Dualities, Extended Geometries and the String Landscape

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Zusammenfassung

Die vorliegende Arbeit befasst sich mit der Rolle von Dualitäten und nichtgeometrischen Hintergründen in der Stringtheorie. Dualitäten definieren nichttriviale Abbildungen, anhand derer scheinbar unterschiedliche Theorien als alternative Beschreibungen derselben physikalischen Gegebenheiten identifiziert werden können. Ihre Existenz deutet oftmals darauf hin, dass den Modellen fundamentale Strukturen zugrunde liegen, welche durch den verwendeten Formalismus nicht vollständig erfasst werden.

In der Stringtheorie diente das Geflecht aus Dualitäten zwischen den fünf konsistenten Superstringtheorien als Motivation, die Existenz einer übergeordneten M-Theorie zu postulieren. Später zeigte sich jedoch, dass dabei gewisse Hintergrundflüsse auf Objekte abgebildet werden, welche in der konventionellen Differentialgeometrie nicht wohldefiniert sind. Derartige nichtgeometrische Hintergründe spielen eine zentrale Rolle im Bereich der Stringphänomenologie.

Die erste Hälfte dieser Arbeit befasst sich mit der Anwendung erweiterter Feldtheorien zur Beschreibung von Stringtheorien auf verallgemeinerten Hintergründen. Im Fokus des Interesses liegen dabei dimensionale Reduktionen der sogenannten Typ-II-Doppelfeldtheorie, welche eine lokale Beschreibung von Typ-II-Supergravitationen mit geometrischen und nichtgeometrischen Flüssen ermöglicht. Wir zeigen anhand der Beispiele von Calabi-Yau-Mannigfaltigkeiten und $K3 \times T^2$ explizit, dass die effektive vierdimensionale Physik derartiger Modelle durch geeichte Supergravitationen beschrieben wird, in welcher alle vorkommenden Moduli stabilisiert sind. Die Rolle der Flüsse im Bezug auf die Struktur der effektiven Wirkung sowie die Relation zu anderen Formalismen der Flusskompaktifizierung werden dabei im Detail diskutiert.

Das Kernthema der zweiten Hälfte stellt die statistische Analyse von Stringvakua in Orientifold-Kompaktifizierungen mit Flüssen dar. Dabei wird insbesondere auf das Zusammenspiel von Dualitäten und der sogenannten Tadpole-Wegkürzungsbedingung eingegangen. Anhand des Beispiels $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ wird demonstriert, dass sich nur ein geringer Anteil der berechneten Vakua in einem Bereich befindet, in welchem sowohl eine perturbative Betrachtung als auch eine Probenapproximierung von D-Branen zuverlässig ist. Wir zeigen zudem, dass sich die physikalischen Vakua oftmals auf Untermannigfaltigkeiten des Moduliraums anhäufen und gewisse Hohlräume existieren, in denen unter den gegebenen Annahmen keine stabilisierten Werte auftreten. Die Problematiken der Modulistabilisierung und Modellgestaltung sind somit eng miteinander verknüpft, und eine einheitliche Betrachtungung könnte entscheidende Einblicke in die Struktur der Stringlandschaft ermöglichen.

Abstract

This thesis is concerned with the role of dualities and nongeometric backgrounds in string theory. Dualities define nontrivial mappings by which seemingly distinct theories can be identified as alternative descriptions of the same physical reality. Their presence often suggests that the dual models are built upon more fundamental structures which cannot be fully captured by the applied formalisms.

In string theory the web of dualities between the five consistent superstring theories served as a motivation to postulate the existence of an underlying M-theory. However, it was later observed that certain background fluxes are thereby mapped to objects which are ill-defined in conventional differential geometry. Such nongeometric backgrounds play an essential role in the field of string phenomenology.

The first half of this work focuses on the application of extended field theories to describe string theories on generalized backgrounds. An emphasis is thereby placed on dimensional reductions of type II double field theory, which allows for a local description of type II supergravities with geometric and nongeometric fluxes. We show explicitly by the examples of Calabi-Yau manifolds and $K3 \times T^2$ that the effective four-dimensional action of such models is described by gauged supergravities in which all appearing moduli are stabilized. The role of the fluxes in respect of the structure of the effective action and the relation to other approaches to flux compactifications are discussed in detail.

The second half of this thesis is built around the statistical analysis of string vacua in orientifold compactifications with fluxes. A major focus is thereby set on the interplay between dualities and the so-called tadpole-cancellation condition. We demonstrate at the example of $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ that only a small fraction of the computed vacua is located in a region for which both a perturbative approach and a probe approximation for D-branes are reliable. In addition, we show that the vacua often accumulate on submanifolds of the full moduli space and that there exist certain voids in which no values are stabilized under the given assumptions. The issues of moduli stabilization and model building are therefore closely intertwined, and a unified treatment might provide valuable insights into the structure of the string landscape.

Contents

Part I

Introduction

Chapter 1

Introduction

1.1 Historical Overview

The quest for *unification* has been a driving force for the development of physics ever since the antiquity. While this long happened on a more subtle level in foundational fields such as classical mechanics and thermodynamics, the systematic search for a unified description of previously independent theories under a general framework is now a common task in various branches of physics. On many occasions, this led to valuable new insights and a deeper understanding of phenomena that had previously been inaccessible by existing theories. Before delving deeper into the main topic of this thesis, let us briefly review how this ongoing process of unification has eventually led to the development of *string theory* as a promising approach to a consistent framework describing all fundamental interactions of our universe.

1.1.1 The Role of Unification in Physics

Commonly seen as a first major milestone is Maxwell's unification of electricity and magnetism in the years 1862 to 1864 [1]. Building upon previous observations of Ørsted and Faraday, Maxwell elaborated a set of equations which later become known as the famous *Maxwell equations* and was the first to predict the existence of electromagnetic waves traveling a finite speed c. This brought the theory of *classical electromagnetism* to its final form and serves as an important foundation for modern particle physics.

Inspired by Maxwell's ideas, Lorentz, Poincaré and Einstein started their efforts to unify the concepts of space and time in the years 1904 and 1905. A first landmark was the formulation of *special relativity* [2], which introduced a fundamentally different view of our universe by abandoning the idea of absolute space and time and assigning a new meaning to the concept of mass. After implementing Newtonian gravity into the framework, Einstein eventually formulated his theory of *general relativity* [3] in 1916. Unlike classical field theory, general relativity describes the force of gravity as as an intrinsic geometric property of four-dimensional spacetime itself, giving rise to the prediction of various now-confirmed phenomena such as black holes, gravitational redshift or gravitational waves. The strong reliance on the formalism of differential geometry furthermore led to a reunification of physics and the mostly isolated field of pure mathematics. As of today, general relativity is considered the most accurate description of gravity.

Initially postulated by Planck in 1900 and later refined by Heisenberg, Schrödinger and others, the ideas *quantum mechanics* started to play a crucial role in the process of unification after Dirac formulated a relativistic description of the electron in 1927 [4]. Following further work on the concept of renormalization by Feynman, Dyson, Schwinger and Tomonaga, this eventually led to the development of quantum electrodynamics as the first *quantum field theory*. In the 1960s, Glashow [5], Weinberg [6] and Salam [7] combined the electromagnetic and weak force into an electroweak force with gauge group $SU(2) \times U(1)_Y$, which is spontaneously broken to the gauge group $U(1)_{\mathrm{em}}$ of quantum electrodynamics by the Higgs mechanism [8–12] at low energies. This model was in turn used to embed the electroweak and strong force into the larger gauge group $SU(3) \times SU(2) \times U(1)_Y$, which builds the base for the *standard model of particle physics* in its modern form.

1.1.2 Physics beyond the Standard Model

As of today, the standard model of particle physics and Einstein's theory of general relativity serve as the foundation to describe the four fundamental interactions of our universe. While this approach has proven outstandingly successful in many aspects, there exist several open problems which motivate the search for new physics at higher energy scales. Frequently discussed issues include but are not limited to the following:

- The standard model does not include gravity and is therefore regarded an effective rather than a fundamental theory. On the other hand, general relativity is a classical theory and might not be valid at small length scales. Naive approaches to formulate a quantum theory of gravity based on general relativity typically lead to nonrenormalizable models suffering from ultraviolet divergences, rendering straightforward implementation into the standard model difficult.

- The current theories fall short of explaining several cosmological observations such as galaxy rotation curves or the accelerated expansion of the universe. Only about 5% of the energy content in our universe can be described by the standard model. Approximately 25% are presumed to manifest in the form of dark matter, the remaining 70% as dark energy arising from a positive cosmological constant. The standard model does neither provide a viable candidate for dark matter nor an explanation for the small value of the cosmological constant.

- Naively, one would expect quantum corrections to make the renormalized mass of the Higgs boson very large. In order to correctly reproduce the comparatively low mass of the Higgs boson, its bare mass has to be fine-tuned to a high degree. This is commonly called the hierarchy problem. While not posing an inconsistency of the theory, the necessity for such fine tuning as well as the large number of free parameters of the standard model are often considered unnatural.

Considering how the unification of existing frameworks lead to new insights and a better understanding of the theories themselves in the past, it is hoped that a unified description of all fundamental forces will help to resolve these issues as well. However, as the underlying theories got more complex in nature, the process of unification turned out to do so as well.

1.1.3 Towards a Theory of Everything

A first natural approach to continue to process of unification was to embed the standard model into a larger gauge group such as $SU(5)$ [13] or $SO(10)$ [14]. Such models are commonly referred to as *Grand Unified Theories* (*GUTs*). One interesting property is that their high degree of symmetry allows them to automatically predict seemingly arbitrary phenomena like the relative strength of the interactions or the quantization of the $U(1)_Y$ charge. While certainly a viable first step, GUTs are strongly constrained by experimental observations such as the minimum lifetime of protons, which have ruled out many of the simpler models.

Another important building block of many modern theories in high energy physics is the idea of *Supersymmetry* (*SUSY*) [15–17]. Supersymmetric models base on a unique extension of the Poincaré algebra, the so-called *super-Poincaré algebra*, which introduces additional fermionic generators giving rise to a new symmetry between bosons and fermions. Most notably, the *Minimally Supersymmetric Standard Model (MSSM)* was long considered a promising candidate to remedy some of the standard model's shortcomings. In this model, the existence of a superpartner to each particle leads to the cancellation of first-order contributions to the Higgs mass, thereby avoiding the necessity for fine tuning. This comes, however, at the high price that the number of free parameters increases to over 100, and several other ad-hoc mechanisms are required to circumvent problems such as the proton decay.

Following the ideas of GUTs and supersymmetry, several more elaborate theories were constructed. A final major step towards the development of modern string theory is based on the idea that the four fundamental interactions might arise from an underlying higher-dimensional *Supergravity* theory compactified to four dimensions [18–22]. While suffering from similar shortcomings as previous approaches to quantum gravity and grand unification, it was the idea to interpret supergravities as low-energy limits of a corresponding string theory which eventually led to the development of *Superstring* and *M-Theory* as promising candidates for a true *Theory of Everything*.

1.1.4 A brief Overview of String Theory

As of today, there exist several approaches to address the issues of quantum gravity and unification. Some promising candidates include the frameworks of of *Loop Quantum Gravity* [23–32] (see also [33, 34] for an introduction to the topic), *Noncommutative Geometry* [35, 36] and *String Theory*, the last of which will be in the focus of this thesis.

Bosonic String Theory

The origins of string theory reach back to the late 1960, at which time it was developed as a model for the strong nuclear force [37–42]. Despite the inital idea becoming obsolete after the advent of quantum chromodynamics, the interest in string theory rekindled after a massless spin-2 excitation encountered in vibrating closed strings was found to match with the properties of the *graviton*, the hypothetical messenger particle of the gravitational interaction [43, 44]. This eventually led to the development of *Bosonic String Theory* formulated in terms of the *Polyakov Action*

$$S_{\text{Polyakov}} = \frac{1}{2\pi\alpha'} \int_{\Sigma} \mathrm{d}^2\sigma \, \sqrt{h} h^{\alpha\beta} g_{mn} \partial_\alpha x^m \partial_\beta x^n \tag{1.1.1}$$

describing the two-dimensional surface of the worldsheet Σ swept out by a string moving through some D-dimensional target space M. Here, $h_{\alpha\beta} = h_{\alpha\beta}\left(\sigma^0, \sigma^1\right)$ defines the worldsheet metric, $g_{mn} = g_{mn}\left(x^0, \ldots, x^{D-1}\right)$ the target space metric, and the slope parameter α' is related to the fundamental string length scale l_s by $l_s = 2\pi\sqrt{\alpha'}$. The worldsheet coordinates $\sigma^\alpha = (\tau, \sigma)$ contain one time- and one space-like direction. The target-space coordinates x^m, $m = 0, \ldots D-1$ can be interpreted as bosonic fields living on the worldsheet and are therefore eponymous for bosonic string theory. In the quantized theory, the Fourier modes of the x^m operators take the role of vibration modes of the string and carry quantum numbers of the D-dimensional Poincaré group. Heuristically, one can thus interpret particles in string theory as manifestations of different excitations of the string.

A peculiar feature of this framework is that preservation of symmetry under Weyl-rescalings $h_{\alpha\beta}(\sigma) \to e^{2\Lambda(\sigma)} h_{\alpha\beta}(\sigma)$ at the quantum level forces strong constraints on the target space. More precisely, cancellation of the Weyl anomaly for bosonic strings requires the dimension of M to take the value $D = 26$. Since higher-dimensional spacetimes are in clear contradiction to experimental observations, a common task in string theory is to find viable methods to compactify extra dimensions in such a way that they become unobservable at low energies. This will also be one of the main issues of this thesis.

Before delving deeper into the details, let us, however, briefly summarize the strengths of bosonic string theory:

- As mentioned in the beginning of the paragraph, the closed string spectrum always contains a massless spin-2 particle which can be identified with the graviton. This raised hopes that string theory is a viable candidate for a quantum theory of gravity.

- By generalizing the Polyakov action (1.1.1) to contain additional background fields, cancellation of the Weyl anomaly forces certain constraints on the target-space metric. At low energies, these conditions reduce to Einstein's equations, showing that the bosonic string can reproduce general relativity. The theory furthermore predicts corrections to Einstein gravity at high energies.

- In replacing point-like fundamental objects by finitely-sized strings, some of the major issues related to field theories can be avoided. This in particular includes the occurrence of ultraviolet divergences and spacetime singularities, both of which had plagued field-theoretical approaches to quantum gravity for a long time.

While these properties qualify the framework as a viable starting point, it also suffers from shortcomings. Most notably, two major problems eventually led to bosonic string theory being replaced by more sophisticated approaches.

- The spectrum of bosonic string theory always contains a state with negative squared mass. Excitations of such states are called *tachyons* and cause the ground state to become unstable if no further modifications of the theory are included.

- The spectrum does not contain fermionic excitations of the target space. The theory is therefore incapable of describing the matter content of our universe.

These issues were eventually addressed by including fermionic degrees of freedom and supersymmetry into the framework, which led to the development of modern *Superstring Theories*.

Superstring Theory

After the GSO-projection [45] enabled string theorists to define a first family of (tachyon-free) consistent *Superstring Theories*, interest in the subject eventually surged in 1984 and 1985 with the development of the Green-Schwarz mechanism [46] and the discovery of *Heterotic String Theories* [47]. This era is today commonly referred to as the *First Superstring Revolution* and marks the beginning of string theory being widely considered one of the most promising candidates for a theory of everything. Similar to bosonic string theory, it was found that cancellation of the Weyl anomaly constrains the target space dimension to the value $D = 10$. By 1985, five stable and consistent supersymmetric extensions to string theory had been constructed:

- **Type I Superstring Theory** describes open and closed unoriented superstrings with $\mathcal{N} = (1, 0)$ supersymmetry. It is thus a chiral theory. Its low-energy description is given by type I supergravity coupled to an $\mathcal{N} = 1$ supersymmetric $SO(32)$ Yang-Mills theory.

- **Type II Superstring Theories** describe oriented closed strings. Type IIA Superstring Theory is a non-chiral theory with $\mathcal{N} = (1, 1)$ supersymmetry, Type IIB Superstring Theory its chiral counterpart with $\mathcal{N} = (2, 0)$ supersymmetry. Their respective low-energy limits are described by type IIA and IIB supergravity.

- **Heterotic String Theories** utilize an oriented closed hybrid of the type I superstring and the bosonic string with $\mathcal{N} = (1, 0)$ supersymmetry. There exist two subtypes, HE and HO, differing in their ten-dimensional gauge groups $E_8 \times E_8$ and $Spin(32)/\mathbb{Z}_2$, respectively. Their low-energy limits are described by corresponding $\mathcal{N} = 1$ supersymmetric Yang-Mills theories coupled to type I supergravity.

Dualities and M-theory

While the known consistent string theories themselves are promising candidates to provide a unified description of all elementary particles and their interactions, the fact that there exist five distinct ones of them might raise doubts whether they are truly the right approach to construct a theory of everything. This issue was eventually settled by a range of discoveries made in the early 1990s, a period which later became known as the *Second Superstring Revolution*.

A first major finding of this era was that the five seemingly distinct super- and heterotic string theories are in fact related by various highly non-trivial transformations, called *dualities*. This not only raised hopes that string theory might indeed provide a unique description of all fundamental forces, but also revealed that an appropriate description of string theory requires a new understanding of some of the most basic principles particle physicists had relied on for decades. Most notably, some manifestations *T-duality* (also referred to as "target-space-duality") [48, 49] gave rise to an equivalence between large and small geometries, while *S-duality* ("strong-weak-duality") [50, 51] relates regimes of strong and weak coupling. These insights had a major impact on the mathematical framework of string theory and raised numerous new questions related to it, some of which will be among the core issues of this thesis.

Following the discovery of T- and S-duality, it was later shown that their corresponding transformations can be embedded into a more general form called *U-duality* ("unified duality"). This finding sparked the idea that all five super- and heterotic string theories might be realizations of a more fundamental theory [52–54], today commonly known as *M-theory* (see also figure 1.1 for an illustration). Since the ten-dimensional supergravities which describe the low-energy limits of the consistent string theories are known to descend from a unique eleven-dimensional supergravity, a natural assumption is that the latter should be reproduced as the low-energy limit of M-theory. Interestingly, it could be concluded from earlier findings [20, 55] that – rather than on strings – M-theory is most likely based on supermembranes, -fivebranes and maybe further objects. As of today, the problem of finding a complete quantum mechanical description of M-theory still remains open.

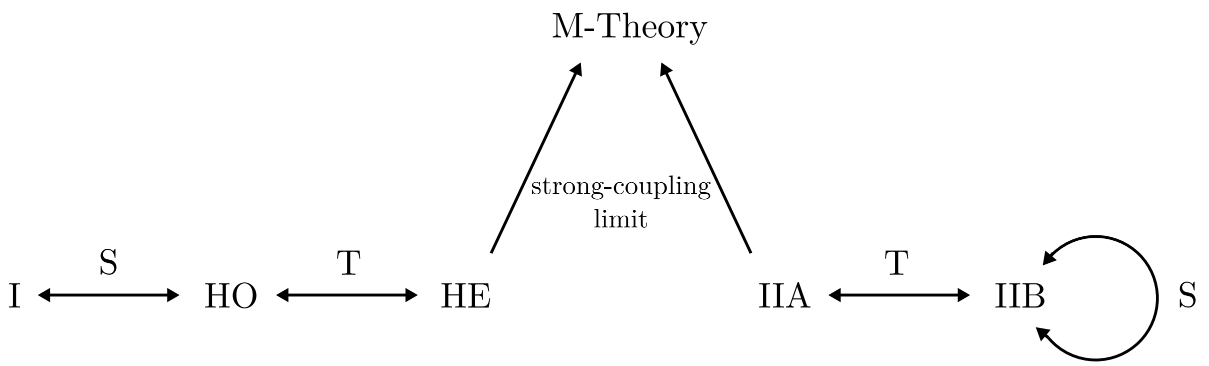


Figure 1.1: M-theory and the *Web of Dualities*. "S" and "T" indicate the respective duality transformation between different models.

Further Developments and Open Problems

Despite the search for M-theory still going on, large progress has been made in other directions throughout the last three decades. Among the most important findings was Polchinski's description of *D-branes* [56], which today play an important role in model engineering and string cosmology. Furthermore, Maldacena's conjectured *AdS/CFT correspondence* [57] had implications far beyond the field of string theory and is now applied to areas reaching from cosmology to as far as solid state physics. Similarly, *Mirror Symmetry* (see [58–60] for detailed reviews on the topic), which is conjectured to be a highly complex manifestation of T-duality [61], has had a great influence on various fields of pure mathematics such as enumerative geometry.

On the other hand, there are a number of open questions which are still subject to research today. Three particular such topics which will be in the focus of this thesis are the issue of *moduli stabilization*, the *landscape problem* and the role of *nongeometry* in string theory.

The term *moduli stabilization* describes a phenomenological issue encountered when trying to relate ten-dimensional string theories to four-dimensional physics. It was found in the 1980s that, in order to obtain $\mathcal{N} = 1$ or $\mathcal{N} = 2$ supersymmetric theories in four dimensions, the six extra dimensions of string theory commonly have to be compactified on a *Calabi-Yau manifold* [62]. While this poses strong constraints on the theory, there is no mechanism which fixes the choice to one specific such Calabi-manifold, and there can exist infinitesimal deformations of a manifold under which the Calabi-Yau property is preserved. Such deformations manifest as massless scalar fields – so-called *moduli* – in the effective four-dimensional theory and pose a severe contradiction to experimental observations. The development of viable methods to get rid of these undesired scalar fields is still an active field of research in string phenomenology.

One approach to achieve this goal are *flux compactifications*. Such models base on the simple idea to relax some of the conditions proposed in conventional Calabi-Yau compactifications and allow for the presence of background fields on the internal manifold. The flux of these fields through the homological cycles of the manifold then gives rise to an additional scalar potential which can fix parts of the moduli [63–68] (see also [69] for a detailed review on the topic). But while these results can be considered an important step, they also raised new questions: As it turns out, the duality relations between different string theories imply the existence of new objects arising as duals to the background fluxes, which seem to play an essential role in acquiring full moduli stabilization. However, these objects elude a description in terms of the commonly used framework of differential geometry and were therefore assigned the name *nongeometric fluxes* [70–72]. The phenomenon of *nongeometry* in string theory later became a subject research itself (see also [73] for a detailed review on the topic), and major parts of this thesis will be devoted to newly-developed frameworks enabling a unified mathematical description of geometric and nongeometric fluxes.

Finally, another issue arising in the context of string compactification is the so-called *landscape problem*. Due to neither a specific compactification manifold nor a corresponding configuration of background fluxes being fixed or preferred, the number of string vacua is thought to be extremely large, with famous estimates reaching from 10^{500} [74] or

10^{1500} [75] to as high as 10^{272000} [76]. This of course has strong implications regarding how much fine tuning is required to construct a theory of everything based on the framework of string theory. As a consequence, one of the main challenges in string phenomenology is to analyze the statistical and mathematical structure of the string landscape in order to resolve the question whether and how realistic solutions can be constructed. This topic will be another major focus of this thesis.

1.2 Outline of Topics

The purpose of this work is twofold. One the one hand, a major aim is to study the physical and mathematical nature of dualities and their potential application to open problems of string theory and beyond. On the other hand, we also want to build upon recent developments in the areas of nongeometry and *extended field theories* to address new questions arising from the very presence of dualities. The thesis is structured as follows:

- Chapter 2 provides a brief overview of the elementary concepts and technical tools which build the foundation for the remaining parts of the work. We begin by outlining the idea of higher-dimensional theories and their compactification to four dimensions. Based on instructive examples, we demonstrate how the issue of moduli stabilization poses a major challenge to the construction of phenomenologically accurate models and how it can be addressed by introducing background fluxes in the compact dimensions. Following a short discussion of dualities in field and string theory, we then present a first instance of a nongeometric background arising naturally from T-duality transformations of such fluxes. The topics are dealt with on a basic level, with the focus being set on heuristic explanations to provide the reader an intuitive understanding of the methods used throughout the following chapters. The overview is concluded by a short summary and a discussion of open problems.

- In chapter 3 we delve deeper into the phenomenology of type II superstring theories. We start with a brief discussion of their low-energy spectra before elaborating on the role of Calabi-Yau geometry in string compactifications. The concepts are then applied to relate the type IIA and IIB actions to that of four-dimensional $\mathcal{N} = 2$ supergravity. The chapter is concluded by a brief discussion of Mirror Symmetry and open problems of naive Calabi-Yau compactifications.

- Chapter 4 introduces the framework of double field theory as a T-duality covariant extension of conventional field theory. We discuss the structure of T-duality transformations in more detail and demonstrate how the notions of differential geometry can be generalized such that dualities become a manifest symmetry of a given theory. Topics of particular importance include generalized diffeomorphisms and Lie derivatives, generalized fields and consistency constraints of the formalism. The concepts are then applied to formulate an Einstein-Hilbert-like action for double field theory and to derive its (projected) equations of motion. The chapter is concluded by a brief outlook on U-duality covariant exceptional field theories and a summary of the presented topics.

- In chapter 5 the framework of double field theory is applied to explicitly perform dimensional reductions of type II theories with all geometric and nongeometric fluxes of the T-duality chain turned on and all moduli stabilized. The elaborations are mainly based on [77]. We first show how the flux formulation of double field theory provides a natural interpretation of fluxes as local operators acting on fields. Building upon previous works [78, 79], the type IIA and IIB scalar potentials are then reduced on Calabi-Yau three-folds and $K3 \times T^2$, thereby generalizing various concepts and introducing an additional set of generalized dilaton fluxes giving rise to non-unimodular gaugings in four dimensions. Following this, we extend our discussion of Calabi-Yau three-folds to the kinetic terms and derive the full four-dimensional action of $\mathcal{N} = 2$ gauged supergravity. The chapter is concluded by a short discussion of related work and an outlook on future developments in the field.

- Chapter 6 focuses on the role of dualities and consistency constraints in type IIB orientifold compactifications and is mainly based on [80]. We start with a brief discussion of orientifold projections as a viable way to obtain phenomenologically favorable $D = 4$ $\mathcal{N} = 1$ supergravities from type IIB theory. We then delve deeper into the mathematical structures and show how an important constraint called the *tadpole cancellation condition*, along with various dualities, greatly reduces the number of physically-distinct and trustable vacua obtained from type IIB theory compactified on the orientifold $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$. We discuss three settings with increasing degree of generality and perform an in-depth analysis of the mathematical structures underlying their space of solutions as well as the statistical distributions of their vacua. The reliability of solutions in light of commonly used approximations and phenomenological implications are discussed in detail. The chapter is again concluded by a short summary of results and an outlook on future directions of research.

- Chapter 7 concludes the thesis and provides an outlook on future developments.

This thesis covers a wide variety of topics in and related to string theory. While the structure is kept self-contained as much as possible, some familiarity with the basic notions of differential geometry and algebraic topology is highly favorable in order to obtain a deeper understanding of the discussed topics. The most important definitions and conventions used throughout this work are provided in appendix A. We will furthermore employ natural units and set the string slope parameter α' to 1 for the remainder of this work.

1.3 Publications

Several publications have grown out of the research conducted for the completion of this thesis. Chapters 5 and 6 are based on and in large part identical to the works

- P. Betzler and E.Plauschinn, “Dimensional reductions of DFT and mirror symmetry for Calabi-Yau three-folds and $K3 \times T^2$”, *Nucl. Phys.* **B933** (2018) 384-432, 1712.08382.
- P. Betzler and E.Plauschinn, “Type IIB flux vacua and tadpole cancellation”, *Fortschr. Phys.* **67** (2019), no. 11 1900065, 1905.08823.

In addition, further results which are only loosely connected to the topic of this thesis are presented in

- P. Betzler and S. Krippendorf, “Connecting dualities and machine learning”, *Fortschr. Phys.* **68** (2020), no. 5 2000022, 2002.05169.

Part II

Conceptual Preliminaries

Chapter 2

(Flux) Compactifications and Dualities

Before delving into the details of current research topics, this chapter shall provide an instructive overview on the mathematical framework which builds the foundation of this thesis. Although often highly technical in nature, many of the ideas encountered in the following chapters can be traced back to concepts introduced in the following sections.

2.1 Kaluza-Klein and Flux Compactifications

While studied most thoroughly in the context of string theory, the idea of constructing physical theories based on compactified higher-dimensional spacetimes originated in the 1920s. In their pioneering works [81–83], Kaluza and Klein elaborated a unified model of gravity and electromagnetism based on a curved five-dimensional spacetime compactified on a circle. The theory showed several strong points, such at its capability to reproduce the four-dimensional Einstein and Maxwell equations and to provide a natural explanation for the quantization of the electric charge. On the other hand, it also came with various flaws such as the prediction of an unidentified scalar field and was therefore discarded shortly after its inital development.

Nevertheless, many of the technical tools used by Kaluza and Klein as well as the conceptual issues of the theory carry over to string compactifications in a more or less straightforward way, rendering it an ideal toy model for the topic.

2.1.1 Massless Scalar Field in Flat Five-Dimensional Spacetime

A well-suited example to get familiar with the idea is the case of a massless scalar field $\hat{\phi}(x^m) = \hat{\phi}(x^\mu, y)$ living in a flat five-dimensional spacetime with one direction $y \sim y + 2\pi R$ compactified on a circle of radius R [84]. The action then takes the simple form

$$S = \frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}y\,\partial_m\hat{\phi}\partial^m\hat{\phi}, \tag{2.1.1}$$

giving rise to the equations of motion

$$\Box\hat{\phi} = 0 \quad \Leftrightarrow \quad \partial_\mu \partial^\mu \hat{\phi} + \partial_y^2 \hat{\phi} = 0. \tag{2.1.2}$$

By periodicity along the y-direction, $\hat{\phi}$ can be expanded as a Fourier series

$$\hat{\phi}\left(x^\mu, y\right) = \sum_{k=-\infty}^{\infty} \phi_k\left(x^\mu\right) e^{\frac{iky}{R}}, \tag{2.1.3}$$

which enables us to reformulate the equations of motion (2.1.2) as

$$\partial_\mu \partial^\mu \phi_k - \frac{k^2}{R^2}\phi_k = 0. \tag{2.1.4}$$

These relations can alternatively be obtained from the four-dimensional action

$$S = \frac{1}{2}\sum_{k=-\infty}^{\infty} \int \mathrm{d}^4 x\, \partial_\mu \phi_k \partial^\mu \overline{\phi}_k + \frac{k^2}{R^2}\overline{\phi}_k \phi_k\,. \tag{2.1.5}$$

In four dimensions, the model thus describes one massless scalar field ϕ_0 and an infinite tower of massive scalar fields $\{\phi_k\}_{k\neq 0}$ with masses $m^2 = \frac{k^2}{R^2}$.

The idea of describing a higher-dimensional theory from a lower-dimensional viewpoint is what is known by the term *compactification* in the narrower sense. Since experimental constraints require the extra dimensions to be unobservable in four dimensions, the radius R needs to be chosen sufficiently large such that the masses $\frac{k^2}{R^2}$ of the fields $\{\phi_k\}_{k\neq 0}$ lie, at least, beyond currently accessible energy scales. Taking this limit is commonly referred to as *dimensional reduction* of a theory.

The modus operandi of this simple example can be readily generalized to settings encountered in string theory. To get an idea of this, notice that the normalized expressions $\frac{1}{\sqrt{2\pi R}} e^{\frac{iky}{R}}$ are precisely the orthonormalized eigenfunctions of the Laplace operator ∂_y^2 on the circle. On a more general level, the Laplace operator $\triangle_D$ of a $D = (d+N)$-dimensional manifold M splits into its external and internal components as

$$\triangle_D = \triangle_d + \triangle_N, \tag{2.1.6}$$

and the massless components of a scalar or p-form gauge field in d dimensions are encoded by the zero modes of the internal Laplacian $\triangle_N$. In the language of differential geometry, the corresponding differential-forms are called *harmonic*, and their space $\mathcal{H}^p\left(M\right)$ is isomorphic to the pth *de Rham cohomology group* $H^p\left(M\right)$ of M. The standard way of performing dimensional reductions in superstring theory is therefore to expand the ten-dimensional fields in terms of the cohomology bases of the compactification manifold, thereby automatically taking into account only those fields which remain massless in four dimensions. A remarkable feature of this approach is that one can study dimensional reductions on manifolds based mainly on their topological properties. This shortcut proves highly valuable in string compactifications, where the metric of the compactification space is rarely known explicitly.

2.1.2 Pure Gravity on $\mathbb{R}^{1,3} \times S^1$

Having the basic tools of dimensional reduction at hand, we are now ready to discuss Kaluza and Klein's theory of five-dimensional gravity. As will become clear soon, this model nicely exemplifies the issue of moduli stabilization, which will be one of the major topics of this thesis. The following discussion mainly follows the lines of [85], with some additional information taken from the original works [81–83].

We once more consider a five-dimensional spacetime manifold with one periodic direction $y \sim y + 2\pi R$ and metric $\hat{g}_{mn}$ parameterized as

$$\hat{g}_{mn} = \varphi^{-\frac{1}{3}} \begin{pmatrix} g_{\mu\nu} + \varphi A_\mu A_\nu & \varphi A_\mu \\ \varphi A_\nu & \varphi \end{pmatrix}, \tag{2.1.7}$$

where $g_{\mu\nu}(x^\mu, y)$ denotes the metric of the four-dimensional external space, $\varphi(x^\mu, y)$ is a scalar and $A_\mu(x^\mu, y)$ a $U(1)$ gauge field. Following the ideas of general relativity, we now split the metric into a ground state and small fluctuations,

$$\hat{g}_{mn} = \langle \hat{g}_{mn} \rangle + \delta \hat{g}_{mn}, \tag{2.1.8}$$

the former of which we assume to describe the product $\mathbb{R}^{1,3} \times S^1$ of four-dimensional Minkowski spacetime and a circle,

$$\langle \hat{g}_{mn} \rangle = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.1.9}$$

The action

$$S = \frac{1}{2\pi} \int \mathrm{d}^4x \, \mathrm{d}y \, \sqrt{-\hat{g}^{(5)}} \hat{R}^{(5)}, \tag{2.1.10}$$

is defined analogously to standard general relativity and depends only on the determinant $\hat{g}^{(5)}$ of the metric $\hat{g}_{mn}$ and the five-dimensional Ricci scalar $\hat{R}^{(5)}$. Similar to the four-dimensional case, the equations of motion,

$$\hat{R}^{(5)}_{mn} = 0, \tag{2.1.11}$$

imply Ricci-flatness of the spacetime. Starting from this, we can now follow the familiar recipe for dimensional reduction and expand the five-dimensional fields in terms of their Fourier modes,

$$\begin{aligned} g_{\mu\nu}(x^\mu, y) &= \sum_{k=-\infty}^{\infty} g_{\mu\nu\,k}(x^\mu)\, e^{\frac{iky}{R}}, \\ \varphi(x^\mu, y) &= \sum_{k=-\infty}^{\infty} \varphi_k(x^\mu)\, e^{\frac{iky}{R}}, \\ A_\mu(x^\mu, y) &= \sum_{k=-\infty}^{\infty} A_{\mu\,k}(x^\mu)\, e^{\frac{iky}{R}}. \end{aligned} \tag{2.1.12}$$

The groundstate (2.1.9) is then reproduced by

$$\mathring{g}_{\mu\nu\,0} = \eta_{\mu\nu}, \qquad \mathring{\varphi}_0 = 1, \qquad \mathring{A}_{\mu\,0} = 0, \tag{2.1.13}$$

with all other modes vanishing. Truncating out all massive modes and integrating over the internal space, the action (2.1.10) eventually reduces to [86]

$$S = \frac{1}{2}\int \mathrm{d}^4x\,\sqrt{-g^{(4)}{}_0}\left[R^{(4)}{}_0 - \frac{1}{4}\varphi_0 F_{\mu\nu\,0}F_0^{\mu\nu} - \frac{1}{6\varphi_0^2}\partial_\mu\varphi_0\partial^\mu\varphi_0\right]. \tag{2.1.14}$$

Here, $F_{\mu\nu\,0} = \partial_\mu A_{\nu\,0} - \partial_\nu A_{\mu\,0}$ denotes the field strength tensor of $A_{\mu\,0}$, and all expressions are to be understood with respect to the fluctuations about the groundstate (2.1.13). As can be seen, this model slightly missed the target of unifying gravity and electromagnetism into a single higher-dimensional theory as it contains an additional massless scalar field φ_0 descending from φ. The appearance of such undesired *moduli* is not isolated to this particular setting and remains an important issue in modern approaches to dimensional reduction.

Taking again a more general point of view, the problem not only carries over but often worsens as the mathematical structure of the theory gets more intricate. As we will see later, Calabi-Yau manifolds used in conventional string compactifications exhibit various permissible deformations of their shape and volume, all of which preserve the important Calabi-Yau property. This manifests in the appearance of a large number of massless scalar particles in four dimensions, posing a severe contradiction to experimental observations. Over the last decades, much effort has been devoted to the search for mechanisms to get rid of the vacuum degeneracy and fix the moduli. One of the most promising approaches to achieve this is known under the term *flux compactifications*.

2.1.3 Flux Compactifications

Conventional approaches to string compactification rely on the simplifying assumption that there do not exist any background fields in the higher-dimensional spacetime. While the presence of such background fields on the external component is indeed prohibited by requiring Poincaré invariance, extending this assumption to the internal component lacks physical justification. As it turns out, relaxing the constraint provides not only valuable insights into the mathematics of string theory, but also important tools to address the problem of moduli stabilization.

For a qualitative picture, consider some p-form field C_p with field strength F_{p+1} living on an arbitrary manifold M with non-trivial $(p+1)$ th homology group $H_{p+1}(M)$ [87]. One can then define the *flux* of F_{p+1} through a non-trivial element Γ of $H_{p+1}(M)$ by

$$\int_\Gamma F_{p+1} = n, \tag{2.1.15}$$

where n denotes a coefficient contained in some field, commonly $\mathbb{Z}$ or multiples thereof. This can be considered a generalization of the magnetic flux through a surface surrounding a corresponding monopole charge, with C_p being sourced by a higher-dimensional Dp-brane. It is, however, important to bear in mind that the mathematical concept extends beyond this heuristic picture, and abstract fluxes can be defined without the pictorial device of charged sources.

Denoting the basis of $H_{p+1}(M)$ by $\left(\sigma^1, \dots \sigma^{b^{p+1}}\right)$, a particular field configuration of F_{p+1} can be uniquely described by a corresponding vector $\left(n_1, \dots n_{b^{p+1}}\right)$, where

$$\int_{\sigma^i} F_{p+1} = n_i. \tag{2.1.16}$$

Now, similarly to Maxwell's theory of electromagnetism, turning on such field strengths comes with an energetic cost

$$V = \int_M F_{p+1} \wedge \star F_{p+1} \tag{2.1.17}$$

depending on the metric of the internal manifold. The presence of F_{p+1} therefore creates a scalar potential term V, possibly fixing certain geometric properties of M when V is minimized.

Again, this abstract example only illustrates the basic concepts of flux compactifications. More details as well as concrete settings of flux compactifications in string theory will be addressed in chapters 5 and 6. Comprehensive reviews on the topic can be found in [69, 87, 88]. It is also important to stress that a naive approach to flux compactifications can only fix parts of the moduli appearing in string compactifications. There exist several methods to construct models with all moduli stabilized. The approach utilized in this thesis builds upon the phenomena of dualities and nongeometry, which shall be discussed next.

2.2 Dualities and Nongeometry

The phenomenon of dualities is encountered in many branches of physics and has been known long before the development of string theory. Heuristically, dualities describe nontrivial one-to-one mappings between seemingly distinct physical theories, effectively rendering them different descriptions of the same physical reality. The consequences are twofold: On the one hand, dualities can map complicated or infeasible tasks in one theory to a simpler task in another theory, thus serving as a valuable tool to facilitate the analysis of existing models. On the other hand, the existence of more than one equivalent description of the same physical situation implies that there is still an essential aspect of the theories which is not yet completely understood. In some instances, the very presence of dualities calls into question some of the most fundamental concepts such as the notions of "large" and "small" geometries in string theory or "hot" and "cold" systems in statistical physics. We will next discuss some of the simplest manifestations of dualities in field and string theory before delving deeper into their mathematical structure in chapter 4.

2.2.1 Electromagnetic and S-Duality

A very simple example of a (self-)duality in classical physics is that of Maxwell's theory of electromagnetism [89]. To get familiar with the concept, consider the Maxwell equations

$$\begin{aligned} \nabla \vec{E} &= \varrho, & \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0, \\ \nabla \vec{B} &= 0, & \nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} &= \vec{j}. \end{aligned} \tag{2.2.1}$$

Assuming for the moment that the theory discribes fields in a vacuum, we can set the source terms $\varrho, \vec{j}$ to zero, in which case it is easy to check that the complete set of equations remains invariant under the transformations

$$\vec{E} \to \vec{B}, \qquad \vec{B} \to -\vec{E}. \tag{2.2.2}$$

In this particular setting, electric and magnetic fields are thus treated on equal footing, and the assigment of names is pure convention. This characteristic feature of electromagnetic waves raised speculations whether there should also exist magnetic source terms $\sigma, \vec{k}$. The Maxwell equations could then be modified to take the forms

$$\begin{aligned} \nabla \vec{E} &= \varrho, & \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= -\vec{k}, \\ \nabla \vec{B} &= \sigma, & \nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} &= \vec{j}, \end{aligned} \tag{2.2.3}$$

and self-duality could be restored by extending the mappings (2.2.2) to

$$\begin{aligned} \vec{E} &\to \vec{B}, & \vec{B} &\to -\vec{E}, \\ \varrho &\to \sigma, & \sigma &\to -\varrho, \\ \vec{j} &\to \vec{k}, & \vec{k} &\to -\vec{j}. \end{aligned} \tag{2.2.4}$$

While a naive implementation of magnetic source terms is in clear contradiction to Gauss' law, there have been shown to exist several structures such as the Dirac string [90] or 'tHooft-Polyakov monopoles [91, 92] which can effectively realize magnetic monopoles at least on the mathematical level. A particular feature of the former is that the existence of such monopoles requires the electric and magnetic elementary charges e, g to satisfy the condition [93]

$$eg = 2\pi n \in \mathbb{Z}, \tag{2.2.5}$$

thereby automatically requiring quantization of the elementary charges. Since the duality transformations (2.2.4) exchange the roles of the electric and magnetic fields, the respective elementary charges effectively get inverted. The electromagnetic duality thus maps between strong and weak coupling regimes and can be considered a simple example of a strong-weak duality.

Delving a bit deeper into the details, the Dirac string should be considered with caution since it defines a singular solution to Maxwell's equations. This flaw can, however, be remedied by considering certain grand unified theories, where structures such as the 'tHooft-Polyakov monopole arise as topological solitons without any singularities. A characteristic feature of such solutions is that they are often finitely-sized objects and behave like particles only asymptotically. It is another common property of strong-weak-type dualities to contain mappings between such topological solitons and point particles.

It was precisely the presence of solitonic monopoles in the spectrum of the Georgi-Glashow model [13] which prompted Montonen and Olive to conjecture the existence of a similar type of electromagnetic duality for grand unified theories [50]. The idea was then refined by Witten [94], which eventually led to the discovery of a more general duality group $SL(2,\mathbb{Z})$ in $\mathcal{N}=4$ supersymmetric Yang-Mills theories. In this setting, the gauge coupling constant e and the theta-angle θ can be combined to form a complex coupling

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}\,, \tag{2.2.6}$$

on which the $SL(2,\mathbb{Z})$ duality group acts as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau := \frac{a\tau+b}{c\tau+d}\,, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\,. \tag{2.2.7}$$

A similar type of transformation was later found to relate various types of superstring theories [51], at which time it was assigned the name *S-duality*. Notice also that the above mappings reduce to the familiar electromagnetic duality $e \to -\frac{4\pi}{e}$ when $\theta = 0$ and $a=0, b=-1, c=1, d=0$.

2.2.2 T-Duality and Mirror Symmetry

In addition to S-duality, string theory contains new types of dualities which originate purely from the extended nature of strings. Among the most important ones is a class of highly non-trivial relations between different target space geometries, known by the name *T-duality*.

Circular Compactifications of the Bosonic String

To illustrate the idea, we begin by considering a closed bosonic string in $D = 26$ dimensions, with one direction X^{25} compactified on a circle of radius R [95]. While sharing many similarities with the settings discussed in section 2.1, a particular property of this model is that closed strings can wind around the internal direction. Such different winding states can be uniquely described by a so-called *winding number* $\tilde{p}$ (see also figure 2.1), which, due to the topology of the circle, is invariant under continuous transformations. They thus define a conserved charge and are an inherent property of the theory. Depending on the number of windings around the circle, a closed string has to satisfy different boundary conditions

$$X^{25}(\tau,\sigma+2\pi) = X^{25}(\tau,\sigma) + 2\pi\tilde{p}R \tag{2.2.8}$$

ensuring periodicity along internal direction. From here, one can proceed by splitting the bosonic fields into linear combinations of left- and right-movers,

$$X^M(\tau,\sigma) = X^M_L(\tau+\sigma) \pm X^M_R(\tau-\sigma)\,, \quad M \in \{0,1,\ldots,25\}\,, \tag{2.2.9}$$

and perform mode expansions analogously to the previously discussed settings. After taking into account the boundary condition (2.2.8) and a somewhat lengthy calculation [95], one eventually arrives at the mass formula

$$m^2 = \left(\frac{p}{R}\right)^2 + \left(\frac{\tilde{p}R}{\alpha'}\right)^2 + \frac{2}{\alpha'}\left(N + \widetilde{N} - 2\right) \tag{2.2.10}$$

and the level-matching condition

$$N - \widetilde{N} = p\tilde{p} \tag{2.2.11}$$

ensuring worldsheet reparameterization invariance under constant shifts of σ. Here, p denotes the (quantized) total momentum of the string along the internal direction, N and $\widetilde{N}$ describe the left- respectively right-moving oscillation modes of the string, and we included the slope parameter $\alpha' = 1$ explicitly for pedagogical reasons. Taking a closer look at (2.2.10), one can see that the mass spectrum contains contributions of a zero-point energy term, the internal momentum, the winding number and the left- and right-moving oscillation modes.

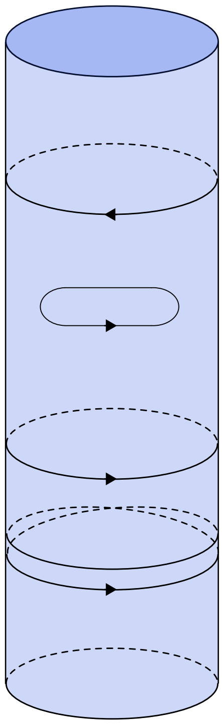


Figure 2.1: Various topologically distinct configurations of a string winding around a two-dimensional cylinder. From top to bottom: $\tilde{p} = -1, 0, 1, 2$.

At this point, one might notice that the contributions originating from the internal momentum p and the winding number $\tilde{p}$ show a very similar structure, the only difference being their inverse scaling behavior with respect to the radius R. This analogy extends so far that the two equations get mapped onto themselves when exchanging the roles of

p and $\tilde{p}$ while at the same time inverting the radius,

$$p \leftrightarrow \tilde{p}, \quad R \leftrightarrow \frac{\alpha'}{R}. \tag{2.2.12}$$

A remarkable feature of this result is that the inversion of the radius renders small and large compactification spaces physically equivalent. In particular, there exists a self-dual radius $R = \alpha'$, which defines a lower bound for all physically-distinct values of R. This is a simple example of a T-duality transformation and nicely demonstrates how some of the most fundamental concepts of geometry can break down in string theory.

Buscher Rules

Circular compactifications describe only one particular instance of T-duality in string theory, and there exist various approaches to generalize the idea. One concept which will be of essential importance for the upcoming sections and chapters are the so-called *Buscher rules* [96, 97]. To illustrate the idea, we follow the lines of [98] and consider a slightly generalized version of the Polyakov action (1.1.1) with a possibly non-vanishing *Kalb-Ramond two-form field B*. Employing conformal gauge $h_{\alpha\beta} = \mathrm{diag}(-1, 1)$ and complex worldsheet coordinates, the action takes the form

$$S = \frac{1}{2\pi} \int_\Sigma \mathrm{d}^2 z \ (g_{mn} + B_{mn})\, \partial x^m \bar{\partial} x^n, \tag{2.2.13}$$

where the bosonic fields x^m can again be interpreted as coordinates of the D-dimensional target space M. Assume now that there exists an Abelian 2π-periodic isometry for g, generated by a corresponding Killing vector field k with $L_k g = 0$. Furthermore, let $L_k B = d\omega$ for some one-form ω on M. One can then show that the transformation given by $\delta x^m = \epsilon k^m$ is a symmetry of the action.

Using diffeomorphism-invariance, the coordinates can be chosen in a way that the isometry k acts as translation in one particular direction x^r. Similarly, the B-field can be brought to a form satisfying $L_k B = \frac{\partial}{\partial r} B = 0$ via spacetime gauge transformations $B \mapsto B + d\chi$ with $\chi \in \Omega^1(M)$, such that both fields g and B do not depend on the isometric direction. Under these assumptions, the original action (2.2.13) can be obtained from another sigma-model where the isometry appears as a gauge symmetry[1]. More precisely, consider the "master action"

$$\begin{aligned} S_{\text{Master}} = \frac{1}{2\pi} \int_\Sigma \mathrm{d}^2 z \ \big[& g_{rr} A\bar{A} + (g_{rp} + B_{rp})\, A \bar{\partial} x^p + (g_{pr} + B_{pr})\, \partial x^p \bar{A} + \\ & + (g_{pq} + B_{pq})\, \partial x^p \bar{\partial} x^q + \theta \left(\partial \bar{A} - \bar{\partial} A \right) \big], \end{aligned} \tag{2.2.14}$$

with some gauge field $\mathcal{A} = A(z)\, dz + \bar{A}(\bar{z})\, d\bar{z}$, a Lagrange multiplier θ and the indices p, q running over all values except for r. Now there are two routes to follow. Integrating out θ yields the equations of motion

$$\partial \bar{A} - \bar{\partial} A = 0, \tag{2.2.15}$$

[1] This method is referred to as "gauging the isometry" in most literature.

which in topologically trivial worldsheets Σ can be solved by letting the fields A and $\bar{A}$ become pure gauge,

$$A = \partial\theta \,, \quad \bar{A} = \bar{\partial}\theta \,. \tag{2.2.16}$$

Inserting these relations into (2.2.14) restores the original action (2.2.13), with θ taking the role of the coordinate x^r. On the other hand, integrating out the gauge field $\mathcal{A}$ gives rise to the equations of motion

$$\begin{aligned} (g_{rp} + B_{rp})\,\bar{\partial}x^p + g_{rr}\bar{A} = 0, \\ (g_{pr} + B_{pr})\,\partial x^p + g_{rr}A = 0, \end{aligned} \tag{2.2.17}$$

and their solutions

$$\begin{aligned} \bar{A} &= -\frac{g_{rp} + B_{rp}}{g_{rr}}\bar{\partial}x^p - \frac{1}{g_{rr}}\bar{\partial}\theta, \\ A &= -\frac{g_{pr} + B_{pr}}{g_{rr}}\partial x^p + \frac{1}{g_{rr}}\partial\theta. \end{aligned} \tag{2.2.18}$$

Substituting these expressions back into the master action (2.2.14) leads to a dual action

$$\tilde{S} = \frac{1}{2\pi}\int_\Sigma \mathrm{d}^2z\,(\tilde{g}_{mn} + \tilde{B}_{mn})\partial x^m\bar{\partial}x^n, \tag{2.2.19}$$

where the newly-introduced fields $\tilde{g}$ and $\tilde{B}$ are related to g and B by the *Buscher rules*

$$\begin{aligned} \tilde{g}_{rr} &= \frac{1}{g_{rr}}, \qquad & \tilde{g}_{rq} &= \frac{B_{rq}}{g_{rr}}, \qquad & \tilde{g}_{pq} &= g_{pq} - \frac{g_{pr}g_{rq} + B_{pr}B_{rq}}{g_{rr}}, \\ & & \tilde{B}_{rq} &= \frac{g_{rq}}{g_{rr}}, & \tilde{B}_{pq} &= B_{pq} - \frac{g_{pr}B_{rq} + B_{pr}g_{rq}}{g_{rr}}. \end{aligned} \tag{2.2.20}$$

Using a more involved approach, the above transformations can be generalized to sigma-models with non-trivial dilaton background. Computing the Buscher rules for such settings commonly requires consideration of path integrals at one loop, for which one obtains the transformation behaviour [96, 97]

$$\phi = \phi - \frac{1}{2}\ln g_{rr} \,. \tag{2.2.21}$$

Taking up the previous example of circular compactifications, the isometric direction is given by the azimutal angle φ, and the above transformations correctly reproduce the T-duality mappings (2.2.12) we encountered earlier. In the case of D-dimensional tori, the T-duality transformations along the D different directions span the group $O\,(D, D; \mathbb{Z})$, which builds the basis for the construction of *double field theory.* This will be discussed in more detail in chapter 4.

We should at this point remark that we kept our discussion of the Buscher roles on a somewhat superficial level, and there exist several subtleties that have to be taken into account to show that the dual models are truly equivalent as conformal field theories. This was addressed in more detail in [98]. A generalization of the above approach to non-Abealian isometries can furthermore be found in [99].

Mirror Symmetry

While usually more intricate in nature, it is a common property of target-space dualities to draw connections between geometries that, at first glance, seem completely unrelated or even antithetic. A particularly important example for this is *Mirror Symmetry* [100, 58], which relates the complex and Kähler structures of Calabi-Yau manifolds and is extensively studied various fields of pure mathematics [58–60]. Interestingly, this highly complex duality could be traced back to simple T-duality transformations by using standard string-theoretic methods. This postulated equivalence is known as the SYZ-conjecture [61].

As we will see in the following sections, T-duality not only unifies different concepts geometry, but also gives rise to new structures which cannot be described in terms of the widely used frameworks. Much of this thesis is devoted to studying recently-developed extensions of field theory and geometry which allow for an integration of such *nongeometric* phenomena.

2.2.3 Nongeometric Fluxes

In this subsection we consider a simple example of a generalized flux background with nongeometric structures. The concept of nongeometry was first introduced in [70] and elaborated further in [71, 72], which also serve as the main references for our review of the topic. Some additional details and explanations are furthermore adopted from [73].

Three-Torus and H-Flux

Taking up our discussion in section 2.1.3, recall that there can exist non-trivial background fluxes through homological cycles of a compactification manifold. Considering the case of a three-dimensional torus T^3 with coordinates x, y, z and line element

$$ds^2 = dx^2 + dy^2 + dz^2 \,, \tag{2.2.22}$$

we allow for a non-trivial three-form flux $H = H_{xyz}\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ with

$$\int_{T^3} H = n \,, \tag{2.2.23}$$

which can be realized by choosing

$$H = \mathrm{d}B, \quad B = nz\, \mathrm{d}x \wedge \mathrm{d}y. \tag{2.2.24}$$

In order to model the topological properties of a torus, we furthermore assume the three directions to be "rolled up" by identifying

$$(x, y, z) \sim (x+1, y, z) \sim (x, y+1, z) \sim (x, y, z+1) \,. \tag{2.2.25}$$

Since no component of the metric or B depends on the coordinates x or y, the setting involves two isometries along the x- and y-directions.

Twisted Torus and geometric F-Flux

As discussed in the previous subsection, T-duality transformations in isometric spaces are described by the Buscher rules (2.2.20), which can be readily applied to the isometric x- and y-directions in this case [70–72]. Doing so for the former will map the line element and the B-field to

$$ds^2 = (dx - F^x{}_{yz}\, z\, dy)^2 + dy^2 + dz^2\,, \qquad B = 0 \tag{2.2.26}$$

with $F^x{}_{yz} = n$. This still shows some resemblance of the original structure, but with the flux-quantum n now appearing in the line element. A particular consequence of this is that the metric becomes globally ill-defined when naively identifying $z \sim z+1$. This can, however, be compensated for by introducing an additional shift of x by $F^x{}_{yz}\, y$, leading to the modified identifications

$$(x, y, z) \sim (x+1, y, z) \sim (x, y+1, z) \sim (x + F^x{}_{yz}\, y, y, z+1)\,. \tag{2.2.27}$$

Heuristically, this structure can be interpreted as a two-torus along the directions (x, y), which gets twisted as one moves around a circle parameterized by z. It thus defines a T^2 fibration over S^1, which is topologically distinct from T^3 and called a *twisted torus*. Since the metric is still globally well-defined, the object $F^x{}_{yz}$ is often referred to as a *geometric flux*.

T-fold and nongeometric Q-Flux

When performing an additional T-duality transformation along y, the line element and B-field are mapped to

$$ds^2 = \frac{1}{1+(Q_z{}^{xy}\, z)^2}(dx^2 + dy^2) + dz^2\,, \qquad B = \frac{Q_z{}^{xy}\, z}{1+(Q_z{}^{xy}\, z)^2} dx \wedge dy\,, \tag{2.2.28}$$

with $Q_z{}^{xy} = n$. Unlike the previous settings, there now does not exist any diffeomorphism which can relate the line element at z and $z+1$, and hence the metric is not globally well-defined (see also [73] for a detailed discussion). For this reason, the obtained structure is called a *(globally) nongeometric background*, and the object Q is accordingly classified as a *nongeometric flux*. Interestingly, the background can still be described in a manifold-like way by additionally allowing for T-duality transformations as transition functions between coordinate patches. Such structures were therefore assigned the name *T-folds* [101].

Nongeometric R-space

While only the x- and y-directions define isometries in our considered setting, the authors of the original works [70–72] suggested that another nongeometric object R^{xyz} might be created by performing an additional T-duality transformation along the z-direction. Such backgrounds are still poorly understood, and various arguments hint at structures which lack even a local geometric description and are related to nonassociative structures [102, 103]. They are therefore often refered to as *locally nongeometric backgrounds*.

T-duality Chain and Discussion

The subsequent creation of new fluxes and simultaneous modification of background (non-)geometries on a torus can be summarized in a T-duality chain [70–72],

$$\begin{array}{ccccccc} H_{xyz} & \overset{T_x}{\longleftrightarrow} & F^{x}{}_{yz} & \overset{T_y}{\longleftrightarrow} & Q_z{}^{xy} & \overset{T_z}{\longleftrightarrow} & R^{xyz} \\ & & & & & & \\ \text{3-form flux} & & \text{twisted torus} & & \text{T-fold} & & \text{non-associative} \\ \text{background} & & & & & & \text{structures.} \end{array} \tag{2.2.29}$$

We should at this point remark that the above example is supposed to serve only as a toy model, and there are various subtleties which require a more rigorous treatment. In particular, any results obtained from T-duality transformations of the present setting need to be treated with caution since the considered background does not solve the string-theoretical equations of motion. One the other hand, there exist numerous arguments supporting the assertion that the structures encountered in this example are indeed relevant for string theory. We will come back to this issue in the later chapters of this thesis. An extensive review of the topic can furthermore be found in [73].

2.3 Summary

Having discussed the basic concepts of string compactifications and the challenges arising from the presence of dualities, let us briefly summarize the most important aspects which will play a major role for the remainder of this thesis:

- Naive approaches to dimensional reduction of field or string theories often come with undesired massless scalar particles in four dimensions and are therefore insufficient to construct physically realistic models. This is called the problem of *moduli stabilization*.
- One way to address the problem of moduli stabilization is to allow for the presence of nontrivial background fields on the compactification space. The *flux* of such fields through homological cycles can give rise to an additional scalar potential term in the action which fixes parts of the moduli.
- *Dualities* connect different physical models or regimes which might seem unrelated at first glance. This can be useful to make settings accessible which would otherwise be hard to study.
- Despite their benefits, dualities also raise new questions regarding underlying structures of dual theories. In particular, their presence in string theory implies the existence of *nongeometric fluxes* which elude a description in the standard framework of differential geometry.

The following chapters will focus on both the utilization of dualities to address open problems in physics as well as the development of new frameworks to grasp their mathematical nature. We will next discuss dimensional reductions of type II string theories in more detail, before applying the newly-developed formalism of *double field theory* (DFT) to study a concrete example of a nongeometric flux compactification with all moduli stabilized. Following this, it will be shown how the physical equivalance of theories induced by dualities can be utilized to address the landscape problem and to estimate the proper number of physically-distinct vacua of so-called type IIB *orientifold compactifications*.

Chapter 3

Calabi-Yau Compactification and Type II Superstrings

We conclude this introductory part by delving deeper into the details of Calabi-Yau compactifications and type II superstring theories. The presented setting will serve as the starting point for our discussion of modern frameworks and recent developments in the following chapters.

3.1 Type II Supergravities

The first important building block for our upcoming analysis are the ten-dimensional low-energy limits of type IIA and IIB superstring theories. Since the focus of this work lies on the topic of dimensional reduction, we will go straight to the corresponding supergravity theories and refer the interested reader to the standard works [104, 95, 105] for a detailed discussion of supersymmetric worldsheet theories. Throughout this thesis, we will furthermore adopt the common convention and restrict our discussion to the bosonic part of the spectra, while the fermionic part is assumed to be accessible by supersymmetry.

3.1.1 Type IIA Supergravity

The bosonic field content of type IIA supergravity can be split into its so-called Neveau-Schwarz-Neveau-Schwarz (NS-NS) and Ramond-Ramond (R-R) sectors, which are named after the corresponding worldsheet boundary conditions they originate from. The former contains the ten-dimensional graviton $\hat{g}_{mn}$, the dilaton $\hat{\phi}$ and an antisymmetric rank-two-tensor $\hat{B}_{mn}$ called *Kalb-Ramond field*. The latter consists of a one-form field $\hat{C}_1$ and a three-form field $\hat{C}_3$. In the string frame, the action takes the form [106]

$$\begin{aligned} S^{(\mathrm{IIA})} = \int \, & e^{-2\hat{\phi}} \left(-\frac{1}{2} \hat{R}^{(10)} \star \mathbf{1}^{(10)} + 2\mathrm{d}\hat{\phi} \wedge \star \mathrm{d}\hat{\phi} - \frac{1}{4} \hat{H}_3 \wedge \star \hat{H}_3 \right) \\ & -\frac{1}{2} \left(\hat{F}_2 \wedge \star \hat{F}_2 + \hat{\tilde{F}}_4 \wedge \star \hat{\tilde{F}}_4 \right) + \mathcal{L}_{\mathrm{CS}}, \end{aligned} \tag{3.1.1}$$

where $\hat{R}^{(10)}$ denotes the ten-dimensional Ricci scalar and $\star\mathbf{1} = \sqrt{-\hat{g}^{(10)}}\,\mathrm{d}^{10}x$ is the ten-dimensional volume form. The differential form field strengths of same degree are summarized as

$$\hat{F}_2 = \mathrm{d}\hat{C}_1, \quad \hat{H}_3 = \mathrm{d}\hat{B}_2, \quad \hat{\tilde{F}}_4 = \mathrm{d}\hat{C}_3 - \hat{B}_2 \wedge \mathrm{d}\hat{C}_1, \tag{3.1.2}$$

and the topological Chern-Simons term is given by

$$\mathcal{L}_{\mathrm{CS}} = -\frac{1}{2}\left[\hat{B}_2 \wedge \mathrm{d}\hat{C}_3 \wedge \mathrm{d}\hat{C}_3 - \left(\hat{B}_2\right)^2 \wedge \mathrm{d}\hat{C}_3 \wedge \mathrm{d}\hat{C}_1 + \frac{1}{3}\left(\hat{B}_2\right)^3 \wedge \mathrm{d}\hat{C}_1 \wedge \mathrm{d}\hat{C}_1\right]. \tag{3.1.3}$$

3.1.2 Type IIB Supergravity

Analogously to the type IIA case, the bosonic spectrum of type IIB supergravity splits into its NS-NS and R-R sectors, with the former being identical to its IIA analogue and the latter containing three even-degree differential form fields $\hat{C}_0, \hat{C}_2, \hat{C}_4$. The action in the string frame reads [106]

$$\begin{aligned} S^{(\mathrm{IIB})} = \int e^{-2\hat{\phi}} &\left(-\frac{1}{2}\hat{R}^{(10)} \star \mathbf{1}^{(10)} + 2\mathrm{d}\hat{\phi} \wedge \star\mathrm{d}\hat{\phi} - \frac{1}{4}\hat{H}_3 \wedge \star\hat{H}_3\right) \\ &-\frac{1}{2}\left(\hat{F}_1 \wedge \star\hat{F}_1 + \hat{\tilde{F}}_3 \wedge \star\hat{\tilde{F}}_3 + \frac{1}{2}\hat{\tilde{F}}_5 \wedge \star\hat{\tilde{F}}_5\right) + \mathcal{L}_{\mathrm{CS}}, \end{aligned} \tag{3.1.4}$$

where

$$\hat{H}_3 = \mathrm{d}\hat{B}_2, \quad \hat{\tilde{F}}_3 = \mathrm{d}\hat{C}_2 - \hat{C}_0\mathrm{d}\hat{B}_2, \quad \hat{\tilde{F}}_5 = \mathrm{d}\hat{C}_4 - \frac{1}{2}\hat{C}_2 \wedge \mathrm{d}\hat{B}_2 + \frac{1}{2}\hat{B}_2 \wedge \mathrm{d}\hat{C}_2 \tag{3.1.5}$$

and

$$\mathcal{L}_{\mathrm{CS}} = -\frac{1}{2}\hat{C}_4 \wedge \hat{H}_3 \wedge \mathrm{d}\hat{C}_2. \tag{3.1.6}$$

Unlike its type IIA counterpart, this action has to be supplemented by a self-duality constraint $\hat{\tilde{F}}_5 = \star\hat{\tilde{F}}_5$. Taking this condition into account is a nontrivial task since a naive implementation into the action (3.1.4) would lead to a vanishing kinetic term of $\tilde{F}_5$ [107]. While there do exist more sophisticated ways to circumvent this problem [108], we here follow the common approach and impose the constraint by hand at a later point in our discussion.

