}} := \{X \in L^0_{\mathbb{P}} \mid \|X\|_{\mathcal{R}} < \infty\}$.

Note that we may interpret $\|\cdot\|_{\mathcal{R}}$ as a Minkowski functional given the level set

$\rho(|\cdot|)^{-1}(-\infty, 1]$,

and its domain $L^{\mathcal{R}}$ is thus called the *Minkowski domain*.

Proposition 4.2.

(i) *For all $c > 0$ there exist constants $A_c, B_c > 0$ such that*

$A_c\|\cdot\|_{c,\mathcal{R}} \le \|\cdot\|_{\mathcal{R}} \le B_c\|\cdot\|_{c,\mathcal{R}}$.

In particular $L^{\mathcal{R}} = \{X \in L^0_{\mathbb{P}} \mid \|X\|_{c,\mathcal{R}} < \infty\}$ for all $c > 0$, and $(\|\cdot\|_{c,\mathcal{R}})_{c>0}$ is a family of equivalent norms on $L^{\mathcal{R}}$. Moreover, $\|X\|_\infty \ge B^{-1}_{\rho_{\mathcal{R}}(1)}\|X\|_{\mathcal{R}}$, $X \in L^\infty_{\mathbb{P}}$, and thus $L^\infty_{\mathbb{P}} \subseteq L^{\mathcal{R}}$.

(ii) *$X \in L^{\mathcal{R}}$ if and only if $\int |X|\,d\mu < \infty$ for all $\mu \in \mathrm{dom}(\rho^*_{\mathcal{R}})$.*

(iii) *$(L^{\mathcal{R}}, \|\cdot\|_{\mathcal{R}})$ is a Banach lattice.*

(iv) *$|X| \le |Y|$ implies $\|X\|_{c,\mathcal{R}} \le \|Y\|_{c,\mathcal{R}}$ and thus $L^{\mathcal{R}}$ is solid. In particular, $L^{\mathcal{R}}$ is invariant under rearrangements of profits and losses, i.e. if $\varphi \in L^\infty_{\mathbb{P}}$ attaining values in $[-1, 1]$, then $\varphi \cdot X \in L^{\mathcal{R}}$ with $\|\varphi X\|_{c,\mathcal{R}} \le \|X\|_{c,\mathcal{R}}$.*

Proof. First we set $\Lambda_c(X) := \{\lambda > 0 \mid \rho(\lambda^{-1}|X|) \le c\}$, i.e. $\|X\|_{c,\mathcal{R}} = \inf \Lambda_c(X)$.
(i) Suppose that $c \in (0, 1)$ and let $X \in L^0_{\mathbb{P}}$. Note that $\|X\|_{\mathcal{R}} = \infty$ if and only if $\Lambda_1(X) = \emptyset$, which implies $\Lambda_c(X) = \emptyset$ or equivalently $\|X\|_{c,\mathcal{R}} = \infty$. Now assume $\|X\|_{\mathcal{R}} < \infty$, and pick $\lambda \in \Lambda_1(X)$. As $\rho^*_{\mathcal{R}} \ge 0$, we have

$$\rho(c|X|/\lambda) = \sup_{\mu\in\mathrm{dom}(\rho^*_{\mathcal{R}})} \int \frac{c}{\lambda}|X|\,d\mu - \rho^*_{\mathcal{R}}(\mu) \le c\rho(|X|/\lambda) \le c,$$

which implies $\|X\|_{\mathcal{R}} \ge c\|X\|_{c,\mathcal{R}}$. Trivially, $\Lambda_c(X) \subseteq \Lambda_1(X)$ and therefore $\|X\|_{c,\mathcal{R}} \ge \|X\|_{\mathcal{R}}$. Hence, we may choose $A_c = c$ and $B_c = 1$. The case $c > 1$ is treated similarly.

Monotonicity implies that $\rho(|X|/\|X\|_\infty) \le \rho_{\mathcal{R}}(1)$ for all $X \in L^\infty_{\mathbb{P}}$, which yields $\|X\|_\infty \ge \|X\|_{\rho_{\mathcal{R}}(1),\mathcal{R}} \ge B^{-1}_{\rho_{\mathcal{R}}(1)}\|X\|_{\mathcal{R}}$ and $L^\infty_{\mathbb{P}} \subseteq L^{\mathcal{R}}$.

$\|\cdot\|_{c,\mathcal{R}}$ is indeed a norm: The verification of the triangle inequality and homogeneity are straightforward. For the definiteness of $\|\cdot\|_{c,\mathcal{R}}$, let $\mu \in \mathrm{dom}(\rho^*_{\mathcal{R}})$ be arbitrary. As for all $\lambda \in \Lambda_c(X)$ we obtain the estimate $\|\lambda^{-1}X\|_{L^1_\mu} - \rho^*_{\mathcal{R}}(\mu) \le \rho(\lambda^{-1}|X|) \le c$, we can infer

$$\frac{1}{c + \rho^*_{\mathcal{R}}(\mu)}\|X\|_{L^1_\mu} \le \|X\|_{c,\mathcal{R}}. \tag{4.2}$$

Choosing $\mu \in \mathrm{dom}(\rho^*_{\mathcal{R}})$ such that $\mu = \gamma\mathbb{P}$ yields definiteness of $\|\cdot\|_{c,\mathcal{R}}$.
(ii) It follows from (4.2) that for all $X \in L^{\mathcal{R}}$ and all $\mu \in \mathrm{dom}(\rho^*_{\mathcal{R}})$ the integrability condition $\int |X|\,d\mu < \infty$ holds. For the converse implication, let $X \in L^0_{\mathbb{P}} \setminus L^{\mathcal{R}}$ be arbitrary, the latter being equivalent to $\rho(t|X|) = \infty$ for all $t > 0$. As before, we set $E_c := \{\mu \in \mathbf{ca}_{\mathbb{P}} \mid \rho^*_{\mathcal{R}}(\mu) \le c\}$, $c \in \mathbb{R}$, and will show that there is a $\nu \in E_1$ such that $\int |X|\,d\nu = \infty$. First assume that $\sup_{\mu\in E_1}\int |X|\,d\mu = \infty$. Choose a sequence $(\mu_n)_{n\in\mathbb{N}} \subseteq E_1$ such that $\int |X|\,d\mu_n \ge 2^{2n}$, $n \in \mathbb{N}$, and set $\nu = \sum_{n=1}^\infty 2^{-n}\mu_n$, which is itself an element of E_1 by (3.8). Moreover,

$$\int |X|\,d\nu = \sum_{n=1}^\infty 2^{-n}\int |X|\,d\mu_n \ge \sum_{n=1}^\infty 2^n = \infty.$$

Hence, X is not ν-integrable. In a second step, we show that the case $\sup_{\mu\in E_1}\int |X|\,d\mu < \infty$ cannot occur. Assume for contradiction that $\sup_{\mu\in E_1}\int |X|\,d\mu < \infty$. If there were a constant $\kappa > 0$ such that for all $\mu \in \mathrm{dom}(\rho^*_{\mathcal{R}}) \setminus E_1$ the estimate

$$\int |X|\,d\mu \le \kappa\rho^*_{\mathcal{R}}(\mu)$$

holds, one could estimate

$$\rho(\kappa^{-1}|X|) \le \frac{1}{\kappa}\sup_{\mu\in E_1}\int |X|\,d\mu < \infty,$$

and thus $X \in L^{\mathcal{R}}$. Thus, there must be a sequence $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathrm{dom}(\rho^*_{\mathcal{R}})$ such that $\rho^*_{\mathcal{R}}(\mu_n) > 1$ and $\int |X|\,d\mu_n \ge 2^{2n}\rho^*_{\mathcal{R}}(\mu_n)$, $n \in \mathbb{N}$. We set $C := \sum_{n=1}^\infty \frac{1}{2^n\rho^*_{\mathcal{R}}(\mu_n)} \in (0, 1)$, and

$$\zeta := \sum_{n=1}^\infty \frac{1}{2^n\rho^*_{\mathcal{R}}(\mu_n)C}\mu_n.$$

As $\mu_n(\Omega) \le \rho_{\mathcal{R}}(1) + \rho^*_{\mathcal{R}}(\mu_n)$, $\zeta(\Omega)$ is finite. Moreover, by $\sigma(\mathbf{ca}_{\mathbb{P}}, L^\infty_{\mathbb{P}})$-lower semicontinuity of $\rho^*_{\mathcal{R}}$, $\rho^*_{\mathcal{R}}(\zeta) \le \frac{1}{C}\sum_{n=1}^\infty 2^{-n} = \frac{1}{C}$. Note that

$$\int |X|\,d\zeta \ge \sum_{n=1}^\infty \frac{2^{2n}\rho^*_{\mathcal{R}}(\mu_n)}{2^n\rho^*_{\mathcal{R}}(\mu_n)C} = \frac{1}{C}\sum_{n=1}^\infty 2^n = \infty,$$

Hence, for $\nu := C\zeta + (1 - C)\mu_0 \in E_1$, where $\mu_0 \in E_0$ is chosen arbitrarily, we also obtain $\int |X|\,d\nu = \infty$. This is the desired CONTRADICTION.
(iii) follows from Owari (2014, Proposition 4.10), and (iv) is an immediate consequence of the monotonicity of $\rho(|\cdot|)$. $\square$

The proof of Proposition 4.6 will clarify the reason for introducing the norms $\|\cdot\|_{c,\mathcal{R}}$ instead of just $\|\cdot\|_{\mathcal{R}}$.

Remark 4.3.

(i) In the coherent case, we can infer from Proposition 4.2(ii) that $\|\cdot\|_{c,\mathcal{R}} = c^{-1}\rho(|X|) = c^{-1}\sup_{\mu\in\mathrm{dom}(\rho^*_{\mathcal{R}})}\|X\|_{L^1_\mu}$.

(ii) The Minkowski norm $\|\cdot\|_{c,\mathcal{R}}$ can be interpreted as a generalisation of the so-called *Aumann–Serrano economic index of riskiness* (see Aumann and Serrano, 2008 and Drapeau and Kupper, 2013, Example 3).

(iii) The Minkowski domain and similar spaces have appeared in Kupper and Svindland (2011), Owari (2014) and Svindland (2010b). The definition of $L^{\mathcal{R}}$ depends on the null sets of the probability measure $\mathbb{P}$ only, and thus is invariant under any choice of the underlying probability measure $\mathbb{P} \approx \mathrm{dom}(\rho^*_{\mathcal{R}})$, in particular under changes of weak and strong reference probability models.

The main purpose for introducing the Minkowski domain $L^{\mathcal{R}}$ is to extend $\rho_{\mathcal{R}}$ to a larger domain than $L^\infty_{\mathbb{P}}$ in a robust way in terms of the fundamentals, i.e. the risk measurement regime $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \mathfrak{p})$. There is a canonical candidate for this given by $\tilde{\mathcal{R}} := (\tilde{\mathcal{A}}, \mathcal{S}, \mathfrak{p})$ where

$$\tilde{\mathcal{A}} := \{X \in L^{\mathcal{R}} \mid \forall\mu \in \mathrm{dom}(\rho^*_{\mathcal{R}}) : \int X\,d\mu \le \rho^*_{\mathcal{R}}(\mu)\}, \tag{4.3}$$

so $\tilde{\mathcal{A}}$ is given by lifting – and thus also preserving – the acceptability criteria $\int X\,d\mu \le \rho^*_{\mathcal{R}}(\mu)$, $\mu \in \mathrm{dom}(\rho^*_{\mathcal{R}})$, from $L^\infty_{\mathbb{P}}$ to $L^{\mathcal{R}}$. Indeed the following Theorem 4.4 shows that $\tilde{\mathcal{R}}$ is a risk measurement regime, and that the corresponding risk measure $\rho_{\tilde{\mathcal{R}}}$ preserves the dual representation of $\rho_{\mathcal{R}}$. Dual approaches to extending convex functions are commonly used in the literature; see, e.g., Filipovic and Svindland (2012) and Owari (2014). Note that $\rho_{\tilde{\mathcal{R}}}$ also preserves any functional form $\rho_{\mathcal{R}}$ may have, as for instance in the case of the entropic risk measure in Example 5.4.

Theorem 4.4. *$\tilde{\mathcal{R}} := (\tilde{\mathcal{A}}, \mathcal{S}, \mathfrak{p})$ is a risk measurement regime on the Banach lattice $(L^{\mathcal{R}}, \|\cdot\|_{\mathcal{R}})$. $\rho_{\tilde{\mathcal{R}}}$ can be expressed as*

$$\rho_{\tilde{\mathcal{R}}}(X) = \sup_{\mu\in\mathrm{dom}(\rho^*_{\mathcal{R}})}\int X\,d\mu - \rho^*_{\mathcal{R}}(\mu), \quad X \in L^{\mathcal{R}}, \tag{4.4}$$

where $\rho^*_{\mathcal{R}}$ is defined as in (3.10). Moreover, for every $\mu \in \mathrm{dom}(\rho^*_{\mathcal{R}})$, the linear functional $\int \cdot\, d\mu$ is bounded on $(L^{\mathcal{R}}, \|\cdot\|_{\mathcal{R}})$. A fortiori, $\rho_{\tilde{\mathcal{R}}}|_{L^\infty_{\mathbb{P}}} = \rho_{\mathcal{R}}$. Moreover, $\rho_{\tilde{\mathcal{R}}}$ is l.s.c. on $(L^{\mathcal{R}}, \|\cdot\|_{\mathcal{R}})$, and satisfies

$$\rho_{\tilde{\mathcal{R}}}(X) = \sup_{m\in\mathbb{N}} \rho_{\tilde{\mathcal{R}}}(X \wedge m). \tag{4.5}$$

Proof. Note that for arbitrary $\mu \in \mathrm{dom}(\rho^*_{\mathcal{R}})$ and all $X \neq 0$, we have

$$\int \frac{|X|}{\|X\|_{\mathcal{R}}} d\mu = \sup_{\varepsilon>0} \int \frac{|X|}{\|X\|_{\mathcal{R}}+\varepsilon} d\mu \le \sup_{\varepsilon>0} \rho\left(\frac{|X|}{\|X\|_{\mathcal{R}}+\varepsilon}\right) + \rho^*_{\mathcal{R}}(\mu) \le 1 + \rho^*_{\mathcal{R}}(\mu),$$

hence $\int \cdot\, d\mu$ is a bounded linear functional on $L^{\mathcal{R}}$. For arbitrary $X \in L^{\mathcal{R}}$ and $\mu \in E_0$ we have

$$\sup\{\mathfrak{p}(Z) \mid Z \in \mathcal{S}, X+Z \in \tilde{\mathcal{A}}\} \le \sup\left\{\mathfrak{p}(Z) \mid Z \in \mathcal{S}, \mathfrak{p}(Z) \le -\int X d\mu\right\} = -\int X d\mu < \infty,$$

where the finiteness of the bound is due to Proposition 4.2(ii). Thus, $\tilde{\mathcal{R}}$ satisfies (2.1) and is indeed a risk measurement regime, because $\tilde{\mathcal{A}}$ is monotone by $\mathrm{dom}(\rho^*_{\mathcal{R}}) \subseteq (\mathbf{ca}_{\mathbb{P}})_+$, and convex as intersection of convex subsets of $L^{\mathcal{R}}$. It is straightforward to show (4.4), so $\rho_{\tilde{\mathcal{R}}}$ is l.s.c. as pointwise supremum of a family of continuous functions. In order to prove (4.5), let $\mu \in \mathrm{dom}(\rho^*_{\mathcal{R}})$ be arbitrary and note that by the Monotone Convergence Theorem and monotonicity of $\rho_{\tilde{\mathcal{R}}}$, we have

$$\int X\, d\mu - \rho^*_{\mathcal{R}}(\mu) = \sup_{m\in\mathbb{N}} \int (X\wedge m) d\mu - \rho^*_{\mathcal{R}}(\mu) \le \sup_{m\in\mathbb{N}} \rho_{\tilde{\mathcal{R}}}(X \wedge m) \le \rho_{\tilde{\mathcal{R}}}(X).$$

Now take the supremum over $\mu \in \mathrm{dom}(\rho^*_{\mathcal{R}})$ on the left-hand side. □

Another way to extend $\rho_{\mathcal{R}}$ could be considering

$$\overline{\mathcal{A}} := \mathrm{cl}_{\|\cdot\|_{\mathcal{R}}}(\mathcal{A}). \tag{4.6}$$

and $\overline{\mathcal{R}} = (\overline{\mathcal{A}}, \mathcal{S}, \mathfrak{p})$. We will discuss this approach in Remark 4.11 where we show that $\overline{\mathcal{R}}$ is no risk measurement regime on $L^{\mathcal{R}}$ in general, and that, where $\rho_{\overline{\mathcal{R}}}$ makes sense, it indeed equals $\rho_{\tilde{\mathcal{R}}}$. As announced in the introduction, we also consider the following extensions of $\rho_{\mathcal{R}}$ given by monotone approximation procedures:

$$\xi(X) := \sup_{m\in\mathbb{N}} \inf_{n\in\mathbb{N}} \rho_{\mathcal{R}}((-n)\vee X \wedge m), \quad X \in L^{\mathcal{R}},$$

and

$$\eta(X) := \inf_{n\in\mathbb{N}} \sup_{m\in\mathbb{N}} \rho_{\mathcal{R}}((-n)\vee X \wedge m), \quad X \in L^{\mathcal{R}}.$$

The question is under which conditions we have

$$\rho_{\tilde{\mathcal{R}}}(X) = \xi(X) = \eta(X) = \lim_{n\to\infty} \rho_{\mathcal{R}}((-n)\vee X\wedge n). \tag{4.7}$$

Note that as a byproduct of (4.5), we obtain the estimate

$$\begin{aligned} &\rho_{\tilde{\mathcal{R}}} \le \xi \le \eta \quad \text{and} \\ &\forall X \in L^{\mathcal{R}}:\ \rho_{\tilde{\mathcal{R}}}(|X|) = \xi(|X|) = \eta(|X|) = \rho(|X|). \end{aligned} \tag{4.8}$$

The following Theorem 4.5 shows that $\rho_{\tilde{\mathcal{R}}}$ possesses some regularity in terms of monotone approximation in that always $\rho_{\tilde{\mathcal{R}}} = \xi$.

Theorem 4.5. *For all $X \in L^{\mathcal{R}}$ and all $U \in L^\infty_{\mathbb{P}}$ we have*

$$\rho_{\tilde{\mathcal{R}}}(X+U) = \sup_{m\in\mathbb{N}} \inf_{n\in\mathbb{N}} \rho_{\mathcal{R}}((-n)\vee X\wedge m + U). \tag{4.9}$$

A fortiori, the equality $\rho_{\tilde{\mathcal{R}}} = \xi$ holds, and $\rho_{\tilde{\mathcal{R}}}$ can equivalently be interpreted as the risk measure associated to the risk measurement regime $\mathcal{R}_\xi := (\mathcal{A}_\xi, \mathcal{S}, \mathfrak{p})$ on $(L^{\mathcal{R}}, \|\cdot\|_{\mathcal{R}})$, where

$$\begin{aligned}\mathcal{A}_\xi &:= \{X \in L^{\mathcal{R}} \mid \xi(X) \le 0\} \\ &= \{X \in L^{\mathcal{R}} \mid \sup_{m\in\mathbb{N}} \inf_{n\in\mathbb{N}} \rho_{\mathcal{R}}((-n)\vee X\wedge m) \le 0\}.\end{aligned}$$

For the sake of brevity, we shall in the remainder of our investigations often use the following piece of notation: for random variables $U, V \in (L^\infty_{\mathbb{P}})_+$ and $X \in L^{\mathcal{R}}$, we set $X_U := X \vee (-U)$ and $X^V := X \wedge V$.

Proof. We show first that $\rho_{\tilde{\mathcal{R}}} = \xi$ holds. Let $X \in L^{\mathcal{R}}$, $m \in \mathbb{N}$ be fixed and $n \in \mathbb{N}$ be arbitrary. Let $\mu \in \mathrm{dom}(\rho^*_{\mathcal{R}})$ be such that

$$\rho_{\tilde{\mathcal{R}}}(-X^-) - 1 \le \rho_{\mathcal{R}}(X^m_n) - 1 \le \int X^m_n d\mu - \rho^*_{\mathcal{R}}(\mu) \le \int (X^+)^m d\mu - \rho^*_{\mathcal{R}}(\mu).$$

Of course, the first and last inequalities in the latter estimate always hold by monotonicity. For $\varepsilon > 0$ arbitrary we can thus estimate

$$\begin{aligned}\rho^*_{\mathcal{R}}(\mu) - 1 &\le \int (X^+)^m d\mu - \rho_{\tilde{\mathcal{R}}}(-X^-) \\ &= \frac{1}{1+\varepsilon}\int (1+\varepsilon)(X^+)^m d\mu - \rho_{\tilde{\mathcal{R}}}(-X^-) \\ &\le \frac{1}{1+\varepsilon}\rho_{\mathcal{R}}((1+\varepsilon)(X^+)^m) \\ &\quad + \frac{1}{1+\varepsilon}\rho^*_{\mathcal{R}}(\mu) - \rho_{\tilde{\mathcal{R}}}(-X^-).\end{aligned}$$

Rearranging this inequality, we obtain

$$\rho^*_{\mathcal{R}}(\mu) \le \frac{1}{\varepsilon}\rho_{\mathcal{R}}((1+\varepsilon)(X^+)^m) + \frac{1+\varepsilon}{\varepsilon}\left(1 - \rho_{\tilde{\mathcal{R}}}(-X^-)\right) =: c,$$

a bound which is independent of $n \in \mathbb{N}$. Since $E_c = \{\mu \in \mathbf{ca}_{\mathbb{P}} \mid \rho^*_{\mathcal{R}}(\mu) \le c\}$ is $\sigma(\mathbf{ca}_{\mathbb{P}}, L^\infty_{\mathbb{P}})$-compact by Lemma 3.6, we conclude for all $n \in \mathbb{N}$ that $\rho_{\mathcal{R}}(X^m_n) = \max_{\mu\in E_c} f(\mu, n)$, where the function f is given by

$$f: E_c \times \mathbb{N} \to \mathbb{R}, \quad f(\mu, n) := \int X^m_n d\mu - \rho^*_{\mathcal{R}}(\mu),$$

Our aim is to apply Fan's Minimax Theorem Fan (1953, Theorem 2) to the function f in order to infer

$$\begin{aligned}\xi(X^m) &= \inf_n \max_{\mu\in E_c} f(\mu, n) = \max_{\mu\in E_c} \inf_{n\in\mathbb{N}} f(\mu, n) \\ &= \max_{\mu\in E_c} \inf_{n\in\mathbb{N}} \int X^m_n d\mu - \rho^*_{\mathcal{R}}(\mu).\end{aligned} \tag{4.10}$$

To this end we have to check the following conditions:

- E_c is a compact Hausdorff space when endowed with the relative $\sigma(\mathbf{ca}_{\mathbb{P}}, L^\infty_{\mathbb{P}})$-topology. This follows from continuity from above.
- f is convex-like on $\mathbb{N}$ in that for all $n_1, n_2 \in \mathbb{N}$ and all $0 \le t \le 1$ there is a $n_0 \in \mathbb{N}$ such that

$$\forall \mu \in E_c:\ f(\mu, n_0) \le tf(\mu, n_1) + (1-t)f(\mu, n_2).$$

Indeed, choose $n_0 := \max\{n_1, n_2\}$ and note that

$$\begin{aligned} tf(\mu, n_1) + (1-t)f(\mu, n_2) &= t\int X^m_{n_1}d\mu + (1-t) \\ &\quad \times \int X^m_{n_2}d\mu - \rho^*_{\mathcal{R}}(\mu) \\ &\geq (t+1-t)\int X^m_{n_0}d\mu - \rho^*_{\mathcal{R}}(\mu) \\ &= f(\mu, n_0). \end{aligned}$$

- f is concave-like on E_c, which is defined analogous to convex-like. Indeed, let $\mu_1, \mu_2 \in E_c$ and define $\mu_0 = t\mu_1 + (1-t)\mu_2 \in E_c$ (by convexity of E_c). Then for all $n \in \mathbb{N}$, convexity of $\rho^*_{\mathcal{R}}$ implies

$$\begin{aligned} tf(\mu_1, n) + (1-t)f(\mu_2, n) &= \int X^m_n d\mu_0 - t\rho^*_{\mathcal{R}}(\mu_1) \\ &\quad - (1-t)\rho^*_{\mathcal{R}}(\mu_2) \\ &\leq \int X^m_n d\mu_0 - \rho^*_{\mathcal{R}}(\mu_0) = f(\mu_0, n). \end{aligned}$$

- For all $n \in \mathbb{N}$, the mapping $\mu \mapsto f(\mu, n)$ is upper semicontinuous. This follows from the continuity of $\mu \mapsto \int X^m_n d\mu$ and the lower semicontinuity of $\rho^*_{\mathcal{R}}$.

From (4.10), by the positivity of μ and, e.g., dominated convergence,

$$\xi(X^m) = \max_{\mu\in E_c}\int X^m d\mu - \rho^*_{\mathcal{R}}(\mu) \leq \rho_{\tilde{\mathcal{R}}}(X^m),$$

and $\rho_{\tilde{\mathcal{R}}}(X^m) = \xi(X^m)$ holds by (4.8). Taking the limit $m \to \infty$, we obtain from the definition of ξ and (4.5) that $\rho_{\tilde{\mathcal{R}}}(X) = \xi(X)$.

Now, let $X \in L^{\mathcal{R}}$ and $U \in L^\infty_{\mathbb{P}}$ be arbitrary and assume $m, n \geq u := \|U\|_\infty$. We obtain

$$\begin{aligned} (X+U)_n &= (X+U)\mathbf{1}_{\{X\geq -U-n\}} - n\mathbf{1}_{\{X<-U-n\}} \\ &= X\mathbf{1}_{\{X\geq -U-n\}} - (n+U)\mathbf{1}_{\{X<-U-n\}} + U \qquad (4.11) \\ &= X_{U+n} + U, \end{aligned}$$

and in addition

$$\begin{aligned} (X+U)^m &= (X+U)\mathbf{1}_{\{X\leq m-U\}} + m\mathbf{1}_{\{X>m-U\}} \\ &= X\mathbf{1}_{\{X\leq m-U\}} + (m-U)\mathbf{1}_{\{X>m-U\}} + U \qquad (4.12) \\ &= X^{m-U} + U. \end{aligned}$$

From these two equations (4.11) and (4.12) we infer

$$\xi(X+U) = \sup_{m\geq u}\inf_{n\geq u}\rho_{\mathcal{R}}((X_{U+n}+U)^m) = \sup_{m\geq u}\inf_{n\geq u}\rho_{\mathcal{R}}(X^{m-U}_{U+n}+U).$$

This implies that

$$\begin{aligned} \sup_{m\in\mathbb{N}}\inf_{n\in\mathbb{N}}\rho_{\mathcal{R}}(X^m_n + U) &= \sup_{m\geq u}\inf_{n\geq u}\rho_{\mathcal{R}}(X^{m-U}_{U+n}+U) \\ &= \sup_{m\geq u}\inf_{n\geq u}\rho_{\mathcal{R}}((X+U)^m_n) \\ &= \xi(X+U) = \rho_{\tilde{\mathcal{R}}}(X+U). \end{aligned}$$

(4.9) is proved. $\xi = \rho_{\tilde{\mathcal{R}}}$ being $\mathcal{S}$-additive, monotone, and proper, directly implies $\mathcal{R}_\xi$ is a risk measurement regime. The equality $\rho_{\tilde{\mathcal{R}}} = \xi = \rho_{\mathcal{R}_\xi}$ obviously holds true. □

Theorem 4.5 appeared as Svindland (2010b, Lemma 2.8) in the context of law-invariant monetary risk measures. Our proof not only serves as an alternative to the one given in Svindland (2010b), relying irreducibly on law-invariance, but also generalises the result to a much wider class of risk measures.

In contrast to Theorem 4.5, we demonstrate in Example 5.2 that $\rho_{\tilde{\mathcal{R}}} \neq \eta$ may happen. Before we study conditions under which $\rho_{\tilde{\mathcal{R}}}$ displays regularity in the sense of (4.7), we show the following properties of η:

Proposition 4.6. *Define the acceptance set*

$$\mathcal{A}_\eta := \{X \in L^{\mathcal{R}} \mid \inf_{n\in\mathbb{N}}\rho_{\tilde{\mathcal{R}}}((-n)\vee X) \leq 0\} \subsetneq L^{\mathcal{R}}.$$

Then η is the risk measure associated to the risk measurement regime $\mathcal{R}_\eta := (\mathcal{A}_\eta, \mathcal{S}, \mathfrak{p})$. Moreover,

$$\forall X \in L^{\mathcal{R}}: \quad \eta(X) = \inf_{n\in\mathbb{N}}\rho_{\tilde{\mathcal{R}}}((-n)\vee X), \qquad (4.13)$$

and

$$\begin{aligned} \Gamma &:= \{X \in L^{\mathcal{R}} \mid \exists\,\varepsilon > 0: \ \rho((1+\varepsilon)X^+) < \infty\} \\ &= \operatorname{int}\operatorname{dom}(\eta) \subseteq \operatorname{int}\operatorname{dom}(\rho_{\tilde{\mathcal{R}}}). \end{aligned}$$

Proof. From (4.5) and $\eta|_{L^\infty_{\mathbb{P}}} = \rho_{\mathcal{R}} = \rho_{\tilde{\mathcal{R}}}|_{L^\infty_{\mathbb{P}}}$, we immediately obtain that for all $X \in L^{\mathcal{R}}$ the equality $\eta(X) = \inf_{n\in\mathbb{N}}\rho_{\tilde{\mathcal{R}}}((-n)\vee X)$ holds. (4.8) shows that $\mathcal{A}_\eta \subsetneq L^{\mathcal{R}}$ and that η is a proper function. In order to prove the theorem, it suffices to check $\mathcal{S}$-additivity, convexity and monotonicity. Let $Z \in \mathcal{S}$ and $X \in L^{\mathcal{R}}$. From the $\mathcal{S}$-additivity of $\rho_{\tilde{\mathcal{R}}}$ and (4.11) we obtain, using the notational conventions introduced before the proof of Theorem 4.5, that

$$\begin{aligned} \eta(X) &= \inf_{n\geq\|Z\|_\infty}\rho_{\tilde{\mathcal{R}}}(X_{Z+n}+Z) = \inf_{n\geq\|Z\|_\infty}\rho_{\tilde{\mathcal{R}}}(X_{Z+n}) + \mathfrak{p}(Z) \\ &= \eta(X) + \mathfrak{p}(Z). \end{aligned}$$

For each $n \in \mathbb{N}$, $f_n(x) := (-n)\vee x$ is convex and monotone, thus $\eta = \lim_n \rho_{\tilde{\mathcal{R}}}\circ f_n$ is convex and monotone. Next we show that $\Gamma \subseteq \operatorname{int}\operatorname{dom}(\eta)$. To this end we first show that $\mathbf{B} := \bigcup_{c>0}\{Y \in L^{\mathcal{R}} \mid \|Y\|_{c,\mathcal{R}} < 1\} \subseteq \operatorname{int}\operatorname{dom}(\eta)$. Indeed for any X with $\|X\|_{c,\mathcal{R}} < 1$, there is $\lambda < 1$ such that $\rho(|X|/\lambda) \leq c$, and thus

$$\eta(X) \leq \eta(|X|) = \rho(|X|) \leq \lambda\rho(|X|/\lambda) \leq \lambda c < \infty,$$

so $\mathbf{B} \subseteq \operatorname{dom}(\eta)$. Moreover, by definition $\mathbf{B}$ is open in $(L^{\mathcal{R}}, \|\cdot\|_{\mathcal{R}})$. Now, let $X \in \Gamma$, and thus $X^+ \in \mathbf{B}$. Hence, there is $\delta > 0$ and a ball $B_\delta(0) := \{Y \in L^{\mathcal{R}} \mid \|Y\|_{\mathcal{R}} < \delta\}$ such that $\{X^+\} + B_\delta(0) \subseteq \operatorname{dom}(\eta)$. By monotonicity of η it now follows that also $\{X\} + B_\delta(0) = \{X^+\} + B_\delta(0) - \{X^-\} \subseteq \operatorname{dom}(\eta)$, so $X \in \operatorname{int}\operatorname{dom}(\eta)$.

In order to show $\Gamma \supseteq \operatorname{int}\operatorname{dom}(\eta)$ let $X \in \operatorname{int}\operatorname{dom}(\eta)$. Then there is $\varepsilon > 0$ such that $(1+2\varepsilon)X \in \operatorname{dom}(\eta)$ and thus also $(1+\varepsilon)X \in \operatorname{dom}(\eta)$, and by (4.13) there must be $n \in \mathbb{N}$ such that $(1+2\varepsilon)((-n)\vee X) \in \operatorname{dom}(\rho_{\tilde{\mathcal{R}}})$ and $(1+\varepsilon)((-n)\vee X) \in \operatorname{dom}(\rho_{\tilde{\mathcal{R}}})$. Let $X_n := (-n)\vee X$ and $Y = (1+\varepsilon)(X^- \wedge n) \in L^\infty_{\mathbb{P}}$, so we have $(1+\varepsilon)X^+ = (1+\varepsilon)X_n + Y$. If $\delta > 0$ satisfies $(1+\delta)(1+\varepsilon) = 1+2\varepsilon$, convexity implies

$$\begin{aligned} \rho((1+\varepsilon)X^+) &= \rho_{\tilde{\mathcal{R}}}((1+\varepsilon)X_n + Y) \\ &= \rho_{\tilde{\mathcal{R}}}\left(\frac{1+\delta}{1+\delta}(1+\varepsilon)X_n + \frac{\delta(1+\delta)}{\delta(1+\delta)}Y\right) \\ &\leq \frac{1}{1+\delta}\rho_{\tilde{\mathcal{R}}}((1+2\varepsilon)X_n) \\ &\quad + \frac{\delta}{(1+\delta)}\rho_{\tilde{\mathcal{R}}}\left(\frac{(1+\delta)}{\delta}Y\right) < \infty. \qquad (4.14) \end{aligned}$$

Hence, $X \in \Gamma$. $\operatorname{int}\operatorname{dom}(\eta) \subseteq \operatorname{int}\operatorname{dom}(\rho_{\tilde{\mathcal{R}}})$ follows from $\rho_{\tilde{\mathcal{R}}} \leq \eta$, see (4.8). □

The following Theorem 4.7 states conditions under which (4.7) holds.

Theorem 4.7. *Let $X \in \Gamma$. Consider the following conditions:*

(i) *there is $s > 0$ such that for all $n \in \mathbb{N}$ we have*

$$\rho_{\tilde{\mathcal{R}}}((-n)\vee X) = \lim_{m\to\infty}\rho_{\tilde{\mathcal{R}}}((-n)\vee X + sX^+\mathbf{1}_{\{X^+\geq m\}});$$

(ii) *there is $s > 0$ such that $\eta(X) = \lim_{m\to\infty}\eta(X + sX^+\mathbf{1}_{\{X^+\geq m\}})$;*

(iii) *for all $n \in \mathbb{N}$ we have $\lim_{m\to\infty}\rho(nX\mathbf{1}_{\{X\geq m\}}) = 0$.*

Any of the conditions (i)–(iii) *implies* (4.7).

The set Γ appears to be a set of reasonable risks in that they can at least be leveraged by a small amount and still remain hedgeable. Risks outside Γ should probably not be considered by any sound agent. Note that the conditions (i)–(iii) are satisfied whenever monotone or dominated convergence results can be applied to $\rho_{\tilde{\mathcal{R}}}$, as is the case for many risk measures used in practice like the entropic risk measure in Example 5.4 or Average Value at Risk based risk measures in Example 5.5. The proof of Theorem 4.7 is based on a study of subgradients of $\rho_{\tilde{\mathcal{R}}}$ and η, respectively, and therefore postponed to the end of Section 4.4. It turns out that the regularity condition (4.7) is closely related to the existence of regular subgradients for η and $\rho_{\tilde{\mathcal{R}}}$.

4.2. The structure of the Minkowski domain

In this section, we will decompose $L^{\mathcal{R}}$ into parts with clear operational meanings.

Definition 4.8. We denote the closure of $L^\infty_{\mathbb{P}}$ in $L^{\mathcal{R}}$ by $M^{\mathcal{R}} := \mathrm{cl}_{\|\cdot\|_{\mathcal{R}}}(L^\infty_{\mathbb{P}})$, and define the HEART of the Minkowski domain to be

$H^{\mathcal{R}} := \{X \in L^{\mathcal{R}} \mid \rho(k|X|) < \infty \text{ for all } k > 0\}.$

$H^{\mathcal{R}}$, a concept which clearly adapts the idea of an Orlicz heart,[6] is the set of risky positions which can be hedged at any quantity with finite cost.

Proposition 4.9. *$M^{\mathcal{R}}$ and $H^{\mathcal{R}}$ are solid Banach sublattices of $L^{\mathcal{R}}$ and $M^{\mathcal{R}} \subseteq H^{\mathcal{R}}$. Moreover, $H^{\mathcal{R}} \subseteq \Gamma$, and both $\rho_{\tilde{\mathcal{R}}}|_{H^{\mathcal{R}}}$ and $\eta|_{H^{\mathcal{R}}}$ are continuous.*

Proof. The first assertions are easily verified. Recall the set $\mathbf{B}$ from the proof of Proposition 4.6 for which we know that $\mathbf{B} \subseteq \Gamma$. For the inclusion $H^{\mathcal{R}} \subseteq \mathbf{B}$, let $0 \neq X \in H^{\mathcal{R}}$ and note that $\rho(2|X|) < \infty$. The latter means $\|X\|_{c,\mathcal{R}} \leq \frac{1}{2} < 1$ for some $c > 0$, and thus $H^{\mathcal{R}} \subseteq \mathbf{B}$. Finally, as $(H^{\mathcal{R}}, \|\cdot\|_{\mathcal{R}})$ is a Banach lattice and both η and $\rho_{\tilde{\mathcal{R}}}$ are convex, monotone and finite-valued on $(H^{\mathcal{R}}, \|\cdot\|_{\mathcal{R}})$, $\rho_{\tilde{\mathcal{R}}}|_{H^{\mathcal{R}}}$ and $\eta|_{H^{\mathcal{R}}}$ are continuous according to Remark 2.4(v). □

From Proposition 4.9 we can derive the following characterisation of $M^{\mathcal{R}}$, a result which can also be found as Owari (2014, Lemma 3.3).

Corollary 4.10. *$M^{\mathcal{R}} = \{X \in L^{\mathcal{R}} \mid \forall \lambda > 0: \ \lim_{k\to\infty}\rho(\lambda|X|\mathbf{1}_{\{|X|\geq k\}}) = 0\}$.*

Proof. Let $X \in M^{\mathcal{R}}$ and $\lambda, \varepsilon > 0$ be arbitrary. Let $\delta > 0$ be such that $\|Y\|_{\mathcal{R}} \leq \delta$, $Y \in H^{\mathcal{R}}$, implies $\rho(|Y|) = \rho_{\tilde{\mathcal{R}}}(|Y|) \leq \varepsilon$. This is possible due to Proposition 4.9. Choose now $Y \in L^\infty_{\mathbb{P}}$ such that $\|\lambda(X-Y)\|_{\mathcal{R}} \leq \frac{\delta}{2}$ and $k \in \mathbb{N}$ such that $\|\lambda Y \mathbf{1}_{\{|X|\geq k\}}\|_{\mathcal{R}} \leq \frac{\delta}{2}$, the latter being due to continuity from above. Then $Z := |X - Y|\mathbf{1}_{\{|X|\geq k\}} + |Y|\mathbf{1}_{\{|X|\geq k\}}$ satisfies $\|\lambda Z\|_{\mathcal{R}} \leq \delta$, and by monotonicity $\rho(\lambda|X|\mathbf{1}_{\{|X|\geq k\}}) \leq \rho_{\tilde{\mathcal{R}}}(\lambda Z) \leq \varepsilon$. The converse inclusion above is obvious. □

As $H^{\mathcal{R}}$ is closed, the set of directions along whose absolute value $\rho_{\tilde{\mathcal{R}}}$ attains the value infinity is thus norm-open. In particular, we can only approximate such vectors with sequences of vectors along which $\rho_{\tilde{\mathcal{R}}}$ behaves equally discontinuous, and limits of well-behaved financial positions are equally well-behaved. Hence shifting to $L^{\mathcal{R}}$ yields a structure which conveniently separates regimes of "good" and "bad" risk behaviour. In that respect consider the set $C^{\mathcal{R}} := \mathrm{dom}(\rho_{\tilde{\mathcal{R}}}) \setminus H^{\mathcal{R}} \subseteq L^{\mathcal{R}}$. $C^{\mathcal{R}}$ is the set of "less bad" positions, and shields $H^{\mathcal{R}}$ from the financial positions that carry infinite risk. It has a nice interpretation in terms of liquidity risk in the sense of Lacker (2015). In that paper the author considers liquidity risk profiles, i.e. curves of the form $\rho_{\tilde{\mathcal{R}}}(tX)_{t\geq 0}$ capturing how risk scales when increasing the leverage. $C^{\mathcal{R}}$ consists of financial positions X such that the liquidity risk profiles of X^+ or X^- breach the infinite risk regimes. Whereas an agent could at least hypothetically hedge any position in $H^{\mathcal{R}}$ at finite cost, no matter what the leverage, she has to be very careful in the case of elements in $C^{\mathcal{R}}$ that have finite risk themselves but which produce potentially completely non-hedgeable losses under incautious scaling.

Recalling that for any $X \in L^{\mathcal{R}}$ there is $\lambda > 0$ such that $\rho(|X|/\lambda) < \infty$, we obtain that $C^{\mathcal{R}} = \emptyset$ if and only if $H^{\mathcal{R}} = L^{\mathcal{R}}$, and $\rho_{\tilde{\mathcal{R}}}$ is continuous. Moreover, if $H^{\mathcal{R}} \subsetneq L^{\mathcal{R}}$, both $H^{\mathcal{R}}$ and $M^{\mathcal{R}}$ are nowhere dense (as true subspaces of $L^{\mathcal{R}}$) and – by Baire's Theorem – $C^{\mathcal{R}} \cup \{\rho_{\tilde{\mathcal{R}}} = \infty\}$ is a dense open set.

Note that the inclusions $M^{\mathcal{R}} \subseteq H^{\mathcal{R}} \subseteq L^{\mathcal{R}}$ can all be strict, as is illustrated by Example 5.3.

Remark 4.11. Having introduced $M^{\mathcal{R}}$ we can now discuss the extension given by the norm closure operation (4.6). Seen as a subset of $L^{\mathcal{R}}$, $\overline{\mathcal{A}}$ is unfortunately not an acceptance set in the sense of Definition 2.1, since $X \leq Y$ and $Y \in \overline{\mathcal{A}}$ does not necessarily imply $X \in \overline{\mathcal{A}}$, so the monotonicity property is violated. However, one can show that $\overline{\mathcal{R}} := (\overline{\mathcal{A}}, \mathcal{S}, \mathfrak{p})$ is a risk measurement regime on the Banach lattice $M^{\mathcal{R}}$. By Proposition 4.9 it follows that $\rho_{\overline{\mathcal{A}}}(X) = \rho_{\tilde{\mathcal{R}}}(X) = \eta(X)$ for all $X \in M^{\mathcal{R}}$, and $\rho_{\overline{\mathcal{A}}}$ is continuous on $M^{\mathcal{R}}$.

4.3. The dual of the Minkowski domain

In this short interlude we discuss a few properties of the norm dual $(L^{\mathcal{R}*}, \|\cdot\|_{\mathcal{R}*})$ of $(L^{\mathcal{R}}, \|\cdot\|_{\mathcal{R}})$, the space of continuous linear functionals on the Minkowski domain, which will be essential when we study subgradients in Section 4.4.

Theorem 4.12. *$L^{\mathcal{R}*}$ is the direct sum of two subspaces CA and PA, i.e.*

$L^{\mathcal{R}*} = CA \oplus PA.$

Elements in CA have the shape $X \mapsto \int X\,d\mu$ for a unique $\mu \in \mathbf{ca}_{\mathbb{P}}$. $\lambda \in PA$ are characterised by $\lambda|_{M^{\mathcal{R}}} = 0$. For $\ell = \mu \oplus \lambda$,[7] μ is the REGULAR PART *of ℓ and λ the* SINGULAR PART. *Moreover, $L^\infty_{\mathbb{P}}$ can be identified with a subspace of $L^{\mathcal{R}*}$.*

Proof. Let $\ell \in L^{\mathcal{R}*}$ and consider the additive set function $\mu = \mu_\ell : \mathcal{F} \to \mathbb{R}$, $\mu(A) := \ell(\mathbf{1}_A)$. It is straightforward to prove that $\mu \in \mathbf{ba}_{\mathbb{P}}$ and that it is unique, given ℓ. Let now $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}$ be a vanishing sequence of sets. For all $\lambda > 0$ continuity from above implies

$$\lim_{n\to\infty}\rho(\lambda^{-1}\mathbf{1}_{A_n}) = \rho(0) = 0,$$

which reads as $\lim_n\|\mathbf{1}_{A_n}\|_{\mathcal{R}} = 0$ and thus $\lim_n\mu(A_n) = \lim_n\ell(\mathbf{1}_{A_n}) = 0$. Hence $\mu \in \mathbf{ca}_{\mathbb{P}}$.

We will now show that the linear functional $X \mapsto \int X\,d\mu$ is bounded. To this end, note first that by its definition, $\ell(X) = \int X\,d\mu_\ell$ holds for all $X \in L^\infty_{\mathbb{P}}$. Moreover, by Aliprantis and Border (1999, Theorem 7.46), $L^{\mathcal{R}*}$ is a Banach lattice in its own right, and the mapping $\ell \mapsto \mu_\ell$ is positive and linear in ℓ, hence it suffices to show $X \mapsto \int X\,d\mu \in L^{\mathcal{R}*}$ is bounded for $\mu = \mu_\ell \in (\mathbf{ca}_{\mathbb{P}})_+$, $\ell \in L^{\mathcal{R}*}_+$. Let $X \in L^{\mathcal{R}}_+$ be arbitrary.

$$\int(X\wedge n)d\mu = |\ell(X\wedge n)| \leq \|\ell\|_{\mathcal{R}*}\|(X\wedge n)\|_{\mathcal{R}} \leq \|\ell\|_{\mathcal{R}*}\|X\|_{\mathcal{R}},$$

[6] For an introduction to Orlicz space theory we refer to Rao and Ren (1991).

[7] We shall stick to the abuse of notation of identifying functionals in CA with the unique measure $\mu \in \mathbf{ca}_{\mathbb{P}}$ in their integral representation.

where the last inequality follows from Proposition 4.2(iv). We apply the Monotone Convergence Theorem and obtain $\int X\,d\mu \le \|\ell\|_{\mathcal{R}*}\|X\|_{\mathcal{R}}$. For a general $X \in L^{\mathcal{R}}$, we get

$$\left|\int X\,d\mu\right| \le \int |X| d\mu \le \|\ell\|_{\mathcal{R}*}\||X|\|_{\mathcal{R}} = \|\ell\|_{\mathcal{R}*}\|X\|_{\mathcal{R}}.$$

$X \mapsto \int X\,d\mu \in L^{\mathcal{R}*}$ follows, and from $L^\infty_{\mathbb{P}}$ being dense in $M^{\mathcal{R}}$, $\ell|_{M^{\mathcal{R}}} = \int \cdot\, d\mu|_{M^{\mathcal{R}}}$ has to hold. Let $CA := \{\int \cdot\, d\mu_\ell \mid \ell \in L^{\mathcal{R}*}\}$, which is a subspace of $L^{\mathcal{R}*}$. For $\ell \in L^{\mathcal{R}*}$, let $\lambda := \ell - \int \cdot\, d\mu \in L^{\mathcal{R}*}$, which satisfies $\lambda|_{M^{\mathcal{R}}} = 0$. Clearly, $\ell = \int \cdot\, d\mu + \lambda$ is a unique decomposition of ℓ as a sum of elements in CA and PA.

If $Z \in L^\infty_{\mathbb{P}}$, the inclusion $L^{\mathcal{R}} \subseteq L^1_{\mathbb{P}}$, Hölder's inequality and (4.2) yield $L^{\mathcal{R}} \ni X \mapsto \mathbb{E}_{\mathbb{P}}[ZX]$ is well-defined and continuous, i.e. $\mathbb{E}_{\mathbb{P}}[Z\cdot] \in L^{\mathcal{R}*}$. $\square$

CA stands for "countably additive", *PA* for "purely additive". One can show that *CA* is a closed subspace of $L^{\mathcal{R}*}$. The following corollary is a direct consequence of Theorem 4.12.

Corollary 4.13. *For all $\lambda \in PA$, $X \in L^{\mathcal{R}}$ and $r > 0$, we have the identity*

$$\lambda(X) = \lambda(X\mathbf{1}_{\{|X|\ge r\}}).$$

Moreover, if $\ell = \mu \oplus \lambda \in L^{\mathcal{R}*}$, $\lim_{r\to\infty}\ell(X\mathbf{1}_{\{|X|\ge r\}}) = \lambda(X)$ holds for all $X \in L^{\mathcal{R}}$.

Theorem 4.12 implies another characterisation of $\rho_{\tilde{\mathcal{R}}}$.

Corollary 4.14. *Consider the following two classes of extensions of $\rho_{\mathcal{R}}$ to $L^{\mathcal{R}}$:*

$$\mathcal{E}_1 = \{g : L^{\mathcal{R}} \to (-\infty,\infty] \mid g \text{ convex}, \sigma(L^{\mathcal{R}}, CA)\text{-l.s.c.}, \\ g|_{L^\infty_{\mathbb{P}}} = \rho_{\mathcal{R}}\},$$
$$\mathcal{E}_2 := \{g : L^{\mathcal{R}} \to (-\infty,\infty] \mid g \text{ monotone}, \\ g = \sup_{m\in\mathbb{N}} g(\cdot \wedge m),\ g|_{L^\infty_{\mathbb{P}}} = \rho_{\mathcal{R}}\}.$$

Then $\rho_{\tilde{\mathcal{R}}}$ is maximal both in $\mathcal{E}_1$ and $\mathcal{E}_2$, i.e. $g \in \mathcal{E}_i$ implies $g \le \rho_{\tilde{\mathcal{R}}}$.

Proof. First assume $g \in \mathcal{E}_1$. By the Fenchel–Moreau Theorem (cf. Ekeland and Témam, 1999, Proposition 4.1) g has a dual representation

$$g(X) = \sup_{\mu\in CA} \int X\,d\mu - g^*(\mu), \quad X \in L^{\mathcal{R}},$$

where $g^*(\mu) = \sup_{X\in L^{\mathcal{R}}} \int X\,d\mu - g(X)$. By $g|_{L^\infty_{\mathbb{P}}} = \rho_{\mathcal{R}}$, we have $\mathrm{dom}(g^*) \subseteq \mathrm{dom}(\rho^*_{\mathcal{R}})$ and $g^*(\mu) \ge \rho^*_{\mathcal{R}}(\mu)$ for all $\mu \in \mathrm{dom}(\rho^*_{\mathcal{R}})$. Hence, for $X \in L^{\mathcal{R}}$ arbitrary, we have

$$\begin{aligned} g(X) &\le \sup_{\mu\in\mathrm{dom}(\rho^*_{\mathcal{R}})} \int X\,d\mu - g^*(\mu) \\ &\le \sup_{\mu\in\mathrm{dom}(\rho^*_{\mathcal{R}})} \int X\,d\mu - \rho^*_{\mathcal{R}}(\mu) = \rho_{\tilde{\mathcal{R}}}(X). \end{aligned}$$

For the second claim, let $g \in \mathcal{E}_2$ and let $X \in L^{\mathcal{R}}$ be arbitrary. Monotonicity of g allows for the following estimate:

$$\begin{aligned} g(X) &= \sup_{m\in\mathbb{N}} g(X\wedge m) \le \sup_{m\in\mathbb{N}}\inf_{n\in\mathbb{N}} \underbrace{g((-n)\vee X\wedge m)}_{=\rho_{\mathcal{R}}((-n)\vee X\wedge m)} \\ &= \xi(X) = \rho_{\tilde{\mathcal{R}}}(X). \quad \square \end{aligned}$$

4.4. Subgradients over the Minkowski domain

In this section we will study subgradients of $\rho_{\tilde{\mathcal{R}}}$ and η, and how to ensure that subgradients correspond to measures on $(\Omega, \mathcal{F})$. Given Theorem 4.12, it does not seem surprising that this is not always the case. The reason for also considering subgradients of η is that existence of regular subgradients of η and $\rho_{\tilde{\mathcal{R}}}$ is closely related to the question (4.7), and the developed results pave the way for the proof of Theorem 4.7.

Definition 4.15. Let $(\mathcal{X}, \tau)$ be a topological vector space with dual space $\mathcal{X}^*$. Given a proper convex function $f : \mathcal{X} \to (-\infty,\infty]$, the SUBGRADIENT of f at $X \in \mathcal{X}$ is the set

$$\begin{aligned} \partial f(X) &:= \{\ell \in \mathcal{X}^* \mid \forall Y \in \mathcal{X} : f(Y) \ge f(X) + \ell(Y - X)\} \\ &= \{\ell \in \mathcal{X}^* \mid f(X) = \ell(X) - f^*(\ell)\}, \end{aligned}$$

where $f^*(\ell) := \sup_{X\in\mathcal{X}} \ell(X) - f(X)$, $\ell \in \mathcal{X}^*$.

Note that if a convex function $f : L^{\mathcal{R}} \to (-\infty,\infty]$ is additionally monotone and $\mathcal{S}$-additive, its subgradients will be positive functionals in $L^{\mathcal{R}*}_+$ that agree with $\mathfrak{p}$ on $\mathcal{S}$.

In the study of risk measures subgradients play an important role, for instance as pricing rules in equilibria. The following easy example serves as an economic motivation.

Example 4.16 (*Optimal Investment*)**.** For some capital constraint $c > 0$ and some linear pricing rule $\ell \in L^{\mathcal{R}*}_+$ consider the following optimisation problem:

$$(*) \quad \rho_{\tilde{\mathcal{R}}}(Y) \to \min, \quad \text{over all } Y \in L^{\mathcal{R}} \text{ with } \ell(-Y) \le c.$$

In order to solve this, by monotonicity, we can without loss of generality focus on Y satisfying $\ell(-Y) = c$. If $X \in L^{\mathcal{R}}$ satisfies $\ell \in \partial\rho_{\tilde{\mathcal{R}}}(X)$ and $\ell(-X) = c$, then X solves $(*)$. Indeed for all $Y \in \ell^{-1}(\{-c\})$, we have

$$\begin{aligned} \rho_{\tilde{\mathcal{R}}}(X) &= \rho_{\tilde{\mathcal{R}}}(X) + \ell(Y - X) - \ell(Y) - c \\ &\le \rho_{\tilde{\mathcal{R}}}(Y) + \ell(-Y) - c = \rho_{\tilde{\mathcal{R}}}(Y). \end{aligned}$$

An important feature of the space $(L^{\mathcal{R}}, \|\cdot\|_{\mathcal{R}})$ is that $\mathrm{dom}(\rho_{\tilde{\mathcal{R}}})$ possesses a particularly rich interior, see Proposition 4.6. Thus we have the following result:

Theorem 4.17. *Suppose $X \in \mathrm{int}\,\mathrm{dom}(\rho_{\tilde{\mathcal{R}}})$, so in particular if $X \in \Gamma$, then $\partial\rho_{\tilde{\mathcal{R}}}(X) \ne \emptyset$. Also $\partial\eta(Y) \ne \emptyset$ whenever $Y \in \Gamma$.*

Proof. It is well-known that a convex, proper and monotone function f on a Banach lattice is subdifferentiable at every point in $\mathrm{int}\,\mathrm{dom}(f)$, see Ruszczyński and Shapiro (2006, Proposition 1). The claim thus follows from Theorem 4.4 and Proposition 4.6. $\square$

We devote the remainder of this subsection to the question under which conditions $\partial\rho_{\tilde{\mathcal{R}}}(X)$ will contain regular (that is σ-additive) elements.[8] To this end, note that by (4.8) we have that $\eta^* \le \rho^*_{\tilde{\mathcal{R}}}$, which implies $\mathrm{dom}(\rho^*_{\tilde{\mathcal{R}}}) \subseteq \mathrm{dom}(\eta^*)$. Moreover, $CA \cap \mathrm{dom}(\rho^*_{\tilde{\mathcal{R}}}) \subseteq CA \cap \mathrm{dom}(\eta^*) \subseteq \mathrm{dom}(\rho^*_{\mathcal{R}})$, so regular subgradients of $\rho_{\tilde{\mathcal{R}}}$ and η are necessarily in $\mathrm{dom}(\rho^*_{\mathcal{R}})$. Indeed, if $\mu \in CA \cap \mathrm{dom}(\eta^*)$, then

$$\begin{aligned} \rho^*_{\mathcal{R}}(\mu) &= \sup_{Y\in L^\infty_{\mathbb{P}}} \int Y\,d\mu - \eta(Y) \\ &\le \sup_{Y\in L^{\mathcal{R}}} \int Y\,d\mu - \eta(Y) = \eta^*(\mu) < \infty. \end{aligned}$$

Conversely, for all $Y \in \mathrm{dom}(\eta)$ the definition of η and $CA \subseteq \mathbf{ca}_{\mathbb{P}}$ shows for $\mu \in CA$

$$\begin{aligned} \int Y\,d\mu - \eta(Y) &= \lim_{n\to\infty}\lim_{m\to\infty} \int Y^m_n d\mu - \rho_{\mathcal{R}}(Y^m_n) \\ &\le \sup_{U\in L^\infty_{\mathbb{P}}} \int U\,d\mu - \rho_{\mathcal{R}}(U) = \rho^*_{\mathcal{R}}(\mu). \end{aligned} \tag{4.15}$$

8 There are immediate – however very strong – sufficient conditions for this to happen, e.g. $L^{\mathcal{R}*} \subseteq \mathbf{ca}_{\mathbb{P}}$, which is the case if and only if $M^{\mathcal{R}} = L^{\mathcal{R}}$, or continuity of $\rho_{\tilde{\mathcal{R}}}$ with respect to the $\sigma(L^{\mathcal{R}}, CA)$-topology.

This shows that $\eta^*(\mu) = \rho^*_{\mathcal{R}}(\mu)$, which provides a first step towards the proof of Theorem 4.7:

Lemma 4.18. *Let $X \in \mathrm{dom}(\eta)$ and suppose that $\mu \oplus \lambda \in \partial\eta(X)$, where $\mu \in CA$ and $\lambda \in PA$. Then $\lambda(X^-) = 0$. If, moreover, $\lambda = 0$, i.e. $\mu \in \partial\eta(X)$, then $\eta(X) = \rho_{\tilde{\mathcal{R}}}(X)$.*

Proof. Let $\mu \oplus \lambda \in \partial\eta(X)$. Define $\tilde{\lambda}$ by $\tilde{\lambda}(Y) = \lambda(Y\mathbf{1}_{\{X\geq 0\}})$, $Y \in L^{\mathcal{R}}$. One verifies that $\tilde{\lambda} \in (L^{\mathcal{R}})^*$. Also we have

$$\begin{aligned}\eta^*(\mu \oplus \tilde{\lambda}) &= \sup_{Y\in L^{\mathcal{R}}} \int Y\,d\mu + \lambda(Y\mathbf{1}_{\{X\geq 0\}}) - \eta(Y)\\ &\leq \sup_{Y\in L^{\mathcal{R}}} \lim_{n\to\infty} \int (-n)\vee Y\,d\mu + \lambda(Y^+) - \eta((-n)\vee Y)\\ &\leq \limsup_{n\to\infty} \sup_{Y\in L^{\mathcal{R}}} \int (-n)\vee Y\,d\mu + \lambda((-n)\vee Y)\\ &\quad - \eta((-n)\vee Y)\\ &\leq \eta^*(\mu\oplus\lambda),\end{aligned}$$

where we used monotonicity of λ. Hence,

$$\begin{aligned}\eta(X) &= \int X\,d\mu + \lambda(X) - \eta^*(\mu\oplus\lambda)\\ &\leq \int X\,d\mu + \tilde{\lambda}(X) - \eta^*(\mu\oplus\tilde{\lambda}) \leq \eta(X),\end{aligned}$$

and the first inequality would be strict if $\lambda(X^-) > 0$. Thus $\lambda(X^-) = 0$ follows. For the last assertion, suppose that $\mu \in \partial\eta(X) \cap CA$. The observations preceding the lemma and (4.8) show

$$\eta(X) = \int X\,d\mu - \rho^*_{\mathcal{R}}(\mu) \leq \rho_{\tilde{\mathcal{R}}}(X) \leq \eta(X). \quad \square$$

Consequently, if $\partial\eta(X) \neq \emptyset$, so for instance for $X \in \Gamma$, then η may display a "jump" $\lambda(X^+)$ produced by the unbounded risk X^+. If that jump is not present, then $\eta(X) = \rho_{\tilde{\mathcal{R}}}(X)$. In the following we will introduce a weak local continuity assumption, *tail continuity*, which quantifies which tails are not too fat to lead to such jumps. In Svindland (2010b), a version of it is studied for law-invariant monetary risk measures.

