Mean-Field Stochastic Differential Equations with Irregular Coefficients: Solutions and Regularity Properties

Dissertation an der Fakultät für Mathematik, Informatik und Statistik der Ludwig-Maximilians-Universität München

Eingereicht von

Martin Bauer

19. Dezember 2019

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Tag der mündlichen Prüfung: 24. Juni 2020

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Hiermit erkläre ich, Martin Bauer, an Eides statt, dass die Dissertation von mir selbstständig und ohne unerlaubte Beihilfe angefertigt worden ist.

München, 3. Juli 2020	Martin Bauer
Ort, Datum	Unterschrift Doktorand

Zusammenfassung

Die vorliegende Dissertation beschäftigt sich mit der Untersuchung von 'mean-field' stochastischen Differentialgleichungen, auch genannt McKean-Vlasov Gleichungen. Zusätzlich zu Zeit und Raum hängen 'mean-field' stochastische Differentialgleichungen von der Verteilung des zugrundeliegenden Prozesses ab and erweitern somit die Klasse der stochastischen Differentialgleichungen. Historisch aus der Modellierung von Partikelsystemen in der mathematischen Physik stammend, verzeichnete das Interesse an 'mean-field' stochastische Differentialgleichungen und des zugehörigen Felds der 'mean-field' Spiele, mit ihren Anwendungen in der Ökonomie und der Finanztheorie (siehe z.B. [17] und [38]), in den letzten Jahren ein große Steigerung.

Das erste Hauptaugenmerk dieser Thesis liegt auf der Existenz und Eindeutigkeit von (schwachen und starken) Lösungen zu 'mean-field' stochastischen Differentialgleichungen mit irregulären Koeffizienten. Genauer gesagt, betrachten wir Driftkoeffizienten, die, ganz im Gegensatz zu der gewöhnlichen Annahme von (Lipschitz) Stetigkeit, lediglich messbar in der Raumvariable sind. Beginnend mit einem eindimensionalen Modell mit additiver Brownscher Störung, zeigen wir die Existenz von schwachen und starken Lösungen und leiten Annahmen her unter denen die Lösungen in der Verteilung bzw. pfadweise eindeutig sind. Anschließend gehen wir über zu mehreren Dimensionen, um äquivalente Ergebnisse für mehrdimensionale 'mean-field' stochastische Differentialgleichungen mit additiver Brownscher Störung zu beweisen. Zuletzt untersuchen wir noch die Existenz und Eindeutigkeit von Lösungen für 'mean-field' stochastische Differentialgleichungen in unendlichdimensionalen separablen Hilberträumen mit zylindrischer fraktionaler Brownscher Bewegung als treibende Störung. Um Existenz- und Eindeutigkeitsresultate für unendlichdimensionale 'mean-field' stochastische Differentialgleichungen herzuleiten, wird zunächst die Klasse der gewöhnlichen stochastischen Differentialgleichungen auf einem unendlichdimensionalen separablen Hilbertraum mit zylindrischer fraktionaler Brownscher Bewegung als treibende Störung betrachtet. Für diese ist im Beweis der Existenz von starken Lösungen die Anwendung eines Kompaktheitskriteriums basierend auf dem Malliavin Kalkül eines der Hauptinstrumente. Dieser Ansatz stammt aus Arbeiten von Meyer-Brandis und Proske, vgl. [4], [43], und

[44], und wird hier auf den unendlichdimensionalen Fall erweitert.

Das andere Hauptziel dieser Thesis ist die Untersuchung von Regularitätseigenschaften der Lösungen zu 'mean-field' stochastischen Differentialgleichungen mit irregulären Koeffizienten. Auf Grund des gewählten Ansatzes mittels des Malliavin Kalküls folgt die Malliavin Differenzierbarkeit der jeweiligen Lösungen sowohl im endlich- als auch im unendlichdimensionalen Fall direkt. Darüberhinaus untersuchen wir im endlichdimensionalen Fall die Abhängigkeit der Lösung bezüglich ihres Startwertes. Mit Hilfe von Approximationstechniken zeigen wir Sobolev Differenzierbarkeit im Startwert sowie Hölder Stetigkeit in Zeit und Startwert. Zuletzt leiten wir Sobolev Differenzierbarkeit für Erwartungswertfunktionale von Lösungen und eine Version einer Bismut-Elworthy-Li Formel her. Für bestimmte Arten von 'mean-field' stochastischen Differentialgleichungen ist es uns darüber hinaus möglich starke Differenzierbarkeit für eine große Klasse von Erwartungswertfunktionalen der betreffenden Lösungen zu zeigen.

Abstract

The dissertation discusses the analysis of mean-field stochastic differential equations, also called McKean-Vlasov equations. Additionally to time and space, mean-field stochastic differential equations depend on the law of the underlying process and thus expand the class of stochastic differential equations. Originating from the modeling of particle systems in mathematical physics, the interest in mean-field stochastic differential equations and the related field of mean-field games with its applications in Economics and Finance (see e.g. [17] and [38]) has experienced a strong increase in recent years.

In this thesis the first main objective is existence and uniqueness of (weak and strong) solutions to mean-field stochastic differential equations with irregular drift coefficients. More precisely, we consider drift coefficients that are merely measurable in the spatial variable as opposed to the frequently used assumption of (Lipschitz) continuity. Starting from a one-dimensional model with additive Brownian noise, we show existence of weak and strong solutions and derive assumptions under which the solutions are unique in law and pathwisely unique. Afterwards we proceed to multiple dimensions in order to prove equivalent results for multi-dimensional mean-field stochastic differential equations with additive Brownian noise. Lastly, we examine existence and uniqueness of solutions for mean-field stochastic differential equations in infinite-dimensional separable Hilbert spaces with cylindrical fractional Brownian motion as driving noise. In order to derive existence and uniqueness results for mean-field stochastic differential equations in the infinite-dimensional case, first the class of ordinary stochastic differential equations on an infinite-dimensional separable Hilbert space with cylindrical fractional Brownian motion as driving noise is considered. Here, one of the main tools to prove existence of strong solutions is a compactness criterion based on Malliavin calculus. This approach originates from works of Meyer-Brandis and Proske, cf. [4], [43], and [44], and is extended to the infinite-dimensional setup.

The other main objective of this thesis is to study regularity properties of solutions of mean-field stochastic differential equations with irregular drift. Due to the chosen approach via Malliavin calculus, Malliavin differentiability of the respective solutions is implied in the finite as well as in the infinite-dimensional case. Further,

we investigate in the finite-dimensional case the dependence of the solutions on their respective initial data. Using approximation techniques we establish Sobolev differentiability in the initial value as well as Hölder continuity in time and initial value. Lastly, we derive Sobolev differentiability for expectation functionals of the solution and a Bismut-Elworthy-Li type formula. For certain types of mean-field stochastic differential equations we are even able to gain strong differentiability for a broad class of expectation functionals of the respective solutions.

Acknowledgment

First and foremost, I would like to thank the Ludwig-Maximilians Universität München and in particular, Prof. Dr. Thilo Meyer-Brandis for giving me the opportunity to work on my doctoral thesis. In the past three years, your extensive support and advice helped me to evolve my mathematical proficiency as well as my personality. I have learned a lot from your expertise, especially, how to present content in a precise and clear but compact manner. Moreover, I would like to express my gratitude to Prof. Dr. Francesca Biagini who together with Prof. Dr. Thilo Meyer-Brandis strongly believed in me when I was a Master student and presented me the opportunity of staying at the LMU for my doctoral thesis. I am very grateful for my time at the department of mathematics and especially, would like to express my gratitude to my colleagues in the research group of Mathematical Finance. You all made my time at the department special and I will keep the best memories. Here, special thanks goes to my office mate and friend Andrea Mazzon. Everything from our chats about mathematical problems to our extensive discussions on football made me enjoy my days at the office even more. Grazie di tutto!

I tillegg sier jeg takk til Universitetet i Oslo og især Prof. Dr. Frank Proske som har gitt meg muligheten til å reise til Oslo og arbeide sammen med forskergruppen i stokastisk analyse. Tallrike diskusjoner har hjulpet for å forbedre doktoravhandlingen min og å fremkalle nye forsknings idéer. Dessuten vil jeg gjerne takke Ousamma Amine og David Baños for mange diskusjoner og gledelige samtaler ved kaffe.

Zum Ende hin möchte ich noch den wohl wichtigsten Menschen auf meinem bisherigen Weg danken, meiner Familie und meiner Freundin. All das, was ich bis hierhin erreicht habe, wäre ohne eure Unterstützung nicht möglich gewesen. Ihr habt mich in vielerlei Hinsicht zu dem gemacht, was ich heute bin. Selbiges gilt für all die Menschen, denen ich auf meinem Weg begegnet bin. Ihr alle habt mich in irgendeiner Weise geformt und ohne euch wäre das Leben bis hierhin wohl nicht einmal halb so schön gewesen. Danke!

Contents

Chapter 1 Introduction

Globalization, digital transformation, and big data are just a few examples of topics the modern world has to deal with. The amount of information publicly available on the internet tremendously increased in the past few decades. Nowadays three billion people use smartphones[1] which gives them the possibility to access all of this information with just a few clicks. But not only access to information is enabled due to modern technology, also connecting to other people all over the world has never been easier than today. Social networks like *Facebook* and *Instagram*, as well as communication applications like *WhatsApp* and *Skype* simplify instant connections between individuals located on opposite sides of the planet. These enormously huge networks boost movements like *Black Lives Matter* or *Fridays for Future* to become influential global campaigns that do have an effect on political, economical, and social changes. Due to the close-knit network the origination and development of such social movements is hard to trace and further progression is intricate to be estimated, although mathematical modeling of the behavior of individuals in large networks has been around for some time. The movement of particles in a gas, population growth of bacteria, or swarm behavior of animals are just a few fields of interest in scientific research in this area. However, the analysis of large networks is getting more complex and faces greater challenges with an increasing size. More precisely, analyzing the interaction between all possible pairs of participants in the network becomes a cumbersome task. Therefore, a macroscopic consideration of the network, where all of the one-to-one interactions are replaced by an average interaction, may simplify the examination. The superordinate theory of this ansatz is better known as *mean-field theory*.

This thesis deals with the study of mean-field stochastic differential equations and especially with the existence and uniqueness of solutions of equations with ir-

[1] `https://de.statista.com/themen/581/smartphones/`

regular drift coefficients. In particular, we consider additive Brownian noise whose regularizing effect enables us to consider inter alia merely measurable and bounded drift functions. Further we examine regularity properties of the respective solutions such as Malliavin and Sobolev differentiability as well as Hölder continuity which forms the second central objective. Subsequently, we give an introduction to the field of mean-field stochastic differential equations, provide an overview of the research done in the course of this dissertation, and expound the main ideas of the conducted mathematical analysis.

1.1 Mean-Field Stochastic Differential Equations

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space. A general d-dimensional mean-field stochastic differential equation, for short mean-field SDE, is given by

$$X_t = X_0 + \int_0^t b\left(s, X_s, \mathbb{P}_{X_s}\right) ds + \int_0^t \sigma\left(s, X_s, \mathbb{P}_{X_s}\right) dB_s, \; t \geq 0, \; X_0 \in \mathbb{R}^d, \qquad (1.1)$$

where $\mathbb{P}_{X_s}$ denotes the law of the stochastic process X_s at time $s \geq 0$ with respect to the probability measure $\mathbb{P}$ and $B = (B_t)_{t\geq 0}$ is n-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. The function $b : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ is called the drift and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^{d\times n}$ the volatility or diffusion coefficient of the mean-field SDE (1.1).

$\mathcal{P}(\mathbb{R}^d)$ denotes the space of probability measures over $\mathbb{R}^d$, whereas $\mathcal{P}_p(\mathbb{R}^d)$ is defined as the spaces of probability measures over $\mathbb{R}^d$ with finite p-th moment, $p \geq 1$. Further, we define the Kantorovich metric on the space $\mathcal{P}_1(\mathbb{R}^d)$ by

$$\mathcal{K}(\mu, \nu) := \sup \left\{ \int_{\mathcal{X}} f(x)(\mu - \nu)(dx) \middle| f \in \operatorname{Lip}_1(\mathcal{X}, \mathbb{R}) \right\}.$$

Generally speaking, mean-field SDEs, also referred to as McKean-Vlasov equations, are SDEs that in addition to time and state further depend on the law of the underlying process.

Historically, equation (1.1) originates in the kinetic theory of gases describing the gas as a large network of particles moving randomly in a medium. A central concept in kinetic theory is the propagation of chaos which characterizes the composition of the medium as the number of particles tends to infinity. More precisely, Boltzmann assumed the so-called “Stosszahlansatz” also known as molecular chaos which states that any two arbitrary particles in the medium are independent, cf. [27]. For its part propagation of chaos provides that this molecular chaos remains as the number of particles increases. This in turn yields the opportunity to relate the central equations of Boltzmann and Vlasov to many body systems. In his paper [35], Kac implements a *toy model* to study the Boltzmann equation, which describes the statistical distribution of particles in a medium, and especially discusses under which properties Boltzmann's assumption of molecular chaos is justified. Further, McKean's motivation in [41] is to explain the connection of Markov processes and certain nonlinear parabolic equations with Boltzmann's and Burger's equations as special cases. He introduces a class of Markov processes with nonconstant transition mechanism which can be pictured *"[...] as the motion of a tagged molecule in a bath of infinitely many like molecules."*([41], p. 1909) Moreover, he says that *"a very seductive conjecture is that chaos increases in the infinite gas. This means that if the initial ∞-molecule distribution is symmetrical but not chaotic [...], it becomes more nearly so as time passes."*([41], p. 1911) Vlasov examines in [54] the collection of charged particles, as for example in plasma or electron gasses. He finds that *"[...] for a system of charged particles the kinetic equation method which considers only binary interactions - interactions through collisions - is an approximation which is strictly speaking inadequate[.]"* ([54], p. 722)

A different approach from merely considering binary interactions is to consider dependency of each particle on the empirical measure of the state of all particles. In greater detail, this approach assumes the behavior of the particles to be given by a system of SDEs of the form

$$X_t^{i,N} = X_0^{i,N} + \int_0^t b\left(X_s^{i,N}, \frac{1}{N}\sum_{k=1}^N \delta_{X_s^{k,N}}\right) ds + \int_0^t \sigma\left(X_s^{i,N}, \frac{1}{N}\sum_{k=1}^N \delta_{X_s^{k,N}}\right) dB_s^i, \quad (1.2)$$

for $t \geq 0$ and $i = 1, \dots N$, where δ_x is the Dirac-measure in $x \in \mathbb{R}^d$.

In general the propagation of chaos result states that N-particle systems as (1.2) or systems with binary interaction like for example

$$Y_t^{i,N} = Y_0^{i,N} + \int_0^t \frac{1}{N}\sum_{k=1}^N b\left(Y_s^{i,N}, Y_s^{k,N}\right) ds + \int_0^t \frac{1}{N}\sum_{k=1}^N \sigma\left(Y_s^{i,N}, Y_s^{k,N}\right) dB_s^i, \quad (1.3)$$

for $t \geq 0$ and $i = 1, \dots N$, converge as the number of particles N tends to infinity to an equation of type (1.1). Hence, the name mean-field. In Example 1.1.1 below we demonstrate how a particle system (1.3) converges to a mean-field SDE (1.1). For more details on the topic of propagation of chaos and the characterisation of the asymptotic behaviour of large interacting particle systems, the reader is referred to [10], [19], [23], [24], [25], [31], [42], [46], [49], and the cited sources therein.

Example 1.1.1 Consider the N-particle system, $N \in \mathbb{N}$,

$$X_t^{i,N} = x_i + \int_0^t \frac{1}{N}\left(\sum_{j=1}^N a_j X_s^{j,N}\right) - X_s^{i,N} ds + B_t^i, \quad t \in [0,T], \quad (1.4)$$

for $i = 1, \dots N$, where $a_j \in \mathbb{R}$, for $j = 1, \dots N$, and $B = (B^1, \dots B^N)$ is N-dimensional Brownian motion. Furthermore, consider the mean-field SDE

$$Y_t = y + \int_0^t a\mathbb{E}[Y_s] - Y_s ds + W_t, \quad t \in [0,T], \tag{1.5}$$

where $a > 0$ and W is one-dimensional Brownian motion. At first we want to derive explicit solutions to the differential equations (1.4) and (1.5). Thus note that the equation system (1.4) can be written as a multidimensional SDE, namely

$$X_t^N = x + \int_0^t \left(\frac{1}{N}A - \mathrm{I}_N\right) X_s^N ds + B_t, \quad t \in [0,T], \tag{1.6}$$

where I_N is the $N \times N$-identity matrix, $x := (x_1, \dots x_N)^\top$, here $\cdot^\top$ denotes the vector transpose, $X^N := (X^{1,N}, \dots X^{N,N})^\top$, and

$$A = \begin{pmatrix} a_1 & \dots & a_N \\ \vdots & \ddots & \vdots \\ a_1 & \dots & a_N \end{pmatrix}.$$

This is an Ornstein-Uhlenbeck SDE with solution

$$X_t^N = e^{t\left(\frac{1}{N}A - \mathrm{I}_N\right)}x + \int_0^t e^{(t-s)\left(\frac{1}{N}A - \mathrm{I}_N\right)} dB_s, \quad t \in [0,T].$$

More precisely, the matrix exponential $e^{t\left(\frac{1}{N}A - \mathrm{I}_N\right)}$ is given by

$$e^{t\left(\frac{1}{N}A - \mathrm{I}_N\right)} = e^{-t}\left(\mathrm{I}_N + \frac{1}{\|A\|}\left(e^{\frac{t}{N}\|A\|} - 1\right)A\right),$$

where $\|A\| := \sum_{j=1}^N a_j$. For mean-field SDE (1.5) note first that

$$\mathbb{E}[Y_t] = \mathbb{E}\left[y + \int_0^t a\mathbb{E}[Y_s] - Y_s ds + W_t\right] = y + (a-1)\int_0^t \mathbb{E}[Y_s] ds.$$

Hence, $\mathbb{E}[Y_t^y] = ye^{t(a-1)}$ and therefore, mean-field SDE (1.5) reduces to an SDE of the form

$$dY_t = aye^{t(a-1)} - Y_t dt + dW_t, \quad Y_0 = y \in \mathbb{R}, \quad t \in [0,T]. \tag{1.7}$$

SDE (1.7) is again an Ornstein-Uhlenbeck process with solution

$$Y_t^y = ye^{t(a-1)} + \int_0^t e^{s-t} dW_s.$$

Furthermore, assume that

(i) $x_i = y$ for all $i = 1, \dots N$,

(ii) $\frac{1}{N}\|A\| = \frac{1}{N}\sum_{j=1}^{N} a_j \xrightarrow[N\to\infty]{} a$,

(iii) $\exists C > 0 : |a_j| \leq C, \quad \forall j \in \mathbb{N}$.

Observe using Itô's isometry and independence of the Brownian motions $\{B^j\}_{j\in\mathbb{N}}$ that under assumption (iii)

$$
\begin{aligned}
&\mathbb{E}\left[\left|\sum_{j=1}^{N}\int_0^t \frac{e^{-(t-s)}}{\|A\|}\left(e^{(t-s)\frac{\|A\|}{N}} - 1\right) a_j dB_s^j\right|^2\right] \\
&= \sum_{j=1}^{N}\frac{a_j^2}{\|A\|^2}\int_0^t e^{-2(t-s)}\left(e^{(t-s)\frac{\|A\|}{N}} - 1\right)^2 ds \\
&\leq \frac{NC^2}{\|A\|^2}\int_0^t e^{-2(t-s)}\left(e^{(t-s)\frac{\|A\|}{N}} - 1\right)^2 ds \xrightarrow[N\to\infty]{} 0,
\end{aligned}
$$

by dominated convergence and assumption (ii). Hence, we get for every $i \in \mathbb{N}$ and $t \in [0, T]$ that

$$
\begin{aligned}
X_t^{i,N} &= ye^{-t\left(1-\frac{\|A\|}{N}\right)} + \int_0^t e^{-(t-s)}dB_s^i + \sum_{j=1}^{N}\int_0^t \frac{e^{-(t-s)}}{\|A\|}\left(e^{(t-s)\frac{\|A\|}{N}} - 1\right) a_j dB_s^j \\
&\xrightarrow[N\to\infty]{d} ye^{-t(1-a)} + \int_0^t e^{-(t-s)}dW_s = Y_t,
\end{aligned}
$$

where d denotes convergence in distribution. Consequently, the initially non-chaotic particle system (1.6) converges as its size increases to an chaotic system in which all the particles are independent of each other and follow a mean-field SDE (1.5). In particular, we observe propagation of chaos.

Nowadays mean-field SDEs, particle systems, and propagation of chaos gain an increased amount of attention also in economic applications due to the pioneering work of Lasry and Lions [38] on mean-field games. More precisely, a mean-field game is an N-player stochastic differential game modeling the evolution of *rational* agents with limited information interacting in a very large network. *"Each player chooses his optimal strategy in view of the global (or macroscopic) information[] that [is] available to him and that result[s] from the actions of all players."* ([38], p. 1) In order to examine this kind of problem they consider the so-called mean-field game system consisting of two (stochastic) partial differential equations. *"[These models are derived] from a "continuum limit" (in other words letting the number of agents go to infinity) which is somehow reminiscent of the classical mean field* *approaches in Statistical Mechanics and Physics[.]"* ([38], p. 2) Carmona and Delarue transferred the analysis of mean-field games based on partial differential equations by Lasry and Lions to a probabilistic environment. For a more detailed insight into the field of mean-field games especially in the probabilistic setup, we refer here first and foremost to the extensive manuscripts [17] but also to [14], [15], [16], [18], and the cited sources therein.

1.2 Existence and Uniqueness of Solutions

The main objective of this dissertation is the study of mean-field SDEs which in the most general form are given by

$$X_t = x + \int_0^t b\left(s, X_s, \mathbb{P}_{X_s}\right) ds + \mathbb{B}_t, \quad t \in [0,T], \quad x \in \mathcal{H}, \tag{1.8}$$

where $\mathcal{H}$ is either $\mathbb{R}$, $\mathbb{R}^d$, or most generally a separable Hilbert space and $\mathbb{B} = (\mathbb{B}_t)_{t\in[0,T]}$ cylindrical fractional Brownian motion defined as

$$\mathbb{B}_t = \sum_{k\geq 1} \lambda_k B_t^{H_k} e_k. \tag{1.9}$$

Here, $\{e_k\}_{k\geq 1}$ is an orthogonal basis of $\mathcal{H}$, $\lambda = \{\lambda_k\}_{k\geq 1} \in \ell^2$, and $\{B^{H_k}\}_{k\geq 1}$ a sequence of fractional Brownian motions with Hurst parameters $\{H_k\}_{k\geq 1} \subset (0,1)$. Fractional Brownian motion, or for short fBm, is a continuous real-valued centered Gaussian process $(B_t^H)_{t\geq 0}$ starting in zero with covariance structure

$$\mathbb{E}\left[B_t^H B_s^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad s,t \geq 0,$$

where $H \in (0,1)$ is called the Hurst parameter. In general, for fractional Brownian motion one distinguishes between three cases regarding the Hurst parameter. The class $H \in (0, \frac{1}{2})$ is called singular and describes fractional Brownian motion with negatively correlated increments. Opposed to that the class $H \in (\frac{1}{2}, 1)$ is called regular and contains fractional Brownian motion with positively correlated increments. The class $H = \frac{1}{2}$ is the class of classical Brownian motion, in particular the increments of the process are independent. In this dissertation we consider in Chapters 2 to 4 the special case of classical Brownian motion, i.e. $\{H_k\}_{k\geq 1} \equiv \frac{1}{2}$, whereas in Chapters 5 and 6 we allow for a general class of Hurst parameters $\{H_k\}_{k\geq 1} \subset (0,1)$.

The first main objective is the study of existence and uniqueness of solutions to mean-field SDE (1.8) for irregular drift coefficients b. Prior to specifying the notion of irregular coefficients, we consider the notions of existence and uniqueness of a solution, which we recall in the following.

Definition 1.2.1 (Weak Solution) A six-tuple $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \mathbb{B}, X)$ is called *weak solution* of mean-field SDE (1.8), if X is a.s.[2] continuous, $\mathbb{F}$-adapted, satisfies $\mathbb{P}$-a.s. equation (1.8), and

$$\int_0^T \mathbb{E}\left[X_t^x\right] dt < \infty.$$

Definition 1.2.2 (Strong Solution) A *strong solution* of mean-field SDE (1.8) is a weak solution $(\Omega, \mathcal{F}, \mathbb{F}^{\mathbb{B}}, \mathbb{P}, \mathbb{B}, X)$ where $\mathbb{F}^{\mathbb{B}}$ is the filtration generated by the cylindrical fractional Brownian motion $\mathbb{B}$ and augmented with the $\mathbb{P}$-null sets.

Definition 1.2.3 (Uniqueness in Law) A weak solution $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \mathbb{B}, X)$ of mean-field SDE (1.8) is said to be *weakly unique* or *unique in law*, if for any other weak solution $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}, \widetilde{\mathbb{P}}, \widetilde{\mathbb{B}}, Y)$ of (1.8) with the same initial condition $X_0 = Y_0$, it holds that

$$\mathbb{P}_X = \widetilde{\mathbb{P}}_Y.$$

Here, $\mathbb{P}_Z$ denotes the law of a random process Z with respect to the probability measure $\mathbb{P}$.

Definition 1.2.4 (Pathwise Uniqueness) A weak solution $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \mathbb{B}, X)$ of mean-field SDE (1.8) is said to be *pathwisely unique*, if for any other weak solution Y with respect to the same stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \mathbb{B})$ with the same initial condition $X_0 = Y_0$, it holds that

$$\mathbb{P}\left(\forall t \geq 0 : X_t = Y_t\right) = 1.$$

In general we merely speak of X as a weak and a strong solution of mean-field SDE (1.8), respectively, if there is no ambiguity concerning the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \mathbb{B})$. Due to its definition every strong solution is in particular a weak solution. In contrast to this the relation between uniqueness in law and pathwise uniqueness is not that direct. However, Yamada and Watanabe have shown in their famous paper [55] that pathwise uniqueness implies uniqueness in law. One of the main objectives in this dissertation is to show the existence of a pathwisely unique strong solution of mean-field SDE (1.8) with irregular drift coefficient b.

Subsequently, we introduce and motivate the class of irregular drift coefficients considered in our analysis of mean-field SDE (1.8).

[2]almost surely

Irregular drift coefficients & stochastic regularization

Consider first an ordinary differential equation, for short ODE, of the form

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds, \quad t \geq t_0 \geq 0, \tag{1.10}$$

where $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$. We say that ODE (1.10) has a solution, if there exists an absolutely continuous function $y : \mathbb{R}_+ \to \mathbb{R}^d$ which satisfies equation (1.10). Well-known results regarding the existence and uniqueness of a solution to ODE (1.10) are due to Peano and Picard-Lindelöf as well as Carathéodory whose theorem is stated subsequently.

Theorem 1.2.5 (Carathéodory) *Let $\mathcal{I} \times \mathcal{R} \subset \mathbb{R}_+ \times \mathbb{R}^d$ such that $(t_0, y_0) \in \mathcal{I} \times \mathcal{R}$ and suppose $f : \mathcal{I} \times \mathcal{R} \to \mathbb{R}^d$ is measurable in $t \in \mathcal{I}$ for all $y \in \mathcal{R}$ and continuous in $y \in \mathcal{R}$ for all $t \in \mathcal{I}$. If there exists $C \in L^1(\mathcal{I}, \mathbb{R})$ such that*

$$\|f(t, y)\| \leq C(t) \left(1 + \|y\|\right), \quad (t, y) \in \mathcal{I} \times \mathcal{R},$$

then ODE (1.10) *has a solution.*

If in addition $f(t, \cdot) \in Lip_{C(t)}(\mathcal{R}, \mathbb{R}^d)$ for every $t \geq 0$, the solution is unique.[3]

[47, Theorem II.3.2 & Theorem II.3.5]

Looking at Theorem 1.2.5 one may ask the question, if the class of functions f can be extended, especially, if the regularity assumptions in the spatial variable can be weakened, and by any chance, drift coefficients that are merely of linear growth can be considered. The answer is at least partially yes but discontinuities are merely allowed on a set of measure zero. For more details on this topic we refer the reader to [29] and [33]. The addition of stochastic noise to ODE (1.10) enables the expansion to an even broader class of functions f. This technique is called regularization by noise and finds apart from ODEs also use in the theory of partial differential equations in connection with fluid dynamics. The term regularization by noise describes the phenomenon that ill-posed differential equations become well-posed due to the addition of some kind of noise. The notion of a well-posed problem due to Hadamard, cf. [32], itself can be seen as a lack of uniqueness. In this case, the regularizing effect of the added noise enables uniqueness of a solution whereas the ODE or PDE[4] may have several solutions. But moreover, the notion of an ill-posed problem can also be seen as the absence of a solution. Here, the

[3] Here, $\mathrm{Lip}_C(\mathcal{R}, \mathbb{R}^d)$ denotes the space of Lipschitz continuous functions from $\mathcal{R}$ to $\mathbb{R}^d$ with Lipschitz constant $C > 0$.

[4] partial differential equation

regularization effect can yield a solution at all. Lastly, a well-posed problem is required to admit a solution which is continuous in the initial condition. For more details on regularization by noise we refer the reader to [30].

Example 1.2.6 Consider the function $b : \mathbb{R} \to \mathbb{R}$, $b(x) := -\operatorname{sgn}(x)$, and the associated ordinary differential equation

$$X_t = x_0 + \int_0^t b(X_s)ds, \ t \in [0,T], \ x_0 \in \mathbb{R}. \tag{1.11}$$

Here, the function sgn is defined as $\operatorname{sgn}(x) = 1$, if $x \geq 0$, and $\operatorname{sgn}(x) = -1$, if $x < 0$. For the initial value $x_0 = 0$, ODE (1.11) has no solution. Contrarily, the SDE

$$Y_t = x_0 + \int_0^t b(Y_s)ds + \varepsilon B_t, \ t \in [0,T], \ x_0 \in \mathbb{R},$$

where $B = (B_t)_{t\in[0,T]}$ is standard Brownian motion, has a unique solution for all $x_0 \in \mathbb{R}$ and all $\varepsilon > 0$.

For SDEs several results exist concerning the existence of solutions to equations with irregular drift coefficients, in particular the pioneering works of Zvonkin [56] and Veretennikov [50], [51], [52] should be mentioned here. Zvonkin shows that in the case of a one-dimensional SDE

$$X_t = x + \int_0^t b(s, X_s)ds + B_t, \ t \in [0,T], \ x \in \mathbb{R},$$

where $B = (B_t)_{t\in[0,T]}$ is standard Brownian motion, it suffices to assume b to be a bounded measurable function in order to guarantee for the existence of a pathwisely unique strong solution. In his proof he uses a transformation to remove the drift which is nowadays denoted as Zvonkin transformation. Veretennikov later on extended the findings of Zvonkin to the multidimensional case and further noted that for drift coefficients which are of at most linear growth a pathwisely unique strong solution exists up to the time of explosion. In particular, both authors use the Yamada-Watanabe theorem to guarantee the existence of a strong solution.

Contrarily and more recently, Meyer-Brandis and Proske developed an approach to show the existence of a pathwisely unique strong solution without using the result of Yamada and Watanabe, see [3], [4], [43], and [44]. This concept is based on Malliavin calculus and uses an $L^2(\Omega)$ compactness criterion rested on a result by Da Prato, Malliavin, and Nualart [22]. Due to the virtue of the approach, Malliavin differentiability of the solution is gained as a by-product alongside the existence of a pathwisely unique strong solution. In the multidimensional case the authors assume the drift function to be merely measurable and bounded whereas in the one-dimensional case they even consider the drift coefficient to be dismountable in a mere measurable and bounded plus some regular but of linear growth part.

In the course of this dissertation the aim is to extend this kind of results, especially the work done by Meyer-Brandis and Proske, to the class of mean-field SDEs (1.8) with irregular drift coefficients and consequently, expand the theory on differential equations with irregular coefficients.

Literature on Existence and Uniqueness

Existence of a (unique) solution for various types of mean-field SDEs is discussed by several authors, see for example [11], [12], [13], [20], [26], [34], and [40]. In particular, we want to emphasize the works of Li and Min [39] as well as Mishura and Veretennikov [45]. In the first part of their paper [39], Li and Min discuss the existence of a weak solution and uniqueness in law. More precisely, they consider the mean-field SDE

$$X_t = X_0 + \int_0^t b\left(s, X_{s\wedge\cdot}, \mathbb{P}_{X_s}\right) ds + \int_0^t \sigma\left(s, X_{s\wedge\cdot}\right) dB_s, \quad t \in [0,T], \tag{1.12}$$

i.e. they look at path dependent mean-field SDEs where the volatility is not dependent on the law. They find that there exists a weak solution, if

(i) b is bounded and measurable,

(ii) σ is measurable, bounded, Lipschitz continuous in the spatial variable, and for every $t \in [0,T]$ and $\varphi \in \mathcal{C}([0,T];\mathbb{R}^d)$ [5] the matrix $\sigma(t,\varphi)$ is invertible such that the inverse matrix $\sigma^{-1}(t,\varphi)$ is bounded in (t,φ), and

(iii) there exists a continuous increasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{x\to 0} \rho(x) = 0$, such that for all $t \in [0,T]$, $\varphi \in \mathcal{C}([0,T];\mathbb{R}^d)$, and $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$,

$$\|b(t, \varphi_{t\wedge\cdot}, \mu) - b(t, \varphi_{t\wedge\cdot}, \nu)\| \leq \rho(\mathcal{K}(\mu,\nu)).$$

Under the additional assumption that there exists a continuous and increasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$, whith $\rho(x) > 0$, for all $x > 0$, and $\int_{0+} \frac{dx}{\rho(x)} = \infty$, such that for all $t \in [0,T]$, $\varphi \in \mathcal{C}([0,T];\mathbb{R}^d)$, and $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$,

$$\|b(t, \varphi_{t\wedge\cdot}, \mu) - b(t, \varphi_{t\wedge\cdot}, \nu)\|^2 \leq \rho\left(\mathcal{K}(\mu,\nu)^2\right),$$

they prove that the weak solution is unique in law. Li and Min show the existence of a weak solution by using a Girsanov type measure change where they

[5] $\mathcal{C}(\mathcal{X};\mathcal{Y})$ denotes the space of continuous functions mapping from $\mathcal{X}$ to $\mathcal{Y}$

consider first equation (1.12) with the law $\mathbb{P}_{X_s}$ replaced by a dummy variable $\mu \in \mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}^d))$ and afterwards apply Schauder's fixed point theorem, cf. [48]. Uniqueness in law is proven by showing the equality of certain Girsanov type measure changes and stochastic exponentials, respectively.

Mishura and Veretennikov consider in their paper [45] the analysis of the general mean-field SDE (1.1) but also of the special case

$$X_t = X_0 + \int_0^t \int_{\mathbb{R}^d} b(s, X_s, y)\, \mathbb{P}_{X_s}(dy) ds + \int_0^t \int_{\mathbb{R}^d} \sigma(s, X_s, y)\, \mathbb{P}_{X_s}(dy) dB_s, \quad t \geq 0. \tag{1.13}$$

In order to show weak existence they use a variation of Krylov's approach for SDEs, in particular an approximation by smooth functions, cf. [36], [37], and [53]. They assume that the drift function b and the volatility σ are of at most linear growth and σ fulfills some nondegeneracy condition. Furthermore, in the general case they additionally assume that b and σ are continuous in the law variable with respect to the topology of weak convergence. For σ being non-dependent on the law as well as Lipschitz continuous in the spatial variable and b being Lipschitz continuous in the law variable in the total variation norm, they show the existence of a pathwisely unique strong solution. At this point it should be noted that the work [45, Version 6] on the general mean-field SDE (1.1) was uploaded on arXiv after our preprint [8] on existence and uniqueness of strong solutions for mean-field SDEs of type (1.8) with irregular drift coefficients.

Our approach

In the course of this dissertation one main objective is to show the existence of a pathwisely unqiue strong solution of mean-field SDE (1.8) with irregular drift coefficient b. In the following we shortly outline the general underlying principles of our approach to existence and uniqueness of strong solutions, cf. Chapters 2, 3 and 6. Details on the assumptions on the drift coefficient b and more chapter specific results can be found in Section 1.4.

In a first step existence of a weakly unique weak solution is established. More precisely, the applied approach to show existence of a weak solution to mean-field equation (1.8) orientates itself mostly towards the idea of Li and Min in [39]. At first an SDE of the form

$$Z_t^\mu = x + \int_0^t b(s, Z_s^\mu, \mu_s)\, ds + \mathbb{B}_t, \quad t \in [0,T], \tag{1.14}$$

is considered, where $\mu \in \mathcal{C}([0,T];\mathcal{P}_1(\mathcal{H}))$. The existence of a weak solution to SDE (1.14) is shown by means of Girsanov's theorem. Afterwards existence of a fixed point of the mapping $\mu \mapsto \mathbb{P}_{Z^\mu}$ is shown using Schauder's fixed point theorem. Combined this yields the existence of a weak solution to mean-field SDE (1.8). Moreover, using once more the ideas of [39] uniqueness in law is proven by showing that certain Girsanov measure changes coincide. In particular, the fact is used that due to the direct approach in the proof of existence of a weak solution there exists a Girsanov measure change such that every weak solution can be transformed to a Brownian motion and vice versa.

Having established the existence of a weak solution the existence of a pathwisely unique strong solution can then be reduced to existence results for SDEs. Indeed, consider the SDE

$$Y_t = x + \int_0^t b^{\mathbb{P}_X}(s, Y_s)ds + \mathbb{B}_t, \quad t \in [0, T], \tag{1.15}$$

where $b^{\mathbb{P}_X}(t, y) := b(t, y, \mathbb{P}_{X_t})$ and X is a weak solution of mean-field SDE (1.8). At first it should be noted that X is as a weak solution of mean-field SDE (1.8) also a weak solution of SDE (1.15). Recall that a strong solution of mean-field SDE (1.8) and SDE (1.15) is a weak solution of mean-field SDE (1.8) and SDE (1.15), respectively, which is adapted to the filtration generated by the driving noise $\mathbb{B}$. Thus, if SDE (1.15) possesses a pathwisely unique strong solution Y, the solution X of mean-field equation (1.8) coincides with Y and therefore, mean-field SDE (1.8) exhibits a strong solution. Under the assumption that X is unique in law, i.e. the law process $\mathbb{P}_X$ is unique, also the associated SDE (1.15) is unique and thus pathwise uniqueness of the solution Y of SDE (1.15) yields pathwise uniqueness of the solution X of mean-field SDE (1.8). As mentioned in the section above on *Irregular coefficients*, the literature on SDEs with irregular drift coefficients is quite broad. In the course of this thesis we use the approach and the results by Meyer-Brandis and Proske et al., cf. [3], [4], [43], and [44], in order to conclude the existence of a pathwisely unique strong solution of the associated SDE (1.15) and thus, in particular of mean-field SDE (1.8). The approach of Meyer-Brandis and Proske yields the advantage to additionally conclude Malliavin differentiability of the strong solution of mean-field SDE (1.8), cf. Section 1.3 for more details. However, in the case of a general separable Hilbert space no adequate results exist on the existence of a pathwisely unique strong solution of SDE (1.15) and in the first instance have to be established, cf. Chapter 5. Here, the approach of Meyer-Brandis and Proske using Malliavin calculus and an $L^2(\Omega)$ compactness argument is adapted and extended to infinite dimensions.

1.3 Regularity Properties

The second main objective of this thesis is to examine regularity properties of the solutions, in particular we establish Malliavin differentiability as well as Sobolev differentiability in the initial data, and Hölder continuity in time and space of the solutions.

Malliavin Calculus

Malliavin calculus, also denoted as stochastic calculus of variations, is an infinite-dimensional differential calculus on the Wiener space. It is applicable to examine regularity properties of stochastic processes, or rather of functionals of Wiener processes, and in particular of SDEs. Malliavin differentiability of solutions to mean-field SDEs is a direct consequence of results on Malliavin calculus in the field of SDEs. More specifically, assume that mean-field SDE (1.1) has a strong solution X. Since the probability law $\mathbb{P}_X$ is deterministic, it does not have an effect on the Malliavin derivative. Thus, consider the SDE

$$Y_t = X_0 + \int_0^t b^{\mathbb{P}_X}(s, Y_s) ds + \int_0^t \sigma^{\mathbb{P}_X}(s, Y_s) dB_s, \quad t \in [0,T], \tag{1.16}$$

where $b^{\mathbb{P}_X}(s,y) := b(s,y,\mathbb{P}_{X_s})$ and $\sigma^{\mathbb{P}_X}(s,y) := \sigma(s,y,\mathbb{P}_{X_s})$. Deducing that any strong solution of equation (1.16) is Malliavin differentiable, yields that X is as a solution to SDE (1.16) Malliavin differentiable. In this sense the stochastic calculus of variation analysis of a mean-field SDE breaks down to the analysis of an ordinary SDE.

Malliavin calculus of mean-field SDEs has been approached by [1] and [21]. In [1], Baños proves Malliavin differentiability of the unique strong solution of mean-field SDE

$$X_t = X_0 + \int_0^t b\left(s, X_s, \mathbb{E}[\varphi(X_s)]\right) ds + \int_0^t \sigma\left(s, X_s, \mathbb{E}[\psi(X_s)]\right) B_t, \quad t \in [0,T], \tag{1.17}$$

for b, σ, φ, and ψ being sufficiently regular. On the other hand, Crisan and McMurray consider in [21] the general mean-field SDE (1.1) for sufficiently smooth b and σ. Both make use of the aforementioned relation of the Malliavin calculus on SDEs and mean-field SDEs.

In the course of the present dissertation we establish Malliavin differentiability of the mean-field SDE (1.8) for the domains $\mathbb{R}$, $\mathbb{R}^d$, and an infinite dimensional separable Hilbert space $\mathcal{H}$ by making the same assumptions as used for the derivation of a unique strong solution.

Regularity in the initial data

In Chapters 2 to 4 we consider the solution of mean-field SDE (1.8) as a function in its initial data, i.e. we look at the function

$$x \mapsto X_t^x, \quad t \in [0,T],$$

where $(X_t^x)_{t\in[0,T]}$ is a solution of mean-field SDE (1.8) with initial value $x \in \mathbb{R}^d$. We are interested in the analysis of the derivative $(\partial_x X_t^x)_{t\in[0,T]}$[6] which is also denoted as the first variation process of X^x. One useful application of the first variation process is found in the Bismut-Elworthy-Li formula. Here, the point of interest is the expectation functional $\mathbb{E}[\Phi(X_T^x)]$ for some functional $\Phi : \mathbb{R}^d \to \mathbb{R}$ which is analyzed in the differentiability with respect to the initial value x. The aim of the Bismut-Elworthy-Li formula is to find a function $\theta = (\theta_t)_{t\in[0,T]}$ such that

$$\partial_x \mathbb{E}\left[\Phi(X_T^x)\right] = \mathbb{E}\left[\Phi(X_T^x) \int_0^T \theta_t dB_t\right]. \tag{1.18}$$

The stochastic integral $\int_0^T \theta_t dB_t$ is called the Malliavin weight. In the derivation of the weight θ the Malliavin differentiability of the strong solution X is crucial, which is a consequence of our approach to establish the existence of a strong solution to mean-field SDE (1.8).

The formula (1.18) itself finds application inter alia in the field of Mathematical Finance. There, the process X^x usually is the solution of an SDE describing the dynamics of an asset X^x and the expectation functional $\mathbb{E}\left[\Phi(X_T^x)\right]$ depicts the risk-neutral price of a derivative Φ on the underlying asset X^x with maturity $T > 0$. The derivative $\partial_x \mathbb{E}\left[\Phi(X_T^x)\right]$ is better known as the *Delta* of the derivative $\Phi(X_T)$ and is used in the course of the famous *Delta hedging* of the derivative. The term *hedging* here denotes an investment position which is intended to compensate potential losses and gains that may occur due to an investment associated with the derivative $\Phi(X_T)$.

