Local Risk-Minimization under Illiquidity and Consistent Specification of Credit Migration Models

Panagiotis Christodoulou

Dissertation an der Fakultät für Mathematik, Informatik und Statistik der Ludwig-Maximilians-Universität München

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Hiermit erkläre ich, Panagiotis Christodoulou, an Eides statt, dass die Dissertation von mir selbstständig und ohne unerlaubte Beihilfe angefertigt worden ist.

München, 11. August 2020	Panagiotis Christodoulou
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Zusammenfassung

In dieser Dissertation befassen wir uns mit der mathematischen Modellierung und Steuerung von zwei unterschiedlichen Risikoarten an Finanzmärkten: Liquiditätsrisiko und Migrationsrisiko. Die klassische Theorie der Finanzmärkte hat sich über das letzte Jahrzehnt in beiderlei Hinsicht weiterentwickelt, und die vorliegende Arbeit trägt zu der dynamischen Entwicklung dieser aktiven Forschungsfelder bei.

Im ersten Teil der Arbeit schlagen wir einen neuen Ansatz zum Hedging von Derivaten vor, wenn Liquidität im Markt ein Problem darstellt und der Handel durch Liquiditätskosten betroffen ist. Mehrere Assets mit unterschiedlichen Liquiditätsniveaus sind zu Hedgezwecken verfügbar. Die angestrebte Hedgestrategie basiert auf folgendem Risikokriterium: das Risiko der Hedgefehlerschwankungen soll unter gleichzeitiger Betrachtung der Liquiditätskosten minimiert werden. Basierend auf Çetin et al. (2004) arbeiten wir in einem arbitragefreien Setting und nehmen an, dass jedes Asset einer sogenannten Angebotskurve (supply curve) entspricht. In diskreter Zeit und nach den Ideen von Schweizer et. al. (Schweizer, 1988, Lamberton et al., 1998) beweisen wir die Existenz einer Lokal-Risiko-minimierenden Strategie unter der Annahme von milden Bedingungen an den Preisprozess. Unter stochastischem und zeitabhängigem Liquiditätsrisiko geben wir im Falle einer linearen Angebotskurve für das Modell die Lösung der optimalen Strategie in geschlossener Form an. Zum Schluss zeigen wir wie unsere Hedging-Methode angewandt werden kann, insbesondere in Energiemärkten, in denen Futures mit unterschiedlichen Fälligkeiten als Finanzinstrumente zum Hedging zur Verfügung stehen. Die Futures, die sich nah an ihrem Lieferzeitraum befinden, sind üblicherweise sehr liquide aber nicht unbedingt, abhängig von dem Claim, die optimalen Hedging-Instrumente. In einer Simulationsstudie untersuchen wir diesen Tradeoff und vergleichen die resultierenden Hedgestrategien mit den klassischen Strategien.

In einem weiteren Kapitel erweitern wir dann darüberhinaus die Theorie auf den Fall von zusätzlichen Preisauswirkungen (price impact). Basierend auf einem ähnlichen, mehrdimensionalen Modell in diskreter Zeit, welches durch limitierte Orderbücher motiviert ist, zeigen wir unter der Annahme milder Bedingungen und stochastischem Liquiditätsrisiko, die Existenz einer optimalen Lösung der Strategie und geben diese in geschlossener Form an. Für allgemeine Derivate und im Falle einer linearen Angebotskurve für jedes Asset, hat unser Risikokriterium das Ziel das Risiko durch Marktschwankungen und Preisauswirkungen zu minimieren und gleichzeitig die Liquiditätskosten niedrig zu halten.

Im zweiten Teil der Dissertation betrachten wir die Modellierung von Migrationsrisiko und beschäftigen uns mit dem intensitätsbasierten Ausfallrisiko-unterliegendem (defaultable) Anleihemodell mit mehreren Bonitätsklassen, welches auf dem bekannten Zinsmodell von Heath-Jarrow-Morton basiert. In der Literatur existieren unterschiedliche Annahmen, um Arbitragefreiheit in dem Modell zu garantieren. Wir untersuchen, wie diese das Modell beeinflussen. Durch die Analyse der fundamentalen und der auf dem internen Rating basierenden arbitrage-

freien Spreadstruktur, zeigen wir mögliche Komplikationen und Einschränkungen des Modells auf und erläutern wie einige von diesen vermieden werden können. Insbesondere zeigen wir, dass unter milden Bedingungen der Spread in endlicher Zeit vor dem Ausfall eines Bonds explodiert. Dies ist eine unerwünschte Eigenschaft, bei der der defaultable Bondpreisprozess den Wert Null noch vor dem Ausfall des Bonds annimmt. In Folge dessen argumentieren wir, dass es eine prinzipientreue Interpretation des Modells ist, die Spreadstruktur der Ratingsklassen zuerst zu spezifizieren und anschließend die Arbitragefreiheit auszunutzen, um die Volatilitäten der Forward Rates zu bestimmen.

Abstract

In this thesis, we address two active research topics in mathematical finance dealing with two different kinds of risk; liquidity risk and credit migration risk. The classical theory of financial markets has been recently developed in both these respects, and this thesis contributes to the dynamic development in these research fields.

In the first part, we propose a hedging approach for general contingent claims when liquidity is a concern and trading is subject to liquidity costs. Multiple assets with different liquidity levels are available for hedging. Our risk criterion targets a tradeoff between minimizing the risk of fluctuations in the stock price and incurring low liquidity costs. Following Çetin et al. (2004) we work in an arbitrage-free setting assuming a supply curve for each asset. In discrete time, following the ideas in Schweizer et. al. (Schweizer, 1988, Lamberton et al., 1998) we prove the existence of a locally risk-minimizing strategy under mild conditions on the price process. Under stochastic and time-dependent liquidity risk we give a closed-form solution for an optimal strategy in the case of a linear supply curve model. Finally, we show how our hedging method can be applied in energy markets where futures with different maturities are available for trading. The futures closest to their delivery period are usually the most liquid but depending on the contingent claim not necessarily optimal in terms of hedging. In a simulation study we investigate this tradeoff and compare the resulting hedge strategies with the classical ones.

Further contributing to the development of the theory of the first part, we extend the previous results when additionally price impact is taken into account. Following a similar discrete, multi-dimensional setting, motivated from a limit order book we prove the existence and give a closed-form solution of an optimal strategy under mild conditions and stochastic liquidity risk. For general contingent claims and in a linear supply curve model for each asset, our risk criterion targets a tradeoff between minimizing the risk incurred by market fluctuations and by lasting price impact while incurring low liquidity costs.

In the second part of the dissertation, we investigate the concept of credit migration risk and revisit the popular intensity-based defaultable bond model with multiple credit rating classes based on the Heath-Jarrow-Morton interest rate term structure. Different conditions appear in the literature for a no-arbitrage framework. We investigate these and how they influence the model. By analyzing the fundamental as well as the inter-rating no-arbitrage spread structure we illustrate possible complications and restrictions of the model and explain how some of these can be avoided. In particular, under some mild conditions the spread explodes in finite time before default occurs, which represents the undesirable property that the risk-neutral defaultable bond price process can drop to identically zero prior to default. As a consequence, we argue that a principled way of interpreting the framework is that of specifying the spread structure first and then exploiting no-arbitrage relations to work out the forward rates volatilities.

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Contents

1 Introduction

Classical arbitrage pricing theory requires the market to be frictionless and infinitely liquid. In other words, no transaction costs are included and no trade restrictions are taken into account. The security's price does not depend on the quantity of the order (buy or sell), which is also known in the literature by the term competitive market. Nevertheless, liquidity issues have long troubled option traders. Following a hedging strategy, the possible profit of a trader depends greatly on liquidity costs due to different positions in the strategy. Liquidity becomes a risk factor when the ability to trade is triggering important changes in asset prices.

Although there is not a clear enough agreed notion of liquidity risk, in Çetin (2003) it is argued that liquidity risk is closely related to the transaction costs literature, since prices are affected by the quantity of an order due to transaction costs (see for example Soner et al. (1995), Jouini and Kallal (1995), Cvitanic and Karatzas (1996), Barles and Soner (1998), Constantinides and Zariphopoulou (1999), Cvitanic et al. (1999), Jouini (2000), Jouini et al. (2001)). In this case, the standard theory can be used, hedging strategies are rebuilt but the intuition behind remains unchanged. Also, the market microstructure literature (see Glosten and Milgron (1985), Kyle (1985), Grossman and Miller (1988)) is lacking with regard to an arbitrage pricing theory that incorporates liquidity risk. The convenience yield approach (see Jarrow and Turnbull (1997), Jarrow (2001)) is an attempt in this direction capturing the liquidity risk inside the model structure. Nonetheless, it does not capture the price inelasticities through quantity impact explicitly.

Recently, in the asset pricing literature, an attempt for a simple yet robust method capturing liquidity features in a market incorporating the impact of different trade sizes on the price is the model of Çetin et al. (2004). A so called *supply curve* model is assumed for the asset price. The existence of different supply curve structures (e.g. linear) is showed in Blais and Protter (2010) where they give evidence through the use of order book data. The authors in Çetin et al. (2004) showed that the first fundamental theorem of asset pricing holds. Briefly stated it says that in an extended liquidity market there exists an equivalent martingale measure if and only if it is free of arbitrage. An appropriate generalization of the second fundamental theorem of asset pricing is proven which briefly states that if the martingale measure is unique then the market is called "approximately complete" (the converse statement fails). One can take continuous strategies of finite variation and approximate in the L^2 sense a perfect hedging strategy avoiding all liquidity costs. Although in practice the necessity of discrete trading is required, the interpretation of this result is that by trading with high frequency in small amounts one can obtain arbitrarily small liquidity charges. See also Jarrow and Protter (2015) in this regard. As a consequence of the model structure of Çetin et al. (2004), the value of a derivative security is identical to its price in a perfectly liquid market which has raised criticism by other authors. Improvements on this can be found in the work of Çetin et al. (2006) and Çetin et al. (2010).

The literature on liquidity risk can mainly be classified into two different approaches. The first model category is commonly known as "large trader models". The way that a trader affects the price is permanent, since after an order of a transaction size the depth of the order book is changed permanently releasing a lasting impact on the security price. The effect of trade size on the limit order book (hereafter LOB) is called the *resilience effect*, that is the ability of the order book to recover itself. Examples of these are Jarrow (1994), Frey (1998), Frey (2000), Platen and Schweizer (1998), Sircar and Papanicolaou (1998), Schönbucher and Wilmott (2000), Bank and Baum (2004), Roch (2011). These kind of models are also known in the literature as models of feedback effects. The second approach are models of zero resilience, commonly known as "small trader models". The changes in the fundamental value of the stock do not depend on the history of the trades since the limit order book immediately fully recovers itself and liquidity costs are taken into account locally in time. Such models are for example Çetin et al. (2004), Çetin et al. (2006), Çetin and Rogers (2007), Rogers and Singh (2005), Rogers and Singh (2010), Agliardi and Gençay (2014).

The first two chapters of this thesis, Chapters 2 and 3, are concerned with the problem of hedging general contingent claims in an illiquid market in the case of a small trader framework as well as in a large trader model, respectively.

In Chapter 2 we consider the small agents approach. Motivated from energy and electricity markets, our framework allows for a general multi-dimensional setting for hedging contingent claims of a general form in an illiquid market environment in discrete time.

In Section 2.2 we start by presenting the setup. The hedge instruments are considered to be non-negative stochastic processes with possibly different levels of liquidity. In particular, analogously to the extended arbitrage-free setting of Çetin et al. (2004) we assume a supply curve for each stock price. By considering an investor who aims at hedging a claim H, among the two main quadratic hedging methods that exist in the literature, we choose to follow the *local risk-minimization* approach which in contrast to the *mean-variance* method does not insist on the self-financing condition of the strategy. This approach was introduced in the seminal paper of Schweizer (1988). We extend his work in a discrete time framework in two directions. Firstly by accounting for illiquidity and secondly by considering a multi-dimensional stock price process. Our proposed local risk-minimization criterion targets a tradeoff between minimizing the risk against price fluctuations and incurring low liquidity costs in a similar spirit of Rogers and Singh (2010) and Agliardi and Gençay (2014). This is described and motivated in Section 2.2.1. Moving forward to Section 2.2.2, we describe explicitly the basic problem by making appropriate definitions and we derive some useful results following the ideas in Lamberton et al. (1998).

Although we manage to prove some important results in a general framework, further assumptions are required in order to get more explicit formulas for the optimal strategy. Thus, in Section 2.3 we work in a linear supply curve model which is motivated from a (multiplicative) limit order book. By means of a backward induction algorithm we explain the idea of computing an optimal strategy. Before proving our main Theorem 2.3.15 we impose some mild conditions on the *marginal* asset price process in Section 2.3.1 and we introduce the *F-property* condition in Definition 2.3.11. As shown in Sections 2.3.1 and 2.3.4 these assumptions hold in both an independent increments as well as independent returns setting together with the assumption of a positive definite covariance matrix of the marginal price process. In our main existence Theorem 2.3.15 we show the existence of a *local risk-minimization strategy under illiquidity* belonging to the appropriate space of trading strategies under stochastic and time-dependent liquidity risk. Moreover, a closed-form solution is provided which in contrast to the existing literature makes our approach a more profound tool for risk management for market-makers. Since a linear design for a supply curve can take negative values for some negative transaction sizes below a barrier, Section 2.3.5 serves the purpose of constructing a setting of a linear supply curve which produces non-negative prices.

Finally, Section 2.4 is dedicated to the application of our previous results to electricity markets. We considering as hedge instruments electricity futures which are exposed to liquidity costs and which might have different maturities possibly terminating before the final time horizon T. Since this is not covered a priori by our framework we show in Section 2.4.1 how this can be embedded in our setting. We conclude by a simulation study in Section 2.4.2 where the tradeoff between liquidity and hedging performance of electricity futures is examined.

Chapter 3 is an extension of Chapter 2 to a framework where both liquidity costs and permanent price impact are taken into account. Motivated by the same underlying question of identifying a locally risk-minimizing strategy, a similar structure as in Chapter 2 is carried out in the case of large trader models. We work directly from the beginning with a linear supply curve model in order to be able to reach similar explicit results as in the small agents setting. Introducing a discrete version of the additive limit order book model of Roch (2011), we show in Section 3.2.2 that under certain assumptions the model is free of arbitrage. In Section 3.3, using the additive structure of the model (additive LOB) for separating price impact and liquidity terms, we give a characterization of the optimal strategy through a minimization problem in the case of a stochastic time-dependent liquidity process. We identify three sources of risk to be hedged; the risk incurred by market price fluctuations, by lasting price impact and by liquidity costs. We introduce appropriate parameters so that depending on how risk averse an investor wants to be against the different types of risk, these can be adjusted accordingly.

Furthermore, in order to prove our main existence Theorem 3.4.17 and give a closed form solution for the *local risk-minimization strategy under illiquidity with price impact* we impose additionally some mild conditions for the liquidity level parameter of the order book and the resilience parameter. In Section 3.5 we conclude with an Idea of a slightly alternative optimality criterion showing some explicit results in the case where price impact and liquidity costs are treated together. In particular, both these risks, are identified as one source of risk coming from the illiquidity of the order book. An explicit existence result of an optimal strategy is provided in the full permanent price impact case.

The second part of this thesis deals with intensity-based credit migration bond models of HJM type. In finance, risk management has mainly the objective of controlling three fundamental factors of financial risk that an investment decision is usually exposed to: liquidity risk, market risk and credit risk. While the first part of this thesis falls within the scope of liquidity risk, the second part lies in the context of credit migration risk.

In the financial industry, the recent credit crisis has considerably increased the need to handle credit risk. The demand of more sophisticated quantitative methodologies in interest rate markets has strained the development of credit risk modeling. In the literature, credit risk modeling can be identified mainly according to two conceptual perspectives: the *structural models* and the *reduced-form models*, also commonly known as the *intensity-based models*. The first broad category of models is based on modeling the stochastic evolution of the balance sheet of the issuer. The default event is modeled by the first time the firm's asset value hits a prespecified barrier, decribed by the fact that the issuer is unable to meet its obligations. Since the default time is given endogenously, this allows a direct connection to the model and hence the powerful machinery developed for interest rate models can be applied. Representatives of this approach are for example Merton (1974), Black and Cox (1976), Geske (1977), Longstaff and Schwartz (1995), Zhou (1997), Cathcart and El-Jahel (1998).

In the second modeling framework, the reduced-form models, one directly models the default event (or other credit events). This is usually done by means of an exogenously specified conditional probability of default. Papers belonging to the traditional literature of this type of models are for example Jarrow and Turnbull (1995), Duffie and Singleton (1997), Lando (1998), Schönbucher (1998), Madan and Unal (1998), Duffie and Singleton (1999). In many cases, a hazard rate or intensity of default is used for modeling the conditional probability of default, which explains that reduced-form models are also referred to as *hazard-rate models* or *intensity-based models*. Examples of intensity-based models have been developed in the papers of Jarrow et al. (1997), Thomas et al. (1998), Lando (2000), Schönbucher (2000), Bielecki and Rutkowski (2000). Moreover, the modeling of credit migration by means of either discrete or continuous conditionally Markov chains, has become very popular in the past years and is explored for example among others in Jarrow and Turnbull (1995), Das and Tufano (1996), Jarrow et al. (1997), Arvanitis et al. (1999), Duffie and Singleton (1999). In this thesis we focus on a more recent markovian model approach of credit migrations based on the methodology of Heath, Jarrow and Morton (hereafter HJM) that was presented in Bielecki and Rutkowski (2000) (see also Bielecki and Rutkowski (2004a)). In this framework, information on rating migration is taken into account and an arbitrage-free model of defaultable bonds is constructed. The real-world risky forward rate dynamics as well as recovery schemes and the intensity matrix process are exogenously specified. The existence of an arbitrage-free model is established by the so called *consistency condition* that all the model specifications must satisfy. Then under an appropriate measure, the discounted defaultable bond under credit migration is a local martingale. As a consequence and in analogy to the HJM methodology, the consistency condition can be seen as a generalized drift condition on the forward rates as we show in Chapter 4 (Sections 4.3.1 and 4.4.1).

The last chapter of this thesis, Chapter 4, is devoted to the analysis of the Bielecki and Rutkowski (2000) (or Bielecki and Rutkowski (2004a)) model. We investigate and show possible complications resulting from the intensity based approach which essentially stem from the constraint of a risk-neutral framework imposing the interplay between the various model components potentially yielding to constraints on the model specifications. In principle if such constraints are not met by the involved coefficients, problems may arise, such as explosive dynamics of the rating spreads of the model or even no migration between the rating classes. We demonstrate how these model inconsistencies may emerge and how some can be avoided. In particular, these occur as a result of a direct analytical connection between the conditional credit state bond price processes, the defaultable bond and the migration intensities of the rates, which is established by the consistency conditions.

In the literature, two different types of consistency conditions appear within the extended credit migration HJM framework that we refer to as *strong* and *weak* consistency condition. Chronologically, the strong consistency condition appears first in the literature in Bielecki and Rutkowski (2000), Bielecki and Rutkowski (2004b) and Bielecki and Rutkowski (2004a). The strong consistency condition guarantees an arbitrage-free model by requiring that forward rates in all rating classes are “active" at all times, and as a result multiple drift conditions are enforced. Here, the term “active forward rates" is used to denote that fact that the bond evolution in all rating classes exhibit a local martingale dynamics simultaneously at any time, and not merely the dynamics of the rating class the bond is actually traded in. On the other hand, the weak consistency condition, appearing later in the literature (see for example Özkan and Schmidt (2005), Schmidt (2006) and Jakubowski and Nieweglowski (2009)), only requires that the forward rate of the rating class the bond is currently traded in satisfies a generalized HJM no-arbitrage drift condition. While the strong consistency condition is merely sufficient for no arbitrage, the weak consistency condition is also necessary.

We begin by introducing in Section 4.2 the Bielecki-Rutkowski credit risk model framework together with the basic model ingredients and definitions. Furthermore, in Section 4.3 the weak consistency condition is presented and given in terms of an extended HJM no-arbitrage drift condition. We prove that this relation is an equivalent relation. Throughout this section we analyze the model and we use the spread dynamics coming from the consistency condition as an instrument to investigate what the model restrictions may be. Two kind of spreads are of relevance in this chapter: the *fundamental spread*, defined as the difference between the risky forward rates and the risk-free one, and the *inter-rating spread* which is the difference between two different risky forward rates. Under the weak consistency we analyze the model exploiting the fundamental spreads and show in Theorem 4.3.14, Section 4.3.2 that under some additional conditions the fundamental spread explodes in finite time with positive probability prior to default. This is shown with the use of the *ordering condition* of the forward rates, a natural economic assumption that reflects the fact that the price of a bond must decrease as the default risk increases. Although this condition is not necessarily linked directly to no arbitrage it is assumed in many papers (for example Bielecki and Rutkowski (2000), Eberlein and Özkan (2003) and in Schmidt (2006)). Nevertheless, in Corollary 4.3.19, we additionally prove that in the special case of one risky forward rate (one risky rating class) and a default class the fundamental spread still has the undesirable property that it explodes even without assuming any ordering condition. All these results are shown in a setting where the recovery rate is not zero. That is, in case of default, the bond holder receives a reduced payment of the bond, where the recovery rate is the fixed fraction of the face value at maturity date T. A possible methodology for constructing a non-explosive, non-zero recovery model from a zero recovery one is shown in Section 4.3.3. The idea hinges upon a transformation of zero recovery spreads to positive recovery ones, relying on simple no-arbitrage arguments typical of defaultable models.

Similar results hold also for the inter-rating spreads as we show in Section 4.4 under the strong consistency in a zero recovery setting. Considering the zero recovery case does not necessarily mean restriction of the generalities of the explosion results: since strong consistency implies weak consistency, when recovery rates are positive, explosions under strong consistency are obtained as a consequence of explosions under weak consistency in a non-zero recovery setting. Furthermore, an alternative interpretation of the strong consistency condition can be given and proven within the framework of a multiple-issuer migration model where it becomes more appropriate than the weak one. The short Section 4.4.3 serves this purpose.

Section 4.5 is devoted to the equal volatility specification model. Closed form solutions of inter-rating spreads under the strong consistency can be obtained. This shows that despite the previously stated explosion theorems, meaningful consistent simple models are still possible in the case of deterministic inter-rating spreads. As a consequence this is linked to a "vanishing at zero" property on the initial spread value, which when this is violated then explosions still occur as shown in Theorem 4.5.10 of Section 4.5.2.

In Section 4.6 we move on to proportional volatility spread structure models where we show that the spread has positive dynamics but it explodes in finite time with positive probability. In some sense, the situation portrayed is thus similar to the classical HJM setup without migration: in this classical setup forward rates are known to explode for unbounded volatilities (e.g. proportional volatilities). Finally, we present in Section 4.7 a further condition that is formulated as a desirable and natural property in the framework for example in Bielecki and Rutkowski (2000) and Bielecki and Rutkowski (2004a): the assumption that the the risk premium processes for the HJM underlying forward rates is independent of the maturity. However, we prove in Proposition 4.7.2, that assuming this condition together with the weak (and thus strong) consistency condition leads to a model where there is no possibility of migration between the credit classes.

2 Local risk-minimization with multiple assets under illiquidity with applications in energy markets

Contributions of the thesis' author:

This chapter is a joint work of P. Christodoulou, Prof. Dr. Thilo Meyer-Brandis and Prof. Dr. Nils Detering. It is based on Christodoulou et al. (2018). P. Christodoulou was significantly involved in the development of all parts of that paper. The final framework for incorporating liquidity into the model as well as the interpretation of the results has been discussed and established jointly. The development and extension to the multidimensional case together with the conditions needed for the framework were done by P. Christodoulou. In particular, P. Christodoulou made major contributions to all the proofs in Sections 2.2 and 2.3 together with the main existence Theorem 2.3.15. The application Section 2.4 was established in a close cooperation of the three authors, but with major parts done by P. Christodoulou. The proofs of Proposition 2.4.2 and Corollary 2.4.3 were done by P. Christodoulou. The numerical study has been designed by all the three authors and the simulation has been implemented by P. Christodoulou.

2.1 Introduction

The problem of hedging general contingent claims under illiquidity is handled in this chapter. The main motivation comes from energy markets. Hedging with multiple assets with possibly different liquidity levels incurs stochastic liquidity costs. For example consider an Asian-style call option written on the average spot price $S = (S_u)_{0 \leq u \leq T_2}$ of energy and an agent hedging the option. The payoff of such an option is

$$\left(\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} S_u du - K \right)^+, \tag{2.1.1}$$

for some strike K and some *delivery period* $[T_1, T_2]$. Usually, futures delivering over the same or a different time period are considered as instruments available for hedging such options. The challenge of such a setting is the fact that such futures are either not trading at all in their delivery period (see for example Benth and Detering, 2015) or are very illiquid and hence they incur large transaction costs due to hedging. Additionally and for $t \ll T_1$ futures are usually very illiquid, so their liquidity time-structure is particularly delicate. Multiple futures with different delivery periods (week, month, quarter, year) and different levels of liquidity are available as hedging instruments in the market. Our paper results can be applied to general hedging options with multiple futures by accounting for their different liquidity levels in energy markets. The Asian-style option is a particular example but also other payoffs such as Quanto options (see Benth et al., 2015) can be handled equally in the same way.

A very active research topic at the moment in mathematical finance is the effect of illiquidity on optimal trading and hedging. Roughly speaking, liquidity risk is the additional risk due to timing and size of a trade. Nevertheless there is neither an agreed notion in the literature nor a standard hedging approach under liquidity costs. For a nice overview on existing liquidity models in discrete and continuous time see the paper of Gökay et al. (2011).

Modeling liquidity risk is basically done in two different ways. The first approach is when the trade volume has a lasting impact on the asset price. This is a class of models incorporating feedback effects. This is also known as lasting impact or permanent price impact (see e.g. Bank and Baum, 2004). The second one considers agents whose transactions have no lasting impact on the price of the underlying. These are also some times called small agent models (see e.g. Çetin et al., 2004, and the references therein).

The liquidity modeling approach considered in this chapter is the small agents approach. Additionally, the transaction costs incurred from a hedging strategy by trading through a fast recovering limit order book is what we understand as liquidity costs. More precisely we follow the arbitrage-free *supply curve* model first introduced in Çetin et al. (2004) where the authors developed an extended arbitrage pricing theory for the case when the asset price is a function of the trade size.

The majority of papers on illiquid markets is dealing with the optimal execution. See for example Bertsimas and Lo (1998), Almgren and Chriss (2001), Gatheral and Schied (2011), Schied (2013) and the references therein. In addition, most of the papers investigating hedging under illiquidity consider super-replication as for example in Bank and Baum (2004), Çetin et al. (2010), Gökay and Soner (2012). In our case we consider a quadratic risk criterion since super-replication is often too expensive. Two approaches for quadratic hedging are mainly considered in the literature in the classical frictionless theory without transaction costs. A good survey can be found in Schweizer (2001). The first approach, introduced first in Föllmer and Sondermann (1986), is the *mean-variance* approach, which relies on self-financing strategies producing as final outcome the portfolio $V_T := c + \int_0^T X_u dS_u$ where $c \in \mathbb{R}$ is some initial value and X is a trading strategy in the risky asset $S = (S_u)_{0 \le u \le T}$. The aim of this method is to find the best square approximation of a contingent claim H by the terminal portfolio value V_T. In particular, under the real world probability measure one needs to minimize the quadratic hedge error

$$\mathbb{E}\left[\left(H - \left(c + \int_0^T X_u dS_u\right)\right)^2\right], \tag{2.1.2}$$

over an appropriately constrained set of strategies. This approach is also called global risk-minimization by many authors. In a discrete time setting, this problem was handled and solved in Schweizer (1995) by generalizing and relaxing the assumptions imposed earlier in Schäl (1994). Later on and under proportional transaction cost, the global risk-minimization was extended further by Motoczyński (2000) and Beutner (2007), where the authors show existence of an optimal strategy in the multidimensional case. An extension of the mean-variance hedging criterion under illiquidity are the papers of Rogers and Singh (2010), Agliardi and Gençay (2014) and Bank et al. (2017). This criterion is based on minimizing the global risk against random fluctuations of the stock price process incurring low liquidity costs.

The second main quadratic hedging method in an incomplete market is called *local risk-minimization* and it was first introduced in Schweizer (1988). An extension by accounting for proportional transaction costs in the discrete time case can be found in Lamberton et al. (1998). In contrast to the mean-variance hedging, this method does not insist on the self-financing condition. For discrete time $k = 0, \ldots, T$ the goal is to find a strategy $(X, Y) = (X_{k+1}, Y_k)_{k=0,\ldots,T}$ with book value $V_k = X_{k+1}S_k + Y_k$ (assume that the risk-free asset is constantly equal to 1) such that $H = V_T$, $C_k = V_k - \sum_{m=1}^{k} X_m(S_m - S_{m-1})$ is a martingale and the variance of the incremental cost is minimized, where C_k is the cost process. Here, we denote by Y_k the units held in the bank account and the strategy X_{k+1} represents the number of shares held in the risky asset in the time interval $(k, k+1]$. In this chapter, we extend the work of Schweizer (1988) and the local risk-minimizing quadratic criterion to an illiquid market in the spirit of Rogers and Singh (2010) and Agliardi and Gençay (2014) in discrete time. Secondly we extend it to a multidimensional asset price process setting.

Our approach and in contrast to the existing literature is designed in an appropriate general setting to address the above mentioned problem in energy markets. First, a multi-dimensional setup is needed to allow for multiple futures used as hedge instruments. Then, for capturing the characteristics of energy markets, the assets price dynamics has to be general enough and we additionally need a time dependent liquidity structure. The choice of our risk criterion allows for more explicit formulas for the optimal strategy compared to existing approaches in the literature. Moreover, they are also computationally tractable as we show in a case study. By requiring only mild conditions on the asset price process, our main result is the existence of a locally risk-minimizing strategy under illiquidity. These conditions might seem quite technical but we show that they can be reduced to sufficient conditions on the covariance matrix of the price process. These can be checked easily for most processes relevant in practice. Furthermore and by using a *least-squares Monte Carlo* algorithm, the optimal strategies can be calculated backwards in time.

In our setup it is possible to investigate the tradeoff between liquidity and hedge quality of available hedge instruments. Consider for example the situation where different futures with different delivery periods and different liquidity levels are available for hedging an Asian-style option (2.1.1) in a market. Futures with delivery period well matching the delivery period of the option payout exist in the market. Despite the strong correlation that these may have between the future and the option to hedge, in certain time periods these *hedge-optimal* futures are very illiquid. Hence futures which are more liquid and less correlated might be better for hedging and more optimal. In our framework we can provide market-makers with a more profound tool for risk management since our setup allows us to explore this tradeoff.

The current chapter is structured as follows. In Section 2.2 the model framework is explained and described together with the basic problem. Moreover, we focus on a linear supply curve model and impose necessary assumptions on the price process to prove our main existence Theorem 2.3.15 in Section 2.3. We additionally provide sufficient conditions for the assumptions that we impose. An application to energy markets is considered in Section 2.4. In order to explore the tradeoff between liquidity and hedging performance of futures available for hedging, optimal strategies under illiquidity are simulated by means of a least-squares Monte Carlo method.

2.2 The Model

Assume that a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is given and consider a financial market consisting of $d+1$ assets. Let $\mathbb{P}$ the *objective* probability measure. The flow of information is described by the filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\ldots,T}$. The indices $k = 0, 1, \ldots, T$ will be used to refer to a discrete time grid with time points $t_0 < t_1 < \cdots < t_T$. Both notions might be used interchangeably. The discounted (marginal) price of d risky assets (typically futures or stocks) is described by an $\mathbb{F}$-adapted, nonnegative d-dimensional stochastic process $S = (S_k)_{k=0,1,\ldots,T}$. By the price process notion S_k^j we refer to the price of asset j at time t_k. Moreover, assume that a riskless asset (typically a bond) exists and let its discounted price be constantly 1.

Based on the setting of Çetin et al. (2004) we sssume that a hedger observes an exogenously given nonnegative, d-dimensional *supply curve* $S_k(x) = (S_k(x)^1, \ldots, S_k(x)^d)$, $x \in \mathbb{R}^d$, where $S_k(x)^j := S_k^j(x^j)$ denotes the j-th stock price per share at time k for the sale (if $x^j < 0$) or purchase (if $x^j > 0$) of $|x^j|$ shares. $S(0) = S$ is called the *marginal price*. The actual price that market participants pay or receive respectively for a transaction of size x^j (of the j-th asset) at time k is determined and described by the supply curve which additionally is assumed to be independent of the participants past actions. This implies that a trading strategy has no lasting impact on the supply curve. The measurability of the supply curve with respect to the filtration $\mathbb{F}$ is assumed. Furthermore and in order to ensure that the liquidity costs are nonnegative we assume that it is non-decreasing in the number of shares x, i.e. for each k and j, $S_k(x)^j \leq S_k(y)^j$, $\mathbb{P} - a.s.$ for $x^j \leq y^j$.

An extended arbitrage pricing theory was recently developed in Çetin et al. (2004). The authors show that the existence of an equivalent local martingale measure $\mathbb{Q}$ for the marginal price process S rules out arbitrage. Despite that this was shown for a continuous time version of such a supply curve model, since liquidity cost is always positive, a similar result can easily be seen to hold in our setting as well.

However, if one incorporates illiquidity then even a unique martingale measure (and state space restrictions in a discrete setting) do not necessarily ensure completeness of the market. Since perfect hedging is not possible, we aim at minimizing locally the risk of hedging under illiquidity according to an optimality criterion which we introduce in Definition 2.2.4.

We shall consider in the following an investor who aims at hedging an $\mathcal{F}_T$-measurable claim H. For $x \in \mathbb{R}^d$, let $|x|$ denote the Euclidian norm and x^* the transpose of x. Further, $\langle x, y \rangle$ denotes the inner product of $x, y \in \mathbb{R}^d$. We define the investor's possible trading strategies by adapting Schweizer (1988). For this we denote by $\mathbb{L}_T^p(\mathbb{R}^d)$ (in short $\mathbb{L}_T^{p,d}$), the space of all $\mathcal{F}_T$-measurable random variables $Z : \Omega \to \mathbb{R}^d$ satisfying $\|Z\|^p = \mathbb{E}(|Z|^p) < \infty$. We abbreviate $\Delta S_k := S_k - S_{k-1}$. Furthermore, we denote by $\Theta_d(S)$ the space of all $\mathbb{R}^d$-valued predictable strategies $X = (X_k)_{k=1,2,\ldots,T+1}$ so that $X_k^* \Delta S_k \in \mathbb{L}_T^{2,1}$ and $\Delta X_{k+1}^* [S_k(\Delta X_{k+1}) - S_k(0)] \in \mathbb{L}_T^{1,1}$ for $k = 1, 2, \ldots, T$.

Definition 2.2.1. *A pair $\varphi = (X, Y)$ is called a trading strategy if:*

(i) $Y = (Y_k)_{k=0,1,\ldots,T}$ *is a real-valued* $\mathbb{F}$*-adapted process.*

(ii) $X \in \Theta_d(S)$.

(iii) $V_k(\varphi) := X^*_{k+1}S_k + Y_k \in \mathbb{L}^{2,1}_T$ *for* $k = 0, 1, \ldots, T$.

X^j_{k+1} denotes the number of shares held in the risky asset S^j_k and Y_k the units in the riskless asset (bank account) in the time interval $(k, k+1]$. $V_k(\varphi)$ is the so called *marked-to-market value* or the *book value* of the portfolio (X_{k+1}, Y_k) at time k. Note that since the portfolio value that can be realized depends on the liquidation strategy then there is no unique value for a portfolio. This is due to the non-flat supply curve structure.

2.2.1 Cost and Risk process

Let an $\mathbb{L}^{2,1}_T$*-contingent claim* of the form $H = \bar{X}^*_{T+1}S_T + \bar{Y}_T$, with $\bar{X}^*_{T+1}S_T \in \mathbb{L}^{2,1}_T$, $\bar{X}_{T+1} \in \mathbb{L}^{2,d}_T$ where the terms $\bar{X}_{T+1}$ and $\bar{Y}_T$ are $\mathcal{F}_T$-measurable random variables. More precisely, $\bar{X}_{T+1}$ describes the quantity in risky assets that the option seller is committed to provide to the buyer at the expiration date T of the financial contract H. Analogously, the quantity $\bar{Y}_T$ describes the obligated number of bonds at time T.[1].

Assuming that an order of $\Delta Y_k := Y_k - Y_{k-1}$ bonds and $\Delta X_{k+1} := X_{k+1} - X_k$ shares is made at time $k \in \{1, 2, \ldots, T\}$, then

$$\Delta Y_k + \Delta X^*_{k+1}S_k(\Delta X_{k+1}) = \Delta Y_k + \Delta X^*_{k+1}S_k + \Delta X^*_{k+1}[S_k(\Delta X_{k+1}) - S_k(0)]. \quad (2.2.1)$$

is the *total outlay* (under liquidity costs). Note that the last term can be seen as the transaction cost which is coming and resulting from market illiquidity, by the fact that $S_k(0) = S_k$ is the marginal price. Moreover, using the definition of the book value we can write

$$\Delta Y_k + \Delta X^*_{k+1}S_k(\Delta X_{k+1}) = \Delta V_k(\varphi) - X^*_k \Delta S_k + \Delta X^*_{k+1}[S_k(\Delta X_{k+1}) - S_k(0)]. \quad (2.2.2)$$

Note that for a self-financing trading strategy the total outlay at time k would have been zero.

Let us now define the cumulative costs of the strategy $\varphi = (X, Y)$. By letting $\hat{C}_0(\varphi) := V_0(\varphi)$, the *initial cost*[2], we can define the *cost process under illiquidity* $\hat{C}(\varphi) = (\hat{C}_k(\varphi))_{k=0,1,\ldots,T}$ by

$$\hat{C}_k(\varphi) := \sum_{m=1}^{k} \Delta Y_m + \sum_{m=1}^{k} \Delta X^*_{m+1}S_m(\Delta X_{m+1}) + V_0(\varphi). \quad (2.2.3)$$

Using the definition of $V_k(\varphi)$ one can show further that

$$\hat{C}_k(\varphi) = V_k(\varphi) - \sum_{m=1}^{k} X^*_m \Delta S_m + \sum_{m=1}^{k} \Delta X^*_{m+1}[S_m(\Delta X_{m+1}) - S_m(0)]. \quad (2.2.4)$$

[1]For example, in the 1-dimensional case one could set $\bar{X}_{T+1} = 0$ and $Y_T = (S_T - K)^+$ for a call option with strike K without physical delivery.

[2]For simplicity we do not account for any liquidity costs paid to set up the initial portfolio.

By ensuring the square integrability of the cost process, we can define the *quadratic risk process under illiquidity* $\hat{R}(\varphi) = (\hat{R}_k(\varphi))_{k=0,1,\dots,T}$ by

$$\hat{R}_k(\varphi) := \mathbb{E}[(\hat{C}_T(\varphi) - \hat{C}_k(\varphi))^2 | \mathcal{F}_k] \,. \tag{2.2.5}$$

Let us mention at this point that the classical local risk-minimization approach aims at finding a locally risk-minimizing strategy $\varphi = (X, Y)$ such that $V_T(\varphi) = H$ with $X_{T+1} = \bar{X}_{T+1}$ and $Y_T = \bar{Y}_T$. For more details see Section 2.2.2.

Let $C(\varphi) = (C_k(\varphi))_{k=0,1,\dots,T}$ denote the classical cost process without liquidity costs (i.e., $S(x) = S(0)$), more precisely define

$$C_k(\varphi) := V_k(\varphi) - \sum_{m=1}^{k} X_m^* \Delta S_m. \tag{2.2.6}$$

Then the relation

$$\hat{C}_T(\varphi) - \hat{C}_k(\varphi) = C_T(\varphi) - C_k(\varphi) + \sum_{m=k+1}^{T} \Delta X_{m+1}^* [S_m(\Delta X_{m+1}) - S_m(0)], \tag{2.2.7}$$

is obtained. Moreover, let $R(\varphi) = (R_k(\varphi))_{k=0,1,\dots,T}$ denote the classical risk process, defined as in (2.2.5) but with $\hat{C}$ replaced by C.

Another possibility how one could define the risk process of a strategy would be the *linear risk process under illiquidity*

$$\bar{R}_k(\varphi) := \mathbb{E}[|\hat{C}_T(\varphi) - \hat{C}_k(\varphi)| | \mathcal{F}_k], \tag{2.2.8}$$

motivated by Coleman et al. (2003).

Remark 2.2.2. *From a financial perspective a linear local risk-minimization criterion might be more natural than a quadratic one. In fact, the L^2-norm might overemphasize large values even if these will occur with small probability. Nevertheless, it is possible to get explicit results by minimizing over the L^2-norm.*

The *quadratic-linear risk process (QLRP) under illiquidity*

$$T_k(\varphi) := \mathbb{E}[(C_T(\varphi) - C_k(\varphi))^2 | \mathcal{F}_k] + \mathbb{E}[\sum_{m=k+1}^{T} \Delta X_{m+1}^* [S_m(\Delta X_{m+1}) - S_m(0)] | \mathcal{F}_k], \tag{2.2.9}$$

is a combination of measuring the quadratic difference of the classical cost process and linearly the liquidity costs. Later on we will be able to construct and prove an explicit representation of the local risk-minimizing strategy under illiquidity by minimizing the expression in (2.2.9) where large values of liquidity costs are not overemphasized by the L^2-norm.

2.2.2 Description of the basic problem

The classical local risk-minimization is targeting on minimizing locally the conditional mean square incremental cost of a strategy. The aim of our criterion is to minimize locally the risk against random fluctuations of the stock price but at the same time reducing liquidity costs incurred by the strategy. Such an approach yields a tractable problem and is similar to Agliardi and Gençay (2014) or Rogers and Singh (2010) which balances low liquidity costs against poor replication.

Definition 2.2.3 and Definition 2.2.4 give us the optimality criterion on which the minimization problem is based on. The idea is to make the current optimal choice of the strategy by fixing the holdings at past or future times. That is, we only minimize locally in Y_k and X_{k+1}, the risk process at time k.

Definition 2.2.3. *A local perturbation $\varphi' = (X', Y')$ of a strategy $\varphi = (X, Y)$ at time $k \in \{0, 1, \dots, T-1\}$ is a trading strategy such that $X_{m+1} = X'_{m+1}$ and $Y_m = Y'_m$ for all $m \neq k$.*

By a slight abuse of notation define

$$T_k^\alpha(\varphi) := \mathbb{E}[(C_T(\varphi) - C_k(\varphi))^2|\mathcal{F}_k] + \alpha\mathbb{E}[\Delta X_{k+2}^*[S_{k+1}(\Delta X_{k+2}) - S_{k+1}(0)]|\mathcal{F}_k]. \quad (2.2.10)$$

In Definition 2.2.4 we specify what we call local risk minimizing strategy under illiquidity for some $\alpha \in \mathbb{R}^+$.

Definition 2.2.4. *A trading strategy $\varphi = (X, Y)$ is called locally risk-minimizing (LRM) under illiquidity if for any time $k \in \{0, 1, \dots, T-1\}$*

$$T_k^\alpha(\varphi) \leq T_k^\alpha(\varphi'), \qquad \mathbb{P}-a.s., \quad (2.2.11)$$

for any local perturbation φ' of φ at time k.

Note that in Definition 2.2.4 we have taken into account the liquidity costs only at the current time. By the fact that we minimize only locally, this is equivalent to minimizing over T_k in equation (2.2.9). Also note that since Definition 2.2.1 ensures that the classical cost process C is square-integrable and the liquidity costs are integrable for any strategy then Definition 2.2.4 is well defined.

Remark 2.2.5. *An equal concern about the risk to be hedged as incurred by market price fluctuations and the cost of hedging incurred by liquidity costs corresponds to the choice $\alpha = 1$. Otherwise $\alpha > 1$ represents a major risk aversion to the cost of illiquidity and $\alpha < 1$ means a major risk aversion to the risk of miss-hedging. One could also generalize by using a deterministic positive $\mathbb{R}$-valued process $\alpha = (\alpha_k)_{k=0,1,\dots,T}$. In such a case trivially our results will still hold true.*

From now on we will assume α is given and our goal is to find a locally risk-minimizing strategy $\varphi = (X, Y)$ under illiquidity such that $V_T(\varphi) = H$ with $X_{T+1} = \bar{X}_{T+1}$ and $Y_T = \bar{Y}_T$. In the multi-dimensional case, some useful Lemmas follow, which can be shown by means very similar to those used in Lamberton et al. (1998) in the 1-dimensional case. For completeness we provide their proofs.

A main property of a local risk-minimizing strategy, is that its cost process is a martingale. The first Lemma shows that this property generalizes to our setting. The reason is that by changing only the $\mathcal{F}_k$-measurable risk free investment, a strategy φ can be perturbed to φ' such that $C(\varphi')$ is a martingale. This in turn reduces the first term in (2.2.10) but at the same time leaves the second term unchanged.

Lemma 2.2.6. *For a LRM-strategy φ under illiquidity, the cost process $C(\varphi)$ is a martingale. Furthermore, from the martingale property of the cost process we get the representation,*

$$R_k(\varphi) = \mathbb{E}[R_{k+1}(\varphi)|\mathcal{F}_k] + \mathbb{V}ar(\Delta C_{k+1}(\varphi)|\mathcal{F}_k), \qquad \mathbb{P}-a.s., \tag{2.2.12}$$

for $k = 0, 1, \ldots, T-1$.

Proof. The arguments follow those in the proof of Lemma 1 in Lamberton et al. (1998).

Let $\varphi = (X, Y)$ be a LRM-strategy under illiquidity and fix some $k \in \{0, 1, \ldots, T-1\}$. Assuming that $C(\varphi)$ is not a martingale, we can choose a local perturbation $\varphi' = (X', Y')$ of φ at time k by defining $X' := X$ and only modifying the cash holding Y' at time k, by adding the conditional expectation of the incremental cost at time k to Y,

$$Y_k^{'} := \mathbb{E}[C_T(\varphi) - C_k(\varphi)|\mathcal{F}_k] + Y_k\,. \tag{2.2.13}$$

This implies that $\mathbb{E}[C_T(\varphi') - C_k(\varphi')|\mathcal{F}_k] = 0$ and $\mathbb{V}ar(C_T(\varphi') - C_k(\varphi')|\mathcal{F}_k) = \mathbb{V}ar(C_T(\varphi) - C_k(\varphi)|\mathcal{F}_k)$. Since $\mathbb{E}[X^2] = \mathbb{V}ar[X] + (\mathbb{E}[X])^2$ for a random variable X, one can conclude that using the strategy φ' the risk process becomes less, that is,

$$R_k(\varphi') \leq R_k(\varphi)\,. \tag{2.2.14}$$

Since $X' := X$, the liquidity costs of φ' and φ equal. This implies,

$$T_k^{\alpha}(\varphi') \leq T_k^{\alpha}(\varphi)\,. \tag{2.2.15}$$

By the fact that φ is a LRM-strategy under illiquidity, we must have equality on T_k^{α} which implies equality on R_k i.e., $R_k(\varphi') = R_k(\varphi)$. So, the cost process $C(\varphi)$ must be a martingale. □

So, for φ a LRM-strategy under illiquidity, we have the representation

$$T_k^{\alpha}(\varphi) = \mathbb{E}[R_{k+1}(\varphi)|\mathcal{F}_k] + \mathbb{V}ar(\Delta C_{k+1}(\varphi)|\mathcal{F}_k) + \alpha\mathbb{E}[\Delta X_{k+2}^{*}[S_{k+1}(\Delta X_{k+2}) - S_{k+1}(0)]|\mathcal{F}_k], \tag{2.2.16}$$

of the quadratic-linear risk process under illiquidity.

A representation for the QLRP of a perturbed strategy is provided in the next lemma.

Lemma 2.2.7. *If $C(\varphi)$ is a martingale and φ' a local perturbation of φ at time k then*

$$\begin{aligned} T_k^{\alpha}(\varphi') =& \mathbb{E}[R_{k+1}(\varphi)|\mathcal{F}_k] + \mathbb{E}[(\Delta C_{k+1}(\varphi'))^2|\mathcal{F}_k] \\ &+ \alpha\mathbb{E}[(X_{k+2} - X'_{k+1})^{*}[S_{k+1}(X_{k+2} - X'_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k]\,. \end{aligned} \tag{2.2.17}$$

Proof. As in Lamberton et al. (1998) (see proof of Proposition 2), by using Lemma 2.2.6 and the fact that

$$\mathbb{E}[C_T(\varphi') - C_k(\varphi')|\mathcal{F}_k] = \Delta C_{k+1}(\varphi'), \tag{2.2.18}$$

which follows from the martingale property of $C(\varphi)$, one can conclude that

$$R_k(\varphi') = \mathbb{E}[R_{k+1}(\varphi)|\mathcal{F}_k] + \mathbb{E}[(\Delta C_{k+1}(\varphi'))^2|\mathcal{F}_k]\,. \tag{2.2.19}$$

Furthermore since φ' is a local perturbation of φ at time k, we have

$$\begin{aligned}&\mathbb{E}[(X'_{k+2} - X'_{k+1})^*[S_{k+1}(X'_{k+2} - X'_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k]\\ &\quad = \mathbb{E}[(X_{k+2} - X'_{k+1})^*[S_{k+1}(X_{k+2} - X'_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k],\end{aligned} \tag{2.2.20}$$

and the claim follows. □

Remark 2.2.8. *By the fact that $R_{k+1}(\varphi) = R_{k+1}(\varphi')$ holds for any local perturbation φ' of φ at time k, then equation (2.2.17) implies that one needs to minimize over the expression*

$$\mathbb{V}ar(\Delta C_{k+1}(\varphi)|\mathcal{F}_k) + \alpha\mathbb{E}[\Delta X^*_{k+2}[S_{k+1}(\Delta X_{k+2}) - S_{k+1}(0)]|\mathcal{F}_k], \tag{2.2.21}$$

at time k (see Proposition 2.2.9).

Proposition 2.2.9. *A trading strategy $\varphi = (X, Y)$ is LRM under illiquidity if and only if the two following properties are satisfied:*

(i) $C(\varphi)$ is a martingale.

(ii) For each $k \in \{0, 1, \dots, T-1\}$, X_{k+1} minimizes

$$\begin{aligned}&\mathbb{V}ar(V_{k+1}(\varphi) - (X'_{k+1})^*\Delta S_{k+1}|\mathcal{F}_k)\\ &\qquad + \alpha\mathbb{E}[(X_{k+2} - X'_{k+1})^*[S_{k+1}(X_{k+2} - X'_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k],\end{aligned} \tag{2.2.22}$$

over all $\mathcal{F}_k$-measurable random variables X'_{k+1} so that $(X'_{k+1})^*\Delta S_{k+1} \in \mathbb{L}^{2,1}_T$ and $(X_{k+2} - X'_{k+1})^*[S_{k+1}(X_{k+2} - X'_{k+1}) - S_{k+1}(0)] \in \mathbb{L}^{1,1}_T$.

Proof. The proof follows the steps in the proof of Proposition 2 in Lamberton et al. (1998).

Let us first show the " $\Leftarrow$ " direction of the proof. We want to show that $\varphi = (X, Y)$ is a LRM-strategy under illiquidity, according to Definition 2.2.4. So, fix some $k \in \{0, 1, \dots, T-1\}$ and let $\varphi' = (X', Y')$ be a local perturbation of φ at time k.

Since property (i) holds and φ' a local perturbation of φ at time k then by Lemma 2.2.7 we have the equality

$$\begin{aligned}T^\alpha_k(\varphi') =&\mathbb{E}[R_{k+1}(\varphi)|\mathcal{F}_k] + \mathbb{E}[(\Delta C_{k+1}(\varphi'))^2|\mathcal{F}_k]\\ &+ \alpha\mathbb{E}[(X_{k+2} - X'_{k+1})^*[S_{k+1}(X_{k+2} - X'_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k],\end{aligned} \tag{2.2.23}$$

Moreover, from the definition of the conditional variance we have

$$\mathbb{E}[(\Delta C_{k+1}(\varphi'))^2|\mathcal{F}_k] \geq \mathbb{V}ar(\Delta C_{k+1}(\varphi')|\mathcal{F}_k), \tag{2.2.24}$$

and so we can estimate

$$\begin{aligned} T_k^\alpha(\varphi') \geq &\mathbb{E}[R_{k+1}(\varphi)|\mathcal{F}_k] + \mathbb{V}ar(\Delta C_{k+1}(\varphi')|\mathcal{F}_k) \\ &+ \alpha\mathbb{E}[(X_{k+2} - X'_{k+1})^*[S_{k+1}(X_{k+2} - X'_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k]\,. \end{aligned} \tag{2.2.25}$$

Since φ' a local perturbation of φ at time k then $X'_{k+2} = X_{k+2}$ and $Y'_{k+1} = Y_{k+1}$ and so we get

$$\begin{aligned} \mathbb{V}ar(\Delta C_{k+1}(\varphi')|\mathcal{F}_k) = \mathbb{V}ar(C_{k+1}(\varphi')|\mathcal{F}_k) &= \mathbb{V}ar(V_{k+1}(\varphi') - (X'_{k+1})^*\Delta S_{k+1}|\mathcal{F}_k) \\ &= \mathbb{V}ar(V_{k+1}(\varphi) - (X'_{k+1})^*\Delta S_{k+1}|\mathcal{F}_k), \end{aligned} \tag{2.2.26}$$

and we can conclude that

$$\begin{aligned} T_k^\alpha(\varphi') \geq &\mathbb{E}[R_{k+1}(\varphi)|\mathcal{F}_k] + \mathbb{V}ar(V_{k+1}(\varphi) - (X'_{k+1})^*\Delta S_{k+1}|\mathcal{F}_k) \\ &+ \alpha\mathbb{E}[(X_{k+2} - X'_{k+1})^*[S_{k+1}(X_{k+2} - X'_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k]\,. \end{aligned} \tag{2.2.27}$$

Furthermore, since (ii) holds, then

$$\begin{aligned} T_k^\alpha(\varphi') \geq &\mathbb{E}[R_{k+1}(\varphi)|\mathcal{F}_k] + \mathbb{V}ar(V_{k+1}(\varphi) - (X_{k+1})^*\Delta S_{k+1}|\mathcal{F}_k) \\ &+ \alpha\mathbb{E}[(X_{k+2} - X_{k+1})^*[S_{k+1}(X_{k+2} - X_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k]\,. \end{aligned} \tag{2.2.28}$$

On the other hand, we have by definition (see Equation (2.2.10))

$$T_k^\alpha(\varphi) = R_k(\varphi) + \alpha\mathbb{E}[\Delta X_{k+2}^*[S_{k+1}(\Delta X_{k+2}) - S_{k+1}(0)]|\mathcal{F}_k]\,. \tag{2.2.29}$$

Since $C(\varphi)$ is a martingale, we get the representation (2.2.12) for the risk process $R_k(\varphi)$. So we can conclude that

$$\begin{aligned} T_k^\alpha(\varphi) = &\mathbb{E}[R_{k+1}(\varphi)|\mathcal{F}_k] + \mathbb{V}ar(\Delta C_{k+1}(\varphi)|\mathcal{F}_k) \\ &+ \alpha\mathbb{E}[(X_{k+2} - X_{k+1})^*[S_{k+1}(X_{k+2} - X_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k], \end{aligned} \tag{2.2.30}$$

Finally, since (2.2.28) and (2.2.30) hold then $T_k^\alpha(\varphi') \geq T_k^\alpha(\varphi)$ and this shows the " $\Leftarrow$ " direction of the proof.

Now, assuming that φ is a LRM-strategy under illiquidity i.e., $T_k^\alpha(\varphi') \geq T_k^\alpha(\varphi)$ for any local perturbation φ' of φ at time k, we will show the " $\Rightarrow$ " direction of the proof. Property (i) holds from Lemma 2.2.6. So it remains to show Property (ii).

Since $C(\varphi)$ is a martingale and φ' a local perturbation of φ at time k, then from Lemma 2.2.7 we know that equation (2.2.17) holds. On the other hand, since (2.2.30) holds (from the martingale property of $C(\varphi)$) then from the fact that $T_k^\alpha(\varphi') \geq T_k^\alpha(\varphi)$ we have

$$\begin{aligned} &\mathbb{E}[R_{k+1}(\varphi)|\mathcal{F}_k] + \mathbb{E}[(\Delta C_{k+1}(\varphi'))^2|\mathcal{F}_k] \\ &\quad + \alpha\mathbb{E}[(X_{k+2} - X'_{k+1})^*[S_{k+1}(X_{k+2} - X'_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k] \\ &\geq \mathbb{E}[R_{k+1}(\varphi)|\mathcal{F}_k] + \mathbb{V}ar(\Delta C_{k+1}(\varphi)|\mathcal{F}_k) \\ &\quad + \alpha\mathbb{E}[(X_{k+2} - X_{k+1})^*[S_{k+1}(X_{k+2} - X_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k], \end{aligned} \tag{2.2.31}$$

and from the definition of the conditional variance we can conclude that

$$\begin{aligned}
&\mathbb{V}ar(\Delta C_{k+1}(\varphi')|\mathcal{F}_k) + (\mathbb{E}[\Delta C_{k+1}(\varphi')|\mathcal{F}_k])^2 \\
&\quad + \alpha\mathbb{E}[(X_{k+2} - X'_{k+1})^*[S_{k+1}(X_{k+2} - X'_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k] \\
&\geq \mathbb{V}ar(\Delta C_{k+1}(\varphi)|\mathcal{F}_k) + \alpha\mathbb{E}[(X_{k+2} - X_{k+1})^*[S_{k+1}(X_{k+2} - X_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k],
\end{aligned} \tag{2.2.32}$$

for all X'_{k+1} and Y'_k. Fixing X'_{k+1} and choosing Y'_k as in the proof of Lemma 2.2.6 the inequality still holds and the liquidity costs remain unchanged. Since this choice gives us $\mathbb{E}[\Delta C_{k+1}(\varphi')|\mathcal{F}_k] = 0$ (as in the proof of Lemma 2.2.6) and since φ' a local perturbation of φ at time k, we get the inequality

$$\begin{aligned}
&\mathbb{V}ar(V_{k+1}(\varphi) - (X'_{k+1})^*\Delta S_{k+1}|\mathcal{F}_k) \\
&\qquad + \alpha\mathbb{E}[(X_{k+2} - X'_{k+1})^*[S_{k+1}(X_{k+2} - X'_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k] \\
&\geq \mathbb{V}ar(V_{k+1}(\varphi) - (X_{k+1})^*\Delta S_{k+1}|\mathcal{F}_k) \\
&\qquad + \alpha\mathbb{E}[(X_{k+2} - X_{k+1})^*[S_{k+1}(X_{k+2} - X_{k+1}) - S_{k+1}(0)]|\mathcal{F}_k]\,.
\end{aligned} \tag{2.2.33}$$

This shows that Property (ii) holds and the proof is completed. □

Proposition 2.2.9 holds for any supply curve thus is a quite general result. In the next section we will consider a special case of the supply curve, for the existence and recursive construction of a LRM-strategy under illiquidity. This will be motivated from a multiplicative limit order book. For this model we will be able to construct explicitly the optimal strategy and in particular under conditions that ensure that the optimal strategy belongs to the space $\Theta_d(S)$ of trading strategies.