Definition 4.19. Let $f : L^{\mathcal{R}} \to (-\infty, \infty]$ be monotone and proper, and let $X \in \mathrm{dom}(f)$. We call f TAIL CONTINUOUS at X along $Y \in L^{\mathcal{R}}$ if $X + Y^+ \in \mathrm{dom}(f)$ and

$$f(X) = \lim_{r\to\infty} f\left(X + Y\mathbf{1}_{\{Y\geq r\}}\right)$$

holds. $\mathcal{T}^f_X$ denotes the set of tails Y along which f is tail continuous at X. With a slight abuse of language, we call f tail continuous at X if $\mathcal{T}^f_X = \{Y \in L^{\mathcal{R}} \mid X + Y^+ \in \mathrm{dom}(f)\}$.

Note that $\mathcal{T}^f_X$ is monotone in that $Y_1 \leq Y_2$ $\mathbb{P}$-a.s. and $Y_2 \in \mathcal{T}^f_X$ implies $Y_1 \in \mathcal{T}^f_X$. The next proposition shows that sufficient tail continuity can eliminate non-σ-additive elements in the subgradient. We prove this for general monotone functions f, but we clearly have $f = \rho_{\tilde{\mathcal{R}}}$ or $f = \eta$ in mind.

Proposition 4.20. *Let $f : L^{\mathcal{R}} \to (-\infty, \infty]$ be proper, monotone, and convex, and let $X \in \mathrm{dom}(f)$. Suppose that $\{sY \mid s \geq 0, Y \in \mathcal{T}^f_X\}$ is norm-dense (or equivalently $\mathcal{T}^f_X$ separates the points of $L^{\mathcal{R}*}$). Then $\partial f(X) \subseteq CA$. In particular, if f is tail continuous at $X \in \mathrm{int}\,\mathrm{dom}(f)$, then $\partial f(X) \subseteq CA$.*

Proof. Let $\ell = \mu \oplus \lambda \in \partial f(X)$. Assume $\lambda \neq 0$. The density assumption and monotonicity allows to pick $Y \in \mathcal{T}^f_X$, $Y \geq 0$, such that $\lambda(Y) > 0$. Corollary 4.13 and ℓ being a subgradient together with tail continuity along Y yield the CONTRADICTION

$$\begin{aligned}f(X) &< f(X) + \lambda(Y) = \lim_{r\to\infty} f(X) + \lambda(Y\mathbf{1}_{\{Y\geq r\}})\\ &= \lim_{r\to\infty} f(X) + \ell(Y\mathbf{1}_{\{Y\geq r\}}) = \lim_{r\to\infty} \ell(X) - f^*(\ell) + \ell(Y\mathbf{1}_{\{Y\geq r\}})\\ &= \lim_{r\to\infty} \ell(X + Y\mathbf{1}_{\{Y\geq r\}}) - f^*(\ell)\\ &\leq \liminf_{r\to\infty} f(X + Y\mathbf{1}_{\{Y\geq r\}}) = f(X). \quad \square\end{aligned}$$

Unfortunately, in general we only have tail continuity along $M^{\mathcal{R}}$, as is shown in the following Lemma 4.21. As we have already observed, if $L^{\mathcal{R}} = M^{\mathcal{R}}$, then $L^{\mathcal{R}*} = CA$ and therefore trivially $\partial\rho_{\tilde{\mathcal{R}}}(X) \subseteq CA$, so just knowing tail continuity along $M^{\mathcal{R}}$ is not sufficient for the existence of countably additive subgradients in non-trivial cases.

Lemma 4.21. *Let $f : L^{\mathcal{R}} \to (-\infty, \infty]$ be proper, monotone, and convex such that $L^\infty_{\mathbb{P}} \subseteq \mathrm{dom}(f)$. If $X \in \mathrm{int}\,\mathrm{dom}(f)$, then $M^{\mathcal{R}}_+ - L^{\mathcal{R}}_+ \subseteq \mathcal{T}^f_X$.*

Proof. $\mathcal{T}^f_X$ is monotone, hence it suffices to consider $Y \in M^{\mathcal{R}}_+$. The condition $X \in \mathrm{int}\,\mathrm{dom}(f)$ guarantees $X + Y \in \mathrm{dom}(f)$ as in (4.14). From Corollary 4.10 we obtain $\lim_n \|Y\mathbf{1}_{\{Y\geq n\}}\|_{\mathcal{R}} = 0$, hence $X + Y\mathbf{1}_{\{Y\geq n\}} \in \mathrm{int}\,\mathrm{dom}(f)$ for all n large enough. The desired tail continuity follows from the continuity of $f|_{\mathrm{int}\,\mathrm{dom}(f)}$ (see Remark 2.4(v)). $\square$

While in Proposition 4.20 we gave a condition under which the subgradient contains regular dual elements only, we will now turn to conditions guaranteeing the existence of at least one regular element in the subgradient, namely by means of projection.

Proposition 4.22. *Let $X \in \mathrm{dom}(\rho_{\tilde{\mathcal{R}}})$ and $\ell = \mu\oplus\lambda \in \partial\rho_{\tilde{\mathcal{R}}}(X)$. Then also $\mu \in \partial\rho_{\tilde{\mathcal{R}}}(X)$ whenever μ satisfies $\int X\,d\mu \geq \ell(X)$. Similarly, if $X \in \mathrm{dom}(\eta)$ and $\ell = \mu \oplus \lambda \in \partial\eta(X)$, then $\mu \in \partial\eta(X)$ whenever μ satisfies $\int X\,d\mu \geq \ell(X)$. In particular, the assumption $\int X\,d\mu \geq \ell(X)$ is met if $X \in M^{\mathcal{R}}_+ - L^{\mathcal{R}}_+$.*

Proof. By the same argument employed in (4.15) and the equality $\rho_{\tilde{\mathcal{R}}} = \xi$, we obtain that $\eta^*(\mu) = \rho^*_{\tilde{\mathcal{R}}}(\mu) = \sup_{U\in L^\infty_{\mathbb{P}}} \int U\,d\mu - \rho_{\mathcal{R}}(U) = \rho^*_{\mathcal{R}}(\mu)$ holds for all $\mu \in CA$. From this and $\ell|_{M^{\mathcal{R}}} = \int \cdot\, d\mu$ we infer $\rho^*_{\tilde{\mathcal{R}}}(\mu) \leq \rho^*_{\tilde{\mathcal{R}}}(\ell)$, and $\eta^*(\mu) \leq \eta^*(\ell)$. The assumption $\int X\,d\mu \geq \ell(X)$ and $\ell \in \partial\rho_{\tilde{\mathcal{R}}}(X)$ imply

$$\rho_{\tilde{\mathcal{R}}}(X) \geq \int X\,d\mu - \rho^*_{\tilde{\mathcal{R}}}(\mu) \geq \ell(X) - \rho^*_{\tilde{\mathcal{R}}}(\ell) = \rho_{\tilde{\mathcal{R}}}(X).$$

The assertion for η follows in the same way. $\square$

Remark 4.23. In the situation of Proposition 4.22, as $\int X\,d\mu - \rho^*_{\tilde{\mathcal{R}}}(\mu) = \ell(X) - \rho^*_{\tilde{\mathcal{R}}}(\ell)$, $\rho^*_{\tilde{\mathcal{R}}}(\mu) \leq \rho^*_{\tilde{\mathcal{R}}}(\ell)$, and $\int X\,d\mu \geq \ell(X)$, we in fact obtain $\int X\,d\mu = \ell(X)$ and $\rho^*_{\tilde{\mathcal{R}}}(\ell) = \rho^*_{\tilde{\mathcal{R}}}(\mu)$. In other words, singularities in the subgradient cannot be excluded, but they are redundant for X.

The following proposition establishes a handy criterion for $\int X\,d\mu \geq \ell(X)$.

Proposition 4.24. *Suppose that $f : L^{\mathcal{R}} \to (-\infty, \infty]$ is monotone, proper and convex, and that $\ell = \mu\oplus\lambda \in \partial f(X)$. Then $\int X\,d\mu \geq \ell(X)$ whenever $sX^+ \in \mathcal{T}^f_X$ for some $s > 0$.*

Proof. Suppose that $\lambda(X^+) =: \delta > 0$. By monotonicity one obtains for all $n \in \mathbb{N}$

$$\delta = \lambda(X^+) = \lambda(X^+\mathbf{1}_{\{X^+\geq n\}}) \leq \ell(X^+\mathbf{1}_{\{X^+\geq n\}}).$$

Define $X_n = X + sX^+\mathbf{1}_{\{X^+\geq n\}} \geq X$, $n \in \mathbb{N}$, where $s > 0$ is chosen like in the assumption of the proposition. We estimate

$$\begin{aligned}\ell(X) - f(X) = f^*(\ell) &\geq \ell(X_n) - f(X_n)\\ &= \ell(X) + s\ell(X^+\mathbf{1}_{\{X^+\geq n\}}) - f(X_n)\\ &\geq \ell(X) + s\delta - f(X_n).\end{aligned}$$

Consequently, we arrive at the CONTRADICTION $0 = \lim_{n\to\infty} f(X_n) - f(X) \geq s\delta$. Hence, $\lambda(X^+) = 0$, and thus $\int X\,d\mu \geq \ell(X)$. □

We now have the tools at hand to provide the proof of Theorem 4.7.

Proof of Theorem 4.7. Note that condition (i) implies (ii), and suppose that one of them holds. $X \in \Gamma$ implies that $\partial\eta(X) \neq \emptyset$ (Theorem 4.17), and Propositions 4.22 and 4.24 in conjunction with the second part of Lemma 4.18 do the rest.

Condition (iii) is equivalent to $X^+ \in M^{\mathcal{R}}$ by Corollary 4.10. Hence Proposition 4.22 applies, and Lemma 4.18 yields the assertion. □

5. Examples

Example 5.1. Consider $(\Omega, \mathcal{F})$ to be the natural numbers endowed with their power set. Let $\zeta \in \mathbf{ca}_+$ be defined by the discrete density $(2^{-\omega})_{\omega\in\mathbb{N}}$ and let $\nu \in \mathbf{ba}_+$ be the purely finitely additive measure on $(\Omega, \mathcal{F})$ arising from the Banach–Mazur limit (cf. Aliprantis and Border, 1999, Definition 15.46); the reader should keep in mind that $\nu(F) = 0$ for all finite sets $F \subseteq \mathbb{N}$. Moreover, for $\lambda \in [0, 1]$ we set $\mu_\lambda = (1-\lambda)\zeta + \lambda\nu$ and define the closed acceptance set

$$\mathcal{A} := \left\{X \in \mathcal{L}^\infty \middle| \forall\lambda \in [0,1]: \int X\,d\mu_\lambda \leq \lambda\right\}.$$

Clearly, $\mathcal{B}(\mathcal{A}) \setminus \mathbf{ca} \neq \emptyset$. Now let $\emptyset \neq A \subset \mathbb{N}$ be any finite subset, $\mathcal{S} = \{U(\alpha,\beta) := \alpha\mathbf{1}_A + \beta \mid \alpha, \beta \in \mathbb{R}\}$, $\mathfrak{p}(U(\alpha,\beta)) = \int U(\alpha,\beta)\,d\zeta$. We first show that $\mathcal{R} := (\mathcal{A}, \mathcal{S}, \mathfrak{p})$ is a risk measurement regime and $\rho_{\mathcal{R}}$ is finite. To this end, note first that, for arbitrary $X \in \mathcal{L}^\infty$, $X + U(\alpha,\beta) \in \mathcal{A}$ implies

$$0 \geq \int (X + U(\alpha,\beta))d\mu_0 = \int X\,d\zeta + \mathfrak{p}(U(\alpha,\beta)),$$

hence $\mathfrak{p}(U(\alpha,\beta)) \leq -\int X\,d\zeta < \infty$, and $\mathcal{R}$ is a risk measurement regime. Moreover, for any $k \in \mathbb{N}$ we have that $k - U(0,k) \in \mathcal{A}$, which means that $\rho_{\mathcal{R}}(k) \leq k$. By monotonicity, $\rho_{\mathcal{R}}$ does not attain the value $+\infty$.

Next we prove that $\rho_{\mathcal{R}}$ is continuous from above even though $\mathcal{B}(\mathcal{A}) \setminus \mathbf{ca} \neq \emptyset$. We proceed in three steps.

Step 1: $\sigma_{\mathcal{A}}(\mu_\lambda) = \lambda$. Indeed, let $A_n := \{n, n+1, \ldots\}$ and note that

$$\mu_\lambda(A_n) = (1-\lambda)\sum_{i=n}^\infty 2^{-i} + \lambda.$$

Hence $Y_n := \mathbf{1}_{A_n} - \sum_{i=n}^\infty 2^{-i} \in \mathcal{A}$, and

$$\sigma_{\mathcal{A}}(\mu_\lambda) \geq \lim_{n\to\infty}\int Y_n\,d\mu_\lambda = \lim_{n\to\infty} -\lambda\sum_{i=n}^\infty 2^{-i} + \lambda = \lambda.$$

The converse inequality $\sigma_{\mathcal{A}}(\mu_\lambda) \leq \lambda$ is due to the definition of $\mathcal{A}$.

Step 2: $\mathcal{B}(\mathcal{A}) = \mathrm{cone}(\{\zeta,\nu\})$, where cone$(E)$ refers to the smallest convex and pointed cone containing $E \subseteq \mathbf{ba}$ and 0. The inclusion $\mathcal{B}(\mathcal{A}) \supseteq \mathrm{cone}(\{\zeta,\nu\})$ is clear, for the other one note that $\mathcal{B}(\mathcal{A}) = \mathrm{cone}(\mathcal{B}(\mathcal{A})_1)$ always holds, where $\mathcal{B}(\mathcal{A})_1 := \{\frac{1}{\mu(\Omega)}\mu \mid 0 \neq \mu \in \mathcal{B}(\mathcal{A})\}$. Assume we can find $\mu \in \mathcal{B}(\mathcal{A})_1 \setminus co(\{\zeta,\nu\})$, where $co(\{\zeta,\nu\})$ denotes the convex hull of ζ and ν. As $co(\{\zeta,\nu\})$ is $\sigma(\mathbf{ba}, \mathcal{L}^\infty)$-compact and convex, by means of separation we can find a $Y \in \mathcal{L}^\infty$ such that

$$\max_{\lambda\in[0,1]}\left(\int Y\,d\mu_\lambda - \lambda\right) \leq \max_{\lambda\in[0,1]}\int Y\,d\mu_\lambda = 0 < \int Y\,d\mu.$$

The same holds true when Y is replaced by tY, $t > 0$. Thus $\{tY \mid t > 0\} \subseteq \mathcal{A}$, and

$$\sigma_{\mathcal{A}}(\mu) \geq \sup_{t>0}\int tY\,d\mu = \infty.$$

We conclude that $\mathcal{B}(\mathcal{A})_1 = co(\{\zeta,\nu\})$ and thus $\mathcal{B}(\mathcal{A}) = \mathrm{cone}(\{\zeta,\nu\})$.

Step 3: $\mathcal{E}_{\mathfrak{p}} \cap \mathcal{B}(\mathcal{A}) = \{\zeta\}$ and therefore $\rho_{\mathcal{R}}(X) = \int X\,d\zeta$ by Proposition 3.1(i), which is continuous from above. Indeed, $\mu \in \mathcal{E}_{\mathfrak{p}} \cap \mathcal{B}(\mathcal{A})$ only if $\mu \in \mathcal{B}(\mathcal{A})_1$, therefore by Step 2 we can assume $\mu_\lambda \in \mathcal{E}_{\mathfrak{p}} \cap \mathcal{B}(\mathcal{A})$ for some $\lambda \in [0,1]$. We reformulate the condition as for all $\alpha, \beta \in \mathbb{R}$ it has to hold

$$\alpha\zeta(A) + \beta = (1-\lambda)(\alpha\zeta(A) + \beta) + \lambda\beta = (1-\lambda)\alpha\zeta(A) + \beta,$$

which is the case if and only if $\lambda = 0$.

Example 5.2 ($\rho_{\tilde{\mathcal{R}}} \neq \eta$). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the integers $\mathbb{Z}$ endowed with their power set and a probability measure specified below. Let

$$\mathbb{Q}_k := \frac{1}{2k}(\delta_k + \delta_{-k}) + \left(1 - \frac{1}{k}\right)\delta_0, \quad k \in \mathbb{N},$$

and define $\mathbb{P} := \sum_{k\in\mathbb{N}} 2^{-k}\mathbb{Q}_k$. It is straightforward to check that

$$\mathcal{A} := \{X \in L^\infty_{\mathbb{P}} \mid \forall k \in \mathbb{N}: \ \mathbb{E}_{\mathbb{Q}_k}[X] \leq 0\}, \quad \mathcal{S} = \mathbb{R}, \quad \mathfrak{p} = id_{\mathbb{R}},$$

is a risk measurement regime on $L^\infty_{\mathbb{P}}$ such that $\rho_{\mathcal{R}}(X) := \sup_{k\in\mathbb{N}}\mathbb{E}_{\mathbb{Q}_k}[X]$, $X \in L^\infty_{\mathbb{P}}$, is a coherent monetary risk measure which is continuous from above and sensitive with respect to the strong reference model $\mathbb{P}$. We consider $X := id_{\mathbb{Z}}$. We first observe that for all $k \in \mathbb{N}$ it holds that $\mathbb{E}_{\mathbb{Q}_k}[|X|] = 1$, which is sufficient for $X \in L^{\mathcal{R}}$. Using the notational conventions of Theorem 4.5, for all $n \in \mathbb{N}$

$$\rho_{\tilde{\mathcal{R}}}(X_n) \geq \rho_{\mathcal{R}}(X_n^{n^2}) \geq \mathbb{E}_{\mathbb{Q}_{n^2}}[X_n^{n^2}] = \frac{1}{2}\left(1 - \frac{1}{n}\right).$$

Hence $\eta(X) \geq \frac{1}{2}$. However, for $m \in \mathbb{N}$ fixed, we obtain for all $n > m$ that

$$\mathbb{E}_{\mathbb{Q}_k}[X_n^m] = \begin{cases} 0, \text{ if } k \leq m,\\ \dfrac{m}{2k} - \dfrac{1}{2}, \text{ if } m < k \leq n,\\ \dfrac{m-n}{2k}, \text{ if } k > n.\end{cases}$$

This implies

$$\rho_{\tilde{\mathcal{R}}}(X^m) = \xi(X^m) = \lim_{n\to\infty}\rho_{\mathcal{R}}(X_n^m) = 0,$$

and therefore

$$\rho_{\tilde{\mathcal{R}}}(X) = \lim_{m\to\infty}\rho_{\tilde{\mathcal{R}}}(X^m) = 0 < \frac{1}{2} \leq \eta(X).$$

Example 5.3 ($M^{\mathcal{R}} \subsetneq H^{\mathcal{R}} \subsetneq L^{\mathcal{R}}$). Let $(\Omega, \mathcal{F})$ be the real numbers endowed with their Borel sets $\mathbb{B}(\mathbb{R})$. Let $\mathbb{P}_0$ be the probability measure $\mathbb{P}$ from Example 5.2 extended to $\mathbb{B}(\mathbb{R})$, and define $\mathbb{P}_1$ by its Lebesgue density $d\mathbb{P}_1 = e^{1-x}\mathbf{1}_{(1,\infty)}dx$. Let $\mathbb{P} := \frac{1}{2}(\mathbb{P}_0 + \mathbb{P}_1)$, and consider the risk measurement regime

$$\mathcal{A} := \{X \in L^\infty_{\mathbb{P}} \mid \forall k \in \mathbb{N}: \ \mathbb{E}_{\mathbb{Q}_k}[X] \leq 0, \text{and } \mathbb{E}_{\mathbb{P}_1}[e^X] \leq 1\},$$
$$\mathcal{S} = \mathbb{R}, \quad \mathfrak{p} = id_{\mathbb{R}},$$

where the probability measures $(\mathbb{Q}_k)_{k\in\mathbb{N}}$ are chosen as in Example 5.2 and extended to $\mathbb{B}(\mathbb{R})$. One can easily show that $\rho_{\mathcal{R}}(X) = \rho_0(X) \vee \rho_1(X)$, where

$$\rho_0(V) = \sup_{k\in\mathbb{N}}\mathbb{E}_{\mathbb{Q}_k}[V], \quad \rho_1(U) = \log\left(\mathbb{E}_{\mathbb{P}_1}[e^U]\right), \quad U, V \in L^\infty_{\mathbb{P}}.$$

$\rho_{\mathcal{R}}$ is a sensitive finite risk measure on $L^\infty_{\mathbb{P}}$ being continuous from above, and $\mathbb{P}$ is a strong reference probability model.

Consider first $X \in L^0_{\mathbb{P}}$ be generated by $id_{\mathbb{Z}}$. We have already shown in Example 5.2 that $\rho(t|X|) = \rho_0(t|X|) = t$ for all $t \geq 0$, hence $X \in H^{\mathcal{R}}$. Nevertheless, it holds for all $k \in \mathbb{N}$ that

$$1 \geq \rho(|X|\mathbf{1}_{\{|X|>k\}}) \geq \mathbb{E}_{\mathbb{Q}_{k+1}}[|X|\mathbf{1}_{\{|X|>k\}}] = 1.$$

Hence $\lim_k \rho(|X|\mathbf{1}_{\{|X|\geq k\}}) = 1$, which is sufficient for $X \in H^{\mathcal{R}} \setminus M^{\mathcal{R}}$ by Corollary 4.10.

Let now $\lambda > 0$ and define $Y \in L^0_{\mathbb{P}}$ generated by

$$\omega \mapsto \begin{cases} \frac{1}{\lambda}(\omega - 1), \ \omega \in (1, \infty) \setminus \mathbb{Z}, \\ 0 \text{ otherwise.} \end{cases}$$

Y is exponentially distributed under $\mathbb{P}_1$ with parameter λ. Moreover, $Y \in L^{\mathcal{R}}$ and satisfies $\rho_{\tilde{\mathcal{R}}}(Y) < \infty$. However, for every $t > \lambda$, $\rho(t|Y|) = \log(\mathbb{E}_{\mathbb{P}_1}[e^{tY}]) = \infty$, hence $Y \in C^{\mathcal{R}}$.

Example 5.4 (*Entropic Risk Measure*). On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ consider for $\beta > 0$ fixed the *entropic risk measure* $\rho_{\mathcal{R}}(X) := \frac{1}{\beta} \log\left(\mathbb{E}_{\mathbb{P}}[e^{\beta X}]\right)$, $X \in L^\infty_{\mathbb{P}}$. One can easily show

$$L^{\mathcal{R}} = \{X \in L^0_{\mathbb{P}} \mid \exists k > 0 : e^{k|X|} \in L^1_{\mathbb{P}}\},$$
$$\rho_{\tilde{\mathcal{R}}}(X) = \frac{1}{\beta} \log\left(\mathbb{E}_{\mathbb{P}}[e^{\beta X}]\right), \quad X \in L^{\mathcal{R}}.$$

$\rho_{\tilde{\mathcal{R}}}$ is tail continuous. Indeed, choose $X \in \mathrm{dom}(\rho_{\tilde{\mathcal{R}}})$ arbitrary and $Y \in L^{\mathcal{R}}$ such that $X + Y^+ \in \mathrm{dom}(\rho_{\tilde{\mathcal{R}}})$, i.e. $e^{\beta X + \beta Y^+} \in L^1_{\mathbb{P}}$. By continuity of log and dominated convergence, we obtain

$$\lim_r \rho_{\tilde{\mathcal{R}}}(X + Y\mathbf{1}_{\{Y\geq r\}}) = \lim_r \frac{1}{\beta} \log\left(\mathbb{E}_{\mathbb{P}}[e^{\beta(X+Y^+)}\mathbf{1}_{\{Y\geq r\}}] + \mathbb{E}_{\mathbb{P}}[e^{\beta X}\mathbf{1}_{\{Y<r\}}]\right) = \rho_{\tilde{\mathcal{R}}}(X).$$

Example 5.5 (*AVaR-based Risk Measures*). Consider the Average Value at Risk $AVaR_\alpha$ for some $\alpha \in (0, 1]$ on $L^\infty_{\mathbb{P}}$, which is known to have the minimal dual representation

$$AVaR_\alpha(X) = \max_{\mathbb{Q}\in\mathcal{Q}_\alpha} \mathbb{E}_{\mathbb{Q}}[X], \quad X \in L^\infty_{\mathbb{P}},$$

where

$$\mathcal{Q}_\alpha := \left\{\mathbb{Q} \ll \mathbb{P} \mid \frac{d\mathbb{Q}}{d\mathbb{P}} \leq \frac{1}{1-\alpha}\right\},$$

see Föllmer and Schied (2011, Theorem 4.52). Given the acceptance set $\mathcal{A} := \{X \in L^\infty_{\mathbb{P}} \mid AVaR_\alpha(X) \leq 0\}$ we define the risk measurement regime $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \mathfrak{p})$ by $\mathcal{S} = \mathbb{R} \cdot U$ for some $U \in L^\infty_{\mathbb{P}}$ with $\mathbb{P}(U > 0) = 1$, and $\mathfrak{p}(mU) := m$, $m \in \mathbb{R}$. By Farkas et al. (2013, Proposition 4.4) and Proposition 3.1 the resulting risk measure $\rho_{\mathcal{R}}$ is finite, continuous from above with strong reference model $\mathbb{P}$, and $L^{\mathcal{R}} = H^{\mathcal{R}} = M^{\mathcal{R}}$. Hence, Theorem 4.7 applies for all $X \in L^{\mathcal{R}}$.

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Appendix A. Proofs of Proposition 3.1 and Corollary 3.2

Proof of Proposition 3.1. (i) Suppose $\rho_{\mathcal{R}}(0)$ is a real number and let $X = Y+N$ for some $Y \in \mathcal{A}$ and some $N \in \ker(\mathfrak{p})$. By $\mathcal{S}$-additivity, $\rho_{\mathcal{R}}(X) = \rho_{\mathcal{R}}(Y) \leq 0$ holds. $\rho_{\mathcal{R}}$ being l.s.c. implies $\rho_{\mathcal{R}}(X) \leq 0$ for all $X \in \mathrm{cl}_{|\cdot|_\infty}(\mathcal{A} + \ker(\mathfrak{p}))$. Again, $\mathcal{S}$-additivity of $\rho_{\mathcal{R}}$ allows to infer $\mathfrak{p}(Z) = \rho_{\mathcal{R}}(Z) - \rho_{\mathcal{R}}(0) \leq -\rho_{\mathcal{R}}(0)$ for all $Z \in \mathrm{cl}_{|\cdot|_\infty}(\mathcal{A} + \ker(\mathfrak{p})) \cap \mathcal{S}$, and $\mathfrak{p}$ is bounded from above on the latter set. By virtue of Farkas et al. (2015, Theorems 2 and 3), $\mathcal{B}(\mathcal{A}) \cap \mathcal{E}_{\mathfrak{p}} \neq \emptyset$ and

$$\rho_{\mathcal{R}}(X) = \sup_{\mu\in\mathcal{B}(\mathcal{A})\cap\mathcal{E}_{\mathfrak{p}}} \int X\,d\mu - \sigma_{\mathcal{A}}(\mu), \quad X \in \mathcal{L}^\infty,$$

from which the claimed eqs. (3.3) and (3.4) are derived easily. If $\rho_{\mathcal{R}}$ is coherent, its positive homogeneity implies that $\rho^*_{\mathcal{R}}|_{\mathrm{dom}(\rho^*_{\mathcal{R}})} \equiv 0$. As furthermore $\mathrm{dom}(\rho^*_{\mathcal{R}}) = \mathcal{B}(\mathcal{A}) \cap \mathcal{E}_{\mathfrak{p}}$, (3.5) and (3.6) are special cases of (3.3) and (3.4).

(ii) $\rho^*_{\mathcal{R}}$ is by definition a $\sigma(\mathbf{ba}, \mathcal{L}^\infty)$-l.s.c. function, hence its lower level sets are closed in this topology. Let $c \in \mathbb{R}$ and suppose $\mu \in E_c$, thus *a fortiori* $\mu \in \mathbf{ba}_+$. (3.4) implies

$$\forall \mu \in E_c: \quad \mu(\Omega) \leq \rho^*_{\mathcal{R}}(\mu) + \rho_{\mathcal{R}}(1) \leq c + \rho_{\mathcal{R}}(1) < \infty.$$

Thus being a closed subset of a dilation of the closed unit ball of $\mathbf{ba}$, E_c is weakly* compact by virtue of the Banach–Alaoglu Theorem Aliprantis and Border (1999, Theorem 6.25).

(iii) Assume first a risk measure $\rho_{\mathcal{R}}$ associated to the acceptance set $\mathcal{A}$ is finite and continuous from above. $\rho_{\mathcal{R}}$ is in particular norm-continuous by Remark 2.4(v) and statements (i) and (ii) apply. Continuity from above implies $\rho_{\mathcal{R}}(k\mathbf{1}_{A_n}) \downarrow \rho_{\mathcal{R}}(0)$ for all $k > 0$ whenever $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}$ is a sequence of events decreasing to $\emptyset$. For $\mu \in \mathrm{dom}(\rho^*_{\mathcal{R}})$,

$$\begin{aligned} -\rho_{\mathcal{R}}(0) &\leq \sup_{k>0} k \lim_{n\to\infty} \mu(A_n) - \rho_{\mathcal{R}}(0) \\ &= \sup_{k>0} \lim_{n\to\infty} k\mu(A_n) - \rho_{\mathcal{R}}(k\mathbf{1}_{A_n}) \leq \rho^*_{\mathcal{R}}(\mu) < \infty. \end{aligned}$$

This can only hold if $\lim_n \mu(A_n) = 0$, i.e. $\mu \in \mathbf{ca}_+$. By (ii), this is equivalent to all level sets E_c of $\rho^*_{\mathcal{R}}$, $c \in \mathbb{R}$, being $\sigma(\mathbf{ca}, \mathcal{L}^\infty)$-compact.

For the converse, assume that $\mathrm{dom}(\rho^*_{\mathcal{R}}) \subseteq \mathbf{ca}_+$. Let $(X_n)_{n\in\mathbb{N}}$ be any sequence in $\mathcal{L}^\infty$ such that $X_n \downarrow X$ for some $X \in \mathcal{L}^\infty$. Let $Y \in \{X, X_1, X_2, \ldots\}$ and suppose that $\mu \in \mathrm{dom}(\rho^*_{\mathcal{R}})$ satisfies $\rho_{\mathcal{R}}(Y) - 1 \leq \int Y\,d\mu - \rho^*_{\mathcal{R}}(\mu)$. We can thus use the monotonicity of $\rho_{\mathcal{R}}$ and the positivity of μ to estimate

$$\begin{aligned} \rho^*_{\mathcal{R}}(\mu) &\leq \int Y\,d\mu - \rho_{\mathcal{R}}(Y) + 1 \leq \int X_1 d\mu - \rho_{\mathcal{R}}(X) + 1 \\ &\leq \frac{1}{2}\rho_{\mathcal{R}}(2X_1) + \frac{1}{2}\rho^*_{\mathcal{R}}(\mu) - \rho_{\mathcal{R}}(X) + 1. \end{aligned}$$

Rearranging this inequality yields that

$$\rho^*_{\mathcal{R}}(\mu) \leq 2 + \rho_{\mathcal{R}}(2X_1) - 2\rho_{\mathcal{R}}(X) =: c,$$

a bound which is independent of Y. Therefore, for all $Y \in \{X, X_1, X_2, \ldots\}$ it holds that

$$\rho_{\mathcal{R}}(Y) = \sup_{\mu\in E_c} \int Y\,d\mu - \rho^*_{\mathcal{R}}(\mu) = \max_{\mu\in E_c} \int Y\,d\mu - \rho^*_{\mathcal{R}}(\mu),$$

where in the last equality we used the $\sigma(\mathbf{ca}, \mathcal{L}^\infty)$-continuity of $\mu \mapsto \int Y\,d\mu - \rho^*_{\mathcal{R}}(\mu)$ and the compactness of E_c. For each $n \in \mathbb{N}$ choose $\mu_n \in E_c$ such that

$$\rho_{\mathcal{R}}(X_n) = \int X_n\,d\mu_n - \rho^*_{\mathcal{R}}(\mu_n).$$

Note that E_c is $\sigma(\mathbf{ca}, \mathbf{ca}^*)$-compact by virtue of Bogachev (2007, Theorem 4.7.25). The Eberlein–Smulian Theorem (see e.g. Aliprantis and Border, 1999, Theorem 6.38) now implies that we may select a $\sigma(\mathbf{ca}, \mathbf{ca}^*)$-convergent subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ with limit $\bar{\mu} \in E_c$. Choose a measure ν, for instance

$$\nu := \bar{\mu} + \sum_{k\in\mathbb{N}} \frac{1}{2^k}\mu_{n_k},$$

such that for all $\mu \in \mathcal{K} := \{\bar{\mu}, \mu_{n_1}, \mu_{n_2}, \ldots\}$ we have $\mu \ll \nu$. As $\nu(A) \leq \varepsilon$ implies $\mu(A) \leq \varepsilon$ for all $A \in \mathcal{F}$ and the set of Radon–Nikodym derivatives $\{\frac{d\mu}{d\nu} \mid \mu \in \mathcal{K}\}$ is $\|\cdot\|_{L^1_\nu}$-bounded as a subset of $L^1(\nu)$, we conclude that they form a ν-uniformly integrable family by Bogachev (2007, Proposition 4.5.3). Abbreviating $Z_k := \frac{d\mu_{n_k}}{d\nu}$, we obtain for all constants $L > 0$

$$\begin{aligned}
&\limsup_{k\to\infty}\left|\int X\,d\bar\mu - \int X_{n_k}\,d\mu_{n_k}\right| \\
&\le \limsup_{k\to\infty}\left|\int X\,d\bar\mu - \int X\,d\mu_{n_k}\right| + \left|\int (X - X_{n_k})\,d\mu_{n_k}\right| \\
&\le \limsup_{k\to\infty}\int_{\{Z_k\ge L\}}|X_1 - X|Z_k\,d\nu + \int_{\{Z_k<L\}}|X_{n_k} - X|L\,d\nu \\
&= \limsup_{k\to\infty}\int_{\{Z_k\ge L\}}|X_1 - X|Z_k\,d\nu = 0,
\end{aligned}$$

where we applied monotone convergence for the second but last equality and where the last equality follows from the uniform ν-integrability of the densities Z_k and the fact that $|X_1 - X|$ is bounded by a constant. Hence $\lim_k \int X_{n_k}\,d\mu_{n_k} = \int X\,d\bar\mu$, and from lower semicontinuity of $\rho^*_{\mathcal{R}}$, we arrive at

$$\begin{aligned}
\lim_{k\to\infty}\rho_{\mathcal{R}}(X_{n_k}) &= \limsup_{k\to\infty}\int X_{n_k}\,d\mu_{n_k} - \rho^*_{\mathcal{R}}(\mu_{n_k}) \\
&\le \int X\,d\bar\mu - \rho^*_{\mathcal{R}}(\bar\mu) \le \rho_{\mathcal{R}}(X).
\end{aligned}$$

$\rho_{\mathcal{R}}(X) \le \inf_{n\in\mathbb{N}}\rho_{\mathcal{R}}(X_n) = \lim_{k\to\infty}\rho_{\mathcal{R}}(X_{n_k})$ holds *a priori*, however. We infer $\rho_{\mathcal{R}}(X) = \lim_n\rho_{\mathcal{R}}(X_n)$.

Suppose $\mathcal{B}(\mathcal{A}) \subseteq \mathbf{ca}_+$. By (3.3) and statement (ii), the lower level sets of the dual conjugate of any finite risk measure $\rho_{\mathcal{R}}$ associated to $\mathcal{A}$ are $\sigma(\mathbf{ca}, \mathcal{L}^\infty)$-compact, and continuity from above follows from the equivalence proved just before.

(iv) Suppose that the risk measurement regime $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \mathfrak{p})$ is such that $\mathcal{S} = \mathbb{R}\cdot U$ for some $U \in \mathcal{L}^\infty_{++}$ and such that the resulting risk measure is finite. Assume for contradiction the existence of a $0 \ne \mu \in \mathcal{B}(\mathcal{A})$ such that $\int U\,d\mu = 0$. Recall that μ is necessarily positive, and let $k > \frac{\sigma_{\mathcal{A}}(\mu)}{\mu(\Omega)}$. For any $r \in \mathbb{R}$

$$\int (k - rU)d\mu = k\mu(\Omega) > \sigma_{\mathcal{A}}(\mu),$$

which would imply that $k - rU \notin \mathcal{A}$ for any $r \in \mathbb{R}$, and thus $\rho_{\mathcal{R}}(k) = \infty$ in CONTRADICTION to the finiteness of $\rho_{\mathcal{R}}$. As hence $\int U\,d\mu > 0$ has to hold for all $0 \ne \mu \in \mathcal{B}(\mathcal{A})$, we can identify with (3.3)

$$\mathrm{dom}(\rho^*_{\mathcal{R}}) = \mathcal{E}_{\mathfrak{p}} \cap \mathcal{B}(\mathcal{A}) = \left\{\frac{\mathfrak{p}(U)}{\int U\,d\mu}\mu \,\middle|\, 0 \ne \mu \in \mathcal{B}(\mathcal{A})\right\},$$

and by (3.4),

$$\rho_{\mathcal{R}}(X) = \sup_{0\ne\mu}\frac{\mathfrak{p}(U)}{\int U\,d\mu}\left(\int X\,d\mu - \sigma_{\mathcal{A}}(\mu)\right), \quad X \in \mathcal{L}^\infty,$$

is the minimal dual representation of $\rho_{\mathcal{R}}$. From this representation and (iii), we infer that $\rho_{\mathcal{R}}$ is continuous from above if and only if $\mathcal{B}(\mathcal{A}) \subseteq \mathbf{ca}$. □

Proof of Corollary 3.2. As $\rho_{\mathcal{R}}(X) \le 0$ for all $X \in \mathcal{A} + \ker(\mathfrak{p})$, finiteness and $\mathcal{S}$-additivity of $\rho_{\mathcal{R}}$ together with Farkas et al. (2015, Remark 6) show that $\mathcal{A} + \ker(\mathfrak{p})$ is proper and thus an acceptance set. Fix $U \in \mathcal{S} \cap \mathcal{L}^\infty_{++}$ and recall from Farkas et al. (2015, Lemma 3) the identity

$$\rho_{\mathcal{R}}(X) = \inf\{\mathfrak{p}(rU) \mid r \in \mathbb{R}, X - rU \in \mathcal{A} + \ker(\mathfrak{p})\}, \quad X \in \mathcal{L}^\infty.$$

A fortiori, $\mathcal{R}' := (\mathcal{A} + \ker(\mathfrak{p}), \mathbb{R}\cdot U, \mathfrak{p}|_{\mathbb{R}\cdot U})$ is a risk measurement regime and the associated risk measure $\rho_{\mathcal{R}'}$ is continuous from above if and only if $\rho_{\mathcal{R}}$ is continuous from above. As the identity $\mathcal{B}(\mathcal{A} + \ker(\mathfrak{p})) = \mathcal{B}(\mathcal{A}) \cap \ker(\mathfrak{p})^\perp$ is easily verified, the claimed equivalence follows from Proposition 3.1(iv). □

Appendix B. Proof of Theorem 3.10

The proof heavily relies on the following result.

Lemma B.1 (*Grothendieck; see Exercise 1, Chapter 5, part 4 of* Grothendieck, 1973). *A convex subset C of $L^\infty_{\mathbb{P}}$ is closed in the $\sigma(L^\infty_{\mathbb{P}}, \mathbf{ca}_{\mathbb{P}})$-topology if and only if for arbitrary $r > 0$ the set $C_r := \{X \in C \mid \|X\|_\infty \le r\}$ is closed with respect to convergence in probability, i.e. with respect to the metric*

$$d_{\mathbb{P}}(X, Y) := \mathbb{E}_{\mathbb{P}}[|X - Y| \wedge 1].$$

(i) For a set $\Gamma \subseteq L^\infty_{\mathbb{P}}$ we define $\Gamma^\diamond := \{\mu \in \mathbf{ca}_{\mathbb{P}} \mid \forall X \in \Gamma : \int X\,d\mu \le 1\}$, the one-sided polar of Γ. Moreover, $\Gamma^\diamond = (\mathrm{cl}_{\sigma(L^\infty_{\mathbb{P}},\mathbf{ca}_{\mathbb{P}})}(\Gamma))^\diamond$. Having this information at hand, one can easily identify[9]

$$\{c\mu \mid c \ge 0, \mu \in E_0\} = \left(\bigcup\{t\mathcal{A} + \ker(\mathfrak{p}) \mid t \ge 0\}\right)^\diamond = \mathbf{C}^\diamond.$$

From the Bipolar Theorem Aliprantis and Border (1999, Theorem 5.91) we deduce

$$\mathbf{C} = \{X \in L^\infty_{\mathbb{P}} \mid \forall\mu \in E_0 : \int X\,d\mu \le 0\}.$$

Hence $\mathbf{C}$ is an acceptance set. Consider the following risk measurement regime and its implied risk measure:

$$\check{\mathcal{R}} := (\mathbf{C}, \mathcal{S}, \mathfrak{p}), \quad \rho_{\check{\mathcal{R}}}(X) = \sup_{\mu\in E_0}\int X\,d\mu, \; X \in L^\infty_{\mathbb{P}}.$$

$\rho_{\check{\mathcal{R}}}$ is finite, coherent, and continuous from above by Lemma 3.6. Hence, by Lemma 3.8, $\mathcal{P} \ne \emptyset$ is equivalent to $\rho_{\check{\mathcal{R}}}$ being sensitive, i.e. $\mathbf{C}$ does not contain any element in $(L^\infty_{\mathbb{P}})_{++}$.

(ii) Suppose that $\mathbf{C} \cap (L^\infty_{\mathbb{P}})_{++}$ is non-empty. Using the monotonicity and conicity of $\mathbf{C}$, we can find some $B \in \mathcal{F}_+$ such that $\mathbf{1}_B \in \mathbf{C}$. Let us define the sets

$$\mathbf{D} := \{Y = d_{\mathbb{P}}\text{-}\lim_n t_n W_n \mid t_n \ge 0, W_n \in \mathcal{A} + \ker(\mathfrak{p})\},$$

$$\mathbf{D}_r = \{Y \in \mathbf{D} \mid \|Y\|_\infty \le r\}, \; r > 0.$$

$\mathbf{D}$ is a convex cone. It is straightforward to check that $\mathbf{D}_r$ is $d_{\mathbb{P}}$-closed. We apply Grothendieck's Lemma B.1 to infer that $\mathbf{D}$ is a $\sigma(L^\infty_{\mathbb{P}}, \mathbf{ca}_{\mathbb{P}})$-closed cone. Thus the inclusion $\mathbf{C} \subseteq \mathbf{D}$ holds and we must be able to find sequences $(X_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$, $(Z_n)_{n\in\mathbb{N}} \subseteq \ker(\mathfrak{p})$ and $(t_n)_{n\in\mathbb{N}} \subseteq (0, \infty)$ such that $\mathbf{1}_B = d_{\mathbb{P}}\text{-}\lim_n t_n(X_n + Z_n)$. Define $V_n := t_n(X_n + Z_n)$, $n \in \mathbb{N}$. Without loss of generality we can assume that $\|X_n + Z_n\|_\infty \le 1$. Otherwise, note that by normalisation $0 \in \mathbb{A} := \mathrm{cl}_{\|\cdot\|_\infty}(\mathcal{A} + \ker(\mathfrak{p}))$, and $\mathbf{C}$ is also the smallest $\sigma(L^\infty_{\mathbb{P}}, \mathbf{ca}_{\mathbb{P}})$-closed cone that contains $\mathbb{A}$; thus we can shift to

$$\frac{X_n + Z_n}{\|X_n + Z_n\|_\infty} \in \mathbb{A}, \quad \tilde t_n = \|X_n + Z_n\|_\infty t_n.$$

As $\mathbb{A}$ is convex, $(t_n)_{n\in\mathbb{N}}$ cannot be bounded. If there is some $M > 0$ such that $\sup_{n\in\mathbb{N}} t_n \le M$, we can define the sequence $(t_n(X_n + Z_n)/2M)_n \subseteq \mathbb{A}$ which converges in probability and with respect to $\sigma(L^\infty_{\mathbb{P}}, \mathbf{ca}_{\mathbb{P}})$ to $\mathbf{1}_B/2M \in \mathbb{A}$. As $\mathbb{A} = \{Y \mid \rho_{\mathcal{R}}(Y) \le 0\}$, we would

[9] The only difficult part is the following: Recall from Remark 2.4(iv) that $\rho_{\mathcal{R}}(Y) \le 0$ if and only if $Y \in \mathrm{cl}_{\|\cdot\|_\infty}(\mathcal{A} + \ker(\mathfrak{p}))$. Assume $\nu \in \mathbf{C}^\diamond$, then $\nu \in (\mathbf{ca}_{\mathbb{P}})_+$ by normalisation and monotonicity of $\mathcal{A}$. Also, $\mathbf{C}$ being a cone shows

$$\mathbf{C}^\diamond = \{\nu \in \mathbf{ca}_{\mathbb{P}} \mid \forall Y \in \mathbf{C} : \int Y\,d\nu \le 0\}.$$

Let $U \in \mathcal{S} \cap (L^\infty_{\mathbb{P}})_{++}$ and $X \in L^\infty_{\mathbb{P}}$. Since $\rho_{\mathcal{R}}(X - \frac{\rho_{\mathcal{R}}(X)}{\mathfrak{p}(U)}U) = 0$, we obtain that either $c := \int \frac{1}{\mathfrak{p}(U)}U\,d\nu = 0$, which implies $\nu(\Omega) = 0$ and $\nu = 0$, or

$$c\sup_{X\in L^\infty_{\mathbb{P}}}\left(\int X\,d\left(\frac{\nu}{c}\right) - \rho_{\mathcal{R}}(X)\right) \le 0 \implies \frac{\nu}{c} \in E_0.$$

obtain a CONTRADICTION to the sensitivity of $\rho_{\mathcal{R}}$ and can therefore assume $t_n \uparrow \infty$. $d_{\mathbb{P}}(V_n, \mathbf{1}_B) \to 0$ for $n \to \infty$ implies

$$V_n^- \xrightarrow{d_{\mathbb{P}}} 0, \quad V_n^+ \xrightarrow{d_{\mathbb{P}}} \mathbf{1}_B, \quad n \to \infty,$$

and this means that

$$\limsup_{n\to\infty} \mathbb{P}\left(\left\{V_n^+ \geq \frac{1}{2}\right\} \cap B\right) = \mathbb{P}(B).$$

The rule of equal speed of convergence is violated.
(iii) Let $(X_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$, $(Z_n)_{n\in\mathbb{N}} \subseteq \ker(\mathfrak{p})$ and $(t_n)_{n\in\mathbb{N}}$ be sequences violating the rule of equal speed of convergence such that the rescaled sequence $(V_n)_{n\in\mathbb{N}}$ is bounded in the $\|\cdot\|_\infty$-norm. Let B be a measurable set with positive probability such that

$$\limsup_{n\to\infty} \mathbb{P}(\{V_n \geq \varepsilon\} \cap B) = \mathbb{P}(B).$$

Let $\mu \approx \mathbb{P}$ be a finite measure with $\int Z\,d\mu = \mathfrak{p}(Z)$ for all $Z \in \mathcal{S}$, and let $\eta > 0$ be an arbitrary positive number. Note that due to the Dominated Convergence Theorem and the bounded vanishing in probability of V_n^-, we obtain for $n \to \infty$ the behavior $\lim_n \int V_n \mathbf{1}_{\{V_n \leq -\eta\}}\,d\mu \to 0$. For all n large enough such that $|\int V_n \mathbf{1}_{\{V_n \leq -\eta\}}\,d\mu| < \eta$ we can estimate

$$\int V_n\,d\mu \geq \varepsilon\mu(\{V_n \geq \varepsilon\} \cap B) - \eta\mu(V_n \in (-\eta, \varepsilon)) - \eta.$$

Thus for all $\eta > 0$ our assumption yields the estimate

$$\limsup_{n\to\infty} \int V_n\,d\mu \geq \varepsilon\mu(B) - \eta(\mu(\Omega) + 1).$$

Sending $\eta \downarrow 0$, we obtain from $\mu \approx \mathbb{P}$ that $\limsup_{n\to\infty} \int V_n\,d\mu \geq \varepsilon\mu(B) > 0$. After choosing n suitably we have found a vector in $\mathcal{A} + \ker(\mathfrak{p})$ such that $\int (X_n + Z_n)\,d\mu > 0$, hence $\mu \notin E_0$.

References

Aliprantis, C., Border, K., 1999. Infinite Dimensional Analysis–A Hitchhiker'S Guide, second ed.. Springer.
Aumann, R.J., Serrano, R., 2008. An economic index of riskiness. J. Polit. Econ. 116 (5), 810–836.
Bellini, F., Laeven, R., Rosazza Gianin, E., 2017. Robust return risk measures. Preprint.
Bogachev, V.I., 2007. Measure Theory, Volume I. Springer.
Cheridito, P., Li, T., 2009. Risk measures on orlicz hearts. Math. Finance 19 (2), 189–214.
Delbaen, F., 2000. Coherent risk measures. In: Cattedra Galileiana. In: Lecture Notes, Scuola Normale Superiore di Pisa.
Delbaen, F., 2002. Coherent risk measures on general probability spaces. In: Advances in Finance and Stochastics: Essays in Honor of Dieter Sondermann. Springer, pp. 1–38.
Drapeau, S., Kupper, M., 2013. Risk preferences and their robust representations. Math. Oper. Res. 38 (1), 28–62.
Ekeland, I., Témam, R., 1999. Convex Analysis and Variational Problems. SIAM, (Chapter I).
Fan, K., 1953. Minimax theorems. Proc. Natl. Acad. Sci. USA 39, 42–47.
Farkas, E.W., Koch Medina, P., Munari, C.-A., 2013. Beyond cash-additive risk measures: When changing the numéraire fails. Finance Stoch. 18 (1), 145–173.
Farkas, E.W., Koch Medina, P., Munari, C.-A., 2014. Capital requirements with defaultable securities. Insurance Math. Econom. 55, 58–67.
Farkas, E.W., Koch Medina, P., Munari, C.-A., 2015. Measuring risk with multiple eligible assets. Math. Financ. Econ. 9 (1), 3–27.
Filipovic, D., Svindland, G., 2012. The canonical model space for law-invariant convex risk measures is L^1. Math. Finance 22 (3), 585–589.
Föllmer, H., Schied, A., 2002. Convex measures of risk and trading constraints. Finance Stoch. 6 (4), 429–447.
Föllmer, H., Schied, A., 2011. Stochastic Finance: An Introduction in Discrete Time, third ed.. De Gruyter.
Frittelli, M., Rosazza Gianin, E., 2002. Putting order in risk measures. J. Bank. Finance 26 (7), 1473–1486.
Grothendieck, A., 1973. Topological Vector Spaces. Gordon and Breach.
Jaschke, S., Küchler, U., 2001. Coherent risk measures and good deal bounds. Finance Stoch. 5 (2), 181–200.
Jouini, E., Schachermayer, W., Touzi, N., 2008. Optimal risk sharing for law invariant monetary utility functions. Math. Finance 18 (2), 269–292.
Knight, F., 1921. Risk, Uncertainty and Profit. Hart, Schaffner & Marx; Houghton Mifflin Co, Boston, MA.
Kupper, M., Svindland, G., 2011. Dual representation of monotone convex functions on L^0. Proc. Amer. Math. Soc. 139 (11), 4073–4086.
Kusuoka, S., 2001. On law-invariant coherent risk measures. Adv. Math. Econ. 3, 83–95.
Lacker, D., 2015. Liquidity, risk measures, and concentration of measure. Preprint.
Laeven, R., Stadje, M., 2013. Entropy coherent and entropy convex measures of risk. Math. Oper. Res. 38, 265–293.
Munari, C.-A., 2015. Measuring risk beyond the cash-additive paradigm. ETH Zürich, (Ph.D. thesis).
Owari, K., 2014. Maximum lebesgue extensions of monotone convex functions. J. Funct. Anal. 266 (6), 3572–3611.
Rao, M.M., Ren, Z.D., 1991. Theory of Orlicz Spaces. Marcel Dekker, Inc.
Rockafellar, R.T., Uryasev, S., Zabarankin, M., 2006. Generalized deviations in risk analysis. Finance Stoch. 10 (1), 51–74.
Ruszczyński, A., Shapiro, A., 2006. Optimization of convex risk functions. Math. Oper. Res. 31 (3), 433–452.
Svindland, G., 2010a. Continuity properties of law-invariant (quasi-)convex risk functions on L^∞. Math. Financ. Econ. 3 (1), 39–43.
Svindland, G., 2010b. Subgradients of law-invariant convex risk measures on L^1. Statist. Decisions 27 (2), 169–199.

Chapter 3

Risk Sharing for Capital Requirements with Multidimensional Security Markets

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My own contribution: The article is joint work with my supervisor, Gregor Svindland. I developed most parts of the paper, with continual improvements resulting from discussions with him. In particular, the introduction, Assumptions 4.6–4.8 as well as Proposition 5.10 have been developed in joint discussions. I contributed substantially to all results and did most of the editorial work. Moreover, I have developed the results on agent systems, representative agent formulations, and optimal payoffs (Theorems 3.1 and 3.3, Proposition 3.4), continuity of optimal risk allocation maps (Theorems 4.9, 5.7, 5.9), and optimal portfolio splits (Section 6) independently.

RISK SHARING WITH MULTIDIMENSIONAL SECURITY MARKETS

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Abstract. We consider the risk sharing problem for capital requirements induced by capital adequacy tests and security markets. The agents involved in the sharing procedure may be heterogeneous in that they apply varying capital adequacy tests and have access to different security markets. We discuss conditions under which there exists a representative agent. Thereafter, we study two frameworks of capital adequacy more closely, polyhedral constraints and distribution based constraints. We prove existence of optimal risk allocations and equilibria within these frameworks and elaborate on their robustness.

Keywords: capital requirements, polyhedral acceptance sets, law-invariant acceptance sets, multidimensional security spaces, Pareto-optimal risk allocations, equilibria, robustness of optimal allocations.

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1. Introduction

In this paper we consider the risk sharing problem for capital requirements. Optimal capital and risk allocation among economic agents, or business units, has for decades been a predominant subject in the respective academic and industrial research areas. Measuring financial risks with capital requirements goes back to the seminal paper by Artzner et al. [6]. There, risk measures are *by definition* capital requirements determined by two primitives: the *acceptance set* and the *security market*.

The acceptance set, a subset of an ambient space of losses, corresponds to a *capital adequacy test*. A loss is deemed adequately capitalised if it belongs to the acceptance set, and inadequately capitalised otherwise. If a loss does not pass the capital adequacy test, the agent has to take prespecified remedial actions: she can raise capital in order to buy a portfolio of securities in the security market which, when combined with the loss profile in question, results in an adequately capitalised secured loss.

Suppose the security market only consists of one numéraire asset, liquidly traded at arbitrary quantities. After discounting, one obtains a so-called *monetary risk measure*, which is characterised by satisfying the *cash-additivity property*, that is $\rho(X + a) = \rho(X) + a$. Here, ρ denotes the monetary risk measure, X is a loss, and $a \in \mathbb{R}$ is a capital amount which is added to or withdrawn from the loss. Monetary risk measures have been widely studied, see Föllmer & Schied [27, Chap. 4] and the references therein. As observed in Farkas et al. [22, 23, 24] and Munari [35, Chap. 1], there are good reasons for revisiting the original approach to risk measures of Artzner et al. [6]:

(1) Typically, more than one asset is available in the security market. It is also less costly for the agent to invest in a portfolio of securities designed to secure a specific loss rather than restricting the remedial action to investing in a single asset independent of the loss profile.
(2) Even if securitisation is constrained to buying a single asset, discounting with this asset may be impossible because it is not a numéraire; cf. Farkas et al. [23]. Also, as risk is measured *after discounting*, the discounting procedure is implicitly assumed not to add additional risk, which is questionable in view of risk factors such as uncertain future interest rates. For a thorough discussion of this issue see El Karoui & Ravenelli [21]. Often, risk is determined purely in terms of the *distribution* of a risky position, a paradigm we discuss in detail below. Therefore, instability of this crucial *law-invariance property* of a risk measure under discounting is another objection. If the security is not riskless (i.e., is an amount of cash added or withdrawn), losses which originally were identically distributed may not share the same distribution any longer after discounting, while losses that originally display different laws may become identically distributed.
(3) Without discounting, if only a single asset is available in the security market, cash-additivity requires the security to be riskless, and it is questionable whether such a

security is realistically available, at least for longer time horizons. This is a particularly nagging issue in the insurance context.

In this paper we will follow the original ideas in [6] and study the risk sharing problem for risk measures induced by acceptance sets and possibly multidimensional security spaces. We consider a one-period market populated by a finite number $n \geq 2$ of agents who seek to secure losses occurring at a fixed future date, say tomorrow. We attribute to each agent $i \in \{1, \dots, n\}$ an ordered vector space $\mathcal{X}_i$ of *losses net of gains* she may incur, an acceptance set $\mathcal{A}_i \subseteq \mathcal{X}_i$ as capital adequacy test, and a security market consisting of a subspace $\mathcal{S}_i \subseteq \mathcal{X}_i$ of security portfolios as well as observable prices of these securities given by a linear functional $\mathfrak{p}_i : \mathcal{S}_i \to \mathbb{R}$. As the securities in $\mathcal{S}_i$ are deemed suited for hedging, the linearity assumptions on $\mathcal{S}_i$ and $\mathfrak{p}_i$ reflect that they are liquidly traded and their bid-ask spread is zero. The risk attitudes of agent i are fully captured by the resulting *risk measure*

$$\rho_i(X) := \inf\{\mathfrak{p}_i(Z) : Z \in \mathcal{S}_i, X - Z \in \mathcal{A}_i\}, \quad X \in \mathcal{X}_i, \tag{1.1}$$

that is the minimal capital required to secure X with securities in $\mathcal{S}_i$.

The problem we consider is how to reduce the aggregated risk in the system by means of redistribution. Formally, we assume that each individual space $\mathcal{X}_i$ of losses net of gains is a subspace of a larger ambient ordered vector space $\mathcal{X}$. This space models the losses the system in total incurs if $\mathcal{X} = \sum_{i=1}^n \mathcal{X}_i$, which we shall assume *a priori*. *Given such a market loss $X \in \mathcal{X}$, we need to solve the optimisation problem*

$$\sum_{i=1}^n \rho_i(X_i) \to \min \quad \text{subject to } X_i \in \mathcal{X}_i \text{ and } X_1 + \cdots + X_n = X. \tag{1.2}$$

A vector $\mathbf{X} = (X_1, \dots, X_n)$, a so-called allocation of X, which solves the optimisation problem and yields a finite optimal value is *Pareto-optimal*. However, this resembles centralised redistribution which attributes to each agent a certain portion of the aggregate loss in an overall optimal way without considering individual well-being. Redistribution by agents trading portions of losses at a certain price while adhering to individual rationality constraints leads to the notion of *equilibrium allocations* and *equilibrium prices*, a variant of the risk sharing problem above.