The analysis of the first variation process $\partial_x X^x$ is considered for example in the works [1], [13], and [21]. In the manuscript [1] the author shows differentiability of the solution to mean-field SDE (1.17), where b, σ, φ and ψ are assumed to be sufficiently regular. Further, a Bismut-Elworthy-Li formula (1.18) is shown for Φ merely fulfilling some integrability condition. The papers [13] and [21] consider a different approach using a flow property and the notion of Lions derivative. The Lions derivative is a derivative with respect to a measure and thus, gives the opportunity to generalize the analysis of the first variation process. For a definition and further details on Lions derivatives we refer the reader to [14] and [38]. Both papers [13] and [21] give results on the differentiability of the solution to a mean-field SDE (1.1) and a Bismut-Elworthy-Li type formula for quite regular coefficients b and σ.

In our works we consider the unique strong solution X^x of mean-field SDE (1.8) and analyze it as a function in x. Using an approach employing an approximation by smooth functions, we show that X^x is weakly (Sobolev) differentiable in the initial condition. In particular, we do not make use of the Lions derivative and

[6]Here, ∂_x denotes the derivative with respect to the variable $x \in \mathbb{R}^d$.

instead consider the functional $x \mapsto b(t, y, \mathbb{P}_{X_t^x})$ as a function from $\mathbb{R}^d$ to $\mathbb{R}^d$. Showing that this function is Lipschitz continuous in x implies almost everywhere and weak differentiability of the process X^x. We find a direct relation between the first variation process $\partial_x X^x$ and the Malliavin derivative DX, namely

$$\partial_x X_t^x = D_s X_t^x \partial_x X_s^x + \int_s^t D_u X_t^x \partial_x b\left(u, y, \mathbb{P}_{X_u^x}\right)|_{y=X_u^x} du. \tag{1.19}$$

Using the latter we derive a Bismut-Elworthy-Li formula and in particular, prove weak (Sobolev) differentiability of the expectation functional $\mathbb{E}[\Phi(X_T^x)]$ for some Φ merely satisfying some integrability condition. The Bismut-Elworthy-Li formula established is given by

$$\partial_x \mathbb{E}[\Phi(X_T^x)] = \mathbb{E}\left[\Phi(X_T^x) \int_0^T \left(a(s)\partial_x X_s^x + \partial_x b\left(s, y, \mathbb{P}_{X_s^x}\right)|_{y=X_s^x} \int_0^s a(u)du\right) dW_s\right], \tag{1.20}$$

where $a : \mathbb{R} \to \mathbb{R}$ is any bounded, measurable function such that $\int_0^T a(s)ds = 1$.

Throughout the analysis of the mapping $x \mapsto X^x$, we assume that the drift coefficient b satisfies the assumptions made for the existence of a strong solution and additionally that it is Lipschitz continuous in the law variable. In particular, we allow for irregular drift coefficients. For more details on the assumptions on the drift function b, we refer the reader to Section 1.4.

Example 1.3.1 From Example 1.1.1 we have that the solution of mean-field SDE (1.5) explicitly given by

$$Y_t^y = ye^{(a-1)t} + \int_0^t e^{s-t} dW_s,$$

has the first variation process

$$\partial_y Y_t^y = e^{(a-1)t}.$$

From relation (1.19) we get equivalently that with $s = 0$

$$\begin{aligned}\partial_y Y_t^y &= e^{-t} + \int_0^t e^{-(t-u)} \partial_y \left(aye^{u(a-1)} - z\right)|_{z=Y_u^y} du \\ &= e^{-t} + \int_0^t ae^{ua-t} du = e^{(a-1)t}.\end{aligned}$$

Considering the functional $\Phi(x) := x$ we get via direct calculations that

$$\partial_y \mathbb{E}\left[\Phi\left(Y_T^y\right)\right] = e^{(a-1)T}. \tag{1.21}$$

Using the Bismut-Elworthy-Li formula (1.20) we get using the weight function $a(s) \equiv \frac{1}{T}$ that

$$\begin{aligned}\partial_y \mathbb{E}\left[\Phi\left(Y_T^y\right)\right] &= \mathbb{E}\left[Y_T^y \int_0^T \frac{1}{T} e^{(a-1)s} + a e^{(a-1)s} \frac{s}{T} dW_s\right] \\ &= \frac{1}{T}\int_0^T e^{s-T}\left(e^{(a-1)s} + ase^{(a-1)s}\right) ds = e^{(a-1)T},\end{aligned}$$

which is exactly the same as in (1.21).

Hölder continuity

As a further result of the analysis in the initial condition and due to Kolmogorov's continuity theorem, Hölder continuity in time and initial condition of the solution can be established. More precisely, for the unique strong solution $(X_t^x)_{t\in[0,T]}$ of mean-field SDE (1.8) it can be shown that

$$\mathbb{E}\left[\|X_t^x - X_s^y\|^2\right] \leq C\left(|t-s| + \|x-y\|^2\right),$$

where $\|\cdot\|$ denotes the euclidean norm on $\mathbb{R}^d$, $d \geq 1$, $C > 0$ is some constant independent of $s,t \in [0,T]$ and $x,y \in \mathbb{R}^d$. This in turn yields that the random field $(t,x) \mapsto X_t^x$ has a version with Hölder continuous trajectories of order $\alpha < \frac{1}{2}$ in $t \in [0,T]$ and $\alpha < 1$ in $x \in \mathbb{R}^d$. Here, the central point in the proof is the approximational approach used in the calculus of variations which yields the boundedness of the first variation process $\partial_x X^x$ in the $L^p(\Omega)$ norm.

1.4 Structure of the thesis

Closing the introduction we give a brief outline of the papers constituting the further chapters of the dissertation. In the course of this overview we present the central research issues of the several chapters and provide a glimpse to the main theorems of each paper.

Chapter 2: Strong Solutions of Mean-Field Stochastic Differential Equations with irregular drift

The manuscript *Strong Solutions of Mean-Field Stochastic Differential Equations with irregular drift* published in the *Electronic Journal of Probability* [9] treats the analysis of the one-dimensional mean-field SDE

$$X_t^x = x + \int_0^t b\left(s, X_s^x, \mathbb{P}_{X_s^x}\right) ds + B_t,\ t \in [0,T],\ x \in \mathbb{R}, \tag{1.22}$$

where $B = (B_t)_{t\in[0,T]}$ is standard Brownian motion. In a first step it is shown that for a drift coefficient b that is of at most linear growth, i.e. there exists a constant $C > 0$ such that

$$|b(t,y,\mu)| \leq C\left(1 + |y| + \mathcal{K}(\mu,\delta_0)\right), \quad y \in \mathbb{R}, \quad t \in [0,T], \quad \mu \in \mathcal{P}_1(\mathbb{R}), \tag{1.23}$$

and continuous in the law variable, i.e. for all $\mu \in \mathcal{P}_1(\mathbb{R})$ and all $\varepsilon > 0$ exists a $\delta > 0$ such that

$$(\forall \nu \in \mathcal{P}_1(\mathbb{R}) : \mathcal{K}(\mu,\nu) < \delta) \Rightarrow |b(t,y,\mu) - b(t,y,\nu)| < \varepsilon, \tag{1.24}$$

for all $t \in [0,T]$ and $y \in \mathbb{R}$, there exists a weak solution of mean-field SDE (1.22). Further, under the additional assumption that b admits a modulus of continuity in the law variable, i.e. there exists a continuous function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$, with $\theta(y) > 0$ for all $y \in \mathbb{R}$, $\int_0^z \frac{dy}{\theta(y)} = \infty$ for all $z \in \mathbb{R}_+$, and for all $t \in [0,T]$, $y \in \mathbb{R}$, and $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$,

$$|b(t,y,\mu) - b(t,y,\nu)|^2 \leq \theta\left(\mathcal{K}(\mu,\nu)^2\right), \tag{1.25}$$

it is proven that the weak solution is unique in law. Applying the results of [4] on SDEs, more precisely under the assumption that the drift function b admits a modulus of continuity in the law variable and allows for a decomposition

$$b(t,y,\mu) := \widehat{b}(t,y,\mu) + \widetilde{b}(t,y,\mu), \tag{1.26}$$

where $\widehat{b}$ is merely measurable and bounded and $\widetilde{b}$ is of at most linear growth and Lipschitz continuous in the spatial variable, we show that mean-field SDE (1.22) has a Malliavin differentiable pathwisely unique strong solution. Note here that existence of a strong solution can be established merely under the assumption that b is measurable, of at most linear growth (1.23), and continuous in the law variable (1.24), but in order to guarantee for Malliavin differentiability we require that b allows for a decomposition (1.26). In addition, we are able to establish an explicit representation of the Malliavin derivative using integration with respect to local time. Namely, the Malliavin derivative of the unique strong solution X^x of mean-field SDE (1.22) is given by

$$D_s X_t^x = \exp\left\{-\int_s^t \int_{\mathbb{R}} b\left(u,y,\mathbb{P}_{X_u^x}\right) L^X(du,dy)\right\}, \quad 0 \leq s \leq t \leq T,$$

where L^X is the local time of the stochastic process X^x. For more details on integration with respect to local time we refer the reader to [4] and [28].

Moreover, assuming that in addition b is Lipschitz continuous in the law variable, it is shown that the function $x \mapsto X^x$ is weakly (Sobolev) differentiable and the first variation process $(\partial_x X_t^x)_{t\in[0,T]}$ has the explicit representation

$$\partial_x X_t^x = \exp\left\{ -\int_0^t \int_{\mathbb{R}} b\left(u, y, \mathbb{P}_{X_u^x}\right) L^{X^x}(du, dy) \right\} \\ + \int_0^t \exp\left\{ -\int_s^t \int_{\mathbb{R}} b\left(u, y, \mathbb{P}_{X_u^x}\right) L^{X^x}(du, dy) \right\} \partial_x b\left(s, y, \mathbb{P}_{X_s^x}\right) |_{y=X_s^x} ds.$$

Furthermore, under the same assumptions as for the proof of Sobolev differentiability, Hölder continuity in time and the initial condition is established. Lastly, the Bismut-Elworthy-Li formula (1.20) for the derivative $\partial_x \mathbb{E}[\Phi(X_T^x)]$ is derived, where Φ merely fulfills some integrability condition.

Chapter 3: Existence and Regularity of Solutions to Multi-Dimensional Mean-Field Stochastic Differential Equations with Irregular Drift

In the paper *Existence and Regularity of Solutions to Multi-Dimensional Mean-Field Stochastic Differential Equations with Irregular Drift* [5] mean-field SDE (1.22) is examined in the d-dimensional case. The existence of a strong solution is shown for drift coefficients b of at most linear growth (1.23) that are continuous in the law variable (1.24) using a result of Veretennikov for SDEs [50]. Further, it is derived that if the drift function b is additionally bounded and admits a modulus of continuity in the law variable (1.25), then the strong solution is Malliavin differentiable and pathwisely unique. Similar to the one-dimensional case weak (Sobolev) differentiability and Hölder continuity of the strong solution X^x are gained by the assumption that b is additionaly Lipschitz continuous in the law variable. Concluding analogously to [9], a Bismut-Elworthy-Li formula for $\partial_x \mathbb{E}[\Phi(X_T^x)]$ is deduced, where Φ merely fulfills some integrability condition.

Chapter 4: Strong Solutions of Mean-Field SDEs with irregular expectation functional in the drift

The special case of a mean-field SDE

$$X_t = X_0 + \int_0^t b\left(s, X_s, \int_{\mathbb{R}} \varphi(s, X_s, z) \mathbb{P}_{X_s}(dz)\right) ds + B_t, \quad t \in [0, T], \tag{1.27}$$

where $\varphi : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, is the main objective in the paper *Strong Solutions of Mean-Field SDEs with irregular expectation functional in the drift* [7]. This kind of mean-field SDE combines two frequently used versions of the general mean-field SDE (1.1). Namely, equation (1.13) and (1.17) with $\sigma = I_d$, respectively. In a first step, mean-field SDE (1.27) is linked to mean-field SDE (1.22) and first results regarding the existence of solutions and regularity properties are obtained from [5]. However, interesting cases as for example $\varphi(s, y, z) = \mathbb{1}_{\{z \leq u\}}$, for some arbitrary $u \in \mathbb{R}^d$, are not covered by the results obtained from [5] and thus, have to be shown. For the functional φ being merely measurable and of at most linear growth (1.23), the existence of a strong solution is proven under the assumption that the drift function b is measurable, of at most linear growth (1.23), and continuous in the law variable (1.24). Whereas uniqueness is gained under the additional assumption that the drift b is Lipschitz continuous in the law variable. Opposed to the general approach illustrated in Section 1.2 in order to show the existence of a solution, here an approximational ansatz is considered. More precisely, a sequence $\{Y^n\}_{n\in\mathbb{N}}$ of solutions to mean-field SDE (1.27) with sufficiently regular coefficients is introduced and it is shown that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a sequence $\{X^k\}_{k\in\mathbb{N}}$ on this space which is equivalent to $\{Y^n\}_{n\in\mathbb{N}}$ and converges in $L^2(\Omega)$ to some stochastic process X. Further, it is proven that this sequence $\{X^k\}_{k\in\mathbb{N}}$ converges also weakly in $L^2(\Omega)$ to a solution of mean-field SDE (1.27) and thus by uniqueness of the limit, the process X is a solution of mean-field SDE (1.27). Using again the connection of mean-field SDEs to general SDEs yields subsequently the existence of a Malliavin differentiable pathwisely unique strong solution under the assumption that the drift coefficient b is bounded and Lipschitz continuous in the law variable and φ is merely measurable and of at most linear growth.

Additionally the results on existence of a (unique) strong solution are extended to mean-field SDEs of type

$$X_t^x = x + \int_0^t b\left(X_s^x, \int_{\mathbb{R}^d} \varphi(X_s^x, z)\mathbb{P}_{X_s^x}(dz)\right) ds + \int_0^t \sigma(X_s^x)dB_s,\ t \in [0,T],\ x \in \mathbb{R}^d,$$

by means of Itô's formula. Concluding the section on the existence of solutions, a connection to ODEs is pointed out through the mean-field SDE

$$X_t^x = x + \int_0^t b(s, \mathbb{E}[X_s^x])ds + B_t,\ t \in [0,T],\ x \in \mathbb{R}^d,$$

which provides a probabilistic access to Carathéodory's existence theorem for ODEs, cf. Theorem 1.2.5. More precisely, it is obtained that $\mathbb{E}[X_t^x]$ solves the ODE

$$u(t) = u(0) + \int_0^t b(s, u(s))ds,\ t \in [0,T],\ u(0) = x \in \mathbb{R}^d.$$

Again, the strong solution is examined as a function in the initial value. In a first step it is shown that for sufficiently regular coefficients, i.e. for continuously differentiable functions b and φ, the map $x \mapsto X_t^x$ is continuously differentiable, or in other words strongly differentiable. This in turn enables to use an approximational approach in order to show in the one-dimensional case, $d = 1$, that the expectation functional $x \mapsto \mathbb{E}[\Phi(X_T^x)]$ is continuously differentiable for Φ merely satisfying some integrability condition. Here, we merely assume that

$$(b \diamond \varphi)(t, y, \mu) := b\left(s, y, \int_{\mathbb{R}} \varphi(s, y, z)\mu(dz)\right)$$

admits a decomposition

$$(b \diamond \varphi)(t, y, \mu) := \widehat{b}\left(t, y, \int_{\mathbb{R}} \widehat{\varphi}(t, y, z)\mu(dz)\right) + \widetilde{b}\left(t, y, \int_{\mathbb{R}} \widetilde{\varphi}(t, y, z)\mu(dz)\right),$$

where the drift $\widehat{b}$ is merely measurable and bounded and the functional $\widehat{\varphi}$ is merely measurable and of linear growth whereas $\widetilde{b}$ and $\widetilde{\varphi}$ are of linear growth and Lipschitz continuous in the spatial variable, and b as well as φ are continuously differentiable in the law variable, respectively.

Chapter 5: Restoration of Well-Posedness of Infinite-dimensional Singular ODE's via Noise

The main objective in *Restoration of Well-Posedness of Infinite-dimensional Singular ODE's via Noise* [2] is the analysis of the infinite-dimensional SDE

$$X_t = x + \int_0^t b(s, X_s)ds + \mathbb{B}_t, \; t \in [0, T], \; x \in \mathcal{H}, \tag{1.28}$$

where $b : [0, T] \times \mathcal{H} \to \mathcal{H}$ and $\mathcal{H}$ is an infinite-dimensional separable Hilbert space. Here, the driving noise $(\mathbb{B}_t)_{t\in[0,T]}$ is a cylindrical fractional Brownian motion as defined in (1.9). More precisely, the sequence of fractional Brownian motions $\{B^{H_k}\}_{k\in\mathbb{N}}$ is affiliated to the sequence of Hurst parameters $\{H_k\}_{k\geq 1} \subset (0, 1/12)$ with

$$\sum_{k\geq 1} H_k < \frac{1}{6}.$$

Furthermore, the technical assumption $\frac{\lambda}{\sqrt{H}} := \{\frac{\lambda_k}{\sqrt{H_k}}\}_{k\geq 1} \in \ell^1$ is made such that the stochastic process $\mathbb{B}$ has almost surely continuous sample paths on $[0, T]$.

The aim is to generalize the results by Zvonkin [56] and Veretennikov [50] to the infinite-dimensional setting. We use a similar approach as in the papers [4] and [43] in the application of an $L^2(\Omega)$ compactness argument based on Malliavin calculus. In detail, a double sequence of SDEs

$$X_t^{d,\varepsilon} = x + \int_0^t b^{d,\varepsilon}(s, X_s^{d,\varepsilon})ds + \mathbb{B}_t, \; t \in [0, T], \; x \in \mathcal{H}, \tag{1.29}$$

is defined, where $b^{d,\varepsilon} : [0,T] \times \mathcal{H} \to \pi_d(\mathcal{H})$ is a smooth approximation of the truncated drift function $\pi_d b$. Here, the map π_d is the projection on the subspace spanned by the first $d \geq 1$ basis vectors $\{e_k\}_{1\leq k\leq d}$ of $\mathcal{H}$. In particular, the natural isometry between $\pi_d(\mathcal{H})$ and $\mathbb{R}^d$ is used in order to use mollification to approximate the function $\pi_d b$ by a sequence of smooth functions $\{b^{d,\varepsilon}\}_{\varepsilon>0}$. In the first part of the paper it is shown that SDE (1.29) has a Malliavin differentiable unique strong solution for sufficiently regular drift functions $b^{d,\varepsilon}$ for every $d \geq 1$ and $\varepsilon > 0$. More precisely, it is assumed that $b^{d,\varepsilon}$ is a measurable function such that there exist sequences $C \in \ell^1$, $D \in \ell^1$, and $L \in \ell^2$ with

$$\begin{aligned} \sup_{y\in\mathcal{H}} \sup_{t\in[0,T]} |b_k(t,y)| &\leq C_k \lambda_k, \\ \sup_{d\geq 1} \int_{\mathbb{R}^d} \sup_{t\in[0,T]} \left| b_k\left(t, \sqrt{Q}\sqrt{\mathcal{K}}\tau^{-1} z\right)\right| dz &\leq D_k \lambda_k, \text{ and} \\ b_k(t,\cdot) &\in \mathrm{Lip}_{L_k}(\mathcal{H};\mathbb{R}), \end{aligned} \tag{1.30}$$

for every $k \geq 1$ and $t \in [0,T]$. Here, b_k, $k \geq 1$, is the projection of the drift function b on the subspace spanned by the k-th basis vector of $\mathcal{H}$, $\tau : \mathcal{H} \to \mathbb{R}^\infty$ is a change of basis operator, and for $y \in \mathcal{H}$ the operator $\sqrt{Q}\sqrt{\mathcal{K}} : \mathcal{H} \to \mathcal{H}$ is defined by

$$\sqrt{Q}\sqrt{\mathcal{K}} y := \sum_{k\geq 1} \lambda_k \sqrt{\mathfrak{K}_{H_k}} \langle y, e_k \rangle_{\mathcal{H}} e_k,$$

where $\{\mathfrak{K}_{H_k}\}_{k\geq 1}$ is the local non-determinism constant of $\{B^{H_k}\}_{k\geq 1}$, i.e. for every $t \in [0,T]$ and $0 < r \leq t$

$$\mathrm{Var}\Big(B_t^{H_k} \Big| B_s^{H_k} : |t-s| \geq r\Big) \geq \mathfrak{K}_{H_k} r^{2H_k}.$$

Subsequently, using the $L^2(\Omega)$ compactness criterion, one has to show that for $0 < \alpha_m < \beta_m < \frac{1}{2}$ and $\gamma_m > 0$ for all $m \geq 1$, $d \geq 1$, and $\varepsilon > 0$

$$\left\| X_t^{d,\varepsilon} \right\|_{L^2(\Omega;\mathcal{H})} \leq C,$$

$$\sum_{m\geq 1} \gamma_m^{-2} \left\| D^m X_t^{d,\varepsilon} \right\|^2_{L^2(\Omega;L^2([0,T];\mathcal{H}))} \leq C,$$

and

$$\sum_{m\geq 1} \frac{1}{(1-2^{-2(\beta_m-\alpha_m)})\gamma_m^2} \int_0^T \int_0^T \frac{\left\| D_s^m X_t^{d,\varepsilon} - D_u^m X_t^{d,\varepsilon} \right\|^2_{L^2(\Omega;\mathcal{H})}}{|s-u|^{1+2\beta_m}} ds du \leq C.$$

Here, D^m denotes the Malliavin derivative in the direction of the m-th dimension. In the main theorem it is shown that for the existence of a Malliavin differentiable unique strong solution of SDE (1.28) the assumption of Lipschitz continuity in the spatial variable (1.30) can be dropped and thus, irregular drift functions are permitted. The paper is closed by an example, in particular showing that the class of possible drift coefficients is not empty.

Chapter 6: McKean-Vlasov equations on infinite-dimensional Hilbert spaces with irregular drift and additive fractional noise

The infinite dimensional case of mean-field SDE (1.8) is considered in the paper *McKean-Vlasov equations on infinite-dimensional Hilbert spaces with irregular drift and additive fractional noise* [6]. Here, SDE (1.28) is extended to the mean-field SDE

$$X_t = x + \int_0^t b\left(s, X_s, \mathbb{P}_{X_s}\right) ds + \mathbb{B}_t, \; t \in [0,T], \; x \in \mathcal{H}, \tag{1.31}$$

where $b : [0,T] \times \mathcal{H} \times \mathcal{P}_1(\mathcal{H}) \to \mathcal{H}$, $\mathcal{H}$ is an infinite-dimensional separable Hilbert space, and $\mathbb{B}$ is defined as in (1.9). Since results on the existence of strong solutions to the related SDE (1.28) have already been derived in the paper *Restoration of Well-Posedness of Infinite-dimensional Singular ODE's via Noise* [2], here the focus mainly lies on the existence and uniqueness of weak solutions to mean-field SDE (1.31). In addition, a more general class of Hurst parameters permitted for the cylindrical fractional Brownian motion $\mathbb{B}$ is considered. More precisely, a partition $\{I_-, I_0, I_+\}$ of $\mathbb{N}$ is defined such that for the sequence of Hurst parameters $\mathbb{H} := \{H_k\}_{k\geq 1} \subset (0,1)$ it holds that

(i) $k \in I_- : \; H_k \in \left(0, \frac{1}{2}\right)$,

(ii) $k \in I_0 : \; H_k = \frac{1}{2}$, and

(iii) $k \in I_+ : \; H_k \in \left(\frac{1}{2}, 1\right)$.

Furthermore, we assume in the definition of the cylindrical Brownian motion (1.9) that $\lambda \in \ell^1$ and $\sum_{k\in I_-} \frac{\lambda_k}{\sqrt{H_k}} < \infty$.

Similar to [9], the usual approach applying Girsanov's theorem and Schauder's fixed point theorem is applied to show that mean-field SDE (1.31) has a weak solution. Here, it is assumed that $\|b_k\|_\infty \leq C_k \lambda_k$ [7] for all $k \geq 1$, where $\frac{C}{\sqrt{1-\mathbb{H}}} := \{\frac{C_k}{\sqrt{1-H_k}}\}_{k\geq 1} \in \ell^1$ and that

$$\left(\sum_{k\geq 1} \lambda_k^2 (t-s)^{2H_k}\right)^{\frac{1}{2}} \leq \rho |t-s|^\kappa,$$

[7] Here, $\|\cdot\|_\infty$ denotes the sup norm with respect to all respective variables.

where $\rho > 0$ and $0 < \kappa < 1$ are constants. Moreover, in the case $k \in I_+$ it is assumed that

$$|b_k(t,x,\mu) - b_k(s,y,\nu)| \leq C_k\lambda_k \left(|t-s|^{\gamma_k} + \|x-y\|_{\mathcal{H}}^{\alpha_k} + \mathcal{K}(\mu,\nu)^{\beta_k}\right),$$

where $\gamma_k > H_k - \frac{1}{2}$, $2 \geq \kappa\alpha_k > 2H_k - 1$, and $\kappa\beta_k > H_k - \frac{1}{2}$, and in the case $k \in I_- \cup I_0$ that for every $\mu \in \mathcal{C}([0,T];\mathcal{P}_1(\mathcal{H}))$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $k \geq 1$ and $\nu \in \mathcal{C}([0,T];\mathcal{P}_1(\mathcal{H}))$

$$\sup_{t\in[0,T]} \mathcal{K}(\mu_t,\nu_t) < \delta \ \Rightarrow \sup_{t\in[0,T],\ y\in\mathcal{H}} |b_k(t,y,\mu_t) - b_k(t,y,\nu_t)| < \varepsilon C_k\lambda_k.$$

Uniqueness in law is established under the additional assumptions that the drift coefficient is Lipschitz continuous in the law variable and $\sup_{k\in I_+} H_k < 1$. Closing the paper the connection to [2] and SDEs in general is revisited and the existence of pathwisely unique strong solutions is discussed.

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Chapter 2
Strong Solutions of Mean-Field Stochastic Differential Equations with irregular drift

Contribution of the thesis' author

The paper *Strong Solutions of Mean-Field Stochastic Differential Equations with irregular drift* published in *Electronic Journal of Probability* 23(2018), no. 132, is a joint work with Prof. Dr. Thilo Meyer-Brandis and Prof. Dr. Frank Proske. Opposed to the published version three typos have been corrected. Namely, the period in page 48 line -4 after $[0,T]$ has been changed to a comma, instead of ρ we write $\mathbb{P}_X$ in page 61 line 5, and in display (2.21) $\tilde{\mathbb{Q}}_{(X,\widetilde{B})}$ became $\tilde{\mathbb{Q}}_{(Y,\widetilde{B})}$. Furthermore, the definition of a weak solution (Definition 2.1) has been modified. More precisely, the integrability assumption (2.14) has been added.

M. Bauer was significantly involved in the development of all parts of the paper. In particular, M. Bauer made major contributions to the editorial work and the proofs of Theorem 2.3, Theorem 2.7, Theorem 2.12, Theorem 3.3, Theorem 3.12, and Theorem 4.2 as well as the augmenting remarks Remark 2.9 and Remark 2.11.

STRONG SOLUTIONS OF MEAN-FIELD STOCHASTIC DIFFERENTIAL EQUATIONS WITH IRREGULAR DRIFT

MARTIN BAUER, THILO MEYER-BRANDIS, AND FRANK PROSKE

Abstract. We investigate existence and uniqueness of strong solutions of mean-field stochastic differential equations with irregular drift coefficients. Our direct construction of strong solutions is mainly based on a compactness criterion employing Malliavin Calculus together with some local time calculus. Furthermore, we establish regularity properties of the solutions such as Malliavin differentiablility as well as Sobolev differentiability and Hölder continuity in the initial condition. Using this properties we formulate an extension of the Bismut-Elworthy-Li formula to mean-field stochastic differential equations to get a probabilistic representation of the first order derivative of an expectation functional with respect to the initial condition.

Keywords. mean-field stochastic differential equation · McKean-Vlasov equation · strong solutions · irregular coefficients · Malliavin calculus · local-time integral · Sobolev differentiability in the initial condition · Bismut-Elworthy-Li formula.

1. Introduction

Throughout this paper, let $T > 0$ be a given time horizon. Mean-field stochastic differential equations (hereafter mean-field SDE), also referred to as McKean-Vlasov equations, given by

$$dX_t^x = b(t, X_t^x, \mathbb{P}_{X_t^x})dt + \sigma(t, X_t^x, \mathbb{P}_{X_t^x})dB_t, \quad X_0^x = x \in \mathbb{R}^d, \quad t \in [0,T], \tag{2.1}$$

are an extension of stochastic differential equations where the coefficients are allowed to depend on the law of the solution in addition to the dependence on the solution itself. Here $b : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^{d\times n}$ are some given drift and volatility coefficients, $(B_t)_{t\in[0,T]}$ is an n-dimensional Brownian motion,

$$\mathcal{P}_1(\mathbb{R}^d) := \left\{ \mu \,\middle|\, \mu \text{ probability measure on } (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \text{ with } \int_{\mathbb{R}^d} |x| d\mu(x) < \infty \right\}$$

is the space of probability measures over $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with existing first moment, and $\mathbb{P}_{X_t^x}$ is the law of X_t^x with respect to the underlying probability measure $\mathbb{P}$. Based on the works of Vlasov [39], Kac [25] and McKean [33], mean-field SDEs arised from Boltzmann's equation in physics, which is used to model weak interaction between particles in a multi-particle system. Since then the study of mean-field SDEs has evolved as an active research field with numerous applications. Various extensions of the class of mean-field SDEs as for example replacing the driving noise by a Lévy process or considering backward equations have been examined e.g. in [24], [4], [5], and [6]. With their work on mean-field games in [29], Lasry and Lions have set a cornerstone in the application of mean-field SDEs in Economics and Finance, see also [7] for a readily accessible summary of Lions' lectures at Collège de France. As opposed to the analytic approach taken in [29], Carmona and Delarue developed a probabilistic approach to mean-field games, see e.g. [8], [9], [10], [11] and [14]. More recently, the mean-field approach also found application in systemic risk modeling, especially in models for inter-bank lending and borrowing, see e.g. [12], [13], [19], [20], [21], [28], and the cited sources therein.

In this paper we study existence, uniqueness and regularity properties of (strong) solutions of one-dimensional mean-field SDEs of the type

$$dX_t^x = b(t, X_t^x, \mathbb{P}_{X_t^x})dt + dB_t, \quad X_0^x = x \in \mathbb{R}, \quad t \in [0, T]. \tag{2.2}$$

If the drift coefficient b is of at most linear growth and Lipschitz continuous, existence and uniquenss of (strong) solutions of (2.2) are well understood. Under further smoothness assumptions on b, differentiability in the initial condition x and the relation to non-linear PDE's is studied in [6]. We here consider the situation when the drift b is allowed to be irregular. More precisely, in addition to some linear growth condition we basically only require measurability in the second variable and some continuity in the third variable.

The first main contribution of this paper is to establish existence and uniqueness of strong solutions of mean-field SDE (2.2) under such irregularity assumptions on b. To this end, we firstly consider existence and uniqueness of weak solutions of mean-field SDE (2.2). In [16], Chiang proves the existence of weak solutions for time-homogeneous mean-field SDEs with drift coefficients that are of linear growth and allow for certain discontinuities. Using the methodology of martingale problems, Jourdain proves in [23] the existence of a unique weak solution under the assumptions of a bounded drift which is Lipschitz continuous in the law variable. In the time-inhomogeneous case, Mishura and Veretennikov ensure in [37] the existence of weak solutions by requiring in addition to linear growth that the drift is of the form

$$b(t, y, \mu) = \int \overline{b}(t, y, z)\mu(dz), \tag{2.3}$$

for some $\overline{b} : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. In [31], Li and Min show the existence of weak solutions of mean-field SDEs with path-dependent coefficients, supposing that the drift is bounded and continuous in the third variable. We here relax the boundedness requirement in [31] (for the non-path-dependent case) and show existence of a weak solution of (2.2) by merely requiring that b is continuous in the third variable, i.e. for all $\mu \in \mathcal{P}_1(\mathbb{R})$ and all $\varepsilon > 0$ exists a $\delta > 0$ such that

$$(\forall \nu \in \mathcal{P}_1(\mathbb{R}) : \mathcal{K}(\mu, \nu) < \delta) \Rightarrow |b(t, y, \mu) - b(t, y, \nu)| < \varepsilon, \quad t \in [0, T], \quad y \in \mathbb{R}, \tag{2.4}$$

and of at most linear growth, i.e. there exists a constant $C > 0$ such that for all $t \in [0,T]$, $y \in \mathbb{R}$ and $\mu \in \mathcal{P}_1(\mathbb{R})$,

$$|b(t,y,\mu)| \leq C(1 + |y| + \mathcal{K}(\mu, \delta_0)). \tag{2.5}$$

Here δ_0 is the Dirac-measure in 0 and $\mathcal{K}$ the Kantorovich metric:

$$\mathcal{K}(\lambda, \nu) := \sup_{h \in \mathrm{Lip}_1(\mathbb{R})} \left| \int_{\mathbb{R}} h(x)(\lambda - \nu)(dx) \right|, \quad \lambda, \nu \in \mathcal{P}_1(\mathbb{R}),$$

where $\mathrm{Lip}_1(\mathbb{R})$ is the space of Lipschitz continuous functions with Lipschitz constant 1 (for an explicit definition see the notations below). Further we show that if b admits a modulus of continuity in the third variable (see Definition 2.6) in addition to (2.4) and (2.5), then there is weak uniqueness (or uniqueness in law) of solutions of (2.2).

In order to establish the existence of strong solutions of (2.2), we then show that any weak solution actually is a strong solution. Indeed, given a weak solution X^x (and in particular its law) of mean-field SDE (2.2), one can re-interprete X as the solution of a common SDE

$$dX_t^x = b^{\mathbb{P}_X}(t, X_t^x)dt + dB_t, \quad X_0^x = x \in \mathbb{R}, \quad t \in [0,T], \tag{2.6}$$

where $b^{\mathbb{P}_X}(t,y) := b(t,y,\mathbb{P}_{X_t^x})$. This re-interpretation allows to apply the ideas and techniques developed in [2],[34] and [36] on strong solutions of SDEs with irregular coefficients to equation (2.6). In order to deploy these results and to prove that the weak solution X^x is indeed a strong solution, we still assume condition (2.4), i.e. the drift coefficient b is supposed to be continuous in the third variable, but require the following particular form proposed in [2] of the linear growth condition (2.5):

$$b(t,y,\mu) = \hat{b}(t,y,\mu) + \tilde{b}(t,y,\mu), \tag{2.7}$$

where $\hat{b}$ is merely measurable and bounded and $\tilde{b}$ is of at most linear growth (2.5) and Lipschitz continuous in the second variable, i.e. there exists a constant $C > 0$ such that for all $t \in [0,T]$, $y_1, y_2 \in \mathbb{R}$ and $\mu \in \mathcal{P}_1(\mathbb{R})$,

$$|\tilde{b}(t,y_1,\mu) - \tilde{b}(t,y_2,\mu)| \leq C|y_1 - y_2|. \tag{2.8}$$

We remark that while a typical approach to show existence of strong solutions is to establish existence of weak solutions together with pathwise uniqueness (Yamada-Watanabe Theorem), in [2],[34] and [36] the existence of strong solutions is shown by a direct constructive approach based on some compactness criterion employing Malliavin calcuclus. Further, pathwise (or strong) uniqueness is then a consequence of weak uniqueness. We also remark that in [37] the existence of strong solutions of mean-field SDEs is shown in the case that the drift is of the special form (2.3) where $\bar{b}$ fulfills certain linear growth and Lipschitz conditions.

The second contribution of this paper is the study of certain regularity properties of strong solutions of mean-field equation (2.2). Firstly, from the constructive approach to strong solutions based on [2], [34] and [36] we directly gain Malliavin differentiability of strong solutions of SDE (2.6), i.e. Malliavin differentiability of strong solutions of mean-field SDE (2.2). Similar to [2] we provide a probabilistic representation of the Malliavin derivative using the local time-space integral introduced in [18].

Secondly, we investigate the regularity of the dependence of a solution X^x on its initial condition x. For the special case where the mean-field dependence is given via an expectation functional of the form

$$dX_t^x = \bar{b}(t, X_t^x, \mathbb{E}[\varphi(X_t^x)])dt + dB_t, \quad X_0^x = x \in \mathbb{R}, \quad t \in [0,T], \tag{2.9}$$

for some $\bar{b} : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, continuous differentiability of X^x with respect to x can be deduced from [6] under the assumption that $\bar{b}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ are continuously differentiable with bounded Lipschitz derivatives. We here establish weak (Sobolev) differentiability of X^x with respect to x for the general drift b given in (2.2) by assuming in addition to (2.7) that $\mu \mapsto b(t, y, \mu)$ is Lipschitz continuous uniformly in $t \in [0,T]$ and $y \in \mathbb{R}$, i.e. there exists a constant $C > 0$ such that for all $t \in [0,T]$, $y \in \mathbb{R}$ and $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$

$$|b(t, y, \mu) - b(t, y, \nu)| \leq C\mathcal{K}(\mu, \nu). \tag{2.10}$$

Further, also for the Sobolev derivative we provide a probabilistic representation in terms of local-time space integration.

The third main contribution of this paper is a Bismut-Elworthy-Li formula for first order derivatives of expectation functionals $\mathbb{E}[\Phi(X_T^x)]$, $\Phi : \mathbb{R} \to \mathbb{R}$, of a strong solution X^x of mean-field SDE (2.2). Assuming the drift b is in the form (2.7) and fulfills the Lipschitz condition (2.10), we first show Sobolev differentiability of these expectation functionals whenever Φ is continuously differentiable with bounded Lipschitz derivative. We then continue to develop a Bismut-Elworthy-Li type formula, that is we give a probabilistic representation for the first-order derivative of the form

$$\frac{\partial}{\partial x}\mathbb{E}[\Phi(X_T^x)] = \mathbb{E}\left[\Phi(X_T^x)\int_0^T \theta_t dB_t\right], \tag{2.11}$$

where $(\theta_t)_{t\in[0,T]}$ is a certain stochastic process measurable with respect to $\sigma(X_s : s \in [0,T])$. We remark that in [1], the author provides a Bismut-Elworthy-Li formula for multi-dimensional mean-field SDEs with multiplicative noise but smooth drift and volatility coefficients. For one-dimensional mean-field SDEs with additive noise (i.e. $\sigma \equiv 1$), we thus extend the result in [1] to irregular drift coefficients. Moreover, we are able to further develop the formula such that the so-called Malliavin weight $\int_0^T \theta_t dB_t$ is given in terms of an Itô integral and not in terms of an anticipative Skorohod integral as in [1].

Finally, we remark that in [3] we study (strong) solutions of mean-field SDEs and a corresponding Bismut-Elworthy-Li formula where the dependence of the drift b on the solution law $\mathbb{P}_{X_t^x}$ in (2.2) is of the special form

$$dX_t^x = \overline{b}\left(t, X_t^x, \int_{\mathbb{R}} \varphi(t, X_t^x, z)\mathbb{P}_{X_t^x}(dz)\right) dt + dB_t, \quad X_0^x = x \in \mathbb{R}, \tag{2.12}$$

for some $\overline{b}, \varphi : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. For this special class of mean-field SDEs, which includes the two popular drift families given in (2.3) and (2.9), we allow for irregularity of $\overline{b}$ and φ that is not covered by our assumptions on b in this paper. For example, for the indicator function $\varphi(t,x,z) = I_{z\le u}$ we are able to deal in [3] with the important case where the drift $\overline{b}\left(t, X_t^x, F_{X_t^x}(u)\right)$ depends on the distribution function $F_{X_t^x}(\cdot)$ of the solution.

The remaining paper is organized as follows. In the second section we deal with existence and uniqueness of solutions of the mean-field SDE (2.2). The third section investigates the aforementioned regularity properties of strong solutions. Finally, a proof of weak differentiability of expectation functionals $\mathbb{E}[\Phi(X_T^x)]$ is given in the fourth section together with a Bismut-Elworthy-Li formula.

Notation: Subsequently we list some of the most frequently used notations. For this, let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be two metric spaces.

- $\mathcal{C}(\mathcal{X};\mathcal{Y})$ denotes the space of continuous functions $f : \mathcal{X} \to \mathcal{Y}$.
- $\mathcal{C}_0^\infty(U)$, $U \subseteq \mathbb{R}$, denotes the space of smooth functions $f : U \to \mathbb{R}$ with compact support.
- For every $C > 0$ we define the space $\mathrm{Lip}_C(\mathcal{X},\mathcal{Y})$ of functions $f : \mathcal{X} \to \mathcal{Y}$ such that

  $$d_{\mathcal{Y}}(f(x_1), f(x_2)) \le C d_{\mathcal{X}}(x_1, x_2), \quad \forall x_1, x_2 \in \mathcal{X},$$

  as the space of Lipschitz functions with Lipschitz constant $C > 0$. Furthermore, we define $\mathrm{Lip}(\mathcal{X},\mathcal{Y}) := \bigcup_{C>0} \mathrm{Lip}_C(\mathcal{X},\mathcal{Y})$ and denote by $\mathrm{Lip}_C(\mathcal{X}) := \mathrm{Lip}_C(\mathcal{X},\mathcal{X})$ and $\mathrm{Lip}(\mathcal{X}) := \mathrm{Lip}(\mathcal{X},\mathcal{X})$, respectively, the space of Lipschitz functions mapping from $\mathcal{X}$ to $\mathcal{X}$.
- $\mathcal{C}_{b,C}^{1,1}(\mathbb{R})$ denotes the space of continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that its derivative f' satisfies for $C > 0$
  (a) $\sup_{y\in\mathbb{R}} |f'(y)| \le C$, and
  (b) $(y \mapsto f'(y)) \in \mathrm{Lip}_C(\mathbb{R})$.
  We define $\mathcal{C}_b^{1,1}(\mathbb{R}) := \bigcup_{C>0} \mathcal{C}_{b,C}^{1,1}(\mathbb{R})$.
- $\mathcal{C}_b^{1,L}(\mathbb{R} \times \mathcal{P}_1(\mathbb{R}))$ is the space of functions $f : \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ such that there exists a constant $C > 0$ with
  (a) $(y \mapsto f(y,\mu)) \in \mathcal{C}_{b,C}^{1,1}(\mathbb{R})$ for all $\mu \in \mathcal{P}_1(\mathbb{R})$, and
  (b) $(\mu \mapsto f(y,\mu)) \in \mathrm{Lip}_C(\mathcal{P}_1(\mathbb{R}), \mathbb{R})$ for all $y \in \mathbb{R}$.
- Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a generic complete filtered probability space with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ and $B = (B_t)_{t\in[0,T]}$ be a Brownian motion defined on this probability space. Furthermore, we write $\mathbb{E}[\cdot] := \mathbb{E}_{\mathbb{P}}[\cdot]$, if not mentioned differently.

- $L^p(\mathcal{S},\mathcal{X})$ denotes the Banach space of functions on the measurable space $(\mathcal{S},\mathcal{G})$ mapping to the normed space $(\mathcal{X},\|\cdot\|_{\mathcal{X}})$ integrable to some power p, $p\geq 1$.
- $L^p(\Omega,\mathcal{F}_t)$ denotes the space of $\mathcal{F}_t$–measurable functions in $L^p(\Omega)$.
- Let $f:\mathbb{R}\to\mathbb{R}$ be a (weakly) differentiable function. Then we denote by $\partial_y f(y) := \frac{\partial f}{\partial y}(y)$ its first (weak) derivative evaluated at $y\in\mathbb{R}$.
- We denote the Doléans-Dade exponential for a progressively measurable process Y with respect to the corresponding Brownian integral if well-defined for $t\in[0,T]$ by
$$\mathcal{E}\left(\int_0^t Y_u dB_u\right) := \exp\left\{\int_0^t Y_u dB_u - \frac{1}{2}\int_0^t |Y_u|^2 du\right\}.$$
- We define $B_t^x := x + B_t$, $t\in[0,T]$, for any Brownian motion B.
- For any normed space $\mathcal{X}$ we denote its corresponding norm by $\|\cdot\|_{\mathcal{X}}$; the Euclidean norm is denoted by $|\cdot|$.
- We write $E_1(\theta)\lesssim E_2(\theta)$ for two mathematical expressions $E_1(\theta),E_2(\theta)$ depending on some parameter θ, if there exists a constant $C>0$ not depending on θ such that $E_1(\theta)\leq CE_2(\theta)$.
- We denote by L^X the local time of the stochastic process X and furthermore by $\int_s^t\int_{\mathbb{R}} b(u,y)L^X(du,dy)$ for suitable b the local-time space integral as introduced in [18] and extended in [2].
- We denote the Wiener transform of some $Z\in L^2(\Omega,\mathcal{F}_T)$ in $f\in L^2([0,T])$ by
$$\mathcal{W}(Z)(f) := \mathbb{E}\left[Z\mathcal{E}\left(\int_0^T f(s)dB_s\right)\right].$$

2. Existence and Uniqueness of Solutions

The main objective of this section is to investigate existence and uniqueness of strong solutions of the one-dimensional mean-field SDE
$$dX_t^x = b(t,X_t^x,\mathbb{P}_{X_t^x})dt + dB_t,\quad X_0^x = x\in\mathbb{R},\quad t\in[0,T], \tag{2.13}$$
with irregular drift coefficient $b:\mathbb{R}_+\times\mathbb{R}\times\mathcal{P}_1(\mathbb{R})\to\mathbb{R}$. We first consider existence and uniqueness of weak solutions of (2.13) in Section 2.1, which consecutively is employed together with results from [2] to study strong solutions of (2.13) in Section 2.2.