2.3 Linear Supply Curve

In an illiquid environment, liquidity costs are related to the depth of the limit order book (LOB), when trading through it. The ability of the order book to recover itself after a trade is called the resilience effect. We assume that the speed of resilience is infinite, hence we do not take into account any feedback effects from hedging strategies. We choose the supply curve form $S_k(x) = (S^1_k(x^1), \dots, S^d_k(x^d))^*$ to be

$$S^j_k(x^j) = S^j_k + x^j \varepsilon^j_k S^j_k, \tag{2.3.1}$$

and assume that the price process S is a non-negative semimartingale and $\varepsilon = (\varepsilon_k)_{k=0,1,\dots,T}$ is a positive deterministic $\mathbb{R}^d$-valued process. Despite the fact that it is possible for $S_k(x)$ to take negative values for some x, note that in practice this is unlikely to happen for small values of ε_k and reasonable values of x.

For the specific form of the supply curve, let us now describe a (multiplicative) limit order book. A symmetric, 1-dimensional, time independent (for simplicity) LOB is represented by a density function q, where $q(x)dx$ is the bid or ask offers at price level xS_k. Denote further by $F(\rho) = \int_1^\rho q(x)dx$ the quantity to buy up through the LOB at price ρS_k. If an order of $x = F(\rho)$

shares is made by an investor through the LOB at time k then some limit orders are eaten up and the quoted price is shifted up to $S_k(x)^+ := g(x)S_k$ where the function $g(x)$ solves the equation $x = \int_1^{g(x)} q(y)dy$, hence $g(x) = F^{-1}(x)$.[3] After a trade and since we do not account for any price impact, the price returns to S_k.[4] The cost of an order of x shares will be $S_k \int_1^{g(x)} \rho dF(\rho)$. For an appropriate choice of the function q, this should be equal to $xS_k(x) = xS_k + \varepsilon x^2 S_k$. Then by choosing the density

$$q(x) = \frac{1}{2\varepsilon}, \tag{2.3.2}$$

which is independent from price, does the job. Note that when ε is tending to zero the market becomes more liquid and the liquidity cost vanishes. The process ε can be thought as a measure of illiquidity.

Remark 2.3.1. *Recall that the liquidity costs $x[S_k(x) - S_k(0)] \geq 0$ are non-negative by the fact that the supply curve $S_k(x)$ is increasing in the transaction size x. In the special case of a linear supply curve the liquidity costs for a transaction of size x at time k are $\varepsilon S_k|x|^2$. Note that since these depend on the price process then in order to avoid negative liquidity costs it is essential to assume that the marginal price process S is non-negative. Furthermore, since the depth of the order book $q_k(y) = \frac{1}{2\varepsilon_k}$ depends only on ε_k then note that when the price process S_k increases, then naturally also the liquidity cost increases but not the availability of assets in the LOB.*

Proposition 2.2.9 indicate the way how to construct an optimal strategy according to the LRM-criterion under illiquidity. That is, at time k, we need to minimize

$$\begin{aligned} &\mathbb{V}ar(V_{k+1}(\varphi) - (X'_{k+1})^* \Delta S_{k+1}|\mathcal{F}_k) \\ &\qquad + \alpha \mathbb{E}[\sum_{j=1}^{d} \varepsilon^j_{k+1} S^j_{k+1} (X^j_{k+2} - (X'_{k+1})^j)^2 | \mathcal{F}_k], \end{aligned} \tag{2.3.3}$$

over all appropriate X'_{k+1} (see Definition 2.2.1) and choose Y_k so that the cost process C becomes a martingale. This should be implemented and run through a backward in time algorithm.

Before continuing let us first introduce some notation:

$$\begin{aligned} A^0_{k;j} &:= \mathbb{V}ar(\Delta S^j_{k+1}|\mathcal{F}_k), & A^\varepsilon_{k;j} &:= \mathbb{E}[\varepsilon^j_{k+1} S^j_{k+1}|\mathcal{F}_k], & A_{k;j} &:= A^0_{k;j} + A^\varepsilon_{k;j}, \\ b^0_{k;j} &:= \mathbb{C}ov(V_{k+1}, \Delta S^j_{k+1}|\mathcal{F}_k), & b^\varepsilon_{k;j} &:= \mathbb{E}[\varepsilon^j_{k+1} S^j_{k+1} X^j_{k+2}|\mathcal{F}_k], & b_{k;j} &:= b^0_{k;j} + b^\varepsilon_{k;j}, \\ D_{k;j,i} &:= \mathbb{C}ov(\Delta S^j_{k+1}, \Delta S^i_{k+1}|\mathcal{F}_k), & &\text{for } i \neq j, \end{aligned} \tag{2.3.4}$$

for all $i, j = 1, \ldots, d$ and $k = 0, \ldots, T-1$.

Furthermore and assuming for simplicity $\alpha = 1$, we can rewrite the expression (2.3.3) by defining the function $f_k : \mathbb{R}^d \times \Omega \to \mathbb{R}^+$ as

$$f_k(c, \omega) = \sum_{j=1}^{d} |c_j|^2 A_{k;j}(\omega) - 2\sum_{j=1}^{d} c_j b_{k;j}(\omega) + \sum_{j \neq i} c_j c_i D_{k;j,i}(\omega)$$

[3]Note the multiplicative way of shifting up the price. In an additive LOB, as for example in Roch (2011), this would be of the form $S_k(x)^+ := S_k + g(x)$. See for example Løkka (2012) for a description of multiplicative and additive limit order books.

[4]In the literature, this is usually called the resilience effect, measuring the proportion of new bid or ask orders filling up the LOB after a trade. In our case we have infinite resilience.

$$+ \mathbb{V}ar(V_{k+1}|\mathcal{F}_k)(\omega) + \sum_{j=1}^{d} \mathbb{E}[\varepsilon_{k+1}^j S_{k+1}^j |X_{k+2}^j|^2 |\mathcal{F}_k](\omega). \tag{2.3.5}$$

Fixing ω one can easily calculate the gradient of the multidimensional function f_k. We need to solve $grad(f_k) = 0$ to calculate the candidates of extreme points which translates into solving a linear equation system of the form

$$F_k \, c = b_k, \tag{2.3.6}$$

where $F_k \in \mathbb{R}^{d\times d}$ with $F_{k;i,j} = D_{k;i,j}$ for $i \neq j$, $F_{k;i,j} = A_{k;j}$ for $i = j$ and $b_k = (b_{k;1}, \dots, b_{k;d}) \in \mathbb{R}^d$. Let $F_k^\varepsilon = diag(A_{k;1}^\varepsilon, \dots, A_{k;d}^\varepsilon)$ and denote by F_k^0 the matrix F_k with $\varepsilon_{k+1}^j = 0$ for all j, that is the covariance matrix of the marginal price process S. Then the symmetric matrix F_k is the sum of two real symmetric, positive semidefinite matrices $F_k = F_k^0 + F_k^\varepsilon$. This implies that the matrix F_k is also positive semidefinite[5] and therefore also the Hessian matrix which calculates as $H_{f_k}(c) = 2F_k$. So, assuming that the covariance matrix F_k^0 is positive definite, this implies that F_k is invertible and equation (2.3.6) has a unique solution. Furthermore, since also the Hesse matrix is positive definite the function $c \to f_k(c, \omega)$ is strictly convex, which implies that $c^* := F_k^{-1} b_k$ is a global minimizer. Furthermore, since the matrix F_k^{-1} and b_k are both $\mathcal{F}_k$-measurable it is clear that the minimizer c^* is also $\mathcal{F}_k$-measurable.

2.3.1 Properties of the marginal price process S

Slightly stronger assumptions on the matrix F_k are needed for showing that the optimal strategy c^* calculated above belongs to the space $\Theta_d(S)$. These assumptions can be reduced to assumptions on the covariance matrix of S. It will turn out that they hold for independent increments as well as for independent returns. We impose these assumptions now.

Definition 2.3.2. *We say that S has bounded mean-variance tradeoff process if for some constant* $C > 0$

$$\frac{(\mathbb{E}[\Delta S_{k+1}^j|\mathcal{F}_k])^2}{\mathbb{V}ar(\Delta S_{k+1}^j|\mathcal{F}_k)} \leq C \quad \mathbb{P}-a.s. \quad \textit{for all } j = 1, \dots, d \tag{2.3.7}$$

uniformly in k and ω.

Definition 2.3.3. *We say that S has modified above bounded mean-variance tradeoff process if for some constant* $C > 0$

$$\frac{(\mathbb{E}[S_{k+1}^j|\mathcal{F}_k])^2}{\mathbb{V}ar(S_{k+1}^j|\mathcal{F}_k)} \leq C \quad \mathbb{P}-a.s. \quad \textit{for all } j = 1, \dots, d \tag{2.3.8}$$

uniformly in k and ω. Furthermore S has modified below bounded mean-variance tradeoff process if for some constant $\tilde{C} > 0$

$$\frac{(\mathbb{E}[S_{k+1}^j|\mathcal{F}_k])^2}{\mathbb{V}ar(S_{k+1}^j|\mathcal{F}_k)} \geq \tilde{C} \quad \mathbb{P}-a.s. \quad \textit{for all } j = 1, \dots, d \tag{2.3.9}$$

[5] In fact, F_k is positive definite if ε_{k+1}^j is positive for all $j = 1, \dots, d$.

uniformly in k and ω. If both bounds hold then we say that S has modified bounded mean-variance tradeoff.

Remark 2.3.4. *For the case of S being a submartingale, by the fact that S_k^j is positive and since $(a+b)^2 \leq 2a^2 + 2b^2$ then we can estimate*

$$(\mathbb{E}[\Delta S_{k+1}^j | \mathcal{F}_k])^2 \leq 2(\mathbb{E}[S_{k+1}^j | \mathcal{F}_k])^2 + 2|S_k|^2 \leq 4(\mathbb{E}[S_{k+1}^j | \mathcal{F}_k])^2. \tag{2.3.10}$$

Thus the property of modified above bounded mean-variance tradeoff implies that of bounded mean-variance tradeoff when S is a submartingale.

Definition 2.3.5. *We say that S satisfies the F-diagonal condition if for some constant $C > 0$*

$$\sqrt{\mathbb{V}ar(\Delta S_{k+1}^j | \mathcal{F}_k)} + \frac{\mathbb{E}[S_{k+1}^j | \mathcal{F}_k]}{\sqrt{\mathbb{V}ar(S_{k+1}^j | \mathcal{F}_k)}} \geq C \quad \mathbb{P}-a.s. \quad \text{for all } j = 1, \ldots, d \tag{2.3.11}$$

uniformly in k and ω and if for some constant $\tilde{C} > 0$

$$\frac{\sqrt{\mathbb{V}ar(S_{k+1}^j | \mathcal{F}_k)}}{\mathbb{E}[S_{k+1}^j | \mathcal{F}_k]} + \frac{1}{\sqrt{\mathbb{V}ar(\Delta S_{k+1}^j | \mathcal{F}_k)}} \geq \tilde{C} \quad \mathbb{P}-a.s. \quad \text{for all } j = 1, \ldots, d \tag{2.3.12}$$

uniformly in k and ω.

Remark 2.3.6. *The diagonal terms of the matrix F justify the name F-diagonal condition in Definition 2.3.5, since*

$$\begin{aligned}
\frac{F_{k;j,j}^0}{|F_{k;j,j}|^2} &= \left(\sqrt{\mathbb{V}ar(\Delta S_{k+1}^j | \mathcal{F}_k)} + \frac{\mathbb{E}[\varepsilon_{k+1}^j S_{k+1}^j | \mathcal{F}_k]}{\sqrt{\mathbb{V}ar(S_{k+1}^j | \mathcal{F}_k)}} \right)^{-2}, \\
|F_{k;j,j}^\varepsilon|^2 \frac{F_{k;j,j}^0}{|F_{k;j,j}|^2} &= \left(\frac{\sqrt{\mathbb{V}ar(S_{k+1}^j | \mathcal{F}_k)}}{\mathbb{E}[\varepsilon_{k+1}^j S_{k+1}^j | \mathcal{F}_k]} + \frac{1}{\sqrt{\mathbb{V}ar(\Delta S_{k+1}^j | \mathcal{F}_k)}} \right)^{-2}.
\end{aligned} \tag{2.3.13}$$

Writing $S_{k+1}^j = S_k^j(1 + \rho_{k+1}^j)$ for $j = 1, \ldots, d$, we denote by $\rho = (\rho_k)_{k=0,1,\ldots T}$ the d-dimensional return process of S.

Sufficient conditions are given by the next two Propositions 2.3.7 and 2.3.8 for the previous properties on the marginal price process S.

Proposition 2.3.7. *For S satisfying $\tilde{C} \leq \mathbb{V}ar(\Delta S_{k+1}^j | \mathcal{F}_k) \leq C$ for some positive constants $C, \tilde{C}$ and for all $j = 1, \ldots, d$, then the F-diagonal condition holds. In particular, if S has independent increments then S has bounded mean-variance tradeoff and satisfies the F-diagonal condition.*

Proof. The claim follows directly from the fact that $\tilde{C} \leq \mathbb{V}ar(\Delta S_{k+1}^j | \mathcal{F}_k) \leq C$. □

Proposition 2.3.8. *For S having modified bounded mean-variance tradeoff then the F-diagonal condition holds. In particular, if S has independent returns then S has bounded mean-variance tradeoff and satisfies the F-diagonal condition.*

Proof. The claim follows directly from the fact that S has modified bounded mean-variance tradeoff. □

Remark 2.3.9. *Consider the 1-dimensional Black-Scholes model of a geometric Brownian motion W, that is*

$$S_{kh} = S_0 \exp\left(bkh + \sigma W_{kh}\right), \tag{2.3.14}$$

with discretization time step $\Delta t = h$. Then the return process ρ_k can be defined by

$$1 + \rho_k = \frac{S_{kh}}{S_{(k-1)h}}, \tag{2.3.15}$$

and is lognormally distributed. This is also a process of i.i.d. random variables. By Proposition 2.3.8, S has bounded mean-variance tradeoff and satisfies the F-diagonal condition.

2.3.2 Some preliminaries

In order to show that the integrability conditions are fulfilled we are going to state some useful Lemmas needed in the proof of Theorem 2.3.15. We will use the following notation:

$$\begin{aligned} \alpha_{k;i,j} &:= F^0_{k;j,j} F^0_{k;i,i} |F^{-1}_{k;j,i}|^2, & \alpha^\varepsilon_{k;i,j} &:= F^0_{k;j,j} |F^\varepsilon_{k;i,i}|^2 |F^{-1}_{k;j,i}|^2, \\ \beta_{k;i,j} &:= F^0_{k;i,i} |F^{-1}_{k;j,i}|^2, & \beta^\varepsilon_{k;i,j} &:= |F^\varepsilon_{k;i,i}|^2 |F^{-1}_{k;j,i}|^2, \end{aligned} \tag{2.3.16}$$

for $i, j = 1, \ldots, d$ and $k = 0, \ldots, T$ when the inverse matrix F_k^{-1} of F_k exists.

Furthermore, we will denote by $M_{k;i,j}$ the matrix F_k without the i-th row and j-th column. From linear algebra recall also that if the inverse of a symmetric matrix F_k exists then $F^{-1}_{k;j,i} = \frac{(-1)^{i+j}\det(M_{k;i,j})}{\det(F_k)}$ which we use in Lemma 2.3.13.

Lemma 2.3.10. *For all $d \in \mathbb{N}_{\geq 2}$:*

$$\det(M_{k;i,j})^2 \leq C F^0_{k;j,j} F^0_{k;i,i} \prod_{\substack{l=1 \\ l \neq i,j}}^{d} |F_{k;l,l}|^2 \qquad \text{for all } i, j = 1, \ldots, d \text{ with } i \neq j, \tag{2.3.17}$$

$$|F_{k;j,j}|^2 \det(M_{k;j,j})^2 \leq \tilde{C} \det(F_k^A)^2 \qquad \text{for all } j = 1, \ldots, d, \tag{2.3.18}$$

$$F_{k;j,j} F_{k;i,i} \det(M_{k;i,j})^2 \leq \bar{C} \det(F_k^A)^2 \qquad \text{for all } i, j = 1, \ldots, d, \tag{2.3.19}$$

for some positive constants $C, \tilde{C}$ and $\bar{C}$ where $F_k^A := \operatorname{diag}(A_{k;1}, \ldots, A_{k;d})$.

Proof. First note that the last inequality (2.3.19) follows from the first two. Indeed for the case $i \neq j$ and since $F^0_{k;j,j} \leq F_{k;j,j}$ (since ε^j_{k+1} and S^j_{k+1} are non-negative) for all j, then from inequality (2.3.17) we have

$$\det(M_{k;i,j})^2 \leq C F_{k;j,j} F_{k;i,i} \prod_{\substack{l=1 \\ l\neq i,j}}^{d} |F_{k;l,l}|^2 . \tag{2.3.20}$$

Since the matrix F^A_k is a diagonal matrix then it is clear that now inequality (2.3.19) follows for $i \neq j$. The case $i = j$ follows directly from inequality (2.3.18).

For showing the inequalities (2.3.17) and (2.3.18) for $d = 2$ is trivial. We will show for the case $d = 3$ the inequality (2.3.17). Inequality (2.3.18) follows then analogously. Let w.l.o.g. $i = 1$. For $j = 2$ we have

$$\det(M_{k;1,2})^2 = (D_{k;1,2}A_{k;3} - D_{k;2,3}D_{k;1,3})^2 \leq 2|D_{k;1,2}|^2|A_{k;3}|^2 + 2|D_{k;2,3}|^2|D_{k;1,3}|^2, \tag{2.3.21}$$

where we have used the inequality $(a+b)^2 \leq 2a^2+2b^2$. Now, applying the conditional Cauchy-Schwarz inequality we get,

$$\det(M_{k;1,2})^2 \leq 2A^0_{k;1}A^0_{k;2}|A_{k;3}|^2 + 2A^0_{k;2}A^0_{k;3}A^0_{k;1}A^0_{k;3} \leq 4A^0_{k;1}A^0_{k;2}|A_{k;3}|^2 . \tag{2.3.22}$$

The case $j = 3$ follows analogously and so inequality (2.3.17) holds.

A generalization of the proof for an arbitrary d can be done using the Laplace's formula and the symmetry of the matrices F_k and F^0_k. □

In the next Definition that follows we describe the F*-property* condition which is crucial. This property does not only extend the LRM-criterion of Schweizer (1988) to the illiquid case (i.e. $\varepsilon \neq 0$) but also and especially extends the setting to the multidimensional case. For a one-dimensional price process S the F-property translates to $\mathbb{V}ar(\Delta S_{k+1}|\mathcal{F}_k) + \mathbb{E}[\varepsilon_{k+1}S_{k+1}|\mathcal{F}_k] \geq 0$ in the 1-dimensional case. Note that this is always fulfilled.[6] Moreover and in the case of independent components, i.e. S^i and S^j are independent for $i \neq j$, then the condition reduces to $\det(F^A_k) \geq 0$ which also is always fulfilled since the matrix F^A_k is positive semi-definite. Thus the next property is essentially linked to the covariance matrix of the multidimensional price process S. Furthermore, this property can be reduced to a property on the covariance matrix of S, as we will show in Section 2.3.4. In the following, C denotes a generic positive constant that might change from line to line.

Definition 2.3.11. *We say that the process S has the F-property if there exists some $\delta \in (0,1)$ such that*

$$\det(F_k) - (1-\delta)\det(F^A_k) \geq 0, \tag{2.3.23}$$

for all $k = 0, 1, \ldots, T$ where $F^A_k := \operatorname{diag}(A_{k;1}, \ldots, A_{k;d})$.

Lemma 2.3.12. *Assume that S has the F-property and satisfies the F-diagonal condition. Then the terms $\alpha_{k;i,j}$, $\beta_{k;i,j}$, $\alpha^\varepsilon_{k;i,j}$ and $\beta^\varepsilon_{k;i,j}$ are uniformly bounded in k and ω for all $i, j = 1, \ldots, d$.*

[6] Recall the assumption that the process ε and the price process S are both non-negative.

Proof. For the first term $\alpha_{k;i,j}$ we have

$$\alpha_{k;i,j} = F^0_{k;j,j}F^0_{k;i,i}\frac{\det(M_{k;i,j})^2}{\det(F_k)^2} \le C\frac{\det(F^A_k)^2}{\det(F_k)^2} \le C\frac{1}{(1-\delta)^2}, \tag{2.3.24}$$

by using first the inequality (2.3.19) from Lemma 2.3.10 and then the F-property. For the second term $\beta_{k;i,j}$ we can estimate for the case $i = j$

$$\beta_{k;i,j} = F^0_{k;i,i}\frac{\det(M_{k;i,i})^2}{\det(F_k)^2} \le C\frac{F^0_{k;i,i}}{|F_{k;i,i}|^2}\frac{\det(F^A_k)^2}{\det(F_k)^2} \le C\frac{1}{(1-\delta)^2}, \tag{2.3.25}$$

using inequality (2.3.18) from Lemma 2.3.10 and then the F-property and inequality (2.3.11). For the case $i \neq j$ and using inequality (2.3.17) from Lemma 2.3.10 we have

$$\det(F_k)^2\beta_{k;i,j} = F^0_{k;i,i}\det(M_{k;i,j})^2 \le CF^0_{k;j,j}|F^0_{k;i,i}|^2\prod_{\substack{l=1\\ l\neq i,j}}^{d}|F_{k;l,l}|^2 \le C\frac{F^0_{k;j,j}}{|F_{k;j,j}|^2}\det(F^A_k)^2, \tag{2.3.26}$$

and from the F-property and inequality (2.3.11), $\beta_{k;i,j}$ is uniformly bounded. Furthermore and by the same arguments as for the term $\beta_{k;i,j}$ we have for the case $i = j$

$$\alpha^\varepsilon_{k;i,j} = |F^\varepsilon_{k;i,i}|^2F^0_{k;i,i}\frac{\det(M_{k;i,i})^2}{\det(F_k)^2} \le C|F^\varepsilon_{k;i,i}|^2\frac{F^0_{k;i,i}}{|F_{k;i,i}|^2}\frac{\det(F^A_k)^2}{\det(F_k)^2} \le C\frac{1}{(1-\delta)^2}, \tag{2.3.27}$$

using the F-property and inequality (2.3.12). For $i \neq j$ we can estimate

$$\begin{aligned}\det(F_k)^2\alpha^\varepsilon_{k;i,j} = |F^\varepsilon_{k;i,i}|^2F^0_{k;j,j}\det(M_{k;i,j})^2 &\le C|F^\varepsilon_{k;i,i}|^2F^0_{k;i,i}|F^0_{k;j,j}|^2\prod_{\substack{l=1\\ l\neq i,j}}^{d}|F_{k;l,l}|^2\\ &\le C|F^\varepsilon_{k;i,i}|^2\frac{F^0_{k;i,i}}{|F_{k;i,i}|^2}\det(F^A_k)^2,\end{aligned} \tag{2.3.28}$$

and from the F-property and inequality (2.3.12), $\alpha^\varepsilon_{k;i,j}$ is also uniformly bounded. For the last term $\beta^\varepsilon_{k;i,j}$ we have for $i = j$

$$\beta^\varepsilon_{k;i,j} = |F^\varepsilon_{k;i,i}|^2\frac{\det(M_{k;i,i})^2}{\det(F_k)^2} \le C\frac{|F^\varepsilon_{k;i,i}|^2}{|F_{k;i,i}|^2}\frac{\det(F^A_k)^2}{\det(F_k)^2} \le C\frac{1}{(1-\delta)^2}, \tag{2.3.29}$$

by the F-property. Moreover for $i \neq j$

$$\begin{aligned}\det(F_k)^2\beta^\varepsilon_{k;i,j} = |F^\varepsilon_{k;i,i}|^2\det(M_{k;i,j})^2 &\le C|F^\varepsilon_{k;i,i}|^2F^0_{k;i,i}F^0_{k;j,j}\prod_{\substack{l=1\\ l\neq i,j}}^{d}|F_{k;l,l}|^2\\ &= C|F^\varepsilon_{k;i,i}|^2\frac{F^0_{k;i,i}}{|F_{k;i,i}|^2}\frac{F^0_{k;j,j}}{|F_{k;j,j}|^2}\det(F^A_k)^2,\end{aligned} \tag{2.3.30}$$

where from the F-property and the F-diagonal condition the last term $\beta^{\varepsilon}_{k;i,j}$ is uniformly bounded. We also made use of the fact that the process ε is deterministic and that we have a finite number of hedging times. □

Lemma 2.3.13. *Assume that F_k^{-1} exists for $k \in \{0, 1, \dots, T\}$ and S has bounded mean-variance tradeoff. Let (X, Y) be any trading strategy. Then there exists some constant $C > 0$ such that*

$$\begin{aligned} & \mathbb{E}[((F_k^{-1}b_k)_j \Delta S^j_{k+1})^2] \\ \leq\ & C\mathbb{E}[\mathbb{V}ar(V_{k+1}|\mathcal{F}_k)\sum_{i=1}^d \alpha_{k;i,j} + \sum_{i=1}^d (c(\varepsilon_{k+1})\alpha_{k;i,j} + \alpha^{\varepsilon}_{k;i,j})\mathbb{E}[|X^i_{k+2}|^2|\mathcal{F}_k]] \quad (2.3.31) \\ & \mathbb{E}[((F_k^{-1}b_k)_j)^2] \\ \leq\ & C\mathbb{E}[\mathbb{V}ar(V_{k+1}|\mathcal{F}_k)\sum_{i=1}^d \beta_{k;i,j} + \sum_{i=1}^d (c(\varepsilon_{k+1})\beta_{k;i,j} + \beta^{\varepsilon}_{k;i,j})\mathbb{E}[|X^i_{k+2}|^2|\mathcal{F}_k]] \quad (2.3.32) \end{aligned}$$

for all $j = 1, \dots, d$ where $(F_k^{-1}b_k)_j$ is the j-th component of the vector $(F_k^{-1}b_k)$. The term $c(\varepsilon_{k+1})$ denotes a positive constant depending on the process ε at time $k+1$ such that for $\varepsilon_{k+1} \to 0$, $c(\varepsilon_{k+1})$ converges to zero.

Proof. First note that from the definition of the variance and using bounded mean-variance tradeoff, it follows directly that

$$\mathbb{E}[|\Delta S^j_{k+1}|^2|\mathcal{F}_k] = \mathbb{V}ar(\Delta S^j_{k+1}|\mathcal{F}_k) + (\mathbb{E}[\Delta S^j_{k+1}|\mathcal{F}_k])^2 \leq CA^0_{k;j}\,. \quad (2.3.33)$$

Furthermore, denoting $F = F_k$ and $b = b_k$ we have from the tower property and using inequality (2.3.33)

$$\begin{aligned} \mathbb{E}[((F^{-1}b)_j \Delta S^j_{k+1})^2] &= \mathbb{E}[((F^{-1}(b^0 + b^{\varepsilon}))_j)^2 \mathbb{E}[|\Delta S^j_{k+1}|^2|\mathcal{F}_k]] \\ &\leq 2C\mathbb{E}[\sum_{i=1}^d |F^{-1}_{j,i}|^2(|b^0_i|^2 + |b^{\varepsilon}_i|^2)F^0_{j,j}]. \quad (2.3.34) \end{aligned}$$

Moreover, using the conditional Cauchy-Schwarz-Inequality for the term b^0_i and the conditional inequality $(\mathbb{E}[XY|\mathcal{G}])^2 \leq \mathbb{E}[X^2|\mathcal{G}]\mathbb{E}[Y^2|\mathcal{G}]$ on the term b^{ε}_i together with the definition of the variance yields

$$\begin{aligned} & \mathbb{E}[((F^{-1}b)_j \Delta S^j_{k+1})^2] \\ & \leq C\mathbb{E}[\sum_{i=1}^d |F^{-1}_{j,i}|^2(\mathbb{V}ar(V_{k+1}|\mathcal{F}_k)F^0_{i,i} + \mathbb{E}[|\varepsilon^i_{k+1}S^i_{k+1}|^2|\mathcal{F}_k]\mathbb{E}[|X^i_{k+2}|^2|\mathcal{F}_k])F^0_{j,j}] \\ & = C\mathbb{E}[\sum_{i=1}^d |F^{-1}_{j,i}|^2(\mathbb{V}ar(V_{k+1}|\mathcal{F}_k)F^0_{i,i} + |\varepsilon^i_{k+1}|^2F^0_{i,i}\mathbb{E}[|X^i_{k+2}|^2|\mathcal{F}_k] + |F^{\varepsilon}_{i,i}|^2\mathbb{E}[|X^i_{k+2}|^2|\mathcal{F}_k])F^0_{j,j}]\,. \end{aligned} \quad (2.3.35)$$

The other inequality follows analogously.

□

Remark 2.3.14. *Both Lemmas 2.3.12 and 2.3.13 will be used for the Existence of a LRM-strategy under illiquidity. We will need to show basically two integrability properties. The optimal strategy $\hat{X}$ (under the LRM-criterion under illiquidity) must satisfy that $\hat{X}^j_{k+1}\Delta S^j_{k+1} \in \mathbb{L}^{2,1}_T$ and $\hat{X}^j_{k+1} \in \mathbb{L}^{2,1}_T$. Denoting by $\hat{\Theta}_d(S)$, the space of all $\mathbb{R}^d$-valued predictable strategies $X = (X_k)_{k=1,2,\dots,T+1}$ so that $X^*_k \Delta S_k \in \mathbb{L}^{2,1}_T$ for $k = 1, 2, \dots, T$, the first integrability property shows that the strategy $\hat{X}$ belongs to $\hat{\Theta}_d(S)$. The second one is essential for showing the first one. Nevertheless, in order to show that the liquidity costs of the optimal strategy are integrable will will use both integrability properties.*

Furthermore, note that when $\varepsilon = 0$, that is in the infinite liquidity case, the second inequality of Lemma 2.3.13 is not needed by the fact that the terms $c(\varepsilon_{k+1}), \alpha_{k;i,j}$ and $\alpha^{\varepsilon}_{k;i,j}$ vanish. Thus, by using bounded mean-variance tradeoff and the F-property, in the multidimensional case without liquidity costs, one needs to show only that $\hat{X} \in \hat{\Theta}_d(S)$.

Moreover, in the 1-dimensional case ($d = 1$) we have

$$\alpha_{k;1,1} = \frac{|A^0_{k;1}|^2}{|A_{k;1}|^2}, \qquad \beta_{k;1,1} = \frac{A^0_{k;1}}{|A_{k;1}|^2}, \qquad \alpha^{\varepsilon}_{k;1,1} = A^{\varepsilon}_{k;1}\frac{A^0_{k;1}}{|A_{k;1}|^2}, \qquad \beta^{\varepsilon}_{k;1,1} = \frac{|A^{\varepsilon}_{k;1}|^2}{|A_{k;1}|^2}, \tag{2.3.36}$$

where note that the terms $\alpha_{k;1,1}$, $\beta^{\varepsilon}_{k;1,1}$ are bounded by 1 and the terms $\beta_{k;1,1}$, $\alpha^{\varepsilon}_{k;1,1}$ are uniformly bounded by the F-diagonal property. Furthermore for the case when $\varepsilon = 0$ one would need to show only the first inequality of Lemma 2.3.13 which simplifies to

$$\mathbb{E}[((F_k^{-1}b_k)_1 \Delta S^1_{k+1})^2] \le C\mathbb{E}[|V_{k+1}|^2], \tag{2.3.37}$$

as in the 1-dimensional classical case in Schweizer (1988). In this case it is well known that only the assumption of bounded mean-variance tradeoff is essential for proving and constructing the optimal strategy.

In the following we will present our main existence Theorem where under some mild conditions on the marginal price process S we show the existence of a local risk-minimizing strategy under illiquidity.

2.3.3 Existence and recursive construction of an optimal strategy

Under the previous assumptions imposed in the previous Section 2.3.1 the existence of a local risk-minimizing strategy under illiquidity can be proven. Additionally, by means of a backward induction argument and as already explained at the beginning of Section 2.3 we give an explicit representation of the optimal strategy.

Theorem 2.3.15 (Existence result). *Assume that S has the F-property, bounded mean-variance tradeoff and satisfies the F-diagonal condition. Let further the covariance matrix F^0_k be positive definite at all times $k = 0, 1, \dots, T-1$. Then for any contingent claim $H = \bar{X}^*_{T+1}S_T + \bar{Y}_T \in \mathbb{L}^{2,1}_T$ with $\bar{X}^*_{T+1}S_T \in \mathbb{L}^{2,1}_T$ and $\bar{X}_{T+1} \in \mathbb{L}^{2,d}_T$, there exists a local risk-minimizing strategy $\hat{\varphi} = (\hat{X}, \hat{Y})$ under illiquidity with $\hat{X}_{T+1} = \bar{X}_{T+1}$ and $\hat{Y}_T = \bar{Y}_T$. Furthermore, the strategy has the representation*

$$\hat{X}_{k+1} = F_k^{-1}b_k \quad \mathbb{P}\text{-a.s. for } k = 0, \dots, T-1, \tag{2.3.38}$$

$$\hat{Y}_k = \mathbb{E}[\hat{W}_k|\mathcal{F}_k] - \hat{X}^*_{k+1}S_k \quad \mathbb{P}-a.s.\ for\ k = 0, 1, \ldots, T-1, \tag{2.3.39}$$

where $\hat{W}_k = H - \sum_{m=k+1}^{T} \hat{X}^*_m \Delta S_m$.

Proof. The proof is a backward induction argument on $k = 0, 1, \ldots, T$. First set $\hat{X}_{T+1} = \bar{X}_{T+1}$ and $\hat{Y}_T = \bar{Y}_T$. So, fix some $k \in \{0, 1, \ldots, T-2\}$ and assume that at times $l = k, \ldots, T-2$

(i) $\hat{X}^j_{l+2}\Delta S^j_{l+2} \in \mathbb{L}^{2,1}_T$ and $\hat{X}^j_{l+2} \in \mathbb{L}^{2,1}_T$,

(ii) $|\hat{X}^j_{l+2}|^2 S^j_{l+1} \in \mathbb{L}^{1,1}_T$,

(iii) $\hat{X}^*_{l+2}S_{l+1} + \hat{Y}_{l+1} \in \mathbb{L}^{2,1}_T$, $\hat{Y}_{l+1} \in \mathcal{F}_{l+1}$,

for all $j = 1, \ldots, d$ holds. At time k we want to minimize the expression (2.3.3) over all X'_{k+1} and show that the following properties are fulfilled for all $j = 1, \ldots, d$:

(i) $X'^{,j}_{k+1}\Delta S^j_{k+1} \in \mathbb{L}^{2,1}_T$ and $X'^{,j}_{k+1} \in \mathbb{L}^{2,1}_T$,

(ii) $|X'^{,j}_{k+1}|^2 S^j_k \in \mathbb{L}^{1,1}_T$,

(iii) $(X'_{k+1})^* S_k + Y'_k \in \mathbb{L}^{2,1}_T$, $Y'_k \in \mathcal{F}_k$

Properties (i) - (iii) will then ensure that $(\hat{X}, \hat{Y}) \in \Theta_d(S)$. First we define the function f_k as in equation (2.3.5) and note that all the terms in f_k are integrable by induction hypothesis. Since F_k is positive definite then there exists a unique solution to the minimization problem and an $\mathcal{F}_k$-measurable minimizer $\hat{X}_{k+1}$ can be constructed, which equals $F_k^{-1}b_k$. Furthermore define $\hat{Y}_k$ as in equation (2.3.39). Then it is clear that $\hat{Y}_k$ is $\mathcal{F}_k$-measurable. The fact that $\hat{X}^*_{k+1}S_k + \hat{Y}_k = \mathbb{E}[\hat{W}_k|\mathcal{F}_k] \in \mathbb{L}^{2,1}_T$ follows from $H \in \mathbb{L}^{2,1}_T$, the induction hypothesis $\sum_{m=k+2}^{T} \hat{X}^*_m \Delta S_m \in \mathbb{L}^{2,1}_T$ and $\hat{X}^*_{k+1}\Delta S_{k+1} \in \mathbb{L}^{2,1}_T$, which we will show below.

Now let us show first that $\hat{X}^j_{k+1}\Delta S^j_{k+1} \in \mathbb{L}^{2,1}_T$. By inequality (2.3.31) of Lemma 2.3.13 we know that for a constant $C > 0$,

$$\begin{aligned}
&\mathbb{E}[(\hat{X}^j_{k+1}\Delta S^j_{k+1})^2] \\
&\leq C\mathbb{E}[\mathbb{V}ar(\hat{X}^*_{k+2}S_{k+1} + \hat{Y}_{k+1}|\mathcal{F}_k)\sum_{i=1}^{d} \alpha_{k;i,j} + \sum_{i=1}^{d}(c(\varepsilon_{k+1})\alpha_{k;i,j} + \alpha^{\varepsilon}_{k;i,j})\mathbb{E}[|X^i_{k+2}|^2|\mathcal{F}_k]],
\end{aligned} \tag{2.3.40}$$

holds. Since by the induction hypothesis $\hat{X}^*_{k+2}S_{k+1} + \hat{Y}_{k+1}$ and $\hat{X}^i_{k+2}$ both in $\mathbb{L}^{2,1}_T$ for all $i = 1, \ldots, d$, then it remains to show that the terms $\alpha_{k;i,j}$, $\alpha^{\varepsilon}_{k;i,j}$ are uniformly bounded in k and ω. This follows from Lemma 2.3.12. Similarly one can show that $\hat{X}^j_{k+1} \in \mathbb{L}^{2,1}_T$ using inequality (2.3.32) of Lemma 2.3.13.

Next we show that the liquidity costs $\mathbb{E}[\sum_{j=1}^{d} \varepsilon^j_{k+1}S^j_{k+1}|\hat{X}^j_{k+2} - \hat{X}^j_{k+1}|^2|\mathcal{F}_k]$ are integrable. From the minimization problem of expression (2.3.3) and since $\hat{X}_{k+1}$ is a minimizer, we know that (w.l.o.g. $\alpha = 1$):

$$\begin{aligned}&\mathbb{V}ar(\hat{X}^*_{k+2}S_{k+1} + \hat{Y}_{k+1} - (\hat{X}_{k+1})^*\Delta S_{k+1}|\mathcal{F}_k) + \mathbb{E}[\sum_{j=1}^{d} \varepsilon^j_{k+1}S^j_{k+1}|\hat{X}^j_{k+2} - \hat{X}^j_{k+1}|^2|\mathcal{F}_k]\\ &\leq \mathbb{V}ar(\hat{X}^*_{k+2}S_{k+1} + \hat{Y}_{k+1}|\mathcal{F}_k) + \mathbb{E}[\sum_{j=1}^{d} \varepsilon^j_{k+1}S^j_{k+1}|\hat{X}^j_{k+2}|^2|\mathcal{F}_k], \qquad (2.3.41)\end{aligned}$$

holds, where the right hand side corresponds to choosing $X_{k+1} = 0$. Taking expectation on both sides and since by definition the conditional variance is non-negative, we get

$$\mathbb{E}[\sum_{j=1}^{d} \varepsilon^j_{k+1}S^j_{k+1}|\hat{X}^j_{k+2} - \hat{X}^j_{k+1}|^2] \leq \mathbb{E}[|\hat{X}^*_{k+2}S_{k+1} + \hat{Y}_{k+1}|^2] + \mathbb{E}[\sum_{j=1}^{d} \varepsilon^j_{k+1}S^j_{k+1}|\hat{X}^j_{k+2}|^2], \tag{2.3.42}$$

where we have used the fact that $\mathbb{V}ar(X) \leq \mathbb{E}|X|^2$. Now, since by the inductive hypothesis, $\hat{X}^*_{k+2}S_{k+1} + \hat{Y}_{k+1} \in \mathbb{L}^{2,1}_T$ and $S^j_{k+1}|\hat{X}^j_{k+2}|^2 \in \mathbb{L}^{1,1}_T$ for all $j = 1, \dots, d$ then it is clear that the liquidity cost $\sum_{j=1}^{d} \varepsilon^j_{k+1}S^j_{k+1}|\hat{X}^j_{k+2} - \hat{X}^j_{k+1}|^2$ is in $\mathbb{L}^{1,1}_T$. In particular $\varepsilon^j_{k+1}S^j_{k+1}|\hat{X}^j_{k+2} - \hat{X}^j_{k+1}|^2 \in \mathbb{L}^{1,1}_T$ for all $j = 1, \dots, d$. This holds from the fact that the deterministic process ε and the marginal price process S are both non-negative by assumption.

In order to complete the proof, it remains to show that $|\hat{X}^j_{k+1}|^2 S^j_k \in \mathbb{L}^{1,1}_T$. This is needed in order to complete the induction argument and be able to show that the liquidity costs in the next step are again integrable. So, from the equality

$$|\hat{X}^j_{k+1}|^2 S^j_k = -|\hat{X}^j_{k+1}|^2 \Delta S^j_{k+1} + |\hat{X}^j_{k+1}|^2 S^j_{k+1}, \tag{2.3.43}$$

we need to show that $|\hat{X}^j_{k+1}|^2 \Delta S^j_{k+1}$ and $|\hat{X}^j_{k+1}|^2 S^j_{k+1}$ are both in $\mathbb{L}^{1,1}_T$. Since, as already shown, the liquidity costs are integrable for all $j = 1, \dots, d$ and since by induction hypothesis $|\hat{X}^j_{k+2}|^2 S^j_{k+1} \in \mathbb{L}^{1,1}_T$ then the inequality

$$0 \leq |\hat{X}^j_{k+1}|^2 S^j_{k+1} \leq 2|\hat{X}^j_{k+2} - \hat{X}^j_{k+1}|^2 S^j_{k+1} + 2|\hat{X}^j_{k+2}|^2 S^j_{k+1}, \tag{2.3.44}$$

follows. Since $\varepsilon^j_k > 0$ this implies that $|\hat{X}^j_{k+1}|^2 S^j_{k+1}$ is integrable for all $j = 1, \dots, d$. The term $|\hat{X}^j_{k+1}|^2 \Delta S^j_{k+1}$ is also integrable by the fact that $\hat{X}^j_{k+1}\Delta S^j_{k+1}$ and $\hat{X}^j_{k+1}$ are both in $\mathbb{L}^{2,1}_T$. Indeed we have

$$\begin{aligned}\mathbb{E}[|\hat{X}^j_{k+1}|^2 \Delta S^j_{k+1}] &\leq \mathbb{E}[|\hat{X}^j_{k+1}|^2 \mathbf{1}_{\{|\Delta S^j_{k+1}| \leq 1\}}] + \mathbb{E}[|\hat{X}^j_{k+1}\Delta S^j_{k+1}|^2 \mathbf{1}_{\{|\Delta S^j_{k+1}| \geq 1\}}]\\ &\leq \mathbb{E}[|\hat{X}^j_{k+1}|^2] + \mathbb{E}[|\hat{X}^j_{k+1}\Delta S^j_{k+1}|^2], \qquad (2.3.45)\end{aligned}$$

and this proves and completes the induction step at time k.

The base case at time $k = T$ where $\hat{X}^*_{T+1}S_T + \hat{Y}_T = H$ is clear by the same arguments and by the assumptions on H and $\bar{X}_{T+1}$, $\bar{Y}_T$. Indeed, since $\hat{X}^*_{T+1}S_T + \hat{Y}_T$ and $\hat{X}_{T+1}$ are both square integrable, then from Lemma 2.3.13 and Lemma 2.3.12 it follows that $\hat{X}^j_T \Delta S^j_T \in \mathbb{L}^{2,1}_T$ and

$\hat{X}_T^j \in \mathbb{L}_T^{2,1}$ for all j. Moreover, note that with the assumptions $\hat{X}_{T+1}^j S_T^j \in \mathbb{L}_T^{2,1}$, $\hat{X}_{T+1}^j \in \mathbb{L}_T^{2,1}$ one can show that $|\hat{X}_{T+1}^j|^2 S_T^j \in \mathbb{L}_T^{1,1}$. By the same arguments as above, this will imply the integrability of the liquidity costs. The fact that $|\hat{X}_T^j|^2 S_{T-1}^j \in \mathbb{L}_T^{1,1}$ can be shown by using exactly the same arguments as in the proof for the inductive step.

Finally, by defining

$$\hat{Y}_{T-1} = \mathbb{E}[H - \hat{X}_T^* \Delta S_T | \mathcal{F}_k] - \hat{X}_T^* S_{T-1}, \tag{2.3.46}$$

then it is clear that $\hat{Y}_{T-1}$ is $\mathcal{F}_{T-1}$-measurable and $\hat{X}_T^* S_{T-1} + \hat{Y}_{T-1} = \mathbb{E}[H - \hat{X}_T^* \Delta S_T | \mathcal{F}_k]$ belongs to $\mathbb{L}_T^{2,1}$.

The martingale property of $C(\hat{\varphi})$ follows from the construction of $\hat{Y}$ since at each time k we have

$$\mathbb{E}[C_T(\hat{\varphi}) - C_k(\hat{\varphi}) | \mathcal{F}_k] = 0, \tag{2.3.47}$$

and so by Proposition 2.2.9, since both properties are satisfied, then the trading strategy $\hat{\varphi} = (\hat{X}, \hat{Y})$ is local risk-minimizing under illiquidity and the proof is complete. □

Remark 2.3.16. *Consider the* 1*-dimensional case. The explicit representation of the LRM-strategy $\hat{\varphi} = (\hat{X}, \hat{Y})$ under illiquidity is the following:*

$$\hat{X}_{k+1} = \frac{\mathbb{C}ov(V_{k+1}(\hat{\varphi}), \Delta S_{k+1} | \mathcal{F}_k) + \mathbb{E}[\varepsilon_{k+1} S_{k+1} \hat{X}_{k+2} | \mathcal{F}_k]}{\mathbb{V}ar(\Delta S_{k+1} | \mathcal{F}_k) + \mathbb{E}[\varepsilon_{k+1} S_{k+1} | \mathcal{F}_k]}, \tag{2.3.48}$$

$$V_k(\hat{\varphi}) = \mathbb{E}\left[H - \sum_{m=k+1}^{T} \hat{X}_m \Delta S_m \middle| \mathcal{F}_k \right]. \tag{2.3.49}$$

The classical local risk minimization strategy in the case where we do not account for illiquidity is covered by the case of ε_{k+1} tending to zero. Denote this strategy by $\bar{\varphi} = (\bar{X}, \bar{Y})$. In the case when S is a martingale one can easily see that $V_k(\hat{\varphi}) = \mathbb{E}[H|\mathcal{F}_k] = V_k(\bar{\varphi})$. Hence, the two book values are equal.

On the other hand, in the case of infinite liquidity costs, i.e. ε_{k+1} goes to infinity, then it holds

$$\hat{X}_{k+1} \to \mathbb{E}\left[\frac{S_{k+1} \cdots S_T \hat{X}_{T+1}}{\mathbb{E}[S_{k+1}|\mathcal{F}_k] \cdots \mathbb{E}[S_T|\mathcal{F}_{T-1}]} \middle| \mathcal{F}_k \right]. \tag{2.3.50}$$

Furthermore, consider the special but standard in the market case of a cash settlement where the value of the option has to be paid out in cash, i.e. $\hat{X}_{T+1} = 0$ and $\hat{Y}_T = H$. In this case when $\varepsilon_{k+1} \to \infty$ we get $\hat{X}_{k+1} \to 0$ for all $k = 0, 1, \ldots, T$. Interpreting this from a financial point of view, means that in order to avoid infinite liquidity cost the investor's best choice is to invest nothing, which makes sense. In the d-dimensional case, a similar observation holds as well.

2.3.4 A sufficient condition for the F-property in terms of the covariance matrix F^0

For showing the integrability properties of Proposition 2.2.9 for the optimal local risk-minimizing strategy under illiquidity in the proof of Theorem 2.3.15 recall that we have used the F-property from Definition 2.3.11 backwards in time. This condition is related to the covariance matrix F^0 of the price process S as we show in this section. Let us first recall the definition of a principal submatrix (see Horn and Johnson, 2012) which we will use in this section.

Definition of a principal submatrix: In general let $P \in \mathbb{R}^{m,n}$ be a real matrix with m rows and n columns, and let $\alpha \subset \{1, \ldots, m\}$, $\beta \subset \{1, \ldots, n\}$ be index sets. Denote by $P[\alpha, \beta]$ the (sub)matrix of entries that lie in the rows of P indexed by α and the columns indexed by β. For $\alpha = \beta$ denote by $P[\alpha] = P[\alpha, \alpha]$ the (sub)matrix of entries that lie in the rows and columns of P indexed by α. Then $P[\alpha]$ is called a *principal submatrix* of P.[7]

A sufficient criterion in terms of the covariance matrix F^0 is given in the next Lemma 2.3.17.

Lemma 2.3.17. *S has the F-property if there exists some $\delta \in (0, 1)$ such that*

$$\det(P_k^0) - (1 - \delta) \det(P_k^{A^0}) \geq 0, \tag{2.3.51}$$

for all principal submatrices P_k^0 of F_k^0 and principal submatrices $P_k^{A^0}$ of $F_k^{A^0}$ where $F_k^{A^0} := \operatorname{diag}(A_{k;1}^0, \ldots, A_{k;d}^0)$ of size $l \times l$ where $l \in \{2, \ldots, d\}$ and for all $k = 0, 1, \ldots, T$.

Proof. Let $d \in \mathbb{N}_{\geq 2}$, fix $k \in \{0, 1, \ldots, T\}$ and omitting the time k denote $F = F_k$.

Furthermore we denote by $F^{A_m^0;A_l} := F^{A_m^0;A_l}(A_m^0, A_{m+1}^0, \ldots, A_{l-1}^0, A_{l+1}, A_{l+2}, \ldots, A_d)$ for $m, l \in \{1, \ldots, d\}$, $m < l$, the $(d - m) \times (d - m)$ symmetric matrix where for $i = j$, $j \in \{1, \ldots, l - m\}$ we set $F_{i,j}^{A_m^0;A_l} = A_{m+j-1}^0$ and for $j \in \{l - m, \ldots, d - m - 1\}$ we set $F_{i,j}^{A_m^0;A_l} = A_{m+j+1}$ for the diagonal elements of the matrix. Otherwise for $i \neq j$ we set $F_{i,j}^{A_m^0;A_l} = D_{m+i-1,m+j-1}$ for $i, j \in \{1, \ldots, l - m\}$ and $F_{i,j}^{A_m^0;A_l} = D_{m+i+1,m+j+1}$ for $i, j \in \{l - m, \ldots, d - m - 1\}$. For $m = l$ we set $F^{A_l^0;A_l} := F^{A_l^0;A_l}(A_{l+1}, A_{l+2}, \ldots, A_d)$ which is equal to F without the first l rows and columns. Also note that for $l = d$ we have $F^{A_m^0;A_d} := F^{A_m^0;A_d}(A_m^0, A_{m+1}^0, \ldots, A_{d-1}^0)$ which is equal to F^0 without the first $m - 1$ rows and columns and without the last row and the last column.

Since $A_j = A_j^0 + A_j^\varepsilon$ and using the fact that the matrices F and F^0 are symmetric then one can calculate that

$$\begin{aligned}
&\det(F) - (1 - \delta) \det(F^A) \\
&= \det(F^0) - (1 - \delta) \det(F^{A^0}) \\
&+ A_1^\varepsilon [\det(F^{A_1^0;A_1}(A_2, A_3, A_4, \ldots, A_d)) - (1 - \delta) \det(\operatorname{diag}(A_2, A_3, A_4, \ldots, A_d))] \\
&+ A_2^\varepsilon [\det(F^{A_1^0;A_2}(A_1^0, A_3, A_4, \ldots, A_d)) - (1 - \delta) \det(\operatorname{diag}(A_1^0, A_3, A_4, \ldots, A_d))] \\
&+ A_3^\varepsilon [\det(F^{A_1^0;A_3}(A_1^0, A_2^0, A_4, \ldots, A_d)) - (1 - \delta) \det(\operatorname{diag}(A_1^0, A_2^0, A_4, \ldots, A_d))] \\
&+ \cdots + \\
&+ A_d^\varepsilon [\det(F^{A_1^0;A_d}(A_1^0, A_2^0, A_3^0, \ldots, A_{d-1}^0)) - (1 - \delta) \det(\operatorname{diag}(A_1^0, A_2^0, A_3^0, \ldots, A_{d-1}^0))] ,
\end{aligned} \tag{2.3.52}$$

[7] A matrix $P \in \mathbb{R}^{n,n}$ has $\binom{n}{l}$ distinct principal submatrices of size $l \times l$.

where F^0 is the $\binom{d}{d} = 1$ principal submatrix $P^0[\{1, 2, \ldots, d\}]$ of size $d \times d$ and $F^{A_1^0;A_d}(A_1^0, A_2^0, A_3^0, \ldots, A_{d-1}^0) = P^0[\{1, 2, \ldots, d-1\}]$ one of the $\binom{d}{d-1} = d$ principal submatrices of F^0 of size $(d-1) \times (d-1)$. The remaining $d-1$ principal submatrices of size $(d-1) \times (d-1)$ can be calculated recursively as in equation (2.3.52) for the $d-1$ terms in the right hand side of the equation. For example we have

$$
\begin{aligned}
&A_1^\varepsilon[\det(F^{A_1^0;A_1}(A_2, A_3, A_4, \ldots, A_d)) - (1-\delta)\det(\operatorname{diag}(A_2, A_3, A_4, \ldots, A_d))] \\
&= A_1^\varepsilon\Big\{A_2^\varepsilon[\det(F^{A_2^0;A_2}(A_3, A_4, A_5, \ldots, A_d)) - (1-\delta)\det(\operatorname{diag}(A_3, A_4, A_5, \ldots, A_d))] \\
&\quad + A_3^\varepsilon[\det(F^{A_2^0;A_3}(A_2^0, A_4, A_5, \ldots, A_d)) - (1-\delta)\det(\operatorname{diag}(A_2^0, A_4, A_5, \ldots, A_d))] \\
&\quad + A_4^\varepsilon[\det(F^{A_2^0;A_4}(A_2^0, A_3^0, A_5, \ldots, A_d)) - (1-\delta)\det(\operatorname{diag}(A_2^0, A_3^0, A_5, \ldots, A_d))] \\
&\quad + \cdots + \\
&\quad + A_d^\varepsilon[\det(F^{A_2^0;A_d}(A_2^0, A_3^0, A_4^0, \ldots, A_{d-1}^0)) - (1-\delta)\det(\operatorname{diag}(A_2^0, A_3^0, A_4^0, \ldots, A_{d-1}^0))] \\
&\quad + \det(P^0[\{2, 3, \ldots, d\}]) - (1-\delta)\det(P^{A^0}[\{2, 3, \ldots, d\}])\Big\}. \qquad (2.3.53)
\end{aligned}
$$

Note that $F^{A_2^0;A_d}(A_2^0, A_3^0, A_4^0, \ldots, A_{d-1}^0) = P^0[\{2, 3, \ldots, d-1\}]$ is one of the $\binom{d}{d-2}$ principal submatrices of F^0 of size $(d-2) \times (d-2)$. The remaining $\binom{d}{d-2} - 1$ principal submatrices of size $(d-2) \times (d-2)$ can be calculated recursively in the same way as above.

Continuing the calculation recursively (for each of the terms) we get,

$$
\begin{aligned}
&\det(F) - (1-\delta)\det(F^A) \\
&= \det(P^0[\{1, 2, \ldots, d\}]) - (1-\delta)\det(P^{A^0}[\{1, 2, \ldots, d\}]) \\
&+ A_1^\varepsilon\Big\{A_2^\varepsilon\Big\{\ldots A_{d-3}^\varepsilon\Big\{A_{d-2}^\varepsilon[\det(F^{A_{d-2}^0;A_{d-2}}(A_{d-1}, A_d)) - (1-\delta)\det(\operatorname{diag}(A_{d-1}, A_d))] \\
&\quad + A_{d-1}^\varepsilon[\det(F^{A_{d-2}^0;A_{d-1}}(A_{d-2}^0, A_d)) - (1-\delta)\det(\operatorname{diag}(A_{d-2}^0, A_d))] \\
&\quad + A_d^\varepsilon[\det(F^{A_{d-2}^0;A_d}(A_{d-2}^0, A_{d-1}^0)) - (1-\delta)\det(\operatorname{diag}(A_{d-2}^0, A_{d-1}^0))] \\
&\quad + \det(P^0[\{d-2, d-1, d\}]) - (1-\delta)\det(P^{A^0}[\{d-2, d-1, d\}])\Big\}\ldots\Big\} \\
&\quad + \ldots. \qquad (2.3.54)
\end{aligned}
$$

That means, we have rewritten the term $\det(F) - (1-\delta)\det(F^A)$ into terms of $\binom{d}{l}$ (distinct) principal submatrices P^0 of F^0 of size $l \times l$ where $l \in \{3, \ldots, d\}$. Moreover, we are dealing with the determinants of the 2×2 matrices as follows: for example and since $A_d \geq A_d^0$ we have

$$
\begin{aligned}
&\det(F^{A_{d-2}^0;A_{d-1}}(A_{d-2}^0, A_d)) - (1-\delta)\det(\operatorname{diag}(A_{d-2}^0, A_d)) \\
&= \delta A_{d-2}^0 A_d - |D_{d-2,d}|^2 \\
&\geq \delta A_{d-2}^0 A_d^0 - |D_{d-2,d}|^2 \\
&= \det(P^0[\{d-2, d\}]) - (1-\delta)\det(P^{A^0}[\{d-2, d\}]). \qquad (2.3.55)
\end{aligned}
$$

The same holds analogously for the other 2×2 principal submatrices by the fact that $A_j \geq A_j^0$ for $j = 1, \ldots, d$. So, since $A_j^\varepsilon \geq 0$ for $j = 1, \ldots, d$ and since by assumption the inequality

(2.3.51) holds, then we can estimate

$$
\begin{aligned}
&\det(F) - (1-\delta)\det(F^A) \\
&\geq \det(P^0[\{1,2,\dots,d\}]) - (1-\delta)\det(P^{A^0}[\{1,2,\dots,d\}]) \\
+A_1^\varepsilon\Big\{A_2^\varepsilon\Big\{\dots A_{d-3}^\varepsilon\Big\{&A_{d-2}^\varepsilon[\det(P^0[\{A_{d-1}^0, A_d^0\}]) - (1-\delta)\det(P^{A^0}[\{A_{d-1}^0, A_d^0\}])] \\
&+A_{d-1}^\varepsilon[\det(P^0[\{A_{d-2}^0, A_d^0\}]) - (1-\delta)\det(P^{A^0}[\{A_{d-2}^0, A_d^0\}])] \\
&+A_d^\varepsilon[\det(P^0[\{A_{d-2}^0, A_{d-1}^0\}]) - (1-\delta)\det(P^{A^0}[\{A_{d-2}^0, A_{d-1}^0\}])] \\
&+\det(P^0[\{d-2,d-1,d\}]) - (1-\delta)\det(P^{A^0}[\{d-2,d-1,d\}])\Big\}\dots\Big\} \\
&+\dots \\
&\geq 0. \qquad (2.3.56)
\end{aligned}
$$

That means the quantity $\det(F) - (1-\delta)\det(F^A)$ can be estimated from below by the determinants of principal submatrices by terms as in (2.3.51) of F^0 and so by assumption the claim follows. □

An example when the F-property is fulfilled is presented in the next Proposition 2.3.18.

Proposition 2.3.18. *Assume that the covariance matrix F_k^0 is positive definite at all times $k = 0, 1, \dots, T$ and S^j has independent returns for each $j = 1, \dots, d$. Then the F-property holds.*

Proof. Let $d \in \mathbb{N}_{\geq 2}$.

Fix $k \in \{0, 1, \dots, T\}$. First we introduce the notation $\bar{A}_{k;j}^0 := \mathbb{V}ar(\rho_{k+1}^j)$, $\bar{D}_{k;i,j} := \mathbb{C}ov(\rho_{k+1}^i, \rho_{k+1}^j)$ for $i \neq j$ where $\bar{F}_{k;i,j}^0 = \bar{A}_{k;j}^0$ for $i = j$, $\bar{F}_{k;i,j}^0 = \bar{D}_{k;i,j}$ otherwise. Our aim is to make use of Lemma 2.3.17. For simplicity we omit the time k and denote $F = F_k$.