Special instances of this general problem have been extensively studied in the literature. Borch [10], Arrow [5] and Wilson [44] consider the problem for expected utilities. More recent are studies for convex monetary risk measures, starting with Barrieu & El Karoui [8] and Filipović & Kupper [25]. A key assumption which allows to prove existence of optimal risk sharing for convex monetary risk measures is *law-invariance*, i.e., the measured risk is the same for all losses which share the same distribution under a benchmark probability model, see Jouini et al. [31], Filipović & Svindland [26], Acciaio [1], and Acciaio & Svindland [2]. For a thorough discussion of the existing literature on risk sharing with monetary risk measures we refer to Embrechts et al. [19].

Another related line of literature is General Equilibrium Theory in economics. For a survey we refer to Mas Colell & Zame [34] and Aliprantis & Burkinshaw [4, Chap. 8]. A major difference though is that the agents we consider have risk preferences over a vector space of losses net of gains, whereas [34] considers agents with preferences over consumption sets which are bounded from below. Hence, our methods to tackle the problem are very different from the classical ones presented in [34]. More closely related are the contributions of Dana & Le Van [15] and Dana et al. [16], even though they consider different classes of preferences. In [16] consumption sets are unbounded from below like in our work, however the authors assume a finite-dimensional economy. Dana & Le Van [15] allow an infinite dimensional economy, but assume the consumption sets to be bounded from below. As the unbounded infinite-dimensional case is the most relevant in finance and insurance applications, we do not ask for any of those restrictions.

In the following we summarise our main contributions.

Representative agent formulation. We prove a representative agent formulation of the risk sharing problem: the behaviour of the interacting agents in the market is, under mild assumptions, captured by a *market capital requirement* of type (1.1), namely

$$\Lambda(X) = \inf\{\pi(Z) : Z \in \mathcal{M},\ X - Z \in \mathscr{A}_+\},$$

where $\Lambda(X)$ is the infimal level of aggregated risk realised by a redistribution of X as in (1.2), $\mathscr{A}_+$ is a market acceptance set, and $(\mathcal{M}, \pi)$ is a global security market. This allows deriving useful conditions ensuring the existence of optimal risk allocations.

Existence of optimal risk allocations in two case studies. Based on the representative agent formulation, we study two prominent cases, mostly characterised by the involved notions of acceptability, for which we prove that the risk sharing problem (1.2), including the quest for equilibria, admits solutions. In the first instance, individual losses are — in the widest sense — contingent on scenarios of the future state of the economy. A loss is deemed acceptable if certain capital thresholds are not exceeded under a fixed finite set of linear aggregation rules which may vary from agent to agent. The reader may think of a combination of finitely many valuation and stress test rules as studied in Carr et al. [11], see also [27, Sect. 4.8]. The resulting acceptance sets will thus be *polyhedral*.

In the second class of acceptance sets under consideration, whether or not a given loss is deemed adequately capitalised only depends on its distributional properties under a fixed reference probability measure, not on scenariowise considerations: acceptability is a statistical notion. More precisely, losses are modelled as random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and the respective individual acceptance sets $\mathcal{A}_i$, $i \in \{1, \ldots, n\}$, will be *law-invariant*: whether or not a loss X belongs to $\mathcal{A}_i$ only depends on its distribution under the probabilistic reference model $\mathbb{P}$. However, we will not assume that the security spaces $\mathcal{S}_i$ are law-invariant. Hence, securitisation depends on the potentially varying joint distribution of the loss and the security and is thus statewise rather than distributional. This both reflects the practitioner's reality and is mathematically interesting as the resulting capital requirements ρ_i are far from law-invariant. In fact, for non-trivial law-invariant $\mathcal{A}_i$, ρ_i is law-invariant only if the security space is trivial in the sense of being spanned by the cash asset, i.e., $\mathcal{S}_i = \mathbb{R}$. For such risk measures, the risk sharing problem has been solved, cf. [26, 31].

We utilise these results, but should like to emphasise that reducing the general problem for non-trivial $\mathcal{S}_i$ to the law-invariant cash-additive case is impossible.

Robustness of optimal allocations. As a third contribution, we carefully study continuity properties of the set-valued map assigning to an aggregated loss its optimal risk allocations in the mentioned polyhedral and law-invariant acceptability frameworks. These reflect different types of robustness under misspecification of the input. If the map is *upper hemicontinuous*, approximating a complex loss with simpler losses and calculating optimal risk allocations for these will yield an optimal risk allocation for the complex loss as a limit point. It is therefore a useful property from a numerical point of view. *Lower hemicontinuity*, on the other hand, guarantees that *any* given optimal risk allocation stays close to optimal under a slight perturbation of the underlying aggregated loss.

Existence of optimal portfolio splits. At last, we study optimal splitting problems in the spirit of Tsanakas [42] and Wang [43]. The question here is whether, under the presence of market frictions such as transaction costs, a financial institution can split an aggregated loss optimally by introducing subsidiaries subject to potentially varying regulatory regimes having access to potentially varying security markets. Applying our previous results, we will show that this problem admits solutions in our framework.

Structure of the paper. In Sect. 2 we rigorously introduce risk measurement in terms of capital requirements, agent systems, optimal allocations, and equilibria. Sect. 3 presents the representative agent formulation of the risk sharing problem and proves useful meta results. These are key to the discussion of risk sharing involving polyhedral acceptance sets in Sect. 4 and law-invariant acceptance sets in Sect. 5, as well as optimal portfolio splits in Sect. 6. For the convenience of the reader and better accessibility, Sects. 3–5 first present their main results and the discussion thereof. Ancillary results and the proofs of the main results follow in a separate subsection. Technical supplements are relegated to the appendix.

2. Agent systems and optimal allocations

2.1. **Risk measurement regimes.** In a first step of modelling, we assume that the attitude of individual agents towards risk is given by a *risk measurement regime* and corresponding *risk measure*.

Definition 2.1. Let $(\mathcal{X}, \preceq)$ be an ordered vector space, $\mathcal{X}_+$ be its positive cone, i.e., $\mathcal{X}_+ := \{X \in \mathcal{X} : 0 \preceq X\}$, and $\mathcal{X}_{++} := \mathcal{X}_+ \backslash \{0\}$.

- An *acceptance set* is a nonempty proper and convex subset $\mathcal{A}$ of $\mathcal{X}$ which is monotone, i.e. $\mathcal{A} - \mathcal{X}_+ \subseteq \mathcal{A}$.[1]
- A *security market* is a pair $(\mathcal{S}, \mathfrak{p})$ consisting of a finite-dimensional linear subspace $\mathcal{S} \subseteq \mathcal{X}$ and a positive linear functional $\mathfrak{p} : \mathcal{S} \to \mathbb{R}$ such that there is $U \in \mathcal{S} \cap \mathcal{X}_{++}$ with

[1] Here and in the following, given subsets A and B of a vector space $\mathcal{X}$, $A + B$ denotes their Minkowski sum $\{a + b : a \in A, b \in B\}$, and $A - B := A + (-B)$.

$\mathfrak{p}(U) = 1$. The elements $Z \in \mathcal{S}$ are called security portfolios or simply securities, and $\mathcal{S}$ is the *security space*, whereas $\mathfrak{p}$ is called *pricing functional*.

- A triple $\mathcal{R} := (\mathcal{A}, \mathcal{S}, \mathfrak{p})$ is a *risk measurement regime* if $\mathcal{A}$ is an acceptance set and $(\mathcal{S}, \mathfrak{p})$ is a security market such that the following no-arbitrage condition holds:

$$\forall X \in \mathcal{X} : \quad \sup\{\mathfrak{p}(Z) : Z \in \mathcal{S}, X + Z \in \mathcal{A}\} < \infty. \tag{2.1}$$

- The *risk measure* associated to a risk measurement regime $\mathcal{R}$ is the functional

$$\rho_{\mathcal{R}} : \mathcal{X} \to (-\infty, \infty], \quad X \mapsto \inf\{\mathfrak{p}(Z) : Z \in \mathcal{S}, X - Z \in \mathcal{A}\}. \tag{2.2}$$

Risk measure $\rho_{\mathcal{R}}$ is *normalised* if $\rho_{\mathcal{R}}(0) = 0$, or equivalently $\sup_{Z \in \mathcal{A} \cap \mathcal{S}} \mathfrak{p}(Z) = 0$. It is *lower semicontinuous* (l.s.c.) with respect to some vector space topology τ on $\mathcal{X}$ provided every lower level set $\{X \in \mathcal{X} : \rho_{\mathcal{R}}(X) \le c\}$, $c \in \mathbb{R}$, is τ-closed.

Immediate consequences of the definition of $\rho_{\mathcal{R}}$ are the following properties:

- $\rho_{\mathcal{R}}$ is a proper function[2] by (2.1) and $\rho_{\mathcal{R}}(Y) \le 0$ for any choice of $Y \in \mathcal{A}$. Moreover, it is convex, i.e., $\rho_{\mathcal{R}}(\lambda X + (1-\lambda)Y) \le \lambda \rho_{\mathcal{R}}(X) + (1-\lambda)\rho_{\mathcal{R}}(Y)$ holds for all choices of $\lambda \in [0,1]$ and $X, Y \in \mathcal{X}$;
- $\preceq$*-monotonicity*, i.e., $X \preceq Y$ implies $\rho_{\mathcal{R}}(X) \le \rho_{\mathcal{R}}(Y)$;
- $\mathcal{S}$*-additivity*, i.e., $\rho_{\mathcal{R}}(X + Z) = \rho_{\mathcal{R}}(X) + \mathfrak{p}(Z)$ for all $X \in \mathcal{X}$ and all $Z \in \mathcal{S}$.

Note that risk measures as in (2.2) evaluate the risk of *losses net of gains* $X \in \mathcal{X}$. The positive cone $\mathcal{X}_+$ corresponds to pure losses. Therefore, $\rho_{\mathcal{R}}$ is nondecreasing with respect to $\preceq$, not nonincreasing as in most of the literature on risk measures where the risk of *gains net of losses* is measured. The appropriate generalisation of convex risk measures in the usual monotonicity would therefore be the functional $\tilde{\rho}$ defined by $\tilde{\rho}(X) = \rho_{\mathcal{R}}(-X)$, $X \in \mathcal{X}$. In the same vein, the functional $\mathfrak{U}$ defined by $\mathfrak{U}(X) = -\rho_{\mathcal{R}}(-X)$, $X \in \mathcal{X}$, generalises monetary utility functions; cf. Delbaen [18].

In the security market, however, we consider the usual monotonicity, i.e., a security $Z^* \in \mathcal{S}$ is better than $Z \in \mathcal{S}$ if $Z \preceq Z^*$. This also explains positivity of the pricing functional $\mathfrak{p} : \mathcal{S} \to \mathbb{R}$. Combining these two viewpoints, the impact of a security $Z \in \mathcal{S}$ on a loss profile $X \in \mathcal{S}$ is given by $X - Z$, and $\rho_{\mathcal{R}}(X)$ is the infimal price that has to be paid for a security Z in the security market with loss profile $-Z$ in order to reduce the risk of X to an acceptable level. The no-arbitrage condition (2.1) means that one cannot short arbitrarily valuable securities and stay acceptable.

There is a close connection between capital requirements defined by (2.2) and superhedging. Given a risk measurement regime $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \mathfrak{p})$ on an ordered vector space $(\mathcal{X}, \preceq)$, let $\ker(\mathfrak{p}) := \{N \in \mathcal{S} : \mathfrak{p}(N) = 0\}$ denote the kernel of the pricing functional, i.e. the set of fully leveraged security portfolios available at zero cost. Moreover, fix an arbitrary $U \in \mathcal{S} \cap \mathcal{X}_+$ whose price is given by $\mathfrak{p}(U) = 1$. Each $Z \in \mathcal{S}$ can be written as $Z = \mathfrak{p}(Z)U + (Z - \mathfrak{p}(Z)U)$, and $Z - \mathfrak{p}(Z)U \in \ker(\mathfrak{p})$. Hence, $X \in \mathcal{X}$ and $Z \in \mathcal{S}$ satisfy $X - Z \in \mathcal{A}$ if and only if for

[2] Given a nonempty set M, a function $f : M \to [-\infty, \infty]$ is proper if $f^{-1}(\{-\infty\}) = \emptyset$ and $f \not\equiv \infty$.

$r := \mathfrak{p}(Z) \in \mathbb{R}$ we can find $N \in \ker(\mathfrak{p})$ such that

$$rU + N + (-X) \in -\mathcal{A}.$$

The risk $\rho_{\mathcal{R}}(X)$ may thus be expressed as

$$\begin{aligned}\rho_{\mathcal{R}}(X) &= \inf\{\mathfrak{p}(Z) : \, Z \in \mathcal{S}, X - Z \in \mathcal{A}\} \\ &= \inf\{r \in \mathbb{R} : \, \exists\, N \in \ker(\mathfrak{p}) \text{ such that } N + rU + (-X) \in -\mathcal{A})\},\end{aligned}$$

The set $-\mathcal{A}$ is the set of acceptable *gains net of losses*, and $-X$ is the payoff associated to the loss profile X. The elements in $\ker(\mathfrak{p})$ are *zero cost investment opportunities*. If we conservatively choose the acceptance set $\mathcal{A} = -\mathcal{X}_+$,

$$\rho_{\mathcal{R}}(X) = \inf\{r \in \mathbb{R} : \, \exists\, N \in \ker(\mathfrak{p}) \text{ s.t. } N + rU + (-X) \succeq 0\},$$

that is we recover by $\rho_{\mathcal{R}}(X)$ the superhedging price of the payoff $-X$. A general risk measurement regime thus leads to a superhedging functional involving the relaxed notion of superhedging $N + rU + (-X) \in -\mathcal{A}$. In the terminology of superhedging theory, $\rho_{\mathcal{R}}(X)$ is the infimal amount of cash that needs to be invested in the security U such that X can be superhedged when combined with a suitable zero cost trade in the (security) market. Such relaxed superhedging functionals have been recently studied by, e.g., Cheridito et al. [13]. The separation between U and $\ker(\mathfrak{p})$ introduced above will be useful throughout the paper. Let us give a classical example for a risk measurement regime:

Example 2.2. Consider risky future monetary losses net of gains modelled by (equivalence classes) of integrable random variables on an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In other words, $\mathcal{X} := L^1 := L^1(\Omega, \mathcal{F}, \mathbb{P})$. A classical capital adequacy test is given by the *Average Value at Risk* at some level $\beta \in (0, 1)$; that is, $X \in L^1$ belongs to the acceptance set $\mathcal{A}$ and thus passes the capital adequacy test if and only if

$$\sup_{Q \in \mathcal{Q}} \mathbb{E}[QX] \leq 0,$$

where $\mathcal{Q}$ is the set of all densities $Q \in L^\infty_+$ such that $\mathbb{E}[Q] = 1$ and $Q \leq \frac{1}{1-\beta}$ $\mathbb{P}$-almost surely. For the sake of simplicity we will assume that interest rates are trivial. The security market may consist of a defaultable bond, i.e. $\mathbf{1}_A$ for some $A \in \mathcal{F}$ with $0 < \mathbb{P}(A) < 1$, a finite number of assets $\mathbb{X}$ and a finite number of call and put options on these assets. For the latter, we assume for each $X \in \mathbb{X}$ a set of strike prices $\mathbb{K}^X$ to be given, and each of the calls $(X - k)^+$, and puts $(k - X)^-$, $k \in \mathbb{K}^X$, lies in $\mathcal{S}$. Suppose now that $Q^* \in L^\infty_+$ satisfies

$$0 < \delta \leq Q^* \leq \tfrac{1-\delta}{1-\beta} + \delta$$

for some $\delta \in (0, 1)$ and $\mathbb{E}[Q^*] = 1$. We will then see in Sect. 5 that if securities in $\mathcal{S}$ are priced by $\mathfrak{p}(Z) = \mathbb{E}[Q^* Z]$, then $(\mathcal{A}, \mathcal{S}, \mathfrak{p})$ is a risk measurement regime.

We also refer to [24] for more examples of risk measurement regimes in our sense.

2.2. **Agent systems.** In order to introduce the risk sharing problem in precise terms, a notion of the interplay of the individual agents and their respective capital requirements is required; for terminology concerning ordered vector spaces, we refer to [3, Chaps. 8–9]. We consider an abstract one-period market which incurs aggregated losses net of gains modelled by a Riesz space $(\mathcal{X}, \preceq)$. The market comprises $n \geq 2$ agents, and throughout the paper we identify each individual agent with a natural number i in the set $\{1, \dots, n\}$, which we shall denote by $[n]$ for the sake of brevity. The agents might have rather heterogeneous assessments of risks. This is firstly reflected by the assumption that each agent operates on an *(order) ideal*[3] $\mathcal{X}_i \subseteq \mathcal{X}$, $i \in [n]$, which may be a proper subset of $\mathcal{X}$. Without loss of generality we shall impose $\mathcal{X} = \mathcal{X}_1 + \dots + \mathcal{X}_n$. Within each ideal, and thus for each agent, adequately capitalised losses are encoded by an acceptance set $\mathcal{A}_i \subseteq \mathcal{X}_i$. Agent $i \in [n]$ is allowed to secure losses she may incur with securities from a security market $(\mathcal{S}_i, \mathfrak{p}_i)$, where $\mathcal{S}_i \subsetneq \mathcal{X}_i$. We shall impose that each $\mathcal{R}_i := (\mathcal{A}_i, \mathcal{S}_i, \mathfrak{p}_i)$ is a risk measurement regime on $(\mathcal{X}_i, \preceq|_{\mathcal{X}_i \times \mathcal{X}_i})$, $i \in [n]$. In sum, the individual risk assessments are fully captured by the n-tuple of risk measurement regimes $(\mathcal{R}_1, \dots, \mathcal{R}_n)$.

Definition 2.3. An n-tuple $(\mathcal{R}_1, \dots, \mathcal{R}_n)$, where, for each $i \in [n]$, $\mathcal{R}_i$ is a risk measurement regime on $\mathcal{X}_i$, is called an *agent system* if

($\star$) For all $i, j \in [n]$, the pricing functionals $\mathfrak{p}_i$ and $\mathfrak{p}_j$ agree on $\mathcal{S}_i \cap \mathcal{S}_j$. Moreover, if we set $i \sim j$ if $i \neq j$ and $\mathfrak{p}_i$ is non-trivial on $\mathcal{S}_i \cap \mathcal{S}_j$, the resulting graph
$$G = ([n], \{\{i,j\} \subseteq [n] : i \sim j\})$$
is connected.[4]

Axiom ($\star$) clarifies the nature of the interaction of the involved agents: prices for securities accepted by more than one agent have to agree, and any two agents may interact and exchange securities by potentially invoking other agents as intermediaries. Throughout this paper we will assume that the agents $[n]$ form an agent system. Such a situation is not too far-fetched:

Definition 2.4. The space of *jointly accepted securities* is $\check{\mathcal{S}} := \bigcap_{i=1}^n \mathcal{S}_i$. The *global security space* is $\mathcal{M} := \mathcal{S}_1 + \dots + \mathcal{S}_n$.

If, besides agreement of prices, $\check{\mathcal{S}} \neq \{0\}$ and $\mathfrak{p}_i|_{\check{\mathcal{S}}} \neq 0$ for some and thus all $i \in [n]$, then assumption ($\star$) is met. The resulting graph is the complete graph on n vertices. Moreover, if all agents operate on one and the same space $\mathcal{X}_i = \mathcal{X}$, $i \in [n]$, and the available security markets are identical and given by $\mathcal{S}_i = \mathbb{R} \cdot U$, $i \in [n]$, for some $U \in \mathcal{X}_{++}$ and $\mathfrak{p}_i(rU) = r$, $r \in \mathbb{R}$, $(\mathcal{R}_1, \dots, \mathcal{R}_n)$ is an agent system. If we further specify $\mathcal{X}$ to be a sufficiently rich space of random variables and $U = \mathbf{1}$ is the constant random variable with value 1, the results for

[3] An ideal $\mathcal{Y}$ of a Riesz space $(\mathcal{X}, \preceq)$ is a subspace in which the inclusion $\{Z \in \mathcal{X} : |Z| \preceq |Y|\} \subseteq \mathcal{Y}$ holds for all $Y \in \mathcal{Y}$.

[4] That is, between any two vertices $i, j \in [n]$, $i \neq j$, we can find a connecting path over edges of the graph, meaning that either $i \sim j$ or we can find $i_1, \dots, i_m \in [n]$ for a suitable $m \in \mathbb{N}$ such that $i \sim i_1$, $i_1 \sim i_2$, ..., $i_{m-1} \sim i_m$, and $i_m \sim j$. This will for instance be needed in the proof of Proposition 3.6.

risk sharing with convex monetary risk measures can be embedded in our setting of agent systems; cf. [1, 2, 26, 31].
In the following we write ρ_i instead of $\rho_{\mathcal{R}_i}$ for the sake of brevity. Aggregated losses in $\mathcal{X}$ will be denoted by X, Y or W, securities by Z, U or N throughout the paper.

2.3. **The risk sharing problem and its solutions.** In order to introduce the risk sharing associated to an agent system $(\mathcal{R}_1, \dots, \mathcal{R}_n)$, we need the notion of attainable and security allocations:

Definition 2.5. A vector $\mathbf{X} = (X_1, \dots, X_n) \in \prod_{i=1}^n \mathcal{X}_i$ is an *attainable allocation* of an aggregated loss $W \in \mathcal{X}$ if $W = X_1 + \cdots + X_n$. We denote the set of all attainable allocations of W by $\mathbb{A}_W$.
Given a global security $Z \in \mathcal{M}$, we denote by $\mathbb{A}_Z^s := \mathbb{A}_Z \cap \prod_{i=1}^n \mathcal{S}_i$ the set of *security allocations* of Z.

Given a set $S \neq \emptyset$ and a function $f : S \to [-\infty, \infty]$, we set

$$\mathrm{dom}(f) := \{s \in S : \, f(s) < \infty\}$$

to be the *effective domain* of f. We will also abbreviate its lower level sets by $\mathcal{L}_c(f) := \{s \in S : \, f(s) \le c\}$, $c \in \mathbb{R}$.
We are now prepared to introduce the risk sharing problem. Its objective is to minimise the aggregated risk within the system. The allowed remedial action is reallocating an aggregated loss $W \in \mathcal{X}$ among the agents involved; so we study

$$\sum_{i=1}^n \rho_i(X_i) \to \min \quad \text{subject to } \mathbf{X} \in \mathbb{A}_W. \tag{2.3}$$

The optimal value in (2.3) is less than $+\infty$ if and only if $W \in \sum_{i=1}^n \mathrm{dom}(\rho_i)$. It is furthermore well known that (2.3) is closely related to certain notions of economically optimal allocations which we define in the following.

Definition 2.6. Let $(\mathcal{R}_1, \dots, \mathcal{R}_n)$ be an agent system on an ordered vector space $(\mathcal{X}, \preceq)$, let $W \in \mathcal{X}$ be an aggregated loss, and let $\mathbf{W} \in \prod_{i=1}^n \mathcal{X}_i$ be a vector of initial loss endowments.

(1) An attainable allocation $\mathbf{X} \in \mathbb{A}_W$ is *Pareto-optimal* if $\rho_i(X_i) < \infty$, $i \in [n]$, and for any $\mathbf{Y} \in \mathbb{A}_W$ with the property $\rho_i(Y_i) \le \rho_i(X_i)$, $i \in [n]$, in fact $\rho_i(X_i) = \rho_i(Y_i)$ has to hold for all $i \in [n]$.
(2) Suppose $(\mathcal{X}, \preceq, \tau)$ is a *topological Riesz space*, i.e. $\mathcal{X}$ carries a *vector space topology* τ. A tuple $(\mathbf{X}, \phi)$ is an *equilibrium* of $\mathbf{W}$ if
   - $\mathbf{X} \in \mathbb{A}_{W_1 + \cdots + W_n}$,
   - $\phi \in \mathcal{X}^*$ is positive with $\phi|_{\mathcal{S}_i} = \mathfrak{p}_i$, $i \in [n]$,
   - the budget constraints $\phi(-X_i) \le \phi(-W_i)$, $i \in [n]$, hold,[5]
   - and $\rho_i(X_i) = \inf\{\rho_i(Y) : \, Y \in \mathcal{X}_i, \; \phi(-Y) \le \phi(-W_i)\}$ for all $i \in [n]$.

[5] Note that the minus sign in the budget constraints is due to the fact that the elements in $\mathcal{X}_i$ model losses, whereas ϕ prices payoffs.

In that case, $\mathbf{X}$ is called *equilibrium allocation* and ϕ *equilibrium price.*

Now that all pieces are in place, we close this section by commenting on the static nature of the model introduced here. Indeed, we study risk sharing in a generalised one-period framework. The very general notion of market spaces underlying our definitions provide the possibility that loss profiles capture dynamics themselves, being, for instance, trajectories of the evolution of the value of a good over time. However, extending capital requirements to a dynamic multi-period framework poses some difficulties. For instance, in such an extension finite-dimensionality of security spaces might be lost. As we will see, the finite-dimensionality of the security spaces is crucial for important results in this paper. Generalising capital requirements to a multi-period framework will therefore be an interesting topic for future research.

3. Infimal convolutions and the representative agent

This section comprises the formal mathematical treatment of the risk sharing problem on ideals of a Riesz space as introduced in Sect. 2. We shall link risk sharing to the infimal convolution of the individual risk measures, prove its representation as a capital requirement for the market, i.e., for a *representative agent*, and find powerful sufficient conditions for the existence of optimal payoffs, Pareto-optimal allocations, and equilibria. Similar approaches have been undertaken by, e.g., [1, 2, 8, 25, 26, 31, 32].
Beforehand, however, we need to introduce further axioms that an agent system may satisfy in addition to $(\star)$. We shall refer to them at various stages of the paper, they are however not assumed to be met throughout. For $n \geq 2$ let $(\mathcal{R}_1, \ldots, \mathcal{R}_n)$ be an agent system.

(A1) *No security arbitrage.* For some $j \in [n]$ it holds that

$$\Big(\textstyle\sum_{i\neq j} \ker(\mathfrak{p}_i)\Big) \cap \mathcal{S}_j \subseteq \ker(\mathfrak{p}_j);$$

(A2) *Non-redundance of the joint security market.* There is $Z \in \check{\mathcal{S}}$ and $\mathbf{Z} \in \mathbb{A}_Z^s$ such that $\sum_{i=1}^n \mathfrak{p}_i(Z_i) \neq 0$.

(A3) *Supportability.* The underlying space $\mathcal{X}$ carries a locally convex Hausdorff topology τ with dual space $\mathcal{X}^*$. Moreover, there is some $\phi_0 \in \mathcal{X}_+^*$ and a constant $\gamma \in \mathbb{R}$ such that

  (i) for all $\mathbf{Z} \in \prod_{i=1}^n \mathcal{S}_i$ with $\sum_{i=1}^n \mathfrak{p}_i(Z_i) = 0$ we have

$$\phi_0(Z_1 + \cdots + Z_n) = 0,$$

  and for some $\tilde{\mathbf{Z}} \in \prod_{i=1}^n \mathcal{S}_i$ with $\sum_{i=1}^n \mathfrak{p}_i(\tilde{Z}_i) \neq 0$ we have

$$\phi_0(\tilde{Z}_1 + \cdots + \tilde{Z}_n) \neq 0;$$

  (ii) for all $\mathbf{Y} \in \prod_{i=1}^n \mathcal{A}_i$ we have $\phi_0(Y_1 + \cdots + Y_n) \leq \gamma$.

(A4) *Infinite supportability.* $(\mathcal{R}_i)_{i\in\mathbb{N}}$ is a sequence of risk measurement regimes on a common locally convex Hausdorff topological Riesz space $(\mathcal{X}, \preceq, \tau)$ such that $(\mathcal{R}_1, \ldots, \mathcal{R}_n)$ satisfies $(\star)$ for all $n \in \mathbb{N}$ and such that there is some $\phi_0 \in \mathcal{X}^*$ with

$$\textstyle\sum_{i\in\mathbb{N}} \sup_{Y\in\mathcal{A}_i} \phi_0(Y) < \infty \text{ and } \phi_0|_{\mathcal{S}_i} = \mathfrak{p}_i,\ i \in \mathbb{N}.$$

Condition (A1) is violated if each agent would be able to obtain arbitrarily valuable securitisation from the other agents, who can provide it at zero individual cost. That would reveal a mismatch of security markets leading to hypothetical infinite wealth for all agents. Non-redundance of the joint security market is in particular satisfied if there is $Z \in \check{\mathcal{S}}$ such that $\mathfrak{p}_i(Z) \neq 0$ *for some* $i \in [n]$ and thus *for all* $i \in [n]$ by the defining property ($\star$) of an agent system and the agreement of prices. Hence, under (A2) there is a jointly accepted security valuable for the market. Regarding condition (A3), think of ϕ_0 as a pricing functional. (i) is a consistency requirement between ϕ_0 and the individual prices $\mathfrak{p}_i$. (ii) reads as the impossibility to decompose a loss X acceptably for all agents if X is sufficiently poor, that is the value $\phi_0(-X)$ of the corresponding payoff $-X$ under ϕ_0 is less than a certain level $-\gamma$. Condition (A4) is a strengthening of (A3) for all finite subsystems of $(\mathcal{R}_i)_{i\in\mathbb{N}}$.

3.1. **Main results.** According to Proposition 3.6 below, an allocation $\mathbf{X} \in \mathbb{A}_X$ of $X \in \mathcal{X}$ is Pareto-optimal if and only if the *risk sharing functional*

$$\Lambda : \mathcal{X} \to [-\infty, \infty], \quad Y \mapsto \inf_{\mathbf{Y}\in\mathbb{A}_Y} \sum_{i=1}^{n} \rho_i(Y_i),$$

is exact at X, that is, $\Lambda(X) = \sum_{i=1}^{n} \rho_i(X_i) \in \mathbb{R}$. Λ corresponds to the so-called *infimal convolution* of the risk measures $\rho_1, \dots, \rho_n$, and thus inherits properties like $\preceq$-monotonicity and convexity. We refer to Appendix A.2, in particular Lemma A.3, for a brief summary of these facts.

Our next result implies that, if proper, Λ is again a risk measure of type (2.2): the shared risk level is the minimal price the market has to pay for a cumulated security that ensures market acceptability. Thus, market behaviour may be seen as the behaviour of a *representative agent* operating on $\mathcal{X}$. Recall from Definition 2.5 that $\mathbb{A}^s_Z$ denotes the set of security allocations of $Z \in \mathcal{M}$.

Theorem 3.1. *Define* $\pi(Z) := \inf_{\mathbf{Z}\in\mathbb{A}^s_Z} \sum_{i=1}^{n} \mathfrak{p}_i(Z_i)$, $Z \in \mathcal{M}$.

(1) *For any* $Z \in \mathcal{M}$ *and arbitrary* $\mathbf{Z} \in \mathbb{A}^s_Z$, $\pi(Z)$ *may be represented as*

$$\pi(Z) = \sum_{i=1}^{n} \mathfrak{p}_i(Z_i) + \pi(0).$$

Either $\pi(0) = 0$ *or* $\pi(0) = -\infty$. $\pi(0) = 0$ *is equivalent to* (A1)*, and in that case* π *is real-valued, linear, and satisfies* $\pi|_{\mathcal{S}_i} = \mathfrak{p}_i$, $i \in [n]$*. Otherwise* $\pi \equiv -\infty$.

(2) Λ *can be represented as*

$$\Lambda(X) = \inf\left\{\pi(Z) : Z \in \mathcal{M},\ X - Z \in \mathscr{B}\right\}, \quad X \in \mathcal{X},$$

for any monotone and convex set $\mathscr{B} \subseteq \mathcal{X}$ *satisfying* $\mathscr{A}_+ \subseteq \mathscr{B} \subseteq \mathcal{L}_0(\Lambda)$*. Here,*

$$\mathscr{A}_+ := \sum_{i=1}^{n} \mathcal{A}_i$$

denotes the market acceptance set.

(3) *If* (A1) *and* (A3) *hold,* Λ *is proper.*
(4) *If* Λ *is proper, then* (A1) *holds, i.e.,* $\pi(0) = 0$*, and* π *is positive. In that case,* $(\mathscr{A}_+, \mathcal{M}, \pi)$ *is a risk measurement regime on* $\mathcal{X}$ *and* Λ *is the associated risk measure.*

In the situation of Theorem 3.1(4), the behaviour of the representative agent is given by the risk measurement regime $(\mathscr{A}_+, \mathcal{M}, \pi)$. The risk sharing functional is the *market capital requirement* associated to the market acceptance set $\mathscr{A}_+$ and the global security market $(\mathcal{M}, \pi)$.
The preceding theorem also offers a more geometric perspective on the assumption (A3). Suppose the agent system operates on a locally convex Hausdorff topological Riesz space $(\mathcal{X}, \preceq, \tau)$ and satisfies (A1). Moreover, assume we can find a security $Z^* \in \mathcal{M}$ such that

$$Z^* \notin \mathrm{cl}_\tau(\mathscr{A}_+ + \ker(\pi)), \tag{3.1}$$

where here and in the the following $\mathrm{cl}_\tau(\cdot)$ denotes the closure of a set with respect to the topology τ. Then (A3) means Z^* is a security which comes at a true cost for the market; it can be strictly separated from $\mathscr{A}_+ + \ker(\pi)$ using a linear functional $\phi_0 \in \mathcal{X}^*_+$, and this functional is exactly as described in (A3).
We now turn our attention to Pareto-optimal allocations of a loss $W \in \mathcal{X}$. We will see that their existence is closely related to the existence of a market security $Z^W \in \mathcal{M}$ which renders market acceptability $W - Z^W \in \mathscr{A}_+$ at the minimal price $\pi(Z^W) = \Lambda(W)$. Such *optimal payoffs* have recently been studied by Baes et al. [7].

Definition 3.2. $W \in \mathcal{X}$ admits an *optimal payoff* $Z^W \in \mathcal{M}$ if $W - Z^W \in \mathscr{A}_+$ and $\pi(Z^W) = \Lambda(W)$.

Theorem 3.3. *Suppose that* Λ *is proper. If* $X \in \mathcal{X}$ *admits an optimal payoff* $Z^X \in \mathcal{M}$*, then* $X \in \mathrm{dom}(\Lambda)$ *and* Λ *is exact at* X*. In particular, for any* $Y_i \in \mathcal{A}_i$*,* $i \in [n]$*, such that* $\sum_{i=1}^n Y_i = X - Z^X$*, and any* $\mathbf{Z} \in \mathbb{A}^s_{Z^X}$*, the allocation* $(Y_i + Z_i)_{i\in[n]} \in \mathbb{A}_X$ *is Pareto-optimal. If moreover* $\mathcal{L}_0(\rho_i) = \mathcal{A}_i + \ker(\mathfrak{p}_i)$*,* $i \in [n]$*, then* Λ *is exact at* $X \in \mathrm{dom}(\Lambda)$ *if, and only if* X *admits an optimal payoff.*

In a topological setting, the existence of optimal payoffs is intimately connected to the Minkowski sum $\mathscr{A}_+ + \ker(\pi)$ being closed:

Proposition 3.4. *Suppose* $(\mathcal{X}, \preceq, \tau)$ *is a topological Riesz space and* Λ *is proper. Then the following are equivalent:*
(1) $\mathscr{A}_+ + \ker(\pi)$ *is closed.*
(2) Λ *is l.s.c. and every* $X \in \mathrm{dom}(\Lambda)$ *admits an optimal payoff.*

Proposition 3.4 is related to [7, Proposition 4.1]. Together with Theorem 3.3, it is a powerful sufficient condition for the existence of Pareto optima which we shall apply in Sects. 4 and 5. The only non-trivial steps will be to verify the properness of Λ and closedness of $\mathscr{A}_+ + \ker(\pi)$.
We proceed with the discussion of equilibria in the very general case when market losses are modelled by a *Fréchet lattice* $(\mathcal{X}, \preceq, \tau)$. As this notion is ambiguous in the literature,

we emphasise that a Fréchet lattice is a locally convex-solid[6] topological Riesz space whose topology is completely metrisable.
In particular, *Banach lattices* are Fréchet lattices. As a more general example, one may consider the Wiener space $C([0,\infty))$ of all continuous functions on the non-negative half-line with the pointwise oder $\leq$ and the topology τ_D arising from the metric

$$D(f,g) := \sum_{k=1}^{\infty} 2^{-k} \frac{\max_{0\leq r\leq k} |f(r)-g(r)|}{1+\max_{0\leq r\leq k} |f(r)-g(r)|}, \quad f,g \in C([0,\infty)).$$

Clearly, $(C([0,\infty)), \leq, \tau_D)$ is not a Banach lattice, but a Fréchet lattice. Its choice as model space is justified if the primitives in question are continuous trajectories of, e.g., the net value of some good over time.
Recall the definition of the jointly accepted securities, $\check{\mathcal{S}} := \bigcap_{i=1}^n \mathcal{S}_i$. Moreover, we set here and in the following $\operatorname{int}\operatorname{dom}(\Lambda)$ to be the τ-interior of the effective domain of the risk sharing functional Λ.

Theorem 3.5. *Suppose $\mathcal{X}$ is a Fréchet lattice and that Λ is l.s.c. and proper. Moreover, let* (A2) *be satisfied, i.e., there is a $\tilde{Z} \in \check{\mathcal{S}}$ with $\pi(\tilde{Z}) \neq 0$. If a vector of loss endowments $\mathbf{W} \in \prod_{i=1}^n \mathcal{X}_i$ satisfies $W := W_1 + \cdots + W_n \in \operatorname{int}\operatorname{dom}(\Lambda)$ and there exists a Pareto-optimal allocation of W, there is an equilibrium $(\mathbf{X}, \phi)$ of $\mathbf{W}$.*

3.2. **Ancillary results and proofs.** Our first result links Pareto optima, equilibria, and solutions to the risk sharing problem (2.3). Proposition 3.6(2) is indeed the first fundamental theorem of welfare economics adapted to our setting.

Proposition 3.6. *Let $(\mathcal{R}_1, \ldots, \mathcal{R}_n)$ be an agent system on an ordered vector space $(\mathcal{X}, \preceq)$, let $W \in \mathcal{X}$ be an aggregated loss, and let $\mathbf{W} \in \prod_{i=1}^n \mathcal{X}_i$ be a vector of initial loss endowments. The following statements hold true:*

(1) *If $W \in \sum_{i=1}^n \operatorname{dom}(\rho_i)$, then $\mathbf{X} \in \mathbb{A}_W$ is a Pareto-optimal attainable allocation of W if and only if*
$$\sum_{i=1}^n \rho_i(X_i) = \inf_{\mathbf{Y}\in\mathbb{A}_W} \sum_{i=1}^n \rho_i(Y_i). \tag{3.2}$$
(2) *If $(\mathcal{X}, \preceq, \tau)$ is a topological Riesz space and $\mathbf{W}$ satisfies $W_1 + \cdots + W_n \in \sum_{i=1}^n \operatorname{dom}(\rho_i)$, any equilibrium allocation of $\mathbf{W}$ is Pareto-optimal.*

The proof requires the following well-known characterisation of Pareto optima; see, e.g., [37, Proposition 3.2].

Lemma 3.7. *Let $W \in \sum_{i=1}^n \operatorname{dom}(\rho_i)$. If $\mathbf{X}$ is a Pareto-optimal attainable allocation of W, there are so-called Negishi weights $\lambda_i \geq 0$, $i \in [n]$, not all equal to zero, such that*
$$\sum_{i=1}^n \lambda_i \rho_i(X_i) = \inf_{\mathbf{Y}\in\mathbb{A}_W} \sum_{i=1}^n \lambda_i \rho_i(Y_i). \tag{3.3}$$

[6] This means that the topology has a neighbourhood base at 0 consisting of convex and solid sets; cf. [4, Sect. 2.3].

Conversely, if $\mathbf{X} \in \mathbb{A}_X$ *satisfies* (3.3) *for a set of strictly positive weights* $\lambda_i > 0$, $i \in [n]$, *then* $\mathbf{X}$ *is a Pareto-optimal attainable allocation.*

The proof of Proposition 3.6 shows that the agent system property $(\star)$ dictates the values of the Negishi weights.

Proof of Proposition 3.6. (1) By Lemma 3.7 any solution to (3.2) is Pareto-optimal. Conversely, let $W \in \sum_{i=1}^n \mathrm{dom}(\rho_i)$ and let $\mathbf{X} \in \mathbb{A}_W$ be a Pareto-optimal attainable allocation. Let $\lambda \in \mathbb{R}^n_{++}$ be any vector of Negishi weights such that $\mathbf{X}$ is a solution to (3.3). Recall the symmetric relation $\sim$ in $(\star)$ and consider $j, k \in [n]$ such that $j \sim k$. By definition, we find $Z \in \mathcal{S}_j \cap \mathcal{S}_k$ such that $p := \mathfrak{p}_j(Z) = \mathfrak{p}_k(Z) \neq 0$. For $t \in \mathbb{R}$ let

$$\mathbf{X}^t := \mathbf{X} + \tfrac{t}{p} Z e_j - \tfrac{t}{p} Z e_k \in \mathbb{A}_X.$$

Here, Ze_j is the vector whose jth entry is Z, whereas all other entries are 0. Analogously, we define Ze_k. By the $\mathcal{S}_i$-additivity of all the ρ_i, we infer

$$-\infty < \sum_{i=1}^n \lambda_i \rho_i(X_i) \leq \inf_{t \in \mathbb{R}} \sum_{i=1}^n \lambda_i \rho_i(X_i^t) = \sum_{i=1}^n \lambda_i \rho_i(X_i) + \inf_{t \in \mathbb{R}} t(\lambda_j - \lambda_k).$$

This is only possible if $\lambda_j = \lambda_k$. Using that the graph G in $(\star)$ is connected, one inductively shows $\lambda_1 = \cdots = \lambda_n$. As not all the λ_i equal 0, this implies that they all have to be positive. Dividing both sides of (3.3) by λ_1 yields

$$\sum_{i=1}^n \rho_i(X_i) = \inf_{\mathbf{Y} \in \mathbb{A}_X} \sum_{i=1}^n \rho_i(Y_i).$$

(2) Suppose that $\mathbf{W}$ is an initial loss endowment with associated equilibrium $(\mathbf{X}, \phi)$. The equality $\phi(X_i) = \phi(W_i)$ holds for all $i \in [n]$ because $\sum_{i=1}^n X_i = \sum_{i=1}^n W_i$, ϕ is linear, and $\phi(-X_i) \leq \phi(-W_i)$ holds for all $i \in [n]$. Given $Z_i \in \mathcal{S}_i$ such that $\mathfrak{p}_i(Z_i) = 1$ and arbitrary $Y_i \in \mathcal{X}_i$,

$$\phi(Y_i + (\phi(X_i) - \phi(Y_i))Z_i) = \phi(X_i) = \phi(W_i)$$

holds as $\phi = \mathfrak{p}_i$ on $\mathcal{S}_i$. Thus the budget constraint is satisfied, and hence

$$\rho_i(X_i) \leq \rho_i(Y_i + \phi(X_i - Y_i)Z_i) = \rho_i(Y_i) + \phi(X_i) - \phi(Y_i).$$

If we set $W := W_1 + \cdots + W_n$, for any other allocation $\mathbf{Y} \in \mathbb{A}_W$ we obtain

$$\sum_{i=1}^n \rho_i(X_i) \leq \sum_{i=1}^n \rho_i(Y_i) + \phi(X_i) - \phi(Y_i) = \sum_{i=1}^n \rho_i(Y_i)$$

since $\sum_{i=1}^n \phi(X_i) = \sum_{i=1}^n \phi(Y_i) = \phi(W)$. By (1), $\mathbf{X}$ is Pareto-optimal. $\square$

Proof of Theorem 3.1. (1) Let $Z \in \mathcal{M}$ and let $\mathbf{Z} \in \mathbb{A}^s_Z$ be arbitrary, but fixed. The identity $\mathbb{A}^s_Z = \mathbf{Z} + \mathbb{A}^s_0$ implies

$$\pi(Z) = \sum_{i=1}^n \mathfrak{p}_i(Z_i) + \inf_{\mathbf{N} \in \mathbb{A}^s_0} \sum_{i=1}^n \mathfrak{p}_i(N_i) = \sum_{i=1}^n \mathfrak{p}_i(Z_i) + \pi(0).$$

Consider $\mathcal{V} := \{(\mathfrak{p}_i(N_i))_{i\in[n]} : \mathbf{N} \in \mathbb{A}_0^s\}$, which is a subspace of $\mathbb{R}^n$. In the following, we denote by e_l the lth unit vector of $\mathbb{R}^n$. We claim $\pi(0) = 0$ if and only if $\dim(\mathcal{V}) < n$. Indeed, let $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{R}^n$ and observe that $\pi(0) = \inf_{x\in\mathcal{V}}\langle \mathbf{1}, x\rangle$ which is $-\infty$ in case $\dim(\mathcal{V}) = n$. Suppose that $\dim(\mathcal{V}) < n$, i.e. $\mathcal{V}^\perp \neq \{0\}$, and let $0 \neq \lambda \in \mathcal{V}^\perp$. As in the proof of Proposition 3.6(1), $e_j - e_k \in \mathcal{V}$ holds for all $j, k \in [n]$ such that $j \sim k$, which implies $\lambda_j = \lambda_k$. As the relation $\sim$ induces a connected graph, $\lambda \in \operatorname{span}\{\mathbf{1}\} = \mathcal{V}^\perp$. Hence, we obtain that $\langle \mathbf{1}, x\rangle = 0$ for all $x \in \mathcal{V}$ which implies $\pi(0) = 0$, so we have proved equivalence of $\pi(0) = 0$ and $\dim(\mathcal{V}) < n$. But $\dim(\mathcal{V}) < n$ is equivalent to the fact that there is a $j \in [n]$ such that $e_j \notin \mathcal{V}$, which in turn is equivalent to (A1): whenever $Z \in \mathcal{S}_j$ lies in the Minkowski sum $\sum_{i\neq j}\ker(\mathfrak{p}_i)$, $\mathfrak{p}_j(Z) = 0$ has to hold.
(2) We first note that $\mathscr{A}_+$ is convex and monotone. Indeed, let $X, Y \in \mathcal{X}$ such that $Y \in \mathscr{A}_+$ and $X \preceq Y$. Fix $\mathbf{Y} \in \mathbb{A}_Y$ such that $Y_i \in \mathcal{A}_i$, $i \in [n]$, and $\mathbf{X} \in \mathbb{A}_X$ arbitrary. By the Riesz Decomposition Property (cf. [3, Sect. 8.5]), there are $W_1, \dots, W_n \in \mathcal{X}_+$ such that $Y - X = \sum_{i=1}^n W_i$ and $W_i \preceq |Y_i - X_i|$, which means $\mathbf{W} \in \mathbb{A}_{Y-X}$. Hence, for all $i \in [n]$, we obtain $Y_i - W_i \in \mathcal{A}_i$ by the monotonicity of $\mathcal{A}_i$, and thus $X = \sum_{i=1}^n Y_i - W_i \in \mathscr{A}_+$. Moreover, $\mathcal{L}_0(\Lambda)$ is monotone and convex as well, which follows from the corresponding properties of Λ. For $\mathscr{B} \subseteq \mathscr{B}'$, we have

$$\inf\{\pi(Z) : Z \in \mathcal{M},\ X - Z \in \mathscr{B}\} \geq \inf\{\pi(Z) : Z \in \mathcal{M},\ X - Z \in \mathscr{B}'\}.$$

As $\mathscr{A}_+ \subseteq \mathcal{L}_0(\Lambda)$, (2) is proved if for arbitrary $X \in \mathcal{X}$ we can show the two estimates

$$\Lambda(X) \geq \inf\{\pi(Z) : Z \in \mathcal{M},\ X - Z \in \mathscr{A}_+\}, \tag{3.4}$$

and

$$\Lambda(X) \leq \inf\{\pi(Z) : Z \in \mathcal{M},\ X - Z \in \mathcal{L}_0(\Lambda)\}. \tag{3.5}$$

The first assertion trivially holds if $\Lambda(X) = \infty$. If $X \in \operatorname{dom}(\Lambda) = \sum_{i=1}^n \operatorname{dom}(\rho_i)$, choose $\mathbf{X} \in \mathbb{A}_X$ such that $\rho_i(X_i) < \infty$, $i \in [n]$, and $\varepsilon > 0$ arbitrary. Suppose $\mathbf{Z} \in \prod_{i=1}^n \mathcal{S}_i$ is such that $\mathfrak{p}_i(Z_i) \leq \rho_i(X_i) + \frac{\varepsilon}{n}$ and $X_i - Z_i \in \mathcal{A}_i$, $i \in [n]$. Set $Z^* := Z_1 + \dots + Z_n$ and observe $X - Z^* \in \mathscr{A}_+$ as well as

$$\sum_{i=1}^n \rho_i(X_i) + \varepsilon \geq \sum_{i=1}^n \mathfrak{p}_i(Z_i) \geq \pi(Z^*) \geq \inf\{\pi(Z) : Z \in \mathcal{M},\ X - Z \in \mathscr{A}_+\}.$$

This proves (3.4). We now turn to (3.5). If $\Lambda(X) = \infty$, assume for contradiction there is some $Z \in \mathcal{M}$ such that $X - Z \in \mathcal{L}_0(\Lambda) \subseteq \sum_{i=1}^n \operatorname{dom}(\rho_i)$. Choose $\mathbf{Y} \in \mathbb{A}_{X-Z}$ such that $Y_i \in \operatorname{dom}(\rho_i)$ for all i, and let $\mathbf{Z} \in \mathbb{A}_Z^s$ be arbitrary. Then

$$\Lambda(X) \leq \sum_{i=1}^n \rho_i(Y_i + Z_i) = \sum_{i=1}^n \rho_i(Y_i) + \sum_{i=1}^n \mathfrak{p}_i(Z_i) < \infty.$$

This is a *contradiction* and no such $Z \in \mathcal{M}$ can exist. (3.5) holds in this case. Now assume $X \in \operatorname{dom}(\Lambda)$ and suppose $Z \in \mathcal{M}$ satisfies $X - Z \in \mathcal{L}_0(\Lambda)$. Hence, for arbitrary $\varepsilon > 0$ there is $\mathbf{Y} \in \mathbb{A}_{X-Z}$ such that $\sum_{i=1}^n \rho_i(Y_i) \leq \varepsilon$. As $\mathbf{Y} + \mathbf{Z} \in \mathbb{A}_X$ for all $\mathbf{Z} \in \mathbb{A}_Z^s$,

$$\Lambda(X) \leq \inf_{\mathbf{Z} \in \mathbb{A}_Z^s} \sum_{i=1}^n \rho_i(Y_i + Z_i) = \sum_{i=1}^n \rho_i(Y_i) + \pi(Z) \leq \varepsilon + \pi(Z).$$

As $\varepsilon > 0$ was chosen arbitrarily, we obtain (3.5).

(3) Assume (A1) and (A3) are fulfilled, let $\phi_0 \in \mathcal{X}_+^*$ as described in (A3), and note that π is linear by (1). We shall prove that $\phi_0|_{\mathcal{M}} = \kappa\pi$ for some $\kappa > 0$, so by rescaling $\phi_0|_{\mathcal{M}} = \pi$ may be assumed without loss of generality. To this end, we restate requirement (A3)(ii) as $\sup_{Y \in \mathscr{A}_+} \phi_0(Y) < \infty$. Condition (A3)(i) means in particular that $\phi_0|_{\ker(\pi)} \equiv 0$. For each $i \in [n]$ fix $U_i \in \mathcal{S}_i \cap \mathcal{X}_{++}$ such that $U := \sum_{i=1}^n U_i$ satisfies

$$\pi(U) = \sum_{i=1}^n \mathfrak{p}_i(U_i) = 1.$$

As $Z - \pi(Z)U \in \ker(\pi)$ holds for all $Z \in \mathcal{M}$ we infer $\phi_0(Z - \pi(Z)U) = 0$, or equivalently $\phi_0 = \phi_0(U)\pi$ on $\mathcal{M}$. By the second part of (A3)(i), $\phi_0(\tilde{Z}) \neq 0$ for some $\tilde{Z} \in \mathcal{M}$ with $\pi(\tilde{Z}) \neq 0$. Using positivity of ϕ_0, we obtain

$$0 < \frac{\phi_0(\tilde{Z})}{\pi(\tilde{Z})} = \phi_0(U),$$

hence we may set $\kappa := \phi_0(U)$. Finally, if $\kappa = 1$, $X \in \mathcal{X}$ is arbitrary, and $Z \in \mathcal{M}$ is such that $X - Z \in \mathscr{A}_+$,

$$\pi(Z) = \phi_0(Z) = \phi_0(X) - \phi_0(X - Z) \geq \phi_0(X) - \sup_{Y \in \mathscr{A}_+} \phi_0(Y) > -\infty.$$

The bound on the right-hand side is independent of Z. Using the representation of Λ in (2), properness follows.

(4) Note that Λ is $\mathcal{M}$-additive by (2). Since Λ is proper, we cannot have $\pi \equiv -\infty$, hence $\pi(0) = 0$, i.e., (A1) holds by (1). As regards the positivity of π, choose $Y \in \mathcal{X}$ with $\Lambda(Y) \in \mathbb{R}$. For $Z \in \mathcal{M} \cap \mathcal{X}_+$, the monotonicity of Λ then shows $\Lambda(Y) \leq \Lambda(Y + Z) = \Lambda(Y) + \pi(Z)$, which entails $\pi(Z) \geq 0$. It follows that $(\mathscr{A}_+, \mathcal{M}, \pi)$ is a risk measurement regime. □

Proof of Theorem 3.3. As Λ is proper, we have that π is linear, finite valued, and $\pi|_{\mathcal{S}_i} = \mathfrak{p}_i$, $i \in [n]$, by Theorem 3.1. Assume $X \in \mathcal{X}$ and $Z = Z^X \in \mathcal{M}$ are such that $\Lambda(X) = \pi(Z)$ and $X - Z \in \mathscr{A}_+$. As $\pi(Z) \in \mathbb{R}$ and $\Lambda|_{\mathscr{A}_+} \leq 0$, $X \in \mathrm{dom}(\Lambda)$. Choose $Y_i \in \mathcal{A}_i$, $i \in [n]$, such that $X - Z = \sum_{i=1}^n Y_i$. For any $\mathbf{Z} \in \mathbb{A}_Z^s$ we thus have $X = \sum_{i=1}^n Y_i + Z_i$ and

$$\Lambda(X) \leq \sum_{i=1}^n \rho_i(Y_i + Z_i) = \sum_{i=1}^n \rho_i(Y_i) + \sum_{i=1}^n \mathfrak{p}_i(Z_i) \leq \pi(Z) = \Lambda(X),$$

where we have used $\rho_i(Y_i) \leq 0$ and $\pi(Z) = \sum_{i=1}^n \mathfrak{p}_i(Z_i)$ (Theorem 3.1). This shows the exactness of Λ at X.

Now assume $\mathcal{L}_0(\rho_i) = \mathcal{A}_i + \ker(\mathfrak{p}_i)$, $i \in [n]$. Let $X \in \mathrm{dom}(\Lambda)$ and $\mathbf{X} \in \mathbb{A}_X$ such that $\Lambda(X) = \sum_{i=1}^n \rho_i(X_i)$. Further, let $U_i \in \mathcal{S}_i$ with $\mathfrak{p}_i(U_i) = 1$. As $X_i - \rho_i(X_i)U_i \in \mathcal{A}_i + \ker(\mathfrak{p}_i)$, $i \in [n]$, by assumption, we may find a vector $\mathbf{N} \in \prod_{i=1}^n \ker(\mathfrak{p}_i)$ such that $X_i - \rho_i(X_i)U_i + N_i \in \mathcal{A}_i$

for every $i \in [n]$. The fact that $\sum_{i=1}^{n} \rho_i(X_i)U_i - N_i$ is an optimal payoff for X is immediately verified. $\square$

Proof of Proposition 3.4. Suppose first that $\mathscr{A}_+ + \ker(\pi)$ is closed. For lower semicontinuity, we have to establish that $\mathcal{L}_c(\Lambda)$ is closed for every $c \in \mathbb{R}$. To this end, let $U_i \in \mathcal{S}_i \cap \mathcal{X}_{++}$ such that $\mathfrak{p}_i(U_i) > 0$ and set $U := \sum_{i=1}^{n} U_i$. Without loss of generality, we may assume $\pi(U) = 1$. We will show that

$$\mathcal{L}_c(\Lambda) = \{cU\} + \mathscr{A}_+ + \ker(\pi), \tag{3.6}$$

which is closed whenever $\mathscr{A}_+ + \ker(\pi)$ is closed. The right-hand set in (3.6) is included in the left-hand set by the $\mathcal{M}$-additivity of Λ. For the converse inclusion, let $X \in \mathcal{L}_c(\Lambda)$. For every $s > c$, there is a $Z_s \in \mathcal{M}$ such that $c \leq \pi(Z_s) \leq s$ and $X - Z_s \in \mathscr{A}_+$. Consider the decomposition

$$X - sU = X - Z_s + (\pi(Z_s) - s)U + Z_s - \pi(Z_s)U.$$

As $X - Z_s + (\pi(Z_s) - s)U \preceq X - Z_s \in \mathscr{A}_+$ and $Z_s - \pi(Z_s)U \in \ker(\pi)$, the monotonicity of $\mathscr{A}_+$ shows $X - sU \in \mathscr{A}_+ + \ker(\pi)$. Thus,

$$X - cU = \lim_{s \downarrow c} X - sU \in \mathrm{cl}_\tau(\mathscr{A}_+ + \ker(\pi)) = \mathscr{A}_+ + \ker(\pi),$$

and (3.6) is proved. Setting $c = 0$ in (3.6) shows $\mathcal{L}_0(\Lambda) = \mathscr{A}_+ + \ker(\pi)$. Hence, $X - \Lambda(X)U \in \mathscr{A}_+ + \ker(\pi)$ for all $X \in \mathrm{dom}(\Lambda)$, and for a suitable $N \in \ker(\pi)$ depending on X we have $X - \Lambda(X)U + N \in \mathscr{A}_+$ and $\pi(\Lambda(X)U - N) = \Lambda(X)$. Therefore, an optimal payoff for X is given by $\Lambda(X)U - N \in \mathcal{M}$.
Assume now that Λ is l.s.c. and that every $X \in \mathrm{dom}(\Lambda)$ allows for an optimal payoff. Let $(X_i)_{i \in I}$ be a net in $\mathscr{A}_+ + \ker(\pi)$ converging to $X \in \mathcal{X}$. Then $\Lambda(X) \leq 0$ by the lower semicontinuity of Λ. Let $Z^X \in \mathcal{M}$ be an optimal payoff for X, so that $\pi(Z^X) = \Lambda(X) \leq 0$. For U as above and $Y := X - Z^X \in \mathscr{A}_+$ we obtain $Y + \pi(Z^X)U \in \mathscr{A}_+$ by the monotonicity of $\mathscr{A}_+$. Also $Z^X - \pi(Z^X)U \in \ker(\pi)$. Thus $X = \big(Y + \pi(Z^X)U\big) + \big(Z^X - \pi(Z^X)U\big) \in \mathscr{A}_+ + \ker(\pi)$. $\square$

For the proof of Theorem 3.5, we need the notion of the *convex conjugate* of a proper function $f : \mathcal{X} \to (-\infty, \infty]$ on a locally convex Hausdorff topological vector space, which is the function $f^* : \mathcal{X}^* \to (-\infty, \infty]$ defined by

$$f^*(\phi) = \sup_{X \in \mathcal{X}} \phi(X) - f(X).$$

Given $X \in \mathrm{dom}(f)$, $\phi \in \mathcal{X}^*$ is a *subgradient* of f at X if $f(X) = \phi(X) - f^*(\phi)$.