Some further insights into the structure of type IIB theory can be obtained by transforming this action to the Einstein frame. Rearranging the fields into complex quantities $\hat{\tau} = \hat{C}_0 + ie^{-\hat{\phi}}$ and $\hat{\tilde{G}}_3 = \mathrm{d}\hat{C}_2 + \tau\hat{H}_3$, the action can be brought to the form [109]

$$\begin{aligned} S^{(\mathrm{IIB})} = \int \frac{1}{2} &\left(-\hat{R}^{(10)} \star \mathbf{1}^{(10)} - \frac{\mathrm{d}\hat{\tau} \wedge \star\mathrm{d}\hat{\bar{\tau}}}{2\left(\mathrm{Im}\hat{\tau}\right)^2} - \frac{1}{2}\frac{\hat{\tilde{G}}_3 \wedge \star\hat{\bar{\tilde{G}}}_3}{\mathrm{Im}\hat{\tau}} - \frac{1}{2}\hat{\tilde{F}}_5 \wedge \star\hat{\tilde{F}}_5\right) \\ &-\frac{1}{8}\frac{\hat{C}_4 \wedge \hat{\tilde{G}}_3 \wedge \hat{\bar{\tilde{G}}}_3}{\mathrm{Im}\hat{\tau}}. \end{aligned} \tag{3.1.7}$$

This formulation reveals a hidden symmetry of the action under under simultaneous transformations

$$\hat{\tau} \to \frac{a\hat{\tau} + b}{c\hat{\tau} + d}, \qquad \hat{\tilde{G}}_3 \to \frac{\hat{\tilde{G}}_3}{c\hat{\tau} + d}, \tag{3.1.8}$$

with $ad - bc = 1$. This corresponds to a $SL(2, \mathbb{R})$ analogue of the Montonen-Olive-type duality we encountered earlier in section 2.2.1 and reduces to $SL(2, \mathbb{Z})$ after taking into account quantum mechanical effects. The symmetry therefore reflects the self-duality of type IIB superstring theory under S-duality as depicted in figure 1.1.

3.2 Calabi-Yau Compactifications

As we have seen in section 2.1, the properties of an effective four-dimensional theory depend strongly on the geometry and topology of the chosen compactification manifold. An essential question arising in superstring compactifications is therefore whether the six extra dimensions have to be constrained in a certain way in order to obtain realistic four-dimensional models.

Beside obvious tasks like obtaining the correct particle content of the effective theory, a minimal requirement to achieve this goal is that some degree of supersymmetry is preserved at string scale. It was found in [62] that preservation of at least minimal supersymmetry can be ensured by restricting the choice of compactification manifolds to a very specific class called *Calabi-Yau manifolds*. This type of manifolds had been well-studied long before their relevance for string theory became known, and the fields of complex and Kähler geometry provided a whole toolbox which could be readily applied to string compactifications. We will devote the rest of this section to introducing the basic notions of Calabi-Yau geometry, which will eventually enable us to apply the concepts to type II superstring theories and generalize the ideas to nongeometric settings.

3.2.1 Calabi-Yau Manifolds

Due to the heavy mathematical content, we will here follow a top-down approach and start right on an abstract level before discussing the most important notions of Calabi-Yau geometry. For the remainder of this thesis, we adopt the following definition of a Calabi-Yau manifold:

Definition. A Calabi-Yau manifold of complex dimension N (or Calabi-Yau N-fold) CY_N is a compact Kähler manifold of complex dimension N satisfying one of the following equivalent conditions:

- CY_N admits a Kähler metric with global holonomy group $Hol(CY_N) = SU(N)$.
- CY_N admits a unique, globally defined, nowhere vanishing and covariantly constant holomorphic $(N, 0)$-form Ω.
- The canonical bundle of CY_N is trivial.

There exist several alternative definitions in literature. These include the two conditions of Ricci-flatness and vanishing first Chern class, which are, however, weaker than the above. It is furthermore common to relax the restriction of the holonomy group to $Hol\,(CY_N) \subseteq SU\,(N)$, thus allowing for any compactification which preserves some degree of supersymmetry rather than only those which break exactly three quarters of the supersymmetry in ten dimensions. We will here restrict our discussion to the latter, but it will become clear in the following chapters how most of the concepts can be generalized to proper subgroups of $SU\,(N)$. For practical reasons, we furthermore assume all Calabi-Yau N-folds CY_N to be connected in the following sections and chapters.

One hardship faced in Calabi-Yau geometry is that there exist only few nontrivial cases for which the form of the metric is known explicitly. However, many of the important physical properties of Calabi-Yau compactifications can be studied by utilizing only their topological structure and basic tools of complex and Kähler geometry. The following elaborations will heavily rely on these frameworks, and a brief overview is provided in appendix A.

Topology of Calabi-Yau Three-Folds

We learned in section 2.1.1 that the massless spectrum of a compactified theory can be obtained by studying the fields corresponding to zero modes of the internal Laplacian operator. A convenient way to perform dimensional reductions on a Calabi-Yau three-fold CY_3 is therefore to expand the ten-dimensional differential-form fields in terms of the de Rham cohomology bases of CY_3 and integrate the action over the internal space[1]. The Calabi-Yau property thereby forces strong constraints on the topology of a manifold and allows only for very specific combinations of Hodge numbers.

To begin, notice that complex conjugation and Hodge duality imply that the Hodge diamond of CY_3 has to be symmetric along its horizontal and vertical axes, giving rise to the conditions

$$\begin{aligned} h^{p,q} &= h^{q,p}\,, \\ h^{p,q} &= h^{3-p,3-q}\,, \end{aligned} \tag{3.2.1}$$

and fixing 10 of the 16 parameters. The dimension $h^{0,0}$ of the zeroth de Rham cohomology group can be trivially determined to equal one, fixing

$$h^{0,0} = h^{3,3} = 1, \tag{3.2.2}$$

and, by uniqueness of the holomorphic three-form $\Omega \in \Omega^{3,0}\,(CY_3)$ one obtains

$$h^{3,0} = h^{0,3} = 1. \tag{3.2.3}$$

Finally, a slightly more involved argument (see also [84] for a pedagogical discussion) shows that $SU\,(3)$ holonomy requires

$$h^{1,0} = h^{0,1} = 0, \tag{3.2.4}$$

[1]We should remark at this point that the true golden standard way of performing dimensional reductions is to integrate the ten-dimensional equations of motion over the internal space. While this is not necessarily equivalent to doing so for the action, we will nevertheless follow the latter method as it does lead to the correct results in the settings discussed in this chapter and is most commonly used in literature.

leaving a total of two free parameters, which can be arranged in a corresponding Hodge diamond

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} . \tag{3.2.5}$$

3.2.2 Moduli Space of Calabi-Yau Manifolds

Proceeding with phenomenological considerations, recall that in our second example of a dimensional reduction discussed in section 2.1.2, we found that the freedom to infinitesimally change the geometry of the compactification manifold gave rise to the presence of an undesired scalar particle in four dimensions. The problem of free moduli not only persists but actually worsens in naive Calabi-Yau compactifications. In order to acquire a deeper understanding of the issue, we next want to analyze the moduli space of Calabi-Yau manifolds. A detailed discussion of the topic can be found, e.g., in [110, 106, 84].

Heuristically, moduli in string compactifications can be interpreted as deformations that do not change the underlying structure of the compactification manifold in a "relevant" manner. Similar to small changes in the radius of a circle, there exist infinitesimal deformations of Calabi-Yau manifolds which do not spoil the Calabi-Yau property and therefore manifest as massless scalars unless stabilized by fluxes or other mechanisms.

We start our analysis by considering a ten-dimensional metric $\hat{g}_{mn}$ whose vacuum expectation value,

$$\langle \hat{g}_{mn} \rangle = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & \mathring{g}_{ij} \end{pmatrix}, \tag{3.2.6}$$

we assume to describe a spacetime manifold $\mathbb{R}^{1,3} \times CY_3$. Notice that this is indeed a valid ansatz as $\langle \hat{g}_{mn} \rangle$ satisfies the vacuum Einstein equations by Ricci-flatness of $\eta_{\mu\nu}$ and $\mathring{g}_{ij}$. We now proceed analogously to section 2.1.2 and consider small fluctuations $\delta\hat{g}_{mn}$ about the vacuum. Here, the purely external components $\delta g_{\mu\nu}$ give rise to the four-dimensional graviton and are not relevant to our discussion. Furthermore, components $\delta g_{\mu j}$ with mixed external and internal indices correspond to one-forms on CY_3 and therefore become massive in four dimensions. The physically important information on the moduli space is therefore encoded in the purely internal components δg_{ij}, and the moduli correspond precisely to physically admissible deformations which preserve the Calabi-Yau property of CY_3. A minimal requirement for this is that the deformed metric $\mathring{g}_{ij} + \delta g_{ij}$ still satisfies the vacuum Einstein equations,

$$R_{ij} = 0. \tag{3.2.7}$$

Fixing diffeomorphism invariance and expanding up to linear order in δg_{ij}, this condition can be brought to the the Lichnerowicz equation

$$\nabla_{CY_3 k} \nabla^k_{CY_3} \delta g_{ij} + 2R_{ikjl}\delta g^{kl} = 0, \tag{3.2.8}$$

where the covariant derivative ∇_{CY_3} and the raising of indices is to be understood with respect to the non-perturbed metric $\mathring{g}_{ij}$. Now, the Kähler property of CY_3 implies that all components of the Riemann tensor except for those with precisely one holomorphic and one antiholomorphic index in each antisymmetric pair of indices vanish. We thus find that the equations arising for variations with mixed and purely (anti-)holomorphic indices decouple, and the moduli space as a whole splits into two components,

$$M_{CY_3} = M_{KC} \times M_{CS}, \tag{3.2.9}$$

called the *Kähler-class* and the *complex-structure moduli spaces*. Here, the name of the former is due to the $(1,1)$ variations $\delta g_{a\bar{b}}$ corresponding to deformations of the Kähler form $J = i g_{a\bar{b}} \mathrm{d}z^a \wedge \mathrm{d}\bar{z}^{\bar{b}}$, for which the Lichnerowicz equation (3.2.8) reduces to the simple form

$$\triangle_{CY_3} \delta g_{a\bar{b}} = 0. \tag{3.2.10}$$

Employing a basis $\{\omega_{\mathsf{i}}\}$ of the $(1,1)$-Dolbeault cohomology group of $H^{1,1}(CY_3)$, the moduli can thus be expanded as

$$\delta g_{a\bar{b}} = v^{\mathsf{i}} \left(\omega_{\mathsf{i}}\right)_{a\bar{b}}, \quad \mathsf{i} = 1, \dots h^{1,1}, \tag{3.2.11}$$

where the coefficients v^{i} manifest as massless real scalar fields in four dimensions and are called *Kähler-class moduli.* It is common to slightly abuse notation and simply write $J = v^{\mathsf{i}}\omega_{\mathsf{i}}$ rather than denoting the variation explicitly. The exact meaning of such expressions is, however, commonly clear from the context.

An analysis of the complex-structure moduli space is less straightforward and shall only be summarized here, while the interested reader is referred to [84] for a short discussion and [111,112] for technical details. The essential insight is that the complex-structure moduli allow for a similar expansion

$$\delta g_{ab} = \frac{i}{\|\Omega\|^2} \overline{U}^{\mathsf{a}} \left(\overline{\chi}_{\mathsf{a}}\right)_{a\bar{c}\bar{d}} \Omega^{\bar{c}\bar{d}}{}_{b}, \tag{3.2.12}$$

where $\{\chi_{\mathsf{a}}\}$ denotes a basis of harmonic $(2,1)$-forms, Ω is the holomorphic three-form of CY_3 and $\|\Omega\|^2 = \frac{1}{3!}\Omega_{abc}\overline{\Omega}^{abc}$. The expansion coefficients U^{a} are massless complex scalar fields in four dimensions and are called *complex-structure moduli.* The origin of their name lies in a subtlety in the above transformations: Since any change in the purely (anti-)holomorphic components of the metric away from zero would spoil the Kähler property of CY_3, such transformations have to be compensated for by a non-holomorphic transformation. This will, however, alter the complex structure of the manifold, giving rise to the name complex-structure moduli. Such changes in the complex structure are closely tied to the holomorphic three-form Ω of CY_3, thus assigning it a similar role as J in the Kähler-class case. This analogy will be discussed more thoroughly in the next subsection.

3.2.3 Special Geometry of Moduli Spaces

An important tool to handle the intricate structure of Calabi-Yau compactifications is the framework of *special geometry.* We will next briefly review how the moduli spaces of type II theories give rise to *special Kähler structures* that can be utilized to facilitate the procedure of dimensional reduction.

Kähler Class Moduli Space

The first step in performing dimensional reductions of supergravities is to expand the ten-dimensional fields in terms of the cohomology bases of the internal space. Due to symmetry considerations that will become clear soon, it is convenient to split the cohomology groups into even and odd degrees. For the former, we define the bases by

$$\begin{aligned} \left\{\mathbf{1}^{(6)}\right\} &\in H^{0,0}\left(CY_3\right), \\ \left\{\omega_{\mathsf{i}}\right\} &\in H^{1,1}\left(CY_3\right), \\ \left\{\widetilde{\omega}^{\mathsf{i}}\right\} &\in H^{2,2}\left(CY_3\right), \\ \left\{\tfrac{1}{\mathcal{K}}\star\mathbf{1}^{(6)}\right\} &\in H^{3,3}\left(CY_3\right), \end{aligned} \qquad \text{with } \mathsf{i}=1,\dots h^{1,1} \tag{3.2.13}$$

where $\star\mathbf{1}^{(6)} = \sqrt{g_{CY_3}}\,\mathrm{d}x^6$ denotes the volume form and

$$\mathcal{K} = \frac{1}{3!}\int_{CY_3} J^3 \tag{3.2.14}$$

the total volume of CY_3. These can be further summarized by setting $\omega_0 := \mathbf{1}^{(6)}$ and $\widetilde{\omega}^0 := \frac{1}{\mathcal{K}}\star\mathbf{1}^{(6)}$ and using the collective notation

$$\begin{aligned} \omega_{\mathsf{I}} &= \left(\ \omega_0, \quad \omega_{\mathsf{i}}\ \right), \\ \widetilde{\omega}^{\mathsf{I}} &= \left(\ \widetilde{\omega}^0, \quad \widetilde{\omega}^{\mathsf{i}}\ \right). \end{aligned} \qquad \text{with } \mathsf{I}=0,\dots h^{1,1} \tag{3.2.15}$$

The bases are furthermore chosen to satisfy the normalization relations

$$\int_{CY_3} \omega_{\mathsf{I}}\wedge\widetilde{\omega}^{\mathsf{J}} = \delta_{\mathsf{I}}{}^{\mathsf{J}}. \tag{3.2.16}$$

In the context of string compactifications, it is now common to combine the Kähler moduli v^{i} and the expansion coefficients b^{i} of the internal Kalb-Ramond field to define a complexified Kähler form

$$\mathcal{J} = \left(b^{\mathsf{i}} + iv^{\mathsf{i}}\right)\omega_{\mathsf{i}} =: t^{\mathsf{i}}\omega_{\mathsf{i}}. \tag{3.2.17}$$

Notice that, in a sense, this symmetrizes the treatment of the Kähler class and complex-structure moduli spaces by assigning complex degrees of freedom to each of them. The complexified Kähler moduli can now be arranged in a $(h^{2,1}+1)$-dimensional vector

$$X^{\mathsf{I}} = \left(1, t^{\mathsf{i}}\right), \tag{3.2.18}$$

which in turn can be used to define a *holomorphic prepotential*

$$\mathsf{F} = -\frac{1}{3!}\frac{\mathcal{K}_{\mathsf{ijk}}X^{\mathsf{i}}X^{\mathsf{j}}X^{\mathsf{k}}}{X^0}. \tag{3.2.19}$$

Here, the objects

$$\mathcal{K}_{\mathsf{ijk}} = \int_{CY_3} \omega_{\mathsf{i}} \wedge \omega_{\mathsf{j}} \wedge \omega_{\mathsf{k}} \tag{3.2.20}$$

are called the *triple intersection numbers* of CY_3. Employing the shorthand notation $\mathsf{F}_{\mathsf{I}} = \frac{\partial}{\partial X^{\mathsf{I}}}\mathsf{F}$, the prepotential F and the holomorphic coordinates X^{I} uniquely define the Kähler potential K_{KC} by

$$e^{-K_{KC}} = -i\left(X^{\mathsf{I}}\overline{\mathsf{F}}_{\mathsf{I}} - \overline{X}^{\mathsf{I}}\mathsf{F}_{\mathsf{I}}\right) = 8\mathcal{K}. \tag{3.2.21}$$

Finally, K_{KC} gives rise to the metric

$$g_{\mathsf{ij}} := \frac{1}{4\mathcal{K}} \int_{CY_3} \omega_{\mathsf{i}} \wedge \star\omega_{\mathsf{j}} = \partial_{\mathsf{i}}\bar{\partial}_{\mathsf{j}} K_{KC} \tag{3.2.22}$$

for the complexified Kähler cone spanned by the moduli t^{i}. This characteristic is a defining property of *special Kähler manifolds*, which are well-established structures used to describe $\mathcal{N} = 2$ supersymmetric theories in four dimensions. We will only utilize the very basic notions of special geometry in our upcoming analysis, but would like to refer the interested reader to the works [113–115] for a detailed discussion of the topic.

Complex Structure Moduli Space

A notable feature of Calabi-Yau compactifications is that, despite their distinctive features, the very same structures used for the Kähler-class moduli space can also be applied to describe its complex-structure counterpart. We once more start by defining a collective basis

$$\left\{\alpha_{\mathsf{A}}, \beta^{\mathsf{A}}\right\} \in H^3\left(CY_3\right) \qquad \text{with } \mathsf{A} = 0, \dots h^{1,2}, \tag{3.2.23}$$

satisfying

$$\int_{CY_3} \alpha_{\mathsf{A}} \wedge \beta^{\mathsf{B}} = \delta_{\mathsf{A}}{}^{\mathsf{B}}, \tag{3.2.24}$$

this time spanning all nontrivial odd Dolbeault cohomology groups $H^{3,0}\left(CY_3\right)$, $H^{2,1}\left(CY_3\right)$, $H^{1,2}\left(CY_3\right)$ and $H^{0,3}\left(CY_3\right)$ of CY_3. Denoting the so-called periods of CY_3 by X^{A} and F_{A} the holomorphic three-form Ω can be expanded as

$$\Omega = X^{\mathsf{A}}\alpha_{\mathsf{A}} - \mathsf{F}_{\mathsf{A}}\beta^{\mathsf{A}}. \tag{3.2.25}$$

The periods are not independent, but can be obtained from a holomorphic prepotential similarly to the Kähler class case [110],

$$\mathsf{F}_{\mathsf{A}} = \frac{\partial}{\partial X^{\mathsf{A}}}\mathsf{F}\,. \tag{3.2.26}$$

The coordinates X^{A} are related to the complex-structure moduli (3.2.12) via $U^{\mathsf{a}} = \frac{X^{\mathsf{a}}}{X^0}$, and one can choose

$$X^{\mathsf{A}} = \left(1, U^{\mathsf{a}}\right). \tag{3.2.27}$$

The Kähler potential then takes the simple form

$$e^{-K_{CS}} = -i\left(X^{\mathsf{A}}\overline{\mathsf{F}}_{\mathsf{A}} - \overline{X}^{\mathsf{A}}\mathsf{F}_{\mathsf{A}}\right) = i\int_{CY_3} \Omega \wedge \overline{\Omega}, \tag{3.2.28}$$

thus allowing to write the moduli metric g_{ab} as

$$g_{\mathsf{ab}} := -\frac{i}{\int_{CY_3}\Omega\wedge\overline{\Omega}}\int_{CY_3}\chi_{\mathsf{a}}\wedge\bar{\chi}_{\mathsf{b}} = \partial_{\mathsf{a}}\bar{\partial}_{\mathsf{b}}K_{CS}. \tag{3.2.29}$$

3.2.4 Dimensional Reduction of Type II Theories

With the necessary tools of special geometry at hand, we now continue with a brief discussion of conventional Calabi-Yau compactifications of type IIA and IIB supergravities. As we will see in chapter 5, the setting considered here arises as a special case in the double field theory framework we will use for generalized flux backgrounds. We will therefore spare most of the technical steps and only sketch the key principles at this point. The computations of this subsection were first presented in [116–118], but we will keep the structure of our discussion close to the review provided in [69]. A more detailed analysis of type II compactifications and their technical details can be found, e.g., in the series of works [119–123] on the topic.

Throughout this section we will assume that the supergravity theories live on a ten-dimensional spacetime manifold $M^{1,3}\times CY_3$ which decomposes into an external four-dimensional spacetime $M^{1,3}$ and an internal Calabi-Yau three-fold CY_3.

NS-NS Sector

The field content of the NS-NS sector is identical for both theories and consists of the ten-dimensional metric $\hat{g}_{mn}$, the Kalb-Ramond field $\hat{B}_{mn}$ and the dilaton $\hat{\phi}$. The contribution of the former is thereby encoded by the ten-dimensional Ricci scalar $\hat{R}^{(10)}$ and the volume form $\star\mathbf{1} = \sqrt{-\hat{g}^{(10)}}\,\mathrm{d}^{10}x$. Dimensional reduction of the gravitational sector essentially leads to kinetic terms of the four-dimensional metric and the moduli,

$$\hat{g}_{mn} \to \left(g_{\mu\nu}, v^{\mathsf{i}}, U^{\mathsf{a}}\right)\,, \tag{3.2.30}$$

but requires careful treatment due to additional contributions arising from Weyl-rescalings [116, 117]. The Kalb-Ramond field can be expanded in terms of the cohomology bases of CY_3 as

$$\hat{B}_2 = B_2 + b^{\mathsf{i}}\omega_{\mathsf{i}}\,,$$

where B_2 and $b^{\mathsf{i}}\omega_{\mathsf{i}}$ denote the purely external respectively internal components, the latter of which combine with the v^{i} to build the complexified Kähler moduli $t^{\mathsf{i}} = b^{\mathsf{i}} + iv^{\mathsf{i}}$. To obtain the standard form of four-dimensional $\mathcal{N}=2$ supergravity, it is common to describe the external two-form field B_2 in terms of an axion a after integrating over CY_3. More generally, Poincaré duality implies that the physical contribution of any massless differential form field ω_p in D dimensions with $p \leq D-2$ can be formulated in terms of a dual $(D-p-2)$-form $\widetilde{\omega}_{D-p-2}$. This is due to the relation between their physical fields

F_{p+1} and $\widetilde{F}_{D-p-1}$ via the Hodge-star operator,

$$\begin{array}{ccc} \omega_p & \longrightarrow & \tilde{\omega}_{D-p-2} \\ \mathrm{d}\big\downarrow & & \big\downarrow\mathrm{d} \\ F_{p+1} & \xrightarrow{\;\star\;} & \widetilde{F}_{D-p-1}. \end{array} \tag{3.2.31}$$

For type IIA theory, the reformulated action can be obtained by adding a Lagrange multiplier $\mathrm{d}B_2 \wedge \mathrm{d}a$ and integrating out the field B_2. In the type IIB setting, B_2 is first dualized to a scalar s_1 which combines with some of the R-R fields to define an axion a. We will not elaborate this explicitly here, but a detailed discussion of the integration over CY_3 as well as the dualization of undesired four-dimensional fields can be found in the previously mentioned works [119–123].

Finally, the remaining dilaton $\hat{\phi}$ takes the role of a ten-dimensional scalar field, and integration over CY_3 is straightforward. We accordingly define the four-dimensional dilaton ϕ by the relation

$$e^{-2\phi} = \int_{CY_3} e^{-2\hat{\phi}} = \mathcal{K} e^{-2\hat{\phi}}\,. \tag{3.2.32}$$

R-R Sector – Type IIA

The type IIA R-R sector consists of a one-form field $\hat{C}_1$ and a three-form field $\hat{C}_3$. By triviality of the first de Rham cohomology group of CY_3, one only has to take care of those components with precisely zero, two or three internal indices. One can therefore expand[2]

$$\begin{aligned} \hat{C}_1 &= A_1^0 \omega_{\mathsf{0}}\,, \\ \hat{C}_3 &= A_3^0 \omega_{\mathsf{0}} + A_1^{\mathsf{i}} \wedge \omega_{\mathsf{i}} + A_0^{\mathsf{A}} \alpha_{\mathsf{A}} - A_{0\,\mathsf{A}} \beta^{\mathsf{A}}. \end{aligned} \tag{3.2.33}$$

Here, A_1^0 and A_1^{i} denote one-forms and A_3^0 a three-form in four-dimensions. A_0^{A} and $A_{0\,\mathsf{A}}$ correspond to the purely internal components of $\hat{C}_3$ and manifest as scalar fields in four dimensions. It will turn out convenient at a later point of this discussion to employ the notation $\xi^{\mathsf{A}} := A_0^{\mathsf{A}}$ and $\tilde{\xi}_{\mathsf{A}} := A_{0\,\mathsf{A}}$.

In respect of the standard formulation of four-dimensional $\mathcal{N} = 2$ supergravity, the theory contains another undesired degree of freedom $A_{3\,0}$ after integration over CY_3. It can be dualized in a similar way as the external component B_2 of the Kalb-Ramond field, with the dual scalar contributing to the cosmological constant. It was found in [119] that this scalar is related to a particular R-R flux, which is commonly set to zero in conventional Calabi-Yau compactifications. Performing the dualization explicitly, it can

[2] We here employ a somewhat unusual notation to prevent confusion with similar expressions appearing in a different context later in chapter 5.

then be shown that the latter condition in fact requires the complete covariant expression $\mathrm{d}A_3^0 - \mathrm{d}A_{1\,0} \wedge B_2$ to vanish. The overall bosonic field content can then be arranged in various multiplets, which for the type IIA setting are given by

- one gravitational multiplet $(g_{\mu\nu}, A_1^0)$,
- one tensor multiplet $(B_2, \phi, \xi^0, \tilde{\xi}_0)$,
- $h^{1,1}$ vector multiplets $(A_1^{\mathsf{i}}, t^{\mathsf{i}})$,
- $h^{1,2}$ hypermultiplets $(U^{\mathsf{a}}, \xi^{\mathsf{a}}, \tilde{\xi}_{\mathsf{a}})$.

Effective Action – Type IIA

Putting the previous steps together and switching to Einstein frame via Weyl-rescaling, one eventually arrives at the effective four-dimensional action

$$\begin{aligned} S^{(\mathrm{IIA})} = \int_{M^{1,3}} & -\frac{1}{2}R^{(4)} \star \mathbf{1}^{(4)} - g_{\mathsf{ij}}\mathrm{d}t^{\mathsf{i}} \wedge \star\mathrm{d}\bar{t}^{\mathsf{j}} - h_{\mathsf{uv}}\mathrm{d}q^{\mathsf{u}} \wedge \star\mathrm{d}q^{\mathsf{v}} \\ & + \frac{1}{2}\mathrm{Re}\mathcal{N}_{\mathsf{IJ}}F_2^{\mathsf{I}} \wedge F_2^{\mathsf{J}} + \frac{1}{2}\mathrm{Im}\mathcal{N}_{\mathsf{IJ}}F_2^{\mathsf{I}} \wedge \star F_2^{\mathsf{J}}, \end{aligned} \tag{3.2.34}$$

where $F_2^{\mathsf{I}} = \mathrm{d}A_1^{\mathsf{I}}$,

$$\begin{aligned} h_{\mathsf{uv}}\mathrm{d}q^{\mathsf{u}} \wedge \star\mathrm{d}q^{\mathsf{v}} = & \, \mathrm{d}\phi \wedge \star\mathrm{d}\phi + g_{\mathsf{ab}}\mathrm{d}U^{\mathsf{a}} \wedge \star\mathrm{d}\overline{U}^{\mathsf{b}} \\ & + \frac{e^{4\phi}}{4}\left[\mathrm{d}a - \left(\tilde{\xi}_{\mathsf{A}}\mathrm{d}\xi^{\mathsf{A}} - \xi^{\mathsf{A}}\mathrm{d}\tilde{\xi}_{\mathsf{A}}\right)\right] \wedge \star\left[\mathrm{d}a - \left(\tilde{\xi}_{\mathsf{A}}\mathrm{d}\xi^{\mathsf{A}} - \xi^{\mathsf{A}}\mathrm{d}\tilde{\xi}_{\mathsf{A}}\right)\right] \\ & - \frac{e^{2\phi}}{2}\left[\mathrm{Im}\mathcal{M}^{-1}\right]^{\mathsf{AB}}\left(\mathrm{d}\tilde{\xi}_{\mathsf{A}} - \mathcal{M}_{\mathsf{AC}}\mathrm{d}\xi^{\mathsf{C}}\right) \wedge \star\left(\mathrm{d}\tilde{\xi}_{\mathsf{B}} - \overline{\mathcal{M}}_{\mathsf{BD}}\mathrm{d}\xi^{\mathsf{D}}\right). \end{aligned} \tag{3.2.35}$$

and the objects g_{ij} and g_{ab} denote the metrics of the Kähler class respectively the complex-structure moduli space discussed in section 3.2.3. This resembles the standard formulation of four-dimensional $\mathcal{N} = 2$ supergravity, where the fields q^{u} can be interpreted as coordinates spanning a quaternionic manifold with metric h_{uv} [124]. The objects $\mathcal{N}_{\mathsf{IJ}}$ and $\mathcal{M}_{\mathsf{AB}}$ take the role of gauge coupling matrices and can be written in terms of the Kähler prepotentials (3.2.19) and (3.2.26) as

$$\begin{aligned} \mathcal{N}_{\mathsf{IJ}} &= \overline{\mathsf{F}}_{\mathsf{IJ}} + 2i\frac{\mathrm{Im}\left(\mathsf{F}_{\mathsf{IK}}\right)X^{\mathsf{K}}\mathrm{Im}\left(\mathsf{F}_{\mathsf{JL}}\right)X^{\mathsf{L}}}{X^{\mathsf{M}}\mathrm{Im}\left(\mathsf{F}_{\mathsf{MN}}\right)X^{\mathsf{N}}} \\ \mathcal{M}_{\mathsf{AB}} &= \overline{\mathsf{F}}_{\mathsf{AB}} + 2i\frac{\mathrm{Im}\left(\mathsf{F}_{\mathsf{AC}}\right)X^{\mathsf{C}}\mathrm{Im}\left(\mathsf{F}_{\mathsf{BD}}\right)X^{\mathsf{D}}}{X^{\mathsf{E}}\mathrm{Im}\left(\mathsf{F}_{\mathsf{EF}}\right)X^{\mathsf{F}}}. \end{aligned} \tag{3.2.36}$$

Similarly to our discussion in section 2.1.2, the free moduli of the model once more manifest as various massless scalar fields in four dimensions. In addition to phenomenological consequences, their structure has further implications reaching beyond string theory. To elaborate on this, we next consider the type IIB setting.

R-R Sector – Type IIB

The R-R sector of type IIB supergravity contains three even-degree differential form fields $\hat{C}_0, \hat{C}_2, \hat{C}_4$. Similar to the type IIA setting, these can be expanded as

$$\begin{aligned} \hat{C}_0 &= A_0^0 \omega_0 \,, \\ \hat{C}_2 &= A_2^0 \omega_0 + A_0^{\mathsf{i}} \omega_{\mathsf{i}} \,, \\ \hat{C}_4 &= A_2^{\mathsf{i}} \wedge \omega_{\mathsf{i}} + A_1^{\mathsf{A}} \wedge \alpha_{\mathsf{A}} - A_{1\,\mathsf{A}} \wedge \beta^{\mathsf{A}} + A_{0\,\mathsf{i}} \tilde{\omega}^{\mathsf{i}} . \end{aligned} \tag{3.2.37}$$

where the first subscript indices on the right-hand side once more indicate the degree of the corresponding differential form in four dimensions. Notice that we intentionally dropped the purely external component A_4^0 of $\hat{C}_4$ because its four-dimensional exterior derivative $\mathrm{d}A_4^0$ vanishes. The fields can again be arranged in various multiplets, given by

- one gravitational multiplet $(g_{\mu\nu}, A_1^0)$,
- one double-tensor multiplet $(B_2, A_2^0, \phi, A_0^0)$,
- $h^{1,2}$ vector multiplets $(A_1^{\mathsf{a}}, U^{\mathsf{a}})$,
- $h^{1,1}$ hypermultiplets $(v^{\mathsf{i}}, b^{\mathsf{i}}, A_0^{\mathsf{i}}, A_{0\,\mathsf{i}})$.

Similar to the previous cases, the type IIB R-R sector contains several fields after dimensional reduction which do not appear in the standard formulation of four-dimensioanl $\mathcal{N} = 2$ supergravity. Half of the degrees of freedom arising from $\hat{C}_4$ are eliminated by the self-duality constraint $\hat{\tilde{\tilde{F}}}_5 = \star \hat{\tilde{\tilde{F}}}_5$ mentioned in section 3.1.2, which in particular allows us to keep only the fields A_1^{A} and $A_{0\,\mathsf{i}}$ as physical degrees of freedom. The remaining two-form field A_2^0 can be dualized to a scalar s_2, and implementing the redefinitions

$$\begin{aligned} & a = s_1 + A_0^0 s_2 + 2A_{0\,\mathsf{i}} \left(A_0^{\mathsf{i}} - A_0^0 b^{\mathsf{i}} \right) , \\ & \xi^0 = A_0^0, \qquad\qquad \xi^{\mathsf{i}} = A_0^0 b^{\mathsf{i}} - A_0^{\mathsf{i}}, \\ & \tilde{\xi}_0 = s_2 + \frac{A_0^0}{6} \mathcal{K}_{\mathsf{ijk}} b^{\mathsf{i}} b^{\mathsf{j}} b^{\mathsf{k}} - \frac{1}{2} \mathcal{K}_{\mathsf{ijk}} b^{\mathsf{i}} b^{\mathsf{j}} A_0^{\mathsf{k}}, \qquad \tilde{\xi}_{\mathsf{i}} = -2A_{0\,\mathsf{i}} - \frac{A_0^0}{2} \mathcal{K}_{\mathsf{ijk}} b^{\mathsf{j}} b^{\mathsf{k}} + \mathcal{K}_{\mathsf{ijk}} b^{\mathsf{j}} A_0^{\mathsf{k}} \end{aligned} \tag{3.2.38}$$

eventually brings us to a field content structure similar to the type IIA case.

Effective Action – Type IIB

Once more integrating over CY_3 and performing a Weyl-rescaling, the final form of the reduced type IIB action reads

$$\begin{aligned} S^{(\mathrm{IIB})} = \int_{M^{1,3}} & -\frac{1}{2} R^{(4)} \star \mathbf{1}^{(4)} - g_{\mathsf{ab}} \mathrm{d}U^{\mathsf{a}} \wedge \star \mathrm{d}\overline{U}^{\mathsf{b}} - h_{\mathsf{uv}} \mathrm{d}q^{\mathsf{u}} \wedge \star \mathrm{d}q^{\mathsf{v}} \\ & + \frac{1}{2} \mathrm{Re} \mathcal{M}_{\mathsf{AB}} F_2^{\mathsf{A}} \wedge F_2^{\mathsf{B}} + \frac{1}{2} \mathrm{Im} \mathcal{M}_{\mathsf{AB}} F_2^{\mathsf{A}} \wedge \star F_2^{\mathsf{B}} , \end{aligned} \tag{3.2.39}$$

where $F_2^{\mathsf{A}} = \mathrm{d}A_1^{\mathsf{A}}$, and the quaternionic term is given by

$$\begin{aligned} h_{\mathsf{uv}}\mathrm{d}q^{\mathsf{u}} \wedge \star \mathrm{d}q^{\mathsf{v}} = \; & \mathrm{d}\phi \wedge \star \mathrm{d}\phi + g_{\mathsf{i}\bar{\mathsf{j}}}\mathrm{d}t^{\mathsf{i}} \wedge \star \mathrm{d}\bar{t}^{\bar{\mathsf{j}}} \\ & + \frac{e^{4\phi}}{4}\left[\mathrm{d}a - \left(\tilde{\xi}_{\mathsf{I}}\mathrm{d}\xi^{\mathsf{I}} - \xi^{\mathsf{I}}\mathrm{d}\tilde{\xi}_{\mathsf{I}}\right)\right] \wedge \star \left[\mathrm{d}a - \left(\tilde{\xi}_{\mathsf{I}}\mathrm{d}\xi^{\mathsf{I}} - \xi^{\mathsf{I}}\mathrm{d}\tilde{\xi}_{\mathsf{I}}\right)\right] \\ & - \frac{e^{2\phi}}{2}\left[\mathrm{Im}\mathcal{N}^{-1}\right]^{\mathsf{IJ}}\left(\mathrm{d}\tilde{\xi}_{\mathsf{I}} - \mathcal{N}_{\mathsf{IK}}\mathrm{d}\xi^{\mathsf{K}}\right) \wedge \star \left(\mathrm{d}\tilde{\xi}_{\mathsf{J}} - \bar{\mathcal{N}}_{\mathsf{JL}}\mathrm{d}\xi^{\mathsf{L}}\right). \end{aligned} \tag{3.2.40}$$

The gauge coupling matrices $\mathcal{N}_{\mathsf{IJ}}$ and $\mathcal{M}_{\mathsf{AB}}$ take the same form as in (3.2.36).

3.2.5 Mirror Symmetry

Since both effective four-dimensional actions (3.2.34) and (3.2.39) are written in the standard quaternionic formulation of $\mathcal{N} = 2$ supergravity, is is not surprising that they share a similar structure. Yet, it is somewhat remarkable that their content looks almost identical, up to swapped numbers of vector and hypermultiplets and an exchange of roles between the Kähler-class and complex-structure moduli. As we have learned in the course of our discussion in section 2.2, such analogies often imply that two theories are in fact physically equivalent. And indeed, assuming that the type IIA and IIB models are compactified on two Calabi-Yau manifolds CY_3 and $\widetilde{CY}_3$ satisfying

$$\begin{aligned} h^{1,1} &= \tilde{h}^{2,1}\,, \\ h^{2,1} &= \tilde{h}^{1,1}\,, \end{aligned} \tag{3.2.41}$$

the effective actions (3.2.34) and (3.2.39) can be mapped onto each other by the transformations

$$\begin{aligned} U^{\mathsf{a}} &\leftrightarrow t^{\mathsf{i}}\,, \\ g_{\mathsf{ab}} &\leftrightarrow g_{\mathsf{ij}}\,, \\ \mathcal{N}_{\mathsf{IJ}} &\leftrightarrow \mathcal{M}_{\mathsf{AB}}. \end{aligned} \tag{3.2.42}$$

Using the definitions of the gauge coupling matrices (3.2.19) and (3.2.26), one can show that this mapping extends to the corresponding prepotentials and holomorphic coordinates. This in turn implies that the Kähler potentials and, thus, the moduli spaces as a whole are exchanged.

This kind of highly nontrivial duality is also encountered in in heterotic string theories. It was assigned the name *Mirror Symmetry* due to the conditions (3.2.41) indicating that the Hodge diamonds CY_3 and $\widetilde{CY}_3$ are related by a reflection along their diagonal axes (see figure 3.1 for an illustration). It is today extensively studied in both theoretical physics and pure mathematics and enjoys fields of applications ranging from quantum field theory [125,60] and topological string theory [126] to as far as enumerative geometry [127, 128].

As discussed in the context of the SYZ-conjecture [61], there exist certain limits in which a Calabi-Yau three-fold can be written as a toroidal fibration over some base manifold. In such cases, the mirror transformations (3.2.42) can be traced back to a

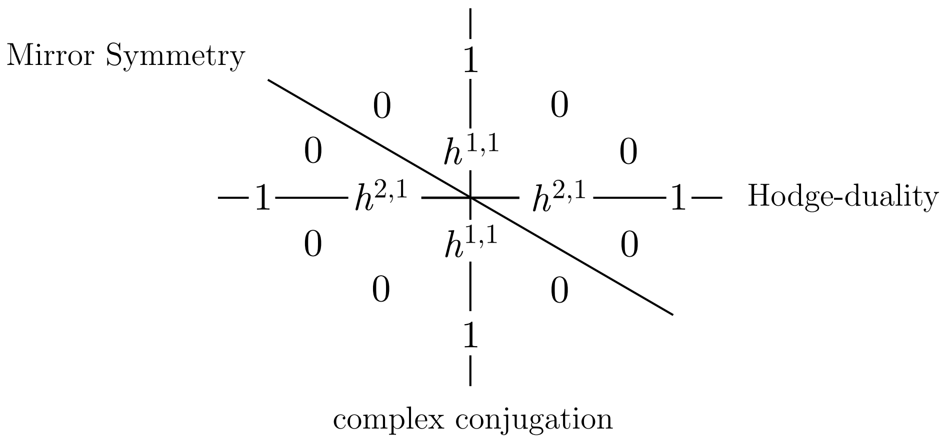


Figure 3.1: Effects of various symmetry and duality transformations on a Calabi-Yau Hodge diamond.

simple application of T-duality transformations on the T^3 fibers. The relation we derived here can thus be considered a more convoluted manifestation of the well-known T-duality relation between type IIA and IIB as well as HE and HO string theories.

3.3 Summary and Discussion

We again conclude the chapter with a summary of the most important results and a brief outlook on their relation to the upcoming parts of this thesis. The main insights of this chapter are the following:

- The low-energy limits of type IIA and IIB superstring theory are described by their corresponding supergravity theories (3.1.1) and (3.1.4). These commonly serve as the starting point for dimensional reductions of superstring theories.

- Phenomenological considerations force strong constraints on the structure of the compactification manifolds used in dimensional reductions of superstring theories. An important class of eligible internal spaces are Calabi-Yau manifolds, however, several relaxations of their defining conditions and more sophisticated constructions are possible.

- While the metric of Calabi-Yau manifolds is usually not known explicitly, dimensional reductions can be performed by utilizing mainly their topological properties. In addition, their moduli spaces are well-understood and can be described in the language of special Kähler geometry.

- The effective action of type IIA and IIB supergravity compactified on a Calabi-Yau three-fold takes the form (3.2.34) respectively (3.2.39) of four-dimensional $\mathcal{N} = 2$ supergravity. In this action, the complexified Kähler-class and complex-structure moduli give rise to $h^{1,1}$ respectively $h^{2,1}$ massless complex scalar fields.

- Mirror Symmetry defines a duality transformation between the effective four-dimensional theories arising from dimensional reductions of type IIA and IIB supergravity on Calabi-Yau three-folds with exchanged Hodge numbers $h^{1,1}$ and $h^{2,1}$. It is implied by the SYZ-conjecture [61] that this takes the role of a Calabi-Yau analogue of IIA $\leftrightarrow$ IIB T-duality known from toroidal compactifications.

While all these results mark important steps towards a better phenomenological understanding of string theory, many questions are still left open at this point. The presence of $h^{1,1} + h^{2,1}$ massless scalar fields (which can be as many as hundreds) in the four-dimensional theory is in contradiction to experiments and necessitates additional mechanisms to stabilize the moduli. Some progress in this context has been made by turning on background NS-NS and R-R fluxes [64, 65], which was worked out explicitly in [129, 119–121] for various settings. It was found that the presence of fluxes gives rise to an additional scalar potential term in the four-dimensional action which fixes parts of the moduli. In some cases this comes, however, at the cost of breaking IIA $\leftrightarrow$ IIB Mirror Symmetry, while at the same time neither setting is sufficient to stabilize all moduli.

An additional hint at the existence of yet-unknown structures comes from the four-dimensional viewpoint. It was found that there exists a more general class of *gauged supergravities* [130–132], which incorporates the known (ungauged) supergravities as special cases. Unlike the latter, there was however no known higher-dimensional origin of such gauged supergravities (see also figure 3.2 for a schematic illustration). While this not necessarily constitutes a flaw of the theory from a phenomenological perspective, the presence of missing links in the otherwise well-structured web of supergravities seemed, at least, unnatural. Interestingly, the effective four-dimensional theories found in [129, 119–121] showed characteristic features of gauged supergravities, thereby hinting a at a relation between flux backgrounds in ten dimensions and gaugings in four-dimensions. The constructed theories did, however, not take the most general form of gauged supergravities, once more implying that more general backgrounds need to be considered.

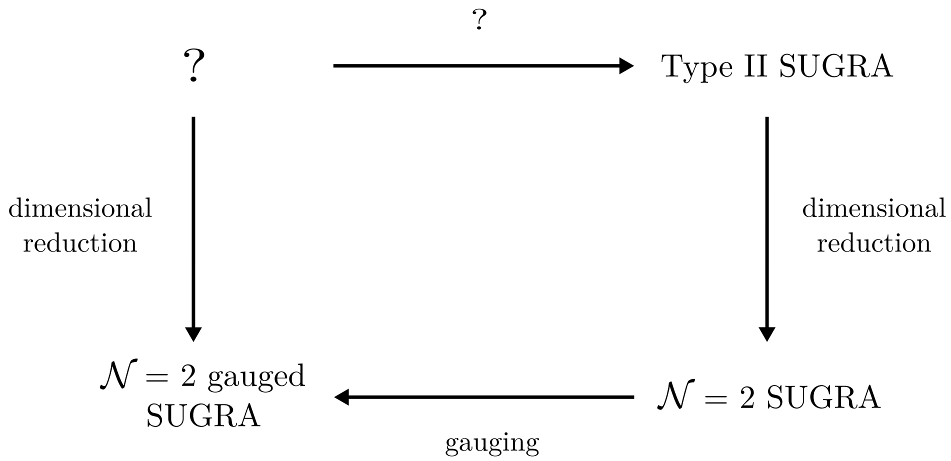


Figure 3.2: Schematic illustration of the relation between ten-dimensional and (gauged) four-dimensional supergravities. Conventional Calabi-Yau compactifications do not provide a ten-dimensional origin for gaugings in four dimensions.

A promising approach to restore the Web of Dualities, fix the remaining moduli and integrate gauged supergravities into string theory is the introduction of nongeometric backgrounds. As already addressed in section 2.2.3, T-duality transformations of background fluxes can give rise to structures which cannot be described in the language of differential geometry but still seem to constitute an inherent part of string theory [70–72]. We will next discuss how the framework of *double field theory* (*DFT*) can be utilized to implement T-duality naturally into the framework of field theory and allow for a mathematical treatment of nongeometric fluxes. This will eventually enable us to perform dimensional reductions of type IIA and IIB theory on generalized flux backgrounds explicitly in chapter 5.

Part III

Modern Developments and Applications

Chapter 4

Double Field Theory

In this chapter we introduce a T-duality covariant extension of conventional field theory, known by the name *double field theory* (*DFT*). It will be demonstrated how the notions of differential geometry can be generalized to incorporate T-duality as a manifest symmetry, thus unifying previously distinct T-dual theories in a single model. This will eventually enable us to formulate type II supergravities with nongeometric fluxes in a geometry-like way and perform dimensional reductions with all moduli stabilized.

The following discussion will provide a brief overview on the properties of double field theory and essentially summarizes the construction established in [133–137]. Some additional insights and explanations can furthermore be found in the reviews [138–142] as well as a more recent series of works [143–146] focusing on the structure and interpretation of solutions to extended field theories.

Throughout the chapter, we consider double field theory on a D-dimensional manifold M with Euclidean signature and do not employ any splitting into external and internal components. The concepts can be straightforwardly adjusted to ten-dimensional product spacetimes $M^{10} = M^{1,3} \times M^6$ by replacing the corresponding symmetry groups and adjusting signs to Lorentzian backgrounds.

4.1 Doubled Space and Geometry

We begin by briefly outlining the ideas of double field theory from a heuristic viewpoint before providing an overview on the concept of *extended geometries*, which builds the foundation of duality covariant frameworks.

4.1.1 Motivation and Basics

Doubled Coordinates and Generalized Tangent Bundle

In section 2.2.2 we saw that simple manifestations of T-duality commonly involve an exchange of roles or mixings between a string's momenta p_m and its winding numbers $\widetilde{p}^m$ along the compact directions. In order to integrate T-duality as a manifest symmetry into the framework of field theory, this property has to be accounted for by enhancing the fundamental point particles with "stringy" properties in a way that both quantities p_m

and $\widetilde{p}^m$ are treated on equal footing. In double field theory this is realized by introducing additional *winding coordinates* $\tilde{x}_m$ arranged in *doubled coordinates*

$$X^M = (x^m, \tilde{x}_m)\,,\ P_M = (p_m, \widetilde{p}^m) \qquad \text{with} \qquad m = 1, \dots D \text{ and } M = 1, \dots 2D, \quad (4.1.1)$$

while at the same time forcing certain *consistency constraints* on the model to retrieve the correct number of physical degrees of freedom. As implied by the notation, the winding coordinates $\tilde{x}_m$ are to be interpreted as variables conjugate to the corresponding winding numbers $\widetilde{p}^m$, which is in complete analogy to the relation between the normal coordinates x^m and the momenta p_m.

From a differential-geometric perspective, momenta on a spacetime manifold M are locally described by its tangent bundle TM, while the winding numbers are encoded in its cotangent bundle T^*M. In double field theory, the manifold M is therefore equipped with a *generalized tangent bundle*

$$E = TM \oplus T^*M, \quad (4.1.2)$$

defined as the direct sum of TM and T^*M. This object also builds the foundation for the formalism of *generalized geometry* [147, 148], which shares many of its mathematical structures with double field theory. The essential difference between the two frameworks is that double field theory additionally incorporates doubled versions of the dual objects themselves, thus allowing to explicitly formulate a T-duality covariant action.

Generalized Metric and Dilaton

Very similar to its effect on coordinates, T-duality also involves mixings between fields that were previously considered independent objects. This is nicely reflected in the Buscher rules (2.2.20) and (2.2.21), which serve as the motivation to arrange the type II NS-NS fields in a *generalized metric*

$$\mathcal{H}_{MN} = \begin{pmatrix} g_{mn} - B_{mp} g^{pq} B_{qn} & B_{mp} g^{pn} \\ -g^{mp} B_{pn} & g^{mn} \end{pmatrix}, \quad (4.1.3)$$

and a *generalized dilaton*

$$d = \phi - \frac{1}{4} \ln g. \quad (4.1.4)$$

The former is an element of the group $O(D,D)$, while the latter defines an $O(D,D)$ scalar. Notice that $O(D,D) := O(D,D;\mathbb{R})$ here denotes the continuous extension of the T-duality group $O(D,D;\mathbb{Z})$ encountered in toroidal compactifications with internal dimension D. It will be clarified in section 4.1.4 how such special cases of isometric T-dualities emerge naturally in this framework. The unification of g and B is also closely related to the generalized tangent bundle (4.1.2) in the sense that diffeomorphisms and (one-form) gauge transformations of B are unified in a single structure.

$O(D,D)$ Doubled Geometry

Our next aim is to utilize the above structures to define a T-duality covariant action for the type II NS-NS fields. For this purpose, we first need to introduce additional objects to generalize the framework of differential geometry. An important component is the $O(D,D)$ *invariant structure*

$$\eta_{MN} = \begin{pmatrix} 0 & \delta_m{}^n \\ \delta^m{}_n & 0 \end{pmatrix}, \qquad \eta^{MN} = \begin{pmatrix} 0 & \delta^m{}_n \\ \delta_m{}^n & 0 \end{pmatrix}, \tag{4.1.5}$$

which is used to raise and lower doubled indices. The generalized metric (4.1.3) satisfies the relations

$$\mathcal{H}^{MN} = \eta^{MP}\mathcal{H}_{PQ}\eta^{QN}, \qquad \mathcal{H}_{MP}\mathcal{H}^{PN} = \delta_M{}^N, \qquad \mathcal{H}_{MP}\eta^{PQ}\mathcal{H}_{QN} = \eta_{MN}, \tag{4.1.6}$$

and one can define a *generalized vielbein* $\mathcal{E}^A{}_M$ via

$$\mathcal{H}_{MN} = \mathcal{E}^A{}_M\mathcal{E}^B{}_N S_{AB}. \tag{4.1.7}$$

Employing a frame formalism, we then distinguish between curved spacetime indices $M,N,\ldots$ and flat tangent space indices $A,B,\ldots$. We will later consider the so-called *flux formulation* of double field theory [149,150], for which one chooses the flat generalized metric

$$S_{AB} = \begin{pmatrix} \delta_{mn} & 0 \\ 0 & \delta^{mn} \end{pmatrix}. \tag{4.1.8}$$

The generalized vielbein $\mathcal{E}^A{}_M$ can then be parameterized in terms of the vielbein $e^a{}_m$ of the spacetime metric g_{mn} as

$$\mathcal{E}^A{}_M = \begin{pmatrix} e^a{}_m & 0 \\ -e_a{}^k B_{km} & e_a{}^m \end{pmatrix}. \tag{4.1.9}$$

Notice that the matrix S_{AB} is invariant under action of the maximal (pseudo-)compact subgroup $O(D)\times O(D)$ of $O(D,D)$. This implies that the generalized metric $\mathcal{H}_{MN}$ is invariant under local $O(D)\times O(D)$ transformations and therefore parameterizes the symmetric coset space $O(D,D)/O(D)\times O(D)$. This is in complete analogy to general relativity, where global reparameterization invariance and local Lorentz invariance give rise to the corresponding coset space $GL(1,D-1)/SO(1,D-1)$.

Implementing T-duality

To integrate T-duality as a manifest symmetry into the framework, the doubled coordinates are organized in the fundamental representation of $O(D,D)$ transforming as

$$X^M \to h^M{}_N X^N, \quad h \in O(D,D). \tag{4.1.10}$$

The generalized metric $\mathcal{H}_{MN}$ defines an element of $O(D,D)$, while the generalized dilaton d is an $O(D,D)$ scalar. One therefore obtains the transformation rules

$$\mathcal{H}_{MN}(X) \to h_M{}^P h_N{}^Q \mathcal{H}_{PQ}(hX), \quad d(X) \to d(hX), \quad h \in O(D,D) \tag{4.1.11}$$

under $O(D,D)$ rotations. As mentioned in [139], these transformations correctly reduce to the Buscher rules (2.2.20, 2.2.21) when h corresponds to T-duality transformations along an isometric direction. A major advantage of double field theory is that it additionally provides a natural description of T-duality along non-isometric directions, which can map between configurations depending on the normal coordinates and such depending on (parts of) the dual coordinates [151–155, 101]. This is in stark contrast to the approach we presented earlier, which breaks down in such instances. The reason for this will become clear in light of the previously mentioned consistency constraints, which will be discussed next.

4.1.2 Generalized Diffeomorphisms and Lie Derivative

In the previous subsection, we built the foundation of double field theory by arranging the fields and the coordinates in representations of the group $O(D,D)$. Similar to general relativity, the global $O(D,D)$ symmetry of double field theory extends to a local one under *generalized diffeomorphisms*. The next step is therefore to adjust the basic notions of differential geometry to construct a suitable $O(D,D)$-based framework of doubled geometry. This will eventually allow us to derive consistency constraints for double field theory which help to retrieve the correct number of physical degrees of freedom. The following discussion will mainly expand on the review provided in [141].

Diffeomorphisms and Lie Derivative in Differential Geometry

We start by briefly reviewing the notions of diffeomorphisms and Lie derivatives in conventional differential geometry, which will then be generalized to the $O(D,D)$ setting. Let U define a vector living in D-dimensional spacetime. The transformation of an arbitrary tensor T under infinitesimal diffeomorphisms generated by U is described by the Lie derivative L. As a tensor derivation, L is uniquely defined by its action on scalars S and vectors V, and the transformation laws for all remaining tensors are determined by the Leibniz rule. Its action on scalars is given by the transport term

$$L_U S = U^m \partial_m S \tag{4.1.12}$$

and the action on vectors by the commutator

$$L_U V^m = [U,V]^m = U^n \partial_n V^m - V^n \partial_n U^m, \tag{4.1.13}$$

where the first term again describes a transportation and the second term defines a $GL(D)$ transformation of the components V^n. As can be shown by direct computation, the algebra of diffeomorphisms closes,

$$[L_U, L_V] = L_{[U,V]}\,. \tag{4.1.14}$$

This property in particular ensures that two successive gauge transformations acting on a given field produce a new gauge transformation of the same field [139]. It is therefore crucial to ensure consistency of a physical model built on top of the framework.

Generalized Diffeomorphisms and Lie Derivative in Doubled Geometry

The essential idea of doubled geometry is to extend the concepts of general relativity such that the global and local symmetry groups can be readily replaced by their double field theory counterparts. This motivates the definition of a *generalized Lie derivative* $\mathcal{L}$ with [133, 134, 156]

$$\mathcal{L}_U V^M = U^N \partial_N V^M - \left(V^N \partial_N U^M - \eta^{MP} \eta_{NQ} V^N \partial_P U^Q\right) + \lambda\left(V\right) \partial_N U^N V^M, \quad (4.1.15)$$

to encode the gauge algebra of double field theory. Here, the first term again describes the transport term, and the expression in the brackets is the $O\left(D, D\right)$ counterpart of the second term in (4.1.13). The definition also includes an additional weight term for the sake of generality, but we will mainly consider settings with vanishing weights in this chapter. Similarly to vectors, the generalized Lie derivative for scalars S reads

$$\mathcal{L}_U S = U^M \partial_M S + \lambda\left(S\right) \partial_N U^N S. \quad (4.1.16)$$

Analogously to differential geometry, the definition of the generalized Lie derivative is uniquely determined by its action on vectors and scalars and naturally extends to arbitrary tensors by requiring satisfaction of the Leibniz rule [136]

$$\mathcal{L}_U \left(A^{N_1 \dots}_{M_1 \dots} B^{Q_1 \dots}_{P_1 \dots}\right) = \left(\mathcal{L}_U A^{N_1 \dots}_{M_1 \dots}\right) B^{Q_1 \dots}_{P_1 \dots} + A^{N_1 \dots}_{M_1 \dots} \left(\mathcal{L}_U B^{Q_1 \dots}_{P_1 \dots}\right). \quad (4.1.17)$$

One can then check that $\mathcal{L}_U$ also preserves the $O\left(D, D\right)$ invariant structure,

$$\mathcal{L}_U \eta_{MN} = 0, \quad (4.1.18)$$

as desired.

4.1.3 Algebra of Generalized Diffeomorphisms and Consistency Constraints

A minimal requirement for $O\left(D, D\right)$ geometry to define a consistent theory is that the generalized Lie derivative gives rise to a closed algebra. Unlike its differential-geometric counterpart (4.1.14), this condition is not satisfied in the generic case. Considering the common case of vectors with vanishing weight, the commutator of two generalized Lie derivatives generated by U and V takes the form [141]

$$\begin{aligned}\left[\mathcal{L}_U, \mathcal{L}_V\right] W^M = \mathcal{L}_{[\![U,V]\!]_{\mathsf{C}}} W^M + \eta^{RS} \eta_{PQ} \Big[&(\partial_R U^P \partial_S V^M - \partial_R U^M \partial_S V^P) W^Q \\ &+ \frac{1}{2} (U^Q \partial_S V^P - V^Q \partial_S U^P) \partial_R W^M \Big]. \end{aligned} \quad (4.1.19)$$

where the C-*bracket* $[\![\cdot, \cdot]\!]_{\mathsf{C}}$ of two generalized vectors appearing on the right-hand side is defined by

$$[\![U, V]\!]^M_{\mathsf{C}} := U^N \partial_N V^M - V^N \partial_N U^M - \frac{1}{2} \eta^{MN} \eta_{PQ} \left(U^P \partial_N V^Q - V^P \partial_N U^Q\right). \quad (4.1.20)$$

In order to restore closure of the algebra, further constraints need to be introduced. The best-studied approach to do so is by imposing the so-called *section condition*

$$\eta^{MN}\partial_M \otimes \partial_N = 0\,, \tag{4.1.21}$$

where the differential operators can act on any field, gauge parameter or product of such. Since each extra term in (4.3.7) involves two derivatives ∂_R, ∂_S contracted with η^{RS}, this condition causes the undesired contribution to vanish, and the relation (4.3.7) reduces to a closed algebra

$$[\mathcal{L}_U, \mathcal{L}_V] = \mathcal{L}_{[\![U,V]\!]_{\mathsf{C}}} \tag{4.1.22}$$

governed by the C-bracket. Notice that this operator does not generically satisfy the Jacobi identity, but has a non-vanishing Jacobiator

$$J\,(U,V,W)_{\mathsf{C}} = [\![[\![U,V]\!]_{\mathsf{C}}\,, W]\!]_{\mathsf{C}} + [\![[\![V,W]\!]_{\mathsf{C}}\,, U]\!]_{\mathsf{C}} + [\![[\![W,U]\!]_{\mathsf{C}}\,, V]\!]_{\mathsf{C}}\ , \tag{4.1.23}$$

for which one finds

$$J\,(U,V,W)^M_{\mathsf{C}} = \eta^{MN}\partial_N \left[\eta_{PQ}\left([\![U,V]\!]^P_{\mathsf{C}}\, W^Q + [\![V,W]\!]^P_{\mathsf{C}}\, U^Q + [\![W,U]\!]^P_{\mathsf{C}}\, V^Q\right)\right]\ . \tag{4.1.24}$$

Such terms of the form

$$\widetilde{U}^M = \eta^{MN}\partial_N S \tag{4.1.25}$$

with some arbitrary scalar S generate only trivial gauge transformations if the section condition (4.1.21) is assumed to hold. They are therefore often referred to as *trivial gauge parameters* in literature. In the present context, the presence of this structure in particular ensures that the additional term (4.1.24) preventing the C-bracket from satisfying the Jacobi generates zero transformations [140].

Another important structure encountered in doubled geometry is the so-called D-*bracket* [136],

$$[\![U,V]\!]^M_{\mathsf{D}} := \mathcal{L}_U V^M = [\![U,V]\!]^M_{\mathsf{C}} + \frac{1}{2}\eta^{MN}\partial_N\left(\eta^{PQ}U_P V_Q\right), \tag{4.1.26}$$

which takes the role of a doubled-geometry analogue of the standard Lie-bracket. Unlike the C-bracket, this operator satisfies the Jacobi identity, but is not antisymmetric. However, the second term in (4.1.26) again defines a trivial gauge parameter, and the properies of the Lie-bracket are effectively restored under the section condition. In this case, the $\mathsf{C}-$ and D-bracket generate the same generalized Lie derivative, and the operators reduce to the Courant bracket [157] respectively the Dorfman bracket [158] known from generalized geometry.

4.1.4 Interpretation of Consistency Constraints

To better understand the structure of doubled spacetime, it is helpful to elaborate further on the yet abstract section condition (4.1.21), which in fact incorporates two distinct constraints. Assuming that Φ and Ψ can denote any field or gauge parameter, one distinguishes between the *weak constraint*

$$\eta^{MN}\partial_M\partial_N\Phi = 0 \tag{4.1.27}$$

and the *strong constraint*

$$\eta^{MN}\partial_M\Phi\partial_N\Psi = 0, \tag{4.1.28}$$

In literature, the latter is often equated with the term *section condition* in the narrower sense, but we here adopt the conventions of [140, 143] and make a distinction between both terms in the following discussion.

The essential statement of the weak and strong constraint is that fields and gauge parameters are only allowed to depend on either the original coordinate x^m or the dual coordinate $\tilde{x}_m$ for any given m, but not on both. The section condition is therefore often interpreted heuristically as choosing a D-dimensional "section" through the $2D$-dimensional spacetime to form the physical spacetime, and different solutions can be thought of as different T-duality frames which are rotated into each other by $O(D, D)$-transformations [151]. In many instances, choosing a physical section is equivalent to solving the strong constraint (4.1.28) in a particular way, but there exist settings in which this analogy does not hold. This was first elaborated in more detail in [143], and we will as well distinguish between the section condition and the strong constraint in this thesis.

To get some intuition, let us note that one obvious solution to the section condition is a model which depends only on the normal coordinates x^m. This leads to a supergravity formulation in terms of Hitchin's generalized geometry [147], in which only the tangent space remains doubled [159–161]. Similarly, a T-dual model can be obtained by choosing the dual coordinates $\tilde{x}_m$ as physical degrees of freedom, and numerous intermediate cases arise from dependencies on subsets of both coordinates.

Section Condition, Strong Constraint and Isometric T-Duality

An interesting insight which can be obtained from the separate treatment of the strong constraint and the section condition is how isometric T-duality arises naturally in the framework of double field theory [143]. Solving the strong constraint effectively requires picking one of each dual pair of coordinates and rendering the complete content of the theory independent of the respective other coordinate. If there are no isometries present, this is done in a canonical way by choosing D of the $2D$ coordinates as physical degrees of freedom, and the strong constraint becomes equivalent to the section condition. However, if there are isometries present, the field content may automatically depend on fewer coordinates than required by the strong constraint. In such cases, different choices of sections can lead to the same overall coordinate dependence, and there is an ambiguity in which D of the $2D$ coordinates are identified as physical. In the supergravity picture, these different sections can be precisely identified as different frames related by isometric T-duality. This has been elaborated explicitly for several examples, and the interested reader is referred to the works [143–145] for more details on the topic.

Weak Constraint

The weak constraint is generally less understood, and finding a physical interpretation is not as straightforward. Nevertheless, some interesting insights can be gained by considering the weak constraint from a string-theoretical perspective. Due to the relation between the doubled coordinates and their conjugate momenta or winding numbers (4.1.1), the weak constraint (4.1.27) essentially encodes the statement

$$p_m \widetilde{p}^m = 0, \tag{4.1.29}$$

which is precisely the level matching condition (2.2.11) for closed strings with equal number of left- and right-moving oscillator modes. It is indeed often stated that the weak constraint has its origins in the level-matching condition, but some caution is needed when making this relation explicit as there exist several subtleties which have to be properly taken into account. More details on this issue are discussed in [139]. Some more recent developments in weakly-constrained double field theory are furthermore presented in [162].

Relaxations

The section condition provides valuable insights into the structure of double field theory, but is only of limited use as the resulting theories do not countain any additional physical content beyond conventional supergravities. Several works have therefore focused on possible relaxations of the section condition. In [149, 150, 163–166] a so-called *Scherk-Schwarz ansatz* is used to construct solutions depending on both the normal and the winding coordinates. Compactifications of such models lead to lower-dimensional gauged supergravities. Double field theory thereby provides a purely geometric origin for gaugings which would arise from nongeometric fluxes in the conventional approach. Chapter 5 of this thesis will in large part build upon a similar approach, with the major aim being to explicitly perform dimensional reductions of double-field-theoretic extensions of type II supergravities.

4.2 Action and Equations of Motion

Having discussed the basic notions of $O(D, D)$ geometry, we are now able to formulate a doubled analogue of supergravity theories in which T-duality is incorporated as a manifest symmetry. We will here do this for the bosonic NS-NS sector of type II theories and elaborate on the R-R sector later throughout our discussion in chapter5. The following review is mainly based on the analysis of [144].