First note that since the covariance matrix F^0 is positive definite then

$$\det(F^0) > 0 \text{ and } \det(F^{A^0}) > 0\,. \qquad (2.3.57)$$

Now using $\Delta S_{k+1}^j = S_k^j \rho_{k+1}^j$, the fact that ρ_{k+1}^j is independent of $\mathcal{F}_k$ for all $j = 1, \dots, d$, the properties of the determinant and the symmetry of the covariance matrix F^0 we get

$$
\begin{aligned}
\det(F^0) &= |S_k^1|^2 \cdots |S_k^d|^2 \det(\bar{F}^0) > 0, \\
\det(F^{A^0}) &= |S_k^1|^2 \cdots |S_k^d|^2 \det(\bar{F}^{\bar{A}^0}) > 0, \qquad (2.3.58)
\end{aligned}
$$

with the obvious notation $\bar{F}_k^{\bar{A}^0} := diag(\bar{A}_{k;1}^0, \dots, \bar{A}_{k;d}^0)$. Since $S_k^j > 0$, this implies

$$\det(F^0) - (1-\delta)\det(F^{A^0}) \geq 0 \iff \det(\bar{F}^0) - (1-\delta)\det(\bar{F}^{A^0}) \geq 0, \qquad (2.3.59)$$

for $\delta \in (0,1)$. Furthermore, since $\bar{F}^0$ and $\bar{F}^{\bar{A}^0}$ are deterministic matrices with $\det(\bar{F}^0) > 0$ and $\det(\bar{F}^{\bar{A}^0}) > 0$, then

$$\det(\bar{F}^0) - (1-\delta)\det(\bar{F}^{\bar{A}^0}) \geq 0, \qquad (2.3.60)$$

for some $\delta \in (0,1)$. For the 1 principal submatrix of F^0 of size $d \times d$ which is again the matrix F^0 we want to show that

$$\det(F^0) + (1-\delta)\det(F^{A^0}) \geq 0, \tag{2.3.61}$$

which for independent returns and positive marginal price process is equivalent to $\det(\bar{F}^0) + (1-\delta)\det(\bar{F}^{\bar{A}^0}) \geq 0$ as shown in the equivalence relation (2.3.59). So it remains to show that for the all (distinct) $\binom{d}{l}$ principal submatrices P^0 of F^0 of size $l \times l$ where $l \in \{2,\ldots,d-1\}$ we have that $\det(P^0) + (1-\delta)\det(P^{A^0}) \geq 0$ for some $\delta \in (0,1)$. Now using again the fact that F_k^0 is positive definite then we know that each principal submatrix P^0 is positive definite (Horn and Johnson, 2012, Observation 7.1.2). That means

$$\det(P^0) > 0 \text{ and } \det(P^{A^0}) > 0\,. \tag{2.3.62}$$

Since all principal submatrices P^0 of F^0 are covariance matrices, then by the same argumentation (and obvious notation) as above we get $\det(\bar{P}^0) - (1-\delta)\det(\bar{P}^{\bar{A}^0}) \geq 0$ for some $\delta \in (0,1)$ which for independent returns and $S_k^j > 0$ is equivalent to

$$\det(P^0) + (1-\delta)\det(P^{A^0}) \geq 0\,. \tag{2.3.63}$$

Finally, from Lemma 2.3.17 the claim follows. □

Proposition 2.3.19. *Assume that the covariance matrix F_k^0 at all times $k = 0,1,\ldots,T$ is positive definite and S^j has independent increments for each $j = 1,\ldots,d$. Then the F-property holds.*

Proof. Follows by analogous arguments as in Proposition 2.3.18. □

Remark 2.3.20. *In the case when $\varepsilon = 0$, principal submatrices do not need to be considered in Lemma 2.3.17. Indeed, rewriting Lemma 2.3.17, the condition simply reduces to the covariance matrix being such*

$$\det(F^0) - (1-\delta)\det(F^{A^0}) \geq 0, \tag{2.3.64}$$

for some constant $\delta \in (0,1)$.

Remark 2.3.21. *Consider the 2-dimensional case. In the case of independent returns (or increments) the covariance matrix F_k^0 is positive definite [8] if $A_{k;1}^0 A_{k;2}^0 - D_{k;1,2}^2 > 0$, $A_{k;1}^0 > 0$, $A_{k;2}^0 > 0$. This can be ensured by having strict Cauchy-Schwarz inequality. Thus, in the case when S^1 and S^2 are linearly independent, Proposition 2.3.18 can be applied.*

2.3.5 Nonnegative supply curve

Recall that the supply curve can also take negative values for some negative trade sizes. More precisely and by considering the 1-dimensional case for simplicity, the (linear) supply curve $S_k(x) = (1 + x\varepsilon_k)S_k$ is possible to take negative values when a negative transaction x is such

[8] Note that a matrix F is positive definite if and only if its leading principal minors are all positive.

that $x \leq -1/\varepsilon$. A natural question that arises in such a setting is if and how is possible to define a function $h : \mathbb{R} \to \mathbb{R}$ so that the supply curve process

$$S_k(x) = h(x)S_k, \tag{2.3.65}$$

will not produce negative values, hence is nonnegative.

In this section we show how one could construct a nonnegative linear supply curve. Despite that we show this for the 1-dimensional case, an extension to the multidimensional case is straightforward.

Consider for example the function

$$h(x) := (1 + x\varepsilon_k)\mathbf{1}_{\{x \geq -z_k\}} + (1 - z_k\varepsilon_k)\mathbf{1}_{\{x < -z_k\}}, \tag{2.3.66}$$

where $z = (z_k)_{k=0,1,\dots,T}$ is some deterministic positive process with $0 < z_k \leq 1/\varepsilon_k$ for all $k = 0, 1, \dots, T$. Then note that $z_k S_k$ corresponds to a lower bound for the price received when an investor is selling a large quantity of shares.

As a consequence, the corresponding cost process under illiquidity $\hat{C}^b(\varphi) = (\hat{C}^b_k(\varphi))_{k=0,1,\dots,T}$ of a strategy $\varphi = (X, Y)$ is then given by

$$\begin{aligned}\hat{C}^b_k(\varphi) := V_k(\varphi) - \sum_{m=1}^{k} X_m \Delta S_m + \sum_{m=1}^{k} \varepsilon_m S_m |\Delta X_{m+1}|^2 \mathbf{1}_{\{\Delta X_{m+1} \geq -z_m\}} \\ - \sum_{m=1}^{k} z_m \varepsilon_m S_m \Delta X_{m+1} \mathbf{1}_{\{\Delta X_{m+1} < -z_m\}}.\end{aligned} \tag{2.3.67}$$

Furthermore and following the same steps as in Section 2.3.2, then by Proposition 2.2.9 at time k we aim at minimizing the expression

$$\begin{aligned}&\mathbb{V}ar(V_{k+1}(\varphi) - X'_{k+1}\Delta S_{k+1}|\mathcal{F}_k) \\ &\quad + \mathbb{E}[\varepsilon_{k+1} S_{k+1} |X_{k+2} - X'_{k+1}|^2 \mathbf{1}_{\{X_{k+2} - X'_{k+1} \geq -z_{k+1}\}} |\mathcal{F}_k] \\ &\quad - \mathbb{E}[z_{k+1}\varepsilon_{k+1} S_{k+1}(X_{k+2} - X'_{k+1}) \mathbf{1}_{\{X_{k+2} - X'_{k+1} < -z_{k+1}\}} |\mathcal{F}_k],\end{aligned} \tag{2.3.68}$$

over all appropriate X'_{k+1} where w.l.o.g. $\alpha = 1$. Equivalently to the above expression, we want to minimize the function $\hat{f}^b_k : \mathbb{R} \times \Omega \to \mathbb{R}^+$ given by

$$\begin{aligned}\hat{f}^b_k(c, \omega) &= |c|^2 \hat{A}^b_k(\omega) - 2c\hat{b}^b_k(\omega) + c\hat{d}^b_k(\omega) \\ &\quad + \mathbb{V}ar(V_{k+1}|\mathcal{F}_k)(\omega) + \mathbb{E}[\varepsilon_{k+1} S_{k+1} |X_{k+2}|^2 \mathbf{1}_{\{X_{k+2} - c \geq -z_{k+1}\}} |\mathcal{F}_k](\omega) \\ &\quad - \mathbb{E}[z_{k+1}\varepsilon_{k+1} S_{k+1} X_{k+2} \mathbf{1}_{\{X_{k+2} - c < -z_{k+1}\}} |\mathcal{F}_k](\omega),\end{aligned} \tag{2.3.69}$$

where we are making use of the following notation:

$$\begin{aligned}\hat{A}^b_k &= \mathbb{V}ar(\Delta S_{k+1}|\mathcal{F}_k) + \mathbb{E}[\varepsilon_{k+1} S_{k+1} \mathbf{1}_{\{X_{k+2} - c \geq -z_{k+1}\}} |\mathcal{F}_k], \\ \hat{b}^b_k &= \mathbb{C}ov(V_{k+1}, \Delta S_{k+1}|\mathcal{F}_k) + \mathbb{E}[\varepsilon_{k+1} S_{k+1} X_{k+2} \mathbf{1}_{\{X_{k+2} - c \geq -z_{k+1}\}} |\mathcal{F}_k],\end{aligned}$$

$$\hat{d}_k^b = \mathbb{E}[z_{k+1}\varepsilon_{k+1}S_{k+1}\mathbf{1}_{\{X_{k+2}-c<-z_{k+1}\}}|\mathcal{F}_k]. \tag{2.3.70}$$

Finally, by the dominated convergence theorem and using similar arguments and assumptions as in Sections 2.3.2 and 2.3.1, then the equation $\frac{d}{dc}\hat{f}_k^b(c) = 0$ implies that the optimal strategy $\hat{\varphi} = (\hat{X}, \hat{Y})$ fulfills an implicit relation, in particular it holds

$$\hat{X}_{k+1} = \frac{\mathbb{C}ov(V_{k+1}, \Delta S_{k+1}|\mathcal{F}_k) + \mathbb{E}[\varepsilon_{k+1}S_{k+1}\hat{X}_{k+2}\mathbf{1}_{\{\hat{X}_{k+2}-\hat{X}_{k+1}\geq -z_{k+1}\}}|\mathcal{F}_k] - \frac{1}{2}Q}{\mathbb{V}ar(\Delta S_{k+1}|\mathcal{F}_k) + \mathbb{E}[\varepsilon_{k+1}S_{k+1}\mathbf{1}_{\{\hat{X}_{k+2}-\hat{X}_{k+1}\geq -z_{k+1}\}}|\mathcal{F}_k]}, \tag{2.3.71}$$

with

$$Q = \mathbb{E}[z_{k+1}\varepsilon_{k+1}S_{k+1}\mathbf{1}_{\{\hat{X}_{k+2}-\hat{X}_{k+1}<-z_{k+1}\}}|\mathcal{F}_k]. \tag{2.3.72}$$

2.4 Application to Electricity Markets

By applying the previous results in this section we aim at hedging an Asian-style electricity option in an illiquid market with electricity futures. These are supposed to be exposed to liquidity costs. Additionally, these futures, used as hedge instruments, might have different maturities. This means that a future is possible to terminate before the option maturity (final time horizon T). Thus hedging in these instruments is restricted to certain subintervals of $[0, T]$. In the previous sections, it is assumed that hedging is possible until T in all hedge instruments, so this situation described here is not a priori covered by our setting. Subsection 2.4.1 serves the purpose to shortly sketch how a setting with hedge instruments having possibly different maturities can be embedded in our setting from the previous sections such that our previous results apply also to such situations. Then, in Subsection 2.4.2 we focus our example on electricity markets.

2.4.1 Hedge instruments with different maturities

Consider on our stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, d available hedge instruments $S^j = (S_k^j)_{k=0,1,\dots,T_j}$ of nonnegative price processes with maturity $T_j \leq T$, $j = 1, \dots, d$ and assume a final time horizon T. Without loss of generality we let $0 < T_1 \leq T_2 \leq \dots \leq T_d \leq T$. We want to fit in our setting the situation when hedging in asset j is only possible until time $T_j \leq T$, $j = 1, \dots, d$. Thus, by artificially keeping each asset S^j constant on the remaining interval $[T_j, T]$, where the asset j is not defined, we introduce an associated d-dimensional price process $\tilde{S} = (\tilde{S}_k)_{k=0,1,\dots,T}$ given by

$$\tilde{S}_k^j = S_k^j \mathbf{1}_{[0,T_j)}(k) + S_{T_j}^j \mathbf{1}_{[T_j,T]}(k), \tag{2.4.1}$$

for $j = 1, \dots, d$ and $k \in \{0, 1, \dots, T\}$. Furthermore, for the extended price dynamics of the asset processes we also introduce an extended positive liquidity process. More precisely, by considering a positive, deterministic $\mathbb{R}^d$-valued liquidity process $\varepsilon = (\varepsilon_k)_{k=0,1,\dots,T}$, we extend this on the subintervals $m \in [T_j, T]$ for all $j \in \{1, \dots, d\}$ by some $\varepsilon_m^j > 0$.

Since, we assume positive liquidity costs during the extended price dynamics, it is then clear already intuitively that a trader will not choose to invest any money in asset j during the interval $[T_j, T]$. This is due to the fact that in this time frame the asset generates zero gains while at the same time incurring positive liquidity costs. Indeed, using the fact that for $k \geq T_l$ it holds $\Delta \tilde{S}^l_{k+1} = 0$, it is straightforward to see from Proposition 2.2.9, Property (ii), that in such a situation a LRM-strategy under illiquidity must be of the form $\tilde{X}^l_m = 0$ for $m = T_l + 1, \dots, T$, $l \in \{1, \dots, d\}$. From a financial point of view this means that the hedger liquidates his position in the j-th asset at time $T_j + 1$. Hence, in our constructed extended market a LRM-strategy $\tilde{X}$ under illiquidity automatically respects and follows the original hedge constraints beyond maturities T_j, $j \in \{1, \dots, d\}$. This implies that this strategy is also a LRM-strategy under illiquidity in our setting with hedge instruments with possibly different maturities. In the following we will distinguish between active and inactive assets. In particular we say the asset $\tilde{S}^j$ is *active* at time k if $k \leq T_j$ and *inactive* at time k if $k > T_j$.

Now, we aim at showing existence and computing a LRM-strategy under illiquidity for hedge instruments with different maturities using the development tools and following the steps of Section 2.3. We consider the extended linear supply curve $\tilde{S}^j_k(x^j) = \tilde{S}^j_k + x^j \varepsilon^j_k \tilde{S}^j_k$ for the extended price processes. Taking into account the fact that a LRM-strategy $\tilde{X}$ under illiquidity fulfills $\tilde{X}^l_m = 0$ for $m = T_l + 1, \dots, T$, $l \in \{1, \dots, d\}$, the minimization problem at step $k \in \{0, 1, \dots, T-1\}$ of the function f_k in (2.3.5) simplifies to the minimization of the function $\tilde{f}_k : \mathbb{R}^{d-l_k} \times \Omega \to \mathbb{R}^+$ given by (w.l.o.g. $\alpha = 1$)

$$\tilde{f}_k(c, \omega) = \sum_{j=l_k+1}^{d} |c_j|^2 A_{k;j}(\omega) - 2 \sum_{j=l_k+1}^{d} c_j b_{k;j}(\omega) + \sum_{j \neq i, j, i \geq l_k+1}^{d} c_j c_i D_{k;j,i}(\omega) \qquad (2.4.2)$$
$$+ \mathbb{V}ar(V_{k+1}|\mathcal{F}_k)(\omega) + \sum_{j=l_k+1}^{d} \mathbb{E}[\varepsilon^j_{k+1} S^j_{k+1} |X^j_{k+2}|^2 | \mathcal{F}_k](\omega),$$

where note that the sums are only over the assets $\tilde{S}^j$, $j = l+1, ..., d$, which are active during the k'th period, i.e. $l_k := \max\{r \in \{1, \dots, d\} : T_r < k\}$. Then we can deduce that the conditions required in Theorem 2.3.15 for showing existence of a LRM-strategy under illiquidity reduce to lower-dimensional conditions which in each period only the active hedge instruments are being concerned. In particular and following the notation from Section 2.3 we define for each period $k \in \{0, 1, \dots, T-1\}$ the symmetric matrix $\tilde{F}_k \in \mathbb{R}^{d-l_k \times d-l_k}$ (a principal submatrix of F_k) by $\tilde{F}_{k;i,j} = D_{k;i+l_k,j+l_k}$ for $i \neq j$, $\tilde{F}_{k;i,j} = A_{k;j+l_k}$ for $i = j$, $i, j \in \{1, \dots, d-l_k\}$ and $\tilde{b}_k := (b_{k;l_k+1}, \dots, b_{k;d})^* \in \mathbb{R}^{d-l_k}$. Then solving the following linear equation system

$$\tilde{F}_k c = \tilde{b}_k \,, \qquad (2.4.3)$$

is equivalent minimizing over (2.4.2) in $c \in \mathbb{R}^{d-l_k}$. Moreover note that we have a reduced form of the covariance matrix of the price process $\tilde{S}$. Indeed it holds $\tilde{F}_k = \tilde{F}^0_k + \tilde{F}^\varepsilon_k$ where $\tilde{F}^\varepsilon = diag(A^\varepsilon_{k;l+1}, \dots, A^\varepsilon_{k;d})$ and $\tilde{F}^0_k$ is the matrix $\tilde{F}_k$ with $\varepsilon^j_{k+1} = 0$ for $j = l+1, \dots, d$, the reduced covariance matrix.

In the context of hedge instruments with different maturities in our extended liquidity market and following the arguments in Section 2.3, then the following version of Theorem 2.3.15 holds for the existence and explicit representation of a LRM-strategy under illiquidity:

Corollary 2.4.1. *Consider a contingent claim $H = \bar{X}^*_{T+1} S_T + \bar{Y}_T \in \mathbb{L}^{2,1}_T$ with $\bar{X}_{T+1} = 0$ and a price process of the form in equation (2.4.1). Assume that for each k-th period, the covariance matrix $\tilde{F}^0_k$ is positive definite. Furthermore assume that bounded mean-variance tradeoff, the F-property and the F-diagonal condition hold for the active assets in the k-th period at time $k \in \{0, 1, \dots, T-1\}$. Then there exists a LRM-strategy $\hat{\varphi} = (\hat{X}, \hat{Y})$ under illiquidity with $\hat{X}_{T+1} = 0$, $\hat{Y}_T = H$. In particular for $k \in \{0, 1, \dots, T-1\}$ we have $\hat{X} = (\bar{0}, \tilde{X})$ with $\bar{0} = (0, \dots, 0) \in \mathbb{R}^{l_k}$ and*

$$\tilde{X}_{k+1} = \tilde{F}^{-1}_k \tilde{b}_k \quad \mathbb{P}-a.s., \tag{2.4.4}$$

in $\mathbb{R}^{d-l_k}$ and for $k \in \{0, 1, \dots, T-1\}$

$$\hat{Y}_k = \mathbb{E}[\hat{W}_k | \mathcal{F}_k] - \hat{X}^*_{k+1} \tilde{S}_k \quad \mathbb{P}-a.s., \tag{2.4.5}$$

where $\hat{W}_k = H - \sum_{m=k+1}^{T} \hat{X}^*_m \Delta \tilde{S}_m$.

2.4.2 LRM strategies in electricity markets

For the remaining next parts of the section we are considering an even more explicit situation in an extended market in the sense of the previous Subsection 2.4.1. In particular, we are going to deal now with an example from electricity markets of hedging an Asian-style electricity option with electricity futures under liquidity costs by a LRM-strategy under illiquidity. Based on a continuous-time multi-factor spot price model proposed in the paper of Benth et al. (2007) we are considering price processes for electricity futures. In Subsection 2.4.2.1 we recall and describe this particular model while in the following Subsection 2.4.2.2 we explicitly compute and simulate LRM-strategies under illiquidity in a more specific example.

2.4.2.1 An electricity market model

The price $E(t)$ of spot electricity at time $t \in [0, T]$ is modeled in Benth et al. (2007) by the equation

$$E(t) = \sum_{i=1}^{n} \Lambda_i(t) Y_i(t), \tag{2.4.6}$$

where for $i = 1, \dots, n$ it is assumed that Λ_i is the positive and deterministic function which accounts for seasonality and Y_i is the solution to an Ornstein-Uhlenbeck stochastic differential equation

$$dY_i(t) = -\lambda_i Y_i(t) dt + \sigma_i(t) dL_i(t) \quad , \quad Y_i(0) = y_i, \tag{2.4.7}$$

where $\sigma_i(t)$ are assumed deterministic, positive bounded functions and $\lambda_i > 0$ are constants. Furthermore, let the L_i's be independent, increasing pure jump Lévy processes with jump measures $N_i(dt, dz)$ which have deterministic predictable compensators of the form $\nu_i(dt, dz) = dt\nu_i(dz)$. The positivity of the processes Y_i's and hence also of the spot price E is ensured through the increasing property of the L_i's. Moreover, let $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$ be a stochastic basis where the model (2.4.6) is defined and assume that the filtration $(\mathcal{F}_t)_{0 \le t \le T}$ is generated by the L_i's.

Electricity futures are known to have the flow character of electricity which instead of delivering spot electricity at a fixed point in time they rather deliver over a delivery period $[T_1^F, T_2^F]$ for $T_1^F < T_2^F \leq T$. Considering such futures as available hedge instruments, the pay-off of the (financially settled) futures at the end of the delivery period is then

$$\frac{1}{T_2^F - T_1^F} \int_{T_1^F}^{T_2^F} E(u)du\,, \tag{2.4.8}$$

and the life of the asset terminates at T_2^F. For computing further the price dynamics of electricity futures we let for simplicity the measure $\mathbb{P}$ be already an equivalent martingale measure. This implies that the price $F(t; T_1^F, T_2^F)$ of the electricity futures at time $t \leq T_2^F$ as a traded asset is given by the equation

$$F(t; T_1^F, T_2^F) = \mathbb{E}\left[\frac{1}{T_2^F - T_1^F} \int_{T_1^F}^{T_2^F} E(u)du \middle| \mathcal{F}_t\right]. \tag{2.4.9}$$

A straightforward computation of the conditional expectation in (2.4.9) by using the fact that

$$Y_i(u) = Y_i(t)e^{-\lambda_i(u-t)} + \int_t^u \sigma_i(s)e^{-\lambda_i(u-s)}dL_i(s), \tag{2.4.10}$$

is the explicit solution for the Ornstein-Uhlenbeck components Y_i, $i = 1, ...n$, implies the following more explicit price of futures contracts in the continuous-time spot model :

Proposition 2.4.2. *The price $F(t, T_1^F, T_2^F)$ at time t of an electricity futures with delivery period $[T_1^F, T_2^F]$ is given by*

$$\begin{aligned} F(t, T_1^F, T_2^F) = {} & \sum_{i=1}^n Y_i(t) \frac{1}{T_2^F - T_1^F} \int_{T_1^F}^{T_2^F} \Lambda_i(u)e^{-\lambda_i(u-t)}du \\ & + \frac{1}{T_2^F - T_1^F} \int_{T_1^F}^{T_2^F} \int_t^u \int_{\mathbb{R}^+} \sigma_i(s)\Lambda_i(u)e^{-\lambda_i(u-s)} z\nu_i(dz)dsdu, \end{aligned} \tag{2.4.11}$$

for $0 \leq t \leq T_1^F$, and

$$\begin{aligned} F(t, T_1^F, T_2^F) = {} & \frac{1}{T_2^F - T_1^F} \int_{T_1^F}^{t} E(u)du + \sum_{i=1}^n Y_i(t) \frac{1}{T_2^F - T_1^F} \int_t^{T_2^F} \Lambda_i(u)e^{-\lambda_i(u-t)}du \\ & + \frac{1}{T_2^F - T_1^F} \int_t^{T_2^F} \int_t^u \int_{\mathbb{R}^+} \sigma_i(s)\Lambda_i(u)e^{-\lambda_i(u-s)} z\nu_i(dz)dsdu, \end{aligned} \tag{2.4.12}$$

for $T_1^F \leq t \leq T_2^F$.

In order to fit this continuous-time spot and futures price model into our discrete-time framework we construct an electricity market model by sampling the continuous-time processes at finitely many trading times $0 = t_0, t_1, ..., T$. That means, our hedge instruments S^j, $j = 1, ...d$ are now given by futures price processes of the following form :

$$S_k^j := F^j(t_k, T_1^{F^j}, T_2^{F^j}) \quad \text{for} \quad 0 \leq t_k \leq T_2^{F^j} \leq T\,. \tag{2.4.13}$$

For simplicity we will assume in the following that delivery period times belong to the discrete time grid, i.e. $T_1^F, T_2^F \in \{t_0, t_1, ..., T\}$. As already indicated, the futures contract continues to exist also after the maturity time T_2^F and investing is not possible anymore. In general, depending on the conventions and rules of the exchange, during the delivery period $[T_1^F, T_2^F]$, trading is either very illiquid and thus restricted or not possible at all. This feature is captured in our setting by specifying high liquidity costs during the period $[T_1^F, T_2^F]$, where the limit case of liquidity costs tend to infinity means that trading is impossible. Another feature that one typically observes on electricity markets is the fact that the shorter the remaining time to delivery period the more liquid becomes the future. This behavior is captured by the following definition of the liquidity structure ε^j for the futures F^j, $j = 1, ...d$:

$$\begin{aligned} &\varepsilon_t^j = a_j(1 - \exp(-(T_1^{F^j} - t))) + \delta_j, \qquad a_j = M_j \frac{1}{1 - \exp(-T_1^{F^j})} \qquad \text{for } 0 \leq t \leq T_1^{F^j}\,, \\ &\varepsilon_t^j = N_j \quad \text{for } T_1^{F^j} < t \leq T_2^{F^j}\,. \end{aligned} \tag{2.4.14}$$

Observe that at time 0, the liquidity structure ε^j for a future F^j thus starts from some constant $M_j > 0$. Then it moves down to a level $\delta_j > 0$ by decreasing exponentially in time until the start of the delivery period. Afterwards and during the delivery period it jumps to a constant (high) level $N_j > 0$ which as explained above, it captures the feature of very high liquidity costs of the future.

Moreover, we additionally consider a constant liquidity structure given by

$$\varepsilon_t^j = M_j \quad \text{for } 0 \leq t \leq T_1^{F^j}\,, \qquad \varepsilon_t^j = N_j \quad \text{for } T_1^{F^j} < t \leq T_2^{F^j}\,, \tag{2.4.15}$$

for $M_j > 0$ and $N_j > 0$. In our simulation study in the next Subsection 2.4.2.2 we compare the time varying liquidity structure in (2.4.14) with the constant one in (2.4.15).

2.4.2.2 LRM-strategies of electricity call options

Based on the electricity market model previously specified in Subsection 2.4.2.1, we are now considering a financially settled Asian call option which is written on an electricity future with delivery period $[T_1^c, T_2^c]$ for $0 < T_1^c < T_2^c \leq T$. We intend to compute explicitly a LRM-strategy under illiquidity for a given claim $H = \bar{Y}_T$ where

$$\bar{Y}_T = \left(\frac{1}{T_2^c - T_1^c} \int_{T_1^c}^{T_2^c} E(u)du - K \right)^+\,, \tag{2.4.16}$$

for some given strike price K. Moreover, the assumption $T_2^c = T$ will always hold from now on, i.e. the option maturity is equal to the terminal time horizon.

Consider two different futures F^1, F^2 with corresponding delivery periods $[T_1^{F^1}, T_2^{F^1}]$ and $[T_1^{F^2}, T_2^{F^2}]$, respectively, where we further specify $T_2^{F^1} \leq T_2^{F^2} \leq T$ and $T_1^{F^1} \neq T_1^{F^2}$.[9] Our purpose is to analyze the case of a trader which can hedge in two different futures by also comparing various specifications. The existence of a LRM-strategy under illiquidity for this case is guaranteed by Corollary 2.4.1. In particular, from the property of independent increments of the futures, it follows from Proposition 2.3.7 that both F-diagonal condition and the bounded mean-variance tradeoff hold for the active assets in each period. Furthermore, from Proposition 2.3.19 and in combination with Remark 2.3.21, it remains to check for the active hedge instruments F^1 and F^2 if the conditional Cauchy-Schwarz-Inequality is strict, that means if the inequality

$$\mathbb{C}ov(\Delta F_{k+1}^1, \Delta F_{k+1}^2 | \mathcal{F}_k)^2 < \mathbb{V}ar(\Delta F_{k+1}^1 | \mathcal{F}_k) \mathbb{V}ar(\Delta F_{k+1}^2 | \mathcal{F}_k)\,, \tag{2.4.17}$$

holds for each $k \in \{0, ..., T_2^{F^1}\}$. This will further ensure two properties: the F-property and the existence of the inverse matrix $\tilde{F}_k^{-1}$. By the fact that $T_1^{F^1} \neq T_1^{F^2}$ the CS-inequality is indeed strict which implies that $\mathbb{P}(F_{k+1}^1 = aF_{k+1}^2) < 1$ for any constant $a \in \mathbb{R}$.[10] Finally, from Corollary 2.4.1 we know that a LRM-strategy $\hat{\varphi} = (\hat{X}, \hat{Y})$ under illiquidity exists and is of the form $\hat{X}_{T+1} = 0$, $\hat{Y}_T = H$ and $\hat{X} = (\bar{0}, \tilde{X})$ with $\bar{0} = (0, \ldots, 0) \in \mathbb{R}^{l_k}$ and

$$\tilde{X}_{k+1} = \tilde{F}_k^{-1} \tilde{b}_k \quad \mathbb{P}-\text{a.s.}, \tag{2.4.18}$$

in $\mathbb{R}^{d-l_k}$ for $k \in \{0, \ldots, T-1\}$. Furthermore, note that the structure of the matrix $\tilde{F}_k^{-1}$ is 2×2-dimensional for $k \in \{0, \ldots, T_2^{F^1} - 1\}$ and 1-dimensional for $k \in \{T_2^{F^1}, \ldots, T_2^{F^2} - 1\}$ because of the active assets structure in each period.

The optimal strategy $\tilde{X}$ consist of conditional expectations of the form $\mathbb{E}[Y|X]$. For computing numerically such conditional expectations for square integrable random variables X and Y a popular method from the literature is the so called *least-squares Monte Carlo* (LSMC) method. This method was first used by Longstaff and Schwartz (2001) in finance and in particular for the valuation of American options. We will employ this method in the following for our simulation. Since it can be found in the literature we will not explain or go into further details but we just refer to Fries (2007) for a nice introduction regarding the LSMC method. For our simulation study we want to mention that we use indicator functions constructed via the *binning* method as basis functions.

Now, for our example in the 2-dimensional case we need to simulate

$$\begin{aligned}
&\tilde{X}_{T+1} = 0, \\
&\tilde{X}_{k+1} = \frac{1}{A_{k;2}} b_{k;2} \quad \text{for } k \in \{T_2^{F^1}, \ldots, T_2^{F^2} - 1\}, \\
&\tilde{X}_{k+1} = (\tilde{X}_{k+1}^1, \tilde{X}_{k+1}^2) \quad \text{for } k \in \{0, \ldots, T_2^{F^1} - 1\}, \quad \text{where}
\end{aligned}$$

[9] Note that, for the conditional Cauchy-Schwarz inequality to be strict, basically one needs that either $T_2^{F^1} \neq T_2^{F^2}$ or $T_1^{F^1} \neq T_1^{F^2}$. See Remark 2.3.21.

[10] That means, with a positive probability both futures are linearly independent.

$$\tilde{X}^1_{k+1} = \frac{1}{A_{k;1}A_{k;2} - |D_{k;1,2}|^2}(A_{k;2}b_{k;1} - D_{k;1,2}b_{k;2}),$$
$$\tilde{X}^2_{k+1} = \frac{1}{A_{k;1}A_{k;2} - |D_{k;1,2}|^2}(A_{k;1}b_{k;2} - D_{k;1,2}b_{k;1})\,. \tag{2.4.19}$$

The square integrability of all random variables in the conditional expectations it is an important property for the LSMC method. Based mostly on Lemma 2.3.13, we are able to show the next Corollary 2.4.3 below, which ensures the square integrability property and thus we are able to implement the LSMC method. We are going to use the notation of Section 2.4.1. Recall that the price process $\tilde{S} = (\tilde{S}^1, \dots, \tilde{S}^d)$ is the one of the (extended) hedge instruments.

Corollary 2.4.3. *Assume that the components of the marginal price process $\tilde{S}$ and the contingent claim H are both in $\mathbb{L}^{4,1}_T$ as well as $\bar{X}_{T+1} = 0$. Under the assumptions of Corollary 2.4.1 there exists a LRM-strategy $\hat{\varphi} = (\hat{X}, \hat{Y})$ under illiquidity such that for some constant $C > 0$*

$$\mathbb{E}[((\tilde{F}_k^{-1}\tilde{b}_k)_j \Delta\tilde{S}^j_{k+1})^4] \leq C(\mathbb{E}|V_{k+1}(\hat{\varphi})|^4 + \sum_{i=1}^d \mathbb{E}|\hat{X}^i_{k+2}|^4), \tag{2.4.20}$$

$$\mathbb{E}[((\tilde{F}_k^{-1}\tilde{b}_k)_j)^4] \leq C(\mathbb{E}|V_{k+1}(\hat{\varphi})|^4 + \sum_{i=1}^d \mathbb{E}|\hat{X}^i_{k+2}|^4), \tag{2.4.21}$$

for $k \in \{0, 1, \dots, T-1\}$ where $V_{k+1}(\hat{\varphi}) = \mathbb{E}[H - \sum_{m=k+2}^T \hat{X}^*_m \Delta\tilde{S}_m | \mathcal{F}_{k+1}]$. In particular, all random variables in the conditional expectations in the terms $A_{k,j}$, $b_{k,j}$ and $D_{k,j,i}$ are square integrable for all $j = l_k + 1, \dots, d$ and $k = 0, 1, \dots, T-1$.

Proof. The existence of a LRM-strategy $\hat{\varphi} = (\hat{X}, \hat{Y})$ under illiquidity follows directly from Corollary 2.4.1. The fact that $V_{k+1}(\hat{\varphi}) = \mathbb{E}[H - \sum_{m=k+2}^T \hat{X}^*_m \Delta\tilde{S}_m | \mathcal{F}_{k+1}]$ follows also directly from $\hat{Y}_k$ defined as in Corollary 2.4.1.

By Lemma 2.3.13 together with Lemma 2.3.12 applied for the active assets at time $k \in \{0, 1, \dots, T-1\}$, we get

$$\mathbb{E}[((\tilde{F}_k^{-1}\tilde{b}_k)_j \Delta\tilde{S}^j_{k+1})^4] \leq C\mathbb{E}[(\mathbb{V}ar(V_{k+1}(\hat{\varphi})|\mathcal{F}_k) + \sum_{i=1}^d \mathbb{E}[|\hat{X}^i_{k+2}|^2|\mathcal{F}_k])^2]\,. \tag{2.4.22}$$

Furthermore, using $\mathbb{V}ar[X] \leq \mathbb{E}[X^2]$ we can estimate,

$$\begin{aligned}
\mathbb{E}[((\tilde{F}_k^{-1}\tilde{b}_k)_j \Delta\tilde{S}^j_{k+1})^4] &\leq C\mathbb{E}[(\mathbb{E}[|V_{k+1}(\hat{\varphi})|^2|\mathcal{F}_k] + \sum_{i=1}^d \mathbb{E}[|\hat{X}^i_{k+2}|^2|\mathcal{F}_k])^2] \\
&\leq C\mathbb{E}[\mathbb{E}[|V_{k+1}(\hat{\varphi})|^4|\mathcal{F}_k] + \sum_{i=1}^d \mathbb{E}[|\hat{X}^i_{k+2}|^4|\mathcal{F}_k]] \\
&= C(\mathbb{E}|V_{k+1}(\hat{\varphi})|^4 + \sum_{i=1}^d \mathbb{E}|\hat{X}^i_{k+2}|^4),
\end{aligned} \tag{2.4.23}$$

where for the last inequality we have used the conditional Jensen Inequality and for the equality we have applied the tower property. Analogously we also get the second inequality of the claim.

This shows that,

$$\mathbb{E}[(\hat{X}^j_{k+1}\Delta\tilde{S}^j_{k+1})^4] \le C(\mathbb{E}|V_{k+1}(\hat{\varphi})|^4 + \sum_{i=1}^{d}\mathbb{E}|\hat{X}^i_{k+2}|^4), \tag{2.4.24}$$

$$\mathbb{E}[(\hat{X}^j_{k+1})^4] \le C(\mathbb{E}|V_{k+1}(\hat{\varphi})|^4 + \sum_{i=1}^{d}\mathbb{E}|\hat{X}^i_{k+2}|^4), \tag{2.4.25}$$

for all $k = 0, 1, \ldots, T-1$, $j = l_k + 1, \ldots, d$. By the definition of $V_{k+1}(\hat{\varphi})$ and since by assumption $H \in \mathbb{L}^{4,1}_T$ and $\hat{X}_{T+1} = 0$, one can argue recursively that both $\hat{X}^j_{k+1}\Delta\tilde{S}^j_{k+1}$ and $\hat{X}^j_{k+1}$ are in $\mathbb{L}^{4,1}_T$.

Furthermore, we have for some $j \in \{l_k+1, \ldots, d\}$ at time $k \in \{0, 1, \ldots, T-1\}$ for the term

$$b^0_{k;j} = \mathbb{C}ov(V_{k+1}(\hat{\varphi}), S^j_{k+1}|\mathcal{F}_k) = \mathbb{E}[V_{k+1}S^j_{k+1}|\mathcal{F}_k] - \mathbb{E}[V_{k+1}|\mathcal{F}_k]\mathbb{E}[S^j_{k+1}|\mathcal{F}_k], \tag{2.4.26}$$

that $V_{k+1}(\hat{\varphi}) \in \mathbb{L}^{2,1}_T$, $S^j_{k+1} \in \mathbb{L}^{2,1}_T$ and $V_{k+1}(\hat{\varphi})S^j_{k+1} \in \mathbb{L}^{2,1}_T$ since $V_{k+1}(\hat{\varphi}) \in \mathbb{L}^{4,1}_T$, $\tilde{S}^j_{k+1} \in \mathbb{L}^{4,1}_T$ and by the Cauchy-Schwarz inequality. For the term

$$b^{\varepsilon}_{k;j} = \mathbb{E}[\varepsilon^j_{k+1}S^j_{k+1}\hat{X}^j_{k+2}|\mathcal{F}_k], \tag{2.4.27}$$

we have $S^j_{k+1}\hat{X}^j_{k+2} \in \mathbb{L}^{2,1}_T$ since $\tilde{S}^j_{k+1} \in \mathbb{L}^{4,1}_T$, $\hat{X}^j_{k+2} \in \mathbb{L}^{4,1}_T$ and using the Cauchy-Schwarz inequality.

So, all random variables in the conditional expectations for the term $b_{k;j}$ are square integrable. Analogously the same holds for the terms $A_{k;j}$ and $D_{k;j,i}$. □

We are now ready to give a concrete specification of our electricity market model for the simulation study that follows. For our model setting, we let the spot price model (2.4.6) be such that is driven by two OU factors ($n = 2$). The first factor Y_1 represents the base regime while the second one Y_2 the spike regime with strong upward moves followed by quick reversion to normal levels. We set the seasonality function equal to a constant, in particular $\Lambda_1 = \Lambda_2 = 1$. Moreover we assume that $Y_1(0) = Y_2(0) = 0.5$, constant volatilities $\sigma^1 = 0.34, \sigma^2 = 0.01$ and $\lambda_1 = 0.01, \lambda_2 = 0.1$ for the mean reversion rates. We are giving the following characterizations for the driving Lévy processes: L_1 is supposed to be a Gamma process where $L_1(t)$ has $\Gamma(\gamma^1 t, \alpha^1)$-distribution and L_2 is assumed to be a compound Poisson process with intensity γ^2 and $\exp(\alpha^2)$-distributed jumps. Finally we set the parameters $\gamma^1 = \gamma^2 = \alpha^1 = 1$, $\alpha^2 = 0.1$. The Euler Scheme is used for the simulation of both OU-processes.[11]

To this end, an equal concern between the risk from market price fluctuations and the cost of liquidity will be considered in our example by assuming $\alpha = 1$ in the performance criterium (2.2.10). Furthermore, let the strike price in (2.4.16) be equal to $K = 1.05$.

[11] Note that for the use of the Least-squares Monte Carlo method, for the purpose of calculating conditional expectations for the simulation, a 2-dimensional basis functions are needed to be simulated using both Markov processes L^1 and L^2.

In our simulation study we will consider two different settings for hedging a call option, each with various pairs of futures with different delivery periods as available hedge instruments. We consider *various combinations of futures that cover the delivery period* $[T_1^c, T_2^c]$ *of the option* in our first simulation setting for hedging the option. For this purpose we assume three futures F^1, F^2, F^3 having delivery periods $[T_1^{F^1}, T_2^{F^1}]$, $[T_1^{F^2}, T_2^{F^2}]$, $[T_1^{F^3}, T_2^{F^3}]$, respectively, where we define $T_1^c = T_1^{F^1} = T_1^{F^2} = 0.0125$, $T_2^c = T_2^{F^2} = T_2^{F^3} = 0.1$, $T_2^{F^1} = T_1^{F^3} = 0.05$. Both time liquidity structures are being considered. For the time varying liquidity structure (2.4.14), we set $M_i = 0.005$, $N_i = 2M_i$, $\delta_i = 0.000001$ and for the constant liquidity structure (2.4.15), we choose $M_i = N_i = 0.01$ for $i = 1, 2, 3$.

For LRM-strategies $\varphi = (X, Y)$ the following criteria are being computed: $T_0(\varphi)$, $\tilde{T}_0(\varphi)$, $L_0(\varphi)$, and $C_0(\varphi)$. Note that $\tilde{T}_0(\varphi) = \mathbb{E}[(C_T(\varphi) - C_0(\varphi))^2]$ is the quadratic hedge criterion, $L_0(\varphi) = \mathbb{E}[\sum_{m=1}^{T} \Delta X_{m+1}^* [S_m(\Delta X_{m+1}) - S_m(0)]]$ the liquidity costs incurred by the strategy, $T_0(\varphi) = \tilde{T}_0(\varphi) + L_0(\varphi)$ our combined LRM minimization (optimality) criterion (2.2.9), and the cost for a strategy φ at time 0 is $C_0(\varphi) = \mathbb{E}[H - \sum_{m=1}^{T} (X_m)^* \Delta S_m]$. In the following Tables 2.4.1 and 2.4.2 the simulation results for a LRM-strategy $\varphi^L = (X^L, Y^L)$ are displayed with time varying liquidity (2.4.14) and constant liquidity (2.4.15), respectively. For comparison purposes we additionally compute the results also with the classical LRM-strategy $\varphi^C = (X^C, Y^C)$ where the liquidity costs are zero (i.e., $\varepsilon^i = 0$) in the classical case. At this point we want to mention and recall that the quantity T_0 is minimized by φ^L while $\tilde{T}_0$ is minimized by φ^C. We use the same trajectories in both cases for comparison purposes.

One first observation that one can make from the simulation results is that the hedging costs together with the corresponding minimization criterion decrease in the number of available hedge instruments. Additionally, note also that the initial cost of the strategy φ^L is more than the one of φ^C. This is due to the fact that the optimal strategy φ^L under liquidity costs will cost more for the trader to generate it.

Using two main examples we focus on the hedge performance with two futures in the case where they cover the delivery period of the option. In our first main example the delivery periods of the futures F^1, F^2 are overlapping, while in the second one, the delivery periods of the futures F^1, F^3 (see Figure 2.4.1b) are different. Observing the Tables 2.4.1 and 2.4.2 and by comparing the quantity $T_0(\varphi^L)$ we can conclude that for the case where the futures F^1, F^3 are used for hedging, the performance is better since they incur less cost. In Table 2.4.2 with the time-varying liquidity results, this can be justify by the fact that F^3 has shorter delivery period than F^2 and can be used for hedging the option longer in time. In the case with constant liquidity, from the results in Table 2.4.1 we see again that it is better to hedge with the two hedge instruments F^1 and F^3. Note that they perform better despite that the future F^2 has a delivery period which coincides perfect with the option H. The same holds also in the classical case, simply by the increased dimension of the hedge instruments. Indeed, from observing the quantity $\tilde{T}_0(\varphi^C)$ one can conclude that under the classical LRM-criterion, the classical LRM-strategy using the futures F^1 and F^3 has a better performance.

Another observation that we can make in our simulation study from the two examples in Tables 2.4.1 and 2.4.2 is the balance between low liquidity costs against poor replication that our quadratic criterion gives. As a matter of fact, from the example, the futures F^1, F^3 have a better performance than F^1, F^2 with less cost $\tilde{T}_0(\varphi^L)$ from market fluctuations but at the same time incurring more liquidity cost $L_0(\varphi^L)$.

The numerical results of Agliardi and Gençay (2014) and Rogers and Singh (2010) show that the optimal strategy under illiquidity is less volatile than the classical one. This is confirmed also in our case in Figure 2.4.1a which corresponds to the result for F^2 in Table 2.4.1. We understand this by the fact that in an illiquid market changing position drastically incurs large liquidity cost which is perfectly intuitive. Furthermore, in Figure 2.4.1b, note that both futures are used actively before the start of the delivery periods while afterwards when entering into the delivery period of F^1 then the trader hedge almost only with the future F^3. This is due to the fact that F^3 is more liquid than F^1 and additionally it expires later.

We now turn our focus in the second setting where in various hedge constellations we consider *trade-off between liquidity costs and hedging performance*. For this purpose assume three futures G^1, G^2, G^3 as hedge instruments having delivery periods $[T_1^{G^1}, T_2^{G^1}]$, $[T_1^{G^2}, T_2^{G^2}]$, $[T_1^{G^3}, T_2^{G^3}]$, respectively, where we set $T_1^c = T_1^{G^1} = T_1^{G^2} = T_2^{G^3} = 0.05$, $T_2^c = T_2^{G^2} = 0.1$, $T_2^{G^1} = 0.075$, $T_1^{G^3} = 0.0125$. Otherwise, we are considering the same model specifications as in the first setting above. We are dealing with two examples, with one common future G^2, which has delivery period coinciding with the one of the option H. According to the quantity $T_0(\varphi^L)$ from Tables 2.4.3 and 2.4.4, note that G^1, G^2 performs better than G^2, G^3. From $\tilde{T}_0(\varphi^C)$ we can observe that in the classical setting this is also the case. This is mostly because of the fact that by comparing the two futures G^1 and G^3 we see that G^1 expires later and its delivery period lies within the delivery period of the option. Moreover and by comparing the quantity $T_0(\varphi^L)$ of both examples note that the difference between them becomes less in Table 2.4.4 than in Table 2.4.3. The reason is based on the liquidity costs. In the period $[0, 0.0125]$ note that G^3 is more liquid than G^1 and hence can be used for hedging at low liquidity cost by the trader in this case. So, a correct specification of the term-structure of liquidity seems therefore not only meaningful but also important. For both cases we illustrate the strategies in Figure 2.4.2 and Figure 2.4.3 for one single trajectory. In the case with time dependent liquidity one can actually observe in Figure 2.4.3b that due to liquidity reasons in the period $[0, 0.0125]$, G^3 is the more active hedge instrument since the future G^2 will incur more liquidity cost.

Hedging Instruments	$T_0(\varphi^L)$	$T_0(\varphi^C)$	$\tilde{T}_0(\varphi^L)$	$\tilde{T}_0(\varphi^C)$	$L_0(\varphi^L)$	$L_0(\varphi^C)$	$C_0(\varphi^L)$	$C_0(\varphi^C)$
F^2	2.19E-3	4.79E-2	2.03E-3	3.40E-4	1.56E-4	4.76E-2	1.09E-2	9.29E-3
F^1, F^2	1.86E-3	3.64E-2	1.67E-3	2.92E-4	1.88E-4	3.61E-2	1.07E-2	9.19E-3
F^1, F^3	1.51E-3	1.59E-2	1.31E-3	2.20E-4	2.01E-4	1.57E-2	1.05E-2	8.92E-3

Table 2.4.1: Simulation results with constant liquidity parameter.

Hedging Instruments	$T_0(\varphi^L)$	$T_0(\varphi^C)$	$\tilde{T}_0(\varphi^L)$	$\tilde{T}_0(\varphi^C)$	$L_0(\varphi^L)$	$L_0(\varphi^C)$	$C_0(\varphi^L)$	$C_0(\varphi^C)$
F^2	1.63E-3	4.11E-2	1.49E-3	3.40E-4	1.40E-4	4.08E-2	1.05E-2	9.29E-3
F^1, F^2	1.56E-3	3.58E-2	1.35E-3	2.92E-4	2.10E-4	3.55E-2	1.04E-2	9.19E-3
F^1, F^3	7.09E-4	1.28E-2	4.50E-4	2.20E-4	2.59E-4	1.26E-2	9.66E-3	8.92E-3

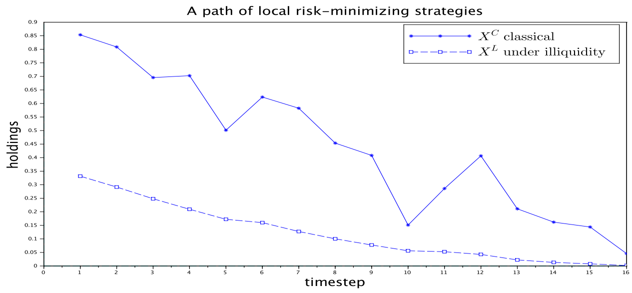
Table 2.4.2: Simulation results with time varying liquidity parameter.

Hedging Instruments	$T_0(\varphi^L)$	$T_0(\varphi^C)$	$\tilde{T}_0(\varphi^L)$	$\tilde{T}_0(\varphi^C)$	$L_0(\varphi^L)$	$L_0(\varphi^C)$	$C_0(\varphi^L)$	$C_0(\varphi^C)$
G^2	3.22E-3	2.30E-2	2.99E-3	7.75E-4	2.28E-4	2.23E-2	1.60E-2	1.41E-2
G^1, G^2	2.33E-3	8.03E-3	2.06E-3	5.21E-4	2.68E-4	7.51E-3	1.55E-2	1.39E-2
G^2, G^3	2.95E-3	1.52E-2	2.69E-3	7.12E-4	2.55E-4	1.45E-2	1.58E-2	1.40E-2

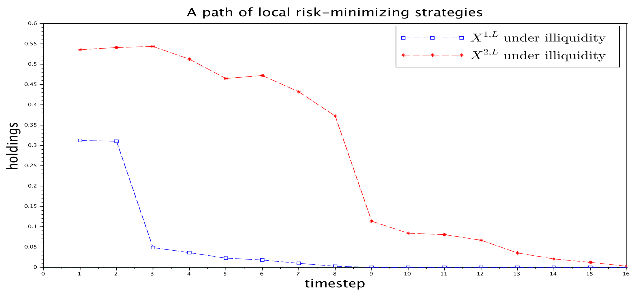
Table 2.4.3: Simulation results with constant liquidity parameter.

Hedging Instruments	$T_0(\varphi^L)$	$T_0(\varphi^C)$	$\tilde{T}_0(\varphi^L)$	$\tilde{T}_0(\varphi^C)$	$L_0(\varphi^L)$	$L_0(\varphi^C)$	$C_0(\varphi^L)$	$C_0(\varphi^C)$
G^2	1.66E-3	1.45E-2	1.49E-3	7.75E-4	1.69E-4	1.37E-2	1.50E-2	1.41E-2
G^1, G^2	1.32E-3	4.64E-3	1.13E-3	5.21E-4	1.92E-4	4.12E-3	1.47E-2	1.39E-2
G^2, G^3	1.63E-3	1.25E-2	1.39E-3	7.12E-4	2.39E-4	1.18E-2	1.49E-2	1.40E-2

Table 2.4.4: Simulation results with time varying liquidity parameter.

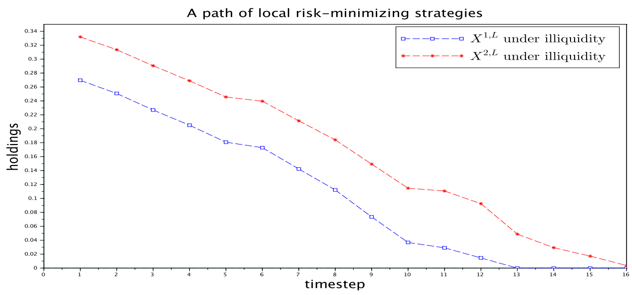


(a) Hedging with only the Future F^2 which has the same delivery period as the claim H. The hedging strategy X^C corresponds to the classical case without liquidity cost and X^L to the case with constant liquidity structure (2.4.15) with parameters $M_2 = N_2 = 0.01$. Observe that the optimal LRM-strategy X^L under illiquidity is less volatile than the classical LRM-strategy X^C.

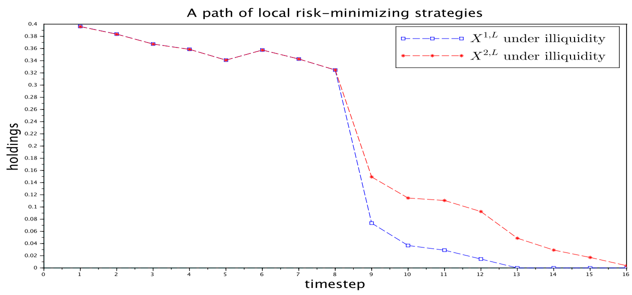


(b) Hedging with the two futures F^1 and F^3 with consecutive delivery periods which together cover the delivery period of the claim H. Both futures have a time-varying liquidity structure (2.4.14) with parameters $M_1 = M_3 = 0.005, N_1 = 2M_1, N_3 = 2M_3$. The optimal LRM-strategy $X^{1,L}$ under illiquidity corresponds to the future F^1 and $X^{2,L}$ to the future F^3 which is more liquid and expires later than F^1. F^3 is used more actively as can be observed from the plot. In the delivery period both futures become very illiquid and thus a rapid drop in the holdings can be observed.

Figure 2.4.1: Comparison of the sample path of optimal LRM-strategies under different liquidity structures and for different hedge instruments. All plots based on the same realization of the underlying.

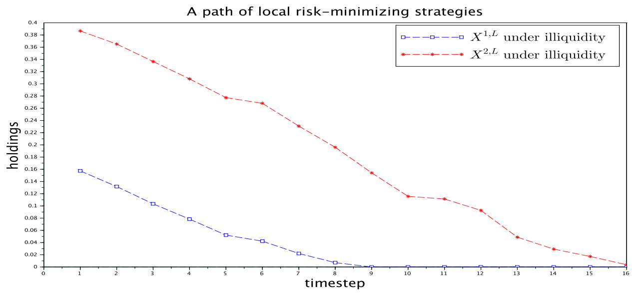


(a) Hedging with the two Futures G^1, G^2 with constant liquidity structure (2.4.15) with parameters $M_i = N_i = 0.01$ for $i = 1, 2$.

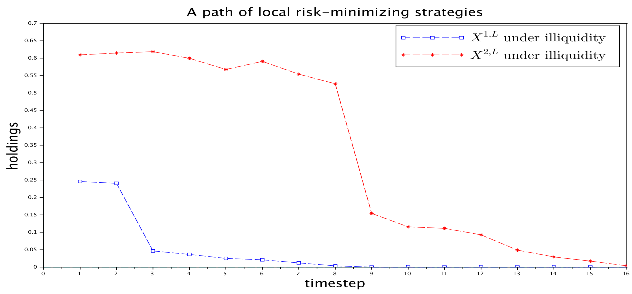


(b) Hedging with two instruments using the Futures G^1, G^2 with time-varying liquidity structure (2.4.14) with parameters $M_i = 0.005, N_i = 2M_i$ for $i = 1, 2$. The sudden drop in the holdings occurs when entering the delivery period where the futures are very illiquid.

Figure 2.4.2: Comparison of the sample path of optimal LRM-strategies under different liquidity structures but with the same hedge instruments. The two futures have overlapping delivery periods starting together but G^1 expires earlier. The delivery period of G^2 is the same as the one of the claim H. The LRM-strategies $X^{1,L}, X^{2,L}$ under illiquidity correspond to the future hedge instruments G^1, G^2 respectively.



(a) Hedging with the two Futures G^2, G^3 with constant liquidity structure (2.4.15) with parameters $M_i = N_i = 0.01$ for $i = 2, 3$.



(b) Hedging with the two Futures G^2, G^3 with time-varying liquidity structure (2.4.14) with parameters $M_i = 0.005, N_i = 2M_i$ for $i = 2, 3$.

Figure 2.4.3: Comparison of the sample path of optimal LRM-strategies under different liquidity structures but with the same hedge instruments. The two futures have consecutive delivery periods with the one of G^3 starting earlier. The delivery period of G^2 and the claim H coincide. The optimal LRM-strategy $X^{1,L}$ under illiquidity corresponds to the hedge instrument G^3 and $X^{2,L}$ to G^2.

3 Local risk-minimization with multiple assets under illiquidity with permanent price impact

Contributions of the thesis' author:

This chapter is a joint work of P. Christodoulou with Prof. Dr. Thilo Meyer-Brandis. It is based on Christodoulou and Meyer-Brandis (2019). The development of the framework for incorporating liquidity and price impact into the model as well as the interpretation of the results has been discussed and established jointly. All the results and proofs in all parts of this paper are mainly derived by P. Christodoulou.

3.1 Introduction

When liquidity is a concern and trading is subject to lasting price impact cost as well as liquidity cost, the problem of hedging general contingent claims is handled in this chapter. Using multiple assets as hedging instruments, accounting for their different levels of liquidity, our main objective is to extend the results of Christodoulou et al. (2018) (Chapter 2) by accounting for price impact.

In this chapter we follow the large trader arbitrage-free model of Roch (2011) which is an extension of Çetin et al. (2004) by incorporating price impact effects. In particular we work in a discrete time version of the model where for each asset we introduce a linear supply curve model under a stochastic, time-dependent liquidity parameter which expresses the density of the limit order book (LOB). Furthermore we extend the setting of Roch (2011) to the multi-dimensional case and additionally by letting the resilience parameter be a stochastic, time-dependent process, instead of a constant and we show that the model is free of arbitrage under certain assumptions in discrete time.

Instead of considering a super-replication problem as for example in Bank and Baum (2004), Çetin et al. (2010), Gökay and Soner (2012) which is usually too expensive, we consider a quadratic risk criterion, the so called *local risk-minimization*.

First introduced in Schweizer (1988) and then extended in discrete time under proportional transaction costs by Lamberton et al. (1998), the local risk-minimization method is the second main approach for quadratic hedging in an incomplete market. Recently, Christodoulou et al. (2018) extended in discrete time the work of Schweizer (1988) in two directions. Firstly, they extended the local risk-minimization criterion under stochastic, time-dependent illiquidity in the spirit of the papers of Rogers and Singh (2010) and Agliardi and Gençay (2014) in the case of a fast recovering limit order book, where the resilience parameter is zero. Secondly, they are considering a multi-dimensional asset price model. The existence of a locally risk-minimizing strategy under illiquidity is proven under mild conditions and is given in a closed-form solution.