Proof of Theorem 3.5. Fix a vector $\mathbf{W} \in \prod_{i=1}^{n} W_i$ with the property

$$W := W_1 + \cdots + W_n \in \mathrm{int}\,\mathrm{dom}(\Lambda).$$

As a Fréchet lattice is a barrelled space, Λ is subdifferentiable at W by [20, Corollary 2.5 & Proposition 5.2], i.e. there is a subgradient $\phi \in \mathcal{X}^*$ of Λ at W satisfying $\Lambda(W) = \phi(W) -$

$\Lambda^*(\phi)$. As Λ is monotone, $\phi \in \mathcal{X}_+^*$, and by Lemma A.4

$$\Lambda^*(\phi) = \sum_{i=1}^n \rho_i^*(\phi|_{\mathcal{X}_i}).$$

Let $\mathbf{Y}$ be any Pareto-optimal allocation of W. As $\Lambda(W) = \sum_{i=1}^n \rho_i(Y_i) \in \mathbb{R}$, $\Lambda(W)$, $\Lambda^*(\phi)$ and $\rho_i^*(\phi|_{\mathcal{X}_i})$, $i \in [n]$, are all real numbers. Also, as

$$\infty > \rho_i^*(\phi|_{\mathcal{X}_i}) \geq \sup_{Z\in\mathcal{S}_i} \phi(Y_i + Z) - \rho_i(Y_i + Z) = \phi(Y_i) - \rho_i(Y_i) + \sup_{Z\in\mathcal{S}_i} \phi(Z) - \mathfrak{p}_i(Z),$$

$\phi|_{\mathcal{S}_i} = \mathfrak{p}_i$, $i \in [n]$, has to hold. This in turn implies $\phi|_{\mathcal{M}} = \pi$ by the linearity of π and Theorem 3.1. By (A2), we may fix $\tilde{Z} \in \check{\mathcal{S}}$ such that $\pi(\tilde{Z}) = 1 = \mathfrak{p}_i(\tilde{Z})$, $i \in [n]$. Let

$$X_i := Y_i + \phi(W_i - Y_i)\tilde{Z}, \quad i \in [n].$$

As $\sum_{i=1}^n W_i = \sum_{i=1}^n Y_i = W$, $\sum_{i=1}^n X_i = W$ holds and $\mathbf{X} \in \mathbb{A}_W$. Moreover, $\mathbf{X}$ is Pareto-optimal:

$$\begin{aligned}\sum_{i=1}^n \rho_i(X_i) &= \sum_{i=1}^n \rho_i(Y_i) + \phi(W_i - Y_i)\pi(\tilde{Z}) = \sum_{i=1}^n \rho_i(Y_i) + \phi(W - W)\\ &= \sum_{i=1}^n \rho_i(Y_i) = \Lambda(W).\end{aligned}$$

Also, as $\phi(X_i) - \rho_i^*(\phi|_{\mathcal{X}_i}) \leq \rho_i(X_i)$ for all $i \in [n]$ and

$$\sum_{i=1}^n \rho_i(X_i) = \Lambda(W) = \phi(W) - \Lambda^*(W) = \sum_{i=1}^n \phi(X_i) - \rho_i^*(\phi|_{\mathcal{X}_i}),$$

$\rho_i(X_i) = \phi(X_i) - \rho_i^*(\phi|_{\mathcal{X}_i})$ has to hold for all $i \in [n]$. We claim that $(\mathbf{X}, \phi)$ is an equilibrium. Indeed, as $\phi(-X_i) = \phi(-W_i)$ holds for all $i \in [n]$, the budget constraints are satisfied. Moreover, if $i \in [n]$ and $Y \in \mathcal{X}_i$ satisfies $\phi(-Y) \leq \phi(-W_i) = \phi(-X_i)$, we obtain

$$\rho_i(Y) \geq \phi(Y) - \rho_i^*(\phi|_{\mathcal{X}_i}) \geq \phi(X_i) - \rho_i^*(\phi|_{\mathcal{X}_i}) = \rho_i(X_i).$$

□

4. Polyhedral agent systems

In the next two sections, we will study two instances of the model introduced in Sect. 2 and employ the methodology discussed in Sect. 3 to find optimal payoffs for the market, Pareto-optimal allocations, and equilibria. Additionally, we will study their robustness. In this section, we shall focus on *polyhedral agent systems.*

4.1. **The setting.** Throughout this section we assume that the agent system $(\mathcal{R}_1, \dots, \mathcal{R}_n)$ operates on a market space $\mathcal{X}$ given by a Fréchet lattice. Each agent $i \in [n]$ operates on a closed ideal $\mathcal{X}_i \subseteq \mathcal{X}$, and $\mathcal{X}_1 + \cdots + \mathcal{X}_n = \mathcal{X}$. The assumption of closedness implies that $(\mathcal{X}_i, \preceq, \tau \cap \mathcal{X}_i)$ is a Fréchet lattice in its own right. We will assume that each acceptance set $\mathcal{A}_i \subseteq \mathcal{X}_i$ is *polyhedral.*

Definition 4.1. Let $(\mathcal{X}, \tau)$ be a locally convex topological vector space. A convex set $\mathcal{C} \subseteq \mathcal{X}$ is called *polyhedral* if there is a finite set $\mathcal{J} \subseteq \mathcal{X}^*$ and $\beta \in \mathbb{R}^{\mathcal{J}}$ such that

$$\mathcal{A} = \{X \in \mathcal{X} : \phi(X) \leq \beta(\phi), \ \forall \phi \in \mathcal{J}\}.$$

If $(\mathcal{X}, \preceq, \tau)$ is a Fréchet lattice, an agent system $(\mathcal{R}_1, \dots, \mathcal{R}_n)$ on $\mathcal{X}$ is *polyhedral* if it has properties (A1) and (A3), and each acceptance set $\mathcal{A}_i$, $i \in [n]$, is polyhedral. That is, for each $i \in [n]$ there is a finite set $\mathcal{J}_i \subseteq \mathcal{X}_i^*$ and $\beta_i \in \mathbb{R}^{\mathcal{J}_i}$ such that

$$\mathcal{A}_i = \{X \in \mathcal{X}_i : \forall \phi \in \mathcal{J}_i \, (\phi(X) \leq \beta_i(\phi))\}.$$

The polyhedrality of a set $\mathcal{C}$ is equivalent to the existence of some $m \in \mathbb{N}$, a continuous linear operator $T : \mathcal{X} \to \mathbb{R}^m$, and $\beta \in \mathbb{R}^m$ such that

$$\mathcal{C} = \{X \in \mathcal{X} : T(X) \leq \beta\},$$

where the defining inequality is understood coordinatewise. In case of an acceptance set, the representing linear operator can be chosen to be positive. Risk measures with polyhedral acceptance sets play a prominent role in Baes et al. [7], where the set of optimal payoffs for a single such risk measure is studied.

Example 4.2. Suppose $\Omega \neq \emptyset$ is a nonempty set of scenarios for the future state of the economy, either suggested by the internal risk management or a regulatory authority. Moreover, suppose $\emptyset \neq \Omega_i \subsetneq \Omega$, $i \in [n]$, are such that $\Omega = \bigcup_{i=1}^n \Omega_i$. Ω_i denotes the set of scenarios relevant for agent $i \in [n]$, whereas $\Omega_i \cap \Omega_j$ is the (possibly empty) set of jointly relevant scenarios for agents i and j. Note that we do not assume the Ω_i to be pairwise disjoint. While Ω collects the scenarios relevant to the whole system, it is both individually and systemically rational of an agent to demand that her stake in the sharing of a market loss is *neutral* in scenarios $\omega \in \Omega \backslash \Omega_i$. The canonical choice of the model spaces is in consequence $\mathcal{X} := \{X \in \mathbb{R}^\Omega : \sup_{\omega \in \Omega} |X(\omega)| < \infty\}$ endowed with the supremum norm and

$$\mathcal{X}_i := \{X \in \mathcal{X} : X(\omega) = 0, \ \omega \notin \Omega_i\}, \quad i \in [n].$$

Consider individual capital adequacy tests defined in terms of scenariowise loss constraints on a prespecified finite set of test scenarios $\Omega^* \subseteq \Omega$. We need to assume $\Omega^* \cap \Omega_i \cap \Omega_j \neq \emptyset$ whenever $\Omega_i \cap \Omega_j \neq \emptyset$, and that $\Omega_i^* := \Omega^* \cap \Omega_i$, $i \in [n]$, is not empty either. The polyhedral acceptance set of agent $i \in [n]$ is then defined by

$$\mathcal{A}_i := \{X \in \mathcal{X}_i : \forall \omega \in \Omega_i^* \ (X(\omega) \leq \beta_i(\omega))\}, \quad i \in [n],$$

where $\beta_i \in \mathbb{R}^{\Omega_i^*}$ is an arbitrary, but fixed vector of individual loss constraints.

Concerning the individual security spaces, let Π be the set of all subsets $A \subseteq \Omega$ which have the shape $A = \Omega_i \cap \Omega_j$ or $A = \Omega_i \backslash \Omega_j$ for some $i \neq j$ and are nonempty. The security space of agent $i \in [n]$ is then set to be

$$\mathcal{S}_i := \operatorname{span}\{\mathbf{1}_A : A \in \Pi,\ A \subseteq \Omega_i\}.$$

At last, for a collection $(\sigma_\omega)_{\omega \in \Omega^*}$ of positive weights we define the linear functional

$$\pi : \mathcal{M} \to \mathbb{R}, \quad Z \mapsto \sum_{\omega \in \Omega^*} \sigma_\omega Z(\omega),$$

and the individual prices $\mathfrak{p}_i := \pi|_{\mathcal{S}_i}$, $i \in [n]$. We additionally assume that the family of intersections $\Omega_i \cap \Omega_j$, $i, j \in [n]$, $i \neq j$, which are nonempty is rich enough such that the family $(\mathcal{R}_i)_{i \in [n]} := ((\mathcal{A}_i, \mathcal{S}_i, \mathfrak{p}_i))_{i \in [n]}$ of risk measurement regimes satisfies $(\star)$.
In total, if the situation is as described, $(\mathcal{R}_i)_{i \in [n]}$ is a polyhedral agent system.

4.2. **Main results.** We first turn to the existence of optimal risk allocations in polyhedral agent systems. By definition, such an agent system satisfies (A1) and (A3). The resulting risk sharing functional Λ is proper by Theorem 3.1(3). By Theorem 3.3 and Proposition 3.4, the existence of Pareto-optimal allocations would be proved if the closedness of $\mathscr{A}_+ + \ker(\pi)$ can be established.
The two main results on optimal risk allocations are the following:

Theorem 4.3. *Let $(\mathcal{R}_1, \dots, \mathcal{R}_n)$ be a polyhedral agent system on a Fréchet lattice $\mathcal{X}$. Then the set $\mathscr{A}_+ + \ker(\pi)$ is proper, polyhedral, and closed, Λ is l.s.c., and every $X \in \operatorname{dom}(\Lambda)$ admits an optimal payoff $Z^X \in \mathcal{M}$, and can thus be allocated Pareto-optimally as in Theorem 3.3.*

Theorem 4.3 is illustrated by an example in Sect. 4.4.

Remark 4.4. The proof of the preceding theorem shows that for each agent $i \in [n]$, the Minkowski sum $\mathcal{A}_i + \ker(\mathfrak{p}_i)$ is proper, polyhedral, and closed. Moreover, for all $X_i \in \operatorname{dom}(\rho_i)$ we can find an optimal payoff $Z^{X_i} \in \mathcal{S}_i$, i.e., $\mathfrak{p}_i(Z^{X_i}) = \rho_i(X_i)$ and $X_i - Z^{X_i} \in \mathcal{A}_i$.

Corollary 4.5. *If a polyhedral agent system $(\mathcal{R}_1, \dots, \mathcal{R}_n)$ on a Fréchet lattice $\mathcal{X}$ satisfies* (A2)*, for every $\mathbf{W} \in \prod_{i=1}^n \mathcal{X}_i$ such that $W_1 + \dots + W_n \in \operatorname{int}\operatorname{dom}(\Lambda)$ there is an equilibrium $(\mathbf{X}, \phi)$.*

By Theorem 4.3, the correspondence $\mathcal{P}$ mapping $X \in \operatorname{dom}(\Lambda)$ to the set of its Pareto-optimal allocations $\mathbf{X} \in \mathbb{A}_X$ takes nonempty subsets of $\mathbb{A}_X$ as values. Invoking Proposition 3.6, we can represent

$$\mathcal{P}(X) = \left\{ \mathbf{X} \in \mathbb{A}_X : \Lambda(X) = \sum_{i=1}^n \rho_i(X_i) \right\}. \tag{4.1}$$

For a brief summary of the terminology concerning correspondences (or set-valued maps) and their properties, we refer to Appendix A.3. Theorem 4.9, the main result on the robustness of $\mathcal{P}$ in the case of a polyhedral agent system, asserts that $\mathcal{P}$ can be shown to be lower hemicontinuous. This requires some technical assumptions though.

Suppose $(\mathcal{R}_i)_{i\in[n]}$ is a polyhedral agent system on a market space $\mathcal{X}$. Then for each $i \in [n]$, there is a positive continuous linear operator $T_i : \mathcal{X}_i \to \mathbb{R}^{m_i}$ for suitable $m_i \in \mathbb{R}$ and a vector $\beta_i \in \mathbb{R}^{m_i}$ such that

$$\mathcal{A}_i := \{Y \in \mathcal{X}_i : T_i(Y) \leq \beta_i\}.$$

In case $\mathcal{X}$ is infinite-dimensional, we will need the following assumptions:

Assumption 4.6. For each $i \in [n]$, $\mathcal{X}_i$ is complemented in $\mathcal{X}$ by a closed subspace $\mathcal{Y}_i$, that is $\mathcal{X} = \mathcal{X}_i \oplus \mathcal{Y}_i$, where $\oplus$ denotes the direct sum of two vector spaces.

Hence, for each $X \in \mathcal{X}$ there is a unique decomposition $X = \tilde{X} + \tilde{Y}$, where $\tilde{X} \in \mathcal{X}_i$ and $\tilde{Y} \in \mathcal{Y}_i$. Moreover, by the closed graph theorem [30, Chap. 3, Theorem 5], the projection $\delta_i : X \mapsto \tilde{X}$ is continuous and $\mathcal{Y}_i = \ker(\delta_i)$.
Let us define $\widetilde{T}_i := T_i \circ \delta_i$ and $\mathcal{Y} := \bigcap_{i=1}^n \ker(\widetilde{T}_i)$. Hence, $\mathcal{Y}$ is complemented in $\mathcal{X}$ by a finite-dimensional closed subspace $\mathcal{Z}$, i.e., $\mathcal{X} = \mathcal{Y} \oplus \mathcal{Z}$. The projections $\gamma_1 : \mathcal{X} \to \mathcal{Y}$ and $\gamma_2 : \mathcal{X} \to \mathcal{Z}$ are continuous.

Assumption 4.7. For each $i \in [n]$, we have $\gamma_1(\mathcal{X}_i) \subseteq \mathcal{X}_i$.

At last, we will assume

Assumption 4.8. For each $i \in [n]$ there is a continuous linear $P_i : \mathcal{Y} \to \gamma_1(\mathcal{X}_i)$ such that $\sum_{i=1}^n P_i = id_{\mathcal{Y}}$.

Theorem 4.9. *Consider a polyhedral agent system $(\mathcal{R}_i)_{i\in[n]}$ on a market space $\mathcal{X}$. Suppose that Assumptions 4.6–4.8 hold in case $\mathcal{X}$ is infinite-dimensional. Then the correspondence $\mathcal{P}$ is lower hemicontinuous on* $\mathrm{dom}(\Lambda)$ *and admits a continuous selection on* $\mathrm{dom}(\Lambda)$.

Example 4.10. (1) Assumptions 4.6–4.8 are automatically satisfied if all individual ideals $\mathcal{X}_i$ agree with the market space $\mathcal{X}$. Indeed, $\mathcal{Y}_i = \{0\}$ can be chosen in Assumption 4.6, $\gamma_1(\mathcal{X}_i) = \gamma_1(\mathcal{X}) \subseteq \mathcal{X} = \mathcal{X}_i$ gives Assumption 4.7, and $P_i := \frac{1}{n} id_{\mathcal{X}}$ is possible in Assumption 4.8.

(2) In the situation described by Example 4.2 Assumptions 4.6–4.8 are satisfied. Indeed, the complementing subspaces $\mathcal{Y}_i$ of $\mathcal{X}_i$ in Assumption 4.6 are given by

$$\mathcal{Y}_i := \{X \in \mathcal{X} : X(\omega) = 0, \ \forall\, \omega \in \Omega_i\}.$$

Recall that Ω^* is the set of scenarios relevant for market acceptability and note that

$$\mathcal{Y} = \{X \in \mathcal{X} : X(\omega) = 0, \ \forall\, \omega \in \Omega^*\},$$
$$\mathcal{Z} = \{X \in \mathcal{X} : X(\omega) = 0, \ \forall\, \omega \in \Omega \backslash \Omega^*\}.$$

As $\gamma_1(X) := X\mathbf{1}_{\Omega\backslash\Omega^*}$ and $\gamma_2(X) = X\mathbf{1}_{\Omega^*}$, $X \in \mathcal{X}$, we verify easily that $\gamma_1(\mathcal{X}_i) \subseteq \mathcal{X}_i$, $i \in [n]$, i.e., Assumption 4.7 is met. Regarding the existence of the mappings $(P_i)_{i\in[n]}$ satisfying Assumption 4.8, we can define them consecutively by

$$P_1(X) := X\mathbf{1}_{\Omega_1}, \quad P_{i+1}(X) := X\mathbf{1}_{\Omega_{i+1}\backslash \bigcup_{j=1}^i \Omega_j}, \quad i \in [n-1], \ X \in \mathcal{Y}.$$

Similar examples can be constructed for other function spaces such as the Hilbert space $L^2(\Omega, \mathcal{F}, \mathbb{P})$, the space of all equivalence classes with respect to almost sure equality of square-integrable random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

One may wonder whether the correspondence $\mathcal{E} : \prod_{i=1}^n \mathcal{X}_i \twoheadrightarrow \prod_{i=1}^n \mathcal{X}_i \times \mathcal{X}^*$ mapping an initial loss endowment $\mathbf{W}$ to all its equilibrium allocations such that $\mathbf{X} \in \mathcal{P}(W_1 + \cdots + W_n)$ and ϕ is a subgradient of Λ at $W_1 + \cdots + W_n$ — as in the proof of Theorem 3.5 — is lower hemicontinuous under suitable conditions. This, however, is not the case. Suppose $\mathcal{X}$ admits two positive functionals $\phi, \psi \in \mathcal{X}_+^*$ such that $\ker(\phi) \backslash \ker(\psi) \neq \emptyset$. We assume $n = 1$ and consider an agent system $\mathcal{R} = (\mathcal{A}, \mathcal{S}, \mathfrak{p})$ such that $\rho_{\mathcal{R}}(X) = \max\{\phi(X), \psi(X)\}$, $X \in \mathcal{X}$. Let $W \in \mathcal{X}$ such that $\phi(W) = 0 < \psi(W)$. Thus, for all $n \in \mathbb{N}$, the equilibrium price at $\frac{1}{n}W$ would be ψ, whereas any element of the convex hull of $\{\phi, \psi\}$ could be chosen as equilibrium price at 0. $\mathcal{E}$ is *not* lower hemicontinuous in this case.

4.3. **Ancillary results and proofs.** For the following lemma, recall that a Fréchet space is a completely metrisable locally convex topological vector space. In particular, every Fréchet lattice is a Fréchet space.

Lemma 4.11. *Let $\mathcal{X}$ be a Fréchet space.*

(1) *A subset $\mathcal{C} \subseteq \mathcal{X}$ is polyhedral if and only if there are closed subspaces $\mathcal{X}^1, \mathcal{X}^2 \subseteq \mathcal{X}$ such that $\mathcal{X} = \mathcal{X}^1 \oplus \mathcal{X}^2$, $\dim(\mathcal{X}^2) < \infty$, and $\mathcal{C} = \mathcal{X}^1 + \mathcal{C}'$ for a polyhedral subset $\mathcal{C}' \subseteq \mathcal{X}^2$.*

(2) *Suppose $\mathcal{Y}$ and $\mathcal{X}$ are Fréchet spaces, $\mathcal{C} \subseteq \mathcal{Y}$ is polyhedral, and $T : \mathcal{Y} \to \mathcal{X}$ is a surjective linear operator. Then $T(\mathcal{C})$ is polyhedral in $\mathcal{X}$.*

Proof. (1) Combine the proof of [46, Corollary 2.1] with the closed graph theorem [30, Chap. 3, Theorem 5].

(2) By (1), there are two closed subspaces $\mathcal{Y}^1, \mathcal{Y}^2 \subseteq \mathcal{Y}$ such that $\dim(\mathcal{Y}^2)$ is finite, $\mathcal{Y} = \mathcal{Y}^1 \oplus \mathcal{Y}^2$, and $\mathcal{C} = \mathcal{Y}^1 + \mathcal{C}'$ for a polyhedral subset $\mathcal{C}'$ of $\mathcal{Y}^2$. Define $\mathcal{X}^2 := T(\mathcal{Y}^2)$, which is finite-dimensional. Every finite-dimensional subspace of a Fréchet space is complemented by a closed subspace. Thus $\mathcal{X} = \mathcal{X}^1 \oplus \mathcal{X}^2$ for a closed subspace $\mathcal{X}^1$. Clearly, $T(\mathcal{C}') \subseteq \mathcal{X}^2$ is polyhedral; see [38, Theorem 19.3]. Moreover, denoting by $\gamma_i : \mathcal{X} \to \mathcal{X}^i$ the projection in $\mathcal{X}$ onto the linear subspaces $\mathcal{X}^i$, surjectivity of T implies $\mathcal{X}^1 = \gamma_1(\mathcal{X}) = \gamma_1(T(\mathcal{Y}^1)) + \gamma_1(T(\mathcal{Y}^2) = \gamma_1(T(\mathcal{Y}^1))$. Moreover,

$$T(\mathcal{C}) = T(\mathcal{Y}^1) + T(\mathcal{C}') = \mathcal{X}^1 + \gamma_2(T(\mathcal{Y}^1)) + T(\mathcal{C}').$$

$\gamma_2(T(\mathcal{Y}^1))$ is polyhedral as subspace of the finite-dimensional space $\mathcal{X}^2$, and so is the sum $\gamma_2(T(\mathcal{Y}^1)) + T(\mathcal{C}')$ of two polyhedral sets. Conclude with (1). ☐

The preceding lemma enables us to prove Theorem 4.3.

Proof of Theorem 4.3. The set $\mathscr{A}_+ + \ker(\pi)$ is proper by assumption (A3). Moreover, it is polyhedral: consider the Fréchet space $\mathcal{Y} := (\prod_{i=1}^n \mathcal{X}_i) \times \ker(\pi)$.[7] By assumption, the set $\mathcal{C} :=$

[7] Space $\mathcal{Y}$ is not a Fréchet lattice, hence the necessity for the above formulation of Lemma 4.11.

$(\prod_{i=1}^n \mathcal{A}_i)\times\ker(\pi)$ is polyhedral, and $T:\mathcal{Y}\to\mathcal{X}$ defined by $T(X_1,\dots,X_n,N)=\sum_{i=1}^n X_i+N$ is surjective and linear. As $\mathcal{X}$ is a Fréchet space, Lemma 4.11(2) yields the polyhedrality of $T(\mathcal{C})=\mathscr{A}_+ +\ker(\pi)$. Being a polyhedral set, it is automatically closed. Since Λ is proper, it is l.s.c. and optimal payoffs exist for every $X\in\mathrm{dom}(\Lambda)$ by Proposition 3.4. □

Proof of Corollary 4.5. Combine Theorems 3.5 and 4.3. □

We now turn our attention to the lower hemicontinuity of the correspondence $\mathcal{P}$ as stated in Theorem 4.9. Its proof requires the following highly technical Lemmas 4.12 and 4.13 whose proofs imitate in parts a technique from Baes et al. [7]. Note that in analogy with Theorem 4.3, $\mathscr{A}_+$ is closed.

Lemma 4.12. *Suppose $\mathcal{X}$ is a finite-dimensional locally convex Hausdorff topological vector space, $\mathcal{X}_i\subseteq\mathcal{X}$, $i\in[n]$, are finite-dimensional subspaces such that $\sum_{i=1}^n\mathcal{X}_i=\mathcal{X}$, and $\mathcal{A}_i\subseteq\mathcal{X}_i$, $i\in[n]$, are polyhedral sets. Set $\mathscr{A}_+:=\sum_{i=1}^n\mathcal{A}_i$ and define the correspondence*

$$\Gamma:\mathscr{A}_+\ni X\to\left\{\mathbf{X}\in\prod_{i=1}^n\mathcal{A}_i:\ X_1+\cdots+X_n=X\right\}.$$

Then Γ is lower hemicontinuous.

Proof. As in Definition 4.1, for each $i\in[n]$ we fix $m_i\in\mathbb{N}$, a linear and continuous operator $T_i:\mathcal{X}\to\mathbb{R}^{m_i}$, and vectors $\beta_i\in\mathbb{R}^{m_i}$, such that

$$\mathcal{A}_i=\{Y\in\mathcal{X}_i:\ T_i(Y)\le\beta_i\}.$$

Step 1. For fixed $X\in\mathscr{A}_+$ we decompose $\Gamma(X)$ as the sum of a universal and an X-dependent component. Recall from Appendix A.1 that the recession cone of $\Gamma(X)$ is given by

$$0^+\Gamma(X):=\{\mathbf{Y}\in\textstyle\prod_{i=1}^n\mathcal{X}_i:\ \mathbf{X}+k\mathbf{Y}\in\Gamma(X),\ \forall\,\mathbf{X}\in\Gamma(X),\ \forall\,k>0\}.$$

The lineality space of $\Gamma(X)$ is

$$0^+\Gamma(X)\cap(-0^+\Gamma(X))=\{\mathbf{Y}\in\mathbb{A}_0:\ T_i(Y_i)=0,\ \forall\,i\in[n]\},$$

a subspace independent of X. By virtue of Lemma A.2, there is a X-independent subspace $\mathcal{V}\subseteq\prod_{i=1}^n\mathcal{X}_i$ such that

$$\Gamma(X)=\alpha(X)+0^+\Gamma(X),\quad \alpha(X):=\mathrm{co}(\mathrm{ext}(\Gamma(X)\cap\mathcal{V})),$$

where $\mathrm{co}(\cdot)$ denotes the convex hull operator and $\mathrm{ext}(\Gamma(X)\cap\mathcal{V})$ the set of extreme points of $\Gamma(X)\cap\mathcal{V}$.

Step 2. In this step, we prove that the correspondence $\alpha:\mathscr{A}_+\to\prod_{i=1}^n\mathcal{A}_i$ maps sets which are bounded with respect to some norm on $\mathcal{X}$ to sets which are bounded with respect to some norm on $\mathcal{V}$. To this end, let $D:=\dim(\mathcal{X})=\dim(\mathcal{X}^*)$ and choose a basis $\psi_1,\dots,\psi_D$ of $\mathcal{X}^*$. Note that $\mathbf{X}\in\Gamma(X)\cap\mathcal{V}$ if and only if

- $\mathbf{X}$ is an allocation of X, i.e. $\psi_j(X_1+\cdots+X_n)=\psi_j(X)$ for all $j\in[D]$, or equivalently
$$\psi_j(X_1+\cdots+X_n)\le\psi_j(X)\text{ and }(-\psi_j)(X_1+\cdots+X_n)\le(-\psi_j)(X);$$
- each X_i lies in $\mathcal{A}_i$, i.e. $T_i(X_i)\le\beta_i$;

- $\mathbf{X} \in \mathcal{V}$.

Clearly, the properties listed above describe a polyhedral subset of $\mathcal{V}$; more precisely, for $m := \sum_{i=1}^{n} m_i + 2D$, m_i defined above, we may find a continuous linear operator $\mathbf{S} : \mathcal{V} \to \mathbb{R}^m$ and an affine function $\mathbf{f} : \mathcal{X} \to \mathbb{R}^m$ such that

$$\Gamma(X) \cap \mathcal{V} = \{\mathbf{X} \in \mathcal{V} : \mathbf{S}(\mathbf{X}) \leq \mathbf{f}(X)\}.$$

Every "row" $\mathbf{S}_i$ of $\mathbf{S}$ corresponds to an element of $\mathcal{V}^*$. By [9, Theorem II.4.2], for every extreme point $\mathbf{X} \in \Gamma(X) \cap \mathcal{V}$ the set

$$I(\mathbf{X}) = \{i \in [m] : \mathbf{S}_i(\mathbf{X}) = \mathbf{f}_i(X)\},$$

whose cardinality is at least $\dim(\mathcal{V})$, satisfies $\operatorname{span}\{\mathbf{S}_i : i \in I(\mathbf{X})\} = \mathcal{V}^*$. Let $\mathbb{F}(X) := \{I(\mathbf{X}) : \mathbf{X} \in \operatorname{ext}(\Gamma(X) \cap \mathcal{V})\}$ be the collection of all such $I(\mathbf{X})$ corresponding to an extreme point. Its cardinality is bounded by the number of subsets of $[m]$ with cardinality at least $\dim(\mathcal{V})$. Moreover, for each $I \in \mathbb{F}(X)$, the linear operator $\mathbf{S}_I : \mathcal{V} \ni \mathbf{Y} \mapsto (\mathbf{S}_i(\mathbf{Y}))_{i \in I}$ is injective and thus invertible on its image. We have shown that $(\mathbf{S}_I)_{I \in \mathbb{F}(X)}$ is a finite family of operators with full rank whose cardinality depends on $\dim(\mathcal{V})$ and m only. Let $\mathcal{B} \subseteq \mathscr{A}_+$ be a bounded set. For each $X \in \mathcal{B}$ and $I \in \mathbb{F}(X)$, $\mathbf{f}_I$ is affine and thus maps $\mathcal{B}$ to a bounded set. Also, $\mathbf{S}_I^{-1}$ is continuous by the closed graph theorem [30, Chap. 3, Theorem 5], whence boundedness of $\{\mathbf{S}_I^{-1}(\mathbf{f}_I(Y)) : Y \in \mathcal{B}, I \in \mathbb{F}(Y)\}$ follows. Recall that $\bigcup_{X \in \mathcal{B}} \mathbb{F}(X)$ is finite. Using Carathéodory's theorem [38, Theorem 17.1], $\operatorname{co}\{\mathbf{S}_I^{-1}(\mathbf{f}_I(X)) : X \in \mathcal{B}, I \in \mathbb{F}(X)\}$ is bounded. As

$$\begin{aligned}\bigcup_{X \in \mathcal{B}} \alpha(X) &= \bigcup_{X \in \mathcal{B}} \operatorname{co}\{\mathbf{S}_I^{-1}(\mathbf{f}_I(X)) : I \in \mathbb{F}(X)\} \\ &\subseteq \operatorname{co}\{\mathbf{S}_I^{-1}(\mathbf{f}_I(X)) : X \in \mathcal{B},\, I \in \mathbb{F}(X)\},\end{aligned}$$

it has to be bounded as well and Step 2 is proved.
Step 3. Γ is lower hemicontinuous. Let $(X^k)_{k \in \mathbb{N}} \subseteq \mathscr{A}_+$ be convergent to $X \in \mathscr{A}_+$ and let $\mathbf{X} \in \Gamma(X)$. We have to show that there is a subsequence $(k_\lambda)_{\lambda \in \mathbb{N}}$ and $\mathbf{X}^\lambda \in \Gamma(X^{k_\lambda})$ such that $\mathbf{X}^\lambda \to \mathbf{X}$; cf. Appendix A.3. To this end, let first $\mathbf{Y}^k \in \alpha(X^k)$, $k \in \mathbb{N}$, which is a bounded sequence by Step 2. After passing to a subsequence $(k_\lambda)_{\lambda \in \mathbb{N}}$, we may assume $\mathbf{Y}^{k_\lambda} \to \mathbf{Y} \in \Gamma(X)$ (as $\mathcal{A}_i$ is closed, $i \in [n]$). If $\Gamma(X)$ is a singleton, $\mathbf{Y} = \mathbf{X}$ has to hold and we may choose $\mathbf{X}^\lambda := \mathbf{Y}^{k_\lambda}$. Otherwise, suppose first that $\mathbf{X}$ lies in the *relative interior* of $\Gamma(X)$, i.e., there is an $\varepsilon > 0$ such that $\mathbf{X} + \varepsilon(\mathbf{X} - \mathbf{Y}) \in \Gamma(X)$, as well. Recall the definition of the linear operators T_i, $i \in [n]$, above and fix $i \in [n]$. Let $1 \leq j \leq m_i$ be arbitrary. We denote by $T_i^j(W)$ the jth entry of $T_i(W)$.
Case 1. $T_i^j(X_i) = \beta_j$. From $Y_i \in \mathcal{A}_i$, we infer

$$0 \geq T_i^j(X_i + \varepsilon(X_i - Y_i)) - \beta_j = \varepsilon(\beta_j - T_i^j(Y_i)) \geq 0,$$

which means $T_i^j(Y_i) = \beta_j$, as well. Set $\lambda(i, j) = 1$.

Case 2. $T_i^j(X_i) < \beta_j$. As $Y_i^{k_\lambda} \to Y_i$ for $\lambda \to \infty$, there must be a $\lambda(i,j) \in \mathbb{N}$ such that for all $\lambda \geq \lambda(i,j)$

$$T_i^j(Y_i^{k_\lambda} - Y_i + X_i) \leq \beta_j.$$

Hence for all $\lambda \geq \max_{i\in[n], 1\leq j\leq m_i} \lambda(i,j)$, one obtains

$$\mathbf{X}^\lambda := \mathbf{Y}^{k_\lambda} - \mathbf{Y} + \mathbf{X} \in \prod_{i=1}^n \mathcal{A}_i \cap \mathbb{A}_{X^{k_\lambda}} = \Gamma(X^{k_\lambda}),$$

and $\mathbf{X}^\lambda \to \mathbf{X}$. It remains to notice that each $\mathbf{X} \in \Gamma(X)$ may be approximated with a sequence in the relative interior of $\Gamma(X)$, cf. [38, Theorem 6.3]. The assertion is proved. $\square$

Lemma 4.13. *Suppose the polyhedral agent system $(\mathcal{R}_i)_{i\in[n]}$ and the infinite-dimensional market space $\mathcal{X}$ conform to Assumptions 4.6–4.8. Then the correspondence $\Gamma : \mathscr{A}_+ \ni X \mapsto \mathbb{A}_X \cap \prod_{i=1}^n \mathcal{A}_i$ is lower hemicontinuous.*

Proof. We will use the terminology introduced in Assumptions 4.6–4.8. In particular,

$$\mathcal{Y} := \bigcap_{i=1}^n \ker(T_i \circ \delta_i),$$

which is complemented in $\mathcal{X}$ by a finite-dimensional closed subspace $\mathcal{Z}$, and the projections $\gamma_1 : \mathcal{X} \to \mathcal{Y}$ and $\gamma_2 : \mathcal{X} \to \mathcal{Z}$ are continuous.

We now consider the ambient space $\mathcal{Z}$ which may be written as

$$\mathcal{Z} = \gamma_2(\mathcal{X}) = \sum_{i=1}^n \gamma_2(\mathcal{X}_i).$$

We will apply Lemma 4.12 to the sets

$$\mathcal{B}_i := \{Y \in \gamma_2(\mathcal{X}_i) : \widetilde{T}_i(Y) \leq \beta_i\}, \quad i \in [n].$$

Suppose $(X^k)_{k\in\mathbb{N}} \subseteq \mathscr{A}_+$ is a sequence which converges to $X \in \mathscr{A}_+$. Let $\mathbf{X} \in \Gamma(X)$. We need to find a subsequence $(k_\lambda)_{\lambda\in\mathbb{N}}$ and a sequence of allocations $\mathbf{X}^{k_\lambda} \in \Gamma(X^{k_\lambda})$ such that $\mathbf{X}^{k_\lambda} \to \mathbf{X}$.

To this end note that $\gamma_2(X^k) \to \gamma_2(X)$ in $\mathcal{Z}$ as $k \to \infty$. Moreover, $(\gamma_2(X_i))_{i\in[n]}$ satisfies $\gamma_2(X_i) \in \gamma_2(\mathcal{X}_i)$, $i \in [n]$, $\sum_{i=1}^n \gamma_2(X_i) = \gamma_2(X)$, and by construction of $\mathcal{Z}$, $\widetilde{T}_i(\gamma_2(X_i)) = \widetilde{T}_i(X_i) \leq \beta_i$. Hence, we may apply Lemma 4.12 to obtain a subsequence $(k_\lambda)_{\lambda\in\mathbb{N}}$ and $\mathbf{Y}^{k_\lambda} \in \prod_{i=1}^n \mathcal{B}_i$ such that the identity $\sum_{i=1}^n Y_i^{k_\lambda} = \gamma_2(X_{k_\lambda})$ holds for all $\lambda \in \mathbb{N}$ and $\mathbf{Y}^{k_\lambda} \to (\gamma_2(X_i))_{i\in[n]}$, $\lambda \to \infty$.

Note that by Assumption 4.7, $\gamma_1(X_i) \in \mathcal{X}_i$ for each $i \in [n]$. Let P_i be the continuous linear operators defined in Assumption 4.8. Consider

$$X_i^{k_\lambda} := \gamma_1(X_i) + P_i\left(\gamma_1(X^{k_\lambda} - X)\right) + Y_i^{k_\lambda} \in \gamma_1(\mathcal{X}_i) + \gamma_2(\mathcal{X}_i) = \mathcal{X}_i, i \in [n],\ \lambda \in \mathbb{N}.$$

Then

$$\sum_{i=1}^n X_i^{k_\lambda} = \gamma_1(X^{k_\lambda}) + \gamma_2(X^{k_\lambda}) = X^{k_\lambda}.$$

Moreover, for each $i \in [n]$, we have

$$T_i(X_i^{k_\lambda}) = T_i(\gamma_2(X_i^{k_\lambda})) = T_i(Y_i^{k_\lambda}) \leq \beta_i.$$

Hence, $X_i^{k_\lambda} \in \mathcal{A}_i$ for each $i \in [n]$. These observations combined yield that $\mathbf{X}^{k_\lambda} \in \Gamma(X^{k_\lambda})$. By the continuity of P_i and γ_1, we obtain

$$\forall\, i \in [n]: \; X_i^{k_\lambda} \to \gamma_1(X_i) + \gamma_2(X_i) = X_i.$$

This finishes the proof. □

At last we can prove Theorem 4.9.

Proof of Theorem 4.9. In addition to the correspondence $\mathcal{P}$ defined by (4.1) consider the following three correspondences:

- $\Gamma_1 : \mathrm{dom}(\Lambda) \twoheadrightarrow \mathscr{A}_+ \times \mathcal{M},\; X \mapsto \{(X-Z, Z) : Z \in \mathcal{M},\, X - Z \in \mathscr{A}_+,\; \Lambda(X) = \pi(Z)\}$, which is lower hemicontinuous on $\mathrm{dom}(\Lambda)$ by virtue of the polyhedrality of $\mathscr{A}_+$ and [7, Theorem 5.11].
- $\Gamma_2 : \mathscr{A}_+ \twoheadrightarrow \prod_{i=1}^n \mathcal{A}_i,\; X \mapsto \mathbb{A}_X \cap \prod_{i=1}^n \mathcal{A}_i$, which is lower hemicontinuous by Lemma 4.12, if $\mathcal{X}$ is finite-dimensional, or Lemma 4.13, in case $\mathcal{X}$ is infinite-dimensional.
- $\Gamma_3 : \mathcal{M} \twoheadrightarrow \prod_{i=1}^n \mathcal{S}_i,\; Z \mapsto \mathbb{A}_Z^s$, which is lower hemicontinuous by Lemma A.5.

Applying [3, Theorem 17.23],

$$\Gamma : \mathrm{dom}(\Lambda) \ni X \mapsto \bigcup_{(X-Z,Z)\in\Gamma_1(X)} (\Gamma_2(X-Z) + \Gamma_3(Z))$$

is lower hemicontinuous as well.
In fact, $\Gamma = \mathcal{P}$ holds. To see this, let $X \in \mathrm{dom}(\Lambda)$ be arbitrary. From the proof of Theorem 3.3, $\Gamma(X) \subseteq \mathcal{P}(X)$ follows. For the converse inclusion, let $\mathbf{X} \in \mathcal{P}(X)$ be arbitrary. Choose $Z_i \in \mathcal{S}_i$, $i \in [n]$, such that $X_i - Z_i \in \mathcal{A}_i$ and $\rho_i(X_i) = \mathfrak{p}_i(Z_i)$, which is possible by Theorem 4.3 in the case $n = 1$; see Remark 4.4. Let $Z = Z_1 + \cdots + Z_n$ and note that

$$\pi(Z) = \textstyle\sum_{i=1}^n \mathfrak{p}_i(Z_i) = \sum_{i=1}^n \rho_i(X_i) = \Lambda(X),$$

i.e. $(X - Z, Z) \in \Gamma_1(X)$. Moreover, as $\mathbf{X} - \mathbf{Z} \in \Gamma_2(X - Z)$, it only remains to note that $\mathbf{X} = (\mathbf{X} - \mathbf{Z}) + \mathbf{Z} \in \Gamma_2(X - Z) + \Gamma_3(Z)$. Equality of sets is established.
Finally, $\mathrm{dom}(\Lambda)$ is metrisable and therefore paracompact; cf. [39]. Moreover, $\prod_{i=1}^n \mathcal{X}_i$ is a Fréchet space, and as $\mathcal{P} : \mathrm{dom}(\Lambda) \twoheadrightarrow \prod_{i=1}^n \mathcal{X}_i$ has nonempty closed convex values, a continuous selection for $\mathcal{P}$ exists by the Michael selection theorem [3, Theorem 17.66]. □

4.4. **An example.** We close this section by showing how Pareto optima can be computed in the situation of Example 4.2. For the sake of simplicity, we assume that $n = 2$, Ω is a finite set, and $A := \Omega_1 \backslash \Omega_2$, $B := \Omega_1 \cap \Omega_2$ and $C := \Omega_2 \backslash \Omega_1$ are all nonempty. The specifications of Example 4.2 lead to the individual security spaces $\mathcal{S}_1 = \mathrm{span}\{\mathbf{1}_A, \mathbf{1}_B\}$ and $\mathcal{S}_2 = \mathrm{span}\{\mathbf{1}_B, \mathbf{1}_C\}$. Let us assume that $\Omega^* = \Omega$. Hence, the individual acceptance sets are of the shape

$$\mathcal{A}_i := \{X \in \mathcal{X}_i : \forall\, \omega \in \Omega_i \; (X(\omega) \leq \beta_i(\omega))\}, \quad i = 1, 2,$$

where $\beta_i \in \mathbb{R}^{\Omega_i}$ is arbitrary, but fixed. For convenience, we assume the set of weights $(\sigma_\omega)_{\omega\in\Omega^*}$ appearing in the definition of the pricing functionals π and $\mathfrak{p}_i$, $i \in [n]$, to be such that $\pi(\mathbf{1}_A) = \pi(\mathbf{1}_B) = \pi(\mathbf{1}_C) = 1$.
Note that for $x, y \in \mathbb{R}$, we have

$$X - x\mathbf{1}_A - y\mathbf{1}_B \in \mathcal{A}_1 \iff \max_{a\in A} X(a) - \beta_1(a) \leq x \text{ and } \max_{b\in B} X(b) - \beta_1(b) \leq y.$$

Consequently,

$$\rho_1(X) := \rho_{\mathcal{R}_1}(X) = \max_{a\in A} X(a) - \beta_1(a) + \max_{b\in B} X(b) - \beta_1(b), \quad X \in \mathcal{X}_1,$$

and it only takes finite values. An analogous computation shows

$$\rho_2(X) := \rho_{\mathcal{R}_2}(X) = \max_{b\in B} X(b) - \beta_2(b) + \max_{c\in C} X(c) - \beta_2(c), \quad X \in \mathcal{X}_2.$$

which also takes only finite values. Set $\tilde{\beta} := \beta_1\mathbf{1}_A + (\beta_1 + \beta_2)\mathbf{1}_B + \beta_2\mathbf{1}_C$. The representative agent of this polyhedral agent system is given by

$$\mathscr{A}_+ = \mathcal{A}_1 + \mathcal{A}_2 = \{X \in \mathcal{X} : X \leq \tilde{\beta}\},\ \mathcal{M} = \operatorname{span}\{\mathbf{1}_A, \mathbf{1}_B, \mathbf{1}_C\},$$
$$\pi(x\mathbf{1}_A + y\mathbf{1}_B + z\mathbf{1}_C) = x + y + z, \quad x, y, z \in \mathbb{R}.$$

Furthermore

$$\ker(\pi) = \{N_{x,y} := x\mathbf{1}_A - (x + y)\mathbf{1}_B + y\mathbf{1}_C : x, y \in \mathbb{R}\}.$$

We now aim to compute the associated risk sharing functional Λ and Pareto-optimal allocations. To this end, for $X \in \mathcal{X}$, we introduce the notation

$$\rho^A(X) := \max_{a\in A} X(a) - \beta_1(a), \quad \rho^B(X) := \max_{b\in B} X(b) - \tilde{\beta}(b),$$
$$\rho^C(X) := \max_{c\in C} X(c) - \beta_2(c).$$

Using the characterisation of $\mathscr{A}_+$, one obtains

$$\mathscr{A}_+ + \ker(\pi) = \{X \in \mathcal{X} : \rho^B(X) \leq -\rho^A(X) - \rho^C(X)\}.$$

A straightforward computation yields

$$\Lambda(X) = \inf\{r \in \mathbb{R} : X - r\mathbf{1}_B \in \mathscr{A}_+ + \ker(\pi)\} = \rho^A(X) + \rho^B(X) + \rho^C(X).$$

Note that $X - \Lambda(X)\mathbf{1}_B - N_{\rho^A(X),\,\rho^C(X)} \in \mathscr{A}_+$, since

$$\big((X - \rho^A(X))\mathbf{1}_A + \beta_1\mathbf{1}_B, (X - \rho^B(X) - \beta_1)\mathbf{1}_B + (X - \rho^C(X))\mathbf{1}_C\big)$$

is an allocation of $X - \Lambda(X)\mathbf{1}_B - N_{\rho^A(X),\,\rho^C(X)}$ which lies in $\mathcal{A}_1 \times \mathcal{A}_2$. For every $\zeta \in \mathbb{R}$, the allocation $(X_1(\zeta), X_2(\zeta))$ given by

$$X_1(\zeta) = X\mathbf{1}_A + (\beta_1 - \rho^A(X) + \zeta\Lambda(X))\mathbf{1}_B,$$
$$X_2(\zeta) = (X - \rho^B(X) - \rho^C(X) - \beta_1 - (\zeta - 1)\Lambda(X))\mathbf{1}_B + X\mathbf{1}_C,$$

is Pareto-optimal. Last, we note that an optimal payoff for X is given by $\rho^A(X)\mathbf{1}_A + (\Lambda(X) - \rho^A(X) - \rho^C(X))\mathbf{1}_B + \rho^C(X)\mathbf{1}_C \in \mathcal{M}$.

5. Law-invariant acceptance sets

In this section we discuss the risk sharing problem for agent systems with *law-invariant acceptance sets*, the second case study exemplifying the results in Sects. 2 and 3.

5.1. **The setting.** Throughout we fix an *atomless* probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e., there is a random variable $U : \Omega \to \mathbb{R}$ such that the cumulative distribution function $\mathbb{R} \ni x \mapsto \mathbb{P}(U \le x)$ of U under $\mathbb{P}$ is continuous. By $L^1 := L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $L^\infty := L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ we denote the spaces of (equivalence classes of) $\mathbb{P}$-integrable and bounded random variables, respectively. They are Banach lattices when equipped with the usual $\mathbb{P}$-almost sure (a.s.) order and the topologies arising from their natural norms $\|\cdot\|_1 : X \mapsto \mathbb{E}[|X|]$ and

$$\|\cdot\|_\infty : X \mapsto \inf\{m > 0 : \mathbb{P}(|X| \le m) = 1\}.$$

All appearing (in)equalities between random variables are understood in the a.s. sense.

Definition 5.1. A subset $\mathcal{C} \subseteq L^1$ is $\mathbb{P}$-*law-invariant* if $X \in \mathcal{C}$ whenever there is $Y \in \mathcal{C}$ which is equal to X in law under $\mathbb{P}$, i.e. the two Borel probability measures $\mathbb{P} \circ X^{-1}$ and $\mathbb{P} \circ Y^{-1}$ on $(\mathbb{R}, \mathbb{B}(\mathbb{R}))$ agree. Given a $\mathbb{P}$-law-invariant set $\emptyset \neq \mathcal{C} \subseteq L^1$ and some other set $S \neq \emptyset$, a function $f : \mathcal{C} \to S$ is called $\mathbb{P}$-law-invariant if $\mathbb{P} \circ X^{-1} = \mathbb{P} \circ Y^{-1}$ implies $f(X) = f(Y)$.

Let us first specify the setting in the case of the ambient market space $\mathcal{X}$ agreeing with the space L^1 of all integrable random variables. This allows a better grasp of its respective aspects. We will consider a more general setting later.
Model space assumptions. Throughout this section, all agents $i \in [n]$ operate on the same model space $\mathcal{X}_i = \mathcal{X} = L^1$ consisting of equivalence classes of integrable random variables. In Sect. 5.2, the results will be generalised to a wide class of model spaces $L^\infty \subseteq \mathcal{X} \subseteq L^1$, always under the assumption that the model spaces coincide, i.e., $\mathcal{X}_i = \mathcal{X}$, $i \in [n]$. The reason for this is that, in principle, the individual model spaces $\mathcal{X}_i$ should be law-invariant and closed ideals in $\mathcal{X}$. This has strong implications however. In fact, if $\mathcal{X}$ is a law-invariant Banach lattice of random variables which carries a law-invariant lattice norm like we assume below, and $\mathcal{X}_i$ is supposed to be a non-trivial closed and law-invariant ideal in $\mathcal{X}$, then $\mathcal{X}_i = \mathcal{X}$.
Acceptance sets. Each agent $i \in [n]$ deems a loss profile adequately capitalised if it belongs to a closed $\mathbb{P}$-law-invariant acceptance set $\mathcal{A}_i \subseteq L^1$ which contains a riskless payoff, i.e.,

$$\mathbb{R} \cap \mathcal{A}_i \neq \emptyset. \tag{5.1}$$

As the dual space of L^1 may be identified with L^∞, we may see the respective *support functions* $\sigma_{\mathcal{A}_i}$, $i \in [n]$, as law-invariant mappings

$$\sigma_{\mathcal{A}_i} : L^\infty \to (-\infty, \infty], \quad Q \mapsto \sup_{Y \in \mathcal{A}_i} \mathbb{E}[QY];$$

cf. Appendix A.1. Due to monotonicity of the sets $\mathcal{A}_i$, $\mathrm{dom}(\sigma_{\mathcal{A}_i}) \subseteq L^\infty_+$ holds. The reader may think of acceptance sets arising, for instance, from the Average Value at Risk (Expected Shortfall) or distortion risk measures.

Security markets. Regarding the security markets, we require there is a linear functional $\pi : \mathcal{M} \to \mathbb{R}$ on the global security space $\mathcal{M}$ such that the individual pricing functionals are given by $\mathfrak{p}_i = \pi|_{\mathcal{S}_i}$, $i \in [n]$; the agents operate on different sub-markets $(\mathcal{S}_i, \pi|_{\mathcal{S}_i})$ of $(\mathcal{M}, \pi)$. In particular, conditions $(\star)$ and (A1) are satisfied. Moreover, we assume

Assumption 5.2. π is of the shape $\pi(Z) = \mathbb{E}[(Q+\delta)Z]$, $Z \in \mathcal{M}$, where $\delta > 0$ is a constant and $Q \in \bigcap_{i\in[n]} \operatorname{dom}(\sigma_{\mathcal{A}_i}) \subseteq L^\infty_+$.

Our assumption on the pricing functionals is very flexible as illustrated by Example 5.14 below; the pricing functional given in Example 2.2 also conforms to Assumption 5.2. Note that the constant function $1 = \mathbf{1}_\Omega$ is an element of $\operatorname{dom}(\sigma_{\mathcal{A}_i})$, $i \in [n]$; cf. (5.4) below. As the intersection $\bigcap_{i\in[n]} \operatorname{dom}(\sigma_{\mathcal{A}_i})$ is a cone, $Q+\delta \in \bigcap_{i\in[n]} \operatorname{dom}(\sigma_{\mathcal{A}_i})$ for every $Q \in \bigcap_{i\in[n]} \operatorname{dom}(\sigma_{\mathcal{A}_i})$ and every $\delta > 0$. In particular, any jointly relevant density with arbitrarily small constant perturbation can be used for pricing.

Recall from the introduction that assuming the individual acceptance sets $\mathcal{A}_i$ to be law-invariant means that being acceptable or not is merely a statistical property of the loss profile. Mathematically, this intuition necessitates introducing the hypothetical physical measure $\mathbb{P}$. Prices in the security market can, e.g., be determined by a suitable equivalent martingale measure $\mathbb{Q}$ though. For the remainder of this section we assume that Assumption 5.2 is satisfied.

Let us at last introduce the notion of *comonotone partitions of the identity*, or *comonotone functions*, i.e., functions in the set

$$\mathfrak{C} := \{\mathbf{f} = (f_1, \ldots, f_n) : \mathbb{R} \to \mathbb{R}^n : \ f_i \text{ nondecreasing}, \ \textstyle\sum_{i=1}^n f_i = id_{\mathbb{R}}\}.$$

5.2. **Main results.** We will first formulate the main results concerning the existence of Pareto-optimal allocations in the case of $\mathcal{X} = \mathcal{X}_i = L^1$, $i \in [n]$.

Theorem 5.3. *Suppose the assumptions of this section are met.*

(1) *The set* $\mathscr{A}_+ + \ker(\pi)$ *is a closed and proper subset of* L^1, *and* Λ *is proper and l.s.c.*
(2) *All* $X \in \operatorname{dom}(\Lambda)$ *admit an optimal payoff* $Z^X \in \mathcal{M}$. *In particular, for any* $X \in \operatorname{dom}(\Lambda)$, *there exists a Pareto-optimal allocation* $\mathbf{X}$ *of the shape*

$$X_i = A_i - N_i + \Lambda(X)U_i, \ A_i := f_i(X - \Lambda(X)U + N) \in \mathcal{A}_i, \ i \in [n], \tag{5.2}$$

where $N \in \ker(\pi)$ *is an* X*-dependent zero cost global security and* $\mathbf{f} \in \mathfrak{C}$ *is* X*-dependent, whereas* $\mathbf{N} \in \mathbb{A}^s_N$ *is arbitrary and* $U_i \in \mathcal{S}_i \cap L^1_{++}$, $i \in [n]$, *are chosen such that* $U := \sum_{i=1}^n U_i$ *satisfies* $\pi(U) = 1$.

Remark 5.4. If $n = 1$, $\Lambda = \rho_{\mathcal{R}}$ and Theorem 5.3 in fact solves the *optimal payoff problem* studied in [7]. The proof of Theorem 5.3(1) shows that for every single agent $i \in [n]$, the Minkowski sum $\mathcal{A}_i + \ker(\mathfrak{p}_i)$ is a closed and proper subset of L^1, and ρ_i is l.s.c.

Corollary 5.5. *In the situation of Theorem 5.3, suppose that the agent system checks* (A2). *Then for every* $\mathbf{W} \in (L^1)^n$ *such that* $W = \sum_{i=1}^n W_i \in \operatorname{int}\operatorname{dom}(\Lambda)$ *there is an equilibrium* $(\mathbf{X}, \phi)$.

Remark 5.6. Finding elements in the interior of $\mathrm{dom}(\Lambda)$ usually requires stronger continuity properties of the involved risk measures and is an important motivation for studying the risk sharing problem on general model spaces endowed with a stronger topology than $\|\cdot\|_1$. We will do this shortly in Theorem 5.8 and Lemma 5.12. Given a loss $W \in L^1$, the trick is to find a suitable model space $(\mathcal{X}, \|\cdot\|)$ such that $W \in \mathrm{int}_{\|\cdot\|}\mathrm{dom}(\Lambda|_{\mathcal{X}})$; see, e.g., [17, 33, 36, 41].

By Lemma A.5 there is a continuous selection $\Psi : \mathcal{M} \to \prod_{i=1}^n \mathcal{S}_i$ of $\mathcal{M} \ni Z \mapsto \mathbb{A}_Z^s$. Hence, the correspondence $\widehat{\mathcal{P}} : L^1 \twoheadrightarrow (L^1)^n$ mapping X to Pareto-optimal allocations of shape (5.2) such that, additionally, the security allocation of $N \in \ker(\pi)$ is given by $\mathbf{N} = \Psi(N)$[8] has nonempty values on $\mathrm{dom}(\Lambda)$ by Theorem 5.3. Although it might be the case that not all Pareto-optimal allocations of $X \in \mathrm{dom}(\Lambda)$ are elements of $\widehat{\mathcal{P}}(X)$, $\widehat{\mathcal{P}}$ has the advantage of being upper hemicontinuous on the interior of the domain of Λ.

Theorem 5.7. *In the situation of Theorem 5.3 suppose $\mathscr{A}_+$ does not agree with one of the level sets $\{X \in L^1 : \mathbb{E}[X] \le c\}$, $c \in \mathbb{R}$. Then $\widehat{\mathcal{P}}$ is upper hemicontinuous at every continuity point $X \in \mathrm{dom}(\Lambda)$ of Λ and,* a fortiori*, on* $\mathrm{int}\,\mathrm{dom}(\Lambda)$.

We already advertised that the assumption that all agents operate on the space $\mathcal{X} = L^1$ does not restrict the generality of Theorems 5.3 and 5.7 and Corollary 5.5. Indeed, the market space $\mathcal{X}$ may be chosen to be any law-invariant ideal within L^1 with respect to the $\mathbb{P}$-a.s. order falling in one of the following two categories:

(BC) *Bounded case:* $\mathcal{X} = L^\infty$ equipped with the supremum norm $\|\cdot\|_\infty$.
(UC) *Unbounded case:* $L^\infty \subseteq \mathcal{X} \subseteq L^1$ is a $\mathbb{P}$-law invariant Banach lattice endowed with an order continuous law-invariant lattice norm $\|\cdot\|$.[9]

In the unbounded case, one can show that the identity embeddings

$$L^\infty \hookrightarrow \mathcal{X} \hookrightarrow L^1$$

are continuous, i.e. there are constants $\kappa, K > 0$ such that

$$\forall\, X \in L^\infty\, \forall\, Y \in \mathcal{X} : \ \|X\| \le \kappa \|X\|_\infty \text{ and } \|Y\|_1 \le K\|Y\|. \tag{5.3}$$

Moreover, for all $\phi \in \mathcal{X}^*$ there is a unique $Q \in L^1$ such that $QX \in L^1$ and $\phi(X) = \mathbb{E}[QX]$ hold for all $X \in \mathcal{X}$. The reader may think here of L^p-spaces with $1 < p < \infty$, or more generally Orlicz hearts equipped with a Luxemburg norm as for instance in [14, 17, 28].
In view of Lemma 5.12 we will assume that

- each individual acceptance set $\mathcal{A}_i \subseteq \mathcal{X}$ is closed, law-invariant and satisfies $\mathcal{A}_i \cap \mathbb{R} \neq \emptyset$;
- the security markets $(\mathcal{S}_i, \mathfrak{p}_i)$ agree with Assumption 5.2.

Our main result is

Theorem 5.8. *Let $\mathcal{X}$ be a Banach lattice satisfying* **(BC)** *or* **(UC)***. Assume the agent system $(\mathcal{R}_1, \dots, \mathcal{R}_n)$ is such that each individual acceptance set $\mathcal{A}_i \subseteq \mathcal{X}$ is closed, law-invariant and*

[8] Recall that $\mathbf{N}$ in (5.2) can be chosen arbitrarily.

[9] As $\mathcal{X}$ will be a super Dedekind complete Riesz space, this translates as the fact that whenever $X_n \downarrow 0$ in order, $\|X_n\| \downarrow 0$ holds as well; cf. [4, Definition 1.43] and [27, Theorem A.33].

satisfies $\mathcal{A}_i \cap \mathbb{R} \neq \emptyset$*, and the security markets* $(\mathcal{S}_i, \mathfrak{p}_i)$ *agree with Assumption 5.2. Then Theorems 5.3 and 5.7 and Corollary 5.5 hold* verbatim *when* $\mathcal{X}$ *replaces* L^1 *and* $\|\cdot\|$ *replaces* $\|\cdot\|_1$.