4.2.1 Action

The doubled NS-NS sector action is commonly written in an Einstein-Hilbert-like form [136, 164]

$$S = \int \mathrm{d}^{2D} X \, e^{-2d} \, \mathcal{R}. \tag{4.2.1}$$

Here, d denotes the generalized dilaton defined in (4.1.4), and the *generalized Ricci scalar* $\mathcal{R}$ is given by

$$\begin{aligned}\mathcal{R} = & \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{PQ}\partial_N\mathcal{H}_{PQ} - \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{PQ}\partial_P\mathcal{H}_{NQ} \\ & +4\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} - 4\mathcal{H}^{MN}\partial_M d\partial_N d + 4\partial_M\mathcal{H}^{MN}\partial_N d \\ & +\frac{1}{2}\eta^{MN}\eta^{PQ}\partial_M\mathcal{E}^A{}_P\partial_N\mathcal{E}^B{}_Q S_{AB},\end{aligned} \tag{4.2.2}$$

where we used the generalized metric (4.1.3) as well as the generalized vielbein from (4.1.7 - 4.1.9). As pointed out in [144], $\mathcal{R}$ only transforms as an $O(D,D)$ scalar if the section condition is imposed. This formulation is a slight generalization of the original double field theory action [133], where the third line in (4.2.2) is not included and $\mathcal{R}$ becomes a proper $O(D,D)$ scalar. The presence of the extra term is, however, crucial to obtain the correct gauged supergravity actions from Scherk-Schwarz reductions of double field theory [149, 150, 163–166].

As a consistency check, one can consider a solution to the strong constraint in which the complete content of the theory is independent of the winding coordinates. In this setting, the action (4.2.1) reduces to

$$S \overset{\tilde{\partial}=0}{\longrightarrow} \int \mathrm{d}^D x\, \sqrt{g} e^{-2\phi}\left(R + 4\partial_m\phi\partial^m\phi - \frac{1}{12}H^2\right), \tag{4.2.3}$$

where R denotes the Ricci scalar of D-dimensional Riemannian geometry, ϕ is the dilaton and $H = \mathrm{d}B$ is the field strength of the Kalb-Ramond field. This precisely reproduces the NS-NS sector of type II supergravities, and analogous structures can be obtained from different solutions to the strong constraint. Notice that the positive sign in front of the determinant g is due to the Euclidean signature of the metric.

4.2.2 Equations of Motion

The equations of motion in double field theory can be derived in a similar way as in conventional field theories, but some additional attention needs to by paid to its $O(D,D)$-based structure. Varying the action with respect to the generalized dilaton d yields [136]

$$\delta S = \int \mathrm{d}^{2D}X\, e^{-2d}\left[(-2\mathcal{R}) + (\text{total derivatives})\right]\delta d. \tag{4.2.4}$$

This expression has to vanish for all δd, and one obtains the equation of motion

$$\mathcal{R} = 0. \tag{4.2.5}$$

For the generalized metric, caution is needed. One might be tempted to take the same route and vary the action with respect to the generalized metric $\mathcal{H}_{MN}$, obtaining [136, 164, 144]

$$\delta S = \int \mathrm{d}^{2D}X\, e^{-2d} K_{MN}\delta\mathcal{H}^{MN}, \tag{4.2.6}$$

with

$$
\begin{aligned}
K_{MN} = &\frac{1}{8}\partial_M \mathcal{H}^{PQ}\partial_N \mathcal{H}_{PQ} + 2\partial_M\partial_N d \\
&+ \Big(\partial_Q - 2\partial_Q d\Big)\left[\mathcal{H}^{PQ}\left(\partial_{(\underline{M}} H_{\underline{N})P} - \frac{1}{4}\partial_P \mathcal{H}_{MN}\right)\right] \\
&+\frac{1}{4}\Big(\mathcal{H}^{PQ}\mathcal{H}^{RS} - 2\mathcal{H}^{PS}\mathcal{H}^{QR}\Big)\partial_P \mathcal{H}_{MR}\partial_Q \mathcal{H}_{NS} \\
&- \eta^{PQ}\eta^{RS}\left(\partial_P d \partial_Q \mathcal{E}^A{}_R - \frac{1}{2}\partial_P\partial_Q \mathcal{E}^A{}_R\right)\mathcal{H}_{(\underline{N}T}\mathcal{E}^T{}_A \mathcal{H}_{\underline{M})S}.
\end{aligned} \tag{4.2.7}
$$

Naively, the equations of motion could then be obtained by setting K_{MN} to zero. However, there is a pitfall here: The generalized metric $\mathcal{H}_{MN}$ is parameterized by the D-dimensional spacetime metric g_{mn} and Kalb-Ramond field B_{mn}, which encode only $\frac{1}{2}D\,(D+1) + \frac{1}{2}D\,(D-1) = D^2$ degrees of freedom. On the contrary, setting the symmetric object K_{MN} to zero gives rise to $2D^2 + D$ equations of motion. This contradiction occurs because the generalized metric $\mathcal{H}_{MN}$ is constrained to parameterize the coset space $O\,(D,D)\,/O\,(D)\times O\,(D)$, and not all of the considered variations $\delta\mathcal{H}_{MN}$ can be realized this way.

In the original work [136], this problem was resolved by imposing the condition (4.1.6) by hand to ensure that variations of $\mathcal{H}_{MN}$ do not alter its $O\,(D,D)$ structure. We will here follow a different approach presented in [144], which is easier to generalize to other duality groups. In this ansatz, the generalized metric $\mathcal{H}_{MN}$ is separately varied with respect to the spacetime metric g_{mn} and the Kalb-Ramond field B_{mn}, and $O\,(D,D)$ covariance is restored at the end of the computation. The first step then yields

$$
\delta S = \int \mathrm{d}^{2D}X\, e^{-2d} K_{MN}\left(\frac{\delta\mathcal{H}^{MN}}{\delta g_{pq}}\delta g_{pq} + \frac{\delta\mathcal{H}^{MN}}{\delta B_{pq}}\delta B_{pq}\right). \tag{4.2.8}
$$

Using the relations

$$
\frac{\delta g_{mn}}{\delta g_{pq}} = \delta_{\underline{m}}^{(\underline{p}}\delta_{\underline{n}}^{\underline{q})}, \qquad \frac{\delta g^{mn}}{\delta g_{pq}} = -g^{m(\underline{p}}g^{\underline{q})n}, \qquad \frac{\delta B_{mn}}{\delta B_{pq}} = \delta_{\underline{m}}^{[\underline{p}}\delta_{\underline{n}}^{\underline{q}]}, \tag{4.2.9}
$$

this can be written out as

$$
\begin{aligned}
\delta S = \int \mathrm{d}^{2D}X\, e^{-2d}\Big\{&\Big[-K_{mn}g^{m(\underline{p}}g^{\underline{q})n} + 2K_m{}^n g^{m(\underline{p}}g^{\underline{q})r}B_{rn} \\
&+K^{mn}\Big(\delta_{\underline{m}}^{(\underline{p}}\delta_{\underline{n}}^{\underline{q})} + B_{mr}g^{r(\underline{p}}g^{\underline{q})s}B_{sn}\Big)\Big]\delta g_{pq} \\
&+\Big[-2K_m{}^n g^{mr}\delta_{\underline{r}}^{[\underline{p}}\delta_{\underline{n}}^{\underline{q}]} - 2K^{mn}B_{mr}g^{rs}\delta_{\underline{s}}^{[\underline{p}}\delta_{\underline{n}}^{\underline{q}]}\Big]\delta B_{pq}\Big\}\,.
\end{aligned} \tag{4.2.10}
$$

Inserting the definition (4.1.3) of the generalized metric and expanding the antisymmetrizing brackets, this can be reformulated as

$$
\begin{aligned}
\delta S = \int \mathrm{d}^{2D}X\, e^{-2d}\Big\{&\Big[-K_{mn}\mathcal{H}^{mp}\mathcal{H}^{qn} + 2K_m{}^n\mathcal{H}^{mp}\mathcal{H}^q{}_n + K^{mn}\Big(\delta_m^p\delta_n^q - \mathcal{H}_m{}^p\mathcal{H}^q{}_n\Big)\Big]\delta g_{pq} \\
&-\Big(K_m{}^n\mathcal{H}^{mr} + K^{mn}\mathcal{H}_m{}^r\Big)\Big(\delta_r^p\delta_n^q - \delta_r^q\delta_n^p\Big)\delta B_{pq}\Big\}\,.
\end{aligned} \tag{4.2.11}
$$

One can now utilize of the $O(D,D)$ invariant structure η_{MN} defined in (4.1.5) to revert to doubled indices and obtain

$$\begin{aligned}\delta S = \int \mathrm{d}^{2D}X\, e^{-2d} \Big\{ & K_{PQ} \left(\eta^{Pp}\eta^{qQ} - \mathcal{H}^{Pp}\mathcal{H}^{qQ}\right)\delta g_{pq} \\ & -K_{PQ}\left(\mathcal{H}^{PR}\eta_{RM}\eta^{QN} - \mathcal{H}^{PR}\delta^N_R\delta^Q_M\right)\eta^{Mp}\delta^q_N\delta B_{pq}\Big\} .\end{aligned} \tag{4.2.12}$$

The result can be written in a form similar to the original approach (4.2.6). Rearranging the terms as

$$\begin{aligned}\delta S = \int \mathrm{d}^{2D}X\, e^{-2d} \Big\{ & K_{PQ}\left(\delta^P_M\delta^Q_N - \mathcal{H}^{PR}\eta_{RM}\eta_{NS}\mathcal{H}^{SQ}\right)\eta^{Mp}\eta^{qN}\delta g_{pq} \\ & -K_{PQ}\left(\mathcal{H}^{PR}\eta_{RM}\eta^{QS}\mathcal{H}_{ST} - \mathcal{H}^{PR}\delta^S_R\delta^Q_M\mathcal{H}_{ST}\right)\mathcal{H}^{TN}\eta^{Mp}\delta^q_N\delta B_{pq}\Big\}\end{aligned} \tag{4.2.13}$$

and defining the projector ${P_{MN}}^{PQ}$

$${P_{MN}}^{PQ} = \frac{1}{2}\left(\delta^{(\underline{P}}_M\delta^{Q)}_N - \mathcal{H}_{MR}\eta^{R(\underline{P}}\eta_{NS}\mathcal{H}^{\underline{Q})S}\right), \tag{4.2.14}$$

one obtains

$$\delta S = \int \mathrm{d}^{2D}X\, e^{-2d}\, 2{P_{MN}}^{PQ}K_{PQ}\left(\eta^{Mp}\eta^{qN}\delta g_{pq} + \eta^{Mp}\mathcal{H}^{qN}\delta B_{pq}\right). \tag{4.2.15}$$

Since the variation of the action must vanish for any δg_{pq} and δB_{pq} independently, the prefactor has to vanish identically, leading to the *projected equations of motion* [144]

$${P_{MN}}^{PQ}K_{PQ} = 0. \tag{4.2.16}$$

Heuristically, the purpose of the operator ${P_{MN}}^{PQ}$ is to project out all components which violate the constraint (4.1.6) and ensure that the coset structure of the generalized metric is preserved. It can also be shown that the kernel of ${P_{MN}}^{PQ}$ has dimension D^2+D, leaving D^2 nontrivial equations of motion as expected.

4.3 Outlook on Exceptional Field Theory

We conclude this chapter with a brief outlook on U-duality covariant *exceptional field theories* (*EFTs*) [167–170, 138]. Most of the upcoming discussion describes a direct generalization of double field theory and can be applied to a wider variety of duality groups. On a more abstract level, both frameworks are therefore often summarized under the term *extended field theories* (*ExFTs*) and *extended geometries* in modern literature. This section mainly follows the lines of [140].

4.3.1 Extended Spaces

Heuristically, exceptional field theories can be thought of as an M-theory-based analogue of double field theory. Similarly to a string winding around a compact direction, M-theory incorporates higher-dimensional M-branes and monopole structures which can carry wrapping numbers or charges that affect the physical properties of a model. In exceptional field theories, these additional degrees of freedom are encoded in *extended coordinates*, and the content of the theory is arranged in representations of the corresponding U-duality groups.

An important difference to T-duality is that the structure of these groups depends strongly on the dimension of the compactification space. There thus exist different exceptional field theories, each based on a corresponding exceptional algebra $E_{D(D)}(\mathbb{Z})$ spanning the U-duality symmetry of eleven-dimensional supergravity compactified on a D-dimensional torus [171, 172]. The notation $E_{D(D)}(\mathbb{Z})$ for $D \leq 5$ thereby refers to the algebra obtained by cutting off further nodes from the Dynkin diagram of E_6 (see also table 4.1). For a particular D, the eleven-dimensional spacetime manifold is split into its external and internal directions as

$$M^{11} = \overbrace{M^{11-D}}^{\text{external}} \times \overbrace{M^{D}}^{\text{internal}} \,. \tag{4.3.1}$$

The latter is extended by additional coordinates to linearly realize the (continuous) exceptional symmetry $E_{D(D)} := E_{D(D)}(\mathbb{R})$, where the dimension of the *extended space* is determined by the relevant representation R_1 of the corresponding U-duality group. The complete spacetime manifold then takes the form

$$M^{11-D+\mathrm{dim}R_1} = M^{11-D} \times M^{\mathrm{dim}R_1}. \tag{4.3.2}$$

Analogously to double field theory, the extended space $M^{\mathrm{dim}R_1}$ is equipped with an *exceptional generalized metric*, which parameterizes the coset space G/H of the global symmetry group $G = E_{D(D)}$ and its maximal compact subgroup H. Finally, the section condition is realized by requiring the projection of two derivatives onto a particular representation R_2 of G to vanish [140],

$$\partial \otimes \partial\,|_{R_2} = 0\,. \tag{4.3.3}$$

A full list of relevant groups and representations of known exceptional field theories is provided in table 4.1. The respective models were first elaborated, in ascending order, in [173–179]. More recently, first steps towards a generalization to the (infinite-dimensional) Kac-Moody algebra $E_{9(9)}$ have been made in [180].

At this point, let us remark that a distinction between external and internal coordinates is mostly exclusive to exceptional field theories. Although T- and U-duality transformations likewise act only on the compact directions, most formulations of double field theory in modern literature utilize a fully doubled spacetime. Due to the dimensional dependence of the U-duality group, this is not as straightforward to realize in exceptional field theory. It is assumed that all exceptional field theories for $D \leq 10$ can

D	G	H	R_1	R_2
D	$O(D,D)$	$O(D)\times O(D)$	$\mathbf{2D}$	$\mathbf{D}$
2	$SL(2)\times\mathbb{R}^+$	$SO(2)$	$(\mathbf{2},\mathbf{1})$	$(\mathbf{1},\mathbf{1})$
3	$SL(3)\times SL(2)$	$SO(3)\times SO(2)$	$(\mathbf{3},\mathbf{2})$	$(\bar{\mathbf{3}},\mathbf{1})$
4	$SL(5)$	$SO(5)$	$\mathbf{10}$	$\bar{\mathbf{5}}$
5	$SO(5,5)$	$SO(5)\times SO(5)$	$\mathbf{16}$	$\mathbf{10}$
6	E_6	$USp(8)$	$\mathbf{27}$	$\overline{\mathbf{27}}$
7	E_7	$SU(8)$	$\mathbf{56}$	$\mathbf{133}$
8	E_8	$SO(16)$	$\mathbf{248}$	$\mathbf{1}\oplus\mathbf{3875}$

Table 4.1: Duality groups G, their maximal compact subgroups H, extended coordinate representations R_1 and section condition representations R_2 for $D = 2, 3, \ldots 8$ [140].

be embedded into a unique $E_{11(11)}$-based framework. In this case, all directions are set to be internal, and a splitting of coordinates can be avoided. The role of this algebra has been extensively studied in the long-running E_{11} program [181–187].

4.3.2 Exceptional Extended Geometry

Let us now see how the notions of doubled geometry can be generalized to define *(exceptional) extended geometries*. Similarly to double field theory, many of the following structures are closely related to the framework of *exceptional generalized geometry*, which describes the corresponding $E_{D(D)}$ extensions of generalized geometry.

Generalizing Double Field Theory to Extended Field Theory

A first step is to define the so-called *Y-tensor*, which can be thought of as a measure for the departure of an extended geometry from conventional differential geometry. Generally, this object is built from the invariant tensors of the considered duality group. In the case of double field theory, it takes the form

$$Y^{MN}{}_{PQ} = \eta^{MN}\eta_{PQ}. \tag{4.3.4}$$

This enables a more compact notation of the generalized Lie derivative (4.1.15) as

$$\mathcal{L}_U V^M = L_U V^M + Y^{MN}{}_{PQ}\partial_N U^P V^Q + \lambda(V)\,\partial_N U^N V^M, \tag{4.3.5}$$

where the first term is the ordinary Lie derivative and the second term describes the deviation from standard $GL(D)$ geometry. The third term again defines a weight-term and is included for generality. Similarly, one obtains for the C-bracket (4.1.20)

$$[\![U,V]\!]_{\mathsf{C}}^M := [U,V]^M - \frac{1}{2}Y^{MN}{}_{PQ}\left(U^P\partial_N V^Q - V^P\partial_N U^Q\right) \tag{4.3.6}$$

and for the algebra of generalized diffeomorphisms

$$[\mathcal{L}_U, \mathcal{L}_V]\, W^M = \mathcal{L}_{[\![U,V]\!]_\mathsf{C}} W^M + Y^{RS}{}_{PQ} \Big[(\partial_R U^P \partial_S V^M - \partial_R U^M \partial_S V^P) W^Q \\ + \frac{1}{2}(U^Q \partial_S V^P - V^Q \partial_S U^P) \partial_R W^M \Big] . \tag{4.3.7}$$

The physical section condition (4.1.21) is equivalent to

$$Y^{MN}{}_{PQ} \partial_M \otimes \partial_N = 0, \tag{4.3.8}$$

and similar relations can be formulated for the D-bracket (4.1.26) and the Jacobiator (4.1.24). Using this notation, it is mostly straightforward to check that the algebra of generalized diffeomorphisms indeed closes under the section condition, and the complete framework reduces to conventional differential geometry when setting $Y^{MN}{}_{PQ} = 0$.

Constructing Exceptional Field Theories

To formulate a particular exceptional field theory in the above framework, the $O(D, D)$-specific structures have to be replaced by their U-duality counterparts. The Y-tensor $Y^{MN}{}_{PQ}$ is thereby constructed from the invariant structure of a corresponding duality group. One can then define the E-*bracket* $[\![\cdot, \cdot]\!]_\mathsf{E}$ as an analogue to the C-bracket from (4.3.6) and follow a similar pattern to derive consistency constraints for the *algebra of exceptional generalized diffeomorphisms* as done in section 4.1.3. A list of Y-tensors for $E_{D(D)}$ exceptional field theories with $D = 2, 3, \ldots 8$ is provided in table 4.2. A more detailed discussion of their structure can furthermore be found in [156, 140].

4.4 Summary

In this chapter we reviewed the fundamental concepts of extended field theories. We thereby discussed how conventional field theories can be enhanced with string- or M-theoretic properties such that duality transformations become a manifest symmetry. The following key insights will be essential for our upcoming discussion:

- In double field theory the normal spacetime coordinates are to extended by additional winding coordinates $\tilde{x}_m$ and arranged in doubled coordinates $X^M = (x^m, \tilde{x}_m)$ to form the fundamental representation of $O(D, D)$. The spacetime manifold M is furthermore equipped with a generalized tangent bundle $E = TM \oplus T^*M$, which essentially reflects a unification of diffeomorphisms and one-form gauge transformations into a single structure.

- The NS-NS fields mixing under T-duality transformations are organized in a generalized metric (4.1.3) and a generalized dilaton (4.1.4). The former defines an element of $O(D, D)$, the latter transforms as an $O(D, D)$ scalar.

D	G	Y
D	$O(D,D)$	$Y^{MN}{}_{PQ} = \eta^{MN}\eta_{PQ}$
2	$SL(2)\times\mathbb{R}^+$	$Y^{\alpha s}{}_{\beta s} = \delta^{\alpha}{}_{\beta}$
3	$SL(3)\times SL(2)$	$Y^{i\alpha,j\beta}{}_{k\gamma,l\delta} = 4\delta^{ij}{}_{kl}\delta^{\alpha\beta}{}_{\gamma\delta}$
4	$SL(5)$	$Y^{MN}{}_{PQ} = \epsilon^{aMN}\epsilon_{aPQ}$
5	$SO(5,5)$	$Y^{MN}{}_{PQ} = \frac{1}{2}(\gamma^u)^{MN}(\gamma_u)_{PQ}$
6	$E_{6(6)}$	$Y^{MN}{}_{PQ} = 10c^{MNR}c_{PQR}$
7	$E_{7(7)}$	$Y^{MN}{}_{PQ} = 12\tilde{c}^{MN}{}_{PQ} + \delta^{(\underline{M}}{}_{P}\delta^{\underline{N})}{}_{Q} + \frac{1}{2}\Omega^{MN}\Omega_{PQ}$

Table 4.2: Y-tensors of double field theory and exceptional field theories for internal dimension $D = 2,3,\ldots 7$ [156, 140, 173]. ϵ^{aMN} denotes the alternating tensor of $SL(5)$. $(\gamma^u)^{MN}$ define the 16×16 Majorana-Weyl representation of the $SO(5,5)$ Clifford algebra. c^{MNR} is the symmetric invariant tensor of $E_{6(6)}$. $\tilde{c}^{MN}$ and Ω_{PQ} denote the symmetric invariant tensor of $E_{7(7)}$ and the symplectic invariant tensor of its representation **56**. For the U-duality groups with product structure, s denotes a singlet index, $\alpha,\beta,\ldots$ are $SL(2)$ indices, and $i,j,\ldots$ $SL(3)$ indices.

- The geometry of double field theory is described by the framework $O(D,D)$ doubled geometry. Its gauge structure is governed by the generalized Lie derivative (4.1.15) and the algebra of generalized diffeomorphisms (4.3.7). Requiring closure of the latter gives rise consistency constraints such as the section condition (4.1.21).

- The term *section condition* can be further refined to distinguish between a weak constraint (4.1.27) and a strong constraint (4.1.28). Standard supergravities and their duals arise as simple solutions to the strong constraint and can be interpreted as physical sections through the doubled spacetime which are rotated into each other by T-duality transformations.

- The action of double field theory can be written in an Einstein-Hilbert-like formulation (4.2.1) and reduces to that of the standard type II NS-NS sector under the strong constraint. Its dynamics are encoded in the (projected) equations of motion (4.2.5) and (4.2.16).

Our discussion so far focused mostly on the fundamental structures and the physical interpretation of double field theory. We will next show how the framework can provide a natural description for the geometric and nongeometric fluxes related by the T-duality chain (2.2.29). This will eventually enable us to perform dimensional reductions of type II theories with all appearing moduli stabilized.

Chapter 5

Dimensional Reductions of Double Field Theory

In this chapter we will build upon the previous discussion to demonstrate how the framework of double field theory can be utilized to explicitly perform dimensional reductions of type II theories with geometric and nongeometric background fluxes. The following elaborations are based on and in large part identical to the author's publication [77], which in turn is a direct extension of the two previous works [78, 79] on the topic. Due to slight overlappings in the beginning of the discussion, it will be clarified in the respective sections which of the presented results are exclusive to [77] and this thesis.

5.1 Introduction

Before delving into the technical details, let us briefly recapitulate the relevant aspects of string phenomenology discussed in the previous chapters. We learned in section 2.1 that naive approaches to compactify higher-dimensional theories to four dimensions often come with undesired massless scalar particles, called *moduli*, which are in contradiction with experimental observations. In section 3.2 and onward, we saw that this in particular applies to conventional Calabi-Yau compactifications of type II superstring theories, where the moduli can parameterize a huge vacuum degeneracy arising from infinitesimal deformations of the Calabi-Yau's complex and Kähler structures.

One way to address this issue is to introduce non-vanishing background fluxes on the compactification manifold (see section 2.1.3). At string tree-level, this creates a scalar potential that stabilizes parts of the moduli. It was later found that successive application of T-duality transformations to flux backgrounds gives rise to geometrically ill-defined objects which play an essential role in obtaining full moduli stabilization [70–72]. Constructing phenomenologically realistic models from flux compactifications therefore requires suitable frameworks to enable a mathematical description of such *nongeometric* backgrounds.

In chapter 4 we introduced the formalism of *double field theory* (*DFT*) and demonstrated how it can be utilized to formulate a type II NS-NS action in which T-duality becomes a manifest symmetry. Our upcoming discussion will build upon this idea and focus on its application to generalized flux backgrounds. We will in particular show that there exists an alternative *flux formulation* of double field theory [149, 150], in which all fluxes of the T-duality chain 2.2.29 arise naturally as constituents of the action. In this framework, nongeometric fluxes are no longer ill-defined, but can be locally described as operators living in the doubled spacetime. This will eventually enable us to perform dimensional reductions of type II theories with all appearing moduli stabilized.

5.1.1 Background and previous Work

This chapter and its main reference [77] build upon a variety of works that have arisen out of long-running efforts to address the issue of moduli stabilization. Some contributions which are of particular importance for our upcoming discussion shall be briefly reviewed in this subsection.

Generalized Geometry and $SU(3)\times SU(3)$ Structure Manifolds

One well-known approach to moduli stabilization are compactifications on manifolds with $SU(3)$-structure. These models have been found to arise as Mirror Symmetry duals of Calabi-Yau backgrounds with NS-NS fluxes [119–121] and commonly come with parts of the moduli stabilized. In type II theories, the concept can be further generalized to a larger class of $SU(3)\times SU(3)$ structure manifolds by defining a separate non-vanishing spinor for each of the two ten-dimensional supercharges. Such structures show a natural connection to Hitchin's generalized geometry [147, 188], where $SU(3)\times SU(3)$ appears as the structure group of the generalized tangent bundle $TM^6 \oplus T^*M^6$ of the internal manifold M^6. Both $SU(3)$ and $SU(3)\times SU(3)$ structures arise from relaxations of the Calabi-Yau conditions and are therefore often summarized under the term *generalized Calabi-Yau structures*. This framework has been extensively studied in the works [119–121, 189–198].

Compactifications on generalized Calabi-Yau structures can give rise to four-dimensional models with parts or all of the moduli stabilized (see [199] for a discussion of the type II case). A particular strength of the double field theory approach is its capability to additionally provide a natural description of the background fluxes in ten dimensions, while their manifestation as (non-)geometric charges is more indirect in the former. Unsurprisingly, both frameworks are nevertheless closely related, and we will highlight their analogies thoughout our discussion in this chapter.

Double Field Theory and Gauged Supergravities

More research on the connection between double field theory and four-dimensional physics was conducted after the target-space formulation became widely applied in the early 2010s. It was found in [149, 150, 164] that compactifications and Scherk-Schwarz reductions of double field theory yield the scalar potential of electrically gauged $\mathcal{N}=4$ supergravity in four dimensions. Using a corresponding $SL(2)$ extension of double field theory [200], the construction could be generalized to electric/magnetic gaugings (see also [201] for an exceptional field theory analysis). A connection between Calabi-Yau compactifications of double field theory and the scalar potential of four-dimensional $\mathcal{N} = 2$ gauged supergravity was derived explicitly in [78].

This chapter will add to the picture by generalizing the considered setting of [78, 79] to a wider class of compactification manifolds and non-vanishing dilaton fluxes. We furthermore extend the formalism to dimensional reductions of the full double field theory action by including the kinetic terms. This will eventually enable us to show how in double field theory IIA $\leftrightarrow$ IIB Mirror Symmetry is restored by the simultaneous presence of geometric and nongeometric fluxes.

5.1.2 Overview

We will discuss the technical details of our computation in some length and therefore want to briefly summarize the main results of our analysis. The chapter is organized as follows:

- In section 5.2 we provide a brief review on the flux formulation of double field theory and discuss the mathematical structures which will be important for the upcoming discussion.

- In section 5.3 we compactify the purely internal part of the type IIA and IIB double field theory action on a Calabi-Yau three-fold. We mainly build upon the elaborations of [78, 79], thereby generalizing the approach to make it applicable to a wider class of compactification manifolds. Both results are related to the scalar potential of four-dimensional $\mathcal{N} = 2$ gauged supergravity constructed in [202], and a first manifestation of Mirror Symmetry is discussed.

- In section 5.4 the discussion of section 5.3 is repeated for the compactification manifold $K3 \times T^2$, where an additional contribution of dilaton fluxes becomes relevant. The necessary steps to generalize the Calabi-Yau setting are highlighted, and the special-geometric properties of $K3 \times T^2$ are discussed in detail. The resulting four-dimensional scalar potential is related to the framework of [202], and a set of mirror mappings is constructed. A double field theory origin of the $\mathcal{N} = 4$ gauged supergravity scalar potential has already been elaborated in the previous works [149, 150] using Scherk-Schwarz reductions. We here follow a different approach by employing the formalism of generalized Calabi-Yau geometry [147] and generalized $K3$ surfaces [203], which give rise to a scalar potential formulated in the language of $\mathcal{N} = 2$ gauged supergravity. While the result involves characteristic structures of $\mathcal{N} = 4$ supergravity, its relation to the results of [149, 150] seems to be nontrivial and will not be covered in this thesis.

- In section 5.5 we extend the Calabi-Yau setting of section 5.3 by including the kinetic terms. We thereby employ a generalized Kaluza-Klein ansatz [149, 150, 204] and treat the NS-NS and R-R sectors separately. For the former, we will mostly rely on the results of section 5.3 and on the standard literature on Calabi-Yau compactifications of type II theories. The latter is more involved and gives rise to democratic type II supergravities with all NS-NS fluxes and R-R fluxes of the T-duality chain turned on. We first reduce the ten-dimensional equations of motion, following a similar pattern as done in [199] for manifolds with $SU(3)\times SU(3)$ structure. The resulting four-dimensional equations of motion can then be shown to originate from the four-dimensional $\mathcal{N}=2$ gauged supergravity action constructed in [202], where a subset of the axions appearing in the standard formulation is dualized to two-forms in order to account for both electric and magnetic charges. This will finally enable us to once more read off a set of mirror mappings between the full reduced type IIA and IIB actions.

- Section 5.6 concludes the discussion by summarizing the results and giving an outlook on open questions and possible future developments.

Throughout this chapter we consider a doubled extension of the spacetime manifold $M^{10} = M^{1,3} \times M^6$, where $M^{1,3}$ denotes a four-dimensional Lorentzian manifold and M^6 is an arbitrary Calabi-Yau three-fold or $K3 \times T^2$. We will furthermore apply the framework of special geometry to describe the complex-structure and Kähler-class moduli spaces of M^6. Due to the large number of distinct indices used in this chapter, we provide an overview of our indexing system in appendix A. Several important identities and conventions which will be used throughout the upcoming calculations are discussed in appendix B.

5.2 Flux Formulation of Double Field Theory

In the course of our discussion in chapter 4, we learned how the basic notions of geometry can be generalized to define a T-duality covariant extension of conventional field theories. This eventually allowed us to formulate an Einstein-Hilbert-like action (4.2.1) which strongly resembles the NS-NS sector of type II supergravities, but does not yet show a direct relation to the background fluxes.

For the analysis presented in this chapter, we will make use of an alternative formulation of double field theory which is physically equivalent to (4.2.1). This framework is known as the *flux formulation* of double field theory [149, 150] and allows a natural (local) description of geometric as well as nongeometric background fluxes.

5.2.1 Action

This subsection will provide a brief overview on the flux formulation of double field theory and show how it can be extended to also incorporate the R-R sector of type II theories. We will furthermore show how geometric and nongeometric fluxes can be treated on equal footing by interpreting fluxes as simple operators acting on differential form fields.

NS-NS Sector

As starting point for the NS-NS sector, we consider the action [137, 149, 150]

$$S_{\text{NS-NS}} = \frac{1}{2}\int \mathrm{d}^{20}X e^{-2\hat{d}}\Big[\hat{\mathcal{F}}_{MNP}\hat{\mathcal{F}}_{M'N'P'}\left(\frac{1}{4}\hat{\mathcal{H}}^{MM'}\eta^{NN'}\eta^{PP'} - \frac{1}{12}\hat{\mathcal{H}}^{MM'}\hat{\mathcal{H}}^{NN'}\hat{\mathcal{H}}^{PP'}\right. \\ \left. -\frac{1}{6}\eta^{MM'}\eta^{N\hat{N}'}\eta^{PP'}\right) + \hat{\mathcal{F}}_{M}\hat{\mathcal{F}}_{M'}\left(\eta^{MM'} - \hat{\mathcal{H}}^{MM'}\right)\Big], \tag{5.2.1}$$

which was found the be physically equivalent to (4.2.1) under the strong constraint. Employing flat coordinates and using the *generalized Weizenböck connection*

$$\hat{\Omega}_{ABC} = \hat{\mathcal{E}}_A{}^M\left(\partial_I\hat{\mathcal{E}}_B{}^M\right)\hat{\mathcal{E}}_{CM} \tag{5.2.2}$$

the *generalized fluxes* $\hat{\mathcal{F}}_A$ and $\hat{\mathcal{F}}_{ABC}$ can be written as

$$\hat{\mathcal{F}}_A = \hat{\Omega}^B{}_{BA} + 2\hat{\mathcal{E}}_A{}^I\partial_I\hat{d} \qquad \text{and} \qquad \hat{\mathcal{F}}_{ABC} = 3\hat{\Omega}_{[\underline{ABC}]}, \tag{5.2.3}$$

where the squared brackets denote the antisymmetrization operator defined in appendix A. It will be explained in subsection 5.2.2 how these are related to the generalized fluxes with curved indices.

When performing dimensional reductions, a first step is to rewrite the action in terms of objects representing four-dimensional fields and assume all fields with external legs to be independent of the internal coordinates. We will do this by applying a generalized Kaluza-Klein ansatz for double field theory [149, 150, 204], for which we split the coordinates into external and internal parts

$$X^M = \left(x^\mu, \tilde{x}_\mu, Y^I\right), \qquad X^A = \left(x^\varepsilon, \tilde{x}_\varepsilon, Y^E\right), \tag{5.2.4}$$

where we used the collective notation $Y^I = (y^i, \tilde{y}_i)$ and $Y^E = (y^e, \tilde{y}_e)$ for the latter. In order to preserve rigid $O(6,6)$ symmetry, we impose the strong constraint only on the external coordinates. More precisely, we assume all fields and gauge parameters to be independent of the dual external coordinates $\tilde{x}_\mu$, while leaving the dependence of purely internal fields on the doubled coordinates Y^I, Y^A untouched for now.

For the ten-dimensional metric and Kalb-Ramond field, we employ the splitting [149]

$$\hat{g}_{mn} = \begin{pmatrix} g_{\mu\nu} + g_{kl}A^k{}_\mu A^l{}_\nu & A^k{}_\mu g_{kj} \\ g_{ik}A^k{}_\nu & g_{ij}\end{pmatrix}, \qquad \hat{B}_{mn} = \begin{pmatrix} B_{\mu\nu} & -B_{\mu j} \\ B_{i\nu} & B_{ij}\end{pmatrix} \tag{5.2.5}$$

and arrange the parts with mixed external and internal indices in a generalized Kaluza-Klein vector

$$\mathcal{A}^I{}_\mu = \left(-A^i{}_\mu, B_{i\mu}\right). \tag{5.2.6}$$

Inserting this ansatz into (5.2.1), the NS-NS contribution to the action can be reformulated as [149, 150, 204]

$$\begin{aligned}
S_{\text{NS-NS}} = & \frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^{12}Y\,\sqrt{-g^{(4)}}\sqrt{g^{(6)}}e^{-2\hat{\phi}}\Bigg[\\
& R^{(4)} + 4g^{\mu\nu}D_\mu\phi D_\nu\phi - \frac{1}{4}g^{\mu\nu}g^{\rho\sigma}\mathcal{H}_{IJ}\widetilde{\mathcal{F}}^I{}_{\mu\rho}\widetilde{\mathcal{F}}^J{}_{\nu\sigma} \\
& -\frac{1}{12}g^{\mu\nu}g^{\rho\sigma}g^{\tau\lambda}\widetilde{\mathcal{H}}_{\mu\rho\tau}\widetilde{\mathcal{H}}_{\nu\sigma\lambda} + g^{\mu\nu}\frac{1}{8}D_\mu\mathcal{H}_{IJ}D_\nu\mathcal{H}^{IJ} \\
& +\mathcal{F}_{IJK}\mathcal{F}_{I'J'K'}\left(-\frac{1}{12}\mathcal{H}^{II'}\mathcal{H}^{JJ'}\mathcal{H}^{KK'} + \frac{1}{4}\mathcal{H}^{II'}\eta^{JJ'}\eta^{KK'} - \frac{1}{6}\eta^{II'}\eta^{JJ'}\eta^{KK'}\right) \\
& +\mathcal{F}_I\mathcal{F}_{I'}\left(\eta^{II'} - \mathcal{H}^{II'}\right)\Bigg]
\end{aligned} \tag{5.2.7}$$

where we defined the field strengths

$$\begin{aligned}
\widetilde{\mathcal{F}}^I{}_{\mu\nu} &= 2\partial_{[\underline{\mu}}A^I{}_{\underline{\nu}]} - \mathcal{F}^I{}_{JK}\mathcal{A}^J{}_\mu\mathcal{A}^K{}_\nu + 2\mathcal{F}_J\mathcal{A}^J{}_{[\underline{\mu}}\mathcal{A}^I{}_{\underline{\nu}]} - 2\mathcal{F}^IB_{\mu\nu}, \\
\widetilde{\mathcal{H}}_{\mu\nu\rho} &= 3\partial_{[\underline{\mu}}B_{\underline{\nu\rho}]} - 3\partial_{[\underline{\mu}}\mathcal{A}^K{}_{\underline{\nu}}\mathcal{A}_{\underline{\rho}]K} - 6\mathcal{F}_K\mathcal{A}^K{}_{[\underline{\mu}}B_{\underline{\nu\rho}]} - \mathcal{F}_{IJK}\mathcal{A}^I{}_\mu\mathcal{A}^J{}_\nu\mathcal{A}^K{}_\rho
\end{aligned} \tag{5.2.8}$$

and the covariant derivatives

$$\begin{aligned}
D_\mu\mathcal{H}_{IJ} = & \ \partial_\mu\mathcal{H}_{IJ} + \mathcal{A}^K{}_\mu\mathcal{F}_{KI}{}^L\mathcal{H}_{JL} + \mathcal{A}^K{}_\mu\mathcal{F}_{KJ}{}^L\mathcal{H}_{IL} \\
& -\mathcal{A}_{\mu I}\mathcal{H}_{JK}\mathcal{F}^K - \mathcal{A}_{\mu J}\mathcal{H}_{IK}\mathcal{F}^K + \mathcal{F}_I\mathcal{H}_{JK}\mathcal{A}^K{}_\mu + \mathcal{F}_J\mathcal{H}_{IK}\mathcal{A}^K{}_\mu, \\
D_\mu\phi = & \ \partial_\mu\phi - \mathcal{F}_K\mathcal{A}^K{}_\mu.
\end{aligned} \tag{5.2.9}$$

R-R Sector

An extension of double field theory incorporating the R-R sector of type II theories was first developed in [205–209]. The fields transform in the spinor representation of $O(10,10)$, and one can expand

$$\hat{\mathfrak{G}} = \sum_n \frac{1}{n!}\hat{\mathfrak{G}}^{(n)}_{m_1\ldots m_n}e_{a_1}{}^{m_1}\ldots e_{a_n}{}^{m_n}\Gamma^{a_1\ldots a_n}\left|0\right\rangle, \tag{5.2.10}$$

where $\Gamma^{a_1\ldots a_n}$ denotes the totally antisymmetrized product of n gamma-matrices. The R-R gauge potentials can be combined into a spinor

$$\hat{\mathcal{C}} = \begin{cases} \sum_{n=0}^4 \hat{C}_{2n+1} & \text{for type IIA theory} \\ \sum_{n=0}^4 \hat{C}_{2n} & \text{for type IIB theory,} \end{cases} \tag{5.2.11}$$

which can be used to write

$$\hat{\mathfrak{G}} = \begin{cases} G_0 + \nabla\!\!\!/\,\hat{\mathcal{C}} & \text{for type IIA theory} \\ \nabla\!\!\!/\,\hat{\mathcal{C}} & \text{for type IIB theory,} \end{cases} \tag{5.2.12}$$

with the *generalized fluxed Dirac operator*

$$\not{\nabla} = \Gamma^A \hat{\mathcal{E}}_A{}^M \partial_M - \frac{1}{2}\Gamma^A \hat{\mathcal{F}}_A - \frac{1}{6}\Gamma^{ABC} \hat{\mathcal{F}}_{ABC}. \tag{5.2.13}$$

Notice that the zero-form R-R flux G_0 in the type IIA case arises as dual to the background field strength of $\hat{C}_9$. A pseudo-action for the R-R sector can be obtained by summing over all relevant components,

$$S_{\text{R-R}} = \frac{1}{2}\int \mathrm{d}^4x\, \mathrm{d}^{12}Y \left(-\frac{1}{2}\hat{\mathfrak{G}} \wedge \star\hat{\mathfrak{G}}\right). \tag{5.2.14}$$

Since all fields $\hat{C}_n$ of a certain theory appear explicitly, this has to be supplemented by duality constraints. Denoting the ten-dimensional n-form contributions by $\hat{\mathfrak{G}}_n$, these take the form [210]

$$\hat{\mathfrak{G}}_n = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \star \hat{\mathfrak{G}}_n, \tag{5.2.15}$$

where the floor operator $\lfloor\cdot\rfloor$ gives as output the greatest integer less than or equal to the argument.

5.2.2 Fluxes in Doubled Geometry

This section will focus on the scalar potential component of (5.2.7) and introduce a double field theory interpretation of the NS-NS fluxes. This was first investigated in [78] and later in [79]. Much of the following elaborations as well as section 5.3 will build upon these works, but we here slightly generalize the approach and resolve some of the questions which were left open at the end of the original discussion.

Fluxes as Fluctuations about the Calabi-Yau Background

The main idea is to treat the generalized fluxes (5.2.3) as manifestations of small deviations from the Calabi-Yau background, arising from perturbations of the internal vielbeins [78]

$$\mathcal{E}^A{}_I = \mathring{\mathcal{E}}^A{}_I + \overline{\mathcal{E}}^A{}_I + \mathcal{O}\left(\overline{\mathcal{E}}^2\right). \tag{5.2.16}$$

Here, $\mathring{\mathcal{E}}^A{}_I$ describes the Calabi-Yau background and $\overline{\mathcal{E}}^A{}_I$ the fluctuations. Inserting this expansion into the generalized fluxes (5.2.3), we can write

$$\mathcal{F}_A = \mathring{\mathcal{F}}_A + \overline{\mathcal{F}}_A + \mathcal{O}\left(\overline{\mathcal{E}}^2\right), \qquad \mathcal{F}_{ABC} = \mathring{\mathcal{F}}_{ABC} + \overline{\mathcal{F}}_{ABC} + \mathcal{O}\left(\overline{\mathcal{E}}^2\right). \tag{5.2.17}$$

As the notation implies, $\mathring{\mathcal{F}}_A$ and $\mathring{\mathcal{F}}_{ABC}$ depend only on $\mathring{\mathcal{E}}^A{}_I$ and do not contribute to the scalar potential since $\mathring{\mathcal{E}}^A{}_I$ satisfies the double field theory equations of motion. By contrast, $\overline{\mathcal{F}}_A$ and $\overline{\mathcal{F}}_{ABC}$ depend linearly on the fluctuations $\overline{\mathcal{E}}^A{}_I$ and therefore have to be taken into account.

In the following, we will use the background component $\mathring{\mathcal{E}}^A{}_I$ of the vielbein to switch between flat and curved indices (defining, e.g. $\overline{\mathcal{F}}_{IJK} = \mathring{\mathcal{E}}^A{}_I \mathring{\mathcal{E}}^B{}_J \mathring{\mathcal{E}}^C{}_K \overline{\mathcal{F}}_{ABC}$). For the case of constant expectation values, the three-indexed object $\overline{\mathcal{F}}_{IJK}$ has been shown to encode the known geometric and nongeometric NS-NS fluxes by

$$\overline{\mathcal{F}}_{ijk} = H_{ijk}, \quad \overline{\mathcal{F}}^{i}{}_{jk} = F^{i}{}_{jk}, \quad \overline{\mathcal{F}}_{i}{}^{jk} = Q_{i}{}^{jk}, \quad \overline{\mathcal{F}}^{ijk} = R^{ijk}. \tag{5.2.18}$$

Similarly, we define for the trace-terms and generalized dilaton fluxes (cf. the first relation of (5.2.3))

$$\overline{\mathcal{F}}_{i} = 2Y_i + F^{m}{}_{mi}, \quad \overline{\mathcal{F}}^{i} = 2Z^i + Q_m{}^{mi}. \tag{5.2.19}$$

As discussed in [211], writing out the generalized metric $\mathcal{H}$ in terms of the internal metric and Kalb-Ramond field gives rise to certain combinations of the latter with the fluxes, for which it is convenient to use the shorthand notation

$$\begin{aligned}
\mathfrak{H}_{ijk} &= H_{ijk} + 3F^{m}{}_{[\underline{ij}}B_{m\underline{k}]} + 3Q_{[\underline{i}}{}^{mn}B_{m\underline{j}}B_{n\underline{k}]} + R^{mnp}B_{m[\underline{i}}B_{n\underline{j}}B_{p\underline{k}]},\\
\mathfrak{F}^{i}{}_{jk} &= F^{i}{}_{jk} + 2Q_{[\underline{j}}{}^{mi}B_{m]\underline{k}} + R^{mni}B_{m[\underline{j}}B_{n\underline{k}]},\\
\mathfrak{Q}_{k}{}^{ij} &= Q_{k}{}^{ij} + R^{mij}B_{mk},\\
\mathfrak{R}^{ijk} &= R^{ijk},\\
\mathfrak{Y}_{i} &= Y_i + Z^m B_{mi},\\
\mathfrak{Z}^{i} &= Z^{i}.
\end{aligned} \tag{5.2.20}$$

Operator Interpretation of Fluxes

Throughout our upcoming discussion, it will be useful to interpret the geometric and nongeometric fluxes as operators acting on differential forms. Employing a local basis $(\mathrm{d}x^1, \dots \mathrm{d}x^6)$ and the contractions $(\iota_1, \dots \iota_6)$ satisfying $\iota_i \mathrm{d}x^j = \delta_i{}^j$, we define [212, 213, 71, 78, 79]

$$\begin{aligned}
H\wedge :\ & \Omega^p(CY_3) \longrightarrow \Omega^{p+3}(CY_3)\\
& \omega_p \mapsto \frac{1}{3!}H_{ijk}\, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k \wedge \omega_p,\\
F\circ :\ & \Omega^p(CY_3) \longrightarrow \Omega^{p+1}(CY_3)\\
& \omega_p \mapsto \frac{1}{2!}F^{k}{}_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \iota_k \wedge \omega_p,\\
Q\bullet :\ & \Omega^p(CY_3) \longrightarrow \Omega^{p-1}(CY_3)\\
& \omega_p \mapsto \frac{1}{2!}Q_{i}{}^{jk}\, \mathrm{d}x^i \wedge \iota_j \wedge \iota_k \wedge \omega_p,\\
R\llcorner :\ & \Omega^p(CY_3) \longrightarrow \Omega^{p-3}(CY_3)\\
& \omega_p \mapsto \frac{1}{3!}R^{ijk}\, \iota_i \wedge \iota_j \wedge \iota_k \wedge \omega_p,\\
Y\wedge :\ & \Omega^p(CY_3) \longrightarrow \Omega^{p+1}(CY_3)\\
& \omega_p \mapsto Y_i\, \mathrm{d}x^i \wedge \omega_p,\\
Z\blacktriangledown :\ & \Omega^p(CY_3) \longrightarrow \Omega^{p-1}(CY_3)\\
& \omega_p \mapsto Z^i\, \iota_i \wedge \omega_p,
\end{aligned} \tag{5.2.21}$$

the last two of which denote the newly-introduced generalized dilaton fluxes first considered in a different context in [214, 215] (see also [216, 217] for a generalized-geometry perspective). These operators can be combined with the exterior derivative $\hat{\mathrm{d}}$ to define the *twisted differential*

$$\hat{\mathcal{D}} = \hat{\mathrm{d}} - H \wedge - F \circ - Q \bullet - R \llcorner - Y \wedge - Z \blacktriangledown . \tag{5.2.22}$$

Notice that the exterior derivative is that of the full ten-dimensional spacetime manifold. In the following, we will often distinguish between internal and external components, for which it makes sense to split the exterior derivative as

$$\hat{\mathrm{d}} = \mathrm{d} + \mathrm{d}_{CY_3} \tag{5.2.23}$$

and define a purely internal twisted differential $\mathcal{D}$ with respect to d_{CY_3}. For later convenience, we can furthermore define analogous operators for the Fraktur fluxes (5.2.20), including the Fraktur twisted differential $\hat{\mathfrak{D}}$. As shown in [78, 79], requiring nilpotency $\hat{\mathcal{D}}^2 = 0$ of the twisted differential (and similarly for $\hat{\mathfrak{D}}$) gives rise to the Bianchi identities

$$\begin{aligned}
0 &= H_{m[\underline{ij}} F^m{}_{\underline{kl}]} - \frac{2}{3} \partial_{[\underline{i}} H_{\underline{jkl}]} , \\
0 &= F^m{}_{[\underline{jk}} F^l{}_{\underline{i}]m} + H_{m[\underline{ij}} Q_{\underline{k}]}{}^{ml} + \partial_{[\underline{k}} F^l{}_{\underline{ij}]} , \\
0 &= F^m{}_{[\underline{ij}]} Q_m{}^{[\underline{kl}]} - 4 F^{[\underline{k}}{}_{m[\underline{i}} Q_{\underline{j}]}{}^{\underline{l}]m} + H_{mij} R^{mkl} - 2 \partial_{[\underline{i}} Q_{\underline{j}]}{}^{kl} , \\
0 &= Q_m{}^{[\underline{jk}} Q_l{}^{\underline{i}]m} + R^{m[\underline{ij}} F^{\underline{k}]}{}_{ml} - \frac{1}{3} \partial_l R^{ijk} , \\
0 &= R^{m[\underline{ij}} Q_m{}^{\underline{kl}]} , \\
0 &= R^{mn[\underline{i}} F^{\underline{j}]}{}_{mn} - R^{m[\underline{ij}]} Y_m - Z^m Q_m{}^{[\underline{ij}]} , \\
0 &= R^{imn} H_{jmn} - F^i{}_{mn} Q_j{}^{mn} - 2 Q_j{}^{mi} Y_m + 2 Z^m F^i{}_{mj} - 2 \partial_j Z^i , \\
0 &= Q_{[\underline{i}}{}^{mn} H_{\underline{j}]mn} - F^m{}_{[\underline{ij}]} Y_m - Z^m H_{m[\underline{ij}]} + 2 \partial_{[\underline{i}} Y_{j]} , \\
0 &= 6 R^{mnp} H_{mnp} + Z^m Y_m ,
\end{aligned} \tag{5.2.24}$$

where the derivative terms vanish in the discussed setting and were included only for the sake of completeness. This form of the Bianchi identities generalizes the result of [78] and matches with the relations presented earlier in [165] when taking into account the definitions (5.2.19) and assuming independence of the dual coordinates.

Another central role will be played by the generalized primitivity constraints

$$H_{ia\bar{a}} g^{a\bar{a}} = 0, \qquad F^i{}_{a\bar{a}} g^{a\bar{a}} = 0, \qquad Q_i{}^{a\bar{a}} g_{a\bar{a}} = 0, \qquad R^{ia\bar{a}} g_{a\bar{a}} = 0, \tag{5.2.25}$$

which extend the corresponding condition for H arising from supersymmetry considerations in traditional approaches to flux compactifications. Here, the first condition is equivalent to requiring the interior product $H \lrcorner J$ of H and the Kähler form J to vanish. Analogous formulations are possible for the remaining fluxes by taking the interior product with $F\lrcorner$ to be with respect to the subscript indices and defining similar contraction-like operators $Q^{\urcorner}, R^{\urcorner}$ for the superscript indices of the nongeometric fluxes. The primitivity constraints can then be recast in the coordinate-independent forms

$$H\lrcorner J = 0, \qquad F\lrcorner J = 0, \qquad Q^{\urcorner} J = 0, \qquad R^{\urcorner} J = 0. \tag{5.2.26}$$

Notice that the interior product of nongeometric fluxes looks very similar to the corresponding operators defined in (5.2.21), but contracts only as many indices as there are in the differential form it acts on. This structure is motivated by that of the Hodge-star operator (A.2.6), and the relations (5.2.26) describe a generalization of the corresponding constraints used in [78, 79]. As we will see in the next section, this slight relaxation is necessary in order to make the framework applicable to more general settings of flux compactifications.

Geometric Tools

To conclude this section, let us briefly introduce several geometric structures which will become important in the following discussion. A useful tool to handle the flux operators is the so-called the *Mukai-pairing* of two differential forms η and ρ. It is defined by

$$\langle \eta, \rho \rangle = \left[\eta \wedge \lambda\left(\rho\right)\right]_6 , \tag{5.2.27}$$

where $[\cdot]_6$ picks the six-form-component and the involution λ acts on an n-form ρ as

$$\lambda\left(\rho\right) = (-1)^{\left\lceil \frac{n}{2} \right\rceil} \rho. \tag{5.2.28}$$

The operator $\lceil\cdot\rceil$ denotes the ceiling function, giving as output the least integer greater than or equal to the argument. Furthermore, we denote the purely external and internal components of Kalb-Ramond field $\hat{B}$ by

$$B = \frac{1}{2!} B_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \qquad \text{and} \qquad b = \frac{1}{2!} B_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j, \tag{5.2.29}$$

respectively. This allows us to define a b-twisted Hodge-star operator $\star_b$ as [218–220]

$$\star_b \eta = e^b \wedge \star\lambda\left(e^{-b}\eta\right), \tag{5.2.30}$$

which enables a natural extension of the framework to the Fraktur fluxes (5.2.20).

5.3 The Scalar Potential on a Calabi-Yau Three-Fold

We start our discussion of dimensional reductions by separately considering the purely internal parts of (5.2.7) and (5.2.14) on a Calabi-Yau three-fold CY_3. This was first done in [78] for the type IIB setting and later extended to the type IIA setting in [79]. We here generalize this analysis in order to prepare for our discussion in section 5.4. The aim of this section is to show that both the type IIA and IIB case correctly give rise to the scalar potential of four-dimensional $\mathcal{N}=2$ gauged supergravity. We furthermore illustrate how the simultaneous presence of geometric and nongeometric fluxes ensures preservation of IIA $\leftrightarrow$ IIB Mirror Symmetry in double field theory.

Due to a shortage of free letters, we will slightly abuse notation in this section and at some points utilize the letters m, n, p to extend our range of indices $i, j, k, \ldots$ for internal legs. These are not to be confused with ten-dimensional indices commonly associated with the same letters in this thesis. Notice, however, that this subsection focuses on the purely internal part of the action, and the role of each index will always be clear from the context.

Throughout our discussion, we impose the strong constraint on the underlying Calabi-Yau background, assuming independence of the dual coordinates $\tilde{y}_i$ for the metric g_{ij} and b. The latter is furthermore taken to satisfy the relation $\mathrm{d}_{CY_3} b = 0$. We do not impose the strong constraint for the fluxes, but only apply the weaker (quadratic) Bianchi identities (5.2.24). This is to ensure that the theory is capable of describing electric and magnetic gaugings and does not merely reduce to ordinary type II supergravities.

5.3.1 NS-NS Sector

When substituting the expansions (5.2.17) into the purely internal terms of (5.2.7), those terms involving only the objects $\mathring{\mathcal{F}}_I$ and $\mathring{\mathcal{F}}_{IJK}$ describe the Calabi-Yau background and do not contribute to the scalar potential since $\mathring{\mathcal{E}}^A{}_I$ satisfies the double field theory equations of motion. Furthermore, mixings between background values and fluctuations describe first order terms in the expansion about the minimum of the scalar potential and can be neglected as well. Considering the action up to second order in the deviations, we are then left with

$$\begin{aligned} S_{\text{NS-NS, scalar}} &= \frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^{12}Y\,\sqrt{-g^{(4)}}\sqrt{g_{CY_3}}e^{-2\phi}\Big[\overline{\mathcal{F}}_{IJK}\overline{\mathcal{F}}_{I'J'K'}\Big(-\frac{1}{12}\mathcal{H}^{II'}\mathcal{H}^{JJ'}\mathcal{H}^{KK'} \\ &\quad +\frac{1}{4}\mathcal{H}^{II'}\eta^{JJ'}\eta^{KK'}-\frac{1}{6}\eta^{II'}\eta^{JJ'}\eta^{KK'}\Big)+\overline{\mathcal{F}}_I\overline{\mathcal{F}}_{I'}\Big(\eta^{II'}-\mathcal{H}^{II'}\Big)\Big]. \end{aligned} \tag{5.3.1}$$

Inserting the relations (5.2.18) and (5.2.19), this can be rewritten in terms of the geometric and nongeometric fluxes as [78, 79]

$$\begin{aligned} S_{\text{NS-NS, scalar}} =& \frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^{12}Y\,\sqrt{-g^{(4)}}\sqrt{g_{CY_3}}e^{-2\phi}\Big[\\ & -\frac{1}{12}\Big(\mathfrak{H}_{ijk}\mathfrak{H}_{i'j'k'}g^{ii'}g^{jj'}g^{kk'}+3\mathfrak{F}^i{}_{jk}\mathfrak{F}^{i'}{}_{j'k'}g_{ii'}g^{jj'}g^{kk'} \\ & +3\mathfrak{Q}_i{}^{jk}\mathfrak{Q}_{i'}{}^{j'k'}g^{ii'}g_{jj'}g_{kk'}+\mathfrak{R}^{ijk}\mathfrak{R}^{i'j'k'}g_{ii'}g_{jj'}g_{kk'}\Big) \\ & -\frac{1}{2}\Big(\mathfrak{F}^m{}_{ni}\mathfrak{F}^n{}_{mi'}g^{ii'}+\mathfrak{Q}_m{}^{ni}\mathfrak{Q}_n{}^{mi'}g_{ii'}-\mathfrak{H}_{mni}\mathfrak{Q}_{i'}{}^{mn}g^{ii'}-\mathfrak{F}^i{}_{mn}\mathfrak{R}^{mni'}g_{ii'}\Big) \\ & -\Big(\mathfrak{F}^m{}_{mi}+2\mathfrak{Y}_i\Big)\Big(\mathfrak{F}^{m'}{}_{m'i'}+2\mathfrak{Y}_{i'}\Big)g^{ii'} \end{aligned}$$

$$- \left(\mathfrak{Q}_m{}^{mi} + 2\mathfrak{Z}^i\right)\left(\mathfrak{Q}_{m'}{}^{m'i'} + 2\mathfrak{Z}^{i'}\right) g_{ii'}\Bigg], \tag{5.3.2}$$

where the topological terms involving only the $O(6,6)$ invariant structure $\eta^{II'}$ cancel by the Bianchi identities (5.2.24). Now a key issue of this action is that the (generally unknown) metric g_{ij} of CY_3 appears explicitly. In conventional Calabi-Yau compactifications, this can be remedied by applying differential form notation and expanding the fields in terms of the cohomology bases. While this framework is not readily applicable to the considered setting, we can resolve this problem in a similar way by employing the operator interpretation (5.2.21) to build a bridge to the special geometry of the Calabi-Yau moduli spaces. To keep the calculation as general as possible, we will include cohomologically trivial terms for the first part of this section and set them to zero only right before performing the dimensional reduction.

Single-Flux Settings

As already demonstrated in [78], it is convenient to first assume vanishing internal B-field components and consider only one flux turned on at a time. It is then straightforward to show that the constructed reformulation is still applicable in more general settings.

Pure H-Flux

Due to its differential form nature, the discussion of the pure H -flux setting is particularly simple and requires only the tools of standard differential geometry. The corresponding Lagrangian of (5.3.2) takes the form

$$\mathcal{L}_{\text{NS-NS, scalar, } H} = \frac{e^{-2\phi}}{4} H_{ijk} H_{i'j'k'} g^{ii'} g^{jj'} g^{kk'}. \tag{5.3.3}$$

This can be written as

$$\star\mathcal{L}_{\text{NS-NS, scalar, } H} = -\frac{e^{-2\phi}}{2} H \wedge \star H, \tag{5.3.4}$$

where the three-form H is related to the first operator of (5.2.21) by formally defining $H := H \wedge \mathbf{1}_{CY_3}$.

Pure F-Flux

The NS-NS scalar potential Lagrangian in the pure F-flux scenario reads

$$\mathcal{L}_{\text{NS-NS, scalar, } F} = -\frac{e^{-2\phi}}{4}\left(F^i{}_{jk} F^{i'}{}_{j'k'} g_{ii'} g^{jj'} g^{kk'} + 2F^m{}_{ni} F^n{}_{mi'} g^{ii'} + 4F^m{}_{mi} F^m{}_{mi'} g^{ii'}\right). \tag{5.3.5}$$

While the three-form interpretation of H does not apply to F, we can construct a similar object by letting the operator $F\circ$ act on the Kähler form J of CY_3. We then obtain

$$-\frac{1}{2}\left(F\circ J\right)\wedge\star\left(F\circ J\right) = \left[\frac{1}{4} F^m{}_{ij} F^{m'}{}_{i'j'} g_{mm'} g^{ii'} g^{jj'} - \frac{1}{2} F^m{}_{ij} F^{m'}{}_{i'j'} I^{j'}{}_m I^j{}_{m'} g^{ii'}\right] \star \mathbf{1}_{CY_3} \tag{5.3.6}$$

and find that only the first terms of (5.3.5) and (5.3.6) match, while the second term

$$\begin{aligned} & -\frac{1}{2} F^m{}_{ij} F^{m'}{}_{i'j'} I^{j'}{}_m I^j{}_{m'} g^{ii'} \\ = & \left(F^c{}_{ab} F^b{}_{\bar{a}c} + F^{\bar{c}}{}_{a\bar{b}} F^{\bar{b}}{}_{\bar{a}\bar{c}} - F^{\bar{c}}{}_{ab} F^b{}_{\bar{a}\bar{c}} - F^c{}_{a\bar{b}} F^{\bar{b}}{}_{\bar{a}c} \right) g^{a\bar{a}} \end{aligned} \tag{5.3.7}$$

comes with reversed signs for the last two components. To see how this can be compensated for, notice that appropriate contraction of indices in the second Bianchi identity of (5.2.24) yields (for vanishing Q-flux) the relation

$$F^k{}_{a\bar{b}} F^{\bar{b}}{}_{\bar{a}k} + F^k{}_{\bar{b}\bar{a}} F^{\bar{b}}{}_{ak} + F^k{}_{\bar{a}a} F^{\bar{b}}{}_{\bar{b}k} = 0. \tag{5.3.8}$$

Multiplying this by $g^{a\bar{a}}$, we find after taking into account the corresponding primitivity constraint of (5.2.25)

$$F^c{}_{a\bar{b}} F^{\bar{b}}{}_{\bar{a}c} g^{a\bar{a}} = F^{\bar{c}}{}_{ab} F^b{}_{\bar{a}\bar{c}} g^{a\bar{a}} \tag{5.3.9}$$

Using this, adding the expression

$$\frac{1}{2}\left(\Omega \wedge F \circ J\right) \wedge \star \left(\overline{\Omega} \wedge F \circ J\right) = -2\left[F^{\bar{c}}{}_{ab} F^c{}_{\bar{a}\bar{b}} g_{c\bar{c}} g^{a\bar{a}} g^{b\bar{b}} - 2 F^{\bar{c}}{}_{ab} F^b{}_{\bar{a}\bar{c}} g^{a\bar{a}}\right] \star \mathbf{1}_{CY_3} \tag{5.3.10}$$

involving the holomorphic three-form Ω of CY_3 gives the correct second term of (5.3.6), but also comes with an additional contribution that has to be cancelled. We once more resolve this by adding

$$-\frac{1}{2}\left(F \circ \Omega\right) \wedge \star \left(F \circ \overline{\Omega}\right) = \left[2 F^{\bar{c}}{}_{ab} F^c{}_{\bar{a}\bar{b}} g_{c\bar{c}} g^{a\bar{a}} g^{b\bar{b}} + \frac{1}{2} F^m{}_{mi} F^m{}_{mi'} g^{ii'}\right] \star \mathbf{1}_{CY_3}. \tag{5.3.11}$$

Finally, the missing trace-term can be obtained by substituting the primitivity constraint (cf. (5.2.25)) into the only remaining non-trivial expression related the Calabi-Yau structure forms,

$$-\frac{1}{2}\left(F \circ \frac{1}{2} J^2\right) \wedge \star \left(F \circ \frac{1}{2} J^2\right) = \left[\frac{1}{2} F^m{}_{mi} F^m{}_{mi'} g^{ii'}\right] \star \mathbf{1}_{CY_3}, \tag{5.3.12}$$

and we find in total

$$\begin{aligned} \star \mathcal{L}_{\text{NS-NS, scalar, } F} = & -\frac{e^{-2\phi}}{2}\left[\left(F \circ J\right) \wedge \star \left(F \circ J\right) + \left(F \circ \frac{1}{2} J^2\right) \wedge \star \left(F \circ \frac{1}{2} J^2\right)\right. \\ & \left. + \left(F \circ \Omega\right) \wedge \star \left(F \circ \overline{\Omega}\right) - \left(\Omega \wedge F \circ J\right) \wedge \star \left(\overline{\Omega} \wedge F \circ J\right)\right]. \end{aligned} \tag{5.3.13}$$

Notice that this is a slight generalization of the corresponding expression found in [78,79] due to the presence of additional trace-terms of F. In particular, the reformulation works only when employing the relaxed primitivity constraints (5.2.25).

Pure Q-Flux

The analysis of the pure Q-flux setting follows a very similar pattern as for the F-flux, and we will only sketch the basic idea here. By proceeding completely analogously to the F-flux case, one can show that the Lagrangian can be reformulated as

$$\star\,\mathcal{L}_{\text{NS-NS, scalar, }Q} = -\frac{e^{-2\phi}}{2}\Bigg[\left(Q\bullet\frac{1}{2}J^2\right)\wedge\star\left(Q\bullet\frac{1}{2}J^2\right)+\left(Q\bullet\frac{1}{3!}J^3\right)\wedge\star\left(Q\bullet\frac{1}{3!}J^3\right) \\ +\left(Q\bullet\Omega\right)\wedge\star\left(Q\bullet\overline{\Omega}\right)-\left(\Omega\wedge Q\bullet\frac{1}{2}J^2\right)\wedge\star\left(\overline{\Omega}\wedge Q\bullet\frac{1}{2}J^2\right)\Bigg]. \tag{5.3.14}$$

The only non-straightforward step in this computation is to take into account the relation

$$Q_k{}^{a\bar{b}}Q_{\bar{b}}{}^{\bar{a}k}+Q_k{}^{\bar{b}\bar{a}}Q_{\bar{b}}{}^{ak}+Q_k{}^{\bar{a}a}Q_{\bar{b}}{}^{\bar{b}k}=0 \tag{5.3.15}$$

obtained by appropriately contracting the fourth Bianchi identity of (5.2.24), which can eventually be recast in the form

$$g_{a\bar{a}}Q_{\bar{b}}{}^{ac}Q_c{}^{\bar{a}\bar{b}}=g_{a\bar{a}}Q_b{}^{a\bar{c}}Q_{\bar{c}}{}^{\bar{a}b} \tag{5.3.16}$$

and used to identify certain contributions arising from the first and third term of (5.3.14). Again, the result describes a slight generalization of the one found in [78,79], and matching for the trace-terms requires using the primitivity constraints (5.2.25).