In this chapter we extend the work of Christodoulou et al. (2018) by incorporating permanent price impact and extending their optimality risk-criterion. Instead of considering a model based on a multiplicative limit order book as in Christodoulou et al. (2018) our approach is based on an additive limit order book. This allows to handle the price impact terms separately. Our proposed optimality criterion allows us to give explicitly the optimal strategy which is also computationally tractable in contrast to existing approaches. Under mild conditions on the asset price, the liquidity level parameter of the limit order book and the resilience parameter, we show the existence and give a closed-form solution of a locally risk-minimizing strategy under illiquidity with price impact. Similar to Christodoulou et al. (2018), the conditions on the asset price can be reduced to conditions on the price process covariance matrix which usually are easy to check.

The structure of the chapter is the following. Section 3.2 presents the model in an illiquid market under the presence of price impact. We show that our setting is free of arbitrage. In Section 3.3 we define the local risk minimization problem and our optimality risk-criterion. We give a characterization of the optimal strategy through a minimization problem. In Section 3.4 we prove and give an explicit solution of an optimal strategy. We impose assumptions on the price process, the liquidity level and the resilience parameter in order to prove the main existence Theorem 3.4.17. Section 3.5 considers a slightly alternative approach of our initial criterion, treating price impact and liquidity cost together. We prove an existing optimal strategy in the case of full permanent price impact.

3.2 The Price Impact Model

3.2.1 Description of the setup

We consider a discrete time setting in a financial market consisting of $d+1$ assets. A filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is given where the flow of information is described by the filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\ldots,T}$ and $\mathbb{P}$ is the *objective* probability measure. We assume the existence of a riskless asset with discounted price equal to 1 and the discounted (marginal) price of d risky assets is modeled by an $\mathbb{F}$-adapted, nonnegative d-dimensional stochastic process $S = (S_k)_{k=0,1,\ldots,T}$.

A nonnegative d-dimensional *supply curve* $S_k(x) = (S_k(x)^1, \ldots, S_k(x)^d)$ for $x \in \mathbb{R}^d$ is given exogenously as in Çetin et al. (2004). We denote by $S_k(x)^j := S_k^j(x^j)$ the j-th stock price per share at time k for the purchase (if $x^j > 0$) or sale (if $x^j < 0$) of $|x^j|$ shares. The price process $S(0) = S$ is called the *marginal price* and the process $S_k(x)$ the *unaffected supply curve*.

Following the 1-dimensional setting in Roch (2011) we extend this to the multidimensional case. We define the process of a d-dimensional *affected observed supply curve* $S_k^\lambda(x)$. This supply curve determines the actual price that market participants pay or receive respectively for a transaction of size x at time k. This curve is also assumed to be dependent of the participants past actions which implies a lasting impact of the trading strategy on the supply curve. The difference between the two supply curves $S_k^\lambda(x)$ and $S_k(x)$ is the lasting price impact. Furthermore we call the process $S^\lambda(0) = S^\lambda$ the *observed quoted price*. We assume that both supply curves are measurable with respect to the filtration $\mathbb{F}$. Additionally and in order to ensure non-negative liquidity costs, we assume that both supply curves are non-decreasing in the number of shares x, that is for each k and j, $S_k(x)^j \leq S_k(y)^j$, $\mathbb{P}-a.s.$ for $x^j \leq y^j$. Similarly holds for $S_k^\lambda(x)$.

Based on the setting of Roch (2011), which is motivated from a symmetric, linear, additive LOB, we assume that there exists a positive d-dimensional semimartingale $M = (M_k)_{k=0,1,\dots,T}$ such that

$$S_k(x)^j = S_k^j + M_k^j x^j, \tag{3.2.1}$$

$$S_k^\lambda(x)^j = S_k^{\lambda,j} + M_k^j x^j, \tag{3.2.2}$$

for all $j = 1, \dots, d$.

Instead of considering a constant resilience parameter as in Roch (2011) we consider a multi-dimensional stochastic resilience process. In particular, for taking into account permanent price impact we introduce the resilience parameter semimartingale process $\lambda = (\lambda_k)_{k=0,1,\dots,T}$ taking values in $[0,1]^d$, meaning the proportion of new bid orders (respectively ask orders) filling up the LOB when a trade to buy (respectively sell) is made at some time k. That means after a trade of size x^j the quoted price $S_k^{\lambda,j}$ is shifted to $S_k^{\lambda,j} + 2\lambda_k^j M_k^j x^j$. More precisely and at time k, in a LOB with density $\frac{1}{2M_k}$ and after an arbitrary market buy order of x^j shares, the price moves from $S_k^{\lambda,j}$ to $S_k^{\lambda,j} + 2M_k^j x^j$. The lowest ask price is then $S_k^{\lambda,j} + 2M_k^j x^j$ and the highest bid remains the same. Then since new orders are filling up the LOB, the quoted price is shifted again downwards to $S_k^{\lambda,j} + 2\lambda_k^j M_k^j x^j$. For more details see Roch (2011). Then it is clear that we have the relation

$$S_m^{\lambda,j} = S_m^j + 2\sum_{k=1}^{m} \lambda_{k-1}^j M_{k-1}^j x_{k-1}^j, \tag{3.2.3}$$

where $|x_{k-1}^j|$ is the number of shares ordered at time $k-1$. Note that in the case of full resilience, i.e. $\lambda_k^j = 0$, for all $k = 0, 1, \dots, T$, $j = 1, \dots, d$, the LOB is immediately filled up to its previous levels after a trade. This is a linear version of the liquidity model in Çetin et al. (2004), where $S_k^\lambda = S_k$. The case of full price impact is when $\lambda_k^j = 1$, for all $k = 0, 1, \dots, T$, $j = 1, \dots, d$.

3.2.2 No Arbitrage

In Çetin et al. (2004) the authors developed an extended arbitrage pricing theory for a continuous time version of a supply curve model without lasting impact. The existence of an equivalent local martingale measure $\mathbb{Q}$ for the marginal price process S rules out arbitrage. This was extended further in Roch (2011) under price impact with constant resilience parameter where they proved that if additionally the price impact process M is a $\mathbb{Q}$-local submartingale then there are no arbitrage opportunities.

As shown in this section an extended similar result holds also in our extended d-dimensional discrete time setting where the resilience parameter is a stochastic process. In particular, if the process $\lambda M := (\lambda_k M_k)_{k=0,1,\dots,T}$ is a $\mathbb{Q}$-local submartingale then there is no arbitrage.

In the following we will use the notation $\Delta Z_k := Z_k - Z_{k-1}$ for $k = 1, \ldots, T$ for a process $Z = (Z_k)_{k=0,1,\ldots,T}$. Moreover, we will consider trading strategies as in Definition 2.2.1 and additionally we set $V_0^- := Y_0^-$ where Y_0^- is the amount in the bank account before the first trade. By convention we define $X_0 := 0$, $Y_{-1} := Y_0^-$ and $V_{-1} := V_0^-$.

Moreover, a strategy $\varphi = (X, Y)$ is called self-financing if

$$\Delta Y_k + \Delta X_{k+1}^* S_k^\lambda(\Delta X_{k+1}) = 0, \tag{3.2.4}$$

for all times $k = 0, 1, \ldots, T$ where recall $\Delta Y_k = Y_k - Y_{k-1}$ and $\Delta X_k = X_k - X_{k-1}$.

In order to define no-arbitrage in an extended market under illiquidity and lasting price impact one needs a unique tractable portfolio value. An economically meaningful possibility used for example in Roch (2011) and Bank and Baum (2004) is the so called (immediate) *liquidation value*. That is when an investor liquidates his position in stocks immediately by a single block trade. More precisely, for a trading strategy $\varphi = (X, Y)$ the (immediate) liquidation value of the portfolio at time $N \in \{0, 1, \ldots, T\}$ if X_N shares are liquidated after the last trade at time $N - 1$ (if $N \geq 1$) is[1]

$$V_N^L(\varphi) := Y_{N-1} + X_N^* S_N^\lambda(-X_N)\,. \tag{3.2.5}$$

Note that the liquidation strategy is $X_{N+1} = 0$ and the amount in the riskless asset is $Y_N = V_N^L(\varphi)$.

The next definition is based on Çetin et al. (2004) (see also Roch (2011)) and is an extension of the notion of arbitrage to an economy under illiquidity.

Definition 3.2.1. *An arbitrage opportunity is a self financing trading strategy $\varphi = (X, Y)$ such that $V_0^L = 0$, $\mathbb{P}(V_T^L \geq 0) = 1$ and $\mathbb{P}(V_T^L > 0) > 0$.*

The next Lemma 3.2.2 gives us a more precise description of the liquidation portfolio value dynamics for self-financing trading strategies. This will be useful to set the conditions under which there are no arbitrage opportunities in the market.

Lemma 3.2.2. *For a self-financing trading strategy $\varphi = (X, Y)$ the liquidation portfolio value at time $N \in \{1, \ldots, T\}$ is*

$$\begin{aligned} V_N^L(\varphi) = V_0^L(\varphi) + \sum_{m=1}^N X_m^* \Delta S_m - \sum_{m=1}^N \sum_{j=1}^d (1 - \lambda_{m-1}^j) M_{m-1}^j (\Delta X_m^j)^2 \\ - \sum_{j=1}^d (1 - \lambda_N^j) M_N^j (X_N^j)^2 - \sum_{m=1}^N \sum_{j=1}^d (X_m^j)^2 \Delta(\lambda_m^j M_m^j), \end{aligned} \tag{3.2.6}$$

where $\Delta(\lambda_m^j M_m^j) = \lambda_m^j M_m^j - \lambda_{m-1}^j M_{m-1}^j$ for all $j = 1, \ldots, d$.

Proof. Let $N \in \{0, 1, \ldots, T - 1\}$ and $\varphi = (X, Y)$ a trading strategy.

First observe that the two immediate liquidation values at times N and $N + 1$ are $V_N^L(\varphi) = Y_{N-1} + X_N^* S_N^\lambda(-X_N)$ and $V_{N+1}^L(\varphi) = Y_N + X_{N+1}^* S_{N+1}^\lambda(-X_{N+1})$ respectively.

[1] In case of $N = 0$, note that the immediate liquidation value equals the initial portfolio, that is $V_0^L(\varphi) = V_0^-$.

On the other hand, by the self-financing condition and by the definition of the supply curve $S^\lambda_N(x)$ for x shares of stock it holds

$$\begin{aligned} Y_N &= Y_{N-1} - \Delta X^*_{N+1} S^\lambda_N(\Delta X_{N+1}) \\ &= Y_{N-1} - X^*_{N+1} S^\lambda_N(\Delta X_{N+1}) + X^*_N S^\lambda_N(\Delta X_{N+1}) \\ &= Y_{N-1} - X^*_{N+1} S^\lambda_N(\Delta X_{N+1}) + X^*_N S^\lambda_N + \sum_{j=1}^d M^j_N X^j_{N+1} X^j_N - \sum_{j=1}^d M^j_N (X^j_N)^2 \,, \end{aligned} \tag{3.2.7}$$

from where we can deduce

$$Y_N = V^L_N(\varphi) - X^*_{N+1} S^\lambda_N(\Delta X_{N+1}) + \sum_{j=1}^d M^j_N X^j_{N+1} X^j_N \,. \tag{3.2.8}$$

Putting things together, using the definition of the liquidation values we get

$$V^L_{N+1}(\varphi) = V^L_N(\varphi) - X^*_{N+1} S^\lambda_N(\Delta X_{N+1}) + \sum_{j=1}^d M^j_N X^j_{N+1} X^j_N + X^*_{N+1} S^\lambda_{N+1}(-X_{N+1}) \,. \tag{3.2.9}$$

Let us calculate now the difference between the two liquidation values at times N and $N+1$. We have

$$\begin{aligned} &V^L_{N+1}(\varphi) - V^L_N(\varphi) \\ &= X^*_{N+1} S^\lambda_{N+1}(-X_{N+1}) - X^*_{N+1} S^\lambda_N(\Delta X_{N+1}) + \sum_{j=1}^d M^j_N X^j_{N+1} X^j_N \\ &= X^*_{N+1} S^\lambda_{N+1} - \sum_{j=1}^d M^j_{N+1} (X^j_{N+1})^2 - X^*_{N+1} S^\lambda_N - \sum_{j=1}^d M^j_N \Delta X^j_{N+1} X^j_N + \sum_{j=1}^d M^j_N X^j_{N+1} X^j_N . \end{aligned} \tag{3.2.10}$$

Then using the relation (3.2.3) between the observed quoted price S^λ and the marginal price process S we obtain

$$\begin{aligned} V^L_{N+1}(\varphi) - V^L_N(\varphi) = {} & X^*_{N+1} S_{N+1} + 2 \sum_{j=1}^d X^j_{N+1} \sum_{m=1}^{N+1} \lambda^j_{m-1} M^j_{m-1} \Delta X^j_m \\ & - X^*_{N+1} S_N - 2 \sum_{j=1}^d X^j_{N+1} \sum_{m=1}^{N} \lambda^j_{m-1} M^j_{m-1} \Delta X^j_m \\ & - \sum_{j=1}^d M^j_N (X^j_{N+1})^2 - \sum_{j=1}^d M^j_{N+1} (X^j_{N+1})^2 + 2 \sum_{j=1}^d M^j_N X^j_{N+1} X^j_N \,, \end{aligned} \tag{3.2.11}$$

from which we can conclude that

$$V^L_{N+1}(\varphi) - V^L_N(\varphi) = X^*_{N+1}\Delta S_{N+1} + 2\sum_{j=1}^d \lambda^j_N M^j_N (X^j_{N+1})^2 - 2\sum_{j=1}^d \lambda^j_N M^j_N X^j_{N+1} X^j_N \tag{3.2.12}$$
$$- \sum_{j=1}^d M^j_N (X^j_{N+1})^2 - \sum_{j=1}^d M^j_{N+1} (X^j_{N+1})^2 + 2\sum_{j=1}^d M^j_N X^j_{N+1} X^j_N \,.$$

Furthermore, note that

$$\begin{aligned} -2\sum_{j=1}^d \lambda^j_N M^j_N X^j_{N+1} X^j_N &= \sum_{j=1}^d \lambda^j_N M^j_N (\Delta X^j_{N+1})^2 - \sum_{j=1}^d \lambda^j_N M^j_N (X^j_{N+1})^2 \\ &\quad - \sum_{j=1}^d \lambda^j_N M^j_N (X^j_N)^2 \,, \\ 2\sum_{j=1}^d M^j_N X^j_{N+1} X^j_N &= -\sum_{j=1}^d M^j_N (\Delta X^j_{N+1})^2 + \sum_{j=1}^d M^j_N (X^j_{N+1})^2 + \sum_{j=1}^d M^j_N (X^j_N)^2 \,. \end{aligned} \tag{3.2.13}$$

By adding a zero $\pm \sum_{j=1}^d \lambda^j_{N+1} M^j_{N+1} (X^j_{N+1})^2$ and using (3.2.13) we calculate

$$\begin{aligned} &V^L_{N+1}(\varphi) - V^L_N(\varphi) \\ &= X^*_{N+1}\Delta S_{N+1} + 2\sum_{j=1}^d \lambda^j_N M^j_N (X^j_{N+1})^2 \\ &+ \sum_{j=1}^d \lambda^j_N M^j_N (\Delta X^j_{N+1})^2 - \sum_{j=1}^d \lambda^j_N M^j_N (X^j_{N+1})^2 - \sum_{j=1}^d \lambda^j_N M^j_N (X^j_N)^2 \\ &- \sum_{j=1}^d M^j_{N+1} (X^j_{N+1})^2 + \sum_{j=1}^d \lambda^j_{N+1} M^j_{N+1} (X^j_{N+1})^2 - \sum_{j=1}^d \lambda^j_{N+1} M^j_{N+1} (X^j_{N+1})^2 \\ &- \sum_{j=1}^d M^j_N (X^j_{N+1})^2 - \sum_{j=1}^d M^j_N (\Delta X^j_{N+1})^2 + \sum_{j=1}^d M^j_N (X^j_{N+1})^2 + \sum_{j=1}^d M^j_N (X^j_N)^2 \,, \end{aligned} \tag{3.2.14}$$

and by simplifying further we get

$$\begin{aligned} &V^L_{N+1}(\varphi) - V^L_N(\varphi) \\ &= X^*_{N+1}\Delta S_{N+1} \\ &- \sum_{j=1}^d (1-\lambda^j_N) M^j_N (\Delta X^j_{N+1})^2 - \sum_{j=1}^d (1-\lambda^j_{N+1}) M^j_{N+1} (X^j_{N+1})^2 \\ &+ \sum_{j=1}^d \lambda^j_N M^j_N (X^j_{N+1})^2 - \sum_{j=1}^d \lambda^j_{N+1} M^j_{N+1} (X^j_{N+1})^2 + \sum_{j=1}^d (1-\lambda^j_N) M^j_N (X^j_N)^2 \,. \end{aligned} \tag{3.2.15}$$

Thus,

$$\begin{aligned} V^L_{N+1}(\varphi) - V^L_N(\varphi) &= X^*_{N+1}\Delta S_{N+1} \\ &\quad - \sum_{j=1}^d (1-\lambda^j_N) M^j_N (\Delta X^j_{N+1})^2 - \sum_{j=1}^d (1-\lambda^j_{N+1}) M^j_{N+1} (X^j_{N+1})^2 \\ &\quad - \sum_{j=1}^d (X^j_{N+1})^2 \Delta(\lambda^j_{N+1} M^j_{N+1}) + \sum_{j=1}^d (1-\lambda^j_N) M^j_N (X^j_N)^2 \,. \end{aligned} \tag{3.2.16}$$

for all $N \in \{0, 1, \dots, T-1\}$. So summing up for $N \in \{1, 2, \dots, T\}$ we have

$$\begin{aligned} &\sum_{m=0}^{N-1} V^L_{m+1}(\varphi) - V^L_m(\varphi) \\ &= \sum_{m=0}^{N-1} X^*_{m+1}\Delta S_{m+1} \\ &- \sum_{m=0}^{N-1}\sum_{j=1}^d (1-\lambda^j_m) M^j_m (\Delta X^j_{m+1})^2 - \sum_{m=0}^{N-1}\sum_{j=1}^d (1-\lambda^j_{m+1}) M^j_{m+1} (X^j_{m+1})^2 \\ &- \sum_{m=0}^{N-1}\sum_{j=1}^d (X^j_{m+1})^2 \Delta(\lambda^j_{m+1} M^j_{m+1}) + \sum_{m=0}^{N-1}\sum_{j=1}^d (1-\lambda^j_m) M^j_m (X^j_m)^2 \,. \end{aligned} \tag{3.2.17}$$

Hence we can conclude that

$$\begin{aligned} V^L_N(\varphi) - V^L_0(\varphi) &= \sum_{m=0}^{N-1} X^*_{m+1}\Delta S_{m+1} \\ &\quad - \sum_{m=0}^{N-1}\sum_{j=1}^d (1-\lambda^j_m) M^j_m (\Delta X^j_{m+1})^2 - (1-\lambda^j_N) M^j_N (X^j_N)^2 \\ &\quad - \sum_{m=0}^{N-1}\sum_{j=1}^d (X^j_{m+1})^2 \Delta(\lambda^j_{m+1} M^j_{m+1}) \,, \end{aligned} \tag{3.2.18}$$

and finally by an index shift in the sum, the claim follows. □

In the following recall the notation of the d-dimensional process $\lambda M = (\lambda_k M_k)_{k=0,1,\dots,T}$ where $\lambda_k M_k := (\lambda^1_k M^1_k, \dots, \lambda^d_k M^d_k)$ for $k = 0, 1, \dots, T$. As shown in Proposition 3.2.3, in order to exclude arbitrage opportunities it is sufficient to assume that the process λM is a $\mathbb{Q}$-local submartingale.

Proposition 3.2.3. *Let a measure $\mathbb{Q}$ such that $\mathbb{Q} \sim \mathbb{P}$. If S is a $\mathbb{Q}$-local martingale and λM is a $\mathbb{Q}$ local submartingale, then there is no arbitrage.*

Proof. For a zero initial portfolio value we want to show that $\mathbb{E}[V_T^L] \leq 0$.

Since S is a $\mathbb{Q}$-local martingale then also the process $X\Delta S := (X_k^* \Delta S_k)_{k=1,\dots,T}$ is a $\mathbb{Q}$-local martingale. So, using (3.2.6) from Lemma 3.2.2, it is sufficient to show that $\mathbb{E}_{\mathbb{Q}}[\tilde{V}_T^L] \geq 0$ where

$$\begin{aligned}\tilde{V}_T^L(\varphi) = & \sum_{m=1}^{T} \sum_{j=1}^{d} (1-\lambda_{m-1}^j) M_{m-1}^j (\Delta X_m^j)^2 \\ & + \sum_{j=1}^{d} (1-\lambda_N^j) M_N^j (X_N^j)^2 + \sum_{m=1}^{T} \sum_{j=1}^{d} (X_m^j)^2 \Delta(\lambda_m^j M_m^j)\,.\end{aligned} \tag{3.2.19}$$

Since the processes λ and M are positive with the process λ taking values in the interval $[0,1]^d$ then the first two terms in (3.2.19) are positive. Also, since by assumption λM is a $\mathbb{Q}$-local submartingale then the process $(\sum_{m=1}^{k} \sum_{j=1}^{d} (X_m^j)^2 \Delta(\lambda_m^j M_m^j))_{k=1,\dots,T}$ is a $\mathbb{Q}$-local submartingale too. Thus, by taking expectation in (3.2.19) with respect to the measure $\mathbb{Q}$ the claim follows. Indeed, since $\mathbb{Q}$ and $\mathbb{P}$ are equivalent measures, then if V_T^L was an arbitrage opportunity then it would satisfy $\mathbb{E}_{\mathbb{Q}}[V_T^L] > 0$, but since we have shown that $\mathbb{E}_{\mathbb{Q}}[V_T^L] \leq 0$ this leads to a contradiction. □

3.2.2.1 Decreasing liquidity level

Note that the assumption in Proposition 3.2.3 on the process λM being a $\mathbb{Q}$-local submartingale for some measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ is kind of restrictive in a deterministic limit order book. More precisely, in the setting of Roch (2011) for example, where the process λ is a constant, then for a deterministic limit order book depth-level parameter process M this will mean that M must be an increasing process.

The next Proposition 3.2.4 enables us to consider a decreasing deterministic liquidity level M in a no arbitrage setting under some additional conditions. To simplify the notation we will assume that $d = 1$, but it is clear that an analogous result holds also in the multi-dimensional case.

Proposition 3.2.4. *Let $d = 1$ for simplicity and assume that the process λ is a constant in $[0,1]$ and M is a positive deterministic process such that $M_i = \alpha_i M_0$ where $0 \leq \alpha_i \leq 1$ for $i = 0, 1, \dots, T$ with $\alpha_0 := 1$. Let M be decreasing i.e., $M_0 \geq M_1 \geq \dots \geq M_T$. Furthermore let $\varphi = (X, Y)$ a trading strategy and $R := \max_{i=1}^{T} |X_i|$. Then if S is a $\mathbb{Q}$-local martingale for a measure $\mathbb{Q}$ such that $\mathbb{Q} \sim \mathbb{P}$ and the condition*

$$\alpha_T \geq \frac{\lambda}{\frac{1}{T}(1-\lambda)+\lambda}\,, \tag{3.2.20}$$

holds, then there are no arbitrage opportunities.

Proof. First note that since the process M is decreasing this implies that $\alpha_0 \geq \alpha_1 \geq \dots \geq \alpha_T$. As in Proposition 3.2.3 it is sufficient to show $\tilde{V}_T^L \geq 0$ where

$$\tilde{V}_T^L(\varphi) = (1-\lambda) \sum_{m=1}^{T} M_{m-1} (\Delta X_m)^2 + (1-\lambda) M_T (X_T)^2 + \lambda \sum_{m=1}^{T} (X_m)^2 \Delta M_m\,. \tag{3.2.21}$$

Using the definition of M and since $M_{m-1} \geq M_T$ for all $m = 2, \dots, T$ then

$$\tilde{V}_T^L(\varphi) \geq (1-\lambda) \sum_{m=1}^{T} \alpha_T M_0 (\Delta X_m)^2 + (1-\lambda) M_T (X_T)^2 + \lambda \sum_{m=1}^{T} (X_m)^2 M_0 (\alpha_m - \alpha_{m-1}) \,. \tag{3.2.22}$$

Furthermore, by the definition of R and since the term $(1-\lambda)M_T(X_T)^2$ is positive then we estimate

$$\begin{aligned} \tilde{V}_T^L(\varphi) &\geq (1-\lambda) \sum_{m=1}^{T} \alpha_T M_0 (\Delta X_m)^2 + \lambda \sum_{m=1}^{T} R^2 M_0 (\alpha_m - \alpha_{m-1}) \qquad (3.2.23) \\ &= (1-\lambda) \alpha_T M_0 \sum_{m=1}^{T} (\Delta X_m)^2 + \lambda R^2 M_0 (\alpha_T - 1) \,. \end{aligned}$$

It remains to estimate the term $\sum_{m=1}^{T} (\Delta X_m)^2$. Since by convention $X_0 = 0$ then intuitively this should be bigger than the squared maximal distance to zero from all the points X_i divided by the number of points T, thus is holds

$$\sum_{m=1}^{T} (\Delta X_m)^2 \geq \sum_{m=1}^{T} \left(\frac{R}{T} \right)^2 . \tag{3.2.24}$$

The proof of inequality (3.2.24) is a technical proof which can be done by induction or directly. Just to give the idea we will give a direct proof for the case $T = 2$. Then a generalization for $T \in \mathbb{N}$ should be clear.

The case $T = 1$ is trivial so assume that $T = 2$. That is, we want to show that $(X_2 - X_1)^2 + (X_1)^2 \geq \frac{R^2}{2}$.[2] Since the case $R = |X_1|$ is clear, let $R = |X_2|$. If R is zero then X is the zero strategy and there is nothing to show. So by dividing with $|X_2|^2$ and assuming that there exist some $\beta \in [0,1]$ so that $\beta |X_2| = |X_1|$, we get

$$(X_2 - X_1)^2 + (X_1)^2 \geq \frac{R^2}{2} \iff (1-\beta)^2 + \beta^2 \geq \frac{1}{2} \,. \tag{3.2.25}$$

Since $(\sqrt{2}\beta - \frac{1}{\sqrt{2}})^2 \geq 0$ is again equivalent to (3.2.25) then the claim follows for $T = 2$. The case $T = 3$ follows analogously by defining for example the functions $f_1, f_2 : [0,1]^2 \to \mathbb{R}^+$ with $f_1(\alpha, \beta) = (1-\alpha)^2 + (1-\beta)^2 + \beta^2$ for the case $R = |X_2|$ and $f_2(\alpha, \beta) = (1-\alpha)^2 + (\alpha - \beta)^2 + \beta^2$ for the case $R = |X_3|$. A generalization of the proof can be done by defining more complex functions and calculating their minimum.

Coming back to our initial proof and continuing our estimation, then using (3.2.24) we can further estimate from below

$$\tilde{V}_T^L(\varphi) \geq (1-\lambda) \alpha_T M_0 \sum_{m=1}^{T} \left(\frac{R}{T} \right)^2 + \lambda R^2 M_0 (\alpha_T - 1) \tag{3.2.26}$$

[2] Recall that $X_0 = 0$.

$$= (1-\lambda)\alpha_T M_0 \frac{R^2}{T} + \lambda R^2 M_0(\alpha_T - 1)\,.$$

So we can conclude that

$$\tilde{V}_T^L(\varphi) \geq 0 \iff (1-\lambda)\alpha_T \frac{1}{T} + \lambda(\alpha_T - 1) \geq 0\,, \tag{3.2.27}$$

from where the claim now follows. □

Remark 3.2.5. *Note that in Proposition 3.2.4 for the zero resilience case $\lambda = 0$ where no lasting price impact is considered, the condition (3.2.20) becomes $\alpha_T \geq 0$ which means that the process M can be decreasing without any condition on α_T. This agrees with the no price impact setting of Çetin et al. (2004).*

3.3 The Local Risk-Minimization Problem

Under liquidity risk (and also due to state space restrictions in a discrete setting), a unique martingale measure does not necessarily imply completeness of the market. So in a market under illiquidity and since perfect hedging is not possible, we choose the optimality criterion introduced later in Definition 2.2.4 for minimizing locally the risk of hedging under illiquidity.

Before we define the optimization problem, we first want to mention that we will make use of the notation introduced in Chapter 2 (Section 2.2) and in particular of Definition 2.2.1 for a trading strategy $\varphi = (X, Y)$ where property (ii) will be replaced later in Section 3.3.1 by the condition $X \in \Theta_d^\lambda(S)$ for a subspace $\Theta_d^\lambda(S)$ of $\Theta_d(S)$ (see Definition 3.3.1).

Furthermore we also define $V_k^\lambda(\varphi) := X_{k+1}^* S_k^\lambda + Y_k$ the *impacted marked-to-market value* or the *impacted book value* of the portfolio (X_{k+1}, Y_k) at time k with respect to the impacted marginal price S_k^λ.

We will denote by $S^{\lambda,\varphi}$ the impacted quoted price S^λ whenever necessary, to emphasize the dependence of strategy φ. Furthermore recall that by convention $X_0 = 0$ and denote $\Delta X_1 := X_1$.

3.3.1 Cost and Risk process under illiquidity with price impact

Let an $\mathbb{L}_T^{2,1}$*-contingent claim* $H = \bar{X}_{T+1}^* S_T^\lambda + \bar{Y}_T$, where $\bar{X}_{T+1}^* S_T^\lambda \in \mathbb{L}_T^{2,1}$, $\bar{X}_{T+1} \in \mathbb{L}_T^{2,d}$ where $\bar{X}_{T+1}, \bar{Y}_T$ are $\mathcal{F}_T$-measurable random variables representing the quantity in risky assets and bonds respectively that the option seller is obligated to provide to the buyer at the expiration date T of the financial contract H.

For an order of $\Delta Y_k = Y_k - Y_{k-1}$ bonds and $\Delta X_{k+1} = X_{k+1} - X_k$ shares, the *total outlay* under illiquidity with price impact will be

$$\Delta Y_k + \Delta X_{k+1}^* S_k^\lambda(\Delta X_{k+1}) = \Delta Y_k + \Delta X_{k+1}^* S_k^\lambda + \Delta X_{k+1}^*[S_k^\lambda(\Delta X_{k+1}) - S_k^\lambda(0)], \tag{3.3.1}$$

at time $k \in \{1, 2, \ldots, T\}$. Note that the last term in (3.3.1) are the current liquidity costs incurred by trading at current time. Note also that the total outlay at time k would be zero for a self-financing trading strategy. Rewriting equation (3.3.1),

$$\Delta Y_k + \Delta X_{k+1}^* S_k^\lambda(\Delta X_{k+1}) = \Delta V_k^\lambda(\varphi) - X_k^* \Delta S_k^\lambda + \Delta X_{k+1}^*[S_k^\lambda(\Delta X_{k+1}) - S_k^\lambda(0)], \tag{3.3.2}$$

by using the definition of the impacted book value.

By letting $\hat{C}_0^\lambda(\varphi) := V_0(\varphi)$, the *initial cost*[3], the cumulative costs of some strategy $\varphi = (X, Y)$ are defined through the *cost process under illiquidity with price impact*

$$\hat{C}_k^\lambda(\varphi) := \sum_{m=1}^{k} \Delta Y_m + \sum_{m=1}^{k} \Delta X_{m+1}^* S_m^\lambda(\Delta X_{m+1}) + V_0(\varphi), \tag{3.3.3}$$

for $k = 0, 1, \ldots, T$. It is easy to check that

$$\hat{C}_k^\lambda(\varphi) = V_k^\lambda(\varphi) - \sum_{m=1}^{k} X_m^* \Delta S_m^\lambda + \sum_{m=1}^{k} \Delta X_{m+1}^* [S_m^\lambda(\Delta X_{m+1}) - S_m^\lambda(0)]. \tag{3.3.4}$$

Moreover we define the *quadratic risk process under illiquidity with price impact* $\hat{R}^\lambda(\varphi) = (\hat{R}_k^\lambda(\varphi))_{k=0,1,\ldots,T}$ by

$$\hat{R}_k^\lambda(\varphi) := \mathbb{E}[(\hat{C}_T^\lambda(\varphi) - \hat{C}_k^\lambda(\varphi))^2 | \mathcal{F}_k], \tag{3.3.5}$$

where

$$\hat{C}_T^\lambda(\varphi) - \hat{C}_k^\lambda(\varphi) = V_T^\lambda(\varphi) - V_k^\lambda(\varphi) - \sum_{m=k+1}^{T} X_m^* \Delta S_m^\lambda + \sum_{m=k+1}^{T} \Delta X_{m+1}^* [S_m^\lambda(\Delta X_{m+1}) - S_m^\lambda(0)]. \tag{3.3.6}$$

Recalling the classical cost process $C(\varphi)$ from (2.2.6) (i.e., $S^\lambda = S$ and $S(x) = S(0)$) we obtain the relation

$$\begin{aligned} \hat{C}_k^\lambda(\varphi) &= C_k(\varphi) \\ &+ 2\sum_{j=1}^{d} X_{k+1}^j \sum_{m=1}^{k} \lambda_{m-1}^j M_{m-1}^j \Delta X_m^j - 2\sum_{m=1}^{k}\sum_{j=1}^{d} X_m^j \lambda_{m-1}^j M_{m-1}^j \Delta X_m^j \\ &+ \sum_{m=1}^{k}\sum_{j=1}^{d} M_m^j |\Delta X_{m+1}^j|^2, \end{aligned} \tag{3.3.7}$$

where we have used the linear structure of the supply curve (3.2.1) and (3.2.2) and the price impact relation (3.2.3). In the same way we also obtain the relation

$$\hat{C}_T^\lambda(\varphi) - \hat{C}_k^\lambda(\varphi) = C_T(\varphi) - C_k(\varphi) + PI_k^\lambda(X) + LC_k(X), \tag{3.3.8}$$

where PI stands for price impact and LC for liquidity costs defined by

$$PI_k^\lambda(X) := 2\sum_{j=1}^{d} X_{T+1}^j \sum_{m=k+1}^{T} \lambda_{m-1}^j M_{m-1}^j \Delta X_m^j \tag{3.3.9}$$

[3] For simplicity we do not account for any liquidity costs paid to set up the initial portfolio. Note also that $V_0(\varphi) = V_0^\lambda(\varphi)$, hence $\hat{C}_0(\varphi) = \hat{C}_0^\lambda(\varphi)$.

$$
\begin{aligned}
&+ 2\sum_{j=1}^{d}(X^j_{T+1} - X^j_{k+1})\sum_{m=1}^{k}\lambda^j_{m-1}M^j_{m-1}\Delta X^j_m \\
&- 2\sum_{m=k+1}^{T}\sum_{j=1}^{d}X^j_m\lambda^j_{m-1}M^j_{m-1}\Delta X^j_m, \\
LC_k(X) := &\sum_{m=k+1}^{T}\sum_{j=1}^{d}M^j_m|\Delta X^j_{m+1}|^2.
\end{aligned}
$$

At this point we would like to emphasize that from now on we will work with the next Definition 3.3.1 of trading strategies. The main difference with Definition 2.2.1 is that we work within the subspace $\Theta^\lambda_d(S)$ of trading strategies under price impact.

Definition 3.3.1. *A pair $\varphi = (X, Y)$ is called a trading strategy under price impact if φ is a trading strategy according to Definition 2.2.1 and additionally $X \in \Theta^\lambda_d(S)$ where $\Theta^\lambda_d(S)$ the space of all $\mathbb{R}^d$-valued predictable strategies $X = (X_k)_{k=1,2,\dots,T+1}$ so that $X^*_k\Delta S_k \in \mathbb{L}^{2,1}_T$, $PI^\lambda_{k-1} \in \mathbb{L}^{1,1}_T$ and $\Delta X^*_{k+1}[S^\lambda_k(\Delta X_{k+1}) - S^\lambda_k(0)] \in \mathbb{L}^{1,1}_T$ for $k = 1, 2, \dots, T$.*

With a slight abuse of notation we will refer to a trading strategy under price impact by the name "trading strategy", since from now on we will work only within the set $\Theta^\lambda_d(S)$.

Motivated from Coleman et al. (2003), it is also possible to define the *linear risk process under illiquidity with price impact*

$$
\bar{R}^\lambda_k(\varphi) := \mathbb{E}[|\hat{C}^\lambda_T(\varphi) - \hat{C}^\lambda_k(\varphi)||\mathcal{F}_k]. \tag{3.3.10}
$$

As already mentioned and similar to Section 2.2.1 of Chapter 2, there are mainly two different possibilities to approach this hedging problem locally. The one is to use the L^2-norm and take the risk process defined as in (3.3.5) and the other is to use the more intuitive (from a financial point of view) linear approach of (3.3.10). It is preferred to minimize over the L^2-norm since it is possible to get explicit formulas for optimal strategies. Nevertheless, from a negative point of view, large values which may occur with small probability might be overemphasized. The possibility of combining the two, that means measuring quadratically the difference of the classical cost process and linearly the variation of the liquidity cost and price impact yields the *quadratic-linear risk process (QLRP) under illiquidity with price impact*

$$
T^\lambda_k(\varphi) := \mathbb{E}[(C_T(\varphi) - C_k(\varphi))^2|\mathcal{F}_k] + \mathbb{E}\left[\left|PI^\lambda_k(X)\right|\big|\mathcal{F}_k\right] + \mathbb{E}\left[LC_k(X)\big|\mathcal{F}_k\right]. \tag{3.3.11}
$$

Note that the lasting price impact term cost is coming from the difference of the hedging costs incurred by $V^\lambda_l(\varphi) - \sum_{m=1}^{l} X^*_m\Delta S^\lambda_m$ at times $l = T$ and $l = k$. In this contest it can be seen as the additional hedging risk coming from price impact price fluctuations.

Without overemphasizing the price impact and liquidity cost terms by the L^2-norm, an explicit representation of the LRM-strategy under illiquidity with price impact, will be proven in Section 3.4 by minimizing the expression (3.3.11).

Note that for the zero resilience case $\lambda = 0$, the price impact term $PI^\lambda_k(X)$ vanishes for all $k \in \{0, 1, \dots, T\}$. Then for the choice of $M^j_k = \varepsilon^j_k S^j_k$ for $j = 1, \dots, d$, where $\varepsilon = (\varepsilon_k)_{k=0,1,\dots,T}$ is a positive deterministic $\mathbb{R}^d$-valued process, the setting is reduced to the model used in Christodoulou et al. (2018).

3.3.2 The optimality criterion

In this section we will describe and give a definition of the optimality risk-criterion.

The aim is to find a locally risk-minimizing strategy $\varphi = (X, Y)$ under illiquidity with price impact such that $V_T(\varphi) = H$ with $X_{T+1} = \bar{X}_{T+1}$ and $Y_T = \bar{Y}_T$.

By a slight abuse of notation let

$$\begin{aligned} T_k^{\alpha,\beta,\lambda}(\varphi) &:= \mathbb{E}[(C_T(\varphi) - C_k(\varphi))^2 | \mathcal{F}_k] \\ &+ \alpha \mathbb{E}\left[\left| PI_k^\lambda(X) \right| \Big| \mathcal{F}_k \right] + \beta \mathbb{E}\left[\sum_{j=1}^{d} M_{k+1}^j |\Delta X_{k+2}^j|^2 \Big| \mathcal{F}_k \right]. \end{aligned} \tag{3.3.12}$$

Definition 3.3.2 specifies the local risk minimizing strategy under illiquidity with price impact for some given $\alpha, \beta \in \mathbb{R}^+$. It uses the prespecified Definition 2.2.3 (Chapter 2) of local perturbation.

Definition 3.3.2. *A trading strategy $\varphi = (X, Y)$ is called locally risk-minimizing (LRM) under illiquidity with price impact if for any time $k \in \{0, 1, \ldots, T-1\}$*

$$T_k^{\alpha,\beta,\lambda}(\varphi) \leq T_k^{\alpha,\beta,\lambda}(\varphi') \qquad \mathbb{P}-a.s. \tag{3.3.13}$$

for any local perturbation φ' of φ at time k.

Note that, taking into account the liquidity costs of $LC_k(X)$ at the current time only, it does not change the optimization problem. In fact, minimizing over T_k^λ in equation (3.3.11) or over $T_k^{\alpha,\beta,\lambda}$ are both equivalent according to Definition 3.3.2. This is due to the fact that the (local) minimization problem takes place only for the current choice of the strategy at the current time.

Remark 3.3.3. *The choice $\alpha = \beta = 1$ represents an equal concern about the risk to be hedged as incurred by market price fluctuations and by the lasting price impact and the cost of hedging incurred by liquidity costs. Otherwise, for example $\beta < 1$ means a major risk aversion to the risk of miss-hedging coming from market price fluctuations and from lasting price impact separately, while $\beta > 1$ means a major risk aversion to the cost of liquidity. Depending on how risk averse an investor wants to be against the different risks or costs, the parameters α and β are adjusted accordingly. One could also generalize by having deterministic $\mathbb{R}^+$-valued processes $\alpha = (\alpha_k)_{k=0,1,\ldots,T}$, $\beta = (\beta_k)_{k=0,1,\ldots,T}$ and trivially our results together with the main existence Theorem 3.4.17 will still hold.*

The following Lemmas are extensions of the corresponding Lemmas in Christodoulou et al. (2018) for the case when price impact is a concern. Their proofs follow very similar arguments and so we will not provide them here.

Since price impact does not occur in the risk-free investment but only in the risky one, then by adjusting the bond holdings by a perturbed strategy it is possible to extend Lemma 2.4 in Christodoulou et al. (2018) such that the cost process is a martingale while accounting for both price impact and liquidity cost.

Lemma 3.3.4. *For a LRM-strategy φ under illiquidity with price impact, the cost process $C(\varphi)$ is a martingale. Furthermore, from the martingale property of the cost process we get the representation,*

$$R_k(\varphi) = \mathbb{E}[R_{k+1}(\varphi)|\mathcal{F}_k] + \mathbb{V}ar(\Delta C_{k+1}(\varphi)|\mathcal{F}_k) \qquad \mathbb{P}-a.s., \tag{3.3.14}$$

for $k = 0, 1, \ldots, T-1$.

Proof. Follows by the same arguments as in Lemma 2.4 in Christodoulou et al. (2018). $\square$

Then, the representation of the QLRP under illiquidity with price impact is

$$\begin{aligned} T_k^{\alpha,\beta,\lambda}(\varphi) &:= \mathbb{E}[R_{k+1}(\varphi)|\mathcal{F}_k] + \mathbb{V}ar(\Delta C_{k+1}(\varphi)|\mathcal{F}_k) \\ &\quad + \alpha\mathbb{E}\left[\left|PI_k^\lambda(X)\right| |\mathcal{F}_k\right] + \beta\mathbb{E}\left[\sum_{j=1}^d M_{k+1}^j |\Delta X_{k+2}^j|^2 \middle| \mathcal{F}_k\right], \end{aligned} \tag{3.3.15}$$

for a LRM-strategy φ under illiquidity with price impact.

Lemma 3.3.5 will supply us with a useful representation for the QLRP under illiquidity with price impact of a perturbed strategy. We will use the following notation: for a local perturbation $\varphi' = (X', Y')$ of $\varphi = (X, Y)$ at time k denote

$$\begin{aligned} PI_k^\lambda(X, X') &:= 2\sum_{j=1}^d X_{T+1}^j \left(\sum_{m=k+2}^T \lambda_{m-1}^j M_{m-1}^j \Delta X_m^j + \lambda_k^j M_k^j \left(X_{k+1}^{\prime j} - X_k^j\right)\right) \\ &\quad + 2\sum_{j=1}^d (X_{T+1}^j - X_{k+1}^{\prime j}) \sum_{m=1}^k \lambda_{m-1}^j M_{m-1}^j \Delta X_m^j \\ &\quad - 2\sum_{m=k+3}^T \sum_{j=1}^d X_m^j \lambda_{m-1}^j M_{m-1}^j \Delta X_m^j \\ &\quad - 2\sum_{j=1}^d X_{k+1}^{\prime j} \lambda_k^j M_k^j (X_{k+1}^{\prime j} - X_k^j) - 2\sum_{j=1}^d X_{k+2}^j \lambda_{k+1}^j M_{k+1}^j (X_{k+2}^j - X_{k+1}^{\prime j}), \\ LC_k(X, X') &:= \sum_{m=k+2}^T \sum_{j=1}^d M_m^j |\Delta X_{m+1}^j|^2 + \sum_{j=1}^d M_{k+1}^j |X_{k+2}^j - X_{k+1}^{\prime j}|^2 . \end{aligned} \tag{3.3.16}$$

Note that $PI_k^\lambda(X, X) = PI_k^\lambda(X)$ and $LC_k(X, X) = LC_k(X)$.

Lemma 3.3.5. *Assume $C(\varphi)$ is a martingale and φ' a local perturbation of φ at time k. Then it holds*

$$\begin{aligned} T_k^{\alpha,\beta,\lambda}(\varphi') &:= \mathbb{E}[R_{k+1}(\varphi)|\mathcal{F}_k] + \mathbb{E}[(\Delta C_{k+1}(\varphi'))^2|\mathcal{F}_k] \\ &\quad + \alpha\mathbb{E}\left[\left|PI_k^\lambda(X, X')\right| |\mathcal{F}_k\right] + \beta\mathbb{E}\left[\sum_{j=1}^d M_{k+1}^j |X_{k+2}^j - X_{k+1}^{\prime j}|^2 \middle| \mathcal{F}_k\right]. \end{aligned} \tag{3.3.17}$$

Proof. Follows by the same arguments as in Lemma 2.5 in Christodoulou et al. (2018). □

The next Proposition 3.3.6 gives us the characterisation of a LRM strategy under illiquidity with price impact through the minimization problem that one needs to consider and is based on minimizing the expression

$$\mathbb{V}ar(\Delta C_{k+1}(\varphi)|\mathcal{F}_k) + \alpha \mathbb{E}\left[\left|PI_k^{\lambda}(X,X)\right| |\mathcal{F}_k\right] + \beta \mathbb{E}\left[\sum_{j=1}^{d} M_{k+1}^j |X_{k+2}^j - X_{k+1}^j|^2 \middle| \mathcal{F}_k\right], \tag{3.3.18}$$

by equation (3.3.17). This is due to the fact that for any local perturbation φ' of φ at time k it holds $R_{k+1}(\varphi) = R_{k+1}(\varphi')$. This Proposition is quite general and holds for any positive semimartingale process $S = (S_k)_{k=0,1,\dots,T}$ and any positive stochastic liquidity level $M = (M_k)_{k=0,1,\dots,T}$ as well as any positive stochastic resilience process $\lambda = (\lambda_k)_{k=0,1,\dots,T}$ in the multidimensional case.

Proposition 3.3.6. *A trading strategy $\varphi = (X, Y)$ is LRM under illiquidity with price impact if and only if the two following properties are satisfied:*

(i) $C(\varphi)$ is a martingale.

(ii) For each $k \in \{0, 1, \dots, T-1\}$, X_{k+1} minimizes

$$\begin{aligned}&\mathbb{V}ar(V_{k+1}(\varphi) - (X'_{k+1})^* \Delta S_{k+1}|\mathcal{F}_k)\\ &\quad + \alpha \mathbb{E}\left[\left|PI_k^{\lambda}(X,X')\right| |\mathcal{F}_k\right] + \beta \mathbb{E}\left[\sum_{j=1}^{d} M_{k+1}^j |X_{k+2}^j - X_{k+1}^{\prime j}|^2 \middle| \mathcal{F}_k\right]\end{aligned} \tag{3.3.19}$$

over all $\mathcal{F}_k$-measurable random variables X'_{k+1} so that $(X'_{k+1})^* \Delta S_{k+1} \in \mathbb{L}_T^{2,1}$, $PI_k^{\lambda}(X,X') \in \mathbb{L}_T^{1,1}$ and $M_{k+1}^j |X_{k+2}^j - X_{k+1}^{\prime j}|^2 \in \mathbb{L}_T^{1,1}$.

Proof. Since the proof follows the same arguments as the proof of Proposition 2.6 in Christodoulou et al. (2018) we give only a sketch of it.

Fix some $k \in \{0, 1, \dots, T-1\}$ and let $\varphi' = (X', Y')$ be a local perturbation of $\varphi = (X, Y)$ at time k. For the " $\Leftarrow$ " direction of the proof, the following equality holds

$$\begin{aligned}T_k^{\alpha,\beta,\lambda}(\varphi') =& \mathbb{E}[R_{k+1}(\varphi)|\mathcal{F}_k] + \mathbb{E}[(\Delta C_{k+1}(\varphi'))^2|\mathcal{F}_k]\\ &+ \alpha \mathbb{E}\left[\left|PI_k^{\lambda}(X,X')\right| |\mathcal{F}_k\right] + \beta \mathbb{E}\left[\sum_{j=1}^{d} M_{k+1}^j |X_{k+2}^j - X_{k+1}^{\prime j}|^2 \middle| \mathcal{F}_k\right],\end{aligned} \tag{3.3.20}$$

by Lemma 3.3.5 and property (i). Moreover, using the definition of the conditional variance to estimate the term $\mathbb{E}[(\Delta C_{k+1}(\varphi'))^2|\mathcal{F}_k]$ and using the fact that $X'_{k+2} = X_{k+2}$ and $Y'_{k+1} = Y_{k+1}$ as well as property (ii), then it holds

$$T_k^{\alpha,\beta,\lambda}(\varphi') \geq \mathbb{E}[R_{k+1}(\varphi)|\mathcal{F}_k] + \mathbb{V}ar(V_{k+1}(\varphi) - (X_{k+1})^* \Delta S_{k+1}|\mathcal{F}_k)$$

$$+ \alpha\mathbb{E}\left[\left|PI_k^\lambda(X,X)\right| |\mathcal{F}_k\right] + \beta\mathbb{E}\left[\sum_{j=1}^d M_{k+1}^j |X_{k+2}^j - X_{k+1}^j|^2 \middle| \mathcal{F}_k\right]. \quad (3.3.21)$$

At the same time, the martingale property of the cost process $C(\varphi)$ yields (by (3.3.17))

$$\begin{aligned} T_k^{\alpha,\beta,\lambda}(\varphi) =& \mathbb{E}[R_{k+1}(\varphi)|\mathcal{F}_k] + \mathbb{V}ar(\Delta C_{k+1}(\varphi)|\mathcal{F}_k) \\ &+ \alpha\mathbb{E}\left[\left|PI_k^\lambda(X,X)\right| |\mathcal{F}_k\right] + \beta\mathbb{E}\left[\sum_{j=1}^d M_{k+1}^j |X_{k+2}^j - X_{k+1}^j|^2 \middle| \mathcal{F}_k\right]. \end{aligned} \quad (3.3.22)$$

So, by (3.3.21) and (3.3.22) we can conclude that $T_k^{\alpha,\beta,\lambda}(\varphi') \geq T_k^{\alpha,\beta,\lambda}(\varphi)$ which implies that φ is a LRM-strategy under illiquidity with price impact, according to Definition 3.3.2.

For the “ $\Rightarrow$ ” direction of the proof, since Property (i) holds clearly, then it remains to show Property (ii). Since $T_k^{\alpha,\beta,\lambda}(\varphi') \geq T_k^{\alpha,\beta,\lambda}(\varphi)$ then by equations (3.3.17) and (3.3.22) it holds

$$\begin{aligned} &\mathbb{V}ar(\Delta C_{k+1}(\varphi')|\mathcal{F}_k) + (\mathbb{E}[\Delta C_{k+1}(\varphi')|\mathcal{F}_k])^2 \\ &\quad + \alpha\mathbb{E}\left[\left|PI_k^\lambda(X,X')\right| |\mathcal{F}_k\right] + \beta\mathbb{E}\left[\sum_{j=1}^d M_{k+1}^j |X_{k+2}^j - X_{k+1}^{\prime j}|^2 \middle| \mathcal{F}_k\right] \\ &\geq \mathbb{V}ar(\Delta C_{k+1}(\varphi)|\mathcal{F}_k) + \alpha\mathbb{E}\left[\left|PI_k^\lambda(X,X)\right| |\mathcal{F}_k\right] + \beta\mathbb{E}\left[\sum_{j=1}^d M_{k+1}^j |X_{k+2}^j - X_{k+1}^j|^2 \middle| \mathcal{F}_k\right], \end{aligned} \quad (3.3.23)$$

where the definition of the conditional covariance was used. As in Lemma 3.3.4, the appropriate choice of Y_k' yields $\mathbb{E}[\Delta C_{k+1}(\varphi')|\mathcal{F}_k] = 0$ and leaves the liquidity and price impact terms unchanged. This choice, together with the definition of a local perturbation strategy shows Property (ii) and the proof is complete. □

In order to get some explicit results and be able to construct a LRM-strategy under illiquidity with price impact such that it belongs to the space of strategies $\Theta_d^\lambda(S)$, we will assume in the next section a special case of the liquidity process M and of the resilience process λ.

3.4 Existence and Explicit LRM Strategy under Illiquidity with Price Impact

Our goal is to prove the existence of a local risk-minimizing strategy under illiquidity with price impact in a setting where feedback effects from the hedging strategies are taken into account and give an explicit representation of this optimal strategy using a backward induction argument. In order to achieve our goal we will consider a simplified model with time-independent liquidity level.

Throughout this section we will assume that $\bar{X}_{T+1} = 0$. That means an $\mathbb{L}_T^{2,1}$-*contingent claim* H has no physical delivery, hence is of the form $H = \bar{Y}_T$. Furthermore we will assume that the processes $M = (M_k)_{k=0,1,\dots,T}$ and $\lambda = (\lambda_k)_{k=0,1,\dots,T}$ are time-independent, that is for all $k = 0, 1, \dots, T$, $M_k := M$, $\lambda_k := \lambda$ for some positive random variables M and λ.

Let us first prove the following technical Lemma 3.4.1.

Lemma 3.4.1. *For a strategy $\varphi = (X, Y)$ it holds*

$$
\begin{aligned}
&2\left(\sum_{j=1}^{d}\lambda^j M^j |X_{k+1}^j|^2 + \sum_{j=1}^{d}\lambda^j M^j \sum_{m=k+2}^{T} X_m^j \Delta X_m^j\right) \qquad (3.4.1)\\
&= \sum_{j=1}^{d}\lambda^j M^j |X_{k+1}^j|^2 + \sum_{j=1}^{d}\lambda^j M^j |X_T^j|^2 + \sum_{j=1}^{d}\lambda^j M^j \sum_{m=k+1}^{T-1} (X_{m+1}^j - X_m^j)^2,
\end{aligned}
$$

for all $k = 0, 1, \dots, T-1$.

Proof. The claim can be proven by backward induction over $k \in \{0, 1, \dots, T-1\}$. The base case for $k = T-1$ is trivial, since clearly both sides of (3.4.1) equal $2\sum_{j=1}^{d}\lambda^j M^j |X_T^j|^2$. Now let $k \in \{0, 1, \dots, T-2\}$ and assume that (3.4.1) holds at time $k+1$. Then we have

$$
\begin{aligned}
&2\left(\sum_{j=1}^{d}\lambda^j M^j |X_k^j|^2 + \sum_{j=1}^{d}\lambda^j M^j \sum_{m=k+1}^{T} X_m^j \Delta X_m^j\right) \qquad (3.4.2)\\
&= 2\left(\sum_{j=1}^{d}\lambda^j M^j |X_k^j|^2 + \sum_{j=1}^{d}\lambda^j M^j X_{k+1}^j \Delta X_{k+1}^j + \sum_{j=1}^{d}\lambda^j M^j \sum_{m=k+2}^{T} X_m^j \Delta X_m^j\right)\\
&= 2\Bigg(\sum_{j=1}^{d}\lambda^j M^j |X_k^j|^2 + \sum_{j=1}^{d}\lambda^j M^j |X_{k+1}^j|^2\\
&+ \sum_{j=1}^{d} -\lambda^j M^j X_{k+1}^j X_k^j + \sum_{j=1}^{d}\lambda^j M^j \sum_{m=k+2}^{T} X_m^j \Delta X_m^j\Bigg).
\end{aligned}
$$

By adding a zero $\pm\sum_{j=1}^{d}\lambda^j M^j |X_{k+1}^j|^2$, using the induction hypothesis and rearranging the terms we obtain

$$
\begin{aligned}
&2\left(\sum_{j=1}^{d}\lambda^j M^j |X_k^j|^2 + \sum_{j=1}^{d}\lambda^j M^j \sum_{m=k+1}^{T} X_m^j \Delta X_m^j\right) \qquad (3.4.3)\\
&= 2\left(\sum_{j=1}^{d}\lambda^j M^j |X_k^j|^2 + \sum_{j=1}^{d} -\lambda^j M^j X_{k+1}^j X_k^j\right)\\
&+ \sum_{j=1}^{d}\lambda^j M^j |X_{k+1}^j|^2 + \sum_{j=1}^{d}\lambda^j M^j |X_T^j|^2 + \sum_{j=1}^{d}\lambda^j M^j \sum_{m=k+1}^{T-1} (X_{m+1}^j - X_m^j)^2\\
&= \sum_{j=1}^{d}\lambda^j M^j |X_k^j|^2 + \sum_{j=1}^{d}\lambda^j M^j (X_{k+1}^j - X_k^j)^2
\end{aligned}
$$

$$+\sum_{j=1}^{d}\lambda^j M^j |X_T^j|^2 + \sum_{j=1}^{d}\lambda^j M^j \sum_{m=k+1}^{T-1}(X_{m+1}^j - X_m^j)^2 \,.$$

So, we can conclude that

$$2\left(\sum_{j=1}^{d}\lambda^j M^j |X_k^j|^2 + \sum_{j=1}^{d}\lambda^j M^j \sum_{m=k+1}^{T} X_m^j \Delta X_m^j\right) \tag{3.4.4}$$
$$= \sum_{j=1}^{d}\lambda^j M^j |X_k^j|^2 + \sum_{j=1}^{d}\lambda^j M^j |X_T^j|^2 + \sum_{j=1}^{d}\lambda^j M^j \sum_{m=k}^{T-1}(X_{m+1}^j - X_m^j)^2 \,.$$

and the claim follows. □

The next Lemma 3.4.2 follows from Lemma 3.4.1.