2.1. **Existence and Uniqueness of Weak Solutions.** We recall the definition of weak solutions.

Definition 2.1 A *weak solution* of the mean-field SDE (2.13) is a six-tuple $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},B,X^x)$ such that

(i) $(\Omega,\mathcal{F},\mathbb{P})$ is a complete probability space and $\mathbb{F}=\{\mathcal{F}_t\}_{t\in[0,T]}$ is a filtration on $(\Omega,\mathcal{F},\mathbb{P})$ satisfying the usual conditions of right-continuity and completeness,

(ii) $X^x = (X_t^x)_{t\in[0,T]}$ is a continuous, $\mathbb{F}$-adapted, $\mathbb{R}$-valued process; $B = (B_t)_{t\in[0,T]}$ is a one-dimensional $(\mathbb{F},\mathbb{P})$-Brownian motion,

(iii) X^x satisfies $\mathbb{P}$-a.s.

$$dX_t^x = b(t, X_t^x, \mathbb{P}_{X_t^x})dt + dB_t, \quad X_0^x = x \in \mathbb{R}, \quad t \in [0,T],$$

where for all $t \in [0,T]$, $\mathbb{P}_{X_t^x} \in \mathcal{P}_1(\mathbb{R})$ denotes the law of X_t^x with respect to $\mathbb{P}$, and

$$\int_0^T \mathcal{K}(\mathbb{P}_{X_t^x}, \delta_0)dt < \infty. \tag{2.14}$$

Remark 2.2. If there is no ambiguity about the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, B)$ we also refer solely to the process X^x as weak solution (or later on as strong solution) for notational convenience.

Remark 2.3. For bounded drift coefficients $b : [0,T]\times\mathbb{R}\times\mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ condition (2.14) is redundant since it is naturally fulfilled. Indeed,

$$\sup_{t\in[0,T]} \mathcal{K}(\mathbb{P}_{X_t^x}, \delta_0) \leq \mathbb{E}[|X_t^x|] \leq |x| + \mathbb{E}\left[\left|\int_0^T b\left(s, X_s^x, \mathbb{P}_{X_s^x}\right) ds\right|\right] + \sup_{t\in[0,T]} \mathbb{E}[|B_t|] < \infty.$$

In a first step we employ Girsanov's theorem in a well-known way to construct weak solutions of certain stochastic differential equations (hereafter SDE) associated to our mean-field SDE (2.13). Assume the drift coefficient $b : [0,T] \times \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ satisfies the linear growth condition (2.5). For a given $\mu \in \mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}))$ we then define $b^\mu : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ by $b^\mu(t,y) := b(t,y,\mu_t)$ and consider the SDE

$$dX_t^x = b^\mu(t, X_t^x)dt + dB_t, \quad X_0^x = x \in \mathbb{R}, \quad t \in [0,T]. \tag{2.15}$$

Let $\tilde{B}$ be a one-dimensional Brownian motion on a suitable filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$. Define $X_t^x := \tilde{B}_t + x$. By Lemma A.2, the density $\frac{d\mathbb{P}^\mu}{d\mathbb{Q}} = \mathcal{E}\left(\int_0^T b^\mu(t, \tilde{B}_t^x)d\tilde{B}_t\right)$ gives rise to a well-defined equivalent probability measure $\mathbb{P}^\mu$, and by Girsanov's theorem $B_t^\mu := X_t^x - x - \int_0^t b^\mu(s, X_s^{x,\mu})ds$, $t \in [0,T]$, defines an $(\mathbb{F},\mathbb{P}^\mu)$-Brownian motion. Hence, $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^\mu, B^\mu, X_t^x)$ is a weak solution of SDE (2.15).

To show existence of weak solutions of the mean-field SDE (2.13) we proceed by employing the weak solutions of the auxiliary SDEs in (2.15) together with a fixed point argument. Compared to the typical construction of weak solutions of SDE's by a straight forward application of Girsanov's theorem, the construction of weak solutions of mean-field SDE's is thus more complex and requires a fixed point argument in addition to the application of Girsanov's theorem due to the fact that the measure dependence in the drift stays fixed under the Girsanov transformation. The upcoming theorem is a modified version of Theorem 3.2 in [31] for non-path-dependent coefficients, where we extend the assumptions on the drift from boundedness to linear growth.

Theorem 2.4 *Let the drift coefficient $b : [0,T]\times\mathbb{R}\times\mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ be a measurable function that satisfies conditions* (2.4) *and* (2.5)*, i.e. b is continuous in the third variable and of at most linear growth. Then there exists a weak solution of the mean-field SDE* (2.13)*. Furthermore, $\mathbb{P}_{X^x_\cdot} \in \mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}))$ for any weak solution X^x of* (2.13)*.*

Proof. We will state the proof just in the parts that differ from the proof in [31]. For $\mu \in \mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}))$ let $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P}^\mu,B^\mu,X^{x,\mu})$ be a weak solution of SDE (2.15). We define the mapping $\psi : \mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R})) \to \mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}))$ by

$$\psi_s(\mu) := \mathbb{P}^\mu_{X^{x,\mu}_s},$$

where $\mathbb{P}^\mu_{X^{x,\mu}_s}$ denotes the law of $X^{x,\mu}_s$ under $\mathbb{P}^\mu$, $s \in [0,T]$. Note that it can be shown equivalently to (ii) below that $\psi_s(\mu)$ is indeed continuous in $s \in [0,T]$. We need to show that ψ has a fixed point, i.e. $\mu_s = \psi_s(\mu) = \mathbb{P}^\mu_{X^{x,\mu}_s}$ for all $s \in [0,T]$. To this end we aim at applying Schauder's fixed point theorem (cf. [38]) to $\psi : E \to E$, where

$$E := \left\{\mu \in \mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R})) : \mathcal{K}(\mu_t,\delta_x) \leq C,\ \mathcal{K}(\mu_t,\mu_s) \leq C|t-s|^{\frac{1}{2}},\ t,s \in [0,T]\right\},$$

for some suitable constant $C > 0$. Therefore we have to show that E is a non-empty convex subset of $\mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}))$, ψ maps E continuously into E and $\psi(E)$ is compact. Due to the proof of Theorem 3.2 in [31] it is left to show that for all $s,t \in [0,T]$ and $\mu \in E$,

(i) ψ is continuous on E,
(ii) $\mathcal{K}(\psi_t(\mu),\psi_s(\mu)) \lesssim |t-s|^{\frac{1}{2}}$,
(iii) $\mathbb{E}_{\mathbb{P}^\mu}[|X^{\mu,x}_t|\mathbb{1}_{\{|X^{\mu,x}_t|\geq r\}}] \xrightarrow[r\to\infty]{} 0$.

(i) First note that E endowed with $\sup_{t\in[0,T]}\mathcal{K}(\cdot,\cdot)$, is a metric space. Let $\tilde{\varepsilon} > 0$, $\mu \in E$ and $C_1 > 0$ be some constant. Moreover, let $C_{p,T} > 0$ be a constant depending on p and T such that by Burkholder-Davis-Gundy's inequality $\mathbb{E}\left[|B_t|^{2p}\right]^{\frac{1}{2p}} \leq \frac{C_{p,T}}{2C_1}$ for all $t \in [0,T]$. Since b is continuous in the third variable and $\cdot^2$ is a continuous function, we can find $\delta_1 > 0$ such that for all $\nu \in E$ with $\sup_{t\in[0,T]}\mathcal{K}(\mu_t,\nu_t) < \delta_1$,

$$\begin{aligned}\sup_{t\in[0,T],y\in\mathbb{R}} |b(t,y,\mu_t) - b(t,y,\nu_t)| &< \frac{\tilde{\varepsilon}}{2C_{p,T}T^{\frac{1}{2}}},\\ \sup_{t\in[0,T],y\in\mathbb{R}} \left||b(t,y,\mu_t)|^2 - |b(t,y,\nu_t)|^2\right| &< \frac{\tilde{\varepsilon}}{C_{p,T}T}.\end{aligned} \tag{2.16}$$

Furthermore, by the proof of Lemma A.3 we can find $\varepsilon > 0$ such that

$$\sup_{\lambda\in E}\mathbb{E}\left[\mathcal{E}\left(-\int_0^T b(t,B^x_t,\lambda_t)dB_t\right)^{1+\varepsilon}\right]^{\frac{1}{1+\varepsilon}} \leq C_1. \tag{2.17}$$

Then, we get by the definition of ψ and $\mathcal{E}_t(\mu) := \mathcal{E}\left(\int_0^t b(s, B_s^x, \mu_s)dB_s\right)$ that

$$
\begin{aligned}
\mathcal{K}(\psi_t(\mu), \psi_t(\nu)) &= \sup_{h\in \mathrm{Lip}_1} \left\{\left|\int_{\mathbb{R}} h(y)\psi_t(\mu)(dy) - \int_{\mathbb{R}} h(y)\psi_t(\nu)(dy)\right|\right\} \\
&= \sup_{h\in \mathrm{Lip}_1} \left\{\left|\int_{\mathbb{R}} (h(y)-h(x))\left(\mathbb{P}^{\mu}_{X_t^{x,\mu}} - \mathbb{P}^{\nu}_{X_t^{x,\nu}}\right)(dy)\right|\right\} \\
&= \sup_{h\in \mathrm{Lip}_1} \left\{\left|\mathbb{E}_{\mathbb{Q}^\mu}\left[(h(X_t^{x,\mu})-h(x))\,\mathcal{E}_t(\mu)\right] - \mathbb{E}_{\mathbb{Q}^\nu}\left[(h(X_t^{x,\nu})-h(x))\,\mathcal{E}_t(\nu)\right]\right|\right\} \\
&\leq \mathbb{E}\left[|\mathcal{E}_t(\mu) - \mathcal{E}_t(\nu)|\,|B_t|\right],
\end{aligned}
$$

where $\frac{d\mathbb{Q}^\mu}{d\mathbb{P}^\mu} = \mathcal{E}\left(-\int_0^t b(s, X_s^{x,\mu}, \mu_s)dB_s^\mu\right)$ defines an equivalent probability measure $\mathbb{Q}^\mu$ by Lemma A.2. Here we have used the fact that $X^{x,\mu}$ is a Brownian motion under $\mathbb{Q}^\mu$ starting in x for all $\mu \in \mathcal{C}([0,T]; \mathcal{P}_1(\mathbb{R}))$. We get by the inequality

$$
|e^y - e^z| \leq |y-z|(e^y + e^z), \quad y, z \in \mathbb{R}, \tag{2.18}
$$

Hölder's inequality with $p := \frac{1+\varepsilon}{\varepsilon}$, $\varepsilon > 0$ sufficiently small with regard to (2.17), and Minkowski's inequality that

$$
\begin{aligned}
&\mathcal{K}(\psi_t(\mu), \psi_t(\nu)) \\
&\leq \mathbb{E}\Bigg[|B_t|\left(\mathcal{E}_t(\mu) + \mathcal{E}_t(\nu)\right) \\
&\quad \times \left|\int_0^t b(s, B_s^x, \mu_s) - b(s, B_s^x, \nu_s)dB_s - \frac{1}{2}\int_0^t |b(s, B_s^x, \mu_s)|^2 - |b(s, B_s^x, \nu_s)|^2 ds\right|\Bigg] \\
&\leq \left(\mathbb{E}\left[\mathcal{E}_t(\mu)^{1+\varepsilon}\right]^{\frac{1}{1+\varepsilon}} + \mathbb{E}\left[\mathcal{E}_t(\nu)^{1+\varepsilon}\right]^{\frac{1}{1+\varepsilon}}\right) \\
&\quad \times \left(\mathbb{E}\left[\left(\int_0^t |b(s, B_s^x, \mu_s) - b(s, B_s^x, \nu_s)|dB_s\right)^{2p}\right]^{\frac{1}{2p}}\right. \\
&\quad \left. + \frac{1}{2}\mathbb{E}\left[\left(\int_0^t \left||b(s, B_s^x, \mu_s)|^2 - |b(s, B_s^x, \nu_s)|^2\right| ds\right)^{2p}\right]^{\frac{1}{2p}}\right) \mathbb{E}\left[|B_t|^{2p}\right]^{\frac{1}{2p}}.
\end{aligned} \tag{2.19}
$$

Consequently, we get by Burkholder-Davis-Gundy's inequality and the bounds in (2.16) and (2.17) that

$$
\begin{aligned}
\sup_{t\in[0,T]} \mathcal{K}(\psi_t(\mu), \psi_t(\nu)) &\leq C_{p,T}\left(\mathbb{E}\left[\left(\int_0^T |b(s, B_s^x, \mu_s) - b(s, B_s^x, \nu_s)|^2 ds\right)^p\right]^{\frac{1}{2p}}\right. \\
&\quad \left. + \frac{1}{2}\mathbb{E}\left[\left(\int_0^T \left||b(s, B_s^x, \mu_s)|^2 - |b(s, B_s^x, \nu_s)|^2\right| ds\right)^{2p}\right]^{\frac{1}{2p}}\right) \\
&< T^{\frac{1}{2}}\frac{\tilde{\varepsilon}}{2T^{\frac{1}{2}}} + \frac{T}{2}\frac{\tilde{\varepsilon}}{T} = \tilde{\varepsilon}.
\end{aligned}
$$

Hence, ψ is continuous on E.

(ii) Define $p := \frac{1+\varepsilon}{\varepsilon}$, $\varepsilon > 0$ sufficiently small with regard to (2.17), and let $\mu \in E$ and $s, t \in [0,T]$ be arbitrary. Then, equivalently to (2.19)

$$\begin{aligned}\mathcal{K}(\psi_t(\mu), \psi_s(\mu)) &\leq \mathbb{E}\left[|\mathcal{E}_t(\mu) - \mathcal{E}_s(\mu)|\,|B_t|\right] \\ &\lesssim \mathbb{E}\left[\left|\int_s^t b(r, B_r^x, \mu_r) dB_r - \frac{1}{2}\int_s^t |b(r, B_r^x, \mu_r)|^2 dr\right|^{2p}\right]^{\frac{1}{2p}}.\end{aligned}$$

Furthermore, by applying Burkholder-Davis-Gundy's inequality, we get

$$\begin{aligned}\mathcal{K}(\psi_t(\mu), \psi_s(\mu)) &\lesssim \mathbb{E}\left[\left(\int_s^t |b(r, B_r^x, \mu_r)|^2 \, dr\right)^p\right]^{\frac{1}{2p}} + \mathbb{E}\left[\left(\int_s^t |b(r, B_r^x, \mu_r)|^2 \, dr\right)^{2p}\right]^{\frac{1}{2p}} \\ &\leq \mathbb{E}\left[|t-s|^p \sup_{r\in[0,T]} |b(r, B_r^x, \mu_r)|^{2p}\right]^{\frac{1}{2p}} + \mathbb{E}\left[|t-s|^{2p} \sup_{r\in[0,T]} |b(r, B_r^x, \mu_r)|^{4p}\right]^{\frac{1}{2p}}.\end{aligned}$$

Finally by Lemma A.1, we get that

$$\mathcal{K}(\psi_t(\mu), \psi_s(\mu)) \leq C_2 \left(|t-s|^{\frac{1}{2}} + |t-s|\right) \lesssim |t-s|^{\frac{1}{2}},$$

for some constant $C_2 > 0$, which is independent of $\mu \in E$.

(iii) The claim holds by Lemma A.1 and dominated convergence for $r \to \infty$.

□

Next, we study uniqueness of weak solutions. We recall the definition of weak uniqueness, also called uniqueness in law.

Definition 2.5 We say a weak solution $(\Omega^1, \mathcal{F}^1, \mathbb{F}^1, \mathbb{P}^1, B^1, X^1)$ of (2.13) is *weakly unique* or *unique in law*, if for any other weak solution $(\Omega^2, \mathcal{F}^2, \mathbb{F}^2, \mathbb{P}^2, B^2, X^2)$ of (2.13) it holds that

$$\mathbb{P}^1_{X^1} = \mathbb{P}^2_{X^2},$$

whenever $X_0^1 = X_0^2$.

In order to establish weak uniqueness we have to make further assumptions on the drift coefficient.

Definition 2.6 Let $b : [0,T] \times \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ be a measurable function. We say b admits θ as a modulus of continuity in the third variable, if there exists a continuous function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$, with $\theta(y) > 0$ for all $y \in \mathbb{R}_+$, $\int_0^z \frac{dy}{\theta(y)} = \infty$ for all $z \in \mathbb{R}_+$, and for all $t \in [0,T]$, $y \in \mathbb{R}$ and $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$,

$$|b(t, y, \mu) - b(t, y, \nu)|^2 \leq \theta(\mathcal{K}(\mu, \nu)^2). \tag{2.20}$$

Remark 2.7. Note that this definition is a special version of the general definition of modulus of continuity. In general one requires θ to satisfy $\lim_{x\to 0} \theta(x) = 0$ and for all $t \in [0,T]$, $y \in \mathbb{R}$ and $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$,

$$|b(t, y, \mu) - b(t, y, \nu)| \leq \theta(\mathcal{K}(\mu, \nu)).$$

It is readily verified that if b admits θ as a modulus of continuity according to Definition 2.6 it also admits one in the sense of the general definition.

Theorem 2.8 *Let the drift coefficient* $b : [0,T] \times \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ *satisfy conditions* (2.5) *and* (2.20)*, i.e. b is of at most linear growth and admits a modulus of continuity in the third variable. Let* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, B, X)$ *and* $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{\mathbb{P}}, W, Y)$ *be two weak solutions of* (2.13)*. Then*

$$\mathbb{P}_{(X,B)} = \hat{\mathbb{P}}_{(Y,W)}.$$

In particular the solutions are unique in law.

Proof. For the sake of readability we just consider the case $x = 0$. The general case follows in the same way. From Lemma A.2 and Girsanov's theorem, we know that there exist measures $\mathbb{Q}$ and $\hat{\mathbb{Q}}$ under which X and Y are Brownian motions, respectively. Similarly to the idea in the proof of Theorem 4.2 in [31], we define by Lemma A.2 an equivalent probability measure $\tilde{\mathbb{Q}}$ by

$$\frac{d\tilde{\mathbb{Q}}}{d\hat{\mathbb{P}}} := \mathcal{E}\left(-\int_0^T \left(b(s, Y_s, \hat{\mathbb{P}}_{Y_s}) - b(s, Y_s, \mathbb{P}_{X_s}) \right) dW_s \right),$$

and the $\tilde{\mathbb{Q}}$-Brownian motion

$$\tilde{B}_t := W_t + \int_0^t b(s, Y_s, \hat{\mathbb{P}}_{Y_s}) - b(s, Y_s, \mathbb{P}_{X_s}) ds, \quad t \in [0,T].$$

Since

$$B_t = X_t - \int_0^t b(s, X_s, \mathbb{P}_{X_s}) ds \quad \text{and} \quad \tilde{B}_t = Y_t - \int_0^t b(s, Y_s, \mathbb{P}_{X_s}) ds,$$

we can find a measurable function $\Phi : [0,T] \times \mathcal{C}([0,T]; \mathbb{R}) \to \mathbb{R}$ such that

$$B_t = \Phi_t(X) \quad \text{and} \quad \tilde{B}_t = \Phi_t(Y).$$

Recall that X and Y are $\mathbb{Q}$– and $\hat{\mathbb{Q}}$–Brownian motions, respectively. Consequently we have for every bounded measurable functional $F : \mathcal{C}([0,T]; \mathbb{R}) \times \mathcal{C}([0,T]; \mathbb{R}) \to \mathbb{R}$

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}[F(B, X)] &= \mathbb{E}_{\mathbb{Q}}\left[\mathcal{E}\left(\int_0^T b(t, X_t, \mathbb{P}_{X_t}) dX_t \right) F(\Phi(X), X) \right] \\
&= \mathbb{E}_{\hat{\mathbb{Q}}}\left[\mathcal{E}\left(\int_0^T b(t, Y_t, \mathbb{P}_{X_t}) dY_t \right) F(\Phi(Y), Y) \right] \\
&= \mathbb{E}_{\tilde{\mathbb{Q}}}[F(\tilde{B}, Y)].
\end{aligned}$$

Hence,

$$\mathbb{P}_{(X,B)} = \tilde{\mathbb{Q}}_{(Y,\tilde{B})}. \tag{2.21}$$

It is left to show that $\sup_{t \in [0,T]} \mathcal{K}(\tilde{\mathbb{Q}}_{Y_t}, \hat{\mathbb{P}}_{Y_t}) = 0$, from which we conclude together with (2.21) that $\sup_{t \in [0,T]} \mathcal{K}(\mathbb{P}_{X_t}, \hat{\mathbb{P}}_{Y_t}) = 0$ and hence $\frac{d\tilde{\mathbb{Q}}}{d\hat{\mathbb{P}}} = 1$. Consequently, $\mathbb{P}_{(X,B)} = \hat{\mathbb{P}}_{(Y,W)}$.

Using Hölder's inequality, we get for $p := \frac{1+\varepsilon}{\varepsilon}$, $\varepsilon > 0$ sufficiently small with regard to Lemma A.4,

$$\begin{aligned}
\mathcal{K}(\tilde{\mathbb{Q}}_{Y_t}, \hat{\mathbb{P}}_{Y_t}) &= \sup_{h \in \mathrm{Lip}_1} \left| \mathbb{E}_{\tilde{\mathbb{Q}}} \left[h(Y_t) - h(0) \right] - \mathbb{E}_{\hat{\mathbb{P}}} \left[h(Y_t) - h(0) \right] \right| \\
&\leq \sup_{h \in \mathrm{Lip}_1} \mathbb{E}_{\hat{\mathbb{P}}} \left[\left| \mathcal{E} \left(- \int_0^t \left(b(s, Y_s, \hat{\mathbb{P}}_{Y_s}) - b(s, Y_s, \mathbb{P}_{X_s}) \right) dW_s \right) - 1 \right| \left| h(Y_t) - h(0) \right| \right] \\
&\leq \mathbb{E}_{\hat{\mathbb{P}}} \left[\left| \mathcal{E} \left(- \int_0^t \left(b(s, Y_s, \hat{\mathbb{P}}_{Y_s}) - b(s, Y_s, \mathbb{P}_{X_s}) \right) dW_s \right) - 1 \right|^{\frac{2(1+\varepsilon)}{2+\varepsilon}} \right]^{\frac{2+\varepsilon}{2(1+\varepsilon)}} \\
&\quad \times \mathbb{E} \left[\mathcal{E} \left(\int_0^t b(s, B_s, \hat{\mathbb{P}}_{Y_s}) dB_s \right)^{1+\varepsilon} \right]^{\frac{\varepsilon}{2(1+\varepsilon)^2}} \mathbb{E} \left[|B_t|^{2p^2} \right]^{\frac{1}{2p^2}} \\
&\lesssim \mathbb{E}_{\hat{\mathbb{P}}} \left[\left| \mathcal{E} \left(- \int_0^t \left(b(s, Y_s, \hat{\mathbb{P}}_{Y_s}) - b(s, Y_s, \mathbb{P}_{X_s}) \right) dW_s \right) - 1 \right|^{\frac{2(1+\varepsilon)}{2+\varepsilon}} \right]^{\frac{2+\varepsilon}{2(1+\varepsilon)}}.
\end{aligned}$$

Using that b admits a modulus of continuity in the third variable, we get by inequality (2.18), Lemma A.4, and Burkholder-Davis-Gundy's inequality that

$$\begin{aligned}
\mathcal{K}(\tilde{\mathbb{Q}}_{Y_t}, \hat{\mathbb{P}}_{Y_t}) &\lesssim \mathbb{E}_{\hat{\mathbb{P}}} \Bigg[\Bigg| \exp \Bigg\{ - \int_0^t \left(b(s, Y_s, \hat{\mathbb{P}}_{Y_s}) - b(s, Y_s, \mathbb{P}_{X_s}) \right) dW_s \\
&\qquad - \frac{1}{2} \int_0^t \left(b(s, Y_s, \hat{\mathbb{P}}_{Y_s}) - b(s, Y_s, \mathbb{P}_{X_s}) \right)^2 ds \Bigg\} - \exp\{0\} \Bigg|^{\frac{2(1+\varepsilon)}{2+\varepsilon}} \Bigg]^{\frac{2+\varepsilon}{2(1+\varepsilon)}} \\
&\lesssim \mathbb{E}_{\hat{\mathbb{P}}} \Bigg[\Bigg| \int_0^t \left(b(s, Y_s, \hat{\mathbb{P}}_{Y_s}) - b(s, Y_s, \mathbb{P}_{X_s}) \right) dW_s \\
&\qquad + \frac{1}{2} \int_0^t \left(b(s, Y_s, \hat{\mathbb{P}}_{X_s}) - b(s, Y_s, \mathbb{P}_{X_s}) \right)^2 ds \Bigg|^{2p} \Bigg]^{\frac{1}{2p}} \\
&\lesssim \mathbb{E}_{\hat{\mathbb{P}}} \left[\left| \int_0^t \left(b(s, Y_s, \hat{\mathbb{P}}_{Y_s}) - b(s, Y_s, \mathbb{P}_{X_s}) \right)^2 ds \right|^{p} \right]^{\frac{1}{2p}} \\
&\quad + \mathbb{E}_{\hat{\mathbb{P}}} \left[\left| \int_0^t \left(b(s, Y_s, \hat{\mathbb{P}}_{Y_s}) - b(s, Y_s, \mathbb{P}_{X_s}) \right)^2 ds \right|^{2p} \right]^{\frac{1}{2p}} \\
&\leq \left(\int_0^t \theta \left(\mathcal{K}(\tilde{\mathbb{Q}}_{Y_s}, \hat{\mathbb{P}}_{Y_s})^2 \right) ds \right)^{\frac{1}{2}} + \int_0^t \theta \left(\mathcal{K}(\tilde{\mathbb{Q}}_{Y_s}, \hat{\mathbb{P}}_{Y_s})^2 \right) ds.
\end{aligned}$$

Assume $\int_0^t \theta \left(\mathcal{K}(\tilde{\mathbb{Q}}_{Y_s}, \hat{\mathbb{P}}_{Y_s})^2 \right) ds \geq 1$. Then,

$$\mathcal{K}(\tilde{\mathbb{Q}}_{Y_t}, \hat{\mathbb{P}}_{Y_t})^2 \lesssim \int_0^t \tilde{\theta} \left(\mathcal{K}(\tilde{\mathbb{Q}}_{Y_s}, \hat{\mathbb{P}}_{Y_s})^2 \right) ds,$$

where for all $z \in \mathbb{R}_+$, $\tilde{\theta} := \theta^2$ satisfies the assumption $\int_0^z \frac{1}{\tilde{\theta}(y)} dy = \infty$.
In the case $0 \leq \int_0^t \theta\left(\mathcal{K}(\tilde{\mathbb{Q}}_{Y_s}, \hat{\mathbb{P}}_{Y_s})^2\right) ds < 1$, we get

$$\mathcal{K}(\tilde{\mathbb{Q}}_{Y_t}, \hat{\mathbb{P}}_{Y_t})^2 \lesssim \int_0^t \theta\left(\mathcal{K}(\tilde{\mathbb{Q}}_{Y_s}, \hat{\mathbb{P}}_{Y_s})^2\right) ds.$$

We know that $t \mapsto \mathcal{K}(\tilde{\mathbb{Q}}_{Y_t}, \hat{\mathbb{P}}_{Y_t})$ is continuous by the proof of [31, Theorem 4.2] and of Theorem 2.4. Hence, by Bihari's inequality (cf. [32, Lemma 3.6]) $\mathcal{K}(\tilde{\mathbb{Q}}_{Y_t}, \hat{\mathbb{P}}_{Y_t}) = 0$ for all $t \in [0, T]$, which completes the proof. $\square$

2.2. **Existence and Uniqueness of Strong Solutions.** We recall the definition of a strong solution.

Definition 2.9 A *strong solution* of the mean-field SDE (2.13) is a weak solution $(\Omega, \mathcal{F}, \mathbb{F}^B, \mathbb{P}, B, X^x)$ where $\mathbb{F}^B$ is the filtration generated by the Brownian motion B and augmented with the $\mathbb{P}$-null sets.

Remark 2.10. Note that according to Definition 2.9, we say that (2.13) has a strong solution as soon as there exists some stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, B)$ with a Brownian-adapted solution X^x, while usually in the literature the definition of a strong solution requires the (a priori stronger) existence of a Brownian-adapted solution of (2.13) on any given stochastic basis. However, in our setting these two definitions are equivalent. Indeed, a given strong solution $(\Omega, \mathcal{F}, \mathbb{F}^B, \mathbb{P}, B, X^x)$ of the mean-field SDE (2.13) can be considered a strong solution of the associated SDE

$$dX_t^x = b^{\mathbb{P}_X}(t, X_t^x)dt + dB_t, \quad X_0^x = x, \quad t \in [0, T], \tag{2.22}$$

where we define the drift coefficient $b^{\mathbb{P}_X} : [0, T] \times \mathbb{R} \to \mathbb{R}$ by

$$b^{\mathbb{P}_X}(t, y) := b(t, y, \mathbb{P}_{X_t^x}).$$

For strong solutions of SDEs it is then well-known that there exists a family of functionals $(F_t)_{t\in[0,T]}$ with $X_t^x = F_t(B)$ (see e.g. [35] for an explicit form of F_t), such that for any other stochastic basis $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{Q}}, \hat{B})$ the process $\hat{X}_t^x := F_t(\hat{B})$ is an $\mathcal{F}^{\hat{B}}$-adapted solution of SDE (2.22). Further, from the functional form of the solutions we obviously get $\mathbb{P}_X = \mathbb{P}_{\hat{X}}$, and thus $b^{\mathbb{P}_X}(t, y) = b^{\mathbb{P}_{\hat{X}}}(t, y) := b(t, y, \mathbb{P}_{\hat{X}_t^x})$, such that $\hat{X}^x$ fulfills

$$d\hat{X}_t^x = b^{\mathbb{P}_{\hat{X}}}(t, \hat{X}_t^x)dt + d\hat{B}_t, \quad \hat{X}_0^x = x, \quad t \in [0, T],$$

i.e. $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{Q}}, \hat{B}, \hat{X}^x)$ is a strong solution of the mean-field SDE (2.13). Hence, the two definitions of strong solutions are equivalent.

In addition to weak uniqueness, a second type of uniqueness usually considered in the context of strong solutions is pathwise uniqueness:

Definition 2.11 We say a weak solution $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, B^1, X^1)$ of (2.13) is *pathwisely unique*, if for any other weak solution $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, B^2, X^2)$ on the same stochastic basis,

$$\mathbb{P}\left(\forall t \geq 0 : X_t^1 = X_t^2\right) = 1.$$

Remark 2.12. Note that in our setting weak uniqueness and pathwise uniqueness of strong solutions of the mean-field SDE (2.13) are equivalent. Indeed, any weakly unique strong solution of (2.13) is a weakly unique strong solution of the same associated SDE (2.22), i.e. the drift coefficient in (2.22) does not vary with the solution since the law of the solution is unique. Due to [15, Theorem 3.2], a weakly unique strong solution of an SDE is always pathwisely unique, and thus a weakly unique strong solution of (2.13) is pathwisely unique. Vice versa, by the considerations in Remark 2.10, any pathwisely unique strong solution $(\Omega, \mathcal{F}, \mathbb{P}, B, X^x)$ of (2.13) can be represented by $X_t^x = F_t(B)$ for some unique family of functionals $(F_t)_{t\in[0,T]}$ that does not vary with the stochastic basis. Consequently, the strong solution is weakly unique. Thus, in the following we will just speak of a *unique strong solution* of (2.13).

In order to establish existence of strong solutions we require in addition to the assumptions in Theorem 2.4 that the drift coefficient exhibits the particular linear growth given by the decomposable form (2.7), that is, the irregular behavior of the drift stays in a bounded spectrum.

Theorem 2.13 *Suppose the drift coefficient b is in the decomposable form* (2.7) *and additionally continuous in the third variable, i.e. fulfills* (2.4)*. Then there exists a strong solution of the mean-field SDE* (2.13)*. More precisely, any weak solution $(X_t^x)_{t\in[0,T]}$ of* (2.13) *is a strong solution, and in addition X_t^x is Malliavin differentiable for every $t \in [0, T]$.*
If moreover b satisfies (2.20)*, i.e. b admits a modulus of continuity in the third variable, the solution is unique.*

Proof. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, B, X^x)$ be a weak solution of the mean-field SDE (2.13), which exists by Theorem 2.4. Then X^x can be interpreted as weak solution of the associated SDE introduced in (2.22).

Now we note that under the assumptions specified in Theorem 2.13 the drift $b^{\mathbb{P}_X}(t, y)$ of the associated SDE in (2.22) admits a decomposition

$$b^{\mathbb{P}_X}(t, y) = \hat{b}^{\mathbb{P}_X}(t, y) + \tilde{b}^{\mathbb{P}_X}(t, y),$$

where $\hat{b}^{\mathbb{P}_X}$ is merely measurable and bounded and $\tilde{b}^{\mathbb{P}_X}$ is of at most linear growth and Lipschitz continuous in the second variable. Thus, $b^{\mathbb{P}_X}$ fulfills the assumptions required in [2, Theorem 3.1], from which it follows that X^x is the unique strong (that is $\mathbb{F}^B$-adapted) solution of SDE (2.22) and is Malliavin differentiable. Thus, X^x is indeed a Malliavin differentiable strong solution of mean-field SDE (2.13). If further b admits a modulus of continuity in the third variable, then by Theorem 2.8,

X^x is a weakly, and by Remark 2.12 also pathwisely, unique strong solution of (2.13). $\square$

3. Regularity properties

In this section we first give a representation of the Malliavin derivative of a strong solution to mean-field SDE (2.13) in terms of a space-time integral with respect to local time in Subsection 3.1 which yields a relation to the first variation process which will be essential in the remainder of the paper. In the remaining parts of the section we then investigate regularity properties of a strong solution of mean-field SDE (2.13) in its initial condition. More precisely, in Subsection 3.2 we establish Sobolev differentiability and give a representation of the first variation process, and in Subsection 3.3 we show Hölder continuity in time and space.

3.1. Malliavin derivative.

If the drift b is Lipschitz continuous in the second variable, it is well-known that the Malliavin derivative of a strong solution to mean-field SDE (2.13) is given by $D_s X_t^x = \exp\left\{\int_s^t \partial_2 b(u, X_u^x, \mathbb{P}_{X_u^x})du\right\}$. For irregular drift b we obtain the following generalized representation of the Malliavin derivative without the derivative of b which is an immediate consequence of Theorem 2.13 and [2, Proposition 3.2]:

Proposition 3.1 *Suppose the drift coefficient b satisfies the assumptions of Theorem 2.13. Then for $0 \leq s \leq t \leq T$, the Malliavin derivative $D_s X_t^x$ of a strong solution X^x to the mean-field SDE* (2.13) *has the following representation:*

$$D_s X_t^x = \exp\left\{-\int_s^t \int_{\mathbb{R}} b(u, y, \mathbb{P}_{X_u^x}) L^{X^x}(du, dy)\right\}$$

Here $L^{X^x}(du, dy)$ denotes integration with respect to local time of X^x in time and space, see [2] *and* [18] *for more details.*

3.2. Sobolev differentiability.

In the remaining section we analyze the regularity of a strong solution X^x of (2.13) in its initial condition x. More precisely, the two main results in this subsection are the existence of a weak (Sobolev) derivative $\partial_x X_t^x$, which also is referred to as the *first variation process*, for irregular drift coefficients in Theorem 3.3 and a representation of $\partial_x X_t^x$ in terms of a local time integral in Proposition 3.4.

We recall the definition of the Sobolev space $W^{1,2}(U)$.

Definition 3.2 Let $U \subset \mathbb{R}$ be an open and bounded subset. The Sobolev space $W^{1,2}(U)$ is defined as the set of functions $u : \mathbb{R} \to \mathbb{R}$, $u \in L^2(U)$, such that its weak derivative belongs to $L^2(U)$. Furthermore, the Sobolev space is endowed with the norm

$$\|u\|_{W^{1,2}(U)} = \|u\|_{L^2(U)} + \|u'\|_{L^2(U)},$$

where u' is the weak derivative of $u \in W^{1,2}(U)$. We say a stochastic process X is Sobolev differentiable in U, if for all $t \in [0, T]$, X_t belongs $\mathbb{P}$-a.s. to $W^{1,2}(U)$.

Theorem 3.3 *Suppose the drift coefficient b is in the decomposable form* (2.7) *and uniformly Lipschitz continuous in the third variable* (2.10)*. Let $(X_t^x)_{t\in[0,T]}$ be the unique strong solution of* (2.13) *and $U \subset \mathbb{R}$ be an open and bounded subset. Then for every $t \in [0,T]$,*

$$(x \mapsto X_t^x) \in L^2\left(\Omega, W^{1,2}(U)\right).$$

Before we turn our attention to the proof of Theorem 3.3, we give a probabilistic representation of the first variation process $\partial_x X_t^x$ which in particular yields a connection to the Malliavin derivative. We remark that we will see in Proposition 3.11 that the derivative $\partial_x b\left(s, y, \mathbb{P}_{X_s^x}\right)$ used in Proposition 3.4 is well-defined.

Proposition 3.4 *Suppose the drift coefficient b is in the decomposable form* (2.7) *and uniformly Lipschitz continuous in the third variable* (2.10)*. For almost all $x \in \mathbb{R}$ the first variation process (in the Sobolev sense) of the unique strong solution $(X_t^x)_{t\in[0,T]}$ of the mean-field SDE* (2.13) *has $dt \otimes d\mathbb{P}$ almost surely the representation*

$$\begin{aligned}\partial_x X_t^x = &\exp\left\{-\int_0^t \int_{\mathbb{R}} b\left(u, y, \mathbb{P}_{X_u^x}\right) L^{X^x}(du, dy)\right\} \\ &+ \int_0^t \exp\left\{-\int_s^t \int_{\mathbb{R}} b\left(u, y, \mathbb{P}_{X_u^x}\right) L^{X^x}(du, dy)\right\} \partial_x b\left(s, y, \mathbb{P}_{X_s^x}\right)|_{y=X_s^x} ds.\end{aligned} \tag{2.23}$$

Furthermore, for $s, t \in [0,T]$, $s \leq t$, the following relationship with the Malliavin derivative holds:

$$\partial_x X_t^x = D_s X_t^x \partial_x X_s^x + \int_s^t D_u X_t^x \partial_x b\left(u, y, \mathbb{P}_{X_u^x}\right)|_{y=X_u^x} du. \tag{2.24}$$

The remaining parts of this subsection are devoted to the proofs of Theorem 3.3 and Proposition 3.4. More precisely, the proof of Theorem 3.3 is structured as follows. First we show Lipschitz continuity of X_t^x in x for smooth coefficients b in Proposition 3.5. Then we define an approximating sequence of mean-field solutions $\{X_t^{n,x}\}_{n\geq 1}$ with smooth drift coefficients which is shown in Proposition 3.8 to converge in $L^2(\Omega, \mathcal{F}_t)$ to the unique strong solution X_t^x of mean-field SDE (2.13) with general drift. Finally, after also establishing weak L^2-convergence of functionals of the approximating sequence in Proposition 3.9 and a technical result in Lemma 3.10 we are ready to prove Theorem 3.3 using a compactness argument.

Proposition 3.5 *Let $b \in L^\infty([0,T], \mathcal{C}_b^{1,L}(\mathbb{R} \times \mathcal{P}_1(\mathbb{R})))$ and X^x be the unique strong solution of mean-field SDE* (2.13)*. Then, for all $t \in [0,T]$ the map $x \mapsto X_t^x$*

is a.s. Lipschitz continuous and consequently weakly and almost everywhere differentiable. Moreover, the first variation process $\partial_x X_t^x$, $t \in [0,T]$, *has the representation*

$$\begin{aligned}\partial_x X_t^x &= \exp\left\{\int_0^t \partial_2 b(s, X_s^x, \mathbb{P}_{X_s^x}) ds\right\} \\ &\quad + \int_0^t \exp\left\{\int_u^t \partial_2 b(s, X_s^x, \mathbb{P}_{X_s^x}) ds\right\} \partial_x b(u, y, \mathbb{P}_{X_u^x})|_{y=X_u^x} du.\end{aligned} \tag{2.25}$$

Remark 3.6. Note that compared to [1] we consider the more general case of mean-field SDEs of type (2.13) and therefore need to deal with differentiability of functions over the metric space $\mathcal{P}_1(\mathbb{R})$ as in [6], [7], and [29]. We avoid using the notion of differentiation with respect to a measure by considering the real function $x \mapsto b(t, y, \mathbb{P}_{X_t^x})$, for which differentiation is understood in the Sobolev sense.