Pure R-Flux

Similarly to the symmetry between the pure F- and Q-flux settings, the reformulation of pure R-flux case shows a strong resemblance of the pure H-flux setting, and it is natural to consider the term $R\llcorner\frac{1}{3!}J^3$. This expression can be handled best by exploiting the relation

$$\frac{1}{3!}J^3=\star\mathbf{1}_{CY_3}=\frac{\sqrt{g_{CY_3}}}{6!}\varepsilon_{i_1\dots i_6}\mathrm{d}x^{i_1}\wedge\dots\wedge\mathrm{d}x^{i_6}, \tag{5.3.17}$$

to show that

$$R\llcorner\left(\frac{1}{3!}J^3\right)=-\frac{\sqrt{g_{CY_3}}}{3!3!}R^{ijk}\varepsilon_{ijklmn}\mathrm{d}x^l\wedge\mathrm{d}x^m\wedge\mathrm{d}x^n. \tag{5.3.18}$$

Inserting line two of (A.2.2) for $D=3$ and $p=3$, we then find

$$\star\mathcal{L}_{\text{NS-NS, scalar, }R}=-\frac{e^{2\phi}}{2}\left(R\llcorner\frac{1}{3!}J^3\right)\wedge\star\left(R\llcorner\frac{1}{3!}J^3\right). \tag{5.3.19}$$

Pure Y- and Z-Flux

While the nature of the generalized dilaton fluxes Y and Z differs from that of their (three-indexed) geometric and nongeometric counterparts, including them into the framework presented here requires only minor modifications. The idea is again to consider different combinations of flux operators with the holomorphic three-form Ω or powers of the Kähler-form J. Direct computation then shows that the Lagrangian (5.3.2) for the (combined) pure Y- and Z-flux settings can be rewritten as

$$\begin{aligned}\star\mathcal{L}_{\text{NS-NS, scalar, }Y} = -\frac{e^{-2\phi}}{2}\Bigg[&\Big(Y\wedge\mathbf{1}_{CY_3}\Big)\wedge\star\Big(Y\wedge\mathbf{1}_{CY_3}\Big)+\Big(Y\wedge J\Big)\wedge\star\Big(Y\wedge J\Big)\\&+\Big(Y\wedge\frac{1}{2}J^2\Big)\wedge\star\Big(Y\wedge\frac{1}{2}J^2\Big)+\Big(Y\wedge\Omega\Big)\wedge\star\Big(Y\wedge\overline{\Omega}\Big)\Bigg]\end{aligned} \tag{5.3.20}$$

and

$$\begin{aligned}\star\,\mathcal{L}_{\text{NS-NS, scalar, }Z} = -\frac{e^{-2\phi}}{2}\Bigg[&\Big(Z\blacktriangledown J\Big)\wedge\star\Big(Z\blacktriangledown J\Big)+\Big(Z\blacktriangledown\frac{1}{2}J^2\Big)\wedge\star\Big(Z\blacktriangledown\frac{1}{2}J^2\Big)\\&+\Big(Z\blacktriangledown\star\frac{1}{3!}J^3\Big)\wedge\star\Big(Z\blacktriangledown\star\frac{1}{3!}J^3\Big)+\Big(Y\wedge\Omega\Big)\wedge\star\Big(Y\wedge\overline{\Omega}\Big)\Bigg],\end{aligned} \tag{5.3.21}$$

respectively. Notice that, although not all corresponding expressions are trivial, we did not include any mixings between J and Ω. The reason for this discrepancy will become clear when considering more general settings in the next subsection.

Generalization

H-,F-,Q- and R-Fluxes

Before turning to the most general setting, it makes sense to first consider the case of all three-indexed fluxes H, F, Q, R being turned on, while still assuming vanishing one-indexed fluxes Y and Z. It was shown in [78] that the Lagrangian (5.3.2) can then be written as

$$\begin{aligned}\star\,\mathcal{L}_{\text{NS-NS, scalar, }HFQR} = -e^{-2\phi}\Bigg[&\frac{1}{2}\chi\wedge\star\overline{\chi}+\frac{1}{2}\Psi\wedge\star\overline{\Psi}\\&-\frac{1}{4}\left(\Omega\wedge\chi\right)\wedge\star\left(\overline{\Omega}\wedge\overline{\chi}\right)-\frac{1}{4}\left(\Omega\wedge\overline{\chi}\right)\wedge\star\left(\overline{\Omega}\wedge\chi\right)\Bigg],\end{aligned} \tag{5.3.22}$$

where

$$\chi = \mathcal{D}e^{iJ}, \qquad \Psi = \mathcal{D}\Omega\,, \tag{5.3.23}$$

and the twisted differential $\mathcal{D}$ is defined in (5.2.22) (with vanishing Y- and Z-components). Notice that this formulation gives rise to various extra terms when trying to reproduce the single-flux settings, which will however either cancel or vanish due to the generalized primitivity constraints (5.2.25). For the generic case, a minimal requirement for matching with the original Lagrangian (5.3.2) is that all mixings between different fluxes except for the HQ- and FR-combinations vanish. Since the only nontrivial contributions of (5.3.22) to the integral over CY_3 are the ones proportional to its volume form $\star\mathbf{1}_{CY_3}$, the relevant combinations of differential forms to check are those where both constituents share the same degree. This in particular excludes all components of the poly-form Ψ.

Furthermore, terms arising from quadratic combinations of χ which involve precisely one even and one odd power of iJ cancel due to the complex conjugation operator reversing the signs only for imaginary differential forms. A somewhat lengthy computation then shows that the remaining terms of (5.3.22) are the desired HQ- and FR-combinations, which read [78]

$$\begin{aligned} T_{HQ} &= -H \wedge \star \left(Q \bullet \frac{1}{2} J^2 \right) + \mathrm{Re} \left(\Omega \wedge H \right) \wedge \star \left(\overline{\Omega} \wedge Q \bullet \frac{1}{2} J^2 \right), \\ T_{FR} &= -F \circ J \wedge \star \left(R \llcorner \frac{1}{3!} J^3 \right) + \mathrm{Re} \left(\Omega \wedge F \circ J \right) \wedge \star \left(\overline{\Omega} \wedge R \llcorner \frac{1}{3!} J^3 \right). \end{aligned} \tag{5.3.24}$$

To show that these correctly reproduce the mixing terms of (5.3.2), one can again follow a similar pattern as in the single-flux settings [78]. The most important step here is to once more make use of the second and fourth Bianchi identities of (5.2.24) in order to relate the above expressions to the original action, which will in particular offset additional contributions arising from modifications of the relations (5.3.8) and (5.3.15) we used in the pure F- and Q-flux settings.

Including the Y- and Z-Fluxes

When trying to incorporate the generalized dilaton fluxes Y and Z into the framework, one immediate problem is that the relation (5.3.22) does not even hold for the single-flux settings. This is due to the appearance of additional mixings between e^{iJ} and Ω arising from the expressions in the second line, which cancel half of the desired terms and leave an overall mismatch by a factor of $\frac{1}{2}$ [79]. We resolve this by slightly modifying the expression in such a way that only the Y- and Z- terms are affected. Using the Mukai-pairing defined in (5.2.27), we find the more general form of the Lagrangian

$$\begin{aligned} \star \mathcal{L}_{\text{NS-NS, scalar}} = -e^{-2\phi} \bigg[& \frac{1}{2} \chi \wedge \star \overline{\chi} + \frac{1}{2} \Psi \wedge \star \overline{\Psi} \\ & - \frac{1}{4} \langle \Omega, \chi \rangle \wedge \star \langle \overline{\Omega}, \overline{\chi} \rangle - \frac{1}{4} \langle \Omega, \overline{\chi} \rangle \wedge \star \langle \overline{\Omega}, \chi \rangle \bigg], \end{aligned} \tag{5.3.25}$$

where χ and Ψ are defined as in (5.3.23) and the twisted differential now takes its general form (5.2.22). Of the newly appearing mixing terms, the non-vanishing ones are precisely the FY- and QZ-combinations, which correctly give rise to the trace-dilaton-mixings found in the last two lines of (5.3.2).

Notice that this formulation of the scalar potential shows a stronger resemblance of its generalized-geometry counterpart found in [199] for compactifications of type II supergravities on manifolds with $SU(3) \times SU(3)$ structure.

Including the Kalb-Ramond Field

In a final step, the above results are once more generalized to the setting of a non-vanishing internal Kalb-Ramond field b. As can be inferred from the structure of the

Lagrangian (5.3.2), this can be achieved by replacing

$$H \to \mathfrak{H}, \quad F \to \mathfrak{F}, \quad Q \to \mathfrak{Q}, \quad R \to \mathfrak{R}, \quad Y \to \mathfrak{Y}, \quad Z \to \mathfrak{Z} \tag{5.3.26}$$

and, thus, for the twisted differential

$$\mathcal{D} \to \mathfrak{D} = \mathrm{d} - \mathfrak{H} \wedge - \mathfrak{F} \circ - \mathfrak{Q} \bullet - \mathfrak{R}\llcorner - \mathfrak{Y} \wedge - \mathfrak{Z} \blacktriangledown . \tag{5.3.27}$$

Similarly, the Kähler and complex structures of Calabi-Yau manifolds with non-vanishing b-field are described by the modified poly-forms

$$e^{iJ} \to e^{b+iJ}, \qquad \Omega \to e^{b}\Omega. \tag{5.3.28}$$

At a later point, it will be convenient to absorb the factor e^b into the twisted differential. We therefore consider the relation [78]

$$\mathfrak{D} = e^{-b}\mathcal{D}e^{b} - \frac{1}{2}\left(\mathfrak{Q}_i{}^{mn}B_{mn}\mathrm{d}x^i + \mathfrak{R}^{imn}B_{mn}\iota_i\right), \tag{5.3.29}$$

which can be derived by direct computation and using closure of b. Imposing primitivity constraints analogous to (5.2.25) for the Fraktur fluxes and the modified Calabi-Yau structure forms (5.3.28),

$$\mathfrak{Q}\urcorner\mathfrak{J} = 0, \qquad \mathfrak{R}\urcorner\mathfrak{J} = 0, \tag{5.3.30}$$

we furthermore obtain the relations

$$\begin{aligned} Q_i{}^{mn}B_{mn} + iR^{mnp}B_{im}J_{np} + R^{mnp}B_{im}B_{np} &= 0, \\ R^{mnp}B_{np} + iR^{mnp}J_{np} &= 0, \end{aligned} \tag{5.3.31}$$

causing the terms in the brackets of (5.3.29) to vanish.We thus find for the NS-NS scalar potential in the most general setting

$$\begin{aligned} \star\,\mathcal{L}_{\text{NS-NS, scalar}} = -e^{-2\phi}\Bigg[&\frac{1}{2}\chi\wedge\star\overline{\chi} + \frac{1}{2}\Psi\wedge\star\overline{\Psi} \\ &-\frac{1}{4}\langle\Omega,\chi\rangle\wedge\star\langle\overline{\Omega},\overline{\chi}\rangle - \frac{1}{4}\langle\Omega,\overline{\chi}\rangle\wedge\star\langle\overline{\Omega},\chi\rangle\Bigg], \end{aligned} \tag{5.3.32}$$

where

$$\chi = e^{-b}\mathcal{D}e^{b+iJ}, \qquad \Psi = e^{-b}\mathcal{D}\left(e^{b}\Omega\right), \tag{5.3.33}$$

This formulation is again a slight generalization of the result obtained in [78, 79]. The expression invariant under exchangings between the poly-forms e^{iJ} and Ω, and the twisted differential $\mathfrak{D}$ could as well be chosen to act on the latter in the last two terms. This will be important for our discussion of Mirror Symmetry at a later point in this section.

5.3.2 R-R Sector

Reformulating the scalar-potential contribution of the R-R action (5.2.14) is more straightforward since only differential form terms are encountered. An instrutive example of how the appearing expressions can be evaluated is provided, for example, in in section 4 of [211]. We will here follow a similar strategy while proceeding separately for the type IIA and IIB cases.

Type IIA Theory

Starting from the purely internal component of (5.2.14) and inserting the definitions (5.2.12) and (5.2.11), we find for the internal components of the poly-form $\hat{\mathfrak{G}}^{(\text{IIA})}$ [79]

$$\begin{aligned}
\mathfrak{G}_0^{(\text{IIA})} &= G_0 - \mathfrak{Q} \bullet C_1 - \mathfrak{R} \llcorner C_3 - \mathfrak{Z} \blacktriangledown C_1, \\
\mathfrak{G}_2^{(\text{IIA})} &= G_2 - b \wedge G_0 - \mathfrak{F} \circ C_1 - \mathfrak{Q} \bullet C_3 - \mathfrak{R} \llcorner C_5 - \mathfrak{Y} \wedge C_1 - \mathfrak{Z} \blacktriangledown C_3, \\
\mathfrak{G}_4^{(\text{IIA})} &= G_4 - b \wedge G_2 + \frac{1}{2} b^2 \wedge G_0 - \mathfrak{H} \wedge C_1 - \mathfrak{F} \circ C_3 - \mathfrak{Q} \bullet C_5 - \mathfrak{Y} \wedge C_3 - \mathfrak{Z} \blacktriangledown C_5, \\
\mathfrak{G}_6^{(\text{IIA})} &= G_6 - b \wedge G_4 + \frac{1}{2} b^2 \wedge G_2 - \frac{1}{3!} b^3 \wedge G_0 - \mathfrak{H} \wedge C_3 - \mathfrak{F} \circ C_5 - \mathfrak{Y} \wedge C_5,
\end{aligned} \tag{5.3.34}$$

and the Lagrangian takes the form

$$\star \mathcal{L}_{\text{R-R}}^{(\text{IIA})} = -\frac{1}{2} \mathfrak{G}^{(\text{IIA})} \wedge \star \mathfrak{G}^{(\text{IIA})}. \tag{5.3.35}$$

Here, $\mathfrak{G}^{(\text{IIA})}$ denotes the purely internal part of $\hat{\mathfrak{G}}^{(\text{IIA})}$ given by

$$\mathfrak{G}^{(\text{IIA})} = e^{-b} \mathcal{G}^{(\text{IIA})} + e^{-b} \mathcal{D} \left(e^b \mathcal{C}^{(\text{IIA})} \right), \tag{5.3.36}$$

with

$$\begin{aligned}
\mathcal{C}^{(\text{IIA})} &= C_1 + C_3 + C_5 + C_7 + C_9, \\
\mathcal{G}^{(\text{IIA})} &= G_0 + G_2 + G_4 + G_6
\end{aligned} \tag{5.3.37}$$

comprising the purely internal components of the C_{2n+1}-fields[1] and the background R-R fluxes G_{2n}. The former are to be understood as fluctuations $\overline{C}_{2n+1}$ about the vacuum expectation values $\mathring{C}_{2n+1}$, and one can equivalently write (5.3.36) as $\mathfrak{G}^{(\text{IIA})} = G_0 + e^{-b} \mathcal{D} \left[e^b \left(\mathring{\mathcal{C}}^{(\text{IIA})} + \overline{\mathcal{C}}^{(\text{IIA})} \right) \right]$. The above formulation will, however, be more convenient since it allows us to treat all R-R fluxes on equal footing and obtain the same structure for the type IIA und IIB settings.

Type IIB Theory

The analysis of the type IIB setting is completely analogous to the type IIA case, and one eventually arrives at [78, 79]

$$\star \mathcal{L}_{\text{R-R}}^{(\text{IIB})} = -\frac{1}{2} \mathfrak{G}^{(\text{IIB})} \wedge \star \mathfrak{G}^{(\text{IIB})} \tag{5.3.38}$$

with

$$\mathfrak{G}^{(\text{IIB})} = e^{-b} \mathcal{G}^{(\text{IIB})} + e^{-b} \mathcal{D} \left(e^b \mathcal{C}^{(\text{IIB})} \right) \tag{5.3.39}$$

and

$$\begin{aligned}
\mathcal{G}^{(\text{IIB})} &= G_1 + G_3 + G_5, \\
\mathcal{C}^{(\text{IIB})} &= C_0 + C_2 + C_4 + C_6 + C_8.
\end{aligned} \tag{5.3.40}$$

[1] Notice that this expression intentionally includes trivial components and those which become massive in the process of compactification to highlight the symmetry between the type IIA and IIB settings.

5.3.3 Dimensional Reduction

The reformulated scalar potential described in (5.3.32), (5.3.35) and (5.3.38) depends only on the Kähler form and the holomorphic three-form of CY_3 and can thus be evaluated by utilizing the framework of special geometry for the Calabi-Yau moduli spaces.

Special Geometry of Calabi-Yau Three-Folds

Since we are interested only in those fields which do not acquire mass in the course of the compactification, we can follow the standard procedure of Calabi-Yau compactifications and expand the appearing fields in terms of the cohomology bases of CY_3. In the setting discussed here, this additionally requires a way to describe the action of the flux operators (5.2.21) on the field expansions.

Much of this can be achieved by applying the tools of special geometry introduced earlier in chapter 3. To make this chapter more self-contained, we will here provide another concise overview on the relevant structures encountered thoughout our previous discussion, before generalizing the concepts to make them readily applicable to the present setting.

Even Cohomology

The nontrivial even cohomology groups are $H^{n,n}(CY_3)$ with $n = 0, 1, 2, 3$. We denote the corresponding bases by

$$\begin{aligned} \left\{\mathbf{1}^{(6)}\right\} &\in H^{0,0}(CY_3)\,, \\ \left\{\omega_{\mathsf{i}}\right\} &\in H^{1,1}(CY_3)\,, \\ \left\{\widetilde{\omega}^{\mathsf{i}}\right\} &\in H^{2,2}(CY_3)\,, \\ \left\{\tfrac{1}{\mathcal{K}}\star\mathbf{1}^{(6)}\right\} &\in H^{3,3}(CY_3)\,, \end{aligned} \qquad \text{with } \mathsf{i} = 1, \dots h^{1,1} \tag{5.3.41}$$

where $\star\mathbf{1}^{(6)}$ is the volume form and $\mathcal{K}$ the total volume of CY_3. For later convenience, it makes sense to set $\omega_0 := \frac{1}{\mathcal{K}}\star\mathbf{1}^{(6)}$ and $\widetilde{\omega}^0 := \mathbf{1}^{(6)}$, allowing us to use the collective notation

$$\begin{aligned} \omega_{\mathsf{I}} &= \left(\ \omega_0,\ \ \omega_{\mathsf{i}}\ \right), \\ \widetilde{\omega}^{\mathsf{I}} &= \left(\ \widetilde{\omega}^0,\ \ \widetilde{\omega}^{\mathsf{i}}\ \right). \end{aligned} \qquad \text{with } \mathsf{I} = 0, \dots h^{1,1} \tag{5.3.42}$$

Notice that this convention differs slightly from the one used in (3.2.15). This structure enables a more straightforward implementation of the involution operator (5.2.28) into the framework but does not affect the overall result. We again choose the two bases such that the normalization condition

$$\int_{CY_3} \omega_{\mathsf{I}} \wedge \widetilde{\omega}^{\mathsf{J}} = \delta_{\mathsf{I}}{}^{\mathsf{J}} \tag{5.3.43}$$

holds. For the Kähler form J of CY_3 and the Kalb-Ramond field $\hat{B}$, we use the expansions

$$J = v^{\mathsf{i}}\omega_{\mathsf{i}} \qquad \text{and} \qquad \hat{B} = B + b = B + b^{\mathsf{i}}\omega_{\mathsf{i}}, \tag{5.3.44}$$

where B denotes the external component of $\hat{B}$ living in $M^{1,3}$ and b its internal part. The internal expansion coefficients b^{i} can be combined with the Kähler moduli v^{i} to define the complexified Kähler form

$$\mathfrak{J} = \left(b^{\mathsf{i}} + iv^{\mathsf{i}}\right)\omega_{\mathsf{i}} =: t^{\mathsf{i}}\omega_{\mathsf{i}}. \tag{5.3.45}$$

We furthermore introduce the shorthand notation

$$\begin{aligned} \mathcal{K}_{\mathsf{ijk}} &= \int_{CY_3} \omega_{\mathsf{i}} \wedge \omega_{\mathsf{j}} \wedge \omega_{\mathsf{k}}, \\ \mathcal{K}_{\mathsf{ij}} &= \int_{CY_3} \omega_{\mathsf{i}} \wedge \omega_{\mathsf{j}} \wedge J &&= \mathcal{K}_{\mathsf{ijk}} v^{\mathsf{k}}, \\ \mathcal{K}_{\mathsf{i}} &= \int_{CY_3} \omega_{\mathsf{i}} \wedge J \wedge J &&= \mathcal{K}_{\mathsf{ijk}} v^{\mathsf{j}} v^{\mathsf{k}}, \\ \mathcal{K} &= \frac{1}{3!}\int_{CY_3} J \wedge J \wedge J &&= \frac{1}{6}\mathcal{K}_{\mathsf{ijk}} v^{\mathsf{i}} v^{\mathsf{j}} v^{\mathsf{k}}, \end{aligned} \tag{5.3.46}$$

where the $\mathcal{K}_{\mathsf{ijk}}$, $\mathcal{K}_{\mathsf{ij}}$ and $\mathcal{K}_{\mathsf{i}}$ are called intersection numbers. Using this, one can expand the first poly-form of (5.3.33) in terms of the complexified Kähler moduli as

$$e^{B+iJ} = e^{\mathfrak{J}} = \tilde{\omega}^0 + t^{\mathsf{i}}\omega_{\mathsf{i}} + \frac{1}{2!}\left(\mathcal{K}_{\mathsf{ijk}} t^{\mathsf{i}} t^{\mathsf{j}}\right)\tilde{\omega}^{\mathsf{k}} + \frac{1}{3!}\left(\mathcal{K}_{\mathsf{ijk}} t^{\mathsf{i}} t^{\mathsf{j}} t^{\mathsf{k}}\right)\omega_0, \tag{5.3.47}$$

where all powers of order ≥ 4 vanish on CY_3.

Odd Cohomology

The nontrivial odd cohomology groups are given by $H^{3,0}(CY_3)$, $H^{2,1}(CY_3)$,$H^{1,2}(CY_3)$ and $H^{0,3}(CY_3)$. For these we introduce a collective basis

$$\left\{\alpha_{\mathsf{A}}, \beta^{\mathsf{A}}\right\} \in H^3(CY_3) \qquad \text{with } \mathsf{A} = 0, \dots h^{1,2}, \tag{5.3.48}$$

which can be normalized to satisfy

$$\int_{CY_3} \alpha_{\mathsf{A}} \wedge \beta^{\mathsf{B}} = \delta_{\mathsf{A}}{}^{\mathsf{B}}. \tag{5.3.49}$$

The complex-structure moduli are encoded by the holomorphic three-form Ω of CY_3, which we expand in terms of the periods X^{A} and F_{A} as

$$\Omega = X^{\mathsf{A}}\alpha_{\mathsf{A}} - \mathsf{F}_{\mathsf{A}}\beta^{\mathsf{A}}. \tag{5.3.50}$$

Notice that there is a minus sign in front of the β^{A}. Throughout this chapter we will apply this convention to all odd cohomology expansions of fields, while the signs are exchanged for field strengths. The periods F_{A} are functions of X^{A} and can be determined from a holomorphic prepotential F by $\mathsf{F}_{\mathsf{A}} = \frac{\partial \mathsf{F}}{\partial X^{\mathsf{A}}}$. Defining $\mathsf{F}_{\mathsf{AB}} = \frac{\partial \mathsf{F}_{\mathsf{A}}}{\partial X^{\mathsf{B}}}$, one can write the period matrix $\mathcal{M}_{\mathsf{AB}}$ as

$$\mathcal{M}_{\mathsf{AB}} = \overline{\mathsf{F}}_{\mathsf{AB}} + 2i\frac{\mathrm{Im}\left(\mathsf{F}_{\mathsf{AC}}\right)X^{\mathsf{C}}\mathrm{Im}\left(\mathsf{F}_{\mathsf{BD}}\right)X^{\mathsf{D}}}{X^{\mathsf{E}}\mathrm{Im}\left(\mathsf{F}_{\mathsf{EF}}\right)X^{\mathsf{F}}}, \tag{5.3.51}$$

which is related to the cohomology bases (5.3.48) by

$$\begin{aligned}\int_{CY_3} \alpha_{\mathsf{A}} \wedge \star\alpha_{\mathsf{B}} &= -\left[(\mathrm{Im}\mathcal{M}) + (\mathrm{Re}\mathcal{M})\,(\mathrm{Im}\mathcal{M})^{-1}\,(\mathrm{Re}\mathcal{M})\right]_{\mathsf{AB}},\\ \int_{CY_3} \alpha_{\mathsf{A}} \wedge \star\beta^{\mathsf{B}} &= -\left[(\mathrm{Re}\mathcal{M})\,(\mathrm{Im}\mathcal{M})^{-1}\right]_{\mathsf{A}}{}^{\mathsf{B}},\\ \int_{CY_3} \beta^{\mathsf{A}} \wedge \star\beta^{\mathsf{B}} &= -\left[\mathrm{Im}\mathcal{M}^{-1}\right]^{AB}.\end{aligned} \tag{5.3.52}$$

Gauge Coupling Matrices

Denoting some arbitrary poly-form field A which can be expanded in terms of the non-trivial cohomology bases of CY_3 by

$$A = A^{\mathsf{I}}\omega_{\mathsf{I}} + A_{\mathsf{I}}\tilde{\omega}^{\mathsf{I}} + A^{\mathsf{A}}\alpha_{\mathsf{A}} - A_{\mathsf{A}}\beta^{\mathsf{A}}, \tag{5.3.53}$$

one can define a collective notation by

$$A^{\mathbb{I}} = \left(\ A^{\mathsf{I}},\ \ A_{\mathsf{I}}\ \right) \qquad \text{and} \qquad A^{\mathbb{A}} = \left(\ A^{\mathsf{A}},\ \ -A_{\mathsf{A}}\ \right). \tag{5.3.54}$$

Similarly, we define the collective cohomology bases

$$\Sigma_{\mathbb{I}} = \left(\ \omega_{\mathsf{I}},\ \ \tilde{\omega}^{\mathsf{I}}\ \right) \qquad \text{and} \qquad \Xi_{\mathbb{A}} = \left(\ \alpha_{\mathsf{A}},\ \ \beta^{\mathsf{A}}\ \right) \tag{5.3.55}$$

and the matrix

$$\mathbb{M}_{\mathbb{AB}} = \int_{CY_3} \begin{pmatrix} -\left\langle \alpha_{\mathsf{A}}, \star_b \alpha_{\mathsf{B}} \right\rangle & \left\langle \alpha_{\mathsf{A}}, \star_b \beta^{\mathsf{B}} \right\rangle \\ \left\langle \beta^{\mathsf{A}}, \star_b \alpha_{\mathsf{B}} \right\rangle & -\left\langle \beta^{\mathsf{A}}, \star_b \beta^{\mathsf{B}} \right\rangle \end{pmatrix}, \tag{5.3.56}$$

which can be expressed in terms of the period matrix (5.3.52) as

$$\mathbb{M} = \begin{pmatrix} \mathbb{1} & -\mathrm{Re}\mathcal{M} \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathrm{Im}\mathcal{M} & 0 \\ 0 & \mathrm{Im}\mathcal{M}^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ -\mathrm{Re}\mathcal{M} & \mathbb{1} \end{pmatrix}. \tag{5.3.57}$$

For consistency of notation, we parameterize the even-cohomology analogue

$$\mathbb{N}_{\mathbb{IJ}} = \int_{CY_3} \begin{pmatrix} \left\langle \omega_{\mathsf{I}}, \star_b \omega_{\mathsf{J}} \right\rangle & \left\langle \omega_{\mathsf{I}}, \star_b \tilde{\omega}^{\mathsf{J}} \right\rangle \\ \left\langle \tilde{\omega}^{\mathsf{I}}, \star_b \omega_{\mathsf{J}} \right\rangle & \left\langle \tilde{\omega}^{\mathsf{I}}, \star_b \tilde{\omega}^{\mathsf{J}} \right\rangle \end{pmatrix} \tag{5.3.58}$$

as

$$\mathbb{N} = \begin{pmatrix} \mathbb{1} & -\mathrm{Re}\mathcal{N} \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathrm{Im}\mathcal{N} & 0 \\ 0 & \mathrm{Im}\mathcal{N}^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ -\mathrm{Re}\mathcal{N} & \mathbb{1} \end{pmatrix}, \tag{5.3.59}$$

where $\mathcal{N}_{\mathsf{IJ}}$ denotes the corresponding period matrix of the special Kähler manifold spanned by the complexified Kähler-class moduli. A detailed discussion of its structure can be found e.g. in [69].

Using the notation (5.3.42), one can also see that the Mukai-pairing (5.2.27) induces a symplectic structure by

$$\int_{CY_3} \langle \Sigma_{\mathbb{I}}, \Sigma_{\mathbb{J}} \rangle = (S_{\text{even}})_{\mathbb{IJ}} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \in Sp\left(2h^{1,1}+2, \mathbb{R}\right) \tag{5.3.60}$$

and

$$\int_{CY_3} \langle \Xi_{\mathbb{A}}, \Xi_{\mathbb{B}} \rangle = (S_{\text{odd}})_{\mathbb{AB}} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \in Sp\left(2h^{1,2}+2, \mathbb{R}\right). \tag{5.3.61}$$

For simplicity, we will omit the subscripts "even" and "odd" from now on. The dimension can, however, easily be inferred from the context or read off from the indices when using component notation.

Fluxes and Cohomology Bases

In the previous subsections, we treated the fluxes as operators in a local coordinate basis. For our subsequent analysis, we need to relate these operators to actions on the cohomology basis elements (5.3.41) and (5.3.48). In toroidal compactifications, this transition from the coordinate basis to the cohomology is straightforward to derive, but for more general manifolds this remains an open problem. However, as in [198], we can propose an action of the fluxes on the cohomology and check whether it leads to the expected results. For the three-index fluxes in the present context this has been done in [78]. For the Y- and Z-fluxes, the existence of such expansions is questionable as there do not exist any homological one- or five-cycles on a Calabi-Yau three-fold. We will therefore stick to the common convention and set Y and Z to zero for the remainder of this discussion, but come back to their role when we consider $K3 \times T^2$ compactifications in section 5.4.

To get familiar with the idea, notice that the H-flux can be expanded in the basis (5.3.48) as

$$H = -h^{\mathsf{A}}\alpha_{\mathsf{A}} + h_{\mathsf{A}}\beta^{\mathsf{A}}\,. \tag{5.3.62}$$

This particular example defines a differential form by itself, however, it can be alternatively interpreted as an operator acting on other differential forms as a wedge product with a three-form. With regard to the cohmology of the Calabi-Yau three-fold, it therefore defines a mapping between the zeroth and third as well as the third and sixth cohomology groups. In a similar way, the remaining fluxes can also be described by their effect on the cohomology basis elements. Following [198], we consider the action of the twisted differential $\mathcal{D}$ on the cohomology of the Calabi-Yau three-fold,

$$\begin{aligned} \mathcal{D}\alpha_{\mathsf{A}} &= \quad O_{\mathsf{A}}{}^{\mathsf{I}}\omega_{\mathsf{I}} + O_{\mathsf{AI}}\tilde{\omega}^{\mathsf{I}}, & \mathcal{D}\beta^{\mathsf{A}} &= \tilde{P}^{\mathsf{AI}}\omega_{\mathsf{I}} + \tilde{P}^{\mathsf{A}}{}_{\mathsf{I}}\tilde{\omega}^{\mathsf{I}}, \\ \mathcal{D}\omega_{\mathsf{I}} &= -\tilde{P}^{\mathsf{A}}{}_{\mathsf{I}}\alpha_{\mathsf{A}} + O_{\mathsf{AI}}\beta^{\mathsf{A}}, & \mathcal{D}\tilde{\omega}^{\mathsf{I}} &= \tilde{P}^{\mathsf{AI}}\alpha_{\mathsf{A}} - O_{\mathsf{A}}{}^{\mathsf{I}}\beta^{\mathsf{A}}, \end{aligned} \tag{5.3.63}$$

where the components

$$\begin{aligned} O_{\mathsf{Ai}} &= f_{\mathsf{Ai}}, & \tilde{P}^{\mathsf{A}}{}_{\mathsf{i}} &= f^{\mathsf{A}}{}_{\mathsf{i}}, \\ O_{\mathsf{A}}{}^{\mathsf{i}} &= q_{\mathsf{A}}{}^{\mathsf{i}}, & \tilde{P}^{\mathsf{Ai}} &= q^{\mathsf{Ai}} \end{aligned} \tag{5.3.64}$$

encode the action of the F- and Q-fluxes and we used the convention (5.3.42) to set

$$O_{\mathsf{A}0} = r_{\mathsf{A}}, \qquad \tilde{P}^{\mathsf{A}}{}_0 = r^{\mathsf{A}}, \\ O_{\mathsf{A}}{}^0 = h_{\mathsf{A}}, \qquad \tilde{P}^{\mathsf{A}0} = h^{\mathsf{A}}. \tag{5.3.65}$$

Similarly to the previous sections, one can arrange the flux coefficients in a collective notation that will greatly simplify calculations at a later point. We define the matrices

$$\mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}} = \begin{pmatrix} -\tilde{P}^{\mathsf{A}}{}_{\mathsf{I}} & \tilde{P}^{\mathsf{AI}} \\ O_{\mathsf{AI}} & -O_{\mathsf{A}}{}^{\mathsf{I}} \end{pmatrix}, \qquad \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}} = \begin{pmatrix} (O^T)^{\mathsf{I}}{}_{\mathsf{A}} & (\tilde{P}^T)^{\mathsf{IA}} \\ (O^T)_{\mathsf{IA}} & (\tilde{P}^T)_{\mathsf{I}}{}^{\mathsf{A}} \end{pmatrix}, \tag{5.3.66}$$

which are related by

$$\widetilde{\mathcal{O}} = -S^{-1}\mathcal{O}^T S. \tag{5.3.67}$$

This can be used to express the action of the twisted differential on the cohomology bases in the shorthand notation

$$\mathcal{D}(\Sigma^T)_{\mathbb{I}} = (\mathcal{O}^T)_{\mathbb{I}}{}^{\mathbb{A}}(\Xi^T)_{\mathbb{A}}, \qquad \mathcal{D}(\Xi^T)_{\mathbb{A}} = (\widetilde{\mathcal{O}}^T)_{\mathbb{A}}{}^{\mathbb{I}}(\Sigma^T)_{\mathbb{I}}. \tag{5.3.68}$$

Nilpotency of the twisted differential then implies that the relations

$$\mathcal{D}^2(\Sigma^T)_{\mathbb{I}} = 0 \qquad \text{and} \qquad \mathcal{D}^2(\Xi^T)_{\mathbb{A}} = 0 \tag{5.3.69}$$

have to be satisfied, giving rise to the constraints

$$\widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}\mathcal{O}^{\mathbb{A}}{}_{\mathbb{J}} = 0, \qquad \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{B}} = 0, \tag{5.3.70}$$

which take the role of a cohomology version of (5.2.24) and will be important in section 5.5.

Integrating over the Internal Space – NS-NS Sector

Proceeding in the same manner as for ordinary type II supergravity theories, we now expand the fields of the scalar potential in the cohomology bases (5.3.42) and (5.3.48) in order to filter out those terms which become massive in four dimensions. For the NS-NS poly-forms, we utilize the expansions (5.3.47) and (5.3.50) and arrange the coefficients in vectors

$$V^{\mathbb{I}} = \begin{pmatrix} \frac{1}{3!}\mathcal{K}_{\mathsf{ijk}}t^{\mathsf{i}}t^{\mathsf{j}}t^{\mathsf{k}}, & t^{\mathsf{i}}, & 1, & \frac{1}{2!}\mathcal{K}_{\mathsf{ijk}}t^{\mathsf{i}}t^{\mathsf{j}} \end{pmatrix} \\ W^{\mathbb{A}} = \begin{pmatrix} X^{\mathsf{A}}, & -F_{\mathsf{A}} \end{pmatrix} \tag{5.3.71}$$

of dimension $(2h^{1,1}+2)$ respectively $(2h^{1,2}+2)$, enabling us to use the shorthand notation

$$e^{b+iJ} = \Sigma_{\mathbb{I}}V^{\mathbb{I}}, \qquad \Omega = \Xi_{\mathbb{A}}W^{\mathbb{A}}. \tag{5.3.72}$$

Using the flux matrices (5.3.66) and the relations (5.3.68), the poly-forms χ and Ψ can now be expressed as

$$\chi = e^{-b}\Xi_{\mathbb{A}}\mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}V^{\mathbb{I}}, \\ \Psi = e^{-b}\Sigma_{\mathbb{I}}\widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}W^{\mathbb{A}}. \tag{5.3.73}$$

When integrating the NS-NS action (5.3.32) over CY_3, the first two terms of (5.3.73) combine to the matrices (5.3.56) and (5.3.58), and one eventually obtains for the scalar potential [78]

$$\begin{aligned} V_{\text{scalar, NS-NS}} = \frac{e^{-2\phi}}{2} \Big[& V^{\mathbb{I}} (\mathcal{O}^T)_{\mathbb{I}}{}^{\mathbb{A}} \mathbb{M}_{\mathbb{A}\mathbb{B}} \mathcal{O}^{\mathbb{B}}{}_{\mathbb{J}} V^{\mathbb{J}} + W^{\mathbb{A}} (\widetilde{\mathcal{O}}^T)_{\mathbb{A}}{}^{\mathbb{I}} \mathbb{N}_{\mathbb{I}\mathbb{J}} \widetilde{\mathcal{O}}^{\mathbb{J}}{}_{\mathbb{B}} \overline{W}^{\mathbb{B}} \\ & - \frac{1}{2\mathcal{K}} W^{\mathbb{A}} S_{\mathbb{A}\mathbb{B}} \mathcal{O}^{\mathbb{B}}{}_{\mathbb{I}} \left(V^{\mathbb{I}} \overline{V}^{\mathbb{J}} + \overline{V}^{\mathbb{I}} V^{\mathbb{J}} \right) (\mathcal{O}^T)_{\mathbb{J}}{}^{\mathbb{C}} (S^T)_{\mathbb{C}\mathbb{D}} \overline{W}^{\mathbb{D}} \Big] . \end{aligned} \tag{5.3.74}$$

Integrating over the Internal Space - R-R Sector

Following a similar pattern for the R-R sector, we start by discarding the cohomologically trivial C-fields and expand

$$\begin{aligned} e^b \mathcal{C}^{(\text{IIA})} &= C^{(3)\mathsf{A}} \alpha_{\mathsf{A}} - C^{(3)}{}_{\mathsf{A}} \beta^{\mathsf{A}}, \\ e^b \mathcal{C}^{(\text{IIB})} &= C^{(0)}{}_0 \tilde{\omega}^0 + C^{(2)\mathsf{I}} \omega_{\mathsf{I}} + C^{(4)}{}_{\mathsf{I}} \tilde{\omega}^{\mathsf{I}} + C^{(6)0} \omega_0. \end{aligned} \tag{5.3.75}$$

The expansion coefficients are again arranged in vectors

$$\begin{aligned} \mathsf{C}_0^{\mathbb{A}} &= \left(C^{(3)\mathsf{A}},\, C^{(3)}{}_{\mathsf{A}} \right) && \text{(type IIA theory)}, \\ \mathsf{C}_0^{\mathbb{I}} &= \left(C^{(6)0},\, C^{(2)\mathsf{I}},\, C^{(0)}{}_0,\, C^{(4)}{}_{\mathsf{I}} \right) && \text{(type IIB theory)}, \end{aligned} \tag{5.3.76}$$

where the subscript index "0" denotes the number of external components and is introduced for consistency with section 5.5. Similarly, we write for the non-trivial R-R fluxes

$$\begin{aligned} \mathcal{G}^{(\text{IIA})} &= G^{(0)}{}_0 \tilde{\omega}^0 + G^{(2)\mathsf{I}} \omega_{\mathsf{I}} + G^{(4)}{}_{\mathsf{I}} \tilde{\omega}^{\mathsf{I}} + G^{(6)0} \omega_0, \\ \mathcal{G}^{(\text{IIB})} &= -G^{(3)\mathsf{A}} \alpha_{\mathsf{A}} + G^{(3)}{}_{\mathsf{A}} \beta^{\mathsf{A}}, \end{aligned} \tag{5.3.77}$$

and

$$\begin{aligned} \mathsf{G}_{\text{flux}}^{\mathbb{I}} &= \left(G^{(6)0},\, G^{(2)\mathsf{I}},\, G^{(0)}{}_0,\, G^{(4)}{}_{\mathsf{I}} \right) && \text{(type IIA theory)}, \\ \mathsf{G}_{\text{flux}}^{\mathbb{A}} &= \left(G^{(3)\mathsf{A}},\, G^{(3)}{}_{\mathsf{A}} \right) && \text{(type IIB theory)}, \end{aligned} \tag{5.3.78}$$

enabling us to reformulate the poly-forms (5.3.36) and (5.3.39) as

$$\begin{aligned} \mathfrak{G}^{(\text{IIA})} &= e^{-b} \Sigma_{\mathbb{I}} \left(\mathsf{G}_{\text{flux}}^{\mathbb{I}} + \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}} \mathsf{C}_0^{\mathbb{A}} \right), \\ \mathfrak{G}^{(\text{IIB})} &= e^{-b} \Xi_{\mathbb{A}} \left(\mathsf{G}_{\text{flux}}^{\mathbb{A}} + \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}} \mathsf{C}_0^{\mathbb{I}} \right). \end{aligned} \tag{5.3.79}$$

Integrating (5.3.35) and (5.3.38) over CY_3 and once more utilizing the relations (5.3.56) and (5.3.58), we eventually arrive at [78, 79]

$$\begin{aligned} V^{(\text{IIA})}_{\text{scalar, R-R}} &= \frac{1}{2} \left(\mathsf{G}_{\text{flux}}^{\mathbb{I}} + \mathsf{C}_0^{\mathbb{A}} (\widetilde{\mathcal{O}}^T)_{\mathbb{A}}{}^{\mathbb{I}} \right) \mathbb{N}_{\mathbb{I}\mathbb{J}} \left(\mathsf{G}_{\text{flux}}^{\mathbb{J}} + \widetilde{\mathcal{O}}^{\mathbb{J}}{}_{\mathbb{B}} \mathsf{C}_0^{\mathbb{B}} \right), \\ V^{(\text{IIB})}_{\text{scalar, R-R}} &= \frac{1}{2} \left(\mathsf{G}_{\text{flux}}^{\mathbb{A}} + \mathsf{C}_0^{\mathbb{I}} (\mathcal{O}^T)_{\mathbb{I}}{}^{\mathbb{A}} \right) \mathbb{M}_{\mathbb{A}\mathbb{B}} \left(\mathsf{G}_{\text{flux}}^{\mathbb{B}} + \mathcal{O}^{\mathbb{B}}{}_{\mathbb{J}} \mathsf{C}_0^{\mathbb{J}} \right). \end{aligned} \tag{5.3.80}$$

Mirror Symmetry

In purely geometric flux compactifications, Mirror Symmetry between type IIA and IIB supergravities is not preserved for the generic case. In light of the conjectured equivalence of Mirror Symmetry and T-duality [61], this is not surprising as the latter is well-known to map parts of the fluxes to nongeometric backgrounds. Since double field theory incorporates all fluxes of the T-duality chain (2.2.29), it is to be hoped that IIA $\leftrightarrow$ IIB Mirror Symmetry is restored in this setting. And indeed, comparing the results (5.3.80) for the type IIA and IIB cases, it is straightforward to verify that the theories are related to each other by the mappings [79]

$$\begin{aligned} \mathbb{M}_{\mathbb{A}\mathbb{B}} &\leftrightarrow \mathbb{N}_{\mathbb{I}\mathbb{J}}, & h^{1,1} &\leftrightarrow h^{1,2}, \\ V^{\mathbb{I}} &\leftrightarrow W^{\mathbb{A}}, & S_{\mathbb{I}\mathbb{J}} &\leftrightarrow S_{\mathbb{A}\mathbb{B}}, \\ \mathsf{C}_0^{\mathbb{I}} &\leftrightarrow \mathsf{C}_0^{\mathbb{A}}, & \mathsf{G}_{\text{flux}}^{\mathbb{I}} &\leftrightarrow \mathsf{G}_{\text{flux}}^{\mathbb{A}}, \\ \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}} &\leftrightarrow \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}. \end{aligned} \tag{5.3.81}$$

These transformations strongly resemble the mirror mappings of conventional Calabi-Yau compactifications we encountered earlier in section 3.2.5. The first two lines again describe an exchange of roles between the Kähler-class and complex-structure moduli spaces, which is complemented by a simple replacement of the theory-specific R-R fields in line three. The last line encodes various mappings between the fluxes, which in particular contain exchanges between the geometric and nongeometric expansion coefficients. Taken as a whole, this relation implies that type IIA double field theory compactified on a Calabi-Yau three-fold CY_3 is physically equivalent to its type IIB analogue compactified on a mirror Calabi-Yau three-fold $\widetilde{CY}_3$, with the Hodge-diamonds of the two manifolds being related by a reflection along their diagonal axes.

The relations involving the expansion coefficients can be lifted to ten dimensions, allowing for a more compact notation

$$e^{iJ} \leftrightarrow \Omega, \qquad \mathfrak{G}^{(\text{IIA})} \leftrightarrow \mathfrak{G}^{(\text{IIB})} \tag{5.3.82}$$

of the mirror mappings as an exchange of the poly-forms (5.3.33), (5.3.36) and (5.3.39) we used to reformulate the double field theory action. Similarly to the component notation, this again describes an exchange of the terms encoding the complexified Kähler-class and complex-structure moduli and a mapping between the IIA and IIB R-R fields.

5.4 The Scalar Potential on $K3 \times T^2$

We next repeat the process of dimensional reduction for double field theory on $K3 \times T^2$ and show how the framework can be straightforwardly generalized to other ompactification manifolds. Much of the following discussion is completely analogous to the Calabi-Yau setting, and we will therefore focus on the specific features of $K3 \times T^2$. We will furthermore simplify computations by setting cohomologically trivial terms to zero right at the beginning of the calculation from now on.

In order to distinguish between $K3$ and T^2 indices, we split the internal indices into $I, J, \ldots$ labeling $K3$ coordinates and $R, S \ldots$ labeling T^2 coordinates. Their complex-geometric (undoubled) analogues are denoted by $a, \bar{a}, b, \bar{b}$ and $g, \bar{g}, h, \bar{h}$, respectively. For convenience, we accordingly split the flux operators (5.2.21) into their distinct cohomologically nontrivial components,

$$\begin{aligned}
H\wedge: \quad \Omega^p\left(K3\times T^2\right) &\longrightarrow \Omega^{p+3}\left(K3\times T^2\right)\\
\omega_p &\mapsto \frac{1}{2!}H_{ijr}\,\mathrm{d}x^i\wedge\mathrm{d}x^j\wedge\mathrm{d}x^r\wedge\omega_p,\\
F\circ: \quad \Omega^p\left(K3\times T^2\right) &\longrightarrow \Omega^{p+1}\left(K3\times T^2\right)\\
\omega_p &\mapsto \left(\frac{1}{2!}F^r{}_{ij}\,\mathrm{d}x^i\wedge\mathrm{d}x^j\wedge\iota_r + F^j{}_{ir}\,\mathrm{d}x^i\wedge\mathrm{d}x^r\wedge\iota_j\right)\wedge\omega_p,\\
Q\bullet: \quad \Omega^p\left(K3\times T^2\right) &\longrightarrow \Omega^{p-1}\left(K3\times T^2\right)\\
\omega_p &\mapsto \left(\frac{1}{2!}Q_r{}^{ij}\,\mathrm{d}x^r\wedge\iota_i\wedge\iota_j + Q_i{}^{jr}\,\mathrm{d}x^i\wedge\iota_j\wedge\iota_r\right)\wedge\omega_p,\\
R\llcorner: \quad \Omega^p\left(K3\times T^2\right) &\longrightarrow \Omega^{p-3}\left(K3\times T^2\right)\\
\omega_p &\mapsto \frac{1}{3!}R^{ijr}\,\iota_i\wedge\iota_j\wedge\iota_r\wedge\omega_p,\\
Y\wedge: \quad \Omega^p\left(K3\times T^2\right) &\longrightarrow \Omega^{p+1}\left(K3\times T^2\right)\\
\omega_p &\mapsto Y_r\,\mathrm{d}x^r\wedge\omega_p,\\
Z\blacktriangledown: \quad \Omega^p\left(K3\times T^2\right) &\longrightarrow \Omega^{p-1}\left(K3\times T^2\right)\\
\omega_p &\mapsto Z^r\,\iota_r\wedge\omega_p.
\end{aligned} \tag{5.4.1}$$

Finally, we again impose the strong constraint for the background and the field fluctuations, but assume only the less restrictive the Bianchi identities (5.2.24) to hold for the fluxes.

5.4.1 Reformulating the Action

The toolbox we used to reformulate the internal NS-NS action on CY_3 builds upon on the mathematical framework of generalized Calabi-Yau structures [147]. For the manifold $K3 \times T^2$, this problem can be approached in a similar way by utilizing the notions of generalized $K3$ surfaces [203] and formally treating T^2 as a complex torus. We start by exploiting the product structure of $K3 \times T^2$ and consider the Kähler class and complex-structure forms

$$e^{b+iJ} = e^{b_{K3}+iJ_{K3}}\wedge e^{b_{T^2}+iJ_{T^2}}, \qquad e^b\wedge\Omega = \left(e^{b_{K3}}\wedge\Omega_{K3}\right)\wedge\left(e^{b_{T^2}}\wedge\Omega_{T^2}\right), \tag{5.4.2}$$

respectively. The reformulation of the scalar potential part of the NS-NS sector (5.2.7) then follows a very similar pattern as in the Calabi-Yau case. As an instructive example, one can easily check that the only non-trivial contribution of the pure H-flux setting is given by

$$\star\mathcal{L}_{\text{NS-NS, scalar, } H} = \frac{e^{-2\phi}}{4} H_{ijr} H_{i'j'r'} g^{ii'} g^{jj'} g^{rr'} \star \mathbf{1}_{K3\times T^2}, \tag{5.4.3}$$

which can again be written as

$$\star\mathcal{L}_{\text{NS-NS, scalar, } H} = -\frac{e^{-2\phi}}{2} H \wedge \star H, \tag{5.4.4}$$

with H now defined as in (5.4.1). The F-flux allows for different nontrivial components and is therefore slightly more involved. From the initial action (5.2.7), we obtain

$$\begin{aligned}\mathcal{L}_{\text{NS-NS, scalar, } F} = -\frac{e^{-2\phi}}{4} \Big(& F^r{}_{ij} F^{r'}{}_{i'j'} g^{ii'} g^{jj'} g_{rr'} + 2F^i{}_{jr} F^{i'}{}_{j'r'} g_{ii'} g^{jj'} g^{rr'} + 2F^m{}_{nr} F^n{}_{mr'} g^{rr'} \\ & +4F^m{}_{mr} F^{m'}{}_{m'r'} g^{rr'} + 4F^r{}_{mi} F^m{}_{ri'} g^{ii'} \Big),\end{aligned} \tag{5.4.5}$$

Denoting the first and second component of $F\circ$ by $F_1\circ$ respectively $F_2\circ$ (based on the split employed in (5.4.1)), the first term can be rewritten similarly to the H-flux contribution as

$$-\frac{e^{-2\phi}}{4} F^r{}_{ij} F^{r'}{}_{i'j'} g^{ii'} g^{jj'} g_{rr'} \star \mathbf{1}_{K3\times T^2} = -\frac{e^{-2\phi}}{2} \left[F_1 \circ (\mathbf{1}_{K3} \wedge iJ_{T^2})\right] \wedge \star \left[F_1 \circ (\mathbf{1}_{K3} \wedge iJ_{T^2})\right] . \tag{5.4.6}$$

A similar structure as in (5.3.13) can be obtained by formally adding corresponding expressions for $F_1 \circ \left(\Omega_{K3} \wedge \Omega_{T^2}\right)$ and $\left(\Omega_{K3} \wedge \Omega_{T^2}\right) \wedge F_1 \circ (iJ_{K3} \wedge \mathbf{1}_{T^2})$, but their contributions cancel in this particular case. Proceeding analogously to the pure F-flux case in the Calabi-Yau setting, one finds for the next three terms

$$\begin{aligned} & -\frac{e^{-2\phi}}{4} \left(2F^i{}_{jr} F^{i'}{}_{j'r'} g_{ii'} g^{jj'} g^{rr'} + 2F^m{}_{nr} F^n{}_{mr'} g^{rr'} + 4F^m{}_{mr} F^{m'}{}_{m'r'} g^{rr'}\right) \star \mathbf{1}_{K3\times T^2} \\ = & -\frac{e^{-2\phi}}{2} \Big\{ \left[F_2 \circ \left(iJ_{K3} \wedge \mathbf{1}_{T^2}\right)\right] \wedge \star \left[F_2 \circ \left(iJ_{K3} \wedge \mathbf{1}_{T^2}\right)\right] \\ & \qquad + \left[F_2 \circ \left(\star \mathbf{1}_{K3} \wedge \mathbf{1}_{T^2}\right)\right] \wedge \star \left[F_2 \circ \left(\star \mathbf{1}_{K3} \wedge \mathbf{1}_{T^2}\right)\right] \\ & \qquad + \left[F_2 \circ \left(\Omega_{K3} \wedge \Omega_{T^2}\right)\right] \wedge \star \left[F_2 \circ \left(\overline{\Omega}_{K3} \wedge \overline{\Omega}_{T^2}\right)\right] \\ & \qquad - \left[\left(\Omega_{K3} \wedge \Omega_{T^2}\right) \wedge F_2 \circ (iJ_{K3} \wedge \mathbf{1}_{T^2})\right] \wedge \star \left[\left(\overline{\Omega}_{K3} \wedge \overline{\Omega}_{T^2}\right) F_2 \circ (iJ_{K3} \wedge \mathbf{1}_{T^2})\right] \Big\} . \end{aligned} \tag{5.4.7}$$

The final term can be reformulated as

$$\begin{aligned} & -e^{-2\phi} F^{r}{}_{mi} F^{m}{}_{ri'} g^{ii'} \star \mathbf{1}_{K3\times T^2} \\ = & -e^{-2\phi} \Big\{ \big[F_1 \circ (\mathbf{1}_{K3} \wedge iJ_{T^2})\big] \wedge \star \big[F_2 \circ (iJ_{K3} \wedge \mathbf{1}_{T^2})\big] \\ & \quad - \big[(\Omega_{K3} \wedge \Omega_{T^2}) \wedge F_1 \circ (\mathbf{1}_{K3} \wedge iJ_{T^2})\big] \wedge \star \big[(\Omega_{K3} \wedge \Omega_{T^2}) \wedge F_2 \circ (iJ_{K3} \wedge \mathbf{1}_{T^2})\big] \Big\}, \end{aligned} \tag{5.4.8}$$

showing that the F-contribution to the scalar potential takes the form (5.3.13) already known from the Calabi-Yau setting. The discussion of the nongeometric and generalized dilaton fluxes as well as the R-R sector is analogous. For the most general setting, we then eventually arrive at the familiar expressions (5.3.32), (5.3.35) and (5.3.38), with the fluxes adjusted according to (5.4.1) and e^{iJ} and Ω as in (5.4.2).

5.4.2 Dimensional Reduction

We next proceed as usual by expanding the fields and fluxes in terms of the cohomology bases of $K3 \times T^2$ before integrating over the internal manifold.

Special Geometry of $K3 \times T^2$

As in the Calabi-Yau case, we treat the even and odd cohomology groups of the compactification manifolds separately to enable a suitable description of the Kähler-class and complex-structure moduli spaces. A simple way to find a viable basis is to employ a splitting into the $K3$ and T^2 components and consider the factorized Hodge diamond

$$\begin{array}{ccc} & & \\ & 1 & \\ 1 & & 1 \\ & 1 & \\ & & \\ & T^2 & \end{array} \quad \times \quad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \\ & & K3\,. & & \end{array} \tag{5.4.9}$$

Since all nontrivial cohomology groups of $K3$ are of even degree, the property of a cohomologically nontrivial differential form on $K3 \times T^2$ being even or odd depends purely on its T^2 part.

Even Cohomology

The even cohomology bases of T^2 are precisely the identity $\mathbf{1}_{T^2}$ for the zero-forms and $\frac{1}{\mathcal{K}_{T^2}} \star \mathbf{1}_{T^2}$ for the two-forms (the latter of which coincides with the normalized Kähler form),

$$\begin{aligned} \Big\{ \mathbf{1}_{T^2} \Big\} &\in H^0\left(T^2\right), \\ \Big\{ \tfrac{1}{\mathcal{K}_{T^2}} \star \mathbf{1}_{T^2} \Big\} &\in H^2\left(T^2\right), \end{aligned} \tag{5.4.10}$$

and we denote them by v_0 respectively v_3 from now on. The bases of the $K3$ de Rham cohomology groups are given by

$$\begin{aligned} \left\{\mathbf{1}_{K3}\right\} &\in H^0(K3)\,, \\ \left\{\sigma_{\mathsf{u}}\right\} &\in H^2(K3)\,, \qquad\qquad \text{with } \mathsf{u}=1,\dots 22\,, \\ \left\{\tfrac{1}{\mathcal{K}_{K3}} \star \mathbf{1}_{K3}\right\} &\in H^4(K3)\,, \end{aligned} \tag{5.4.11}$$

and we define $\sigma_0 = \mathbf{1}_{K3}$ and $\sigma_{23} = \frac{1}{\mathcal{K}_{K3}} \star \mathbf{1}_{K3}$ to employ a collective notation

$$\sigma_{\mathsf{U}} = \begin{pmatrix} \sigma_0 & \sigma_{\mathsf{u}} & \sigma_{23} \end{pmatrix}. \tag{5.4.12}$$

We furthermore define η_{uv} to be the intersection metric

$$\eta_{\mathsf{uv}} = \int_{K3} \sigma_{\mathsf{u}} \wedge \sigma_{\mathsf{v}}\,, \tag{5.4.13}$$

which has signature $(3, 19)$ due to the existence of three antiselfdual two-forms (the Kähler form, the holomorphic two-form and its antiholomorphic counterpart) and 19 selfdual ones. This metric can serve as a building block of a matrix

$$L_{\mathsf{UV}} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \eta_{\mathsf{uv}} & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad L^{\mathsf{UV}} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \eta^{\mathsf{uv}} & 0 \\ -1 & 0 & 0 \end{pmatrix}, \tag{5.4.14}$$

which we use to lower and raise cohomological $K3$ indices,

$$\sigma^{\mathsf{U}} = L^{\mathsf{UV}} \sigma_{\mathsf{V}}. \tag{5.4.15}$$

Putting all the above objects together, we can define a collective basis for the even de Rham cohomology groups of $K3 \times T^2$ by

$$\begin{aligned} \omega_{\mathsf{I}} &= \left(\omega_0\ \omega_{\mathsf{u}}\ \omega_{23}\right) = \left(v_0 \wedge \sigma_0\ v_0 \wedge \sigma_{\mathsf{u}}\ v_0 \wedge \sigma_{23}\right), \\ \widetilde{\omega}^{\mathsf{I}} &= \left(\widetilde{\omega}^0\ \widetilde{\omega}^{\mathsf{u}}\ \widetilde{\omega}^{23}\right) = \left(v_3 \wedge \sigma^0\ v_3 \wedge \sigma^{\mathsf{u}}\ v_3 \wedge \sigma^{23}\right), \end{aligned} \tag{5.4.16}$$

where the labeling $\mathsf{I}, \mathsf{J}, \dots$ was chosen for later convenience. The basis elements satisfy the normalization condition

$$\int_{K3\times T^2} \omega_{\mathsf{I}} \wedge \widetilde{\omega}^{\mathsf{J}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \delta_{\mathsf{u}}{}^{\mathsf{v}} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{5.4.17}$$

and we again use a collective notation

$$\Sigma_{\mathbb{I}} = \begin{pmatrix} \omega_{\mathsf{I}} & \widetilde{\omega}^{\mathsf{I}} \end{pmatrix}. \tag{5.4.18}$$

Analogously to the Calabi-Yau case, this basis defines a symplectic structure by

$$\int_{K3\times T^2} \langle \Sigma_{\mathbb{I}}, \Sigma_{\mathbb{J}} \rangle = (S_{\text{even}})_{\mathbb{I}\mathbb{J}} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \in Sp\,(48, \mathbb{R})\,. \tag{5.4.19}$$

In order to describe the Kähler-class moduli space of $K3 \times T^2$, we combine the Kähler form J and the internal part b of the $\hat{B}$-field to a complexified Kähler form

$$\mathfrak{J} = b + iJ = (b_{T^2} + iJ_{T^2}) + (b_{K3} + iJ_{K3}) = \rho\widetilde{\omega}^0 + t^{\mathsf{u}}\omega_{\mathsf{u}}, \tag{5.4.20}$$

where the latter splitting can be applied due to the vanishing first Betti number of $K3$. The complex parameter $\rho = b^0 + iw^0$ encodes the volume modulus w^0 of T^2 as well as the component b^0 of $\hat{B}$ living purely in T^2. Analogously, the expressions t^{u} encode the moduli w^{u} and b^{u}, spanning the complexified Kähler cone of $K3$. In the upcoming discussion, we will often encounter the poly-form $e^{\mathfrak{J}}$, which we expand as $e^{\mathfrak{J}} = \Sigma_{\mathbb{I}} V^{\mathbb{I}}$ with

$$V^{\mathbb{I}} = \begin{pmatrix} 1, & t^{\mathsf{u}}, & t^{\mathsf{u}}t^{\mathsf{v}}\eta_{\mathsf{uv}}, & \rho t_{\mathsf{u}}t_{\mathsf{v}}\eta^{\mathsf{uv}}, & \rho t_{\mathsf{u}}, & \rho \end{pmatrix}. \tag{5.4.21}$$

Odd Cohomology

A basis for the odd cohomology groups can be constructed in a similar way by replacing the even basis elements of T^2 by two one-form basis elements

$$\big\{v_1, v_2\big\} \in H^1\left(T^2\right) \qquad \text{with} \int_{T^2} v_1 \wedge v_2 = 1 \tag{5.4.22}$$

and defining

$$\begin{aligned} \alpha_{\mathsf{A}} &= \begin{pmatrix} \alpha_0 & \alpha_{\mathsf{u}} & \alpha_{23} \end{pmatrix} = \begin{pmatrix} v_1 \wedge \sigma_0 & v_1 \wedge \sigma_{\mathsf{u}} & v_1 \wedge \sigma_{23} \end{pmatrix}, \\ \beta^{\mathsf{A}} &= \begin{pmatrix} \beta^0 & \beta^{\mathsf{u}} & \beta^{23} \end{pmatrix} = \begin{pmatrix} v_2 \wedge \sigma^0 & v_2 \wedge \sigma^{\mathsf{u}} & v_2 \wedge \sigma^{23} \end{pmatrix}. \end{aligned} \tag{5.4.23}$$

These objects satisfy the normalization condition

$$\int_{K3\times T^2} \alpha_{\mathsf{A}} \wedge \beta^{\mathsf{A}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \delta_{\mathsf{u}}{}^{\mathsf{v}} & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{5.4.24}$$

and can be arranged in a collective basis

$$\Xi_{\mathbb{A}} = \begin{pmatrix} \alpha_{\mathsf{A}} & \beta^{\mathsf{A}} \end{pmatrix} \tag{5.4.25}$$

to define a symplectic structure by

$$\int_{K3\times T^2} \langle \Xi_{\mathbb{A}}, \Xi_{\mathbb{B}} \rangle = (S_{\text{odd}})_{\mathbb{A}\mathbb{B}} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \in Sp\,(48, \mathbb{R})\,. \tag{5.4.26}$$

Notice that we again incorporated a relative minus sign into the expansions in terms of the even and odd cohomology bases for later convenience. More specifically, we expand an arbitrary poly-form field A as

$$A = A^{\mathbb{I}}\Sigma_{\mathbb{I}} + A^{\mathbb{A}}\Xi_{\mathbb{A}} = A^{\mathsf{I}}\omega_{\mathsf{I}} + A_{\mathsf{I}}\tilde{\omega}^{\mathsf{I}} + A^{\mathsf{A}}\alpha_{\mathsf{A}} - A_{\mathsf{A}}\beta^{\mathsf{A}}. \tag{5.4.27}$$

Similarly to its Kähler-class counterpart, the complex-structure moduli space of $K3 \times T^2$ can be described by a three-form Ω, defined as the product of a holomorphic one-form Ω_{T^2} living in T^2 and a holomorphic two-form Ω_{K3} living in $K3$. Viewing T^2 as a one-dimensional complex torus, Ω_{T^2} encodes the modular (complex-structure) parameter τ by

$$\Omega_{T^2} = v_1 - \tau v_2, \tag{5.4.28}$$

where

$$\tau = \int_{T^2} \Omega_{T^2} \wedge v_1. \tag{5.4.29}$$

Similarly, Ω_{K3} can be expanded as

$$\Omega_{K3} = T^{\mathsf{u}} \sigma_{\mathsf{u}}, \tag{5.4.30}$$

In the following, we will be mainly concerned with the expression $e^b\Omega$, which can be expanded in terms of the basis (5.4.23) as $e^b\Omega = \Xi_{\mathbb{A}} W^{\mathbb{A}}$ with

$$W^{\mathbb{A}} = \left(\begin{array}{cccccc} 0, & T^{\mathsf{u}}, & T^{\mathsf{u}} b^{\mathsf{v}} \eta_{\mathsf{uv}}, & \tau T_{\mathsf{u}} b_{\mathsf{v}} \eta^{\mathsf{uv}}, & \tau T_{\mathsf{u}}, & 0 \end{array} \right). \tag{5.4.31}$$

Gauge Coupling Matrices

As in the Calabi-Yau setting, we again define a gauge coupling matrix

$$\mathbb{M}_{\mathbb{AB}} = \int_{K3\times T^2} \begin{pmatrix} -\left\langle \alpha_{\mathsf{A}}, \star_b \alpha_{\mathsf{B}} \right\rangle & \left\langle \alpha_{\mathsf{A}}, \star_b \beta^{\mathsf{B}} \right\rangle \\ \left\langle \beta^{\mathsf{A}}, \star_b \alpha_{\mathsf{B}} \right\rangle & -\left\langle \beta^{\mathsf{A}}, \star_b \beta^{\mathsf{B}} \right\rangle \end{pmatrix}, \tag{5.4.32}$$

which can be written as

$$\mathbb{M}_{\mathbb{AB}} = \frac{1}{\mathrm{Im}\tau} \begin{pmatrix} |\tau|^2 \, \widetilde{\mathbb{N}}_{\mathsf{AB}} & \mathrm{Re}\tau \widetilde{\mathbb{N}}_{\mathsf{A}}{}^{\mathsf{B}} \\ \mathrm{Re}\tau \widetilde{\mathbb{N}}^{\mathsf{A}}{}_{\mathsf{B}} & \widetilde{\mathbb{N}}^{\mathsf{AB}} \end{pmatrix}, \tag{5.4.33}$$

where

$$\widetilde{\mathbb{N}}_{\mathbb{AB}} = \int_{K3} \begin{pmatrix} \left\langle \sigma_{\mathsf{U}}, \star_{b_{K3}} \sigma_{\mathsf{V}} \right\rangle & \left\langle \sigma_{\mathsf{U}}, \star_{b_{K3}} \sigma^{\mathsf{V}} \right\rangle \\ \left\langle \sigma^{\mathsf{U}}, \star_{b_{K3}} \sigma_{\mathsf{V}} \right\rangle & \left\langle \sigma^{\mathsf{U}}, \star_{b_{K3}} \sigma^{\mathsf{V}} \right\rangle \end{pmatrix} \tag{5.4.34}$$

is the $K3$ analogue of (5.3.58) (recall that the indices $\mathsf{A}, \mathsf{B}, \ldots$, $\mathsf{I}, \mathsf{J}, \ldots$ and $\mathsf{U}, \mathsf{V}, \ldots$ run over the same values). Similarly, we define for the even cohomology groups

$$\mathbb{N}_{\mathbb{IJ}} = \int_{K3\times T^2} \begin{pmatrix} \left\langle \omega_{\mathsf{I}}, \star_b \omega_{\mathsf{J}} \right\rangle & \left\langle \omega_{\mathsf{I}}, \star_b \tilde{\omega}^{\mathsf{J}} \right\rangle \\ \left\langle \tilde{\omega}^{\mathsf{I}}, \star_b \omega_{\mathsf{J}} \right\rangle & \left\langle \tilde{\omega}^{\mathsf{I}}, \star_b \tilde{\omega}^{\mathsf{J}} \right\rangle \end{pmatrix}, \tag{5.4.35}$$

which can be reformulated as

$$\mathbb{N}_{\mathbb{IJ}} = \frac{1}{\mathrm{Im}\rho} \begin{pmatrix} |\rho|^2 \, \widetilde{\mathbb{N}}_{\mathsf{IJ}} & \mathrm{Re}\rho \widetilde{\mathbb{N}}_{\mathsf{I}}{}^{\mathsf{J}} \\ \mathrm{Re}\rho \widetilde{\mathbb{N}}^{\mathsf{I}}{}_{\mathsf{J}} & \widetilde{\mathbb{N}}^{\mathsf{IJ}} \end{pmatrix}, \tag{5.4.36}$$

with $\widetilde{\mathbb{N}}_{\mathbb{IJ}}$ taking the same form as (5.4.34).