For the final result on upper hemicontinuity of the equilibrium correspondence, recall that the finite risk measure $\rho_{\mathcal{R}} : L^\infty \to \mathbb{R}$ resulting from a risk measurement regime $\mathcal{R}$ on L^∞ is *continuous from above* if $\rho_{\mathcal{R}}(X_n) \downarrow \rho_{\mathcal{R}}(X)$ whenever $(X_n)_{n\in\mathbb{N}} \subseteq L^\infty$ and $X \in L^\infty$ are such that $X_n \downarrow X$ a.s.

Theorem 5.9. *Assume that* (A2) *is satisfied and that in case* **(BC)** ρ_1 *is continuous from above, whereas in case* **(UC)** $\mathcal{X}$ *is reflexive. Suppose furthermore that* $\mathscr{A}_+$ *does not agree with a level set* $\mathcal{L}_c(\mathbb{E}[\cdot])$ *and consider the correspondence* $\mathcal{E} : \mathcal{X}^n \twoheadrightarrow \mathcal{X}^n \times \mathcal{X}^*$ *mapping* $\mathbf{W}$ *to equilibrium allocations* $(\mathbf{X}, \phi)$ *of shape*

$$X_i = Y_i + \frac{\phi(W_i - Y_i)}{\phi(\tilde{Z})}\tilde{Z}, \quad i \in [n],$$

where $\mathbf{Y} \in \widehat{\mathcal{P}}(W_1 + \cdots + W_n)$*,* $\tilde{Z} \in \check{\mathcal{S}}$ *with* $\pi(\tilde{Z}) \neq 0$*, and* ϕ *is a subgradient of* Λ *at* $W_1+\cdots+W_n$*. Then* $\mathcal{E}$ *is upper hemicontinuous in the following sense: whenever* $(\mathbf{W}^k)_{k\in\mathbb{N}} \subseteq \prod_{i=1}^n \operatorname{int}\operatorname{dom}(\rho_i)$ *and* $\mathbf{W}^k \to \mathbf{W} \in \prod_{i=1}^n \operatorname{int}\operatorname{dom}(\rho_i)$ *as* $k \to \infty$ *and* $(\mathbf{X}^k, \phi_k) \in \mathcal{E}(\mathbf{W}^k)$ *is chosen arbitrarily,* $k \in \mathbb{N}$*, there is a subsequence* $(k_\lambda)_{\lambda\in\mathbb{N}}$ *such that* $(\mathbf{X}, \phi) := \lim_{\lambda\to\infty}(\mathbf{X}^{k_\lambda}, \phi_{k_\lambda}) \in \mathcal{E}(\mathbf{W})$.

5.3. **Ancillary results and proofs.** We begin with the existence of optimal payoffs and Pareto-optimal allocations. This is a direct consequence of Proposition 3.4 in the case $\mathcal{X} = L^1$, provided we can prove the properness of Λ and closedness of $\mathscr{A}_+ + \ker(\pi)$. First, we characterise the recession cone $0^+\mathcal{A}$ of a convex law-invariant acceptance set $\mathcal{A}$, a result of independent interest. For the definition of a recession cone and of the support function $\sigma_{\mathcal{A}}$, we refer to Appendix A.1. With slight modifications, Proposition 5.10 also holds true for general closed, convex and law-invariant sets $\mathcal{C}$ which do not agree with one of the sets $\{X \in L^1 : c_- \leq \mathbb{E}[X] \leq c_+\}$, where $-\infty \leq c_- \leq c_+ \leq \infty$.

Proposition 5.10. *Suppose* $\emptyset \neq \mathcal{A} \subsetneq L^1$ *is a law-invariant and closed acceptance set.*

(1) $0^+\mathcal{A}$ *is law-invariant.*
(2) *Suppose furthermore that* $\mathcal{A}$ *does not agree with one of the level sets* $\mathcal{L}_c(\mathbb{E}[\cdot])$ *for some* $c \in \mathbb{R}$*. Let* $Q \in \operatorname{dom}(\sigma_{\mathcal{A}})$ *and* $\delta > 0$ *be arbitrary. If* $U \in 0^+\mathcal{A}$ *satisfies* $\mathbb{E}[(Q+\delta)U] = 0$*, then* $U = 0$.

Proof. (1) As $\mathcal{A}$ is norm closed and convex, we may apply the Hahn-Banach separation theorem to obtain the representation

$$\mathcal{A} = \{X \in L^1 : \mathbb{E}[QX] \leq \sigma_{\mathcal{A}}(Q), \ \forall\, Q \in \operatorname{dom}(\sigma_{\mathcal{A}})\},$$

where $\sigma_{\mathcal{A}}$ is the support function of $\mathcal{A}$. It is well-known that $\operatorname{dom}(\sigma_{\mathcal{A}})$ is a law-invariant and convex cone in L^∞_+. The law-invariance of $\operatorname{dom}(\sigma_{\mathcal{A}})$ combined with Lemma A.1 shows that the recession cone $0^+\mathcal{A}$ is law-invariant and closed as well.

(2) By (1) and [40, Lemma 1.3], for any $U \in 0^+\mathcal{A}$, $Q \in \mathrm{dom}(\sigma_{\mathcal{A}})$, and sub-$\sigma$-algebra $\mathcal{H} \subseteq \mathcal{F}$, we have

$$\mathbb{E}[U|\mathcal{H}] \in 0^+\mathcal{A} \quad \text{and} \quad \mathbb{E}[Q|\mathcal{H}] \in \mathrm{dom}(\sigma_{\mathcal{A}}). \tag{5.4}$$

Choosing $\mathcal{H} = \{\emptyset, \Omega\}$, we obtain that $E[Q] \in \mathrm{dom}(\sigma_{\mathcal{A}})$ for all $Q \in \mathrm{dom}(\sigma_{\mathcal{A}})$. Moreover, by choosing $Q \in \mathrm{dom}(\sigma_{\mathcal{A}}) \subseteq L^\infty_+$ appropriately, we obtain that $1 \in \mathrm{dom}(\sigma_{\mathcal{A}})$.
Now suppose there is no $c \in \mathbb{R}$ such that $\mathcal{A} = \mathcal{L}_c(\mathbb{E}[\cdot])$, and assume a direction $U \in 0^+\mathcal{A}$ is not constant. In a first step, we will exclude the possibility that $\mathbb{E}[U] = 0$. To this end let $Q \in \mathrm{dom}(\sigma_{\mathcal{A}})$ be non-constant. Such a Q exists because $\mathcal{A}$ does not agree with one of the lower level sets of $\mathbb{E}[\cdot]$. As $(\Omega, \mathcal{F}, \mathbb{P})$ is non-atomic, for $k \geq 2$ large enough there is a finite measurable partition[10] $\Pi := (A_1, \dots, A_k)$ of Ω such that $\mathbb{P}(A_j) = \frac{1}{k}$, $j \in [k]$, and

$$\widehat{U} = \mathbb{E}[U|\sigma(\Pi)] = \textstyle\sum_{i=1}^k u_i \mathbf{1}_{A_i} \text{ and } \widehat{Q} = \mathbb{E}[Q|\sigma(\Pi)] = \sum_{i=1}^k q_i \mathbf{1}_{A_i}$$

are both non-constant. For any permutation $\tau : [k] \to [k]$ the random variable given by $\widehat{U}_\tau := \sum_{i=1}^k u_{\tau(i)} \mathbf{1}_{A_i}$ has the same distribution under $\mathbb{P}$ as $\widehat{U}$. Hence, by (5.4) and (1), $\widehat{U}_\tau \in 0^+\mathcal{A}$ follows. Similarly, $\widehat{Q}_\tau := \sum_{i=1}^k q_{\tau(i)} \mathbf{1}_{A_i} \in \mathrm{dom}(\sigma_{\mathcal{A}})$. For our argument we will hence assume without loss of generality that the vectors u and q satisfy $u_1 \leq \cdots \leq u_k$ and $q_1 \leq \cdots \leq q_k$. In both chains of inequalities, at least one inequality has to be strict. We estimate

$$\mathbb{E}[\widehat{Q}]\mathbb{E}[\widehat{U}] = \left(\frac{1}{k}\sum_{i=1}^k q_i\right) \cdot \left(\frac{1}{k}\sum_{i=1}^k u_i\right) < \frac{1}{k}\sum_{i=1}^k q_i u_i = \mathbb{E}[\widehat{Q} \cdot \widehat{U}] \leq 0.$$

Here, the first strict inequality is due to Chebyshev's sum inequality [29, Theorem 43] and u and q being non-constant. The last inequality is due to $\widehat{U} \in 0^+\mathcal{A}$, $\widehat{Q} \in \mathrm{dom}(\sigma_{\mathcal{A}})$, and Lemma A.1. $\mathbb{E}[\widehat{U}] = \mathbb{E}[U] = 0$ is hence impossible.
In a second step, let $Q \in \mathrm{dom}(\sigma_{\mathcal{A}})$ and $\delta > 0$ be arbitrary. Suppose $U \in 0^+\mathcal{A}$ satisfies $\mathbb{E}[(Q + \delta)U] = \mathbb{E}[QU] + \delta\mathbb{E}[U] = 0$. As $Q, \delta \in \mathrm{dom}(\sigma_{\mathcal{A}})$, Lemma A.1 yields $\mathbb{E}[QU] \leq 0$ and $\delta\mathbb{E}[U] \leq 0$. Combining these facts leads to the identities $\mathbb{E}[U] = \mathbb{E}[QU] = 0$, whence $U = 0$ follows with the first step. $\square$

We continue with a result for comonotone functions. One easily verifies that for $\mathbf{f} \in \mathfrak{C}$ and $i \in [n]$ the coordinate function f_i is Lipschitz continuous with Lipschitz constant 1. Moreover, for $\gamma > 0$, we set

$$\mathfrak{C}_\gamma := \{\mathbf{f} \in \mathfrak{C} : \mathbf{f}(0) \in [-\gamma, \gamma]^n\}.$$

From [26, Lemma B.1] we recall the following compactness result:

Lemma 5.11. *For every $\gamma > 0$, $\mathfrak{C}_\gamma \subseteq (\mathbb{R}^n)^{\mathbb{R}}$ is sequentially compact in the topology of pointwise convergence: for any sequence $(\mathbf{f}^k)_{k\in\mathbb{N}} \subseteq \mathfrak{C}_\gamma$ there is a subsequence $(k_\lambda)_{\lambda\in\mathbb{N}}$ and $\mathbf{f} \in \mathfrak{C}_\gamma$ such that*

$$\forall\, x \in \mathbb{R} : \ \mathbf{f}^{k_\lambda}(x) \to \mathbf{f}(x), \quad \lambda \to \infty.$$

We are now ready to prove Theorem 5.3.

[10] That is the sets are pairwise disjoint, measurable, and their union is Ω.

Proof of Theorem 5.3. (1) The individual acceptance sets $\mathcal{A}_i$ may be used to define $\mathbb{P}$-law-invariant l.s.c. monetary base risk measures ξ_i by

$$\xi_i(X) := \inf\{m \in \mathbb{R} : X - m \in \mathcal{A}_i\} \in (-\infty, \infty], \quad X \in L^1.$$

By (5.1), $\xi_i(Y) \in \mathbb{R}$ holds for all bounded random variables $Y \in L^\infty$. The lower level sets $\mathcal{L}_c(\xi_i)$, $c \in \mathbb{R}$, may be written as $\mathcal{L}_c(\xi_i) = c + \mathcal{A}_i$. The risk measures ξ_i admit a dual representation

$$\xi_i(X) = \sup_{Q \in \mathrm{dom}(\xi_i^*)} \mathbb{E}[QX] - \xi_i^*(Q), \quad X \in L^1, \tag{5.5}$$

where cash-additivity implies that

$$\mathrm{dom}(\xi_i^*) \subseteq \{Q \in L^\infty_+ : \mathbb{E}[Q] = 1\} \text{ and } \xi_i^*(Q) = \sigma_{\mathcal{A}_i}(Q),\ Q \in \mathrm{dom}(\xi_i^*). \tag{5.6}$$

Moreover, the infimal convolution $\xi := \square_{i=1}^n \xi_i > -\infty$ is a $\mathbb{P}$-law-invariant monetary risk measure on L^1 as well and $\xi^* = \sum_{i=1}^n \xi_i^*$ by Lemma A.4. Now, by [26, Corollary 2.7], ξ is l.s.c., and for each $X \in \mathrm{dom}(\xi)$ there is $\mathbf{f} \in \mathfrak{C}$ such that

$$\xi(X) = \sum_{i=1}^n \xi_i(f_i(X)). \tag{5.7}$$

Suppose now $X \in L^1$ satisfies $\xi(X) \leq 0$ and let $\mathbf{f}$ as in (5.7). For all $i \in [n]$ we may choose $c_i \in \mathbb{R}$ such that $\xi_i(f_i(X) - c_i) = \xi_i(f_i(X)) - c_i \leq 0$ and $\sum_{i=1}^n c_i = 0$. If $g_i := f_i - c_i$, $g_i(X) \in \mathcal{L}_0(\xi_i) = \mathcal{A}_i$, $i \in [n]$. Hence,

$$X = \sum_{i=1}^n g_i(X) \in \sum_{i=1}^n \mathcal{A}_i = \mathscr{A}_+.$$

We have thus shown that

$$\mathcal{L}_0(\xi) = \mathscr{A}_+.$$

As ξ is l.s.c. the left-hand set (and thus also the right-hand set) is norm closed.
Let $\pi(\cdot) = \mathbb{E}[(Q+\delta)\cdot]$ as in Assumption 5.2. Suppose first that, for some $c \in \mathbb{R}$, $\mathscr{A}_+ = \mathcal{L}_c(\mathbb{E}[\cdot])$. Then $Q \in \mathbb{R}_+$ holds and $\pi = p\mathbb{E}[\cdot]$ for a suitable $p > 0$. We obtain that $0^+\mathscr{A}_+ \cap \ker(\pi) = \ker(\pi)$ is a subspace. By Dieudonné's theorem [45, Theorem 1.1.8], $\mathscr{A}_+ + \ker(\pi)$ is closed.
If $\mathscr{A}_+$ does not agree with one of the lower level sets of $\mathbb{E}[\cdot]$, Proposition 5.10(2) allows us to infer that $0^+\mathscr{A}_+ \cap \ker(\pi) = \{0\}$, a subspace. Again, Dieudonné's theorem yields the closedness of $\mathscr{A}_+ + \ker(\pi)$.
For properness of Λ, let $X \in L^1$ be arbitrary. Suppose $Z \in \mathcal{M}$ is such that $X - Z \in \mathscr{A}_+$, i.e. $\xi(X - Z) \leq 0$. Let $Q \in L^\infty_+$ and $\delta > 0$ be chosen as in Assumption 5.2. By (5.6),

$$Q^* := \frac{Q + \delta}{\mathbb{E}[Q] + \delta} \in \mathrm{dom}(\xi^*).$$

Moreover,

$$0 \geq \mathbb{E}\left[Q^*(X - Z)\right] - \xi^*\left(Q^*\right) = \mathbb{E}\left[Q^* X\right] - \xi^*\left(Q^*\right) - (\mathbb{E}[Q] + \delta)\pi(Z),$$

which implies

$$\pi(Z) \geq \frac{\mathbb{E}\left[Q^* X\right] - \xi^*\left(Q^*\right)}{\mathbb{E}[Q] + \delta} > -\infty.$$

The properness follows with the representation of Λ given in Theorem 3.1(2). The lower semicontinuity of Λ is due to Proposition 3.4.
(2) By (1), Λ is proper and $\mathscr{A}_+ + \ker(\pi)$ is closed. By Proposition 3.4, every $X \in \mathrm{dom}(\Lambda)$ admits an optimal payoff Z^X and thus a Pareto-optimal allocation by Theorem 3.3. For the concrete shape of Z^X and the Pareto-optimal allocation, let $\mathbf{U} \in \prod_{i=1}^n \mathcal{S}_i$ be as in the assertion. As in the proof of (1), we may find $\mathbf{f} \in \mathfrak{C}$ such that $f_i(X - Z^X) \in \mathcal{A}_i$, $i \in [n]$. As $\pi(Z^X) = \Lambda(X)$, we have $N := \Lambda(X)U - Z^X \in \ker(\pi)$. For any $\mathbf{N} \in \mathbb{A}_N^s$, the security allocation $\mathbf{Z}^X := \Lambda(X)\mathbf{U} - \mathbf{N}$ lies in $\mathbb{A}_{Z^X}^s$. According to Theorem 3.3,

$$\mathbf{f}(X - Z^X) + \mathbf{Z}^X = \mathbf{f}(X - \Lambda(X)U + N) + \Lambda(X)\mathbf{U} - \mathbf{N}$$

is a Pareto-optimal allocation of X with $\mathbf{f}(X - \Lambda(X)U + N) \in \prod_{i=1}^n \mathcal{A}_i$. This proves (5.2). $\square$

Proof of Corollary 5.5. Combine Theorem 5.3 and Theorem 3.5. $\square$

Proof of Theorem 5.7. We start with any sequence $(X^k)_{k\in\mathbb{N}} \subseteq \mathrm{int}\,\mathrm{dom}(\Lambda)$ that converges to $X \in \mathrm{int}\,\mathrm{dom}(\Lambda)$. For all $k \in \mathbb{N}$ let $\mathbf{X}^k = (X_i^k)_{i\in[n]} \in \widehat{\mathcal{P}}(X^k)$. By Appendix A.3, it is enough to show that there is a subsequence $(k_\lambda)_{\lambda\in\mathbb{N}}$ and an allocation $\mathbf{X} \in \widehat{\mathcal{P}}(X)$ such that $\mathbf{X}^{k_\lambda} \to \mathbf{X}$ coordinatewise for $\lambda \to \infty$. To this end, we first recall the construction of $\mathbf{X}^k$, $k \in \mathbb{N}$, from Theorem 5.3: There are sequences $(N^k)_{k\in\mathbb{N}} \subseteq \ker(\pi)$ and $(\mathbf{f}^k)_{k\in\mathbb{N}} \subseteq \mathfrak{C}$ such that

- $A_i^k := f_i^k(X^k - \Lambda(X^k)U + N^k) \in \mathcal{A}_i$, $i \in [n]$;
- $\mathbf{X}^k = \mathbf{A}^k + \Lambda(X^k)\mathbf{U} - \mathbf{N}^k$, where $\mathbf{N}^k = \Psi(N^k)$.

We will establish in three steps that $(N^k)_{k\in\mathbb{N}}$ and $(\mathbf{f}^k)_{k\in\mathbb{N}}$ lie in suitable relatively sequentially compact sets, which will allow us to choose a convergent subsequence.
First, as Λ is continuous on $\mathrm{int}\,\mathrm{dom}(\Lambda)$ by [20, Corollary 2.5], the sequence $(X^k - \Lambda(X^k)U)_{k\in\mathbb{N}}$ is bounded.
The second step is to prove that $(N^k)_{k\in\mathbb{N}}$ is a norm bounded sequence as well. We assume for contradiction we can select a subsequence $(k_\lambda)_{\lambda\in\mathbb{N}}$ such that $1 \leq \|N^{k_\lambda}\|_1 \uparrow \infty$. Using compactness of the unit sphere in the finite-dimensional space $\ker(\pi)$ and potentially passing to another subsequence, we may furthermore assume

$$\frac{1}{\|N^{k_\lambda}\|_1} N^{k_\lambda} \to N^* \in \ker(\pi)\backslash\{0\},\ \lambda \to \infty,$$

Let $Y \in \mathscr{A}_+$ be arbitrary and note that

$$Y + N^* = \lim_{\lambda\to\infty} (1 - \|N^{k_\lambda}\|_1^{-1})Y + \|N^{k_\lambda}\|_1^{-1}\left(X^{k_\lambda} - \Lambda(X^{k_\lambda})U + N^{k_\lambda}\right) \in \mathscr{A}_+,$$

as the latter set is closed and convex and the sequence $\left(X^{k_\lambda} - \Lambda(X^{k_\lambda})U\right)_{\lambda\in\mathbb{N}}$ is norm bounded. Hence, $N^* \in 0^+\mathscr{A}_+ \cap \ker(\pi)$ which is trivial by Assumption 5.2 and Proposition 5.10(2), leading to the desired *contradiction.* $(N^k)_{k\in\mathbb{N}}$ has to be bounded and $\{N^k : k \in \mathbb{N}\} \subseteq \ker(\pi)$ is relatively (sequentially) compact by the finite dimension of the latter space.

In a third step, we establish relative sequential compactness for the set $\{\mathbf{f}^k : k \in \mathbb{N}\}$. To this end, recall the definition of the monetary risk measures ξ_i in the proof of Theorem 5.3. As $1 \in \mathrm{dom}(\sigma_{\mathcal{A}_i})$ by the proof of Proposition 5.10 and $\mathbb{E}[1] = 1$, (5.6) implies $\xi_i^*(1) < \infty$ for all $i \in [n]$. Now fix $k \in \mathbb{N}$ and let $I := \{i \in [n] : f_i^k(0) > 0\}$ and $J := [n] \backslash I$. If I is empty, $f_i^k(0) = 0$ has to hold for all $i \in [n]$. Now suppose we can choose $i \in I$. We abbreviate $W^k := X^k - \Lambda(X^k)U + N^k$ and estimate

$$\begin{aligned} -\mathbb{E}[|W^k|] &\leq -\mathbb{E}[|f_i^k(W^k) - f_i^k(0)|] \leq \mathbb{E}[f_i^k(W^k) - f_i^k(0)] \\ &\leq \xi_i(f_i^k(W^k)) + \xi_i^*(1) - f_i^k(0) \leq \xi_i^*(1) - f_i^k(0), \end{aligned}$$

where we used that $A_i^k = f_i^k(W^k) \in \mathcal{A}_i$. Hence,

$$\forall i \in I: \ |f_i^k(0)| \leq \xi_i^*(1) + \|W^k\|_1. \tag{5.8}$$

If $j \in J$, we obtain from the requirement $f_1^k + \cdots + f_n^k = id_{\mathbb{R}}$

$$|f_j^k(0)| = -f_j^k(0) \leq -\sum_{i \in J} f_i^k(0) = \sum_{i \in I} f_i^k(0) \leq \sum_{i \in [n]} \xi_i^*(1) + n\|W^k\|_1 =: \gamma_k.$$

Thus, $\mathbf{f}^k \in \mathfrak{C}_{\gamma_k}$. As the bound γ_k depends on k only in terms of $\|W^k\|_1$ which is uniformly bounded in k by the first and the second step, $\gamma := \sup_{k \in \mathbb{N}} \gamma_k < \infty$ and $(\mathbf{f}^k)_{k \in \mathbb{N}} \subseteq \mathfrak{C}_\gamma$. After passing to subsequences two times, we can find a subsequence $(k_\lambda)_{\lambda \in \mathbb{N}}$ such that

- $\ker(\pi) \ni N := \lim_{\lambda \to \infty} N^{k_\lambda}$ exists and thus $\Psi(N^{k_\lambda}) \to \Psi(N)$ for $\lambda \to \infty$.
- for a suitable $\mathbf{f} \in \mathfrak{C}_\gamma$ it holds that $\max_{i \in [n]} |f_i^{k_\lambda} - f_i| \to 0$ pointwise for $\lambda \to \infty$, cf. Lemma 5.11.

It remains to show that $(f_i(X - \Lambda(X)U + N) + \Lambda(X)U_i + \Psi(N)_i)_{i \in [n]} \in \widehat{\mathcal{P}}(X)$ and that it is the limit of the subsequence of the Pareto-optimal allocations chosen initially. To this end, we set $\mathbf{A} := \mathbf{f}(X - \Lambda(X)U + N)$ and $g_i^{(k_\lambda)} := f_i^{(k_\lambda)} - f_i^{(k_\lambda)}(0)$. $\mathbb{P}$-a.s., the estimate

$$\begin{aligned} \left|A_i - A_i^{k_\lambda}\right| \leq & \left|(g_i - g_i^{k_\lambda})(X - \Lambda(X)U + N)\right| \\ & + \left|f_i^{k_\lambda}(X - \Lambda(X)U + N) - f_i^{k_\lambda}(X^{k_\lambda} - \Lambda(X^{k_\lambda})U + N^{k_\lambda})\right| \\ & + \left|f_i(0) - f_i^{k_\lambda}(0)\right| \end{aligned} \tag{5.9}$$

holds. The third term vanishes for $\lambda \to \infty$. The first term vanishes in norm due to dominated convergence. From the estimate

$$\begin{aligned} & \left\|\left|f_i^{k_\lambda}(X - \Lambda(X)U + N) - f_i^{k_\lambda}(X^{k_\lambda} - \Lambda(X^{k_\lambda})U + N^{k_\lambda})\right|\right\|_1 \\ \leq & \left\|X - X^{k_\lambda} - (\Lambda(X) - \Lambda(X^{k_\lambda}))U + N - N^{k_\lambda}\right\|_1, \end{aligned}$$

we infer the second term vanishes in norm, as well. Set $\mathbf{N} := \Psi(N)$. Lower semicontinuity of ρ_i — which follows from Theorem 5.3 applied in the case $n=1$, see Remark 5.4 — yields

$$\begin{aligned}\sum_{i=1}^n \rho_i(A_i + \Lambda(X)U_i - N_i) &\le \liminf_{\lambda\to\infty} \sum_{i=1}^n \rho_i(A_i^{k_\lambda} + \Lambda(X^{k_\lambda})U_i - N_i^{k_\lambda}) \\ &= \liminf_{\lambda\to\infty} \Lambda(X^{k_\lambda}) = \Lambda(X).\end{aligned}$$

The definition of Λ eventually yields that the inequality is actually an equality, i.e.

$$\sum_{i=1}^n \rho_i(A_i + \Lambda(X)U_i - N_i) = \Lambda(X).$$

We have proved that $(A_i + \Lambda(X)U_i - N_i)_{i\in[n]} \in \widehat{\mathcal{P}}(X)$ and thus upper hemicontinuity, cf. Appendix A.3.
The same proof applies if $X \in \operatorname{dom}(\Lambda)$ is such that Λ is continuous at X. $\square$

We now turn our attention to localising the results to the case when for all $i \in [n]$, $\mathcal{X} = \mathcal{X}_i$ and the space $\mathcal{X}$ conforms with one of the cases **(BC)** or **(UC)**. As a first step, we need the following crucial extension result:

Lemma 5.12. *Let $\mathcal{R} := (\mathcal{A}, \mathcal{S}, \mathfrak{p})$ be a risk measurement regime on a Banach lattice $\mathcal{X}$ satisfying* **(BC)** *or* **(UC)**. *Suppose that $\mathcal{A}$ is $\|\cdot\|$-closed, law-invariant and satisfies $\mathcal{A}\cap\mathbb{R} \neq \emptyset$, and $\mathfrak{p}(Z) = \mathbb{E}[QZ]$ for some $Q \in \operatorname{dom}(\sigma_{\mathcal{A}}) \cap L^\infty$. If we set $\mathcal{B} := \operatorname{cl}_{\|\cdot\|_1}(\mathcal{A})$, $\mathfrak{R} := (\mathcal{B}, \mathcal{S}, \mathfrak{p})$ is a risk measurement regime on L^1 and $\rho_{\mathfrak{R}}|_{\mathcal{X}} = \rho_{\mathcal{R}}$.*

Proof. We first prove that

$$\mathcal{A} = \{Y \in \mathcal{X} : \forall\, Q \in \operatorname{dom}(\sigma_{\mathcal{A}}) \cap L^\infty \ (\mathbb{E}[QY] \le \sigma_{\mathcal{A}}(Q))\}. \tag{5.10}$$

In case **(BC)** this follows from $\mathcal{A}$ being closed in the $\sigma(L^\infty, L^\infty)$-topology, the weak topology associated to the dual pairing $\langle L^\infty, L^\infty\rangle$; cf. [40, Proposition 1.2]. Now consider the case **(UC)** and define the *convex indicator* of $\mathcal{A}$ to be

$$\delta_{\mathcal{A}} : \mathcal{X} \ni X \mapsto \begin{cases}\infty, & X \notin \mathcal{A},\\ 0, & X \in \mathcal{A}.\end{cases}$$

This is a convex and law-invariant function. Without loss of generality we may assume $\mathcal{X} \neq L^1$. As $\delta_{\mathcal{A}}$ has the Fatou property in the sense of [12], $\delta_{\mathcal{A}}$ is l.s.c. in the $\sigma(\mathcal{X}, L^\infty)$-topology by [12, Proposition 2.11]. This directly implies (5.10).
Furthermore, the identities $\operatorname{dom}(\sigma_{\mathcal{B}}) = \operatorname{dom}(\sigma_{\mathcal{A}}) \cap L^\infty$ and $\sigma_{\mathcal{B}} = \sigma_{\mathcal{A}}|_{L^\infty}$ are easily verified. By Lemma A.1,

$$\begin{aligned}\mathcal{B} &= \{Y \in L^1 : \mathbb{E}[QY] \le \sigma_{\mathcal{B}}(Q),\ \forall\, Q \in \operatorname{dom}(\sigma_{\mathcal{B}})\}\\ &= \{Y \in L^1 : \mathbb{E}[QY] \le \sigma_{\mathcal{A}}(Q),\ \forall\, Q \in \operatorname{dom}(\sigma_{\mathcal{A}}) \cap L^\infty\}.\end{aligned}$$

This shows that $\mathcal{B}$ is an acceptance set and that $\mathcal{A} = \mathcal{B} \cap \mathcal{X}$.

In order to verify (2.1), suppose $X \in L^1$ and $Z \in \mathcal{S}$ satisfy $X + Z \in \mathcal{B}$. Then

$$\mathfrak{p}(Z) = \mathbb{E}[QZ] = \mathbb{E}[Q(X+Z)] - \mathbb{E}[QX] \le \sigma_{\mathcal{B}}(Q) - \mathbb{E}[QX] < \infty.$$

Hence, $\mathfrak{R}$ is a risk measurement regime on L^1. For the identity $\rho_{\mathfrak{R}}|_{\mathcal{X}} = \rho_{\mathcal{R}}$, note that for $X \in \mathcal{X}$ and for $Z \in \mathcal{S}$, $X - Z \in \mathcal{B}$ if and only if $X - Z \in \mathcal{B} \cap \mathcal{X} = \mathcal{A}$. We infer $\rho_{\mathfrak{R}}(X) = \rho_{\mathcal{R}}(X)$, $X \in \mathcal{X}$. □

For $\mathbf{f} \in \mathfrak{C}$, $i \in [n]$ and $X \in \mathcal{X}$, 1-Lipschitz continuity of the function f_i yields $|f_i(X)| \le |X| + |f_i(0)| \in \mathcal{X}$ $\mathbb{P}$-a.s. As $\mathcal{X}$ is an ideal, $f_i(X) \in \mathcal{X}$ holds as well; hence, $\mathbf{f}(\mathcal{X}) \subseteq \mathcal{X}^n$, and if we plug in $X \in \mathcal{X}$ in (5.2), the resulting Pareto-optimal allocation lies in $\mathcal{X}^n$ because $\mathbf{U}, \mathbf{N} \in \mathcal{X}^n$ as $\mathcal{S}_i \subseteq \mathcal{X}$ for all $i \in [n]$. We can now give the proof of Theorem 5.8.

Proof of Theorem 5.8. Let $\mathfrak{R}_i$ denote the extension of the risk measurement regime $\mathcal{R}_i$ to L^1 as in Lemma 5.12. Apply Theorem 5.3 to $\rho_{\mathfrak{R}_1}, \dots, \rho_{\mathfrak{R}_n}$ and $X \in \mathcal{X}$ to obtain generalised versions of Theorem 5.3(2) and Corollary 5.5. This in conjunction with Proposition 3.4 generalises Theorem 5.3(1). The proof of Theorem 5.7 only needs to be altered at (5.8) and (5.9). We may replace $\|W^k\|_1$ by $K\|W^k\|$ in the first and use the order continuity of $\|\cdot\|$ in the second instance, where the constant K is chosen as in (5.3). □

Finally we turn to the upper hemicontinuity of the equilibrium correspondence $\mathcal{E}$ as formulated in Theorem 5.9, and we prove this theorem.

Proof of Theorem 5.9. Let $\mathbf{W}$ be such that $W := \sum_{i=1}^n W_i \in \operatorname{int}\operatorname{dom}(\Lambda)$. From the proof of Theorem 3.5 we infer that, indeed, every $(\mathbf{X}, \phi) \in \mathcal{E}(\mathbf{W})$ is an equilibrium of $\mathbf{W}$. For upper hemicontinuity, we shall first establish that the equilibrium prices of an approximating sequence lie in a sequentially relatively compact set in the dual $\mathcal{X}^*$. We shall hence prove that there is $\varepsilon > 0$ and constants c_1 and c_2 only depending on W such that, given any $X \in \mathcal{X}$ with $\|X - W\| \le \varepsilon$ and any subgradient ϕ of Λ at X, it holds that

$$\|\phi\|_* \le c_1 \text{ and } \Lambda^*(\phi) = \textstyle\sum_{i=1}^n \rho_i^*(\phi) \le c_2.$$

As we shall elaborate later, these bound imply that all subgradients of Λ at vectors in a closed ball around W lie in a $\sigma(\mathcal{X}^*, \mathcal{X})$-sequentially compact set.
In order to prove the assertion, continuity of Λ on $\operatorname{int}\operatorname{dom}(\Lambda)$ (see [20, Corollary 2.5]) allows us to choose $\varepsilon > 0$ such that $|\Lambda(W+Y) - \Lambda(W)| \le 1$ whenever $\|Y\| \le 2\varepsilon$. Let now $\delta > 0$ be such that $\delta\varepsilon + \delta\|W\| \le \varepsilon$ and fix X such that $\|X - W\| \le \varepsilon$ and a subgradient ϕ of Λ at X. Moreover, suppose $Y \in \mathcal{X}$ is such that $\|Y\| \le 1$. We obtain from the subgradient inequality

$$\Lambda(X) + \varepsilon\phi(Y) \le \Lambda(X + \varepsilon Y) \le \Lambda(W) + 1.$$

Rearranging this inequality yields

$$\|\phi\|_* = \sup_{\|Y\| \le 1} \phi(Y) \le \frac{\Lambda(W) + 1 - \Lambda(X)}{\varepsilon} \le \frac{2}{\varepsilon} =: c_2.$$

Moreover,

$$\begin{aligned}\Lambda(X) = \phi(X) - \Lambda^*(\phi) &= \frac{1}{1+\delta}\left(\phi((1+\delta)X) - \Lambda^*(\phi)\right) - \frac{\delta}{1+\delta}\Lambda^*(\phi)\\ &\leq \frac{1}{1+\delta}\Lambda((1+\delta)X) - \frac{\delta}{1+\delta}\Lambda^*(\phi).\end{aligned}$$

By rearranging this inequality we obtain

$$\sum_{i=1}^n \rho_i^*(\phi) = \Lambda^*(\phi) \leq \frac{1}{\delta}\Lambda((1+\delta)X) - \frac{1+\delta}{\delta}\Lambda(X) \leq \frac{2+\delta}{\delta} - \Lambda(W) =: c_1,$$

where we have used $\|(1+\delta)X - W\| \leq 2\varepsilon$ following from the choice of δ.
Now consider a sequence $(\mathbf{W}^k)_{k\in\mathbb{N}} \subseteq \prod_{i=1}^n \operatorname{int}\operatorname{dom}(\rho_i)$ such that, for all $i \in [n]$, $W_i^k \to W_i$, $k \to \infty$, holds. Without loss of generality, we may assume that $W^k := W_1^k + \cdots + W_n^k$ lies in the ball around W with radius ε. For each $k \in \mathbb{N}$ assume that $(\mathbf{X}^k, \phi_k) \in \mathcal{E}(\mathbf{W}^k)$, $k \in \mathbb{N}$. We set

$$X_i^k = Y_i^k + \frac{\phi_k(W_i^k - Y_i^k)}{\phi(\tilde{Z})}\tilde{Z}, \quad i \in [n].$$

As $\mathbf{Y}^k \in \widehat{\mathcal{P}}(W^k)$ and $W^k \to W$, $k \to \infty$, we may assume, after passing to a subsequence, that $\mathbf{Y}^k \to \mathbf{Y} \in \widehat{\mathcal{P}}(W)$ by Theorem 5.7.
We shall now select a convergent subsequence $(\phi^k)_{k\in\mathbb{N}}$. In case **(BC)**, we conclude from [33, Proposition 3.1(iii)] and Lemma A.4 that

$$\operatorname{dom}(\Lambda^*) \subseteq \operatorname{dom}(\rho_1^*) \subseteq L^1,$$

which implies that all subgradients ψ of Λ have the shape $\psi = \mathbb{E}[\bar{Q}\,\cdot]$ for a unique $\bar{Q} \in L^1_+$. Hence, the equilibrium prices are given by $\phi_k = \mathbb{E}[Q_k \cdot]$ for a unique $Q_k \in L^1_+$. Moreover, all subgradients Q_k lie in the $\sigma(L^1, L^\infty)$-compact set $\mathcal{L}_{c_1}(\rho_1^*)$. We may invoke the Eberlein-Šmulian theorem [3, Theorem 6.34] to find a subsequence $(k_\lambda)_{\lambda\in\mathbb{N}}$ such that $Q_{k_\lambda} \to Q \in L^1$ weakly, or equivalently $\phi_{k_\lambda} \to \phi = \mathbb{E}[Q\,\cdot]$ in $\sigma(\mathcal{X}^*, \mathcal{X})$. In case **(UC)**, reflexivity of $\mathcal{X}$, the Banach-Alaoglu theorem and the bounds above imply the existence of a sequentially relatively compact set Γ such that $\phi \in \Gamma$ whenever $\|X - W\| \leq \varepsilon$ and ϕ is a subgradient of Λ at X. Hence there is a $\sigma(\mathcal{X}^*, \mathcal{X})$-convergent subsequence $(\phi_{k_\lambda})_{\lambda\in\mathbb{N}}$.
Consequently, in both cases,

$$\phi_{k_\lambda}(W_i^{k_\lambda} - Y_i^{k_\lambda}) \to \phi(W_i - Y_i), \ \lambda \to \infty.$$

It remains to prove that ϕ is a subgradient of Λ at W. But as Λ^* is l.s.c. in the $\sigma(\mathcal{X}^*, \mathcal{X})$-topology and $\phi_{k_\lambda}(W^{k_\lambda}) \to \phi(W)$, we obtain

$$\begin{aligned}\Lambda(W) &= \limsup_{\lambda\to\infty} \phi_{k_\lambda}(W^{k_\lambda}) - \Lambda^*(\phi_{k_\lambda}) = \phi(W) - \liminf_{\lambda\to\infty} \Lambda^*(\phi_{k_\lambda})\\ &\leq \phi(W) - \Lambda^*(\phi),\end{aligned}$$

which implies that, necessarily, $\Lambda(W) = \phi(W) - \Lambda^*(\phi)$ and ϕ is a subgradient of Λ at W. $\square$

5.4. **Examples.** We conclude with two examples.

Example 5.13. We consider the model space $\mathcal{X} := L^1$ on which two agents operate with acceptability criteria given by the *entropic risk measure*. More precisely, we choose $0 < \beta \leq \gamma$ arbitrary and define

$$\mathcal{A}_1 := \{X \in L^1 : \xi_\beta(X) \leq 0\}, \quad \mathcal{A}_2 := \{X \in L^1 : \xi_\gamma(X) \leq 0\},$$

where, for $\alpha > 0$, $\xi_\alpha(X) := \frac{1}{\alpha}\log\left(\mathbb{E}[e^{\alpha X}]\right)$, $X \in L^1$. It is well-known, cf. [26, Example 2.9], that

$$\xi := \xi_\beta \Box \xi_\gamma = \xi_{\frac{\beta\gamma}{\beta+\gamma}}.$$

The convex conjugate ξ^*_α of ξ_α is given in terms of the relative entropy: for all $Q \in L^\infty_+$ such that $\mathbb{E}[Q] = 1$, we have

$$\xi^*_\alpha(Q) = \tfrac{1}{\alpha}\mathbb{E}[Q\log(Q)] < \infty.$$

In order to satisfy Assumption 5.2, we may hence choose any pricing density $Q^* \in L^\infty_+$ such that $Q^* \geq \delta > 0$ for some $\delta > 0$. The pricing functionals are given by $\mathfrak{p}_i = \mathbb{E}[Q^* \cdot]$. Moreover, we choose $A \in \mathcal{F}$ such that $\mathbb{E}[Q^*\mathbf{1}_A] = \mathbb{E}[Q^*\mathbf{1}_{A^c}]$, $\mathcal{S}_1 = \mathcal{M} = \operatorname{span}\{\mathbf{1}_A, \mathbf{1}_{A^c}\}$, and $\mathcal{S}_2 = \operatorname{span}\{\mathbf{1}_A\}$. Given these specifications, $(\mathcal{R}_1, \mathcal{R}_2)$ is an agent system.

Note that $\ker(\pi) = \{N_r := r\mathbf{1}_A - r\mathbf{1}_{A^c} : r \in \mathbb{R}\}$. We will now characterise $\mathscr{A}_+ + \ker(\pi)$ and set, for the sake of brevity,

$$\alpha := \frac{\beta\gamma}{\beta+\gamma}.$$

Given the characterisation of $\mathscr{A}_+$, $X - N_r \in \mathscr{A}_+$ for some $r \in \mathbb{R}$ if and only if $\mathbb{E}[e^{\alpha X}\mathbf{1}_A] \cdot \mathbb{E}[e^{\alpha X}\mathbf{1}_{A^c}] \leq \frac{1}{4}$, as there is then a solution $r \in \mathbb{R}$ to

$$0 \geq \frac{1}{\alpha}\log\left(\mathbb{E}[e^{\alpha(X-N_r)}]\right) = \frac{1}{\alpha}\log\left(e^{-\alpha r}\mathbb{E}[e^{\alpha X}\mathbf{1}_A] + e^{\alpha r}\mathbb{E}[e^{\alpha X}\mathbf{1}_{A^c}]\right).$$

Now, for arbitrary $X \in \operatorname{dom}(\Lambda) = \operatorname{dom}(\xi_\alpha)$, we note that

$$\begin{aligned}
\Lambda(X) &= \inf\{\pi(r\mathbf{1}) : r \in \mathbb{R},\ X - r\mathbf{1} \in \mathscr{A}_+ + \ker(\pi)\} \\
&= \inf\left\{r\mathbb{E}[Q^*] : r \in \mathbb{R},\ e^{-\alpha r}\mathbb{E}[e^{\alpha X}\mathbf{1}_A] \cdot \mathbb{E}[e^{\alpha X}\mathbf{1}_{A^c}] \leq \frac{1}{4}\right\} \\
&= \frac{\mathbb{E}[Q^*]}{\alpha}\left(\log\mathbb{E}[e^{\alpha X}\mathbf{1}_A] + \log\mathbb{E}[e^{\alpha X}\mathbf{1}_{A^c}] + 2\log(2)\right).
\end{aligned}$$

Hereafter, we choose a solution r^* of

$$e^{-\alpha r}\mathbb{E}[e^{\alpha(X-\Lambda(X))}\mathbf{1}_A] + e^{\alpha r}\mathbb{E}[e^{\alpha(X-\Lambda(X))}\mathbf{1}_{A^c}] = 1,$$

e.g.

$$r^* := \log\left(\frac{2\mathbb{E}[e^{\alpha(X-\Lambda(X))}\mathbf{1}_A]}{\sqrt{1 - 4\mathbb{E}[e^{\alpha(X-\Lambda(X))}\mathbf{1}_A] \cdot \mathbb{E}[e^{\alpha(X-\Lambda(X))}\mathbf{1}_{A^c}]} + 1}\right).$$

Using the results from [26, Example 2.9],

$$\left(\frac{\gamma}{\beta+\gamma}(X - \Lambda(X)\mathbf{1} - N_{r^*}), \frac{\beta}{\beta+\gamma}(X - \Lambda(X)\mathbf{1} - N_{r^*})\right) \in \mathcal{A}_1 \times \mathcal{A}_2.$$

Consequently, the following is a Pareto-optimal allocation of X:

$$\left(\frac{\gamma}{\beta+\gamma}(X-\Lambda(X)\mathbf{1}-N_{r^*})+\Lambda(X)\mathbf{1}+N_{r^*},\frac{\beta}{\beta+\gamma}(X-\Lambda(X)\mathbf{1}-N_{r^*})\right)$$

Example 5.14. Here, we choose the model space $\mathcal{X}=L^\infty$ and illustrate the existence of Pareto-optimal allocations for two agents with acceptance sets less similar than in Example 5.13. To this end, we fix two parameters $\beta\in(0,1)$ and $\gamma>0$ and suppose that acceptability for agent 1 is based on the *Average Value at Risk*, i.e.

$$\begin{aligned}\mathcal{A}_1&=\{X\in L^\infty:\xi_1(X):=\mathrm{AVaR}_\beta(X)\le 0\}\\&=\{X\in L^\infty:\forall\, Q\in\mathcal{Q}\ (\mathbb{E}[QX]\le 0)\},\end{aligned}$$

where $\mathcal{Q}=\{Q\in L^\infty_+:\ 0\le Q\le\frac{1}{1-\beta}\ \mathbb{P}\text{-a.s.},\ \mathbb{E}[Q]=1\}$. The acceptance set of agent 2 is, as in Example 5.13, given by an entropic risk measure, i.e.

$$\mathcal{A}_2:=\{X\in L^\infty:\ \xi_2(X):=\tfrac{1}{\gamma}\log\left(\mathbb{E}[e^{\gamma X}]\right)\le 0\}.$$

By [27, Example 4.34 & Theorem 4.52], the support function of $\mathscr{A}_+=\mathcal{A}_1+\mathcal{A}_2$ is given for $Q\in L^\infty_+$ by

$$\sigma_{\mathscr{A}_+}(Q)=(\sigma_{\mathcal{A}_1}+\sigma_{\mathcal{A}_2})(Q)=\begin{cases}0,\ Q=0\\ \frac{1}{\gamma}\mathbb{E}\left[Q\log\left(\frac{Q}{\mathbb{E}[Q]}\right)\right],\ \text{if } Q\neq 0 \text{ and } \frac{Q}{\mathbb{E}[Q]}\in\mathcal{Q},\\ \infty \text{ otherwise.}\end{cases}$$

As in Example 2.2, we choose a pricing density $Q^*\in L^\infty_+$ such that, for some $\delta\in(0,1)$, $\delta\le Q^*\le\frac{1-\delta}{1-\beta}+\delta$ holds and such that $\mathbb{E}[Q^*]=1$. In this case, $Q^*=\delta+(1-\delta)Q$, where $Q=\frac{Q^*-\delta}{1-\delta}\in\mathcal{Q}$, hence Q^* satisfies Assumption 5.2.
Suppose the security spaces $\mathcal{S}_i$, $i=1,2$, are given as in Example 5.13 for some nonempty $A\in\mathcal{F}$. As pricing rules we set $\mathfrak{p}_i:=\mathbb{E}[Q^*\cdot]$, $i=1,2$, which results in

$$\ker(\pi)=\mathrm{span}\{N:=\mathbf{1}_A-r^*\mathbf{1}_{A^c}\},\quad r^*=\frac{\mathbb{E}[Q^*\mathbf{1}_A]}{1-\mathbb{E}[Q^*\mathbf{1}_A]}.$$

Let $X\in L^\infty$ be any aggregated loss. Using [24, Theorem 3], we obtain the dual representation

$$\Lambda(X)=\max_{Q\in\widetilde{\mathcal{Q}}}\mathbb{E}[QX]-\frac{1}{\gamma}\mathbb{E}[Q\log(Q)],$$

where $\widetilde{\mathcal{Q}}=\{Q\in\mathcal{Q}:\ \mathbb{E}[Q\mathbf{1}_A]=\mathbb{E}[Q^*\mathbf{1}_A]\}$. We will now compute the right scaling factor $s\in\mathbb{R}$ such that $X-\Lambda(X)-sN\in\mathscr{A}_+$. This is the case if and only if we have for all $Q\in\mathcal{Q}\backslash\widetilde{\mathcal{Q}}$

$$\mathbb{E}[QX]-\frac{1}{\gamma}\mathbb{E}[Q\log(Q)]-\Lambda(X)\le s\mathbb{E}[QN].$$

We obtain

$$s\ge\sup_{Q\in\mathcal{Q}\backslash\widetilde{\mathcal{Q}}:\ \mathbb{E}[Q\mathbf{1}_A]>\mathbb{E}[Q^*\mathbf{1}_A]}\frac{\mathbb{E}[QX]-\frac{1}{\gamma}\mathbb{E}[Q\log(Q)]+\Lambda(X)}{\mathbb{E}[QN]}$$

and

$$s \leq \inf_{\mathbb{Q}\in\mathcal{Q}\setminus\widetilde{\mathcal{Q}}:\,\mathbb{E}[Q\mathbf{1}_A]<\mathbb{E}[Q^*\mathbf{1}_A]} \frac{\frac{1}{\gamma}\mathbb{E}[Q\log(Q)]+\Lambda(X)-\mathbb{E}[QX]}{|\mathbb{E}[QN]|},$$

and the bounds describe an *a priori* nonempty interval. Choose any s^* in this interval. Combining [31, Proposition 3.2 & Sect. 3.5], we obtain that

$$\big((X-\Lambda(X)-s^*N-\zeta)^+,(X-\Lambda(X)-s^*N)\wedge\zeta\big)\in\mathcal{A}_1\times\mathcal{A}_2.$$

for a suitable $\zeta\in\mathbb{R}$. Thus (X_1,X_2) given by

$$\begin{aligned}X_1(\zeta)&=(X-\Lambda(X)-s^*N-\zeta)^+-s^*r^*\mathbf{1}_{A^c}+\Lambda(X),\\ X_2(\zeta)&=(X-\Lambda(X)-s^*N)\wedge\zeta+s^*\mathbf{1}_A,\end{aligned}$$

is a Pareto-optimal allocation of X.

6. Optimal portfolio splits

In this section we study the existence of optimal portfolio splits. For a thorough discussion of this problem, we refer to Tsanakas [42], although the problem we consider is rather akin to Wang [43]. A financial institution holds a portfolio which yields the future loss W. In order to diversify the risk posed by W, it may consider dividing the portfolio into n sub-portfolios $X_1,\dots,X_n\in\mathcal{X}$, $X_1+\cdots+X_n=W$, and transfer these sub-portfolios to, e.g., distinct legal entities such as subsidiaries which operate under potentially varying regulatory regimes. As observed by Tsanakas, for convex, but not positively homogeneous risk measures, *without* market frictions like transaction costs risk can usually be reduced arbitrarily by introducing more subsidiaries, and hence, there is no incentive to stop this splitting procedure. However, since n can be arbitrarily large, transaction costs should not be neglected in this setting, and we will study the problem of finding *cost-optimal portfolio splits* under market frictions.
To be more precise, we model the subsidiaries as a family $(\rho_i)_{i\in\mathbb{N}}$ of *normalised* risk measures on one and the same Fréchet lattice $(\mathcal{X},\preceq,\tau)$ – which entails $\rho_i^*\geq 0$ for all $i\in\mathbb{N}$ – such that the associated risk measurement regimes $(\mathcal{R}_i)_{i\in\mathbb{N}}$ check infinite supportability (A4): as one and the same parent company splits the losses into n sub-portfolios, assuming that, for each $n\in\mathbb{N}$, the set of subsidiaries $(\rho_i)_{i\in[n]}$ forms an agent system satisfying (A3) seems natural. Let further $\mathfrak{c}:\mathbb{N}\to[0,\infty)$ be a nondecreasing cost function. The transaction costs of introducing subsidiaries $i\in[n]$ and splitting a portfolio among them are given by $\mathfrak{c}(n)$. The condition $\lim_{n\to\infty}\mathfrak{c}(n)=\infty$ prevents infinite splitting. At last we introduce $\Lambda_n(X):=\inf_{\mathbf{X}\in\mathbb{A}_X}\sum_{i=1}^n\rho_i(X_i)$, $X\in\mathcal{X}$, the usual risk sharing functional associated to $(\mathcal{R}_1,\dots,\mathcal{R}_n)$. Note that for all $X\in\mathcal{X}$, $n\in\mathbb{N}$, and every $\mathbf{X}\in\mathcal{X}^n$ with $\sum_{i=1}^n X_i=X$, the estimate $\sum_{i=1}^n\rho_i(X_i)=\sum_{i=1}^n\rho_i(X_i)+\rho_{n+1}(0)\geq\Lambda_{n+1}(X)$ holds, which entails $\Lambda_n(X)\geq\Lambda_{n+1}(X)$, $n\in\mathbb{N}$. In this setting, optimal portfolio splits exist if each Λ_n is exact on $\mathrm{dom}(\Lambda_n)$:

Theorem 6.1. *Suppose $(\mathcal{R}_i)_{i\in\mathbb{N}}$ is a sequence of risk measurement regimes on a Fréchet lattice $\mathcal{X}$ which checks* (A4) *and results in all ρ_i being normalised. Moreover, assume that* *the cost function satisfies*

$$\lim_{n\to\infty} \mathfrak{c}(n) = \infty$$

and that Λ_n *is exact on* $\mathrm{dom}(\Lambda_n)$ *for all* $n \in \mathbb{N}$ *and let* $W \in \sum_{i=1}^m \mathrm{dom}(\rho_i)$ *for some* $m \in \mathbb{N}$. *Then there is* $(n_*, X_1, \dots, X_{n_*})$*, where* $n_* \in \mathbb{N}$ *and* $(X_1, \dots, X_{n_*})$ *is an attainable allocation of* W*, which is a solution of*

$$\sum_{i=1}^{n} \rho_i(X_i) + \mathfrak{c}(n) \to \min \quad \textit{subject to } n \in \mathbb{N} \textit{ and } \mathbf{X} \in \mathcal{X}^n \textit{ with } \sum_{i=1}^{n} X_i = W. \tag{6.1}$$

Proof. Note that (A4) can be rewritten as

$$\exists\, \phi_0 \in \bigcap_{i=1}^{\infty} \mathrm{dom}(\rho_i^*) : \ \sum_{i=1}^{\infty} \rho_i^*(\phi_0) < \infty. \tag{6.2}$$

Let

$$m_* := \min\{m \in \mathbb{N} : \Lambda_m(W) < \infty\} = \min\{m \in \mathbb{N} : W \in \sum_{i=1}^{m} \mathrm{dom}(\rho_i)\} < \infty.$$

By (6.2), we have $\Lambda_n(W) \geq \phi_0(W) - \sum_{i=1}^{\infty} \rho_i^*(\phi_0) > -\infty$ for all $n \geq m_*$. Thus, $\Lambda_n(W) + \mathfrak{c}(n) = \infty$ whenever $n < m_*$ and

$$\liminf_{n\to\infty} \Lambda_n(W) + \mathfrak{c}(n) \geq \phi_0(W) - \sum_{i=1}^{\infty} \rho_i^*(\phi_0) + \lim_{n\to\infty} \mathfrak{c}(n) = \infty.$$

Therefore, we can find $n_* \in \mathbb{N}$ such that

$$\Lambda_{n_*}(W) + \mathfrak{c}(n_*) = \inf_{n\in\mathbb{N}} \Lambda_n(W) + \mathfrak{c}(n) \in \mathbb{R}.$$

In order to obtain a solution to (6.1), choose an attainable allocation $\mathbf{X} \in \mathcal{X}^{n_*}$ of X such that $\Lambda_{n_*}(X) = \sum_{i=1}^{n_*} \rho_i(X_i)$. □

Corollary 6.2. *Suppose* $(\mathcal{R}_i)_{i\in\mathbb{N}}$ *is a sequence of risk measurement regimes on a Fréchet lattice* $\mathcal{X}$ *such that all* ρ_i *are normalised. Then the assertion of Theorem 6.1 holds under each of the following conditions:*

1. *The risk measures* $(\rho_1, \dots, \rho_n)$ *comply with Theorem 5.8 for each* $n \in \mathbb{N}$ *and the pricing functionals are given by* $\mathfrak{p}_i = \mathbb{E}[(Q+\delta)\cdot]|_{\mathcal{S}_i}$ *for a fixed* $\delta > 0$ *and* $Q \in L^\infty_+$ *with* $\sup_{Y\in\mathcal{A}_i} \mathbb{E}[(Q+\delta)Y] \leq 0$, $i \in \mathbb{N}$.
2. (A4) *is satisfied, and for each* $n \in \mathbb{N}$, $(\mathcal{R}_1, \dots, \mathcal{R}_n)$ *is a polyhedral agent system.*

Proof. (1) Let $Q \in L^\infty_+$ and $\delta > 0$ be as described in the assertion and set $Q^* := Q + \delta$. Assumption 5.2 is satisfied. Let $i \in \mathbb{N}$ be arbitrary and recall the definition of the cash-additive risk measures ξ_i in the proof of Theorem 5.3. By (5.6), $\xi_i^*(\frac{Q^*}{\mathbb{E}[Q^*]}) \leq 0$. Theorem 5.8 in the case $n = 1$ (see Remark 5.4) yields that each $X \in \mathrm{dom}(\rho_i)$ admits an optimal payoff

$Z^X \in \mathcal{S}_i$, i.e. $X - Z^X \in \mathcal{A}_i$ and $\mathbb{E}[Q^* Z^X] = \mathfrak{p}_i(Z^X) = \rho_i(X)$. Hence,

$$\rho_i^*(Q^*) = \sup_{X \in \mathrm{dom}(\rho_i)} \mathbb{E}[Q^* X] - \rho_i(X) = \sup_{X \in \mathrm{dom}(\rho_i)} \mathbb{E}[Q^*(X - Z^X)]$$
$$\leq \mathbb{E}[Q^*]\xi_i^*\left(\tfrac{Q^*}{\mathbb{E}[Q^*]}\right) \leq 0.$$

Conversely, as ρ_i is normalised, we have $\rho_i^*(Q^*) \geq 0$. Hence, (A4) holds and ϕ_0 in (6.2) may be chosen as $\phi_0 = \mathbb{E}[Q^* \cdot]$. The solvability of (6.1) under (1) follows from Theorems 5.8 and 6.1.

(2) By Theorem 4.3 Λ_n is exact on $\mathrm{dom}(\Lambda_n)$ for every $n \in \mathbb{N}$. Apply Theorem 6.1. □

Remark 6.3. Suppose that in the situation of Corollary 6.2(1) each of the monetary base risk measures $\xi_i(X) := \inf\{m \in \mathbb{R} : X - m \in \mathcal{A}_i\}$, $X \in \mathcal{X}$, is normalised. Then each $\delta > 0$ and each $Q \in \mathbb{R}_+$ satisfy the assumptions of part (1). This follows from the fact that $\xi_i^*(1) = 0$ holds for every $i \in \mathbb{N}$. Indeed, by arguments similar to the proof of Proposition 5.10, $\xi_i^*(\mathbb{E}[Q|\mathcal{H}]) \leq \xi_i^*(Q)$ holds for all $Q \in \mathrm{dom}(\xi_i^*)$ and all sub-σ-algebras $\mathcal{H} \subseteq \mathcal{F}$. Hence,

$$\xi_i^*(1) = \inf_{Q \in \mathrm{dom}(\xi_i^*)} \xi_i^*(Q) = -\sup_{Q \in \mathrm{dom}(\xi_i^*)} -\xi_i^*(Q) = -\xi_i(0) = 0, \quad i \in \mathbb{N}.$$

Appendix A. Technical supplements

A.1. **The geometry of convex sets.** Fix a nonempty convex subset $\mathcal{C}$ of a locally convex Hausdorff topological Riesz space $(\mathcal{X}, \preceq, \tau)$ with dual space $\mathcal{X}^*$. The *support function* of $\mathcal{C}$ is the functional

$$\sigma_{\mathcal{C}} : \mathcal{X}^* \to (-\infty, \infty], \quad \phi \mapsto \sup_{Y \in \mathcal{C}} \phi(Y).$$

The *recession cone* of $\mathcal{C}$ is the set

$$0^+\mathcal{C} := \{U \in \mathcal{X} : Y + kU \in \mathcal{C},\ \forall\, Y \in \mathcal{C},\ \forall\, k \geq 0\}.$$

A vector U lies in $0^+\mathcal{C}$ if and only if $Y + U \in \mathcal{C}$ holds for all $Y \in \mathcal{C}$. U is then called a *direction* of $\mathcal{C}$. The *lineality space* of $\mathcal{C}$ is the vector space $\mathrm{lin}(\mathcal{C}) := 0^+\mathcal{C} \cap (-0^+\mathcal{C})$. In the case of an acceptance set $\mathcal{A}$, monotonicity implies $\mathrm{dom}(\sigma_{\mathcal{A}}) \subseteq \mathcal{X}_+^*$. If $\mathcal{C}$ is closed, the Hahn-Banach separation theorem shows that

$$\mathcal{C} = \{Y \in \mathcal{X} : \phi(Y) \leq \sigma_{\mathcal{C}}(\phi),\ \forall\, \phi \in \mathrm{dom}(\sigma_{\mathcal{C}})\}.$$

Combining this identity with the definition of the recession cone and the lineality space yields

Lemma A.1. *If $\mathcal{C} \subseteq \mathcal{X}$ is closed and convex and $\mathcal{J} \subseteq \mathrm{dom}(\sigma_{\mathcal{C}})$ is such that*

$$\mathcal{C} = \{X \in \mathcal{X} : \phi(X) \leq \sigma_{\mathcal{C}}(\phi),\ \forall\, \phi \in \mathcal{J}\},$$

then

$$0^+\mathcal{C} = \bigcap_{\phi \in \mathcal{J}} \mathcal{L}_0(\phi) = \{U \in \mathcal{X} : \phi(U) \leq 0,\ \forall\, \phi \in \mathcal{J}\} \quad \text{and} \quad \mathrm{lin}(\mathcal{A}) = \bigcap_{\phi \in \mathcal{J}} \ker(\phi).$$

Last we state a decomposition result for closed convex sets specific to finite-dimensional spaces. It follows from arguments in the proofs of [9, Lemmas II.16.2 and II.16.3].