Proof of Proposition 3.5. In order to prove Lipschitz continuity we have to show that there exists a constant $C > 0$ such that for almost every $\omega \in \Omega$ and for all $t \in [0,T]$ the map $(x \mapsto X_t^x) \in \mathrm{Lip}_C(\mathbb{R})$. For notational reasons we hide ω in our computations and obtain using $b \in C_b^{1,L}(\mathbb{R} \times \mathcal{P}_1(\mathbb{R}))$ that

$$\begin{aligned}|X_t^x - X_t^y| &= \left| x - y + \int_0^t b(s, X_s^x, \mathbb{P}_{X_s^x}) - b(s, X_s^y, \mathbb{P}_{X_s^y}) ds \right| \\ &\lesssim |x - y| + \int_0^t |X_s^x - X_s^y| + \mathcal{K}(\mathbb{P}_{X_s^x}, \mathbb{P}_{X_s^y}) ds.\end{aligned} \tag{2.26}$$

Hence, we immediately get that

$$\mathcal{K}(\mathbb{P}_{X_t^x}, \mathbb{P}_{X_t^y}) \le \mathbb{E}[|X_t^x - X_t^y|] \lesssim |x - y| + \int_0^t \mathbb{E}[|X_s^x - X_s^y|] ds,$$

and therefore by Grönwall's inequality with respect to $\mathbb{E}[|X_t^x - X_t^y|]$ we have that

$$\mathcal{K}(\mathbb{P}_{X_s^x}, \mathbb{P}_{X_s^y}) \lesssim |x - y|. \tag{2.27}$$

Consequently, (2.26) simplifies to

$$|X_t^x - X_t^y| \lesssim |x - y| + \int_0^t |X_s^x - X_s^y| ds, \tag{2.28}$$

and again by Grönwall's inequality we get that $(x \mapsto X_t^x) \in \mathrm{Lip}_C(\mathbb{R})$. Note that due to (2.27) and the assumptions on b also $x \mapsto b(t, y, \mathbb{P}_{X_t^x})$ is weakly differentiable for every $t \in [0,T]$ and $y \in \mathbb{R}$.
Regarding representation (2.25), note first that by taking the derivative with respect to x in (2.13), $\partial_x X_t^x$ has the representation

$$\partial_x X_t^x = 1 + \int_0^t \partial_2 b(s, X_s^x, \mathbb{P}_{X_s^x}) \partial_x X_s^x + \partial_x b(s, y, \mathbb{P}_{X_s^x})|_{y=X_s^x} ds. \tag{2.29}$$

It is readily seen that (2.25) solves this ODE ω-wise and therefore is a representation of the first variation process of X_t^x. $\square$

As an immediate consequence of Proposition 3.5 and the representation of the Malliavin derivative $D_s X_t^x$, $0 \leq s \leq t \leq T$, given in Proposition 3.1, we get the following connection between the first variation process and the Malliavin derivative:

Corollary 3.7 *Let $b \in L^\infty([0,T], \mathcal{C}_b^{1,L}(\mathbb{R} \times \mathcal{P}_1(\mathbb{R})))$. Then, for every $0 \leq s \leq t \leq T$,*

$$\partial_x X_t^x = D_s X_t^x \partial_x X_s^x + \int_s^t D_u X_t^x \partial_x b(u, y, \mathbb{P}_{X_u^x})|_{y=X_u^x} du. \tag{2.30}$$

Now let b be a general drift coefficient that allows for a decomposition $b = \tilde{b} + \hat{b}$ as in (2.7) and is uniformly Lipschitz continuous in the third variable (2.10). Let $(X_t^x)_{t\in[0,T]}$ be the corresponding strong solution of (2.13) ascertained by Theorem 2.13. In order to extend Proposition 3.5 we apply a compactness criterion to an approximating sequence of weakly differentiable mean-field SDEs. By standard approximation arguments there exists a sequence of approximating drift coefficients

$$b_n := \tilde{b}_n + \hat{b}, \quad n \geq 1, \tag{2.31}$$

where $\tilde{b}_n \in L^\infty([0,T], \mathcal{C}_b^{1,L}(\mathbb{R} \times \mathcal{P}_1(\mathbb{R})))$ with $\sup_{n\geq 1} \|\tilde{b}_n\|_\infty \leq C < \infty$, where $\|\cdot\|_\infty$ is the sup norm on all variables, such that $b_n \to b$ pointwise in every μ and a.e. in (t, y) with respect to the Lebesgue measure. Furthermore, we denote $b_0 := b$ and choose the approximating coefficients b_n such that they fulfill the uniform Lipschitz continuity in the third variable (2.10) uniformly in $n \geq 0$. Under these conditions the corresponding mean-field SDEs, defined by

$$dX_t^{n,x} = b_n(t, X_t^{n,x}, \mathbb{P}_{X_t^{n,x}})dt + dB_t, \quad X_0^{n,x} = x \in \mathbb{R}, \quad t \in [0,T], \quad n \geq 1, \tag{2.32}$$

have unique strong solutions which are Malliavin differentiable by Theorem 2.13. Likewise the strong solutions $\{X^{n,x}\}_{n\geq 1}$ are weakly differentiable with respect to the initial condition by Proposition 3.5. In the next step we verify that $(X_t^{n,x})_{t\in[0,T]}$ converges to $(X_t^x)_{t\in[0,T]}$ in $L^2(\Omega, \mathcal{F}_t)$ as $n \to \infty$.

Proposition 3.8 *Suppose the drift coefficient b is in the decomposable form* (2.7) *and uniformly Lipschitz continuous in the third variable* (2.10)*. Let $(X_t^x)_{t\in[0,T]}$ be the unique strong solution of* (2.13)*. Furthermore, $\{b_n\}_{n\geq 1}$ is the approximating sequence of b as defined in* (2.31) *and $(X_t^{n,x})_{t\in[0,T]}$, $n \geq 1$, the corresponding unique strong solutions of* (2.32)*. Then, there exists a subsequence $\{n_k\}_{k\geq 1} \subset \mathbb{N}$ such that*

$$X_t^{n_k,x} \xrightarrow[k\to\infty]{} X_t^x, \quad t \in [0,T],$$

strongly in $L^2(\Omega, \mathcal{F}_t)$.

Proof. In the case of SDEs it is shown in [2, Theorem A.4] that for every $t \in [0,T]$, the sequence $\{X_t^{n,x}\}_{n\geq 1}$ is relatively compact in $L^2(\Omega, \mathcal{F}_t)$. The proof therein can be extended to the assumptions of Proposition 3.8 and the case of mean-field SDEs due to Proposition 3.1. Consequently, for every $t \in [0,T]$ there exists a subsequence $\{n_k(t)\}_{k\geq 1} \subset \mathbb{N}$ such that $X_t^{n_k(t),x}$ converges to some Y_t strongly in $L^2(\Omega, \mathcal{F}_t)$. We need to show that the converging subsequence can be chosen independent of t. To this end we consider the Hida test function space $\mathcal{S}$ and the Hida distribution space $\mathcal{S}^*$ as defined in Definition B.1 and prove that $\{t \mapsto X_t^{n,x}\}_{n\geq 1}$ is relatively compact in $\mathcal{C}([0,T];\mathcal{S}^*)$, which is well-defined since

$$\mathcal{S} \subset L^2(\Omega) \subset \mathcal{S}^*.$$

In order to show this, we use Theorem B.2 and show instead that $\{t \mapsto X_t^{n,x}[\phi]\}_{n\geq 1}$ is relatively compact in $\mathcal{C}([0,T];\mathbb{R})$ for any $\phi \in \mathcal{S}$, where $X_t^{n,x}[\phi] := \mathbb{E}[X_t^{n,x}\phi]$. Since $X^{n,x}$ is a solution of (2.32), using Cauchy-Schwarz' inequality and Lemma A.4 yields

$$\begin{aligned}
|X_t^{n,x}[\phi] - X_s^{n,x}[\phi]| &= |\mathbb{E}[(X_t^{n,x} - X_s^{n,x})\phi]| \\
&= \left|\mathbb{E}\left[\left(\int_s^t b_n(u, X_u^{n,x}, \mathbb{P}_{X_u^{n,x}})du + B_t - B_s\right)\phi\right]\right| \\
&\leq \left(\int_s^t \mathbb{E}\left[b_n(u, X_u^{n,x}, \mathbb{P}_{X_u^{n,x}})^2\right]^{\frac{1}{2}} du + |t-s|^{\frac{1}{2}}\right) \|\phi\|_{L^2(\Omega)} \leq C\|\phi\|_{L^2(\Omega)}|t-s|^{\frac{1}{2}},
\end{aligned} \tag{2.33}$$

where $C > 0$ is a constant depending on T and in particular is independent of n which shows equicontinuity of $\{t \mapsto X_t^{n,x}[\phi]\}_{n\geq 1}$. Moreover, due to Lemma A.4

$$\sup_{n\geq 1} X_0^{n,x}[\phi] = \sup_{n\geq 1} \mathbb{E}[X_0^{n,x}\phi] \leq \sup_{n\geq 1} x\|\phi\|_{L^2(\Omega)} < \infty,$$

and therefore $X_t^{n,x}[\phi]$ is uniformly bounded in $n \geq 1$. Thus, by the version of the Arzelà-Ascoli theorem given in Theorem B.3 the family $\{t \mapsto X_t^{n,x}[\phi]\}_{n\geq 1}$ is relatively compact in $\mathcal{C}([0,T];\mathbb{R})$. Since ϕ was arbitrary, we have proven using Theorem B.2 that $\{t \mapsto X_t^{n,x}\}_{n\geq 1}$ is relatively compact in $\mathcal{C}([0,T];\mathcal{S}^*)$, i.e. there exists a subsequence $(n_k)_{k\geq 1}$ and $\{t \mapsto Z_t\} \in \mathcal{C}([0,T];\mathcal{S}^*)$ such that

$$\{t \mapsto X_t^{n_k,x}\} \xrightarrow[k\to\infty]{} \{t \mapsto Z_t\} \tag{2.34}$$

in $\mathcal{C}([0,T];\mathcal{S}^*)$. Furthermore, we have shown that for every $t \in [0,T]$ there exists a subsequence $(n_{k_m}(t))_{m\geq 1} \subset (n_k)_{k\geq 1}$ such that in $L^2(\Omega, \mathcal{F}_t)$,

$$X_t^{n_{k_m}(t),x} \xrightarrow[m\to\infty]{} Y_t.$$

Note that for every $t \in [0,T]$, we get by (2.34)

$$X_t^{n_{k_m}(t),x} \xrightarrow[m\to\infty]{} Z_t$$

in $\mathcal{S}^*$. By uniqueness of the limit $Y_t = Z_t$ for every $t \in [0,T]$ and hence, the convergence in $L^2(\Omega, \mathcal{F}_t)$ holds for the t independent subsequence $(n_k)_{k\geq 1}$.
In the last step, which is deferred to the subsequent lemma, we show for all $t \in$

$[0,T]$ that $X_t^{n,x}$ converges weakly in $L^2(\Omega,\mathcal{F}_t)$ to the unique strong solution $\overline{X}_t^x$ of SDE

$$d\overline{X}_t^x = b(t,\overline{X}_t^x,\mathbb{P}_{Y_t})dt + dB_t, \quad \overline{X}_0^x = x \in \mathbb{R}, \quad t\in[0,T]. \tag{2.35}$$

Consequently, $X_t^{n,x}$ converges to X_t^x in $L^2(\Omega,\mathcal{F}_t)$. Indeed, we have shown that $X_t^{n,x}$ converges in $L^2(\Omega,\mathcal{F}_t)$ to Y_t for all $t\in[0,T]$. Moreover $X_t^{n,x}$ converges weakly in $L^2(\Omega,\mathcal{F}_t)$ to $\overline{X}_t^x$ for all $t\in[0,T]$. Hence, by uniqueness of the limit, $Y_t \stackrel{d}{=} \overline{X}_t^x$ for all $t\in[0,T]$. Thus (2.35) is identical to (2.13) and we can write $X=\overline{X}$, which shows Proposition 3.8. □

In the following we assume without loss of generality that the whole sequence $\{X_t^{n,x}\}_{n\geq 1}$ converges to X_t^x strongly in $L^2(\Omega,\mathcal{F}_t)$ for every $t\in[0,T]$. Then, in addition to strong L^2-convergence of the solutions, we also get weak L^2-convergence of $\phi(X_t^{n,x})$ to $\phi(X_t^x)$ for functions ϕ in certain L^p-spaces. To this end, we define the weight function $\omega_T:\mathbb{R}\to\mathbb{R}$ by

$$\omega_T(y) := \exp\left\{-\frac{|y|^2}{4T}\right\}, \quad y\in\mathbb{R}. \tag{2.36}$$

Proposition 3.9 *Suppose the drift coefficient b is in the decomposable form* (2.7) *and uniformly Lipschitz continuous in the third variable* (2.10). *Let $(X_t^x)_{t\in[0,T]}$ be the unique strong solution of* (2.13). *Furthermore, $\{b_n\}_{n\geq1}$ is the approximating sequence of b as defined in* (2.31) *and $(X_t^{n,x})_{t\in[0,T]}$, $n\geq 1$, the corresponding unique strong solutions of* (2.32). *Then, for every $t\in[0,T]$ and function $\phi\in L^{2p}(\mathbb{R};\omega_T)$ with $p:=\frac{1+\varepsilon}{\varepsilon}$, $\varepsilon>0$ sufficiently small with regard to Lemma A.4,*

$$\phi(X_t^{n,x}) \xrightarrow[n\to\infty]{} \phi(X_t^x)$$

weakly in $L^2(\Omega,\mathcal{F}_t)$.

Proof. As described in the proof of Proposition 3.8 it suffices to show for all $t\in[0,T]$ that $\phi(X_t^{n,x})$ converges weakly to $\phi(\overline{X}_t^x)$, where $\overline{X}_t^x$ is the unique strong solution of SDE (2.35). This can be shown equivalently to [2, Lemma A.3]. First note that $\phi(X_t^{n,x}),\phi(\overline{X}_t^x)\in L^2(\Omega,\mathcal{F}_t)$, $n\geq 0$. Hence, in order to show weak convergence it suffices to show that

$$\mathcal{W}(\phi(X_t^{n,x}))(f) \xrightarrow[n\to\infty]{} \mathcal{W}(\phi(\overline{X}_t^x))(f),$$

for every $f\in L^2([0,T])$. One can show by Hölder's inequality, inequality (2.18) and Lemma A.4 that

$$\begin{aligned}
&\left|\mathcal{W}(\phi(X_t^{n,x}))(f) - \mathcal{W}(\phi(\overline{X}_t^x))(f)\right| = \\
&\lesssim \mathbb{E}\left[\left(\mathcal{E}\left(\int_0^T b_n(s,B_s^x,\mathbb{P}_{X_s^{n,x}}) + f(s)dB_s\right) - \mathcal{E}\left(\int_0^T b(s,B_s^x,\mathbb{P}_{Y_s}) + f(s)dB_s\right)\right)^q\right]^{\frac{1}{q}} \\
&\lesssim A_n,
\end{aligned}$$

where $q := \frac{2(1+\varepsilon)}{2+\varepsilon}$ and

$$A_n := \mathbb{E}\left[\left(\int_0^T \left(b_n(s, B_s^x, \mathbb{P}_{X_s^{n,x}}) - b(s, B_s^x, \mathbb{P}_{Y_s})\right) dB_s\right.\right.$$
$$\left.\left. -\frac{1}{2}\int_0^T \left((b_n(s, B_s^x, \mathbb{P}_{X_s^{n,x}}) + f(s))^2 - (b(s, B_s^x, \mathbb{P}_{Y_s}) + f(s))^2\right) ds\right)^{2p}\right]^{\frac{1}{2p}}.$$

Using Minkowski's inequality and Burkholder-Davis-Gundy's inequality yields

$$\begin{aligned}
A_n &\leq \mathbb{E}\left[\left|\int_0^T b_n(s, B_s^x, \mathbb{P}_{X_s^{n,x}}) - b(s, B_s^x, \mathbb{P}_{Y_s}) dB_s\right|^{2p}\right]^{\frac{1}{2p}} \\
&\quad + \mathbb{E}\left[\left|\frac{1}{2}\int_0^T (b_n(s, B_s^x, \mathbb{P}_{X_s^{n,x}}) + f(s))^2 - (b(s, B_s^x, \mathbb{P}_{Y_s}) + f(s))^2 ds\right|^{2p}\right]^{\frac{1}{2p}} \\
&\lesssim \mathbb{E}\left[\left(\int_0^T \left|b_n(s, B_s^x, \mathbb{P}_{X_s^{n,x}}) - b(s, B_s^x, \mathbb{P}_{Y_s})\right|^2 ds\right)^p\right]^{\frac{1}{2p}} \\
&\quad + \mathbb{E}\left[\left(\int_0^T \left|(b_n(s, B_s^x, \mathbb{P}_{X_s^{n,x}}) + f(s))^2 - (b(s, B_s^x, \mathbb{P}_{Y_s}) + f(s))^2\right| ds\right)^{2p}\right]^{\frac{1}{2p}} \\
&=: D_n + E_n.
\end{aligned}$$

Looking at the first summand, we see using the triangle inequality that

$$\begin{aligned}
D_n &= \mathbb{E}\left[\left(\int_0^T \left|b_n(s, B_s^x, \mathbb{P}_{X_s^{n,x}}) - b(s, B_s^x, \mathbb{P}_{Y_s})\right|^2 ds\right)^p\right]^{\frac{1}{2p}} \\
&\leq \mathbb{E}\left[\left(\int_0^T \left|b_n(s, B_s^x, \mathbb{P}_{X_s^{n,x}}) - b_n(s, B_s^x, \mathbb{P}_{Y_s})\right|^2 ds\right)^p\right]^{\frac{1}{2p}} \\
&\quad + \mathbb{E}\left[\left(\int_0^T |b_n(s, B_s^x, \mathbb{P}_{Y_s}) - b(s, B_s^x, \mathbb{P}_{Y_s})|^2 ds\right)^p\right]^{\frac{1}{2p}}.
\end{aligned}$$

Since there exists a constant $C > 0$ such that $(\mu \mapsto b_n(t, y, \mu)) \in \mathrm{Lip}_C(\mathcal{P}_1(\mathbb{R}))$ for all $n \geq 0$, $t \in [0, T]$, $y \in \mathbb{R}$ and $X_s^{n,x} \xrightarrow[n\to\infty]{L^2(\Omega, \mathcal{F}_s)} Y_s$ for all $s \in [0, T]$ by the proof of Proposition 3.8, we get by dominated convergence that D_n converges to 0 as $n \to \infty$. Equivalently one can show that also E_n converges to 0 as n tends to infinity. Therefore $\left|\mathcal{W}(\phi(X_t^{n,x}))(f) - \mathcal{W}(\phi(\overline{X}_t^x))(f)\right|$ converges to 0 as $n \to \infty$ and the claim holds. $\square$

The following lemma will be used in the application of the compactness argument in the proof of Theorem 3.3.

Lemma 3.10 *Let* $\{(X_t^{n,x})_{t\in[0,T]}\}_{n\geq 1}$ *be the unique strong solutions of* (2.32)*. Then, for any compact subset* $K\subset\mathbb{R}$ *and* $p\geq 2$*,*

$$\sup_{n\geq 1}\sup_{t\in[0,T]}\operatorname*{ess\,sup}_{x\in K}\mathbb{E}\left[|\partial_x X_t^{n,x}|^p\right]\leq C,$$

for some constant $C>0$.

Proof. By Corollary 3.7, we have

$$\partial_x X_t^{n,x} = D_0 X_t^{n,x} + \int_0^t D_u X_t^{n,x}\partial_x b_n(u,y,\mathbb{P}_{X_u^{n,x}})|_{y=X_u^{n,x}}du. \tag{2.37}$$

Using Proposition 3.1 as well as Girsanov's theorem and Hölder's inequality with $q:=\frac{1+\varepsilon}{\varepsilon}$, $\varepsilon>0$ sufficiently small with regard to Lemma A.4, yields together with Lemma A.5 that

$$\begin{aligned}\mathbb{E}\left[|D_s X_t^{n,x}|^p\right] &= \mathbb{E}\left[\exp\left\{-p\int_s^t\int_{\mathbb{R}} b_n(u,y,\mathbb{P}_{X_u^{n,x}})L^{X^x}(du,dy)\right\}\right]\\ &\lesssim \mathbb{E}\left[\exp\left\{-qp\int_s^t\int_{\mathbb{R}} b_n(u,y,\mathbb{P}_{X_u^{n,x}})L^{B^x}(du,dy)\right\}\right]^{\frac{1}{q}}\leq C_1,\end{aligned}\tag{2.38}$$

for some constant $C_1>0$ independent of $n\geq 0$, $x\in K$ and $s,t\in[0,T]$. Hence, we get for every $n\geq 1$ and almost every $x\in K$ with Minkowski's and Hölder's inequality using that $(\mu\mapsto b(t,y,\mu))\in \mathrm{Lip}_C(\mathcal{P}_1(\mathbb{R}))$ for every $t\in[0,T]$ and $y\in\mathbb{R}$ that

$$\begin{aligned}\mathbb{E}\left[|\partial_x X_t^{n,x}|^p\right]^{\frac{1}{p}} &= \mathbb{E}\left[\left|D_0 X_t^{n,x} + \int_0^t D_u X_t^{n,x}\partial_x b_n(u,y,\mathbb{P}_{X_u^{n,x}})|_{y=X_u^{n,x}}du\right|^p\right]^{\frac{1}{p}}\\ &\lesssim \sup_{0\leq u\leq T}\mathbb{E}\left[|D_u X_t^{n,x}|^{2p}\right]^{\frac{1}{2p}}\left(1+\mathbb{E}\left[\left(\int_0^t\left|\partial_x b_n\left(u,y,\mathbb{P}_{X_u^{n,x}}\right)\right|_{y=X_u^{n,x}}du\right)^{2p}\right]^{\frac{1}{2p}}\right)\\ &\lesssim 1+\mathbb{E}\left[\left(\int_0^t\left|\lim_{x_0\to x}\frac{b_n\left(u,X_u^{n,x},\mathbb{P}_{X_u^{n,x}}\right)-b_n\left(u,X_u^{n,x},\mathbb{P}_{X_u^{n,x_0}}\right)}{|x-x_0|}\right|du\right)^{2p}\right]^{\frac{1}{2p}}\\ &\lesssim 1+\liminf_{x_0\to x}\frac{1}{|x-x_0|}\int_0^t\mathcal{K}\left(\mathbb{P}_{X_u^{n,x}},\mathbb{P}_{X_u^{n,x_0}}\right)du.\end{aligned}\tag{2.39}$$

Denote by $\overline{\operatorname{conv}(K)}$ the closed convex hull of K and note that $\overline{\operatorname{conv}(K)}$ is again a compact set. Moreover, we can bound the Kantorovich metric of $\mathbb{P}_{X_u^{n,x}}$ and $\mathbb{P}_{X_u^{n,x_0}}$ for arbitrary $x,x_0\in\overline{\operatorname{conv}(K)}$ by using the second fundamental theorem of calculus and representation (2.25):

$$\begin{aligned}
\mathcal{K}\left(\mathbb{P}_{X_u^{n,x}},\mathbb{P}_{X_u^{n,x_0}}\right) &\leq \mathbb{E}\left[\left|\int_0^u b_n(s,X_s^{n,x},\mathbb{P}_{X_s^{n,x}}) - b_n(s,X_s^{n,x_0},\mathbb{P}_{X_s^{n,x_0}})ds\right|\right] \\
&= |x-x_0|\mathbb{E}\left[\left|\int_0^u\int_0^1 \partial_2 b_n\left(s,X_s^{n,x+\tau(x_0-x)},\mathbb{P}_{X_s^{n,x+\tau(x_0-x)}}\right)\partial_\tau X_s^{n,x+\tau(x_0-x)}\right.\right. \\
&\quad \left.\left.+\partial_\tau b_n\left(s,z,\mathbb{P}_{X_s^{n,x+\tau(x_0-x)}}\right)|_{z=X_s^{n,x+\tau(x_0-x)}}d\tau ds\right|\right] \\
&\leq |x-x_0|\int_0^1\mathbb{E}\left[\left|\int_0^u \partial_2 b_n\left(s,X_s^{n,x+\tau(x_0-x)},\mathbb{P}_{X_s^{n,x+\tau(x_0-x)}}\right)\partial_\tau X_s^{n,x+\tau(x_0-x)}\right.\right. \qquad (2.40) \\
&\quad \left.\left.+\partial_\tau b_n\left(s,z,\mathbb{P}_{X_s^{n,x+\tau(x_0-x)}}\right)|_{z=X_s^{n,x+\tau(x_0-x)}}ds\right|\right]d\tau \\
&= |x-x_0|\int_0^1\mathbb{E}\left[\left|\partial_\tau X_u^{n,x+\tau(x_0-x)}-(1-\tau)\right|\right]d\tau \\
&\lesssim |x-x_0| + |x-x_0|\operatorname*{ess\,sup}_{x\in\overline{\mathrm{conv}(K)}}\mathbb{E}\left[|\partial_x X_u^{n,x}|\right].
\end{aligned}$$

Putting all together we can find a constant $C_2>0$ independent of $n\geq 1$, $t\in[0,T]$ and $x\in\overline{\mathrm{conv}(K)}$ such that

$$\operatorname*{ess\,sup}_{x\in\overline{\mathrm{conv}(K)}}\mathbb{E}\left[|\partial_x X_t^{n,x}|^p\right]^{\frac{1}{p}} \leq C_2 + C_2\int_0^t \operatorname*{ess\,sup}_{x\in\overline{\mathrm{conv}(K)}}\mathbb{E}\left[|\partial_x X_u^{n,x}|^p\right]^{\frac{1}{p}}du.$$

Note that by (2.39) and (2.27) we can find constants $C_3(n),C_4(n)>0$ for every $n\geq 1$ independent of $t\in[0,T]$ and $x\in\overline{\mathrm{conv}(K)}$ such that

$$\mathbb{E}\left[|\partial_x X_t^{n,x}|^p\right]^{\frac{1}{p}} \leq C_3(n)\left(1+\liminf_{x_0\to x}\frac{1}{|x-x_0|}\int_0^t\mathcal{K}\left(\mathbb{P}_{X_u^{n,x}},\mathbb{P}_{X_u^{n,x_0}}\right)du\right)\leq C_4(n)<\infty.$$

Hence, $t\mapsto \operatorname{ess\,sup}_{x\in\overline{\mathrm{conv}(K)}}\mathbb{E}\left[|\partial_x X_t^{n,x}|^p\right]^{\frac{1}{p}}$ is integrable over $[0,T]$. Since it is also Borel measurable, we can apply Jones' generalization of Grönwall's inequality [22, Lemma 5] to get

$$\operatorname*{ess\,sup}_{x\in K}\mathbb{E}\left[|\partial_x X_t^{n,x}|^p\right]^{\frac{1}{p}} \leq \operatorname*{ess\,sup}_{x\in\overline{\mathrm{conv}(K)}}\mathbb{E}\left[|\partial_x X_t^{n,x}|^p\right]^{\frac{1}{p}} \leq C_2 + C_2^2\int_0^t e^{C_2(t-s)}ds<\infty.$$

□

Finally, we are able to give the proof of Theorem 3.3.

Proof of Theorem 3.3. Let $(X_t^{n,x})_{t\in[0,T]}$ be the unique strong solutions of (2.32). The main idea of this proof is to show that $\{X_t^n\}_{n\geq 1}$ is weakly relatively compact in $L^2(\Omega,W^{1,2}(U))$ and to identify the weak limit $Y:=\lim_{k\to\infty}X^{n_k}$ in $L^2(\Omega,W^{1,2}(U))$ with X, where $\{n_k\}_{k\geq 1}$ is a suitable subsequence.

Due to Lemma A.4 and Lemma 3.10

$$\sup_{n\geq 1}\mathbb{E}\left[\|X_t^{n,x}\|^2_{W^{1,2}(U)}\right]<\infty,$$

and thus, the sequence $X_t^{n,x}$ is weakly relatively compact in $L^2(\Omega, W^{1,2}(U))$, see e.g. [30, Theorem 10.44]. Consequently, there exists a sub-sequence n_k, $k \geq 0$ such that $X_t^{n_k,x}$ converges weakly to some $Y_t \in L^2(\Omega, W^{1,2}(U))$ as $k \to \infty$. Let $\phi \in \mathcal{C}_0^\infty(U)$ be an arbitrary test function and denote by ϕ' if well-defined its first derivative. Define

$$\langle X_t^n, \phi \rangle := \int_U X_t^{n,x} \phi(x) dx.$$

Then for all measurable sets $A \in \mathcal{F}$ and $t \in [0,T]$ we get by Lemma A.4 that

$$\mathbb{E}\left[\mathbb{1}_A \langle X_t^n - X_t, \phi' \rangle\right] \leq \|\phi'\|_{L^2(U)} |U|^{\frac{1}{2}} \sup_{x \in \overline{U}} \mathbb{E}\left[\mathbb{1}_A |X_t^{n,x} - X_t^x|^2\right]^{\frac{1}{2}} < \infty,$$

where $\overline{U}$ is the closure of U, and consequently by Proposition 3.8 we get that $\lim_{n\to\infty} \mathbb{E}\left[\mathbb{1}_A \langle X_t^n - X_t, \phi' \rangle\right] = 0$. Therefore,

$$\mathbb{E}[\mathbb{1}_A \langle X_t, \phi' \rangle] = \lim_{k\to\infty} \mathbb{E}[\mathbb{1}_A \langle X_t^{n_k}, \phi' \rangle] = -\lim_{k\to\infty} \mathbb{E}\left[\mathbb{1}_A \left\langle \partial_x X_t^{n_k}, \phi \right\rangle\right] = -\mathbb{E}\left[\mathbb{1}_A \left\langle \partial_x Y_t, \phi \right\rangle\right].$$

Thus,

$$\mathbb{P}\text{-a.s.} \quad \langle X_t, \phi' \rangle = -\left\langle \partial_x Y_t, \phi \right\rangle. \tag{2.41}$$

Finally, we have to show as in [2, Theorem 3.4] that there exists a measurable set $\Omega_0 \subset \Omega$ with full measure such that $X_t^\cdot$ has a weak derivative on this subset. To this end, choose a sequence $\{\phi_n\}_{n\geq1} \subset \mathcal{C}_0^\infty(\mathbb{R})$ dense in $W^{1,2}(U)$ and a measurable subset $\Omega_n \subset \Omega$ with full measure such that (2.41) holds on Ω_n with ϕ replaced by ϕ_n. Then $\Omega_0 := \bigcap_{n\geq1} \Omega_n$ satisfies the desired property. □

We conclude this subsection with the proof of Proposition 3.4 that generalizes the probabilistic representation (2.25) of the first variation process $(\partial_x X_t^x)_{t\in[0,T]}$ and the connection to the Malliavin derivative given in Corollary 3.7 to irregular drift coefficients. To this end we first verify the weak differentiability of the function $\left(x \mapsto b\left(t, y, \mathbb{P}_{X_t^x}\right)\right)$ in the next proposition.

Proposition 3.11 *Suppose the drift coefficient b is in the decomposable form* (2.7) *and uniformly Lipschitz continuous in the third variable* (2.10). *Let $(X_t^x)_{t\in[0,T]}$ be the unique strong solution of* (2.13) *and $U \subset \mathbb{R}$ be an open and bounded subset. Then for every $1 < p < \infty$, $t \in [0,T]$ and $y \in \mathbb{R}$,*

$$\left(x \mapsto b\left(t, y, \mathbb{P}_{X_t^x}\right)\right) \in W^{1,p}(U).$$

Proof. Let $\{b_n\}_{n\geq1}$ be the approximating sequence of b as defined in (2.31) and $(X_t^{n,x})_{t\in[0,T]}$, $n \geq 1$, the corresponding unique strong solutions of (2.32). For notational simplicity we define $b_n(x) := b_n\left(t, y, \mathbb{P}_{X_t^{n,x}}\right)$ for every $n \geq 0$. We proceed similar to the proof of Theorem 3.3 and thus start by showing that $\{b_n\}_{n\geq1}$

is weakly relatively compact in $W^{1,p}(U)$. Due to Lemma A.4 and the proof of Lemma 3.10

$$\sup_{n\geq 1} \|b_n\|_{W^{1,p}(U)} < \infty.$$

Hence, $\{b_n\}$ is bounded in $W^{1,p}(U)$ and thus weakly relatively compact by [30, Theorem 10.44]. Therefore, we can find a sub-sequence $\{n_k\}_{k\geq 1}$ and $g \in W^{1,p}(U)$ such that b_{n_k} converges weakly to g as $k \to \infty$.

Let $\phi \in \mathcal{C}_0^\infty(U)$ be an arbitrary test-function and denote by ϕ' if well-defined its first derivative. Define

$$\langle b_n, \phi \rangle := \int_U b_n(x)\phi(x)dx.$$

Due to Lemma A.4

$$\langle b_n - b, \phi' \rangle \leq \|\phi'\|_{L^p(U)} |U|^{\frac{1}{p}} \sup_{x\in\overline{U}} |b_n(x) - b(x)| < \infty,$$

where $\overline{U}$ is the closure of U, and since by Proposition 3.8

$$\begin{aligned} &\left|b_n\left(t, y, \mathbb{P}_{X_t^{n,x}}\right) - b\left(t, y, \mathbb{P}_{X_t^x}\right)\right| \\ &\quad \leq \left|b_n\left(t, y, \mathbb{P}_{X_t^{n,x}}\right) - b_n\left(t, y, \mathbb{P}_{X_t^x}\right)\right| + \left|b_n\left(t, y, \mathbb{P}_{X_t^x}\right) - b\left(t, y, \mathbb{P}_{X_t^x}\right)\right| \\ &\quad \leq C\mathcal{K}\left(\mathbb{P}_{X_t^{n,x}}, \mathbb{P}_{X_t^x}\right) + \left|b_n\left(t, y, \mathbb{P}_{X_t^x}\right) - b\left(t, y, \mathbb{P}_{X_t^x}\right)\right| \xrightarrow[n\to\infty]{} 0, \end{aligned}$$

we get $\lim_{n\to\infty} \langle b_n - b, \phi' \rangle = 0$. Thus,

$$\langle b, \phi' \rangle = \lim_{k\to\infty} \langle b_{n_k}, \phi' \rangle = -\lim_{k\to\infty} \left\langle b'_{n_k}, \phi \right\rangle = -\langle g', \phi \rangle,$$

where b'_{n_k} and g' are the first variation processes of b_{n_k} and g, respectively. $\square$

Proof of Proposition 3.4. Let $(b_n)_{n\geq 1}$ be the approximating sequence of b as defined in (2.31) and $(X_t^{n,x})_{t\in[0,T]}$ be the corresponding unique strong solutions of (2.32). We define for $n \geq 0$

$$\begin{aligned} \Psi_n &:= \exp\left\{-\int_0^t \int_{\mathbb{R}} b_n\left(u, y, \mathbb{P}_{X_u^{n,x}}\right) L^{X^{n,x}}(du, dy)\right\} \\ &\quad + \int_0^t \exp\left\{-\int_s^t \int_{\mathbb{R}} b_n\left(u, y, \mathbb{P}_{X_u^{n,x}}\right) L^{X^{n,x}}(du, dy)\right\} \partial_x b_n\left(s, y, \mathbb{P}_{X_s^{n,x}}\right)|_{y=X_s^{n,x}} ds, \end{aligned}$$

which is well-defined for all $n \geq 0$ due to Lemma A.5 and Proposition 3.11. For every $t \in [0,T]$ the sequence $\{X_t^{n,x}\}_{n\geq 1}$ converges weakly in $L^2(\Omega, W^{1,2}(U))$ to X_t^x by the proof of Theorem 3.3. Hence, it suffices to show for every $f \in L^2([0,T])$ and $g \in C_0^\infty(U)$ that

$$\langle \mathcal{W}\left(\Psi_n - \Psi_0\right)(f), g \rangle \xrightarrow[n\to\infty]{} 0.$$

Define for every $n \geq 0$

$$L_n(s,t,x) := \exp\left\{-\int_s^t \int_{\mathbb{R}} b_n\left(u, y, \mathbb{P}_{X_s^{n,x}}\right) L^{B^x}(du, dy)\right\}, \text{ and}$$

$$\mathcal{E}_n(x) := \mathcal{E}\left(\int_0^T b_n\left(u, B_u^x, \mathbb{P}_{X_s^{n,x}}\right) + f(u)dB_u\right).$$

Applying Girsanov's theorem and Minkowski's inequality yields

$$\begin{aligned}
&\langle \mathcal{W}(\Psi_n - \Psi_0)(f), g\rangle \\
&\quad \le \int_U g(x)\mathbb{E}\left[|L_n(0,t,x) - L_0(0,t,x)|\,\mathcal{E}_n(x)\right]dx \\
&\qquad + \int_U g(x)\mathbb{E}\left[|\mathcal{E}_n(x) - \mathcal{E}_0(x)|\,L_0(0,t,x)\right]dx \\
&\qquad + \int_U\int_0^t g(x)\mathbb{E}\left[|L_n(s,t,x) - L_0(s,t,x)|\left|\partial_x b_n\left(s,y,\mathbb{P}_{X_s^{n,x}}\right)\right|_{y=B_s^x}\mathcal{E}_n(x)\right]dsdx \\
&\qquad + \int_U\int_0^t g(x)\mathbb{E}\left[|\mathcal{E}_n(x) - \mathcal{E}_0(x)|\,L_0(s,t,x)\left|\partial_x b_n\left(s,y,\mathbb{P}_{X_s^{n,x}}\right)\right|_{y=B_s^x}\right]dsdx \\
&\qquad + \int_U\int_0^t g(x)\mathbb{E}\left[\left|\partial_x b_n\left(s,y,\mathbb{P}_{X_s^{n,x}}\right) - \partial_x b\left(s,y,\mathbb{P}_{X_s^x}\right)\right|_{y=B_s^x}L_0(s,t,x)\mathcal{E}_0(x)\right]dsdx.
\end{aligned}$$

Note that for any $1 < p < \infty$,

$$\sup_{n\ge 0}\sup_{s\in[0,T]}\operatorname*{ess\,sup}_{x\in U}\mathbb{E}\left[\left|\partial_x b_n\left(s,y,\mathbb{P}_{X_s^{n,x}}\right)|_{y=B_s^x}\right|^p\right] < \infty, \tag{2.42}$$

due to Lemma 3.13 and the proof of Lemma 3.10. Hence, we get by Hölder's inequality, Lemma A.4, and Lemma A.5 that for $q := \frac{2(1+\varepsilon)}{2+\varepsilon}$ and $p := \frac{2(1+\varepsilon)}{\varepsilon}$, where $\varepsilon > 0$ is sufficiently small with regard to Lemma A.4,

$$\begin{aligned}
&\langle \mathcal{W}(\Psi_n - \Psi_0)(f), g\rangle \\
&\quad \lesssim \int_U g(x)\left(\sup_{s,t\in[0,T]}\mathbb{E}\left[|L_n(s,t,x) - L_0(s,t,x)|^p\right]^{\frac{1}{p}} + \mathbb{E}\left[|\mathcal{E}_n(x) - \mathcal{E}_0(x)|^q\right]^{\frac{1}{q}}\right)dx \\
&\qquad + \int_U\int_0^t g(x)\mathbb{E}\left[\left|\partial_x b_n\left(s,y,\mathbb{P}_{X_s^{n,x}}\right) - \partial_x b\left(s,y,\mathbb{P}_{X_s^x}\right)\right|_{y=B_s^x}^p\right]^{\frac{1}{p}}dsdx.
\end{aligned}$$

The first two summands converge due to Lemma A.6, Lemma A.7, and dominated convergence. For the third summand we use that $\left(x \mapsto b\left(t,y,\mathbb{P}_{X_t^x}\right)\right) \in W^{1,p}(U)$. Consequently, by dominated convergence and [40, Lemma 2.1.3] we get that

$$\int_U\int_0^t g(x)\mathbb{E}\left[\left|\partial_x b_n\left(s,y,\mathbb{P}_{X_s^{n,x}}\right) - \partial_x b\left(s,y,\mathbb{P}_{X_s^x}\right)\right|_{y=B_s^x}^p\right]^{\frac{1}{p}}dsdx \xrightarrow[n\to\infty]{} 0.$$

Representation (2.24) is a direct consequence of equation (2.23) and Proposition 3.1. □

3.3. **Hölder continuity.** We complete Section 3 by proving Hölder continuity of the unique strong solution $(X_t^x)_{t\in[0,T]}$ to mean-field SDE (2.13) in time and space.

Theorem 3.12 *Suppose the drift coefficient b is in the decomposable form* (2.7) *and uniformly Lipschitz continuous in the third variable* (2.10). *Let $(X_t^x)_{t\in[0,T]}$ be the unique strong solution of the mean-field SDE* (2.13). *Then for every compact* *subset $K \subset \mathbb{R}$ there exists a constant $C > 0$ such that for all $s, t \in [0, T]$ and $x, y \in K$,*

$$\mathbb{E}[|X_t^x - X_s^y|^2] \leq C(|t-s| + |x-y|^2). \tag{2.43}$$

In particular, there exists a continuous version of the random field $(t, x) \mapsto X_t^x$ with Hölder continuous trajectories of order $\alpha < \frac{1}{2}$ in $t \in [0, T]$ and $\alpha < 1$ in $x \in \mathbb{R}$.

To prove Theorem 3.12 we need the following extension of Lemma 3.10 to include also $\partial_x X_t^x$.

Lemma 3.13 *Suppose the drift coefficient b is in the decomposable form* (2.7) *and uniformly Lipschitz continuous in the third variable* (2.10)*. Let $(X_t^x)_{t\in[0,T]}$ be the unique strong solution of* (2.13)*. Then for any compact subset $K \subset \mathbb{R}$ and $p \geq 1$, there exists a constant $C > 0$ such that*

$$\sup_{t\in[0,T]} \operatorname*{ess\,sup}_{x\in K} \mathbb{E}\left[(\partial_x X_t^x)^p\right] \leq C.$$

Proof. The proof follows by Lemma 3.10 and the application of Fatou's lemma:

$$\mathbb{E}\left[(\partial_x X_t^x)^p\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[(\partial_x X_t^{n,x})^p\right] \leq C.$$

□

Proof of Theorem 3.12. Let $s, t \in [0, T]$ and $x, y \in K$ be arbitrary. Consider the approximating sequence $\{X^{n,x}\}_{n\geq 1}$ as defined in (2.32). Note first that similar to (2.40) it can be shown that for every $n \geq 1$

$$\mathbb{E}\left[|X_t^{n,x} - X_t^{n,y}|^2\right]^{\frac{1}{2}} \lesssim |x-y| + |x-y| \operatorname*{ess\,sup}_{x\in\overline{\mathrm{conv}(K)}} \mathbb{E}\left[|\partial_x X_t^{n,x}|^2\right]^{\frac{1}{2}}.$$

Since $\operatorname{ess\,sup}_{x\in\overline{\mathrm{conv}(K)}} \mathbb{E}\left[|\partial_x X_u^{n,x}|^2\right]$ is bounded uniformly in $n \geq 1$ and $t \in [0, T]$ due to (3.10), there exists a constant $C_1 > 0$ such that for all $n \geq 1$ and $t \in [0, T]$

$$\mathbb{E}\left[|X_t^{n,x} - X_t^{n,y}|^2\right]^{\frac{1}{2}} \leq C_1|x-y|.$$

Moreover, we have similar to (2.33) that there exists a constant $C_2 > 0$ such that for every $n \geq 1$ and $x \in K$

$$\mathbb{E}\left[|X_t^{n,x} - X_s^{n,x}|^2\right]^{\frac{1}{2}} \leq C_2|t-s|^{\frac{1}{2}}.$$

Consequently, there exists a constant $C > 0$ such that for all $n \geq 1$

$$\mathbb{E}\left[|X_t^{n,x} - X_s^{n,y}|^2\right] \leq C(|t-s| + |x-y|^2).$$

Finally, using Fatou's lemma applied to a subsequence and that $X_t^{n,x}$ converges to X_t^x in $L^2(\Omega)$ by Proposition 3.8, yields the result. □

4. Bismut-Elworthy-Li formula

In this section we turn our attention to finding a Bismut-Elworthy-Li type formula, i.e. with the help of Proposition 3.4 we give a probabilistic representation of type (2.11) for $\partial_x \mathbb{E}[\Phi(X_T^x)]$ for functions Φ merely satisfying some integrability condition. The following lemma prepares the grounds for the main result in Theorem 4.2.

Lemma 4.1 *Suppose the drift coefficient b is in the decomposable form* (2.7) *and uniformly Lipschitz continuous in the third variable* (2.10). *Let $(X_t^x)_{t\in[0,T]}$ be the unique strong solution of the corresponding mean-field SDE* (2.13) *and $U \subset \mathbb{R}$ be an open and bounded subset. Furthermore, consider the functional $\Phi \in \mathcal{C}_b^{1,1}(\mathbb{R})$. Then for every $t \in [0,T]$ and $1 < p < \infty$,*

$$(x \mapsto \mathbb{E}\left[\Phi(X_t^x)\right]) \in W^{1,p}(U).$$

Moreover, for almost all $x \in U$

$$\partial_x \mathbb{E}\left[\Phi(X_t^x)\right] = \mathbb{E}\left[\Phi'(X_t^x)\partial_x X_t^x\right], \tag{2.44}$$

where Φ' denotes the first derivative of Φ.