Fluxes and Cohomology Bases

To relate the flux operators (5.4.1) to the gaugings of four-dimensional supergravity, we once more proceed analogously to the Calabi-Yau setting. The action of the twisted differential (5.2.22) on the cohomology bases can be summarized by the relations

$$\mathcal{D}(\Sigma^T)_{\mathbb{I}} = (\mathcal{O}^T)_{\mathbb{I}}{}^{\mathbb{A}}(\Xi^T)_{\mathbb{A}}, \qquad \mathcal{D}(\Xi^T)_{\mathbb{A}} = (\widetilde{\mathcal{O}}^T)_{\mathbb{A}}{}^{\mathbb{I}}(\Sigma^T)_{\mathbb{I}}, \tag{5.4.37}$$

where the charge matrices

$$\mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}} = \begin{pmatrix} -\tilde{P}^{\mathsf{A}}{}_{\mathsf{I}} & \tilde{P}^{\mathsf{AI}} \\ O_{\mathsf{AI}} & -O_{\mathsf{A}}{}^{\mathsf{I}} \end{pmatrix}, \qquad \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}} = \begin{pmatrix} (O^T)^{\mathsf{I}}{}_{\mathsf{A}} & (\tilde{P}^T)^{\mathsf{IA}} \\ (O^T)_{\mathsf{IA}} & (\tilde{P}^T)_{\mathsf{I}}{}^{\mathsf{A}} \end{pmatrix} \tag{5.4.38}$$

comprise the flux expansion coefficients. Their components read

$$\begin{aligned}
\tilde{P}^{\mathsf{A}}{}_{\mathsf{I}} &= \begin{pmatrix} (f+y)^0{}_0 & q^0{}_{\mathsf{u}} & 0 \\ h^{\mathsf{u}}{}_0 & (f+y)^{\mathsf{u}}{}_{\mathsf{u}} & q^{\mathsf{u}}{}_{23} \\ 0 & h^{23}{}_{\mathsf{u}} & (f+y)^{23}{}_{23} \end{pmatrix}, \\
\tilde{P}^{\mathsf{AI}} &= \begin{pmatrix} 0 & r^{0\mathsf{u}} & (q+z)^{\,0\,23} \\ r^{\mathsf{u}0} & (q+z)^{\,\mathsf{uu}} & f^{\mathsf{u}\,23} \\ (q+z)^{\,23\,0} & f^{23\,\mathsf{u}} & 0 \end{pmatrix}, \\
O_{\mathsf{AI}} &= \begin{pmatrix} 0 & h_{0\mathsf{u}} & (f+y)_{0\,23} \\ h_{\mathsf{u}0} & (f+y)_{\mathsf{uu}} & q_{\mathsf{u}\,23} \\ (f+y)_{23\,0} & q_{23\,\mathsf{u}} & 0 \end{pmatrix}, \\
O_{\mathsf{A}}{}^{\mathsf{I}} &= \begin{pmatrix} (q+z)_0{}^0 & f_0{}^{\mathsf{u}} & 0 \\ r_{\mathsf{u}}{}^0 & (q+z)_{\mathsf{u}}{}^{\mathsf{u}} & f_{\mathsf{u}}{}^{23} \\ 0 & r_{23}{}^{\mathsf{u}} & (q+z)_{23}{}^{23} \end{pmatrix},
\end{aligned} \tag{5.4.39}$$

and again satisfy the relation

$$\widetilde{\mathcal{O}} = -S^{-1}\mathcal{O}^T S. \tag{5.4.40}$$

The notation was chosen such that the small letters in the charge matrices indicate the fluxes they descend from. While their origin should be clear for most cases, there are some caveats for the F- and Q-fluxes. Here, the coefficients with different indices arise from the flux components with two sub- or superscript $K3$ indices, while the coefficients with matching indices originate from the components with one sub- and one superscript index in $K3$.

Integrating over the Internal Space

With everything formulated in an analogous framework as in the Calabi-Yau setting, it is now straightforward to integrate over the internal manifold. Similar considerations as in subsection 5.3.3 and 5.3.3 eventually lead to the results

$$V^{(\mathrm{IIA})}_{\text{scalar, NS-NS}} = \frac{e^{-2\phi}}{2}\Big[V^{\mathbb{I}}(\mathcal{O}^T)_{\mathbb{I}}{}^{\mathbb{A}}\mathbb{M}_{\mathbb{A}\mathbb{B}}\mathcal{O}^{\mathbb{B}}{}_{\mathbb{J}}V^{\mathbb{J}} + W^{\mathbb{A}}(\widetilde{\mathcal{O}}^T)_{\mathbb{A}}{}^{\mathbb{I}}\mathbb{N}_{\mathbb{I}\mathbb{J}}\widetilde{\mathcal{O}}^{\mathbb{J}}{}_{\mathbb{B}}\overline{W}^{\mathbb{B}} - \frac{1}{2\mathcal{K}}W^{\mathbb{A}}S_{\mathbb{A}\mathbb{B}}\mathcal{O}^{\mathbb{B}}{}_{\mathbb{I}}\left(V^{\mathbb{I}}\overline{V}^{\mathbb{J}} + \overline{V}^{\mathbb{I}}V^{\mathbb{J}}\right)(\mathcal{O}^T)_{\mathbb{J}}{}^{\mathbb{C}}(S^T)_{\mathbb{C}\mathbb{D}}\overline{W}^{\mathbb{D}}\Big] + \frac{1}{2}\left(\mathsf{G}^{\mathbb{I}}_{\text{flux}} + \mathsf{C}^{\mathbb{A}}_0(\widetilde{\mathcal{O}}^T)_{\mathbb{A}}{}^{\mathbb{I}}\right)\mathbb{N}_{\mathbb{I}\mathbb{J}}\left(\mathsf{G}^{\mathbb{J}}_{\text{flux}} + \widetilde{\mathcal{O}}^{\mathbb{J}}{}_{\mathbb{B}}\mathsf{C}^{\mathbb{B}}_0\right), \tag{5.4.41}$$

for the type IIA case and

$$V^{(\mathrm{IIB})}_{\text{scalar, NS-NS}} = \frac{e^{-2\phi}}{2}\Big[V^{\mathbb{I}}(\mathcal{O}^T)_{\mathbb{I}}{}^{\mathbb{A}}\mathbb{M}_{\mathbb{A}\mathbb{B}}\mathcal{O}^{\mathbb{B}}{}_{\mathbb{J}}V^{\mathbb{J}} + W^{\mathbb{A}}(\widetilde{\mathcal{O}}^T)_{\mathbb{A}}{}^{\mathbb{I}}\mathbb{N}_{\mathbb{I}\mathbb{J}}\widetilde{\mathcal{O}}^{\mathbb{J}}{}_{\mathbb{B}}\overline{W}^{\mathbb{B}} - \frac{1}{2\mathcal{K}}W^{\mathbb{A}}S_{\mathbb{A}\mathbb{B}}\mathcal{O}^{\mathbb{B}}{}_{\mathbb{I}}\left(V^{\mathbb{I}}\overline{V}^{\mathbb{J}} + \overline{V}^{\mathbb{I}}V^{\mathbb{J}}\right)(\mathcal{O}^T)_{\mathbb{J}}{}^{\mathbb{C}}(S^T)_{\mathbb{C}\mathbb{D}}\overline{W}^{\mathbb{D}}\Big] + \frac{1}{2}\left(\mathsf{G}^{\mathbb{A}}_{\text{flux}} + \mathsf{C}^{\mathbb{I}}_0(\mathcal{O}^T)_{\mathbb{I}}{}^{\mathbb{A}}\right)\mathbb{M}_{\mathbb{A}\mathbb{B}}\left(\mathsf{G}^{\mathbb{B}}_{\text{flux}} + \mathcal{O}^{\mathbb{B}}{}_{\mathbb{J}}\mathsf{C}^{\mathbb{J}}_0\right) \tag{5.4.42}$$

for the type IIB case. Comparing the reduced potentials reveals the same set of mirror mappings (5.3.81) already encountered in the Calabi-Yau setting (including a reflection of the Hodge diamond (5.4.9) onto itself). One can furthermore see from the structure of the $K3\times T^2$ gauge coupling matrices (5.4.33) and (5.4.36) that the mappings $\mathbb{M}_{\mathbb{A}\mathbb{B}} \leftrightarrow \mathbb{N}_{\mathbb{I}\mathbb{J}}$ involve a characteristic exchange of the volume and complex-structure modular parameters in a complex torus

$$\tau \leftrightarrow \rho\,. \tag{5.4.43}$$

Assessing the effect of Mirror Symmetry on the $K3$-part is less straightforward since the moduli space does not factorize into a complex-structure and a Kähler component as in the case of Calabi-Yau three-folds. In this setting, the complex-structure moduli can be interpreted as coefficients arising from the variations [221]

$$\delta g_{ab} \sim \Omega_{ac}g^{c\bar{d}}\chi_{b\bar{d}} + \Omega_{bc}g^{c\bar{d}}\chi_{a\bar{d}}\,, \tag{5.4.44}$$

where χ denotes some closed $(1,1)$-form on $K3$. If χ is proportional to the Kähler-form, this variation vanishes, and one is left with a total of 19 complex-structure moduli which naturally combine with the $(2,0)$- and $(0,2)$-components of the B-field. On the other hand, the Kähler moduli and the $(1,1)$-components of the B-field again form 20 complexified Kähler moduli. Mirror Symmetry thus involves a similar exchange of roles between the "extended" complex-structure and the complexified Kähler moduli as in the CY_3-setting, although the moduli space does not factorize.

5.5 Obtaining the full Action of $\mathcal{N}=2$ Gauged Supergravity

We next show how the framework can be extended to the kinetic terms of type II theories. This will allow us to derive the full four-dimensional action of $\mathcal{N}=2$ gauged supergravity from the Calabi-Yau setting. In doing so, we again set cohomologically trivial terms to zero at the beginning of the calculation. A more thorough analysis similar to section 5.3 and dimensional reductions on $K3 \times T^2$ are more involved due to the appearance of additional Kaluza-Klein-like terms and will not be covered in this thesis.

The primary objective of our upcoming analysis is to relate the full double field theory action to a particular formulation of four-dimensional $\mathcal{N} = 2$ gauged supergravity first presented in [202]. In accordance with the original work, we will adopt the assumptions $h^{1,1} \leq h^{1,2}$ for the type IIA case and $h^{1,1} \geq h^{1,2}$ for the type IIB case. We furthermore assume the matrix $\mathcal{O}^{\mathbb{A}}{}_{\mathsf{I}}$ or $\widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathsf{A}}$ to have maximal rank $h^{1,1}+1$ or $h^{1,2}+1$ in the respective settings.

5.5.1 NS-NS Sector

Due to the vanishing first and fifth Betti numbers of Calabi-Yau three-folds, there do not exist any non-trivial one- or five-cycles on CY_3. It follows that all fields with effectively one or five free internal indices acquire mass in four dimensions and can be ignored in the low-energy limit. One immediate effect is that all components of the metric and the Kalb-Ramond field with mixed indices can be discarded, which drastically simplifies the expressions (5.2.8) and (5.2.9) building up the NS-NS contribution (5.2.7) to the action,

$$\widetilde{\mathcal{F}}^I{}_{\mu\nu} \to 0, \qquad \widetilde{\mathcal{H}}_{\mu\nu\rho} \to \partial_{[\underline{\mu}} B_{\underline{\nu\rho}]}, \qquad D_\mu \mathcal{H}_{IJ} \to \partial_\mu \mathcal{H}_{IJ}, \qquad \mathcal{F}_I \to 0, \tag{5.5.1}$$

and leaves us with

$$\begin{aligned} S_{\text{NS-NS}} = & \frac{1}{2}\int \mathrm{d}^4x\, \mathrm{d}^{12}Y\, \sqrt{-g^{(4)}}\sqrt{g_{CY_3}}e^{-2\hat{\phi}}\Big[\\ & R^{(4)} + 4g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{12}g^{\mu\nu}g^{\rho\sigma}g^{\tau\lambda}\partial_{[\underline{\mu}}B_{\underline{\rho\tau}]}\partial_{[\underline{\nu}}B_{\underline{\sigma\lambda}]} + \frac{1}{8}g^{\mu\nu}\partial_\mu\mathcal{H}_{IJ}\partial_\nu\mathcal{H}^{IJ} \\ & + \mathcal{F}_{IJK}\mathcal{F}_{I'J'K'}\left(-\frac{1}{12}\mathcal{H}^{II'}\mathcal{H}^{JJ'}\mathcal{H}^{KK'} + \frac{1}{4}\mathcal{H}^{II'}\eta^{JJ'}\eta^{KK'} - \frac{1}{6}\eta^{II'}\eta^{JJ'}\eta^{KK'}\right)\Big]. \end{aligned} \tag{5.5.2}$$

The first three terms are known from conventional type II supergravities, while the last two lines were shown to correctly reduce to the scalar potential of $\mathcal{N} = 2$ gauged supergravity in section 5.3. It is therefore to be expected that the remaining term $\frac{1}{8}g^{\mu\nu}\partial_\mu\mathcal{H}_{IJ}\partial_\nu\mathcal{H}^{IJ}$ gives rise to the kinetic terms of the Kähler-class and complex-structure moduli. And indeed, inserting (4.1.3) and using antisymmetry of the Kalb-Ramond field, one obtains

$$\frac{1}{8}g^{\mu\nu}\partial_\mu\mathcal{H}_{IJ}\partial_\nu\mathcal{H}^{IJ} = \frac{1}{4}g^{\mu\nu}\left(\partial_\mu g_{ij}\partial_\nu g^{ij} - g^{ik}g^{jl}\partial_\mu b_{ij}\partial_\nu b_{kl}\right). \tag{5.5.3}$$

Here, the first expression on the right-hand side encodes the dynamics of the internal metric and its fluctuations about the vacuum. Analogously to conventional Calabi-Yau compactifications, these fluctuations can be described in terms of the Kähler class and complex-structure moduli [110]. For the Kalb-Ramond field, one can proceed analogously by using the expansion (5.3.44), which combines with the Kähler-class moduli to form the complexified Kähler moduli.

Using this as a starting point, the rest of the dimensional reduction follows the same principles as in conventional Calabi-Yau compactifications of type II theories. Explicit computations can be found in [116–118], a similar discussion concerning manifolds with $SU(3)\times SU(3)$ structure is presented in [220, 199]. Defining the four-dimensional dilaton as

$$e^{-2\phi} = \int_{CY_3} e^{-2\hat{\phi}} \tag{5.5.4}$$

and switching to Einstein frame via Weyl-rescaling of the external metric,

$$g_{\mu\nu} \to e^{-2\phi} g_{\mu\nu}\,, \tag{5.5.5}$$

one eventually arrives at

$$S_{\text{NS-NS, kin}} = \int_{M^{1,3}} \frac{1}{2} R^{(4)} \star \mathbf{1}^{(4)} - \mathrm{d}\phi \wedge \star \mathrm{d}\phi - \frac{1}{4} e^{-4\phi} \mathrm{d}B \wedge \star \mathrm{d}B - g_{\mathsf{ij}} \mathrm{d}t^{\mathsf{i}} \wedge \star \mathrm{d}\bar{t}^{\mathsf{j}} - g_{\mathsf{ab}} \mathrm{d}U^{\mathsf{a}} \wedge \star \mathrm{d}\overline{U}^{\mathsf{b}}, \tag{5.5.6}$$

where we again switched to differential form notation for later convenience. In the above formulation, the expansion coefficients t^{i} (cf. (5.3.45)) again parameterize the complexified Kähler-class moduli space M_{KC} with metric g_{ij}, and U^{a} the complex-structure moduli space M_{CS} with metric g_{ab}.

5.5.2 R-R Sector

A natural way to proceed for the R-R sector would be to evaluate the corresponding action of (5.2.14) in four dimensions and then implement the duality relations (5.2.15) in order to recover the action of $\mathcal{N}=2$ gauged supergravity. Since handling these duality relations in four dimensions turns out rather complicated, we will, however, pursue a different approach and consider the reduced equations of motion instead. Notice that this has been done for compactifications on $SU(3)\times SU(3)$ structure manifolds in [199], and many of the following technical steps are close to the ones employed in this work.

Type IIA Setting

Relation to Democratic Type IIA Supergravity

Starting from (5.2.14), a first step is to write down the pseudo-action explicitly in terms of poly-form fields and obtain a form similar to (5.3.35). In doing so, we again neglect all cohomologically trivial expressions and, thus, take into account only those components with zero, two, three, four or six internal indices. Applying the methods presented in [211] and arranging the (now ten-dimensional) $\hat{C}$-fields and R-R fluxes in poly-forms

$$\begin{aligned} \hat{\mathcal{C}}^{(\text{IIA})} &= \hat{C}_1 + \hat{C}_3 + \hat{C}_5 + \hat{C}_7 + \hat{C}_9, \\ \mathcal{G}^{(\text{IIA})} &= G_0 + G_2 + G_4 + G_6, \end{aligned} \tag{5.5.7}$$

we can define

$$\hat{\mathfrak{G}}^{(\mathrm{IIA})} = e^{-\hat{B}}\mathcal{G}^{(\mathrm{IIA})} + \hat{\mathfrak{D}}\hat{\mathcal{C}}^{(\mathrm{IIA})} = e^{-\hat{B}}\mathcal{G}^{(\mathrm{IIA})} + e^{-\hat{B}}\hat{\mathcal{D}}\left(e^{\hat{B}}\hat{\mathcal{C}}^{(\mathrm{IIA})}\right), \tag{5.5.8}$$

with the ten-dimensional twisted differential

$$\hat{\mathcal{D}} = \hat{\mathrm{d}} - H\wedge - F\circ - Q\bullet - R_{\llcorner}, \tag{5.5.9}$$

to write the complete type IIA R-R pseudo-Lagrangian (5.2.14) as

$$\star\mathcal{L}_{\text{R-R}} = -\frac{1}{2}\hat{\mathfrak{G}}^{(\mathrm{IIA})}\wedge\star\hat{\mathfrak{G}}^{(\mathrm{IIA})}. \tag{5.5.10}$$

Notice that this resembles the R-R sector of democratic type IIA supergravity [210], up to an exchange of signs in the exponential factors and the inclusion of additional background fluxes. Since the action depends on all R-R potentials explicitly, the duality relations (5.2.15) have to be imposed by hand. For the type IIA case, these are equivalent to

$$\hat{\mathfrak{G}}^{(\mathrm{IIA})} = \lambda\left(\star\hat{\mathfrak{G}}^{(\mathrm{IIA})}\right), \tag{5.5.11}$$

where λ denotes the involution operator defined in (5.2.28). Varying the corresponding action of (5.5.10) with respect to the R-R fields, one obtains the poly-form equation

$$\left(\hat{\mathrm{d}} - \mathrm{d}\hat{B}\wedge + \mathfrak{H}\wedge + \mathfrak{F}\circ + \mathfrak{Q}\bullet + \mathfrak{R}_{\llcorner}\right)\star\hat{\mathfrak{G}}^{(\mathrm{IIA})} = 0. \tag{5.5.12}$$

Employing the duality relations (5.5.11), this can be recast to a set of Bianchi identities

$$e^{-\hat{B}}\hat{\mathcal{D}}\left(e^{\hat{B}}\hat{\mathfrak{G}}^{(\mathrm{IIA})}\right) = 0, \tag{5.5.13}$$

where the prefactor of $e^{-\hat{B}}$ was included for later convenience. These relations are automatically satisfied when imposing nilpotency of the twisted differential by hand, and the nontrivial equations of motion in four dimensions now arise from the duality constraints (5.5.11).

Reduced Equations of Motion

In order to evaluate the equations of motion in four dimensions, we next express the appearing objects in a way that the framework of special geometry presented in subsection 5.3.3 can be applied. This can be achieved by switching to an alternative basis [210], for which we define

$$e^{\hat{B}}\mathcal{C}^{(\mathrm{IIA})} = \left(\mathsf{C}_1^{\mathsf{I}} + \mathsf{C}_3^{\mathsf{I}}\right)\omega_{\mathsf{I}} + \left(\mathsf{C}_0^{\mathsf{A}} + \mathsf{C}_2^{\mathsf{A}} + \mathsf{C}_4^{\mathsf{A}}\right)\alpha_{\mathsf{A}} - \left(\mathsf{C}_{0\,\mathsf{A}} + \mathsf{C}_{2\,\mathsf{A}} + \mathsf{C}_{4\,\mathsf{A}}\right)\beta^{\mathsf{A}} + \left(\mathsf{C}_{1\,\mathsf{I}} + \mathsf{C}_{3\,\mathsf{I}}\right)\widetilde{\omega}^{\mathsf{I}} \tag{5.5.14}$$

and

$$G_0 = \mathsf{G}_{\text{flux}\,0}\widetilde{\omega}^0, \quad G_2 = \mathsf{G}^{\mathsf{i}}_{\text{flux}}\omega_{\mathsf{i}}, \quad G_4 = \mathsf{G}_{\text{flux}\,\mathsf{i}}\widetilde{\omega}^{\mathsf{i}}, \quad G_6 = \mathsf{G}^0_{\text{flux}}\,\omega_0, \tag{5.5.15}$$

where the objects C_n now denote differential n-forms living in four dimensional spacetime. The R-R poly-form (5.5.8) can then be expressed as

$$\hat{\mathfrak{G}}^{(\text{IIA})} = e^{-\hat{B}}\hat{\mathsf{G}}^{(\text{IIA})} = e^{-\hat{B}}\left(\hat{\mathsf{G}}_0^{(\text{IIA})} + \hat{\mathsf{G}}_2^{(\text{IIA})} + \hat{\mathsf{G}}_4^{(\text{IIA})} + \hat{\mathsf{G}}_6^{(\text{IIA})} + \hat{\mathsf{G}}_8^{(\text{IIA})} + \hat{\mathsf{G}}_{10}^{(\text{IIA})}\right). \tag{5.5.16}$$

Using the flux matrices (5.3.66) and the relations (5.3.68), the appearing poly-forms can be expanded in terms of four-dimensional differential form fields,

$$\begin{aligned}
\hat{\mathsf{G}}_0^{(\text{IIA})} &= \mathsf{G}_{0\,0}\widetilde{\omega}^0,\\
\hat{\mathsf{G}}_2^{(\text{IIA})} &= \mathsf{G}_{2\,0}\widetilde{\omega}^0 + \mathsf{G}_0^{\mathsf{i}}\omega_{\mathsf{i}},\\
\hat{\mathsf{G}}_4^{(\text{IIA})} &= \mathsf{G}_{4\,0}\widetilde{\omega}^0 + \mathsf{G}_2^{\mathsf{i}}\wedge\omega_{\mathsf{i}} - \mathsf{G}_1^{\mathsf{A}}\wedge\alpha_{\mathsf{A}} + \mathsf{G}_{1\,\mathsf{A}}\wedge\beta^{\mathsf{A}} + \mathsf{G}_{0\,\mathsf{i}}\widetilde{\omega}^{\mathsf{i}},\\
\hat{\mathsf{G}}_6^{(\text{IIA})} &= \mathsf{G}_4^{\mathsf{i}}\wedge\omega_{\mathsf{i}} - \mathsf{G}_3^{\mathsf{A}}\wedge\alpha_{\mathsf{A}} + \mathsf{G}_{3\,\mathsf{A}}\wedge\beta^{\mathsf{A}} + \mathsf{G}_{2\,\mathsf{i}}\wedge\widetilde{\omega}^{\mathsf{i}} + \mathsf{G}_0^0\wedge\omega_0,\\
\hat{\mathsf{G}}_8^{(\text{IIA})} &= \mathsf{G}_{4\,\mathsf{i}}\wedge\widetilde{\omega}^{\mathsf{i}} + \mathsf{G}_2^0\wedge\omega_0,\\
\hat{\mathsf{G}}_{10}^{(\text{IIA})} &= \mathsf{G}_4^0\wedge\omega_0,
\end{aligned} \tag{5.5.17}$$

with the expansion coefficients given by

$$\begin{aligned}
\mathsf{G}_0^{\mathbb{I}} &= \mathsf{G}_{\text{flux}}^{\mathbb{I}} + \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}\mathsf{C}_0^{\mathbb{A}},\\
\mathsf{G}_1^{\mathbb{A}} &= \mathrm{d}\mathsf{C}_0^{\mathbb{A}} + \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{C}_1^{\mathbb{I}},\\
\mathsf{G}_2^{\mathbb{I}} &= \mathrm{d}\mathsf{C}_1^{\mathbb{I}} + \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}\mathsf{C}_2^{\mathbb{A}},\\
\mathsf{G}_3^{\mathbb{A}} &= \mathrm{d}\mathsf{C}_2^{\mathbb{A}} + \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{C}_3^{\mathbb{I}},\\
\mathsf{G}_4^{\mathbb{I}} &= \mathrm{d}\mathsf{C}_3^{\mathbb{I}} + \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}\mathsf{C}_4^{\mathbb{A}}.
\end{aligned} \tag{5.5.18}$$

This expansion can be used as a starting point to compute the reduced equations of motion descending from (5.5.13). Substituting the definition (5.5.16) into (5.5.13), one obtains in A-basis notation

$$\hat{\mathcal{D}}\hat{\mathsf{G}}^{(\text{IIA})} = 0. \tag{5.5.19}$$

After separating different components and integrating over CY_3, this gives rise to the four-dimensional equations of motion

$$\begin{aligned}
\mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{G}_0^{\mathbb{I}} &= 0,\\
\mathrm{d}\mathsf{G}_0^{\mathbb{I}} - \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}\mathsf{G}_1^{\mathbb{A}} &= 0,\\
\mathrm{d}\mathsf{G}_1^{\mathbb{A}} - \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{G}_2^{\mathbb{I}} &= 0,\\
\mathrm{d}\mathsf{G}_2^{\mathbb{I}} - \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}\mathsf{G}_3^{\mathbb{A}} &= 0,\\
\mathrm{d}\mathsf{G}_3^{\mathbb{A}} - \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{G}_4^{\mathbb{I}} &= 0.
\end{aligned} \tag{5.5.20}$$

Since the Kalb-Ramond field couples with the C-fields, one furthermore has to take into account the (non-trivial) equation of motion obtained by varying the complete ten-dimensional action with respect to $\hat{B}$, which yields an eight-form equation

$$\mathrm{d}\left(e^{-2\hat{\phi}}\star\hat{\mathrm{d}}\hat{B}\right)+\left[\hat{\mathfrak{G}}^{(\mathrm{IIA})}\wedge\star\hat{\mathfrak{G}}^{(\mathrm{IIA})}\right]_8=0. \tag{5.5.21}$$

Reduced Duality Constraints

Our aim is now to implement the duality constraints (5.5.11) into the equations of motion (5.5.20) and (5.5.21) in an appropriate way in order to recover the $D=4$ $\mathcal{N}=2$ gauged supergravity action found in formula (35) of [202]. In particular, we want the fundamental (but not necessarily propagating) degrees of freedom to be given by[2] $2h^{1,2}+2$ scalars $\hat{Z}^{\mathbb{A}}$, $h^{1,1}+1$ one-forms A_1^{I}, $2h^{1,2}+2$ two-forms $B^{\mathbb{A}}$ and the external Kalb-Ramond field B.

Up to conventions, the reduced duality constraints can be obtained in a way completely analogous to the approach of [199]. Inserting the expansion

$$e^{-\hat{B}}\hat{\mathsf{G}}^{(\mathrm{IIA})}=e^{-b}\left(K^{\mathsf{I}}\omega_{\mathsf{I}}+K_{\mathsf{I}}\widetilde{\omega}^{\mathsf{I}}+L^{\mathsf{A}}\alpha_{\mathsf{A}}-L_{\mathsf{A}}\beta^{\mathsf{A}}\right) \tag{5.5.22}$$

into (5.5.11), one obtains

$$K^{\mathsf{I}}\omega_{\mathsf{I}}+K_{\mathsf{I}}\widetilde{\omega}^{\mathsf{I}}+L^{\mathsf{A}}\alpha_{\mathsf{A}}-L_{\mathsf{A}}\beta^{\mathsf{A}}=-\star\lambda\left(K^{\mathsf{I}}\right)\star_b\omega_{\mathsf{I}}-\star\lambda\left(K_{\mathsf{I}}\right)\star_b\widetilde{\omega}^{\mathsf{I}}-\star\lambda\left(L^{\mathsf{A}}\right)\star_b\alpha_{\mathsf{A}}+\star\lambda\left(L_{\mathsf{A}}\right)\star_b\beta^{\mathsf{A}}. \tag{5.5.23}$$

Applying the operators $\int_{CY_3}\left\langle\widetilde{\omega}^{\mathsf{I}},\star_b\cdot\right\rangle$ and $\int_{CY_3}\left\langle\beta^{\mathsf{A}},\star_b\cdot\right\rangle$ to both sides of the equation and using (5.3.57 - 5.3.59), one can separate different internal components and obtain the reduced duality constraints

$$\begin{aligned}K_{\mathsf{I}}&=-\mathrm{Im}\mathcal{N}_{\mathsf{IJ}}\star\lambda\left(K^{\mathsf{J}}\right)+\mathrm{Re}\mathcal{N}_{\mathsf{IJ}}K^{\mathsf{J}},\\L_{\mathsf{A}}&=-\mathrm{Im}\mathcal{M}_{\mathsf{AB}}\star\lambda\left(L^{\mathsf{B}}\right)+\mathrm{Re}\mathcal{M}_{\mathsf{AB}}L^{\mathsf{B}}.\end{aligned} \tag{5.5.24}$$

The K- and L-poly-forms still contain four-dimensional differential forms of different degrees. Separating components by hand and performing a Weyl-rescaling (5.5.5) according to (5.5.5), we eventually arrive at

$$\begin{aligned}\mathsf{G}_{2\,\mathsf{I}}-B\mathsf{G}_{0\,\mathsf{I}}&=\mathrm{Im}\mathcal{N}_{\mathsf{IJ}}\star\left(\mathsf{G}_2^{\mathsf{J}}-B\wedge\mathsf{G}_0^{\mathsf{J}}\right)+\mathrm{Re}\mathcal{N}_{\mathsf{IJ}}\left(\mathsf{G}_2^{\mathsf{J}}-B\wedge\mathsf{G}_0^{\mathsf{J}}\right),\\\mathsf{G}_4^{\mathbb{I}}-B\wedge\mathsf{G}_2^{\mathbb{I}}+\frac{1}{2}B^2\mathsf{G}_0^{\mathbb{I}}&=-e^{4\phi}\left(S^{-1}\right)^{\mathbb{IJ}}\mathbb{N}_{\mathbb{JK}}\mathsf{G}_0^{\mathbb{K}}\star\mathbf{1}^{(4)},\\\mathsf{G}_3^{\mathbb{A}}-B\wedge\mathsf{G}_1^{\mathbb{A}}&=e^{2\phi}\left(S^{-1}\right)^{\mathbb{AB}}\mathbb{M}_{\mathbb{BC}}\star\mathsf{G}_1^{\mathbb{C}}.\end{aligned} \tag{5.5.25}$$

Evaluating the Equations of Motion – Constraints on Fluxes

Before implementing the duality constraints, it makes sense to take a closer look at the first line of (5.5.20). Unlike the remaining equations of motion, the left-hand side does

[2] We preliminarily adopt the notation of [202] and identify the correct definitions in the course of the following discussion.

not vanish trivially when imposing the nilpotency conditions (5.3.70). Instead, we are left with a set of additional constraints, which take the form

$$\mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{G}^{\mathbb{I}}_{\text{flux}} = 0 \tag{5.5.26}$$

and resemble the conditions found in (37) of [202]. Notice that these arise automatically from the double field theory framework and do not have to be imposed by hand in our considered setting.

Evaluating the Equations of Motion – $\mathsf{C}_1^{\mathsf{I}}$

The simplest equations of motion to derive are those of the one-forms $\mathsf{C}_1^{\mathsf{I}}$, which we will be able to identify with the fields A_1^{I} from [202] at the end of this subsection. In order to get some intuition for the way of proceeding, we will treat this example in more detail. A similar strategy can then be followed for the remaining degrees of freedom.

Many of the technical steps in the following discussion are again very close to the ones employed in [199]. The essential difference is that in the present setting, the expressions (5.5.18) are fixed by the double field theory action, whereas in the case of [199], their structure was described solely in terms of the equations of motion (5.5.20). This leads to slight redefinitions of the encountered objects, but will eventually lead to the same physical degrees of freedom as in the $SU(3)\times SU(3)$ framework.

To motivate our ansatz, it makes sense to first take a look at the equations of motion obtained by varying the action found in [202] with respect to the A_1^{I},

$$\mathrm{d}\left(\mathrm{Im}\mathcal{N}_{\mathsf{IJ}}\star\mathsf{F}_2^{\mathsf{J}}+\mathrm{Re}\mathcal{N}_{\mathsf{IJ}}\mathsf{F}_2^{\mathsf{J}}-e_{\mathsf{I}\mathbb{A}}B^{\mathbb{A}}-c_{\mathsf{I}}B\right)=0. \tag{5.5.27}$$

The first two terms appearing on the left-hand side have a very characterstic structure and strongly resemble the first line of (5.5.25). Furthermore, the term $B\mathsf{G}_{0\mathsf{I}}$ already shows some resemblance of the expression $c_{\mathsf{I}}B$ from the equation of motion (5.5.27). A viable ansatz is therefore to replace $\mathsf{G}_{2\mathsf{I}}$ in the lower-index components of the fourth equation of motion from (5.5.20) by using line one of (5.5.25). Applying the nilpotency constraint (5.3.70) of $\mathcal{D}$, the former can be written as

$$\mathrm{d}\mathsf{G}_{2\mathsf{I}}-\widetilde{\mathcal{O}}_{\mathsf{I}\mathbb{A}}\mathrm{d}\mathsf{C}_2^{\mathbb{A}}=0. \tag{5.5.28}$$

Substituting the first line of (5.5.25) into $\mathsf{G}_{2\mathsf{I}}$ yields

$$\mathrm{d}\left(\mathrm{Im}\mathcal{N}_{\mathsf{IJ}}\star\mathsf{F}_2^{\mathsf{J}}+\mathrm{Re}\mathcal{N}_{\mathsf{IJ}}\mathsf{F}_2^{\mathsf{J}}-\widetilde{\mathcal{O}}_{\mathsf{I}\mathbb{A}}\mathsf{C}_2^{\mathbb{A}}+B\wedge\mathsf{G}_{0\mathsf{I}}\right)=0, \tag{5.5.29}$$

where

$$\mathsf{F}_2^{\mathsf{I}}:=\mathsf{G}_2^{\mathsf{I}}-B\wedge\mathsf{G}_0^{\mathsf{I}}. \tag{5.5.30}$$

This can be further simplified by pulling out a factor of $B\wedge$ from the definition (5.5.14) of $\mathsf{C}_2^{\mathbb{A}}$. We do this by employing the alternative expansion

$$
\begin{aligned}
e^{b}\hat{\mathcal{C}}^{(\mathrm{IIA})} =& \left(\widetilde{\mathsf{C}}_1^{\mathsf{I}} + \widetilde{\mathsf{C}}_3^{\mathsf{I}}\right)\omega_{\mathsf{I}} \\
&+ \left(\widetilde{\mathsf{C}}_0^{\mathsf{A}} + \widetilde{\mathsf{C}}_2^{\mathsf{A}} + \widetilde{\mathsf{C}}_4^{\mathsf{A}}\right)\alpha_{\mathsf{A}} - \left(\widetilde{\mathsf{C}}_{0\,\mathsf{A}} + \widetilde{\mathsf{C}}_{2\,\mathsf{A}} + \widetilde{\mathsf{C}}_{4\,\mathsf{A}}\right)\beta^{\mathsf{A}} \\
&+ \left(\widetilde{\mathsf{C}}_{1\,\mathsf{I}} + \widetilde{\mathsf{C}}_{3\,\mathsf{I}}\right)\widetilde{\omega}^{\mathsf{I}},
\end{aligned}
\tag{5.5.31}
$$

from which we infer the relation

$$
\mathsf{C}_2^{\mathbb{A}} = \widetilde{\mathsf{C}}_2^{\mathbb{A}} + B \wedge \mathsf{C}_0^{\mathbb{A}},
\tag{5.5.32}
$$

while the other fields appearing in (5.5.29) remain unaffected. Inserting the definitions (5.5.18) for the $\mathsf{G}_{0\,\mathsf{I}}$, we are left with

$$
\mathsf{F}_2^{\mathsf{I}} = \mathrm{d}\mathsf{C}_1^{\mathsf{I}} + \widetilde{\mathcal{O}}^{\mathsf{I}}{}_{\mathbb{A}}\widetilde{\mathsf{C}}_2^{\mathbb{A}} - B \wedge \mathsf{G}^{\mathsf{I}}_{\mathrm{flux}}
\tag{5.5.33}
$$

and the equations of motion

$$
\mathrm{d}\left(\mathrm{Im}\mathcal{N}_{\mathsf{IJ}} \star \mathsf{F}_2^{\mathsf{J}} + \mathrm{Re}\mathcal{N}_{\mathsf{IJ}}\mathsf{F}_2^{\mathsf{J}} - \widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}\widetilde{\mathsf{C}}_2^{\mathbb{A}} + B \wedge \mathsf{G}_{\mathsf{I}\,\mathrm{flux}}\right) = 0.
\tag{5.5.34}
$$

Up to sign convention for B, these take precisely the form of (5.5.27) when identifying $A_1^{\mathsf{I}} = \mathsf{C}_1^{\mathsf{I}}$, $B^{\mathbb{A}} = \widetilde{\mathsf{C}}_2^{\mathbb{A}}$, $e_{\mathsf{I}\,\mathbb{A}} = \widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}$ and $c_{\mathsf{I}} = \mathsf{G}_{\mathsf{I}\,\mathrm{flux}}$.

Evaluating the Equations of Motion – $\widetilde{\mathsf{C}}_2^{\mathbb{A}}$

A similar analysis for the fields $B^{\mathbb{A}}$ in [202] implies that a viable strategy is to use lines one and three of the duality constraints (5.5.25) to reformulate the third equation of motion in (5.5.20). For this, we introduce a new matrix $\check{\mathcal{O}}^{\mathsf{I}}{}_{\mathbb{A}}$ defined to satisfy [202]

$$
\check{\mathcal{O}}^{\mathsf{I}}{}_{\mathbb{A}}\mathcal{O}^{\mathbb{A}}{}_{\mathsf{J}} = \delta^{\mathsf{I}}{}_{\mathsf{J}}\,,
\tag{5.5.35}
$$

which in turn can be used to construct the projector

$$
\mathcal{P}^{\mathbb{A}}{}_{\mathbb{B}} := \mathcal{O}^{\mathbb{A}}{}_{\mathsf{I}}\check{\mathcal{O}}^{\mathsf{I}}{}_{\mathbb{B}}
\tag{5.5.36}
$$

on the $(h^{1,1}+1)$-dimensional subspace corresponding to the non-vanishing minor of $\mathcal{O}^{\mathbb{A}}{}_{\mathsf{I}}$. We can then formally split the $2(h^{1,2}+1)$ scalars $\mathsf{C}_0^{\mathbb{A}}$ into two components

$$
\mathsf{C}_0^{\mathbb{A}} = \mathcal{P}^{\mathbb{A}}{}_{\mathbb{B}}\mathsf{C}_0^{\mathbb{B}} + \widetilde{\mathsf{C}}_0^{\mathbb{A}}
\tag{5.5.37}
$$

and identify $\widetilde{\mathsf{C}}_0^{\mathbb{A}} := (\delta^{\mathbb{A}}{}_{\mathbb{B}} - \mathcal{P}^{\mathbb{A}}{}_{\mathbb{B}})\mathsf{C}_0^{\mathbb{B}}$ with the $2(h^{1,2}+1) - (h^{1,1}+1)$ propagating degrees of freedom encoded by the scalars $\hat{Z}^{\mathbb{A}}$ from [202]. Our aim is now to rewrite the third equation of motion from (5.5.20) in such a way that only the fields $\widetilde{\mathsf{C}}_0^{\mathbb{A}}$, $\mathsf{C}_1^{\mathsf{I}}$, $\widetilde{\mathsf{C}}_2^{\mathbb{A}}$ and B appear explicitly. This can be done by first left-multiplying line three of (5.5.25) with $\widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}$, yielding

$$
\widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}\mathrm{d}\mathsf{C}_2^{\mathbb{A}} - B \wedge \mathrm{d}(\widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}\widetilde{\mathsf{C}}_0^{\mathbb{A}}) = e^{2\phi}\widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}\left(S^{-1}\right)^{\mathbb{A}\mathbb{B}}\mathbb{M}_{\mathbb{B}\mathbb{C}} \star \mathsf{G}_1^{\mathbb{C}}\,,
\tag{5.5.38}
$$

where we in particular used that $\widetilde{\mathcal{O}}_{\mathsf{I}\mathbb{A}}\mathcal{P}^{\mathbb{A}}{}_{\mathbb{B}}$ vanishes due to (5.3.70). Employing the expansion (5.5.31) and using that $\mathcal{P}^{\mathbb{A}}{}_{\mathbb{B}}\mathcal{O}^{\mathbb{B}}{}_{\mathbb{I}} = \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}$, we obtain

$$\mathcal{P}^{\mathbb{A}}{}_{\mathbb{B}}\mathsf{C}_0^{\mathbb{B}} + \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{C}_1^{\mathbb{I}} = -\mathcal{O}^{\mathbb{A}}{}_{\mathsf{I}}(\Delta^{-1})^{\mathsf{IJ}}\left(\star\mathrm{d}(\widetilde{\mathcal{O}}_{\mathsf{J}\mathbb{B}}\widetilde{\mathsf{C}}_2^{\mathbb{B}}) + \widetilde{\mathcal{O}}_{\mathsf{J}\mathbb{B}}\widetilde{\mathsf{C}}_0^{\mathbb{B}}\star\mathrm{d}B + e^{2\phi}(\mathcal{O}^T)_{\mathsf{J}}{}^{\mathbb{B}}\mathbb{M}_{\mathbb{B}\mathbb{C}}\mathrm{d}\widetilde{\mathsf{C}}_0^{\mathbb{C}}\right), \tag{5.5.39}$$

with

$$\Delta_{\mathsf{IJ}} = e^{2\phi}(\mathcal{O}^T)_{\mathsf{I}}{}^{\mathbb{A}}\mathbb{M}_{\mathbb{A}\mathbb{B}}\mathcal{O}^{\mathbb{B}}{}_{\mathsf{J}}. \tag{5.5.40}$$

Starting from line three of (5.5.20), we separate components to get

$$\mathrm{d}\mathsf{G}_1^{\mathbb{A}} - \mathrm{d}(\mathcal{O}^{\mathbb{A}}{}_{\mathsf{I}}\mathsf{C}_1^{\mathsf{I}}) - \mathcal{O}^{\mathbb{A}}{}_{\mathsf{I}}\widetilde{\mathcal{O}}^{\mathsf{I}}{}_{\mathbb{B}}\mathsf{C}_2^{\mathbb{B}} - \mathcal{O}^{\mathbb{A}\,\mathsf{I}}\mathsf{G}_{2\,\mathsf{I}} = 0. \tag{5.5.41}$$

In this formulation, the third term can be substituted by the identity

$$\mathcal{O}^{\mathbb{A}}{}_{\mathsf{I}}\widetilde{\mathcal{O}}^{\mathsf{I}}{}_{\mathbb{B}}\mathsf{C}_2^{\mathbb{B}} = -\mathcal{O}^{\mathbb{A}\,\mathsf{I}}\widetilde{\mathcal{O}}_{\mathsf{I}\mathbb{B}}\mathsf{C}_2^{\mathbb{B}} \tag{5.5.42}$$

derived from (5.3.70) and the fourth term by line two of (5.5.25). Inserting then the previously derived relation (5.5.20) into $\mathsf{G}_1^{\mathbb{A}}$, we obtain after left-multiplication with $S_{\mathbb{A}\mathbb{B}}$

$$\begin{aligned}0 = &-\mathrm{d}\left[(\widetilde{\mathcal{O}}^T)_{\mathbb{A}\mathsf{I}}(\Delta^{-1})^{\mathsf{IJ}}\left(\star\mathrm{d}(\widetilde{\mathcal{O}}_{\mathsf{J}\mathbb{B}}\widetilde{\mathsf{C}}_2^{\mathbb{B}}) + \widetilde{\mathcal{O}}_{\mathsf{J}\mathbb{B}}\widetilde{\mathsf{C}}_0^{\mathbb{B}}\star\mathrm{d}B + e^{2\phi}(\mathcal{O}^T)_{\mathsf{J}}{}^{\mathbb{B}}\mathbb{M}_{\mathbb{B}\mathbb{C}}\mathrm{d}\widetilde{\mathsf{C}}_0^{\mathbb{C}}\right)\right]\\ &-\mathrm{d}(\widetilde{\mathcal{O}}^T)_{\mathbb{A}\mathsf{I}}\mathsf{C}_1^{\mathsf{I}} + (\widetilde{\mathcal{O}}^T)_{\mathbb{A}}{}^{\mathsf{I}}\left(\mathrm{Im}\mathcal{N}_{\mathsf{IJ}}\star\mathsf{F}_2^{\mathsf{J}} + \mathrm{Re}\mathcal{N}_{\mathsf{IJ}}\mathsf{F}_2^{\mathsf{J}} + B\wedge\mathsf{G}_{\mathsf{I}\,\mathrm{flux}} - \widetilde{\mathcal{O}}_{\mathsf{I}\mathbb{B}}\widetilde{\mathsf{C}}_2^{\mathbb{B}}\right).\end{aligned} \tag{5.5.43}$$

Evaluating the Equations of Motion – $\widetilde{\mathsf{C}}_0^{\mathbb{A}}$

Following the same procedure once more, we implement lines two and three of (5.5.25) into the fifth equation of motion of (5.5.20). Simplifying via equations of motion one and three, we obtain

$$\mathrm{d}\left[e^{2\phi}(S^{-1})^{\mathbb{A}\mathbb{B}}\mathbb{M}_{\mathbb{B}\mathbb{C}}\star\mathsf{G}_1^{\mathbb{C}}\right] + \mathrm{d}B\wedge\mathsf{G}_1^{\mathbb{A}} + e^{4\phi}\mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\left(S^{-1}\right)^{\mathbb{I}\mathbb{J}}\mathbb{N}_{\mathbb{J}\mathbb{K}}\mathsf{G}_0^{\mathbb{K}}\star\mathbf{1}^{(4)} = 0. \tag{5.5.44}$$

Inserting (5.5.39) and lowering symplectic indices with $S_{\mathbb{A}\mathbb{B}}$, we arrive at

$$\begin{aligned}0 = &-\mathrm{d}\left[\widetilde{\Delta}_{\mathbb{A}\mathbb{B}}\star\mathrm{d}\widetilde{\mathsf{C}}_0^{\mathbb{B}} - e^{2\phi}\mathbb{M}_{\mathbb{A}\mathbb{B}}\mathcal{O}^{\mathbb{B}}{}_{\mathsf{I}}(\Delta^{-1})^{\mathsf{IJ}}\left(\mathrm{d}(\widetilde{\mathcal{O}}_{\mathsf{J}\mathbb{C}}\widetilde{\mathsf{C}}_2^{\mathbb{C}}) + \widetilde{\mathcal{O}}_{\mathsf{J}\mathbb{C}}\widetilde{\mathsf{C}}_0^{\mathbb{C}}\mathrm{d}B\right)\right]\\ &-\mathrm{d}B\wedge\Big[S_{\mathbb{A}\mathbb{B}}\mathrm{d}\widetilde{\mathsf{C}}_0^{\mathbb{B}} - (\widetilde{\mathcal{O}}^T)_{\mathbb{A}\mathsf{I}}(\Delta^{-1})^{\mathsf{IJ}}\\ &\qquad\cdot\left(\star\mathrm{d}(\widetilde{\mathcal{O}}_{\mathsf{J}\mathbb{C}}\widetilde{\mathsf{C}}_2^{\mathbb{C}}) + \widetilde{\mathcal{O}}_{\mathsf{J}\mathbb{C}}\widetilde{\mathsf{C}}_0^{\mathbb{C}}\star\mathrm{d}B + e^{2\phi}(\mathcal{O}^T)_{\mathsf{J}}{}^{\mathbb{C}}\mathbb{M}_{\mathbb{C}\mathbb{D}}\mathrm{d}\widetilde{\mathsf{C}}_0^{\mathbb{D}}\right)\Big]\\ &+e^{4\phi}(\widetilde{\mathcal{O}}^T)_{\mathbb{A}}{}^{\mathbb{I}}\mathbb{N}_{\mathbb{I}\mathbb{J}}\left(\mathsf{G}^{\mathbb{J}}_{\mathrm{flux}} + \widetilde{\mathcal{O}}^{\mathbb{J}}{}_{\mathbb{B}}\widetilde{\mathsf{C}}_0^{\mathbb{B}}\right)\star\mathbf{1}^{(4)},\end{aligned} \tag{5.5.45}$$

where

$$\widetilde{\Delta}_{\mathbb{A}\mathbb{B}} = e^{2\phi}\left(\mathbb{M}_{\mathbb{A}\mathbb{B}} - e^{2\phi}\mathbb{M}_{\mathbb{A}\mathbb{C}}\mathcal{O}^{\mathbb{C}}{}_{\mathsf{I}}(\Delta^{-1})^{\mathsf{IJ}}(\mathcal{O}^T)_{\mathsf{J}}{}^{\mathbb{D}}\mathbb{M}_{\mathbb{D}\mathbb{B}}\right). \tag{5.5.46}$$

Evaluating the Equations of Motion – B

The equation of motion (5.5.21) of $\hat{B}$ is already non-trivial and only needs to be reformulated in a way that the undesired degrees of freedom disappear. We here consider the part with two external and six internal components. Using the expansion (5.5.22) and manually inserting involution operators (5.2.28), we can use (5.3.57) and (5.3.59) to integrate over CY_3, and after another Weyl-rescaling according to (5.5.5), we arrive at

$$\frac{1}{2}\mathrm{d}\left(e^{-4\phi}\star \mathrm{d}B\right) - \mathsf{G}_0^{\mathsf{I}}\mathsf{G}_{2\,\mathsf{I}} + \mathsf{G}_{0\,\mathsf{I}}\mathsf{G}_2^{\mathsf{I}} + \mathsf{G}_{1\,\mathsf{A}}\wedge \mathsf{G}_1^{\mathsf{A}} = 0. \tag{5.5.47}$$

Substituting the corresponding expressions from (5.5.18), we eventually find

$$\begin{aligned}0 = &\frac{1}{2}\mathrm{d}\left(e^{-4\phi}\star \mathrm{d}B\right) - \mathsf{G}^{\mathsf{I}}_{\text{flux}}\left(\mathrm{Im}\mathcal{N}_{\mathsf{IJ}}\star \mathsf{F}_2^{\mathsf{J}} + \mathrm{Re}\mathcal{N}_{\mathsf{IJ}}\mathsf{F}_2^{\mathsf{J}}\right) + \mathsf{G}_{\mathsf{I}\,\text{flux}}\mathsf{F}_2^{\mathsf{I}} + \frac{1}{2}\mathrm{d}\widetilde{\mathsf{C}}_0^{\mathbb{A}}S_{\mathbb{A}\mathbb{B}}\mathrm{d}\widetilde{\mathsf{C}}_0^{\mathbb{B}}\\ &-\mathrm{d}\left[\widetilde{\mathsf{C}}_0^{\mathbb{A}}(\widetilde{\mathcal{O}}^T)_{\mathbb{A}\,\mathsf{I}}(\Delta^{-1})^{\mathsf{IJ}}\left(\star\mathrm{d}(\widetilde{\mathcal{O}}_{\mathsf{J}\,\mathbb{B}}\widetilde{\mathsf{C}}_2^{\mathbb{B}}) - \widetilde{\mathcal{O}}_{\mathsf{J}\,\mathbb{B}}\widetilde{\mathsf{C}}_0^{\mathbb{B}}\star \mathrm{d}B + e^{2\phi}(\mathcal{O}^T)_{\mathsf{J}}{}^{\mathbb{B}}\mathbb{M}_{\mathbb{B}\mathbb{C}}\mathrm{d}\widetilde{\mathsf{C}}_0^{\mathbb{C}}\right)\right].\end{aligned} \tag{5.5.48}$$

This will be identified as the equation of motion for the external Kalb-Ramond field B in the next paragraph.

Reconstructing the Action of $D=4$ $\mathcal{N}=2$ Gauged Supergravity

Building upon our results for the scalar potential (5.3.80) and the kinetic NS-NS sector (5.5.6), we can now utilize the previously derived equations of motion to reconstruct the full four-dimensional action, which takes the form [202]

$$\begin{aligned}S_{\text{IIA}} = \int_{M^{1,3}} &\frac{1}{2}R^{(4)}\star\mathbf{1}^{(4)} - \mathrm{d}\phi\wedge\star\mathrm{d}\phi - \frac{e^{-4\phi}}{4}\mathrm{d}B\wedge\star\mathrm{d}B - g_{\mathsf{ij}}\mathrm{d}t^{\mathsf{i}}\wedge\star\mathrm{d}\bar{t}^{\bar{\mathsf{j}}} - g_{\mathsf{ab}}\mathrm{d}U^{\mathsf{a}}\wedge\star\mathrm{d}\overline{U}^{\mathsf{b}}\\ &+\frac{1}{2}\mathrm{Re}\mathcal{N}_{\mathsf{IJ}}\mathsf{F}_2^{\mathsf{I}}\wedge\mathsf{F}_2^{\mathsf{J}} + \frac{1}{2}\mathrm{Im}\mathcal{N}_{\mathsf{IJ}}\mathsf{F}_2^{\mathsf{I}}\wedge\star\mathsf{F}_2^{\mathsf{J}} + \frac{1}{2}\widetilde{\Delta}_{\mathbb{A}\mathbb{B}}\mathrm{d}\widetilde{\mathsf{C}}_0^{\mathbb{A}}\wedge\star\mathrm{d}\mathsf{C}_0^{\mathbb{B}}\\ &+\frac{1}{2}(\Delta^{-1})^{\mathsf{IJ}}\left(\mathrm{d}(\widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}\widetilde{\mathsf{C}}_2^{\mathbb{A}}) + \widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}\widetilde{\mathsf{C}}_0^{\mathbb{A}}\mathrm{d}B\right)\wedge\star\left(\mathrm{d}(\widetilde{\mathcal{O}}_{\mathsf{J}\,\mathbb{B}}\widetilde{\mathsf{C}}_2^{\mathbb{B}}) + \widetilde{\mathcal{O}}_{\mathsf{J}\,\mathbb{B}}\widetilde{\mathsf{C}}_0^{\mathbb{B}}\mathrm{d}B\right)\\ &+\left(\mathrm{d}(\widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}\widetilde{\mathsf{C}}_2^{\mathbb{A}}) + \widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}\widetilde{\mathsf{C}}_0^{\mathbb{A}}\mathrm{d}B\right)\wedge\left(e^{2\phi}(\Delta^{-1})^{\mathsf{IJ}}(\mathcal{O}^T)_{\mathsf{J}}{}^{\mathbb{B}}\mathbb{M}_{\mathbb{B}\mathbb{C}}\mathrm{d}\widetilde{\mathsf{C}}_0^{\mathbb{C}}\right) - \frac{1}{2}\mathrm{d}B\wedge\widetilde{\mathsf{C}}_0^{\mathbb{A}}S_{\mathbb{A}\mathbb{B}}\mathrm{d}\widetilde{\mathsf{C}}_0^{\mathbb{B}}\\ &-\left(\widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}\widetilde{\mathsf{C}}_2^{\mathbb{A}} - \mathsf{G}_{\mathsf{I}\,\text{flux}}B\right)\wedge\left(\mathrm{d}\mathsf{C}_1^{\mathsf{I}} + \tfrac{1}{2}\widetilde{\mathcal{O}}^{\mathsf{I}}{}_{\mathbb{B}}\widetilde{\mathsf{C}}_2^{\mathbb{B}} - \tfrac{1}{2}\mathsf{G}^{\mathsf{I}}_{\text{flux}}B\right) + V_{\text{scalar}}\star\mathbf{1}^{(4)},\end{aligned} \tag{5.5.49}$$

with

$$\begin{aligned}V_{\text{scalar}} &= V_{\text{NSNS}} + V_{\text{RR}}\\ &= +\frac{e^{2\phi}}{2}V^{\mathbb{I}}(\mathcal{O}^T)_{\mathbb{I}}{}^{\mathbb{A}}\mathbb{M}_{\mathbb{A}\mathbb{B}}\mathcal{O}^{\mathbb{B}}{}_{\mathbb{J}}V^{\mathbb{J}} + \frac{e^{2\phi}}{2}W^{\mathbb{A}}(\widetilde{\mathcal{O}}^T)_{\mathbb{A}}{}^{\mathbb{I}}\mathbb{N}_{\mathbb{I}\mathbb{J}}\widetilde{\mathcal{O}}^{\mathbb{J}}{}_{\mathbb{B}}\overline{W}^{\mathbb{B}}\\ &\quad-\frac{e^{2\phi}}{4\mathcal{K}}W^{\mathbb{A}}S_{\mathbb{A}\mathbb{C}}\mathcal{O}^{\mathbb{C}}{}_{\mathbb{I}}\left(V^{\mathbb{I}}\overline{V}^{\mathbb{J}} + \overline{V}^{\mathbb{I}}V^{\mathbb{J}}\right)(\mathcal{O}^T)_{\mathbb{J}}{}^{\mathbb{D}}S_{\mathbb{D}\mathbb{B}}\overline{W}^{\mathbb{B}}\\ &\quad+\frac{e^{4\phi}}{2}\left(\mathsf{G}^{\mathbb{I}}_{\text{flux}} + \widetilde{\mathsf{C}}_0^{\mathbb{A}}(\widetilde{\mathcal{O}}^T)_{\mathbb{A}}{}^{\mathbb{I}}\right)\mathbb{N}_{\mathbb{I}\mathbb{J}}\left(\mathsf{G}^{\mathbb{J}}_{\text{flux}} + \widetilde{\mathcal{O}}^{\mathbb{J}}{}_{\mathbb{B}}\widetilde{\mathsf{C}}_0^{\mathbb{B}}\right).\end{aligned} \tag{5.5.50}$$

One can verify by direct calculation and use of the relations (5.3.67) and (5.5.26) that the equations of motion are correctly recovered when varying with respect to the corresponding fields. Up to different conventions and additional terms from the remaining sectors, this replicates the structure of (35) from [202].

A similar result was derived for compactifications of ordinary type II theories on $SU(3) \times SU(3)$ structure manifolds in [199], where a slightly different formulation in terms of the actual propagating degrees of freedom was obtained. Indeed, in our present discussion the fundamental fields $\widetilde{\mathsf{C}}_2^{\mathbb{A}}$ appear only in particular combinations with the fluxes (or charges), and the actual propagating degrees of freedom are given by $\widetilde{\mathcal{O}}_{\mathsf{I}\mathbb{A}}\widetilde{\mathsf{C}}_2^{\mathbb{A}}$. In a similar way, the fields $\mathsf{C}_0^{\mathbb{A}}$ enter the equations of motion exclusively in form of the projections $\widetilde{\mathsf{C}}_0^{\mathbb{A}}$, which encode only a part of the original degrees of freedom. Taking the corresponding definitions into account, one can verify that the results for both frameworks are indeed equivalent as expected.

To tie up loose ends, let us also note that we utilized only parts of the relations arising from (5.5.20) to derive the four-dimensional supergravity equations of motion. One can show by careful use of the Bianchi identities (5.3.70) that the remaining components automatically depend only on the fields appearing in the effective action (5.5.49) and are trivially satisfied when inserting the definitions (5.5.18). This is again in accordance with the result of [199], where the corresponding relations were used to express the appearing G-fields directly in terms of the propagating degrees of freedom.

Relation to the Standard Formulation of $D = 4$ $\mathcal{N} = 2$ Gauged Supergravity

To conclude our discussion of the type IIA setting, let us briefly discuss how this result relates to the standard formulation of $D = 4$ $\mathcal{N} = 2$ gauged supergravity. In the original work [202], the authors first constructed an alternative formulation of the theory in which a subset of the scalars is dualized to two-forms. In this framework, the external component B of the Kalb-Ramond field appears explicitly, and there exist certain combinations of electric charges and new two-form fields which are not present in the initial action. It was then found that this partially dualized formulation permits a natural extension involving additional magnetic charges, which cannot be straightforwardly included into the standard formulation.

In the framework applied throughout this chapter, the electric and magnetic charges descend from the ten-dimensional generalized NS-NS fluxes. The magnetic charges are thereby represented by the expressions $\mathcal{O}^{\mathbb{A}\mathsf{I}}$ and $\widetilde{\mathcal{O}}^{\mathsf{I}}{}_{\mathbb{A}}$, which encode precisely half of the flux coefficients. In the generic case, this leads to a partially dualized $\mathcal{N} = 2$ gauged supergravity action along the lines of [202]. However, there also exist certain special cases for which the dualization procedure becomes reversibe and the original formulation can be recovered. A similar role is played by the R-R fluxes, which were already found in [119–121] to prevent the four-dimensional Kalb-Ramond field from being dualized to an axion.

We will next discuss some of these particular settings in more detail and show how their action can be related to the standard formulation of $\mathcal{N} = 2$ gauged and ungauged supergravity. Notice that a similar analysis was also presented in [199], where the considered special cases were shown to be equivalent to compactifications on $SU(3)$ structure manifolds.