Lemma 3.4.2. *For a strategy $\varphi = (X, Y)$ it holds*

$$\mathbb{E}[|PI_k^\lambda(X)||\mathcal{F}_k] \tag{3.4.5}$$
$$= \mathbb{E}\left[\sum_{j=1}^{d}\lambda^j M^j |X_{k+1}^j|^2 + \sum_{j=1}^{d}\lambda^j M^j |X_T^j|^2 + \sum_{j=1}^{d}\lambda^j M^j \sum_{m=k+1}^{T-1}(X_{m+1}^j - X_m^j)^2 \middle| \mathcal{F}_k\right].$$

Proof. Since $\bar{X}_{T+1} = 0$ then from (3.3.16) it follows that

$$PI_k^\lambda(X) = -2\sum_{j=1}^{d} X_{k+1}^j \sum_{m=1}^{k}\lambda_{m-1}^j M_{m-1}^j \Delta X_m^j - 2\sum_{j=1}^{d}\sum_{m=k+1}^{T} X_m^j \lambda_{m-1}^j M_{m-1}^j \Delta X_m^j. \tag{3.4.6}$$

Furthermore, since for all $k = 0, 1, \ldots, T$, $M_k = M$, $\lambda_k = \lambda$ and since by assumption $X_0 = 0$ then

$$PI_k^\lambda(X) = -2\sum_{j=1}^{d} X_{k+1}^j \lambda^j M^j X_k^j - 2\sum_{j=1}^{d}\lambda^j M^j \sum_{m=k+1}^{T} X_m^j \Delta X_m^j \tag{3.4.7}$$
$$= -2\sum_{j=1}^{d}\lambda^j M^j |X_{k+1}^j|^2 - 2\sum_{j=1}^{d}\lambda^j M^j \sum_{m=k+2}^{T} X_m^j \Delta X_m^j.$$

Moreover and using Lemma 3.4.1,

$$PI_k^\lambda(X) = -\left(\sum_{j=1}^{d}\lambda^j M^j |X_{k+1}^j|^2 + \sum_{j=1}^{d}\lambda^j M^j |X_T^j|^2 + \sum_{j=1}^{d}\lambda^j M^j \sum_{m=k+1}^{T-1}(X_{m+1}^j - X_m^j)^2\right). \tag{3.4.8}$$

Applying conditional expectation and using the definition of the absolute value, equation (3.4.5) follows. □

Applying Lemma 3.4.2 in equation (2.2.9) it follows that the QLRP under illiquidity with price impact has the representation

$$\begin{aligned}
&T_k^{\alpha,\beta,\lambda}(\varphi) \qquad (3.4.9)\\
&= \mathbb{E}[(C_T(\varphi) - C_k(\varphi))^2|\mathcal{F}_k]\\
&+ \alpha\mathbb{E}\left[\sum_{j=1}^{d} \lambda^j M^j |X_{k+1}^j|^2 + \sum_{j=1}^{d} \lambda^j M^j |X_T^j|^2 + \sum_{j=1}^{d} \lambda^j M^j \sum_{m=k+1}^{T-1} (X_{m+1}^j - X_m^j)^2 \middle| \mathcal{F}_k\right]\\
&+ \beta\mathbb{E}\left[\sum_{j=1}^{d} M^j |\Delta X_{k+2}^j|^2 \middle| \mathcal{F}_k\right].
\end{aligned}$$

As in equation (3.3.12), taking into account only the current liquidity costs and the current price impact terms, let

$$\begin{aligned}
\hat{T}_k^{\alpha,\beta,\lambda}(\varphi) := {}& \mathbb{E}[(C_T(\varphi) - C_k(\varphi))^2|\mathcal{F}_k] \qquad (3.4.10)\\
&+ \alpha\mathbb{E}\left[\sum_{j=1}^{d} \lambda^j M^j |X_{k+1}^j|^2 + \sum_{j=1}^{d} \lambda^j M^j (X_{k+2}^j - X_{k+1}^j)^2 \middle| \mathcal{F}_k\right]\\
&+ \beta\mathbb{E}\left[\sum_{j=1}^{d} M^j |X_{k+2}^j - X_{k+1}^j|^2 \middle| \mathcal{F}_k\right].
\end{aligned}$$

Then our objective is to minimize $\hat{T}_k^{\alpha,\beta,\lambda}(\varphi)$ among all local perturbations, that means, a trading strategy $\varphi = (X, Y)$ is called *locally risk-minimizing under illiquidity with price impact* if

$$\hat{T}_k^{\alpha,\beta,\lambda}(\varphi) \leq \hat{T}_k^{\alpha,\beta,\lambda}(\varphi') \qquad \mathbb{P}-\text{a.s.} \qquad (3.4.11)$$

for any time $k \in \{0, 1, \ldots, T-1\}$ and any local perturbation φ' of φ at time k. Note that this is equivalent to Definition 3.3.2.

The minimization problem is characterized by Proposition 3.3.6. For the case of time independent liquidity and resilience parameters and by assuming $\alpha = \beta = 1$ for simplicity, it can be clearly simplified as follows:

Proposition 3.4.3. *A trading strategy $\varphi = (X, Y)$ is LRM under illiquidity with price impact if and only if the two following properties are satisfied:*

(i) $C(\varphi)$ is a martingale.

(ii) For each $k \in \{0, 1, \ldots, T-1\}$, X_{k+1} minimizes

$$\begin{aligned}
&\mathbb{V}ar(V_{k+1}(\varphi) - (X'_{k+1})^* \Delta S_{k+1}|\mathcal{F}_k) \qquad (3.4.12)\\
&+ \mathbb{E}\left[\sum_{j=1}^{d} \lambda^j M^j |X_{k+1}^{\prime j}|^2 \middle| \mathcal{F}_k\right] + \mathbb{E}\left[\sum_{j=1}^{d} (1+\lambda^j) M^j (X_{k+2}^j - X_{k+1}^{\prime j})^2 \middle| \mathcal{F}_k\right].
\end{aligned}$$

over all $\mathcal{F}_k$-measurable random variables X'_{k+1} so that $(X'_{k+1})^* \Delta S_{k+1} \in \mathbb{L}_T^{2,1}$, $\lambda^j M^j |X_{k+1}^{\prime j}|^2 \in \mathbb{L}_T^{1,1}$ and $(1+\lambda^j) M^j |X_{k+2}^j - X_{k+1}^{\prime j}|^2 \in \mathbb{L}_T^{1,1}$.

The most important part of Proposition 3.4.3 is expression (3.4.12). The idea for constructing an optimal strategy under illiquidity with price impact is to minimize (3.4.12) going backward in time while choosing Y_k such that the cost process C is a martingale.

Let us now introduce the following notation:

$$\begin{aligned} A^M_{k;j} &:= \mathbb{E}[(1+2\lambda^j)M^j|\mathcal{F}_k], & A^\lambda_{k;j} &:= A^0_{k;j} + A^M_{k;j}, \\ b^M_{k;j} &:= \mathbb{E}[(1+\lambda^j)M^j X^j_{k+2}|\mathcal{F}_k], & b^\lambda_{k;j} &:= b^0_{k;j} + b^M_{k;j}, \end{aligned} \tag{3.4.13}$$

for all $i, j = 1, \dots, d$ and $k = 0, \dots, T-1$ and recall $A^0_{k;j}, b^0_{k;j}, D_{k;j,i}$ from notation (2.3.4) of Chapter 2.

Define the function $f^\lambda_k : \mathbb{R}^d \times \Omega \to \mathbb{R}^+$ by

$$\begin{aligned} f^\lambda_k(c,\omega) = {} & \sum_{j=1}^d |c_j|^2 A^\lambda_{k;j}(\omega) - 2\sum_{j=1}^d c_j b^\lambda_{k;j}(\omega) + \sum_{j\neq i} c_j c_i D_{k;j,i}(\omega) \\ & + \mathbb{V}ar(V_{k+1}|\mathcal{F}_k)(\omega) + \sum_{j=1}^d \mathbb{E}[(1+\lambda^j)M^j|X^j_{k+2}|^2|\mathcal{F}_k](\omega). \end{aligned} \tag{3.4.14}$$

Note that (3.4.14) is an equivalent expression for (3.4.12). We aim at calculating the extreme points of the multidimensional function f^λ_k. By fixing ω one needs to solve $grad(f^\lambda_k) = 0$. For doing so, we fix some more notation.

Let $F^\lambda_k \in \mathbb{R}^{d\times d}$ with $F^\lambda_{k;i,j} = D_{k;i,j}$ for $i \neq j$, $F^\lambda_{k;i,j} = A^\lambda_{k;j}$ for $i = j$ and $b^\lambda_k = (b^\lambda_{k;1}, \dots, b^\lambda_{k;d}) \in \mathbb{R}^d$. Let $F^M_k = diag(A^M_{k;1}, \dots, A^M_{k;d})$ and denote by F^0_k the matrix F^λ_k with $M^j = 0$ for all j, that is the covariance matrix of the marginal price process S. For calculating the candidates of extreme points one needs to solve the linear equation system

$$F^\lambda_k\, c = b^\lambda_k. \tag{3.4.15}$$

Then we can write the symmetric matrix F^λ_k as the sum of two real symmetric, positive semidefinite matrices $F^\lambda_k = F^0_k + F^M_k$. Since F^λ_k is positive semidefinite[4], the Hessian matrix $H_{f^\lambda_k}(c) = 2F^\lambda_k$ also. Hence by letting the covariance matrix F^0_k be positive definite, then the matrix F^λ_k is invertible and equation (3.4.15) has a unique solution. Moreover, since the function $c \to f^\lambda_k(c,\omega)$ is strictly convex (by the fact that the Hesse matrix is positive definite) then $c^* := (F^\lambda_k)^{-1} b^\lambda_k$ is a global minimizer, which additionally is $\mathcal{F}_k$-measurable since the matrix $(F^\lambda_k)^{-1}$ and b^λ_k are both $\mathcal{F}_k$-measurable.

3.4.1 Assumptions and Properties of the marginal price process S

It needs to be ensured that the optimal strategy is in the space of strategies $\Theta^\lambda_d(S)$. So, assumptions on the matrix F^λ_k and the marginal price process S will be imposed in this section. These will hold for independent increments as well as for independent returns. Only in the case of independent returns the additional $ML.1$ condition, concerning the liquidity level and the resilience parameter needs to be assumed.

[4] In fact, F^λ_k is positive definite if M^j is positive for all $j = 1, \dots, d$.

Recall also the bounded mean-variance tradeoff condition on S from Defiinition 2.3.2 which we do not address again here.

Definition 3.4.4. *We say that S satisfies the MS-condition from above if for some constant $C > 0$*

$$\frac{(\mathbb{E}[(1+2\lambda^j)M^j|\mathcal{F}_k])^2}{\mathbb{V}ar(S^j_{k+1}|\mathcal{F}_k)} \leq C \quad \mathbb{P}-a.s. \quad \textit{for all } j=1,\dots,d \tag{3.4.16}$$

uniformly in k and ω. Furthermore S satisfies the MS-condition from below if for some constant $\tilde{C} > 0$

$$\frac{(\mathbb{E}[(1+2\lambda^j)M^j|\mathcal{F}_k])^2}{\mathbb{V}ar(S^j_{k+1}|\mathcal{F}_k)} \geq \tilde{C} \quad \mathbb{P}-a.s. \quad \textit{for all } j=1,\dots,d \tag{3.4.17}$$

uniformly in k and ω. If both bounds hold then we say that S satisfies the MS-condition.

Definition 3.4.5. *We say that the $ML.1$-condition is satisfied if for some constant $C > 0$*

$$\frac{\mathbb{E}[|(1+\lambda^j)M^j|^2|\mathcal{F}_k]}{(\mathbb{E}[(1+2\lambda^j)M^j|\mathcal{F}_k])^2} \leq C \quad \mathbb{P}-a.s. \quad \textit{for all } j=1,\dots,d \tag{3.4.18}$$

uniformly in k and ω. If $(1+\lambda^j)M^j$ is independent of the filtration $\mathbb{F}$ for all $j=1,\dots,d$ then we say that the $ML.2$-condition is satisfied.

Note that the $ML.1$ condition holds for example for deterministic parameters M and λ or for stochastic ones in the case when these are independent from the underlying filtration.

Definition 3.4.6. *We say that S satisfies the F^λ-diagonal condition with respect to the matrix F^λ if for some constant $C > 0$*

$$\sqrt{\mathbb{V}ar(\Delta S^j_{k+1}|\mathcal{F}_k)} + \frac{\mathbb{E}[(1+2\lambda^j)M^j|\mathcal{F}_k]}{\sqrt{\mathbb{V}ar(S^j_{k+1}|\mathcal{F}_k)}} \geq C \quad \mathbb{P}-a.s. \quad \textit{for all } j=1,\dots,d \tag{3.4.19}$$

uniformly in k and ω and if for some constant $\tilde{C} > 0$

$$\frac{\sqrt{\mathbb{V}ar(S^j_{k+1}|\mathcal{F}_k)}}{\mathbb{E}[(1+2\lambda^j)M^j|\mathcal{F}_k]} + \frac{1}{\sqrt{\mathbb{V}ar(\Delta S^j_{k+1}|\mathcal{F}_k)}} \geq \tilde{C} \quad \mathbb{P}-a.s. \quad \textit{for all } j=1,\dots,d \tag{3.4.20}$$

uniformly in k and ω.

Remark 3.4.7. *As already mentioned in Remark 2.3.6, the choice of the name F^λ-diagonal condition in Definition 3.4.6 is because of the diagonal terms of the matrix F^λ,*

$$\begin{aligned}
\frac{F^0_{k;j,j}}{|F^\lambda_{k;j,j}|^2} &= \left(\sqrt{\mathbb{V}ar(\Delta S^j_{k+1}|\mathcal{F}_k)} + \frac{\mathbb{E}[(1+2\lambda^j)M^j|\mathcal{F}_k]}{\sqrt{\mathbb{V}ar(S^j_{k+1}|\mathcal{F}_k)}}\right)^{-2} \\
|F^M_{k;j,j}|^2\frac{F^0_{k;j,j}}{|F^\lambda_{k;j,j}|^2} &= \left(\frac{\sqrt{\mathbb{V}ar(S^j_{k+1}|\mathcal{F}_k)}}{\mathbb{E}[(1+2\lambda^j)M^j|\mathcal{F}_k]} + \frac{1}{\sqrt{\mathbb{V}ar(\Delta S^j_{k+1}|\mathcal{F}_k)}}\right)^{-2}.
\end{aligned} \tag{3.4.21}$$

Recall the d-dimensional return process $\rho = (\rho_k)_{k=0,1,\dots T}$ of S from Section 2.3.1 (Chapter 2).

Sufficient condition for the previous assumptions to hold, are given in the next Propositions 3.4.8 and 3.4.9.

Proposition 3.4.8. *If S satisfies $\tilde{C} \leq \mathbb{V}ar(\Delta S^j_{k+1}|\mathcal{F}_k) \leq C$ for some positive constants $C, \tilde{C}$ and for all $j = 1, \dots, d$, then the F^λ-diagonal condition holds. In particular, if S has independent increments then S has bounded mean-variance tradeoff and satisfies the F^λ-diagonal condition.*

Proof. The claim follows directly from the fact that $\tilde{C} \leq \mathbb{V}ar(\Delta S^j_{k+1}|\mathcal{F}_k) \leq C$. □

Proposition 3.4.9. *If the ML.2 condition holds then the F^λ-diagonal condition holds. In particular, if S has independent returns and the ML.2 condition is satisfied then S has bounded mean-variance tradeoff and satisfies the F^λ-diagonal condition. Additionally, the ML.1-condition is also satisfied.*

Proof. The claim follows directly from the fact that S has independent returns, by the independence property of the process M and the filtration $\mathbb{F}$ and that the functions $f_1(x) := x + \frac{a}{x}$, $f_2(x) := \frac{x}{a} + \frac{1}{x}$ are bounded from below for all positive constants a. □

Remark 3.4.10. *A simple example with discretization time mesh h is the 1-dimensional Black-Scholes model from Remark 2.3.9. By Proposition 3.4.9, if the processes M and λ are independent of the filtration $\mathbb{F}$ then S has bounded mean-variance tradeoff and satisfies the F^λ-diagonal condition.*

3.4.2 Some preliminaries

For showing the integrability conditions fulfilled by the strategy in the proof of the main Theorem 3.4.17 in this subsection we state some useful technical Lemmas.[5] Consider the following notation:

$$\begin{aligned} \alpha^\lambda_{k;i,j} &:= F^0_{k;j,j} F^0_{k;i,i} |(F^\lambda_{k;j,i})^{-1}|^2, & \alpha^M_{k;i,j} &:= F^0_{k;j,j} |F^M_{k;i,i}|^2 |(F^\lambda_{k;j,i})^{-1}|^2, \\ \beta^\lambda_{k;i,j} &:= F^0_{k;i,i} |(F^\lambda_{k;j,i})^{-1}|^2, & \beta^M_{k;i,j} &:= |F^M_{k;i,i}|^2 |(F^\lambda_{k;j,i})^{-1}|^2, \end{aligned} \tag{3.4.22}$$

for $i, j = 1, \dots, d$ and $k = 0, \dots, T$ when the inverse matrix $(F^\lambda_k)^{-1}$ of F^λ_k exists.

Denote by $M^\lambda_{k;i,j}$ the matrix F^λ_k without the i-th row and j-th column. Moreover we will use the fact that $(F^\lambda_{k;j,i})^{-1} = \frac{(-1)^{i+j}\det(M^\lambda_{k;i,j})}{\det(F^\lambda_k)}$ for the case that the inverse of a symmetric matrix F^λ_k exists.

[5] Compare also with Section 2.3.2 of Chapter 2.

Lemma 3.4.11. *For all $d \in \mathbb{N}_{\geq 2}$:*

$$\det(M^{\lambda}_{k;i,j})^2 \leq C F^0_{k;j,j} F^0_{k;i,i} \prod_{\substack{l=1 \\ l \neq i,j}}^{d} |F^{\lambda}_{k;l,l}|^2 \qquad \text{for all } i, j = 1, \dots, d \text{ with } i \neq j, \tag{3.4.23}$$

$$|F^{\lambda}_{k;j,j}|^2 \det(M^{\lambda}_{k;j,j})^2 \leq \tilde{C} \det(F^{A^{\lambda}}_{k})^2 \qquad \text{for all } j = 1, \dots, d \tag{3.4.24}$$

$$F^{\lambda}_{k;j,j} F^{\lambda}_{k;i,i} \det(M^{\lambda}_{k;i,j})^2 \leq \bar{C} \det(F^{A^{\lambda}}_{k})^2 \qquad \text{for all } i, j = 1, \dots, d \tag{3.4.25}$$

for some positive constants $C, \tilde{C}$ and $\bar{C}$ where $F^{A^{\lambda}}_{k} := \operatorname{diag}(A^{\lambda}_{k;1}, \dots, A^{\lambda}_{k;d})$.

Proof. Follows by the same arguments as in Lemma 3.6 in Christodoulou et al. (2018).

□

As already introduced and indicated in Christodoulou et al. (2018) the Definition of the F^{λ}-*property* that follows has a very essential use. Firstly, it is the key for extending the LRM-criterion of Schweizer (1988) to the multidimensional case and secondly to the illiquid case.

Definition 3.4.12. *We say that the process S has the F^{λ}-property with respect to the matrix F^{λ} if there exists some $\delta \in (0, 1)$ such that*

$$\det(F^{\lambda}_{k}) - (1 - \delta) \det(F^{A^{\lambda}}_{k}) \geq 0, \tag{3.4.26}$$

for all $k = 0, 1, \dots, T$ where $F^{A^{\lambda}}_{k} := \operatorname{diag}(A^{\lambda}_{k;1}, \dots, A^{\lambda}_{k;d})$.

Remark 3.4.13. *For the simple 1-dimensional case, the F^{λ}-property is always fulfilled since this is equivalent to $\mathbb{V}ar(\Delta S_{k+1}|\mathcal{F}_k) + \mathbb{E}[(1+\lambda)M|\mathcal{F}_k] \geq 0$ where recall that M is a non-negative process. As already noted in Section 2.3.2, for the case where S^i and S^j are independent for $i \neq j$ the F^{λ}-property is again fulfilled. This is due to the fact that in this particular case, this is equivalent to $\det(F^{A^{\lambda}}_{k}) \geq 0$ where the matrix $F^{A^{\lambda}}_{k}$ is positive semi-definite. This shows the strong relation between this property and the covariance matrix of the price process S.*

In Section 3.4.4 we show that this property is strongly related to a sufficient condition in terms of the covariance matrix of S.

Lemma 3.4.14. *Assume that S has the F^{λ}-property and satisfies the F^{λ}-diagonal condition. Then the terms $\alpha^{\lambda}_{k;i,j}$, $\beta^{\lambda}_{k;i,j}$, $\alpha^{M}_{k;i,j}$ and $\beta^{M}_{k;i,j}$ are uniformly bounded in k and ω for all $i, j = 1, \dots, d$.*

Proof. Follows by the same arguments as in Lemma 3.8 in Christodoulou et al. (2018) by using Lemma 3.4.11 □

Lemma 3.4.15. *Assume that $(F^{\lambda}_{k})^{-1}$ exists for $k \in \{0, 1, \dots, T\}$ and S has bounded mean-variance tradeoff. Let (X, Y) be any trading strategy. Then there exists some constant $C > 0$ such that*

$$\mathbb{E}[(((F^{\lambda}_{k})^{-1} b^{\lambda}_{k})_j \Delta S^{j}_{k+1})^2]$$

$$\leq C\mathbb{E}\left[\mathbb{V}ar(V_{k+1}|\mathcal{F}_k)\sum_{i=1}^{d}\alpha^{\lambda}_{k;i,j}+\sum_{i=1}^{d}\alpha^{M}_{k;i,j}\frac{1}{|F^{M}_{k;i,i}|^2}\mathbb{E}[|(1+\lambda^i)M^i|^2|\mathcal{F}_k]\mathbb{E}[|X^i_{k+2}|^2|\mathcal{F}_k]\right], \tag{3.4.27}$$

$$\mathbb{E}[(((F^{\lambda}_k)^{-1}b^{\lambda}_k)_j)^2]$$
$$\leq C\mathbb{E}\left[\mathbb{V}ar(V_{k+1}|\mathcal{F}_k)\sum_{i=1}^{d}\beta^{\lambda}_{k;i,j}+\sum_{i=1}^{d}\beta^{M}_{k;i,j}\frac{1}{|F^{M}_{k;i,i}|^2}\mathbb{E}[|(1+\lambda^i)M^i|^2|\mathcal{F}_k]\mathbb{E}[|X^i_{k+2}|^2|\mathcal{F}_k]\right], \tag{3.4.28}$$

for all $j=1,\dots,d$ where $((F^{\lambda}_k)^{-1}b^{\lambda}_k)_j$ is the j-th component of the vector $((F^{\lambda}_k)^{-1}b^{\lambda}_k)$.

Proof. This proof follows similar arguments as in Lemma 3.9 in Christodoulou et al. (2018) but for the sake of completeness we give a short proof. Using bounded mean-variance tradeoff then by the definition of the variance it follows

$$\mathbb{E}[|\Delta S^j_{k+1}|^2|\mathcal{F}_k]\leq CA^0_{k;j}\,, \tag{3.4.29}$$

for some positive constant C. Furthermore, from the tower property and inequality (3.4.29) it holds

$$\mathbb{E}[(((F^{\lambda})^{-1}b^{\lambda})_j\Delta S^j_{k+1})^2]\leq 2C\mathbb{E}[\sum_{i=1}^{d}|(F^{\lambda}_{j,i})^{-1}|^2(|b^0_i|^2+|b^M_i|^2)F^0_{j,j}], \tag{3.4.30}$$

where we have denoted $F^{\lambda}=F^{\lambda}_k$ and $b^{\lambda}=b^{\lambda}_k$. By the Cauchy-Schwarz-Inequality for the term b^0_i and the conditional inequality $(\mathbb{E}[XY|\mathcal{G}])^2\leq\mathbb{E}[X^2|\mathcal{G}]\mathbb{E}[Y^2|\mathcal{G}]$ on the term b^M_i together with the definition of the variance yields

$$\mathbb{E}[(((F^{\lambda})^{-1}b^{\lambda})_j\Delta S^j_{k+1})^2]$$
$$\leq C\mathbb{E}[\sum_{i=1}^{d}|(F^{\lambda}_{j,i})^{-1}|^2(\mathbb{V}ar(V_{k+1}|\mathcal{F}_k)F^0_{i,i}+\mathbb{E}[|(1+\lambda^i)M^i|^2|\mathcal{F}_k]\mathbb{E}[|X^i_{k+2}|^2|\mathcal{F}_k])F^0_{j,j}]\,. \tag{3.4.31}$$

The second inequality follows analogously. □

Remark 3.4.16. *The two Lemmas 3.4.14 and 3.4.15 will play a central role for the existence proof of a LRM-strategy $\hat{X}^{\lambda}$ under illiquidity with price impact which belongs to the space of strategies $\Theta^{\lambda}_d(S)$. The main integrability properties that we will need to show are $\hat{X}^{\lambda,j}_{k+1}\Delta S^j_{k+1}\in\mathbb{L}^{2,1}_T$ and $\hat{X}^{\lambda,j}_{k+1}\in\mathbb{L}^{2,1}_T$. Moreover, these two properties will be essential for showing that the liquidity terms (liquidity cost and price impact terms) of the optimal strategy fulfill certain integrability properties.*

3.4.3 Explicit form of the optimal strategy

Under the conditions imposed in Section 3.4.1 and using a backward induction argument, we give an explicit representation of a local risk-minimizing strategy under illiquidity with price impact and thus prove our main existence theorem.

Theorem 3.4.17 (Existence result). *Assume that S has the F^λ-property, bounded mean-variance tradeoff and satisfies the F^λ-diagonal condition. Additionally assume that the $ML.1$ condition is satisfied. Let further the covariance matrix F_k^0 be positive definite at all times $k = 0, 1, \dots, T-1$. Then for any contingent claim $H = \bar{Y}_T \in \mathbb{L}_T^{2,1}$, there exists a local risk-minimizing strategy $\hat{\varphi}^\lambda = (\hat{X}^\lambda, \hat{Y}^\lambda)$ under illiquidity with price impact with $\hat{X}_{T+1}^\lambda = \bar{X}_{T+1}$ (=0) and $\hat{Y}_T^\lambda = \bar{Y}_T$. Moreover, the optimal strategy can be written down in explicit form, in particular*

$$\hat{X}_{k+1}^\lambda = (F_k^\lambda)^{-1} b_k^\lambda \quad \mathbb{P}-a.s.\ for\ k = 0, \dots, T-1, \tag{3.4.32}$$

$$\hat{Y}_k^\lambda = \mathbb{E}[\hat{W}_k^\lambda | \mathcal{F}_k] - \hat{X}_{k+1}^{\lambda,*} S_k \quad \mathbb{P}-a.s.\ for\ k = 0, 1, \dots, T-1, \tag{3.4.33}$$

where $\hat{W}_k^\lambda = H - \sum_{m=k+1}^T \hat{X}_m^{\lambda,*} \Delta S_m$.

Proof. The idea of the proof is a backward induction argument on $k = 0, 1, \dots, T$. First set $\hat{X}_{T+1}^\lambda = \bar{X}_{T+1}$ and $\hat{Y}_T^\lambda = \bar{Y}_T$. So, fix some $k \in \{0, 1, \dots, T-2\}$ and assume that at times $l = k, \dots, T-2$

(i) $\hat{X}_{l+2}^{\lambda,j} \Delta S_{l+2}^j \in \mathbb{L}_T^{2,1}$ and $\hat{X}_{l+2}^{\lambda,j} \in \mathbb{L}_T^{2,1}$,

(ii) $|\hat{X}_{l+2}^{\lambda,j}|^2 (1 + \lambda^j) M^j \in \mathbb{L}_T^{1,1}$,

(iii) $\hat{X}_{l+2}^{\lambda,*} S_{l+1} + \hat{Y}_{l+1}^\lambda \in \mathbb{L}_T^{2,1}$, $\hat{Y}_{l+1}^\lambda \in \mathcal{F}_{l+1}$,

for all $j = 1, \dots, d$ holds. At time k we want to minimize the expression (3.4.12) over all $X_{k+1}^{'}$ and show that the following properties are fulfilled for all $j = 1, \dots, d$:

(i) $X_{k+1}^{',j} \Delta S_{k+1}^j \in \mathbb{L}_T^{2,1}$ and $X_{k+1}^{',j} \in \mathbb{L}_T^{2,1}$,

(ii) $|X_{k+1}^{',j}|^2 (1 + \lambda^j) M^j \in \mathbb{L}_T^{1,1}$,

(iii) $(X_{k+1}^{'})^* S_k + Y_k^{'} \in \mathbb{L}_T^{2,1}$, $Y_k^{'} \in \mathcal{F}_k$,

Properties (i) - (iii) will then guarantee that $(\hat{X}^\lambda, \hat{Y}^\lambda) \in \Theta_d^\lambda(S)$.

Let the function f_k^λ as in equation (3.4.14). Note that by the induction hypothesis all the terms in f_k^λ are integrable. By the positive definite property of F_k^λ there exists a unique solution to the minimization problem and an $\mathcal{F}_k$-measurable minimizer $\hat{X}_{k+1}^\lambda = (F_k^\lambda)^{-1} b_k$ can be computed explicitly. Moreover let $\hat{Y}_k^\lambda$ defined as in equation (3.4.33). Then $\hat{Y}_k^\lambda$ is $\mathcal{F}_k$-measurable. By the fact that $H \in \mathbb{L}_T^{2,1}$, the induction hypothesis $\sum_{m=k+2}^T \hat{X}_m^{\lambda,*} \Delta S_m \in \mathbb{L}_T^{2,1}$ and $\hat{X}_{k+1}^{\lambda,*} \Delta S_{k+1} \in \mathbb{L}_T^{2,1}$ (we show this below) we can conclude that $\hat{X}_{k+1}^{\lambda,*} S_k + \hat{Y}_k^\lambda = \mathbb{E}[\hat{W}_k^\lambda | \mathcal{F}_k] \in \mathbb{L}_T^{2,1}$.

We show first that $\hat{X}_{k+1}^{\lambda,j} \Delta S_{k+1}^j \in \mathbb{L}_T^{2,1}$. Inequality (3.4.27) of Lemma 3.4.15 imply that for a constant $C > 0$,

$$\begin{aligned} & \mathbb{E}[(\hat{X}_{k+1}^{\lambda,j} \Delta S_{k+1}^j)^2] \\ \leq \ & C\mathbb{E}\Bigg[\mathbb{V}ar(\hat{X}_{k+2}^{\lambda,*} S_{k+1} + \hat{Y}_{k+1}^\lambda | \mathcal{F}_k) \sum_{i=1}^d \alpha_{k;i,j}^\lambda \end{aligned} \tag{3.4.34}$$

$$+\sum_{i=1}^{d}\alpha^{M}_{k;i,j}\frac{1}{|F^{M}_{k;i,i}|^2}\mathbb{E}[|(1+\lambda^i)M^i|^2|\mathcal{F}_k]\mathbb{E}[|\hat{X}^{\lambda,i}_{k+2}|^2|\mathcal{F}_k]\Bigg],$$

holds. Furthermore, since the $ML.1$ condition is satisfied then for a constant $C>0$

$$\begin{aligned}&\mathbb{E}[(\hat{X}^{\lambda,j}_{k+1}\Delta S^{j}_{k+1})^2]\\ \leq\ & C\mathbb{E}\left[\mathbb{V}ar(\hat{X}^{\lambda,*}_{k+2}S_{k+1}+\hat{Y}^{\lambda}_{k+1}|\mathcal{F}_k)\sum_{i=1}^{d}\alpha^{\lambda}_{k;i,j}+\sum_{i=1}^{d}\alpha^{M}_{k;i,j}\mathbb{E}[|\hat{X}^{\lambda,i}_{k+2}|^2|\mathcal{F}_k]\right].\end{aligned}\tag{3.4.35}$$

Since from Lemma 3.4.14 the terms $\alpha^{\lambda}_{k;i,j}$, $\alpha^{M}_{k;i,j}$ are uniformly bounded in k and ω and since by induction hypothesis $\hat{X}^{\lambda,*}_{k+2}S_{k+1}+\hat{Y}^{\lambda}_{k+1}$ and $\hat{X}^{\lambda,i}_{k+2}$ both in $\mathbb{L}^{2,1}_{T}$ for all $i=1,\dots,d$ then $\hat{X}^{\lambda,j}_{k+1}\Delta S^{j}_{k+1}\in\mathbb{L}^{2,1}_{T}$. In the same way and using inequality (3.4.28) of Lemma 3.4.15 we can show that $\hat{X}^{\lambda,j}_{k+1}\in\mathbb{L}^{2,1}_{T}$. Indeed we have

$$\begin{aligned}&\mathbb{E}[(\hat{X}^{\lambda,j}_{k+1})^2]\\ \leq\ & C\mathbb{E}\left[\mathbb{V}ar(\hat{X}^{\lambda,*}_{k+2}S_{k+1}+\hat{Y}^{\lambda}_{k+1}|\mathcal{F}_k)\sum_{i=1}^{d}\beta^{\lambda}_{k;i,j}+\sum_{i=1}^{d}\beta^{M}_{k;i,j}\mathbb{E}[|\hat{X}^{\lambda,i}_{k+2}|^2|\mathcal{F}_k]\right].\end{aligned}\tag{3.4.36}$$

Next we show that the liquidity costs and price impact terms are integrable, that is

$$\sum_{j=1}^{d}\lambda^jM^j|\hat{X}^{\lambda,j}_{k+1}|^2+\sum_{j=1}^{d}(1+\lambda^j)M^j(\hat{X}^{\lambda,j}_{k+2}-\hat{X}^{\lambda,j}_{k+1})^2\in\mathbb{L}^{1,1}_{T}\,.\tag{3.4.37}$$

Since $\hat{X}^{\lambda}_{k+1}$ is a minimizer, then from expression (3.4.12) and by choosing $X_{k+1}=0$ for the right hand side, it holds

$$\mathbb{V}ar(\hat{X}^{\lambda,*}_{k+2}S_{k+1}+\hat{Y}^{\lambda}_{k+1}-(\hat{X}^{\lambda}_{k+1})^*\Delta S_{k+1}|\mathcal{F}_k)\tag{3.4.38}$$

$$+\mathbb{E}\left[\sum_{j=1}^{d}\lambda^jM^j|\hat{X}^{\lambda,j}_{k+1}|^2\middle|\mathcal{F}_k\right]+\mathbb{E}\left[\sum_{j=1}^{d}(1+\lambda^j)M^j(\hat{X}^{\lambda,j}_{k+2}-\hat{X}^{\lambda,j}_{k+1})^2\middle|\mathcal{F}_k\right]$$

$$\leq\mathbb{V}ar(\hat{X}^{\lambda,*}_{k+2}S_{k+1}+\hat{Y}^{\lambda}_{k+1}|\mathcal{F}_k)+\mathbb{E}\left[\sum_{j=1}^{d}(1+\lambda^j)M^j|\hat{X}^{\lambda,j}_{k+2}|^2\middle|\mathcal{F}_k\right].\tag{3.4.39}$$

Since the conditional variance is non-negative then applying expectation yields

$$\mathbb{E}\left[\sum_{j=1}^{d}\lambda^jM^j|\hat{X}^{\lambda,j}_{k+1}|^2\middle|\mathcal{F}_k\right]+\mathbb{E}\left[\sum_{j=1}^{d}(1+\lambda^j)M^j(\hat{X}^{\lambda,j}_{k+2}-\hat{X}^{\lambda,j}_{k+1})^2\middle|\mathcal{F}_k\right]\tag{3.4.40}$$

$$\leq\mathbb{E}[|\hat{X}^{\lambda,*}_{k+2}S_{k+1}+\hat{Y}^{\lambda}_{k+1}|^2]+\mathbb{E}\left[\sum_{j=1}^{d}(1+\lambda^j)M^j|\hat{X}^{\lambda,j}_{k+2}|^2\middle|\mathcal{F}_k\right],$$

where we have used the inequality $\mathbb{V}ar(X) \le \mathbb{E}|X|^2$. Then the liquidity and price impact terms $\sum_{j=1}^d \lambda^j M^j |\hat{X}_{k+1}^{\lambda,j}|^2 + \sum_{j=1}^d (1+\lambda^j) M^j (\hat{X}_{k+2}^{\lambda,j} - \hat{X}_{k+1}^{\lambda,j})^2$ are in $\mathbb{L}_T^{1,1}$. Indeed this is due to the inductive hypothesis, $\hat{X}_{k+2}^{\lambda,*} S_{k+1} + \hat{Y}_{k+1}^{\lambda} \in \mathbb{L}_T^{2,1}$ and $(1+\lambda^j) M^j |\hat{X}_{k+2}^{\lambda,j}|^2 \in \mathbb{L}_T^{1,1}$ for all $j = 1, \dots, d$. More precisely and since λ^j and M^j are both non-negative then it holds that $\lambda^j M^j |\hat{X}_{k+1}^{\lambda,j}|^2 \in \mathbb{L}_T^{1,1}$ and $(1+\lambda^j) M^j (\hat{X}_{k+2}^{\lambda,j} - \hat{X}_{k+1}^{\lambda,j})^2 \in \mathbb{L}_T^{1,1}$ for all $j = 1, \dots, d$.

The last step for showing and completing the induction argument is to show that $|\hat{X}_{k+1}^{\lambda,j}|^2 (1 + \lambda^j) M^j \in \mathbb{L}_T^{1,1}$. Thus the liquidity and price impact terms in the next time step will be again integrable. Since λ^j is bounded in $[0,1]$ by assumption then $|\hat{X}_{k+1}^{\lambda,j}|^2 (1+\lambda^j) M^j \le 2 |\hat{X}_{k+1}^{\lambda,j}|^2 M^j$, where $|\hat{X}_{k+1}^{\lambda,j}|^2 M^j \in \mathbb{L}_T^{1,1}$ by inequality (3.4.40) and since $|\hat{X}_{k+2}^{\lambda,j}|^2 M^j \in \mathbb{L}_T^{1,1}$ by the induction hypothesis.

For showing the base case at time $k = T$ where $\hat{Y}_T^{\lambda} = H$ one just needs the same arguments and the conditions on H and $\bar{X}_{T+1}$, $\bar{Y}_T$. In particular, since $H = \hat{Y}_T^{\lambda} \in \mathbb{L}_T^{2,1}$ and $\hat{X}_{T+1}^{\lambda} = 0$, then by Lemma 3.4.15 and 3.4.14 it holds that $\hat{X}_T^{\lambda,j} \Delta S_T^j \in \mathbb{L}_T^{2,1}$ and $\hat{X}_T^{\lambda,j} \in \mathbb{L}_T^{2,1}$ for all j. Moreover, note that with the assumptions $\hat{X}_{T+1}^{\lambda,j} = 0, \hat{Y}_T^{\lambda} \in \mathbb{L}_T^{2,1}$ one can show the integrability of the liquidity and price impact terms using (3.4.40).

Lastly, by defining the $\mathcal{F}_{T-1}$-measurable random variable

$$\hat{Y}_{T-1}^{\lambda} = \mathbb{E}[H - \hat{X}_T^{\lambda,*} \Delta S_T | \mathcal{F}_k] - \hat{X}_T^{\lambda,*} S_{T-1}, \tag{3.4.41}$$

then $\hat{X}_T^{\lambda,*} S_{T-1} + \hat{Y}_{T-1}^{\lambda} = \mathbb{E}[H - \hat{X}_T^{\lambda,*} \Delta S_T | \mathcal{F}_k]$ belongs to $\mathbb{L}_T^{2,1}$.

Since at each time k we have

$$\mathbb{E}[C_T(\hat{\varphi}^{\lambda}) - C_k(\hat{\varphi}^{\lambda}) | \mathcal{F}_k] = 0. \tag{3.4.42}$$

from the definition of $\hat{Y}^{\lambda}$, then we can conclude the martingale property of $C(\hat{\varphi}^{\lambda})$. Then Proposition 3.3.6 completes the proof, since both properties are satisfied. The trading strategy $\hat{\varphi}^{\lambda} = (\hat{X}^{\lambda}, \hat{Y}^{\lambda})$ is local risk-minimizing under illiquidity with price impact. □

Remark 3.4.18. *In the simple case of $d = 1$, the LRM-strategy $\hat{\varphi}^{\lambda} = (\hat{X}^{\lambda}, \hat{Y}^{\lambda})$ under illiquidity with price impact equals*

$$\hat{X}_{k+1}^{\lambda} = \frac{\mathbb{C}ov(V_{k+1}(\hat{\varphi}^{\lambda}), \Delta S_{k+1} | \mathcal{F}_k) + \mathbb{E}[(1+\lambda) M \hat{X}_{k+2}^{\lambda} | \mathcal{F}_k]}{\mathbb{V}ar(\Delta S_{k+1} | \mathcal{F}_k) + \mathbb{E}[(1+2\lambda) M | \mathcal{F}_k]}, \tag{3.4.43}$$

$$V_k(\hat{\varphi}^{\lambda}) = \mathbb{E}\left[H - \sum_{m=k+1}^{T} \hat{X}_m^{\lambda} \Delta S_m \middle| \mathcal{F}_k \right]. \tag{3.4.44}$$

For λ equal zero we have the local risk minimization strategy $\hat{\varphi}^0 = (\hat{X}^0, \hat{Y}^0)$ under illiquidity without accounting for lasting impact. For M tending to zero we get the classical local risk minimization strategy without accounting for illiquidity. Let us denote this by $\bar{\varphi} = (\bar{X}, \bar{Y})$. Also, one can easily note that in the case where S is a martingale, then $V_k(\hat{\varphi}^{\lambda}) = V_k(\hat{\varphi}^0) = V_k(\bar{\varphi}) = \mathbb{E}[H | \mathcal{F}_k]$. That means in all cases the book values are equal.

3.4.4 F^{λ}-property in terms of the covariance matrix F^0

The main existence Theorem 3.4.17 was proven using the crucial F^{λ}-property from Definition 3.4.12. Similar to Christodoulou et al. (2018) (Chapter 2, Section 2.3.4), this condition can be reduce to a sufficient criterion in terms of a the covariance matrix F^0. In the following we are making use of the definition of a principal submatrix (Chapter 2, Section 2.3.4).

A sufficient condition in terms of the covariance matrix F^0 is given in the next Lemma 3.4.19.

Lemma 3.4.19. *S has the F^{λ}-property if there exists some $\delta \in (0,1)$ such that*

$$\det(P_k^0) - (1-\delta)\det(P_k^{A^0}) \geq 0, \tag{3.4.45}$$

for all principal submatrices P_k^0 of F_k^0 and principal submatrices $P_k^{A^0}$ of $F_k^{A^0}$ where $F_k^{A^0} := \operatorname{diag}(A_{k;1}^0, \dots, A_{k;d}^0)$ of size $l \times l$ where $l \in \{2,\dots,d\}$ and for all $k = 0,1,\dots,T$.

Proof. The proof follows exactly the same steps as Lemma 3.11 in Christodoulou et al. (2018). □

An example, when the F^{λ}-property holds, is given in the next Propositions.

Proposition 3.4.20. *Assume that the covariance matrix F_k^0 is positive definite at all times $k = 0,1,\dots,T$ and S^j has independent returns for each $j = 1,\dots,d$. Then the F^{λ}-property holds.*

Proof. The proof follows exactly the same steps as Proposition 3.12 in Christodoulou et al. (2018). □

Proposition 3.4.21. *Assume that the covariance matrix F_k^0 at all times $k = 0,1,\dots,T$ is positive definite and S^j has independent increments for each $j = 1,\dots,d$. Then the F^{λ}-property holds.*

Proof. Follows by analogous arguments as in Proposition 3.4.20. □

3.5 An Alternative Optimality Criterion

As we saw in Section 3.2 the optimality criterion under which we work is treating the liquidity cost and price impact terms separately. An alternative approach is to treat these terms together and consider their variation coming from the current liquidity cost and the price impact cost incurred by the strategy. Since the structure together with the proofs of this section are similar to the previous Sections 3.3 and 3.4 we will avoid going into detail at some places.

Instead of considering the optimality criterion function (3.3.12) we will consider

$$T_k^{\alpha,\lambda}(\varphi) := \mathbb{E}[(C_T(\varphi) - C_k(\varphi))^2 | \mathcal{F}_k] + \alpha \mathbb{E}\left[\left| PI_k^{\lambda}(X) + TC_k(X) \right| | \mathcal{F}_k \right], \tag{3.5.1}$$

for some given $\alpha \in \mathbb{R}^+$.

Note that (3.5.1) is clearly justified analogously as in Subsection 3.3.1. A similar observation as in Remark 3.3.3 holds also here.

Definition 3.5.1 specifies the LRM strategy under illiquidity with price impact under the alternative optimality criterion function.

Definition 3.5.1. *A trading strategy $\varphi = (X, Y)$ is called locally risk-minimizing under illiquidity with price impact if*

$$T_k^{\alpha,\lambda}(\varphi) \leq T_k^{\alpha,\lambda}(\varphi') \qquad \mathbb{P}-a.s. \tag{3.5.2}$$

for any time $k \in \{0, 1, \dots, T-1\}$ and any local perturbation φ' of φ at time k.

Remark 3.5.2. *By a slight abuse of notation, the name "locally risk-minimizing under illiquidity with price impact" of the optimal strategy in Definition 3.5.1 remains the same as the one in Definition 3.3.2 for this Section 3.5. It is meant to be understood under the alternative optimality criterion function (3.5.1).*

The following Lemma 3.5.3 and Proposition 3.5.4 is similar to Lemma 3.3.5 and Proposition 3.3.6 and so we provide them without any proof.

Lemma 3.5.3. *Assume $C(\varphi)$ is a martingale and φ' a local perturbation of φ at time k. Then it holds*

$$\begin{aligned} T_k^{\alpha,\lambda}(\varphi') := \mathbb{E}[R_{k+1}(\varphi)|\mathcal{F}_k] + \mathbb{E}[(\Delta C_{k+1}(\varphi'))^2|\mathcal{F}_k] \\ + \alpha\mathbb{E}\left[\left|PI_k^\lambda(X, X') + LC_k(X, X')\right| |\mathcal{F}_k\right]. \end{aligned} \tag{3.5.3}$$

Proof. See Lemma 3.3.5. □

Proposition 3.5.4. *A trading strategy $\varphi = (X, Y)$ is LRM under illiquidity with price impact if and only if the two following properties are satisfied:*

(i) *$C(\varphi)$ is a martingale.*

(ii) *For each $k \in \{0, 1, \dots, T-1\}$, X_{k+1} minimizes*

$$\mathbb{V}ar(V_{k+1}(\varphi) - (X'_{k+1})^* \Delta S_{k+1}|\mathcal{F}_k) + \alpha\mathbb{E}\left[\left|PI_k^\lambda(X, X') + LC_k(X, X')\right| |\mathcal{F}_k\right], \tag{3.5.4}$$

over all $\mathcal{F}_k$-measurable random variables X'_{k+1} so that $(X'_{k+1})^* \Delta S_{k+1} \in \mathbb{L}_T^{2,1}$ and $PI_k^\lambda(X, X')$ $+LC_k(X, X') \in \mathbb{L}_T^{1,1}$.

Proof. See Proposition 3.3.6. □

Note that in Proposition 3.5.4 no additional assumptions in the processes λ and M are required. Nevertheless, in order to get more explicit results for the optimal strategy further conditions will be assumed in the next Section 3.5.1.

3.5.1 Explicit-form optimal LRM strategy

In this section we will assume from now on that the resilience process λ is a 1-dimensional random variable taking values in $[0,1]$ and the liquidity level process M is an $\mathbb{R}^d_{\geq 0}$-valued, time-independent process. Also, let $\bar{X}_{T+1} = 0$. Then the following Lemma 3.5.5 holds.

Lemma 3.5.5. *For a strategy $\varphi = (X, Y)$, it holds*

$$\begin{aligned}
&\mathbb{E}[|PI_k^\lambda(X) + LC_k(X)||\mathcal{F}_k] \\
&= \mathbb{E}\left[\left| -\lambda \sum_{j=1}^d M^j |X_{k+1}^j|^2 + (1-\lambda)\sum_{j=1}^d M^j \sum_{m=k+1}^T (X_{m+1}^j - X_m^j)^2 \right| \mathcal{F}_k\right].
\end{aligned} \tag{3.5.5}$$

Proof. As in the proof of Lemma 3.4.2 and using Lemma 3.4.1, since $\bar{X}_{T+1} = 0$ it holds

$$PI_k^\lambda(X) = -\lambda\left(\sum_{j=1}^d M^j |X_{k+1}^j|^2 + \sum_{j=1}^d M^j |X_T^j|^2 + \sum_{j=1}^d M^j \sum_{m=k+1}^{T-1} (X_{m+1}^j - X_m^j)^2\right). \tag{3.5.6}$$

Then we can calculate

$$\begin{aligned}
PI_k^\lambda(X) + LC_k(X) &= -\lambda\left(\sum_{j=1}^d M^j |X_{k+1}^j|^2 + \sum_{j=1}^d M^j \sum_{m=k+1}^T (X_{m+1}^j - X_m^j)^2\right) \\
&\quad + \sum_{j=1}^d M^j \sum_{m=k+1}^T (X_{m+1}^j - X_m^j)^2,
\end{aligned} \tag{3.5.7}$$

hence,

$$PI_k^\lambda(X) + LC_k(X) = -\lambda \sum_{j=1}^d M^j |X_{k+1}^j|^2 + (1-\lambda)\sum_{j=1}^d M^j \sum_{m=k+1}^T (X_{m+1}^j - X_m^j)^2. \tag{3.5.8}$$

By applying conditional expectation, equation (3.5.5) follows. □

In the next Corollary 3.5.6 we are able to get rid of the absolute value in (3.5.5) in the full permanent price impact case.

Corollary 3.5.6. *Assume that λ is a constant and equals* 1. *Then for a strategy $\varphi = (X, Y)$, it holds*

$$\mathbb{E}[|PI_k^\lambda(X) + LC_k(X)||\mathcal{F}_k] = \mathbb{E}\left[\sum_{j=1}^d M^j |X_{k+1}^j|^2 \middle| \mathcal{F}_k\right]. \tag{3.5.9}$$

Proof. Follows clearly from Lemma 3.5.5. □

By applying Corollary 3.5.6 on the optimality criterion function (3.5.1) then Proposition 3.5.4 is simplified to the following Proposition 3.5.7 from where we will be able to construct explicitly an optimal strategy under illiquidity with price impact with respect to the Definition 3.5.1. For simplicity we will assume that $\alpha = 1$ which means an equal concern between the risk to be hedged as incurred from market price fluctuations and the expected variation of liquidity costs with price impact.

Proposition 3.5.7. *Assume that λ is a constant and equals 1. A trading strategy $\varphi = (X, Y)$ is LRM under illiquidity with price impact if and only if the two following properties are satisfied:*

(i) $C(\varphi)$ is a martingale.

(ii) For each $k \in \{0, 1, \ldots, T-1\}$, X_{k+1} minimizes

$$\mathbb{V}ar(V_{k+1}(\varphi) - (X'_{k+1})^* \Delta S_{k+1} | \mathcal{F}_k) + \mathbb{E}\left[\sum_{j=1}^{d} M^j |X'^j_{k+1}|^2 \middle| \mathcal{F}_k \right], \qquad (3.5.10)$$

over all $\mathcal{F}_k$-measurable random variables X'_{k+1} so that $(X'_{k+1})^* \Delta S_{k+1} \in \mathbb{L}^{2,1}_T$ and $M^j |X'^j_{k+1}| \in \mathbb{L}^{1,1}_T$.

As already mentioned at the beginning of this section, since we are going to use similar “machinery” and arguments as in Section 3.4 we omit some details and explanations because of the ones already provided.

Let us introduce some additional notation where the superscript in λ^1 is to emphasize the full permanent price impact case $\lambda = 1$, which we are currently investigating:

$$\bar{A}^M_{k;j} := \mathbb{E}[M^j | \mathcal{F}_k], \qquad \bar{A}^{\lambda^1}_{k;j} := A^0_{k;j} + \bar{A}^M_{k;j}, \qquad (3.5.11)$$

for all $i, j = 1, \ldots, d$ and $k = 0, \ldots, T-1$.

Moreover define the function $\bar{f}^{\lambda^1}_k : \mathbb{R}^d \times \Omega \to \mathbb{R}^+$ by

$$\begin{aligned} \bar{f}^{\lambda^1}_k (c, \omega) = {} & \sum_{j=1}^{d} |c_j|^2 \bar{A}^{\lambda^1}_{k;j}(\omega) - 2 \sum_{j=1}^{d} c_j b^0_{k;j}(\omega) + \sum_{j \neq i} c_j c_i D_{k;j,i}(\omega) \\ & + \mathbb{V}ar(V_{k+1} | \mathcal{F}_k)(\omega). \end{aligned} \qquad (3.5.12)$$

Let $\bar{F}^{\lambda^1}_k \in \mathbb{R}^{d \times d}$ with $\bar{F}^{\lambda^1}_{k;i,j} = D_{k;i,j}$ for $i \neq j$, $\bar{F}^{\lambda^1}_{k;i,j} = \bar{A}^{\lambda^1}_{k;j}$ for $i = j$. Let $\bar{F}^M_k = diag(\bar{A}^M_{k;1}, \ldots, \bar{A}^M_{k;d})$ where note that $\bar{F}^{\lambda^1}_k = F^0_k + \bar{F}^M_k$. Then the linear equation system

$$\bar{F}^{\lambda^1}_k \, c = b^0_k, \qquad (3.5.13)$$

needs to be solved in order to calculate the extreme points.

The Hessian matrix is $H_{\bar{f}^{\lambda^1}_k}(c) = 2\bar{F}^{\lambda^1}_k$. Moreover, if the covariance matrix F^0_k is positive definite, then the matrix $\bar{F}^{\lambda^1}_k$ is invertible and $\bar{c}^* := (\bar{F}^{\lambda^1}_k)^{-1} b^0_k$ is an $\mathcal{F}_k$-measurable global minimizer.

For ensuring that the optimal strategy belongs to the space of strategies $\Theta_d^{\lambda}(S)$ we will need the bounded mean-variance tradeoff property of the process S and two more additional conditions similar to the MS-condition and the F^{λ}-diagonal condition from Section 3.4.1. These are the following Definitions 3.5.8 and 3.5.9.

Definition 3.5.8. *We say that S satisfies the M-condition from above if for some constant $C > 0$*

$$\frac{(\mathbb{E}[M^j|\mathcal{F}_k])^2}{\mathbb{V}ar(S^j_{k+1}|\mathcal{F}_k)} \leq C \quad \mathbb{P}-a.s. \quad \text{for all } j = 1, \dots, d \tag{3.5.14}$$

uniformly in k and ω. Furthermore S satisfies the M-condition from below if for some constant $\tilde{C} > 0$

$$\frac{(\mathbb{E}[M^j|\mathcal{F}_k])^2}{\mathbb{V}ar(S^j_{k+1}|\mathcal{F}_k)} \geq \tilde{C} \quad \mathbb{P}-a.s. \quad \text{for all } j = 1, \dots, d \tag{3.5.15}$$

uniformly in k and ω. If both bounds hold then we say that S satisfies the M-condition.

Definition 3.5.9. *We say that S satisfies the $\bar{F}^{\lambda^1}$-diagonal condition with respect to the matrix $\bar{F}^{\lambda^1}$ if for some constant $C > 0$*

$$\sqrt{\mathbb{V}ar(\Delta S^j_{k+1}|\mathcal{F}_k)} + \frac{\mathbb{E}[M^j|\mathcal{F}_k]}{\sqrt{\mathbb{V}ar(S^j_{k+1}|\mathcal{F}_k)}} \geq C \quad \mathbb{P}-a.s. \quad \text{for all } j = 1, \dots, d \tag{3.5.16}$$

uniformly in k and ω and if for some constant $\tilde{C} > 0$

$$\frac{\sqrt{\mathbb{V}ar(S^j_{k+1}|\mathcal{F}_k)}}{\mathbb{E}[M^j|\mathcal{F}_k]} + \frac{1}{\sqrt{\mathbb{V}ar(\Delta S^j_{k+1}|\mathcal{F}_k)}} \geq \tilde{C} \quad \mathbb{P}-a.s. \quad \text{for all } j = 1, \dots, d \tag{3.5.17}$$

uniformly in k and ω.

Remark 3.5.10. *A similar condition to the $ML.1$-condition will not be needed. The reason is because the strategy X_{k+2} at time $k+2$ does not appear in the liquidity terms $\mathbb{E}[\sum_{j=1}^d M^j|X^j_{k+1}|^2|\mathcal{F}_k]$. Hence, a similar term to b^M as in (3.4.13) will not be needed to be defined. So, a further estimation of such a term as we did in the proof of Theorem 3.4.17 using Lemma 3.4.15 will not be necessary.*

Clearly, similar Propositions to the ones of Section 3.4.1 can be formulated.

Proposition 3.5.11. *If S satisfies $\tilde{C} \leq \mathbb{V}ar(\Delta S^j_{k+1}|\mathcal{F}_k) \leq C$ for some positive constants $C, \tilde{C}$ and for all $j = 1, \dots, d$, then the $\bar{F}^{\lambda^1}$-diagonal condition holds. In particular, if S has independent increments then S has bounded mean-variance tradeoff and satisfies the $\bar{F}^{\lambda^1}$-diagonal condition.*

Proof. As in Proposition 3.4.8. □

Proposition 3.5.12. *If the M-condition holds then the $\bar{F}^{\lambda^1}$-diagonal condition holds. In particular, if S has independent returns and the M-condition is satisfied then S has bounded mean-variance tradeoff and satisfies the $\bar{F}^{\lambda^1}$-diagonal condition.*

Proof. As in Proposition 3.4.9 □

The critical $\bar{F}^{\lambda^1}$-property in terms of the matrix $\bar{F}^{\lambda^1}$ can also be defined as in the following Definition 3.5.13.

Definition 3.5.13. *We say that the process S has the $\bar{F}^{\lambda^1}$-property with respect to the matrix $\bar{F}^{\lambda^1}$ if there exists some $\delta \in (0,1)$ such that*

$$\det(\bar{F}_k^{\lambda^1}) - (1-\delta)\det(\bar{F}_k^{\bar{A}^{\lambda^1}}) \geq 0, \tag{3.5.18}$$

for all $k = 0, 1, \ldots, T$ where $\bar{F}_k^{\bar{A}^{\lambda^1}} := \mathrm{diag}(\bar{A}_{k;1}^{\lambda^1}, \ldots, \bar{A}_{k;d}^{\lambda^1})$.

Recall from Section 3.4.4 that this property can be reduced to a sufficient condition in terms of the covariance matrix of the process S. Then it is clear that the following Propositions 3.5.14 and 3.5.15 hold in terms of the $\bar{F}^{\lambda^1}$-property with respect to the matrix $\bar{F}^{\lambda^1}$.

Proposition 3.5.14. *Assume that the covariance matrix F_k^0 is positive definite at all times $k = 0, 1, \ldots, T$ and S^j has independent returns for each $j = 1, \ldots, d$. Then the $\bar{F}^{\lambda^1}$-property holds.*

Proof. As in Proposition 3.4.20. □

Proposition 3.5.15. *Assume that the covariance matrix F_k^0 at all times $k = 0, 1, \ldots, T$ is positive definite and S^j has independent increments for each $j = 1, \ldots, d$. Then the $\bar{F}^{\lambda^1}$-property holds.*

Proof. As in Proposition 3.4.21. □

Furthermore, using the appropriate notation, similar Lemmas as in Subsection 2.3.2 hold. We will just provide them with the current notation, without giving their proofs since these are very similar to the ones already stated previously.

Define the following:

$$\bar{\alpha}_{k;i,j}^{\lambda^1} := F_{k;j,j}^0 F_{k;i,i}^0 |(\bar{F}_{k;j,i}^{\lambda^1})^{-1}|^2, \qquad \bar{\beta}_{k;i,j}^{\lambda^1} := F_{k;i,i}^0 |(\bar{F}_{k;j,i}^{\lambda^1})^{-1}|^2, \tag{3.5.19}$$

for $i, j = 1, \ldots, d$ and $k = 0, \ldots, T$ when the inverse matrix $(\bar{F}_k^{\lambda^1})^{-1}$ of $\bar{F}_k^{\lambda^1}$ exists.

Lemma 3.5.16. *Assume that S has the $\bar{F}^{\lambda^1}$-property and satisfies the $\bar{F}^{\lambda^1}$-diagonal condition. Then the terms $\bar{\alpha}_{k;i,j}^{\lambda^1}$ and $\bar{\beta}_{k;i,j}^{\lambda^1}$ are uniformly bounded in k and ω for all $i, j = 1, \ldots, d$.*

Proof. Follows as in the proof of Lemma 3.4.14 by using a similar result as in Lemma 3.4.11 with the appropriate terms. □

Lemma 3.5.17. *Assume that $(\bar{F}_k^{\lambda^1})^{-1}$ exists for $k \in \{0, 1, \ldots, T\}$ and S has bounded mean-variance tradeoff. Let (X, Y) be any trading strategy. Then there exists some constant $C > 0$ such that*

$$\mathbb{E}[(((\bar{F}_k^{\lambda^1})^{-1} b_k^0)_j \Delta S_{k+1}^j)^2] \leq C \mathbb{E}\left[\mathbb{V}ar(V_{k+1}|\mathcal{F}_k) \sum_{i=1}^{d} \bar{\alpha}_{k;i,j}^{\lambda^1}\right], \tag{3.5.20}$$

$$\mathbb{E}[(((\bar{F}_k^{\lambda})^{-1}b_k^0)_j)^2] \leq C\mathbb{E}\left[\mathbb{V}ar(V_{k+1}|\mathcal{F}_k)\sum_{i=1}^{d}\bar{\beta}_{k;i,j}^{\lambda^1}\right], \tag{3.5.21}$$

for all $j = 1, \ldots, d$ where $((\bar{F}_k^{\lambda^1})^{-1}b_k^0)_j$ is the j-th component of the vector $((\bar{F}_k^{\lambda^1})^{-1}b_k^0)$.