Proof. It is readily seen that $(x \mapsto \mathbb{E}[X_t^x]) \in \mathrm{Lip}_{C_1}(U,\mathbb{R})$ for some constant $C_1 > 0$ due to (2.40) and Proposition 3.8. Therefore, we get with the assumptions on the functional Φ that there exists a constant $C_2 > 0$ such that $(x \mapsto \mathbb{E}[\Phi(X_t^x)]) \in \mathrm{Lip}_{C_2}(U,\mathbb{R})$. Hence, $\mathbb{E}[\Phi(X_t^x)]$ is almost everywhere and weakly differentiable on U and for almost all $x \in U$

$$\begin{aligned}\partial_x \mathbb{E}[\Phi(X_t^x)] &= \lim_{h\to 0} \frac{\mathbb{E}[\Phi(X_t^{x+h})] - \mathbb{E}[\Phi(X_t^x)]}{h} = \mathbb{E}\left[\lim_{h\to 0} \frac{\Phi(X_t^{x+h}) - \Phi(X_t^x)}{h}\right] \\ &= \mathbb{E}\left[\Phi'(X_t^x)\partial_x X_t^x\right],\end{aligned}$$

where we used dominated convergence and the chain rule. Finally, we can conclude from (2.44) using Lemma 3.13 and the boundedness of Φ' that $(x \mapsto \mathbb{E}\left[\Phi(X_t^x)\right]) \in W^{1,p}(U)$ for every $1 < p < \infty$. □

Theorem 4.2 *Suppose the drift coefficient b is in the decomposable form* (2.7) *and uniformly Lipschitz continuous in the third variable* (2.10). *Let $(X_t^x)_{t\in[0,T]}$ be the unique strong solution of the corresponding mean-field SDE* (2.13), *$K \subset \mathbb{R}$ be a compact subset and $\Phi \in L^{2p}(\mathbb{R};\omega_T)$, where $p := \frac{1+\varepsilon}{\varepsilon}$, $\varepsilon > 0$ sufficiently small with regard to Lemma A.4, and ω_T is as defined in* (2.36). *Then, for every open subset $U \subset K$, $t \in [0,T]$ and $1 < q < \infty$,*

$$(x \mapsto \mathbb{E}\left[\Phi(X_t^x)\right]) \in W^{1,q}(U),$$

and for almost all $x \in K$

$$\partial_x \mathbb{E}[\Phi(X_T^x)] = \mathbb{E}\left[\Phi(X_T^x)\int_0^T \left(a(s)\partial_x X_s^x + \partial_x b\left(s,y,\mathbb{P}_{X_s^x}\right)|_{y=X_s^x}\int_0^s a(u)du\right)dB_s\right], \tag{2.45}$$

where $\partial_x X_s^x$ *is given in* (2.23) *and* $a : \mathbb{R} \to \mathbb{R}$ *is any bounded, measurable function such that*

$$\int_0^T a(s)ds = 1.$$

Remark 4.3. Note that in the case of an SDE the derivative (2.45) collapses to the representation

$$\mathbb{E}\left[\Phi(X_T^x)\int_0^T a(s)\partial_x X_s^x dB_s\right]$$

established in [2], where the first variation process $\partial_x X^x$ has the representation

$$\partial_x X_t^x = \exp\left\{-\int_0^t \int_{\mathbb{R}} b(u,y)L^{X^x}(du,dy)\right\}.$$

Hence, one can speak of a derivative free representation. Regarding mean-field SDEs, the derivative $\partial_x b\left(s,y,\mathbb{P}_{X_s^x}\right)$ still appears in the representation of $\partial_x X^x$.

Remark 4.4. In [3] we show that for the special case of mean-field SDEs of type (2.12), the expectation functional $\mathbb{E}[\Phi(X_t^x)]$ is even continuously differentiable in x for irregular drift coefficients under certain additional assumptions on the functions $\hat{b}$ and φ given in (2.12).

Proof of Theorem 4.2. We start by showing the result for $\Phi \in \mathcal{C}_b^{1,1}(\mathbb{R})$. In this case the derivative $\partial_x \mathbb{E}[\Phi(X_T^x)]$ exists by Lemma 4.1 and admits representation (2.44). Furthermore, by (2.24) for any $s \leq T$,

$$\partial_x X_T^x = D_s X_T^x \partial_x X_s^x + \int_s^T D_u X_T^x \partial_x b\left(u,y,\mathbb{P}_{X_u^x}\right)|_{y=X_u^x} du.$$

Recall that $D_s X_T^x = 0$ for $s \geq T$. Thus for any bounded function $a : \mathbb{R} \to \mathbb{R}$ with $\int_0^T a(s)ds = 1$,

$$\begin{aligned}
\partial_x X_T^x &= \int_0^T a(s)\left(D_s X_T^x \partial_x X_s^x + \int_s^T D_u X_T^x \partial_x b\left(u,y,\mathbb{P}_{X_u^x}\right)|_{y=X_u^x} du\right) ds \\
&= \int_0^T a(s) D_s X_T^x \partial_x X_s^x ds + \int_0^T \int_s^T a(s) D_u X_T^x \partial_x b\left(u,y,\mathbb{P}_{X_u^x}\right)|_{y=X_u^x} du ds.
\end{aligned}$$

We look at each summand individually starting with the first one. Since $\Phi \in \mathcal{C}_b^{1,1}(\mathbb{R})$, $\Phi(X_T^x)$ is Malliavin differentiable and

$$\mathbb{E}\left[\Phi'(X_T^x)\int_0^T a(s) D_s X_T^x \partial_x X_s^x ds\right] = \mathbb{E}\left[\int_0^T a(s) D_s \Phi(X_T^x) \partial_x X_s^x ds\right].$$

Due to the fact that $s \mapsto a(s)\partial_x X_s^x$ is an adapted process satisfying

$$\mathbb{E}\left[\int_0^T \left(a(s)\partial_x X_s^x\right)^2 ds\right] < \infty$$

by Lemma 3.13, we can apply the duality formula [17, Corollary 4.4] and get

$$\mathbb{E}\left[\int_0^T a(s) D_s \Phi(X_T^x) \partial_x X_s^x ds\right] = \mathbb{E}\left[\Phi(X_T^x) \int_0^T a(s) \partial_x X_s^x dB_s\right].$$

For the second summand note that by (2.38) and the proof of Lemma 3.10

$$\sup_{u,s\in[0,T]} \mathbb{E}\left[\left|\Phi'(X_T^x) a(s) D_u X_T^x \partial_x b\left(u, y, \mathbb{P}_{X_u^x}\right)|_{y=X_u^x}\right|\right] < \infty.$$

Hence, the integral

$$\int_0^T \int_0^T \mathbb{E}\left[\left|\Phi'(X_T^x) a(s) D_u X_T^x \partial_x b\left(u, y, \mathbb{P}_{X_u^x}\right)|_{y=X_u^x}\right|\right] duds$$

exists and is finite by Tonelli's Theorem. Consequently, we can interchange the order of integration to deduce

$$\begin{aligned} &\mathbb{E}\left[\Phi'(X_T^x) \int_0^T \int_s^T a(s) D_u X_T^x \partial_x b\left(u, y, \mathbb{P}_{X_u^x}\right)|_{y=X_u^x} duds\right] \\ &\quad = \mathbb{E}\left[\int_0^T D_u \Phi(X_T^x) \partial_x b\left(u, y, \mathbb{P}_{X_u^x}\right)|_{y=X_u^x} \int_0^u a(s) dsdu\right]. \end{aligned} \tag{2.46}$$

Furthermore, $u \mapsto \partial_x b\left(u, y, \mathbb{P}_{X_u^x}\right)|_{y=X_u^x}$ is an $\mathcal{F}$-adapted process. Hence, we can apply the duality formula [17, Corollary 4.4] and get

$$\begin{aligned} &\mathbb{E}\left[\int_0^T D_u \Phi(X_T^x) \partial_x b\left(u, y, \mathbb{P}_{X_u^x}\right)|_{y=X_u^x} \int_0^u a(s) dsdu\right] \\ &\quad = \mathbb{E}\left[\Phi(X_T^x) \int_0^T \partial_x b\left(u, y, \mathbb{P}_{X_u^x}\right)|_{y=X_u^x} \int_0^u a(s) dsdB_u\right]. \end{aligned}$$

Putting all together provides representation (2.45) for $\Phi \in \mathcal{C}_b^{1,1}(\mathbb{R})$.

By standard arguments, we can now approximate $\Phi \in L^{2p}(\mathbb{R}; \omega_T)$ by a smooth sequence $\{\Phi_n\}_{n\geq 1} \subset C_0^\infty(\mathbb{R})$ such that $\Phi_n \to \Phi$ in $L^{2p}(\mathbb{R}; \omega_T)$ as $n \to \infty$. Define

$$\begin{aligned} u_n(x) &:= \mathbb{E}\left[\Phi_n(X_T^x)\right] \quad \text{and} \\ \overline{u}(x) &:= \mathbb{E}\left[\Phi(X_T^x) \int_0^T \left(a(s)\partial_x X_s^x + \partial_x b(s, X_s^x, \mathbb{P}_{X_s^x})|_{y=X_s^x} \int_0^s a(u) du\right) dB_s\right]. \end{aligned}$$

First, we obtain that $\overline{u}$ is well-defined using Hölder's inequality, Itô's isometry and Lemma A.4. Indeed,

$$
\begin{aligned}
|\overline{u}(x)| &\leq \mathbb{E}\left[\Phi(X_T^x)^2\right]^{\frac{1}{2}} \\
&\quad \times \mathbb{E}\left[\left(\int_0^T \left(a(s)\partial_x X_s^x + \partial_x b(s, X_s^x, \mathbb{P}_{X_s^x})|_{y=X_s^x} \int_0^s a(u)du\right) dB_s\right)^2\right]^{\frac{1}{2}} \\
&\leq \mathbb{E}\left[\Phi(B_T^x)^2 \mathcal{E}\left(\int_0^T b(u, B_u^x, \mathbb{P}_{X_u^x}) dB_u\right)\right]^{\frac{1}{2}} \\
&\quad \times \mathbb{E}\left[\int_0^T \left(a(s)\partial_x X_s^x + \partial_x b(s, X_s^x, \mathbb{P}_{X_s^x})|_{y=X_s^x} \int_0^s a(u)du\right)^2 du\right]^{\frac{1}{2}} \\
&\lesssim \mathbb{E}\left[|\Phi(B_T^x)|^{2p}\right]^{\frac{1}{2p}} < \infty,
\end{aligned}
\tag{2.47}
$$

where the last inequality holds due to Lemma 3.10 and the proof of Proposition 3.11. Similar to the proof of Proposition 3.11 it is left to show that $\langle u_n' - \overline{u}, \phi\rangle_U$ for any test-function $\phi \in \mathcal{C}_0^\infty(U)$ as $n \to \infty$, where $U \subset K$ is an open set. Since the bounds in (2.47) hold for almost all $x \in U \subset K$, we get exactly in the same way that

$$
\begin{aligned}
|u'(x) - \overline{u}(x)| &\leq C(x)\mathbb{E}\left[|\Phi_n(B_T^x) - \Phi(B_T^x)|^{2p}\right]^{\frac{1}{2p}} \\
&= C(x)\left(\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi T}} |\Phi_n(y) - \Phi(y)|^{2p} e^{-\frac{(y-x)^2}{2T}} dy\right)^{\frac{1}{2p}} \\
&\leq C(x)\left(\frac{e^{\frac{x^2}{2T}}}{\sqrt{2\pi T}} \int_{\mathbb{R}} |\Phi_n(y) - \Phi(y)|^{2p} e^{-\frac{y^2}{4T}} dy\right)^{\frac{1}{2p}} \\
&= C(x)\left(\frac{e^{\frac{x^2}{2T}}}{\sqrt{2\pi T}}\right)^{\frac{1}{2p}} \|\Phi_n - \Phi\|_{L^{2p}(\mathbb{R};\omega_T)},
\end{aligned}
$$

where $C(x) > 0$ is bounded for almost every $x \in K$ and where we have used

$$
e^{-\frac{(y-x)^2}{2t}} = e^{-\frac{y^2}{4t}} e^{-\frac{(y-2x)^2}{4t}} e^{\frac{x^2}{2t}} \leq e^{-\frac{y^2}{4t}} e^{\frac{x^2}{2t}}.
$$

Hence, for any open subset $U \subset K$, we get

$$
\lim_{n\to\infty} \langle u_n'(x) - \overline{u}(x), \phi\rangle_U = 0.
$$

Thus $u' = \overline{u}$ for almost every $x \in K$. □

Remark 4.5. Note that for one-dimensional mean-field SDEs with additive noise (i.e. $\sigma \equiv 1$) Theorem 4.2 extends the Bismut-Elworthy-Li formula in [1] to irregular drift coefficients. More precisely, by changing the order of integration in (2.46) we are actually able to further develop the formula in [1] such that the Malliavin weight is given in terms of an Itô integral as opposed to an anticipative Skorohod integral in [1].

Appendix A. Technical Results

Lemma A.1 *Let $b : [0,T]\times\mathbb{R}\times\mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ be a measurable function satisfying the linear growth condition* (2.5). *Furthermore, let $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},B,X^x)$ be a weak solution of* (2.15). *Then, for $1 \leq p < \infty$, and every compact set $K \subset \mathbb{R}$,*

$$\sup_{x\in K}\mathbb{E}\left[\sup_{t\in[0,T]}|b(t,X_t^x,\mu_t)|^p\right] < \infty. \tag{2.48}$$

In particular, $b(\cdot,X_\cdot^x,\mu_\cdot) \in L^p([0,T]\times\Omega)$, $1 \leq p < \infty$. Furthermore,

$$\sup_{x\in K}\mathbb{E}\left[\sup_{t\in[0,T]}|X_t^x|^p\right] < \infty. \tag{2.49}$$

Proof. Note first that $\sup_{t\in[0,T]}\mathcal{K}(\mu_t,\delta_0)dt$ is well-defined and finite. Indeed, since $\mu \in \mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}))$ and $\mathcal{K}(\cdot,\delta_0)$ is continuous, the supremum over $t \in [0,T]$ of $\mathcal{K}(\mu_t,\delta_0)$ is attained. Furthermore, we can write

$$\begin{aligned}\mathcal{K}(\mu_t,\delta_0) &= \sup_{h\in\mathrm{Lip}_1}\left|\int_{\mathbb{R}} h(y)\mu_t(dy) - h(0)\right| \leq \sup_{h\in\mathrm{Lip}_1}\int_{\mathbb{R}}|h(y)-h(0)|d\mu_t(dy)\\ &\leq \int_{\mathbb{R}}|y|\mu_t(dy) < \infty,\end{aligned} \tag{2.50}$$

where the last term is finite by the definition of $\mathcal{P}_1(\mathbb{R})$. Therefore, we get due to the linear growth of b that

$$|X_t^x| = \left|x + \int_0^t b(s,X_s^x,\mu_s)ds + B_t\right| \lesssim |x| + T + |B_t| + \int_0^t |X_s^x|ds.$$

Thus, Grönwall's inequality yields that there exist constants C_1 and C_2 such that

$$\begin{aligned}|X_t^x| &\leq C_1\left(1+|x|+\sup_{s\in[0,T]}|B_s^x|\right), \text{ and}\\ |b(t,X_t^x,\mu_t)| &\leq C_2\left(1+|x|+\sup_{s\in[0,T]}|B_s^x|\right).\end{aligned} \tag{2.51}$$

The boundedness of (2.48) is a direct consequence of (2.51) and Doob's maximal inequality. □

We define the complete probability space $(\Omega,\mathcal{F},\mathbb{Q})$ carrying a Brownian motion B. In the following lemma we will prove the existence of an equivalent measure $\mathbb{P}^\mu$ induced by the drift coefficient b.

Lemma A.2 *Let $b : [0,T] \times \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ be a measurable function satisfying the linear growth condition* (2.5)*. Then the Radon-Nikodym derivative*

$$\frac{d\mathbb{P}^\mu}{d\mathbb{Q}} = \mathcal{E}\left(\int_0^T b(s, B_s^x, \mu_s) dB_s\right) \tag{2.52}$$

is well-defined and yields a probability measure $\mathbb{P}^\mu \sim \mathbb{Q}$. If $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^\mu, B^\mu, X^x)$ is a weak solution of (2.15)*, the Radon-Nikodym derivative*

$$\frac{d\mathbb{Q}^\mu}{d\mathbb{P}^\mu} = \mathcal{E}\left(-\int_0^T b(s, X_s^x, \mu_s) dB_s^\mu\right) \tag{2.53}$$

is well-defined and yields a probability measure $\mathbb{Q}^\mu$ equivalent to $\mathbb{P}^\mu$. Moreover, $(X_t^x)_{t\in[0,T]}$ is a $\mathbb{Q}^\mu$-Brownian motion starting in x.

Proof. This is a direct consequence of Beneš' result (cf. [27, Corollary 3.5.16]) and (2.51). ☐

Lemma A.3 *Let $b : [0,T] \times \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ be a measurable function satisfying the linear growth condition* (2.5)*. Then, there exists an $\varepsilon > 0$ such that for any $\mu \in \mathcal{C}([0,T]; \mathcal{P}_1(\mathbb{R}))$,*

$$\mathbb{E}\left[\mathcal{E}\left(\int_0^T b(u, B_u^x, \mu_u) dB_u\right)^{1+\varepsilon}\right] < \infty. \tag{2.54}$$

Proof. First, we rewrite

$$\begin{aligned}
&\mathbb{E}\left[\mathcal{E}\left(\int_0^T b(u, B_u^x, \mu_u) dB_u\right)^{1+\varepsilon}\right] \\
&= \mathbb{E}\left[\exp\left\{\int_0^T (1+\varepsilon) b(u, B_u^x, \mu_u) dB_u - \frac{1}{2}\int_0^T (1+\varepsilon)|b(u, B_u^x, \mu_u)|^2 du\right\}\right] \\
&= \mathbb{E}\left[\mathcal{E}\left(\int_0^T (1+\varepsilon) b(u, B_u^x, \mu_u) dB_u\right) \exp\left\{\frac{1}{2}\int_0^T \varepsilon(1+\varepsilon)|b(u, B_u^x, \mu_u)|^2 du\right\}\right] \\
&= \mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T \varepsilon(1+\varepsilon)|b(u, X_u^{\varepsilon,x}, \mu_u)|^2 du\right\}\right],
\end{aligned}$$

where in the last step by Girsanov's theorem $X^{\varepsilon,x}$ denotes a weak solution of

$$dX_t^{\varepsilon,x} = (1+\varepsilon) b(t, X_t^{\varepsilon,x}, \mu_t) dt + dB_t, \quad X_0^{\varepsilon,x} = x \in \mathbb{R}, \quad t \in [0,T].$$

Since b satisfies the linear growth condition (2.5), we have that

$$\begin{aligned}
|X_t^{\varepsilon,x}| &\leq |x| + (1+\varepsilon)\int_0^t |b(u, X_u^{\varepsilon,x}, \mu_u)| du + |B_t| \\
&\leq |x| + C(1+\varepsilon)\int_0^t (1 + |X_u^{\varepsilon,x}| + \mathcal{K}(\mu_u, \delta_0)) du + |B_t|.
\end{aligned}$$

Therefore, Grönwall's inequality gives us

$$|X_t^{\varepsilon,x}| \leq (1+\varepsilon)\left(T + |x| + \sup_{s\in[0,T]} |B_s| + \sup_{u\in[0,T]} \mathcal{K}(\mu_u, \delta_0)\right) e^{C(1+\varepsilon)T},$$

and thus, we can find a constant $C_{\varepsilon,\mu}$ depending on ε, μ and T such that $\lim_{\varepsilon\to 0} C_{\varepsilon,\mu}$ exists, is finite, and

$$|b(t, X_t^{\varepsilon,x}, \mu_t)| \leq C_{\varepsilon,\mu}\left(1 + |x| + \sup_{s\in[0,T]} |B_s|\right).$$

Hence,

$$\begin{aligned}
&\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T \varepsilon(1+\varepsilon)|b(u, X_u^{\varepsilon,x}, \mu_u)|^2 du\right\}\right] \\
&\quad \leq \mathbb{E}\left[\exp\left\{\frac{1}{2}T\varepsilon(1+\varepsilon)C_{\varepsilon,\mu}^2\left(1 + |x| + \sup_{s\in[0,T]} |B_s|\right)^2\right\}\right].
\end{aligned}$$

Clearly, $\lim_{\varepsilon\to 0}\varepsilon(1+\varepsilon)C_{\varepsilon,\mu}^2 = 0$ and therefore we can choose $\varepsilon > 0$ sufficiently small such that (2.54) holds. ☐

Lemma A.4 *Let $b : [0,T]\times\mathbb{R}\times\mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ be a measurable function satisfying the linear growth condition* (2.5)*. Furthermore, let $(\Omega, \mathcal{F}, \mathbb{G}, \mathbb{P}, B, X^x)$ be a weak solution of the mean-field SDE* (2.13)*. Then,*

$$|b(t, X_t^x, \mathbb{P}_{X_t^x})| \leq C\left(1 + |x| + \sup_{s\in[0,T]} |B_s|\right) \tag{2.55}$$

for some constant $C > 0$. Consequently, for any compact set $K \subset \mathbb{R}$, and $1 \leq p < \infty$, there exists $\varepsilon > 0$ such that the following boundaries hold:

$$\begin{aligned}
\sup_{x\in K}\mathbb{E}\left[\sup_{t\in[0,T]} |b(t, X_t^x, \mathbb{P}_{X_t^x})|^p\right] &< \infty \\
\sup_{x\in K}\sup_{t\in[0,T]} \mathbb{E}\left[|X_t^x|^p\right] &< \infty \\
\sup_{x\in K}\mathbb{E}\left[\mathcal{E}\left(\int_0^T b(u, B_u^x, \mathbb{P}_{X_u^x})dB_u\right)^{1+\varepsilon}\right] &< \infty
\end{aligned}$$

Proof. Due to the proofs of Lemma A.1 and Lemma A.3, it suffices to show (2.55). Note first that $\mathcal{K}(\mathbb{P}_{X_t^x}, \delta_0) \leq \mathbb{E}[|X_t^x|]$ for every $t \in [0,T]$ by (2.50). Hence, it is enough to show that $\mathbb{E}[|X_t^x|] \leq C(1+|x|)$ for every $t \in [0,T]$ and some constant $C > 0$. Since $(X_t^x)_{t\in[0,T]}$ is a weak solution of (2.13) and b fulfills the linear growth condition (2.5), we get

$$\mathbb{E}[|X_t^x|] \lesssim |x| + \int_0^t 1 + \mathbb{E}[|X_s^x|] + \mathcal{K}(\mathbb{P}_{X_s^x}, \delta_0)ds + \mathbb{E}[|B_t|] \lesssim 1 + |x| + \int_0^t \mathbb{E}[|X_s^x|]ds.$$

Consequently $\mathbb{E}[|X_t^x|] \leq C(1+|x|)$ by Grönwall's inequality which concludes the proof. $\square$

Lemma A.5 *Suppose the drift coefficient* $b : [0,T] \times \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ *is in the decomposable form* (2.7) *and uniformly Lipschitz continuous in the third variable* (2.10)*. Let* $(X_t^x)_{t\in[0,T]}$ *be the unique strong solution of* (2.13)*. Furthermore,* $\{b_n\}_{n\geq 1}$ *is the approximating sequence of* b *as defined in* (2.31) *and* $(X_t^{n,x})_{t\in[0,T]}$*,* $n \geq 1$*, the corresponding unique strong solutions of* (2.32)*. Then, for all* $\lambda \in \mathbb{R}$ *and any compact subset* $K \subset \mathbb{R}$*,*

$$\sup_{n\geq 0} \sup_{s,t\in[0,T]} \sup_{x\in K} \mathbb{E}\left[\exp\left\{-\lambda \int_s^t \int_{\mathbb{R}} b_n\left(s,y,\mathbb{P}_{X_s^{n,x}}\right) L^{B^x}(ds,dy)\right\}\right] < \infty.$$

Proof. Recall that b_n can be decomposed into $b_n = \tilde{b}_n + \hat{b}$ for all $n \geq 0$. Here $\tilde{b}_n$ is uniformly bounded in $n \geq 0$. Hence, by [2, Lemma A.2]

$$\sup_{n\geq 0} \sup_{s,t\in[0,T]} \sup_{x\in K} \mathbb{E}\left[\exp\left\{-\lambda \int_s^t \int_{\mathbb{R}} \tilde{b}_n\left(s,y,\mathbb{P}_{X_s^{n,x}}\right) L^{B^x}(ds,dy)\right\}\right] < \infty.$$

Moreover, $\|\partial_2 \hat{b}\|_\infty < \infty$ by definition. Consequently,

$$\begin{aligned}
&\sup_{n\geq 0} \sup_{s,t\in[0,T]} \sup_{x\in K} \mathbb{E}\left[\exp\left\{-\lambda \int_s^t \int_{\mathbb{R}} \hat{b}\left(s,y,\mathbb{P}_{X_s^{n,x}}\right) L^{B^x}(ds,dy)\right\}\right] \\
&\qquad = \sup_{n\geq 0} \sup_{s,t\in[0,T]} \sup_{x\in K} \mathbb{E}\left[\exp\left\{\lambda \int_s^t \partial_2 \hat{b}\left(s,B_s^x,\mathbb{P}_{X_s^{n,x}}\right) ds\right\}\right] < \infty.
\end{aligned}$$

$\square$

Lemma A.6 *Suppose the drift coefficient* $b : [0,T] \times \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ *is in the decomposable form* (2.7) *and uniformly Lipschitz continuous in the third variable* (2.10)*. Let* $(X_t^x)_{t\in[0,T]}$ *be the unique strong solution of* (2.13)*. Furthermore,* $\{b_n\}_{n\geq 1}$ *is the approximating sequence of* b *as defined in* (2.31) *and* $(X_t^{n,x})_{t\in[0,T]}$*,* $n \geq 1$*, the corresponding unique strong solutions of* (2.32)*. Then for any compact subset* $K \subset \mathbb{R}$ *and* $q := \frac{2(1+\varepsilon)}{2+\varepsilon}$*,* $\varepsilon > 0$ *sufficiently small with regard to Lemma A.4,*

$$\sup_{x\in K} \mathbb{E}\left[\left|\mathcal{E}\left(\int_0^T b_n(t,B_t^x,\mathbb{P}_{X_t^{n,x}})dB_t\right) - \mathcal{E}\left(\int_0^T b(t,B_t^x,\mathbb{P}_{X_t^x})dB_t\right)\right|^q\right]^{\frac{1}{q}} \xrightarrow[n\to\infty]{} 0.$$

Proof. For the sake of readability we use the abbreviation $\mathfrak{b}_n(X_t^{k,x}) = b_n(t,B_t^x,\mathbb{P}_{X_t^{k,x}})$ for $n,k \geq 0$. First using inequality (2.18), Lemma A.4 and Burkholder-Davis-Gundy's inequality yields

$$\begin{aligned}
A_n(T,x) &:= \mathbb{E}\left[\left|\mathcal{E}\left(\int_0^T \mathfrak{b}_n(X_t^{n,x})dB_t\right) - \mathcal{E}\left(\int_0^T \mathfrak{b}(X_t^x)dB_t\right)\right|^q\right]^{\frac{1}{q}} \\
&\leq \mathbb{E}\left[\left|\int_0^T \mathfrak{b}_n(X_t^{n,x}) - \mathfrak{b}(X_t^x)dB_t + \frac{1}{2}\int_0^T \mathfrak{b}_n(X_t^{n,x})^2 - \mathfrak{b}(X_t^x)^2 dt\right|^q\right.
\end{aligned}$$

$$
\begin{aligned}
&\left(\mathcal{E}\left(\int_0^T \mathfrak{b}_n(X_t^{n,x})dB_t\right) + \mathcal{E}\left(\int_0^T \mathfrak{b}(X_t^x)dB_t\right)\right)^q\Bigg]^{\frac{1}{q}} \\
&\lesssim \mathbb{E}\left[\left|\int_0^T \left(\mathfrak{b}_n(X_t^{n,x}) - \mathfrak{b}(X_t^x)\right)^2 dt\right|^{\frac{p}{2}}\right]^{\frac{1}{p}} + \mathbb{E}\left[\left|\int_0^T \mathfrak{b}_n(X_t^{n,x})^2 - \mathfrak{b}(X_t^x)^2 dt\right|^p\right]^{\frac{1}{p}},
\end{aligned}
$$

where $p := \frac{1+\varepsilon}{\varepsilon}$. Due to its definition b_n is of linear growth uniformly in $n \geq 0$ and thus we get with Lemma A.4 that

$$
\mathbb{E}\left[\left|\mathfrak{b}_n(X_t^{n,x})^2 - \mathfrak{b}(X_t^x)^2\right|^p\right]^{\frac{1}{p}} \lesssim \mathbb{E}\left[|\mathfrak{b}_n(X_t^{n,x}) - \mathfrak{b}(X_t^x)|^{2p}\right]^{\frac{1}{2p}}
$$

and by Minkowski's integral as well as Cauchy-Schwarz' inequality, we have

$$
\begin{aligned}
&A_n(T,x) \\
&\lesssim \left(\int_0^T \mathbb{E}\left[|\mathfrak{b}_n(X_t^{n,x}) - \mathfrak{b}(X_t^x)|^{2p}\right]^{\frac{2}{2p}} dt\right)^{\frac{1}{2}} + \int_0^T \mathbb{E}\left[|\mathfrak{b}_n(X_t^{n,x}) - \mathfrak{b}(X_t^x)|^{2p}\right]^{\frac{1}{2p}} dt \\
&\lesssim \left(\int_0^T \mathbb{E}\left[|\mathfrak{b}_n(X_t^{n,x}) - \mathfrak{b}(X_t^x)|^{2p}\right]^{\frac{2}{2p}} dt\right)^{\frac{1}{2}}.
\end{aligned}
$$

Using the triangle inequality and $(\mu \mapsto b(t,y,\mu)) \in \mathrm{Lip}_C(\mathcal{P}_1(\mathbb{R}))$ for every $t \in [0,T]$ and $y \in \mathbb{R}$ yields

$$
\begin{aligned}
&\mathbb{E}\left[|\mathfrak{b}_n(X_t^{n,x}) - \mathfrak{b}(X_t^x)|^{2p}\right]^{\frac{1}{2p}} \\
&\quad \leq \mathbb{E}\left[|\mathfrak{b}_n(X_t^{n,x}) - \mathfrak{b}_n(X_t^x)|^{2p}\right]^{\frac{1}{2p}} + \mathbb{E}\left[|\mathfrak{b}_n(X_t^x) - \mathfrak{b}(X_t^x)|^{2p}\right]^{\frac{1}{2p}} \\
&\quad \leq C\mathcal{K}\left(\mathbb{P}_{X_t^{n,x}}, \mathbb{P}_{X_t^x}\right) + D_n(t,x) \leq C\mathbb{E}\left[|X_t^{n,x} - X_t^x|\right] + D_n(t,x),
\end{aligned}
$$

where $D_n(t,x) := \mathbb{E}\left[|\mathfrak{b}_n(X_t^x) - \mathfrak{b}(X_t^x)|^{2p}\right]^{\frac{1}{2p}}$, $t \in [0,T]$. With Girsanov's Theorem and Jensen's inequality we get

$$
\begin{aligned}
\mathbb{E}\left[|X_t^{n,x} - X_t^x|\right] &= \mathbb{E}\left[|B_t^x|\left|\mathcal{E}\left(\int_0^t \mathfrak{b}_n(X_s^{n,x})dB_s\right) - \mathcal{E}\left(\int_0^t \mathfrak{b}(X_s^x)dB_s\right)\right|\right] \\
&\lesssim \mathbb{E}\left[\left|\mathcal{E}\left(\int_0^t \mathfrak{b}_n(X_s^{n,x})dB_s\right) - \mathcal{E}\left(\int_0^t \mathfrak{b}(X_s^x)dB_s\right)\right|^q\right]^{\frac{1}{q}} = A_n(t,x).
\end{aligned}
$$

Consequently, $A_n(T,x) \lesssim \left(\int_0^T (A_n(t,x) + D_n(t,x))^2 dt\right)^{\frac{1}{2}}$ and therefore

$$
A_n^2(T,x) \lesssim \int_0^T A_n^2(t,x)dt + \int_0^T D_n^2(t,x)dt.
$$

Hence, we get with Grönwall's inequality

$$
A_n^2(T,x) \leq C \int_0^T D_n^2(t,x)dt,
$$

for some constants $C > 0$ independent of $x \in K$, $n \geq 0$ and $t \in [0,T]$ and as a consequence it suffices to show

$$\sup_{x \in K} \int_0^T D_n^2(t,x)dt \xrightarrow[n\to\infty]{} 0. \tag{2.56}$$

Note first

$$\begin{aligned} D_n^2(t,x) &= \mathbb{E}\left[\left|b_n\left(t, B_t^x, \mathbb{P}_{X_t^x}\right) - b\left(t, B_t^x, \mathbb{P}_{X_t^x}\right)\right|^{2p}\right]^{\frac{2}{2p}} \\ &= \left(\int_{\mathbb{R}} \left|b_n\left(t, y, \mathbb{P}_{X_t^x}\right) - b\left(t, y, \mathbb{P}_{X_t^x}\right)\right|^{2p} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} dy\right)^{\frac{2}{2p}} \\ &\leq e^{\frac{x^2}{2pt}} \left(\int_{\mathbb{R}} \left|b_n\left(t, y, \mathbb{P}_{X_t^x}\right) - b\left(t, y, \mathbb{P}_{X_t^x}\right)\right|^{2p} \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{4t}} dy\right)^{\frac{2}{2p}}, \end{aligned}$$

where we have used $e^{-\frac{(y-x)^2}{2t}} = e^{-\frac{y^2}{4t}} e^{-\frac{(y-2x)^2}{4t}} e^{\frac{x^2}{2t}} \leq e^{-\frac{y^2}{4t}} e^{\frac{x^2}{2t}}$. Furthermore, by Theorem 3.12 there exists a constant $C > 0$ such that for all $t \in [0,T]$ and $x, y \in K$

$$\mathcal{K}\left(\mathbb{P}_{X_t^x}, \mathbb{P}_{X_t^y}\right) \leq \mathbb{E}\left[|X_t^x - X_t^y|^2\right]^{\frac{1}{2}} \leq C|x-y|.$$

Consequently the function $x \mapsto \mathbb{P}_{X_t^x}$ is continuous for all $t \in [0,T]$. Thus $\mathbb{P}_{X_t^K} := \{\mathbb{P}_{X_t^x} : x \in K\} \subset \mathcal{P}_1(\mathbb{R})$ is compact as an image of a compact set under a continuous function. Therefore due to the definition of the approximating sequence

$$\sup_{x \in K} \left|b_n\left(t, y, \mathbb{P}_{X_t^x}\right) - b\left(t, y, \mathbb{P}_{X_t^x}\right)\right| = \sup_{\mu \in \mathbb{P}_{X_t^K}} |b_n(t,y,\mu) - b(t,y,\mu)| \xrightarrow[n\to\infty]{} 0,$$

and hence $D_n^2(t,x)$ converges to 0 uniformly in $x \in K$. Consequently, $\int_0^T D_n^2(t,x)dt$ converges uniformly to 0 by Lemma A.4 and dominated convergence, which proves the result. □

Lemma A.7 *Suppose the drift coefficient $b : [0,T] \times \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ is in the decomposable form* (2.7) *and uniformly Lipschitz continuous in the third variable* (2.10)*. Let $(X_t^x)_{t\in[0,T]}$ be the unique strong solution of* (2.13)*. Furthermore, $\{b_n\}_{n\geq 1}$ is the approximating sequence of b as defined in* (2.31) *and $(X_t^{n,x})_{t\in[0,T]}$, $n \geq 1$, the corresponding unique strong solutions of* (2.32)*. Then for any compact subset $K \subset \mathbb{R}$, $s,t \in [0,T]$, $s \leq t$ and $p \geq 1$,*

$$\sup_{x\in K} \mathbb{E}\left[\left|\exp\left\{-\int_s^t \int_{\mathbb{R}} b_n^{\mathbb{P}_{X^n}}(u,y) L^{B^x}(du,dy)\right\} - \exp\left\{-\int_s^t \int_{\mathbb{R}} b^{\mathbb{P}_{X^n}}(u,y) L^{B^x}(du,dy)\right\}\right|^p\right]^{\frac{1}{p}} \xrightarrow[n\to\infty]{} 0,$$

where $b_n^{\mathbb{P}_{X^n}}(u,y) := b_n\left(u, y, \mathbb{P}_{X_u^{n,x}}\right)$ for all $n \geq 0$.

Proof. We first use inequality (2.18) to obtain with Lemma A.5

$$\mathbb{E}\left[\left|\exp\left\{-\int_s^t \int_{\mathbb{R}} b_n^{\mathbb{P}_{X^n}}(u,y) L^{B^x}(du,dy)\right\} - \exp\left\{-\int_s^t \int_{\mathbb{R}} b^{\mathbb{P}_{X^n}}(u,y) L^{B^x}(du,dy)\right\}\right|^p\right]^{\frac{1}{p}}$$

$$
\begin{aligned}
&\leq \mathbb{E}\left[\left|\int_s^t \int_{\mathbb{R}} b_n^{\mathbb{P}_{X^n}}(u,y) L^{B^x}(du,dy) - \int_s^t \int_{\mathbb{R}} b^{\mathbb{P}_{X^n}}(u,y) L^{B^x}(du,dy)\right|^p \right.\\
&\left.\times \left(\exp\left\{-\int_s^t \int_{\mathbb{R}} b_n^{\mathbb{P}_{X^n}}(u,y) L^{B^x}(du,dy)\right\} + \exp\left\{-\int_s^t \int_{\mathbb{R}} b^{\mathbb{P}_{X^n}}(u,y) L^{B^x}(du,dy)\right\}\right)^p\right]^{\frac{1}{p}}\\
&\lesssim \mathbb{E}\left[\left|\int_s^t \int_{\mathbb{R}} b_n^{\mathbb{P}_{X^n}}(u,y) L^{B^x}(du,dy) - \int_s^t \int_{\mathbb{R}} b^{\mathbb{P}_{X^n}}(u,y) L^{B^x}(du,dy)\right|^{2p}\right]^{\frac{1}{2p}}.
\end{aligned}
$$

We define the time-reversed Brownian motion $\hat{B}_t := B_{T-t}$, $t \in [0,T]$, and the Brownian motion W_t, $t \in [0,T]$, with respect to the natural filtration of $\hat{B}$. By [2, Theorem 2.10], Burkholder-Davis-Gundy's inequality and Cauchy-Schwarz' inequality

$$
\begin{aligned}
&\mathbb{E}\left[\left|\int_s^t \int_{\mathbb{R}} b_n^{\mathbb{P}_{X^n}}(u,y) - b^{\mathbb{P}_{X^n}}(u,y) L^{B^x}(du,dy)\right|^{2p}\right]^{\frac{1}{2p}}\\
&= \mathbb{E}\left[\left|\int_s^t b_n^{\mathbb{P}_{X^n}}(u,B_u^x) - b^{\mathbb{P}_{X^n}}(u,B_u^x) dB_u + \int_{T-t}^{T-s} b_n^{\mathbb{P}_{X^n}}(T-u,\hat{B}_u^x) - b^{\mathbb{P}_{X^n}}(T-u,\hat{B}_u^x) dW_u \right.\right.\\
&\qquad \left.\left. - \int_{T-t}^{T-s} \left(b_n^{\mathbb{P}_{X^n}}(T-u,\hat{B}_u^x) - b^{\mathbb{P}_{X^n}}(T-u,\hat{B}_u^x)\right) \frac{\hat{B}_u}{T-u} du\right|^{2p}\right]^{\frac{1}{2p}}\\
&\lesssim \mathbb{E}\left[\left(\int_s^t \left(b_n^{\mathbb{P}_{X^n}}(u,B_u^x) - b^{\mathbb{P}_{X^n}}(u,B_u^x)\right)^2 du\right)^p\right]^{\frac{1}{2p}}\\
&\quad + \mathbb{E}\left[\left(\int_{T-t}^{T-s} \left(b_n^{\mathbb{P}_{X^n}}(T-u,\hat{B}_u^x) - b^{\mathbb{P}_{X^n}}(T-u,\hat{B}_u^x)\right)^2 du\right)^p\right]^{\frac{1}{2p}}\\
&\quad + \int_{T-t}^{T-s} \left\| b_n^{\mathbb{P}_{X^n}}(T-u,\hat{B}_u^x) - b^{\mathbb{P}_{X^n}}(T-u,\hat{B}_u^x)\right\|_{L^{4p}(\Omega)} \left\|\frac{\hat{B}_u}{T-u}\right\|_{L^{4p}(\Omega)} du.
\end{aligned}
$$

Similar to the proof of Lemma A.6 one obtains the result. □

Appendix B. Hida spaces

In order to prove Proposition 3.8, we need the definition of the Hida test function and distribution space (cf. [17, Definition 5.6]). Furthermore we state the central theorem used in the proof of Proposition 3.8, followed by a further helpful criterion for relative compactness using modulus of continuity.

Definition B.1 Let $\mathcal{I}$ be the set of all finite multi-indices and $\{H_\alpha\}_{\alpha \in \mathcal{I}}$ be an orthogonal basis of the Hilbert space $L^2(\Omega)$ defined by

$$
H_\alpha(\omega) := \prod_{j=1}^m h_{\alpha_j}\left(\int_{\mathbb{R}} e_j(t) dW_t(\omega)\right),
$$

where h_n is the n-th hermitian polynomial, e_n the n-th hermitian function and W a standard Brownian motion. Furthermore, we define for every $\alpha = (\alpha_1, \dots \alpha_m) \in \mathcal{I}$,

$$(2\mathbb{N})^\alpha := \prod_{j=1}^m (2j)^{\alpha_j}.$$

(i) We define the Hida test function Space $\mathcal{S}$ as

$$\mathcal{S} := \left\{ \phi = \sum_{\alpha \in \mathcal{I}} a_\alpha H_\alpha \in L^2(\Omega) : \|\phi\|_k < \infty \ \forall k \in \mathbb{R} \right\},$$

where the norm $\|\cdot\|_k$ is defined by

$$\|\phi\|_k := \sqrt{\sum_{\alpha \in \mathcal{I}} \alpha! a_\alpha^2 (2\mathbb{N})^{\alpha k}}.$$

Here, $\mathcal{S}$ is equipped with the projective topology.

(ii) The Hida distribution space $\mathcal{S}^*$ is defined by

$$\mathcal{S}^* := \left\{ \phi = \sum_{\alpha \in \mathcal{I}} a_\alpha H_\alpha \in L^2(\Omega) : \exists k \in \mathbb{R} \text{ s.t. } \|\phi\|_{-k} < \infty \right\},$$

where the norm $\|\cdot\|_{-k}$ is defined by

$$\|\phi\|_{-k} := \sqrt{\sum_{\alpha \in \mathcal{I}} \alpha! a_\alpha^2 (2\mathbb{N})^{-\alpha k}}.$$

Here, $\mathcal{S}^*$ is equipped with the inductive topology.

Theorem B.2 (Mitoma) *The following statements are equivalent:*

(i) $\mathcal{A}$ is relatively compact in $\mathcal{C}([0,T];\mathcal{S}^*)$,

(ii) For any $\phi \in \mathcal{S}$, $\{f(\cdot)[\phi] : f \in \mathcal{A}\}$ is relatively compact in $\mathcal{C}([0,T];\mathbb{R})$.

Proof. [26, Theorem 2.4.4] □

In the following we state a version of the Arzelà-Ascoli theorem which is used in the proof of Proposition 3.8 and can be found in [27, Theorem 2.4.9]

Theorem B.3 *The set $\mathcal{A} \subset \mathcal{C}([0,T],\mathbb{R})$ is relatively compact if and only if*

$$\sup_{f \in \mathcal{A}} |f(0)| < \infty, \quad \textit{and}$$

$$\lim_{\delta \to 0} \sup_{f \in \mathcal{A}} \sup\{\|f(t) - f(s)\| : s,t \in [0,T], |t-s| < \delta\} = 0.$$

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Chapter 3

Existence and Regularity of Solutions to Multi-Dimensional Mean-Field Stochastic Differential Equations with Irregular Drift

Contribution of the thesis' author

The paper *Existence and Regularity of Solutions to Multi-Dimensional Mean-Field Stochastic Differential Equations with Irregular Drift* is a joint project with Prof. Dr. Thilo Meyer-Brandis.

M. Bauer was significantly involved in the development of all parts of the paper. In particular, M. Bauer made major contributions to the editorial work and the proofs of Theorem 2.7, Theorem 3.1, Theorem 3.3, Theorem 3.12, and Theorem 4.1.

EXISTENCE AND REGULARITY OF SOLUTIONS TO MULTI-DIMENSIONAL MEAN-FIELD STOCHASTIC DIFFERENTIAL EQUATIONS WITH IRREGULAR DRIFT

MARTIN BAUER AND THILO MEYER-BRANDIS

Abstract. We examine existence and uniqueness of strong solutions of multi-dimensional mean-field stochastic differential equations with irregular drift coefficients. Furthermore, we establish Malliavin differentiability of the solution and show regularity properties such as Sobolev differentiability in the initial data as well as Hölder continuity in time and the initial data. Using the Malliavin and Sobolev differentiability we formulate a Bismut-Elworthy-Li type formula for mean-field stochastic differential equations, i.e. a probabilistic representation of the first order derivative of an expectation functional with respect to the initial condition.