Since the magnetic charges in four dimensions arise from the fluxes $\mathcal{O}^{\mathbb{A}\,\mathsf{I}}$ and $\widetilde{\mathcal{O}}^{\mathsf{I}}{}_{\mathbb{A}}$, a natural ansatz is to reconsider the ten-dimensional equations of motion under the additional assumption

$$\mathcal{O}^{\mathbb{A}\,\mathsf{I}} = 0, \qquad \widetilde{\mathcal{O}}^{\mathsf{I}}{}_{\mathbb{A}} = 0. \tag{5.5.51}$$

In this setting, parts of the undesired degrees of freedom automatically disappear from the equations of motion, and the four-dimensional action can be formulated without additional two-form fields $\widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}\widetilde{\mathsf{C}}_2^{\mathbb{A}}$. This can be achieved by substituting lines one and three of (5.5.25) into the lower-index components of the fourth equation of motion from (5.5.20), which yields a new non-trivial equation of motion

$$\mathrm{d}\left(\mathrm{Im}\mathcal{N}_{\mathsf{IJ}} \star \mathsf{F}_2^{\mathsf{J}} + \mathrm{Re}\mathcal{N}_{\mathsf{IJ}}\mathsf{F}_2^{\mathsf{J}}\right) + \left(\mathsf{G}_{\mathsf{I}\,\mathrm{flux}} + \widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}\mathsf{C}_0^{\mathbb{A}}\right)\mathrm{d}B + e^{2\phi}(\mathcal{O}^T)_{\mathsf{I}}{}^{\mathbb{A}}\mathbb{M}_{\mathbb{A}\mathbb{B}} \star \left(\mathrm{d}\mathsf{C}_0^{\mathbb{A}} + \mathcal{O}^{\mathbb{A}}{}_{\mathsf{I}}\mathsf{C}_1^{\mathsf{I}}\right) = 0 \tag{5.5.52}$$

with

$$\mathsf{F}_2^{\mathsf{I}} = \mathrm{d}\mathsf{C}_1^{\mathsf{I}} - B \wedge \mathsf{G}^{\mathsf{I}}_{\mathrm{flux}}. \tag{5.5.53}$$

From here on, the relations (5.5.44) and (5.5.47) can be derived analogously to the general case, and no further reformulations or substitutions are required for the scalar fields $\mathsf{C}_0^{\mathbb{A}}$. The resulting equations of motion can then be derived from a different four-dimensional action

$$\begin{aligned} S_{\mathrm{IIA}} = \int_{M^{1,3}} & \frac{1}{2}R^{(4)} \star \mathbf{1}^{(4)} - \mathrm{d}\phi \wedge \star\mathrm{d}\phi - \frac{e^{-4\phi}}{4}\mathrm{d}B \wedge \star\mathrm{d}B - g_{\mathsf{ij}}\mathrm{d}t^{\mathsf{i}} \wedge \star\mathrm{d}\bar{t}^{\bar{\mathsf{j}}} - g_{\mathsf{ab}}\mathrm{d}U^{\mathsf{a}} \wedge \star\mathrm{d}\overline{U}^{\mathsf{b}} \\ & + \frac{1}{2}\mathrm{Re}\mathcal{N}_{\mathsf{IJ}}\mathsf{F}_2^{\mathsf{I}} \wedge \mathsf{F}_2^{\mathsf{J}} + \frac{1}{2}\mathrm{Im}\mathcal{N}_{\mathsf{IJ}}\mathsf{F}_2^{\mathsf{I}} \wedge \star\mathsf{F}_2^{\mathsf{J}} + \frac{e^{2\phi}}{2}\mathbb{M}_{\mathbb{A}\mathbb{B}}D\mathsf{C}_0^{\mathbb{A}} \wedge \star D\mathsf{C}_0^{\mathbb{B}} \\ & - \frac{1}{2}\mathrm{d}B \wedge \left[\mathsf{C}_0^{\mathbb{A}}S_{\mathbb{A}\mathbb{B}}D\mathsf{C}_0^{\mathbb{B}} + \left(2\mathsf{G}_{\mathsf{I}\,\mathrm{flux}} + \widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}\mathsf{C}_0^{\mathbb{A}}\right)\mathsf{C}_1^{\mathsf{I}}\right] - \frac{1}{2}\mathsf{G}_{\mathsf{I}\,\mathrm{flux}}\mathsf{G}^{\mathsf{I}}_{\mathrm{flux}}B \wedge B \\ & + V_{\mathrm{scalar}} \star \mathbf{1}^{(4)}, \end{aligned} \tag{5.5.54}$$

in which the physical degrees of freedom arising from the two-form fields $\widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}\widetilde{\mathsf{C}}_2^{\mathbb{A}}$ are now encoded by a new set of scalar fields. The scalar potential V_{scalar} takes the same form as in (5.5.50) and the covariant derivative D is defined by

$$D\mathsf{C}_0^{\mathbb{A}} = \mathrm{d}\mathsf{C}_0^{\mathbb{A}} + \mathcal{O}^{\mathbb{A}}{}_{\mathsf{I}}\mathsf{C}_1^{\mathsf{I}}, \tag{5.5.55}$$

the right-hand side of which matches with the field strength $\mathsf{G}_1^{\mathbb{A}}$ in this particular setting. A similar result was found in [199] and identified as the effective action of compactifications on $SU(3)$ structure manifolds.

Parts of the action (5.5.54) already resemble the standard formulation of $D = 4$ $\mathcal{N} = 2$ gauged supergravity. In a final step, we would like to dualize the four-dimensional Kalb-Ramond field B to an axion a. As already discussed in the context of [119–121], this is not as straightforward to realize for the general case since the presence of R-R fluxes gives rise to an additional mass term for B. Similarly to the magnetic charges in

(5.5.49), this problem can, however, be resolved by setting half of the corresponding flux coefficients to zero,

$$\mathsf{G}^{\mathsf{I}}_{\text{flux}} = 0. \tag{5.5.56}$$

One can then follow the standard strategy by adding a Lagrange multiplier $\mathrm{d}B_2 \wedge \mathrm{d}a$ and integrating out B. This eventually leads to

$$\begin{aligned} S_{\text{IIA}} = \int_{M^{1,3}} & \frac{1}{2}R^{(4)} \star \mathbf{1}^{(4)} - \mathrm{d}\phi \wedge \star\mathrm{d}\phi - g_{\mathsf{ij}}\mathrm{d}t^{\mathsf{i}} \wedge \star\mathrm{d}\bar{t}^{\mathsf{j}} - g_{\mathsf{ab}}\mathrm{d}U^{\mathsf{a}} \wedge \star\mathrm{d}\overline{U}^{\mathsf{b}} \\ & + \frac{1}{2}\mathrm{Re}\mathcal{N}_{\mathsf{IJ}}\mathsf{F}^{\mathsf{I}}_2 \wedge \mathsf{F}^{\mathsf{J}}_2 + \frac{1}{2}\mathrm{Im}\mathcal{N}_{\mathsf{IJ}}\mathsf{F}^{\mathsf{I}}_2 \wedge \star\mathsf{F}^{\mathsf{J}}_2 + \frac{e^{2\phi}}{2}\mathbb{M}_{\mathbb{AB}}D\mathsf{C}^{\mathbb{A}}_0 \wedge \star D\mathsf{C}^{\mathbb{B}}_0 \\ & - \frac{e^{4\phi}}{4}\left(Da + \mathsf{C}^{\mathbb{A}}_0 S_{\mathbb{AB}} D\mathsf{C}^{\mathbb{B}}_0\right) \wedge \star\left(Da + \mathsf{C}^{\mathbb{A}}_0 S_{\mathbb{AB}} D\mathsf{C}^{\mathbb{B}}_0\right) \\ & + V_{\text{scalar}} \star \mathbf{1}^{(4)}, \end{aligned} \tag{5.5.57}$$

where the covariant derivative of the axion reads

$$Da = \mathrm{d}a - \left(2\mathsf{G}_{\mathsf{I}\,\text{flux}} + \widetilde{\mathcal{O}}_{\mathsf{I}\,\mathbb{A}}\mathsf{C}^{\mathbb{A}}_0\right)\mathsf{C}^{\mathsf{I}}_1. \tag{5.5.58}$$

The field content of this action now strongly resembles that of normal $D = 4$ $\mathcal{N} = 2$ supergravity, albeit with additional gaugings arising from the remaining non-vanishing fluxes. Setting the latter to zero, the contributions of $\mathsf{G}_{\mathsf{I}\,\text{flux}}$ as well as the matrices $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ vanish, and one obtains (up to slight changes in conventions) the standard quanternionic action (3.2.35) known from conventional Calabi-Yau compactifications of type II theories.

Type IIB Setting

The discussion for type IIB theory follows a very similar pattern, and we will only sketch the most important steps here.

Relation to Democratic Type IIB Supergravity

Our ansatz is again to reformulate the type IIB R-R pseudo-action (5.2.14) in poly-form notation. The computations are mostly analogous to the type IIA case, and we obtain

$$\star\mathcal{L}^{(\text{IIB})}_{RR} = -\frac{1}{2}\hat{\mathfrak{G}}^{(\text{IIB})} \wedge \star\hat{\mathfrak{G}}^{(\text{IIB})} \tag{5.5.59}$$

with

$$\hat{\mathfrak{G}}^{(\text{IIB})} = e^{-\hat{B}}\mathcal{G}^{(\text{IIB})} + \hat{\mathfrak{D}}\hat{\mathcal{C}}^{(\text{IIB})} = e^{-\hat{B}}\mathcal{G}^{(\text{IIB})} + e^{-\hat{B}}\hat{\mathcal{D}}\left(e^{\hat{B}}\hat{\mathcal{C}}^{(\text{IIB})}\right), \tag{5.5.60}$$

and

$$\begin{aligned} \mathcal{G}^{(\text{IIB})} &= G_3, \\ \hat{\mathcal{C}}^{(\text{IIB})} &= \hat{C}_0 + \hat{C}_2 + \hat{C}_4 + \hat{C}_6 + \hat{C}_8. \end{aligned} \tag{5.5.61}$$

Notice that we consider only the three-form R-R flux since the one- and five-forms appear exclusively in cohomologically trivial expressions on CY_3. The factor $e^{-\hat{B}}$ in front of $\hat{\mathcal{G}}^{(\text{IIB})}$ thus has no effect and is included only for later convenience. The duality constraints (5.2.15) for the type IIB case can be written as

$$\hat{\mathfrak{G}}^{(\text{IIB})} = -\lambda\left(\star\hat{\mathfrak{G}}^{(\text{IIB})}\right), \tag{5.5.62}$$

and varying the action with respect to the C-field components yields the equations of motion

$$\left(\mathrm{d} - \mathrm{d}\hat{B}\wedge + \mathfrak{H}\wedge + \mathfrak{F}\circ + \mathfrak{Q}\bullet + \mathfrak{R}\llcorner\right)\star\hat{\mathfrak{G}}^{(\text{IIB})} = 0, \tag{5.5.63}$$

which are equivalent to the Bianchi identities

$$e^{-\hat{B}}\hat{\mathcal{D}}\left(e^{\hat{B}}\hat{\mathfrak{G}}^{(\text{IIB})}\right) = 0. \tag{5.5.64}$$

Reduced Equations of Motion and Duality Constraints

In order to employ the framework of special geometry, we again rewrite the above expressions in A-basis notation. We define

$$e^{\hat{B}}\mathcal{C}^{(\text{IIB})} = \left(\mathsf{C}_0^{\mathsf{I}} + \mathsf{C}_2^{\mathsf{I}} + \mathsf{C}_4^{\mathsf{I}}\right)\omega_{\mathsf{I}} + \left(\mathsf{C}_1^{\mathsf{A}} + \mathsf{C}_3^{\mathsf{A}}\right)\alpha_{\mathsf{A}} - \left(\mathsf{C}_{1\,\mathsf{A}} + \mathsf{C}_{3\,\mathsf{A}}\right)\beta^{\mathsf{A}} + \left(\mathsf{C}_{0\,\mathsf{I}} + \mathsf{C}_{2\,\mathsf{I}} + \mathsf{C}_{4\,\mathsf{I}}\right)\widetilde{\omega}^{\mathsf{I}} \tag{5.5.65}$$

and

$$G_3 = -\mathsf{G}_{\text{flux}}^{\mathsf{A}}\alpha_{\mathsf{A}} + \mathsf{G}_{\text{flux}\,\mathsf{A}}\beta^{\mathsf{A}}, \tag{5.5.66}$$

which can be utilized to reformulate the type IIB R-R poly-form (5.5.60) as

$$\hat{\mathfrak{G}}^{(\text{IIB})} = e^{-\hat{B}}\hat{\mathsf{G}}^{(\text{IIB})} = e^{-\hat{B}}\left(\hat{\mathsf{G}}_1^{(\text{IIB})} + \hat{\mathsf{G}}_3^{(\text{IIB})} + \hat{\mathsf{G}}_5^{(\text{IIB})} + \hat{\mathsf{G}}_7^{(\text{IIB})} + \hat{\mathsf{G}}_9^{(\text{IIB})}\right). \tag{5.5.67}$$

Notice that these objects strongly resemble the corresponding expressions of the type IIA case appearing in (5.5.14), (5.5.15) and (5.5.16), but with exchanged roles of the even and odd cohomology components. We once more employ a shorthand notation

$$\begin{aligned}
\hat{\mathsf{G}}_1^{(\text{IIB})} &= \mathsf{G}_{1\,0}\widetilde{\omega}^0, \\
\hat{\mathsf{G}}_3^{(\text{IIB})} &= \mathsf{G}_{3\,0}\widetilde{\omega}^0 + \mathsf{G}_1^{\mathsf{i}}\omega_{\mathsf{i}} - \mathsf{G}_0^{\mathsf{A}}\wedge\alpha_{\mathsf{A}} + \mathsf{G}_{0\,\mathsf{A}}\wedge\beta^{\mathsf{A}}, \\
\hat{\mathsf{G}}_5^{(\text{IIB})} &= \mathsf{G}_3^{\mathsf{i}}\wedge\omega_{\mathsf{i}} - \mathsf{G}_2^{\mathsf{A}}\wedge\alpha_{\mathsf{A}} + \mathsf{G}_{2\,\mathsf{A}}\wedge\beta^{\mathsf{A}} + \mathsf{G}_{1\,\mathsf{i}}\widetilde{\omega}^{\mathsf{i}}, \\
\hat{\mathsf{G}}_7^{(\text{IIB})} &= -\mathsf{G}_4^{\mathsf{A}}\wedge\alpha_{\mathsf{A}} + \mathsf{G}_{4\,\mathsf{A}}\wedge\beta^{\mathsf{A}} + \mathsf{G}_{3\,\mathsf{i}}\wedge\widetilde{\omega}^{\mathsf{i}} + \mathsf{G}_1^0\wedge\omega_0, \\
\hat{\mathsf{G}}_9^{(\text{IIB})} &= \mathsf{G}_3^0\wedge\omega_0,
\end{aligned} \tag{5.5.68}$$

where the expansion coefficients

$$\begin{aligned}
\mathsf{G}_0^{\mathbb{A}} &= \mathsf{G}_{\text{flux}}^{\mathbb{A}} + \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{C}_0^{\mathbb{I}},\\
\mathsf{G}_1^{\mathbb{I}} &= \mathrm{d}\mathsf{C}_0^{\mathbb{I}} + \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}\mathsf{C}_1^{\mathbb{A}},\\
\mathsf{G}_2^{\mathbb{A}} &= \mathrm{d}\mathsf{C}_1^{\mathbb{A}} + \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{C}_2^{\mathbb{I}},\\
\mathsf{G}_3^{\mathbb{I}} &= \mathrm{d}\mathsf{C}_2^{\mathbb{I}} + \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}\mathsf{C}_3^{\mathbb{A}},\\
\mathsf{G}_4^{\mathbb{A}} &= \mathrm{d}\mathsf{C}_3^{\mathbb{A}} + \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{C}_4^{\mathbb{I}}
\end{aligned} \tag{5.5.69}$$

can be derived by using the flux matrix relations (5.3.66 - 5.3.68). The equations of motion (5.5.64) reduce to

$$\hat{\mathcal{D}}\hat{\mathsf{G}}^{(\text{IIB})} = 0, \tag{5.5.70}$$

giving rise to the set of four-dimensional relations

$$\begin{aligned}
\widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}\mathsf{G}_0^{\mathbb{A}} &= 0,\\
\mathrm{d}\mathsf{G}_0^{\mathbb{A}} - \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{G}_1^{\mathbb{I}} &= 0,\\
\mathrm{d}\mathsf{G}_1^{\mathbb{I}} - \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}\mathsf{G}_2^{\mathbb{A}} &= 0,\\
\mathrm{d}\mathsf{G}_2^{\mathbb{A}} - \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}}\mathsf{G}_3^{\mathbb{I}} &= 0,\\
\mathrm{d}\mathsf{G}_3^{\mathbb{I}} - \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}\mathsf{G}_4^{\mathbb{A}} &= 0
\end{aligned} \tag{5.5.71}$$

after applying the same methods we already used to derive (5.5.20). The relevant equation of motion for $\hat{B}$ reads

$$\frac{1}{2}\mathrm{d}\left(e^{-4\phi}\star\mathrm{d}B\right) - \mathsf{G}_0^{\mathsf{A}}\mathsf{G}_{2\,\mathsf{A}} + \mathsf{G}_{0\,\mathsf{A}}\mathsf{G}_2^{\mathsf{A}} + \mathsf{G}_{1\,\mathsf{I}}\wedge\mathsf{G}_I^{\mathsf{A}} = 0. \tag{5.5.72}$$

For the duality constraints (5.5.62), we follow the same pattern as for (5.5.11) and obtain

$$\begin{aligned}
\mathsf{G}_{2\,\mathsf{A}} - B\mathsf{G}_{0\,\mathsf{A}} &= \mathrm{Im}\mathcal{M}_{\mathsf{AB}}\star\left(\mathsf{G}_2^{\mathsf{B}} - B\wedge\mathsf{G}_0^{\mathsf{B}}\right) + \mathrm{Re}\mathcal{M}_{\mathsf{AB}}\left(\mathsf{G}_2^{\mathsf{B}} - B\wedge\mathsf{G}_0^{\mathsf{B}}\right),\\
\mathsf{G}_4^{\mathbb{A}} - B\wedge\mathsf{G}_2^{\mathbb{A}} + \tfrac{1}{2}B^2\mathsf{G}_0^{\mathbb{A}} &= -e^{4\phi}\left(S^{-1}\right)^{\mathbb{AB}}\mathbb{M}_{\mathbb{BC}}\mathsf{G}_0^{\mathbb{C}}\star\mathbf{1}^{(4)},\\
\mathsf{G}_3^{\mathbb{I}} - B\wedge\mathsf{G}_1^{\mathbb{I}} &= e^{2\phi}\left(S^{-1}\right)^{\mathbb{IJ}}\mathbb{N}_{\mathbb{JK}}\star\mathsf{G}_1^{\mathbb{K}}.
\end{aligned} \tag{5.5.73}$$

Reconstructing the Action

As the structural analogies between the two settings suggest, the equations of motion can be evaluated by following the same pattern as in the type IIA case. Defining

$$\widetilde{\mathsf{C}}_2^{\mathbb{I}} = \mathsf{C}_2^{\mathbb{I}} - B\wedge\mathsf{C}_0^{\mathbb{I}}, \tag{5.5.74}$$

and

$$\widetilde{\mathsf{C}}_0^{\mathbb{I}} = (\delta^{\mathbb{I}}{}_{\mathbb{J}} - \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathsf{A}}\check{\mathcal{O}}^{\mathsf{A}}{}_{\mathbb{J}})\mathsf{C}_0^{\mathbb{J}}\,, \qquad \check{\mathcal{O}}^{\mathsf{A}}{}_{\mathbb{I}}\widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathsf{B}} = \delta^{\mathsf{A}}{}_{\mathsf{B}}\,, \tag{5.5.75}$$

this strategy eventually brings us to the effective four-dimensional action

$$\begin{aligned} S_{\text{IIB}} = \int_{M^{1,3}} & \frac{1}{2}R^{(4)} \star \mathbf{1}^{(4)} - \mathrm{d}\phi \wedge \star\mathrm{d}\phi - \frac{e^{-4\phi}}{4}\mathrm{d}B \wedge \star\mathrm{d}B - g_{\mathsf{ij}}\mathrm{d}t^{\mathsf{i}} \wedge \star\mathrm{d}\bar{t}^{\mathsf{j}} - g_{\mathsf{ab}}\mathrm{d}U^{\mathsf{a}} \wedge \star\mathrm{d}\overline{U}^{\mathsf{b}} \\ & + \frac{1}{2}\mathrm{Re}\mathcal{M}_{\mathsf{AB}}\mathsf{F}_2^{\mathsf{A}} \wedge \mathsf{F}_2^{\mathsf{B}} + \frac{1}{2}\mathrm{Im}\mathcal{M}_{\mathsf{AB}}\mathsf{F}_2^{\mathsf{A}} \wedge \star\mathsf{F}_2^{\mathsf{B}} + \frac{1}{2}\widetilde{\Delta}_{\mathbb{IJ}}\mathrm{d}\widetilde{\mathsf{C}}_0^{\mathbb{I}} \wedge \star\mathrm{d}\widetilde{\mathsf{C}}_0^{\mathbb{J}} \\ & + \frac{1}{2}(\Delta^{-1})^{\mathsf{AB}}\left(\mathrm{d}(\mathcal{O}_{\mathsf{A}\,\mathbb{I}}\widetilde{\mathsf{C}}_2^{\mathbb{I}}) + \mathcal{O}_{\mathsf{A}\,\mathbb{I}}\widetilde{\mathsf{C}}_0^{\mathbb{I}}\mathrm{d}B\right) \wedge \star\left(\mathrm{d}(\mathcal{O}_{\mathsf{B}\,\mathbb{J}}\widetilde{\mathsf{C}}_2^{\mathbb{J}}) + \mathcal{O}_{\mathsf{B}\,\mathbb{J}}\widetilde{\mathsf{C}}_0^{\mathbb{J}}\mathrm{d}B\right) \\ & + \left(\mathrm{d}(\mathcal{O}_{\mathsf{A}\,\mathbb{I}}\widetilde{\mathsf{C}}_2^{\mathbb{I}}) + \mathcal{O}_{\mathsf{A}\,\mathbb{I}}\widetilde{\mathsf{C}}_0^{\mathbb{I}}\mathrm{d}B\right) \wedge \left(e^{2\phi}(\Delta^{-1})^{\mathsf{AB}}(\widetilde{\mathcal{O}}^T)_{\mathsf{B}}{}^{\mathbb{J}}\mathbb{N}_{\mathbb{JK}}\mathrm{d}\widetilde{\mathsf{C}}_0^{\mathbb{K}}\right) + \frac{1}{2}\mathrm{d}B \wedge \widetilde{\mathsf{C}}_0^{\mathbb{I}}S_{\mathbb{IJ}}\mathrm{d}\widetilde{\mathsf{C}}_0^{\mathbb{J}} \\ & - \left(\mathcal{O}_{\mathsf{A}\,\mathbb{I}}\widetilde{\mathsf{C}}_2^{\mathbb{I}} - \mathsf{G}_{\mathsf{A}\,\text{flux}}B\right) \wedge \left(\mathrm{d}\mathsf{C}_1^{\mathsf{A}} + \tfrac{1}{2}\mathcal{O}^{\mathsf{A}}{}_{\mathbb{J}}\widetilde{\mathsf{C}}_2^{\mathbb{J}} - \tfrac{1}{2}\mathsf{G}_{\text{flux}}^{\mathsf{A}}B\right) + V_{\text{scalar}} \star \mathbf{1}^{(4)} \end{aligned} \tag{5.5.76}$$

with

$$\begin{aligned} V_{\text{scalar}} &= V_{\text{NSNS}} + V_{\text{RR}} \\ &= +\frac{e^{2\phi}}{2}V^{\mathbb{I}}(\mathcal{O}^T)_{\mathbb{I}}{}^{\mathbb{A}}\mathbb{M}_{\mathbb{AB}}\mathcal{O}^{\mathbb{B}}{}_{\mathbb{J}}V^{\mathbb{J}} + \frac{e^{2\phi}}{2}W^{\mathbb{A}}(\widetilde{\mathcal{O}}^T)_{\mathbb{A}}{}^{\mathbb{I}}\mathbb{N}_{\mathbb{IJ}}\widetilde{\mathcal{O}}^{\mathbb{J}}{}_{\mathbb{B}}\overline{W}^{\mathbb{B}} \\ &\quad -\frac{e^{2\phi}}{4\mathcal{K}}W^{\mathbb{A}}S_{\mathbb{AC}}\mathcal{O}^{\mathbb{C}}{}_{\mathbb{I}}\left(V^{\mathbb{I}}\overline{V}^{\mathbb{J}} + \overline{V}^{\mathbb{I}}V^{\mathbb{J}}\right)(\mathcal{O}^T)_{\mathbb{J}}{}^{\mathbb{D}}S_{\mathbb{DB}}\overline{W}^{\mathbb{B}} \\ &\quad +\frac{e^{4\phi}}{2}\left(\mathsf{G}_{\text{flux}}^{\mathbb{A}} + \widetilde{\mathsf{C}}_0^{\mathbb{I}}(\mathcal{O}^T)_{\mathbb{I}}{}^{\mathbb{A}}\right)\mathbb{M}_{\mathbb{AB}}\left(\mathsf{G}_{\text{flux}}^{\mathbb{B}} + \mathcal{O}^{\mathbb{B}}{}_{\mathbb{J}}\widetilde{\mathsf{C}}_0^{\mathbb{J}}\right) \end{aligned} \tag{5.5.77}$$

Comparing this action to its IIA counterpart (5.5.49), one can again construct a set of mirror mappings by extending (5.3.81) to

$$\begin{aligned} t^{\mathsf{i}} &\leftrightarrow U^{\mathsf{a}}, & g_{\mathsf{ij}} &\leftrightarrow g_{\mathsf{ab}}, \\ \mathbb{M}_{\mathbb{AB}} &\leftrightarrow \mathbb{N}_{\mathbb{IJ}}, & h^{1,1} &\leftrightarrow h^{1,2}, \\ V^{\mathbb{I}} &\leftrightarrow W^{\mathbb{A}}, & S_{\mathbb{IJ}} &\leftrightarrow S_{\mathbb{AB}} \\ \mathsf{C}_n^{\mathbb{I}} &\leftrightarrow \mathsf{C}_n^{\mathbb{A}}, & \mathsf{G}_{\text{flux}}^{\mathbb{I}} &\leftrightarrow \mathsf{G}_{\text{flux}}^{\mathbb{A}}, \\ \mathcal{O}^{\mathbb{A}}{}_{\mathbb{I}} &\leftrightarrow \widetilde{\mathcal{O}}^{\mathbb{I}}{}_{\mathbb{A}}, \end{aligned} \tag{5.5.78}$$

once more confirming preservation of IIA ↔ IIB Mirror Symmetry in the simultaneous presence of geometric and nongeometric fluxes.

5.6 Discussion

In this chapter we have analyzed dimensional reductions of type II double field theories with geometric and nongeometric fluxes. We again conclude our discussion by summarizing our most important results and providing a short outlook on possible future directions of research.

Scalar Potential

In section 5.2 we derived the scalar potential of four-dimensional $\mathcal{N} = 2$ gauged supergravity from dimensional reductions of the purely internal type IIA and IIB double field theory action on a Calabi-Yau three-fold CY_3. Building upon the elaborations of [78, 79], we extended the discussed setting by formally including generalized dilaton fluxes and relaxing the primitivity constraints. This modification revealed a more general form of the reformulated double field theory action, which shows a strong structural resemblance of supergravity compactifications on $SU(3)\times SU(3)$ structure manifolds [199].

In section 5.3 it was then exemplified through $K3 \times T^2$ how the framework can be generalized beyond the Calabi-Yau setting. This was done by utilizing the features of generalized Calabi-Yau and $K3$ structures [147, 203] to enable a special-geometric description of the $K3 \times T^2$ moduli space. The dimensional reduction led to a scalar potential term resembling that of $\mathcal{N} = 4$ gauged supergravity formulated in the $\mathcal{N} = 2$ framework of [202]. An important stechnical step here was to exploit the properties of $K3$ and T^2 to formally construct $K3 \times T^2$ analogues to the structure forms of CY_3,

$$\begin{aligned} e^{b_{CY_3}+iJ_{CY_3}} &\longleftrightarrow e^{b_{K3}+iJ_{K3}} \wedge e^{b_{T^2}+iJ_{T^2}} , \\ e^{b_{CY_3}} \wedge \Omega_{CY_3} &\longleftrightarrow \left(e^{b_{K3}} \wedge \Omega_{K3}\right) \wedge \left(e^{b_{T^2}} \wedge \Omega_{T^2}\right) , \end{aligned} \tag{5.6.1}$$

where J denotes the Kähler form of the respective manifold and Ω its holomorphic volume form. While the constructed scalar potential involves characteristic structures of $\mathcal{N} = 4$ gauged supergravity, relating the result to its standard formulation explicitly turned out to be a nontrivial task and might be an interesting direction for future research. It is to be expected that the discussion for arbitrary manifolds allowing for a generalized Calabi-Yau structure in the sense of [147, 203] follows a similar pattern.

An essential novelty of the approach discussed in these sections is its capability of describing generalized dilaton fluxes and non-vanishing trace-terms of the NS-NS fluxes. While their inclusion into Calabi-Yau compactifications is only a formal generalization, their contribution becomes nontrivial in the $K3 \times T^2$ setting. In light of the previous works [214, 215], it is to be expected that such fluxes serve as a ten-dimensional origin of non-unimodular gaugings in the $\mathcal{N} = 4$ gauged supergravity framework. This was also briefly discussed in section 4.2.3 of [149] in a double field theory context. Integrating the dilaton flux operators into the twisted differential of double field theory did not require including a rescaling charge operator as done in [215], which is in accordance with the result of [199] for $SU(3)\times SU(3)$ structure manifolds.

Relation to four-dimensional $\mathcal{N} = 2$ Gauged Supergravity

In section 5.5 we reconstructed the full bosonic part of the four-dimensional $\mathcal{N} = 2$ gauged supergravity action by including the kinetic terms into the Calabi-Yau setting. Our results replicate the action constructed in [202] and illustrate how the simultaneous presence of all NS-NS and R-R fluxes not only gives rise to gaugings in the effective four-dimensional theory, but also requires dualizing a subset of the axions in order to account for all fluxes. Turning off half of the fluxes correctly led to the standard formulation of

$\mathcal{N} = 2$ gauged supergravity, which could be further reduced to its ungauged version when setting the remaining fluxes to zero.

Our analysis of the R-R sector strongly resembles that of [199] for $SU(3)\times SU(3)$ manifolds. An essential difference of the approach considered in this chapter is that the field strengths are automatically determined by the double field theory action. This leads to a slightly altered formulation of the action in which the ten-dimensional origin of the four-dimensional fields becomes evident. A subset of these fields thereby appears only in particular combinations with the fluxes, which eventually leads to the same physical degrees of freedom as obtained in the $SU(3)\times SU(3)$ framework.

Taking up our discussion at the end of section 3.3, our result shows that double field theory provides a natural ten-dimensional origin for previously isolated gauged supergravities. In the considered setting, it can thus serve as the missing link to complete the "web of (gauged) supergravities" from figure 3.2 to a new form as illustrated in figure 5.1.

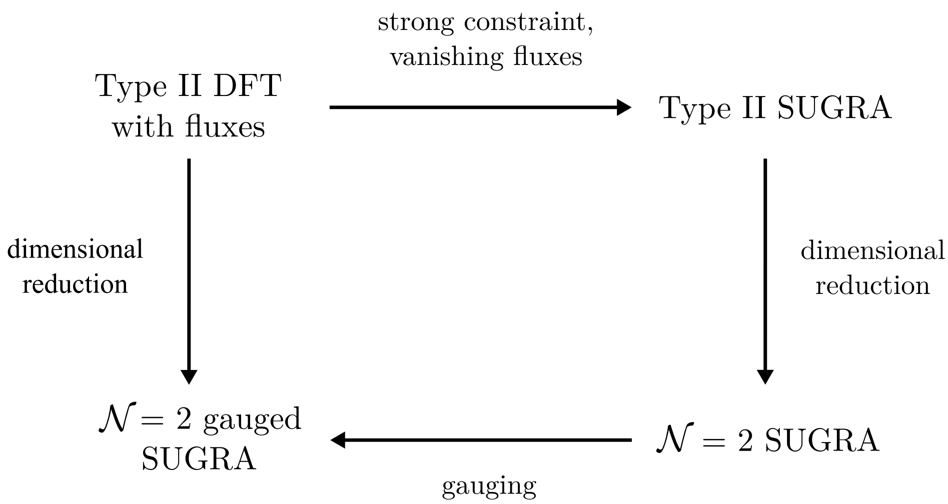


Figure 5.1: A "web of supergravities". Double field theory serves as the missing link between ten-dimensional supergravities and gauged four-dimensional supergravities.

Mirror Symmetry

A final interesting result of our analysis is the recovery of Mirror Symmetry. Both the CY_3 and the $K3 \times T^2$ setting featured a set of IIA $\leftrightarrow$ IIB mirror mappings of their effective actions that involved a characteristic exchange of roles between the Kähler-class and complex-structure moduli. As was to be expected in light of the conjectured equivalence of T-duality and Mirror Symmetry [61], this was also accompanied by mappings between the geometric and non-geometric fluxes. In all cases, the double field theory framework provided a nicely-interpretable notation of the mirror mappings as simple interchangings between ten-dimensional poly-forms encoding the different types of moduli and the theory-specific R-R fields.

Open Questions and Future Directions

An interesting task to pursue in future research would be to to derive the remaining four-dimensional gauged supergravities from double field theory. A natural next step is thereby to analyze how the framework can be applied to the full action compactified on $K3 \times T^2$. Since dimensional reductions on Calabi-Yau three-folds lead to a partially dualized formulation of gauged $\mathcal{N} = 2$ supergravity, an important question in this context is whether a similar construction has to be applied to its $\mathcal{N} = 4$ analogue. Similarly to the Calabi-Yau setting, it would make sense to also address these questions with a view to compactifications on $SU(2)$ structure manifolds [222–224] and to elaborate the analogies between both frameworks. Other possible directions include extensions of the orientifold setting discussed in [78] or dimensional reductions of heterotic double field theory.

Chapter 6

Type IIB Flux Vacua and Tadpole Cancellation

We have seen in the previous chapters that (flux) compactifications of type II theories give rise to (gauged) $\mathcal{N} = 2$ supergravities in four dimensions. As discussed in the beginning of this thesis, the most commonly used methods of model building in string theory require the amount of supersymmetry to be reduced to $\mathcal{N} = 1$. One way this problem can be addressed is by introducing *orientifold projections* and D-branes. In conjunction with constraints arising from the presence of fluxes, such compactifications come with a variety of further restrictions which greatly affect the space of allowed background configurations. This chapter will focus on the role of such consistency constraints in type IIB theory compactified on the orientifold $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$. Up to minor changes, the contents of this chapter are mostly quoted in verbatim from the author's work [80].

6.1 Overview

We again begin this chapter with a brief outline of the main ideas and most important results our discussion. Two particular phenomenological considerations will be in the focus of our analysis:

1. As discussed in the previous chapters, compactifications of string theory to four dimensions are typically performed on Calabi-Yau three-folds. The resulting effective theory contains a number of massless scalar fields corresponding to deformations of the background, which should be absent due to experimental constraints. A way to achieve this for type II theories is to deform the background geometry by Neveu-Schwarz–Neveu-Schwarz (NS-NS) and Ramond-Ramond (R-R) fluxes, which generate a potential in the four-dimensional theory and provide mass-terms for the moduli fields. However, especially in the type IIB setting some of the moduli cannot be stabilized by geometric fluxes. One therefore includes nongeometric fluxes or non-perturbative effects which lead to the KKLT [225] and large-volume [226] scenarios. Moduli stabilization often results in anti-de-Sitter or Minkowski vacua, while it is difficult to obtain de-Sitter solutions [227].

2. A gauge-theory sector for type II theories can be engineered using D-branes. D-branes filling four-dimensional spacetime and wrapping submanifolds in the compact space have a gauge theory localized on their worldvolume. Chiral matter can be localized at the intersection loci of different D-branes in the compact space, and in this way four-dimensional gauge theories can be constructed in a geometric way (for a review see for instance [228]). However, when introducing D-branes one typically has to perform an orientifold projection of the theory. The fixed-loci of this projection correspond to orientifold planes which generically have negative mass and negative charge.

Moduli stabilization and the construction of a gauge-theory sector are two important aspects of connecting string theory to realistic four-dimensional physics. These tasks are often approached independently, however, as emphasized for instance in [229], there is a complicated interplay between them. This interplay can prevent moduli from being stabilized or can modify the stabilization procedure. Following this line of thought, the purpose of this chapter is to study how parts 1) and 2) are connected via the tadpole-cancellation condition (schematically)

$$\text{fluxes} = \text{D-branes} + \text{O-planes}\,. \tag{6.1.1}$$

We approach this question by analyzing how properties of the space of flux vacua depend on the contribution of fluxes to the left-hand side of (6.1.1). We perform our analysis for the type IIB orientifold of $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ and consider the R-R three-form flux, the geometric NS-NS H-flux as well as the nongeometric NS-NS Q-flux. The geometric fluxes generically stabilize the complex-structure moduli and the axio-dilaton, while the nongeometric fluxes allow for stabilization of the Kähler moduli. We then determine distributions for how the values of the stabilized moduli depend on the tadpole contribution of the fluxes. Note that distributions of flux vacua have been discussed extensively in the literature before. For instance, for type IIB compactifications various aspects have been studied in [230–239, 71, 240] and for type IIA related work can be found in [68, 241]. In the context of M-theory, similar questions have been discussed in [242], and for F-theory see [243]. Recently also topological data analysis has been used to investigate properties of flux vacua in [244, 245]. The main results of our analysis can be summarized as follows:

- We observe that the space of flux vacua is not homogenous but shows characteristic structures such as circular voids [233]. We show furthermore that solutions can be accumulated on submanifolds in the moduli space.

- We find that flux configurations which stabilize moduli in a weak-coupling, large complex-structure and/or large-volume regime make up only a very small fraction of all possible configurations. The number of reliable flux vacua is therefore much smaller than naively expected.

- In order to stabilize moduli in a perturbatively-controlled regime at weak coupling, large complex structure and large volume, the flux contribution to the left-hand side of tadpole-cancellation condition (6.1.1) has to be larger than a certain threshold.

The more reliable these vacua are required to be, the larger this threshold has to be. However, the contribution of D-branes and orientifold planes to the right-hand side of (6.1.1) is typically small. It is therefore difficult to perform moduli stabilization in a perturbatively-controlled regime and to satisfy the tadpole-cancellation condition.

Our findings for the structure of the space of flux vacua agree with for instance [233, 245] for the axio-dilaton, but we extend their analysis by including the complex-structure moduli. Our observation concerning the difficulty of obtaining reliable flux vacua is consistent with for instance [246], who find that type IIB solutions at weak string-coupling are rare. Similarly, in [247] the authors argue that in order to avoid a certain runaway behaviour large fluxes have to be considered. Also in [248] large fluxes are needed to obtain reliable solutions, and related difficulties are encountered in [249].

This chapter is organized as follows: in section 6.2 we review type IIB orientifold compactifications with geometric and nongeometric fluxes, we discuss the corresponding tadpole-cancellation conditions, we specialize to the example of the $T^6/\mathbb{Z}_2\times\mathbb{Z}_2$ orientifold and determine the relevant dualities. In section 6.3 we study moduli stabilization for the axio-dilaton, in section 6.4 we discuss the combined moduli stabilization of the axio-dilaton and the complex-structure moduli, and in section 6.5 we stabilize all of the closed-string moduli at tree-level. At the end of sections 6.3, 6.4 and 6.5 we have included brief summaries for each section, which may help the reader to get an overview of the main results. Conclusions and implications of our results are discussed in section 6.6.

6.2 Flux Compactifications on Orientifolds

In this chapter we are interested in compactifications of type IIB string theory on Calabi-Yau orientifolds with geometric and nongeometric fluxes. In order to fix our notation, we start in sections 6.2.1, 6.2.2 and 6.2.3 by briefly reviewing orientifold compactifications and tadpole-cancellation conditions for general Calabi-Yau three-folds. In section 6.2.4 we specialize to the example of $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$, and in section 6.2.5 we discuss duality transformations for this background.

6.2.1 Orientifold Projection

Orientifold compactifications owe their name to the involvement of a so-called *worldsheet parity operator* Ω_{P} which essentially reverses the orientation of the string worldsheet by exchanging left- and righ-moving modes of closed strings or the two ends of open strings. Its purpose in string compactifications is to mod out the worldsheet parity, thus giving rise to an unoriented theory. In the case of type II strings, this leads to the low-energy spectrum of four-dimensional $\mathcal{N}=2$ supergravity being truncated to that of its $\mathcal{N}=1$ counterpart.

To define the full orientifold projection,

$$\Omega_{\mathrm{P}}(-1)^{F_{\mathrm{L}}}\,, \tag{6.2.1}$$

$\Omega_{\rm P}$ is combined with a holomorphic involution σ on CY_3 and an additional factor $(-1)^{F_{\rm L}}$ involving the left-moving fermion operator $F_{\rm L}$. The former can be chosen to act on Ω and J as

$$\sigma^* J = +J\,, \qquad \sigma^* \Omega = -\Omega\,. \tag{6.2.2}$$

A concept which will be of particular importance for our considerations are so-called *orientifold planes* or *Op-planes*, which are defined as the fixed loci of σ. Since the holomorphic involution leaves the non-compact four-dimensioanl part invariant, such planes cover the entire external space. Their presence generally poses a challenge from the phenomenological viewpoint as Op-planes can carry negative R-R charges inducing closed string excitations out of the vacuum. Owed to the characteristic shape of the corresponding Feynman diagrams, this is commonly referred to as the problem of *tadpole cancellation*. A way to address this issue is to include additional Dp-branes carrying positive R-R charges into the theory. This comes, however, with a variety of consistency constraints summarized under the term *tadpole cancellation conditions* which will play a crucial role in our upcoming discussion and will be discussed in more detail in section 6.2.3.

For our choice (6.2.2), the allowed configurations of Op-planes are O3- and O7-planes, whose tadpole contributions have to be cancelled by the presence of D3 and D7-branes. There exists, however, also an alternative definition of (6.2.2) giving rise to O5- and O9-planes [250].

6.2.2 Calabi-Yau Orientifolds

Type IIB orientifold compactifications on Calabi-Yau three-folds give rise to a $\mathcal{N}=1$ supergravity theory in four dimensions. This theory can be characterized in terms of a superpotential, Kähler potential and D-term potential, which we determine in the following.

Cohomology

The combined worldsheet parity and left-moving fermion operator act on the bosonic fields as

$$\begin{aligned} \Omega_{\rm P}\,(-1)^{F_{\rm L}}\, g &= +\,g\,, \qquad & \Omega_{\rm P}\,(-1)^{F_{\rm L}}\, B &= -B\,, \\ \Omega_{\rm P}\,(-1)^{F_{\rm L}}\, \phi &= +\,\phi\,, \qquad & \Omega_{\rm P}\,(-1)^{F_{\rm L}}\, C_{2p} &= (-1)^p\, C_{2p}\,, \end{aligned} \tag{6.2.3}$$

with g the metric, B the Kalb-Ramond field, ϕ the dilaton and C_{2p} the type IIB Ramond-Ramond potentials. Since σ is an involution, the cohomology groups of the Calabi-Yau three-fold CY_3 split into even and odd eigenspaces as $H^{p,q}(CY_3) = H^{p,q}_+(CY_3) \oplus H^{p,q}_-(CY_3)$ [250], and for our discussion we assume that the corresponding Hodge numbers satisfy

$$h^{2,1}_+ = 0\,, \qquad h^{1,1}_- = 0\,. \tag{6.2.4}$$

However, more general cases can be considered as well. For the even Dolbeault cohomology groups we introduce bases analogously to (5.3.41)

$$\begin{aligned} \left\{\mathbf{1}^{(6)}\right\} &\in H_+^{0,0}(CY_3)\,, \\ \left\{\omega_{\hat{\mathsf{i}}}\right\} &\in H_+^{1,1}(CY_3)\,, \\ \left\{\widetilde{\omega}^{\hat{\mathsf{i}}}\right\} &\in H_+^{2,2}(CY_3)\,, \\ \left\{\tfrac{1}{\mathcal{K}}\star\mathbf{1}^{(6)}\right\} &\in H_+^{3,3}(CY_3)\,, \end{aligned} \qquad \text{with } \hat{\mathsf{i}} = 1,\ldots,h^{1,1} \tag{6.2.5}$$

where $\star\mathbf{1}^{(6)}$ denotes the volume form and $\mathcal{K}$ the total volume of CY_3. These bases can again be chosen to satisfy an analogous normalization condition to (5.3.43). Defining $\omega_0 := \frac{1}{\mathcal{K}}\star\mathbf{1}^{(6)}$ and $\widetilde{\omega}^0 := \mathbf{1}^{(6)}$, we can then construct a collective basis

$$\{\omega_{\hat{\mathsf{I}}},\tilde{\omega}^{\hat{\mathsf{I}}}\}\,, \qquad \int_{CY_3}\omega_{\hat{\mathsf{I}}}\wedge\tilde{\omega}^{\hat{\mathsf{J}}} = \delta_{\hat{\mathsf{I}}}^{\hat{\mathsf{J}}}\,, \qquad \hat{\mathsf{I}},\hat{\mathsf{J}} = 0,\ldots,h_+^{1,1}\,. \tag{6.2.6}$$

For the odd component of the third de-Rham cohomology group $H_-^3(CY_3)$ we can follow the same pattern as in (5.3.48) and choose a symplectic basis

$$\{\alpha_{\check{\mathsf{A}}},\beta^{\check{\mathsf{A}}}\}\,, \qquad \int_{CY_3}\alpha_{\check{\mathsf{A}}}\wedge\beta^{\check{\mathsf{B}}} = \delta_{\check{\mathsf{A}}}{}^{\check{\mathsf{B}}}\,, \qquad \check{\mathsf{A}},\check{\mathsf{B}} = 0,\ldots,h_-^{2,1}\,, \tag{6.2.7}$$

with all other pairings vanishing.

Moduli

When compactifying string theory from ten to four dimensions, the deformations of the six-dimensional background become dynamical fields in the four-dimensional theory. These moduli fields are contained in the poly-forms [219]

$$\begin{aligned} \Phi^+ &= e^{-\phi}\,e^{B+iJ}\,, \\ \Phi^- &= \Omega\,, \end{aligned} \qquad \Phi_{\mathrm{c}}^+ = e^B\,\mathcal{C}^{(\mathrm{IIB})} + i\,\mathrm{Re}\Phi^+\,, \tag{6.2.8}$$

where the sum over all R-R potentials $\mathcal{C}^{(\mathrm{IIB})} = \sum_{p=0}^4 C_{2p}$ is again to be understood as the moduli contribution of the C_{2p}-fields. Complex scalar fields τ and $T_{\hat{\mathsf{i}}}$ can be determined by expanding Φ_{c}^+ into its zero- and four-form components as

$$\Phi_{\mathrm{c}}^+ = \tau + T_{\hat{\mathsf{i}}}\tilde{\omega}^{\hat{\mathsf{i}}} =: T_{\hat{\mathsf{I}}}\tilde{\omega}^{\hat{\mathsf{I}}}\,, \tag{6.2.9}$$

where $T_0 := \tau$ is called the axio-dilaton and the objects $T_{\hat{\mathsf{i}}}$ contain the Kähler moduli of CY_3. In general, this expression also contains two-forms anti-invariant under σ, which are however vanishing due to our choice (6.2.4). The complex-structure moduli $U^{\check{\mathsf{a}}}$ with $\check{\mathsf{a}} = 1,\ldots,h_-^{2,1}$ are contained in the holomorphic three-form Ω.

Fluxes

We furthermore consider non-vanishing fluxes for the internal space. These are the R-R three-form flux $G_3 = \mathrm{d}\mathring{\mathcal{C}}^{(\mathrm{IIB})}\big|_3$ as well as geometric and nongeometric fluxes (5.2.21) in the NS-NS sector. For the latter, we will restrict our discussion to those coming with a nontrivial contribution in cohomology, which are the three-form flux H, the geometric flux F, and the nongeometric Q- and R-fluxes. The internal component of the twisted differential operator (5.2.22) therefore takes the form

$$\mathcal{D} = \mathrm{d} + H\wedge - F\circ + Q\bullet - R\llcorner \,. \tag{6.2.10}$$

in the considered setting. The precise action of the fluxes on the split cohomology bases will be specified below. We furthermore summarize the action of the combined worldsheet parity and left-moving fermion number on the fluxes as [70, 78]

$$\begin{aligned} \Omega_{\mathrm{P}}(-1)^{F_{\mathrm{L}}}\, H &= -\, H\,, \\ \Omega_{\mathrm{P}}(-1)^{F_{\mathrm{L}}}\, F &= +\, F\,, \\ \Omega_{\mathrm{P}}(-1)^{F_{\mathrm{L}}}\, Q &= -\, Q\,, \\ \Omega_{\mathrm{P}}(-1)^{F_{\mathrm{L}}}\, R &= +\, R\,. \end{aligned} \qquad \Omega_{\mathrm{P}}(-1)^{F_{\mathrm{L}}}\, G_3 = -G_3\,, \tag{6.2.11}$$

For our assumption (6.2.4), this implies that F and R are vanishing. We also note that the R-R and NS-NS three-form fluxes have to satisfy quantization conditions of the form (see e.g. [69])

$$\int_\Gamma G_3 \in \mathbb{Z}\,, \qquad\qquad \int_\Gamma H \in \mathbb{Z}\,, \tag{6.2.12}$$

where $\Gamma \in H_3(CY_3, \mathbb{Z})$ is an arbitrary three-cycle on the Calabi-Yau three-fold CY_3. For orbifolds and orientifolds this condition can be modified, and we come back to this point on page 127 below. Furthermore, as will be explained in section 6.2.5, the NS-NS fluxes are related among each other through T-duality transformations, and hence also the geometric F- and the nongeometric Q- and R-fluxes should be appropriately quantized.

Supergravity Data

When compactifying type IIB string theory on orientifolds of Calabi-Yau three-folds, the resulting four-dimensional effective theory can be described in terms of $\mathcal{N} = 1$ supergravity [250]. In particular, the Kähler potential takes the form

$$K = -\log\big[-i(\tau-\overline{\tau})\big] - 2\log\hat{\mathcal{K}} - \log\left[-i\int_{CY_3}\Omega\wedge\overline{\Omega}\right], \tag{6.2.13}$$

where $\hat{\mathcal{K}}$ denotes the volume of the Calabi-Yau manifold in Einstein frame. The superpotential is generated by the fluxes and can be expressed using the Mukai pairing (5.2.27) of the poly-forms (6.2.8) and the generalized derivative (6.2.10) in the form [70, 213, 71]

$$\begin{aligned} W &= \int_{CY_3} \big\langle \Phi^-, G_3 - \mathcal{D}\Phi^+_{\rm c} \big\rangle \\ &= \int_{CY_3} \Omega \wedge \Big[G_3 - \tau H - \big(Q \bullet \tilde{\omega}^{\hat{\rm i}} \big) T_{\hat{\rm i}} \Big] . \end{aligned} \tag{6.2.14}$$

In general, the fluxes (6.2.10) also generate a D-term potential which can be expressed using the three-form part of $\mathcal{D}(\mathrm{Im}\,\Phi^+)$ [78]. However, due to (6.2.11) the latter belongs to the σ-even third cohomology and vanishes when taking into account our requirements (6.2.4). In our setting therefore no D-term potential is generated.

Bianchi Identities and Tadpole-Cancellation Conditions

Finally, the R-R and NS-NS fluxes have to satisfy a number of Bianchi identities. These can be expressed using the generalized derivative $\mathcal{D}$ as

$$\mathcal{D}^2 = \text{NS-NS sources}\,, \qquad\qquad \mathcal{D}G_3 = \text{R-R sources}\,, \tag{6.2.15}$$

where NS-NS sources stand for NS5-branes, Kaluza-Klein monopoles or non-geometric 5^2_2-branes (see for instance [73] for a review and collection of references). However, in this discussion we assume these to be absent and therefore require $\mathcal{D}^2 = 0$. The R-R sources stand for orientifold planes and D-branes, and the second constraint in (6.2.15) is also known as the tadpole cancellation condition. We discuss this condition in more detail in the following section.

6.2.3 Tadpole-Cancellation Condition

The tadpole-cancellation condition is an important consistency condition for type I string theories. It links the closed-string to the open-string sector and puts strong constraints on the allowed D-brane configurations (for a review see for instance [228]). From a conformal-field-theory point of view, the tadpole-cancellation condition ensures the absence of UV divergencies in one-loop amplitudes (see e.g. [251, 252] for textbook reviews) and therefore plays an important role for string theory being a consistent theory of gravity. From an effective-field-theory point of view, the tadpole-cancellation condition is the integrated version of the equation of motion for the R-R potentials and ensures the absence of certain anomalies in type II orientifold compactifications via the generalized Green-Schwarz mechanism [253]. The tadpole-cancellation condition is thus an important consistency condition for string compactifications.

Explicit Expressions

We now formulate the tadpole-cancellation condition for the setting of the previous section. The contribution of the R-R-sources can be described using the charges [254, 255]

$$\mathcal{Q}_{\mathrm{D}p} = \mathrm{ch}\,(\mathcal{F}) \wedge \sqrt{\frac{\hat{\mathcal{A}}(\mathcal{R}_T)}{\hat{\mathcal{A}}(\mathcal{R}_N)}} \wedge [\Gamma_{\mathrm{D}p}]\,, \qquad \mathcal{Q}_{\mathrm{O}p} = Q_p \sqrt{\frac{\mathcal{L}(\mathcal{R}_T/4)}{\mathcal{L}(\mathcal{R}_N/4)}} \wedge [\Gamma_{\mathrm{O}p}]\,, \tag{6.2.16}$$

where $[\Gamma_{\mathrm{D}p}]$ and $[\Gamma_{\mathrm{O}p}]$ denote the Poincaré duals of the cycles wrapped by D-branes and O-planes in $\mathbb{R}^{1,3} \times CY_3$. The open-string gauge flux on the D-branes $\mathcal{F}$ appears in the Chern character, the tangential and normal part of the curvature two-form $\mathcal{R}$ appear in the $\hat{\mathcal{A}}$-genus and the Hirzebruch polynomial $\mathcal{L}$, and $Q_p = -2^{p-4}$ denotes the charge of an orientifold p-plane. For more details, the reader is referred, for instance, to section 8.6 in [73]. Denoting the orientifold image of a Dp-brane with a prime, the Bianchi identity for the R-R fluxes then reads

$$\mathcal{D}G_3 = \sum_{\mathrm{D}p+\mathrm{D}p'} \mathcal{Q}_{\mathrm{D}p} + \sum_{\mathrm{O}p} \mathcal{Q}_{\mathrm{O}p}\,, \tag{6.2.17}$$

where the sum is over all D-branes and orientifold planes present in the background. The Freed-Witten anomaly-cancellation condition [256] for D-branes takes the general form [257, 73]

$$\mathcal{D}\mathcal{Q}_{\mathrm{D}p} = 0\,, \qquad \mathcal{D}\mathcal{Q}_{\mathrm{O}p} = 0\,, \tag{6.2.18}$$

where we included the corresponding expression for an orientifold p-plane. Equation (6.2.17) can therefore be interpreted as a relation in $\mathcal{D}$-cohomology.

For the setting discussed in this chapter, the orientifold projection satisfies (6.2.2) and therefore leads to spacetime-filling O3- and O7-planes. Taking into account (6.2.4) and that G_3 in (6.2.17) is a three-form flux, we find the explicit expressions

$$\begin{aligned} Q \bullet G_3 &= -\,2 \sum_{\mathrm{D}7_a} N_{\mathrm{D}7_a} [\Gamma_{\mathrm{D}7_a}] + 8 \sum_{\mathrm{O}7_b} [\Gamma_{\mathrm{O}7_b}]\,, \\ H \wedge G_3 &= -\,2 \left(N_{\mathrm{D}3} - \frac{N_{\mathrm{O}3}}{4} \right) \omega_0 \\ &\qquad - \sum_{\mathrm{D}7_a} \mathrm{tr}\big(\mathcal{F}_a\big)^2 \wedge [\Gamma_{\mathrm{D}7_a}] + 2 \left(\sum_{\mathrm{D}7_a} N_{\mathrm{D}7_a} \frac{\chi(\Gamma_{\mathrm{D}7_a})}{24} + \sum_{\mathrm{O}7_b} \frac{\chi\big(\Gamma_{\mathrm{O}7_b}\big)}{12} \right) \omega_0\,. \end{aligned} \tag{6.2.19}$$

Here, $N_{\mathrm{D}3}$ and $N_{\mathrm{O}3}$ denote the number of D3-branes and O3-planes, $N_{\mathrm{D}7_a}$ denotes the number of coincident D7-branes in a stack a, $[\Gamma_{\mathrm{D}7_a}] = n_a^{\hat{\imath}} \omega_{\hat{\imath}}$ is the Poincaré dual of the cycle $\Gamma_{\mathrm{D}7_a}$ expanded in the basis (6.2.6), $\mathcal{F}_a$ is the (quantized) two-form gauge flux on the stack of D7-branes a in the fundamental representation, and $\chi(\Gamma)$ denotes the Euler number of the cycle Γ. The D-brane sums are over all D7-branes, and due to $h^{1,1}_- = 0$ the orientifold images give a factor of two. For details on the derivation of these expressions see for instance [258].

Orientifold Contributions

Let us now discuss the contribution of orientifold planes to the right-hand sides in (6.2.19). Typically orientifold planes give a positive contribution while D-branes give a negative contribution. For some classes of models the orientifold-contributions can be estimated as follows:

- For orientifolds of $T^6/\mathbb{Z}_M$ or $T^6/\mathbb{Z}_M \times \mathbb{Z}_N$ the numbers of O3- and O7-planes have been computed for instance in [259] for some examples. Here, the authors find that $N_{\rm O3}, N_{\rm O7} \lesssim 60$ and the contributions of the Euler numbers to (6.2.19) are vanishing. The contribution of orientifold planes to the right-hand sides of (6.2.19) is therefore typically positive and of order $\mathcal{O}(10)$.
- For del-Pezzo surfaces the possible orientifold projections have been classified in [260]. The number of orientifold three- and seven-planes are of order $\mathcal{O}(10)$, and in some examples the Euler numbers of the four-cycles are of order $\mathcal{O}(100)$. Also here, the contribution of orientifold planes to the right-hand side of (6.2.19) is positive and of order $\mathcal{O}(10)$.
- In F-theory the geometry of Calabi-Yau four-folds CY_4 encodes the geometry of D7-branes and orientifold planes in Calabi-Yau three-folds CY_3. If a lift from type IIB orientifolds to F-theory is possible, one finds that

$$\frac{\chi(CY_4)}{24} = \frac{N_{\rm O3}}{4} + \sum_{{\rm D7}_a} N_{{\rm D7}_a} \frac{\chi(\Gamma_{{\rm D7}_a})}{24} + \sum_{{\rm O7}_b} \frac{\chi\big(\Gamma_{{\rm O7}_b}\big)}{12}\,, \qquad (6.2.20)$$

  where $\chi(CY_4)$ denotes the Euler number of the Calabi-Yau four-fold CY_4. In [261, 262] a manifold $CY_{4\rm max}$ was identified with the largest known Euler number for a Calabi-Yau four-fold

$$\chi(CY_{4\rm max}) = 1\,820\,448\,, \qquad (6.2.21)$$

  and more details for the present context can be found in [76]. Hence, for this example the contribution to the D3-tadpole in (6.2.19) is of order $\mathcal{O}(10^5)$.

D-brane Contributions

We furthermore note that the tadpole-cancellation conditions (6.2.19) are the integrated versions of the R-R Bianchi identities (6.2.15). The former are therefore less restrictive than the latter, but for a proper string-theory solution also the Bianchi identities with localized sources have to be solved. When placing D-branes directly on top of orientifold planes, solutions may be constructed more easily, but in general this a difficult task (see for instance [263]).

However, we can make the following general argument: because D-branes have a non-vanishing mass, their probe approximation breaks down when too many D-branes are placed into a compact space (away from the orientifold planes). In this case the back-reaction of D-branes on the geometry has to be taken into account, and an extreme case for this mechanism is the formation of black holes. It would be desirable to make this more precise, but we can argue that for ignoring back-reaction effects the contribution of D-branes to the right-hand sides in (6.2.19) should not be arbitrarily large.

Flux Contributions

Turning now to the flux contribution on the left-hand sides in (6.2.19), we note that for vanishing Q-flux the $H \wedge G_3$-term typically has to be positive in order to obtain physically-relevant solutions. Since the right-hand sides are bounded from above by the orientifold contributions, the flux contributions should not be larger than $\mathcal{O}(10)$ to $\mathcal{O}(10^5)$. In the presence of nongeometric Q-flux the left-hand sides in (6.2.19) can be negative – but since also the D-brane contributions are bounded, again the flux contributions should not be too large. This is an important point for our approach, which we summarize as

> In order to solve the tadpole-cancellation condition (6.2.19) and ignore the back-reaction of D-branes, the contribution of fluxes to the left-hand sides in (6.2.19) should not be too large. Depending on the setting, known bounds are of orders $\mathcal{O}(10)$ to $\mathcal{O}(10^5)$.

6.2.4 $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ Orientifold

Let us now turn to a specific example for a compactification space. We consider the orbifold $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$, which provides a simple example of a Calabi-Yau three-fold with only few moduli. For our purposes it is sufficient to stay in the orbifold limit and not blow-up the fixed-point singularities, that is we ignore the twisted sectors. This model has been extensively studied in the literature, and we refer for instance to [264–267] for more details in the present context.

For this model the contribution of orientifold planes to the tadpole-cancellation condition (6.2.19) only allows for a small number of different flux choices. In order to be able to study general properties of the space of solutions, in the following we therefore ignore the precise form of the tadpole cancellation condition and allow for arbitrarily-large values of $H \wedge G_3$ and $Q \bullet G_3$. We do however keep in mind that these tadpole contributions are bounded by the D-brane and orientifold contributions.

Compactification Space

We start from the following six-dimensional orbifold construction which has the properties of a Calabi-Yau three-fold:

$$CY_3 = \frac{T^2 \times T^2 \times T^2}{\mathbb{Z}_2 \times \mathbb{Z}_2} \,. \tag{6.2.22}$$

On each of the two-tori we introduce complex coordinates as

$$z^{\check{a}} = x^{\check{a}} + U^{\check{a}} y^{\check{a}} \,, \qquad \check{a} = 1, 2, 3 \,, \tag{6.2.23}$$

where $x^{\check{a}}$ and $y^{\check{a}}$ denote real coordinates with identifications $x^{\check{a}} \sim x^{\check{a}} + 1$ and $y^{\check{a}} \sim y^{\check{a}} + 1$, $U^{\check{a}}$ denote the complex structures on each of the T^2, and no summation is performed in (6.2.23). The orbifold action is given by

$$\Theta : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \to \begin{pmatrix} -z_1 \\ -z_2 \\ +z_3 \end{pmatrix} , \qquad \Theta' : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \to \begin{pmatrix} +z_1 \\ -z_2 \\ -z_3 \end{pmatrix} , \tag{6.2.24}$$

where Θ and Θ' are the two generators of the orbifold group $\mathbb{Z}_2 \times \mathbb{Z}_2$. In addition, we perform an orientifold projection

$$\sigma : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \to \begin{pmatrix} -z_1 \\ -z_2 \\ -z_3 \end{pmatrix} . \tag{6.2.25}$$

Cohomology

Next, we turn to the cohomology of (6.2.22). We note that there are no one- or five-forms invariant under the orbifold action (6.2.24), and that the invariant three-forms are given by the combinations

$$\begin{aligned} \alpha_0 &= \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 , & \beta^0 &= +\mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}y^3 , \\ \alpha_1 &= \mathrm{d}y^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 , & \beta^1 &= -\mathrm{d}x^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}y^3 , \\ \alpha_2 &= \mathrm{d}x^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}x^3 , & \beta^2 &= -\mathrm{d}y^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}y^3 , \\ \alpha_3 &= \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}y^3 , & \beta^3 &= -\mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}x^3 . \end{aligned} \tag{6.2.26}$$

Choosing the orientation of the six-dimensional space (6.2.22) such that we have $\int \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}y^3 = 1$, the three-forms in (6.2.26) satisfy the intersection relation (6.2.7). We can furthermore define a holomorphic three-form

$$\Omega = dz^1 \wedge dz^2 \wedge dz^3 , \tag{6.2.27}$$

which – when expanded in the basis (6.2.26) – takes the form [1]

$$\begin{aligned} \Omega = \alpha_0 + \big(\quad U^1\, \alpha_1 + \quad U^2\, \alpha_2 + \quad U^3\, \alpha_3 \big) \\ - \big(U^2U^3\, \beta^1 + U^3U^1\, \beta^2 + U^1U^2\, \beta^3 \big) + U^1U^2U^3\,\beta^0 . \end{aligned} \tag{6.2.28}$$

Turning to the orientifold action (6.2.25), we see that all three-forms (6.2.26) are odd under σ, and therefore $h^{2,1}_{-} = 3$ and $h^{2,1}_{+} = 0$. We also note that Ω is odd under the orientifold action as required by (6.2.2).

For the even cohomology we observe that the zero- and six-form cohomologies are even under the orbifold action (6.2.24). For the second cohomology we find the invariant $(1,1)$-forms

$$\omega_{\hat{\mathsf{i}}} = \frac{i}{2\,\mathrm{Im}\,U^{\hat{\mathsf{i}}}}\, dz^{\hat{\mathsf{i}}} \wedge d\bar{z}^{\hat{\mathsf{i}}} , \qquad \hat{\mathsf{i}} = 1,2,3 , \tag{6.2.29}$$

[1] The expression (6.2.28) is the classical result; in the quantum theory the coefficients of $\alpha_{\check{\mathsf{A}}}$ and $\beta^{\check{\mathsf{A}}}$ (i.e. the periods) typically receive quantum corrections. However, in the large complex-structure limit $\mathrm{Im}\,U^{\check{a}} \gg 1$ these corrections can be ignored.

with no summation over $\hat{\mathsf{i}}$, and we define invariant $(2,2)$-forms as

$$\tilde{\omega}^1 = -\omega_2 \wedge \omega_3 \,, \qquad \tilde{\omega}^2 = -\omega_3 \wedge \omega_1 \,, \qquad \tilde{\omega}^3 = -\omega_1 \wedge \omega_2 \,. \tag{6.2.30}$$

Note that these satisfy the relations shown in (6.2.5). For the orbifold (6.2.22) we can now define a real Kähler form as

$$J = v^1\omega_1 + v^2\omega_2 + v^3\omega_3 \,, \tag{6.2.31}$$

where the $v^{\hat{\mathsf{i}}}$ are the (real) Kähler moduli. The forms (6.2.29) and (6.2.30) are all even under the orientifold projection (6.2.25), and therefore $h^{1,1}_+ = 3$ and $h^{1,1}_- = 0$. We also note that J is even under σ, in agreement with (6.2.2).

Moduli

With the explicit expressions for the even cohomologies discussed above, we can now determine the moduli fields contained in Φ_{c}^+ via equation (6.2.9). For the R-R zero- and four-form potentials (purely in the internal space) we use the conventions

$$C_0 = C^{(0)}{}_0\tilde{\omega}^0 \,, \qquad C_4 = C^{(4)}{}_1\tilde{\omega}^1 + C^{(4)}{}_2\tilde{\omega}^2 + C^{(4)}{}_3\tilde{\omega}^3 \,, \tag{6.2.32}$$

introduced in the previous chapter. Evaluating (6.2.9) in the present situation leads to

$$T_0 = \tau = C^{(0)}{}_0 + ie^{-\phi} \,, \qquad \begin{aligned} T_1 &= C^{(4)}{}_1 + i\mathsf{v}^2\mathsf{v}^3 \,, \\ T_2 &= C^{(4)}{}_2 + i\mathsf{v}^3\mathsf{v}^1 \,, \\ T_3 &= C^{(4)}{}_3 + i\mathsf{v}^1\mathsf{v}^2 \,, \end{aligned} \tag{6.2.33}$$

with the Einstein-frame Kähler moduli defined as $\mathsf{v}^{\hat{\mathsf{i}}} = e^{-\phi/2}v^{\hat{\mathsf{i}}}$. We also note that the R-R two-form potential C_2 is odd under the combined worldsheet parity and left-moving fermion number (cf. (6.2.3)) and should therefore be expanded in the σ-odd $(1,1)$-cohomology, which however vanishes. Finally, the complex-structure moduli $U^{\check{\mathsf{a}}}$ are contained in Ω as can be seen from (6.2.28).