Proof. Follows as in the proof of Lemma 3.4.15. □

Now we are ready to state the main existence theorem which provides us with an explicit formula of a local risk minimization strategy under illiquidity with full permanent price impact according to Definition 3.5.1. The proof is very similar to the one of Theorem 3.4.17.

Theorem 3.5.18 (Existence result). *Let λ equal to 1. Assume that S has the $\bar{F}^{\lambda^1}$-property, bounded mean-variance tradeoff and satisfies the $\bar{F}^{\lambda^1}$-diagonal condition. Let further the covariance matrix F_k^0 be positive definite at all times $k = 0, 1, \ldots, T-1$. Then for any contingent claim $H = \bar{Y}_T \in \mathbb{L}_T^{2,1}$, there exists a local risk-minimizing strategy $\hat{\varphi}^{\lambda^1} = (\hat{X}^{\lambda^1}, \hat{Y}^{\lambda^1})$ under illiquidity with full permanent price impact according to Definition 3.5.1 with $\hat{X}_{T+1}^{\lambda^1} = \bar{X}_{T+1}(= 0)$ and $\hat{Y}_T^{\lambda^1} = \bar{Y}_T$. Moreover, the optimal strategy can be written down in explicit form, in particular*

$$\hat{X}_{k+1}^{\lambda^1} = (\bar{F}_k^{\lambda^1})^{-1}b_k^0 \quad \mathbb{P}-a.s. \text{ for } k = 0, \ldots, T-1, \tag{3.5.22}$$

$$\hat{Y}_k^{\lambda^1} = \mathbb{E}[\hat{W}_k^{\lambda^1}|\mathcal{F}_k] - \hat{X}_{k+1}^{\lambda^1,*}S_k \quad \mathbb{P}-a.s. \text{ for } k = 0, 1, \ldots, T-1, \tag{3.5.23}$$

where $\hat{W}_k^{\lambda^1} = H - \sum_{m=k+1}^{T}\hat{X}_m^{\lambda^1,*}\Delta S_m$.

Proof. As in Theorem 3.4.17. □

Remark 3.5.19. *From Theorem 3.5.18 and for the 1-dimensional case, $d = 1$, the LRM-strategy $\hat{\varphi}^{\lambda^1} = (\hat{X}^{\lambda^1}, \hat{Y}^{\lambda^1})$ under illiquidity with full permanent price impact with respect to Definition 3.5.1 has the explicit form*

$$\hat{X}_{k+1}^{\lambda^1} = \frac{\mathbb{C}ov(V_{k+1}(\hat{\varphi}^{\lambda^1}), \Delta S_{k+1}|\mathcal{F}_k)}{\mathbb{V}ar(\Delta S_{k+1}|\mathcal{F}_k) + \mathbb{E}[M|\mathcal{F}_k]}, \tag{3.5.24}$$

$$V_k(\hat{\varphi}^{\lambda^1}) = \mathbb{E}\left[H - \sum_{m=k+1}^{T}\hat{X}_m^{\lambda^1}\Delta S_m \middle| \mathcal{F}_k\right]. \tag{3.5.25}$$

Note that for M equal to zero we get the classical local risk minimization strategy without accounting for illiquidity.

4 Complications and consistent specification of intensity-based credit migration bond models of HJM type

Contributions of the thesis' author:

This chapter is a joint work of P. Christodoulou, Prof. Dr. Thilo Meyer-Brandis, Prof. Dr. Christian Fries and Dr. Lorenzo Torricelli. It is based on Christodoulou et al. (2019). P. Christodoulou was significantly involved in the development of all parts of that paper. In particular, P. Christodoulou made major contributions in all of the proofs in Sections 4.3 and 4.4 together with the main results of Theorems 4.3.14 and 4.4.4 as well as Corollaries 4.3.19 and 4.4.9. Section 4.4.3 as well as the following results are not included in the original paper: Remark 4.4.8, Example 4.3.2, Remark 4.3.20, Corollary 4.3.19, Remark 4.3.18, Proposition 4.3.17. The authors developed jointly Section 4.3.3 which deals with the transformation of the spreads in different settings. Section 4.5 was developed in a close cooperation of the three authors but with major parts done by P. Christodoulou. The results of Proposition 4.5.2, Remark 4.5.3, Corollary 4.5.7, Lemma 4.5.9, Remarks 4.5.12 and 4.5.13, Lemma 4.5.14 as well as Corollary 4.5.15 are not included in the original paper. Section 4.6 was developed independently by P. Christodoulou and is not included in the original paper. The results of Section 4.7 were mainly established by P. Christodoulou with the support of Dr. Lorenzo Torricelli.

4.1 Introduction

The modeling of changes over time in the credit quality of a Bond can be done mainly in two ways. One way is using structural credit (pricing) models. Representatives of this approach are for example the models of Black and Cox (1976) and Leland (1994).

The second broad category is the reduced-form approach (or intensity-based models) such as the models for example in Jarrow and Turnbull (1995) and Eberlein and Grbac (2013). Within the framework of the intensity-based methodology, the modeling of credit migrations between different credit rating classes can be done in terms of Markov chains. To our knowledge, the first paper using a discrete-time parameter Markov chain C (as well as a continuous one) was the paper of Jarrow et al. (1997). Extensions of this can be found for example in the paper of Das and Tufano (1996) and others. An important issue of all these models is the specification of the transition probabilities of the matrix while in the continuous-time version we have the transition intensities. Such models, in continuous time, are for example the models of Jarrow et al. (1997) and Bielecki and Rutkowski (2000) which also lie inside the context of the Heath-Jarrow-Morton (HJM) methodology for modeling the defaultable term structure first introduced by Jarrow and Turnbull (1995). The development of an arbitrage-free term structure model of defaultable bonds within the HJM methodology was extended by Bielecki and Rutkowski (2000) and Bielecki and Rutkowski (2004b) accounting for multiple ratings. Further, this was generalized by Eberlein and Özkan (2003) using Lévy motion as driving processes for the model. Also Özkan and Schmidt (2005), Schmidt (2006) and Jakubowski and Nieweglowski (2009) consider some further extensions where infinite-dimensional processes are considered. This approach has the characteristic that the credit migration process is endogenously given in the model which results in a conditionally Markovian model of credit risk.

In this chapter we show some difficulties resulting from the intensity based approach in continuous time. These complications essentially stem from the constraint of a risk-neutral framework imposing the interplay between the HJM term structure of the individual forward rates and the specification of the credit migration process together with the migration intensity parameters: the so-called *consistency conditions*. In the literature there are mainly two consistency conditions which appear. The one enforces only the current forward rate to be “active", i.e., an extended HJM no-arbitrage drift condition must hold for the forward rate in which the bond holder is currently trading, while the other consistency condition requires that all the forward rates in all rating classes are active at each time.

We investigate the possible complications concerning both consistency conditions in both a zero and a non-zero recovery framework. We use the no-arbitrage spread dynamics coming from the consistency conditions in order to show that the model entails some difficulties and restrictions. Within this chapter we define two kind of spreads, the *fundamental spreads* measuring the difference between the risky forward rates and the risk-free one and the *inter-rating spreads* which is the difference between the risky foward rates. More precisely in a non-zero recovery setting, i.e., if a bond issuer defaults prior to maturity the bond holder receives a reduced amount of the face value of the bond, and using the fundamental spreads we show that under some mild conditions this explodes in finite time prior to default. We show that this problematic in the model can be fixed in the following way: One starts with a zero recovery model and using an appropriate transformation a non-exploding non-zero recovery model can be constructed.

Additionally, in Bielecki and Rutkowski (2000) and Bielecki and Rutkowski (2004a) the authors propose some rather optional conditions in regard to the structure of the model. We show that under these, serious model complications can be arise and migration between the rating classes is no longer viable. For example an independent from maturity drift of the risky bond price dynamics for each rating class will imply a model without migrations between the classes.

The structure of the chapter is the following: In Section 4.2 we present the multiple rating credit risk model of Bielecki and Rutkowsky. We introduce the main ingredients of the model and give the necessary and sufficient no arbitrage conditions. Sections 4.3 and 4.4 highlight and demonstrate possible problems and restrictions of the model under the different no arbitrage consistency conditions that appear in the literature: weak and strong consistency, respectively. This is showed mainly using the fundamental as well as the inter-rating no-arbitrage spread dynamics. Under mild conditions we show explosion of the spreads in finite time with positive probability prior to default. Moreover, through a transformation of the spread we present in Section 4.3 an approach how to construct a non-explosive admissible model. In Section 4.5 we discuss the special case of equal volatility models under strong consistency and give closed form solutions of spread's ODEs. Additionally in this case we show how the spread has explosive dynamics under bounded assumptions on the initial spread value which gives us the non-admissible models. Furthermore, the case of proportional volatility spread models in analogy with the classical proportional volatility forward rate HJM model, which as is well known it explodes, is discussed in Section 4.6. The issue of the migration intensity vanishings when the market price of risk bears no dependence on maturity is illustrated in Section 4.7.

4.2 The Bielecki-Rutkowski Credit Risk Model Framework

We start by presenting the multiple ratings credit risk model in Bielecki and Rutkowski (2000) and Bielecki and Rutkowski (2004a).

For a maximum maturity $T^* > 0$, consider a standard d-dimensional Brownian motion W_t defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{P}$ is the *objective* probability measure and the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T^*]}$ supports the process W_t.

Throughout the chapter the Euclidean inner product is denoted by juxtaposition, and thus:

$$\int_0^t \beta_s dW_s = \sum_{i=1}^d \int_0^t \beta_s^i dW_s^i \quad , \quad \alpha_t\beta_t = \sum_{i=1}^d \alpha_t^i \beta_t^i \, , \tag{4.2.1}$$

for processes α_t, β_t. For $x \in \mathbb{R}^d$, $|x|$ denotes its Euclidean norm.

Define the set of rating classes $\mathcal{K} = \{1, 2, \ldots, K\}$, where K is the number of the possible credit classes, with 1 being the best rating class (AAA for S&P credit rating agency) and $K-1$ the lowest (D for S&P credit rating agency). The class K represents the default event. For each class rate $i = 1, \ldots, K-1$ denote the corresponding non-random recovery rate by $\delta_i \in [0, 1)$. In case of default from the i-th rating class, the bond holder receives a reduced payment of the bond at maturity T, with δ_i the fixed fraction of the face value at the maturity date T, where $T \le T^*$.

4.2.1 The defaultable term structure with multiple rating classes

The model in Bielecki and Rutkowski (2000) relies on the following elements.

For all maturities $T \le T^*$, based on the Heath-Jarrow-Morton approach to term structure modeling (Heath et al. (1992), Morton (1988)), the *default-free instantaneous forward rates* $f(t, T)$ for $t \le T$ are modeled by the SDEs:

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t, \quad f(0,T) > 0, \tag{4.2.2}$$

where $\alpha(\cdot, T)$ and $\sigma(\cdot, T)$ are $\mathbb{F}$-adapted stochastic processes with values in $\mathbb{R}$ and $\mathbb{R}^d$ respectively. In addition, denoting $\mathcal{T} = \{(t, T) : 0 \le t \le T \le T^*\}$, $\alpha : \mathcal{T} \to \mathbb{R}$ must be continuous on $\mathcal{T}$, non-negative and for $l = 1, \ldots, d$, $\sigma^l : \mathbb{R} \times \mathcal{T} \to \mathbb{R}$ is positive, uniformly bounded and Lipschitz continuous in its first argument. The initial value $f(0, \cdot)$ is a non-random, non-negative, Lipschitz continuous function on $[0, T^*]$.

Similarly, for $i \in \{1, 2, \dots, K-1\}$, the *instantaneous forward rate* $g_i(t,T)$, *for the rating class* i satisfies:

$$dg_i(t,T) = \alpha_i(t,T)dt + \sigma_i(t,T)dW_t, \quad g_i(0,T) > 0, \tag{4.2.3}$$

where $\alpha_i(\cdot,T)$ and $\sigma_i(\cdot,T)$ are $\mathbb{F}$-adapted stochastic processes with values in $\mathbb{R}$ and $\mathbb{R}^d$ respectively.

Note that the volatilities of the risk-free forward rate $f(t,T)$ as well as the risky forward rates $g_i(t,T)$ might depend on the rates it self. For abbreviation we write $\sigma(t,T) = \sigma(f(t,T),t,T)$ and $\sigma_i(t,T) = \sigma_i(g_i(t,T),t,T)$.

The price of a T-maturity default-free zero coupon bond is defined for $0 \leq t \leq T$ as

$$B(t,T) = \exp\left(-\int_t^T f(t,u)du\right). \tag{4.2.4}$$

Analogously, the conditional price of a bond at time $0 \leq t \leq T$ given that it trades in the rating class $i \in \mathcal{K} \setminus K$ in the time interval $[0,t]$ is defined as

$$D_i(t,T) = \exp\left(-\int_t^T g_i(t,u)du\right). \tag{4.2.5}$$

The following two Lemmas in Bielecki and Rutkowski (2000) give the dynamics of the default-free bond $B(t,T)$ and the conditional rating-based bond $D_i(t,T)$:

Lemma 4.2.1. *The default-free bond price dynamics for* $B(t,T)$ *under* $\mathbb{P}$ *are*

$$dB(t,T) = B(t,T)\left(a(t,T)dt + b(t,T)dW_t\right), \tag{4.2.6}$$

where

$$a(t,T) = f(t,t) - \int_t^T \alpha(t,u)du + \frac{1}{2}\left|\int_t^T \sigma(t,u)du\right|^2, \quad b(t,T) = -\int_t^T \sigma(t,u)du. \tag{4.2.7}$$

Proof. Lemma 2.1 in Bielecki and Rutkowski (2000). □

Lemma 4.2.2. *The conditional rating-based bond price dynamics for* $D_i(t,T)$ *under* $\mathbb{P}$ *satisfy*

$$dD_i(t,T) = D_i(t,T)(a_i(t,T)dt + b_i(t,T)dW_t), \tag{4.2.8}$$

where

$$a_i(t,T) = g_i(t,t) - \int_t^T \alpha_i(t,u)du + \frac{1}{2}\left|\int_t^T \sigma_i(t,u)du\right|^2, \quad b_i(t,T) = -\int_t^T \sigma_i(t,u)du. \tag{4.2.9}$$

Proof. See Bielecki and Rutkowski (2000). □

The short-term interest (spot) rate is $r(t) := f(t,t)$. Thus, the risk-free savings account accumulates returns at the spot rate and is denoted as

$$B(t) = \exp\left(\int_0^t r(s)ds\right). \tag{4.2.10}$$

Discounting the default-free bond price process and the conditional rating-based bond, the dynamics of $Z(t,T) := B(t)^{-1}B(t,T)$ and $Z_i(t,T) := B(t)^{-1}D_i(t,T)$ are given in the following Lemma 4.2.3.

Lemma 4.2.3. *The discounted default-free and conditional rating-based bond price processes satisfy under* $\mathbb{P}$

$$dZ(t,T) = Z(t,T)\left(\left(\frac{1}{2}\left|\int_t^T \sigma(t,u)du\right|^2 - \int_t^T \alpha(t,u)du\right)dt + b(t,T)dW_t\right), \tag{4.2.11}$$

and

$$\begin{aligned} dZ_i(t,T) = Z_i(t,T)\Bigg(&\left(\frac{1}{2}\left|\int_t^T \sigma_i(t,u)du\right|^2 - \int_t^T \alpha_i(t,u)du + g_i(t,t) - r(t)\right)dt \\ &+ b_i(t,T)dW_t\Bigg), \end{aligned} \tag{4.2.12}$$

respectively.

Proof. Follows from Lemma 4.2.1 and Lemma 4.2.2. □

In order to exclude arbitrage opportunities in the default-free setting for all bonds of all maturities $T \le T^*$ we assume in the following:

There exists an adapted $\mathbb{R}^d$-valued process γ such that

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\int_0^{T^*} \gamma_s dW_s - \frac{1}{2}\int_0^{T^*} |\gamma_s|^2 ds\right)\right] = 1\,, \tag{4.2.13}$$

and

$$\gamma_t \int_t^T \sigma(t,u)du = \frac{1}{2}\left|\int_t^T \sigma(t,u)du\right|^2 - \int_t^T \alpha(t,u)du\,, \tag{4.2.14}$$

for all maturities $T \le T^*$.

Furthermore, and for all $0 \le t \le T$ we have the drift condition

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)du - \gamma_t\sigma(t,T), \tag{4.2.15}$$

of the risk-free forward rate $f(t,T)$.

Then using the process γ_t as market price of risk one can define a probability measure $\mathbb{P}^*$ (the *spot martingale measure*) through the formula

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left(\int_0^{T^*} \gamma_s dW_s - \frac{1}{2}\int_0^{T^*} |\gamma_s|^2 ds\right), \tag{4.2.16}$$

$\mathbb{P}$ a.s.. The corresponding Brownian motion W_t^* under $\mathbb{P}^*$ is defined through

$$W_t^* = W_t - \int_0^t \gamma_s ds, \tag{4.2.17}$$

for $t \in [0, T^*]$. Then for any fixed maturity $T \le T^*$ the discounted default-free bond price process $Z(t,T)$ is a $\mathbb{P}^*$-martingale, since

$$dZ(t,T) = Z(t,T)b(t,T)dW_t^*. \tag{4.2.18}$$

However, under the measure $\mathbb{P}^*$ and by defining for all $0 \le t \le T$

$$\eta_i(t,T) := a_i(t,T) - r(t) + b_i(t,T)\gamma_t\,, \tag{4.2.19}$$

for $i = 1, \ldots, K-1$, the conditional rating-based bond price dynamics satisfy

$$dZ_i(t,T) = Z_i(t,T)\left(\eta_i(t,T)dt + b_i(t,T)dW_t^*\right). \tag{4.2.20}$$

Note that the processes $Z_i(t,T)$ do not need to be (local) $\mathbb{P}^*$-martingales to exclude arbitrage as they do not correspond to prices of traded assets.

Moreover, the credit migration process is introduced. Since one needs a framework were both the discounted default-free *and* the discounted defaultable bond are martingales, the idea is to enlarge the underlying probability space $(\Omega, \mathbb{F}, \mathbb{P}^*)$ and to construct a new probability measure $\mathbb{Q}^*$, accommodating a credit rating migration process, whose infinitesimal generator must then be specified in a suitable way.

Thus, let C^1 be a conditionally Markov chain on the state space of rating classes $\mathcal{K}$. Its formal (canonical) construction can be found in Bielecki and Rutkowski (2004a) and is done following a canonical filtration-enlargement argument from $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t\in[0,T^*]}, \mathbb{P}^*)$ to another probability space $(\tilde{\Omega}, \tilde{\mathbb{F}}, (\tilde{\mathcal{F}}_t)_{t\in[0,T^*]}, \mathbb{Q}^*)$: in synthesis, $\mathbb{Q}^*$ is obtained as the product measure between $\mathbb{P}^*$, the Hilbert cube probability space $[0,1]^{\mathbb{N}}$ and the initial law μ of the Markov chain C^1. The sample space and the original filtration are extended accordingly. The $\mathbb{F}$-conditional infinitesimal generator of C^1 under $\mathbb{Q}^*$ at time t is given by

$$\Lambda_t^* = \begin{pmatrix} \lambda_{1,1}^*(t) & \cdots & \lambda_{1,K}^*(t) \\ \vdots & \cdots & \vdots \\ \lambda_{K-1,1}^*(t) & \cdots & \lambda_{K-1,K}^*(t) \\ 0 & \cdots & 0 \end{pmatrix}, \tag{4.2.21}$$

where for $i \neq j$ the intensity parameters $\lambda_{i,j}^*(t)$ are $\mathbb{F}$-adapted, non-negative processes and $\mathbb{Q}^*$–a.s. integrable on every interval $[0,t]$ such that for $i = 1, \ldots, K-1$, $\lambda_{i,i}^*(t) = -\sum_{j\in\mathcal{K}\setminus\{i\}} \lambda_{i,j}^*(t)$.

The last row of the intensity matrix Λ_t^* being zero corresponds to the fact that K is the absorbing state as the defaulted firm goes bankrupt.

Furthermore define the credit migration process $C = (C_t^1, C_t^2)$, where the process C_t^1 is the current rating at time t and C_t^2 is the previous rating before the current state C_t^1.

Remark 4.2.4. *In some of the literature, for example in Bielecki and Rutkowski (2000) and Bielecki and Rutkowski (2004a), the migration intensity parameter processes $\lambda_{i,j}^*$ are assumed to be strictly positive. We do not require this here for the entries of the intensity matrix Λ_t^*.*

It should be stressed out that in order to have a uniformly notation then by a slight abuse of notation, all the stochastic processes that we consider, they maintain their names on the enlarged probability space. So, for example the $\mathbb{P}^*$ brownian motion W^* follows again a standard brownian motion under Q^* with respect to the filtration $(\tilde{\mathcal{F}}_t)_{t\in[0,T^*]}$.

The T*-maturity defaultable bond price process* $D_{C_t}(t,T)$ is defined by the equation

$$D_{C_t}(t,T) := \mathbf{1}_{\{C_t^1 \neq K\}} D_{C_t^1}(t,T) + \mathbf{1}_{\{C_t^1 = K\}} \delta_{C_t^2} B(t,T), \tag{4.2.22}$$

for all $0 \leq t \leq T$. Note that the process $D_i(t,T)$ is not the process of a defaultable bond which is traded but instead $D_{C_t}(t,T)$ is a traded bond. Also note that the recovery modeling of definition (4.2.22) is commonly known in the literature as fractional recovery of treasury value model or rating based recovery of treasury.

Furthermore, equation (4.2.22) is equivalent to

$$D_{C_t}(t,T) = \sum_{i=1}^{K-1} H_i(t) D_i(t,T) + \delta_i H_{i,K}(t) B(t,T), \tag{4.2.23}$$

where we used the notation

$$H_i(t) := \mathbf{1}_{\{s\geq 0: C_s^1 = i\}}(t) \text{ , for } i \in \mathcal{K}, \quad H_{i,j}(t) := \sum_{0\leq u\leq t} H_i(u-) H_j(u) \text{ , for } i \neq j, \tag{4.2.24}$$

as in Bielecki and Rutkowski (2004a). $H_{i,j}(t)$ is the number of transitions from rating i to rating j over the time interval $[0,t]$ for $i \neq j$.

The T*-maturity discounted defaultable bond price process* is therefore

$$\hat{Z}(t,T) := B(t)^{-1} D_{C_t}(t,T) = \mathbf{1}_{\{C_t^1 \neq K\}} Z_{C_t^1}(t,T) + \mathbf{1}_{\{C_t^1 = K\}} \delta_{C_t^2} Z(t,T), \tag{4.2.25}$$

for all $0 \leq t \leq T$. The process $\hat{Z}(t,T)$ "switches" its dynamics between the various $Z_i(t,T)$ according to the states of the credit migration process $C = (C_t^1, C_t^2)$.

Assume also that the initial value is given by

$$\hat{Z}(0,T) := \sum_{i=1}^{K-1} H_i(0) Z_i(0,T)\,, \tag{4.2.26}$$

which means that at time $t = 0$, no bankruptcy has been observed. Note that, this is equivalent to $C_0^1 \neq K$.

Finally we observe that the bond *default time* $\tau : \tilde{\Omega} \to \mathbb{R}_+$ is the $\tilde{\mathcal{F}}_t$-stopping time given by

$$\tau := \inf\{t \geq 0 : C_t^1 = K\}, \tag{4.2.27}$$

where $\inf \emptyset := +\infty$.

4.3 The Weak Consistency Condition

No arbitrage in a credit migration bond model connects both the class-conditional bond prices and the migration process, since the defaultable price dynamics depend upon both these elements. Specifically, absence of arbitrage is guaranteed by a set of equations establishing a relationship between the class-specific price dynamics and the migration intensities. Such constraints are known in the literature as *consistency condtions* and have appeared in two forms: a *strong* one which *implies* no arbitrage, and a *weak* one, which is *equivalent* to no arbitrage.

In this section we will analyze the model under the weak consistency condition. Such a condition is assumed in e.g. Özkan and Schmidt (2005), Schmidt (2006) and Jakubowski and Nieweglowski (2009). This condition is a relationship which is both sufficient and necessary for no-arbitrage. The key property is a no-arbitrage drift condition only on the current forward rate in which the bond holder is currently trading. In other words, only the current forward rate becomes “active”.

This condition introduce an interplay between the various model components potentially yielding to constraints on the model specifications. In principle if such constraints are not met by the involved coefficients, problems may arise. We shall show in Section 4.3.2 that the fundamental spread of the last credit class (and hence of the current class) explodes in finite time with positive probability, under some additional conditions that we impose. Nevertheless we propose in Section 4.3.3 a method for producing consistent model specifications via a transformation argument.

Before we proceed let us state two important results.

Lemma 4.3.1. *For all $i = 1, \dots, K-1$ and $0 \leq t \leq T$ we have*

$$H_i(t) = H_i(0) - \sum_{j=1, j\neq i}^{K-1} H_{i,j}(t) + \sum_{j=1, j\neq i}^{K-1} H_{j,i}(t) - H_{i,K}(t)\,. \tag{4.3.1}$$

Proof. See Remark 3.2 in Eberlein and Özkan (2003). □

Lemma 4.3.2. *For all $i, j = 1, \dots, K-1$ with $i \neq j$ and $0 \leq t \leq T$, define the adapted process,*

$$M_{i,j}(t) := H_{i,j}(t) - \int_0^t \lambda^*_{i,j}(u) H_i(u) du\,, \tag{4.3.2}$$

with respect to the filtration $(\tilde{\mathcal{F}}_t)_{t\in[0,T^*]}$. Then $M_{i,j}(t)$ is a $\mathbb{Q}^*$-martingale.

Proof. See Proposition 2.1 in Bielecki and Rutkowski (2000). □

The *(weak) consistency condition* N.1 which follows, is a critical no-arbitrage assumption establishing a direct analytical connection between the conditional credit state bond price processes, the defaultable bond and the migration intensities.

Condition N.1. Assume that the entries of Λ^* satisfy for all $0 \le t \le T$ on the set $\{C_t^1 \neq K\}$:

$$\sum_{j=1, j\neq C_t^1}^{K-1} \lambda^*_{C_t^1,j}(t)(Z_j(t,T) - Z_{C_t^1}(t,T)) + \lambda^*_{C_t^1,K}(t)(\delta_{C_t^1} Z(t,T) - Z_{C_t^1}(t,T)) \tag{4.3.3}$$
$$= -\eta_{C_t^1}(t,T) Z_{C_t^1}(t,T) .$$

Remark 4.3.3. *Note that the condition* N.1 *can also be written as*

$$\sum_{i=1}^{K-1} H_i(t) \Bigg\{ \sum_{j=1, i\neq j}^{K-1} \lambda^*_{i,j}(t)(Z_j(t,T) - Z_i(t,T)) + \lambda^*_{i,K}(t)(\delta_i Z(t,T) - Z_i(t,T)) \tag{4.3.4}$$
$$+ \eta_i(t,T) Z_i(t,T) \Bigg\} = 0 ,$$

for $0 \le t \le T$*, which is an equivalent form of (4.3.3).*

Remark 4.3.4. *The consistency condition* N.1 *is not the condition which is assumed for example in Bielecki and Rutkowski (2004a) and Eberlein and Özkan (2003) but instead a stronger one is being adopted and used. In particular they use condition* N.2 *which we introduce in Section 4.4.2.*

As the following Theorem 4.3.5 shows, the condition N.1 is a necessary and sufficient condition for the (local) martingale property of $\hat{Z}(t,T)$ under $\mathbb{Q}^*$.

Theorem 4.3.5. *The discounted defaultable bond* $\hat{Z}(\cdot,T)$ *is a local martingale under* $\mathbb{Q}^*$ *if and only if the consistency condition* N.1 *holds.*

Proof. The proof follows from Lemma 4.3.1, Lemma 4.3.2 and using the same arguments as in the proof of Theorem 4.1 in Eberlein and Özkan (2003). □

4.3.1 No-arbitrage drift condition on the current forward rate.

The consistency condition N.1 is related to a no-arbitrage drift condition of the instantaneous forward rates. This can be written in terms of the spreads and can be understood as an extended HJM no-arbitrage drift condition. This relation is an equivalence relation, as shown in Proposition 4.3.6.

Let us first define the *instantaneous forward inter-rating spreads*

$$s_{i,j}(t,T) := g_i(t,T) - g_j(t,T) , \tag{4.3.5}$$

for $i, j = 1, \dots, K-1$ with $i \neq j$, as well as the *instantaneous fundamental spreads*

$$s_i^f(t,T) := g_i(t,T) - f(t,T)\,, \tag{4.3.6}$$

for $i = 1, \dots, K-1$. Additionally, we set for $i = 2, \dots, K-1$

$$s_i(t,T) := g_i(t,T) - g_{i-1}(t,T)\,. \tag{4.3.7}$$

Proposition 4.3.6. *Assume that the consistency condition* N.1 *holds and fix some maturity $T \leq T^*$. The condition* N.1 *is equivalent to the following: for all $0 \leq t \leq T$ the drift condition*

$$\begin{aligned}\alpha_{C_t^1}(t,T) &= \sigma_{C_t^1}(t,T)\int_t^T \sigma_{C_t^1}(t,u)du - \gamma_t\sigma_{C_t^1}(t,T) \\ &+ \sum_{j=1,j\neq C_t^1}^{K-1} \lambda^*_{C_t^1,j}(t) s_{C_t^1,j}(t,T)\exp\left(\int_t^T s_{C_t^1,j}(t,u)du\right) \\ &+ \lambda^*_{C_t^1,K}(t)\delta_{C_t^1} s^f_{C_t^1}(t,T)\exp\left(\int_t^T s^f_{C_t^1}(t,u)du\right)\,,\end{aligned} \tag{4.3.8}$$

of the current forward rate $g_{C_t^1}(t,T)$ holds, together with the condition

$$s^f_{C_t^1}(t,t) = \lambda^*_{C_t^1,K}(t)(1-\delta_{C_t^1})\,, \tag{4.3.9}$$

on the set $\{C_t^1 \neq K\}$.

Proof. Using the definition of $Z_i(t,T)$ and $Z(t,T)$ we can rewrite the consistency condition N.1 in terms of spreads so that we have

$$\begin{aligned}&\sum_{j=1,j\neq C_t^1}^{K-1} \lambda^*_{C_t^1,j}(t)\left(1-\exp\left(-\int_t^T g_j(t,u) - g_{C_t^1}(t,u)du\right)\right) \\ &+ \lambda^*_{C_t^1,K}(t)\left(1-\delta_{C_t^1}\exp\left(-\int_t^T f(t,u) - g_{C_t^1}(t,u)du\right)\right) = \eta_{C_t^1}(t,T)\,.\end{aligned} \tag{4.3.10}$$

Then, by the definition of $\eta_i(t,T)$ in equation (4.2.19) and the definition of the spreads (4.3.5) and (4.3.6) we calculate,

$$\begin{aligned}&g_{C_t^1}(t,t) - f(t,t) + \frac{1}{2}\left|\int_t^T \sigma_{C_t^1}(t,u)du\right|^2 - \int_t^T \alpha_{C_t^1}(t,u)du - \gamma_t\int_t^T \sigma_{C_t^1}(t,u)du \\ &= \sum_{j=1,j\neq C_t^1}^{K-1} \lambda^*_{C_t^1,j}(t)\left(1-\exp\left(\int_t^T s_{C_t^1,j}(t,u)du\right)\right) \\ &+ \lambda^*_{C_t^1,K}(t)\left(1-\delta_{C_t^1}\exp\left(\int_t^T s^f_{C_t^1}(t,u)du\right)\right)\,,\end{aligned} \tag{4.3.11}$$

for all $0 \leq t \leq T$. Furthermore, we have for $T = t$

$$g_{C_t^1}(t,t) - f(t,t) = \lambda^*_{C_t^1,K}(t)(1-\delta_{C_t^1})\,, \tag{4.3.12}$$

which is (4.3.9). Then since the migration intensity parameters are independent of the maturity T we can conclude that

$$\begin{aligned}
&\lambda^*_{C_t^1,K}(t)(1-\delta_{C_t^1}) + \frac{1}{2}\left|\int_t^T \sigma_{C_t^1}(t,u)du\right|^2 - \int_t^T \alpha_{C_t^1}(t,u)du - \gamma_t \int_t^T \sigma_{C_t^1}(t,u)du \\
&= \sum_{j=1, j\neq C_t^1}^{K-1} \lambda^*_{C_t^1,j}(t)\left(1-\exp\left(\int_t^T s_{C_t^1,j}(t,u)du\right)\right) \\
&+ \lambda^*_{C_t^1,K}(t)\left(1-\delta_{C_t^1}\exp\left(\int_t^T s^f_{C_t^1}(t,u)du\right)\right)\,,
\end{aligned} \tag{4.3.13}$$

which is equivalent to

$$\begin{aligned}
&\frac{1}{2}\left|\int_t^T \sigma_{C_t^1}(t,u)du\right|^2 - \int_t^T \alpha_{C_t^1}(t,u)du - \gamma_t \int_t^T \sigma_{C_t^1}(t,u)du \\
&= \sum_{j=1, j\neq C_t^1}^{K-1} \lambda^*_{C_t^1,j}(t)\left(1-\exp\left(\int_t^T s_{C_t^1,j}(t,u)du\right)\right) \\
&+ \lambda^*_{C_t^1,K}(t)\delta_{C_t^1}\left(1-\exp\left(\int_t^T s^f_{C_t^1}(t,u)du\right)\right)\,.
\end{aligned} \tag{4.3.14}$$

Now, taking the derivative with respect to T in equation (4.3.14) we get

$$\begin{aligned}
&\sigma_{C_t^1}(t,T)\int_t^T \sigma_{C_t^1}(t,u)du - \alpha_{C_t^1}(t,T) - \gamma_t\sigma_{C_t^1}(t,T) \\
&= -\sum_{j=1, j\neq C_t^1}^{K-1} \lambda^*_{C_t^1,j}(t)s_{C_t^1,j}(t,T)\exp\left(\int_t^T s_{C_t^1,j}(t,u)du\right) \\
&- \lambda^*_{C_t^1,K}(t)\delta_{C_t^1}s^f_{C_t^1}(t,T)\exp\left(\int_t^T s^f_{C_t^1}(t,u)du\right)\,,
\end{aligned} \tag{4.3.15}$$

and rewriting equation (4.3.15) in terms of the drift term $\alpha_{C_t^1}(t,T)$ of the forward rate $g_{C_t^1}(t,T)$, equation (4.3.8) follows. Since all manipulations above are equivalences the reverse also holds. □

Remark 4.3.7. *From Proposition 4.3.6 we clearly see that the active drift $\alpha_{C_t^1}(t,T)$ from (4.3.8) and the active defaultable intensity $\lambda^*_{C_t^1,K}(t)$ from (4.3.9) together with a spread structure of the drifts of the forward rates $g_i(t,T)$, determines the drifts of all the forward rates. So, the model is fully specified on $t \leq \tau$ given the volatilities $\sigma_i(t,T)$ (or equivalently given the spread* *volatilities $\sigma_i(t,T)-\sigma_{i-1}(t,T)$ and the active volatility $\sigma_{C_t^1}(t,T)$), the migration intensities $\lambda^*_{i,j}$ for $i,j=1,\ldots,K-1$ with $i\neq j$, the recovery rates δ_i, the spread structure, condition (4.3.9) and the no-arbitrage drift condition (4.3.8) where τ is the time of default given in equation (4.2.27). After default, at time $t\geq\tau$, the drifts of the risky forward rates can be chosen freely.*

The following Example 4.3.1 illustrates and presents a fully specified defautlable simple model that satisfies N.1, with constant inter-rating spreads $s_i(t,T)$. The model is constructed by following Remark 4.3.7.

Example 4.3.1. *Consider a model where $K\geq 3$, with zero recovery rate, i.e., $\delta_i=0$ for $i=1,\ldots,K-1$. Furthermore assume $\lambda^*_{i,j}(t)=\lambda^*$ for all $i,j=1,\ldots,K-1$ with $i\neq j$. Let a 1-dimensional Brownian motion driving all the forward rates and assume constant volatilities, such that $\sigma(t,T)=\sigma$ and $\sigma_i(t,T)=\sigma_1$ for all $i=1,\ldots,K-1$. Then we have*

$$\begin{aligned} df(t,T) &= \alpha(t,T)dt+\sigma dW^*_t\,, \\ dg_i(t,T) &= \alpha_i(t,T)dt+\sigma_1 dW^*_t \quad , \quad i=1,\ldots,K-1\,, \\ ds_i(t,T) &= (\alpha_i(t,T)-\alpha_{i-1}(t,T))dt \quad , \quad i=2,\ldots,K-1\,, \end{aligned} \tag{4.3.16}$$

Note that, inter-rating spreads get constant, if $\alpha_i(t,T)=\alpha_j(t,T)$ for all $i,j=1,\ldots,K-1$.

To specify constant inter-rating spreads we require that for all $0\leq t\leq T$,

$$s_i(t,T)=c \quad , \quad \text{for all } i=2,\ldots,K-1\,. \tag{4.3.17}$$

Then by the no-arbitrage consistency condition N.1 *we have from Proposition 4.3.6, for almost all $0\leq t\leq T$ the drift condition*

$$\begin{aligned} \alpha_{C_t^1}(t,T) &= \sigma_1^2(T-t)-\gamma_t\sigma_1+(K-2)\lambda^* c\exp\left(c(T-t)\right)\,, \\ \alpha(t,T) &= \sigma^2(T-t)-\gamma_t\sigma\,, \end{aligned} \tag{4.3.18}$$

of the current forward rate $g_{C_t^1}(t,T)$ on the set $\{C_t^1\neq K\}$ and of the risk-free forward rate $f(t,T)$ respectively.

So, by setting all the drifts equal to the active one, that is for almost all $t\leq T$

$$\begin{aligned} \alpha_i(t,T) &= \alpha_{C_t^1}(t,T) \quad , \quad \text{for } t<\tau\,, \\ \alpha_i(t,T) &= 0 \quad , \quad \text{for } t\geq\tau\,, \end{aligned} \tag{4.3.19}$$

for all $i=1,\ldots,K-1$, where recall from (4.2.27) that τ is the time of default, then one can have a model with constant inter-rating spreads under the no-arbitrage consistency condition N.1.

By making this choice of the drift $\alpha_i(t,T)$, then by (4.3.19), the forward rates $g_i(t,T)$ are given by

$$\begin{aligned} g_i(t,T) = g_i(0,T)+\sigma_1^2(tT-\frac{1}{2}t^2)-\sigma_1\int_0^t\gamma_s ds+\lambda^*\exp(cT)[1-\exp(-ct)] \\ +\sigma_1 W^*_t \quad , \quad i=1,2\,, \end{aligned} \tag{4.3.20}$$

for $t < \tau$.

It is also quite clear that one can now easily derive and determine the fundamental spreads $s_i^f(t,T) = g_i(t,T) - f(t,T)$ *for* $i = 1, \dots, K-1$ *as well as the defaultable intensity parameters* $\lambda^*_{C_t^1,K}(t)$ *from (4.3.9). Finally, specifying* $f(0,T)$ *and* $g_i(0,T)$ *for some* $i \in \{1, \dots, K-1\}$*, then all the other initial values are derived from (4.3.17).*

Using Proposition 4.3.6, the next Corollary 4.3.8 gives a more structural approach of Theorem 4.3.5, bringing out a more direct and clear way for the construction of a no-arbitrage defaultable model.

Corollary 4.3.8. *The discounted defaultable bond* $\hat{Z}(\cdot,T)$ *is a local martingale under* $\mathbb{Q}^*$ *if and only if the no-arbitrage conditions (4.3.8) and (4.3.9) hold.*

Proof. Follows from Theorem 4.3.5 and Proposition 4.3.6. □

4.3.2 Fundamental spreads and explosions under the weak consistency

In this section we want to show that under the consistency condition N.1 the current fundamental spread $s^f_{C_t^1}(t,T)$ defined as in (4.3.6), explodes in finite time with positive probability on the set $\{C_t^1 \neq K\}$ for the non-zero recovery case. We obtain this result by using a natural condition of positivity of the spreads, the so called *ordering condition* introduced in (4.3.41). Such a condition is assumed in many papers in the literature, but it seems that there is no a direct link to no-arbitrage. However, we shall show in the special case of $K = 2$ that even without the ordering condition, explosions may still occur.

The next Corollary 4.3.9 gives us the dynamics of the fundamental spread $s^f_{C_t^1}(t,T)$ under an equivalent measure $\mathbb{Q}$ characterized by the property that the restriction of $\mathbb{Q}$ to the original probability space coincides with $\mathbb{P}$. Specifically:

$$\frac{d\mathbb{Q}}{d\mathbb{Q}^*}\Big|_{\tilde{\mathcal{F}}_t} = L_t. \tag{4.3.21}$$

where

$$dL_t = -L_t \gamma_t dW_t^*. \tag{4.3.22}$$

The measure $\mathbb{Q}$ can be seen as the physical measure on the extended probability space such that the change to the pricing measure $\mathbb{Q}^*$ entails no market price of credit risk (Bielecki and Rutkowski (2000), Section 2.5). Recall that all the processes (e.g. W) retain their names in the enlarged probability space.

Corollary 4.3.9. *For any fixed maturity* $T \le T^*$ *and under the consistency condition* N.1*, we have the dynamics*

$$ds^f_{C_t^1}(t,T) = \Bigg\{ \sigma_{C_t^1}(t,T) \int_t^T \sigma_{C_t^1}(t,u)du - \sigma(t,T) \int_t^T \sigma(t,u)du \tag{4.3.23}$$

$$
\begin{aligned}
&+ \sum_{j=1, j\neq C_t^1}^{K-1} \lambda^*_{C_t^1,j}(t) s_{C_t^1,j}(t,T) \exp\left(\int_t^T s_{C_t^1,j}(t,u)du \right) \\
&+ \lambda^*_{C_t^1,K}(t) \delta_{C_t^1} s^f_{C_t^1}(t,T) \exp\left(\int_t^T s^f_{C_t^1}(t,u)du \right) \Bigg\} dt \\
&+ (\sigma_{C_t^1}(t,T) - \sigma(t,T)) dW_t \,,
\end{aligned}
$$

for the current fundamental spread $s^f_{C_t^1}(t,T)$, for $t < \tau_1$, where $\tau_1 := \inf\{t > 0 : C_t^1 \neq C_0^1\}$ is the first time where a jump in another class occurs.

Proof. From Proposition 4.3.6 we have the dynamics

$$
\begin{aligned}
ds^f_{C_t^1}(t,T) = &\Bigg\{ \sigma_{C_t^1}(t,T) \int_t^T \sigma_{C_t^1}(t,u)du - \sigma(t,T) \int_t^T \sigma(t,u)du \\
&+ \sum_{j=1, j\neq C_t^1}^{K-1} \lambda^*_{C_t^1,j}(t) s_{C_t^1,j}(t,T) \exp\left(\int_t^T s_{C_t^1,j}(t,u)du \right) \\
&+ \lambda^*_{C_t^1,K}(t) \delta_{C_t^1} s^f_{C_t^1}(t,T) \exp\left(\int_t^T s^f_{C_t^1}(t,u)du \right) \\
&- \gamma_t (\sigma_{C_t^1}(t,T) - \sigma(t,T)) \Bigg\} dt \\
&+ (\sigma_{C_t^1}(t,T) - \sigma(t,T)) dW^*_t \,,
\end{aligned}
\tag{4.3.24}
$$

for $t < \tau_1$. Then by a change of measure using Girsanov's Theorem the claim follows. □

Remark 4.3.10. *The measure $\mathbb{Q}$ defined in (4.3.21) and used in Corollary 4.3.9 is the physical measure on the extended probability space. Of course, because of the presence of the conditional Markov chain C_t, more complicated martingale densities (which do admit a market price of credit risk) can be devised, but this is not necessary for our purposes. A general measure $\mathbb{Q}$ can be defined as follows: Recall (4.3.2) from Lemma 4.3.2 and define the $\mathbb{Q}^*$-local martingale M as*

$$
dM_t = \sum_{i\neq j} (\varphi_{i,j}(t) - 1)\, dM_{i,j}(t), \tag{4.3.25}
$$

where $\varphi_{i,j}$ is an arbitrary nonnegative $\mathbb{F}$-predictable process such that

$$
\int_0^{T^*} \varphi_{i,j}(t) \lambda^*_{i,j}(t) dt < \infty, \tag{4.3.26}
$$

$\mathbb{Q}^*$-a.s. for all $i \neq j$. Furthermore set the $\mathbb{Q}^*$-local positive martingale L equal to

$$
dL_t = -L_t \gamma_t dW^*_t + L_{t-} dM_t, \tag{4.3.27}
$$

and finally define the measure $\mathbb{Q}$ by the formula

$$\frac{d\mathbb{Q}}{d\mathbb{Q}^*}|_{\tilde{\mathcal{F}}_t} = L_t. \tag{4.3.28}$$

See Bielecki and Rutkowski (2000) for more details.

The next Lemma 4.3.11 and Lemma 4.3.12 are important technical tools for showing the spread explosions. In particular, Lemma 4.3.12 is a comparison theorem taken from Morton (1988).

Lemma 4.3.11. *Let $R, T \geq 0$ with $R \leq T$, $a > 0$, $R \leq t \leq T$, $t < a/T$. Then the function*

$$h^a(t,T) = \frac{2a}{(a - tT)^2}\,, \tag{4.3.29}$$

satisfies the differential equation

$$h^a(t,T) = \exp\left\{\int_R^t \int_s^T h(s,u)duds\right\} \frac{2a}{|a-t^2|} \frac{|a-R^2|}{(a-RT)^2}\,. \tag{4.3.30}$$

Proof. The proof relies on using partial fraction decomposition. We can calculate

$$\begin{aligned} \exp\left\{\int_0^t \int_s^T h(s,u)duds\right\} &= \exp\left\{\int_0^t \int_s^T \frac{2a}{(a-su)^2}duds\right\} \\ &= \exp\left\{2a\int_0^t \frac{1}{(as-Ts^2)} - \frac{1}{(as-s^3)}ds\right\}, \end{aligned} \tag{4.3.31}$$

since

$$\frac{d}{du}\left(\frac{1}{s(a-su)}\right) = \frac{1}{(a-su)^2}, \tag{4.3.32}$$

Furthermore, calculating the zeros of $as - Ts^2$ and $as - s^3$ we get the following equations

$$\begin{aligned} \frac{1}{as-Ts^2} &= \frac{A_1}{s} + \frac{A_2}{s-\frac{a}{T}}, \\ \frac{1}{as-s^3} &= \frac{B_1}{s} + \frac{B_2}{s-\sqrt{a}} + \frac{B_3}{s+\sqrt{a}}, \end{aligned} \tag{4.3.33}$$

for some constants A_i, B_j for $i = 1,2$ and $j = 1,2,3$. By equating the coefficients we have

$$A_1 = B_1 = \frac{1}{a} \quad , \quad A_2 = -\frac{1}{a} \quad , \quad B_2 = B_3 = -\frac{1}{2a}\,. \tag{4.3.34}$$

Thus,

$$\exp\left\{\int_0^t \int_s^T h(s,u)duds\right\} \tag{4.3.35}$$

$$
\begin{aligned}
&= \exp\left\{\int_0^t -2\frac{1}{(s-\frac{a}{T})} + \frac{1}{(s-\sqrt{a})} + \frac{1}{(s+\sqrt{a})}ds\right\} \\
&= \exp\left\{\ln\left(\left|t-\frac{a}{T}\right|^{-2}\right)\right\}\exp\left\{\ln\left(\left|-\frac{a}{T}\right|^{2}\right)\right\}\exp\left\{\ln\left(\left|t-\sqrt{a}\right|\right)\right\} \\
&\quad \exp\left\{\ln\left(\left|-\sqrt{a}\right|^{-1}\right)\right\}\exp\left\{\ln\left(\left|t+\sqrt{a}\right|\right)\right\}\exp\left\{\ln\left(\left|\sqrt{a}\right|^{-1}\right)\right\} \\
&= \frac{1}{\left(\left|t-\frac{a}{T}\right|\right)^2}\left(\frac{a}{T}\right)^2\left|t-\sqrt{a}\right|\left(\frac{1}{\sqrt{a}}\right)\left|t+\sqrt{a}\right|\left(\frac{1}{\sqrt{a}}\right),
\end{aligned}
$$

and so one can conclude that

$$\exp\left\{\int_0^t\int_s^T h(s,u)duds\right\} = \frac{1}{\left(\left|t-\frac{a}{T}\right|\right)^2}\frac{a}{T^2}\left|t^2-a\right|. \tag{4.3.36}$$

and the claim follows. □

Lemma 4.3.12 (Morton's Lemma). *Let R, T^* with $0 \le R \le T^*$ and $K = \{(t,T) : R \le t \le T \le T^*\}$. Suppose $f_1(\cdot,\cdot), f_2(\cdot,\cdot), \hat{g}_1(\cdot,\cdot), \hat{g}_2(\cdot,\cdot)$ each map $K \to \mathbb{R}$ with $\hat{g}_1$ and $\hat{g}_2$ continuous, positive and satisfying $\hat{g}_1 \ge \hat{g}_2 > 0$. Furthermore*

$$f_1(t,T) \ge \exp\left\{\int_R^t\int_s^T f_1(s,u)duds\right\}\hat{g}_1(t,T), \tag{4.3.37}$$

and

$$f_2(t,T) = \exp\left\{\int_R^t\int_s^T f_2(s,u)duds\right\}\hat{g}_2(t,T), \tag{4.3.38}$$

for all $(t,T) \in K$. Then $f_1 \ge f_2$.

Proof. See Lemma 4.7.2 in Morton (1988). □

A reverse version of the well known Grönwall's inequality is given in the next Lemma 4.3.13.

Lemma 4.3.13 (Reverse Grönwall's inequality). *Let $I = [a,b]$ with $a < b$ and $X, N : I \to \mathbb{R}$, $E : I \to [0,\infty)$ continuous functions. Furthermore assume*

$$X(t) \ge N(t) + \int_a^t E(s)X(s)ds \quad \textit{for all } t \in I. \tag{4.3.39}$$

Then it holds

$$X(t) \ge N(t) + \int_a^t N(s)E(s)\exp\left(\int_s^t E(u)du\right)ds \quad \textit{for all } t \in I. \tag{4.3.40}$$

Proof. The proof can be found in the literature.

□

As in Bielecki and Rutkowski (2000) we introduce the following natural *ordering condition*:

Condition O. Assume:

$$f(t,T) < g_1(t,T) < \cdots < g_{K-2}(t,T) < g_{K-1}(t,T)\,. \tag{4.3.41}$$

Condition O reflects the fact that the price of a bond must decrease as the default risk increases. It is a natural economic assumption to have a system of strictly positive inter-rating class spreads: the riskier the bond the higher the forward rate. This condition is assumed for example in Bielecki and Rutkowski (2000), Eberlein and Özkan (2003) and in Schmidt (2006) but it is not necessary for the model framework and does not directly relate to no arbitrage.

Now the main Theorem 4.3.14 follows under some mild conditions. The Idea of the proof is to bound the fundamental spread from below by a function with positive probability using the reverse Grönwall's inequality from Lemma 4.3.13. Then applying the Morton's comparison Lemma 4.3.12 we can conclude.

We fix the following notation for $i = 1, \dots, K-1$:

$$N_i(t,T) := \tilde{N}_i(t,T) + \sum_{j=1, j\neq i}^{K-1} \int_0^t \lambda^*_{i,j}(s) s_{i,j}(s,T) \exp\left(\int_s^T s_{i,j}(s,u)du\right) ds, \tag{4.3.42}$$

where

$$\begin{aligned} \tilde{N}_i(t,T) &:= s_i^f(0,T) + \int_0^t \int_s^T \{\sigma_i(s,T)\sigma_i(s,u) - \sigma(s,T)\sigma(s,u)\}\, duds \\ &\quad + \int_0^t (\sigma_i(s,T) - \sigma(s,T))dW_s\,, \end{aligned} \tag{4.3.43}$$

and for $\lambda^*_{i,K}(t)\delta_i > 0$ define the sets

$$\begin{aligned} A_i^{R,S,a} &:= \left\{\omega : \tilde{N}_i(t,T) \geq \frac{2a}{(a-S^2)} \frac{(a-R^2)}{(a-RT)^2} \frac{1}{\hat{\lambda}^*_{i,K}(t)\delta_i} \text{ for all } R \leq t \leq S \right\}, \\ B_i^S &:= \left\{\omega : C_t^1 = i \text{ for all } 0 \leq t \leq S\right\}, \\ \hat{A}_i^{R,S,a} &:= A_i^{R,S,a} \cap B_i^S\,, \\ \hat{K}^{R,S} &:= \{(t,T) : R \leq t \leq S \text{ and } t \leq T \leq T^*\}\,, \end{aligned} \tag{4.3.44}$$

where a, R, S are positive constants with $a > S^2$, $a > RT$, $R \leq S \leq T$ and $\hat{\lambda}^*_{i,K}(t) := \inf_\omega \lambda^*_{i,K}(t)$.

Theorem 4.3.14. *Assume conditions* N.1 *and* O. *Furthermore, assume that* $\delta_{K-1} > 0$ *and*

(i) $\lambda^*_{K-1,K}(t)$ *is uniformly bounded from below in* ω*;*

(ii) For all $T \leq T^*$ *it holds* $\mathbb{Q}(A_\infty) > 0$ *with*

$$A_\infty = \hat{A}^{R,S,a}_{K-1}, \quad R = \frac{T}{4}, \quad S = \frac{T}{2}, \quad a = \frac{T^2}{2}. \tag{4.3.45}$$

If a solution to (4.3.23) exists, then for all $T \leq T^*$, $\lim_{t\to\frac{T}{2}} s^f_{K-1}(t,T) = +\infty$ *with positive probability under* $\mathbb{Q}$ *on* B^S_{K-1}. *In particular*

$$\lim_{t\to\frac{T}{2}} s^f_{C^1_t}(t,T) = +\infty\,, \tag{4.3.46}$$

with positive probability under $\mathbb{Q}$ *prior to default.*

Proof. First note that for the case $K \geq 2$ we have from Corollary 4.3.9 and by the ordering condition O

$$s^f_{K-1}(t,T) \geq N_{K-1}(t,T) + \int_R^t \lambda^*_{K-1,K}(s)\delta_{K-1}s^f_{K-1}(s,T)\exp\left(\int_s^T s^f_{K-1}(s,u)du\right)ds\,, \tag{4.3.47}$$

for $t \geq R$ and for all $\omega \in B^S_{K-1}$.

Now, define for $t \in [0,T]$ with $t < \tau_1$ the continuous functions

$$\begin{aligned} X(t) &:= s^f_{K-1}(t,T), \quad N(t) := N_{K-1}(t,T), \\ E(t) &:= \lambda^*_{K-1,K}(t)\delta_{K-1}\exp\left(\int_t^T s^f_{K-1}(t,u)du\right). \end{aligned} \tag{4.3.48}$$

Since from the model assumptions δ_{K-1} and $\lambda^*_{K-1,K}(t)$ are non-negative then $E(t)$ is a non-negative function. Also, since from condition (4.3.9), $\lambda^*_{K-1,K}(t)$ is continuous then $E(t)$ is continuous as well.

Then by Lemma 4.3.13 (reverse Grönwall's inequality) using the specifications in (4.3.48) we have

$$s^f_{K-1}(t,T) \geq N_{K-1}(t,T) + \int_R^t N_{K-1}(s,T)\lambda^*_{K-1,K}(s)\delta_{K-1}\exp\left(\int_s^T s^f_{K-1}(s,u)du\right) \tag{4.3.49}$$

$$\exp\left(\int_s^t \lambda^*_{K-1,K}(u)\delta_{K-1}\exp\left(\int_u^T s^f_{K-1}(u,v)dv\right)du\right)ds\,.$$

Furthermore, note that by the ordering condition O we have $N_{K-1}(t,T) \geq \tilde{N}_{K-1}(t,T)$.

So we can further estimate

$$s^f_{K-1}(t,T) \geq \tilde{N}_{K-1}(t,T) + \int_R^t \tilde{N}_{K-1}(s,T)\hat{\lambda}^*_{K-1,K}(s)\delta_{K-1}\exp\left(\int_s^T s^f_{K-1}(s,u)du\right)ds\,, \tag{4.3.50}$$

for $R \leq t$ and where recall $\hat{\lambda}^*_{K-1,K}(t) = \inf_\omega \lambda^*_{K-1,K}(t)$ from (4.3.44).

Thus, we can conclude that for $R \leq t$ and with positive probability we can estimate

$$s^f_{K-1}(t,T) \geq \int_R^t \frac{2a}{(a-S^2)}\frac{(a-R^2)}{(a-RT)^2}\exp\left(\int_s^T s^f_{K-1}(s,u)du\right)ds\,. \tag{4.3.51}$$

More precisely, inequality (4.3.51) holds for $(t,T) \in \hat{K}^{R,S}$ and for $\omega \in A_\infty$ with a, R, S given by (4.3.45).

Then, since $t \leq S$ we have $a - S^2 \leq a - t^2$ and we can further estimate

$$s^f_{K-1}(t,T) \geq \frac{2a}{(a-t^2)} \frac{(a-R^2)}{(a-RT)^2} \int_R^t \exp\left(\int_s^T s^f_{K-1}(s,u)du \right) ds \,. \tag{4.3.52}$$

Moreover for $R \leq t$ and using Jensen's inequality,

$$s^f_{K-1}(t,T) \geq \frac{2a}{(a-t^2)} \frac{(a-R^2)}{(a-RT)^2} (t-R) \exp\left\{ \int_R^t \int_s^T \frac{1}{(t-R)} s^f_{K-1}(s,u)duds \right\} . \tag{4.3.53}$$

By defining $\tilde{s}^f_{K-1}(t,T) := \frac{1}{(t-R)} s^f_{K-1}(t,T)$ we get

$$\tilde{s}^f_{K-1}(t,T) \geq \frac{2a}{(a-t^2)} \frac{(a-R^2)}{(a-RT)^2} \exp\left\{ \int_R^t \int_s^T \tilde{s}^f_{K-1}(s,u)duds \right\} , \tag{4.3.54}$$

for $(t,T) \in \hat{K}^{R,S}$ and for $\omega \in A_\infty$.

Then by Morton's Lemma 4.3.12 on the set A_∞ and for $(t,T) \in \hat{K}^{R,S}$ we finally have

$$\tilde{s}^f_{K-1}(t,T) \geq h^a(t,T) \,, \tag{4.3.55}$$

where $h^a(t,T)$ satisfies the differential equation

$$h^a(t,T) = \exp\left\{ \int_R^t \int_s^T h^a(s,u)duds \right\} \frac{2a}{(a-t^2)} \frac{(a-R^2)}{(a-RT)^2} \,. \tag{4.3.56}$$

Moreover, by Lemma 4.3.11 we know that $h^a(t,T) = \frac{2a}{(a-tT)^2}$ satisfies (4.3.56).