Keywords. McKean-Vlasov equation · mean-field stochastic differential equation · weak solution · strong solution · uniqueness in law · pathwise uniqueness · singular coefficients · Malliavin derivative · Sobolev derivative · Hölder continuity · Bismut-Elworthy-Li formula.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space. Throughout the manuscript let $T > 0$ be a finite time horizon. Consider the mean-field stochastic differential equation, hereafter for short mean-field SDE,

$$dX_t^x = b\left(t, X_t^x, \mathbb{P}_{X_t^x}\right) dt + \sigma\left(t, X_t^x, \mathbb{P}_{X_t^x}\right) dB_t, \quad t \in [0,T], \quad X_0^x = x \in \mathbb{R}^d, \quad (3.1)$$

where $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^d$ is the drift coefficient, $\sigma : [0,T] \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^{d\times n}$ the diffusion coefficient, and $\mathbb{P}_{X_t^x} \in \mathcal{P}_1(\mathbb{R}^d)$ denotes the law of X_t^x with respect to the measure $\mathbb{P}$. Here, $B = (B_t)_{t\in[0,T]}$ is n-dimensional Brownian motion and $\mathcal{P}_1(\mathbb{R}^d)$ is the space of probability measures over $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with finite first moment.

Mean-field SDE (3.1), also called McKean-Vlasov equation, originates in the study on multi-particle systems with weak interaction and traces back to works of Vlasov [41], Kac [29], and McKean [36]. In recent years the interest in mean-field SDEs increased due to the work of Lasry and Lions [32] on mean-field games and the related application in the fields of Economics and Finance, for example in the study of systemic risk, see e.g. [16], [17], [23], [24], [25], [30], and the cited sources therein. Carmona and Delarue developed subsequently the theory on mean-field games in a mere probabilistic environment, cf. [11], [12], [13], [14], [15], and [18].

In this paper the focus lies on existence and uniqueness as well as regularity properties of solutions to multi-dimensional mean-field SDEs with additive noise, i.e. equations of the form

$$dX_t^x = b\left(t, X_t^x, \mathbb{P}_{X_t^x}\right) dt + dB_t, \quad t \in [0,T], \quad X_0^x = x \in \mathbb{R}^d, \tag{3.2}$$

where B is d-dimensional Brownian motion. In particular, we are interested in irregular drift coefficients b that are merely measurable in the spatial variable.

Existence and uniqueness of solutions to mean-field SDEs have been discussed in several works, cf. for example [6], [5], [7], [8], [9], [19], [21], [26], [28], [34], [35], [38], and [39]. Li and Min show in [34] the existence of a weak solution for a path dependent mean-field SDE, where the drift b is assumed to be bounded and continuous in the law variable. Under the additional assumption that b admits a modulus of continuity they prove uniqueness in law of the solution. In [38], the authors derive existence of a pathwisely unique strong solution for drift coefficients b of at most linear growth that are continuous in the law variable with respect to the total variation metric. In order to prove their result, Mishura and Veretennikov use an approach similar to Krylov in his analysis of stochastic differential equations, cf. [31]. In [39] it is shown that mean-field SDE (3.1) has a strong solution for b fulfilling some integrability condition and being weakly continuous in the law variable. The one-dimensional case of mean-field SDE (3.2) is considered in [6]. There, we show that mean-field SDE (3.2) has a Malliavin differentiable pathwisely unique strong solution for drift coefficients b admitting a modulus of continuity in the law variable and having a decomposition

$$b(t,y,\mu) := \hat{b}(t,y,\mu) + \tilde{b}(t,y,\mu), \tag{3.3}$$

where $\hat{b}$ is merely measurable and bounded and $\tilde{b}$ is of at most linear growth and Lipschitz continuous in the spatial variable. We remark that in [6] the decomposition (3.3) is required to establish regularity properties such as Malliavin differentiability of the strong solution, whereas for mere existence of a strong solution it suffices to assume the drift coefficient to be of at most linear growth and continuous in the law variable, see also Theorem 3.8 below. In [5] a special class of mean-field SDEs is considered, where the dependence on the law is in form of a Lebesgue integral. Inter alia for this kind of mean-field SDE the existence of a unique strong solution is shown for singular drift coefficients that are not necessarily continuous in the law variable. We remark here that weak existence of a solution has been established in [2], [3], and [4], for another class of mean-field SDEs that are related to Fokker-Plank equations where the drift coefficient might allow for discontinuities in the law variable.

Regularity properties of solutions to mean-field SDEs are investigated for example in [6], [9], and [20]. In [9] and [20], the authors derive Malliavin differentiability of solutions to mean-field SDE (3.1) for regular coefficients b and σ. Further, they examine in the case of regular coefficients differentiability of the solution with respect to the initial value. In their analysis they use the notion of Lions derivative which denotes the derivative with respect to a measure. We derive in [6] Malliavin differentiability, Sobolev differentiability in the initial data, and Hölder continuity in time and initial data for the one-dimensional mean-field SDE (3.2) but for drift coefficients that are merely Lipschitz continuous in the law variable and admit a decomposition (3.3). In particular, we prove Sobolev differentiability in the initial data without using the notion of Lions derivative. Lastly, we show that the expectation functional $\mathbb{E}[(\Phi(X_T^x)]$ is Sobolev differentiable with respect to x, where X^x is the unique strong solution of mean-field SDE (3.2) and $\Phi : \mathbb{R} \to \mathbb{R}$ satisfies merely some integrability condition. Further, we derive a Bismut-Elworthy-Li type formula for the derivative $\nabla_x \mathbb{E}[(\Phi(X_T^x)]$.[1]

The main objective of this paper is to extend the results obtained in [6] to the multi-dimensional case. More precisely, at first we show existence of a strong solution for drift coefficients b that are merely measurable, of at most linear growth, and continuous in the law variable. Here, we proceed as in [6] to show first existence of a weak solution by applying Girsanov's theorem and Schauder's fixed point theorem, and then resort to existence results of SDE's to guarantee the existence of a strong solution. Under the additional assumption that b admits a modulus of continuity in the law variable pathwise uniqueness of the solution is derived. If the drift coefficient b is bounded and continuous in the law variable, we further show that the strong solution of the multi-dimensional mean-field SDE (3.2) is Malliavin differentiable. Finally, for b being merely bounded and Lipschitz continuous in the law variable, Sobolev differentiability in the initial data and Hölder continuity in time and intitial data as well as a Bismut-Elworthy-Li type formula are derived.

The main difference compared to the one-dimensional case in [6] in the courses of the proofs of Sobolev differentiability, Hölder continuity, and the Bismut-Elworthy-Li formula is that there does not exist a representation of the Malliavin derivative by means of integration with respect to local time. Instead, we derive in a first step for regular drift coefficients b the relation

$$\nabla_x X_t^x = D_s X_t^x \nabla_x X_s^x + \int_s^t D_r X_t^x \nabla_x b(r, y, \mathbb{P}_{X_r^x})\Big|_{y=X_r^x} dr, \ 0 \leq s \leq t \leq T,$$

where $(D_s X_t^x)_{0 \leq s \leq t \leq T}$ is the Malliavin derivative and $(\nabla_x X_t^x)_{0 \leq t \leq T}$ the Sobolev derivative of the strong solution X^x of mean-field SDE (3.2). Afterwards we use this relation to derive the pursued regularity properties for irregular drift coefficients b by applying an approximational approach.

The paper is structured as follows. In Section 2 we give the definitions of the assumptions applied on the drift function b. Section 3 contains the main result on existence of a pathwisely unique solution. Afterwards, we discuss the properties of Malliavin and Sobolev differentiability as well as Hölder continuity in Sections 4.1

[1]Here, ∇_x denotes the Jacobian with respect to the variable x.

to 4.3, respectively. The paper is closed by deriving a Bismut-Elworthy-Li type formula in Section 5.

2. Notation and Assumptions

Subsequently we list some of the most frequently used notations.

- $\{e_k\}_{1\leq k\leq d}$ is the standard basis of $\mathbb{R}^d$ consisting of the unit vectors.
- $\mathcal{C}_b^{1,1}(\mathbb{R}^d)$ is the space of continuously differentiable functions $f:\mathbb{R}^d\to\mathbb{R}^d$ with bounded and Lipschitz continuous partial derivatives.
- $\mathcal{C}_0^\infty(\mathbb{R}^d)$ denotes the space of smooth functions with compact support.
- $L^\infty\left([0,T],\mathcal{C}_b^{1,L}\left(\mathbb{R}^d\times\mathcal{P}_1\left(\mathbb{R}^d\right)\right)\right)$ is the space of functions $f:[0,T]\times\mathbb{R}^d\times\mathcal{P}_1\left(\mathbb{R}^d\right)\to\mathbb{R}^d$ such that
  - $t\mapsto f(t,y,\mu)$ is bounded uniformly in $y\in\mathbb{R}$ and $\mu\in\mathcal{P}_1\left(\mathbb{R}^d\right)$
  - $(y\mapsto f(t,y,\mu))\in\mathcal{C}_b^{1,1}(\mathbb{R}^d)$ uniformly in $t\in[0,T]$ and $\mu\in\mathcal{P}_1\left(\mathbb{R}^d\right)$
  - $\mu\mapsto f(t,y,\mu)$ is Lipschitz continuous uniformly in $t\in[0,T]$ and $y\in\mathbb{R}^d$.
- δ_0 denotes the Dirac measure in 0.
- $\mathrm{Lip}_1\left(\mathbb{R}^d,\mathbb{R}\right)$ denotes the set of functions $f:\mathbb{R}^d\to\mathbb{R}$ that are Lipschitz continuous with Lipschitz constant 1.
- The Kantorovich metric on the space $\mathcal{P}_1(\mathbb{R}^d)$ is defined by

$$\mathcal{K}(\mu,\nu):=\sup_{h\in\mathrm{Lip}_1(\mathbb{R}^d,\mathbb{R})}\left|\int_{\mathbb{R}^d}h(y)(\mu-\nu)(dy)\right|,\quad \mu,\nu\in\mathcal{P}_1\left(\mathbb{R}^d\right).$$

- We write $E_1(\theta)\lesssim E_2(\theta)$ for two mathematical expressions $E_1(\theta),E_2(\theta)$ depending on some parameter θ, if there exists a constant $C>0$ not depending on θ such that $E_1(\theta)\leq CE_2(\theta)$.
- $\|\cdot\|_\infty$ sup norm over all variables
- $\|\cdot\|$ is the euclidean norm
- ∇_x is the Jacobian in the direction of the variable $x\mathbb{R}^d$, ∇_k is the Jacobian in the direction of the k-th variable, ∂_x is the (weak) partial derivative in the direction of the variable $x\in\mathbb{R}$, ∂_k is the (weak) partial derivative in the direction of e_k.
- We define the weight function

$$\omega_T(y):=\exp\left\{-\frac{\|y\|^2}{4T}\right\},\quad y\in\mathbb{R}^d, \tag{3.4}$$

and the weighted L^2-space $L^2(\mathbb{R}^d;\omega_T)$ as the space of functions $f:\mathbb{R}^d\to\mathbb{R}^d$ such that

$$\left(\int_{\mathbb{R}^d}\|f(y)\|^2\omega_T(y)dy\right)^{\frac{1}{2}}<\infty.$$

In the following we give conditions on the drift function

$$b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^d$$

that we use frequently throughout the paper.

We say that the function b is of *linear growth*, if there exists a constant $C > 0$ such that for every $t \in [0,T]$, $y \in \mathbb{R}^d$, and $\mu \in \mathcal{P}_1(\mathbb{R}^d)$

$$\|b(t,y,\mu)\| \leq C \left(1 + \|y\| + \mathcal{K}(\mu, \delta_0)\right). \tag{3.5}$$

The function b is said to be *continuous in the third variable* (uniformly with respect to the first and second variable), if for every $\mu \in \mathcal{P}_1(\mathbb{R}^d)$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\nu \in \mathcal{P}_1(\mathbb{R}^d)$ with $\mathcal{K}(\mu,\nu) < \delta$, we have for all $t \in [0,T]$ and $y \in \mathbb{R}^d$

$$\|b(t,y,\mu) - b(t,y,\nu)\| < \varepsilon. \tag{3.6}$$

The drift coefficient b admits a *modulus of continuity* (in the third variable), if there exists a continuous function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\int_0^z (\theta(y))^{-1} dy = \infty$ for all $z \in \mathbb{R}_+$ such that for every $t \in [0,T]$, $y \in \mathbb{R}^d$, and $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$

$$\|b(t,y,\mu) - b(t,y,\nu)\|^2 \leq \theta\left(\mathcal{K}(\mu,\nu)^2\right). \tag{3.7}$$

We say the drift coefficient b is *Lipschitz continuous in the third variable* (uniformly with respect to the first and second variable), if there exists a constant $C > 0$ such that for all $t \in [0,T]$, $y \in \mathbb{R}^d$, and $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$

$$\|b(t,y,\mu) - b(t,y,\nu)\| \leq C\mathcal{K}(\mu,\nu). \tag{3.8}$$

3. Existence and Uniqueness of Solutions

In this section we investigate under which of the assumptions specified in Section 2 on the drift coefficient b mean-field SDE (3.2) has a (strong) solution and moreover, in which case this solution is unique. Let us recall the definitions of weak and strong solutions as well as weak and pathwise uniqueness.

Definition 3.1 (Weak Solution) A six-tuple $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, B, X^x)$ is called *weak solution* of mean-field SDE (3.2), if

(i) $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a complete filtered probability space and $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ satisfies the usual conditions of right-continuity and completeness,
(ii) $B = (B_t)_{t\in[0,T]}$ is d-dimensional $(\mathbb{F},\mathbb{P})$-Brownian motion,
(iii) $X^x = (X_t^x)_{t\in[0,T]}$ is a continuous, $\mathbb{F}$-adapted, $\mathbb{R}^d$-valued process; $B = (B_t)_{t\in[0,T]}$ is a d-dimensional $(\mathbb{F},\mathbb{P})$-Brownian motion,
(iv) X^x satisfies $\mathbb{P}$-a.s.

$$dX_t^x = b(t, X_t^x, \mathbb{P}_{X_t^x})dt + dB_t, \quad X_0^x = x \in \mathbb{R}^d, \quad t \in [0,T],$$

where for all $t \in [0,T]$, $\mathbb{P}_{X_t^x} \in \mathcal{P}_1(\mathbb{R}^d)$ denotes the law of X_t^x with respect to $\mathbb{P}$, and

$$\int_0^T \mathcal{K}(\mathbb{P}_{X_t^x}, \delta_0)dt < \infty. \tag{3.9}$$

Remark 3.2. For bounded drift coefficients $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^d$ condition (3.9) is redundant since it is naturally fulfilled. Indeed,

$$\begin{aligned}\sup_{t\in[0,T]} \mathcal{K}(\mathbb{P}_{X_t^x}, \delta_0) &\leq \mathbb{E}[\|X_t^x\|] \\ &\leq \|x\| + \mathbb{E}\left[\left\|\int_0^T b\left(s, X_s^x, \mathbb{P}_{X_s^x}\right) ds\right\|\right] + \sup_{t\in[0,T]} \mathbb{E}[\|B_t\|] < \infty.\end{aligned}$$

Definition 3.3 (Strong Solution) A *strong solution* of mean-field SDE (3.2) is a weak solution $(\Omega, \mathcal{F}, \mathbb{F}^B, \mathbb{P}, B, X^x)$ where $\mathbb{F}^B$ is the filtration generated by the Brownian motion B and augmented with the $\mathbb{P}$-null sets.

Remark 3.4. In the following we merely speak of X^x as a weak and a strong solution of mean-field SDE (3.2), respectively, if there is no ambiguity concerning the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, B)$.

Definition 3.5 (Uniqueness in Law) A weak solution $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, B, X^x)$ of mean-field SDE (3.2) is said to be *weakly unique* or *unique in law*, if for any other weak solution $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}, \widetilde{\mathbb{P}}, \widetilde{B}, Y^x)$ of (3.2) with the same initial condition $X_0^x = Y_0^x$, it holds that

$$\mathbb{P}_{X^x} = \widetilde{\mathbb{P}}_{Y^x}.$$

Definition 3.6 (Pathwise Uniqueness) A weak solution $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, B, X^x)$ of mean-field SDE (3.2) is said to be *pathwisely unique*, if for any other weak solution Y^x with respect to the same stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, B)$ with the same initial condition $X_0^x = Y_0^x$, it holds that

$$\mathbb{P}\left(\forall t \geq 0 : X_t^x = Y_t^x\right) = 1.$$

Remark 3.7. Since for strong solutions of mean-field SDE's of type (3.2) the notions of pathwise uniqueness and uniqueness in law are equivalent (cf. [6, Remark 2.11]), we merely speak of a unique strong solution, if a strong solution is unique in any of the two senses.

The following result provides sufficient conditions allowing for irregular drift coefficients b such that mean-field SDE (3.2) has a (unique) strong solution. Note that in [38, Proposition 2] a similar result on the existence of a strong solution of mean-field SDE (3.2) is derived where the authors assume drift coefficients of at most linear growth that are continuous in the law variable with respect to the topology of weak convergence. Here, in contrast to [38], we assume continuity in the law variable merely with respect to the Kantorovich metric and provide a more direct alternative of proof that is not based on approximation arguments.

Theorem 3.8 *Suppose the drift coefficient* $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^d$ *is of at most linear growth* (3.5) *and continuous in the third variable* (3.6)*. Then, mean-field SDE* (3.2) *has a strong solution.*

If in addition b is admitting a modulus of continuity (3.7)*, the solution is unique.*

Proof. First note that identically to [6, Theorem 2.3] one can show that under the assumptions of linear growth (3.5) and continuity in the third variable (3.6) on the drift coefficient b, mean-field SDE (3.2) has a weak solution $(X_t^x)_{t\in[0,T]}$ for any finite time horizon $T > 0$. In particular, $\mathbb{P}_{X^x} \in \mathcal{C}([0,T]; \mathcal{P}_1(\mathbb{R}^d))$ and due to Lemma A.1 for every $p \geq 1$

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t^x\|^p\right] < \infty. \tag{3.10}$$

In order to show the existence of a strong solution, consider the stochastic differential equation

$$dY_t^x = b^{\mathbb{P}_X}(t, Y_t^x)\, dt + dB_t, \quad t \in [0,T], \quad Y_0^x = x \in \mathbb{R}^d, \tag{3.11}$$

where $b^{\mathbb{P}_X}(t,y) := b(t,y,\mathbb{P}_{X_t^x})$ for all $t \in [0,T]$ and $y \in \mathbb{R}^d$. Due to the work of Veretennikov [40] it is well-known that SDE (3.11) has a unique strong solution $(Y_t)_{t\in[0,\tau]}$ up to the time of explosion $\tau > 0$. Since X^x is a weak solution of SDE (3.11) on the interval $[0,T]$, both processes X^x and Y^x must coincide on the interval $[0,\tau]$, due to uniqueness of the solution Y to SDE (3.11). But due to condition (3.10), X^x is almost surely finite on the interval $[0,T]$ and thus Y^x is also almost surely finite on the interval $[0,T]$. Consequently, Y^x is a strong solution of SDE (3.11) on the interval $[0,T]$ which coincides pathwisely and in law with X^x. In particular, for all $t \in [0,T]$

$$\mathbb{P}_{Y_t} = \mathbb{P}_{X_t},$$

and thus, SDE (3.11) and mean-field SDE (3.2) coincide and Y^x is a strong solution of mean-field SDE (3.2).

If in addition b admits a modulus of continuity (3.7), it can be shown analogously to [6, Theorem 2.7] that the weak solution of mean-field equation (3.2) is unique in law. This in fact yields a unique associated SDE (3.11). In addition with the uniqueness of the strong solution to SDE (3.11), this yields a unique strong solution of mean-field equation (3.2). □

4. Regularity Properties

4.1. **Malliavin Differentiability.** Similar to the existence of a strong solution, the property of being Malliavin differentiable transfers directly from the solution Y^x of SDE (3.11) to the solution X^x of mean-field SDE (3.2). Thus, we immediately get from [37, Theorem 3.3] the following result.

Theorem 4.1 *Suppose the drift coefficient* $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^d$ *is continuous in the third variable* (3.6) *and bounded. Then, the strong solution* $(X_t^x)_{t\in[0,T]}$ *of mean-field SDE* (3.2) *is Malliavin differentiable.*

4.2. **Sobolev Differentiability.** In this section we consider the unique strong solution of mean-field SDE (3.2) as a function in the initial value x, i.e. for every $t \in [0,T]$ we consider the function $x \mapsto X_t^x$. More precisely, we are interested in the existence of the first variation process $(\nabla_x X_t^x)_{t\in[0,T]}$ in a weak (Sobolev) sense. Let us first recall the definition of the Sobolev space $W^{1,2}(U)$ and then state the main result of this section.

Definition 4.2 Let $U \subset \mathbb{R}^d$ be an open and bounded subset. The Sobolev space $W^{1,2}(U)$ is defined as the set of functions $u : \mathbb{R}^d \to \mathbb{R}^d$, $u \in L^2(U)$, such that its weak derivative belongs to $L^2(U)$. Furthermore, the Sobolev space is endowed with the norm

$$\|u\|_{W^{1,2}(U)} = \|u\|_{L^2(U)} + \sum_{k=1}^{d} \|\partial_k u\|_{L^2(U)}.$$

We say a stochastic process X is Sobolev differentiable in U, if for all $t \in [0,T]$, $X_t^{\cdot}$ belongs $\mathbb{P}$-a.s. to $W^{1,2}(U)$.

Theorem 4.3 *Suppose the drift coefficient* $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^d$ *is Lipschitz continuous in the third variable* (3.8) *and bounded. Let* $(X_t^x)_{t\in[0,T]}$ *be the unique strong solution of mean-field SDE* (3.2) *and* $U \subset \mathbb{R}^d$ *be an open and bounded subset. Then, for every* $t \in [0,T]$

$$(x \mapsto X_t^x) \in L^2\left(\Omega, W^{1,2}(U)\right).$$

The remaining part of this subsection is devoted to the proof of Theorem 4.3. We start by showing that the result does hold for regular drift coefficients b. Subsequently, we define a sequence $\{b_n\}_{n\geq 1}$ of regular functions that approximate the irregular drift coefficient b from Theorem 4.3 and prove that the strong solutions $\{X^{n,x}\}_{n\geq 1}$ to the corresponding mean-field SDEs converge strongly in $L^2(\Omega)$ to the solution X^x of (3.2). Concluding we get by showing that $\{X^{n,x}\}_{n\geq 1}$ is weakly relatively compact in the space $L^2(\Omega, W^{1,2}(U))$ that X^x is Sobolev differentiable as a function in the initial value x.

Proposition 4.4 *Let the drift coefficient* $b \in L^\infty\left([0,T], \mathcal{C}_b^{1,L}\left(\mathbb{R}^d \times \mathcal{P}_1\left(\mathbb{R}^d\right)\right)\right)$ *and let* $(X_t^x)_{t\in[0,T]}$ *be the unique strong solution of mean-field SDE* (3.2)*. Then, for all* $t \in [0,T]$ *the map* $x \mapsto X_t^x$ *is a.s. Lipschitz continuous and consequently weakly and almost everywhere differentiable.*

Proof. The proof is equivalent to the proof of [6, Proposition 3.5]. □

Corollary 4.5 *The map* $x \mapsto b(s, y, \mathbb{P}_{X_s^x})$ *is Lipschitz continuous for all* $t \in [0,T]$ *and* $y \in \mathbb{R}^d$ *under the assumptions of Proposition 4.4 and thus weakly and* *almost everywhere differentiable. Moreover, for every* $0 \leq s < t \leq T$

$$\nabla_x X_t^x = D_s X_t^x \nabla_x X_s^x + \int_s^t D_r X_t^x \nabla_x b(r, y, \mathbb{P}_{X_r^x})\Big|_{y=X_r^x} dr. \tag{3.12}$$

Proof. Similar to the proof of [6, Proposition 3.5] it can be shown that $x \mapsto b(s, y, \mathbb{P}_{X_s^x})$ is Lipschitz continuous for all $t \in [0, T]$ and $y \in \mathbb{R}^d$. Furthermore, consider the linear affine ODE

$$Z_t = I_d + \int_0^t \nabla_2 b\left(s, X_s^x, \mathbb{P}_{X_s^x}\right) Z_s + \nabla_x b\left(s, y, \mathbb{P}_{X_s^x}\right)\Big|_{y=X_s^x} ds. \tag{3.13}$$

First note that $\nabla_x X_t^x$ is a solution of ODE (3.13). Moreover, by assumption $\|\nabla_2 b\|_\infty \leq C_1 < \infty$ for some constant $C_1 > 0$ and since $x \mapsto X_s^x$ is Lipschitz continuous for all $s \in [0, T]$ we get

$$\begin{aligned}
\left\| \nabla_x b\left(s, y, \mathbb{P}_{X_s^x}\right)\Big|_{y=X_s^x} \right\| &\leq \sum_{k=1}^d \lim_{x_0^{(k)} \to x^{(k)}} \left\| \frac{b\left(s, X_s^x, \mathbb{P}_{X_s^x}\right) - b\left(s, X_s^x, \mathbb{P}_{X_s^{\overline{x_0}^{(k)}}}\right)}{x^{(k)} - x_0^{(k)}} \right\| \\
&\lesssim \sum_{k=1}^d \lim_{x_0^{(k)} \to x^{(k)}} \frac{\mathcal{K}\left(\mathbb{P}_{X_s^x}, \mathbb{P}_{X_s^{\overline{x_0}^{(k)}}}\right)}{\left|x^{(k)} - x_0^{(k)}\right|} \lesssim 1,
\end{aligned}$$

where $\overline{x_0}^{(k)} = x + \langle x_0 - x, e_k \rangle$. Therefore, $\|\nabla_x b(s, y, \mathbb{P}_{X_s^x})|_{y=X_s^x}\|_\infty \leq C_2 < \infty$ for some constant $C_2 > 0$ and consequently, ODE (3.13) has the unique solution $\nabla_x X_t^x$. On the other hand, the Malliavin derivative $D_s X_t^x$, $0 \leq s < t \leq T$, is the unique solution to the homogeneous ODE

$$D_s X_t^x = I_d + \int_s^t \nabla_2 b\left(r, X_r^x, \mathbb{P}_{X_r^x}\right) D_s X_r^x dr.$$

Consequently, we get that the Malliavin derivative has the explicit representation

$$D_s X_t^x = \exp\left\{ \int_s^t \nabla_2 b\left(r, X_r^x, \mathbb{P}_{X_r^x}\right) dr \right\},$$

and the first variation process has the representation

$$\nabla_x X_t^x = D_0 X_t^x \left(I_d + \int_0^t (D_0 X_r)^{-1} \nabla_x b\left(r, y, \mathbb{P}_{X_r^x}\right)\Big|_{y=X_r^x} dr \right).$$

Thus, we get

$$\begin{aligned}
D_s X_t^x \nabla_x X_s^x &= D_0 X_t^x \left(I_d + \int_0^s (D_0 X_r)^{-1} \nabla_x b\left(r, y, \mathbb{P}_{X_r^x}\right)\Big|_{y=X_r^x} dr \right) \\
&= D_0 X_t^x + \int_0^s D_r X_t \nabla_x b\left(r, y, \mathbb{P}_{X_r^x}\right)\Big|_{y=X_r^x} dr \\
&= \nabla_x X_t^x - \int_s^t D_r X_t \nabla_x b\left(r, y, \mathbb{P}_{X_r^x}\right)\Big|_{y=X_r^x} dr.
\end{aligned}$$

Rearranging yields equation (3.12). $\square$

Now consider a general drift coefficient b which fulfills the assumptions of Theorem 4.3, namely Lipschitz continuity in the third variable (3.8) and boundedness, and let X^x be the corresponding unique strong solution of mean-field SDE (3.2). Due to standard approximation arguments there exists a sequence of approximating drift coefficients

$$b_n \in L^\infty\left([0,T], \mathcal{C}_b^{1,L}\left((\mathbb{R}^d \times \mathcal{P}_1\left(\mathbb{R}^d\right)\right)\right), \quad n \geq 1, \tag{3.14}$$

with $\sup_{n\geq 1} \|b_n\|_\infty \leq C < \infty$ such that $b_n \to b$ pointwise in every μ and a.e. in (t,y) with respect to the Lebesgue measure. We denote $b_0 := b$ and assume that the drift coefficients b_n are Lipschitz continuous in the third variable (3.8) uniformly in $n \geq 0$. We define the corresponding mean-field SDEs

$$dX_t^{n,x} = b_n\left(t, X_t^{n,x}, \mathbb{P}_{X_t^{n,x}}\right) dt + dB_t, \quad t \in [0,T], \quad X_0^{n,x} = x \in \mathbb{R}^d, \tag{3.15}$$

which admit unique Malliavin differentiable strong solutions due to Theorem 3.8 and Theorem 4.1. Moreover, the solutions $\{X^{n,x}\}_{n\geq 1}$ are Sobolev differentiable in the initial condition x by Proposition 4.4. Subsequently, we show that $(X_t^{n,x})_{t\in[0,T]}$ converges to $(X_t^x)_{t\in[0,T]}$ in $L^2(\Omega, \mathcal{F}_t)$ as $n \to \infty$.

Proposition 4.6 *Suppose the drift coefficient* $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^d$ *is Lipschitz continuous in the third variable* (3.8) *and bounded. Let* $(X_t^x)_{t\in[0,T]}$ *be the unique strong solution of mean-field SDE* (3.2)*. Furthermore,* $\{b_n\}_{n\geq 1}$ *is the approximating sequence as defined in* (3.14) *and* $(X_t^{n,x})_{t\in[0,T]}$, $n \geq 1$, *the corresponding unique strong solutions of* (3.15)*. Then, there exists a subsequence* $\{n_k\}_{k\geq 1} \subset \mathbb{N}$ *such that*

$$X_t^{n_k,x} \xrightarrow[k\to\infty]{} X_t^x, \quad t \in [0,T],$$

strongly in $L^2(\Omega, \mathcal{F}_t)$.

Proof. In [37, Corollary 3.6] it is shown in the case of SDEs that for every $t \in [0,T]$ the sequence $\{X^{n,x}\}_{n\geq 1}$ is relatively compact in $L^2(\Omega, \mathcal{F}_t)$. Due to Theorem 4.1 the proof therein can be extended to the case of mean-field SDEs under the assumptions of Proposition 4.6. Thus, for every $t \in [0,T]$ we can find a subsequence $\{n_k(t)\}_{k\geq 1}$ such that $X_t^{n_k(t),x}$ converges to some Y_t strongly in $L^2(\Omega, \mathcal{F}_t)$. Following the same ideas as in the proof of [6, Proposition 3.8] it can be shown that the subsequence $\{n_k(t)\}_{k\geq 1}$ can be chosen independent of $t \in [0,T]$. Moreover, the proof of [6, Proposition 3.9] can be readily extended to the multi-dimensional case which yields that $\{X_t^{n_k,x}\}_{k\geq 1}$ converges weakly in $L^2(\Omega, \mathcal{F}_t)$ to the unique strong solution $\overline{X}_t^x$ of the SDE

$$d\overline{X}_t^x = b\left(t, \overline{X}_t^x, \mathbb{P}_{Y_t}\right) dt + dB_t, \ t \in [0,T], \ \overline{X}_0^x = x \in \mathbb{R}^d. \tag{3.16}$$

Due to uniqueness of the limit we get that $Y_t^x \stackrel{d}{=} \overline{X}_t^x$ for all $t \in [0,T]$. Consequently, SDE (3.16) is identical to mean-field SDE (3.2) and thus $\{X_t^{n_k,x}\}_{k\geq 1}$ converges strongly in $L^2(\Omega, \mathcal{F}_t)$ to $X_t = Y_t = \overline{X}_t$ for every $t \in [0,T]$. □

Remark 4.7. For the sake of readability we assume subsequently without loss of generality that for every $t \in [0,T]$ the whole sequence $\{X_t^{n,x}\}$ converges strongly in $L^2(\Omega, \mathcal{F}_t)$ to X_t^x.

Lemma 4.8 *Let $(X_t^{n,x})_{t\in[0,T]}$, $n \geq 1$, be the unique strong solutions of mean-field SDEs* (3.15). *Then, for any compact subset $K \subset \mathbb{R}^d$ and $p \geq 2$,*

$$\sup_{n\geq 1} \sup_{t\in[0,T]} \operatorname*{ess\,sup}_{x\in K} \mathbb{E}[\|\nabla_x X_t^{n,x}\|^p] \leq C,$$

for some constant $C > 0$.

Proof. In the course of this proof we make use of representation (3.12), namely

$$\nabla_x X_t^{n,x} = D_0 X_t^{n,x} + \int_0^t D_r X_t^{n,x} \nabla_x b(r, y, \mathbb{P}_{X_r^{n,x}})\Big|_{y=X_r^{n,x}} dr, \quad n \geq 1.$$

First note that due to [37, Lemma 3.5] and the uniform boundedness of b_n in $n \geq 1$, we have that

$$\sup_{n\geq 1} \sup_{s,t\in[0,T]} \sup_{x\in K} \mathbb{E}[\|D_s X_t^{n,x}\|^p] \leq C_1 < \infty, \tag{3.17}$$

for some constant $C_1 > 0$. Moreover, we get that

$$\begin{aligned}
&\mathbb{E}\left[\left\|\int_0^t \nabla_x b_n(r, y, \mathbb{P}_{X_r^{n,x}})\Big|_{y=X_r^{n,x}} dr\right\|^{2p}\right] \\
&\quad \lesssim \sum_{k=1}^d \sum_{j=1}^d \mathbb{E}\left[\left(\int_0^t \left|\partial_{x^{(k)}} b_n^{(j)}(r, y, \mathbb{P}_{X_r^{n,x}})\right|_{y=X_r^{n,x}} dr\right)^{2p}\right].
\end{aligned}$$

Following the proof of [6, Lemma 3.10], we get due to the assumption $(\mu \mapsto b_n(t,y,\mu)) \in \mathrm{Lip}(\mathcal{P}_1(\mathbb{R}^d))$ for every $t \in [0,T]$ and $y \in \mathbb{R}^d$ uniformly in $n \geq 1$ that

$$\mathbb{E}\left[\left(\int_0^t \left|\partial_{x^{(k)}} b_n^{(j)}(r, y, \mathbb{P}_{X_r^{n,x}})\right|_{y=X_r^{n,x}} dr\right)^{2p}\right] \lesssim 1 + \int_0^t \operatorname*{ess\,sup}_{x\in\overline{\mathrm{conv}(K)}} \mathbb{E}\left[\left|\partial_{x^{(k)}} X_r^{n,(j),x}\right|\right] dr.$$

All things considered we get that

$$\operatorname*{ess\,sup}_{x\in\overline{\mathrm{conv}(K)}} \mathbb{E}[\|\nabla_x X_t^{n,x}\|^p]^{\frac{1}{p}} \lesssim 1 + \int_0^t \operatorname*{ess\,sup}_{x\in\overline{\mathrm{conv}(K)}} \mathbb{E}[\|\nabla_x X_r^{n,x}\|^p]^{\frac{1}{p}} dr.$$

Here, $\overline{\mathrm{conv}(K)}$ is the closure of the convex hull of the set K. Noting that $t \mapsto \operatorname{ess\,sup}_{x\in\overline{\mathrm{conv}(K)}} \mathbb{E}[\|\nabla_x X_t^{n,x}\|^p]$ is integrable over $[0,T]$ and Borel measurable, cf. [6, Lemma 3.10] for more details, allows for the application of Jones' generalization of Grönwall's inequality [27, Lemma 5], and thus we get that

$$\operatorname*{ess\,sup}_{x\in K} \mathbb{E}[\|\nabla_x X_t^{n,x}\|^p]^{\frac{1}{p}} \leq \operatorname*{ess\,sup}_{x\in\overline{\mathrm{conv}(K)}} \mathbb{E}[\|\nabla_x X_t^{n,x}\|^p]^{\frac{1}{p}} < \infty.$$

$\square$

Proof of Theorem 4.3. The proof is equivalent to the proof of [6, Theorem 3.3] but for the sake of completeness we present it in the following. Consider the unique strong solutions $\{X^{n,x}\}_{n\geq 1}$ of mean-field SDEs (3.15) and the unique strong solution X^x of mean-field SDE (3.2). Subsequently, we show that $\{X^{n,x}\}_{n\geq 1}$ is weakly relatively compact in $L^2(\Omega, W^{1,2}(U))$ and then identify the weak limit $Y := \lim_{k\to\infty} X^{n_k}$ in $L^2(\Omega, W^{1,2}(U))$ with X^x, where $\{n_k\}_{k\geq 1}$ is a suitable subsequence.

Note first that due to Lemma A.1 and Lemma 4.8

$$\sup_{n\geq 1} \sup_{t\in[0,T]} \mathbb{E}\left[\|X_t^{n,x}\|^2_{W^{1,2}(U)}\right] < \infty,$$

and therefore, $\{X_t^{n,x}\}_{n\geq 1}$ is weakly relatively compact in $L^2(\Omega, W^{1,2}(U))$, see e.g. [33, Theorem 10.44]. Thus, there exists a subsequence $\{n_k\}_{k\geq 0}$, such that $X_t^{n_k,x}$ converges weakly to some $Y_t \in L^2(\Omega, W^{1,2}(U))$ as $k\to\infty$. Define for every $t\in[0,T]$

$$\langle X_t^n, \phi\rangle := \int_U X_t^{n,x}\phi(x)dx,$$

for some arbitrary test function $\phi \in \mathcal{C}_0^\infty(U)$ and denote by ϕ' its first derivative. Then we get by Lemma A.1 that for all measurable sets $A\in\mathcal{F}$ and $t\in[0,T]$

$$\mathbb{E}\left[\mathbb{1}_A\langle X_t^n - X_t, \phi'\rangle\right] \leq \|\phi'\|_{L^2(U)}|U|^{\frac{1}{2}} \sup_{x\in\overline{U}} \mathbb{E}\left[\mathbb{1}_A\|X_t^{n,x} - X_t^x\|^2\right]^{\frac{1}{2}} < \infty,$$

where $\overline{U}$ is the closure of U. Hence, we get by Proposition 4.6 that

$$\lim_{n\to\infty} \mathbb{E}\left[\mathbb{1}_A\langle X_t^n - X_t, \phi'\rangle\right] = 0,$$

and thus,

$$\mathbb{E}[\mathbb{1}_A\langle X_t, \phi'\rangle] = \lim_{k\to\infty} \mathbb{E}[\mathbb{1}_A\langle X_t^{n_k}, \phi'\rangle] = -\lim_{k\to\infty} \mathbb{E}\left[\mathbb{1}_A\left\langle \nabla_x X_t^{n_k}, \phi\right\rangle\right] = -\mathbb{E}\left[\mathbb{1}_A\left\langle \nabla_x Y_t, \phi\right\rangle\right].$$

Consequently,

$$\mathbb{P}\text{-a.s.} \quad \langle X_t, \phi'\rangle = -\left\langle \nabla_x Y_t, \phi\right\rangle. \tag{3.18}$$

It is left to show as in [1, Theorem 3.4] that there exists a measurable set $\Omega_0 \subset \Omega$ with full measure such that $(x\mapsto X_t^x)$ has a weak derivative on the subset Ω_0. In order to show this we choose a sequence $\{\phi_n\}_{n\geq 1} \subset \mathcal{C}_0^\infty(\mathbb{R})$ which is dense in $W^{1,2}(U)$ and a measurable subset $\Omega_n \subset \Omega$ with full measure such that (3.18) if fulfilled on Ω_n where ϕ is replaced by ϕ_n. Then $\Omega_0 := \bigcap_{n\geq 1}\Omega_n$ is a full measure set such that $(x\mapsto X_t^x)$ has a weak derivative on it. $\square$

Closing the part on Sobolev differentiability we consider the function $x \mapsto b\left(t, y, \mathbb{P}_{X_t^x}\right)$ and show that it is weakly differentiable. In Section 5 the weak derivative $\nabla_x b\left(t, y, \mathbb{P}_{X_t^x}\right)$ is then used in the Bismut-Elworthy-Li formula. Further, we give a remark on the connection to the Lions derivative.

Proposition 4.9 *Suppose the drift coefficient* $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^d$ *is Lipschitz continuous in the third variable* (3.8) *and bounded. Let* $(X_t^x)_{t\in[0,T]}$ *be the unique strong solution of mean-field SDE* (3.2) *and* $U \subset \mathbb{R}^d$ *be an open and bounded subset. Then for every* $1 < p < \infty$, $t \in [0,T]$, *and* $y \in \mathbb{R}^d$,

$$\left(x \mapsto b\left(t, y, \mathbb{P}_{X_t^x}\right)\right) \in W^{1,p}(U).$$

Proof. Using the proof of Lemma 4.8 the result follows equivalently to [6, Proposition 3.11]. Nevertheless for completeness we give the proof here.

Consider the approximating sequence $\{b_n\}_{n\geq 1}$ of the drift function b as defined in (3.14) and let $(X_t^{n,x})_{t\in[0,T]}$, $n \geq 1$, be the corresponding unique strong solutions of mean-field SDEs (3.15). For the sake of readability we denote $b_n(x) := b_n\left(t, y, \mathbb{P}_{X_t^{n,x}}\right)$ for every $n \geq 0$. First note that $\{b_n\}_{n\geq 1}$ is weakly relatively compact in $W^{1,p}(U)$, since by Lemma A.1 and the proof of Lemma 4.8

$$\sup_{n\geq 1} \|b_n\|_{W^{1,p}(U)} < \infty,$$

and thus the sequence is weakly relatively compact by [33, Theorem 10.44]. Thus, there exists a subsequence $\{n_k\}_{k\geq 1}$ and $g \in W^{1,p}(U)$ such that b_{n_k} converges weakly to g as $k \to \infty$.

Let $\phi \in \mathcal{C}_0^\infty(U)$ be an arbitrary test-function with first derivative ϕ'. Define

$$\langle b_n, \phi \rangle := \int_U b_n(x)\phi(x)dx.$$

We get due to Lemma A.1 that

$$\langle b_n - b, \phi' \rangle \leq \|\phi'\|_{L^p(U)} |U|^{\frac{1}{p}} \sup_{x\in\overline{U}} \|b_n(x) - b(x)\| < \infty.$$

Here, $\overline{U}$ is the closure of U. Further, by Proposition 4.6

$$\begin{aligned} &\left\|b_n\left(t, y, \mathbb{P}_{X_t^{n,x}}\right) - b\left(t, y, \mathbb{P}_{X_t^x}\right)\right\| \qquad (3.19)\\ &\leq \left\|b_n\left(t, y, \mathbb{P}_{X_t^{n,x}}\right) - b_n\left(t, y, \mathbb{P}_{X_t^x}\right)\right\| + \left\|b_n\left(t, y, \mathbb{P}_{X_t^x}\right) - b\left(t, y, \mathbb{P}_{X_t^x}\right)\right\| \\ &\leq C\mathcal{K}\left(\mathbb{P}_{X_t^{n,x}}, \mathbb{P}_{X_t^x}\right) + \left\|b_n\left(t, y, \mathbb{P}_{X_t^x}\right) - b\left(t, y, \mathbb{P}_{X_t^x}\right)\right\| \xrightarrow[n\to\infty]{} 0, \end{aligned}$$

which yields $\lim_{n\to\infty}\langle b_n - b, \phi' \rangle = 0$. Therefore,

$$\langle b, \phi' \rangle = \lim_{k\to\infty} \langle b_{n_k}, \phi' \rangle = -\lim_{k\to\infty} \left\langle b'_{n_k}, \phi \right\rangle = -\langle g', \phi \rangle,$$

where b'_{n_k} and g' are the first variation processes of b_{n_k} and g, respectively. □

Remark 4.10. Note that by the proof of Proposition 4.9 the process $\nabla_x b$ is bounded, i.e.

$$\|\nabla_x b\|_\infty \leq C < \infty, \qquad (3.20)$$

for some constant $C > 0$.

Remark 4.11. Due to Lemma A.1 the law of the unique strong solution X^x of mean-field SDE (3.2) is in the space $\mathcal{P}_2(\mathbb{R}^d)$ of probability measures with finite second moment. Thus, restraining the domain of the drift function b to $[0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ enables the introduction of the Lions derivative $\nabla_\mu b(t,y,\cdot)$ for every $t \in [0,T]$ and $y \in \mathbb{R}^d$. For an introduction to this topic we refer the reader to [10]. The analysis in [9] and [20] of the first variation process $\nabla_x X_t^x$ suggests that the representation

$$\nabla_x b\left(t, y, \mathbb{P}_{X_t^x}\right) = \mathbb{E}\Big[\nabla_\mu b\left(t, y, \mathbb{P}_{X_t^x}\right)(X_t^x)\nabla_x X_t^x\Big]$$

holds. Note that the Lions derivative entails an additional variable which is here denoted by $\nabla_\mu b(\cdot)(X_t^x)$.