Fluxes

Let us now turn to the fluxes. Using the basis of three-forms (6.2.26), the NS-NS and R-R three-form fluxes can be expanded as

$$G_3 = \mathfrak{f}^{\check{\mathsf{A}}}\alpha_{\check{\mathsf{A}}} + \mathfrak{f}_{\check{\mathsf{A}}}\beta^{\check{\mathsf{A}}} \,, \qquad H = h^{\check{\mathsf{A}}}\alpha_{\check{\mathsf{A}}} + h_{\check{\mathsf{A}}}\beta^{\check{\mathsf{A}}} \,, \tag{6.2.34}$$

where $\check{\mathsf{A}} = 0, \ldots, 3$ and we used the definitions $\mathfrak{f}^{\check{\mathsf{A}}} := G^{(3)\check{\mathsf{A}}}, \mathfrak{f}_{\check{\mathsf{A}}} := G^{(3)}{}_{\check{\mathsf{A}}}$ to simplify notation. The expansion coefficients $\mathfrak{f}^{\check{\mathsf{A}}}, \mathfrak{f}_{\check{\mathsf{A}}}, h^{\check{\mathsf{A}}}, h_{\check{\mathsf{A}}}$ are quantized due to the flux quantization conditions for G_3 and H shown in (6.2.12). For the remaining fluxes in the NS-NS sector we note that due to (6.2.4) and (6.2.11), the geometric F- and the nongeometric R-flux vanish. The Q-flux is in general non-vanishing, and we specify it by its action on the third and fourth cohomology as

$$\begin{aligned} Q \bullet \alpha_{\check{\mathsf{A}}} &= -q_{\check{\mathsf{A}}}{}^{\hat{\mathsf{i}}}\omega_{\hat{\mathsf{i}}}\,, \\ Q \bullet \beta^{\check{\mathsf{A}}} &= +q^{\check{\mathsf{A}}\hat{\mathsf{i}}}\omega_{\hat{\mathsf{i}}}\,, \end{aligned} \qquad Q \bullet \tilde{\omega}^{\hat{\mathsf{i}}} = q^{\check{\mathsf{A}}\hat{\mathsf{i}}}\alpha_{\check{\mathsf{A}}} + q_{\check{\mathsf{A}}}{}^{\hat{\mathsf{i}}}\beta^{\check{\mathsf{A}}}\,. \tag{6.2.35}$$

Here we have again $\hat{\mathsf{i}} = 1, 2, 3$, and the flux quanta are integers. In order to shorten the notation for our subsequent discussion, we combine the H-flux with the Q-flux by defining

$$q_{\check{\mathsf{A}}}{}^{0} = h_{\check{\mathsf{A}}}\,, \qquad\qquad q^{\check{\mathsf{A}}0} = h^{\check{\mathsf{A}}}\,. \tag{6.2.36}$$

Let us briefly discuss a subtlety concerning the flux quantization condition (6.2.12). It was first pointed out in [268] that on orbifold (or orientifold) spaces besides bulk cycles inherited from the covering space, twisted cycles of shorter length can exist. This implies that the quantization condition of the fluxes shown above is slightly modified. For the present example of the type IIB $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold this observation has been mentioned in [269, 71] and has been analyzed in detail for instance in [270, 271]. More concretely, for $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold actions with and without discrete torsion (see [272]) one finds that fluxes on generic bulk cycles have to satisfy

$$\begin{aligned} \text{without discrete torsion} \qquad & \mathfrak{f}^{\check{\mathsf{A}}}, \mathfrak{f}_{\check{\mathsf{A}}}, q^{\check{\mathsf{A}}\hat{\mathsf{I}}}, q_{\check{\mathsf{A}}}{}^{\hat{\mathsf{I}}} \in 8\mathbb{Z}\,, \\ \text{with discrete torsion} \qquad & \mathfrak{f}^{\check{\mathsf{A}}}, \mathfrak{f}_{\check{\mathsf{A}}}, q^{\check{\mathsf{A}}\hat{\mathsf{I}}}, q_{\check{\mathsf{A}}}{}^{\hat{\mathsf{I}}} \in 4\mathbb{Z}\,. \end{aligned} \tag{6.2.37}$$

As mentioned at the beginning of this subsection, in our discussion we ignore the twisted sector which effectively implies that we consider models without discrete torsion [270]. Fluxes will therefore be quantized in multiples of eight. In the literature similar orientifolds have been studied [269, 230, 233, 237, 71], although with slightly different quantization conditions.

Bianchi Identities

Turning to the Bianchi identities (6.2.15), we recall from (6.2.34) and (6.2.35) that the R-R and NS-NS flux quanta are given by $\mathfrak{f}_{\check{\mathsf{A}}}$, $\mathfrak{f}^{\check{\mathsf{A}}}$, $q_{\check{\mathsf{A}}}{}^{\hat{\mathsf{I}}}$ and $q^{\check{\mathsf{A}}\hat{\mathsf{I}}}$. For the left-hand side of the Bianchi identities we then introduce the general notation

$$\begin{aligned} \mathcal{D}G_3 &= \mathsf{Q}^{\hat{\mathsf{I}}}\ \omega_{\hat{\mathsf{I}}}\,, \\ \mathcal{D}H &= \mathsf{Q}^{0\hat{\mathsf{J}}}\omega_{\hat{\mathsf{J}}}\,, \\ \mathcal{D}(Q\bullet\tilde{\omega}^{\hat{\mathsf{I}}}) &= \mathsf{Q}^{\hat{\mathsf{I}}\hat{\mathsf{J}}}\ \omega_{\hat{\mathsf{J}}}\,, \end{aligned} \qquad \begin{aligned} \mathsf{Q}^{\hat{\mathsf{I}}} &= \mathfrak{f}_{\check{\mathsf{A}}}q^{\check{\mathsf{A}}\hat{\mathsf{I}}} - \mathfrak{f}^{\check{\mathsf{A}}}q_{\check{\mathsf{A}}}{}^{\hat{\mathsf{I}}}\,, \\ \mathsf{Q}^{\hat{\mathsf{I}}\hat{\mathsf{J}}} &= q_{\check{\mathsf{A}}}{}^{\hat{\mathsf{I}}}q^{\check{\mathsf{A}}\hat{\mathsf{J}}} - q^{\check{\mathsf{A}}\hat{\mathsf{I}}}q_{\check{\mathsf{A}}}{}^{\hat{\mathsf{J}}}\,, \end{aligned} \tag{6.2.38}$$

where $\hat{\mathsf{I}}, \hat{\mathsf{J}} = 0, \ldots, 3$. Note that these expressions can be combined into an anti-symmetric five-by-five matrix of the form

$$\mathbf{Q} = \begin{pmatrix} 0 & +\mathsf{Q}^{\hat{\mathsf{J}}} \\ -\mathsf{Q}^{\hat{\mathsf{I}}} & \mathsf{Q}^{\hat{\mathsf{I}}\hat{\mathsf{J}}} \end{pmatrix}, \qquad\qquad \hat{\mathsf{I}}, \hat{\mathsf{J}} = 0, 1, \ldots, 3\,. \tag{6.2.39}$$

The right-hand side of the Bianchi identities (6.2.15) corresponds to NS-NS and R-R sources, and schematically we have the relations

$$\begin{aligned} \mathsf{Q}^0 &\longleftrightarrow \text{O3-planes/D3-branes,} \\ \mathsf{Q}^{\hat{\mathsf{i}}} &\longleftrightarrow \text{O7-planes/D7-branes,} \\ \mathsf{Q}^{0\hat{\mathsf{i}}} &\longleftrightarrow \text{NS5-branes,} \\ \mathsf{Q}^{\hat{\mathsf{i}}\hat{\mathsf{j}}} &\longleftrightarrow 5^2_2\text{-branes,} \end{aligned} \tag{6.2.40}$$

where in particular $\mathsf{Q}^{\hat{\mathsf{I}}}$ for $\hat{\mathsf{I}} = 0, \ldots, 3$ are the contributions to the R-R tadpole cancellation conditions (6.2.19). As mentioned above, in this discussion we do not consider NS5-branes or nongeometric 5^2_2-branes, which leads to the requirement $\mathsf{Q}^{\hat{\mathsf{I}}\hat{\mathsf{J}}} = 0$ for $\hat{\mathsf{I}}, \hat{\mathsf{J}} = 0, \ldots, 3$.

Supergravity Data

Let us finally determine the Kähler and superpotential for the $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold compactification. Evaluating (6.2.13) we find for the Kähler potential

$$K = -\sum_{\hat{\mathsf{I}}=0}^{3} \log\Big[-i(T_{\hat{\mathsf{I}}} - \overline{T}_{\hat{\mathsf{I}}})\Big] - \sum_{\check{\mathsf{a}}=1}^{3} \log\Big[-i(U^{\check{\mathsf{a}}} - \overline{U}^{\check{\mathsf{a}}})\Big] \,, \tag{6.2.41}$$

up to an irrelevant constant term. Turning to the superpotential (6.2.14), the expansions of the fluxes in (6.2.34) and (6.2.35) give rise to

$$\begin{aligned} W = \; & \mathfrak{f}_0 - q_0{}^{\hat{\mathsf{I}}}\, T_{\hat{\mathsf{I}}} \\ & +U^{\check{\mathsf{a}}} \left(\mathfrak{f}_{\check{\mathsf{a}}} - q_{\check{\mathsf{a}}}{}^{\hat{\mathsf{I}}}\, T_{\hat{\mathsf{I}}} \right) \\ & +\tfrac{1}{2}\sigma_{\check{\mathsf{a}}\check{\mathsf{b}}\check{\mathsf{c}}} U^{\check{\mathsf{a}}} U^{\check{\mathsf{b}}} \left(\mathfrak{f}^{\check{\mathsf{c}}} - q^{\check{\mathsf{c}}\hat{\mathsf{I}}}\, T_{\hat{\mathsf{I}}} \right) \\ & -\tfrac{1}{6}\sigma_{\check{\mathsf{a}}\check{\mathsf{b}}\check{\mathsf{c}}} U^{\check{\mathsf{a}}} U^{\check{\mathsf{b}}} U^{\check{\mathsf{c}}} \left(\mathfrak{f}^{0} - q^{0\hat{\mathsf{I}}}\, T_{\hat{\mathsf{I}}} \right) , \end{aligned} \tag{6.2.42}$$

where a summation over $\hat{\mathsf{I}} = 0, \ldots, 3$ and $\check{\mathsf{a}} = 1, 2, 3$ is understood. For ease of notation, we also defined the symmetric symbol $\sigma_{\check{\mathsf{a}}\check{\mathsf{b}}\check{\mathsf{c}}}$, which has the only non-vanishing components

$$\sigma_{123} = \sigma_{132} = \sigma_{231} = \sigma_{213} = \sigma_{312} = \sigma_{321} = +1 \,. \tag{6.2.43}$$

The scalar F-term potential is determined in terms of the Kähler potential K and superpotential W according to

$$V_F = e^K \Big[D_\alpha W\, \mathcal{G}^{\alpha\overline{\beta}} D_{\overline{\beta}} \overline{W} - 3|W|^2 \Big] \,, \tag{6.2.44}$$

where ϕ^α collectively labels the complex scalar fields of the theory. The Kähler metric is computed from the Kähler potential as $g_{\alpha\overline{\beta}} = \partial_\alpha \partial_{\overline{\beta}} K$, and the covariant derivative reads

$$D_\alpha W = \partial_\alpha W + \big(\partial_\alpha K\big) W \,. \tag{6.2.45}$$

We also note that due to our assumption $h^{2,1}_+ = 0$ shown in (6.2.4), no D-term potential is generated by the fluxes.

6.2.5 Dualities

We now want to discuss dualities for the orientifold of $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ introduced in the previous section. We are interested in transformations which leave the physical properties of a system invariant but which are not necessarily symmetries of the action. In particular, we note that an extremum of the F-term potential (6.2.44) is reached for vanishing F-terms

$$0 = \partial_\alpha W + \big(\partial_\alpha K\big) W \,, \tag{6.2.46}$$

and in our subsequent analysis we are interested in duality transformations which map solutions of (6.2.46) to new solutions.

Overall Sign-Change

Let us start by noting that the F-term potential (6.2.44) as well as the F-term equations (6.2.46) are invariant under changing the sign of all fluxes [71]

$$\big(\mathfrak{f}_{\check{\mathsf{A}}}, \mathfrak{f}^{\check{\mathsf{A}}}, q_{\check{\mathsf{A}}}{}^{\hat{\mathsf{I}}}, q^{\check{\mathsf{A}}\hat{\mathsf{I}}}\big) \;\longrightarrow\; \big(-\mathfrak{f}_{\check{\mathsf{A}}}, -\mathfrak{f}^{\check{\mathsf{A}}}, -q_{\check{\mathsf{A}}}{}^{\hat{\mathsf{I}}}, -q^{\check{\mathsf{A}}\hat{\mathsf{I}}}\big) \,. \tag{6.2.47}$$

This $\mathbb{Z}_2$ transformation maps $W \to -W$, which indeed leaves the scalar potential (6.2.44), the equations (6.2.46) and the tadpole contributions (6.2.38) invariant.

$SL(2,\mathbb{Z})$ for Complex-Structure Moduli $U^{\check{\mathsf{a}}}$

Next, we consider the group of large diffeomorphisms for each of the two-tori in (6.2.22) [71]. For a single T^2 this group is $SL(2,\mathbb{Z})$, which is generated by T- and S-transformations of the form

$$T: \, U^{\check{\mathsf{a}}} \to U^{\check{\mathsf{a}}} + 1 \,, \qquad\qquad S: \, U^{\check{\mathsf{a}}} \to -1/U^{\check{\mathsf{a}}} \,, \tag{6.2.48}$$

with $\check{\mathsf{a}} = 1,2,3$. In order for the F-term equation (6.2.46) to stay invariant under T-transformations, the fluxes have to transform as

$$U^{\check{\mathsf{a}}} \to U^{\check{\mathsf{a}}} + b^{\check{\mathsf{a}}} \,, \qquad \begin{aligned} A_0 &\to A_0 - b^{\check{\mathsf{a}}} A_{\check{\mathsf{a}}} + \tfrac{1}{2}\sigma_{\check{\mathsf{a}}\check{\mathsf{b}}\check{\mathsf{c}}}\, b^{\check{\mathsf{a}}} b^{\check{\mathsf{b}}} A^{\check{\mathsf{c}}} + \tfrac{1}{6}\sigma_{\check{\mathsf{a}}\check{\mathsf{b}}\check{\mathsf{c}}}\, b^{\check{\mathsf{a}}} b^{\check{\mathsf{b}}} b^{\check{\mathsf{c}}} \,, \\ A_{\check{\mathsf{a}}} &\to A_{\check{\mathsf{a}}} - \sigma_{\check{\mathsf{a}}\check{\mathsf{b}}\check{\mathsf{c}}}\, b^{\check{\mathsf{b}}} A^{\check{\mathsf{c}}} - \tfrac{1}{2}\sigma_{\check{\mathsf{a}}\check{\mathsf{b}}\check{\mathsf{c}}}\, b^{\check{\mathsf{b}}} b^{\check{\mathsf{c}}} A^0 \,, \\ A^{\check{\mathsf{a}}} &\to A^{\check{\mathsf{a}}} + b^{\check{\mathsf{a}}} A^0 \,, \\ A^0 &\to A^0 \,, \end{aligned} \tag{6.2.49}$$

where $(A_{\check{\mathsf{A}}}, A^{\check{\mathsf{A}}})$ stands collectively for $(\mathfrak{f}_{\check{\mathsf{A}}}, \mathfrak{f}^{\check{\mathsf{A}}})$ and $(q_{\check{\mathsf{A}}}{}^{\hat{\mathsf{I}}}, q^{\check{\mathsf{A}}\hat{\mathsf{I}}})$, and $\sigma_{\check{\mathsf{a}}\check{\mathsf{b}}\check{\mathsf{c}}}$ was defined in (6.2.43). Under S-transformations of the complex-structure moduli, the fluxes transform as

$$U^1 \to -1/U^1 \,, \qquad \begin{aligned} A_0 &\to -A_1 \,, & A^0 &\to -A^1 \,, \\ A_1 &\to +A_0 \,, & A^1 &\to +A^0 \,, \\ A_2 &\to -A^3 \,, & A^2 &\to +A_3 \,, \\ A_3 &\to -A^2 \,, & A^3 &\to +A_2 \,, \end{aligned} \tag{6.2.50}$$

and similarly for U^2 and U^3. Note that for the fluxes this is not a $\mathbb{Z}_2$ but a $\mathbb{Z}_4$ action, which is however reduced to $\mathbb{Z}_2$ using (6.2.47). We also note that for a simultaneous S-transformation of all three complex-structure moduli, the transformation reads

$$U^{\check{a}} \to -1/U^{\check{a}}\,, \qquad \begin{aligned} A_{\check{A}} &\to +A^{\check{A}}\,, \\ A^{\check{A}} &\to -A_{\check{A}}\,. \end{aligned} \tag{6.2.51}$$

Furthermore, all Bianchi identities and tadpole contributions $\mathsf{Q}^{\hat{I}}$ and $\mathsf{Q}^{\hat{I}\hat{J}}$ are invariant under these transformations.

T-duality

We now turn to T-duality transformations. It is well-known that performing an odd number of T-dualities for type IIB string theory results in the type IIA theory and vice versa, and applying two or six T-dualities to type IIB string theory with O3-/O7-planes results in type IIB with O5-/O9-planes. For T-duality to map the present setting of type IIB with O3-/O7-planes to itself, we therefore have to perform four collective T-duality transformations.

Let us now consider more closely the $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold with O3-/O7-planes. Using the Buscher rules (2.2.20, 2.2.21), a collective T-duality transformation [273] say along the first and second T^2 results in the following transformation of the moduli:

$$\text{T-duality along } z^1, z^2\colon \quad \left\{ \begin{array}{lll} \tau \;\to\; T_3\,, & & \\ T_1 \to T_2\,, & & U^1 \;\to\; -1/U^1\,, \\ T_2 \to T_1\,, & & U^2 \;\to\; -1/U^2\,, \\ T_3 \to \tau\,. & & \end{array} \right. \tag{6.2.52}$$

In (6.2.52) we have only shown how the moduli transform, but also the fluxes transform in a non-trivial way under T-duality. However, (6.2.52) contains an S-transform of the complex-structure moduli U^1 and U^2. To better show the underlying structure, let us undo the $U^{\check{a}}$ transformation in (6.2.52) using (6.2.50). We then obtain the transformation

$$\begin{array}{lll} \tau \;\to\; T_3\,, & & q_{\check{A}}{}^1 \;\leftrightarrow\; q_{\check{A}}{}^2\,, \\ T_1 \to T_2\,, & & q_{\check{A}}{}^3 \;\leftrightarrow\; h_{\check{A}}\,, \\ T_2 \to T_1\,, & & q^{\check{A}1} \;\leftrightarrow\; q^{\check{A}2}\,, \\ T_3 \to \tau\,, & & q^{\check{A}3} \;\leftrightarrow\; h^{\check{A}}\,, \end{array} \tag{6.2.53}$$

which by a slight abuse of notation we will refer to as T-duality in the following. Similar transformations are obtained for T-duality along the second & third and first & third two-torus.[2]

[2]We also mention that the transformation of the moduli under T-duality shown in (6.2.53) was to be expected: for type IIB orientifolds the R-R zero- and four-form potentials C_0 and C_4 are the real parts of τ and $T_{\hat{\imath}}$, respectively. Under a single T-duality the R-R potentials transform as $C_p \to C_{p\pm1}$ [274], where the upper/lower sign is for a transformation transversal/longitudinal to C_p. For a collective T-duality along four directions we therefore map $C_0 \to C_4$ and some components of $C_4 \to C_0$. This agrees with (6.2.53).

We furthermore observe that the F-term equations (6.2.46) are invariant under a permutation of the Kähler moduli $T_{\hat{\imath}}$ and fluxes $(q_{\check{A}}{}^{\hat{\imath}}, q^{\check{A}\hat{\imath}})$. This is just a re-labelling of indices and corresponds to the permutation group $\mathcal{S}_3$. Using now the T-duality action (6.2.53) together with the permutation of Kähler moduli, we see that $\mathcal{S}_3$ is enhanced to $\mathcal{S}_4$ acting on $T_{\hat{I}} = (T_0, T_{\hat{\imath}})$ and fluxes $(q_{\check{A}}{}^{\hat{I}}, q^{\check{A}\hat{I}})$. Indeed, for $S_{\hat{I}}{}^{\hat{J}} \in \mathcal{S}_4$ the superpotential (6.2.42) is invariant under

$$T_{\hat{I}} \to S_{\hat{I}}{}^{\hat{J}} T_{\hat{J}} \,, \qquad \begin{aligned} q_{\check{A}}{}^{\hat{I}} &\to q_{\check{A}}{}^{\hat{J}} (S^{-1})_{\hat{J}}{}^{\hat{I}} \,, \\ q^{\check{A}\hat{I}} &\to q^{\check{A}\hat{J}} (S^{-1})_{\hat{J}}{}^{\hat{I}} \,, \end{aligned} \tag{6.2.54}$$

and the flux contributions to the Bianchi identities shown in (6.2.38) transform as

$$\mathbf{Q} \to \mathbf{S}^{-T} \mathbf{Q}\, \mathbf{S}^{-1} \,, \qquad \mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & S_{\hat{I}}{}^{\hat{J}} \end{pmatrix} . \tag{6.2.55}$$

We emphasize that four collective T-duality transformations for type II orientifold compactification are permutations of moduli and fluxes. They do not correspond to transformations which invert $T_{\hat{\imath}}$.

S-duality

We finally consider the $SL(2,\mathbb{Z})$ duality of type IIB string theory. For vanishing Q-flux its action on the axio-dilaton and the G_3- and H-flux takes the form

$$\tau \to \frac{a\tau + b}{c\tau + d} \,, \qquad \begin{pmatrix} G_3 \\ H \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} G_3 \\ H \end{pmatrix} , \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \,, \tag{6.2.56}$$

where $a, b, c, d \in \mathbb{Z}$. In particular, this transformation leaves the F-term equations (6.2.46) invariant. However, for non-vanishing Q-flux only part of this duality survives.

- For constant shifts of the axio-dilaton and the Kähler moduli $T_{\hat{\imath}}$, the Kähler potential (6.2.41) as well as the superpotential (6.2.42) are invariant under

$$T_{\hat{I}} \to T_{\hat{I}} + b_{\hat{I}} \,, \qquad \begin{aligned} \mathfrak{f}_{\check{A}} &\to \mathfrak{f}_{\check{A}} + q_{\check{A}}{}^{\hat{I}} b_{\hat{I}} \,, \\ \mathfrak{f}^{\check{A}} &\to \mathfrak{f}^{\check{A}} + q^{\check{A}\hat{I}} b_{\hat{I}} \,, \\ q_{\check{A}}{}^{\hat{I}} &\to q_{\check{A}}{}^{\hat{I}} \,, \\ q^{\check{A}\hat{I}} &\to q^{\check{A}\hat{I}} \,, \end{aligned} \tag{6.2.57}$$

  where the parameter $b_{\hat{I}}$ has to be quantized as $b_{\hat{I}} \in \mathbb{Z}$. This corresponds to a gauge transformation of the R-R zero- and four-form potentials.

- For an S-transformation $\tau \to -1/\tau$ in the presence of nongeometric fluxes, the F-term equations (6.2.46) are in general not invariant. To restore the $SL(2,\mathbb{Z})$ duality the authors of [212] introduced additional nongeometric P-fluxes as the counterpart of the Q-fluxes. In this discussion we do not consider such P-fluxes, but refer for instance to [275–278, 257, 279–281] for more details on this topic.

6.3 Moduli Stabilization I

As a first example for moduli stabilization on the $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold, we consider a specific choice of fluxes which stabilizes the axio-dilaton τ, fixes the complex-structure moduli $U^{\check{\mathsf{a}}}$ to a symmetric point but leaves the Kähler moduli $T_{\hat{\mathsf{i}}}$ unstabilized. This setting has been studied for instance in [230, 233, 237], and here we use it as a toy model for the more involved settings in the subsequent sections.

6.3.1 Setting

We start by specifying the superpotential (6.2.42). We consider a configuration with vanishing nongeometric fluxes (6.2.35),

$$q_{\check{\mathsf{A}}}{}^{\hat{\mathsf{i}}} = 0\,, \qquad q^{\check{\mathsf{A}}\hat{\mathsf{i}}} = 0\,, \tag{6.3.1}$$

which implies that the Kähler moduli $T_{\hat{\mathsf{i}}}$ do not appear in the potential and hence are not stabilized. The remaining R-R and NS-NS fluxes (6.2.34) are chosen as

$$\begin{aligned}
&\mathfrak{f}^0 = 3\tilde{\mathfrak{f}}^0\,, && \mathfrak{f}_1 = \mathfrak{f}_2 = \mathfrak{f}_3 = -\tilde{\mathfrak{f}}^0\,, && \tilde{\mathfrak{f}}^0 \in 8\mathbb{Z}\,,\\
&\mathfrak{f}_0 = 3\tilde{\mathfrak{f}}_0\,, && \mathfrak{f}^1 = \mathfrak{f}^2 = \mathfrak{f}^3 = +\tilde{\mathfrak{f}}_0\,, && \tilde{\mathfrak{f}}_0 \in 8\mathbb{Z}\,,\\
&h^0 = 3\tilde{h}^0\,, && h_1 = h_2 = h_3 = -\tilde{h}^0\,, && \tilde{h}^0 \in 8\mathbb{Z}\,,\\
&h_0 = 3\tilde{h}_0\,, && h^1 = h^2 = h^3 = +\tilde{h}_0\,, && \tilde{h}_0 \in 8\mathbb{Z}\,,
\end{aligned} \tag{6.3.2}$$

where $\tilde{h}^0$ and $\tilde{h}_0$ should not be zero simultaneously. Since the superpotential W is independent of the Kähler moduli, the F-term equations (6.2.46) take the simple form

$$\begin{aligned}
0 = F_{T_{\hat{\mathsf{i}}}} &= \phantom{\partial_{U^{\check{\mathsf{a}}}}W +{}} \partial_{T_{\hat{\mathsf{i}}}}K\, W\,, & & & 0 &= W\,,\\
0 = F_{U^{\check{\mathsf{a}}}} &= \partial_{U^{\check{\mathsf{a}}}}W + \partial_{U^{\check{\mathsf{a}}}}K\, W\,, & &\Rightarrow & 0 &= \partial_{U^{\check{\mathsf{a}}}}W\,,\\
0 = F_{\tau} &= \partial_{\tau}W + \partial_{\tau}K\, W\,, & & & 0 &= \partial_{\tau}W\,.
\end{aligned} \tag{6.3.3}$$

Ignoring unphysical values for $U^{\check{\mathsf{a}}}$ and τ with negative or vanishing imaginary part, we obtain the following solution to (6.3.3):

$$U^1 = U^2 = U^3 = i\,, \qquad \tau = \frac{\tilde{\mathfrak{f}}_0 - i\tilde{\mathfrak{f}}^0}{\tilde{h}_0 - i\tilde{h}^0}\,. \tag{6.3.4}$$

The fluxes in (6.3.2) are not arbitrary but are subject to the Bianchi identities (6.2.38). Since all nongeometric Q-fluxes vanish, the only nontrivial condition is the D3-tadpole contribution

$$\mathsf{Q}^0 = 12\Big(\tilde{\mathfrak{f}}_0\tilde{h}^0 - \tilde{\mathfrak{f}}^0\tilde{h}_0\Big) > 0\,, \tag{6.3.5}$$

where the requirement of Q^0 being positive is related to having $\mathrm{Im}\,\tau > 0$. Note that due to the quantization condition for the fluxes in (6.3.2), Q^0 is a multiple of 768.

6.3.2 Finite Number of Solutions for fixed Q^0

Next, we briefly review the arguments of [233,237] showing that the number of physically-distinct solutions (6.3.4) is finite for finite Q^0. We restrict the values of the axio-dilaton τ to the fundamental domain of the corresponding $SL(2,\mathbb{Z})$ duality (6.2.56)

$$\mathcal{F}_\tau = \left\{ -\frac{1}{2} \leq \tau_1 \leq 0,\, |\tau|^2 \geq 1 \;\cup\; 0 < \tau_1 < +\frac{1}{2},\, |\tau|^2 > 1 \right\}. \tag{6.3.6}$$

For ease of notation, we express the axio-dilaton in terms of its real and imaginary part as $\tau = \tau_1 + i\tau_2$, and for the solution (6.3.4) to the F-term equations we have

$$\tau = \frac{\tilde{h}_0 \tilde{\mathfrak{f}}_0 + \tilde{h}^0 \tilde{\mathfrak{f}}^0}{(\tilde{h}_0)^2 + (\tilde{h}^0)^2} + \frac{i}{12} \frac{\mathsf{Q}^0}{(\tilde{h}_0)^2 + (\tilde{h}^0)^2} \,. \tag{6.3.7}$$

For a fixed positive value of Q^0, the imaginary part of τ is bounded from above as $\tau_2 \leq \frac{\mathsf{Q}^0}{768}$ since $\tilde{h}_0$ and $\tilde{h}^0$ are integer multiples of eight which cannot be zero simultaneously. We now argue along the following lines:

- The tadpole contribution Q^0 is invariant under the $SL(2,\mathbb{Z})$ transformations (6.2.56). Using then T-transformations acting on the axio-dilaton as $\tau \to \tau + b$ with $b \in \mathbb{Z}$, we can bring τ_1 into the region $-\frac{1}{2} \leq \tau_1 < +\frac{1}{2}$. This T-transformation is a duality transformation, and therefore we have the equivalence

  $$\tilde{\mathfrak{f}}_0 \sim \tilde{\mathfrak{f}}_0 + b\,\tilde{h}_0 \,, \qquad\qquad \tilde{\mathfrak{f}}^0 \sim \tilde{\mathfrak{f}}^0 + b\,\tilde{h}^0 \,. \tag{6.3.8}$$

  Choosing without loss of generality $\tilde{h}_0 \neq 0$, for fixed $\tilde{h}_0$ there are only finitely-many inequivalent values for $\tilde{f}_0$ given by

  $$\tilde{\mathfrak{f}}_0 = 0, \ldots, \tilde{h}_0 - 1 \,. \tag{6.3.9}$$

- Next, using an S-transformation $\tau \to -1/\tau$ (possibly together with additional T-transformations) we can bring τ into the fundamental domain $\mathcal{F}_\tau$. In $\mathcal{F}_\tau$ a lower bound for the imaginary part τ_2 is obtained by considering $\tau_1 = -\frac{1}{2}$ for which $\tau_2 \geq \sqrt{3}/2$. Using (6.3.7) we then find

  $$0 < (\tilde{h}_0)^2 + (\tilde{h}^0)^2 \leq \frac{\mathsf{Q}^0}{6\sqrt{3}} \,, \tag{6.3.10}$$

  which leaves only finitely many possibilities for the integers $\tilde{h}_0, \tilde{h}^0$. Together with (6.3.9), this implies also a finite number of choices for $\tilde{\mathfrak{f}}_0$.

- The remaining flux $\tilde{\mathfrak{f}}^0$ is now determined via the tadpole contribution Q^0 shown in (6.3.5).

In summary, for a fixed positive value of the D3-tadpole contribution Q^0, the F-term equations (6.3.3) have only a finite number of of physically-distinct solutions for τ.

6.3.3 Space of Solutions

In [233, 237] it was shown that the solutions (6.3.4) mapped to the fundamental domain for τ are not distributed homogeneously. In particular, the space of solutions contains voids with large degeneracies in their centers. In this section we review these findings and provide some new results on the dependence of these distributions on the D3-tadpole contribution Q^0. Our data has been obtained using a computer algorithm to generate all physically-distinct flux vacua for a given upper bound on the D3-brane tadpole contribution.

Distribution of Solutions

As we argued above, for a fixed value of Q^0, the number of physically inequivalent solutions for the axio-dilaton τ is finite. Using the $SL(2,\mathbb{Z})$ duality (6.2.56), we can map these solutions to the fundamental domain (6.3.6). The corresponding space of solutions is shown in figures 6.1 and 6.2.

- For figure 6.1 we have included all flux configurations for which the tadpole contribution satisfies $0 < \frac{\mathsf{Q}^0}{768} \leq 300$, and in order to have a symmetric plot we have added points on the boundary of the fundamental domain at $\tau_1 = +\frac{1}{2}$. We see that the space of solutions for (6.3.4) is bounded as $\tau_2 \leq 300$, and that solutions are located on lines with fixed τ_1.
- In figure 6.2 we show a zoom of figure 6.1 for a small range of τ_2. Here we see a characteristic structure of voids [233] with accumulated points in their centers. Large voids are typically encircled by smaller ones, leading to the appearance of gradually finer void structures as one zooms further into the plot. Notice that the higher density of points near $|\tau|^2 = 1$ is not a physical property as we have not taken into account the metric on moduli space.

Let us next note that the moduli space of the axio-dilaton τ is hyperbolic. Indeed from the Kähler potential (6.2.41) we can derive the corresponding Kähler metric with components

$$g_{\tau_1\tau_1} = g_{\tau_2\tau_2} = \frac{1}{4{\tau_2}^2}\,, \qquad g_{\tau_1\tau_2} = 0\,. \tag{6.3.11}$$

A convenient way to visualize this hyperbolic space is by mapping the Poincaré half-plane to the Poincaré disk via the conformal transformation

$$(\tau_1,\tau_2) \to (\tilde{\tau}_1,\tilde{\tau}_2) = \left(\frac{2\tau_1}{\tau_1^2+(1+\tau_2)^2}\,,\,\frac{\tau_1^2+\tau_2^2-1}{\tau_1^2+(1+\tau_2)^2}\right). \tag{6.3.12}$$

The space of solutions for the axio-dilaton mapped to the Poincaré disk is shown in figure 6.3, which is the mapping of figure 6.1 under (6.3.12).

- In figure 6.3 the characteristic structure of voids is visible. In this plot effects of the moduli-space metric are incorporated.

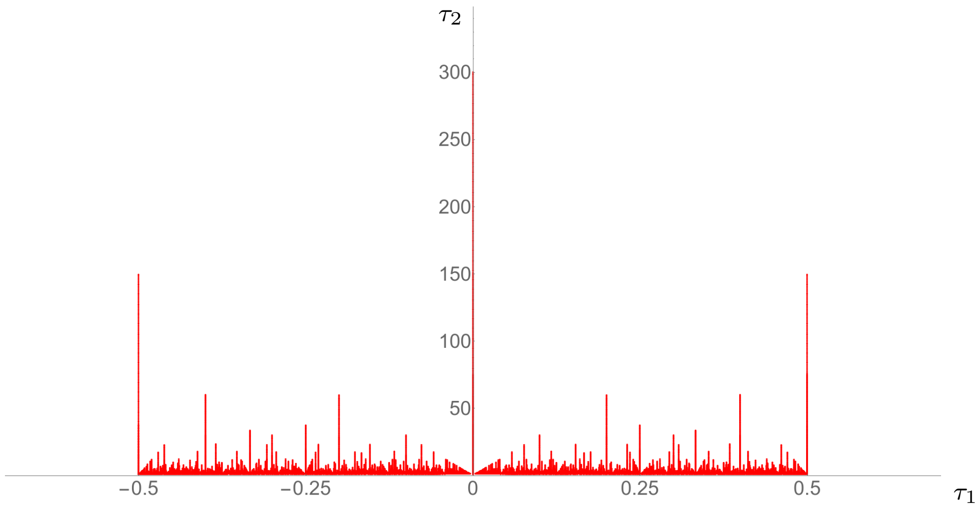


Figure 6.1: Space of solutions for the axio-dilaton τ with fluxes (6.3.2), mapped to the fundamental domain $\mathcal{F}_\tau$. All solutions satisfy the bound $\frac{Q^0}{768} \leq \frac{Q^0_{\max}}{768} = 300$.

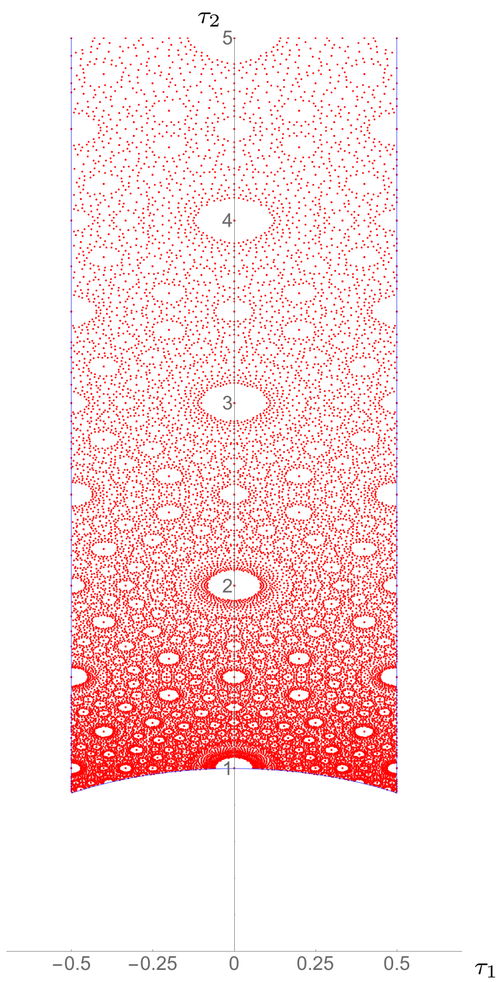


Figure 6.2: Zoom of figure 6.1 for $0 \leq \tau_2 \leq 5$.

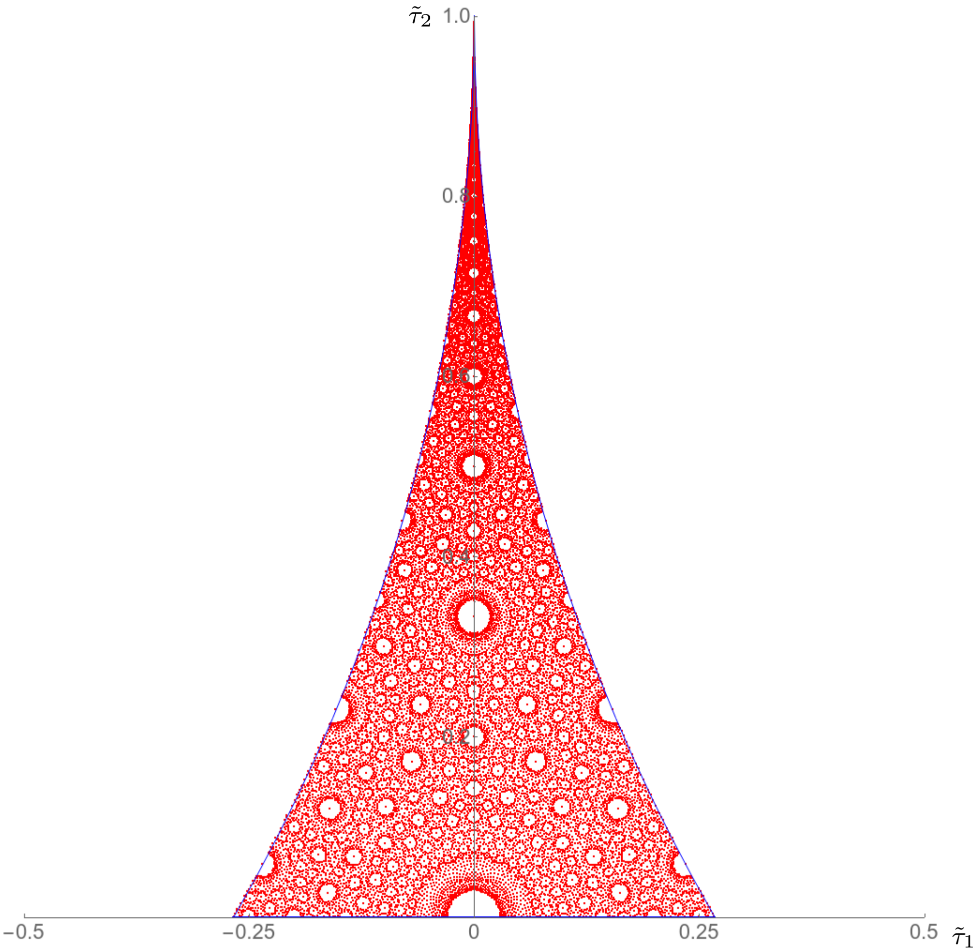


Figure 6.3: Space of solutions for the axio-dilaton τ with fluxes of the form (6.3.2), restricted to the fundamental domain and mapped to the Poincaré disk. All solutions satisfy the bound $\frac{Q^0}{768} \leq 300$.

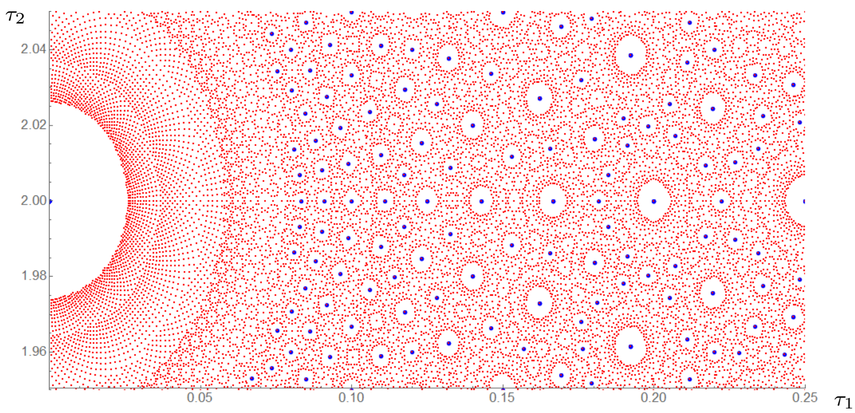


Figure 6.4: Space of solutions for the axio-dilaton τ with fluxes of the form (6.3.2) near $\tau = 2i$ on the Poincaré plane for $\frac{Q^0}{768} \leq 300$ (blue) and $\frac{Q^0}{768} \leq 3000$ (red).

Analysis of Voids

The number of physically-distinct solutions for the axio-dilaton is finite for fixed tadpole-contribution Q^0. The number of solutions N with $\mathsf{Q}^0 \leq \mathsf{Q}^0_{\text{max}}$ can be determined numerically, which leads to the following scaling behaviour

$$\mathsf{N} \approx 0.823 \left[\frac{\mathsf{Q}^0_{\text{max}}}{768}\right]^2 \tag{6.3.13}$$

for large $\mathsf{Q}^0_{\text{max}}$. We now want to study how the voids change depending on N, or, equivalently, depending on $\mathsf{Q}^0_{\text{max}}$. In particular, we are interested how the size of the voids depends on $\mathsf{Q}^0_{\text{max}}$. Qualitatively, this behaviour is illustrated in figure 6.4:

- In figure 6.4 the space of solutions for the axio-dilaton around the point $\tau = 2i$ is shown. The blue points correspond to solutions which satisfy $\frac{\mathsf{Q}^0}{768} \leq 300$, and the red points correspond to solutions with $\frac{\mathsf{Q}^0}{768} \leq 3000$. For larger $\mathsf{Q}^0_{\text{max}}$ the void around $\tau = 2i$ therefore becomes smaller, and finer void structures appear. These results are in agreement with the ones found using topological data analysis in [245].

Let us denote the origin of a void by τ_{void}, and define its size by the distance to the nearest solution τ_{sol} (not located at τ_{void}). The geodesic distance d is measured using the metric (6.3.11) on the axio-dilaton moduli space, for which we have

$$d\left(\tau\,, \widetilde{\tau}\right) = \frac{1}{2}\text{arccosh}\left[1 + \frac{\left(\widetilde{\tau}_1 - \tau_1\right)^2 + \left(\widetilde{\tau}_2 - \tau_2\right)^2}{2\,\widetilde{\tau}_2\,\tau_2}\right]. \tag{6.3.14}$$

As we can see for instance from figure 6.3, in the proper distance the voids can be approximated by a circle whose radius we define as

$$R_{\text{void}} = \min_{\tau_{\text{sol}} \neq \tau_{\text{void}}} d(\tau_{\text{void}}, \tau_{\text{sol}})\,. \tag{6.3.15}$$

The scaling behaviour of R_{void} with $\mathsf{Q}^0_{\text{max}}$ has been obtained for instance in [233, 237] as $R^2_{\text{void}} \sim 1/\mathsf{Q}^0_{\text{max}}$, and below we have determined the prefactors for some families of voids numerically. For voids located in the fundamental domain on the Poincaré plane we have the following relation between the radius of the void R_{void}, the tadpole contribution $\mathsf{Q}^0_{\text{max}}$ and the number of solutions located at the center of the void n_{void}:

$$R^2_{\text{void}} \approx \frac{1}{C\,\tau_{2\text{void}}}\left[\frac{768}{\mathsf{Q}^0_{\text{max}}}\right] \qquad n_{\text{void}} \approx \frac{2\pi}{C\,\tau_{2\text{void}}}\left[\frac{\mathsf{Q}^0_{\text{max}}}{768}\right] \tag{6.3.16}$$

$\tau_{1\text{void}}$	$\tau_{2\text{void}}$	C
0	n	4
0	$n+0.5$	16
± 0.2	$n+0.4$	20
± 0.2	$n+0.6$	20

Here, $n \in \mathbb{Z}_+$. The constant C depends on the family of voids under consideration and can be read off from the table in (6.3.16) for several examples. Note also that the number of solutions located at the center of the void divided by the area of the void takes the simple form

$$\frac{n_{\text{void}}}{2\pi R^2_{\text{void}}} \approx \left[\frac{\mathsf{Q}^0_{\text{max}}}{768}\right]^2 . \tag{6.3.17}$$

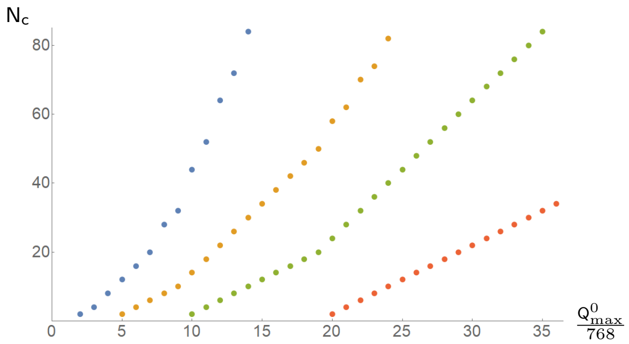


Figure 6.5: Number of solutions $\mathsf{N_c}$ (for the axio-dilaton τ with fluxes of the form (6.3.2)) which satisfy $\mathsf{c} \le \tau_2 \le \frac{\mathsf{Q}^0_{\rm max}}{768}$ for $\mathsf{c} = 2, 5, 10, 20$ in colors blue, orange, green and red, respectively.

Solutions at Small Coupling

The imaginary part of the axio-dilaton τ is bounded from above by the D3-tadpole contribution Q^0, which via (6.2.33) implies a restriction on the string coupling g_s as

$$\tau_2 = e^{-\phi} = \frac{1}{g_s} \le \frac{\mathsf{Q}^0}{768} \qquad \Rightarrow \qquad g_s \ge \frac{768}{\mathsf{Q}^0}\,. \tag{6.3.18}$$

Recall that in our conventions Q^0 is a multiple of 768. In the following we determine the number of physically-distinct solutions $\mathsf{N_c}$ which satisfy $\tau_2 \ge \mathsf{c}$ for some cutoff $\mathsf{c} > 0$ so that we have

$$\frac{768}{\mathsf{Q}^0_{\rm max}} \le g_s \le \frac{1}{\mathsf{c}}\,. \tag{6.3.19}$$

Note that in order to ignore string-loop corrections and corrections from worldsheet instantons, we need to stabilize the axio-dilaton at small g_s. This implies that Q^0 and c should be sufficiently large. Using then the exact data for the space of solutions, we can obtain fits for $\mathsf{N_c}$ for values of $\mathsf{Q}^0_{\rm max}$ of the order $\frac{\mathsf{Q}^0_{\rm max}}{768} = \mathcal{O}(10^3)$. In particular, with the scaling of the total number of solutions N shown in (6.3.13) we have

$$\frac{\mathsf{Q}^0_{\rm max}}{768} \gg 1 \tag{6.3.20}$$

c	$\mathsf{N_c}$	$\mathsf{N_c}/\mathsf{N}$
2	$0.393\left[\frac{\mathsf{Q}^0_{\rm max}}{768}\right]^2$	0.478
5	$0.157\left[\frac{\mathsf{Q}^0_{\rm max}}{768}\right]^2$	0.191
10	$0.078\left[\frac{\mathsf{Q}^0_{\rm max}}{768}\right]^2$	0.095
20	$0.039\left[\frac{\mathsf{Q}^0_{\rm max}}{768}\right]^2$	0.047

We observe that in this limit the percentage of solutions with $g_s \ll 1$ is small and independent of $\mathsf{Q}^0_{\rm max}$. For instance, only about 5% of the solutions have a string coupling satisfying $g_s \le 0.05$. However, the region of small tadpole contributions $\frac{\mathsf{Q}^0}{768} = \mathcal{O}(1)$ is more interesting. Here the number of solutions $\mathsf{N_c}$ does not follow a simple quadratic behavior, and the precise numbers are shown in figure 6.5. We see that for a particular c in $g_s \le 1/\mathsf{c}$, the D3-tadpole contributions $\frac{\mathsf{Q}^0}{768}$ has to be larger than some threshold. Furthermore, above this threshold the number of solutions is not large but only $\mathcal{O}(10)$.

6.3.4 Summary

Let us summarize the results and observations of this section for moduli stabilization of the axio-dilaton with the choice of fluxes given in equation (6.3.4):

- As already known before, for a fixed D3-brane tadpole contribution Q^0, the number of physically-distinct solutions to the F-term equations for the axio-dilaton τ is finite due to the corresponding $SL(2,\mathbb{Z})$ duality [233, 237].
- The solutions for the axio-dilaton in the fundamental domain are not distributed homogeneously, but show characteristic void structures as illustrated in figures 6.2 and 6.3.
- With increasing upper bound $\mathsf{Q}^0_{\max}$ on the tadpole contribution, the area of these voids shrinks and the number of solutions located at the center n_{void} increases as shown in (6.3.16). The precise behaviour for the radius R_{void} and n_{void} is proportional to a constant depending on the location of the void, however, the ratio $n_{\text{void}}/2\pi R^2_{\text{void}}$ is universal.
- The string coupling is bounded from below by the tadpole contribution as $\frac{768}{\mathsf{Q}^0} \leq g_{\text{s}}$. In order to ignore string corrections and trust the solutions (6.3.4), we have to demand $g_{\text{s}} \ll 1$ which implies $\frac{\mathsf{Q}^0}{768} \gg 1$. This is in contrast to our discussion of the tadpole-cancellation condition on page 124, which requires $\frac{\mathsf{Q}^0}{768}$ to be small, and illustrates the difficulty of obtaining reliable solutions to the F-term equations.
- We have furthermore analyzed the number of physically-distinct solutions satisfying $g_{\text{s}} \leq 1/\mathsf{c}$. Requiring a small string coupling of for instance $g_{\text{s}} \leq 1/10$, we find that only about 10% of the solutions satisfy this condition. If we require g_{s} to be smaller, the corresponding fraction of solutions becomes smaller.

6.4 Moduli Stabilization II

In this section we extend our previous discussion by including the complex-structure moduli $U^{\check{\mathsf{a}}}$. We choose flux configurations which stabilize the axio-dilaton and fix the complex-structure moduli at an isotropic minimum with $U^1 = U^2 = U^3$. Such vacua have previously been studied for instance in [269, 237].

6.4.1 Setting

We start again by specifying the superpotential (6.2.42) and set to zero the nongeometric fluxes (6.2.35)

$$q_{\check{\mathsf{A}}}{}^{\hat{\mathsf{i}}} = 0\,, \qquad q^{\check{\mathsf{A}}\hat{\mathsf{i}}} = 0\,. \tag{6.4.1}$$

For the R-R and remaining NS-NS fluxes (6.2.34) we choose the restricted setting

$$\begin{aligned} \mathfrak{f}_1 &= \mathfrak{f}_2 = \mathfrak{f}_3\,, & h_1 &= h_2 = h_3\,, \\ \mathfrak{f}^1 &= \mathfrak{f}^2 = \mathfrak{f}^3\,, & h^1 &= h^2 = h^3\,, \end{aligned} \qquad \mathfrak{f}^{\check{\mathsf{A}}}, \mathfrak{f}_{\check{\mathsf{A}}}, h^{\check{\mathsf{A}}}, h_{\check{\mathsf{A}}} \in 8\mathbb{Z}\,. \tag{6.4.2}$$

Since the superpotential is independent of the Kähler moduli $T_{\hat{\imath}}$, the F-term equations (6.2.46) simplify as in (6.3.3) and we obtain

$$0 = W\,, \qquad 0 = \partial_{U^{\check{a}}} W\,, \qquad 0 = \partial_\tau W\,. \tag{6.4.3}$$

Due to the isotropic choice of fluxes in (6.4.2), the complex-structure moduli $U^{\check{a}}$ are stabilized such that

$$U^1 = U^2 = U^3 =: U\,, \tag{6.4.4}$$

and the F-term equations (6.4.3) reduce to

$$\begin{aligned} -\mathfrak{f}_0 - 3U\mathfrak{f}_1 - 3(U)^2\mathfrak{f}^1 + (U)^3\mathfrak{f}^0 &= 0\,, \\ -h_0 - 3U h_1 - 3(U)^2 h^1 + (U)^3 h^0 &= 0\,, \end{aligned} \tag{6.4.5}$$

$$(\mathfrak{f}_1 - \tau h_1) + 2U(\mathfrak{f}^1 - \tau h^1) - (U)^2(\mathfrak{f}^0 - \tau h^0) = 0\,. \tag{6.4.6}$$

The R-R and NS-NS fluxes in (6.4.2) are furthermore subject to the Bianchi identities (6.2.38). Due to the vanishing Q-fluxes, the only nontrivial relation is again given by the D3-brane tadpole contribution

$$\mathsf{Q}^0 = \mathfrak{f}_0 h^0 - \mathfrak{f}^0 h_0 + 3\left(\mathfrak{f}_1 h^1 - \mathfrak{f}^1 h_1\right) > 0\,. \tag{6.4.7}$$

Note that due to the quantization condition for the fluxes, the tadpole contribution Q^0 is an integer multiple of 64. However, as it has been explained in footnote 10 of [269], in order to obtain physically-viable solutions Q^0 receives an additional factor of three. The tadpole contribution is therefore always a multiple of 192, which is also what we observe explicitly in our data.

6.4.2 Finite Number of Solutions for fixed Q^0

The two equations for the complex-structure modulus shown in (6.4.5) define an overdetermined cubic system for U, which in general does not allow for a solution in closed form. Since the coefficients in (6.4.5) are real, one can, however, bring these equations into the form

$$(U - u_0)(U - u_1)(U - \bar{u}_1) = 0\,, \qquad u_0 \in \mathbb{R}\,,\ u_1 \in \mathbb{C}\,, \tag{6.4.8}$$

where $u_0, u_1, \bar{u}_1$ denote the roots. Physically-acceptable solutions have to satisfy $\mathrm{Im}\,U > 0$, and therefore the F-term equations (6.4.5) have at most one solution for U of interest to us. The equation (6.4.6) can be solved for the axio-dilaton as

$$\tau = \frac{\mathfrak{f}_1 + 2U\mathfrak{f}^1 - (U)^2\mathfrak{f}^0}{h_1 + 2Uh^1 - (U)^2 h^0}\,, \tag{6.4.9}$$

which still depends on the complex-structure modulus U. More details on these solutions can be found in appendix C, where we follow the discussion of [269, 237]. As reviewed in section 6.2.5, in the absence of nongeometric Q-fluxes the axio-dilaton and the complex-structure moduli enjoy $SL(2,\mathbb{Z})$ dualities. These can be used to bring τ and U into their fundamental domains

$$\begin{aligned} \mathcal{F}_\tau &= \left\{ -\frac{1}{2} \le \tau_1 \le 0,\ |\tau|^2 \ge 1 \ \cup\ 0 < \tau_1 < +\frac{1}{2},\ |\tau|^2 > 1 \right\}, \\ \mathcal{F}_U &= \left\{ -\frac{1}{2} \le U_1 \le 0,\ |U|^2 \ge 1 \ \cup\ 0 < U_1 < +\frac{1}{2},\ |U|^2 > 1 \right\}, \end{aligned} \tag{6.4.10}$$

where we again split τ and U into their real and imaginary parts as $\tau = \tau_1 + i\tau_2$ and $U = U_1 + iU_2$. We furthermore note that the two $SL(2,\mathbb{Z})$ dualities leave the D3-tadpole contribution Q^0 invariant. Now, as first demonstrated in [269, 237] and reviewed in appendix C, the dualities can be used to show that the number of physically-distinct vacua in the fundamental domain is finite for fixed Q^0. In the following we explore how the properties of the space of solutions for τ and U depend on Q^0.

6.4.3 Space of Solutions

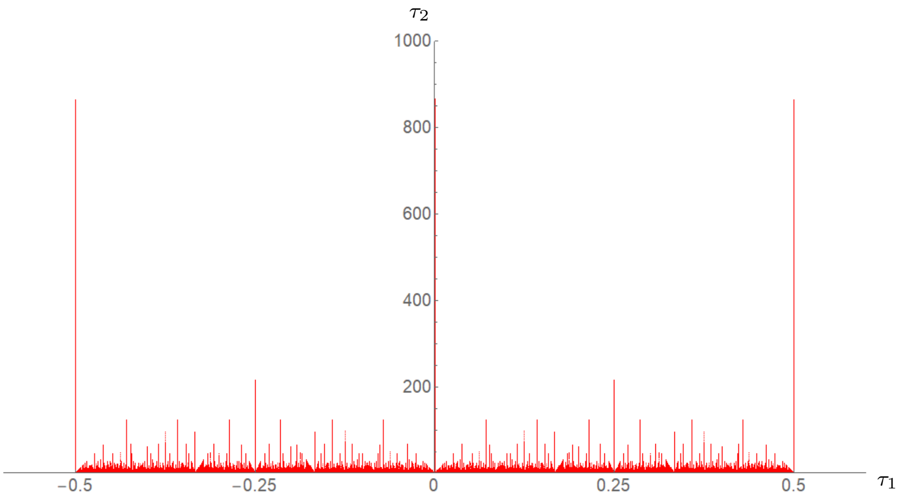
In this section we study the space of solutions to the F-term equations (6.2.46) for the combined axio-dilaton and complex-structure-modulus system. Since for the axio-dilaton system we found two-dimensional circular voids in the two-dimensional moduli space, it might seem natural to expect four-dimensional spherical voids in the four-dimensional moduli space. It turns out that this is not the case, and the space of solutions involves more intricate structures. Our data has again been obtained using a computer algorithm, which generated all physically-distinct flux vacua for a given upper bound on the D3-brane tadpole contribution Q^0.

Distribution of Solutions

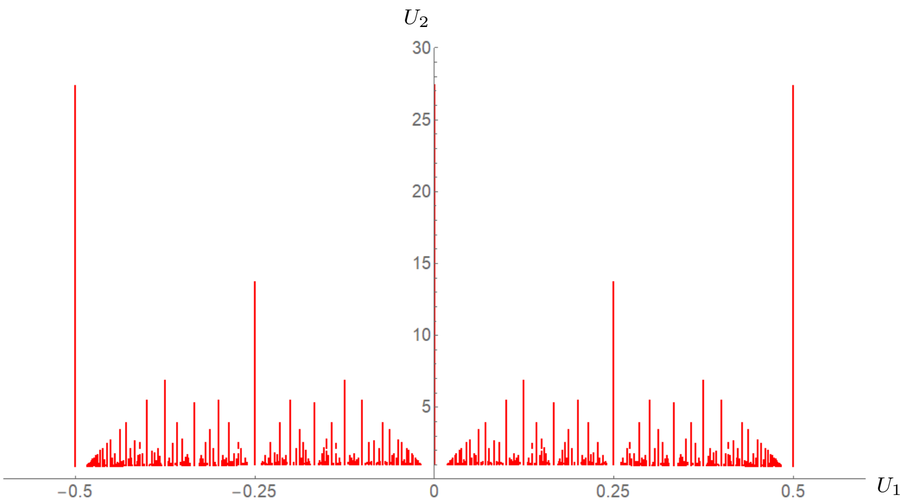
In [269, 237] (as well as in appendix C) it is shown that for fixed Q^0 the number of physically-distinct solutions is finite. We have determined all solutions for the setting described in section 6.4.1 numerically and visualized them in the following figures.

- Figure 6.6 shows the solutions for the fluxes of the form (6.4.2) projected onto the τ and onto the U-plane [237]. All solutions satisfy the bound on the tadpole contribution $\frac{\mathsf{Q}^0}{192} \le 1000$, and in order to have a symmetric plot we included points on the boundary of the fundamental domains. These plots are similar to the one in figure 6.1. When comparing figures 6.6a and 6.6b, we note that for the same Q^0 the maximum values for τ_2 and U_2 differ significantly. Furthermore, we note that the number of different values for τ_1 is much larger than for U_1.

- In figure 6.7 we show sections through the four-dimensional space of solutions for $\tau_2 \le 2$, characterized by different values of the complex-structure modulus. All solutions satisfy $\frac{\mathsf{Q}^0}{192} \le 1000$, and these plots show void structures similar as in figure 6.2. We note however that although the location of the voids stays the same when going from $U = i$ to $U = 2i$ and similarly from $U = \sqrt{2}i$ to $U = 2\sqrt{2}i$,



(a) Projection of solutions onto the τ-plane.



(b) Projection of solutions onto the U-plane.

Figure 6.6: Space of solutions for the setting described in section 6.4.1, mapped to the fundamental domains $\mathcal{F}_\tau$ and $\mathcal{F}_U$ and projected onto the τ- and U-plane. All solutions satisfy the bound $\frac{Q^0}{192} \leq 1000$.

the density of solutions decreases. This appears to be a general feature which we observe in the data.

- Figure 6.8 contains three-dimensional sections of the four-dimensional space of solutions for $U_1 = 0$. All solutions have been mapped to the fundamental domain. Figures 6.8a and 6.8b show two different points of view, which illustrate that the three-dimensional section of the space of solutions is not homogenous. Solutions are accumulated on planes for particular values of U_2, while the space between these planes is only sparsely populated. This is in agreement with our observations in figures 6.7, which also imply a varying density of solutions.

  The lines in figures 6.8a and 6.8b connect voids for different values of U_2 and are described by the following equations for $t \in \mathbb{R}_+$:

$$\begin{array}{lll} \text{orange} & l_1 & (\tau_1, \tau_2, U_1, U_2) = (\ \ 0\,,\ 1+t\,,\ 0\,,\ 1+\tfrac{1}{1}t\,)\,, \\ \text{red} & l_2 & (\tau_1, \tau_2, U_1, U_2) = (\ \ 0\,,\ 2+t\,,\ 0\,,\ 1+\tfrac{1}{2}t\,)\,, \\ \text{purple} & l_3 & (\tau_1, \tau_2, U_1, U_2) = (\ \ 0\,,\ 3+t\,,\ 0\,,\ 1+\tfrac{1}{3}t\,)\,, \\ \text{green} & l_4 & (\tau_1, \tau_2, U_1, U_2) = (-\tfrac{1}{2}\,,\ \tfrac{3}{2}+t\,,\ 0\,,\ 1+\tfrac{2}{3}t\,)\,. \end{array} \qquad (6.4.11)$$

- Figure 6.9 shows the same three-dimensional section of the space of solutions as in figure 6.8. The point of view in figure 6.9a is along the line l_1 (orange) of (6.4.11) and the point of view in figure 6.9b is along the line l_2 (red). In these three-dimensional sections of the four-dimensional space of solutions we therefore have an oblique-cylindrical void centered around the lines in (6.4.11).