Since

$$\lim_{t \to \frac{T}{2}} h^a(t,T) = \lim_{t \to \frac{T}{2}} \frac{T^2}{\left(\frac{T^2}{2} - tT\right)^2} = +\infty \,, \tag{4.3.57}$$

then with positive probability we can conclude that $\lim_{t \to \frac{T}{2}} \tilde{s}^f_{K-1}(t,T) = +\infty$ and hence

$$\lim_{t \to \frac{T}{2}} s^f_{K-1}(t,T) = +\infty \,, \tag{4.3.58}$$

and so the claim follows under the measure $\mathbb{Q}$.

□

Remark 4.3.15. *Since the measures $\mathbb{Q}$, $\mathbb{Q}^*$ and $\mathbb{P}^*$ are all equivalent measures then the corresponding statements of Theorem 4.3.14 hold under any of these measures.*

Remark 4.3.16. *Note that, assumption (ii) of Theorem 4.3.14 on the set A_∞ is closely related to the volatility structure of the forward rates. More precisely and intuitively speaking, it is possible for a class of volatility structures $\sigma_{K-1}(t,T), \sigma(t,T)$ that assumption (ii) holds true and thus the spread explodes. This shows the importance of the model specification that one should be aware of and that the free parameters of the model should be specified carefully. In Subsection 4.3.3 we give a possible solution.*

At a first glance, assumptions that the set A_∞ is a positive probability set and $\lambda^*_{K-1,K}(t)$ is uniformly bounded from below in Theorem 4.3.14 are for example satisfied in the case in which the intensities $\lambda^*_{i,j}(t)$ are all continuous deterministic functions, and the volatilities $\sigma_{K-1}(T,t)$ and $\sigma(t,T)$ do not have the pathological specification of being such that for some $S < T$ the probability of an excursion of the process $\tilde{N}_{K-1}(t,T)$ over the threshold in the set $A^{R,S,a}_{K-1}$ on the interval $S \le t \le T$ is zero. Indeed in such a case $A^{R,S,a}_{K-1}$ and B^S_{K-1} are genuinely independent positive probability events, and therefore $\mathbb{Q}(A_\infty) > 0$ is clear. However, let us remark that in view of the constraint (4.3.9), $\lambda^*_{K-1,K}(t)$ cannot be legitimately interpreted as a free parameter, so the example below provides a more complete picture.

Proposition 4.3.17. *Assume that for all $i, j = 1, \ldots, K-1$ with $i \neq j$, the intensity parameters $\lambda^*_{i,j}(t)$ are deterministic functions. Moreover assume also that the defaultable intensities $\lambda^*_{i,K}(t)$ for $i = 1, \ldots, K-2$ are deterministic and the volatilities $\sigma_i(t,T), \sigma(t,T)$ are such that $\sigma_i(t,t) = \sigma(t,t)$ and $\mathbb{Q}\left(A^{R,S,a}_{K-1}\right) > 0$ for all $T \le T^*$ for a, R, S as in (4.3.45). Furthermore, assume $\alpha_i(t,t)$ is such that the process $\alpha_{K-1}(t,t) - \alpha_i(t,t)$ is deterministic on the set $\{C^1_t = K-1\}$ for $t < \tau_1$.*[1] *Then it holds $\mathbb{Q}(A_\infty) > 0$ for all $T \le T^*$.*

Proof. From Corollary 4.3.9 and for $T = t$ we have

$$ds^f_{K-1}(t,t) = \left\{ \sum_{j=1, j\neq K-1}^{K-1} \lambda^*_{K-1,j}(t) s_{K-1,j}(t,t) + \lambda^*_{K-1,K}(t)\delta_{K-1} s^f_{K-1}(t,t) \right\} dt\,, \tag{4.3.59}$$

on B^S_{K-1} where recall that from condition (4.3.9) we have $s^f_{K-1}(t,t) = \lambda^*_{K-1,K}(t)(1-\delta_{K-1})$[2]. Since by assumption the volatilities are all equal for $T = t$ and since the drift part of the spreads $s_{K-1,j}(t,t)$ for $j = 1, \ldots, K-1$ is deterministic for $T = t$ by assumption, then from (4.3.59) the fundamental spread $s^f_{K-1}(t,t)$ is deterministic. This will imply that the defaultable intensity $\lambda^*_{K-1,K}(t)$ is a deterministic function.

From the construction of the Markov process C^1_t in Bielecki and Rutkowski (2004a) we see that the only factor that depends on the initial space and hence might depend on the brownian motion is the intensity matrix Λ^*. Since all the intensity parameters are deterministic then the Markov process C^1_t is independent of the brownian motion and hence the sets $A^{R,S,a}_{K-1}$ and B^S_{K-1} are independent. Thus $\mathbb{Q}(A_\infty) = \mathbb{Q}\left(A^{R,S,a}_{K-1}\right)\mathbb{Q}\left(B^S_{K-1}\right) > 0$. Note that the set B^S_{K-1} has positive probability by the initial distribution of the Markov chain C^1_t.

□

[1]This is in view of Remark 4.3.7.

[2]Note that here either $s^f_{K-1}(t,t)$ or $\lambda^*_{K-1,K}(t)$ is a free parameter by Remark 4.3.7.

Remark 4.3.18. *For the case $K = 2$ in Proposition 4.3.17 it is quite clear from (4.3.59) that the only assumption that is needed is $\sigma_1(t,t) = \sigma(t,t)$. This is the case for example when choosing exponential volatility functions such that $\sigma_1(t,T) = \sigma e^{-\gamma_1(T-t)}$ and $\sigma(t,T) = \sigma e^{-\gamma(T-t)}$ where σ, γ_1, γ are some positive constants.*

The special case $K = 2$ with two rating classes, the risky class 1 and the defaultable class K, is an important case which gives a better intuition about the explosion Theorem 4.3.14. In particular the ordering condition O is not used, which shows that the spread explosion arises mostly from the N.1 condition. The basic idea of the proof is similar to the proof of Theorem 4.3.14.

Corollary 4.3.19. *Assume condition* N.1. *Furthermore, assume that $\delta_1 > 0$ and*

(i) $\lambda^*_{1,K}(t)$ is uniformly bounded from below in ω;

(ii) For all $T \leq T^*$ it holds $\mathbb{Q}(A_\infty) > 0$ with

$$\begin{aligned} A_\infty &= \hat{A}_1^{R,S,a} \cap A_1^{R,0}, \quad R = \frac{T}{4}, \quad S = \frac{T}{2}, \quad a = \frac{T^2}{2}, \\ A_i^{R,0} &:= \left\{ \omega : \tilde{N}_i(t,T) \geq 0 \, \textit{for all} \, 0 \leq t \leq R \right\} . \end{aligned} \tag{4.3.60}$$

If a solution to (4.3.23) exists, then for all $T \leq T^*$, $\lim_{t \to \frac{T}{2}} s_1^f(t,T) = +\infty$ with positive probability under $\mathbb{Q}$ on B_1^S. In particular

$$\lim_{t \to \frac{T}{2}} s^f_{C_t^1}(t,T) = +\infty \,, \tag{4.3.61}$$

with positive probability under $\mathbb{Q}$ prior to default.

Proof. Since the proof is similar to the proof of Theorem 4.3.14 we only sketch it.

First note that for the case $K = 2$ we have from Corollary 4.3.9 :

$$s_1^f(t,T) = N_1(t,T) + \int_0^t \lambda^*_{1,K}(s)\delta_1 s_1^f(s,T) \exp\left(\int_s^T s_1^f(s,u)du \right) ds, \tag{4.3.62}$$

for all $\omega \in B_1^S$ with S in (4.3.60) and $0 \leq t \leq T$. Then for $t \in [0,T]$ define the continuous functions

$$X(t) := s_1^f(t,T), \quad N(t) := N_1(t,T), \quad E(t) := \lambda^*_{1,K}(t)\delta_1 \exp\left(\int_t^T s_1^f(t,u)du \right) . \tag{4.3.63}$$

Then the reverse Grönwall's inequality 4.3.13 applied with the specification (4.3.63) yields:

$$s_1^f(t,T) \geq N_1(t,T) + \int_0^t N_1(s,T)\lambda^*_{1,K}(s)\delta_1 \exp\left(\int_s^T s_1^f(s,u)du \right) \tag{4.3.64}$$

$$\exp\left(\int_s^t \lambda^*_{1,K}(u)\delta_1 \exp\left(\int_u^T s_1^f(u,v)dv\right)du\right)ds.$$

So, we can further estimate

$$s_1^f(t,T) \geq N_1(t,T) + \int_0^t N_1(s,T)\hat{\lambda}^*_{1,K}(s)\delta_1 \exp\left(\int_s^T s_1^f(s,u)du\right)ds, \tag{4.3.65}$$

for $0 \leq t \leq T$ and on $A_1^{R,0}$. Furthermore and for $t \geq R$ we can write

$$\begin{aligned} s_1^f(t,T) \geq N_1(t,T) &+ \int_0^R N_1(s,T)\hat{\lambda}^*_{1,K}(s)\delta_1 \exp\left(\int_s^T s_1^f(s,u)du\right)ds \\ &+ \int_R^t N_1(s,T)\hat{\lambda}^*_{1,K}(s)\delta_1 \exp\left(\int_s^T s_1^f(s,u)du\right)ds. \end{aligned} \tag{4.3.66}$$

Thus, for $(t,T) \in \hat{K}^{R,S}$ and $\omega \in A_\infty$:

$$s_1^f(t,T) \geq \int_R^t \frac{2a}{(a-S^2)}\frac{(a-R^2)}{(a-RT)^2}\exp\left(\int_s^T s_1^f(s,u)du\right)ds, \tag{4.3.67}$$

with a, R, S given in (4.3.60).
Now the claim follows as in the proof of Theorem 4.3.14.

□

Remark 4.3.20. *The ordering condition O is used in Theorem 4.3.14 only in order to show that $N_{K-1}(t,T) \geq \tilde{N}_{K-1}(t,T)$ and $\lambda^*_{K-1,K}(t) > 0$. For the case $K = 2$ we have that $N_1(t,T) = \tilde{N}_1(t,T)$. Furthermore in (4.3.59) for the case $K = 2$ we have $s_1^f(t,t) > 0$, hence from the no-arbitrage condition (4.3.9) we have $\lambda^*_{1,K}(t) > 0$. Hence the ordering condition O is not needed to be assumed in Corollary 4.3.19.*

The next Example is a special case where Corollary 4.3.19 can be applied.

Example 4.3.2. *Assume that the volatilities $\sigma_1(t,T), \sigma(t,T)$ are deterministic and such that $\sigma_1(t,t) = \sigma(t,t)$ and $\sigma_1(t,T) \neq \sigma(t,T)$ for $t < T$ so that for the set $A_1^{S,R,a} \cap A_1^{R,0}$ from 4.3.60 it holds $\mathbb{Q}(A_1^{S,R,a} \cap A_1^{R,0}) > 0$ for all $T \leq T^*$.*

From Corollary 4.3.9 and for $T = t$ we have

$$ds_1^f(t,t) = \lambda^*_{1,K}(t)\delta_1 s_1^f(t,t)dt, \tag{4.3.68}$$

on B_1^T. Then from (4.3.68) the fundamental spread $s_1^f(t,t)$ is deterministic. Therefore condition (4.3.9), that is $s_1^f(t,t) = \lambda^*_{1,K}(t)(1-\delta_1)$, will imply that the defaultable intensity $\lambda^*_{1,K}(t)$ is a deterministic function. As in the proof of Proposition 4.3.17, since all the intensity parameters are deterministic then the sets $A_1^{R,S,a} \cap A_1^{R,0}$ and B_1^S are independent and thus $\mathbb{Q}(A_\infty) = \mathbb{Q}(B_1^S)\mathbb{Q}(A_1^{S,R,a} \cap A_1^{R,0}) > 0$ for all $T \leq T^*$.

4.3.3 Transformation from zero recovery to non-zero recovery models.

As shown in the main Theorem 4.3.14 of the previous Section 4.3.2 the fundamental spread $s^f_{C^1_t}(t,T)$ explodes in finite time under some mild conditions for the non-zero recovery case. So, a natural question that arises from this is if there exist any kind of non-exploding non-zero recovery models and if yes, how can we specify such a class of models.

In this section we are showing how to construct a non-explosive, non-zero recovery model from a zero recovery one. This is ensured via a transformation given in Proposition 4.3.22, which relies on no-arbitrage arguments, typical of defaultable models.

Assume a zero recovery model from Section 4.2.1, that is let $\delta_i = 0$, for all $i = 1, \dots, K-1$, and consider the following notation:

For $\delta_i = 0$, for all $i = 1, \dots, K-1$, denote the *zero recovery fundamental spread* by $s^{f,0}_{C^1_t}(t,T)$. The resulting (through the transformation) *non-zero recovery fundamental spread* will be denoted by $s^{f,\delta}_{C^1_t}(t,T)$.

Remark 4.3.21. *Recall the discounted defaultable bond price process $\hat{Z}(t,T)$ from (4.2.25) and define the zero-recovery discounted defaultable bond price process $\hat{Z}^0(t,T)$ for all $0 \le t \le T$ by*

$$\hat{Z}^0(t,T) := \mathbf{1}_{\{C^1_t \neq K\}} Z_{C^1_t}(t,T)\,. \tag{4.3.69}$$

Then we can write

$$\hat{Z}(t,T) = \hat{Z}^0(t,T) + \mathbf{1}_{\{C^1_t = K\}} \delta_{C^2_t} Z(t,T)\,, \tag{4.3.70}$$

for all $0 \le t \le T$.

Furthermore note that by using the definitions $\hat{Z}(t,T) = B_t^{-1}\hat{D}(t,T)$ and $Z(t,T) = B_t^{-1}B(t,T)$ as well as the definitions (4.2.4) and (4.2.5), then we get the relations

$$\int_t^T s^{f,\delta}_{C^1_t}(t,u)du = -\log\left(\frac{\hat{Z}(t,T)}{Z(t,T)}\right) = -\log\left(\frac{\hat{D}(t,T)}{B(t,T)}\right), \tag{4.3.71}$$

and

$$\int_t^T s^{f,0}_{C^1_t}(t,u)du = -\log\left(\frac{\hat{Z}^0(t,T)}{Z(t,T)}\right) = -\log\left(\frac{\hat{D}^0(t,T)}{B(t,T)}\right), \tag{4.3.72}$$

where $\hat{D}^0(t,T) = B_t \hat{Z}^0(t,T)$.

Proposition 4.3.22. *For any fixed maturity $T \le T^*$, we get the spread relation*

$$s^{f,\delta}_{C^1_t}(t,T) = \frac{\exp\left(-\int_t^T s^{f,0}_{C^1_t}(t,u)du\right) s^{f,0}_{C^1_t}(t,T) - \frac{\partial}{\partial T}\mathbb{E}^*_T\left[\delta_{C^2_T}\mathbf{1}_{\{\tau\le T\}}|\tilde{\mathcal{F}}_t\right]}{\exp\left(-\int_t^T s^{f,0}_{C^1_t}(t,u)du\right) + \mathbb{E}^*_T\left[\delta_{C^2_T}\mathbf{1}_{\{\tau\le T\}}|\tilde{\mathcal{F}}_t\right]}, \tag{4.3.73}$$

on $\{C^1_t \neq K\}$ where $\mathbb{E}^*_T$ denotes the expectation taken with respect to the terminal measure $\mathbb{Q}^*_T$ using as numeraire $N_t = B(t,T)$. Furthermore, if $\delta_i = \delta$ for all $i = 1, \dots, K-1$, for some *constant* $\delta \in [0,1)$*, then it holds*

$$s^{f,\delta}_{C^1_t}(t,T) = \frac{(1-\delta)\exp\left(-\int_t^T s^{f,0}_{C^1_t}(t,u)du\right)}{(1-\delta)\exp\left(-\int_t^T s^{f,0}_{C^1_t}(t,u)du\right)+\delta} s^{f,0}_{C^1_t}(t,T)\,. \tag{4.3.74}$$

Proof. We want to transform a zero recovery model to a non-zero recovery one using the following transformation:

$$y = -\log\left(\exp(-x) + \mathbb{E}^*_T\left[\delta_{C^2_T}\mathbf{1}_{\{\tau\le T\}}|\tilde{\mathcal{F}}_t\right]\right)\,, \tag{4.3.75}$$
$$x = \int_t^T s^{f,0}_{C^1_t}(t,u)du \quad , \quad y = \int_t^T s^{f,\delta}_{C^1_t}(t,u)du\,.$$

The transformation (4.3.75) is justified as follows: Choose as numeraire $N_t := B(t,T)$ and change from the measure $\mathbb{Q}^*$ to the terminal measure $\mathbb{Q}^*_T$. Consider the default indicator function $\mathbf{1}_{\{\tau>T\}}$ where note that $\{\tau > T\} = \{C^1_T = K\}$. Then for the zero recovery defaultable bond $\hat{D}^0(t,T)$ it holds

$$\hat{D}^0(t,T) = N_t\,\mathbb{E}^*_T\left[\mathbf{1}_{\{\tau>T\}}N_T^{-1}|\tilde{\mathcal{F}}_t\right] = B(t,T)\,\mathbb{E}^*_T\left[\mathbf{1}_{\{\tau>T\}}|\tilde{\mathcal{F}}_t\right]\,, \tag{4.3.76}$$

from where one could also imply that $\frac{\hat{D}^0(t,T)}{B(t,T)} = \mathbb{Q}^*_T(\tau > T|\tilde{\mathcal{F}}_t)$ is the conditional survival probability under the terminal measure $\mathbb{Q}^*_T$.

Furthermore, the non-zero recovery defaultable bond $\hat{D}(t,T)$ with recovery is given by

$$\begin{aligned}\hat{D}(t,T) &= N_t\,\mathbb{E}^*_T\left[\mathbf{1}_{\{\tau>T\}}N_T^{-1}|\tilde{\mathcal{F}}_t\right] + N_t\,\mathbb{E}^*_T\left[\delta_{C^2_T}\mathbf{1}_{\{\tau\le T\}}N_T^{-1}|\tilde{\mathcal{F}}_t\right] \\ &= B(t,T)\,\mathbb{E}^*_T\left[\mathbf{1}_{\{\tau>T\}}|\tilde{\mathcal{F}}_t\right] + B(t,T)\,\mathbb{E}^*_T\left[\delta_{C^2_T}\mathbf{1}_{\{\tau\le T\}}|\tilde{\mathcal{F}}_t\right] \\ &= \hat{D}^0(t,T) + B(t,T)\,\mathbb{E}^*_T\left[\delta_{C^2_T}\mathbf{1}_{\{\tau\le T\}}|\tilde{\mathcal{F}}_t\right]\,.\end{aligned} \tag{4.3.77}$$

Then (4.3.77) implies the relation

$$\frac{\hat{D}(t,T)}{B(t,T)} = \frac{\hat{D}^0(t,T)}{B(t,T)} + \mathbb{E}^*_T\left[\delta_{C^2_T}\mathbf{1}_{\{\tau\le T\}}|\tilde{\mathcal{F}}_t\right]\,, \tag{4.3.78}$$

which is equivalent to

$$\exp\left(-\int_t^T s^{f,\delta}_{C^1_t}(t,u)du\right) = \exp\left(-\int_t^T s^{f,0}_{C^1_t}(t,u)du\right) + \mathbb{E}^*_T\left[\delta_{C^2_T}\mathbf{1}_{\{\tau\le T\}}|\tilde{\mathcal{F}}_t\right]\,, \tag{4.3.79}$$

under the terminal measure $\mathbb{Q}^*_T$. So, (4.3.79) gives us the transformation (4.3.75). In particular, from (4.3.75) and differentiating with respect to the maturity T, we get the spread relation (4.3.73).

For the case of $\delta_i = \delta$, equation (4.3.74) follows by using the fact that

$$\mathbb{E}^*_T\left[\delta_{C^2_T}\mathbf{1}_{\{\tau\le T\}}|\tilde{\mathcal{F}}_t\right] = \delta\left(1-\exp\left(-\int_t^T s^{f,0}_{C^1_t}(t,u)du\right)\right)\,, \tag{4.3.80}$$

$$\frac{\partial}{\partial T}\mathbb{E}_T^*\left[\delta_{C_T^2}\mathbf{1}_{\{\tau\le T\}}|\tilde{\mathcal{F}}_t\right] = \delta s^{f,0}_{C_t^1}(t,T)\exp\left(-\int_t^T s^{f,0}_{C_t^1}(t,u)du\right).$$

□

Remark 4.3.23. *The transformation (4.3.75) gives us the correct approach how to construct a non-zero recovery model from a zero recovery one. Indeed, since* $\lim_{x\to\infty} y = -\log\left(\mathbb{E}_T^*\left[\delta_{C_T^2}\mathbf{1}_{\{\tau\le T\}}|\tilde{\mathcal{F}}_t\right]\right)$ *and since the function $y(x)$ from (4.3.75) is increasing in x then the integrated fundamental non-zero recovery spread* $\int_t^T s^{f,\delta}_{C_t^1}(t,u)du$ *cannot explode, thus* $s^{f,\delta}_{C_t^1}(t,T)$ *has no explosive dynamics. For the case $\delta_i = \delta$ we have* $\lim_{x\to\infty} y = -\log(\delta)$.

Remark 4.3.24. *From the proof of Proposition 4.3.22 note that since*

$$\frac{\hat{D}^0(t,T)}{B(t,T)} = \exp\left(-\int_t^T g_i(t,u) - f(t,u)du\right), \tag{4.3.81}$$

on $\{C_t^1 = i\}$ *and since* $\frac{\hat{D}^0(t,T)}{B(t,T)} = \mathbb{Q}_T^*(\tau > T|\tilde{\mathcal{F}}_t)$, *this enforces the condition* $\int_t^T g_{C_t^1}(t,u)du > \int_t^T f(t,u)du$ *which seems to be necessary. Note that this implies* $g_{C_t^1}(t,T) > f(t,T)$. *The condition* $g_i(t,T) > f(t,T)$ *for all* $i = 1,\ldots,K-1$, *which follows from the ordering condition O is only sufficient.*

Remark 4.3.25. *Another observation from the spread relation (4.3.73) in Proposition 4.3.22 is the following: the assumptions used in the explosion Theorem 4.3.14 show a class of non-admissible models for the non-zero recovery case. The transformation (4.3.73) gives us a class of admissible non-zero recovery models where note that the volatility has a "vanishing at zero structure" when the spread goes to infinity. This kills the explosion by killing the random term which is the integral with respect to the brownian motion.*

Remark 4.3.26. *Despite that the spread relation (4.3.73) is more general, the special case of* $\delta_i = \delta$ *(see equation (4.3.74)), has a more intuitive interpretation. Indeed from (4.3.77) we get*

$$\hat{D}(t,T) = (1-\delta)\hat{D}^0(t,T) + \delta B(t,T), \tag{4.3.82}$$

thus the defaultable bond with recovery is a convex combination of the zero recovery bond and the default-free bond price. Note also that (4.3.82) is equivalent to

$$\hat{D}(t,T) = B(t,T)\left[\delta + (1-\delta)\mathbb{Q}_T^*(\tau > T|\tilde{\mathcal{F}}_t)\right], \tag{4.3.83}$$

which is similar to (13.46) *in Bielecki and Rutkowski (2004a).*

4.4 The Strong Consistency Condition.

Recall that the (weak) consistency condition N.1 introduced and analysed in Section 4.3, is not only sufficient but also necessary for an arbitrage free model setting according to Theorem 4.3.5.

In this section we introduce the (strong) consistency condition N.2 which is only sufficient and is assumed first in the literature, for example in Bielecki and Rutkowski (2000), Bielecki and Rutkowski (2004b) and Eberlein and Özkan (2003). It also appears in Özkan and Schmidt (2005) and Schmidt (2006) as an alternative condition if one would rather have a condition which does not rely on a particular realization of the Markov chain C^1.

In contrast to the weak consistency, the no-arbitrage drift condition under the strong consistency requires that all the forward rates become "active". That means, an extended HJM no-arbitrage drift condition holds for the current forward rate as well as for all other forward rates. We analyze the model under the strong consistency and show in Section 4.4.2 similar explosion results to those of Section 4.3.2 for the inter-rating spreads. Nevertheless, in a multiple-issuer migration model framework, the strong consistency meets the weak one as shown in Section 4.4.3.

Let us start by introducing the *(strong) consistency condition* N.2.

Condition N.2. Assume that the entries of Λ^* satisfy for all $0 \leq t \leq T$ and for all $i = 1, \ldots, K-1$:

$$\sum_{j=1, j\neq i}^{K-1} \lambda_{i,j}^*(t)(Z_j(t,T) - Z_i(t,T)) + \lambda_{i,K}^*(t)(\delta_i Z(t,T) - Z_i(t,T)) = -\eta_i(t,T)Z_i(t,T)\,. \tag{4.4.1}$$

It is clear that since consistency condition N.2 is a condition over each rating class at any time $t \in [0, T]$, this implies condition N.1 (see (4.3.3)) and thus sufficient for Theorem 4.3.5. So, under condition N.2 the defaultable bond $\hat{Z}(t, T)$ is a (local) martingale as shown in Corollary 4.4.1.

Corollary 4.4.1. *The discounted defaultable bond $\hat{Z}(\cdot, T)$ is a local martingale under $\mathbb{Q}^*$ if the consistency condition* N.2 *holds.*

Proof. The proof follows as in Theorem 4.3.5. □

4.4.1 No-arbitrage drift condition on all forward rates.

Similar to Proposition 4.3.6, let us now derive the no-arbitrage drift condition for the forward rates which follows from the consistency condition N.2. Recall also the notation of the spreads of Section 4.3.1.

Proposition 4.4.2. *Assume that the consistency condition* N.2 *holds and fix some maturity $T \leq T^*$. Then condition* N.2 *is equivalent to the following: for all $0 \leq t \leq T$ and $i = 1, \ldots, K-1$ the drift condition*

$$\begin{aligned} \alpha_i(t,T) &= \sigma_i(t,T)\int_t^T \sigma_i(t,u)du - \gamma_t\sigma_i(t,T) \\ &\quad + \sum_{j=1, j\neq i}^{K-1} \lambda_{i,j}^*(t)s_{i,j}(t,T)\exp\left(\int_t^T s_{i,j}(t,u)du\right) \end{aligned} \tag{4.4.2}$$

$$+ \lambda^*_{i,K}(t)\delta_i s^f_i(t,T) \exp\left(\int_t^T s^f_i(t,u)du\right) ,$$

of the forward rate $g_i(t,T)$ *holds, together with the condition*

$$s^f_i(t,t) = \lambda^*_{i,K}(t)(1-\delta_i) , \tag{4.4.3}$$

for all $i = 1, \ldots, K-1$.

Proof. Follows exactly the same steps as the proof of Proposition 4.3.6. □

4.4.2 Inter-rating spreads and explosions under the strong arbitrage condition.

For what follows in this section we will assume that all the recovery rates are zero, i.e.,

$$\delta_i = 0 \quad , \quad \text{for all } i = 1, \ldots, K-1. \tag{4.4.4}$$

In Section 4.3.2 we showed that in the non-zero recovery case the fundamental spread explodes in finite time prior to default with positive probability under the condition N.1. Since, condition N.2 implies N.1 it is reasonable to assume (4.4.4) in this section such that we are in a zero recovery setting where the spreads do not necessarily explode. In other words, this assumption is ultimately not restrictive of the generalities of the explosion results: since N.2 implies N.1, when recovery rates are positive, explosions under N.2 are obtained as a consequence of explosions under N.1.

By deriving the no-arbitrage inter-rating spread dynamics, we show in Theorem 4.4.4 that under condition N.2 the inter-rating spread $s_{K-1,1}(t,T)$, defined as in (4.3.5), explodes in finite time with positive probability for general K in the zero-recovery case. This is due to the fact that under condition N.2 the spread $s_{i,j}(t,T)$ has both forward rates $g_i(t,T)$ as well as $g_j(t,T)$ active in the sense that both must satisfy the drift condition (4.4.2). The ordering condition is also assumed in the explosion proof. However, similar to Section 4.3.2, for the special case of $K = 3$, explosion still occurs even without assuming ordering of the forward rates.

Similarly to Corollary 4.3.9, the next Corollary 4.4.3 gives us the dynamics of the inter-rating spread $s_{i,j}(t,T)$, $i,j = 1,\ldots,K-1$, $i \neq j$, under the equivalent "physical" measure $\mathbb{Q}$.

Corollary 4.4.3. *Assume* $\delta_i = 0$ *for all* $i = 1,\ldots,K-1$. *For any fixed maturity* $T \leq T^*$ *and under the consistency condition* N.2*, we have the dynamics*

$$ds_{i,j}(t,T) = \Bigg\{\sigma_i(t,T)\int_t^T \sigma_i(t,u)du - \sigma_j(t,T)\int_t^T \sigma_j(t,u)du \tag{4.4.5}$$

$$+ \sum_{l=1, l\neq i,j}^{K-1} \Bigg[\lambda^*_{i,l}(t)s_{i,l}(t,T)\exp\left(\int_t^T s_{i,l}(t,u)du\right)$$

$$
\begin{aligned}
&\qquad\qquad - \lambda^*_{j,l}(t) s_{j,l}(t,T) \exp\left(\int_t^T s_{j,l}(t,u)du\right)\Bigg] \\
&+ \lambda^*_{i,j}(t) s_{i,j}(t,T) \exp\left(\int_t^T s_{i,j}(t,u)du\right) \\
&+ \lambda^*_{j,i}(t) s_{i,j}(t,T) \exp\left(-\int_t^T s_{i,j}(t,u)du\right)\Bigg\}dt \\
&+ (\sigma_i(t,T) - \sigma_j(t,T))dW_t\,,
\end{aligned}
$$

for $0 \le t \le T$, *for the inter-rating spread* $s_{i,j}(t,T)$ *for* $i, j = 1, \ldots, K-1$ *with* $i \ne j$.

Proof. From Proposition 4.4.2 we have the dynamics

$$
\begin{aligned}
ds_{i,j}(t,T) = &\Bigg\{\sigma_i(t,T) \int_t^T \sigma_i(t,u)du - \sigma_j(t,T) \int_t^T \sigma_j(t,u)du \qquad (4.4.6)\\
&+ \sum_{l=1, l\ne i}^{K-1} \lambda^*_{i,l}(t) s_{i,l}(t,T) \exp\left(\int_t^T s_{i,l}(t,u)du\right) \\
&- \sum_{l=1, l\ne j}^{K-1} \lambda^*_{j,l}(t) s_{j,l}(t,T) \exp\left(\int_t^T s_{j,l}(t,u)du\right) \\
&- \gamma_t(\sigma_i(t,T) - \sigma_j(t,T))\Bigg\}dt \\
&+ (\sigma_i(t,T) - \sigma_j(t,T))dW^*_t\,,
\end{aligned}
$$

for $0 \le t \le T$. Then by a change of measure using Girsanov's Theorem the claim follows. □

A similar result as for the fundamental spreads in Theorem 4.3.14 holds also for the inter-rating spreads as shown in Theorem 4.4.4. The idea and the arguments of the proof are not surprisingly similar. Thus we give only a sketch of the proof.

We fix the following notation for $i, j = 1, \ldots, K-1$ with $i \ne j$:

$$
\begin{aligned}
N_{i,j}(t,T) := &\tilde{N}_{i,j}(t,T) \qquad (4.4.7)\\
&+ \sum_{j=1, l\ne i,j}^{K-1} \Bigg[\int_0^t \lambda^*_{i,l}(s) s_{i,l}(s,T) \exp\left(\int_s^T s_{i,l}(s,u)du\right) ds \\
&\qquad - \int_0^t \lambda^*_{j,l}(s) s_{j,l}(s,T) \exp\left(\int_s^T s_{j,l}(s,u)du\right) ds\Bigg] \\
&+ \int_0^t \lambda^*_{j,i}(s) s_{i,j}(s,T) \exp\left(-\int_s^T s_{i,j}(s,u)du\right) ds,
\end{aligned}
$$

where

$$
\tilde{N}_{i,j}(t,T) := s_{i,j}(0,T) + \int_0^t \int_s^T \{\sigma_i(s,T)\sigma_i(s,u) - \sigma_j(s,T)\sigma_j(s,u)\}\, duds \qquad (4.4.8)
$$

$$+\int_0^t (\sigma_i(s,T) - \sigma_j(s,T))dW_s\,,$$

and for $\lambda^*_{i,j}(t) > 0$ define the set

$$A^{R,S,a}_{i,j} := \left\{\omega : \tilde{N}_{i,j}(t,T) \geq \frac{2a}{(a-S^2)}\frac{(a-R^2)}{(a-RT)^2}\frac{1}{\hat{\lambda}^*_{i,j}(t)} \text{ for all } R \leq t \leq S\right\}, \qquad (4.4.9)$$

where a, R, S are positive constants with $a > S^2$, $a > RT$, $R \leq S \leq T$ and $\hat{\lambda}^*_{i,j}(t) := \inf_\omega \lambda^*_{i,j}(t)$.

Theorem 4.4.4. *Assume conditions* N.2 *and* O. *Furthermore, assume that* $K \geq 3$ *and* $\delta_i = 0$ *for all* $i = 1, \ldots, K-1$. *Moreover let*

(i) $\lambda^*_{K-1,1}(t)$ *is positive, continuous and uniformly bounded from below in* ω;

(ii) For all $T \leq T^*$ *it holds* $\mathbb{Q}(A_\infty) > 0$ *with*

$$A_\infty = A^{R,S,a}_{K-1,1}, \quad R = \frac{T}{4}, \quad S = \frac{T}{2}, \quad a = \frac{T^2}{2}. \qquad (4.4.10)$$

If a solution to (4.4.5) exists, then for all $T \leq T^*$, $\lim_{t\to\frac{T}{2}} s_{K-1,1}(t,T) = +\infty$ *with positive probability under* $\mathbb{Q}$.

Proof. First note that for the case $K \geq 3$ we have from Corollary 4.4.3 and by the ordering condition O

$$s_{K-1,1}(t,T) \geq N_{K-1,1}(t,T) + \int_R^t \lambda^*_{K-1,1}(s) s_{K-1,1}(s,T)\exp\left(\int_s^T s_{K-1,1}(s,u)du\right)ds\,, \qquad (4.4.11)$$

for $0 \leq t \leq T$.

Moreover, for $t \in [0,T]$ define the continuous functions

$$\begin{aligned} X(t) &:= s_{K-1,1}(t,T), \quad N(t) := N_{K-1,1}(t,T), \\ E(t) &:= \lambda^*_{K-1,1}(t)\exp\left(\int_t^T s_{K-1,1}(t,u)du\right). \end{aligned} \qquad (4.4.12)$$

Since from the model assumptions $\lambda^*_{K-1,1}(t)$ is non-negative then $E(t)$ is also a non-negative function. In addition, $E(t)$ is continuous by the continuity assumption on $\lambda^*_{K-1,1}(t)$.

It is clear that by using similar arguments as in the proof of Theorem 4.3.14 and Reverse Grönwall's Lemma 4.3.13 with the specification (4.4.12), we can show that on the set A_∞ and for $(t,K) \in \hat{K}^{R,S}$ we get

$$\tilde{s}_{K-1,1}(t,T) \geq h^a(t,T)\,, \qquad (4.4.13)$$

where $\tilde{s}_{K-1,1}(t,T) := \frac{1}{(t-R)} s_{K-1,1}(t,T)$.

By choosing again the parameters a, R, S as in (4.4.10), we can conclude that

$$\lim_{t \to \frac{T}{2}} s_{K-1,1}(t,T) = +\infty\,, \tag{4.4.14}$$

with positive probability.

□

Remark 4.4.5. *Recall that, since the measures $\mathbb{Q}$, $\mathbb{Q}^*$ and $\mathbb{P}^*$ are all equivalent measures then the corresponding statements of Theorem 4.4.4 hold under any of these measures.*

Remark 4.4.6. *Note that, as in Remark 4.3.16, the specification of the volatilities of the forward rates is of essential importance in achieving a healthy model without spread explosions.*

Remark 4.4.7. *Recalling Remark 4.3.7 note that in Theorem 4.4.4 the migration intensity parameter $\lambda^*_{K-1,1}(t)$ is a free parameter while in Theorem 4.3.14 the parameter $\lambda^*_{K-1,K}(t)$ must fulfil condition (4.3.9) on the set $\{C^1_t = K-1\}$.*

Remark 4.4.8. *In this Remark we show how an example can be constructed such that the inter-rating spread from Theorem 4.4.4 explodes prior to default. Similar to Proposition 4.3.17, assuming that for all $i, j = 1, \ldots, K-1$ with $i \neq j$, the intensity parameters $\lambda^*_{i,j}(t)$ are deterministic continuous functions and the volatilities $\sigma_i(t,T), \sigma(t,T)$ are such that $\mathbb{Q}\left(A^{R,S,a}_{K-1,1}\right) > 0$ for all $T \leq T^*$ with $\sigma_i(t,t) = \sigma(t,t)$, then since all the forward rates $g_i(t,T)$ are active, i.e., they must all fulfill the drift condition (4.4.2), then it is clear that in the zero recovery case all the inter-rating spreads $s_{i,j}(t,T)$ at time $T = t$ are deterministic. In particular since in this case we have the dynamics*

$$ds^f_i(t,t) = \sum_{j=1, j\neq i}^{K-1} \lambda^*_{i,j}(t) s_{i,j}(t,t)dt\,, \tag{4.4.15}$$

for $i = 1, \ldots, K-1$, and since by condition (4.4.3), $s^f_i(t,t) = \lambda^*_{i,K}(t)$, then the defaultable intensity parameters $\lambda^*_{i,K}(t)$ are deterministic functions. As in the proof of Proposition 4.3.17 this ensures the independence between the Markov chain C^1_t and the brownian motion. So, in the proof of Theorem 4.4.4 we have $\mathbb{Q}\left(A^{R,S,a}_{K-1,1} \cap \{\tau > T\}\right) = \mathbb{Q}\left(A^{R,S,a}_{K-1,1}\right)\mathbb{Q}\left(\{\tau > T\}\right) > 0$. Hence, in this case, in Theorem 4.4.4 we can conclude that for all $T \leq T^*$, $s_{K-1,1}(\frac{T}{2}, T) = +\infty$ with positive probability prior to default.

For the special case $K = 3$ with two risky rating classes and a default class K, there is only one inter-rating spread $s_2(t,T)$. As in Corollary 4.3.19 we show in Corollary 4.4.9 that the inter-rating spread $s_2(t,T)$ explodes in finite time without using the ordering condition O (see also Remark 4.3.20). The proof is similar to Theorem 4.4.4.

Corollary 4.4.9. *Assume condition* N.2. *Furthermore, assume that $K = 3$ and $\delta_i = 0$ for $i = 1, 2$. Moreover let* *(i)* $\lambda^*_{2,1}(t)$ *is positive, continuous and uniformly bounded from below in* ω*;*

(ii) For all $T \leq T^*$ *it holds* $\mathbb{Q}(A_\infty) > 0$ *with*

$$A_\infty = A_{2,1}^{R,S,a} \cap A_{2,1}^{R,0}, \quad R = \frac{T}{4}, \quad S = \frac{T}{2}, \quad a = \frac{T^2}{2}, \tag{4.4.16}$$
$$A_{i,j}^{R,0} := \left\{\omega : \tilde{N}_{i,j}(t,T) \geq 0 \, \textit{for all} \, 0 \leq t \leq R\right\}.$$

If a solution to (4.4.5) exists, then for all $T \leq T^*$*,* $\lim_{t\to\frac{T}{2}} s_2(t,T) = +\infty$ *with positive probability under* $\mathbb{Q}$*.*

Proof. First note that for the case $K = 3$ we have from Corollary 4.4.3,

$$s_2(t,T) = \tilde{N}_{2,1}(t,T) \tag{4.4.17}$$
$$+ \int_0^t s_2(s,T) \left[\lambda^*_{2,1}(s) \exp\left(\int_s^T s_2(s,u)du\right) + \lambda^*_{1,2}(s) \exp\left(-\int_s^T s_2(s,u)du\right)\right] ds,$$

for $0 \leq t \leq T$.

Moreover define for $t \in [0,T]$ the processes

$$X(t) := s_2(t,T), \quad N(t) := \tilde{N}_{2,1}(t,T), \tag{4.4.18}$$
$$E(t) := \lambda^*_{2,1}(t) \exp\left(\int_t^T s_2(t,u)du\right) + \lambda^*_{1,2}(t) \exp\left(-\int_t^T s_2(t,u)du\right),$$

where note that $E(t)$ is continuous and positive. Then Reverse Grönwall's inequality 4.3.13 applied with the specification (4.4.18) yields:

$$s_2(t,T) \geq \tilde{N}_{2,1}(t,T) + \int_0^t \tilde{N}_{2,1}(s,T)E(t) \exp\left(\int_s^t E(u)du\right) ds. \tag{4.4.19}$$

So, we can further estimate

$$s_2(t,T) \geq \tilde{N}_{2,1}(t,T) + \int_0^t \tilde{N}_{2,1}(s,T)\hat{\lambda}^*_{2,1}(s) \exp\left(\int_s^T s_2(s,u)du\right) ds, \tag{4.4.20}$$

for $0 \leq t \leq T$ and on $A_{2,1}^{R,0}$. Furthermore and for $t \geq R$ we can write

$$s_2(t,T) \geq \tilde{N}_{2,1}(t,T) + \int_0^R \tilde{N}_{2,1}(s,T)\hat{\lambda}^*_{2,1}(s) \exp\left(\int_s^T s_2(s,u)du\right) ds \tag{4.4.21}$$
$$+ \int_R^t \tilde{N}_{2,1}(s,T)\hat{\lambda}^*_{2,1}(s) \exp\left(\int_s^T s_2(s,u)du\right) ds.$$

Thus, for $(t,T) \in \hat{K}^{R,S}$ and $\omega \in A_\infty$:

$$s_2(t,T) \geq \int_R^t \frac{2a}{(a - S^2)} \frac{(a - R^2)}{(a - RT)^2} \exp\left(\int_s^T s_2(s,u)du\right) ds, \tag{4.4.22}$$

with a, R, S given in (4.4.16).

So the claim follows as in the proof of Theorem 4.4.4. □

The next Example is an application of Corollary 4.4.9 for a special case.[3].

Example 4.4.1. *In a zero recovery setup for $K = 3$ assume that the volatilities $\sigma_2(t,T), \sigma_1(t,T)$ are deterministic, for example constant, such that $\sigma_2(t,T) = \sigma_2$, $\sigma_1(t,T) = \sigma_1$ with $\sigma_1 \neq \sigma_2$. Furthermore assume that $\lambda^*_{2,1}(t) > 0$, deterministic and continuous. We can minimize $\lambda^*_{2,1}(t)$ and therefore maximize on $[0,T^*]$ the right hand side of the inequality in the definition (4.4.9) of $A^{R,S,a}_{2,1}$. Denote such maximum by M: then for all $x > M$ we have, using the Markov property of $\tilde{N}_{2,1}(t,T)$*

$$\begin{aligned}\mathbb{Q}(A^{R,S,a}_{2,1} \cap A^{R,0}_{2,1}) =& \mathbb{Q}(A^{R,S,a}_{2,1} | A^{R,0}_{2,1})\mathbb{Q}(A^{R,0}_{2,1}) \\ >& \mathbb{Q}(A^{R,S,a}_{2,1} | \tilde{N}_{2,1}(R,T) = x)\mathbb{Q}(A^{R,0}_{2,1})\,.\end{aligned} \tag{4.4.23}$$

That both of the factors above are positive follows by the arcsine law for the rescaled Brownian motion with drift.

4.4.3 A multiple-issuer migration model.

As we already mentioned and showed in previous sections, the consistency conditions N.1 and N.2 are both sufficient for an arbitrage free model, but only N.1 is necessary.

In our view, condition N.1 serves a *single issuer migration model*, with one active class C^1_t, the current one, where the issuer's bond is located at the current time t. It is clear that for such a single issuer model, if all the classes were active, which this would be the interpretation of condition N.2, does not make much sense.

Nevertheless condition N.2 is more appropriate in a *multiple-issuer migration model*. That is, different issuers of possibly different classes exist where at each time t there is always at least one issuer at each class i, so that all the rating classes are active.

Using the notation and definitions of Section 2.2 such a model is constructed as follows: motivated from Section 4.4.2 (see introduction of the section), assume we have a zero recovery model, i.e., $\delta_i = 0$ for all $i = 1, \dots, K-1$ and M number of issuers such that $M \geq K - 1$. Furthermore assume that we have M independent copies of a Markov-chain C^m_t for $m = 1, \dots, M$, with the same migration intensity parameters $\lambda^*_{i,j}(t)$, i.e., $\lambda^{*,m}_{i,j}(t) = \lambda^*_{i,j}(t)$ for all $i, j = 1, \dots, K$, hence $\Lambda^{*,m} = \Lambda^*$. We call $C^m = (C^{m,1}, C^{m,2})$ the credit migration process, where $C^{m,1}_t$ is the current rating at time t of issuer m and $C^{m,2}_t$ is the previous rating before the current state $C^{m,1}_t$ of issuer m.

The following condition P is the mathematical interpretation of this model as we just described above.

Condition P. For all rating classes $i \in \{1, \dots, K-1\}$ there exists some issuer $m_i \in \{1, \dots, M\}$ such that $C^{m_i,1}_t = i$.

Define for each issuer $m \in \{1, \dots, M\}$, defaultable bonds

$$\hat{D}_{C^m_t}(t,T) := D_{C^{m,1}_t}(t,T)\mathbf{1}_{\{C^{m,1}_t \neq K\}} = \sum_{i=1}^{K-1} \mathbf{1}_{\{C^{m,1}_t = i\}} D_i(t,T)\,. \tag{4.4.24}$$

[3]Compare also with the Example 4.3.2

Since by definition $Z_i(t,T) = B_t^{-1} D_i(t,T)$ then for each issuer $m \in \{1, \ldots, M\}$, we have M-tradeable (discounted) defaultable bonds

$$\hat{Z}_{C_t^m}(t,T) := Z_{C_t^{m,1}}(t,T) \mathbf{1}_{\{C_t^{m,1} \neq K\}} = \sum_{i=1}^{K-1} \mathbf{1}_{\{C_t^{m,1}=i\}} Z_i(t,T) \,. \tag{4.4.25}$$

Since for all $m = 1, \ldots, M$, $\hat{Z}_{C_t^m}(t,T)$ are tradeable assets, then N.1 is a necessary and sufficient condition so that $\hat{Z}_{C_t^m}(t,T)$ is a (local) martingale for each m according to Theorem 4.3.5. This implies an arbitrage-free model.

So, we introduce the following consistency condition N.3. Motivated from Remark 4.3.7 we express N.3 in terms of the spreads.

Condition N.3: Assume that the entries of $\Lambda^{*,m}$ satisfy for almost all $0 \leq t \leq T$ on the set $\{C_t^{m,1} \neq K\}$:

$$\eta_{C_t^{m,1}}(t,T) = \lambda^*_{C_t^{m,1},K}(t) + \sum_{j=1, j \neq C_t^{m,1}}^{K-1} \lambda^*_{C_t^{m,1},j}(t) \left(1 - \exp\left(\int_t^T s_{C_t^{m,1},j}(t,u) du\right)\right) \,, \tag{4.4.26}$$

for each $m = 1, \ldots, M$.

As the next Corollary 4.4.10 shows, condition N.3 and condition N.2 are equivalent in a multiple-issuer model, . In particular the consistency condition N.2 is necessary and sufficient for no-arbitrage, which proves that N.2 is the right and appropriate consistency condition for this model.

Corollary 4.4.10. *Assume condition* P. *Then the discounted defaultable bond $\hat{Z}_{C_t^m}(\cdot,T)$ is a local martingale under $\mathbb{Q}^*$ for each $m = 1, \ldots, M$ if and only if the consistency condition* N.3 *holds. Moreover the conditions* N.3 *and* N.2 *are equivalent.*

Proof. The first claim of the Corollary is clear from Theorem 4.3.5 applied to each $\hat{Z}_{C_t^m}(\cdot,T)$ for $m = 1, \ldots, M$. Moreover, the equivalence between N.3 and N.2 is also clear from the fact that $\Lambda^{*,m} = \Lambda^*$ for each $m = 1, \ldots, M$ and since by the model assumption we have that for all $i \in \{1, \ldots, K-1\}$ there exists some $m_i \in \{1, \ldots, M\}$ such that $C_t^{m_i,1} = i$ at each time $0 \leq t \leq T$. □

Remark 4.4.11. *Note that, in a setting where at each time t and for each class $i \in \{1, \ldots, K-1\}$ there is at least one issuer $m \in \{1, \ldots, M\}$ in this class i, then* N.3 *becomes* N.2.

4.5 Dynamics of the Equal Volatility Specification

As already indicated in previous sections (see for example Remark 4.3.16), the volatility structure of the forward rates is important and a consistent specification is essential in order to avoid model specifications which lead to spread explosions.

In this section we discuss the special case of equal volatility models under strong consistency and describe the bounded properties of their initial spread term structure. More precisely, in Section 4.5.1 we give some closed-form solutions for the spreads when they are deterministic. A property on the initial spread value is derived as a necessary requirement of such models. It has a "vanishing at zero" property, meaning that, the initial spread value tends to zero as T tends to infinity. It turns out that, at least in the present equal volatilities setup, non-vanishing of the initial spread term structure yields once again to explosions, as clarified by the following Section 4.5.2.

4.5.1 Closed-form solutions for the spreads

We know by Theorem 4.4.4 that the inter-rating spread $s_{K-1,1}(t,T)$ explodes in finite time under some mild conditions. In this Section 4.5.1 we show that in some special deterministic cases a closed-form solution of the inter-rating spread exists. This shows that despite the spread explosion proven in Theorem 4.4.4, one can still define meaningful, consistent simple models under the strong consistency.

We explore the zero-recovery case when the inter-rating spread volatility is zero, i.e., when the forward rate volatilities are all equal.

Throughout this section we consider

$$K = 3, \quad \delta_1 = \delta_2 = 0, \quad \sigma_1(t,T) = \sigma_2(t,T). \tag{4.5.1}$$

Recall that this is a setting with one inter-rating spread $s_2(t,T)$ which is deterministic. The next Corollary 4.5.1 gives us the spread dynamics.

Corollary 4.5.1. *For any fixed maturity $T \leq T^*$ and under the consistency condition* N.2*, we have the no-arbitrage spread dynamics*

$$\begin{aligned} ds_2(t,T) = {} & \lambda^*_{2,1}(t) s_2(t,T) \exp\left(\int_t^T s_2(t,u)du\right) dt \\ & + \lambda^*_{1,2}(t) s_2(t,T) \exp\left(-\int_t^T s_2(t,u)du\right) dt, \end{aligned} \tag{4.5.2}$$

for $0 \leq t \leq T$ for the inter-rating spread $s_2(t,T)$.

Proof. Follows from Corollary 4.4.3 and by (4.5.1). □

In the next Proposition 4.5.2 we give closed-form solutions for some cases of the inter-rating spread dynamics (4.5.2).

Proposition 4.5.2. *Assume condition* N.2. *and let $T \leq T^*$. Moreover assume that the migration intensity parameters are non-negative real constants, that is $\lambda^*_{2,1}(t) = \lambda_1, \lambda^*_{1,2}(t) = \lambda_2$ with $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}$. Then for the case $\lambda_1 = 0, \lambda_2 > 0$ we have the closed-form solution*

$$s_2(t,T) = \lambda_2 \frac{c(1+c)}{1+c-\exp(-c\lambda_2(T-t))}, \tag{4.5.3}$$

for $0 \leq t \leq T$ *and for a constant* $c \in [-1, \infty)$. *In particular, for each non-negative initial value* $s_2(0, T)$ *there exists some* $c \in [-1, \infty)$ *such that* $s_2(t, T)$ *from (4.5.3) solves the ODE (4.5.2).*

For the case $\lambda_1 > 0, \lambda_2 = 0$ *we have the closed-form solution*

$$s_2(t,T) = \lambda_1 \frac{c(c-1)}{c - 1 + \exp(c\lambda_1(T-t))}\,, \tag{4.5.4}$$

for $0 \leq t \leq T$ *and for a constant* $c \in [1, \infty)$. *In particular, there is a positive constant* $m(T, \lambda_1)$ *depending on the maturity* T *and on the intensity parameter* λ_1 *such that for all initial values* $s_2(0,T) \in [0, m(T, \lambda_1)]$ *there exists some* $c \in [1, \infty)$ *such that* $s_2(t, T)$ *from (4.5.4) solves the ODE (4.5.2).*

Proof. Let us first consider the case $\lambda_1 = 0, \lambda_2 > 0$.

Making the ansatz $s_2(t,T) = y'(-(T-t))$ for some function $y' : [-T, 0] \to \mathbb{R}_{\geq 0}$ we have for $x = -(T-t)$,

$$s_2(t,T) = y'(x) \quad , \quad \frac{ds_2(t,T)}{dt} = y''(x)\,, \tag{4.5.5}$$

and

$$\int_t^T s_2(t,u)du = \int_0^{T-t} s_2(t, u+t)du = \int_0^{T-t} y'(-u)du = -\int_0^{-(T-t)} y'(u)du = -y(x)\,, \tag{4.5.6}$$

with $y(0) = 0$ (since $x = 0 \iff t = T$).

So, for $x \in [-T, 0]$ we have the ODE

$$y''(x) = \lambda_2 y'(x) \exp(y(x)) \quad , \quad y(0) = 0\,. \tag{4.5.7}$$

Note that, w.l.o.g. we may consider $\lambda_2 = 1$. Indeed, assume y is a solution of $y''(x) = y'(x)\exp(y(x))$. Then for $f(x) := y(\lambda_2 x)$ we have $f'(x) = \lambda_2 y'(\lambda_2 x)$ and $f''(x) = |\lambda_2|^2 y''(\lambda_2 x)$. So, we have that $y''(\lambda_2 x) = y'(\lambda_2 x)\exp(y(\lambda_2 x))$ which is equivalent to $|\lambda_2|^2 y''(\lambda_2 x) = |\lambda_2|^2 y'(\lambda_2 x)\exp(y(\lambda_2 x))$. That means f solves the ODE $f''(x) = \lambda_2 f'(x)\exp(f(x))$. More precisely, if $y(x)$ solves (4.5.8) (see below) then $y(\lambda_2 x)$ solves (4.5.7).

So, w.l.o.g. we consider the ODE

$$y''(x) = y'(x)\exp(y(x)) \quad , \quad y(0) = 0\,. \tag{4.5.8}$$

The solution of the ODE (4.5.8) is given by

$$y(x) = -\log\left(\frac{(1+c)\exp(-cx)}{c} - \frac{1}{c}\right)\,, \tag{4.5.9}$$

where c is some real constant. Taking the derivative with respect to x we calculate

$$y'(x) = \frac{c(1+c)}{1 + c - \exp(cx)}\,, \tag{4.5.10}$$

and by the ansatz $s_2(t,T) = y'(x)$ we get (4.5.10) as the closed form solution for the spread. More precisely we have the closed-from solution

$$s_2(t,T) = \frac{c(1+c)}{1+c-\exp(-c(T-t))} \,. \tag{4.5.11}$$

Furthermore, note that from (4.5.11) we have the initial and terminal conditions

$$s_2(0,T) = \frac{c(1+c)}{1+c-\exp(-cT)} \quad , \quad s_2(T,T) = 1+c\,. \tag{4.5.12}$$

Note that since $s_2(T,T) \geq 0$ we must have $c \geq -1$.

The map $\hat{f} : [-1,\infty) \to \mathbb{R}_{\geq 0}$, $c \mapsto s_2(0,T)$ is monoton and continuous with $\hat{f}(-1) = 0$ and $\hat{f}(\infty) = \infty$, so $\hat{f}$ is a bijection. So, for each non-negative initial value $s_2(0,T)$ there exists some $c \in [-1,\infty)$ such that $s_2(t,T)$ defined as in (4.5.11) solves the ODE (4.5.8).

For the general case $\lambda_2 \neq 1$ we have the closed-form solution (4.5.3), using the transformation $f(x) = y(\lambda_2 x)$.

Let us now consider the case $\lambda_1 > 0, \lambda_2 = 0$ which follows analogously as the first one.

Making the same ansatz $s_2(t,T) = y'(-(T-t))$ as in the first case, for some function $y' : [-T,0] \to \mathbb{R}_{\geq 0}$ and w.l.o.g. for $\lambda_1 = 1$ we consider the ODE

$$y''(x) = y'(x)\exp(-y(x)) \quad , \quad y(0) = 0\,, \tag{4.5.13}$$

for $x = -(T-t)$. The solution of the ODE (4.5.13) is given by

$$y(x) = \log\left(\frac{(c-1)\exp(cx)}{c} + \frac{1}{c}\right)\,, \tag{4.5.14}$$

for some real constant c. Furthermore we have

$$y'(x) = \frac{c(c-1)}{c-1+\exp(-cx)}\,, \tag{4.5.15}$$

which implies

$$s_2(t,T) = \frac{c(c-1)}{c-1+\exp(c(T-t))}\,. \tag{4.5.16}$$

From (4.5.16) we have the initial and terminal conditions

$$s_2(0,T) = \frac{c(c-1)}{c-1+\exp(cT)} \quad , \quad s_2(T,T) = c-1\,. \tag{4.5.17}$$

Since $s_2(T,T) \geq 0$ we get $c \geq 1$.

The map $\tilde{f} : [1,\infty) \to \mathbb{R}_{\geq 0}$, $c \mapsto s_2(0,T)$ is continuous with $\tilde{f}(1) = 0$ and $\tilde{f}(\infty) = 0$, so $\tilde{f}$ has a maximum $m(T)$ that depends on T and where $m(T)$ is a decreasing function in T[4]. Thus

[4]Since $\frac{ds_2(0,T)}{dT} < 0$ and with $\lim_{T\to 0} m(T) = +\infty$.

the function $\tilde{f}$ is not surjective and hence not for all non-negative initial values $s_2(0,T)$ we can find some $c \in [1,\infty]$ such that a solution to (4.5.13) and hence to the spread dynamics exists.

So, for the general case $\lambda_1 \neq 1$ we have the closed-form solution (4.5.4), for a specific range of initial values $s_2(0,T) \in [0, m(T,\lambda_1)]$ where $m(T,\lambda_1)$ is a positive constant depending on the maturity T and on the intensity parameter λ_1 and is the maximum of the function g given by

$$g : [1,\infty) \to \mathbb{R}_{\geq 0} \quad , \quad c \mapsto s_2(0,T) = \lambda_1 \frac{c(c-1)}{c-1+\exp(c\lambda_1 T)} \,. \tag{4.5.18}$$

So, the claim follows.

□

Remark 4.5.3. *From Proposition 4.5.2 note that in the case where $\lambda_1 > 0, \lambda_2 = 0$ the initial condition $s_2(0,T)$ converges to zero as the maturity T approaches infinity. We show in Section 4.5.2 that models without this property are problematic with exploding dynamics. More precisely, if the initial condition $s_2(0,T)$ is bounded from below uniformly, then the spread $s_2(t,T)$ in (4.5.2) explodes in finite time with positive probability prior to default.*

Remark 4.5.4. *For $K = 2$ and assuming $C_t^1 = 1$ and by looking at (4.3.74), for constant zero-recovery fundamental spreads $s_1^{f,0}(t,T) = 1$ we get the closed form solution*

$$s_1^{f,\delta}(t,T) = \frac{(1-\delta_1)\exp\left(-(T-t)\right)}{(1-\delta_1)\exp\left(-(T-t)\right)+\delta_1} \,, \tag{4.5.19}$$

for the non-zero-recovery fundamental spread $s_1^{f,\delta}(t,T)$ which solves the ODE

$$ds_1^{f,\delta}(t,T) = \delta_1 s_1^{f,\delta}(t,T) \exp\left(\int_t^T s_1^{f,\delta}(t,u)du\right) \,, \tag{4.5.20}$$

for some $\delta_1 \in (0,1]$.