4.3. **Hölder continuity.** Concluding the section on regularity properties, we show Hölder continuity in time and space of the unique strong solution $(X_t^x)_{t\in[0,T]}$ of mean-field SDE (3.2).

Theorem 4.12 *Suppose the drift coefficient $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^d$ is Lipschitz continuous in the third variable* (3.8) *and bounded. Let $(X_t^x)_{t\in[0,T]}$ be the unique strong solution of mean-field SDE* (3.2)*. Then, for every compact subset $K \subset \mathbb{R}^d$ there exists a constant $C > 0$ such that for all $s,t \in [0,T]$ and $x, y \in K$,*

$$\mathbb{E}\left[\|X_t^x - X_s^y\|^2\right] \leq C\left(|t-s| + \|x-y\|^2\right).$$

In particular, there exists a continuous version of the random field $(t,x) \mapsto X_t^x$ with Hölder continuous trajectories of order $\alpha < \frac{1}{2}$ in $t \in [0,T]$ and $\alpha < 1$ in $x \in \mathbb{R}^d$.

The proof of Theorem 4.12 is analogous to the proof of [6, Theorem 3.12] and uses the following lemma.

Lemma 4.13 *Suppose the drift coefficient $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^d$ is Lipschitz continuous in the third variable* (3.8) *and bounded. Let $(X_t^x)_{t\in[0,T]}$ be the unique strong solution of mean-field SDE* (3.2)*. Then, for every compact subset $K \subset \mathbb{R}^d$ and $p \geq 1$, there exists a constant $C > 0$ such that*

$$\sup_{t\in[0,T]} \operatorname*{ess\,sup}_{x\in K} \mathbb{E}\left[\|\nabla_x X_t^x\|^p\right] \leq C.$$

Proof. The result follows immediately by Lemma 4.8 and Fatou's lemma. □

5. Bismut-Elworthy-Li type formula

In this section we establish an integration by parts formula of Bismut-Elworthy-Li type. More precisely, we consider the functional $x \mapsto \mathbb{E}\left[\Phi(X_t^x)\right]$, where Φ merely fulfills some integrability condition, and show that it is weakly differentiable. Moreover, we give a probabilistic representation of the derivative $\nabla_x \mathbb{E}\left[\Phi(X_t^x)\right]$.

Theorem 5.1 *Suppose the drift coefficient* $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^d$ *is Lipschitz continuous in the third variable* (3.8) *and bounded. Let* $(X_t^x)_{t\in[0,T]}$ *be the unique strong solution of mean-field SDE* (3.2)*,* $K \subset \mathbb{R}^d$ *be a compact subset,* $\Phi \in L^2(\mathbb{R}^d; \omega_T)$*, and* ω_T *is as defined in* (3.4)*. Then, for every open subset* $U \subset K$*,* $t \in [0,T]$*, and* $1 < q < \infty$*,*

$$(x \mapsto \mathbb{E}\left[\Phi(X_t^x)\right]) \in W^{1,q}(U),$$

and for almost all $x \in K$

$$\nabla_x \mathbb{E}[\Phi(X_T^x)] = \mathbb{E}\left[\Phi(X_T^x)\int_0^T \left(a(s)\nabla_x X_s^x + \nabla_x b\left(s, y, \mathbb{P}_{X_s^x}\right)|_{y=X_s^x} \int_0^s a(u)du\right) dB_s\right], \tag{3.21}$$

where $a : \mathbb{R} \to \mathbb{R}$ *is any bounded, measurable function such that*

$$\int_0^T a(s)ds = 1.$$

Proof. First assume that $\Phi \in \mathcal{C}_b^{1,1}(\mathbb{R}^d)$ and let $\{b_n\}_{n\geq 1}$ and $\{X^{n,x}\}_{n\geq 1}$ be defined as in (3.14) and (3.15), respectively. Due to Proposition 4.6 it is readily seen that

$$\mathbb{E}[\Phi(X_T^{n,x})] \xrightarrow[n\to\infty]{} \mathbb{E}[\Phi(X_T^x)], \tag{3.22}$$

for every $t \in [0,T]$ and $x \in K$. Equivalently to [6, Lemma 4.1] it can be shown that $\mathbb{E}[\Phi(X_t^{n,x})]$ is weakly differentiable in x and

$$\nabla_x \mathbb{E}[\Phi(X_T^{n,x})] = \mathbb{E}[\Phi'(X_T^{n,x})\nabla_x X_T^{n,x}].$$

Furthermore, using the representation (3.12) we get for any bounded measurable function $a : \mathbb{R} \to \mathbb{R}$ with $\int_0^T a(s)ds = 1$ that

$$\begin{aligned}
\nabla_x X_T^{n,x} &= \int_0^T a(s)\left(D_s X_T^{n,x}\nabla_x X_s^{n,x} + \int_s^T D_r X_T^{n,x} \nabla_x b_n(r, y, \mathbb{P}_{X_r^{n,x}})\big|_{y=X_r^{n,x}} dr\right) ds \\
&= \int_0^T a(s) D_s X_T^{n,x} \nabla_x X_s^{n,x} ds + \int_0^T \int_s^T a(s) D_r X_T^{n,x} \nabla_x b_n(r, y, \mathbb{P}_{X_r^{n,x}})\big|_{y=X_r^{n,x}} drds.
\end{aligned}$$

Now we look at each term individually starting by the first one. Note first that $\Phi(X_T^{n,x})$ is Malliavin differentiable and thus using the chain rule yields

$$\mathbb{E}\left[\Phi'(X_T^{n,x})\int_0^T a(s) D_s X_T^{n,x} \nabla_x X_s^{n,x} ds\right] = \mathbb{E}\left[\int_0^T a(s) D_s \Phi(X_T^{n,x})\nabla_x X_s^{n,x} ds\right].$$

Since $s \mapsto a(s)\nabla_x X_s^{n,x}$ is an adapted process and by Lemma 4.8

$$\mathbb{E}\left[\int_0^T \|a(s)\nabla_x X_s^{n,x}\|^2 \, ds\right] < \infty,$$

the application of the duality formula [22, Corollary 4.4] yields

$$\mathbb{E}\left[\int_0^T a(s) D_s \Phi(X_T^{n,x})\nabla_x X_s^{n,x} ds\right] = \mathbb{E}\left[\Phi(X_T^{n,x})\int_0^T a(s)\nabla_x X_s^{n,x} dB_s\right].$$

Considering the second term note first that due to (3.17) and (3.20)

$$\sup_{r,s\in[0,T]} \mathbb{E}\left[\left\|\Phi'(X_T^{n,x})a(s)D_rX_T^{n,x}\nabla_x b_n(r,y,\mathbb{P}_{X_r^{n,x}})\Big|_{y=X_r^{n,x}}\right\|\right] < \infty.$$

Consequently, the integral

$$\int_0^T\int_0^T \mathbb{E}\left[\Phi'(X_T^{n,x})a(s)D_rX_T^{n,x}\nabla_x b_n(r,y,\mathbb{P}_{X_r^{n,x}})\Big|_{y=X_r^{n,x}}\right]drds$$

exists and is finite by Tonelli's Theorem. Thus, the order of integration can be swapped and we obtain by using once more the duality formula [22, Corollary 4.4] that

$$\begin{aligned}
&\mathbb{E}\left[\Phi'(X_T^{n,x})\int_0^T\int_s^T a(s)D_rX_T^{n,x}\nabla_x b_n(r,y,\mathbb{P}_{X_r^{n,x}})\Big|_{y=X_r^{n,x}}drds\right]\\
&\quad= \mathbb{E}\left[\int_0^T D_r\Phi(X_T^{n,x})\nabla_x b_n(r,y,\mathbb{P}_{X_r^{n,x}})\Big|_{y=X_r^{n,x}}\int_0^r a(s)dsdr\right]\\
&\quad= \mathbb{E}\left[\Phi(X_T^{n,x})\int_0^T \nabla_x b_n(r,y,\mathbb{P}_{X_r^{n,x}})\Big|_{y=X_r^{n,x}}\int_0^r a(s)dsdB_r\right].
\end{aligned}$$

Putting all together we obtain representation (3.21) for $\Phi\in\mathcal{C}_b^{1,1}(\mathbb{R}^d)$ where b and X^x are substituted by b_n and $X^{n,x}$, respectively.

Next, we show that representation (3.21) is valid also for b and X^x. Let $\varphi\in\mathcal{C}_0^\infty(U)$. We prove subsequently that

$$\begin{aligned}
&\int_U \varphi'(x)\mathbb{E}[\Phi(X_T^x)]dx\\
&\quad= -\int_U\varphi(x)\mathbb{E}\left[\Phi(X_T^x)\int_0^T\left(a(s)\nabla_xX_s^x+\nabla_x b\left(s,y,\mathbb{P}_{X_s^x}\right)|_{y=X_s^x}\int_0^s a(u)du\right)dB_s\right]dx.
\end{aligned}$$

Using (3.22) we have that

$$\begin{aligned}
&\int_U \varphi'(x)\mathbb{E}[\Phi(X_T^x)]dx\\
&\quad= -\lim_{n\to\infty}\int_U\varphi(x)\mathbb{E}\left[\Phi(X_T^{n,x})\int_0^T\left(a(s)\nabla_xX_s^{n,x}+\nabla_x b_n(s,x)\int_0^s a(u)du\right)dB_s\right]dx\\
&\quad= -\lim_{n\to\infty}\int_U\varphi(x)\mathbb{E}\left[\Phi(X_T^{n,x})\int_0^T a(s)\nabla_xX_s^{n,x}dB_s\right]dx\\
&\qquad-\lim_{n\to\infty}\int_U\varphi(x)\mathbb{E}\left[\Phi(X_T^{n,x})\int_0^T \nabla_x b_n(s,x)\int_0^s a(u)dudB_s\right]dx\\
&\quad=: -\lim_{n\to\infty}A_n-\lim_{n\to\infty}C_n,
\end{aligned}$$

where $b_n(s,x):=b_n\left(s,y,\mathbb{P}_{X_s^{n,x}}\right)|_{y=X_s^{n,x}}$, $n\geq 0$. For A_n we further get that

$$A_n=\int_U\varphi(x)\mathbb{E}\left[\left(\Phi(X_T^{n,x})-\Phi(X_T^x)\right)\int_0^T a(s)\nabla_xX_s^{n,x}dB_s\right]dx$$

$$+\int_U \varphi(x)\mathbb{E}\left[\Phi(X_T^x)\int_0^T a(s)\left(\nabla_x X_s^{n,x}-\nabla_x X_s^x\right)dB_s\right]dx$$
$$+\int_U \varphi(x)\mathbb{E}\left[\Phi(X_T^x)\int_0^T a(s)\nabla_x X_s^x dB_s\right]dx$$
$$=: A_n(I)+A_n(II)+\int_U \varphi(x)\mathbb{E}\left[\Phi(X_T^x)\int_0^T a(s)\nabla_x X_s^x dB_s\right]dx.$$

Note that $A_n(I)$ and $A_n(II)$ converge to 0 due to Proposition 4.6 and Lemma 4.8, and the proof of Theorem 4.3, respectively.

For B_n let us first define the measure change

$$\frac{d\mathbb{Q}^n}{d\mathbb{P}} := \mathcal{E}\left(-\int_0^T b_n\left(s,X_s^{n,x},\mathbb{P}_{X_s^{n,x}}\right)dB_s\right), \quad n\geq 0.$$

Note that under $\mathbb{Q}^n$ the processes $X^{n,x}$ is Brownian motion. Hence, we get with

$$\mathcal{E}_T^n := \mathcal{E}\left(\int_0^T b_n\left(s,B_s^x,\mathbb{P}_{X_s^{n,x}}\right)dB_s\right), \quad n\geq 0,$$

that

$$C_n - C_0 = \int_U \varphi(x)\left(\mathbb{E}\left[\Phi(B_T^x)\int_0^T \nabla_x b_n(s,x)\int_0^s a(u)dudB_s\ \mathcal{E}_T^n\right]\right.$$
$$\left.-\ \mathbb{E}\left[\Phi(B_T^x)\int_0^T \nabla_x b_0(s,x)\int_0^s a(u)dudB_s\ \mathcal{E}_T^0\right]\right)dx$$
$$-\int_U \varphi(x)\left(\mathbb{E}\left[\Phi(B_T^x)\int_0^T \nabla_x b_n(s,x)\int_0^s a(u)du\ b_n\left(s,B_s^x,\mathbb{P}_{X_s^{n,x}}\right)ds\ \mathcal{E}_T^n\right]\right.$$
$$\left.-\ \mathbb{E}\left[\Phi(B_T^x)\int_0^T \nabla_x b_0(s,x)\int_0^s a(u)du\ b\left(s,B_s^x,\mathbb{P}_{X_s^x}\right)ds\ \mathcal{E}_T^0\right]\right)dx$$
$$=: C_n(I)-C_n(II),$$

where $b_n(s,x) := b_n\left(s,y,\mathbb{P}_{X_s^{n,x}}\right)|_{y=B_s^x}$, $n\geq 0$. Considering $C_n(I)$ we have due to (3.20)

$$C_n(I) \lesssim \int_U \varphi(x)\left(\mathbb{E}\left[\int_0^T \left|\nabla_x b_n\left(s,y,\mathbb{P}_{X_s^{n,x}}\right)-\nabla_x b\left(s,y,\mathbb{P}_{X_s^x}\right)\right|^2_{y=B_s^x}ds\right]^{\frac{1}{2}}\right.$$
$$\left.+\ \mathbb{E}\left[\left|\mathcal{E}_T^n-\mathcal{E}_T^0\right|^2\right]\right)dx.$$

The first term converges to 0 due to the proof of Proposition 4.9 whereas the second term converges to 0 due to Lemma A.2. Furthermore, for $C_n(II)$ we have

$$C_n(II) \lesssim \int_U \varphi(x)\left(C_n(I)+\mathbb{E}\left[\Phi(B_T^x)\int_0^T \left|b_n\left(s,B_s^x,\mathbb{P}_{X_s^{n,x}}\right)-b\left(s,B_s^x,\mathbb{P}_{X_s^x}\right)\right|ds\right]\right)dx,$$

which converges to 0 due to (3.19) and dominated convergence. Thus equation (3.21) holds for $\Phi \in \mathcal{C}_b^{1,1}(\mathbb{R}^d)$.

Lastly, we show that equation (3.21) holds true for $\Phi \in L^2(\mathbb{R}^d;\omega_T)$. In order to show this, define a sequence $\{\Phi_n\} \subset \mathcal{C}_b^{1,1}(\mathbb{R}^d)$ by standard arguments which approximates Φ with respect to the norm $L^2(\mathbb{R}^d;\omega_T)$. Note first that

$$\begin{aligned}
&\mathbb{E}\left[\left\|\Phi(X_T^x)\int_0^T \left(a(s)\nabla_x X_s^x + \nabla_x b\left(s,y,\mathbb{P}_{X_s^x}\right)|_{y=X_s^x}\int_0^s a(u)du\right)dB_s\right\|\right] \qquad (3.23)\\
&\quad\le \mathbb{E}\left[\|\Phi(X_T^x)\|^2\right]^{\frac{1}{2}}\\
&\qquad\times \mathbb{E}\left[\left\|\int_0^T \left(a(s)\nabla_x X_s^x + \nabla_x b(s,X_s^x,\mathbb{P}_{X_s^x})|_{y=X_s^x}\int_0^s a(u)du\right)dB_s\right\|^2\right]^{\frac{1}{2}}\\
&\quad\le \mathbb{E}\left[\|\Phi(B_T^x)\|^2\,\mathcal{E}\left(\int_0^T b(s,B_s^x,\mathbb{P}_{X_s^x})dB_s\right)\right]^{\frac{1}{2}}\\
&\qquad\times \mathbb{E}\left[\int_0^T \left\|a(s)\nabla_x X_s^x + \nabla_x b(s,X_s^x,\mathbb{P}_{X_s^x})|_{y=X_s^x}\int_0^s a(u)du\right\|^2 du\right]^{\frac{1}{2}}\\
&\quad\lesssim \mathbb{E}\left[\|\Phi(B_T^x)\|^2\right]^{\frac{1}{2}} < \infty,
\end{aligned}$$

where we have used Lemma 4.8 and (3.20). Thus, expression (3.23) is well-defined. Furthermore, it is readily seen that

$$\mathbb{E}[\Phi_n(X_T^x)] \xrightarrow[n\to\infty]{} \mathbb{E}[\Phi(X_T^x)].$$

Thus, for any test function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ we have that

$$\begin{aligned}
&\int_U \varphi(x)\mathbb{E}[\Phi(X_T^x)]dx\\
&\quad= -\lim_{n\to\infty}\int_U \varphi'(x)\mathbb{E}\left[\Phi_n(X_T^x)\int_0^T \left(a(s)\nabla_x X_s^x + \nabla_x b(s,x)\int_0^s a(u)du\right)dB_s\right]dx\\
&\quad\lesssim -\lim_{n\to\infty}\int_U \varphi'(x)\mathbb{E}\left[(\Phi_n(X_T^x)-\Phi(X_T^x))^2\right]^{\frac{1}{2}}dx\\
&\qquad -\int_U \varphi'(x)\mathbb{E}\left[\Phi(X_T^x)\int_0^T \left(a(s)\nabla_x X_s^x + \nabla_x b(s,x)\int_0^s a(u)du\right)dB_s\right]dx\\
&\quad= -\int_U \varphi'(x)\mathbb{E}\left[\Phi(X_T^x)\int_0^T \left(a(s)\nabla_x X_s^x + \nabla_x b(s,x)\int_0^s a(u)du\right)dB_s\right]dx,
\end{aligned}$$

where $b(s,x) := b(s,y,\mathbb{P}_{X_s^x})|_{y=X_s^x}$. Consequently, equation (3.21) holds for $\Phi \in L^2(\mathbb{R}^d;\omega_T)$. □

Appendix A. Technical Results

Consider the (mean-field) stochastic differential equation

$$dX_t^{x,\mu} = b\left(t, X_t^{x,\mu}, \mu_t\right) dt + dB_t,\ t \in [0,T],\ X_0^{x,\mu} = x \in \mathbb{R}^d, \tag{3.24}$$

where $\mu \in \mathcal{C}([0,T]; \mathcal{P}_1(\mathbb{R}^d))$. The following lemmas can be proven similar to [6, Lemma A.1 & Lemma A.6].

Lemma A.1 *Suppose the drift coefficient $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^d$ is of at most linear growth* (3.5) *and $X^{x,\mu}$ is a solution of SDE* (3.24)*. Then, for every $p \geq 1$ and any compact subset $K \subset \mathbb{R}^d$*

$$\sup_{x \in K} \mathbb{E}\left[\sup_{t \in [0,T]} \|b\left(t, X_t^{x,\mu}, \mu_t\right)\|^p\right] < \infty.$$

In particular,

$$\sup_{x \in K} \mathbb{E}\left[\sup_{t \in [0,T]} \|X_t^{x,\mu}\|^p\right] < \infty.$$

Moreover, for a set of measures $E_C := \{\mu \in \mathcal{C}([0,T]; \mathcal{P}_1(\mathbb{R})) : \sup_{t \in [0,T]} \mathcal{K}(\mu_t, \delta_0) \leq C\}$, where $C > 0$ is some constant, and every $p \geq 1$

$$\sup_{x \in K} \mathbb{E}\left[\sup_{t \in [0,T]} \sup_{\mu \in E_C} \|b\left(t, X_t^{x,\mu}, \mu_t\right)\|^p\right] < \infty.$$

Lemma A.2 *Suppose the drift coefficient $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^d$ is Lipschitz continuous in the third variable* (3.8) *and bounded. Let $(X_t^x)_{t \in [0,T]}$ be the unique strong solution of mean-field SDE* (3.2)*. Furthermore, $\{b_n\}_{n \geq 1}$ is the approximating sequence of b as defined in* (3.14) *and $(X_t^{n,x})_{t \in [0,T]}$, $n \geq 1$, the corresponding unique strong solutions of mean-field SDEs* (3.15)*. Then for any $p \geq 1$*

$$\mathbb{E}\left[\left|\mathcal{E}\left(\int_0^T b_n(t, B_t^x, \mathbb{P}_{X_t^{n,x}}) dB_t\right) - \mathcal{E}\left(\int_0^T b(t, B_t^x, \mathbb{P}_{X_t^x}) dB_t\right)\right|^p\right]^{\frac{1}{p}} \xrightarrow[n \to \infty]{} 0.$$

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Chapter 4
Strong Solutions of Mean-Field SDEs with irregular expectation functional in the drift

Contribution of the thesis' author

The paper *Strong Solutions of Mean-Field SDEs with irregular expectation functional in the drift* is a joint work with Prof. Dr. Thilo Meyer-Brandis.

M. Bauer was significantly involved in the development of all parts of the paper. In particular, M. Bauer made major contributions to the editorial work and the proofs of Theorem 3.1, Theorem 3.2, Theorem 3.5, Theorem 3.6, Proposition 4.1, and Theorem 4.3.

STRONG SOLUTIONS OF MEAN-FIELD SDES WITH IRREGULAR EXPECTATION FUNCTIONAL IN THE DRIFT

MARTIN BAUER AND THILO MEYER-BRANDIS

Abstract. We analyze multi-dimensional mean-field stochastic differential equations where the drift depends on the law in form of a Lebesgue integral with respect to the pushforward measure of the solution. We show existence and uniqueness of Malliavin differentiable strong solutions for irregular drift coefficients, which in addition to singularities in the space variable might also allow for discontinuities in the law variable. In particular, the case where the drift depends on the cumulative distribution function of the solution is covered. Moreover, we examine the solution as a function in its initial condition and introduce sufficient conditions on the drift to guarantee differentiability. Under these assumptions we then show that the Bismut-Elworthy-Li formula proposed in [7] holds in a strong sense, i.e. we give a probabilistic representation of the strong derivative with respect to the initial condition of expectation functionals of strong solutions to our type of mean-field equations in one-dimension.

Keywords. McKean-Vlasov equation · mean-field stochastic differential equation · strong solution · uniqueness in law · pathwise uniqueness · irregular coefficients · Malliavin derivative · Sobolev derivative · Hölder continuity · Bismut-Elworthy-Li formula · expectation functional.

1. Introduction

Throughout this paper, let $T > 0$ be a given time horizon. As an extension of stochastic differential equations, mean-field stochastic differential equations (hereafter mean-field SDEs), also referred to as McKean-Vlasov equations, given by

$$dX_t^x = \overline{b}\left(t, X_t^x, \mathbb{P}_{X_t^x}\right) dt + \overline{\sigma}\left(t, X_t^x, \mathbb{P}_{X_t^x}\right) dB_t,\ t \in [0,T],\ X_0^x = x \in \mathbb{R}^d, \qquad (4.1)$$

allow the coefficients to depend on the law of the solution in addition to the solution process. Here, $\overline{b} : [0,T] \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^d$ and $\overline{\sigma} : [0,T] \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^{d\times n}$ are some given drift and volatility coefficients, $(B_t)_{t\in[0,T]}$ is n-dimensional Brownian motion,

$$\mathcal{P}_1(\mathbb{R}^d) := \left\{\mu \,\middle|\, \mu \text{ probability measure on } (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \text{ with } \int_{\mathbb{R}^d} \|x\| d\mu(x) < \infty \right\}$$

is the space of probability measures over $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with existing first moment, and $\mathbb{P}_{X_t^x}$ is the law of X_t^x with respect to the underlying probability measure $\mathbb{P}$.

Mean-field SDEs arised from Boltzmann's equation in physics, which is used to model weak interaction between particles in a multi-particle system, and were first studied by Vlasov [40], Kac [29] and McKean [33]. Nowadays the study of mean-field SDEs is an active research field with numerous applications. Various extensions such as replacing the driving noise by a Lévy process or considering backward equations have been examined e.g. in [8], [9], and [28]. A cornerstone in the application of mean-field SDEs in Economics and Finance was set by Lasry and Lions with their work on mean-field games in [31], see also [11] for a readily accessible summary of Lions' lectures at Collège de France. Carmona and Delarue developed a probabilistic approach to mean-field games opposed to the analytic one taken in [31], see e.g. [12], [13], [14], [16], and [19] as well as the monographs [15]. A more recent application of the concept of mean-fields is in the modeling of systemic risk, in particular in models for inter-bank lending and borrowing, see e.g. [17], [18], [23], [24], [25], [30], and the cited sources therein.

In this paper we analyze (strong) solutions of multi-dimensional mean-field SDEs of the form

$$dX_t^x = b\left(t, X_t^x, \int_{\mathbb{R}^d} \varphi\left(t, X_t^x, z\right) \mathbb{P}_{X_t^x}(dz)\right) dt + dB_t,\ t \in [0,T],\ X_0^x = x \in \mathbb{R}^d, \tag{4.2}$$

for $b, \varphi : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$. This mean-field SDEs generalize two commonly used models in the literature, where the first one considers $b(t,y,z) = z$, see e.g. [34] and [37], or [8] where the authors consider backward mean-field SDEs, and in the second model $\varphi(t,y,z) = \overline{\varphi}(z)$ for some $\overline{\varphi} : \mathbb{R}^d \to \mathbb{R}^d$, see e.g. [1] and [21]. Note that putting $\overline{\sigma} \equiv 1$ and

$$\overline{b}(t,y,\mu) = (b \diamond \varphi)(t,y,\mu) := b\left(t, y, \int_{\mathbb{R}^d} \varphi(t,y,z)\mu(dz)\right), \tag{4.3}$$

yields that mean-field SDE (4.2) is recognized as a special case of the general mean-field SDE (4.1).

The first main contribution of this paper is to establish existence and uniqueness of weak and strong solutions of mean-field SDE (4.2) with irregular drift. Further, we show that the strong solutions are Malliavin differentiable. For coefficients $\overline{b}$ and $\overline{\sigma}$ in the general mean-field SDE(4.1) fulfilling typical regularity assumptions such as linear growth and Lipschitz continuity, existence and uniqueness is well-studied, see e.g [10]. In [20] the existence of strong solutions is shown for time-homogeneous mean-field SDEs (4.1) with drift coefficients $\overline{b}$ that are of linear growth and allow for certain discontinuities in the space variable y and are Lipschitz in the law variable μ. In the time-inhomogeneous case it is shown in [34] that there exists a strong solution of mean-field SDE (4.2) in the special case $b(t,y,z) = z$ under the assumption that φ is of linear growth. The special case of mean-field SDE (4.2), where $\varphi(t,y,z) = \overline{\varphi}(z)$, is treated in [21]. Here the author assumes that the drift coefficient b is bounded and continuously differentiable in the third variable z and $\overline{\varphi}$ is α-Hölder continuous for some $0 < \alpha \leq 1$. In [26] the authors consider mean-field SDEs of the form (4.2) and prove existence of a strong solution in the case where b fulfills an integrability condition and is continuous in the third variable and φ is bounded and Lipschitz continuous in the third variable. The case $b(t,y,z)=z$ is considered in [37], where φ is considered to merely fulfill some integrability condition in order to show the existence of a strong solution. The work that is the closest to our analysis presented in the following are [6] and [7], where for additive noise, i.e. $\overline{\sigma} \equiv 1$, existence and uniqueness of weak and Malliavin differentiable strong solutions of mean-field SDE (4.1) is shown for irregular drift coefficients $\overline{b}$ including the case of bounded coefficients $\overline{b}$ that are allowed to be merely measurable in the space variable y and continuous in the law variable μ.

Considering mean-field SDE (4.2), first existence and uniqueness results of solutions for irregular drifts are inherited from results in [6] on the general mean-field SDE (4.1) by specifying b and φ such that $\overline{b}$ in (4.3) fulfills the assumptions in [6]. We derive these conditions in Section 2. However, in order to guarantee continuity in the law variable μ required in [6] we cannot allow for irregular φ, in particular we need that φ is Lipschitz continuous in the third variable. This excludes interesting examples where φ is irregular, as for example the case when $\varphi(t,x,z) = \mathbb{1}_{\{z \leq u\}}$, $u \in \mathbb{R}$, and thus the case where the drift $b\left(t, X_t^x, F_{X_t^x}(u)\right)$ depends on the distribution function $F_{X_t^x}(\cdot)$ of the solution is not covered. The objective of this paper is thus to show existence and uniqueness of weak and Malliavin differentiable strong solutions of mean-field SDE (4.2) where we relax the conditions on φ even further and merely assume that φ is measurable and of at most linear growth. The assumptions on the drift function b are inherited from [6] which includes the case of merely measurable coefficients of at most linear growth that are continuous in the third variable z. In particular, this implies drift coefficients $\overline{b}$ that are not necessarily continuous in the law variable. We also remark here that for a special class of mean-field SDEs related to Fokker-Plank equations the existence of a weak solution where the drift coefficient is not necessarily required to be continuous in the law variable has been established in [3], [4], and [5]. As one application we obtain a global version of Carathéodory's existence theorem for ODEs.

In the second part of the paper the main objective is to study the differentiability in the initial condition x of the expectation functional $\mathbb{E}[\Phi(X_T^x)]$ and to give a Bismut-Elworthy-Li type representation of $\partial_x \mathbb{E}[\Phi(X_T^x)]$[1], where $\Phi : \mathbb{R} \to \mathbb{R}$ and $(X_t^x)_{t\in[0,T]}$ is the unique strong solution of the one-dimensional mean-field SDE (4.2), i.e. $d=1$. In [7] it is shown that $\mathbb{E}[\Phi(X_T^x)]$ is Sobolev differentiable in its initial condition for a broad range of irregular drift coefficients and for Φ fulfilling merely some integrability condition, and a Bismut-Elworthy-Li formula is derived. However, for various purposes it is of interest to understand when the derivative $\partial_x \mathbb{E}[\Phi(X_T^x)]$ exists in a strong sense. For example, the weak derivative does not allow for a satisfactory interpretation of $\partial_x \mathbb{E}[\Phi(X_T^x)]$ as a sensitivity measure in the sense of the so-called *Delta* from Mathematical Finance. For the case

[1]Here, ∂_x denotes the Jacobian with respect to the variable x.

$\varphi(t,y,z) = \overline{\varphi}(z)$ and for smooth coefficients, [1] provides a Bismut-Elworthy-Li formula for the continuous derivative $\partial_x \mathbb{E}[\Phi(X_T^x)]$. We here show that $\mathbb{E}[\Phi(X_T^x)]$ is continuously differentiable for a large family of irregular drift coefficients. More precisely, we require b and φ in addition to the assumptions for existence and uniqueness of strong solutions to be sufficiently regular in the third variable z. For these coefficients the Bismut-Elworthy-Li representation from [7] thus holds in a strong sense. As a first step to obtain this result, we also need to study strong differentiability of X^x in its initial condition x. In particular, we show that if b and φ are continuously differentiable in the space variable y and the third variable z then X_t^x is continuously differentiable in x.

The paper is structured as follows. In Section 2 we recall the results from [6] and [7] and apply it to the case of mean-field SDEs of type (4.2). These results will be employed in the remaining parts of the paper. In Section 3 we weaken the assumptions on φ and show existence, uniqueness, and Malliavin differentiability of solutions of mean-field equation (4.2). Finally, Section 4 deals with the first variation process $(\partial_x X_t^x)_{t\in[0,T]}$ and provides a Bismut-Elworthy-Li formula for the continuous derivative $\partial_x \mathbb{E}[\Phi(X_T^x)]$ for irregular drift coefficients in the one-dimensional case.

Notation: Subsequently we list some of the most frequently used notations. For this, let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be two metric spaces.

- By $\|\cdot\|$ we denote the euclidean norm.
- $\mathcal{C}(\mathcal{X};\mathcal{Y})$ denotes the space of continuous functions $f:\mathcal{X}\to\mathcal{Y}$. If $\mathcal{X}=\mathcal{Y}$ we write $\mathcal{C}(\mathcal{X}) := \mathcal{C}(\mathcal{X};\mathcal{X})$.
- $\mathcal{C}_0^\infty(\mathcal{X})$ denotes the space of smooth functions $f:\mathcal{X}\to\mathbb{R}$ with compact support.
- For every $C>0$ we define the space $\mathrm{Lip}_C(\mathcal{X};\mathcal{Y})$ of functions $f:\mathcal{X}\to\mathcal{Y}$ such that
$$d_{\mathcal{Y}}(f(x_1), f(x_2)) \leq C d_{\mathcal{X}}(x_1,x_2), \quad \forall x_1,x_2\in\mathcal{X}$$
as the space of Lipschitz functions with Lipschitz constant $C>0$. Furthermore, we define $\mathrm{Lip}(\mathcal{X};\mathcal{Y}) := \bigcup_{C>0} \mathrm{Lip}_C(\mathcal{X};\mathcal{Y})$ and denote by $\mathrm{Lip}_C(\mathcal{X}) := \mathrm{Lip}_C(\mathcal{X};\mathcal{X})$ and $\mathrm{Lip}(\mathcal{X}) := \mathrm{Lip}(\mathcal{X};\mathcal{X})$, respectively, the space of Lipschitz functions mapping from $\mathcal{X}$ to $\mathcal{X}$.
- $\mathcal{C}_{b,C}^{1,1}(\mathbb{R}^d)$ denotes the space of continuously differentiable functions $f:\mathbb{R}^d\to\mathbb{R}^d$ such that there exists a constant $C>0$ with
  (a) $\sup_{y\in\mathbb{R}^d}\|f'(y)\|\leq C$, and
  (b) $(y\mapsto f'(y))\in\mathrm{Lip}_C(\mathbb{R}^d)$.
  Here f' denotes the Jacobian of f. We define $\mathcal{C}_b^{1,1}(\mathbb{R}^d) := \bigcup_{C>0}\mathcal{C}_{b,C}^{1,1}(\mathbb{R}^d)$.
- $\mathfrak{C}([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)$ is the space of functions $f:[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d$ such that there exists a constant $C>0$ with

(a) $(y \mapsto f(t,y,z)) \in \mathcal{C}^{1,1}_{b,C}(\mathbb{R}^d)$ for all $t \in [0,T]$ and $z \in \mathbb{R}^d$, and
(b) $(z \mapsto f(t,y,z)) \in \mathcal{C}^{1,1}_{b,C}(\mathbb{R}^d)$ for all $t \in [0,T]$ and $y \in \mathbb{R}^d$.

- We say a function $f : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^d$ is in the space $\mathcal{L}([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)$, if there exists a constant $C > 0$ such that for every $t \in [0,T]$ and $y \in \mathbb{R}^d$ the function $(z \mapsto f(t,y,z)) \in \mathcal{C}^{1,1}_{b,C}(\mathbb{R}^d)$.
- Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a generic complete filtered probability space with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ and $B = (B_t)_{t\in[0,T]}$ be d-dimensional Brownian motion defined on this probability space. Furthermore, we write $\mathbb{E}[\cdot] := \mathbb{E}_{\mathbb{P}}[\cdot]$, if not mentioned differently.
- $L^p(\Omega)$ denotes the Banach space of functions on the measurable space $(\Omega, \mathcal{F})$ integrable to some power p, $p \geq 1$.
- $L^p(\Omega, \mathcal{F}_t)$ denotes the space of $\mathcal{F}_t$-measurable functions in $L^p(\Omega)$.
- We define the weighted L^p-space over $\mathbb{R}$ with weight function $\omega : \mathbb{R} \to \mathbb{R}$ as
$$L^p(\mathbb{R};\omega) := \left\{ f : \mathbb{R} \to \mathbb{R} \text{ measurable} : \int_{\mathbb{R}} |f(y)|^p \omega(y) dy < \infty \right\}.$$
- Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a (weakly) differentiable function. Then we denote by $\partial_y f(y) := \frac{\partial f}{\partial y}(y)$ its first (weak) derivative evaluated at $y \in \mathbb{R}^d$ and ∂_k is the Jacobian in the direction of the k-th variable.
- We denote the Doléan-Dade exponential for a progressive process Y with respect to the corresponding Brownian integral if well-defined for $t \in [0,T]$ by
$$\mathcal{E}\left(\int_0^t Y_u dB_u\right) := \exp\left\{\int_0^t Y_u dB_u - \frac{1}{2}\int_0^t \|Y_u\|^2 du\right\}.$$
- We define $B_t^x := x + B_t$, $t \in [0,T]$, for any Brownian motion B.
- We write $E_1(\theta) \lesssim E_2(\theta)$ for two mathematical expressions $E_1(\theta), E_2(\theta)$ depending on some parameter θ, if there exists a constant $C > 0$ not depending on θ such that $E_1(\theta) \leq C E_2(\theta)$.
- We denote the Wiener transform of some $Z \in L^2(\Omega, \mathcal{F}_T)$ in $f \in L^2([0,T])$ by
$$\mathcal{W}(Z)(f) := \mathbb{E}\left[Z\mathcal{E}\left(\int_0^T f(s) dB_s\right)\right].$$

2. Results derived from the general mean-field SDE

In this section we recall sufficient conditions on b and φ such that $\overline{b}$ as defined in (4.3) fulfills the corresponding assumptions for existence, uniqueness, and regularity properties of strong solutions required in [6] and [7]. These results will subsequently be applied in Sections 3 and 4 in order to weaken the assumptions on φ such that mean-field SDE (4.2) has a Malliavin differentiable strong solution and to show strong differentiability of this unique strong solution under sufficient conditions on b and φ. We start by giving the definitions of some frequently used assumptions.

Let $f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be a measurable function. The function f is said to be of linear growth, if there exists a constant $C > 0$ such that for all $t \in [0,T]$ and $y, z \in \mathbb{R}^d$,

$$\|f(t, y, z)\| \leq C(1 + \|y\| + \|z\|). \tag{4.4}$$

We say f is continuous in the third variable (uniformly with respect to the first and second variable), if for all $z_1 \in \mathbb{R}^d$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $t \in [0,T]$ and $y \in \mathbb{R}^d$

$$\left(\forall z_2 \in \mathbb{R}^d : \|z_1 - z_2\| < \delta\right) \Rightarrow \|f(t, y, z_1) - f(t, y, z_2)\| < \varepsilon. \tag{4.5}$$

Moreover, we say f admits a modulus of continuity in the third variable, if there exists $\theta \in \{\vartheta \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}) : \vartheta(z) > 0 \text{ and } \int_0^z \frac{dy}{\vartheta(y)} = \infty \ \forall z \in \mathbb{R}_+\}$ such that for all $t \in [0,T]$ and $y, z_1, z_2 \in \mathbb{R}^d$

$$\|f(t, y, z_1) - f(t, y, z_2)\|^2 \leq \theta\left(\|z_1 - z_2\|^2\right). \tag{4.6}$$

The function f is said to be Lipschitz continuous in the second, respectively third, variable (uniformly with respect to the other two variables), if there exists a constant $C > 0$ such that for all $t \in [0,T]$ and $y_1, y_2, z \in \mathbb{R}^d$

$$\|f(t, y_1, z) - f(t, y_2, z)\| \leq C\|y_1 - y_2\|, \tag{4.7}$$

respectively, such that for all $t \in [0,T]$ and $y, z_1, z_2 \in \mathbb{R}^d$

$$\|f(t, y, z_1) - f(t, y, z_2)\| \leq C\|z_1 - z_2\|. \tag{4.8}$$

Concluding, we say the function f is Lipschitz continuous in the second and third variable (uniformly with respect to the first variable), if it fulfills the Lipschitz assumptions (4.7) and (4.8), i.e. there exists a constant $C > 0$ such that for all $t \in [0,T]$ and $y_1, y_2, z_1, z_2 \in \mathbb{R}^d$

$$\|f(t, y_1, z_1) - f(t, y_2, z_2)\| \leq C\left(\|y_1 - y_2\| + \|z_1 - z_2\|\right). \tag{4.9}$$

Note that when we talk about (Lipschitz) continuity in a certain variable, we always understand the continuity to hold uniformly with respect to the other variables.

We start by deriving sufficient conditions on b and φ from [6] and [7] for existence and uniqueness of solutions of mean-field SDE (4.2). For detailed definitions of the notions weak and strong solution as well as pathwise uniqueness and uniqueness in law - as used subsequently - we refer the reader to these same papers.

From [6, Theorems 3.7] we obtain in the following corollary the assumptions on b and φ to ensure the existence of a strong solution of (4.2).

Corollary 2.1 *Let* $b : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ *be a measurable function of at most linear growth* (4.4) *and continuous in the third variable* (4.5)*. Furthermore, assume that* $\varphi : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ *is a measurable functional which is of at most linear growth* (4.4) *and Lipschitz continuous in the third variable* (4.8)*. Then mean-field SDE* (4.2) *has a strong solution.*

If in addition b *admits a modulus of continuity in the third variable* (4.6)*, the solution is pathwisely unique.*

Concerning Malliavin differentiability of the solution we obtain from [6, Theorem 4.1]:

Corollary 2.2 *Let* b *be a bounded measurable function which is continuous in the third variable* (4.5)*. Furthermore, assume that* φ *is a measurable functional which is of at most linear growth* (4.4) *and Lipschitz continuous in the third variable* (4.8)*. Then mean-field SDE* (4.2) *has a Malliavin differentiable strong solution.*

Remark 2.3. In the one-dimensional case, $d = 1$, Corollary 2.2 can be generalized in the following way due to [7]. Let $(b \diamond \varphi)$ allow for a decomposition of the form

$$(b \diamond \varphi)(t,y,\mu) := \hat{b}\left(t, y, \int_{\mathbb{R}} \hat{\varphi}(t,y,z)\mu(dz)\right) + \tilde{b}\left(t, y, \int_{\mathbb{R}} \tilde{\varphi}(t,y,z)\mu(dz)\right), \quad (4.10)$$

where the drift $\hat{b}$ is merely measurable and bounded and the functional $\hat{\varphi}$ is of linear growth (4.4) and Lipschitz continuous in the third variable (4.8). Moreover, the drift $\tilde{b}$ is of at most linear growth (4.4) and Lipschitz continuous in the second variable (4.7) whereas the functional $\tilde{\varphi}$ is of at most linear growth (4.4) and Lipschitz continuous in the second and third variable (4.9). Then, mean-field SDE (4.2) has a Malliavin differentiable unique strong solution and the Malliavin derivative admits for $0 \leq s \leq t \leq T$ the representation

$$D_s X_t^x = \exp\left\{ - \int_s^t \int_{\mathbb{R}} b\left(u, y, \int_{\mathbb{R}} \varphi(u,y,z)\mathbb{P}_{X_u^x}(dz)\right) L^{X^x}(du,dy) \right\}. \quad (4.11)$$

Here, $L^{X^x}(du,dy)$ denotes integration with respect to local time of X^x in time and space, see [2] and [22] for more details. If in addition b is continuously differentiable with respect to the second and third variable and φ is continuously differentiable with respect to the second variable, representation (4.11) can be written as

$$\begin{aligned} D_s X_t^x = \exp\Bigg\{ \int_s^t & \partial_2 b\left(u, X_u^x, \int_{\mathbb{R}} \varphi(u, X_u^x, z)\mathbb{P}_{X_u^x}(dz)\right) \\ & + \partial_3 b\left(u, X_u^x, \int_{\mathbb{R}} \varphi(u, X_u^x, z)\mathbb{P}_{X_u^x}(dz)\right) \int_{\mathbb{R}} \partial_2 \varphi(u, X_u^x, z)\mathbb{P}_{X_u^x}(dz) du \Bigg\}. \end{aligned}$$

Here, ∂_2 and ∂_3 denotes the derivative with respect to the second and third variable, respectively.

Next we state a result on the regularity of a strong solution of (4.2) in its initial condition which is due to [6, Theorem 4.3].