Lemma A.2. *Let $\mathcal{C} \subseteq \mathbb{R}^d$ be convex and closed and $\mathcal{V} := \operatorname{lin}(\mathcal{C})^\perp$. If $\operatorname{ext}(\mathcal{C} \cap \mathcal{V})$ denotes the set of extreme points of $\mathcal{C} \cap \mathcal{V}$ and $\operatorname{co}(\cdot)$ is the convex hull operator, $\mathcal{C}$ can be written as*

$$\mathcal{C} = \operatorname{co}(\operatorname{ext}(\mathcal{C} \cap \mathcal{V})) + 0^+\mathcal{C}.$$

A.2. **Infimal convolution.** Let $(\mathcal{X}, \preceq)$ be a Riesz space and suppose that functions $g_i : \mathcal{X} \to (-\infty, \infty]$, $i \in [n]$, are given. The *infimal convolution* or *epi-sum* of $g_1, \dots, g_n$ is the function $\square_{i=1}^n g_i : \mathcal{X} \to [-\infty, \infty]$ defined by

$$\square_{i=1}^n g_i(X) := \inf \left\{ \sum_{i=1}^n g_i(X_i) : X_1, \dots, X_n \in \mathcal{X}, \sum_{i=1}^n X_i = X \right\}, \quad X \in \mathcal{X}.$$

The convolution is said to be *exact* at $X \in \mathcal{X}$ if $(\square_{i=1}^n g_i)(X) \in \mathbb{R}$ and there is $X_1, \dots, X_n \in \mathcal{X}$ with $\sum_{i=1}^n X_i = X$ such that

$$\sum_{i=1}^n g_i(X_i) = (\square_{i=1}^n g_i)(X).$$

Lemma A.3. *Suppose $\mathcal{X}_i \subseteq \mathcal{X}$, $i \in [n]$, are ideals in a Riesz space $(\mathcal{X}, \preceq)$ such that $\mathcal{X} = \sum_{i=1}^n \mathcal{X}_i$.*

(1) *If all $g_i : \mathcal{X} \to (-\infty, \infty]$ are convex, then $\square_{i=1}^n g_i$ is convex.*
(2) *If g_i is monotone on $\mathcal{X}_i$ with respect to $\preceq$ for all $i \in [n]$, i.e., $X, Y \in \mathcal{X}_i$, $X \preceq Y$, implies $g_i(X) \le g_i(Y)$, and $g_i|_{\mathcal{X} \setminus \mathcal{X}_i} \equiv \infty$, then $\square_{i=1}^n g_i$ is monotone on $\mathcal{X}$.*

Proof. We only prove (2). Let $X, Y \in \mathcal{X}$, $X \preceq Y$, and let $\mathbf{X}, \mathbf{Y} \in \prod_{i=1}^n \mathcal{X}_i$ with $\sum_{i=1}^n X_i = X$ and $\sum_{i=1}^n Y_i = Y$. We thus have

$$0 \preceq Y - X = |Y - X| \preceq \textstyle\sum_{i=1}^n |Y_i - X_i|.$$

By the Riesz space property of $\mathcal{X}$ and the Riesz Decomposition Property (cf. [3, Sect. 8.5]), there is a vector $\mathbf{Z} \in (\mathcal{X}_+)^n$ such that $Y - X = \sum_{i=1}^n Z_i$ and such that $Z_i = |Z_i| \preceq |Y_i - X_i|$, $i \in [n]$. $\mathcal{X}_i$ being an ideal yields that in fact $\mathbf{Z} \in \prod_{i=1}^n \mathcal{X}_i$. By monotonicity of g_i on $\mathcal{X}_i$, $i \in [n]$, we obtain

$$(\square_{i=1}^n g_i)(X) \le \sum_{i=1}^n g_i(Y_i - Z_i) \le \sum_{i=1}^n g_i(Y_i).$$

As $(\square_{i=1}^n g_i)(Y) = \inf\{\sum_{i=1}^n g_i(Y_i) : \mathbf{Y} \in \prod_{i=1}^n \mathcal{X}_i\}$ by the assumption that $g_i|_{\mathcal{X} \setminus \mathcal{X}_i} \equiv \infty$, taking the infimum over suitable $\mathbf{Y}$ on the right-hand side proves the assertion. □

Note that the risk sharing functional satisfies $\Lambda = \square_{i=1}^n g_i$, where the functions g_i are defined by $g_i(X) = \rho_i(X)$ if $X \in \mathcal{X}_i$ and $g_i(X) = \infty$ otherwise, $X \in \mathcal{X}$, $i \in [n]$. These functions g_i inherit convexity on $\mathcal{X}$ and monotonicity on $\mathcal{X}_i$ from ρ_i.

Lemma A.4. *Given a locally convex Hausdorff topological Riesz space $(\mathcal{X}, \preceq, \tau)$ and proper functions $g_i : \mathcal{X} \to (-\infty, \infty]$, $i \in [n]$, the following identities hold:*

$$(\square_{i=1}^n g_i)^* = \sum_{i=1}^n g_i^* \quad \text{and} \quad \operatorname{dom}((\square_{i=1}^n g_i)^*) = \bigcap_{i=1}^n \operatorname{dom}(g_i^*).$$

A.3. **Correspondences.** Given two nonempty sets A and B, a map $\Gamma : A \to 2^B$ mapping elements of A to subsets of B is called a *correspondence* and will be denoted by $\Gamma : A \twoheadrightarrow B$. Assume now that $(\mathcal{X}, \tau)$ and $(\mathcal{Y}, \sigma)$ are topological spaces, and let $\Gamma : \mathcal{X} \twoheadrightarrow \mathcal{Y}$ be a correspondence.

A continuous function $\Psi : \mathcal{X} \to \mathcal{Y}$ is a *continuous selection* for the correspondence Γ if $\Psi(x) \in \Gamma(x)$ holds for all $x \in \mathcal{X}$.

If $(\mathcal{X}, \sigma)$ is first countable, Γ is *upper hemicontinuous* at $x \in \mathcal{X}$ if, whenever $(x_k)_{k\in\mathbb{N}}$ is a sequence σ-convergent to x and $(y_k)_{k\in\mathbb{N}} \subseteq \mathcal{Y}$ is such that, for each $k \in \mathbb{N}$, $y_k \in \Gamma(x_k)$, there is a limit point $y \in \Gamma(x)$ of $(y_k)_{k\in\mathbb{N}}$. If both topological spaces are first countable, Γ is *lower hemicontinuous* at $x \in \mathcal{X}$ if, whenever $(x_k)_{k\in\mathbb{N}}$ is a sequence σ-convergent to x and $y \in \Gamma(x)$, there is a subsequence $(k_\lambda)_{\lambda\in\mathbb{N}}$ and $y_\lambda \in \Gamma(x_{k_\lambda})$, $\lambda \in \mathbb{N}$, such that $y_\lambda \to y$ with respect to τ as $\lambda \to \infty$.[11] An example of a lower hemicontinuous correspondence relevant for our investigations is the security allocation map

$$\mathbb{A}^s_{\cdot} : \mathcal{M} \ni Z \mapsto \mathbb{A}_Z \cap \prod_{i=1}^n \mathcal{S}_i.$$

Lemma A.5. *The correspondence $\mathbb{A}^s_{\cdot}$ is lower hemicontinuous on the global security market $\mathcal{M}$ and admits a continuous selection $\Psi : \mathcal{M} \to \prod_{i=1}^n \mathcal{S}_i$ with respect to any norm on $\mathcal{M}$.*

Proof. Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathcal{M}$. Set $\mathcal{S}_0 := \{0\}$. We claim that there are natural numbers $0 = m_0 < m_1 \leq \cdots \leq m_n$ and $Z_1, \ldots, Z_{m_n} \in \bigcup_{i=1}^n \mathcal{S}_i$ such that for all $i \in [n]$, it holds that $\{Z_{m_{i-1}+1}, \ldots, Z_{m_i}\}$ is an orthonormal basis of $\big\{X \in \mathcal{S}_i : X \perp \operatorname{span}\{Z_1, \ldots, Z_{m_{i-1}}\}\big\}$. Note that every $Z \in \mathcal{M}$ can be expressed as $Z = \sum_{i=1}^{m_n} \langle Z_i, Z \rangle Z_i$, hence the mapping $\Psi : Z \mapsto \mathbb{A}^s_Z$ defined by

$$\Psi(Z)_i := \textstyle\sum_{i=m_{i-1}+1}^{m_i} \langle Z_i, Z \rangle Z_i, \quad i \in [n],$$

is a selection of $\mathbb{A}^s_{\cdot}$ and continuous with respect to the unique locally convex Hausdorff topology on $\mathcal{M}$. Lower hemicontinuity follows immediately. $\square$

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References

[1] Acciaio, B. (2009), Short Note on Inf-Convolution Preserving the Fatou Property. *Annals of Finance*, 5:281-287.

[2] Acciaio, B., and G. Svindland (2009), Optimal Risk Sharing with Different Reference Probabilities. *Insurance: Mathematics and Economics*, 44(3):426-433.

[3] Aliprantis, C. and K. Border (2006), *Infinite Dimensional Analysis – A Hitchhiker's Guide*. 3rd edition, Springer.

[4] Aliprantis, C., and O. Burkinshaw (2003), *Locally Solid Riesz Spaces with Applications to Economics*. American Mathematical Society.

[5] Arrow, K. J. (1963), Uncertainty and the Welfare Economics of Medical Care. *American Economic Review*, 5:941-973.

[11] We define lower and upper hemicontinuity using sequences rather than nets and tacitly use that this is sufficient under the given assumptions, cf. [3, Theorems 17.20 & 17.21].

[6] Artzner, P., F. Delbaen, J.-M. Eber, and D. Heath (1999), Coherent Measures of Risk. *Mathematical Finance*, 9(3):203-228.

[7] Baes, M., P. Koch-Medina, and C. Munari (2019), Existence, Uniqueness, and Stability of Optimal Portfolios of Eligible Assets. Forthcoming in *Mathematical Finance*. Available online at https://doi.org/10.1111/mafi.12205.

[8] Barrieu, P., and N. El Karoui (2005), Inf-Convolution of Risk Measures and Optimal Risk Transfer. *Finance and Stochastics*, 9(2):269-298.

[9] Barvinok, A. (2012), *A Course in Convexity*. Springer.

[10] Borch, K. (1962), Equilibrium in a Reinsurance Market. *Econometrica*, 30:424-444.

[11] Carr, P., H. Geman, and D. Madan (2001), Pricing and Hedging in Incomplete Markets. *Journal of Financial Economics*, 62:131-167.

[12] Chen, S., N. Gao, and F. Xanthos (2018), The Strong Fatou Property of Risk Measures. *Dependence Modeling*, 6(1):183-196.

[13] Cheridito, P., M. Kupper, and L. Tangpi (2017), Duality Formulas for Robust Pricing and Hedging in Discrete Time. *SIAM Journal on Financial Mathematics*, 8(1):738-765.

[14] Cheridito, P., and T. Li (2009), Risk Measures on Orlicz Hearts. *Mathematical Finance*, 19(2):189-214.

[15] Dana, R.-A., and C. Le Van (2000), Arbitrage, Duality and Asset Equilibria. *Journal of Mathematical Economics* 34(3):397-413.

[16] Dana, R.-A., C. Le Van, and F. Magnien (1999), On the Different Notions of Arbitrage and Existence of Equilibrium. *Journal of Economic Theory*, 87(1):169-193.

[17] Delbaen, F. (2002), Coherent Risk Measures on General Probability Spaces. In: *Advances in Finance and Stochastics: Essays in Honor of Dieter Sondermann*, pp. 1-37. Springer.

[18] Delbaen, F. (2012), *Monetary Utility Functions*. Osaka University CSFI Lecture Note Series, Vol. 3.

[19] Embrechts, P., H. Liu, and R. Wang (2017), Quantile-Based Risk Sharing. *Operations Research*, 66(4):936-949.

[20] Ekeland, I. and Témam, R. (1999), *Convex Analysis and Variational Problems*. SIAM.

[21] El Karoui, N., and C. Ravanelli (2009), Cash Subadditive Risk Measures under Interest Rate Ambiguity. *Mathematical Finance*, 19:561-590.

[22] Farkas, E. W., P. Koch Medina, and C. Munari (2013), Beyond Cash-Additive Risk Measures: When Changing the Numéraire Fails. *Finance and Stochastics*, 18(1):145-173.

[23] Farkas, E. W., P. Koch Medina, and C. Munari (2014), Capital Requirements with Defaultable Securities. *Insurance: Mathematics and Economics*, 55:58-67.

[24] Farkas, E. W., P. Koch Medina, and C. Munari (2015), Measuring Risk with Multiple Eligible Assets. *Mathematics and Financial Economics*, 9(1):3-27.

[25] Filipović, D., and M. Kupper (2008), Equilibrium Prices for Monetary Utility Functions. *International Journal of Theoretical and Applied Finance*, 11:325-343.

[26] Filipović, D., and G. Svindland (2008), Optimal Capital and Risk Allocations for Law- and Cash-Invariant Convex Functions. *Finance and Stochastics*, 12(3):423-439.

[27] Föllmer, H., and A. Schied (2011), *Stochastic Finance: An Introduction in Discrete Time*. 3rd edition, De Gruyter.

[28] Gao, N., D. Leung, C. Munari, and F. Xanthos (2018), Fatou Property, Representations, and Extensions of Law-Invariant Risk Measures on General Orlicz Spaces. *Finance and Stochastics*, 22(2):395-415.

[29] Hardy, G. H., J. E. Littlewood, and G. Pólya (1952), *Inequalities*. 2nd edition, Cambridge University Press.

[30] Husain, T. (1965), *The Open Mapping and Closed Graph Theorems in Topological Vector Spaces*. Vieweg.

[31] Jouini, E., W. Schachermayer, and N. Touzi (2008), Optimal Risk Sharing for Law Invariant Monetary Utility Functions. *Mathematical Finance*, 18:269-292.

[32] Klöppel, S., and M. Schweizer (2007), Dynamic Indifference Valuation via Convex Risk Measures. *Mathematical Finance*, 17(4):599-627.

[33] Liebrich, F.-B., and G. Svindland (2017), Model Spaces for Risk Measures. *Insurance: Mathematics and Economics*, 77:150-165.

[34] Mas-Colell, A., and W. Zame (1991), Equilibrium Theory in Infinite Dimensional Spaces. In: Hildenbrand, W., and H. Sonnenschein (eds.), *Handbook of Mathematical Economics*, Vol. 4, pp. 1835-1898, Elsevier.

[35] Munari, C. (2015), *Measuring Risk Beyond the Cash-Additive Paradigm*. PhD thesis, ETH Zürich. Available online at https://doi.org/10.3929/ethz-a-010475509.

[36] Pichler, A. (2013), The Natural Banach Space for Version Independent Risk Measures. *Insurance: Mathematics and Economics*, 53:405-415.

[37] Ravanelli, C., and G. Svindland (2014), Comonotone Pareto Optimal Allocations for Law Invariant Robust Utilities on L^1. *Finance and Stochastics*, 18:249-269.

[38] Rockafellar, R.T. (1970), *Convex Analysis*. Princeton University Press.

[39] Rudin, M. E. (1969), A New Proof that Metric Spaces are Paracompact. *Proceedings of the American Mathematical Society*, 20:603.

[40] Svindland, G. (2010), Continuity Properties of Law-Invariant (Quasi-)Convex Risk Functions on L^∞. *Mathematics and Financial Economics*, 3(1):39-43.

[41] Svindland, G. (2010), Subgradients of law-invariant convex risk measures on L^1. *Statistics & Decisions*, 27(2):169-199.

[42] Tsanakas, A. (2009), To Split or Not to Split: Capital Allocation with Convex Risk Measures. *Insurance: Mathematics and Economics*, 44(2):268-277.

[43] Wang, R. (2016), Regulatory Arbitrage of Risk Measures. *Quantitative Finance*, 16(3):337-347.

[44] Wilson, R. (1968), The Theory of Syndicates. *Econometrica*, 36:119-132.

[45] Zălinsecu, C. (2002), *Convex Analysis in General Vector Spaces*. World Scientific.

[46] Zheng, X. Y. (2009), Pareto Solutions of Polyhedral-valued Vector Optimization Problems in Banach Spaces. *Set-Valued Analysis*, 17:389-408.

Chapter 4

Efficient Allocations under Law-Invariance: A Unifying Approach

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My own contribution: The article is joint work with my supervisor, Gregor Svindland. I developed most parts and ideas of the paper, with continual improvements resulting from discussions with him. I did most of the editorial work. The introduction, the interpretation of coercivity, and using localisation as the key to structuring the results on general rearrangement invariant Banach lattices have been developed in joint discussions. Moreover, I have developed the notion of coercive aggregation functions (Section 3.2 & Appendix), the results on properties of quasi-concave law-invariant functions on general rearrangement invariant spaces (Proposition 20, Theorem 18), and the case studies (Section 5) independently.

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Efficient allocations under law-invariance: A unifying approach

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ABSTRACT

We study the problem of optimising the aggregated utility within a system of agents under the assumption that individual utility assessments are *law-invariant*: they rank Savage acts merely in terms of their distribution under a fixed reference probability measure. We present a unifying framework in which optimisers can be found which are *comonotone allocations* of an aggregated quantity. Our approach can be localised to arbitrary rearrangement invariant commodity spaces containing at least all bounded wealths. The aggregation procedure is a substantial degree of freedom in our study. Depending on the choice of aggregation, the optimisers of the optimisation problems are allocations of a wealth with desirable economic efficiency properties, such as (weakly, biased weakly, and individually rationally) Pareto efficient allocations, core allocations, and systemically fair allocations.

1. Introduction

A substantial driver in the development of mathematical economics has been the theory of general equilibrium, surveyed, among many others, by Debreu (1982) and Mas-Colell and Zame (1991). It mainly analyses whether and how a *pure exchange economy* populated by a finite number of agents can share a commodity in an efficient way. Efficiency is always to be understood against the backdrop of potentially varying *individual preferences* the agents have concerning the shares of the commodity they receive.

We shall refer to such sharing schemes as *allocations*. The most prominent notion of their efficiency is *Pareto efficiency*, a systemic notion of stability and efficiency of an economy which means that no agent can improve her share without worsening the share of another agent. Formally, suppose the agents are represented by the set $\{1, \ldots, n\}$, share a common good X, and entertain preferences $\preceq_i$, $i \in \{1, \ldots, n\}$, concerning the share they are to receive. Then a sharing $\mathbf{X} = (X_1, \ldots, X_n)$ is Pareto efficient if any other sharing scheme $\mathbf{Y} = (Y_1, \ldots, Y_n)$ which satisfies $X_i \prec Y_i$ for some agent i necessarily satisfies $Y_j \prec X_j$ for another agent $j \neq i$.

Pareto efficient allocations can be analysed particularly well if the individual preferences $\preceq_i$, $i \in \{1, \ldots, n\}$, admit a numerical representation: if $\mathcal{X}$ denotes the set of commodities agent i accepts as her share, a function $\mathfrak{U}_i : \mathcal{X} \to [-\infty, \infty)$ is a numerical representation of the preference relation $\preceq_i$ if i weakly prefers Y to X if, and only if, $\mathfrak{U}_i(X) \leq \mathfrak{U}_i(Y)$. We will refer to $\mathfrak{U}_i$ as a *utility function*.[1]

Let us assume now that the commodity space involved in such a problem is a vector space $\mathcal{X}$. Given a good $X \in \mathcal{X}$ which is to be shared, an allocation of X is any vector $\mathbf{X} = (X_1, \ldots, X_n)$ with the property $X_1 + \cdots + X_n = X$. This assumption of perfect substitution means that in principle, any sharing of X is hypothetically feasible for the agents. Suppose furthermore that each individual preference relation $\preceq_i$ can be numerically represented by a utility function $\mathfrak{U}_i : \mathcal{X} \to [-\infty, \infty)$, $i \in \{1, \ldots, n\}$.

A key observation which will be the guiding thread of our investigations, initially due to Negishi, is the following: suppose that suitable positive weights $w_1, \ldots, w_n > 0$ can be found such that the allocation $\mathbf{X}^* = (X_1^*, \ldots, X_n^*)$ satisfies

$$\sum_{i=1}^n w_i \mathfrak{U}_i(X_i) \leq \sum_{i=1}^n w_i \mathfrak{U}_i(X_i^*) \in \mathbb{R},$$

where $\mathbf{X} = (X_1, \ldots, X_n) \in \mathcal{X}^n$ is an arbitrary allocation of X. Then $\mathbf{X}^*$ is indeed a Pareto efficient allocation of X.

Let us abstract this example which we shall get back to at a later stage of the paper. The allocation $\mathbf{X}^*$ is a maximiser for the optimisation problem

$$\Lambda(\mathfrak{U}(\mathbf{X})) \to \max \quad \text{subject to} \quad \mathbf{X} \in \Gamma_X, \tag{1}$$

where Γ_X is the set of all allocations $\mathbf{X} \in \mathcal{X}^n$ with the property $X_1 + \cdots + X_n = X$, whereas $\mathfrak{U}(\mathbf{X}) := (\mathfrak{U}_1(X_1), \ldots, \mathfrak{U}_n(X_n))$ denotes the vector of individual utilities resulting for the agents from the

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[1] As usual, we exclude the case of infinite utility, whereas infinite disutility cannot be excluded *a priori.*

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sharing $\mathbf{X}$. These individual utilities are aggregated to a single quantity using the aggregation function

$$\Lambda(y) := \sum_{i=1}^{n} w_i y_i, \quad -\infty \le y_i < \infty, \tag{2}$$

and the optimal value is finite. As the aggregation function Λ in problem (1) may be chosen freely, it introduces substantial flexibility which we shall exploit in Section 5. Given a parameter $0 < \alpha \le 1$, we will use the aggregation function

$$\Lambda_\alpha(y) := \alpha \min_{1\le i\le n} y_i + (1-\alpha)\max_{1\le i\le n} y_i, \quad -\infty \le y_i < \infty,$$

to obtain *(biased) weakly Pareto efficient* allocations as maximisers for problem (1). Similarly, if we choose the aggregation function

$$\Xi_\alpha(y) := \sum_{\emptyset\neq S\subset\{1,\dots,n\}} \alpha \min_{1\le i\le n} y_i + (1-\alpha)\max_{i\in S} y_i, \quad -\infty \le y_i < \infty,$$

maximisers will be so-called *core allocations*, which reflect certain notions of fairness in game theory. The reader should keep in mind that both the optimal value and the optimisation problem itself will be of secondary importance.

We shall assume throughout our study that all goods are risky future quantities or *Savage acts*, i.e. real-valued random variables contingent on a measurable space $(\Omega, \mathcal{F})$ of future states of the world. One may also think of them in the interpretation of Mas-Colell and Zame (1991) as consumption patterns. Riskiness in the realisation of the states $\omega \in \Omega$ is assumed to be governed by a reference probability measure $\mathbb{P}$ such that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is *non-atomic*.[2] As usual, we shall identify two Savage acts X and Y if the event $\{\omega \in \Omega \mid X(\omega) = Y(\omega)\}$ has full $\mathbb{P}$-probability. Substantial results have been achieved solving problem (1) with aggregation function Λ chosen as in (2) in a framework of Savage acts and involving *law-invariant* preferences, c.f. Acciaio (2007), Carlier and Dana (2008), Chen et al. (2018), Dana (2011), Filipović and Svindland (2008), Jouini et al. (2008), Ravanelli and Svindland (2014).

We adopt the assumption that the agents involved have law-invariant preferences, i.e. the values of the utility functions $\mathfrak{U}_i$, $i \in \{1, \dots, n\}$, only depend on the *distribution* of the commodity under the reference probability measure $\mathbb{P}$: if two Savage acts X and Y induce the same lottery over the real line under $\mathbb{P}$, i.e. if the Borel probability measures $\mathbb{P}\circ X^{-1}$ and $\mathbb{P}\circ Y^{-1}$ on $\mathbb{R}$ are identical, then $\mathfrak{U}_i(X) = \mathfrak{U}_i(Y)$ holds for all $i = 1, \dots, n$, the reasoning being that utility only depends on statistical properties of the commodity. Along the lines of Dana (2011) and Jouini et al. (2008), we shall refer to such utility assessments as law-invariant. Under the name of *probabilistic sophistication* it is a well-known property of preference relations which was introduced by Machina and Schmeidler (1992); we refer to Cerreia-Vioglio et al. (2012) as well as the references in Cerreia-Vioglio et al. (2012, footnote 2); however, these references typically study preference relations in an Anscombe–Aumann framework with general sets of consequences. Strzalecki (2011), on the other hand, studies probabilistic sophistication for general finitely additive reference probabilities. We use the term law-invariance to emphasise that we are working in a Savage setting with a numerical representation of a preference relation whose values only depend on the law under a countably additive reference probability measure.

Normatively, law-invariance can be interpreted as a form of *consequentialism* of the agents in that they are indifferent between two Savage acts yielding the same *consequences* — by inducing the same lottery under the reference probability measure $\mathbb{P}$. Practically, this consequentialism mostly relies on the fact that Savage acts can be grasped only in terms of empirical distributions of certain quantities, an observation which also explains the requirement of non-atomicity of the state space $(\Omega, \mathcal{F}, \mathbb{P})$. There is a one-to-one correspondence between law-invariant utility functions over Savage acts contingent on a non-atomic space and preference relations on (suitable sets of) lotteries on the real line.

Preferences expressed by law-invariant utility functions have another economically appealing feature. Under a mild continuity assumption and quasi-concavity of the utility function $\mathfrak{U}_i$ – that is, convexity of the preference relation expressed by $\mathfrak{U}_i$ – law-invariance of $\mathfrak{U}_i$ is equivalent to two standard notions of risk aversion: (i) monotonicity in the concave order which was introduced to the economics literature by Rothschild and Stiglitz (1970): if every risk averse expected utility agent weakly prefers X to Y, agent i with utility $\mathfrak{U}_i$ weakly prefers X to Y; (ii) dilatation monotonicity: if Π is a finite measurable partition of the state space, agent i weakly prefers the act associated to more information encoded by Π, i.e. the conditional expectation $\mathbb{E}_{\mathbb{P}}[X|\sigma(\Pi)]$, to X itself. This will be discussed in detail in Theorem 18, to the best of our knowledge the most general version of this result in the literature and one of the main results of the paper.

Our established equivalence between law-invariance and concave order monotonicity has the important consequence that, in many situations, *comonotone* maximisers for (1) can be found. An allocation $\mathbf{X}$ of $X \in \mathcal{X}$ is comonotone if there are n non-decreasing functions $f_i : \mathbb{R} \to \mathbb{R}$ summing up to the identity – $f_1(x) + \cdots + f_n(x) = x$ holds for all $x \in \mathbb{R}$ – such that $X_i = f_i(X)$, $i = 1, \dots, n$. The commodity $f_i(X)$ can be interpreted as a contract contingent on the common risk driver X. Such comonotone allocations are desirable and have been widely studied. Empirical investigations of comonotonicity as a property of (optimal) allocations can be found in Attanasio and Davis (1996) and Townsend (1994). According to Carlier et al. (2012), who study multivariate comonotonicity, it is a property which – statistically – is "testable and tractable". Key to solving (1) are so-called *comonotone improvement* results as given by Landsberger and Meilijson (1994), Ludkovski and Rüschendorf (2008), Carlier et al. (2012), and Filipović and Svindland (2008). For comonotonicity in a multivariate setting we refer to Carlier et al. (2012) and the references therein. For its use beyond the risk sharing problem, see Cheung et al. (2014) and Jouini and Napp (2003) as well as the references therein.

Before we outline our main contributions, we give a brief overview of the rich existing literature of sharing problems as described above. For its treatment in general equilibrium theory, we refer to the survey articles by Debreu (1982) and Mas-Colell and Zame (1991) as well as the Khan and Yannelis (1991). More closely related and involving law-invariant criteria are the problems studied by Carlier and Dana (2008), which focuses on Rank Dependent Expected Utility agents and uses additional conditions, and Dana (2011), which studies optimal allocations and equilibria for concave, monotone and law-invariant preferences with strong order continuity properties on bounded wealths. Jouini et al. (2008) and Acciaio (2007) study the problem for law-invariant utility functions under the additional assumption of cash-additivity. The comonotonicity of solutions to such sharing problems has been subject of, e.g., Chateauneuf et al. (2000) for Choquet expected utility agents, Strzalecki and Werner (2011) in the case of more general ambiguity averse preferences, and Ravanelli and Svindland (2014) who study agents with *variational preferences* as axiomatised by Maccheroni et al. (2006). There is also a rich strand of literature on sharing problems when the objective is not to maximise utility, but to minimise risk. The functionals involved are thus not utility functions, but *risk measures*. The case of agents with convex law-invariant and

[2] That is, there is a random variable U whose distribution function $\mathbb{R} \ni x \mapsto \mathbb{P}(U \le x)$ is a continuous function.

cash-additive risk measures has been studied by Filipović and Svindland (2008) on Lebesgue spaces, and by Chen et al. (2018) on general rearrangement invariant spaces. While Acciaio and Svindland (2009) treat the case of law-invariance for *different* reference probability measures, Liebrich and Svindland (2018) consider the problem for convex risk measures beyond law-invariance of the involved functionals. Finally, Mastrogiacomo and Rosazza Gianin (2015) study weak Pareto optima involving quasi-convex risk measures.

Our main contribution is to prove the existence of comonotone maximisers in (1), and thus of economically desirable allocations, in a wide range of situations by laying the groundwork in clear-cut meta results and then applying these in concrete cases which encompass, but go beyond Pareto efficiency, such as the application in game theory mentioned above. We prove that maximisers in problem (1) exist for agents with heterogeneous preferences as long as their utilities are law-invariant with respect to the reference probability measure $\mathbb{P}$ and suitable bounds hold on the one-dimensional subspace of riskless commodities. This approach distinguishes it from other contributions in this direction which restrict preferences to certain classes of law-invariant utilities. It therefore qualifies as unifying. We would like to point out a few noteworthy directions in which we were able to obtain general results:

- *The range of applications:* By making suitable choices for the aggregation function Λ in (1), we show the existence of comonotone biased weakly Pareto efficient, Pareto efficient, and individually rational Pareto efficient allocations under mild assumptions. Moreover, we discuss applications in game theory and the systemic fairness of allocations.
- *Concavity assumptions*: Apart from Mastrogiacomo and Rosazza Gianin (2015), preceding studies of instances of the optimisation problem (1) assume full concavity of the utility functions $\mathfrak{U}_i$.[3] However, this requirement is a very strong form of convexity of the preference relation $\preceq_i$ on $\mathcal{X}$, which means that diversification does (comparatively) not *decrease* utility.[4] Convexity of the preference relation is equivalently characterised by *quasi-concavity* of the numerical representations $\mathfrak{U}_i$: all upper level sets $\{X \in \mathcal{X} \mid \mathfrak{U}_i(X) \geq c\}$, $c \in \mathbb{R}$, are convex sets.
- *Monotonicity assumptions:* Particularly in the financial context, law-invariant utilities are widely assumed to be *monotone in the $\mathbb{P}$-a.s. order*: if for two Savage acts X and Y the event $\{\omega \in \Omega \mid X(\omega) \leq Y(\omega)\}$ has $\mathbb{P}$-probability 1, the individual utility assessments satisfy $\mathfrak{U}_i(X) \leq \mathfrak{U}_i(Y)$. This assumption of "more is better" is not always convincing, in particular in light of finiteness of resources as well as the adverse collateral and ecological effects of economic activity. Our analysis does therefore not rely on monotonicity whatsoever. In all our applications, we only assume that the utility of strictly negative riskless commodities satisfies

  $$\lim_{c \downarrow -\infty} \mathfrak{U}_i(c) = -\infty.$$

  Such an assumption of loss aversion does not seem far-fetched.

  Non-monotonicity of individual preferences is also the reason why we distinguish between sharing *with* and *without* free disposal. In the first case, the aggregated good $X \in \mathcal{X}$ has to be shared without any remainder, i.e. one considers allocations $\mathbf{X} = (X_1, \ldots, X_n) \in \mathcal{X}^n$ with the property $X_1 + \cdots + X_n = X$. In the second case, a unanimously rejected remainder term may be left aside in the sharing scheme, i.e. relevant allocations have the property $X_1 + \cdots + X_n \leq X$ with $\mathbb{P}$-probability 1. In other words, we will study the problem

  $$\Lambda(\mathfrak{U}(\mathbf{X})) \to \max \quad \text{subject to}$$
  $$\mathbf{X} \in \widehat{\Gamma}_X := \{\mathbf{X} \in \mathcal{X}^n \mid \mathbb{P}(X_1 + \cdots + X_n \leq X) = 1\}.$$
- *Choice of the commodity space*: Most applications – apart from Chen et al. (2018) – assume order continuity of the model space norm, or, even more specifically, the commodity space being an L^p-space of random variables. We show that any rearrangement invariant Banach lattice of $\mathbb{P}$-integrable random variables containing all riskless commodities may be considered. This is a consequence of law-invariance in conjunction with slight regularity assumptions, a combination which *implies* that the problem in question may be viewed as the *localised* version of a problem posed on the space L^1 of all integrable random variables. To phrase this differently, the problem can be solved on the level of all integrable random variables if, and only if, it can be solved on any rearrangement invariant commodity space. In this sense, Section 4 and Proposition 20 contain some of the main results of our investigations.
- *Methodology:* Throughout our investigations, we will solve problem (1) in the most classical fashion: we show that a maximising sequence of allocations has a subsequence which converges to a maximiser. This seems interesting against the backdrop of general equilibrium theory in infinite dimensional spaces. As elaborated in Mas-Colell and Zame (1991), infinite dimensionality poses multiple challenges which are usually overcome using fixed point arguments. Whereas it is not clear if the fixed point argument works in our setting, the methodology we use is a powerful addition to the toolkit of general equilibrium theory.

Structure of the paper. In Section 2, we thoroughly describe the setting in which we shall study the sharing problem. In Section 3, we study the problem on the commodity space L^1 of all $\mathbb{P}$-integrable random variables. We isolate the core difficulties of the problem, find powerful meta results applicable in a range of situations as wide as possible, and give a set of straightforward criteria guaranteeing that problems of the shape (1) have maximisers. Section 4 has two parts: Section 4.1 collects the main contributions of our paper on the structural properties of quasi-concave functions on general rearrangement invariant Banach lattices of integrable functions. These findings are of interest beyond the existence of efficient allocations. In Section 4.2 we provide suitable local versions of the results in Section 3 on such general rearrangement invariant commodity spaces. Section 5 illustrates the range of economically relevant allocations which can be obtained with our method. Technical but straightforward estimates necessary for the applications are relegated to Appendix.

2. Preliminaries

We begin with a few crucial pieces of terminology in use throughout our investigations and introduce the setting of the paper.

Given a non-empty set $\mathcal{X}$, a function $f : \mathcal{X} \to [-\infty, \infty] := \mathbb{R} \cup \{-\infty, \infty\}$ and a level $c \in \mathbb{R}$, the UPPER LEVEL SET of f at level c is the set

$$E_c(f) := \{x \in \mathcal{X} \mid f(x) \geq c\}.$$

[3] That is, for all $i = 1, \ldots, n$, for all $X, Y \in \mathcal{X}$, and all $0 < \lambda < 1$, $\mathfrak{U}_i(\lambda X + (1-\lambda)Y) \geq \lambda \mathfrak{U}_i(X) + (1-\lambda)\mathfrak{U}_i(Y)$ holds.

[4] That is, for all $X, Y, Z \in \mathcal{X}$ and $0 < \lambda < 1$, $X \preceq_i Y$ and $X \preceq_i Z$ together imply $X \preceq_i \lambda Y + (1-\lambda)Z$.

If $\mathcal{X}$ is endowed with a topology τ, f is UPPER SEMICONTINUOUS with respect to τ if the sets $E_c(f)$ are τ-closed, $c \in \mathbb{R}$. If $\mathcal{X}$ is a real vector space, f is called QUASI-CONCAVE if each set $E_c(f)$, $c \in \mathbb{R}$, is convex, i.e. for all choices of $x, y \in E_c(f)$ and all $0 < \lambda < 1$, we have $\lambda x + (1-\lambda)y \in E_c(f)$.

The effective domain of f is defined by

$$\mathrm{dom}(f) := \{x \in \mathcal{X} \mid f(x) > -\infty\}.$$

If $f^{-1}(\{\infty\}) = \emptyset$ and $\mathrm{dom}(f) \neq \emptyset$, f is called PROPER.

Throughout the text, bolded symbols will refer to vectors of objects. Hence, whenever $\mathcal{X}$ is a set and $n \in \mathbb{N}$ is a dimension, objects in $\mathcal{X}^n$ will be denoted by $\mathbf{x} = (x_1, \dots, x_n)$. If $f : \mathcal{X} \to [-\infty, \infty]$ is a function, we denote by $f(\mathbf{x}) := (f(x_1), \dots, f(x_n))$ the vector in $[-\infty, \infty]^n$ arising from a coordinatewise evaluation with f. Similarly, if $\mathbf{g} : \mathcal{X}^n \to [-\infty, \infty]^n$ is a vector-valued function, $\mathbf{g}(\mathbf{x})$ is defined as $\mathbf{g}(\mathbf{x}) := (g_1(x_1), \dots, g_n(x_n))$.

We shall fix an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e. there is a random variable U with continuous cumulative distribution function $\mathbb{R} \ni x \mapsto \mathbb{P}(U \le x)$. Given a real-valued random variable $X : \Omega \to \mathbb{R}$, $\mathbb{P} \circ X^{-1}$ denotes its distribution or law under $\mathbb{P}$, i.e. the probability measure $\mathbb{P}(\{\omega \in \Omega \mid X(\omega) \in \cdot\})$ on Borel sets of the real line.

We will usually identify random variables if they agree $\mathbb{P}$-almost surely ($\mathbb{P}$-a.s.). The space of (equivalence classes of) $\mathbb{P}$-integrable random variables is, as usual, denoted by L^1. Similarly, L^∞ is the space of equivalence classes of $\mathbb{P}$-a.s. bounded random variables. By $\mathbb{E}[\cdot] := \mathbb{E}_{\mathbb{P}}[\cdot]$ and $\mathbb{E}[\cdot|\mathcal{H}] := \mathbb{E}_{\mathbb{P}}[\cdot|\mathcal{H}]$ we abbreviate the (conditional) expectation operator (with respect to a sub-σ-algebra $\mathcal{H} \subset \mathcal{F}$) under $\mathbb{P}$. The following notions will be of the utmost importance for our investigations:

Definition 1. A set $\mathcal{C} \subset L^1$ is called REARRANGEMENT INVARIANT with respect to $\mathbb{P}$ if $X \in \mathcal{C}$ and Y being equal to X in law under $\mathbb{P}$, i.e. $\mathbb{P} \circ X^{-1} = \mathbb{P} \circ Y^{-1}$, implies $Y \in \mathcal{C}$.

Given a rearrangement invariant set $\mathcal{C}$ and a function $f : \mathcal{C} \to [-\infty, \infty]$, f is LAW-INVARIANT with respect to $\mathbb{P}$ if $f(X) = f(Y)$ whenever $\mathbb{P} \circ X^{-1} = \mathbb{P} \circ Y^{-1}$.

Whenever we speak of law-invariance in the following, we mean law-invariance with respect to the underlying reference probability measure $\mathbb{P}$ unless specified otherwise. The term is also widely used in the theory of risk measures; c.f. Föllmer and Schied (2011).

Economically, we shall model all appearing goods as random and contingent on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. They are *Savage acts* and represent *state-dependent wealth*.

The set of all goods towards which the agents in question have preferences will be assumed to be an ideal $\mathcal{X}$ of L^1 with respect to the $\mathbb{P}$-a.s. order between random variables[5] which contains all bounded random variables, i.e.

$$L^\infty \subset \mathcal{X} \subset L^1.$$

Although the precise formal properties of these commodity spaces will be elaborated later in Section 4, we remark at this point already that $\mathcal{X}$ is assumed to be rearrangement invariant and normed by a law-invariant lattice norm $\|\cdot\|$. Hence, the commodity spaces cover a very general range of spaces of random variables. One of the crucial messages of our investigations, however, is that we can treat the problem in the setting $\mathcal{X} = L^1$, and we shall do so in Section 3. The general case follows by means of localisation as elaborated in Section 4.

We close this section by recalling the *concave order*, a crucial notion of risk aversion.

[5] That is, $\mathcal{X} \subset L^1$ is a vector space, and if $X, Y \in L^1$ satisfy $|X| \le |Y|$ a.s. and $Y \in \mathcal{X}$, then also $X \in \mathcal{X}$.

Definition 2. Let $X, Y \in L^1$. Y dominates X in the CONCAVE ORDER ($X \preceq_c Y$) if, and only if, $\mathbb{E}[u(X)] \le \mathbb{E}[u(Y)]$ for all concave $u : \mathbb{R} \to \mathbb{R}$. Given a subset $\mathcal{C} \subset L^1$, a function $f : \mathcal{C} \to [-\infty, \infty]$ is non-decreasing in the concave order if $f(X) \le f(Y)$ holds for all $X, Y \in \mathcal{C}$ such that $X \preceq_c Y$.

3. The cornerstones of optimisation involving law-invariance

In the following, for a natural number $n \in \mathbb{N}$, $[n]$ denotes the set of the first n natural numbers, i.e. $[n] = \{1, 2, \dots, n\}$. Throughout this section, we assume the commodity space $\mathcal{X}$ to be given by L^1 endowed with its natural norm $\|\cdot\|_1$, that is

$$\|X\|_1 := \mathbb{E}[|X|], \quad X \in L^1.$$

We identify each agent with some $i \in [n]$ and assume that their preferences over L^1 are represented by a vector $\mathfrak{U} = (\mathfrak{U}_i)_{i \in [n]}$ of functions $\mathfrak{U}_i : L^1 \to [-\infty, \infty)$, $i \in [n]$, with the following properties:

Assumption 3. For each $i \in [n]$, the function $\mathfrak{U}_i : L^1 \to [-\infty, \infty)$ is proper, quasi-concave, upper semicontinuous, and law-invariant.

As discussed in the introduction, law-invariance is a consequentialist assumption which means that two commodities produce the same utility if their distribution under the reference measure $\mathbb{P}$ is the same; that is, $\mathfrak{U}_i(X) = \mathfrak{U}_i(Y)$, $i \in [n]$, if the two distributions $\mathbb{P} \circ X^{-1}$ and $\mathbb{P} \circ Y^{-1}$ on the real line agree. Alternatively, we will see later in Theorem 18(iii) that law-invariance of $\mathfrak{U}_i$ is equivalent to risk aversion in the sense of $\mathfrak{U}_i$ being non-decreasing in the concave order. We remark here that without law-invariance, the existence of solutions to the optimisation problems studied in this paper cannot be guaranteed.

Recall that we abbreviate $\mathfrak{U} : (L^1)^n \to [-\infty, \infty)^n$, $\mathfrak{U}(\mathbf{X}) = (\mathfrak{U}_1(X_1), \dots \mathfrak{U}_n(X_n))$. As mentioned above, vectors in $(L^1)^n$ will be denoted by $\mathbf{X}$, whereas their individual coordinates are denoted by X_i, $i \in [n]$.

As our aim is to maximise aggregated utility within a system arising from distributing a good $X \in L^1$, we first have to clarify what a feasible distribution scheme, an *allocation*, is. To this end, we introduce two types of ATTAINABLE SETS relevant throughout the remainder of the paper. For $X \in L^1$, we consider

$$\Gamma_X := \{\mathbf{X} \in (L^1)^n \mid X_1 + \dots + X_n = X\}$$

and

$$\widehat{\Gamma}_X := \{\mathbf{X} \in (L^1)^n \mid X_1 + \dots + X_n \le X\}.$$

The vectors $\mathbf{X}$ in Γ_X or $\widehat{\Gamma}_X$, respectively, are called ALLOCATIONS of X. $\mathbf{X} \in \Gamma_X$ allocates X without free disposal, whereas $\mathbf{X} \in \widehat{\Gamma}_X$ is an allocation of X when free disposal is allowed. It is apparent from the definition that we study an economy *without production*. Due to potential non-monotonicity of utilities in the almost sure order, $\widehat{\Gamma}_X$ is more relevant in situations in which the economy is not subject to external constraints and a unanimously rejected remainder of X may thus be left aside. Second, we need to introduce the notion of an aggregation function.

Definition 4. A function

$$\Lambda : [-\infty, \infty)^n \to [-\infty, \infty)$$

is an AGGREGATION FUNCTION if it is non-decreasing with respect to the pointwise order on $[-\infty, \infty)^n$, i.e. $\Lambda(\mathbf{y}) \le \Lambda(\mathbf{z})$ for all $\mathbf{y}, \mathbf{z} \in [-\infty, \infty)^n$ such that $y_i \le z_i$ for all $i \in [n]$.

Given a vector $\mathfrak{U}$ of utility functions satisfying Assumption 3 and an aggregation function Λ, we will hence be interested in the quantity

$$\Lambda(\mathfrak{U}(\mathbf{X})) = \Lambda(\mathfrak{U}_1(X_1), \ldots, \mathfrak{U}_n(X_n))$$

representing the aggregated individual utilities in the system given by an allocation $\mathbf{X} \in \Gamma_X$ or $\mathbf{X} \in \widehat{\Gamma}_X$ of a commodity $X \in L^1$.

3.1. Comonotone allocations

The first step towards tackling the optimisation problem (1) would be to reduce the attention to a well-behaved subset of the attainable set. This will turn out to be the set of *comonotone allocations*.

Definition 5. Given $n \in \mathbb{N}$, the set of COMONOTONE n-PARTITIONS OF THE IDENTITY (or COMONOTONE FUNCTIONS) is the set $\mathbb{C}(n)$ of functions $\mathbf{f} = (f_1, \ldots, f_n) : \mathbb{R} \to \mathbb{R}^n$ such that each coordinate f_i is non-decreasing and $\sum_{i=1}^n f_i = id_{\mathbb{R}}$ holds.

For $\gamma > 0$, we set $\mathbb{C}(n)_\gamma$ to be the subset of $\mathbf{f} \in \mathbb{C}(n)$ which satisfy $\sum_{i=1}^n |f_i(0)| \le \gamma$. Moreover, for $\mathbf{f} \in \mathbb{C}(n)$, we set $\tilde{\mathbf{f}} := \mathbf{f} - \mathbf{f}(0) \in \mathbb{C}(n)$.

For each $\mathbf{f} \in \mathbb{C}(n)$, $i \in [n]$, and $x, y \in \mathbb{R}$, the equality

$$\sum_{i=1}^n |f_i(x) - f_i(y)| = |x - y|, \tag{3}$$

holds.[6] In particular, it entails that each f_i, $i \in [n]$, is a Lipschitz continuous function with Lipschitz constant 1. As a consequence, $f_i(X) \in L^1$ holds for all $X \in L^1$. Given $X \in L^1$, recall the abbreviation $\mathbf{f}(X) := (f_1(X), \ldots, f_n(X)) \in (L^1)^n$. Clearly, $\mathbf{f}(X) \in \Gamma_X$ holds by definition. Moreover, if $\mathbf{f} \in \mathbb{C}(n+1)$ and $f_{n+1}(X) \ge 0$ a.s., $(f_1(X), \ldots, f_n(X)) \in \widehat{\Gamma}_X$ holds.

The following results on comonotone functions are essential; statement (ii) is usually referred to as *comonotone order improvement*.

Proposition 6.

(i) For every $\gamma > 0$, $\mathbb{C}(n)_\gamma \subset (\mathbb{R}^n)^{\mathbb{R}}$ is sequentially compact in the topology of pointwise convergence. That is, for each sequence $(\mathbf{f}^k)_{k\in\mathbb{N}} \subset \mathbb{C}(n)_\gamma$ there is a subsequence $(k_\lambda)_{\lambda\in\mathbb{N}}$ and $\mathbf{f} \in \mathbb{C}(n)_\gamma$ such that for all $x \in \mathbb{R}$

$$\mathbf{f}^{k_\lambda}(x) \to \mathbf{f}(x), \quad n \to \infty.$$

(ii) Let $\mathbf{X} \in (L^1)^n$ and set $X := X_1 + \cdots + X_n$. Then there is $\mathbf{f} \in \mathbb{C}(n)$ such that $X_i \preceq_c f_i(X)$ holds for all $i \in [n]$.

Proof. For (i), note that $\mathbb{C}(n)_\gamma$ is a closed subset of the set $\{\mathbf{f} \in \mathbb{C}(n) \mid \mathbf{f}(0) \in [-\gamma, \gamma]^n\}$ and the latter is sequentially compact in the topology of pointwise convergence by Filipović and Svindland (2008, Lemma B.1). (ii) is proved in Filipović and Svindland (2008, Proposition 5.1). In the case $\mathbf{X} \in (L^\infty)^n$, it is Carlier et al. (2012, Theorem 3.1). We also refer to Landsberger and Meilijson (1994) and Ludkovski and Rüschendorf (2008). □

The next two results are essential for optimisation with law-invariant inputs: Proposition 7 shows that in the optimisation problems we consider an optimal allocation can be found if, and only if, an optimal comonotone allocation can be found. Proposition 8 shows that, under further mild conditions, such optimal comonotone allocations actually exist because the set of comonotone allocations is particularly well-behaved.

Proposition 7. *Suppose $\mathfrak{U}$ checks Assumption 3, Λ is an aggregation function, and $X \in L^1$ is arbitrary. Then the identities*

$$\sup_{\mathbf{Y}\in\Gamma_X} \Lambda(\mathfrak{U}(\mathbf{Y})) = \sup_{\mathbf{f}\in\mathbb{C}(n)} \Lambda(\mathfrak{U}(\mathbf{f}(X))) \tag{4}$$

and

$$\sup_{\mathbf{Y}\in\widehat{\Gamma}_X} \Lambda(\mathfrak{U}(\mathbf{Y})) = \sup\{\Lambda(\mathfrak{U}(f_1(X), \ldots, f_n(X))) \mid \mathbf{f} \in \mathbb{C}(n+1),\ f_{n+1}(X) \ge 0\} \tag{5}$$

hold.

Proof. Fix an arbitrary $X \in L^1$. In order to prove (4), let $\mathbf{Y} \in \Gamma_X$ be arbitrary. By Proposition 6(ii), there is $\mathbf{g} \in \mathbb{C}(n)$ such that $Y_i \preceq_c g_i(X)$ holds for all $i \in [n]$. By Theorem 18, $\mathfrak{U}(\mathbf{Y}) \le \mathfrak{U}(\mathbf{g}(X))$ holds in the pointwise order on $[-\infty, \infty)^n$. As Λ is non-decreasing by assumption,

$$\Lambda(\mathfrak{U}(\mathbf{Y})) \le \Lambda(\mathfrak{U}(\mathbf{g}(X))) \le \sup_{\mathbf{f}\in\mathbb{C}(n)} \Lambda(\mathfrak{U}(\mathbf{f}(X))),$$

and thus

$$\sup_{\mathbf{Y}\in\Gamma_X} \Lambda(\mathfrak{U}(\mathbf{Y})) \le \sup_{\mathbf{f}\in\mathbb{C}(n)} \Lambda(\mathfrak{U}(\mathbf{f}(X))).$$

The converse inequality, however, follows from the observation that $\mathbf{f}(X) \in \Gamma_X$ holds for all $\mathbf{f} \in \mathbb{C}(n)$, and (4) is proved.

For the second assertion, consider the slightly altered aggregation function

$$\Xi : [-\infty, \infty)^{n+1} \to [-\infty, \infty),$$
$$\Xi(x_1, \ldots, x_{n+1}) = \Lambda(x_1, \ldots, x_n) + x_{n+1},$$

which, indeed, is non-decreasing in the pointwise order on $[-\infty, \infty)^{n+1}$. Moreover, let

$$\mathfrak{U}_{n+1} = \delta(\cdot|L^1_+) : L^1 \to [-\infty, \infty), \quad X \mapsto \begin{cases} 0, & X \in L^1_+, \\ -\infty, & X \notin L^1_+, \end{cases}$$

be the concave indicator of $L^1_+ := \{Y \in L^1 \mid Y \ge 0 \text{ a.s.}\}$, the positive cone of L^1 in the almost sure order. The function $\mathfrak{U}_{n+1}$ is proper, concave, upper semicontinuous, and law-invariant. We set

$$\bar{\mathfrak{U}} : (L^1)^{n+1} \to [-\infty, \infty)^{n+1},$$
$$\mathbf{X} \mapsto (\mathfrak{U}_1(X_1), \ldots, \mathfrak{U}_n(X_n), \mathfrak{U}_{n+1}(X_{n+1})),$$

and obtain

$$\sup_{\mathbf{Y}\in\widehat{\Gamma}_X} \Lambda(\mathfrak{U}(\mathbf{Y})) = \left\{\Xi(\bar{\mathfrak{U}}(\mathbf{Z})) \mid \mathbf{Z} \in (L^1)^{n+1},\ Z_1 + \cdots + Z_{n+1} = X\right\}$$
$$= \sup_{\mathbf{f}\in\mathbb{C}(n+1)} \Xi\left(\bar{\mathfrak{U}}(\mathbf{f}(X))\right).$$

The last equality here is due to (4). In the last supremum only vectors $\mathbf{f}(X)$, $\mathbf{f} \in \mathbb{C}(n+1)$, are relevant for which $f_{n+1}(X) \in L^1_+$. This in turn implies $(f_1(X), \ldots, f_n(X)) \in \widehat{\Gamma}_X$ for these $\mathbf{f} \in \mathbb{C}(n+1)$. Since they also satisfy $\mathfrak{U}_{n+1}(f_{n+1}(X)) = 0$, we obtain (5) as follows:

$$\sup_{\mathbf{Y}\in\widehat{\Gamma}_X} \Lambda(\mathfrak{U}(\mathbf{Y})) = \sup\left\{\Xi\left(\bar{\mathfrak{U}}(\mathbf{f}(X))\right) \mid \mathbf{f} \in \mathbb{C}(n+1),\ f_{n+1}(X) \ge 0\right\}$$
$$= \sup\{\Lambda(\mathfrak{U}_1(f_1(X)), \ldots, \mathfrak{U}_n(f_n(X))) \mid \mathbf{f} \in \mathbb{C}(n+1),\ f_{n+1}(X) \ge 0\}. \quad \square$$

[6] Each f_i is non-decreasing. Hence, for $x, y \in \mathbb{R}$, $x \ge y$, we have,

$$\sum_{i=1}^n |f_i(x) - f_i(y)| = \sum_{i=1}^n f_i(x) - f_i(y) = x - y = |x - y|.$$

Proposition 8. *In the situation of Proposition 7 assume Λ is additionally upper semicontinuous. For $X \in L^1$ we define the quantities*

$$\eta(X) := \sup_{\mathbf{Y}\in\Gamma_X} \Lambda(\mathfrak{U}(\mathbf{Y})) \quad \text{and} \quad \widehat{\eta}(X) := \sup_{\mathbf{Y}\in\widehat{\Gamma}_X} \Lambda(\mathfrak{U}(\mathbf{Y})).$$

(i) If $\eta(X) < \infty$ and there is a constant $\gamma(X) > 0$ such that for $\mathbf{f} \in \mathbb{C}(n)$,

$$\Lambda(\mathfrak{U}(\mathbf{f}(X))) \geq \eta(X) - 1 \quad \Longrightarrow \quad \mathbf{f} \in \mathbb{C}(n)_{\gamma(X)},$$

there is a $\mathbf{g} \in \mathbb{C}(n)_{\gamma(X)}$ such that

$$\eta(X) = \Lambda\left(\mathfrak{U}(\mathbf{g}(X))\right).$$

(ii) If $\widehat{\eta}(X) < \infty$ and there is some constant $\widehat{\gamma}(X) > 0$ such that for $\mathbf{f} \in \mathbb{C}(n+1)$ with $f_{n+1}(X) \geq 0$,

$$\Lambda(\mathfrak{U}_1(f_1(X)), \dots, \mathfrak{U}_n(f_n(X))) \geq \widehat{\eta}(X) - 1 \quad \Longrightarrow \quad \mathbf{f} \in \mathbb{C}(n+1)_{\widehat{\gamma}(X)},$$

there is a $\mathbf{g} \in \mathbb{C}(n+1)_{\widehat{\gamma}(X)}$ such that $g_{n+1}(X) \geq 0$ a.s. and

$$\widehat{\eta}(X) = \Lambda\left(\mathfrak{U}_1(g_1(X)), \dots, \mathfrak{U}_n(g_n(X))\right).$$

Proof.

(i) By Proposition 7, we may choose a maximising sequence $(\mathbf{f}^k)_{k\in\mathbb{N}} \subset \mathbb{C}(n)$, i.e.

$$\Lambda\left(\mathfrak{U}(\mathbf{f}^k(X))\right) \uparrow \eta(X) < \infty.$$

Combining the assumption and Proposition 6(i) there is a subsequence $(k_\lambda)_{\lambda\in\mathbb{N}}$ and $\mathbf{g} \in \mathbb{C}(n)_{\gamma(X)}$ such that

$$\forall x \in \mathbb{R}: \ \mathbf{f}^{k_\lambda}(x) \to \mathbf{g}(x), \quad \lambda \to \infty.$$

Moreover, by 1-Lipschitz continuity, for all $\lambda \in \mathbb{N}$ and $i \in [n]$,

$$\begin{aligned} |f_i^{k_\lambda}(X) - g_i(X)| &\leq |\tilde{f}_i^{k_\lambda}(X)| + |\tilde{g}_i(X)| + |f_i^{k_\lambda}(0) - g_i(0)| \\ &\leq 2|X| + 2\gamma(X) \ \mathbb{P}\text{-a.s.} \end{aligned}$$

By the Dominated Convergence Theorem,

$$\mathbf{f}^{k_\lambda}(X) \to \mathbf{g}(X) \text{ in } (L^1)^n, \quad \lambda \to \infty.$$

As the *limes superior* is realised as limit along a subsequence, we may, after potentially passing to another subsequence, assume without loss of generality that for each $\mathfrak{U}_i$, $i \in [n]$, the identity $\lim_{\lambda\to\infty} \mathfrak{U}_i(f_i^{k_\lambda}(X)) = \limsup_{\lambda\to\infty} \mathfrak{U}_i(f_i^{k_\lambda}(X))$ holds. As $\mathfrak{U}_i$ is upper semicontinuous, we also obtain $\lim_{\lambda\to\infty} \mathfrak{U}_i(f_i^{k_\lambda}(X)) \leq \mathfrak{U}_i(g_i(X))$. Λ being non-decreasing in the pointwise order implies

$$\begin{aligned} &\Lambda(\mathfrak{U}(\mathbf{g}(X))) \\ &\quad\geq \Lambda\left(\limsup_{\lambda\to\infty} \mathfrak{U}_1(f_1^{k_\lambda}(X)), \dots, \limsup_{\lambda\to\infty} \mathfrak{U}_n(f_n^{k_\lambda}(X))\right) \\ &\quad= \Lambda\left(\lim_{\lambda\to\infty} \mathfrak{U}_1(f_1^{k_\lambda}(X)), \dots, \lim_{\lambda\to\infty} \mathfrak{U}_n(f_n^{k_\lambda}(X))\right). \end{aligned}$$

Moreover, upper semicontinuity of Λ shows

$$\begin{aligned} &\Lambda\left(\lim_{\lambda\to\infty} \mathfrak{U}_1(f_1^{k_\lambda}(X)), \dots, \lim_{\lambda\to\infty} \mathfrak{U}_n(f_n^{k_\lambda}(X))\right) \\ &\quad\geq \limsup_{\lambda\to\infty} \Lambda\left(\mathfrak{U}(\mathbf{f}^{k_\lambda}(X))\right) \\ &\quad= \eta(X). \end{aligned}$$

Together the inequalities read as $\Lambda\left(\mathfrak{U}(\mathbf{g}(X))\right) \geq \eta(X)$. As the converse inequality holds *a priori*, the proof is complete.