On the other hand for $K = 3$ and for $\lambda_1 \in (0,1], \lambda_2 = 0$ we know from Proposition 4.5.2 that by choosing $c = \frac{1}{\lambda_1}$ we have

$$s_2(t,T) = \frac{1}{1+\frac{\exp((T-t))}{\frac{1}{\lambda_1}-1}} \,. \tag{4.5.21}$$

So we can conclude that for $\lambda_1 = \delta_1$ the two solutions (4.5.19) and (4.5.21) equal. For the case $\delta_1 = \lambda_1 = 1$ we have the zero solution.

Note also, that either $s_2(0,T)$ or λ_1 (or respectively δ_1) is a free parameter.

According to Proposition 4.5.2, by setting one of the two intensity parameters equal to zero in (4.5.2) we get closed form solutions of the spread ODE of the form $s_2(t,T) = k(-(T-t))$ for some function $k : [-T,0] \to \mathbb{R}_{\geq 0}$. In addition, if both the intensity parameters are positive then the ODE (4.5.2) admits again a solution of this form. This is proved in the next Proposition 4.5.5.

Proposition 4.5.5. *Assume condition* N.2 *and let* $T \leq T^*$. *Moreover assume that the migration intensity parameters are positive real constants, that is* $\lambda^*_{2,1}(t) > 0, \lambda^*_{1,2}(t) > 0$. *Then we have the closed-form solution*

$$s_2(t,T) = \lambda^*_{1,2}(t)(z^+_\lambda - z^-_\lambda)R^\pm_\lambda(-(T-t)) \left[\frac{1}{1 - R^\pm_\lambda(-(T-t))} - \frac{z^-_\lambda}{z^+_\lambda - z^-_\lambda R^\pm_\lambda(-(T-t))}\right], \tag{4.5.22}$$

for $0 \leq t \leq T$ *and for a constant* $c \in [0, \infty)$, *where*

$$R^\pm_\lambda(x) := \exp(xr^\pm_\lambda)\frac{1 - z^+_\lambda}{1 - z^-_\lambda} \quad , \quad r^\pm_\lambda := \lambda_2(z^+_\lambda - z^-_\lambda)\,, \tag{4.5.23}$$
$$z^\pm_\lambda := \frac{1}{2\lambda_2}\left(-C(\lambda) \pm \sqrt{C(\lambda)^2 + 4\lambda_2\lambda_1}\right),$$
$$C(\lambda) := \lambda_1 - \lambda_2 + c \quad , \quad \lambda_1 := \lambda^*_{2,1}(t) \quad , \quad \lambda_2 := \lambda^*_{1,2}(t)\,.$$

In particular, there is a positive constant $m(T, \lambda^*_{1,2}(0), \lambda^*_{2,1}(0))$ *depending on the maturity* T *and on the intensity parameters* $\lambda^*_{1,2}(0), \lambda^*_{2,1}(0)$ *at time* $t = 0$ *such that for all initial values* $s_2(0,T) \in [0, m(T, \lambda^*_{1,2}(0), \lambda^*_{2,1}(0))]$ *there exists some* $c \in [0, \infty)$ *such that* $s_2(t,T)$ *from (4.5.22) solves the ODE (4.5.2).*

Proof. We consider the ODE (4.5.2) and as in the proof of Proposition 4.5.2 we make the ansatz $s_2(t,T) = y'(-(T-t))$ for some function $y' : [-T, 0] \to \mathbb{R}_{\geq 0}$. Then by (4.5.5) and (4.5.6) we have on $[-T, 0]$ the ODE

$$y''(x) = \lambda^*_{2,1}(t)y'(x)\exp(-y(x)) + \lambda^*_{1,2}(t)y'(x)\exp(y(x)) \quad , \quad y(0) = 0\,, \tag{4.5.24}$$

for $x = -(T-t)$.

Furthermore assume there exists a differentiable function u such that

$$u(y) = y'(x) \quad , \quad u(y(0)) = u(0) = c\,, \tag{4.5.25}$$

for some constant c with $c = y'(0) = s_2(T,T)$ (recall $x = -(T-t)$). Note that c must be strictly positive.

Then since $y''(x) = u'(y)y'(x) = u'(y)u(y)$ we have from (4.5.24) that

$$u'(y) = \lambda^*_{2,1}(t)\exp(-y(x)) + \lambda^*_{1,2}(t)\exp(y(x)) \quad , \quad u'(y(0)) = u'(0) = \tilde{c}\,, \tag{4.5.26}$$

for $-y > 0$ (see (4.5.6)) and for some positive constant $\tilde{c}$ with $\tilde{c} = u'(0) = \lambda^*_{2,1}(t) + \lambda^*_{1,2}(t)$. So, integrating on $[y, 0]$ we have

$$\begin{aligned} c - u(y) = \int_y^0 u'(s)ds &= \int_y^0 \lambda^*_{2,1}(t)\exp(-s) + \lambda^*_{1,2}(t)\exp(s)ds \\ &= \lambda^*_{2,1}(t)(-1 + e^{-y}) + \lambda^*_{1,2}(t)(1 - e^y) \\ &= -\lambda^*_{1,2}(t)e^y + \lambda^*_{2,1}(t)e^{-y} - \lambda^*_{2,1}(t) + \lambda^*_{1,2}(t)\,, \end{aligned} \tag{4.5.27}$$

which results to

$$\begin{aligned} & y'(x) = \lambda^*_{1,2}(t)e^{y(x)} - \lambda^*_{2,1}(t)e^{-y(x)} + \lambda^*_{2,1}(t) - \lambda^*_{1,2}(t) + c \\ \iff & y'(x) = \lambda^*_{1,2}(t)e^{y(x)} - \lambda^*_{2,1}(t)e^{-y(x)} + C(\lambda) \\ \iff & y'(x)e^{y(x)} = \lambda^*_{1,2}(t)e^{2y(x)} - \lambda^*_{2,1}(t) + C(\lambda)e^{y(x)} \\ \iff & 1 = \frac{y'(x)e^{y(x)}}{\lambda_2 e^{2y(x)} + C(\lambda)e^{y(x)} - \lambda_1}\,, \end{aligned} \tag{4.5.28}$$

where recall the notation of $C(\lambda), \lambda_1, \lambda_2$ from (4.5.23).

Integrating on $[x, 0]$ (recall $x = -(T-t)$) we have

$$-x = \int_x^0 1 ds = \int_x^0 \frac{y'(s)e^{y(s)}}{\lambda_2 e^{2y(s)} + C(\lambda)e^{y(s)} - \lambda_1} ds\,. \tag{4.5.29}$$

Substituting $z(s) := e^{y(s)}$ and since $z(0) = 1$, $z(x) = e^{y(x)}$, $dz(s) = y'(s)e^{y(s)}ds$ then

$$x = \int_1^{z(x)} \frac{1}{\lambda_2 z^2 + C(\lambda)z - \lambda_1} dz = \frac{1}{\lambda_2}\int_1^{z(x)} \frac{1}{z^2 + \frac{C(\lambda)}{\lambda_2}z - \frac{\lambda_1}{\lambda_2}} dz\,. \tag{4.5.30}$$

Now, we are going to use partial fraction decomposition. The zeros of $z^2 + \frac{C(\lambda)}{\lambda_2} - \frac{\lambda_1}{\lambda_2}$ are

$$z^{\pm}_{\lambda} = \frac{1}{2\lambda_2}\left(-C(\lambda) \pm \sqrt{C(\lambda)^2 + 4\lambda_2\lambda_1}\right)\,, \tag{4.5.31}$$

where note that these are real numbers for all non-negative values of λ_1, λ_2. So we have

$$\frac{1}{z^2 + \frac{C(\lambda)}{\lambda_2}z - \frac{\lambda_1}{\lambda_2}} = \frac{A}{z - z^+_{\lambda}} + \frac{B}{z - z^-_{\lambda}}\,, \tag{4.5.32}$$

which by comparing the coefficients gives us $A + B = 0$ and $A = -\frac{1}{z^-_{\lambda} - z^+_{\lambda}}$. So we have $A = \frac{1}{z^+_{\lambda} - z^-_{\lambda}}$ and $B = \frac{1}{z^-_{\lambda} - z^+_{\lambda}}$.

Then we can calculate

$$x = \frac{1}{\lambda_2}\frac{1}{z^+_{\lambda} - z^-_{\lambda}}\int_1^{z(x)} \frac{1}{z - z^+_{\lambda}} dz + \frac{1}{\lambda_2}\frac{1}{z^-_{\lambda} - z^+_{\lambda}}\int_1^{z(x)} \frac{1}{z - z^-_{\lambda}} dz \tag{4.5.33}$$

$$\begin{aligned} \iff & x\lambda_2(z^+_{\lambda} - z^-_{\lambda}) = \int_1^{z(x)} \frac{1}{z - z^+_{\lambda}} dz - \int_1^{z(x)} \frac{1}{z - z^-_{\lambda}} dz \\ \iff & x\lambda_2(z^+_{\lambda} - z^-_{\lambda}) = \log\left|\frac{z(x) - z^+_{\lambda}}{z(x) - z^-_{\lambda}}\right| - \log\left|\frac{1 - z^+_{\lambda}}{1 - z^-_{\lambda}}\right| \\ \iff & \log\left|\frac{z(x) - z^+_{\lambda}}{z(x) - z^-_{\lambda}}\right| = x\lambda_2(z^+_{\lambda} - z^-_{\lambda}) + \log\left|\frac{1 - z^+_{\lambda}}{1 - z^-_{\lambda}}\right|\,. \end{aligned}$$

Applying the $\exp(\cdot)$ function we have

$$\left|\frac{z(x)-z_\lambda^+}{z(x)-z_\lambda^-}\right| = \exp(x\lambda_2(z_\lambda^+ - z_\lambda^-))\left|\frac{1-z_\lambda^+}{1-z_\lambda^-}\right| . \tag{4.5.34}$$

Note that the integrals in (4.5.33) are defined for $z_\lambda^+ < z(x)$ or $z_\lambda^+ > 1$ and analogously $z_\lambda^- < z(x)$ or $z_\lambda^- > 1$ where $z(x) \in (0,1)$. On the other hand we have $z_\lambda^+ z_\lambda^- = -\frac{\lambda_1}{\lambda_2}$ which implies $z_\lambda^- < 0$ and $z_\lambda^+ > 0$. In particular, for all $\lambda_1, \lambda_2, c > 0$ we have $z_\lambda^+ \in (0,1)$. Indeed, we have

$$\begin{aligned} z_\lambda^+ < 1 \iff & \frac{1}{2\lambda_2}\left(-C(\lambda) + \sqrt{C(\lambda)^2 + 4\lambda_2\lambda_1}\right) < 1 \\ & \iff \sqrt{C(\lambda)^2 + 4\lambda_2\lambda_1} < \lambda_1 + \lambda_2 + c\,, \end{aligned} \tag{4.5.35}$$

where $\lambda_1 + \lambda_2 + c > 0$. Furthermore we can calculate

$$z_\lambda^+ < 1 \iff (\lambda_1 - \lambda_2 + c)^2 + 4\lambda_2\lambda_1 < (\lambda_1 + \lambda_2 + c)^2 \iff 4\lambda_2 c > 0\,. \tag{4.5.36}$$

This shows that in all the cases we have $z_\lambda^+ < 1$. So, we can conclude that for all $\lambda_1, \lambda_2, c > 0$ we have $z_\lambda^+ \in (0,1)$ and $z_\lambda^- < 0$ and so the integrals in (4.5.33) are well defined for $z(x) \in (0,1)$ if we additionally have that $z_\lambda^+ < z(x)$. This is true and we show it below.

Furthermore and because of the absolute value in (4.5.34), we have the following four possibilities:

$$\begin{aligned} \left|\frac{z(x)-z_\lambda^+}{z(x)-z_\lambda^-}\right| &= \begin{cases} \frac{z(x)-z_\lambda^+}{z(x)-z_\lambda^-} & \text{for } z_\lambda^- > z(x) \text{ or } z_\lambda^+ < z(x), \\ -\frac{z(x)-z_\lambda^+}{z(x)-z_\lambda^-} & \text{for } z_\lambda^- < z(x) < z_\lambda^+, \end{cases} \\ \frac{1-z_\lambda^+}{1-z_\lambda^-} &= \begin{cases} \frac{1-z_\lambda^+}{1-z_\lambda^-} & \text{for } z_\lambda^- > 1 \text{ or } z_\lambda^+ < 1, \\ -\frac{1-z_\lambda^+}{1-z_\lambda^-} & \text{for } z_\lambda^- < 1 < z_\lambda^+. \end{cases} \end{aligned} \tag{4.5.37}$$

Since we know that $z_\lambda^+ < 1$ and $z_\lambda^- < 0$ then from (4.5.34) and (4.5.37) we have

$$\frac{z(x)-z_\lambda^+}{z(x)-z_\lambda^-} = R_\lambda^\pm(x) \iff z(x) = \frac{z_\lambda^+ - z_\lambda^- R_\lambda^\pm(x)}{1-R_\lambda^\pm(x)}\,. \tag{4.5.38}$$

Then since $z(x) = \exp(y(x))$ we get

$$y(x) = \log\left(\frac{z_\lambda^+ - z_\lambda^- R_\lambda^\pm(x)}{1-R_\lambda^\pm(x)}\right)\,. \tag{4.5.39}$$

Furthermore and since $y'(x) = s_2(t,T)$ we want to calculate the derivative, so

$$y'(x) = \frac{(1-R_\lambda^\pm(x))}{(z_\lambda^+ - z_\lambda^- R_\lambda^\pm(x)}\left[\frac{-z_\lambda^- r_\lambda^\pm R_\lambda^\pm(x)(1-R_\lambda^\pm(x)) - (z_\lambda^+ - z_\lambda^- R_\lambda^\pm(x))(-r_\lambda^\pm R_\lambda^\pm(x)))}{(1-R_\lambda^\pm(x))^2}\right] \tag{4.5.40}$$

$$= \frac{r_\lambda^\pm R_\lambda^\pm(x)}{(z_\lambda^+ - z_\lambda^- R_\lambda^\pm(x)(1-R_\lambda^\pm(x))} \left[(z_\lambda^+ - z_\lambda^- R_\lambda^\pm(x)) - z_\lambda^-(1-R_\lambda^\pm(x))\right] ,$$

and thus we can conclude that

$$y'(x) = r_\lambda^\pm R_\lambda^\pm(x) \left[\frac{1}{1-R_\lambda^\pm(x)} - \frac{z_\lambda^-}{z_\lambda^+ - z_\lambda^- R_\lambda^\pm(x)}\right] . \tag{4.5.41}$$

Then (4.5.41) is the closed form solution of the ODE (4.5.24) and hence of the spread if the condition

$$0 < \frac{z_\lambda^+ - z_\lambda^- R_\lambda^\pm(x)}{1-R_\lambda^\pm(x)} < 1 , \tag{4.5.42}$$

holds, since we have $z(x) \in (0,1)$.

Let us now check condition (4.5.42) using the fact that we have $z_\lambda^+ \in (0,1)$ and $z_\lambda^- < 0$. First note that we have $R_\lambda^\pm(x) \in (0,1)$. Also since $z_\lambda^- < z_\lambda^+$ then $z_\lambda^-(1-z_\lambda^+) < z_\lambda^+(1-z_\lambda^-)$ and hence $z_\lambda^-(1-z_\lambda^+)\exp(xr_\lambda^\pm) < z_\lambda^+(1-z_\lambda^-)$ because of the fact that $\exp(xr_\lambda^\pm) \in (0,1)$ where recall $x = -(T-t)$. So, we have $z_\lambda^+ - z_\lambda^- R_\lambda^\pm(x) > 0$ and $1 - R_\lambda^\pm(x) > 0$ which shows that the left hand side inequality of (4.5.42) holds true. In particular we have $z_\lambda^+ < z(x)$.

On the other hand, the right hand side inequality of (4.5.42) also holds. Indeed we have the following equivalence relation

$$\begin{aligned} \frac{z_\lambda^+ - z_\lambda^- R_\lambda^\pm(x)}{1-R_\lambda^\pm(x)} < 1 &\iff z_\lambda^+ + R_\lambda^\pm(x)(1-z_\lambda^-) < 1 \\ &\iff z_\lambda^+ + \exp(-(T-t)\lambda_1(z_\lambda^+ - z_\lambda^-))(1-z_\lambda^+) < 1 , \end{aligned} \tag{4.5.43}$$

which this is a function of the form $f(x) = x + \delta(1-x)$ for $0 < x < 1$ and $\delta \in (0,1)$ with the property $f < 1$. So, condition (4.5.42) always holds true and hence (4.5.41) is the closed form solution of the ODE (4.5.24).

Furthermore, from (4.5.41) we have the initial condition

$$\begin{aligned} s_2(0,T) = \lambda_{1,2}^*(0)(z_\lambda^+ - z_\lambda^-) &\left[\frac{1}{\exp\left(T\lambda_{1,2}^*(0)(z_\lambda^+ - z_\lambda^-)\right)\frac{1-z_\lambda^-}{1-z_\lambda^+} - 1}\right. \\ &\left. + \frac{-z_\lambda^-}{z_\lambda^+ \exp\left(T\lambda_{1,2}^*(0)(z_\lambda^+ - z_\lambda^-)\right)\frac{1-z_\lambda^-}{1-z_\lambda^+} - z_\lambda^-}\right] . \end{aligned} \tag{4.5.44}$$

Moreover, note that the functions $c \mapsto z_\lambda^+ - z_\lambda^-$, $c \mapsto \frac{1-z_\lambda^-}{1-z_\lambda^+}$ are increasing in c. Also, recall that $z_\lambda^+ \in (0,1)$ and note that $-\frac{1}{z_\lambda^-} \to 0$ as $c \to \infty$.

Then the map $\tilde{f} : [0,\infty) \to \mathbb{R}_{\geq 0}$, $c \mapsto s_2(0,T)$ is continuous with $\tilde{f}(0) = a$ and $\tilde{f}(\infty) = 0$, for a positive constant a. So $\tilde{f}$ has a maximum $m(T) := m(T, \lambda_{1,2}^*(0), \lambda_{2,1}^*(0))$ depending on the maturity T and on the intensity parameters $\lambda_{1,2}^*(0), \lambda_{2,1}^*(0)$ at time $t = 0$ where $m(T)$ is a decreasing function in T.[5] Thus the function $\tilde{f}$ is not surjective and hence not for all non-negative initial values $s_2(0,T)$ we can find some $c \in [0,\infty]$ such that a solution to (4.5.24) and hence to the spread dynamics exists but only for the initial values in the interval $[0, m(T)]$. So, the claim follows. $\square$

Remark 4.5.6. *It is easy to see that a similar observation as in Remark 4.5.3 holds. Indeed, from the proof of Proposition 4.5.5 and in particular from (4.5.44) the initial condition $s_2(0,T)$ converges to zero as the maturity T goes to infinity. See Section 4.5.2 for more on this property.*

As the next Corollary 4.5.7 shows, the special case $\lambda^*_{2,1}(t) = 0$ in Proposition 4.5.5 coincides with the one from Proposition 4.5.2.

Corollary 4.5.7. *Assume condition* N.2 *and let $T \leq T^*$. Moreover assume that the migration intensity parameters are non-negative real constants, that is $\lambda^*_{2,1}(t) = \lambda_1, \lambda^*_{1,2}(t) = \lambda_2$ with $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}$. Then for the case $\lambda_1 = 0, \lambda_2 > 0$ the two closed form solutions (4.5.3) and (4.5.22) coincide.*

Proof. We start from the spread solution (4.5.22). Setting $\lambda_1 = 0$ we have

$$z^{\pm}_{\lambda} = \frac{1}{2\lambda_2}\left(-C(\lambda) \pm |C(\lambda)|\right) \text{ with } C(\lambda) = -\lambda_2 + c\,. \tag{4.5.45}$$

We will consider two cases. As the first case we consider $C(\lambda) > 0$. Then we have $z^+_\lambda = 0$ and $z^-_\lambda = 1 - \frac{c}{\lambda_2}$. Plugging this into (4.5.22) we get

$$s_2(t,T) = \lambda_2 z^-_\lambda \frac{\frac{c}{\lambda_2}}{\exp\left((T-t)\lambda_2 z^-_\lambda\right) - \frac{c}{\lambda_2}}\,. \tag{4.5.46}$$

On the other hand, setting the constant $\hat{c}$ from the closed form solution (4.5.3) as $1 + \hat{c} := \frac{c}{\lambda_2}$ then we have $\hat{c} = -z^-_\lambda$, which also implies $\hat{c} \in (0,\infty)$ and the two solutions are equal.

The second case follows analogously. If $C(\lambda) < 0$ then we have $z^-_\lambda = 0$ and $z^+_\lambda = 1 - \frac{c}{\lambda_2}$. Plugging this into (4.5.22) we have

$$s_2(t,T) = \lambda_2 z^+_\lambda \frac{\frac{c}{\lambda_2}}{\exp\left((T-t)\lambda_2 z^-_\lambda\right) - \frac{c}{\lambda_2}}\,. \tag{4.5.47}$$

Then defining the constant $\hat{c}$ from the closed form solution (4.5.3) again as $1+\hat{c} := \frac{c}{\lambda_2}$, we have $\hat{c} = -z^+_\lambda$, which also implies $\hat{c} \in (-1, 0)$ and so the two closed form solutions are identical. $\square$

Remark 4.5.8. *Note that the special case $\lambda^*_{1,2}(t) = 0$ in Proposition 4.5.5 cannot be recovered. So, Proposition 4.5.2 gives us the closed form solution for this particular case.*

[5] Since $\frac{ds_2(0,T)}{dT} < 0$ and with $\lim_{T\to 0} m(T) = +\infty$.

4.5.2 The vanishing property on the initial spread value

From Remarks 4.5.3 and 4.5.6 we know that the initial value $s_2(0, T)$ goes to zero as $T \to \infty$. This property seems to be crucial since models without this condition explode in finite time as we show in Theorem 4.5.10. That is, models with bounded initial spread value are not admissible.

The next technical Lemma is needed for the proof of Theorem 4.5.10.

Lemma 4.5.9. *Let $C > 0$, $T > 0$. Then for $a = \frac{T^2}{2}$, $R = \frac{T}{4}$ we have for all $\frac{T}{4} \leq t \leq \frac{T}{2}$*

$$C \geq \frac{2a}{(a-t^2)} \frac{(a-R^2)}{(a-RT)^2} \iff T \geq \sqrt{\frac{28}{C}}, \tag{4.5.48}$$

and

$$\frac{2a}{(a-t^2)} \frac{(a-R^2)}{(a-RT)^2} > 0. \tag{4.5.49}$$

Proof. Let us show first the second assertion. We have

$$a - t^2 > 0 \iff t < \sqrt{a}, \tag{4.5.50}$$

Since $t \leq \frac{T}{2}$ then $t < \sqrt{a}$. Furthermore, by the definition of R and a it is clear that $a - R^2 > 0$ and hence (4.5.49) follows.

For the second assertion we have the following equivalence relation

$$\begin{aligned} T \geq \sqrt{\frac{28}{C}} &\iff \frac{28}{C} \leq T^2 \iff 7\frac{T^2}{16} \leq \frac{C}{4}\frac{T^4}{16} \\ &\iff 7\frac{T^2}{16} \leq \frac{C}{4}\left(\frac{T^2}{4}\right)^2 \iff T^2 \frac{7\frac{T^2}{16}}{\left(\frac{T^2}{4}\right)^2} \leq C\frac{T^2}{4} \\ &\iff T^2 \frac{8\frac{T^2}{16} - \frac{T^2}{16}}{\left(2\frac{T^2}{4} - \frac{T^2}{4}\right)^2} \leq C\left(2\frac{T^2}{4} - \frac{T^2}{4}\right). \end{aligned} \tag{4.5.51}$$

Thus, it holds,

$$T \geq \sqrt{\frac{28}{C}} \iff 2a\frac{(a-R^2)}{(a-RT)^2} \leq C\left(a - \frac{T^2}{4}\right). \tag{4.5.52}$$

Finally, since $t \leq \frac{T}{2}$ then we have the relation

$$a - \frac{T^2}{4} \leq a - t^2\,. \tag{4.5.53}$$

Putting things together and using all the previous equivalence relations, now the claim can be proven. Indeed, the direction " $\Leftarrow$ " is quite clear, becuase by (4.5.52) together with (4.5.53) we get the left hand side of (4.5.48).

For " $\Rightarrow$ " assume that $C \geq \frac{2a}{(a-t^2)} \frac{(a-R^2)}{(a-RT)^2}$ holds for all $t \leq \frac{T}{2}$. Then in particular for $t = \frac{T}{2}$ and by (4.5.52) this direction is also clear. Thus the claim is proven. $\square$

Theorem 4.5.10. *Assume conditions* N.2 *and* O. *Furthermore assume that* $K \geq 3$ *and* $\delta_i = 0$ *for all* $i = 1, \dots, K-1$. *Moreover assume that* $\sigma_{K-1}(t,T) = \sigma_1(t,T)$ *and* $\lambda^*_{K-1,1}(t)$ *is continuous. Assume also that* $s_{K-1,1}(0,T) \geq M_1$ *and* $\lambda^*_{K-1,1}(t) \geq M_2$ *where* M_1, M_2 *are positive constants independent of* T *and* t. *Then there exists some* $T^0 \in (0,\infty)$ *so that if* $T^* > T^0$ *then* $\lim_{t \to \frac{T}{2}} s_{K-1,1}(t,T) = +\infty$ *for all* $T^0 < T < T^*$ *with positive probability under* $\mathbb{Q}$ *and any other equivalent measure prior to default.*

Proof. Using the same arguments (Grönwall's inequality, ordering condition e.t.c.) as in Theorem 4.4.4 we can estimate

$$s_{K-1,1}(t,T) \geq \tilde{N}_{K-1,1}(t,T) + \int_R^t \tilde{N}_{K-1,1}(s,T)\lambda^*_{K-1,1}(s)\exp\left(\int_s^T s_{K-1,1}(s,u)du\right)ds\,. \tag{4.5.54}$$

Since by assumption the initial value is positive, then we can further estimate

$$s_{K-1,1}(t,T) \geq s_{K-1,1}(0,T) + \int_R^t s_{K-1,1}(0,T)\lambda^*_{K-1,1}(s)\exp\left(\int_s^T s_{K-1,1}(s,u)du\right)ds\,. \tag{4.5.55}$$

Using Jensen's inequality as in Theorem 4.4.4, setting $C := M_1M_2$ and defining $\tilde{s}_{K-1,1}(t,T) := \frac{1}{(t-R)}s_{K-1,1}(t,T)$ we get

$$\tilde{s}_{K-1,1}(t,T) \geq C\exp\left\{\int_R^t\int_s^T \tilde{s}_{K-1,1}(s,u)duds\right\}\,, \tag{4.5.56}$$

on the set $\{\tau > T\}$ and for $(t,T) \in \hat{K}^{R,S}$.

Furthermore we know from Lemma 4.5.9 that for all $\frac{T}{4} \leq t \leq \frac{T}{2}$

$$C \geq \frac{2a}{(a-t^2)}\frac{(a-R^2)}{(a-RT)^2} \iff T \geq T^0\,, \tag{4.5.57}$$

where

$$a := \frac{T^2}{2} \quad , \quad R := \frac{T}{4} \quad , \quad T^0 := \sqrt{\frac{28}{C}}\,. \tag{4.5.58}$$

Define the set

$$\hat{K}^0 := \left\{(t,T) : \frac{T}{4} \leq t \leq \frac{T}{2} \leq T^* \text{ where } T \geq T^0\right\}\,. \tag{4.5.59}$$

Then on the set $\{\tau > T\}$ and for $(t,T) \in \hat{K}^0$ we can estimate

$$\tilde{s}_{K-1,1}(t,T) \geq \frac{2a}{(a-t^2)}\frac{(a-R^2)}{(a-RT)^2}\exp\left\{\int_{\frac{T}{4}}^t\int_s^T \tilde{s}_{K-1,1}(s,u)duds\right\}\,. \tag{4.5.60}$$

So if $T^* > T^0$ then the set $\hat{K}^0$ is not an empty set and inequality (4.5.60) holds with positive probability on $\{\tau > T\}$.

Then by using Morton's Lemma 4.3.12 on the set $\{\tau > T\}$ and for $(t,T) \in \hat{K}^0$ we can conclude as in Theorem 4.4.4 that

$$\lim_{t \to \frac{T}{2}} s_{K-1,1}(t,T) = +\infty\,, \tag{4.5.61}$$

with positive probability prior to default using the function $h^a(t,T)$ from Lemma 4.3.11.

□

Remark 4.5.11. *In Theorem 4.5.10 notice the assumption on the initial condition that this needs to be bounded from below by a positive constant. Such models with this condition are not admissible models. On the other hand, for example in Proposition 4.5.2 for the case $\lambda^*_{1,2}(t) = 0$, $\lambda^*_{2,1} = \lambda_1$, where λ_1 is a positive constant, the initial value in (4.5.4) tends to zero, as the maturity T goes to infinity. For this case a non-exploding solution for the spread exists.*

Remark 4.5.12. *For the case $K = 3$ and analogously to Remark 4.3.20, the ordering condition O in Theorem 4.5.10 is not needed to be assumed. A similar statement as in Corollary 4.4.9 can be proven.*

Remark 4.5.13. *Without getting into details and comparing with Theorem 4.3.14, it is clear that under similar conditions as in Theorem 4.5.10, it can be proven that the non-zero recovery (deterministic) fundamental spread will explode in finite time with positive probability prior to default.*

Now, we are going to introduce Lemma 4.5.14 which will help us to extend the explosion result in Theorem 4.5.10 to all maturities $0 < T < T^0$ which are not originally covered by the statement. We call this, the "parametrization Lemma".

Lemma 4.5.14 (Parametrization Lemma). *Assume condition* N.2. *Furthermore assume that $K \geq 3$ and $\delta_i = 0$ for all $i = 1, \ldots, K-1$. Then for all $\hat{a} > 0$, with $\hat{t} := \hat{a}t, \hat{T} := \hat{a}T$, there exist a spread $\hat{s}_{i,j}(\hat{t},\hat{T})$ with $\hat{s}_{i,j}(\hat{t},\hat{T}) = \frac{1}{\hat{a}} s_{i,j}(t,T)$ satisfying $d\hat{s}_{i,j}(\hat{t},\hat{T}) = d\frac{1}{\hat{a}} s_{i,j}(t,T)$ for $0 \leq t \leq T$ and $i,j = 1,\ldots,K-1$ with $i \neq j$.*

Proof. Let $\hat{a} > 0$ be some arbitrary constant and $\hat{t} := \hat{a}t, \hat{T} := \hat{a}T$.

From Corollary 4.4.3 let $s_{i,j}(t,T)$ satisfying (4.4.5) for $i,j = 1,\ldots,K-1$ with $i \neq j$. Furthermore define the functions $\hat{s}_{i,j}, \hat{\sigma}_i, \hat{\lambda}^*_{i,j}$ such that

$$\hat{s}_{i,j}(\hat{a}t,\hat{a}T) := \frac{1}{\hat{a}} s_{i,j}(t,T), \qquad \hat{\sigma}_i(\hat{a}t,\hat{a}T) := \frac{1}{\hat{a}\sqrt{\hat{a}}}\sigma_i(t,T), \qquad \hat{\lambda}^*_{i,j}(\hat{a}t) \;:= \frac{1}{\hat{a}}\lambda^*_{i,j}(t)\,, \tag{4.5.62}$$

for $i,j = 1,\ldots,K$ with $i \neq j$ and $0 \leq t \leq T$.

Using the transformation $v = \hat{a}u$ and the definition of $\hat{s}_{i,j}$ we can calculate

$$\int_t^T s_{i,j}(t,u)du = \int_{\frac{\hat{t}}{\hat{a}}}^{\frac{\hat{T}}{\hat{a}}} \hat{a}\hat{s}_{i,j}(\hat{a}t,\hat{a}u)du = \int_{\hat{t}}^{\hat{T}} \hat{s}_{i,j}(\hat{t},v)dv\,. \tag{4.5.63}$$

So, we can conclude that

$$s_{i,j}(t,T)\exp\left(\int_t^T s_{i,j}(t,u)du\right) = \hat{a}\hat{s}_{i,j}(\hat{t},\hat{T})\exp\left(\int_{\hat{t}}^{\hat{T}} \hat{s}_{i,j}(\hat{t},u)du\right). \tag{4.5.64}$$

Again by the same transformation $v = \hat{a}u$ and the definition of $\hat{\sigma}_i$ we have

$$\begin{aligned}\sigma_i(t,T)\int_t^T \sigma_i(t,u)du &= \hat{a}\sqrt{\hat{a}}\hat{\sigma}_i(\hat{a}t,\hat{a}T)\int_{\frac{\hat{t}}{\hat{a}}}^{\frac{\hat{T}}{\hat{a}}} \hat{a}\sqrt{\hat{a}}\hat{\sigma}_i(\hat{a}t,\hat{a}u)du \\ &= \hat{a}^2\hat{\sigma}_i(\hat{t},\hat{T})\int_{\hat{t}}^{\hat{T}} \hat{\sigma}_i(\hat{t},v)dv\,.\end{aligned} \tag{4.5.65}$$

So, we conclude

$$\begin{aligned}&\sigma_i(t,T)\int_t^T \sigma_i(t,u)du - \sigma_i(t,T)\int_t^T \sigma_i(t,u)du \\ &= \hat{a}^2\hat{\sigma}_i(\hat{t},\hat{T})\int_{\hat{t}}^{\hat{T}} \hat{\sigma}_i(\hat{t},v)dv - \hat{a}^2\hat{\sigma}_i(\hat{t},\hat{T})\int_{\hat{t}}^{\hat{T}} \hat{\sigma}_i(\hat{t},v)dv\,.\end{aligned} \tag{4.5.66}$$

Furthermore it is clear that we have

$$dt = \frac{1}{\hat{a}}d\hat{t}, \qquad dW_t \overset{d}{=} \frac{1}{\sqrt{\hat{a}}}dW_{\hat{t}}\,. \tag{4.5.67}$$

So, by (4.4.5) from Corollary 4.4.3 and using (4.5.62) one can calculate the dynamics of the SDE $\hat{a}\hat{s}_{i,j}(\hat{a}t,\hat{a}T)$ as follows

$$\begin{aligned}d\hat{a}\hat{s}_{i,j}(\hat{a}t,\hat{a}T) &= ds_{i,j}(t,T) \\ &= \Bigg\{\hat{a}^2\left(\hat{\sigma}_i(\hat{t},\hat{T})\int_{\hat{t}}^{\hat{T}} \hat{\sigma}_i(\hat{t},v)dv - \hat{\sigma}_j(\hat{t},\hat{T})\int_{\hat{t}}^{\hat{T}} \hat{\sigma}_j(\hat{t},v)dv\right) \\ &+ \sum_{l=1,l\neq i,j}^{K-1} \hat{a}^2\Bigg[\hat{\lambda}^*_{i,l}(\hat{t})\hat{s}_{i,l}(\hat{t},\hat{T})\exp\left(\int_{\hat{t}}^{\hat{T}} \hat{s}_{i,l}(\hat{t},u)du\right) \\ &\qquad - \hat{\lambda}^*_{j,l}(\hat{t})\hat{s}_{j,l}(\hat{t},\hat{T})\exp\left(\int_{\hat{t}}^{\hat{T}} \hat{s}_{j,l}(\hat{t},u)du\right)\Bigg] \\ &+ \hat{a}^2\hat{\lambda}^*_{i,j}(\hat{t})\hat{s}_{i,j}(\hat{t},\hat{T})\exp\left(\int_{\hat{t}}^{\hat{T}} \hat{s}_{i,j}(\hat{t},u)du\right) \\ &+ \hat{a}^2\hat{\lambda}^*_{j,i}(\hat{t})\hat{s}_{i,j}(\hat{t},\hat{T})\exp\left(-\int_{\hat{t}}^{\hat{T}} \hat{s}_{i,j}(\hat{t},u)du\right)\Bigg\}\frac{1}{\hat{a}}d\hat{t} \\ &+ \hat{a}\sqrt{\hat{a}}(\hat{\sigma}_i(\hat{t},\hat{T}) - \hat{\sigma}_j(\hat{t},\hat{T}))\frac{1}{\sqrt{\hat{a}}}dW_{\hat{t}}\,.\end{aligned} \tag{4.5.68}$$

So, we can conclude that the dynamics of $\hat{s}_{i,j}(\hat{t},\hat{T})$ are

$$\begin{aligned} d\hat{s}_{i,j}(\hat{t},\hat{T}) = \Bigg\{ & \hat{\sigma}_i(\hat{t},\hat{T}) \int_{\hat{t}}^{\hat{T}} \hat{\sigma}_i(\hat{t},v)dv - \hat{\sigma}_j(\hat{t},\hat{T}) \int_{\hat{t}}^{\hat{T}} \hat{\sigma}_j(\hat{t},v)dv \\ & + \sum_{l=1, l\neq i,j}^{K-1} \Bigg[\hat{\lambda}^*_{i,l}(\hat{t}) \hat{s}_{i,l}(\hat{t},\hat{T}) \exp\left(\int_{\hat{t}}^{\hat{T}} \hat{s}_{i,l}(\hat{t},u)du \right) \\ & \qquad - \hat{\lambda}^*_{j,l}(\hat{t}) \hat{s}_{j,l}(\hat{t},\hat{T}) \exp\left(\int_{\hat{t}}^{\hat{T}} \hat{s}_{j,l}(\hat{t},u)du \right) \Bigg] \\ & + \hat{\lambda}^*_{i,j}(\hat{t}) \hat{s}_{i,j}(\hat{t},\hat{T}) \exp\left(\int_{\hat{t}}^{\hat{T}} \hat{s}_{i,j}(\hat{t},u)du \right) \\ & + \hat{\lambda}^*_{j,i}(\hat{t}) \hat{s}_{i,j}(\hat{t},\hat{T}) \exp\left(-\int_{\hat{t}}^{\hat{T}} \hat{s}_{i,j}(\hat{t},u)du \right) \Bigg\} d\hat{t} \\ & + (\hat{\sigma}_i(\hat{t},\hat{T}) - \hat{\sigma}_j(\hat{t},\hat{T}))dW_{\hat{t}}\,, \end{aligned} \tag{4.5.69}$$

and so the claim follows. □

The next Corollary 4.5.15 is an extension of Theorem 4.5.10 and shows that if we can show explosion for one maturity $T^0 \in (0,T^*)$ then the explosion holds for all maturities $T \in (0,T^*)$.

Corollary 4.5.15. *Let the assumptions of Theorem 4.5.10 hold true. Then if there exists some $T^0 \in (0,T^*)$ with $\lim_{t\to\frac{T^0}{2}} s_{K-1,1}(t,T) = +\infty$ with positive probability under $\mathbb{Q}$ then for all $0 < T < T^*$ we have $\lim_{t\to\frac{T}{2}} s_{K-1,1}(t,T) = +\infty$ with positive probability under $\mathbb{Q}$ and any other equivalent measure prior to default.*

Proof. The idea in this proof is to use the parametrisation Lemma 4.5.14. This Lemma tells us that for all $\hat{a} > 0$ and by parametrizing t,T for $0 \le t \le T$ with $\hat{t} = \hat{a}t$ and $\hat{T} = \hat{a}T$ we have

$$\hat{s}_{K-1,1}(\hat{t},\hat{T}) = \frac{1}{\hat{a}} s_{K-1,1}\left(\frac{\hat{t}}{\hat{a}},\frac{\hat{T}}{\hat{a}}\right). \tag{4.5.70}$$

Also note the relation

$$\hat{s}_{K-1,1}\left(\frac{\hat{T}}{2},\hat{T}\right) = \frac{1}{\hat{a}} s_{K-1,1}\left(\frac{\hat{T}}{2\hat{a}},\frac{\hat{T}}{\hat{a}}\right) = \frac{1}{\hat{a}} s_{K-1,1}\left(\frac{T}{2},T\right), \tag{4.5.71}$$

for $\hat{t} = \frac{\hat{T}}{2}$.

Note that it is clear that the assumptions of Theorem 4.5.10 are satisfied for $\hat{s}_{K-1,1}(\hat{t},\hat{T})$ if they are satisfied for $s_{K-1,1}\left(\frac{\hat{t}}{\hat{a}},\frac{\hat{T}}{\hat{a}}\right)$.

From Theorem 4.5.10 let $T^0 \le T^*$ be such that $\hat{s}_{K-1,1}(\frac{\hat{T}^0}{2},\hat{T}^0) = +\infty$ with positive probability. Then by (4.5.71) we have that for all $\hat{a} > 0$, $\frac{1}{\hat{a}} s_{K-1,1}\left(\frac{\hat{T}^0}{2\hat{a}},\frac{\hat{T}^0}{\hat{a}}\right) = +\infty$ with positive probability. Since we can make $\hat{a} > 0$ arbitrary big or small then for any $T \in (0, T^*)$ we can find some $\hat{a} > 0$ such that $T = \frac{\hat{T}^0}{\hat{a}}$ with $\frac{1}{\hat{a}} s_{K-1,1}\left(\frac{T}{2}, T\right) = +\infty$ with positive probability. Hence the claim follows. □

4.6 Proportional Volatility Spread Models

It is well known that an HJM forward rate model with proportional volatility will result to a forward rate with positive dynamics. It is also well known and proved in Morton (1988) that such a model will explode in finite time with positive probability, hence there is no global solution for the forward rate SDE in this case. Nevertheless as shown in Morton (1988), an existence result for the solution of the forward rate SDE under the assumption of bounded volatility can be obtained. In the case of a migration model where the consistency conditions N.1 or N.2 are assumed, additional exponential terms appear into the dynamics of the forward rates (see (4.3.8) and (4.4.2)) and hence the existence theorem of Morton (1988) cannot be applied even in the case of constant volatilities.

In this section we show a similar result to the classical HJM case. That is, under a proportional volatility spread structure the spread has positive dynamics but it explodes in finite time with positive probability.

In order to make our point clear, we will investigate the zero-recovery inter-rating spread under condition N.2 and for the case $K = 3$ for simplicity. Then it will become obvious that also for the fundamental spreads and under condition N.1 one can get similar results.

Also, we want to point out that to the best of our knowledge, there is no example in the literature for an HJM migration model where the forward rates are positive and ordered. Such an example seems to be hard to construct due to the additional exponential terms of the forward rates (see (4.3.8) and (4.4.2)) and also since comparison Theorems are available only for SDEs with equal volatilities.

The next Corollary 4.6.1 gives us the inter-rating spread dynamics for the proportional volatility spread structure

$$\sigma_i(t,T) - \sigma_j(t,T) = \sigma^s_{i,j}(t,T) s_{i,j}(t,T) \,, \tag{4.6.1}$$

for $0 \le t \le T$ and for all $i, j = 1, \dots, K-1$, where $\sigma^s_{i,j}(\cdot, T)$ is an $\mathbb{F}$-adapted stochastic process with values in $\mathbb{R}^d$.

Corollary 4.6.1. *Assume a spread volatility structure of the form (4.6.1). For any fixed maturity $T \le T^*$ and under the consistency condition* N.2*, and assuming $\delta_i = 0$ for all $i = 1, \dots, K-1$, we have the spread dynamics*

$$ds_{i,j}(t,T) = s_{i,j}(t,T)\Bigg(\Bigg\{\sigma^s_{i,j}(t,T)\int_t^T \sigma^s_{i,j}(t,u)s_{i,j}(t,u)du + \sigma^s_{i,j}(t,T)\int_t^T \sigma_j(t,u)du \tag{4.6.2}$$

$$+ \frac{\sigma_j(t,T)}{s_{i,j}(t,T)}\int_t^T \sigma^s_{i,j}(t,u)s_{i,j}(t,u)du$$

$$+ \frac{1}{s_{i,j}(t,T)} \sum_{l=1, l\neq i,j}^{K-1} \Bigg[\lambda^*_{i,l}(t) s_{i,l}(t,T) \exp\left(\int_t^T s_{i,l}(t,u)du \right)$$
$$- \lambda^*_{j,l}(t) s_{j,l}(t,T) \exp\left(\int_t^T s_{j,l}(t,u)du \right) \Bigg]$$
$$+ \lambda^*_{i,j}(t) \exp\left(\int_t^T s_{i,j}(t,u)du \right) + \lambda^*_{j,i}(t) \exp\left(- \int_t^T s_{i,j}(t,u)du \right) \Bigg\} dt$$
$$+ \sigma^s_{i,j}(t,T) dW_t \Bigg),$$

for $0 \le t \le T$ *for the inter-rating spread* $s_{i,j}(t,T)$ *for* $i,j = 1,\dots,K-1$ *with* $i \neq j$*, in the case of* $g_i(t,T) \neq g_j(t,T)$ *for all* $i,j = 1,\dots,K-1$.[6] *Furthermore and assuming* $K = 3$ *and* $\sigma_1(t,T) = 0$, $\sigma^s_{2,1}(t,T) = \sigma^s > 0$ *then the spread* $s_2(t,T)$ *is positive and has the dynamics*

$$s_2(t,T) = s_2(0,T) \exp\Bigg(|\sigma^s|^2 \int_0^t \int_s^T s_2(s,u)duds \tag{4.6.3}$$
$$+ \int_0^t \lambda^*_{2,1}(s) \exp\left(\int_s^T s_2(s,u)du \right) ds + \int_0^t \lambda^*_{1,2}(s) \exp\left(- \int_s^T s_2(s,u)du \right) ds$$
$$+ \sigma^s W_t - \frac{|\sigma^s|^2}{2} t \Bigg).$$

Proof. The dynamics (4.6.2) follow from Corollary 4.4.3 and from (4.6.1).

Furthermore, for $K = 3$ and for $\sigma_1(t,T) = 0$, $\sigma^s_{2,1}(t,T) = \sigma^s > 0$ we have

$$ds_2(t,T) = s_2(t,T) \Bigg(\Bigg\{ |\sigma^s|^2 \int_t^T s_2(t,u)du \tag{4.6.4}$$
$$+ \lambda^*_{2,1}(t) \exp\left(\int_t^T s_2(t,u)du \right) + \lambda^*_{1,2}(t) \exp\left(- \int_t^T s_2(t,u)du \right) \Bigg\} dt$$
$$+ \sigma^s dW_t \Bigg).$$

So, in this case the inter-rating spread has positive dynamics and is of the form (4.6.3). □

The special case of (4.6.3) will imply a model with positive spreads where the forward rates are ordered. Nevertheless we show in the next Proposition 4.6.3 that the spread admits no solution since it explodes in finite time. The next technical Lemma 4.6.2 is used in the proof of Proposition 4.6.3.

[6] Note that such an assumption where $g_i(t,T) \neq g_j(t,T)$ for all $i,j = 1,\dots,K-1$ means that non of the forward rates can cross each other, which by continuity of the rates will imply an ordering of the forward rates.

Lemma 4.6.2. *Let W a brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and $f : [0,T] \mapsto \mathbb{R}^+$, $g : [0,T] \mapsto \mathbb{R}$ deterministic functions. Furthermore let R, S positive constants such that $0 < R < S < T$. Then it holds*

$$\mathbb{P}\left(f(t)W_t \geq g(t) \text{ for all } t \in [R,S]\right) > 0. \tag{4.6.5}$$

Proof. It holds

$$\begin{aligned}\mathbb{P}\left(W_t \geq \frac{g(t)}{f(t)} \text{ for all } t \in [R,S]\right) &\geq \mathbb{P}\left(\inf_{t\in[R,S]} W_t \geq \sup_{t\in[R,S]} \frac{g(t)}{f(t)}\right) \qquad (4.6.6)\\ &= \mathbb{P}\left(\sup_{t\in[R,S]} W_t^- \leq -G^{R,S}\right),\end{aligned}$$

where $W^- := -W$ is again a $\mathbb{P}$–brownian motion and $G^{R,S} := \sup_{t\in[R,S]} \frac{g(t)}{f(t)}$. Since $\sup_{t\in[0,S-R]} W^-_{R+t}$ has the arcsine distribution the claim follows.

□

Proposition 4.6.3. *Assume condition* N.2. *Furthermore assume $K = 3$, $\delta_1 = \delta_2 = 0$ and a spread volatility structure of the form (4.6.1) with $\sigma_1(t,T) = 0$ and $\sigma^s_{2,1}(t,T) = \sigma^s$ where σ^s is a positive real constant. Then for all $T > 0$, $\lim_{t\to\frac{T}{2}} s_2(t,T) = +\infty$ with positive probability under $\mathbb{Q}$ and any other equivalent measure.*

Proof. The idea of the proof is similar to Morton (1988). From (4.6.3) and since the migration intensity parameters $\lambda^*_{2,1}(t)$, $\lambda^*_{1,2}(t)$ are non-negative, then we can estimate

$$s_2(t,T) \geq s_2(0,T)\exp\left(|\sigma^s|^2 \int_0^t \int_s^T s_2(s,u)duds\right)\exp\left(\sigma^s \hat{W}^*_t - \frac{|\sigma^s|^2}{2}t\right). \tag{4.6.7}$$

Define $\tilde{s}_2(t,T) := |\sigma^s|^2 s_2(t,T)$. Then it holds

$$\tilde{s}_2(t,T) \geq |\sigma^s|^2 s_2(0,T)\exp\left(\int_0^t \int_s^T \tilde{s}_2(s,u)duds\right)\exp\left(\sigma^s \hat{W}^*_t - \frac{|\sigma^s|^2}{2}t\right). \tag{4.6.8}$$

Moreover, note that the set

$$\begin{aligned}A^{R,S,a} := \Bigg\{\omega : \exp\left(\sigma^s W_t\right) \geq \frac{2a}{(a-S^2)}\frac{(a-R^2)}{(a-RT)^2}\frac{1}{|\sigma^s|^2 s_2(0,T)}\exp\left(\frac{|\sigma^s|^2}{2}t\right) \qquad (4.6.9)\\ \text{for all } R \leq t \leq S\Bigg\},\end{aligned}$$

has positive probability by Lemma 4.6.2.[7] Hence for $(t,T) \in \hat{K}^{R,S}$ and $\omega \in A^{R,S,a}$,

$$\tilde{s}_2(t,T) \geq \frac{2a}{(a-S^2)} \frac{(a-R^2)}{(a-RT)^2} \exp\left(\int_R^t \int_s^T \tilde{s}_2(s,u) duds \right). \tag{4.6.10}$$

The claim follows as in the proof of Theorem 4.4.4 , by choosing the parameters $R = \frac{T}{4}, S = \frac{T}{2}, a = \frac{T^2}{2}$ and applying Morton's Lemma 4.3.12 together with Lemma 4.3.11. □

4.7 Vanishing Migration Intensities

In this section we introduce a condition of independence of maturity of the risk premium processes for the HJM underlying forward rates structure. As indicated in Bielecki and Rutkowski (2000) and Bielecki and Rutkowski (2004a), this condition is not necessary for the development of the model but rather optional. Nevertheless is required for the derivation of the risk-neutral valuation formula for the defaultable bond.

Assuming condition M.2 together with condition N.1 (or alternatively N.2) leads to model complications as we show in Proposition 4.7.2. In particular, this will lead to a model without migration between the classes. We call this condition M.2 in order to avoid any confusion with condition M.1 in Bielecki and Rutkowski (2000) and Bielecki and Rutkowski (2004a) which in our case is the condition on the process γ in Section 4.2.1.

Condition M.2. Let γ be the stochastic process as in Section 4.2.1 (see (4.2.13)). For $i = 1, \dots, K-1$, the process η_i, given by (4.2.19), does not depend on the maturity T.

The next Corollary 4.7.1 is an implication of condition M.2 which follows from Proposition 4.3.6.

Corollary 4.7.1. *Assume condition* M.2. *Furthermore assume that the consistency condition* N.1 *holds and fix some maturity $T \leq T^*$. The condition* N.1 *is equivalent to the following: for all $0 \leq t \leq T$ the drift condition*

$$\alpha_{C_t^1}(t,T) = \sigma_{C_t^1}(t,T) \int_t^T \sigma_{C_t^1}(t,u)du - \gamma_t \sigma_{C_t^1}(t,T)\,, \tag{4.7.1}$$

of the current forward rate $g_{C_t^1}(t,T)$ holds, together with the condition

$$s^f_{C_t^1}(t,t) = \lambda^*_{C_t^1,K}(t)(1-\delta_{C_t^1})\,, \tag{4.7.2}$$

on the set $\{C_t^1 \neq K\}$.

[7] Note that for the case of a d-dimensional brownian motion, then $\sigma^s W_t =: Z_t$ can be written as a linear combination of independent brownian motions, which has a standard normal distribution with zero mean and some standard deviation $v(t)$, where $v(t)$ is a positive function (follows by the sum of independent normal random variables). Then one could normalize by using the standard deviation function $v(t)$ where now, $\frac{1}{v(t)} Z_t$ is a one dimensional brownian motion and Lemma 4.6.2 can be applied directly.

Proof. From the proof of Proposition 4.3.6 recall equation (4.3.14)

$$\begin{aligned}
&\frac{1}{2}\left|\int_t^T \sigma_{C_t^1}(t,u)du\right|^2 - \int_t^T \alpha_{C_t^1}(t,u)du - \gamma_t \int_t^T \sigma_{C_t^1}(t,u)du \\
&= \sum_{j=1, j\neq C_t^1}^{K-1} \lambda^*_{C_t^1,j}(t)\left(1-\exp\left(\int_t^T s_{C_t^1,j}(t,u)du\right)\right) \\
&+ \lambda^*_{C_t^1,K}(t)\delta_{C_t^1}\left(1-\exp\left(\int_t^T s^f_{C_t^1}(t,u)du\right)\right),
\end{aligned} \tag{4.7.3}$$

where the left hand side of (4.7.3) equals with $\eta_{C_t^1}(t,T)$. Now, taking the derivative with respect to T in equation (4.7.3) and since by condition M.2, $\eta_{C_t^1}(t,T)$ is independent of the maturity T, then we get

$$\sigma_{C_t^1}(t,T)\int_t^T \sigma_{C_t^1}(t,u)du - \alpha_{C_t^1}(t,T) - \gamma_t\sigma_{C_t^1}(t,T) = 0\,, \tag{4.7.4}$$

and rewriting equation (4.7.4) in terms of the drift term $\alpha_{C_t^1}(t,T)$ of the forward rate $g_{C_t^1}(t,T)$ the claim follows. □

From Corollary 4.7.1 we see that under condition M.2, the active risky forward rate $g_{C_t^1}(t,T)$ has the well-known, classical, HJM drift condition as the risk-free forward rate $f(t,T)$. Nevertheless, as seen in the next Proposition 4.7.2 this implies restrictions on the intensity parameters of the matrix Λ^*_t and hence complications for the model. Under some mild conditions on the current bond forward rate process, imposing M.2 trivializes the intensity matrix structures.

Proposition 4.7.2. *Assume conditions* M.2 *and* N.1. *Furthermore assume that* $g_{C_t^1}(t,T) \neq f(t,T)$ *and* $g_{C_t^1}(t,T) \neq g_j(t,T)$ *for all* $j = 1\dots,K-1$ *with* $j \neq C_t^1$*. Then for almost all* $0 \leq t \leq T$ *and on the set* $\{C_t^1 \neq K\}$ *we have* $\lambda^*_{C_t^1,j}(t) = 0$ *for all* $j = 1,\dots,K-1$ *with* $j \neq C_t^1$ *and either* $\lambda^*_{C_t^1,K}(t) = 0$ *or* $\delta_{C_t^1} = 0$*. This implies no migration between the classes. In particular, for the non-zero recovery case where* $\delta_i \neq 0$ *for all* $i = 1,\dots,K-1$*, there is no migration nor default for the active class, that is* $C_0^1 = C_t^1$ *for all* $0 \leq t \leq T$*.*

Proof. Using the definition of $Z_i(t,T)$ and $Z(t,T)$ we can rewrite the consistency condition N.1 in terms of spreads and since by condition M.2, $\eta_i(t,T) = \eta_i(t)$ then we have

$$\begin{aligned}
&\sum_{j=1, j\neq C_t^1}^{K-1} \lambda^*_{C_t^1,j}(t)\exp\left(\int_t^T s_{C_t^1,j}(t,u)du\right) + \lambda^*_{C_t^1,K}(t)\delta_{C_t^1}\exp\left(\int_t^T s^f_{C_t^1}(t,u)du\right) \\
&+ \sum_{j=1, j\neq C_t^1}^{K-1} -\lambda^*_{C_t^1,j}(t) - \lambda^*_{C_t^1,K}(t) + \eta_{C_t^1}(t) = 0\,.
\end{aligned} \tag{4.7.5}$$

So, on $\{C_t^1 = i\}$, since by assumption the spreads $\int_t^T s_{i,j}(t,u)du$ and $\int_t^T s^f_i(t,u)du$ for all $j \neq i$ with $j = 1,\dots,K-1$ are never zero, then by fixing t we have a system of linear independent functions of the form

$$\sum_{j=1,j\neq i}^{K-1} \alpha^*_{i,j} \exp\left(\int_t^x s_{i,j}(t,u)du\right) + \beta^*_{i,K} \exp\left(\int_t^x s_i^f(t,u)du\right) + \gamma^*_{i,K} e^0 = 0\,, \tag{4.7.6}$$

for $x \in \mathbb{R}^+$, which holds if and only if $\alpha^*_{i,j} = \beta^*_{i,K} = \gamma^*_{i,K} = 0$ for all $j = 1,\dots,K-1$ with $j \neq i$. This is again equivalent to $\lambda^*_{i,j}(t) = \lambda^*_{i,K}(t)\delta_i = \sum_{j=1,j\neq i}^{K-1} -\lambda^*_{i,j}(t) - \lambda^*_{i,K}(t) + \eta_i(t) = 0$ for all $j = 1,\dots,K-1$ with $j \neq i$, on $\{C_t^1 = i\}$. So, this gives us $\lambda^*_{i,j}(t) = 0$ for all $i \neq j$ and either $\lambda^*_{i,K}(t) = 0$ or $\delta_i = 0$ on $\{C_t^1 = i\}$.[8] □

Remark 4.7.3. *Note that in Proposition 4.7.2 we do not use the ordering condition O, which shows how strongly the condition M.*2 *affects the whole model.*

Note also that the assumption $g_{C_t^1}(t,T) \neq f(t,T)$ *on the forward rates is quite natural. In particular this is implied by the necessary condition on the rates from Remark 4.3.24. Furthermore, the condition* $g_{C_t^1}(t,T) \neq g_j(t,T)$ *for all* $j = 1\dots,K-1$ *with* $j \neq C_t^1$ *is rather intuitive and keeps the natural interpretation of the rates, where the active forward rate is never equal to any of the other forward rates.*

Remark 4.7.4. *It is clear that since condition N.*2 *implies condition N.*1 *then a similar result as in Proposition 4.7.2 holds also under the conditions N.*2 *and M.*2.

[8] If for some $j \in \{1,\dots,K-1\}$ with $j \neq C_t^1$, $\int_t^T s_{C_t^1,j}(t,u)du = \int_t^T s^f_{C_t^1}(t,u)du$ or if for some $j,l \in \{1,\dots,K-1\}$ with $j \neq l \neq C_t^1$, $\int_t^T s_{C_t^1,j}(t,u)du = \int_t^T s_{C_t^1,l}(t,u)du$ then one can easily use the fact that the migration intensities are non-negative functions in order to complete the argument and hence the proof.

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