Solutions at small Coupling and large Complex Structure

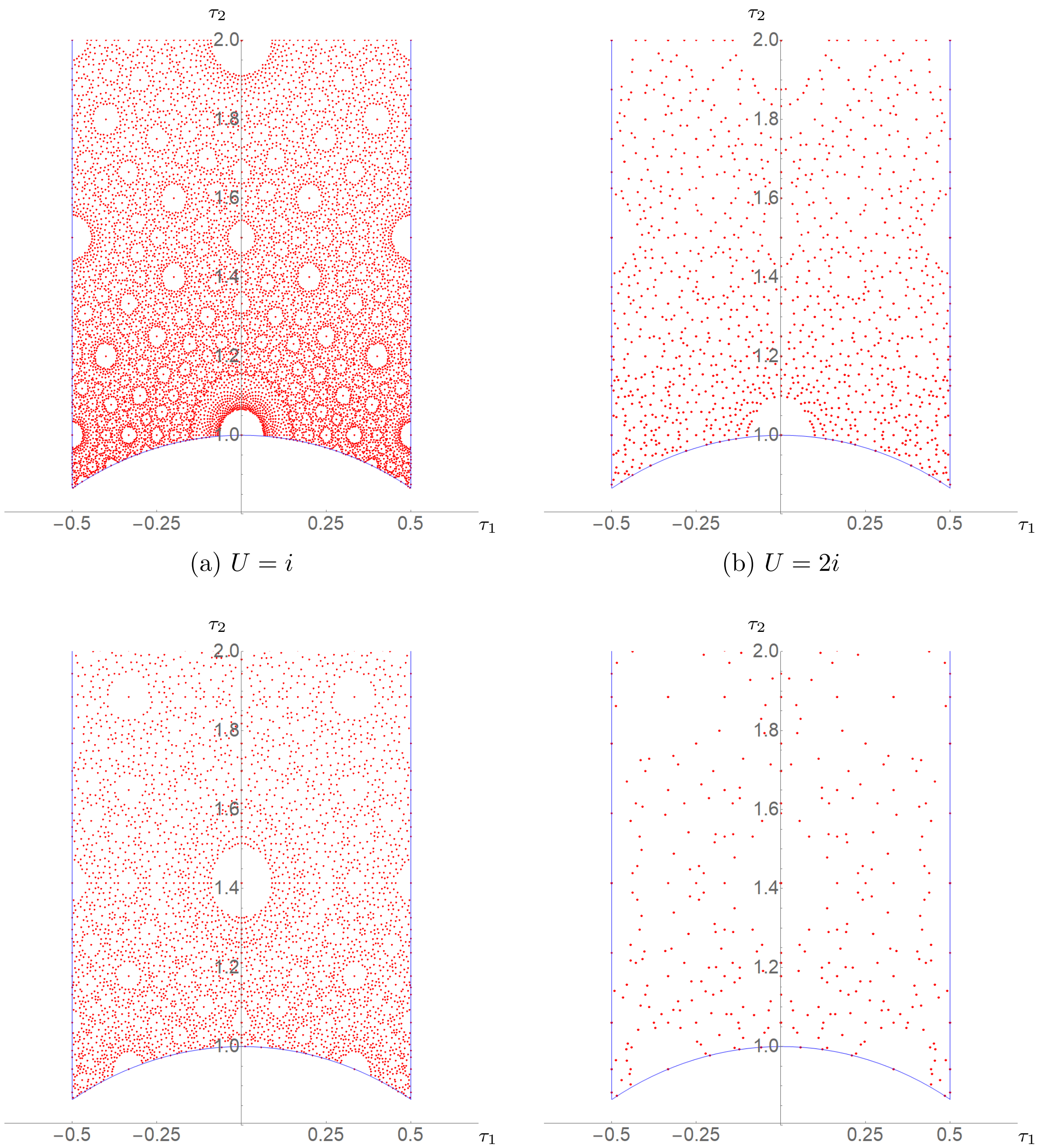
We now consider the number N of physically-distinct solutions for the combined axio-dilaton and complex-structure moduli system defined in section 6.4.1. This number is finite for fixed D3-tadpole contribution Q^0, and since we have the numerical data we can determine this number explicitly. For large Q^0 the dependence takes the form

$$\mathsf{N} \approx 1.2501 \left[\frac{\mathsf{Q}^0}{192}\right]^2 . \qquad (6.4.12)$$

We next note that in the fundamental domains, the imaginary parts of the axio-dilaton and complex-structure moduli satisfy a lower bound similarly as in the previous example. An upper bound can be obtained from the numerical data, which can be expressed as [3]

$$\frac{\sqrt{3}}{2} \leq \tau_2 \leq \frac{\sqrt{3}}{2}\left[\frac{\mathsf{Q}^0}{192}\right], \qquad \frac{\sqrt{3}}{2} \leq U_2 \leq \frac{\sqrt{3}}{2}\left[\frac{\mathsf{Q}^0}{192}\right]^{1/2}. \qquad (6.4.13)$$

[3] More precisely, with $x = \frac{\mathsf{Q}^0}{192}$ the bound on U_2 can be expressed as $U_2 \leq \sqrt{C\,x}$, where the constant C takes values $C = 3/4$ for $x = 0, 1 \mod 4$, $C = 3/8$ for $x = 2 \mod 8$, $C = 1/4$ for $x = 3, 7 \mod 8$, and $C = 1/8$ for $x = 6 \mod 8$.



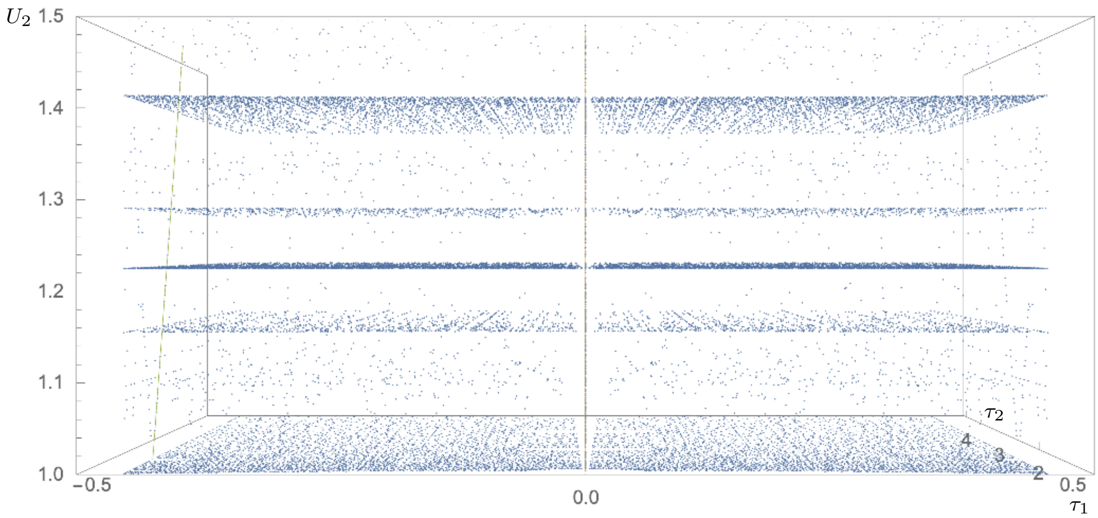
(a) $U = i$

(b) $U = 2i$

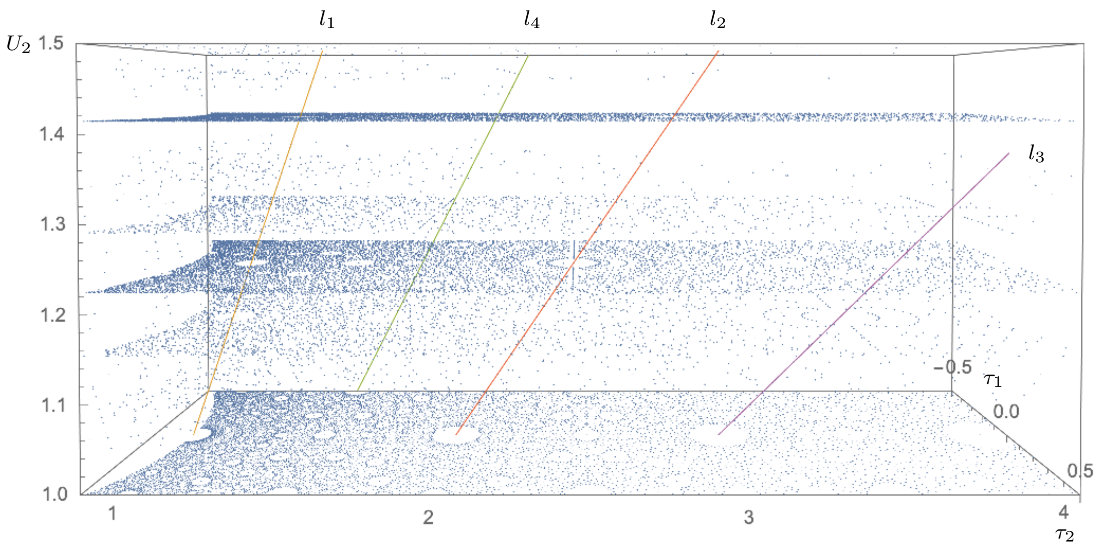
(c) $U = \sqrt{2}i$

(d) $U = 2\sqrt{2}i$

Figure 6.7: Section through the four-dimensional space of solutions for the setting described in section 6.4.1. The solutions have been mapped to the fundamental domains, and the sections are for fixed complex-structure modulus at $U = i$, $U = 2i$, $U = \sqrt{2}i$ and $U = 2\sqrt{2}i$. All solutions satisfy the bound $\frac{Q^0}{192} \leq 1000$.

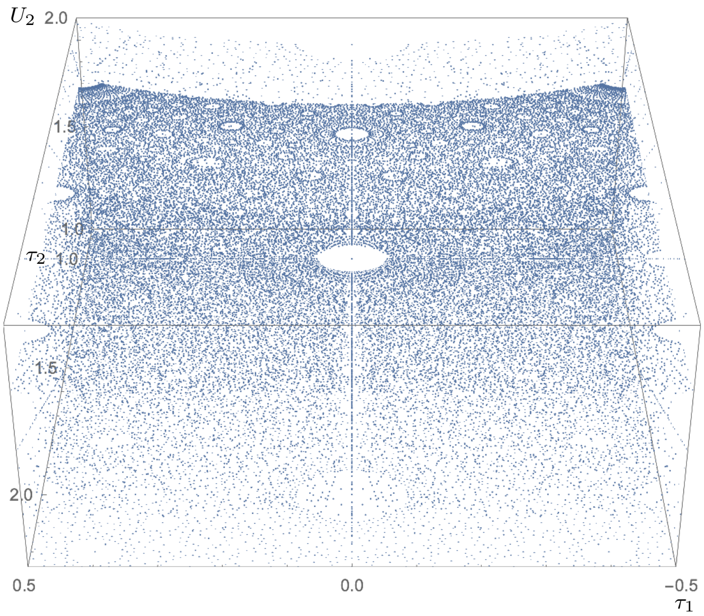


(a) Point of view along the τ_2-direction.

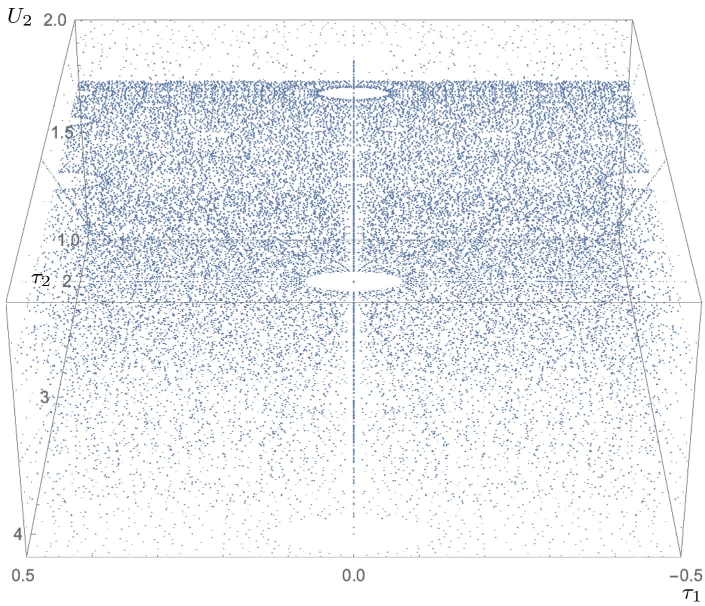


(b) Point of view along the τ_1-direction.

Figure 6.8: Section through the four-dimensional space of solutions with $U_1 = 0$ for the setting described in section 6.4.1. All solutions satisfy the bound $\frac{Q^0}{192} \leq 1000$ and have been mapped to the fundamental domains. The lines in 6.8a and 6.8b connect voids for different values of U_2 and are described by the expressions in equation (6.4.11).



(a) View along the line l_1 (orange) in figures 6.8.



(b) View along the line l_2 (red) in figures 6.8.

Figure 6.9: Section through the four-dimensional space of solutions with $U_1 = 0$ for the setting described in section 6.4.1. All solutions satisfy the bound $\frac{Q^0}{192} \leq 1000$ and have been mapped to the fundamental domains. The points of view are along the line l_1 (figure 6.9a) and line l_2 (figure 6.9b) in figures 6.8, which show a void structure around l_1 and l_2.

Note that in our conventions the tadpole contribution Q^0 is a multiple of 192. However, as we have seen in (6.4.9), the solution for the axio-dilaton depends on the complex-structure modulus. Although this dependence is difficult to analyze analytically, the numerical data gives a bound on the solutions,

$$\tau_2\, U_2 \leq \frac{3}{4}\frac{\mathsf{Q}^0}{192}\,. \tag{6.4.14}$$

This bound is stronger than in (6.4.13), and it implies that for fixed Q^0 the imaginary parts of τ and U cannot be made simultaneously large. In particular, in order to have solutions at small coupling $g_s = \frac{1}{\tau_2} \ll 1$ and large complex structure $U_2 \gg 1$, the tadpole contribution has to be sufficiently large. Let us make this more precise and determine numerically the number of solutions N_c with $\mathsf{Q}^0 \leq \mathsf{Q}^0_{\max}$ for which

$$g_s \leq \frac{1}{\mathsf{c}} \qquad \text{and} \qquad U_2 \geq \mathsf{c}\,. \tag{6.4.15}$$

In the limit of large $\frac{\mathsf{Q}^0_{\max}}{192}$ we obtained the following approximations:

$\frac{\mathsf{Q}^0_{\max}}{192} \gg 1$ (6.4.16)

c	N_c	$\mathsf{N}_\mathsf{c}/\mathsf{N}$
2	$0.0553\left[\frac{\mathsf{Q}^0_{\max}}{192}\right]^2 - 3.4617\left[\frac{\mathsf{Q}^0_{\max}}{192}\right]$	0.041
5	$0.0047\left[\frac{\mathsf{Q}^0_{\max}}{192}\right]^2 - 0.7627\left[\frac{\mathsf{Q}^0_{\max}}{192}\right]$	0.004
10	$0.0009\left[\frac{\mathsf{Q}^0_{\max}}{192}\right]^2 - 0.3569\left[\frac{\mathsf{Q}^0_{\max}}{192}\right]$	0.001

These approximations do not describe the data at a high precision, but are sufficient for our purposes here. In particular, we see that at leading order N_c depends quadratically on $\mathsf{Q}^0_{\max}$ and that the ratios $\mathsf{N}_\mathsf{c}/\mathsf{N}$ are rather small. Thus, only a small percentage of the solutions to the F-term equations are in a perturbatively-controlled regime. More interesting is the limit of small $\frac{\mathsf{Q}^0}{192}$, which we have illustrated in figure 6.10. We see that for $\mathsf{c} = 2$ (blue) there are solutions starting at $\frac{\mathsf{Q}^0}{192} = 16$. For $\mathsf{c} = 5$ (orange) we find solutions starting at $\frac{\mathsf{Q}^0}{192} = 100$, and for $\mathsf{c} = 10$ (green) solutions can be obtained starting at $\frac{\mathsf{Q}^0}{192} = 400$. The main conclusion we want to draw from this analysis is that for solutions at weak coupling $g_s \ll 1$ and large complex structure $U_2 \gg 1$, the D3-tadpole contribution $\frac{\mathsf{Q}^0}{192}$ has to be large. As discussed on page 124, this is in tension with the tadpole cancellation condition.

6.4.4 Summary

Let us summarize the results obtained in this section for the space of solutions of the combined axio-dilaton and complex-structure-moduli system with fluxes characterized by the setting described in section 6.4.1:

- As shown before in [269, 237], for a fixed D3-brane tadpole contribution Q^0 the number of physically-distinct solutions to the F-term equations (6.2.46) for the axio-dilaton and complex-structure moduli is finite. This is again due to the $SL(2,\mathbb{Z})$ dualities for the axio-dilaton and the complex-structure moduli.

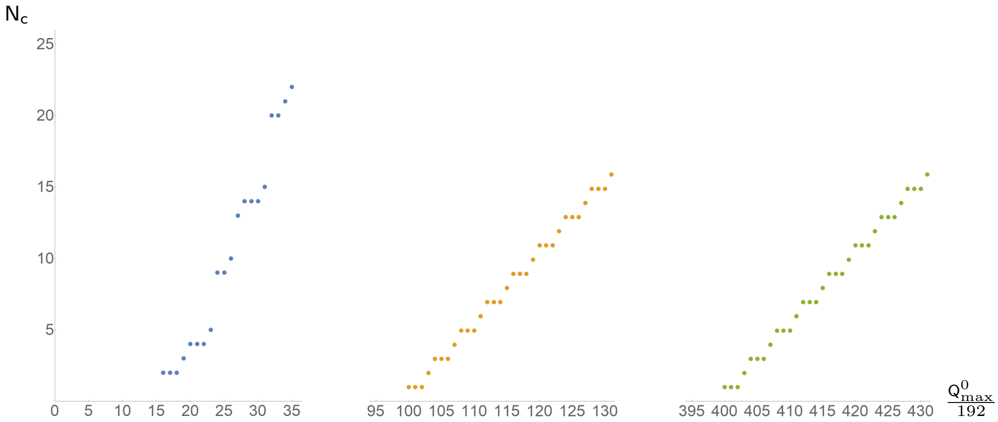


Figure 6.10: Number of solutions $\mathsf{N_c}$ as a function of $\frac{\mathsf{Q}^0_{\max}}{192}$ (for the setting described in section 6.4.1) which satisfy $\tau_2, U_2 \geq \mathsf{c}$ for $\mathsf{c} = 2, 5, 10$ in colors blue, orange and green, respectively. Notice the different ranges for $\frac{\mathsf{Q}^0_{\max}}{192}$ on the horizontal axis.

- The values of the fixed moduli (mapped to their fundamental domains) are not distributed homogeneously in the space of solutions. As shown in figures 6.9, the solutions are accumulated on submanifolds in the four-dimensional space with few points in between. We also find void structures in the space of solutions, which are however not spherical but take a cylindrical form in three-dimensional sections (cf. figures 6.9).

- The imaginary parts of the axio-dilaton and the complex-structure moduli τ_2 and U_2 are bounded from above and below as shown in equation (6.4.13). In our data we additionally find a stronger bound on the product $\tau_2\, U_2 \leq \frac{3}{4}\frac{\mathsf{Q}^0}{192}$, which implies that in the weak-coupling and large-complex-structure regime the tadpole contribution $\frac{\mathsf{Q}^0}{192}$ has to be large. This is again in contrast to our arguments regarding the tadpole-cancellation condition on page 124, which requires $\frac{\mathsf{Q}^0}{192}$ to be small, and illustrates the tension between the closed- and open-string sectors for obtaining reliable solutions.

- We have furthermore shown that the fraction of reliable flux solutions within all solutions for fixed tadpole $\frac{\mathsf{Q}^0}{192}$ is only of orders $\mathcal{O}(10^{-3})$, which is a reduction as compared to the setting of section 6.3.

6.5 Moduli Stabilization III

We now generalize the setting from section 6.4 by including nongeometric Q-fluxes. The fluxes are restricted such that the complex-structure and Kähler moduli are fixed isotropically as $T_1 = T_2 = T_3 =: T$ and $U^1 = U^2 = U^3 =: U$, which reduces the system to the three complex moduli fields τ, U and T. Such vacua have previously been analyzed for instance in [71].

6.5.1 Setting

We specify the superpotential (6.2.42) by imposing the following restrictions on the R-R and NS-NS fluxes (6.2.34) and (6.2.35)

$$\begin{aligned}
&\mathfrak{f}^1 = \mathfrak{f}^2 = \mathfrak{f}^3\,, && h^1 = h^2 = h^3\,,\\
&\mathfrak{f}_1 = \mathfrak{f}_2 = \mathfrak{f}_3\,, && h_1 = h_2 = h_3\,,\\
&q^{01} = q^{02} = q^{03}\,, && \\
&{q_0}^1 = {q_0}^2 = {q_0}^3\,, && \\
&q^{11} = q^{22} = q^{33}\,, && q^{12} = q^{13} = q^{21} = q^{23} = q^{31} = q^{32} =: \tilde{q}^{11}\,,\\
&{q_1}^1 = {q_2}^2 = {q_3}^3\,, && {q_1}^2 = {q_1}^3 = {q_2}^1 = {q_2}^3 = {q_3}^1 = {q_3}^2 =: {\tilde{q}_1}{}^1\,,
\end{aligned} \tag{6.5.1}$$

which leaves four independent R-R G_3-flux components, four independent H-flux components and six independent Q-flux components. As discussed around equation (6.2.37), these fluxes are subject to the quantization conditions

$$\mathfrak{f}^0,\, \mathfrak{f}^1,\, \mathfrak{f}_0,\, \mathfrak{f}_1, \quad h^0,\, h^1,\, h_0,\, h_1, \quad q^{01},\, q^{11},\, \tilde{q}^{11},\, {q_0}^1,\, {q_1}^1,\, {\tilde{q}_1}{}^1 \in 8\mathbb{Z}\,. \tag{6.5.2}$$

Together with the Kähler potential (6.2.41), the F-term equations (6.2.46) can then be determined explicitly. Since in the present situation the superpotential W depends on the Kähler moduli $T_{\mathfrak{i}}$, the condition $W = 0$ is in general not obtained, and the resulting system of equations is more involved. However, provided that solutions to the F-term equations with non-vanishing imaginary parts exist, then for the fluxes (6.5.1) the moduli are stabilized isotropically,

$$U^1 = U^2 = U^3 =: U\,, \qquad T_1 = T_2 = T_3 =: T\,. \tag{6.5.3}$$

A necessary condition to achieve this stabilization is that ${q_1}^1 \neq {\tilde{q}_1}{}^1$ and $q^{11} \neq \tilde{q}^{11}$. The system of seven F-term equations for τ, $U^{\check{\mathfrak{a}}}$, $T_{\mathfrak{i}}$ then reduces to the three equations

$$\begin{aligned}
0 =\; & \big[\mathfrak{f}_0 - h_0\,\overline{\tau} - {q_0}^1\,3T\big] + 3U\,\big[\mathfrak{f}_1 - h_1\,\overline{\tau} - (2{\tilde{q}_1}{}^1 + {q_1}^1)\,T\big]\\
& -(U)^3\big[\mathfrak{f}^0 - h^0\,\overline{\tau} - q^{01}\,3T\big] + 3(U)^2\big[\mathfrak{f}^1 - h^1\,\overline{\tau} - (2\tilde{q}^{11} + q^{11})\,T\big]\,,
\end{aligned} \tag{6.5.4a}$$

$$\begin{aligned}
0 =\; & \big[\mathfrak{f}_0 - h_0\,\tau - {q_0}^1\,(\overline{T} + 2T)\big] + 3U\,\big[\mathfrak{f}_1 - h_1\,\tau - (2{\tilde{q}_1}{}^1 + {q_1}^1)\,\tfrac{1}{3}(\overline{T} + 2T)\big]\\
& -(U)^3\big[\mathfrak{f}^0 - h^0\,\tau - q^{01}\,(\overline{T} + 2T)\big] + 3(U)^2\big[\mathfrak{f}^1 - h^1\,\tau - (2\tilde{q}^{11} + q^{11})\,\tfrac{1}{3}(\overline{T} + 2T)\big]\,,
\end{aligned} \tag{6.5.4b}$$

$$\begin{aligned}
0 =\; & \big[\mathfrak{f}_0 - h_0\,\tau - {q_0}^1\,3T\big] + (\overline{U} + 2U)\,\big[\mathfrak{f}_1 - h_1\,\tau - (2{\tilde{q}_1}{}^1 + {q_1}^1)\,T\big]\\
& -\overline{U}(U)^2\big[\mathfrak{f}^0 - h^0\,\tau - q^{01}\,3T\big] + (2\overline{U}U + (U)^2)\big[\mathfrak{f}^1 - h^1\,\tau - (2\tilde{q}^{11} + q^{11})\,T\big]\,.
\end{aligned} \tag{6.5.4c}$$

The R-R and NS-NS fluxes are furthermore subject to the Bianchi identities (6.2.38), and using the restrictions (6.5.1), we find for the tadpole contributions

$$\begin{aligned}
\mathsf{Q}^0 &= \mathfrak{f}_0 h^0 - \mathfrak{f}^0 h_0 + 3\big(\mathfrak{f}_1 h^1 - \mathfrak{f}^1 h_1\big)\,,\\
\mathsf{Q}^1 &= \mathfrak{f}_0 q^{01} - \mathfrak{f}^0 {q_0}^1 + 2\big(\mathfrak{f}_1 \tilde{q}^{11} - \mathfrak{f}^1 {\tilde{q}_1}{}^1\big) + \mathfrak{f}_1 q^{11} - \mathfrak{f}^1 {q_1}^1\,,\\
\mathsf{Q}^{01} &= h_0 q^{01} - h^0 {q_0}^1 + 2\big(h_1 \tilde{q}^{11} - h^1 {\tilde{q}_1}{}^1\big) + h_1 q^{11} - h^1 {q_1}^1 \overset{!}{=} 0\,.
\end{aligned} \tag{6.5.5}$$

Finally, as we discussed in section 6.2.5, the present setting is duality invariant under $SL(2,\mathbb{Z})$ transformations of the complex-structure modulus U whereas the $SL(2,\mathbb{Z})$ duality (6.2.56) of the axio-dilaton is broken to constant shifts (6.2.57) due to the non-vanishing Q-flux. Furthermore, T-duality (6.2.54) is in general broken because of the isotropic choice of fluxes in (6.5.1).

6.5.2 Infinite Number of Solutions for fixed $\mathsf{Q}^{\hat{I}}$

In contrast to the settings of sections 6.3 and 6.4, for non-vanishing Q-flux the number of solutions for fixed tadpole contributions $\mathsf{Q}^{\hat{I}}$ is infinite [71]. This can be illustrated with the following example from [71]: the D7-brane tadpole contribution is fixed as $\mathsf{Q}^1 = 0$ and the fluxes are chosen as

$$\begin{aligned}
&\mathfrak{f}^0 = 0\,, && \mathfrak{f}^1 = 0\,, && \\
&\mathfrak{f}_0 = \tfrac{\mathsf{Q}^0}{b}\,, && \mathfrak{f}_1 = 0\,, && \\
&h^0 = b\,, && h^1 = b\,, && \\
&h_0 = b\,, && h_1 = -b\,, && \\
&q^{01} = 0\,, && q^{11} = -m\,, && \tilde{q}^{11} = 0\,, \\
&q_0{}^1 = -m-n\,, && q_1{}^1 = n\,, && \tilde{q}_1{}^1 = m\,,
\end{aligned} \qquad (6.5.6)$$

where $m,n \in 8\mathbb{Z}$ and $b \in 8\mathbb{Z}$ is restricted such that $\mathfrak{f}_0 \in 8\mathbb{Z}$. To obtain non-trivial solutions we require $m,n,b \neq 0$, and the above choice of fluxes always satisfies the Bianchi identities (6.5.5). A solution to the equations of motion (6.5.4) is given by

$$\tau = \frac{m^3 \mathsf{Q}^0}{8b^2n^3}(-4 \pm i)\,, \qquad U = \frac{m+n}{m} \pm \frac{n}{m}\, i\,, \qquad T = \frac{m\mathsf{Q}^0}{4bn^2}(2 \pm i)\,, \qquad (6.5.7)$$

where the sign takes the same value for all three moduli. In order for the imaginary parts to be positive, we require that this sign is chosen appropriately and that $\mathsf{Q}^0 > 0$ and $b\,n > 0$.

Note that (6.5.7) describes an infinite set of vacua since m,n are not bounded, which is in contrast to the situations studied in sections 6.3 and 6.4. However, in order to trust these solutions we have to require that $\mathrm{Im}\tau$, $\mathrm{Im}U$, $\mathrm{Im}T > 1$ which translates into the conditions

$$1 < \left(\pm\frac{n}{m}\right)^3 < \frac{\mathsf{Q}^0}{8b^2}\,, \qquad 1 < \pm\frac{n}{m} < \frac{\mathsf{Q}^0}{4bn}\,. \qquad (6.5.8)$$

For a fixed Q^0 there is only a finite number of choices for (m,n,b) which satisfy (6.5.8), and therefore the number of reliable solutions for the particular flux choice (6.5.6) is finite for fixed Q^0. We remark however that duality transformations can change the form of (6.5.6), and therefore a similar analysis has to be performed for the transformed flux choices. It remains an open question whether this leads to a finite number of reliable flux solutions. Addressing this issue promises to be an interesting direction for future research and might provide valuable insights into the structure of nongeometric flux compactifications.

6.5.3 Space of Solutions

Since the number of physically-distinct solutions for fixed tadpole contributions $\mathsf{Q}^{\hat{\mathsf{I}}}$ is in general infinite, for the present setting we cannot construct a complete data set of flux vacua even for fixed tadpole contributions. However, some insights into the space of solutions can be obtained using Monte-Carlo sampling.

Dataset

Our data set of flux vacua for the setting described in section 6.5.1 has been obtained in the following way:

- We restrict the contributions to the D3- and D7-brane tadpoles Q^0 and Q^1 shown in equations (6.5.5) as

$$\left|\frac{\mathsf{Q}^0}{64}\right| \leq 1000\,, \qquad \left|\frac{\mathsf{Q}^1}{64}\right| \leq 1000\,, \qquad \mathsf{Q}^{01} = 0\,. \tag{6.5.9}$$

  Note that due to the flux-quantization condition (6.5.2) the $\mathsf{Q}^{\hat{\mathsf{I}}}$ are always a multiple of 64, and that Q^0 and Q^1 can be negative while still leading to positive imaginary parts for τ, U, T.

- The fluxes in (6.5.1) are chosen randomly with a uniform distribution. The restriction on the value of the fluxes reads

$$\left|\frac{\text{flux quantum}}{8}\right| \leq 100\,. \tag{6.5.10}$$

- We have generated $1.3 \cdot 10^7$ flux configurations for which 1) all moduli τ, U, T are fixed, 2) all imaginary parts of the moduli fields are strictly positive, and 3) the vacua are physically distinct (i.e. not related by $SL(2,\mathbb{Z})$ transformations of the complex-structure moduli nor by T-transformations of the axio-dilaton or Kähler moduli).

- For these flux contributions all moduli are fixed, however, not all of these extrema are stable. The number of vacua with all moduli fixed and without tachyonic or flat directions is $2.9 \cdot 10^6$.

Solutions at small Coupling, large Complex Structure and large Volume

Since we do not have a complete set of solutions for fixed tadpole contributions Q^0 and Q^1, the same analysis as for the previous cases cannot be performed. However, for our data set we have determined the number of solutions N_c for which $\tau_2 = \mathrm{Im}\,\tau$, $U_2 = \mathrm{Im}\,U$ and $T_2 = \mathrm{Im}\,T$ satisfy

$$\tau_2,\, U_2,\, T_2 \geq \mathsf{c}\,. \tag{6.5.11}$$

For sufficiently large c, this corresponds to the weak-coupling, large-complex-structure and large-volume regime. We furthermore denote by $|\mathsf{Q}^{\hat{\mathsf{I}}}/64|_{\min}$ the lowest value of the expression $|\mathsf{Q}^{\hat{\mathsf{I}}}/64| = \sqrt{(\mathsf{Q}^0)^2 + 3(\mathsf{Q}^1)^2}/64$ in the set of vacua determined by (6.5.11). For configurations which fix all moduli but may contain tachyonic directions we find:

all (6.5.12)

c	$\mathsf{N_c}$	$\mathsf{N_c}/\mathsf{N_0}$	$|\mathsf{Q}^{\hat{\mathsf{I}}}/64|_{\min}$
0	$1.33 \cdot 10^7$	1	1
1	$5.97 \cdot 10^5$	$4.5 \cdot 10^{-2}$	16.7
2	$2.11 \cdot 10^4$	$1.6 \cdot 10^{-3}$	42.1
3	190	$1.4 \cdot 10^{-5}$	129.8
5	6	$4.5 \cdot 10^{-7}$	811.8

From here we see that vacua with large imaginary parts τ_2, U_2, T_2 are extremely rare but not excluded. For larger c – that is for more reliable solutions – the tadpole contributions have to be larger. These observations are in line with the results of the previous sections. For the stabilized vacua (without flat or tachyonic directions) we obtain a similar behaviour:

stable (6.5.13)

c	$\mathsf{N_c}$	$\mathsf{N_c}/\mathsf{N_0}$	$|\mathsf{Q}^{\hat{\mathsf{I}}}/64|_{\min}$
0	$2.94 \cdot 10^6$	1	1
1	$5.64 \cdot 10^5$	$1.9 \cdot 10^{-1}$	16.7
2	$2.03 \cdot 10^4$	$6.9 \cdot 10^{-3}$	42.1
3	163	$5.5 \cdot 10^{-5}$	129.8
5	1	$3.4 \cdot 10^{-7}$	1707.4

In table 6.1 we have collected some concrete examples for fully stabilized vacua with imaginary parts greater than one.

Distribution of Solutions in the $\mathsf{Q}^{\hat{\mathsf{I}}}$-Plane

For moduli stabilization of the axio-dilaton and complex-structure moduli studied in sections 6.3 and 6.4, we observed that the tadpole-contribution Q^0 has to be positive in order to obtain physical solutions with positive imaginary parts τ_2 and U_2. However, when including nongeometric fluxes we see that positive as well as negative values of Q^0 and Q^1 can result in positive imaginary parts τ_2, U_2 and T_2.

Having a large data set available, we have analyzed the distributions of vacua over Q^0 and Q^1. For the set of stable vacua without flat or tachyonic directions we obtain

stable (6.5.14)

Q^0	Q^1	fraction of all vacua
≤ 0	≤ 0	0.518765
≤ 0	> 0	0.155262
> 0	≤ 0	0.325822
> 0	> 0	0.000151

It is somewhat surprising that the fraction of vacua with tadpole contributions $\mathsf{Q}^0 > 0$ and $\mathsf{Q}^1 > 0$ is suppressed by three orders of magnitude compared to having at least one

data	example 1	example 2	example 3	example 4	example 5	example 6
$\mathfrak{f}_0$	-40	224	-336	-112	176	296
$\mathfrak{f}_1$	-88	24	96	104	24	-368
$\mathfrak{f}^0$	-56	8	32	40	48	8
$\mathfrak{f}^1$	0	0	24	0	24	-16
h_0	-88	-64	-16	-96	72	200
h_1	0	64	96	0	0	624
h^0	0	-16	-24	0	0	-192
h^1	0	0	-8	-8	0	-24
q^{01}	0	8	8	0	0	72
q^{11}	56	-104	112	24	-136	64
$\tilde{q}^{11}$	-16	48	-64	-16	48	0
${q_0}^1$	72	64	80	48	-112	-192
${q_1}^1$	-64	8	96	-32	-40	488
${\tilde{q}_1}^{\ 1}$	32	-56	-64	16	24	-608
$\mathsf{Q}^0/64$	-77	-48	-10	21	-54	-31
$\mathsf{Q}^1/64$	30	17	-94	-43	66	-193
τ	$+0.45+19.3i$	$-0.02+4.39i$	$-0.40+3.42i$	$+0.47+8.25i$	$+0.32+12.1i$	$-0.48+5.15i$
U	$+3.53i$	$-0.33+3.52i$	$-0.10+3.13i$	$+3.04i$	$+0.10+3.13i$	$+0.32+3.14i$
T	$+7.48i$	$+0.20+3.03i$	$+0.30+4.07i$	$-4.47+8.49i$	$-0.48+3.10i$	$-0.26+4.57i$

Table 6.1: Examples for stable vacua (no tachyonic or flat directions) with $\tau_2, U_2, T_2 > 1$.

$\mathsf{Q}^{\hat{I}}$ negative. For our data set of solutions which include potentially tachyonic directions no such drastic difference is found:

all

Q^0	Q^1	fraction of all vacua
≤ 0	≤ 0	0.360664
≤ 0	> 0	0.256953
> 0	≤ 0	0.254567
> 0	> 0	0.127816

(6.5.15)

We also note that both data sets do not contain any solution with $\mathsf{Q}^0 = \mathsf{Q}^1 = 0$.

Distribution of Solutions in Moduli Space

We have also analyzed the distribution of solutions to the F-term equations (6.2.46) within the moduli space. Since the density of solutions is very small, we were not able to identify any patterns or structures.

6.5.4 Summary

We briefly summarize the main results obtained in this section for the combined moduli stabilization of the axio-dilaton, complex-structure moduli and Kähler moduli by the fluxes shown in equation (6.5.1):

- For fixed D3- and D7-brane tadpole contributions Q^0 and Q^1, the number of physically-distinct vacua is in general infinite [71]. We therefore were not able to generate a complete data set but used Monte-Carlo sampling to randomly generate $1.3 \cdot 10^7$ solutions to the F-term equations which fix all moduli.

- We have shown that reliable solutions at weak string coupling, large complex structure and large volume are only a small fraction of all vacua. For instance, stable solutions with $\tau_2, U_2, T_2 \geq 5$ make only a fraction of $3.4 \cdot 10^{-7}$ of all solutions. Requiring the solutions to be more reliable requires the tadpole contributions $\mathsf{Q}^{\hat{I}}$ to be larger, which is in tension with the tadpole-cancellation condition as discussed on page 124.

- In table 6.1 we have shown some concrete examples of stable vacua with axio-dilaton, complex-structure moduli and Kähler moduli fixed at imaginary parts greater than one.

- Finally, we have pointed out that stable vacua with all tadpole contributions Q^0 and Q^1 positive are statistically disfavored. The reasons and implications of this behaviour are not yet clear and might be an interesting direction for future research.

6.6 Discussion

In this chapter we have studied moduli stabilization with R-R and NS-NS fluxes in type IIB string theory for the example of the $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold. We have analyzed the interplay between moduli stabilization and tadpole cancellation. In particular, we have shown how properties of the vacua depend on the flux contribution to the tadpole-cancellation condition.

Summary of Results

More concretely, the axio-dilaton and complex-structure moduli are fixed by geometric fluxes while the Kähler moduli are fixed at tree-level by the nongeometric Q-flux. In section 6.3 we have focussed on the axio-dilaton only and mainly ignored the complex-structure and Kähler moduli. In section 6.4 we included the complex-structure moduli, and in section 6.5 we studied moduli stabilization for all closed-string moduli. We analyzed the space of solutions to the F-term equations for these settings and found that it is not homogenous:

- For the axio-dilaton the space of solutions contains characteristic void structures (see figure 6.2) [233,237]. The radius of these voids depends on the flux contribution Q^0 to the tadpole-cancellation condition, and for larger Q^0 the radii become smaller.

  When including the complex-structure moduli, we observe that vacua are accumulated on submanifolds within the space of solutions (see figure 6.8). On these planes we again find void structures, which are connected by lines between different planes. We therefore find cylindrical voids in (three-dimensional sections of) this four-dimensional space of solutions.

Furthermore, in section 6.2.3 we have argued that the flux contribution to the tadpole-cancellation condition cannot be arbitrarily large. In particular, for many known examples this contribution is small. We have then contrasted this observation with the requirement of having reliable solutions at weak string-coupling, large complex structure and large volume:

- We have seen that the fraction of vacua with small string coupling $\tau_2 \gg 1$, large complex structure $U_2 \gg 1$ and large volume $T_2 \gg 1$ is small. For instance, within the approach followed in this discussion around 20% of the solutions satisfy $\tau_2 \geq 5$, around 0.4% of the solutions satisfy $\tau_2, U_2 \geq 5$, and a fraction of around 10^{-7} of the solutions satisfy $\tau_2, U_2, T_2 \geq 5$. This suggests that for a large number of moduli, only a very small fraction of the solutions can be trusted (within the tree-level approach used in this work).

- We have also observed that in order to find vacua at weak string-coupling, large complex structure and large volume, the flux contribution to the tadpole-cancellation condition has to be large. Within the approach followed in this discussion, for $\tau_2 \geq 5$ one needs $\mathsf{Q}^0 \geq 3840$, for $\tau_2, U_2 \geq 5$ one needs $\mathsf{Q}^0 \geq 38208$, and for $\tau_2, U_2, T_2 \geq 5$ we have indications that one needs $|\mathsf{Q}^{\hat{\imath}}| \gtrsim \mathcal{O}(10^5)$. This suggests that in order to stabilize a large number of moduli in a perturbatively-controlled regime, a large flux contribution is needed. However, this conclusion is in stark contrast to the tadpole-cancellation condition which strongly disfavors large flux contributions.

To conclude, in order to stabilize moduli in a reliable way a large flux contribution is needed – which is however strongly restricted by the tadpole-cancellation condition. We therefore see that moduli stabilization and model building in string theory cannot be approached independently but have to be addressed simultaneously. This is a difficult task.

Limitations and Future Directions

We now comment on the limitations of the analysis performed in this chapter and on future directions:

- Our conclusions in this chapter are based on the study of a single compactification space. We believe that the $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold captures main features of the problem, but these have to be confirmed by other examples.

- In this discussion we have stabilized moduli at tree-level. Corrections to the effective theory can usually be ignored in the weak-coupling, large-complex-structure and large-volume regime, however, many of the obtained solutions are not in this regime. We therefore should repeat our analysis and include various corrections from the start, which in turn will modify the space of solutions.

- We have found that only a small fraction of solutions stabilize moduli in a perturbatively-controlled regime. This observation has implications for the landscape of string vacua, in particular, it suggests that the landscape may be smaller than naively expected. It would be desirable to make this statement more precise.

- The $SL(2, \mathbb{Z})$ duality of the axio-dilaton was broken by nongeometric Q-fluxes. Including so-called P-fluxes will restore this duality and may again modify the corresponding space of solutions.

- The contribution of orientifold planes to the tadpole-cancellation condition could only be estimated based on known examples. It would be desirable to have a criterium which can put a bound on the orientifold contribution for a particular compactification space.

Part IV

Conclusion

Chapter 7

Summary and Outlook

Dualities and nongeometric backgrounds are closely intertwined and form and integral part of string an M-theory. Their very presence hints at the existence of underlying structures which remain elusive for established mathematical frameworks. In this thesis we presented the formalisms of extended geometries and field theories in which dualities are implemented as a manifest symmetry into the model. This enabled us to approach the problem of moduli stabilization from a more general viewpoint and to gain deeper insights into the intricate structure of dualities and the string landscape.

7.1 Summary

Dualities – nontrival relations between seemingly different theories or structures – are widely known in the fields of physics and mathematics. In many instances, they serve as a valuable tool to address previously inaccessible problems from a different point of view in which the intended task becomes feasible. On the other hand, the blurring between unrelated or contrasting concepts raises suspicions that there is some fundamental aspect of the models which is not yet fully understood. Throughout this thesis, we have seen that the nature of dualities is not a purely philosophical question, but can be crucial for the development of physically realistic models of our universe.

Starting Point

In string theory, the presence of mathematically ill-defined duals to flux backgrounds motivates idea of generalized flux backgrounds, which play an essential role in both obtaining full moduli stabilization and linking gauged supergravities to a higher-dimensional origin. Describing such models requires the consideration of structures beyond those of differential geometry. Extended field theories achieve this goal by enhancing point-particles with additional dual coordinates, thus allowing dual theories to be embedded into extended geometries in which duality transformations become a manifest symmetry. A particularly appealing feature of such models is that previously ill-defined objects are provided a geometric interpretation. A major part of this thesis is built around the utilization of this property to obtain a better understanding of compactifications on non-geometric backgrounds.

Dimensional Reductions of Double Field Theory

In chapter 5 we performed dimensional reductions of type II double field theory on generalized flux backrounds explicitly for Calabi-Yau three-folds and $K3 \times T^2$. The flux formulation of double field theory provides a T-duality covariant extension of ordinary supergravities formulated in terms of abstract generalized fluxes whose fluctuations can can be identified with the known fluxes of the T-duality chain. Enforcing the strong constraint on the background values, the setting effectively describes type II theories on generalized backgrounds in which all geometric and non-geometric fluxes are implemented as simple operators acting on differential form fields.

A major strength of double field theory is its capability to provide a natural ten-dimensional origin for lower-dimensional gauged supergravities, which long appeared to be isolated from string theory. We showed this explicitly by performing dimensional reductions of the presented setting on Calabi-Yau three-folds, which gave rise to four-dimensional $\mathcal{N} = 2$ supergravity with electric and magnetic gaugings. In the most general case, the presence of non-geometric fluxes causes the four-dimensional action to manifest in a partially dualized form [202]; in the purely geometric setting, this dualization becomes reversible, and the theory takes a more familiar form which eventually reduces to that of ungauged supergravity as the remaining fluxes are turned off. These results relate directly to those of an alternative approach [199] employing $SU(3) \times SU(3)$ structure manifolds, and many of the objects encountered in our computations could be related to this framework.

Another appealing feature of our formalism is its transferability to a wider class of compactification manifolds. This was carried out explicitly for the case $K3 \times T^2$. By generalizing the concepts and relaxing some of the constraints introduced previously in [78, 79], we constructed a more general form of the scalar potential which correctly incorporates the contribution of fluxes not present on Calabi-Yau three-folds. A similar special-geometric framework as in the Calabi-Yau case could be constructed by employing the ideas of generalized Calabi-Yau and $K3$ structures [147,203], which eventually enabled us to perform the dimensional reduction in a mostly analogous way.

A final important aspect is the role of Mirror Symmetry in double field theory. Purely geometric flux compactifications break the duality between type IIA and IIB theory due to parts of the fluxes being mapped out of the geometric regime by T-duality transformations. By including all fluxes of the T-duality chain into the framework, Mirror Symmetry is restored, with mappings between geometric and nongeometric fluxes clearly reflected in the mirror mappings. Double field theory thereby catches the structural properties of the duality in an elegant way as a simple exchange of roles between two poly-forms encoding the complex and Kähler structures of the compactification manifold.

Dualities, Tadpole Cancellation and Type IIB Orientifolds

In the second major research part of this thesis, we focused on concrete examples of moduli stabilization in type IIB orientifold compactifications with fluxes. At the heart of such models lies the application of an orientifold projection to mod out the orientation of the string and half of the supersymmetry generators, thus giving rise to $\mathcal{N} = 1$ supergravity in four dimensions. This comes with a variety of consistency constraints on the theory, most notably the tadpole-cancellation condition, which strongly restrict the allowed combinations of flux quanta. At the same time, the presence of dualities renders certain orbits of of configurations equivalent, thus drastically reducing the number of physically-distinct vacua. In chapter 6 we performed an in-depth analysis of the space of vacua for the type IIB orientifold of $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$, thereby keeping a close eye on the interplay between moduli stabilizaiton, tadpole cancellation and dualities.

An essential insight of our discussion is that flux configurations which stabilize moduli in a pertubatively-controlled regime compose only a very small fraction of all possible configurations. In these cases, large tadpole contributions are necessary or strongly favored. On the other hand, the non-vanishing mass of D-branes generally requires low tadpole contributions in order to prevent the probe approximation from breaking down due to backreactions on the geometry. Obtaining physical solutions which are reliable from both perspectives is thus a difficult task, and the landscape of the considered models might ultimately be much smaller than expected if all approximations are taken into account correctly.

Another interesting aspect is the effect of consistency constraints and dualities on the structure of the space of vacua. It was first observed in [233, 237] that the fundamental domains of the $SL(2, \mathbb{Z})$ duality groups for the different moduli contain voids or regions of varying density. We found similar stuctures in more general settings. The vacua then accumulate on submanifolds within the total space, which themselves contain higher-dimensional void structures. The size of these voids generally shrinks as the flux contribution to the tadpoles is increased, which is however again limited by the probe approximation of D-branes. Their presence thus suggests that parts of the solution space might remain inaccessible under the assumptions the considered models are based on.

Our results highlight the strong interplay between moduli stabilization and model building in string theory. While the established approximations and frameworks certainly serve as viable starting points to study the structure of flux vacua, we think it is of essential importance to treat these two issues in a unified way in order to obtain more realiable results.

7.2 Outlook

Dualities and extended geometries are still an active field of research, and promising developments have occured in the recent past. We conclude this thesis with a brief outlook on future developments and possible directions of further research.

Extended Field Theories and Beyond

Despite recent success in providing a better description of dualities and nongeometry in string theory, much is still to be understood in the area of extended field theories. Strong efforts are currently being made in applying the concepts of exceptional field theories to larger U-duality groups, and interesting insights into the structure of generalized flux backgrounds and exotic branes are to be expected. More recently, the original idea of a T-duality covariant worldsheet [282–287, 151, 288–291] was generalized to certain U-duality groups [292, 293], and a study of open-string boundary conditions and D-branes in extended spaces was initiated in [294, 295].

From a phenomenological point of view, there exists a wide variety of settings to analyze. One possible direction is to explcitly elaborate dimensional reductions of exceptional field theories; another interesting option would be to consider nongeometric flux compactifications on various classes of manifolds and their relation to other framworks such as $SU(2)$ structures [222–224].

Flux Vacua and String Phenomenology

Concerning the issues of moduli stabilization and model building, a suggestive next step would be to check to which degree the ideas can be transferred to a wider class of compactification manifolds and how the inclusion of corrections to the effective theories affects the results. Ultimately, it would be desirable to make more general statements on the validity of flux vacua and the size of the string landscape.

On a more general level, there are various further directions of research to pursue. Two particular fields of string phenomenology are currently experiencing a surge of interest. A first approach which was also adopted in this thesis is the utilization of techniques from data analysis and machine learning to study the structure of the string landscape. While we sticked to a rather conventional statistics framework, there are ongoing efforts to make more modern techniques such as topological data analysis or neural networks applicable to problems in string theory [244, 245]. These methods are still in an early stage of development, however, they bear a great potential to handle the large datasets encountered in string theory and to discover patterns which might remain hidden for the human eye.

A different strategy is to address the problem from the opposite direction and try to find criteria for effective field theories to be incompatible with string theory. Research on this so-called string swampland was already initiated in the mid-2000s [296–298] and recently moved back into the focus of interest after a number of new swampland conjectures was elaborated [299–301, 227, 302–308]. While not a subject of this thesis, it is certainly a promising approach to gain both a deeper insights into the phenomenology of string theory as well as to address the issue of falsifiability.

Conclusion

Coming back to the starting point of this thesis, we saw throughout the recent chapters that the quest for unification still acts as a driving force in modern high energy physics. By implementing dualities as a manifest symmetry, extended field theories define a unified framework for dual supergravity theories, which ultimately enabled a mathematical description of nongeometric backgrounds as well as an inclusion of gauged supergravities into the web of dualities. Similarly, addressing the problems of moduli stabilization and model building simultaneously allowed us to gain deeper insights into the structure of the string landscape and the reliability of commonly used approximations. All this shows that the striving for unification is not an end unto itself, but can help to obtain a better understanding of concepts that are already well-established and lead to the development of new ideas which might be essential to address some of the open problems in modern physics.

Part V

Appendices

Appendix A

Notation and Conventions

A.1 Spacetime Geometry and Indices

Throughout this thesis we make use of various kinds of indices, which are structured as follows:

- We distinguish between serif letters $A, a, \ldots$ denoting spacetime indices and sanserif letters $\mathsf{A}, \mathsf{a}, \ldots$ labeling the coordinates of moduli spaces. We furthermore introduce blackboard typeface capital letters $\mathbb{A}, \mathbb{B}, \ldots, \mathbb{I}, \mathbb{J}, \ldots$ as collective notation summarizing certain de Rham cohomology bases, which are specified in subsection 5.3.3 and 5.4.2.

- In orientifold compactifications, cohomological indices of even eigenspaces of the orientifold projection are indicated by "hat" symbols $\hat{\mathsf{A}}, \hat{\mathsf{a}}, \ldots$ and those of odd eigenspaces by "check" symbols $\check{\mathsf{A}}, \check{\mathsf{a}}, \ldots$.

- For spacetime indices, capital letters denote doubled coordinates, and small letters denote normal coordinates.

Using this as a guideline, we define the following indices:

- Capital Latin letters $M, N, \ldots$ and $A, B, \ldots$ label the curved respectively tangent coordinates of twenty-dimensional doubled spacetime.

- Small Latin letters $m, n, \ldots$ and $a, b, \ldots$ label the curved respectively tangent coordinates of ten-dimensional spacetime.

- Small Greek letters $\mu, \nu, \ldots$ and $\varepsilon, \zeta, \ldots$ label the curved respectively tangent coordinates of four-dimensional external spacetime.

- Capital Latin letters $I, J, \ldots$ and $A, B, \ldots$ label the curved respectively tangent coordinates of twelve-dimensional doubled internal space.

- Small Latin letters $i, j, \ldots$ and $a, b, \ldots$ label the curved respectively tangent coordinates of six-dimensional internal space.

- On the product manifold $K3 \times T^2$, the above internal indices are used for the $K3$ component, whereas specific indices $R, S, \ldots, r, s, \ldots$ are used for curved coordinates and $X, Y, \ldots, x, y, \ldots$ for tangent coordinates of the T^2-component.

- On CY_3, small Latin letters $a, \bar{a}, b, \bar{b} \ldots$ denote complex curved coordinates of six-dimensional internal spacetime. On $K3 \times T^2$, $a, \bar{a}, b, \bar{b} \ldots$ denote complex curved coordinates of $K3$ and $g, \bar{g}, h, \bar{h} \ldots$ those of T^2.

Due to the variety of considered compactification manifolds, moduli-space and cohomological indices are specified in the sections where the bases are defined. Notice also that some indices such as $A, B, \ldots$ and $a, b, \ldots$ are assigned multiple roles. Their meaning will, however, always be clarified explicitly or obvious from the context.

A.2 Tensor Formalism and Differential Forms

For general tensors, differential forms and related operators, we apply the following conventions:

- For instances in which the spacetime splits into external and internal components, fields living on the full spacetime manifold are indicated by "hat" symbols $\hat{A}, \ldots$ to prevent confusion with purely external or internal fields $A, \ldots$.

- The antisymmetrization of a tensor A is is defined by

$$A_{[\underline{m_1} \underline{m_2} \ldots \underline{m_n}]} := \frac{1}{n!} \sum_{\pi \in S_n} (-1)^{\mathrm{sign}(\pi)} A_{\pi(m_1)\pi(m_2)\ldots\pi(m_n)}, \tag{A.2.1}$$

where S_n denotes the set of permutations of $\{1, 2, \ldots n\}$.

- The Levi-Civita tensor $\varepsilon^{m_1 \ldots m_D}$ in D dimensions is defined as the totally antisymmetric tensor with $\varepsilon^{012\ldots(D-1)} = 1$ (Lorentzian signature) or $\varepsilon^{123\ldots D} = 1$ (Euclidean signature). It satisfies the relations

$$\begin{aligned}
\varepsilon^{m_1 \ldots m_D} \varepsilon_{n_1 \ldots n_D} &= D! \delta^{[m_1}_{n_1} \ldots \delta^{m_D]}_{n_D} &&= \delta^{m_1 \ldots m_D}_{n_1 \ldots n_D}, \\
\varepsilon^{m_1 \ldots m_p m_{p+1} \ldots m_D} \varepsilon_{m_1 \ldots m_p n_{p+1} \ldots n_D} &= p!\,(D-p)! \delta^{[m_{p+1}}_{n_{p+1}} \ldots \delta^{m_D]}_{n_D} &&= p! \delta^{m_{p+1} \ldots m_D}_{n_{p+1} \ldots n_D}, \\
\varepsilon^{m_1 \ldots m_D} \varepsilon_{m_1 \ldots m_D} &= D!.
\end{aligned} \tag{A.2.2}$$

- The components of a differential p-form ω_p are defined as

$$\omega_p = \frac{1}{p!} \omega_{m_1 \ldots m_p} \mathrm{d}x^{m_1} \wedge \ldots \wedge \mathrm{d}x^{m_p}. \tag{A.2.3}$$

- The exterior product of a p-form ω_p and a q-form χ_q is given by

$$\begin{aligned} \wedge : \quad \Omega^p(M) \times \Omega^q(M) &\rightarrow \Omega^{p+q}(M) \\ (\omega_p, \chi_q) &\mapsto \omega_p \wedge \omega_q = \frac{(p+q)!}{p!q!}\omega_{[\underline{m_1}\ldots\underline{m_p}}\chi_{\underline{n_1}\ldots\underline{n_q}]}\,\mathrm{d}x^{m_1}\wedge\ldots \\ &\qquad \ldots\wedge \mathrm{d}x^{m_p}\wedge\mathrm{d}x^{n_1}\wedge\ldots\wedge\mathrm{d}x^{n_q}. \end{aligned} \tag{A.2.4}$$

In this context, we choose the notation $(\omega_p)^n = \overbrace{\omega_p\wedge\omega_p\wedge\ldots\wedge\omega_p}^{n\text{ factors}}$ for exterior products of a p-form ω_p with itself.

- The exterior derivative d is given by

$$\begin{aligned} \mathrm{d} : \Omega^p(M) &\rightarrow \Omega^{p+1}(M) \\ \omega_p &\mapsto \mathrm{d}\omega_p = \frac{1}{p!}\frac{\partial\omega_{m_1\ldots m_p}}{\partial x^n}\mathrm{d}x^n\wedge\mathrm{d}x^{m_1}\wedge\ldots\wedge\mathrm{d}x^{m_p}. \end{aligned} \tag{A.2.5}$$

- The Hodge star operator $\star$ is defined by

$$\begin{aligned} \star : \quad \Omega^p(M) &\rightarrow \Omega^{D-p}(M) \\ \omega_p &\mapsto \star\omega_p = \frac{1}{\sqrt{g}p!\,(D-p)!}\varepsilon_{m_1\ldots m_p m_{p+1}\ldots m_D}g^{m_1n_1}\ldots g^{m_pn_p}\omega_{n_1\ldots n_p}\mathrm{d}^{D-p}x. \end{aligned} \tag{A.2.6}$$

This induces a pairing of two p-forms ω_p and χ_p given by

$$\omega_p\wedge\star\overline{\chi}_p = \frac{\sqrt{g}}{p!}\omega_{m_1\ldots m_p}\overline{\chi}_{n_1\ldots n_p}g^{m_1n_1}\ldots g^{m_pn_p}\mathrm{d}^Dx. \tag{A.2.7}$$

On $D-$dimensional manifolds, $\star$ satisfies the bijectivity condition

$$\star\star\,\omega_p = \alpha\,(-1)^{p(D-p)}\,\omega_p\,, \tag{A.2.8}$$

where α takes the value 1 for Euclidean and -1 for Lorentzian signatures. Using this, one can show that the b-twisted Hodge star operator (5.2.30) squares to -1,

$$\star_b\star_b = -1. \tag{A.2.9}$$

When splitting a differential p-form $\omega_p = \eta_{p-n}\wedge\rho_n$ living in $M^{10} = M^{1,3}\times M^6$ into two components $\eta_{p-n}\in\Omega^{p-n}(M^{1,3})$ and $\rho_n\in\Omega^n(M^6)$, the Hodge-star operator splits as

$$\star\omega_p = (-1)^{n(p-n)}\star\eta_{p-n}\wedge\star\rho_n. \tag{A.2.10}$$

As a consequence, one obtains for the involution operator (5.2.28)

$$\star\lambda(\omega_p) = \star\lambda(\eta_{p-n})\wedge\star\lambda(\rho_n)\,. \tag{A.2.11}$$

- For differential poly-forms, we define the projectors $[\cdot]_n$ to give as output the n-form components of the argument.

Appendix B

Important Identities of Complex and Kähler Geometry

This appendix provides an overview of important geometric identities for Calabi-Yau three-folds and $K3 \times T^2$ used throughout the calculations in sections 5.3 and 5.4. Most of the technical steps are based on the notions complex and Kähler geometry, which shall be briefly discussed here.

Both CY_3 and $K3 \times T^2$ are complex manifolds, allowing for a standard complex structure I satisfying

$$\begin{aligned} I^a{}_b &= i\delta^a{}_b, & I^{\bar{a}}{}_{\bar{b}} &= -i\delta^{\bar{a}}{}_{\bar{b}}, \\ I^a{}_{\bar{b}} &= 0, & I^{\bar{a}}{}_b &= 0. \end{aligned} \tag{B.0.1}$$

Both manifolds are also Kähler and, thus, Hermitian. The only non-vanishing components of their metric are therefore $g_{a\bar{b}} = \overline{g}_{\bar{a}b}$. They are related to the Kähler form J by

$$J_{a\bar{b}} = ig_{a\bar{b}}, \quad J_{\bar{a}b} = -ig_{\bar{a}b} \tag{B.0.2}$$

and, in real coordinates,

$$J_{ij} = g_{im}I^m{}_j. \tag{B.0.3}$$

For the holomorphic three-form of CY_3, we employ the normalization

$$\frac{i}{8}\Omega \wedge \star\overline{\Omega} = \frac{1}{3!}J^3, \tag{B.0.4}$$

leading to the relations

$$\begin{aligned} \Omega_{abc}\overline{\Omega}_{\bar{a}\bar{b}\bar{c}}g^{c\bar{c}} &= 8\left(g_{a\bar{a}}g_{b\bar{b}} - g_{a\bar{b}}g_{b\bar{a}}\right), \\ \Omega_{abc}\overline{\Omega}_{\bar{a}\bar{b}\bar{c}}g^{b\bar{b}}g^{c\bar{c}} &= 16g_{a\bar{a}}, \\ \Omega_{abc}\overline{\Omega}_{\bar{a}\bar{b}\bar{c}}g^{a\bar{a}}g^{b\bar{b}}g^{c\bar{c}} &= 48. \end{aligned} \tag{B.0.5}$$

The same normalization is applied to the holomorphic volume form $\Omega := \Omega_{K3} \times \Omega_{T^2}$ of

$K3 \times T^2$ (with $J := J_{K3} + J_{T^2}$), and one obtains similarly

$$
\begin{aligned}
\Omega_{gab}\overline{\Omega}_{\bar{g}\bar{a}\bar{b}} g^{g\bar{g}} &= 8\left(g_{a\bar{a}} g_{b\bar{b}} - g_{a\bar{b}} g_{b\bar{a}}\right), \\
\Omega_{gab}\overline{\Omega}_{\bar{g}\bar{a}\bar{b}} g^{b\bar{b}} &= 8 g_{g\bar{g}} g_{a\bar{a}}, \\
\Omega_{gab}\overline{\Omega}_{\bar{g}\bar{a}\bar{b}} g^{a\bar{a}} g^{b\bar{b}} &= 16 g_{g\bar{g}}, \\
\Omega_{gab}\overline{\Omega}_{\bar{g}\bar{a}\bar{b}} g^{g\bar{g}} g^{a\bar{a}} g^{b\bar{b}} &= 16.
\end{aligned}
\tag{B.0.6}
$$

Appendix C

Finite Number of Solutions for Geometric Isotropic Torus

In this appendix we follow the proof of [269, 237] to show that for the setting of section 6.4.1 the number of physically-distinct solutions is finite for fixed Q^0. The most important tools to do this are the $SL(2,\mathbb{Z})$ dualities of the axio-dilaton and complex-structure moduli summarized in section 6.2.5. Splitting the moduli into real and imaginary parts as

$$\tau = \tau_1 + i\tau_2\,, \qquad\qquad U = U_1 + iU_2\,, \tag{C.0.1}$$

we recall that the two equations (6.4.5) define an overdetermined cubic system for U and therefore do not allow for a closed-form solution in the generic case. We will now follow the lines of [269, 237] to demonstrate how a closed solution can still be obtained for the physically relevant cases.

In order for a physical solution to exist, both equations have to share a common root with non-vanishing imaginary part. Since all coefficients are real, there then exists a second solution given by its complex conjugate, and the two equations share a common quadratic factor. In this case, the two cubic polynomials (6.4.5) can be factorized as

$$\begin{aligned} (\,rU + s\,)\,P(U) &= 0\,, \\ (\,uU + v\,)\,P(U) &= 0\,, \end{aligned} \tag{C.0.2}$$

where $P(U)$ defines the common quadratic factor,

$$P(U) = l\,(U)^2 + mU + n\,, \tag{C.0.3}$$

and the seven new variables $l, m, n, r, s, u, v \in \mathbb{Z}$ are defined by an overdetermined system of equations

$$\begin{aligned} rm + sl &= -3\mathfrak{f}^1\,, & rl &= \mathfrak{f}^0\,, \\ rn + sm &= -3\mathfrak{f}_1\,, & sn &= -\mathfrak{f}_0\,, \\ um + vl &= -3h^1, & ul &= h^0, \\ un + vm &= -3h_1, & vn &= -h_0\,. \end{aligned} \tag{C.0.4}$$

The set of admissible septuples is furthermore restricted by requiring the flux quanta to to satisfy the tadpole cancellation condition (6.4.7), which can be reformulated as

$$\big(rv - su\big)\,\big(m^2 - 4ln\big) = -3\mathsf{Q}^0\,. \tag{C.0.5}$$

As shown in [269], this condition can only be satisfied if Q^0 is a multiple of three, yielding an overall factor of 192 when taking into account the flux quantization conditions. Since the prefactors appearing in (C.0.2) are linear in U with real coefficients, the two solutions with non-vanishing imaginary part can be obtained by choosing U such that

$$P(U) = 0\,. \tag{C.0.6}$$

Requiring furthermore the imaginary part of U to be positive, we arrive at the physical solutions

$$\begin{aligned} U &= \frac{-m + \sqrt{m^2 - 4ln}}{2l} \qquad && \text{if } l > 0 \text{ and } n > 0\,, \\ U &= \frac{-m - \sqrt{m^2 - 4ln}}{2l} \qquad && \text{if } l < 0 \text{ and } n < 0\,. \end{aligned} \tag{C.0.7}$$

The F-term equation (6.4.6) is linear in τ and can be solved analytically, leading to the stabilized value

$$\tau = \frac{s\,(m + 2lU) + r\big[n + U\,(2m + 3lU)\big]}{v\,(m + 2lU) + u\big[n + U\,(2m + 3lU)\big]}\,. \tag{C.0.8}$$

We will now proceed similarly to section 6.3.2 to show that using the dualities for the axio-dilaton and complex-structure moduli, for fixed Q^0 only a finite number of solutions can be found. Without loss of generality we focus on the case $l > 0$ and $n > 0$. The situation $l < 0$ and $n < 0$ is completely analogous.

- As can be read off from the first line in (C.0.7), the shift symmetry (6.2.49) of U gives rise to an equivalence

$$m \sim m + 2bl\,, \qquad b \in \mathbb{Z}\,. \tag{C.0.9}$$

  It therefore follows that all inequivalent values of m are contained in the range

$$m = -l, \ldots, l-1\,. \tag{C.0.10}$$

- Considering the boundary $U_2 = -\frac{1}{2}$, a minimal requirement for U to be located in the fundamental domain $\mathcal{F}_U$ is given by $U_2 \geq \sqrt{3}/2$. This is equivalent to requiring

$$m^2 - 4ln \leq -3l^2\,. \tag{C.0.11}$$

On the other hand, both of the factors on the left-hand side of the tadpole-cancellation condition (C.0.5) have to be integers, giving rise to a lower bound

$$m^2 - 4ln \geq -3\mathsf{Q}^0 \,. \tag{C.0.12}$$

This restricts the inequivalent values of both l and n to finite ranges

$$1 \leq l \leq \sqrt{\mathsf{Q}^0}\,, \qquad \frac{3l^2 + m^2}{4l} \leq n \leq \frac{3\mathsf{Q}^0 + m^2}{4l}\,. \tag{C.0.13}$$

- Employing the same arguments for the axio-dilaton, one finds an additional equivalence

$$s \sim s + bv\,, \qquad b \in \mathbb{Z}\,, \tag{C.0.14}$$

as well as upper bounds for u and v.

- The remaining degree of freedom r is fixed by the tadpole cancellation condition (C.0.5).

The above conditions leave only a finite number of inequivalent solutions for a fixed D3-tadpole contribution Q^0.

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