(ii) This is proved in complete analogy with (i). □

Proposition 8 may appear technical at first sight, but it is precisely the instrument which allows us to prove the existence of efficient allocations in Theorem 12. This, however requires to characterise when the interplay between the aggregation function Λ and the individual utilities $(\mathfrak{U}_i)_{i\in[n]}$ is such that the additional assumptions are met, i.e. the bounds $\gamma(X)$ and $\widehat{\gamma}(X)$, respectively, can be found. To this end we suggest the notion of *coercive aggregation functions.*

3.2. Coercive aggregation rules

Recall that for a vector $\mathbf{u}$ of scalar functions $u_i : \mathbb{R} \to [-\infty, \infty)$, we define $\mathbf{u} : \mathbb{R}^n \to [-\infty, \infty)^n$ by $\mathbf{u}(\mathbf{y}) = (u_1(y_1), \dots, u_n(y_n))$, $\mathbf{y} \in \mathbb{R}^n$.

Definition 9. Let $n \in \mathbb{N}$, $\mathbf{u} : \mathbb{R}^n \to [-\infty, \infty)^n$ be a vector-valued function, and $\Lambda : [-\infty, \infty)^n \to [-\infty, \infty)$ be an aggregation function. For $x, m \in \mathbb{R}$, we define the set

$$S(x, m) := \left\{\mathbf{y} \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^n y_i \leq x, \ \ \Lambda(\mathbf{u}(\mathbf{y})) \geq m\right\}. \tag{6}$$

We say Λ is COERCIVE[7] for $\mathbf{u}$ if there are functions $G : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$ such that for all $x, m \in \mathbb{R}$, the condition $S(x, m) \neq \emptyset$ implies

$$\sum_{i=1}^n |y_i| \leq G(x, m), \quad \mathbf{y} \in S(x, m), \tag{7}$$

and

$$m \leq H(x). \tag{8}$$

From an economic point of view, we can interpret a vector $\mathbf{y} \in \mathbb{R}^n$ as a collection of (deterministic) endowments of agents $i = 1, \dots, n$. The sum $\sum_{i=1}^n y_i$ then is the total endowment of the system $[n]$. If we think of the vector-valued function $\mathbf{u}$ as the individual utility assessments, the quantity $\Lambda(\mathbf{u}(\mathbf{y}))$ is the aggregated utility in the system. Suppose the aggregation function Λ is coercive for $\mathbf{u}$. Condition (8) means that a bounded total endowment x cannot lead to arbitrarily large aggregated utility. Regarding condition (7), consider a fixed total endowment x allocated over the system such that $y_i \to \infty$ and $y_j \to -\infty$ for at least two agents $i, j \in [n]$. This implies substantial disutility for agent j, and condition (7) ensures that such allocations will eventually not lead to optimal utility if the spread $y_i - y_j$ is too large. However, as the functions G and H do not have to fulfil any specific requirements, they only pose very soft constraints.

We will use in Section 5 and show in the Appendix that the following aggregation functions are coercive for suitable vector-valued functions $\mathbf{u} : \mathbb{R}^n \to [-\infty, \infty)^n$:

- $\Lambda_\alpha(\mathbf{y}) := \alpha \min_{i\in[n]} y_i + (1-\alpha)\max_{i\in[n]} y_i$, $\mathbf{y} \in [-\infty, \infty)^n$, where $0 < \alpha \leq 1$ is a fixed parameter;
- $\Xi_\alpha(\mathbf{y}) := \sum_{\emptyset\neq S\subset[n]} \alpha \min_{i\in[n]} y_i + (1-\alpha)\max_{i\in S} y_i$, $\mathbf{y} \in [-\infty, \infty)^n$, where $0 < \alpha < 1$ is a fixed parameter;
- $\Lambda_{\mathbf{w}}(\mathbf{y}) := \sum_{i=1}^n w_i y_i$, $\mathbf{y} \in [-\infty, \infty)^n$, where $\mathbf{w} = (w_1, \dots, w_n) \in (0, \infty)^n$ is a family of positive weights.

Let us also give an example of a vector-valued function $\mathbf{u} : \mathbb{R}^2 \to \mathbb{R}^2$ and an aggregation function $\Lambda : [-\infty, \infty)^2 \to [0, \infty)$ such that Λ is *not* coercive for $\mathbf{u}$.

[7] We remark that our use of the term *coercive* is not canonical, however, it does not have a unique meaning in the literature anyway. For instance, a function $f : (\mathcal{X}, \|\cdot\|_{\mathcal{X}}) \to (\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ between two normed spaces is called coercive if $\|x\|_{\mathcal{X}} \to \infty$ implies $\|f(x)\|_{\mathcal{Y}} \to \infty$. As coercive functions in optimisation usually play a similar role as coercive aggregation functions in our setting, we decided to use this suggestive terminology.

Example 10. Let $A_1, A_2 \in \mathbb{R}$ and $B_1, B_2 > 0$. We set $\mathbf{u}(\mathbf{y}) := (A_1 + B_1 y_1, A_2 + B_2 y_2)$, $\mathbf{y} \in \mathbb{R}^2$. Moreover, we consider the aggregation function $\Lambda(\mathbf{z}) := e^{z_1} + e^{z_2}$, $\mathbf{z} \in [-\infty, \infty)^2$ (here, $e^{-\infty} := 0$). Let $x, m \in \mathbb{R}$ be arbitrary. As

$$\Lambda(\mathbf{u}(x-n, n)) = e^{A_1 + B_1 x - B_1 n} + e^{A_2 + B_2 n} \to \infty, \quad n \to \infty,$$

and $x - n + n = x$, we have for all $m \in \mathbb{R}$ and all $n \geq n_0$ for some $n_0 \in \mathbb{N}$ depending on x and m that $(x-n, n) \in S(x, m)$. This implies

$$\forall\, m > 0: \quad \sup_{\mathbf{y} \in S(x,m)} |y_1| + |y_2| \geq \sup_{n \geq n_0} |x - n| + |n| = \infty.$$

Therefore the function G as in (7) cannot exist in this situation. Similarly, as $S(x, m) \neq \emptyset$ holds for all $m \in \mathbb{R}$, the function H in (8) cannot exist either. Note that Λ not being coercive for $\mathbf{u}$ is a result of the very different and conflicting nature of utility assessment and aggregation.

The class of coercive aggregation functions is usually rich though and closed under certain algebraic and order operations:

Proposition 11. *Suppose $n \in \mathbb{N}$ and $\mathbf{u} : \mathbb{R}^n \to [-\infty, \infty)^n$ is a vector-valued function. Moreover, assume $\Lambda, \Xi, \Gamma : [-\infty, \infty)^n \to [-\infty, \infty)$ are aggregation functions such that Λ and Ξ are coercive for $\mathbf{u}$ and such that $\Gamma \leq \Lambda$. Then the following functions are coercive for $\mathbf{u}$, as well:*

(i) $\alpha\Lambda$, $\alpha > 0$ arbitrary;
(ii) $\Lambda + \Xi$;
(iii) Γ.

Proof. Let $G, G' : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $H, H' : \mathbb{R} \to \mathbb{R}$ be functions as in Definition 9 for Λ or Ξ, respectively.

(i) The functions $G_\alpha(m, x) := G(x, \frac{m}{\alpha})$ and $H_\alpha := \alpha H$ satisfy (7) and (8).
(ii) If $\mathbf{y} \in \mathbb{R}^n$ satisfies $(\Lambda + \Xi)(\mathbf{u}(\mathbf{y})) \geq m$, this is only possible if $\max\{\Lambda(\mathbf{u}(\mathbf{y})), \Xi(\mathbf{u}(\mathbf{y}))\} \geq \frac{m}{2}$. Hence, the function $G_+(x, m) := \max\{G(x, \frac{m}{2}), G'(x, \frac{m}{2})\}$ satisfies (7). Similarly, the function $H_+ := 2\max\{H, H'\}$ satisfies (8).
(iii) As $\Gamma \leq \Lambda$, the inclusion

$$\{\mathbf{y} \in \mathbb{R}^n \mid \sum_{i=1}^n y_i \leq x,\ \Gamma(\mathbf{u}(\mathbf{y})) \geq m\} \subset \{\mathbf{y} \in \mathbb{R}^n \mid \sum_{i=1}^n y_i \leq x,\ \Lambda(\mathbf{u}(\mathbf{y})) \geq m\}$$

holds. Hence, the same functions G and H work for Γ in (7) and (8), as well. □

3.3. The existence theorem

The aim of this section is to combine the results obtained above and give a unifying criterion for the existence of comonotone solutions to optimisation problems involving agents with law-invariant preferences.

Beforehand, we define the regions of relevance for the optimisation problem in question. Given a vector of utilities $\mathfrak{U} : (L^1)^n \to [-\infty, \infty)^n$ and an aggregation function $\Lambda : [-\infty, \infty)^n \to [-\infty, \infty)$, the relevant region corresponding to the attainable set Γ_X is

$$\Delta := \{X \in L^1 \mid \sup_{\mathbf{Y} \in \Gamma_X} \Lambda(\mathfrak{U}(\mathbf{Y})) > -\infty\},$$

whereas the region corresponding to the attainable set $\widehat{\Gamma}_X$ is

$$\widehat{\Delta} := \{X \in \mathcal{X} \mid \sup_{\mathbf{Y} \in \widehat{\Gamma}_X} \Lambda(\mathfrak{U}(\mathbf{Y})) > -\infty\}.$$

Theorem 12. *Suppose $\mathfrak{U}$ checks Assumption 3. Let $\Lambda : [-\infty, \infty)^n \to [-\infty, \infty)$ be an upper semicontinuous aggregation function which is coercive for $\mathbf{u} := (u_1, \dots, u_n)$, where $u_i := \mathfrak{U}_i|_{\mathbb{R}}$, $i \in [n]$. Then:*

(i) For all $X \in L^1$, the optimal values satisfy

$$\eta(X) := \sup_{\mathbf{Y} \in \Gamma_X} \Lambda(\mathfrak{U}(\mathbf{Y})) < \infty \quad \text{and} \quad \widehat{\eta}(X) := \sup_{\mathbf{Y} \in \widehat{\Gamma}_X} \Lambda(\mathfrak{U}(\mathbf{Y})) < \infty. \tag{9}$$

(ii) There is a function $\gamma : \Delta \to \mathbb{R}$ such that for all $X \in \Delta$ and all $\mathbf{f} \in \mathbb{C}(n)$,

$$\Lambda\,(\mathfrak{U}(\mathbf{f}(X))) \geq \eta(X) - 1 \quad \Longrightarrow \quad \mathbf{f} \in \mathbb{C}(n)_{\gamma(X)}.$$

Moreover, the first supremum in (9) is attained by $\mathbf{g}(X) \in \Gamma_X$, $\mathbf{g} \in \mathbb{C}(n)$ suitably chosen.

(iii) There is a function $\widehat{\gamma} : \widehat{\Delta} \to \mathbb{R}$ such that for all $X \in \widehat{\Delta}$ and all $\mathbf{f} \in \mathbb{C}(n+1)$ with $f_{n+1}(X) \geq 0$

$$\Lambda\,(\mathfrak{U}_1(f_1(X)), \dots, \mathfrak{U}_n(f_n(X))) \geq \widehat{\eta}(X) - 1 \quad \Longrightarrow \quad f \in \mathbb{C}(n)_{\widehat{\gamma}(X)}.$$

Moreover, the second supremum in (9) is attained by $(g_1(X), \dots, g_n(X)) \in \widehat{\Gamma}_X$, $\mathbf{g} \in \mathbb{C}(n+1)$ suitably chosen.

(iv) If the function $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ in (7) is non-decreasing in the first coordinate and non-increasing in the second, the optimal value mappings $X \mapsto \eta(X)$ and $X \mapsto \widehat{\eta}(X)$ are upper semicontinuous on Δ and $\widehat{\Delta}$, respectively.

Proof. Recall that for $\mathbf{Y} \in \mathcal{X}^n$ we abbreviate $\mathbb{E}[\mathbf{Y}] := (\mathbb{E}[Y_1], \dots, \mathbb{E}[Y_n]) \in \mathbb{R}^n$.

(i) By Corollary 19 and the assumptions on Λ, for all $X \in \mathcal{X}$ and all $\mathbf{Y} \in \widehat{\Gamma}_X$ we have

$$\Lambda(\mathfrak{U}(\mathbf{Y})) \leq \Lambda(\mathfrak{U}(\mathbb{E}[\mathbf{Y}])) = \Lambda(\mathbf{u}(\mathbb{E}[\mathbf{Y}])).$$

Moreover, $\sum_{i=1}^n Y_i \leq X$ implies $\sum_{i=1}^n \mathbb{E}[X_i] \leq \mathbb{E}[X]$. Hence, coercivity of Λ for $\mathbf{u}$ yields

$$\eta(X) = \sup_{\mathbf{Y} \in \Gamma_X} \Lambda(\mathfrak{U}(\mathbf{Y})) \leq \sup_{\mathbf{Y} \in \widehat{\Gamma}_X} \Lambda(\mathfrak{U}(\mathbf{Y})) = \widehat{\eta}(X)$$
$$\leq \sup_{\mathbf{Y} \in \widehat{\Gamma}_X} \Lambda(\mathbf{u}(\mathbb{E}[\mathbf{Y}])) \leq H(\mathbb{E}[X]) < \infty.$$

(ii) Let $X \in \Delta$ and suppose $\mathbf{f} \in \mathbb{C}(n)$ is such that $\eta(X) - 1 \leq \Lambda\,(\mathfrak{U}(\mathbf{f}(X)))$. By Corollary 19, $\eta(X) - 1 \leq \Lambda\,(\mathbf{u}(\mathbb{E}[\mathbf{f}(X)]))$, which means that

$$\sum_{i=1}^n |f_i(0)| - |\mathbb{E}[\tilde{f}_i(X)]| \leq \sum_{i=1}^n |\mathbb{E}[f_i(X)]| \leq G(\mathbb{E}[X], \eta(X) - 1).$$

Rearranging this inequality yields

$$\sum_{i=1}^n |f_i(0)| \leq G(\mathbb{E}[X], \eta(X) - 1) + \sum_{i=1}^n |\mathbb{E}[\tilde{f}_i(X)]|$$
$$\leq G(\mathbb{E}[X], \eta(X) - 1) + \sum_{i=1}^n \mathbb{E}[\tilde{f}_i(|X|)]$$
$$\leq G(\mathbb{E}[X], \eta(X) - 1) + \mathbb{E}[|X|] =: \gamma(X).$$

The existence of a maximiser $\mathbf{g}(X)$, $\mathbf{g} \in \mathbb{C}(n)$, follows with Proposition 8.

(iii) Let $X \in \widehat{\Delta}$. If $\mathbf{f} \in \mathbb{C}(n+1)$ is such that $f_{n+1}(X) \geq 0$ and

$$\widehat{\eta}(X) - 1 \leq \Lambda(\mathfrak{U}_1(f_1(X)), \dots, \mathfrak{U}_n(f_n(X))),$$

using the same arguments as in (ii) yields

$$\sum_{i=1}^n |f_i(0)| \leq G(\mathbb{E}[X], \widehat{\eta}(X) - 1) + \mathbb{E}[|X|].$$

As $f_{n+1}(0) = -\sum_{i=1}^{n} f_i(0)$,

$$\sum_{i=1}^{n+1} |f_i(0)| \leq 2\sum_{i=1}^{n} |f_i(0)| \leq 2G(\mathbb{E}[X], \widehat{\eta}(X)-1)+2\mathbb{E}[|X|] =: \widehat{\gamma}(X).$$

The existence of a maximiser $\mathbf{g}(X)$, $\mathbf{g} \in \mathbb{C}(n+1)$, follows with Proposition 8.

(iv) This will be proved in the context of Theorem 23. □

4. Commodity spaces and law-invariance

As mentioned above, we will now demonstrate how the results in the preceding section can be generalised – or rather *localised* – to general rearrangement invariant commodity spaces $\mathcal{X}$ with the property $L^\infty \subset \mathcal{X} \subset L^1$. For the terminology concerning ordered vector spaces, we refer to Aliprantis and Burkinshaw (2003). The space $\mathcal{X}$ is assumed to have the following properties:

(a) As a subset of L^1, $\mathcal{X}$ is rearrangement invariant;
(b) with respect to the $\mathbb{P}$-a.s. order on L^1, it is a solid Riesz subspace;
(c) $\mathcal{X}$ carries a lattice norm $\|\cdot\|$ which makes it into a Banach lattice and is law-invariant as a function $\|\cdot\| : \mathcal{X} \to [0,\infty)$.

The preceding assumptions entail that the embeddings $L^\infty \hookrightarrow \mathcal{X} \hookrightarrow L^1$ are continuous, i.e. there are positive constants $\kappa, K > 0$ such that for all $X \in L^\infty$ and all $Y \in \mathcal{X}$ the estimates

$$\|Y\| \leq \kappa\|Y\|_\infty \quad \text{and } \mathbb{E}[|Y|] \leq K\|Y\| \tag{10}$$

hold. For the aforementioned facts on rearrangement invariant function spaces, we refer to Chen et al. (2018, Appendix A) and the references therein.

Let $\mathcal{X}^*$ denote the dual space of $\mathcal{X}$. A linear functional $\phi \in \mathcal{X}^*$ is *order continuous* if $\lim_{n\to\infty} \phi(X_n) = 0$ holds for every sequence $(X_n)_{n\in\mathbb{N}} \subset \mathcal{X}$ such that $X_n \downarrow 0$ $\mathbb{P}$-a.s.[8] The space $\mathcal{X}_n^\sim$ of all order continuous functionals may be identified with a subspace of L^1. More precisely, for every $\phi \in \mathcal{X}_n^\sim$ there is a unique $Q \in L^1$ such that $\mathbb{E}[|QX|] < \infty$ holds for all $X \in \mathcal{X}$ and $\phi(X) = \mathbb{E}[QX]$. Moreover, for each $Q \in L^\infty$, $X \mapsto \mathbb{E}[QX]$ defines an order continuous bounded linear functional by (10). Using the Hardy–Littlewood inequalities as stated in Chong and Rice (1971, Theorem 13.4), one can prove that $\mathcal{X}_n^\sim \subset L^1$ is rearrangement invariant, as well.

4.1. Structural properties of law-invariant functions

We assume that $(\mathcal{X}, \|\cdot\|)$ is either $(L^1, \|\cdot\|_1)$ or a rearrangement invariant Banach lattice $L^\infty \subset \mathcal{X} \subsetneq L^1$ as introduced above.

Before we can generalise the results on the existence of efficient allocations to general commodity spaces $\mathcal{X}$, we point out that the potential lack of order continuity of the norm $\|\cdot\|$ is the main problem which needs to be overcome. It results in the fact that $\mathcal{X}_n^\sim \subsetneq \mathcal{X}^*$ is possible. Hence, many of the structural properties of (quasi-)concave and law-invariant functions do not transfer directly.

This necessitates to study structural properties of law-invariant functions on general commodity spaces more closely. We shall see that the localisation procedure works if the individual utilities in question have minimal order continuity properties on $\mathcal{X}$. For the (strong) Fatou property introduced in the following we particularly refer to the recent contributions of Chen et al. (2018) and Gao et al. (2018).

Definition 13. Let $L^\infty \subset \mathcal{X} \subset L^1$ be a rearrangement invariant Banach lattice as elaborated above. A function $f : \mathcal{X} \to [-\infty,\infty)$ is said to have

- the FATOU PROPERTY if every order convergent sequence $(X_n)_{n\in\mathbb{N}} \subset \mathcal{X}$ with limit $X \in \mathcal{X}$ satisfies[9]

  $f(X) \geq \limsup_{n\to\infty} f(X_n)$;

- the STRONG FATOU PROPERTY if the preceding estimate holds for every norm bounded sequence $(X_n)_{n\in\mathbb{N}} \subset \mathcal{X}$ which converges to $X \in \mathcal{X}$ a.s.

Note that the space $\mathcal{X}$ is closed under suitable conditional expectations: if a σ-algebra $\mathcal{H}$ is finitely generated, i.e. $\mathcal{H} = \sigma(\Pi)$ for some finite measurable partition $\Pi = \{A_1, \dots, A_n\} \subset \mathcal{F}$ of Ω, and $X \in \mathcal{X}$, then $\mathbb{E}[X|\mathcal{H}]$ is well-defined and a simple – and thus bounded – function.

Lemma 14. *Given $X \in \mathcal{X}$ and a finitely generated sub-σ-algebra $\mathcal{H} \subset \mathcal{F}$, the conditional expectation $\mathbb{E}[X|\mathcal{H}]$ lies again in $\mathcal{X}$ and satisfies $\|\mathbb{E}[X|\mathcal{H}]\| \leq \|X\|$.*

Proof. Fix arbitrary $X \in \mathcal{X}$ and a finitely generated sub-σ-algebra $\mathcal{H} \subset \mathcal{F}$. Moreover, let $A_k := \{|X| \leq k\} \in \mathcal{F}$. For all $k \in \mathbb{N}$, $\mathbb{E}[X\mathbf{1}_{A_k}|\mathcal{H}] \in L^\infty \subset \mathcal{X}$, and the set $\{Y \in L^\infty \mid \|Y\| \leq \|X\mathbf{1}_{A_k}\|\}$, is $\|\cdot\|_\infty$-closed by (10). Applying Svindland (2010, Lemma 1.3) for the first and the lattice norm property for the second inequality yields

$$\|\mathbb{E}[X\mathbf{1}_{A_k}|\mathcal{H}]\| \leq \|X\mathbf{1}_{A_k}\| \leq \|X\|.$$

As $\mathcal{H}$ is finitely generated, $\mathbb{E}[X\mathbf{1}_{A_k}|\mathcal{H}] \to \mathbb{E}[X|\mathcal{H}]$ in L^∞. Again by (10),

$$\|\mathbb{E}[X|\mathcal{H}]\| = \lim_{k\to\infty} \|\mathbb{E}[X\mathbf{1}_{A_k}|\mathcal{H}]\| \leq \|X\|. \quad \square$$

This observation allows us to define *dilatation monotonicity* of a function on $\mathcal{X}$.

Definition 15. A function $f : \mathcal{X} \to [-\infty,\infty]$ is DILATATION MONOTONE if for every $X \in \mathcal{X}$ and every finite measurable partition Π we have

$$f(X) \leq f\left(\mathbb{E}[X|\sigma(\Pi)]\right).$$

By Jensen's inequality, $\mathbb{E}[X|\sigma(\Pi)]$ dominates X in the concave order. As an immediate consequence, a function $f : \mathcal{X} \to [-\infty,\infty]$ is dilatation monotone if it is non-decreasing in the concave order.

Our main goal here, however, is to link law-invariance of a quasi-concave function to monotonicity properties such as dilatation monotonicity or being non-decreasing in the concave order, c.f. Definition 2, and mild order continuity properties such as the Fatou property, strong Fatou property, and $\sigma(\mathcal{X}, L^\infty)$-upper semicontinuity.

As a first step, we recall the version of Chen et al. (2018, Proposition 2.11) suited to our purposes.

Proposition 16. *Suppose $\mathcal{X} \subsetneq L^1$ and $\|\cdot\|$ is order continuous. For a proper, quasi-concave, and law-invariant function $f : \mathcal{X} \to [-\infty,\infty)$, the following are equivalent:*

(i) f has the strong Fatou property;
(ii) f has the Fatou property;
(iii) f is $\sigma(\mathcal{X}, L^\infty)$-upper semicontinuous.

[8] As the $\mathbb{P}$-a.s. order on L^1 (and also $\mathcal{X}$) renders *super Dedekind complete* spaces, order convergent sequences suffice to characterise order continuity.

[9] That is, $\mathbb{P}(X_n \to X) = 1$ and there is some $X_0 \in \mathcal{X}_+$ such that $\sup_{n\in\mathbb{N}} |X_n| \leq X_0$ holds a.s.

Hence, in the situation of the preceding proposition, the three aforementioned order continuity properties agree.

In the following, we shall denote the *left-continuous quantile function* of $X \in L^1$ by q_X, i.e.

$q_X(s) := \inf\{x \in \mathbb{R} \mid \mathbb{P}(X \leq x) \geq s\}, \quad 0 < s < 1.$

The concave order can be characterised in terms of quantiles; c.f. Dana (2005, Lemma 2.2):

Lemma 17. *For $X, Y \in L^1$, $X \preceq_c Y$ if, and only if, the estimate*

$$\int_0^1 q_X(s)g(s)ds \leq \int_0^1 q_Y(s)g(s)ds$$

holds for any choice of a non-increasing bounded function $g : (0, 1) \to \mathbb{R}$.

The following theorem – which is furthermore of independent interest – encompasses all relevant structural properties needed for the utility inputs in the optimisation problems in question. We will discuss the relation to dilatation monotonicity of $f : \mathcal{X} \to [-\infty, \infty]$ for the sake of completeness.

Theorem 18. *Let $f : \mathcal{X} \to [-\infty, \infty]$ be a function.*

(i) Suppose f is quasi-concave, $\sigma(\mathcal{X}, L^\infty)$-upper semicontinuous, and law-invariant. Then it is non-decreasing in the concave order.

(ii) If f does not attain the value $+\infty$, is dilatation monotone and has the strong Fatou property, then it is law-invariant. The same assertion holds if $\|\cdot\|$ is order continuous and f is $\|\cdot\|$-upper semicontinuous.
If f is additionally quasi-concave, it is $\sigma(\mathcal{X}, L^\infty)$-upper semicontinuous in both cases.

(iii) Suppose a quasi-concave function $f : \mathcal{X} \to [-\infty, \infty)$ has the strong Fatou property, or, if $\|\cdot\|$ is order continuous, is $\|\cdot\|$-upper semicontinuous. Then the following statements are equivalent:

(a) f is non-decreasing in the concave order;
(b) f is dilatation monotone;
(c) f is law-invariant.

Under any of the equivalent conditions (a)–(c), f is $\sigma(\mathcal{X}, L^\infty)$-upper semicontinuous.

Proof.

(i) Suppose $f : \mathcal{X} \to [-\infty, \infty]$ is quasi-concave, $\sigma(\mathcal{X}, L^\infty)$-upper semicontinuous, and law-invariant. For $r \in \mathbb{R}$, we set $\sigma_r : \mathcal{X}^* \to [-\infty, \infty]$ to be the support function of the superlevel set $E_r(f) = \{Y \in \mathcal{X} \mid f(Y) \geq r\}$, i.e.

$$\sigma_r(\phi) = \inf_{Y \in E_r(f)} \phi(Y), \quad \phi \in \mathcal{X}^*.$$

Suppose $E_r(f) \neq \emptyset$. The Hahn–Banach Theorem and $\sigma(\mathcal{X}, L^\infty)$-closedness or the superlevel sets show that $Y \in E_r(f)$ holds if, and only if,

$$\forall Q \in \mathrm{dom}(\sigma_r) \cap L^\infty : \ \mathbb{E}[QY] \geq \sigma_r(Q). \quad (11)$$

Moreover, the superlevel sets of f are rearrangement invariant. This property transfers to law-invariance of $\sigma_r|_{L^\infty}$ and rearrangement invariance of $\mathrm{dom}(\sigma_r) \cap L^\infty$.
Let now $X, Y \in \mathcal{X}$ be arbitrary with the property $X \preceq_c Y$. We have to show that $f(X) \leq f(Y)$. This inequality holds trivially if $f(X) = -\infty$. Otherwise, if $f(X) > -\infty$, there is $r \in \mathbb{R}$ such that $X \in E_r(f)$. Pick any such r and let $Q \in \mathrm{dom}(\sigma_r) \cap L^\infty$ be arbitrary. By Chong and Rice (1971, Theorem 13.4),

$$\mathbb{E}[QY] \geq \inf_{\tilde{Q} \sim Q} \mathbb{E}[\tilde{Q}Y] = -\sup_{Q' \sim -Q} \mathbb{E}[Q'Y] = \int_0^1 (-q_{-Q}(s))q_Y(s)ds. \quad (12)$$

As $-Q$ is bounded, $-q_{-Q} : (0, 1) \to \mathbb{R}$ is a non-increasing bounded function. Lemma 17 yields the estimate

$$\int_0^1 (-q_{-Q}(s))q_X(s)ds \leq \int_0^1 (-q_{-Q}(s))q_Y(s)ds. \quad (13)$$

Combining (12) and (13) yields

$$\mathbb{E}[QY] \geq \int_0^1 (-q_{-Q}(s))q_Y(s)ds \geq \int_0^1 (-q_{-Q}(s))q_X(s)ds = \inf_{\tilde{Q} \sim Q} \mathbb{E}[\tilde{Q}X].$$

Using law-invariance of σ_r on L^∞, we obtain

$$\mathbb{E}[QY] \geq \inf_{\tilde{Q} \sim Q} \sigma_r(\tilde{Q}) = \sigma_r(Q).$$

As $Q \in \mathrm{dom}(\sigma_r) \cap L^\infty$ was chosen arbitrarily, $f(Y) \geq r$ whenever $f(X) \geq r$, which in turn implies $f(X) \leq f(Y)$.

(ii) In a first step, we show that f is law-invariant on the level of simple functions. To this end, suppose two simple functions X and Y are equal in law, i.e. $\mathbb{P} \circ X^{-1} = \mathbb{P} \circ Y^{-1}$. By Cherny and Grigoriev (2007, Lemma 2.4), for every $\varepsilon > 0$ there is a $K_\varepsilon \in \mathbb{N}$ and finitely generated sub-σ-algebras $\mathcal{H}_1, \ldots, \mathcal{H}_{K_\varepsilon} \subset \mathcal{F}$ such that

$$\|X - \mathbb{E}[\mathbb{E}[\ldots\mathbb{E}[Y|\mathcal{H}_1]|\mathcal{H}_2]\ldots|\mathcal{H}_{K_\varepsilon}]\|_\infty < \varepsilon.$$

Setting $X_n := \mathbb{E}[\mathbb{E}[\ldots\mathbb{E}[Y|\mathcal{H}_1]|\mathcal{H}_2]\ldots|\mathcal{H}_{K_{\varepsilon_n}}]$ for a sequence $\varepsilon_n \downarrow 0$, we infer that X is approximated by a sequence $(X_n)_{n\in\mathbb{N}} \subset L^\infty$ uniformly which is bounded in norm $\|\cdot\|$ and converges a.s.
If f has the strong Fatou property, this yields

$$f(X) \geq \limsup_{n\to\infty} f(X_n) \geq \liminf_{n\to\infty} f(X_n) \geq f(Y),$$

where the last inequality is due to dilatation monotonicity applied to each $n \in \mathbb{N}$. The argument is symmetric in the roles of X and Y, hence $f(X) = f(Y)$.
In the second case, i.e. $\|\cdot\|$ is order continuous and f is upper semicontinuous, using $\lim_{n\to\infty} \|X_n - X\| = 0$ yields the same assertion.
In a second step, let X and Y be arbitrary in $\mathcal{X}$ with the property of being equal in law. Note that there are two sequences of finitely generated sub-σ-algebras $(\mathcal{H}_n)_{n\in\mathbb{N}}$ and $(\mathcal{G}_n)_{n\in\mathbb{N}}$ such that $\mathbb{E}[X|\mathcal{H}_n] \to X$ and $\mathbb{E}[Y|\mathcal{G}_n] \to Y$ a.s. and $\mathbb{P} \circ \mathbb{E}[X|\mathcal{H}_n]^{-1} = \mathbb{P} \circ \mathbb{E}[Y|\mathcal{G}_n]^{-1}$, $n \in \mathbb{N}$. Moreover, by Lemma 14,

$$\sup_{n\in\mathbb{N}} \|\mathbb{E}[X|\mathcal{H}_n]\| \leq \|X\| \quad \text{and} \quad \sup_{n\in\mathbb{N}} \|\mathbb{E}[Y|\mathcal{G}_n]\| \leq \|Y\|.$$

Combining dilatation monotonicity and the strong Fatou property yields

$$f(X) = \lim_{n\to\infty} f(\mathbb{E}[X|\mathcal{H}_n]) \quad \text{and} \quad f(Y) = \lim_{n\to\infty} f(\mathbb{E}[Y|\mathcal{G}_n]). \quad (14)$$

In the second case, order continuity of $\|\cdot\|$ yields $\mathbb{E}[X|\mathcal{H}_n] = X$ and $\lim_{n\to\infty} \mathbb{E}[Y|\mathcal{G}_n] = Y$, whence the statement of (14) follows by upper semicontinuity and dilatation monotonicity. Finally, in both cases, the already proved law-invariance on the level of simple functions combined with $\mathbb{P} \circ \mathbb{E}[X|\mathcal{H}_n]^{-1} = \mathbb{P} \circ \mathbb{E}[Y|\mathcal{G}_n]^{-1}$, $n \in \mathbb{N}$ proves the assertion.

$\sigma(\mathcal{X}, L^\infty)$-upper semicontinuity of f is implied by our assumptions and law-invariance by Chen et al. (2018, Theorem 2.6) in the case of the strong Fatou property. If $\|\cdot\|$ is order continuous, upper semicontinuity of f entails that it enjoys the Fatou property. If $\mathcal{X} \neq L^1$, f is hence both $\sigma(\mathcal{X}, L^\infty)$-upper semicontinuous and has the strong Fatou Property by Chen et al. (2018, Proposition 2.11). If $\mathcal{X} = L^1$, $\sigma(\mathcal{X}, L^\infty)$-upper semicontinuity is weak upper semicontinuity and hence already captured by the assumption of $\|\cdot\|$-upper semicontinuity.

(iii) Combine (i) and (ii). □

From the preceding proposition, we immediately obtain the following corollary which allows control of the global behaviour of a quasi-concave function f as in Theorem 18(i) or (iii) in terms of the behaviour on deterministic random variables.

Corollary 19. *In the situation of Theorem 18(i) or (iii),*

$$\forall X \in \mathcal{X}: \ f(X) \le f(\mathbb{E}[X]).$$

We conclude this interlude on the structural properties of law-invariant functions with an important generalisation of Chen et al. (2018, Theorem 2.6). It will be the key to localising Theorem 12 to general commodity spaces in Theorem 23.

Proposition 20. *If $f : \mathcal{X} \to [-\infty, \infty]$ is quasi-concave, $\sigma(\mathcal{X}, L^\infty)$-upper semicontinuous and law-invariant, there exists a unique extension $f^\sharp : L^1 \to [-\infty, \infty]$ which is quasi-concave, upper semicontinuous with respect to $\|\cdot\|_1$, and law-invariant. Moreover, $f^\sharp$ is non-decreasing in the concave order, and properness of f implies properness of $f^\sharp$.*

Proof. By assumption on f, each upper level set $E_r(f)$ is $\sigma(\mathcal{X}, L^\infty)$-closed and (11) holds. Define

$$\mathcal{A}_r := \mathrm{cl}_{L^1}(E_r(f)).$$

If $E_r(f) \neq \emptyset$, the representation

$$\mathcal{A}_r = \{W \in L^1 \mid \forall Q \in \mathrm{dom}(\sigma_r) \cap L^\infty : \ \mathbb{E}[QW] \ge \sigma_r(Q)\}$$

holds, where σ_r is as in (11), and

$$f^\sharp(X) := \sup\{r \in \mathbb{R} \mid X \in \mathcal{A}_r\}, \quad X \in L^1.$$

$f^\sharp$ is law-invariant and quasi-concave. Indeed, for law-invariance, note that $f^\sharp(X) \ge r$ is equivalent to $\mathbb{E}[QX] \ge \sigma_r(Q)$ for all Q in the rearrangement invariant set $L^\infty \cap \mathrm{dom}(\sigma_r)$ and in terms of the law-invariant function σ_r. For quasi-concavity, let $X, Y \in L^1$ and $\lambda \in (0, 1)$, $X, Y \in \mathcal{A}_r$ implies $\lambda X + (1-\lambda)Y \in \mathcal{A}_r$ by convexity of the latter set. As $\min\{f^\sharp(X), f^\sharp(Y)\} = \sup\{r \in \mathbb{R} \mid \{X, Y\} \subset \mathcal{A}_r\}$, the inequality $f^\sharp(\lambda X + (1-\lambda)Y) \ge \min\{f^\sharp(X), f^\sharp(Y)\}$ follows.

Moreover, $f^\sharp$ is upper semicontinuous. Indeed, suppose $X_n \to X$ in L^1. Without loss of generality, we may assume that $s := \lim_{n\to\infty} f^\sharp(X_n) \in [-\infty, \infty]$ exists. Suppose $r \in \mathbb{R}$ is such that $r < s$. $X_n \in \mathcal{A}_r$ has to hold for all n large enough, hence, for all $Q \in L^\infty \cap \mathrm{dom}(\sigma_r)$,

$$\mathbb{E}[QX] = \lim_{n\to\infty} \mathbb{E}[QX_n] \ge \sigma_r(Q),$$

which means $X \in \mathcal{A}_r$. This shows upper semicontinuity.

We now show that $f^\sharp$ extends f. Clearly, $f^\sharp|_{\mathcal{X}} \ge f$, and we hence assume for contradiction the existence of some $X \in \mathcal{X}$ such that $f^\sharp(X) > f(X)$. This allows us to find some $r \in \mathbb{R}$ such that $X \in \mathcal{A}_r$, whereas $X \notin E_r(f)$. The latter set is $\sigma(\mathcal{X}, L^\infty)$-closed. We can thus find some $Q \in L^\infty$ which gives a separating hyperplane in that

$$\mathbb{E}[QX] < \inf_{Y \in E_r(f)} \mathbb{E}[QY] = \inf_{W \in \mathcal{A}_r} \mathbb{E}[QW],$$

where we have used $\mathcal{A}_r = \mathrm{cl}_{L^1}(E_r(f))$ in the last equality. This contradicts $X \in \mathcal{A}_r$. $f^\sharp(X) > f(X)$ has to be absurd.

$f^\sharp$ is the unique extension of f to L^1 which is quasi-concave, upper semicontinuous, and law-invariant. Indeed, let $\hat{f} : L^1 \to [-\infty, \infty]$ be any extension of f with these properties. As $L^\infty \subset \mathcal{X}$, the restrictions $f^\sharp|_{L^\infty}$ and $\hat{f}|_{L^\infty}$ agree. Let $X \in L^1$ and let $(\mathcal{G}_n)_{n\in\mathbb{N}}$ be a sequence of finitely generated sub-σ-algebras such that $\lim_{n\to\infty} \mathbb{E}[X|\mathcal{G}_n] = X$ holds in L^1. $f^\sharp$ and $\hat{f}$ being non-decreasing in the concave order follows from Theorem 18(i). Together with upper semicontinuity, we obtain

$$\hat{f}(X) = \lim_{n\to\infty} \hat{f}(\mathbb{E}[X|\mathcal{G}_n]) = \lim_{n\to\infty} f(\mathbb{E}[X|\mathcal{G}_n])$$
$$= \lim_{n\to\infty} f^\sharp(\mathbb{E}[X|\mathcal{G}_n]) = f^\sharp(X).$$

Thus, both extensions $f^\sharp$ and $\hat{f}$ agree.

It remains to prove that properness of f implies properness of $f^\sharp$. By Corollary 19, $f^\sharp(\mathbb{E}[X]) = \infty$ whenever $X \in L^1$ satisfies $f^\sharp(X) = \infty$. As $f^\sharp(\mathbb{E}[X]) = f(\mathbb{E}[X])$, $f^\sharp$ only attains the value $+\infty$ if f does. □

4.2. The existence theorem for general commodity spaces

We shall assume $\mathcal{X} \subsetneq L^1$ here. If $\|\cdot\|$ is order continuous, Proposition 16 states that for a quasi-concave and law-invariant function $f : \mathcal{X} \to [-\infty, \infty)$, the Fatou property, the strong Fatou property, and upper semicontinuity with respect to the $\sigma(\mathcal{X}, L^\infty)$-topology are all equivalent. This leads to following assumption:

Assumption 21. The commodity space satisfies $\mathcal{X} \subsetneq L^1$ as well as properties (a)–(c) above. For each agent $i \in [n]$, its individual utility assessment is given by a function $\mathfrak{U}_i : \mathcal{X} \to [-\infty, \infty)$, which is proper, quasi-concave, law-invariant, and has the strong Fatou property.

By Proposition 20, each $\mathfrak{U}_i$ has a canonical extension $\mathfrak{U}_i^\sharp : L^1 \to [-\infty, \infty)$ which is proper, quasi-concave, upper semicontinuous, and law-invariant. Consequently, the family of functions $\mathfrak{U}^\sharp := (\mathfrak{U}_i^\sharp)_{i\in[n]}$ checks Assumption 3.

The following two results prove that we can extend the optimisation problem to L^1 and solve it in the larger space. Note that we extend the definition of the attainable sets Γ_X and $\widehat{\Gamma}_X$, $X \in \mathcal{X}$, by

$$\Gamma_X^{\mathcal{X}} = \Gamma_X \cap \mathcal{X}^n \quad \text{and} \quad \widehat{\Gamma}_X^{\mathcal{X}} = \widehat{\Gamma}_X \cap \mathcal{X}^n.$$

Lemma 22. *Suppose $\mathfrak{U} := (\mathfrak{U}_i)_{i\in[n]} : \mathcal{X}^n \to [-\infty, \infty)^n$ is a vector of utility functions satisfying Assumption 21. Let $\Lambda : [-\infty, \infty)^n \to [-\infty, \infty)$ be an aggregation function and $X \in \mathcal{X}$. Then:*

(i) $\mathbf{f}(X) \in \mathcal{X}^n$ holds for every comonotone function $\mathbf{f} \in \mathbb{C}(n)$.

(ii) The identities

$$\sup_{\mathbf{X}\in\Gamma_X^{\mathcal{X}}} \Lambda(\mathfrak{U}(\mathbf{X})) = \sup_{\mathbf{Y}\in\Gamma_X} \Lambda(\mathfrak{U}^\sharp(\mathbf{Y}))$$

and

$$\sup_{\mathbf{X}\in\widehat{\Gamma}_X^{\mathcal{X}}} \Lambda(\mathfrak{U}(\mathbf{X})) = \sup_{\mathbf{Y}\in\widehat{\Gamma}_X} \Lambda(\mathfrak{U}^\sharp(\mathbf{Y}))$$

hold.

(iii) Λ is coercive for $(\mathfrak{U}_i|_{\mathbb{R}})_{i\in[n]}$ if, and only if, it is coercive for $(\mathfrak{U}_i^\sharp|_{\mathbb{R}})_{i\in[n]}$.

Proof.

(i) By (3), the estimate $|f_i(X)| \leq |f_i(0)| + |X|$ holds for all $i \in [n]$. The right-hand side is an element of $\mathcal{X}$ and this space is solid as a subset of L^1. We infer that the left-hand side has to be an element of $\mathcal{X}$ as well.

(ii) As none of the functions $\mathfrak{U}_i^\sharp$ attains the value $+\infty$, the expression

$$\Lambda(\mathfrak{U}^\sharp(\mathbf{Y})) = \Lambda(\mathfrak{U}_1^\sharp(Y_1), \ldots, \mathfrak{U}_n^\sharp(Y_n))$$

is well-defined for all $\mathbf{Y} \in (L^1)^n$. As the family $\mathfrak{U}^\sharp = (\mathfrak{U}_i^\sharp)_{i\in[n]}$ checks Assumption 3, Proposition 7 together with (i) implies

$$\sup_{\mathbf{Y}\in\Gamma_X} \Lambda(\mathfrak{U}^\sharp(\mathbf{Y})) = \sup_{\mathbf{f}\in\mathbb{C}(n)} \Lambda\left(\mathfrak{U}^\sharp(\mathbf{f}(X))\right) = \sup_{\mathbf{f}\in\mathbb{C}(n)} \Lambda\left(\mathfrak{U}(\mathbf{f}(X))\right).$$

As for $\mathbf{f} \in \mathbb{C}(n)$ the vector $\mathbf{f}(X)$ lies in $\Gamma_X^{\mathcal{X}}$ we obtain

$$\sup_{\mathbf{Y}\in\Gamma_X} \Lambda(\mathfrak{U}^\sharp(\mathbf{Y})) \leq \sup_{\mathbf{X}\in\Gamma_X^{\mathcal{X}}} \Lambda\left(\mathfrak{U}(\mathbf{X})\right).$$

The converse inequality holds *a priori*. Note that the second equality is derived from Proposition 7 in an analogous way.

(iii) This follows from the equality $\mathfrak{U}(r) = \mathfrak{U}^\sharp(r)$, which holds for all $r \in \mathbb{R}$. □

The following local version of the existence theorem, Theorem 12, is an immediate consequence of the preceding lemma.

Theorem 23. *Theorem 12 holds true* verbatim *if we replace L^1 by $\mathcal{X}$, Γ_X by $\Gamma_X^{\mathcal{X}}$, and $\widehat{\Gamma}_X$ by $\widehat{\Gamma}_X^{\mathcal{X}}$.*

Proof. It only remains to verify (iv) from Theorem 12. Note that the following proof works in L^1 as well as in the setting of a general commodity space $\mathcal{X}$ of this section.

Let $(X_k)_{k\in\mathbb{N}} \subset \Delta$ be a sequence and $X_\infty \in \Delta$ such that $\lim_{k\to\infty} \|X_k - X_\infty\| = 0$. We shall prove that

$$\eta(X_\infty) \geq \limsup_{k\to\infty} \eta(X_k).$$

The proof for $\widehat{\Delta}$ and $\widehat{\eta}$ is completely analogous. We proceed similarly to the proof of Proposition 8, however under the additional problem that not a fixed X is considered, but a sequence thereof.

First of all, we may assume that $\lim_{k\to\infty} \eta(X_k) = \limsup_{k\to\infty} \eta(X_k)$ up to passing to a subsequence and that $\limsup_k \eta(X_k) > -\infty$ – otherwise, the desired inequality is trivial. Now, for all $k \in \mathbb{N}$ choose $\mathbf{g}^k \in \mathbb{C}(n)$ such that $\Lambda\left(\mathfrak{U}(\mathbf{g}^k(X_k))\right) = \eta(X_k)$. The proof of Theorem 12(ii) together with (10) yields

$$\begin{aligned}\sum_{i=1}^n |g_i^k(0)| &\leq G(\mathbb{E}[X_k], \eta(X_k) - 1) + \mathbb{E}[|X_k|] \leq G(\mathbb{E}[X_k], \eta(X_k) - 1) \\ &\quad + K\|X_k\| \\ &\leq G(\mathbb{E}[X_k], \eta(X_k) - 1) + K \sup_{k\in\mathbb{N}} \|X_k\|.\end{aligned}$$

By the convergence of $\eta(X_k)$, $L := \inf_{k\in\mathbb{N}} \eta(X_k) > -\infty$. The estimate $\mathbb{E}[X_k] \leq K \sup_{k\in\mathbb{N}} \|X_k\|$ holds for all $k \in \mathbb{N}$. In conjunction with G being non-decreasing in the first coordinate and non-increasing in the second, we obtain the bound

$$\sum_{i=1}^n |g_i^k(0)| \leq G(K \sup_{k\in\mathbb{N}} \|X_k\|, L - 1) + K \sup_{k\in\mathbb{N}} \|X_k\| =: \rho < \infty,$$

a constant which is in particular independent of k. We conclude $(\mathbf{g}^k)_{k\in\mathbb{N}} \subset \mathbb{C}(n)_\rho$.

Recall that $X_k \to X_\infty$ holds in L^1, as well. After passing to subsequences twice, we may hence infer by Proposition 6(i) that for a suitable subsequence $(k_\lambda)_{\lambda\in\mathbb{N}}$ and a suitable $\mathbf{g} \in \mathbb{C}(n)_\rho$, we have

$$\mathbf{g}^{k_\lambda}(x) \to \mathbf{g}(x), \ \lambda \to \infty,$$

for all $x \in \mathbb{R}$, and

$$\mathbb{P}(\lim_{\lambda\to\infty} X_{k_\lambda} = X_\infty) = 1.$$

Hence,

$$\mathbf{g}_{k_\lambda}(X_{k_\lambda}) \to \mathbf{g}(X_\infty) \text{ a.s.}$$

If the norm $\|\cdot\|$ is order continuous, this convergence holds in norm as well. At last, choosing the constant κ as in (10), for all $\lambda \in \mathbb{N}$ and all $i \in [n]$ we can estimate

$$\|g_i^{k_\lambda}(X_{k_\lambda})\| \leq \kappa |g_i^{k_\lambda}(0)| + \|X_{k_\lambda}\| \leq \kappa\rho + \sup_{k\in\mathbb{N}} \|X_k\|.$$

This allows us to reason as in the proof of Proposition 8, however invoking the strong Fatou property of the individual utility functions if necessary. We obtain

$$\begin{aligned}\limsup_{k\to\infty} \eta(X_k) &= \limsup_{\lambda\to\infty} \Lambda\left(\mathfrak{U}(\mathbf{g}^{k_\lambda}(X_{k_\lambda}))\right) \leq \Lambda\left(\mathfrak{U}(\mathbf{g}(X_\infty))\right) \\ &\leq \sup_{\mathbf{f}\in\mathbb{C}(n)} \Lambda\left(\mathfrak{U}(\mathbf{f}(X_\infty))\right) = \eta(X_\infty). \quad \square\end{aligned}$$

The crucial message of this section is that in a situation of law-invariant utilities with minimal order continuity properties it does not matter on which commodity space we solve the optimisation problem. Without loss of generality, it may be solved on the canonical commodity space L^1 as the solution automatically localises to the commodity space in question. This is due to the *homogeneity* of the only solutions which are guaranteed to exist, being comonotone transformations of the aggregate wealth.

5. Applications

In this section, the main existence theorem, Theorem 12, will be applied to a number of optimisation problems involving individual preferences within a system of agents and an aggregation thereof. We shall see that the solutions have various economic interpretations.

For the sake of clarity, we assume that a system of $n \geq 2$ agents $i \in [n]$ is given who all have preferences over the commodity space L^1. These can be represented numerically by a vector of utility functions $\mathfrak{U} : (L^1)^n \to [-\infty, \infty)^n$ satisfying Assumption 3. The localisation procedure discussed in Section 4 shows that we are simultaneously solving the problem in all spaces $\mathcal{X}$ and for all individual utility functions $\mathfrak{U}_i : \mathcal{X} \to [-\infty, \infty)$ which satisfy the assumptions of Section 4, in particular Assumption 21.

Before we can discuss the promised applications, we need to introduce more notation: Given a vector $\mathbf{x} \in [-\infty, \infty]^n$, $\mathbf{x}^*$ denotes the maximum and $\mathbf{x}_*$ the minimum of the entries of $\mathbf{x}$, respectively;

$$\mathbf{x}^* := \max_{i\in[n]} x_i \quad \text{and} \quad \mathbf{x}_* := \min_{i\in[n]} x_i.$$

If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} \cdot \mathbf{y}$ denotes the Hadamard product of the two vectors, i.e. $\mathbf{x} \cdot \mathbf{y} = (x_i y_i)_{i\in[n]}$. Also, in order to make fruitful use of the concept of a coercive aggregation function, we shall focus on utilities whose behaviour on riskless commodities can be controlled in the following way:

Definition 24. A function $u : \mathbb{R} \to [-\infty, \infty)$ is an (A, B, C)-FUNCTION, $(A, B, C) \in \mathbb{R} \times (0, \infty) \times (0, \infty)$, if

$$u(x) \leq A + Bx^+ - Cx^-, \quad x \in \mathbb{R},$$

where $x^+ := \max\{x, 0\}$ and $x^- := \max\{-x, 0\}$.

Remark 25.

(i) $u : \mathbb{R} \to [-\infty, \infty)$ is an (A, B, C)-function for some $(A, B, C) \in \mathbb{R} \times (0, \infty) \times (0, \infty)$ if, and only if, there are $\alpha_\pm \in \mathbb{R}$ and $\beta_\pm > 0$ such that for all $x \in \mathbb{R}$

$$u(x) \leq \begin{cases} \alpha_+ + \beta_+ x, & x \geq 0, \\ \alpha_- - \beta_- |x|, & x < 0, \end{cases}$$

that is, u can be controlled from above by affine functions.

(ii) Any proper concave function $u : \mathbb{R} \to [-\infty, \infty)$ is an (A, B, C)-function.

(iii) Suppose (A, B, C) and $(\hat{A}, \hat{B}, \hat{C})$ are elements of $\mathbb{R} \times (0, \infty) \times (0, \infty)$ with the property that $A \leq \hat{A}$, $B \leq \hat{B}$ and $\hat{C} \leq C$. One easily sees that every (A, B, C)-function is also a $(\hat{A}, \hat{B}, \hat{C})$-function.

In this section, we will work under the following assumption:

Assumption 26. The vector of utility functions $\mathfrak{U} = (\mathfrak{U}_i)_{i \in [n]}$ satisfies Assumption 3. Moreover, setting $u_i(x) := \mathfrak{U}_i(x)$, $x \in \mathbb{R}$, we impose that each u_i is an (A_i, B_i, C_i)-function, $i \in [n]$.

We remark that in some of the case studies below, the additional control of the behaviour on riskless commodities rules out cash-additivity of the corresponding utility functions.[10]

5.1. (Biased) weak Pareto efficiency

In this section, we prove the existence of (biased) weak Pareto optima as solutions to a suitable optimisation problem.

Definition 27. Let $\Gamma \subset (L^1)^n$ be an attainable set. $\mathbf{X} \in \Gamma$ is called WEAKLY PARETO EFFICIENT or a WEAK PARETO OPTIMUM, if there is no $\mathbf{Y} \in \Gamma$ with $\mathfrak{U}_i(Y_i) > \mathfrak{U}_i(X_i)$, $i \in [n]$.

The aggregation function we shall be interested in,

$$\Lambda_\alpha(\mathbf{y}) := \alpha \min_{i \in [n]} y_i + (1 - \alpha) \max_{i \in [n]} y_i, \quad \mathbf{y} \in [-\infty, \infty)^n,$$

where $0 < \alpha \leq 1$ is a parameter, is easily seen to be upper semicontinuous. We remark that the function

$$L^1 \ni X \mapsto \sup_{\mathbf{Y} \in \Gamma_X} \Lambda_1(\mathfrak{U}(\mathbf{Y}))$$

is the *quasi-concave sup-convolution* of the individual utilities; c.f. Mastrogiacomo and Rosazza Gianin (2015).

The following observation is immediate:

Lemma 28. *Suppose $0 < \alpha \leq 1$. If $\mathbf{X} \in \Gamma_X$ is such that*

$$\Lambda_\alpha(\mathfrak{U}(\mathbf{X})) = \sup_{\mathbf{Y} \in \Gamma_X} \Lambda_\alpha(\mathfrak{U}(\mathbf{Y})) \in \mathbb{R},$$

then $\mathbf{X}$ is a weakly Pareto efficient allocation of X within Γ_X. The analogous result holds when Γ_X is replaced by $\widehat{\Gamma}_X$.

Theorem 29. *Suppose $\mathfrak{U} = (\mathfrak{U}_i)_{i \in [n]}$ fulfils Assumption 26 and let $0 \leq \alpha < 1$. If $\alpha > 0$, assume that $\mathbf{B} = (B_i)_{i \in [n]}$ and $\mathbf{C} = (C_i)_{i \in [n]}$ additionally satisfy*

$$\frac{B_2}{C_1} < \frac{\alpha}{1 - \alpha} < \frac{C_2}{B_1} \quad \text{if } n = 2, \tag{15}$$

or

$$\frac{(n-1)\mathbf{B}^*}{\mathbf{C}_*} < \frac{\alpha}{1 - \alpha} \quad \text{if } n \geq 3. \tag{16}$$

(i) For all $X \in \sum_{i=1}^n \mathrm{dom}(\mathfrak{U}_i)$ there is $\mathbf{g} \in \mathbb{C}(n)$ such that

$$\Lambda_\alpha(\mathfrak{U}(\mathbf{g}(X))) = \sup_{\mathbf{Y} \in \Gamma_X} \Lambda_\alpha(\mathfrak{U}(\mathbf{Y})) \in \mathbb{R}.$$

$\mathbf{g}(X)$ is a weakly Pareto efficient allocation of X in case free disposal is not allowed.

(ii) For all $X \in \sum_{i=1}^n \mathrm{dom}(\mathfrak{U}_i) + L^1_+$ there is $\mathbf{g} \in \mathbb{C}(n+1)$ such that $g_{n+1}(X) \geq 0$ and

$$\Lambda_\alpha(\mathfrak{U}_1(g_1(X)), \ldots, \mathfrak{U}_n(g_n(X))) = \sup_{\mathbf{Y} \in \widehat{\Gamma}_X} \Lambda_\alpha(\mathfrak{U}(\mathbf{Y})) \in \mathbb{R}.$$

$(g_1(X), \ldots, g_n(X))$ is a weakly Pareto efficient allocation of X in case free disposal is allowed.

Proof. (i) is an immediate consequence of Lemmas A.1 and A.2 and Theorem 12(ii) if one notices that $X \in \Delta$ if, and only if, $X \in \sum_{i=1}^n \mathrm{dom}(\mathfrak{U}_i)$.

(ii) follows from Lemmas A.1 and A.2 and Theorem 12(iii) as $X \in \hat{\Delta}$ if, and only if, $X \geq Y$ for some $Y \in \Delta = \sum_{i=1}^n \mathrm{dom}(\mathfrak{U}_i)$, or equivalently, $X \in \sum_{i=1}^n \mathrm{dom}(\mathfrak{U}_i) + L^1_+$. □

The reason not only to look at the quasi-concave sup-convolution given by Λ_1, but also at the weighted aggregation functions Λ_α for $0 < \alpha < 1$, is the following: Λ_1 only depends on the worst utility achieved by redistribution. It is unaffected by positive deviations from the worst utility other agents may achieve. Therefore it has a bias to overemphasise and sanction negative deviations from a systemic mean utility. The smaller one chooses $\alpha \in (0, 1)$, the more the optimal value under aggregation Λ_α depends on the best utility achieved by redistribution. Within the set of weak Pareto optima, Λ_α therefore has a bias towards those allowing for well-performing agents, while the situation of the worst-performing agents is not too dire at the same time.

5.2. Game theory and core allocations

For this application, we consider the case $n \geq 3$ for the sake of non-triviality. The following notions are adopted from Aliprantis and Burkinshaw (2003, Section 8.10).

Definition 30. Given a vector $\mathbf{W} \in (L^1)^n$ of initial endowments set $W := W_1 + \cdots + W_n$. A CORE ALLOCATION of $\mathbf{W}$ is a vector $\mathbf{X} \in \Gamma_W$ such that no $\emptyset \neq S \subset [n]$ and $\mathbf{Y} \in \Gamma_W$ with the following properties can be found:

- $\sum_{i \in S} Y_i = \sum_{i \in S} W_i$;
- for all $i \in S$, $\mathfrak{U}_i(Y_i) > \mathfrak{U}_i(X_i)$.

The set of all core allocations of $\mathbf{W}$ is denoted by core($\mathbf{W}$).

Core allocations are fair redistributions of a vector of initial endowments: no subsystem $S \subset [n]$ of agents is disadvantaged in that they would be better off by withdrawing their resources from the larger system $[n]$ and distributing them among themselves.

We are interested in the closely related question whether an aggregated quantity $W \in L^1$ can be split into initial endowments such that, relative to these, the allocation is already perceived as fair in the sense of core allocations. More precisely, we ask if there is an allocation $\mathbf{W} \in \Gamma_W$ such that $\mathbf{W} \in \mathrm{core}(\mathbf{W})$. We prove that solutions to a suitable optimisation problem of type (1) do exactly satisfy this. For $0 < \alpha < 1$ consider the aggregation function

$$\begin{aligned}\Xi_\alpha(\mathbf{y}) &:= \sum_{\emptyset \neq S \subset [n]} \alpha \min_{i \in [n]} y_i + (1 - \alpha) \max_{i \in S} y_i \\ &= (2^n - 1)\alpha \min_{i \in [n]} y_i + (1 - \alpha) \sum_{\emptyset \neq S \subset [n]} \max_{i \in S} y_i, \\ &\quad \mathbf{y} \in [-\infty, \infty)^n,\end{aligned}$$

Ξ_α is easily seen to be upper semicontinuous.

[10] $\mathfrak{U}_i$ would be cash-additive if $\mathfrak{U}_i(X + r) = \mathfrak{U}_i(X) + r$ holds for all $X \in L^1$ and all $r \in \mathbb{R}$. In that case, quasi-concavity automatically implies concavity.

Theorem 31. *Suppose $\mathfrak{U} = (\mathfrak{U}_i)_{i\in[n]}$ satisfies Assumption 26. Then there is $0 < \alpha < 1$ such that for any $W \in \sum_{i=1}^n \mathrm{dom}(\mathfrak{U}_i)$ there is $\mathbf{g} \in \mathbb{C}(n)$ with the property*

$$\Xi_\alpha\,(\mathfrak{U}(\mathbf{g}(W))) = \sup_{\mathbf{W}\in\Gamma_W} \Xi_\alpha(\mathfrak{U}(\mathbf{W})) \in \mathbb{R}.$$

In particular, $\mathbf{g}(W) \in \mathrm{core}(\mathbf{g}(W))$ holds.

Proof. Let $W \in \sum_{i=1}^n \mathrm{dom}(\mathfrak{U}_i)$. As $\lim_{\alpha\uparrow 1} \frac{\alpha}{1-\alpha} = \infty$, we may choose $\alpha \in (0, 1)$ with the property that

$$\frac{(n-1)\mathbf{B}^*}{\mathbf{C}_*} < \frac{\alpha}{1-\alpha},$$

i.e. (16) is satisfied. As for all $\emptyset \neq S \subset [n]$ and all $\mathbf{y} \in [-\infty, \infty)^n$, we have

$$\alpha \min_{i\in[n]} y_i + (1-\alpha)\max_{i\in S} y_i \leq \Lambda_\alpha(\mathbf{y})$$

and the latter function is coercive for (A_i, B_i, C_i)-functions satisfying (16) by Lemma A.1, the mapping

$$[-\infty, \infty)^n \ni \mathbf{y} \mapsto \alpha \min_{i\in[n]} y_i + (1-\alpha)\max_{i\in S} y_i$$

is coercive for $\mathbf{u} = (u_1, \ldots, u_n)$ by Proposition 11(iii). By Proposition 11(ii), Ξ_α is coercive for $\mathbf{u}$. By Theorem 12(ii), there is $\mathbf{g} \in \mathbb{C}(n)$ with the claimed properties.

It remains to prove that $\mathbf{g}(W) \in \mathrm{core}(\mathbf{g}(W))$. To this end, assume there is $\emptyset \neq S^* \subset [n]$ and $\mathbf{Y} \in \Gamma_W$ such that

$$\sum_{i\in S^*} Y_i = \sum_{i\in S^*} g_i(W) \quad \text{and} \quad \forall\, i \in S^* : \ \mathfrak{U}_i(g_i(W)) < \mathfrak{U}_i(Y_i).$$

Without loss of generality, we may assume $Y_i = g_i(W)$ for all $i \notin S^*$. This implies

$$\min_{i\in[n]} \mathfrak{U}_i(Y_i) \geq \min_{i\in[n]} \mathfrak{U}_i(g_i(W)) \quad \text{and}$$
$$\max_{i\in S} \mathfrak{U}_i(Y_i) \geq \max_{i\in S} \mathfrak{U}_i(g_i(W)), \ \ \emptyset \neq S \subset [n].$$

As furthermore $\max_{i\in S^*} \mathfrak{U}_i(Y_i) > \max_{i\in S^*} \mathfrak{U}_i(g_i(W))$, we obtain $\Xi_\alpha\,(\mathfrak{U}(\mathbf{g}(W))) < \Xi_\alpha(\mathfrak{U}(\mathbf{Y}))$ which is a CONTRADICTION to $\mathbf{g}(W)$ being a maximiser. Hence, $\mathbf{g}(W)$ has to be a core allocation of itself. □

5.3. Pareto efficiency with and without free disposal

In this section we turn to the more restrictive and, compared to weak Pareto efficiency, economically more desirable property of *Pareto efficiency*.

Definition 32. Let $\Gamma \subset (L^1)^n$ be a attainable set. $\mathbf{X} \in \Gamma$ is called PARETO EFFICIENT or a PARETO OPTIMUM, if $\mathbf{Y} \in \Gamma$ and $\mathfrak{U}_i(Y_i) \geq \mathfrak{U}_i(X_i)$, $i \in [n]$, implies $\mathfrak{U}_i(X_i) = \mathfrak{U}_i(Y_i)$, $i \in [n]$.

Clearly, every Pareto efficient allocation is weakly Pareto efficient. Suppose now $\mathbf{w} := (w_1, \ldots, w_n) \in (0, \infty)^n$ is a vector of positive weights. We define the upper semicontinuous aggregation function

$$\Lambda_{\mathbf{w}}(\mathbf{y}) := \sum_{i=1}^n w_i y_i, \quad \mathbf{y} \in [-\infty, \infty)^n.$$

As elaborated in the introduction, if $\mathbf{X} \in \Gamma_X$ satisfies

$$\sum_{i=1}^n w_i \mathfrak{U}_i(X_i) = \Lambda_{\mathbf{w}}(\mathfrak{U}(\mathbf{X})) = \sup_{\mathbf{Y}\in\Gamma_X} \Lambda_{\mathbf{w}}(\mathfrak{U}(\mathbf{Y})) \in \mathbb{R},$$

then $\mathbf{X}$ is Pareto efficient within Γ_X. Analogously, $\mathbf{X} \in \widehat{\Gamma}_X$ is Pareto efficient within $\widehat{\Gamma}_X$ whenever $\sum_{i=1}^n w_i\mathfrak{U}_i(X_i) = \sup_{\mathbf{Y}\in\widehat{\Gamma}_X} \Lambda_{\mathbf{w}}(\mathfrak{U}(\mathbf{Y})) \in \mathbb{R}$. A natural question is whether for a particular choice of $\mathbf{w} \in (0, \infty)^n$ the function $\Lambda_{\mathbf{w}}$ checks the assumptions of Theorem 12.

Theorem 33. *Suppose $\mathfrak{U} = (\mathfrak{U}_i)_{i\in[n]}$ fulfils Assumption 26. If $\mathbf{w} \in (0, \infty)^n$, $\mathbf{B} = (B_i)_{i\in[n]}$ and $\mathbf{C} = (C_i)_{i\in[n]}$ satisfy*

$$\frac{B_2}{C_1} < \frac{w_1}{w_2} < \frac{C_2}{B_1} \quad \text{if } n = 2, \tag{17}$$

or

$$(\mathbf{w}\cdot\mathbf{B})^* < (\mathbf{w}\cdot\mathbf{C})_* \quad \text{if } n \geq 3, \tag{18}$$

the following assertions hold:

(i) If $X \in \sum_{i=1}^n \mathrm{dom}(\mathfrak{U}_i)$ there is $\mathbf{g} \in \mathbb{C}(n)$ such that

$$\Lambda_{\mathbf{w}}\,(\mathfrak{U}(\mathbf{g}(X))) = \sup_{\mathbf{Y}\in\Gamma_X} \Lambda_{\mathbf{w}}(\mathfrak{U}(\mathbf{Y})) \in \mathbb{R}.$$

Consequently, $\mathbf{g}(X)$ is a Pareto efficient allocation of X in case free disposal is not allowed.

(ii) If $X \in \sum_{i=1}^n \mathrm{dom}(\mathfrak{U}_i) + L^1_+$ there is $\mathbf{g} \in \mathbb{C}(n+1)$ such that $g_{n+1}(X) \geq 0$ and

$$\Lambda_{\mathbf{w}}\,(\mathfrak{U}_1(g_1(X)), \ldots, \mathfrak{U}_n(g_n(X))) = \sup_{\mathbf{Y}\in\widehat{\Gamma}_X} \Lambda_{\mathbf{w}}(\mathfrak{U}(\mathbf{Y})) \in \mathbb{R}.$$

Consequently, $(g_1(X), \ldots, g_n(X))$ is a Pareto efficient allocation of X in case free disposal is allowed.

Proof. Both (i) and (ii) follow from Lemma A.3 and Theorem 12 if one notices that $\Delta = \sum_{i=1}^n \mathrm{dom}(\mathfrak{U}_i)$ and $\widehat{\Delta} = \sum_{i=1}^n \mathrm{dom}(\mathfrak{U}_i) + \mathcal{X}_+$. □

Example 34. In this example, we consider two agents with different law-invariant utility assessments. First, given some fixed constant $\beta > 1$, we set $\mathcal{Q} := \{Q \in L^\infty_+ \mid \mathbb{E}[Q] = 1,\ Q \leq \beta\ \mathbb{P}\text{-a.s.}\}$. The preferences of agent 1 over L^1 are of Yaari type and given by the concave law-invariant and positively homogeneous utility function

$$\mathfrak{U}_1(X) := \inf_{Q\in\mathcal{Q}} \mathbb{E}[QX], \quad X \in L^1,$$

for which $\mathfrak{U}_1|_{\mathbb{R}}$ is a $(0, 1, 1)$-function; c.f. Yaari (1987). Regarding agent 2, we assume that she has law-invariant variational preferences. More precisely, assume that $u_2 : \mathbb{R} \to [-\infty, \infty)$ is a utility function, i.e. u_2 is concave, right-continuous, non-decreasing, $\mathrm{dom}(u_2) \neq \emptyset$ and such that there are $x, y \in \mathrm{dom}(u_2)$ such that $u_2(x) \neq u_2(y)$. Moreover, let $\mathcal{Q}_2 \subset L^\infty_+$ be law-invariant with the property that $\mathbb{E}[Q] = 1$ for all $Q \in \mathcal{Q}_2$, i.e. $\mathcal{Q}_2$ is a set of probability densities with respect to $\mathbb{P}$. We furthermore suppose a convex and law-invariant function $\alpha_2 : \mathcal{Q}_2 \to \mathbb{R}$ with the property $\iota := \inf_{Q\in\mathcal{Q}_2} \alpha_2(Q) > -\infty$ is given. Eventually, the preferences of agent 2 are given by the utility function

$$\mathfrak{U}_2(X) = \inf_{Q\in\mathcal{Q}_2} \mathbb{E}[Qu_2(X)] + \alpha_2(Q).$$

The right derivative $u_2'(x) := \lim_{y\downarrow x} \frac{u_2(y)-u_2(x)}{y-x} \in [0, \infty]$, $x \in \mathbb{R}$, exists and is non-decreasing. Let us assume we can find $z_- < z_+$ such that $\infty > u_2'(z_-) > u_2'(z_+)$. Then $\mathfrak{U}_2|_{\mathbb{R}}$ is an $(A, u_2'(z_+), u_2'(z_-))$-function, where

$$A := \max\{u_2(z_+) - u_2'(z_+)z_+ + \iota,\ u_2(z_-) - u_2'(z_-)z_- + \iota\}.$$

If we choose $w_1, w_2 > 0$ such that

$$u_2'(z_+) < \frac{w_1}{w_2} < u_2'(z_-),$$

(17) is satisfied. By Theorem 33(i), for every $X \in \mathrm{dom}(\mathfrak{U}_1) + \mathrm{dom}(\mathfrak{U}_2) = L^1 + \mathrm{dom}(\mathfrak{U}_2)$ there is $\mathbf{g} \in \mathbb{C}(2)$ such that

$$\infty > w_1\mathfrak{U}_1(g_1(X)) + w_2\mathfrak{U}_2(g_2(X)) = \sup_{\mathbf{X}\in\Gamma_X} w_1\mathfrak{U}_1(X_1) + w_2\mathfrak{U}_2(X_2).$$

If we want to say more about the concrete shape of $\mathbf{g}$, we need to make further assumptions on $\mathfrak{U}_2$. Hence, let us assume

- $\mathrm{dom}(\mathfrak{U}_2) = \mathrm{dom}(\mathfrak{U}_2) + \mathbb{R}$;
- $\mathfrak{U}_2$ is strictly monotone with respect to the a.s. order, i.e. $X \le Y$ a.s. and $\mathbb{P}(X < Y) > 0$ implies $\mathfrak{U}_2(X) < \mathfrak{U}_2(Y)$;
- $\mathfrak{U}_2$ is *strictly risk averse conditional on lower tail events*, that is,

  $$\mathfrak{U}_2(X) < \mathfrak{U}_2\left(X\mathbf{1}_{A^c} + \frac{\mathbb{E}[X\mathbf{1}_A]}{\mathbb{P}(A)}\mathbf{1}_A\right)$$

  whenever A is a lower tail event for X. The latter means that $\mathbb{P}(A) > 0$ and

  $$\begin{aligned}&\sup\{m \in \mathbb{R} \mid \mathbb{P}(\{X \le m\} \cap A) = 0\}\\ &\quad < \inf\{m \in \mathbb{R} \mid \mathbb{P}(\{X \le m\} \cap A) = \mathbb{P}(A)\}\\ &\quad \le \sup\{m \in \mathbb{R} \mid \mathbb{P}(\{X \le m\} \cap A^c) = 0\}.\end{aligned}$$

  Interpretationally, the infimal value X attains on A is strictly less than the supremal value it attains on A, which is bounded from above by the infimal value attained on A^c. As an illustrating example, assume that for some $m \in \mathbb{R}$ and some $\delta > 0$ the three probabilities $\mathbb{P}(X \le m)$, $\mathbb{P}(m < X \le m+\delta)$, and $\mathbb{P}(X > m+\delta)$ are all positive. Then $\{X \le m+\delta\}$ is a lower tail event for X.

If these additional conditions are met, then Ravanelli and Svindland (2014, Proposition 5.2) shows that $\mathbf{g}$ is of the shape

$$\mathbf{g}(x) = (-(x-\ell)_- + k, \max\{x,\ell\} - k), \quad x \in \mathbb{R},$$

for suitable constants $k, \ell \in \mathbb{R}$.

5.4. Pareto efficiency under individual rationality constraints

Here we solve the problem of finding Pareto efficient allocations under individual rationality constraints as posed in Ravanelli and Svindland (2014).

As in Section 5.2 we assume all agents $i \in [n]$ enter the system with an initial endowment. These are given by a vector $\mathbf{W} \in (L^1)^n$. Now, by means of redistribution, they aim to improve the aggregated situation within the system, but in that redistribution they are not willing to accept a loss in utility beyond a certain threshold compared to the utility of their initial endowment. Given these thresholds $c_i \in [-\infty,\infty)^n$ and some sensible positive weights $\mathbf{w} \in (0,\infty)^n$, we thus consider the optimisation problem

$$\sum_{i=1}^n w_i\mathfrak{U}_i(Y_i) \to \max \quad \text{subject to}$$

$$\mathbf{Y} \in \Gamma_W \text{ (or } \mathbf{Y} \in \widehat{\Gamma}_W\text{)},\ \mathfrak{U}_i(Y_i) \ge \mathfrak{U}_i(W_i) + c_i,\ i \in [n],$$

where $W := W_1 + \cdots + W_n$, depending on whether free disposal is allowed in the redistribution or not. Any solution of this optimisation problem will be a Pareto efficient allocation of the aggregated initial endowment W. We will model this situation by altering the attainable sets, which are now defined by

$$\mathbb{A}_{\mathbf{c}}(\mathbf{W}) := \{\mathbf{X} \in \Gamma_W \mid \mathfrak{U}_i(X_i) \ge \mathfrak{U}_i(W_i) + c_i\},$$

if free disposal is not allowed, or, provided free disposal is allowed,

$$\widehat{\mathbb{A}}_{\mathbf{c}}(\mathbf{W}) := \{\mathbf{X} \in \widehat{\Gamma}_W \mid \mathfrak{U}_i(X_i) \ge \mathfrak{U}_i(W_i) + c_i\},$$

where $\mathbf{W} \in (L^1)^n$ is the vector of initial endowments. Clearly, the inclusion $\mathbb{A}_{\mathbf{c}}(\mathbf{W}) \subset \widehat{\mathbb{A}}_{\mathbf{c}}(\mathbf{W})$ holds. However, it is not *a priori* clear whether for a given vector of initial endowments $\mathbf{W}$ any of these two sets is non-empty.

For the next theorem, we set $I_\infty := \{i \in [n] \mid c_i = -\infty\}$, a possibly empty set. However, we may assume without loss of generality that $I_\infty \subsetneq [n]$, as otherwise we are in the situation of Theorem 33. We also define the upper semicontinuous aggregation function

$$\Lambda(\mathbf{y}) := \sum_{i=1}^n y_i, \quad \mathbf{y} \in [-\infty,\infty)^n.$$

Theorem 35. *Suppose $\mathfrak{U} = (\mathfrak{U}_i)_{i\in[n]}$ fulfils Assumption 26 and assume $\mathbf{B} = (B_i)_{i\in[n]}$ and $\mathbf{C} = (C_i)_{i\in[n]}$ satisfy*

$$\max_{i\in[n]} B_i - \inf_{i\in I_\infty} C_i < 0,$$

where $\inf_{i\in I_\infty} C_i := \infty$ if $I_\infty = \emptyset$. Furthermore, let $\mathbf{W} \in \prod_{i=1}^n \mathrm{dom}(\mathfrak{U}_i)$, $W := W_1 + \cdots + W_n$, and let $\mathbf{c} \in [-\infty,\infty)^n$ be a vector of individual rationality constraints.

(i) If $\mathbb{A}_{\mathbf{c}}(\mathbf{W}) \neq \emptyset$ and $\sup_{\mathbf{Y}\in\mathbb{A}_{\mathbf{c}}(\mathbf{W})} \Lambda(\mathfrak{U}(\mathbf{Y})) > -\infty$, there is $\mathbf{g} \in \mathbb{C}(n)$ such that

$$\Lambda\left(\mathfrak{U}(\mathbf{g}(W))\right) = \sup_{\mathbf{Y}\in\mathbb{A}_{\mathbf{c}}(\mathbf{W})} \Lambda(\mathfrak{U}(\mathbf{Y})) \in \mathbb{R}.$$

$\mathbf{g}(W)$ is a Pareto efficient allocation of W which respects the individual rationality constraints $\mathbf{c}$ in case free disposal is not allowed.

(ii) If $\widehat{\mathbb{A}}_{\mathbf{c}}(\mathbf{W}) \neq \emptyset$ and $\sup_{\mathbf{Y}\in\widehat{\mathbb{A}}_{\mathbf{c}}(\mathbf{W})} \Lambda(\mathfrak{U}(\mathbf{Y})) > -\infty$ there is $\mathbf{g} \in \mathbb{C}(n+1)$ such that $g_{n+1}(W) \ge 0$ and

$$\Lambda\left(\mathfrak{U}_1(g_1(W)), \ldots, \mathfrak{U}_n(g_n(W))\right) = \sup_{\mathbf{Y}\in\widehat{\mathbb{A}}_{\mathbf{c}}(\mathbf{W})} \Lambda(\mathfrak{U}(\mathbf{Y})) \in \mathbb{R}.$$

$(g_1(W), \ldots, g_n(W))$ is a Pareto efficient allocation of W which respects the individual rationality constraints $\mathbf{c}$ in case free disposal is allowed.

Proof. Both in (i) and (ii), if $(g_1(W), \ldots, g_n(W))$ is a maximiser, its Pareto efficiency within $\mathbb{A}_{\mathbf{c}}(\mathbf{W})$ – or $\widehat{\mathbb{A}}_{\mathbf{c}}(\mathbf{W})$, respectively – is immediately verified.

(i) We aim to apply Theorem 33 and therefore have to verify condition (18). Consider the utility functions $\widetilde{\mathfrak{U}}_i := \mathfrak{U}_i + \delta(\cdot|\mathcal{C}_i)$, where $\mathcal{C}_i := \{Y \in L^1 \mid \mathfrak{U}_i(Y) \ge \mathfrak{U}_i(W_i) + c_i\}$ is closed and $\delta(\cdot|\mathcal{C}_i)$ is the concave indicator of this set. The family $\widetilde{\mathfrak{U}}$ of new utility functions $\widetilde{\mathfrak{U}}_i$ checks Assumption 3. Furthermore, $\widetilde{\mathfrak{U}}_i \le \mathfrak{U}_i$. Hence, $\widetilde{\mathfrak{U}}_i|_{\mathbb{R}}$ is also an (A_i, B_i, C_i)-function, and Assumption 26 is checked.

We shall now demonstrate that for all $i \in [n]$, the parameter C_i can be assumed to satisfy $\mathbf{B}^* < C_i$ after potential manipulation. This would entail that $(\widetilde{\mathfrak{U}}_1, \ldots, \widetilde{\mathfrak{U}}_n)$ checks the hypotheses of Theorem 33, namely (18) if we choose $\mathbf{w} = (1, 1, \ldots, 1)$.

To this end, note first that for all $i \in I_\infty$ the estimate $\mathbf{B}^* < C_i$ holds by assumption. Second, if $i \in [n]\backslash I_\infty$, assume $r \in \mathbb{R}$ satisfies $\widetilde{\mathfrak{U}}_i(r) > -\infty$. This implies $\widetilde{\mathfrak{U}}_i(r) = u_i(r) \ge \mathfrak{U}_i(W_i) + c_i > -\infty$. Hence, if additionally $r < 0$,

$$\mathfrak{U}_i(W_i) + c_i \le u_i(-|r|) \le A_i - C_i|r|,$$

which can be rearranged as

$$|r| \le \left|\frac{\mathfrak{U}_i(W_i) + c_i - A_i}{C_i}\right| =: \sigma_i.$$

For $n_i \in \mathbb{N}$ large enough, we have for all $y \in [-\sigma_i, \infty)$ that

$$A_i + B_iy^+ - C_iy^- \le n_i + B_iy^+ - (\mathbf{B}^* + 1)y^-.$$

Hence, $\mathfrak{U}_i$ is also a $(n_i, B_i, \mathbf{B}^* + 1)$-function.

Now we can conclude with Theorem 33 the existence of some $\mathbf{g} \in \mathbb{C}(n)$ such that

$$\sum_{i=1}^n \widetilde{\mathfrak{U}}_i(g_i(W)) = \sup_{\mathbf{Y}\in\Gamma_W} \sum_{i=1}^n \widetilde{\mathfrak{U}}_i(Y_i) \in \mathbb{R}.$$

The left-hand side would be $-\infty$ if $\mathbf{g}(X)$ were not in the attainable set $\mathbb{A}_{\mathbf{c}}(\mathbf{W})$, whence we infer $\mathbf{g}(W) \in \mathbb{A}_{\mathbf{c}}(\mathbf{W})$. This implies $\tilde{\mathfrak{U}}_i(g_i(W)) = \mathfrak{U}_i(g_i(W))$ for all $i \in [n]$. Using again that $\sum_{i=1}^n \tilde{\mathfrak{U}}_i(Y_i) = -\infty$ if $\mathbf{Y} \in \Gamma_W \backslash \mathbb{A}_{\mathbf{c}}(\mathbf{W})$,

$$\sup_{\mathbf{Y}\in\Gamma_W} \sum_{i=1}^n \tilde{\mathfrak{U}}_i(Y_i) = \sup_{\mathbf{Y}\in\mathbb{A}_{\mathbf{c}}(\mathbf{W})} \sum_{i=1}^n \tilde{\mathfrak{U}}_i(Y_i) = \sup_{\mathbf{Y}\in\mathbb{A}_{\mathbf{c}}(\mathbf{W})} \sum_{i=1}^n \mathfrak{U}_i(Y_i).$$

(ii) The argument is in complete analogy with the argument for (i). □

5.5. Aggregation with a view towards systemic risk

Throughout this section let $\mathbf{p}, \mathbf{q}, \mathbf{r} \in \mathbb{R}_+^n$ be such that $\sum_{i=1}^n p_i = 1$. We consider the upper semicontinuous aggregation function

$$\Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}(\mathbf{y}) := \sum_{i=1}^n -p_i y_i^- + q_i(y_i - r_i)^+, \quad \mathbf{y} \in [-\infty, \infty)^n. \tag{19}$$

It was suggested by Brunnermeier and Cheridito (2013) as a way to aggregate individual profits net of losses in a system of agents in a meaningful way to account for systemic risk; c.f. Hoffmann et al. (2016, Example 4.3).

In the present setting, we may consider the quantity $\Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}(\mathfrak{U}(\mathbf{X}))$, $\mathbf{X} \in (L^1)^n$, and use $\Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}$ to aggregate the individual utilities of the n agents. In such an application, it would be more appropriate to think of $\Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}$ to account for *systemic fairness*.

Note that for some $\mathbf{X} \in (L^1)^n$, $\Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}(\mathfrak{U}(\mathbf{X}))$ has a clear-cut interpretation: $-p_i\mathfrak{U}_i(X_i)^-$, $i \in [n]$, only appears in the aggregation if i incurs a negative utility and is then weighted according to the impact or importance of agent i. Conversely, the term $q_i(\mathfrak{U}_i(X_i) - r_i)^+$ accounts, likewise in a weighted way, for the positive utility agent i gains as far as it exceeds a certain individual threshold r_i.

One easily sees that $\Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}$ is an upper semicontinuous aggregation function.

Theorem 36. *Suppose $\mathfrak{U} = (\mathfrak{U}_i)_{i\in[n]}$ fulfils Assumption 26 and assume $\mathbf{B} = (B_i)_{i\in[n]}$, $\mathbf{C} = (C_i)_{i\in[n]}$, and $\mathbf{p}, \mathbf{q} \in \mathbb{R}_+^n$ satisfy*

$$(\mathbf{q} \cdot \mathbf{B})^* < (\mathbf{p} \cdot \mathbf{C})_*. \tag{20}$$

Let $\mathbf{r} \in \mathbb{R}_+^n$ be arbitrary and define $\Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}$ as in (19).

(i) For all $X \in \sum_{i=1}^n \mathrm{dom}(\mathfrak{U}_i)$ there is $\mathbf{g} \in \mathbb{C}(n)$ such that

$$\Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}(\mathfrak{U}(\mathbf{g}(X))) = \sup_{\mathbf{Y}\in\Gamma_X} \Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}(\mathfrak{U}(\mathbf{Y})) \in \mathbb{R}.$$

(ii) For all $X \in \sum_{i=1}^n \mathrm{dom}(\mathfrak{U}_i) + L_+^1$ there is $\mathbf{g} \in \mathbb{C}(n+1)$ such that $g_{n+1}(X) \geq 0$ and

$$\Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}(\mathfrak{U}_1(g_1(X)), \ldots, \mathfrak{U}_n(g_n(X))) = \sup_{\mathbf{Y}\in\widehat{\Gamma}_X} \Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}(\mathfrak{U}(\mathbf{Y})) \in \mathbb{R}.$$

Proof. Combine Lemma A.4 with Theorem 12. □

Appendix. Coercivity results

In the following, given a vector $\mathbf{u} = (u_1, \ldots, u_n)$ of (A_i, B_i, C_i)-functions, we set $\mathbf{A} := (A_i)_{i\in[n]}$, $\mathbf{B} := (B_i)_{i\in[n]}$, and $\mathbf{C} := (C_i)_{i\in[n]}$.

Lemma A.1. *Assume $\mathbf{u} = (u_1, \ldots, u_n)$ is a vector of (A_i, B_i, C_i)-functions $u_i : \mathbb{R} \to [-\infty, \infty)$, $i \in [n]$. Let $0 < \alpha < 1$ be a parameter which satisfies (15) if $n = 2$, or, provided $n \geq 3$, (16). Then the aggregation function*

$$\Lambda_\alpha(\mathbf{y}) := \alpha \min_{i\in[n]} y_i + (1-\alpha) \max_{i\in[n]} y_i, \quad \mathbf{y} \in [-\infty, \infty)^n,$$

is upper semicontinuous and coercive for $\mathbf{u}$.

Proof. The aggregation function Λ_α is clearly upper semicontinuous and non-decreasing in the pointwise order on $[-\infty, \infty)^n$. Fix $x, m \in \mathbb{R}$ and consider the set $S(x, m)$ defined in (6). In order to find the bound $G(x, m)$ as in (7), we choose $\mathbf{y} \in S(x, m)$ arbitrary, set $I := \{i \in [n] \mid y_i < 0\}$, and distinguish the following cases:
Case 1: $I = \emptyset$. Then $\sum_{i=1}^n |y_i| = \sum_{i=1}^n y_i \leq x$.
Case 2: $\mathbf{y} \in \mathbb{R}_-^n$. Then

$$m \leq \alpha \min_{i\in[n]} A_i - C_i|y_i| + (1-\alpha) \max_{i\in[n]} A_i - C_i|y_i| \leq \mathbf{A}^* - \alpha \mathbf{C}_* \max_{i\in[n]} |y_i|.$$

Rearranging this inequality yields

$$\sum_{i=1}^n |y_i| \leq n \max_{i\in[n]} |y_i| \leq \frac{n(\mathbf{A}^* - m)}{\alpha \mathbf{C}_*}. \tag{A.1}$$

Case 3: $I \neq \emptyset$ and $J := \{i \in [n] \mid y_i > 0\} \neq \emptyset$.
Case 3.1: If $n = 2$ and $I = \{1\}$, we have $J = \{2\}$. From the property $y_1 + y_2 \leq x$ we infer $|y_2| = y_2 \leq x + |y_1|$. Using this and each u_i being an (A_i, B_i, C_i)-function, we can estimate

$$\begin{aligned}
m &\leq \alpha \min_{i=1,2} u_i(y_i) + (1-\alpha) \max_{i=1,2} u_i(y_i) \\
&\leq \alpha u_1(y_1) + (1-\alpha) \max_{i=1,2} u_i(y_i) \\
&\leq \alpha(A_1 - C_1|y_1|) + (1-\alpha) \max\{A_1 - C_1|y_1|, A_2 + B_2 y_2\} \\
&\leq \alpha(\mathbf{A}^* - C_1|y_1|) + (1-\alpha)(\mathbf{A}^* + B_2 y_2) \\
&= \mathbf{A}^* - \alpha C_1|y_1| + (1-\alpha)B_2(x + |y_1|) \\
&= \mathbf{A}^* + (1-\alpha)B_2 x + ((1-\alpha)B_2 - \alpha C_1)\,|y_1|.
\end{aligned}$$

By the first inequality in (15), $(1-\alpha)B_2 - \alpha C_1 < 0$. Hence, rearranging terms yields

$$|y_1| \leq \frac{\mathbf{A}^* + (1-\alpha)B_2 x - m}{|(1-\alpha)B_2 - \alpha C_1|},$$

and eventually

$$|y_1| + |y_2| \leq x + 2|y_1| \leq x + \frac{2(\mathbf{A}^* + (1-\alpha)B_2 x - m)}{|(1-\alpha)B_2 - \alpha C_1|}. \tag{A.2}$$

Case 3.2: If $n = 2$ and $I = \{2\}$, we obtain completely analogously to Case 3.1 that

$$|y_1| + |y_2| \leq x + 2|y_2| \leq x + \frac{2(\mathbf{A}^* + \alpha B_1 x - m)}{|\alpha B_1 - (1-\alpha)C_2|}. \tag{A.3}$$

Case 3.3: $n \geq 3$. As above,

$$\max_{j\in J} |y_j| \leq \sum_{j\in J} y_j \leq x + (n-1) \max_{i\in I} |y_i|.$$

This allows us to infer

$$\begin{aligned}
m &\leq \alpha \min_{i\in[n]} A_i + B_i y_i^+ - C_i y_i^- + (1-\alpha) \max_{i\in[n]} A_i + B_i y_i^+ - C_i y_i^- \\
&\leq \alpha \min_{i\in I}(\mathbf{A}^* - C_i|y_i|) + (1-\alpha) \max_{j\in J}(\mathbf{A}^* + \mathbf{B}_j y_j) \\
&\leq \mathbf{A}^* - \alpha \mathbf{C}_* \max_{i\in I} |y_i| + (1-\alpha)\mathbf{B}^* \max_{j\in J} |y_j| \\
&\leq \mathbf{A}^* + (1-\alpha)\mathbf{B}^* x + \big((1-\alpha)(n-1)\mathbf{B}^* - \alpha \mathbf{C}_*\big) \max_{i\in I} |y_i|.
\end{aligned}$$

Rearranging the preceding inequality and using that, by (16), $(1-\alpha)(n-1)\mathbf{B}^* - \alpha \mathbf{C}_* < 0$, we conclude

$$\max_{i\in I} |y_i| \leq \frac{\mathbf{A}^* + (1-\alpha)\mathbf{B}^* x - m}{|(1-\alpha)(n-1)\mathbf{B}^* - \alpha \mathbf{C}_*|},$$

and eventually

$$\begin{aligned}
\sum_{i=1}^n |y_i| &\leq x + 2\sum_{i\in I} |y_i| \leq x + 2(n-1) \max_{i\in I} |y_i| \\
&\leq x + \frac{2(n-1)(\mathbf{A}^* + (1-\alpha)\mathbf{B}^* x - m)}{|(1-\alpha)(n-1)\mathbf{B}^* - \alpha \mathbf{C}_*|}.
\end{aligned} \tag{A.4}$$

Let $\tilde{G}(x, m)$ be defined as the maximum of the bounds (A.1)–(A.3) (if $n = 2$) or of (A.1), and (A.4) (if $n \geq 3$). Then

$$G(x, m) := \max\{\tilde{G}(x, m), x\}, \quad (x, m) \in \mathbb{R} \times \mathbb{R},$$

gives the desired bound (7).

Note that for all $x \in \mathbb{R}$, $\tilde{G}(x, m) \downarrow -\infty$ as $m \to \infty$. We may hence consider the real-valued function

$$H(x) := \max\{\inf\{s \in \mathbb{R} \mid \tilde{G}(x, s) \leq -1\}, \mathbf{A}^* + \mathbf{B}^* x + 1\}, \quad x \in \mathbb{R}.$$

Fix $m \geq H(x)$ and suppose we can choose $\mathbf{y} \in S(x, m)$. By construction, $\mathbf{y} \in \mathbb{R}^n_+$ has to hold. We estimate

$$m \leq \max_{i \in [n]} A_i + B_i y_i < \mathbf{A}^* + \mathbf{B}^* x + 1 \leq H(x) \leq m.$$

No such $\mathbf{y}$ can exist, and the function H has property (8). Λ_α is coercive for $\mathbf{u}$. $\square$

Lemma A.2. *Assume $\mathbf{u} = (u_1, \dots, u_n)$ is a vector of (A_i, B_i, C_i)-functions $u_i : \mathbb{R} \to [-\infty, \infty)$, $i \in [n]$. Then the aggregation function $\Lambda_1(\mathbf{y}) = \min_{i \in [n]} y_i$, $\mathbf{y} \in [-\infty, \infty)^n$, is upper semicontinuous and coercive for $\mathbf{u}$.*

Proof. Λ_1 is upper semicontinuous and non-decreasing in the pointwise order on $[-\infty, \infty)^n$. For coercivity in the case $n \geq 3$, note that $\frac{(n-1)\mathbf{B}^*}{\mathbf{C}_*} < \infty$. Hence, for $0 < \alpha < 1$ close enough to 1, the estimate

$$\frac{(n-1)\mathbf{B}^*}{\mathbf{C}_*} < \frac{\alpha}{1-\alpha}$$

holds. Hence, $\Lambda_1(\mathbf{y}) \leq \Lambda_\alpha(\mathbf{y})$ holds for all $\mathbf{y} \in [-\infty, \infty)^n$, and the latter function is coercive for $\mathbf{u}$ by Lemma A.1. Coercivity of Λ_1 follows with Proposition 11(iii).

It remains to treat the case $n = 2$. Let $x, m \in \mathbb{R}$ be arbitrary and suppose $\mathbf{y} \in S(x, m)$. Set $I := \{i \in [n] \mid y_i < 0\}$ and consider the following cases:

Case 1: $I = \emptyset$. Then $|y_1| + |y_2| = y_1 + y_2 \leq x$.

Case 2: $\mathbf{y} \in \mathbb{R}^n_-$. Then

$$m \leq \Lambda_1(\mathbf{u}(\mathbf{y})) \leq \min\{A_1 - C_1|y_1|, A_2 - C_2|y_2|\} \leq \mathbf{A}^* - \mathbf{C}_* \max_{i=1,2} |y_i|.$$

From a rearrangement of this inequality, we infer

$$|y_1| + |y_2| \leq 2 \max_{i=1,2} |y_i| \leq \frac{2(\mathbf{A}^* - m)}{\mathbf{C}_*}.$$

Case 3: $|I| = 1$.

Case 3.1: $I = \{1\}$. We use again that $|y_2| = y_2 \leq x - y_1 = x + |y_1|$. Note that

$$m \leq \Lambda_1(\mathbf{u}(\mathbf{y})) \leq \min\{A_1 - C_1|y_1|, A_2 + B_2 y_2\} \leq \mathbf{A}^* - \mathbf{C}_*|y_1|.$$

From a rearrangement of this inequality, we obtain

$$|y_1| + |y_2| \leq x + 2|y_1| \leq x + \frac{2(\mathbf{A}^* - m)}{\mathbf{C}_*}.$$

Case 3.2: $I = \{2\}$. In this case, we obtain the same bound as in Case 3.1.

Consequently, the function

$$G(x, m) := \max\{x, x^+ + \tfrac{2(\mathbf{A}^* - m)}{\mathbf{C}_*}\}, \quad (x, m) \in \mathbb{R} \times \mathbb{R},$$

has the desired property (7).

Now consider the function

$$H(x) := \max\left\{\mathbf{A}^* + \mathbf{C}_*(\tfrac{1}{2}x^+ + 1), A_1 + B_1 x + 1\right\}, \quad x \in \mathbb{R}.$$

If $m \geq H(x)$, $\max\{\frac{2(\mathbf{A}^* - m)}{\mathbf{C}_*}, x + \frac{2(\mathbf{A}^* - m)}{\mathbf{C}_*}\} \leq -1$. Thus, if we could choose $\mathbf{y} \in S(x, m)$, Case 1 above would have to hold, i.e. $\mathbf{y} \in \mathbb{R}^2_+$. Using that each u_i is an (A_i, B_i, C_i)-function,

$$\Lambda_1(\mathbf{u}(\mathbf{y})) \leq u_1(y_1) \leq A_1 + B_1 y_1 < A_1 + B_1 x + 1 \leq H(x) \leq m.$$

This is a CONTRADICTION, and $S(x, m) = \emptyset$ has to hold. Hence, the function H has the desired property (8), and Λ_1 is coercive for $\mathbf{u}$. $\square$

Lemma A.3. *Assume $\mathbf{u} = (u_1, \dots, u_n)$ is a vector of (A_i, B_i, C_i)-functions $u_i : \mathbb{R} \to [-\infty, \infty)$, $i \in [n]$. Moreover, suppose $\mathbf{w} \in (0, \infty)^n$ satisfies (17) if $n = 2$, or, if $n \geq 3$, (18). Then the aggregation function*

$$\Lambda_{\mathbf{w}}(\mathbf{y}) := \sum_{i=1}^n w_i y_i, \quad \mathbf{y} \in [-\infty, \infty)^n,$$

is upper semicontinuous and coercive for $\mathbf{u}$.

Proof. The function $\Lambda_{\mathbf{w}}$ is clearly upper semicontinuous and non-decreasing in the pointwise order on $[-\infty, \infty)^n$. In order to find a function G with property (7), fix $x, m \in \mathbb{R}$ and assume $\mathbf{y} \in S(x, m)$ is arbitrarily chosen.

Case 1: $\mathbf{y} \in \mathbb{R}^n_+$. Then $\sum_{i=1}^n |y_i| = \sum_{i=1}^n y_i \leq x$.

Case 2: $\mathbf{y} \in \mathbb{R}^n_-$. Then

$$\begin{aligned} m \leq \Lambda_{\mathbf{w}}(\mathbf{u}(\mathbf{y})) &= \sum_{i=1}^n w_i u_i(y_i) \leq \sum_{i=1}^n w_i(A_i - C_i|y_i|) \\ &\leq n(\mathbf{w} \cdot \mathbf{A})^* - (\mathbf{w} \cdot \mathbf{C})_* \sum_{i=1}^n |y_i|. \end{aligned}$$

Rearranging this inequality yields

$$\sum_{i=1}^n |y_i| \leq \frac{n(\mathbf{w} \cdot \mathbf{A})^* - m}{(\mathbf{w} \cdot \mathbf{C})_*}. \tag{A.5}$$

Case 3: $\mathbf{y} \in \mathbb{R}^n \backslash (\mathbb{R}^n_+ \cup \mathbb{R}^n_-)$. We set $I = \{i \in [n] \mid y_i < 0\}$.

Case 3.1: $n = 2$ and $I = \{1\}$. We have

$$m \leq w_1 u_1(y_1) + w_2 u_2(y_2) \leq w_1 A_1 - w_1 C_1|y_1| + w_2 A_2 + w_2 B_2 |y_2|.$$

Using $|y_2| = y_2 \leq x + |y_1|$, one obtains

$$m \leq 2(\mathbf{w} \cdot \mathbf{A})^* + w_2 B_2 x + (w_2 B_2 - w_1 C_1)|y_1|.$$

By the first inequality in (17), $w_2 B_2 - w_1 C_1 < 0$. Hence, rearranging this inequality yields

$$|y_1| + |y_2| \leq x + 2|y_1| \leq x + \frac{4(\mathbf{w} \cdot \mathbf{A})^* + 2(\mathbf{w} \cdot \mathbf{B})^*|x| - 2m}{|w_2 B_2 - w_1 C_1|}. \tag{A.6}$$

Case 3.2: If $n = 2$ and $I = \{2\}$, we obtain completely analogously to Case 3.1 that

$$|y_1| + |y_2| \leq x + 2|y_2| \leq x + \frac{4(\mathbf{w} \cdot \mathbf{A})^* + 2(\mathbf{w} \cdot \mathbf{B})^*|x| - 2m}{|w_1 B_1 - w_2 C_2|}. \tag{A.7}$$

Combining Cases 1–3.2 implies that the function

$$\begin{aligned} G(x, m) := \max\Bigg\{x, \ &\frac{2(\mathbf{w} \cdot \mathbf{A})^* - m}{(\mathbf{w} \cdot \mathbf{C})_*}, \\ &x + \frac{4(\mathbf{w} \cdot \mathbf{A})^* + 2(\mathbf{w} \cdot \mathbf{B})^*|x| - 2m}{\xi}\Bigg\} \end{aligned}$$

has property (7). Here, $\xi := \min\{|w_1 B_1 - w_2 C_2|, \ |w_2 B_2 - w_1 C_1|\}$.

Case 3.3: $n \geq 3$. As in preceding proofs, $\sum_{i \in [n] \backslash I} y_i \leq x + \sum_{i \in I} |y_i|$. This allows us to infer

$$\begin{aligned} m &\leq \sum_{i=1}^n w_i u_i(y_i) \leq \sum_{i \in [n] \backslash I} w_i(A_i + B_i|y_i|) + \sum_{i \in I} w_i(A_i - C_i|y_i|) \\ &\leq \sum_{i=1}^n w_i A_i + (\mathbf{w} \cdot \mathbf{B})^* \sum_{i \in [n] \backslash I} |y_i| - (\mathbf{w} \cdot \mathbf{C})_* \sum_{i \in I} |y_i| \\ &\leq n(\mathbf{w} \cdot \mathbf{A})^* + (\mathbf{w} \cdot \mathbf{B})^* x + \big((\mathbf{w} \cdot \mathbf{B})^* - (\mathbf{w} \cdot \mathbf{C})_*\big) \sum_{i \in I} |y_i| \end{aligned}$$

Rearranging this inequality using (18) yields

$$\sum_{i\in I}|y_i| \le \frac{n(\mathbf{w}\cdot\mathbf{A})^* + (\mathbf{w}\cdot\mathbf{B})^* x - m}{|(\mathbf{w}\cdot\mathbf{B})^* - (\mathbf{w}\cdot\mathbf{C})_*|},$$

and, eventually,

$$\sum_{i=1}^n |y_i| \le x + 2\sum_{i\in I}|y_i| \le x + \frac{2\,(n(\mathbf{w}\cdot\mathbf{A})^* + (\mathbf{w}\cdot\mathbf{B})^* x - m)}{|(\mathbf{w}\cdot\mathbf{B})^* - (\mathbf{w}\cdot\mathbf{C})_*|}. \quad (A.8)$$

Combining equations (A.5) and (A.8) shows that the function

$$G(x,m) := \max\left\{x,\ \frac{n(\mathbf{w}\cdot\mathbf{A})^* - m}{(\mathbf{w}\cdot\mathbf{C})_*},\ x + \frac{2\,(n(\mathbf{w}\cdot\mathbf{A})^* + (\mathbf{w}\cdot\mathbf{B})^* x - m)}{|(\mathbf{w}\cdot\mathbf{B})^* - (\mathbf{w}\cdot\mathbf{C})_*|}\right\}$$

has property (7), provided $n \ge 3$.

We now turn our attention to the existence of a function H with property (8). Fix $x \in \mathbb{R}$ and let $\tilde{H}(x) := n(\mathbf{w}\cdot\mathbf{A})^* + (\mathbf{w}\cdot\mathbf{B})^* x + 1$. If $n = 2$, consider

$$H(x) := \max\left\{\tilde{H}(x),\ (\mathbf{w}\cdot\mathbf{C})_* + 2(\mathbf{w}\cdot\mathbf{A})^*,\ \tfrac{\xi(x+1)}{2} + 2(\mathbf{w}\cdot\mathbf{A})^* + (\mathbf{w}\cdot\mathbf{B})^*|x|\right\},$$

where ξ is defined as above. If $m \ge H(x)$, the right-hand sides of (A.5)–(A.7) are less or equal to -1. If $n \ge 3$, consider

$$H(x) := \max\left\{\tilde{H}(x),\ n(\mathbf{w}\cdot\mathbf{A})^* + (\mathbf{w}\cdot\mathbf{C})_*,\ \tfrac{|(\mathbf{w}\cdot\mathbf{B})^* - (\mathbf{w}\cdot\mathbf{C})_*|(x+1)}{2} + n(\mathbf{w}\cdot\mathbf{A})^* + (\mathbf{w}\cdot\mathbf{B})^* x\right\}.$$

If $m \ge H(x)$ in this case, the right-hand sides of (A.5) and (A.8) are less or equal to -1.

Suppose now $m \ge H(x)$ and $S(x,m) \ne \emptyset$. Then for all $\mathbf{y} \in S(x,m)$, Case 1 from above has to hold, i.e. $\mathbf{y} \in \mathbb{R}^n_+$. Moreover,

$$\sum_{i=1}^n w_i u_i(y_i) \le \sum_{i=1}^n w_i(A_i + B_i y_i) < n(\mathbf{w}\cdot\mathbf{A})^* + (\mathbf{w}\cdot\mathbf{B})^* x + 1 = \tilde{H}(x) \le H(x) \le m,$$

which is absurd. Hence, the function H has property (8). $\square$

Lemma A.4. *Assume $\mathbf{u} = (u_1,\dots,u_n)$ is a vector of (A_i,B_i,C_i)-functions $u_i : \mathbb{R} \to [-\infty,\infty)$, $i \in [n]$. If $\mathbf{p},\mathbf{q} \in \mathbb{R}^n_+$ satisfy (20) and $\mathbf{r} \in \mathbb{R}^n_+$ is arbitrary, the aggregation function*

$$\Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}(y) := \sum_{i=1}^n -p_i y_i^- + q_i(y_i - r_i)^+, \quad y \in [-\infty,\infty)^n,$$

is upper semicontinuous and coercive for $\mathbf{u}$.

Proof. By Remark 25(iii), we may assume without loss of generality that $A_i \ge 0$ holds for all $i \in [n]$.

As already observed, the function $\Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}$ is upper semicontinuous and non-decreasing in the pointwise order on $[-\infty,\infty)^n$. In order to find the function G, fix $x, m \in \mathbb{R}$ and let $\mathbf{y} \in S(x,m)$ be arbitrary. Again, we set $I := \{i \in [n] \mid y_i < 0\}$.

Case 1: $I = \emptyset$, i.e. $\mathbf{y} \in \mathbb{R}^n_+$. As in the preceding proofs, $\sum_{i=1}^n |y_i| \le x$.

Case 2: $\mathbf{y} \in \mathbb{R}^n_-$. Then

$$\begin{aligned} m &\le \Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}(\mathbf{u}(\mathbf{y})) \le \sum_{i=1}^n -p_i(A_i - C_i|y_i|)^- + q_i(A_i - C_i|y_i| - r_i)^+ \\ &\le \sum_{i=1}^n p_i(A_i - C_i|y_i|) + q_i(A_i - r_i)^+ \\ &\le n(\mathbf{p}\cdot\mathbf{A})^* - (\mathbf{p}\cdot\mathbf{C})_* \sum_{i=1}^n |y_i| + \sum_{i=1}^n q_i(A_i - r_i)^+. \end{aligned}$$

As $p_i C_i > 0$ for all $i \in [n]$, we obtain

$$\sum_{i=1}^n |y_i| \le \frac{n(\mathbf{p}\cdot\mathbf{A})^* + \sum_{i=1}^n q_i(A_i - r_i)^+ - m}{(\mathbf{p}\cdot\mathbf{C})_*}. \quad (A.9)$$

Case 3: $I \ne \emptyset$ and $y_j > 0$ for some $j \in J := [n]\backslash I$.

Setting $I' := \{i \in I \mid A_i - C_i|y_i| > r_i\}$ and $J' := \{j \in J \mid A_j + B_j y_j > r_j\}$, we have

$$\begin{aligned} m &\le \Lambda_{\mathbf{p},\mathbf{q},\mathbf{r}}(\mathbf{u}(\mathbf{y})) \le \sum_{i\in I} -p_i(A_i - C_i|y_i|)^- + q_i(A_i - C_i|y_i| - r_i)^+ \\ &\quad + \sum_{j\in J} -p_j(A_j + B_j|y_j|)^- + q_j(A_j + B_j|y_j| - r_j)^+ \\ &\le \sum_{i\in I} p_i(A_i - C_i|y_i|) + \sum_{i\in I'} q_i(A_i - C_i|y_i|) \\ &\quad + \sum_{j\in J'} q_j(A_j + B_j|y_j|). \end{aligned}$$

In the last step, we have used that for $j \in J$, our assumption $A_j \ge 0$ implies $(A_j + B_j y_j)^- = 0$.

- The estimate
$$\sum_{i\in I} p_i(A_i - C_i|y_i|) \le n(\mathbf{p}\cdot\mathbf{A})^* - (\mathbf{p}\cdot\mathbf{C})_* \sum_{i\in I}|y_i| =: \rho_1 - (\mathbf{p}\cdot\mathbf{C})_* \sum_{i\in I}|y_i|$$
is immediate.
- We have $\sum_{i\in I'} q_i(A_i - C_i|y_i|) \le n(\mathbf{q}\cdot\mathbf{A})^* =: \rho_2$.
- From $\sum_{j\in J'}|y_j| \le \sum_{j\in J} y_j \le x + \sum_{i\in I}|y_i|$, we conclude
$$\sum_{j\in J'} q_j(A_j + B_j|y_j|) \le \rho_2 + (\mathbf{q}\cdot\mathbf{B})^*\left(x + \sum_{i\in I}|y_i|\right).$$

Combining all estimates above, we obtain

$$\begin{aligned} m &\le \rho_1 - (\mathbf{p}\cdot\mathbf{C})_* \sum_{i\in I}|y_i| + 2\rho_2 + (\mathbf{q}\cdot\mathbf{B})^*\left(x + \sum_{i\in I}|y_i|\right) \\ &=: \rho_3 + \left((\mathbf{q}\cdot\mathbf{B})^* - (\mathbf{p}\cdot\mathbf{C})_*\right)\sum_{i\in I}|y_i|. \end{aligned}$$

The constant $\rho_3 \in \mathbb{R}$ is independent of $\mathbf{y}$. We rearrange the inequality and use (20) in order to obtain

$$\sum_{i\in I}|y_i| \le \frac{\rho_3 - m}{|(\mathbf{q}\cdot\mathbf{B})^* - (\mathbf{p}\cdot\mathbf{C})_*|}.$$

Consequently, the bound

$$\sum_{i\in[n]}|y_i| \le x + 2\sum_{i\in I}|y_i| \le x + \frac{2(\rho_3 - m)}{|(\mathbf{q}\cdot\mathbf{B})^* - (\mathbf{p}\cdot\mathbf{C})_*|} \quad (A.10)$$

holds. Let $\tilde{G}(x,m)$ be defined as the maximum of the bounds in (A.9) and (A.10). As in the preceding proofs, the function $G(x,m) := \max\{x, \tilde{G}(x,m)\}$, $(x,m) \in \mathbb{R}\times\mathbb{R}$, has property (7).

Note that for all $x \in \mathbb{R}$ we have $\tilde{G}(x,m) \downarrow -\infty$ as $m \to \infty$. This allows us to define a function H which has property (8) by

$$H(x) := \max\{\inf\{s \in \mathbb{R} \mid \tilde{G}(x,s) \leq -1\},$$
$$n(\mathbf{q} \cdot \mathbf{A})^* + (\mathbf{q} \cdot \mathbf{B})^* x + 1\}, \quad x \in \mathbb{R}.$$

The proof of the assertion is completely analogous to the preceding cases. □

References

Acciaio, B., 2007. Optimal risk sharing with non-monotone monetary functionals. Finance Stoch. 11, 267–289.

Acciaio, B., Svindland, G., 2009. Optimal risk sharing with different reference probabilities. Insurance Math. Econom. 44 (3), 426–433.

Aliprantis, C., Burkinshaw, O., 2003. Locally Solid Riesz Spaces with Applications to Economics. American Mathematical Society.

Attanasio, O., Davis, S., 1996. Relative wage movements and the distribution of consumption. J. Political Econ. 104 (6), 1227–1262.

Brunnermeier, M.K., Cheridito, P., 2013. Measuring and allocating systemic risk. Preprint.

Carlier, G., Dana, R.-A., 2008. Two-persons efficient risk-sharing and equilibria for concave law-invariant utilities. Econom. Theory 36 (2), 189–223.

Carlier, G., Dana, R.-A., Galichon, A., 2012. Pareto efficiency for the concave order and multivariate comonotonicity. J. Econom. Theory 147 (1), 207–229.

Cerreia-Vioglio, S., Maccheroni, F., Marinacci, M., Montrucchio, L., 2012. Probabilistic sophistication, second order stochastic dominance and uncertainty aversion. J. Math. Econom. 48, 271–283.

Chateauneuf, A., Dana, R.-A., Tallon, J.-M., 2000. Optimal risk-sharing rules and equilibria with Choquet-expected-utility. J. Math. Econom. 34 (2), 191–214.

Chen, S., Gao, N., Xanthos, F., 2018. The strong Fatou property of risk measures. Dependence Model. 6 (1), 183–196.

Cherny, A.S., Grigoriev, P.G., 2007. Dilatation monotone risk measures are law invariant. Finance Stoch. 11 (2), 291–298.

Cheung, K.C., Rong, Y., Yam, S.C.P., 2014. Borch's Theorem from the perspective of comonotonicity. Insurance Math. Econom. 54, 144–151.

Chong, K.M., Rice, N.M., 1971. Equimeasurable Rearrangements of Functions. In: Queens Papers in Pure and Applied Mathematics, vol. 28.

Dana, R.-A., 2005. A representation result for concave Schur concave functions. Math. Finance 15, 613–634.

Dana, R.-A., 2011. Comonotonicity, efficient risk-sharing and equilibria in markets with short-selling for concave law-invariant utilitie. J. Math. Econom. 47, 328–335.

Debreu, G., 1982. Existence of competitive equilibrium. In: Arrow, K.J., Intriligator, M.D. (Eds.), Handbook of Mathematical Economics, Vol. 2. Elsevier, pp. 697–743.

Filipović, D., Svindland, G., 2008. Optimal capital and risk allocations for law- and cash-invariant convex functions. Finance Stoch. 12 (3), 423–439.

Föllmer, H., Schied, A., 2011. Stochastic Finance: An Introduction in Discrete Time, third ed. De Gruyter.

Gao, N., Leung, D., Munari, C., Xanthos, F., 2018. Fatou property, representations, and extensions of law-invariant risk measures on general Orlicz spaces. Finance Stoch. 22, 395–415.

Hoffmann, H., Meyer-Brandis, T., Svindland, G., 2016. Risk-consistent conditional systemic risk measures. Stochastic Process. Appl. 126 (7), 2014–2037.

Jouini, E., Napp, C., 2003. Comonotonic processes. Insurance Math. Econom. 32, 255–265.

Jouini, E., Schachermayer, W., Touzi, N., 2008. Optimal risk sharing for law invariant monetary utility functions. Math. Finance 18, 269–292.

Khan, M.A., Yannelis, N.C., 1991. Equilibrium Theory in Infinite Dimensional Spaces. In: Studies in Economic Theory, vol. 1, Springer.

Landsberger, M., Meilijson, I., 1994. Comonotone allocations, Bickel–Lehmann dispersion and the Arrow–Pratt measure of risk aversion. Ann. Oper. Res. 52 (2), 97–106.

Liebrich, F.-B., Svindland, G., 2018. Risk sharing for Capital requirements with multidimensional security markets. Preprint.

Ludkovski, M., Rüschendorf, V., 2008. On comonotonicity of pareto optimal risk sharing. Statist. Probab. Lett. 78 (10), 1181–1188.

Maccheroni, F., Marinacci, M., Rustichini, A., 2006. Ambiguity aversion, robustness, and the variational representation of preferences. Econometrica 74, 1447–1498.

Machina, M.J., Schmeidler, D., 1992. A more robust definition of subjective probability. Econometrica 60, 745–780.

Mas-Colell, A., Zame, W., 1991. Equilibrium theory in infinite dimensional spaces. In: Hildenbrand, W., Sonnenschein, H. (Eds.), Handbook of Mathematical Economics, Vol. 4. Elsevier, pp. 1835–1898.

Mastrogiacomo, E., Rosazza Gianin, E., 2015. Pareto optimal allocations and optimal risk sharing for quasiconvex risk measures. Math. Financ. Econ. 9, 149–167.

Ravanelli, C., Svindland, G., 2014. Comonotone Pareto optimal allocations for law invariant robust utilities on L^1. Finance Stoch. 18, 249–269.

Rothschild, M., Stiglitz, J.E., 1970. Increasing risk: I. A definition. J. Econ. Theory 2, 225–243.

Strzalecki, T., 2011. Probabilistic sophistication and variational preferences. J. Econom. Theory 146 (5), 2117–2125.

Strzalecki, T., Werner, J., 2011. Efficient allocations under ambiguity. J. Econom. Theory 146, 1173–1194.

Svindland, G., 2010. Continuity properties of law-invariant (quasi-)convex risk functions on L^∞. Math. Financ. Econ. 3 (1), 39–43.

Townsend, R.M., 1994. Risk and Insurance in village India. Econometrica 62, 539–592.

Yaari, M., 1987. The dual theory of choice under risk. Econometrica 55, 95–115.