Corollary 2.4 *Let b be a bounded measurable function which is Lipschitz continuous in the third variable* (4.8)*. Furthermore, assume that φ is a measurable functional which is of at most linear growth* (4.4) *and Lipschitz continuous in the third variable* (4.8)*. Then, the unique strong solution $(X_t^x)_{t\in[0,T]}$ of mean-field SDE* (4.2) *is Sobolev differentiable in the initial condition x.*

Remark 2.5. In the one-dimensional case, $d = 1$, we further get due to [7, Theorem 3.3 & Proposition 3.4] for $(b\diamond\varphi)$ allowing for a decomposition (4.10) that the first variation process $(\partial_x X_t^x)_{t\in[0,T]}$ has for almost all $x \in K$, where $K \subset \mathbb{R}$ is a compact subset, the representation

$$\begin{aligned}\partial_x X_t^x = \exp\left\{-\int_0^t \int_{\mathbb{R}} (b\diamond\varphi)\left(s, y, \mathbb{P}_{X_s^x}\right) L^{X^x}(ds, dy)\right\} \qquad\qquad (4.12)\\ + \int_0^t \exp\left\{-\int_u^t \int_{\mathbb{R}} (b\diamond\varphi)\left(s, y, \mathbb{P}_{X_s^x}\right) L^{X^x}(ds, dy)\right\} \partial_x (b\diamond\varphi)\left(s, y, \mathbb{P}_{X_u^x}\right)\Big|_{y=X_u^x} du.\end{aligned}$$

Moreover, for $0 \leq s \leq t \leq T$ the following relationship with the Malliavin Derivative holds:

$$\partial_x X_t^x = D_s X_t^x \partial_x X_s^x + \int_s^t D_u X_t^x \partial_x (b\diamond\varphi)\left(u, y, \mathbb{P}_{X_u^x}\right)\Big|_{y=X_u^x} du\,. \qquad (4.13)$$

Furthermore, the unique strong solution is Hölder continuous in time and the initial condition which is due to [6, Theorem 4.12].

Corollary 2.6 *Let b be a bounded measurable function which is Lipschitz continuous in the third variable* (4.8)*. Furthermore, assume that φ is a measurable functional which is of at most linear growth* (4.4) *and Lipschitz continuous in the third variable* (4.8)*. Let $(X_t^x)_{t\in[0,T]}$ be the unique strong solution of mean-field SDE* (4.2)*. Then for every compact subset $K \subset \mathbb{R}^d$ there exists a constant $C > 0$ such that for all $s, t \in [0, T]$ and $x, y \in K$,*

$$\mathbb{E}[\|X_t^x - X_s^y\|^2] \leq C(|t-s| + \|x-y\|^2). \qquad (4.14)$$

In particular, there exists a continuous version of the random field $(t, x) \mapsto X_t^x$ with Hölder continuous trajectories of order $\alpha < \frac{1}{2}$ in $t \in [0, T]$ and $\alpha < 1$ in $x \in \mathbb{R}^d$.

Finally, from [6, Theorem 5.1] we get the following Bismut-Elworthy-Li type formula under the same assumptions as in Corollary 2.4.

Corollary 2.7 *Let b be a bounded measurable function which is Lipschitz continuous in the third variable* (4.8)*. Furthermore, assume that φ is a measurable functional which is of at most linear growth* (4.4) *and Lipschitz continuous in the third variable* (4.8)*. Furthermore, let $\Phi \in L^{2p}(\mathbb{R}^d; \omega_T)$ with $p := \frac{1+\varepsilon}{\varepsilon}$, $\varepsilon > 0$ sufficiently small with regard to Lemma A.2, and $\omega_T(y) := \exp\left\{-\frac{\|y\|^2}{4T}\right\}$. Then, the expectation functional $\mathbb{E}\left[\Phi(X_t^x)\right]$ is Sobolev differentiable in the initial condition* *and the derivative* $\partial_x \mathbb{E}\left[\Phi(X_T^x)\right]$ *admits for almost all* $x \in K$*, where* $K \subset \mathbb{R}^d$ *is a compact subset, the representation*

$$\partial_x \mathbb{E}[\Phi(X_T^x)] = \mathbb{E}\left[\Phi(X_T^x)\left(\int_0^T a(s)\partial_x X_s^x + \partial_x(b \diamond \varphi)\left(s, y, \mathbb{P}_{X_s^x}\right)|_{y=X_s^x} \int_0^s a(u)dudB_s\right)\right], \tag{4.15}$$

where $a : \mathbb{R} \to \mathbb{R}$ *is any bounded, measurable function such that*

$$\int_0^T a(s)ds = 1.$$

3. Existence and Uniqueness of Solutions

The results in Section 2 presume Lipschitz continuity of the function φ. In this section we are interested in showing existence and uniqueness of strong solutions under weakened regularity assumptions on φ. In particular, the drift coefficient $\bar{b}$ might exhibit discontinuities in the law variable μ. For example, this will allow to consider mean-field SDEs where the drift depends on the solution law in form of indicator and distribution functions, respectively.

Theorem 3.1 *Let* $b : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ *be of at most linear growth* (4.4) *and continuous in the third variable* (4.5)*. Further, let* $\varphi : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ *be of at most linear growth* (4.4)*. Then mean-field SDE* (4.2) *has a strong solution.*

If in addition b is Lipschitz continuous in the third variable (4.8)*, the solution is unique.*

Proof. This proof is organized as follows. First we introduce a sequence $\{Y^n\}_{n\geq 1}$ of solutions to mean-field SDE (4.1) with approximating coefficients and show that we can find a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that the equivalent sequence $\{X^k\}_{k\geq 1}$ of $\{Y^n\}_{n\geq 1}$ on this probability space converges in $L^2(\Omega)$ to some stochastic process X. We then prove that X^k further converges weakly in $L^2(\Omega)$ to a solution of (4.2) and thus by uniqueness of the limit X is a weak solution of mean-field SDE (4.2). Afterwards we conclude the existence of a strong solution and prove uniqueness of the solution.

By standard arguments using mollifiers, we can define sequences $\{b_n\}_{n\geq 1}$ and $\{\varphi_n\}_{n\geq 1}$ in $\mathcal{C}_0^\infty([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$ such that b_n converges to b and φ_n converges to φ, respectively, pointwise in $(t,y,z) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d$ a.e. with respect to the Lebesgue measure. We denote the original functions b and φ by b_0 and φ_0, respectively. Due to continuity assumption (4.5) on the coefficient b, we can further assume that the family of coefficients $\{b_n\}_{n\geq 0}$ is pointwisely equicontinuous in the third variable, i.e. for every $\varepsilon > 0$ and $z_1 \in \mathbb{R}^d$ exists a $\delta > 0$ such that for all $n \geq 0$, $t \in [0,T]$, and $y \in \mathbb{R}^d$ we get

$$\left(\forall z_2 \in \mathbb{R}^d : \|z_1 - z_2\| < \delta\right) \Rightarrow \|b_n(t,y,z_1) - b_n(t,y,z_2)\| < \varepsilon. \tag{4.16}$$

Then, by Corollary 2.2, for $n \geq 1$ mean-field SDEs

$$
\begin{aligned}
dY_t^n &= b_n \left(t, Y_t^n, \int_{\mathbb{R}^d} \varphi_n \left(t, Y_t^n, z \right) \mathbb{P}_{Y_t^n}(dz) \right) dt + dW_t, \ t \in [0,T], \\
Y_0^n &= x \in \mathbb{R}^d,
\end{aligned}
\tag{4.17}
$$

where $W = (W_t)_{t\in[0,T]}$ is Brownian motion, have unique strong solutions $\{Y^n\}_{n\geq 1}$ on some complete probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Moreover, due to Lemma A.2 there exists some constant $C > 0$ such that

(i) $\sup_{n\geq 1} \sup_{0\leq t\leq T} \mathbb{E}_{\tilde{\mathbb{P}}}\left[\|Y_t^n\|^2\right] \leq C(1 + \|x\|^2)$,

(ii) $\sup_{n\geq 1} \sup_{0\leq s\leq t\leq T; t-s\leq h} \mathbb{E}_{\tilde{\mathbb{P}}}\left[\|Y_t^n - Y_s^n\|^2\right] \leq Ch$.

Next, we show that the properties (i) and (ii) imply the assumptions of Theorem B.1 and thus there exists a subsequence $\{n_k\}_{k\geq 1} \subset \mathbb{N}$ and a sequence of stochastic processes $\{(X_t^k)_{t\in[0,T]}\}_{k\geq 1}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that the finite dimensional distributions of the processes Y^{n_k} and X^k coincide for every $k \geq 1$, c.f. Remark B.2, and X_t^k converges in probability to X_t as k goes to infinity. Note first that the stochastic processes $\{Y^n\}_{n\geq 1}$ are almost surely continuous as a solution of mean-field SDE (4.17). Furthermore, we get by Chebyshev's inequality that due to (i)

$$
\tilde{\mathbb{P}}(\|Y_t^n\| > K) \leq \frac{1}{K^2}\mathbb{E}_{\tilde{\mathbb{P}}}\left[\|Y_t^n\|^2\right] \leq \frac{1}{K^2}C(1 + \|x\|^2), \quad K > 0,
$$

and thus

$$
\lim_{K\to\infty} \lim_{n\to\infty} \sup_{t\in[0,T]} \tilde{\mathbb{P}}(\|Y_t^n\| > K) \leq \lim_{K\to\infty} \lim_{n\to\infty} \sup_{t\in[0,T]} \frac{1}{K^2}C(1 + \|x\|^2) = 0.
$$

Equivalently, we get due to property (ii) that for every $\varepsilon > 0$

$$
\tilde{\mathbb{P}}(\|Y_t^n - Y_s^n\| > \varepsilon) \leq \frac{1}{\varepsilon^2}\mathbb{E}_{\tilde{\mathbb{P}}}\left[\|Y_t^n - Y_s^n\|^2\right] \leq \frac{Ch}{\varepsilon^2},
$$

and thus

$$
\lim_{h\to 0} \lim_{n\to\infty} \sup_{|t-s|\leq h} \tilde{\mathbb{P}}(\|Y_t^n - Y_s^n\| > \varepsilon) \leq \lim_{h\to 0} \lim_{n\to\infty} \sup_{|t-s|\leq h} \frac{Ch}{\varepsilon^2} = 0.
$$

Consequently, the assumptions of Theorem B.1 are fulfilled. For the sake of readability, we assume in the following without loss of generality that $n_k = k$. Further note that due to the uniform integrability of $\{\|X_t^k\|^2\}$ by property (i), we get that for every $t \in [0,T]$ the sequence $\{X_t^k\}_{k\geq 1}$ converges to X_t in $L^2(\Omega)$. Due to property (ii) we further get in connection with Kolmogorov's continuity theorem that $(X_t)_{t\in[0,T]}$ can be assumed to have almost surely continuous path. Using approximation by Riemann sums, we further have that

$$
\int_0^t b_k \left(s, X_s^k, \int_{\mathbb{R}^d} \varphi_k \left(s, X_s^k, z \right) \mathbb{P}_{X_s^k}(dz) \right) ds
$$

and

$$\int_0^t b_k\left(s, Y_s^k, \int_{\mathbb{R}^d} \varphi_k\left(s, Y_s^k, z\right) \mathbb{P}^k_{Y_s^k}(dz)\right) ds$$

have the same distribution for every $k \geq 1$. Again by virtue of Theorem B.1 we get that

$$B_t^k := X_t^k - \int_0^t b_k\left(s, X_s^k, \int_{\mathbb{R}^d} \varphi_k\left(s, X_s^k, z\right) \mathbb{P}_{X_s^k}(dz)\right) ds$$

is d-dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and thus X^k solves (4.17) on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, B^k)$.

Let us define the stochastic differential equation

$$d\overline{X}_t = b\left(t, \overline{X}_t, \int_{\mathbb{R}^d} \varphi\left(t, \overline{X}_t, z\right) \mathbb{P}_{X_t}(dz)\right) dt + dB_t, \; t \in [0,T], \; \overline{X}_0 = x \in \mathbb{R}^d. \tag{4.18}$$

Due to the result of Veretennikov given in [39], SDE (4.18) has a unique strong solution on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Therefore it is left to show that for every $t \in [0,T]$ the sequence $\{X_t^k\}_{k\geq 1}$ converges weakly in $L^2(\Omega)$ to $\overline{X}_t$. Indeed, if this holds true, we get by the uniqueness of the limit that $\mathbb{P}_{X_t} = \mathbb{P}_{\overline{X}_t}$ for all $t \in [0,T]$ and consequently mean-field SDE (4.2) and (4.18) coincide. Hence, we have found a weak solution of (4.2). In order to prove weak convergence in $L^2(\Omega)$ we use the Wiener transform and show for every $f \in L^2([0,T])$,

$$\left\| \mathcal{W}\left(X_t^k\right)(f) - \mathcal{W}\left(\overline{X}_t\right)(f) \right\| \xrightarrow[n\to\infty]{} 0.$$

Using inequality

$$|e^x - e^y| \leq |x-y|(e^x + e^y), \quad \forall x, y \in \mathbb{R}, \tag{4.19}$$

Burkholder-Davis-Gundy's inequality and Minkowski's integral inequality, we get for $p := \frac{1+\varepsilon}{\varepsilon}$, $\varepsilon > 0$ sufficiently small with respect to Lemma A.2, that

$$\begin{aligned}
&\left\| \mathcal{W}\left(X_t^k\right)(f) - \mathcal{W}\left(\overline{X}_t\right)(f) \right\| \\
&\quad \leq \mathbb{E}\left[\|B_t^x\| \left| \mathcal{E}\left(\int_0^T b_k\left(t, B_t^x, \int_{\mathbb{R}^d} \varphi_k\left(t, B_t^x, z\right) \mathbb{P}_{X_t^k}(dz)\right) + f(t) dB_t \right) \right.\right. \\
&\qquad \left.\left. - \mathcal{E}\left(\int_0^T b\left(t, B_t^x, \int_{\mathbb{R}^d} \varphi\left(t, B_t^x, z\right) \mathbb{P}_{X_t}(dz)\right) + f(t) dB_t \right) \right| \right] \\
&\quad \lesssim \left(\int_0^T \mathbb{E}\left[\left\| b_k\left(t, B_t^x, \int_{\mathbb{R}^d} \varphi_k\left(t, B_t^x, z\right) \mathbb{P}_{X_t^k}(dz)\right) \right.\right.\right. \\
&\qquad \left.\left.\left. - \; b\left(t, B_t^x, \int_{\mathbb{R}^d} \varphi\left(t, B_t^x, z\right) \mathbb{P}_{X_t}(dz)\right) \right\|^p \right]^{\frac{2}{p}} dt \right)^{\frac{1}{2}} + C_n \\
&\quad =: A_n + C_n,
\end{aligned}$$

where

$$C_n := \int_0^T \mathbb{E}\left[\left|\left\|b_k\left(t, B_t^x, \int_{\mathbb{R}^d} \varphi_k\left(t, B_t^x, z\right) \mathbb{P}_{X_t^k}(dz)\right) + f(t)\right\|^2 \right.\right.$$
$$\left.\left. - \left\|b\left(t, B_t^x, \int_{\mathbb{R}^d} \varphi\left(t, B_t^x, z\right) \mathbb{P}_{X_t}(dz)\right) + f(t)\right\|^2\right|^p\right]^{\frac{1}{p}} dt.$$

We show using dominated convergence that A_n converges to 0 as k tends to infinity. Since the family of coefficients $\{b_k\}_{k\geq 0}$ is pointwisely equicontinuous in the third variable (4.16), it suffices to show that for all $t \in [0,T]$ and $y \in \mathbb{R}^d$

$$\left\|\int_{\mathbb{R}^d} \varphi_k\left(t, y, z\right) \mathbb{P}_{X_t^k}(dz) - \int_{\mathbb{R}^d} \varphi\left(t, y, z\right) \mathbb{P}_{X_t}(dz)\right\| \xrightarrow[k\to\infty]{} 0, \text{ and}$$
$$\mathbb{E}\left[\left\|\overline{b}_k\left(t, B_t^x, \mathbb{P}_{X_t}\right) - \overline{b}\left(t, B_t^x, \mathbb{P}_{X_t}\right)\right\|^p\right]^{\frac{1}{p}} \xrightarrow[k\to\infty]{} 0.$$

The second convergence is an immediate consequence of the definition of b_k, Lemma A.2, and dominated convergence. Thus, it remains to show the first convergence. Let $\delta > 0$. Since φ_k is of at most linear growth (4.4) for all $k \geq 0$, we get by (i) that

$$\sup_{k\geq 0} \mathbb{E}\left[\left\|\varphi_k\left(t, y, X_t^k\right)\right\|\right] \leq C\left(1 + \|y\| + \sup_{k\geq 0} \mathbb{E}\left[\left\|X_t^k\right\|\right]\right) \leq C_1,$$

where $C_1 > 0$ is some constant independent of $k \geq 0$. Hence, due to dominated convergence we can find $N_1 \in \mathbb{N}$ sufficiently large such that

$$\sup_{k\geq N_1} \mathbb{E}\left[\left\|\varphi_k\left(t, y, X_t\right) - \varphi\left(t, y, X_t\right)\right\|\right] < \frac{\delta}{3}.$$

Note further that for $\varepsilon > 0$ sufficiently small with respect to Lemma A.2,

$$\sup_{k\geq 0} \mathbb{E}\left[\mathcal{E}\left(\int_0^T b_k\left(t, B_t^x, \int_{\mathbb{R}^d} \varphi_k(t, B_t^x, z)\mathbb{P}_{X_t^k}(dz)\right) dB_t\right)^{1+\varepsilon}\right]^{\frac{1}{1+\varepsilon}} \leq C_2 < \infty,$$

where $C_2 > 0$ is some constant. Thus we can find by Girsanov's theorem and again by dominated convergence an integer $N_2 \in \mathbb{N}$ such that

$$\sup_{m,k\geq N_2} \mathbb{E}\left[\left\|\varphi_k(t, y, X_t^k) - \varphi_m(t, y, X_t^k)\right\|\right]$$
$$\leq \sup_{m,k\geq N_2} C_2 \mathbb{E}\left[\|\varphi_k(t, y, B_t^x) - \varphi_m(t, y, B_t^x)\|^p\right]^{\frac{1}{p}} < \frac{\delta}{3},$$

where $p := \frac{1+\varepsilon}{\varepsilon}$. Therefore, using Minkowski's and Hölder's inequality we get for $N := \max\{N_1, N_2\}$ and $k \geq N$

$$\left\|\mathbb{E}\left[\varphi_k(t, y, X_t^k)\right] - \mathbb{E}\left[\varphi(t, y, X_t)\right]\right\|$$
$$\leq \mathbb{E}\left[\left\|\varphi_k(t, B_t^x, X_t^k) - \varphi_N(t, B_t^x, X_t^k)\right\|\right] + D_k$$

$$+\,\mathbb{E}\left[\|\varphi_N(t,y,X_t)-\varphi(t,y,X_t)\|\right]$$
$$\le D_k+\frac{2\delta}{3},$$

where

$$D_k := \left\|\mathbb{E}\left[\varphi_N(t,y,X_t^k)\right]-\mathbb{E}\left[\varphi_N(t,y,X_t)\right]\right\|.$$

Since φ_N is smooth and has compact support by the definition of mollification, φ_N is also bounded. Hence, using the fact that X_t^k converges in distribution to X_t for every $t\in[0,T]$, we can find $k\ge N$ sufficiently large such that $D_k<\frac{\delta}{3}$. Analogously one can show that C_k converges to 0 as k tends to infinity and therefore, X is a weak solution of the mean-field SDE (4.2). Due to the proof of [6, Theorem 3.7] we get as a direct consequence the existence of a strong solution of mean-field equation (4.2) for the more general class of functionals φ.

Let $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P},X,B)$ and $(\hat{\Omega},\hat{\mathcal{F}},\hat{\mathbb{F}},\hat{\mathbb{P}},Y,W)$ be two weak solutions of mean-field SDE (4.2) and assume that the drift coefficient b is Lipschitz continuous in the third variable (4.8). In the following we show that X and Y have the same law, i.e. $\mathbb{P}_X=\hat{\mathbb{P}}_Y$. For the sake of readability we just consider the case $x=0$. The general case follows analogously. From [6] we know that there exist measures $\mathbb{Q}$ and $\hat{\mathbb{Q}}$ such that under these measures the processes X and Y are Brownian motions, respectively. Similar to the proofs of [6, Theorem 3.7] and [7, Theorem 2.7] we use the idea of Li and Min in the proof of Theorem 4.2 in [32] and define the equivalent probability measure $\widetilde{\mathbb{Q}}$ by

$$\frac{\mathrm{d}\widetilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} := \mathcal{E}\left(-\int_0^T\left(\bar{b}\left(t,X_t,\mathbb{P}_{X_t}\right)-\bar{b}\left(t,X_t,\hat{\mathbb{P}}_{Y_t}\right)\right)dB_t^X\right).$$

Due to [6] and [7],

$$\hat{\mathbb{P}}_{(Y,W)}=\widetilde{\mathbb{Q}}_{(X,B)}.$$

Thus, it is left to show that $\sup_{t\in[0,T]}\mathcal{K}\left(\widetilde{\mathbb{Q}}_{X_t},\mathbb{P}_{X_t}\right)=0$, from which we conclude that $\sup_{t\in[0,T]}\mathcal{K}\left(\hat{\mathbb{P}}_{Y_t},\mathbb{P}_{X_t}\right)=0$ and hence $\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}}=1$. Consequently, $\hat{\mathbb{P}}_{(Y,W)}=\mathbb{P}_{(X,B)}$. Here, $\mathcal{K}$ denotes the Kantorovich metric defined by

$$\mathcal{K}(\mu,\nu)=\sup_{h\in\mathrm{Lip}_1(\mathbb{R}^d;\mathbb{R})}\left|\int_{\mathbb{R}^d}h(x)(\mu-\nu)(dx)\right|,\ \mu,\nu\in\mathcal{P}_1(\mathbb{R}^d).$$

Using Hölder's inequality for $p:=\frac{1+\varepsilon}{\varepsilon}$, where $\varepsilon>0$ is sufficiently small with regard to Lemma A.2, inequality (4.19), Burkholder-Davis-Gundy's inequality, and the Lipschitz continuity of b we get

$$\mathcal{K}\left(\widetilde{\mathbb{Q}}_{X_t},\mathbb{P}_{X_t}\right)=\sup_{h\in\mathrm{Lip}_1(\mathbb{R}^d;\mathbb{R})}\left|\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[h(X_t)-h(0)\right]-\mathbb{E}\left[h(X_t)-h(0)\right]\right|$$
$$\le\mathbb{E}\left[\|X_t\|\left|\mathcal{E}\left(-\int_0^t\left(\bar{b}\left(s,X_s,\mathbb{P}_{X_s}\right)-\bar{b}\left(s,X_s,\hat{\mathbb{P}}_{Y_s}\right)\right)dB_s\right)-1\right|\right]$$

$$
\begin{aligned}
&\lesssim \mathbb{E}\left[\left|\mathcal{E}\left(-\int_0^t \left(\overline{b}\left(s,X_s,\mathbb{P}_{X_s}\right)-\overline{b}\left(s,X_s,\hat{\mathbb{P}}_{Y_s}\right)\right)dB_s\right)-1\right|^{\frac{2(1+\varepsilon)}{2+\varepsilon}}\right]^{\frac{2+\varepsilon}{2(2+\varepsilon)}}\\
&\lesssim \mathbb{E}\left[\left|\int_0^t \left(\overline{b}\left(s,X_s,\mathbb{P}_{X_s}\right)-\overline{b}\left(s,X_s,\hat{\mathbb{P}}_{Y_s}\right)\right)dB_s\right.\right.\\
&\quad\left.\left.+\frac{1}{2}\int_0^t \left\|\overline{b}\left(s,X_s,\mathbb{P}_{X_s}\right)-\overline{b}\left(s,X_s,\hat{\mathbb{P}}_{Y_s}\right)\right\|^2 ds\right|^{2p}\right]^{\frac{1}{2p}}\\
&\lesssim \mathbb{E}\left[\left|\int_0^t \left\|\overline{b}\left(s,X_s,\mathbb{P}_{X_s}\right)-\overline{b}\left(s,X_s,\hat{\mathbb{P}}_{Y_s}\right)\right\|^2 ds\right|^{p}\right]^{\frac{1}{2p}}\\
&\quad+\mathbb{E}\left[\left|\int_0^t \left\|\overline{b}\left(s,X_s,\mathbb{P}_{X_s}\right)-\overline{b}\left(s,X_s,\hat{\mathbb{P}}_{Y_s}\right)\right\|^2 ds\right|^{2p}\right]^{\frac{1}{2p}}\\
&\lesssim \max_{q=1,2}\mathbb{E}\left[\left(\int_0^t \left\|\int_{\mathbb{R}^d}\varphi(s,X_s,z)\left(\mathbb{P}_{X_s}-\hat{\mathbb{P}}_{Y_s}\right)(dz)\right\|^2 ds\right)^{qp}\right]^{\frac{1}{2p}}\\
&=\max_{q=1,2}\mathbb{E}\left[\left(\int_0^t \left\|\int_{\mathbb{R}^d}\varphi(s,B_s,z)\left(\mathbb{P}_{X_s}-\hat{\mathbb{P}}_{Y_s}\right)(dz)\right\|^2 ds\right)^{qp}\right.\\
&\quad\left.\times\mathcal{E}\left(-\int_0^s \overline{b}(u,B_u,\mathbb{P}_{X_u})dB_u\right)\right]^{\frac{1}{2p}}\\
&\lesssim \max_{q=1,2}\mathbb{E}\left[\left(\int_0^t \left\|\int_{\mathbb{R}^d}\varphi(s,B_s,z)\left(\mathbb{P}_{X_s}-\hat{\mathbb{P}}_{Y_s}\right)(dz)\right\|^2 ds\right)^{qp^2}\right]^{\frac{1}{2p^2}}.
\end{aligned}
$$

Equivalent to the steps before we get using $\sup_{t\in[0,T]}\mathbb{E}\left[\|\overline{b}(t,B_t,\mu_t)\|^2\right]<\infty$ for all $\mu\in\mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}^d))$ that

$$
\begin{aligned}
&\left\|\int_{\mathbb{R}^d}\varphi(s,B_s,z)\left(\mathbb{P}_{X_s}-\hat{\mathbb{P}}_{Y_s}\right)(dz)\right\|^2\\
&\quad=\left\|\mathbb{E}\left[\varphi(s,y,X_s)\right]-\mathbb{E}_{\hat{\mathbb{P}}}\left[\varphi(s,y,Y_s)\right]\right\|^2_{y=B_s}\\
&\quad=\mathbb{E}\left[\|\varphi(s,y,B_s)\|\right.\\
&\qquad\left.\times\left|\mathcal{E}\left(-\int_0^s\overline{b}(u,B_u,\mathbb{P}_{X_u})dB_u\right)-\mathcal{E}\left(-\int_0^s\overline{b}(u,B_u,\hat{\mathbb{P}}_{Y_u})dB_u\right)\right|\right]^2_{y=B_s}\\
&\quad\lesssim(1+B_s)^2\,\mathbb{E}\left[\left|\int_0^s\left(\overline{b}(u,B_u,\mathbb{P}_{X_u})-\overline{b}(u,B_u,\hat{\mathbb{P}}_{Y_u})\right)dB_u\right.\right.\\
&\qquad\left.\left.+\frac{1}{2}\int_0^s\left(\left\|\overline{b}(u,B_u,\mathbb{P}_{X_u})\right\|^2-\left\|\overline{b}(u,B_u,\hat{\mathbb{P}}_{Y_u})\right\|^2\right)du\right|^{2p}\right]^{\frac{1}{p}}\\
&\quad\lesssim(1+B_s)^2\,\mathbb{E}\left[\left(\int_0^s\left\|\overline{b}(u,B_u,\mathbb{P}_{X_u})-\overline{b}(u,B_u,\hat{\mathbb{P}}_{Y_u})\right\|^2du\right)^{p}\right]^{\frac{1}{p}}
\end{aligned}
$$

$$+ (1 + B_s)^2 \, \mathbb{E}\left[\left(\int_0^s \left| \left\| \overline{b}(u, B_u, \mathbb{P}_{X_u}) \right\|^2 - \left\| \overline{b}(u, B_u, \hat{\mathbb{P}}_{Y_u}) \right\|^2 \right| du \right)^{2p} \right]^{\frac{1}{p}}$$

$$\lesssim (1 + B_s)^2 \, \mathbb{E}\left[\left(\int_0^s \left\| \overline{b}(u, B_u, \mathbb{P}_{X_u}) - \overline{b}(u, B_u, \hat{\mathbb{P}}_{Y_u}) \right\|^2 du \right)^{p} \right]^{\frac{1}{p}}$$

$$\lesssim (1 + B_s)^2 \, \mathbb{E}\left[\left(\int_0^s \left\| \int_{\mathbb{R}^d} \varphi(s, B_s, z) \left(\mathbb{P}_{X_s} - \hat{\mathbb{P}}_{Y_s}\right)(dz) \right\|^2 du \right)^{p} \right]^{\frac{1}{p}}.$$

Applying the $L^{p^2}(\Omega)$ norm on both sides yields

$$\mathbb{E}\left[\left\| \int_{\mathbb{R}} \varphi(s, B_s, z) \left(\mathbb{P}_{X_s} - \hat{\mathbb{P}}_{Y_s}\right)(dz) \right\|^{2p^2} \right]^{\frac{1}{p^2}}$$

$$\lesssim \mathbb{E}\left[(1 + B_s)^{2p^2}\right] \mathbb{E}\left[\left(\int_0^s \left\| \int_{\mathbb{R}^d} \varphi(s, B_s, z) \left(\mathbb{P}_{X_s} - \hat{\mathbb{P}}_{Y_s}\right)(dz) \right\|^2 du \right)^{p^2} \right]^{\frac{1}{p^2}}$$

$$\lesssim \int_0^s \mathbb{E}\left[\left\| \int_{\mathbb{R}^d} \varphi(s, B_s, z) \left(\mathbb{P}_{X_s} - \hat{\mathbb{P}}_{Y_s}\right)(dz) \right\|^{2p^2} \right]^{\frac{1}{p^2}} du$$

Using a Grönwall argument yields that

$$\mathbb{E}\left[\left\| \int_{\mathbb{R}^d} \varphi(s, B_s, z) \left(\mathbb{P}_{X_s} - \hat{\mathbb{P}}_{Y_s}\right)(dz) \right\|^{2p^2} \right]^{\frac{1}{p^2}} = 0.$$

In particular,

$$\left\| \int_{\mathbb{R}^d} \varphi(s, B_s, z) \left(\mathbb{P}_{X_s} - \hat{\mathbb{P}}_{Y_s}\right)(dz) \right\| = 0, \quad \mathbb{P}\text{-a.s.}$$

and consequently, $\mathcal{K}\left(\widetilde{\mathbb{Q}}_{X_t}, \mathbb{P}_{X_t}\right) = 0$. □

Due to [6, Theorem 4.1] we immediately get Malliavin differentiability of the strong solution of mean-field equation (4.2) for a more general class of functionals φ.

Theorem 3.2 *Let $b : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be bounded and continuous in the third variable* (4.5)*. Further, let $\varphi : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be of at most linear growth* (4.4)*. Then, the strong solution of mean-field SDE* (4.2) *is Malliavin differentiable.*

Remark 3.3. In the one-dimensional case, $d = 1$, the class of drift coefficients b and functionals φ can be further extended in order to obtain Malliavin differentiability of the strong solution. Consider the decomposition

$$(b \diamond \varphi)(t, y, \mu) := \hat{b}\left(t, y, \int_{\mathbb{R}} \hat{\varphi}(t, y, z)\mu(dz)\right) + \tilde{b}\left(t, y, \int_{\mathbb{R}} \tilde{\varphi}(t, y, z)\mu(dz)\right), \quad (4.20)$$

where the drift $\hat{b}$ is merely measurable and bounded and the functional $\hat{\varphi}$ is merely measurable and of linear growth whereas $\tilde{b}$ and $\tilde{\varphi}$ are of linear growth (4.4) and Lipschitz continuous in the second variable (4.7). If b is continuous in the third variable (4.5), the strong solution of mean-field SDE (4.2) is Malliavin differentiable due to [7, Theorem 2.12].

Example 3.4 Let $b : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a measurable and bounded function which is continuous in the third variable (4.5). Furthermore, define the functional $\varphi(t,y,z) = \mathbb{1}_{\{z \leq u\}}$, where $u \in \mathbb{R}$ is some parameter. Then, the mean-field stochastic differential equation

$$dX_t^x = b\left(t, X_t^x, F_{X_t^x}(u)\right) dt + dB_t,\ t \in [0,T],\ X_0^x = x \in \mathbb{R},$$

where $F_{X_t^x}$ denotes the cumulative distribution function of X_t^x, has a Malliavin differentiable strong solution due to Theorem 3.2. If b is Lipschitz continuous in the third variable (4.8), the solution is unique. Note that it is also possible to choose $u = t$ or $u = y$, where the later one yields the mean-field SDE

$$dX_t^x = b\left(t, X_t^x, F_{X_t^x}(X_t^x)\right) dt + dB_t,\ t \in [0,T],\ X_0^x = x \in \mathbb{R}.$$

Using Itô's formula we are able to extend our results on mean-field SDE (4.2) to more general diffusion coefficients. For notational simplicity we just consider the time-homogenous and one-dimensional case. However the time-inhomogeneous and multi-dimensional cases can be shown analogously.

Theorem 3.5 *Consider the time-homogeneous mean-field SDE*

$$dX_t^x = b\left(X_t^x, \int_{\mathbb{R}} \varphi(X_t^x, z)\mathbb{P}_{X_t^x}(dz)\right) dt + \sigma(X_t^x) dB_t,\ t \in [0,T],\ X_0^x = x \in \mathbb{R}, \tag{4.21}$$

with measurable drift $b : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, functional $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and volatility $\sigma : \mathbb{R} \to \mathbb{R}$. Moreover, let $\Lambda : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable bijection with derivatives Λ' and Λ'', such that for all $y \in \mathbb{R}$,

$$\Lambda'(y)\sigma(y) = 1,$$

as well as Λ^{-1} is Lipschitz continuous. Suppose that $(b^* \diamond \varphi^*) : \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$, defined by

$$(b^* \diamond \varphi^*)(y,\mu) :=$$
$$\Lambda'\left(\Lambda^{-1}(y)\right) b\left(\Lambda^{-1}(y), \int_{\mathbb{R}^d} \varphi(\Lambda^{-1}(y), \Lambda^{-1}(z))\mu(dz)\right) + \frac{1}{2}\Lambda''\left(\Lambda^{-1}(y)\right)\sigma\left(\Lambda^{-1}(y)\right)^2,$$

fulfills the assumptions of Theorem 3.1 and Theorem 3.2, respectively. Then, there exists a (Malliavin differentiable) strong solution $(X_t^x)_{t\in[0,T]}$ of (4.21). If moreover b^* is Lipschitz continuous in the second variable (4.8), the solution is unique.

Proof. Since $(b^* \diamond \varphi^*)$ satisfies the conditions of Theorem 3.1 and Theorem 3.2, respectively, mean-field SDE

$$dZ_t^x = b^*\left(Z_t^x, \int_{\mathbb{R}} \varphi^*\left(Z_t^x, z\right) \mathbb{P}_{Z_t^x}(dz)\right) dt + dB_t,\ t \in [0,T],\ Z_0^x = \Lambda(x),$$

has a (Malliavin differentiable) (unique) strong solution. Thus $X_t^x := \Lambda^{-1}(Z_t^x)$ is a (unique) strong solution of (4.21) by the application of Itô's formula, and since Λ^{-1} is Lipschitz continuous, X^x is Malliavin differentiable. □

We conclude this section by applying our existence result on solutions of mean-field SDEs to construct solutions of ODEs. More precisely, consider the mean-field SDE

$$dX_t^x = b(t, \mathbb{E}[X_t^x])dt + dB_t,\ t \in [0,T],\ X_0^x = x \in \mathbb{R}^d, \tag{4.22}$$

i.e. the drift coefficient only depends on the solution via the expectation $\mathbb{E}[X_t^x]$. By Theorem 3.1, mean-field SDE (4.22) has a strong solution if $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is of at most linear growth and continuous in the second variable. Now, by taking expectation on both sides, we loose the randomness and get that $u(t) := \mathbb{E}[X_t^x]$ solves the ODE

$$d\,u(t) = b(t, u(t))dt,\ t \in [0,T],\ u(0) = x \in \mathbb{R}^d. \tag{4.23}$$

We thus have developed a probabilistic approach to the following version of the theorem on existence of solutions of ODEs by Carathéodory, see e.g. [35, Theorem 1.1] or for a direct proof [36, Chapter II, Theorem 3.2]:

Theorem 3.6 *Let $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be of at most linear growth and continuous in the second variable, i.e. b fulfills the corresponding assumptions* (4.4) *and* (4.5). *Furthermore, let $(X_t^x)_{t\in[0,T]}$ be a strong solution of* (4.22). *Then $u(t) := \mathbb{E}[X_t^x]$ is a solution of ODE* (4.23).

4. Regularity in the initial value

The aim of this section is to study the regularity of a strong solution of mean-field SDE (4.2) as a function in its initial condition. More precisely, we investigate under which assumptions on b and φ the strong solution X_t^x of (4.2) is not just Sobolev differentiable but continuously differentiable as a function in x. These results will then be used to develop the Bismut-Elworthy-Li formula (4.15).

4.1. **Strong Differentiability.** First recall that due to Corollary 2.4 the unique strong solution X^x of mean-field SDE (4.2) is Sobolev differentiable under the assumption that b is measurable, bounded, and Lipschitz continuous in the third variable (4.8), and φ is measurable, of at most linear growth (4.4), and Lipschitz continuous in the third variable (4.8). Our aim is to find sufficient assumptions on b and φ such that the unique strong solution X^x of (4.2) is continuously differentiable in the initial condition.

Proposition 4.1 *Suppose $b, \varphi \in \mathfrak{C}([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$. Let $(X_t^x)_{t\in[0,T]}$ be the unique strong solution of mean-field SDE (4.2). Then for every compact subset $K \subset \mathbb{R}^d$ there exists some constant $C > 0$ such that for every $t \in [0,T]$ and $x, y \in K$*

$$\mathbb{E}[\|\partial_x X_t^x - \partial_y X_t^y\|] \leq C\|x-y\|.$$

In particular, the map $x \mapsto X_t^x$ is continuously differentiable for every $t \in [0,T]$ and for every $1 \leq p < \infty$

$$\sup_{t\in[0,T]} \sup_{x\in K} \mathbb{E}[\|\partial_x X_t^x\|^p]^{\frac{1}{p}} < \infty. \tag{4.24}$$

Proof. Since X^x is Sobolev differentiable by Corollary 2.4 and

$$\sup_{t\in[0,T]} \operatorname*{ess\,sup}_{x\in K} \mathbb{E}[\|\partial_x X_t^x\|^p]^{\frac{1}{p}} < \infty,$$

by [6, Lemma 4.13], it suffices to show that $\partial_x X^x$ is almost surely continuous in $x \in K$. Note that we can choose an element of the equivalence class of weak derivatives $\partial_x X^x$ such that (4.24) holds. For the remainder of this proof we just consider this particular element and denote it without loss of generality by $\partial_x X^x$. Let $x, y \in K$ and $t \in [0,T]$ be arbitrary. Note that the first variation process $\partial_x X^x$ has the representation

$$\partial_x X_t^x = 1 + \int_0^t \partial_2 b(s, X_s^x, \rho(X_s^x))\partial_x X_s^x + \partial_3 b(s, X_s^x, \rho(X_s^x))\partial_x \rho(X_s^x) ds,$$

where $\rho(X_t^x) := \int_{\mathbb{R}^d} \varphi(t, X_t^x, z)\mathbb{P}_{X_t^x}(dz)$. Thus, using Minkowski's and Hölder's inequalities we get

$$\begin{aligned}
&\mathbb{E}[\|\partial_x X_t^x - \partial_y X_t^y\|] \\
&\quad\leq \int_0^t \mathbb{E}[\|\partial_2 b(s, X_s^x, \rho(X_s^x))\partial_x X_s^x - \partial_2 b(s, X_s^y, \rho(X_s^y))\partial_y X_s^y\|] \\
&\qquad + \mathbb{E}[\|\partial_3 b(s, X_s^x, \rho(X_s^x))\partial_x \rho(X_s^x) - \partial_3 b(s, X_s^y, \rho(X_s^y))\partial_y \rho(X_s^y)\|] ds \\
&\quad\lesssim \int_0^t \mathbb{E}\Big[\|\partial_2 b(s, X_s^x, \rho(X_s^x)) - \partial_2 b(s, X_s^y, \rho(X_s^y))\|^2\Big]^{\frac{1}{2}} \mathbb{E}\Big[\|\partial_x X_s^x\|^2\Big]^{\frac{1}{2}} \\
&\qquad + \mathbb{E}\Big[\|\partial_3 b(s, X_s^x, \rho(X_s^x)) - \partial_3 b(s, X_s^y, \rho(X_s^y))\|^2\Big]^{\frac{1}{2}} \mathbb{E}\Big[\|\partial_x \rho(X_s^x)\|^2\Big]^{\frac{1}{2}} \\
&\qquad + \mathbb{E}[\|\partial_x X_s^x - \partial_y X_s^y\|] + \mathbb{E}[\|\partial_x \rho(X_s^x) - \partial_y \rho(X_s^y)\|] ds \\
&\quad\lesssim \int_0^t \left(\mathbb{E}\Big[\|X_s^x - X_s^y\|^2\Big]^{\frac{1}{2}} + \mathbb{E}\Big[\|\rho(X_s^x) - \rho(X_s^y)\|^2\Big]^{\frac{1}{2}}\right) \\
&\qquad \times \left(\mathbb{E}\Big[\|\partial_x X_s^x\|^2\Big]^{\frac{1}{2}} + \mathbb{E}\Big[\|\partial_x \rho(X_s^x)\|^2\Big]^{\frac{1}{2}}\right) \\
&\qquad + \mathbb{E}[\|\partial_x X_s^x - \partial_y X_s^y\|] + \mathbb{E}[\|\partial_x \rho(X_s^x) - \partial_y \rho(X_s^y)\|] ds.
\end{aligned}$$

Using the assumptions on φ we get

$$\begin{aligned}
\mathbb{E}\Big[\|\rho(X_t^x)-\rho(X_t^y)\|^2\Big]^{\frac{1}{2}} &= \mathbb{E}\Big[\Big\|\mathbb{E}[\varphi(t,z_1,X_t^x)-\varphi(t,z_2,X_t^y)]_{z_1=X_t^x;z_2=X_t^y}\Big\|^2\Big]^{\frac{1}{2}}\\
&\lesssim \mathbb{E}\Big[\Big\|\mathbb{E}[\|z_1-z_2\|+\|X_t^x-X_t^y\|]_{z_1=X_t^x;z_2=X_t^y}\Big\|^2\Big]^{\frac{1}{2}}\\
&\leq \mathbb{E}\Big[\|X_t^x-X_t^y\|^2\Big]^{\frac{1}{2}}. \qquad (4.25)
\end{aligned}$$

Furthermore, using the chain rule we have that

$$\begin{aligned}
&\mathbb{E}\Big[\|\partial_x\rho(X_t^x)\|^2\Big]^{\frac{1}{2}}\\
&\quad= \mathbb{E}\Big[\Big\|\mathbb{E}[\partial_2\varphi(t,z,X_t^x)]_{z=X_t^x}\partial_x X_t^x+\mathbb{E}[\partial_3\varphi(t,z,X_t^x)\partial_x X_t^x]_{z=X_t^x}\Big\|^2\Big]^{\frac{1}{2}}\\
&\quad\lesssim \mathbb{E}\Big[\|\partial_x X_t^x\|^2\Big]^{\frac{1}{2}}+\mathbb{E}[\|\partial_x X_t^x\|]\leq \mathbb{E}\Big[\|\partial_x X_t^x\|^2\Big]^{\frac{1}{2}}.
\end{aligned}$$

Equivalently we obtain that

$$\begin{aligned}
&\mathbb{E}[\|\partial_x\rho(X_t^x)-\partial_y\rho(X_t^y)\|]\\
&\quad\leq \mathbb{E}\Big[\Big\|\mathbb{E}[\partial_2\varphi(t,z,X_t^x)]_{z=X_t^x}\partial_x X_t^x-\mathbb{E}[\partial_2\varphi(t,z,X_t^y)]_{z=X_t^y}\partial_y X_t^y\Big\|\Big]\\
&\qquad+\mathbb{E}\Big[\Big\|\mathbb{E}[\partial_3\varphi(t,z,X_t^x)\partial_x X_t^x]_{z=X_t^x}-\mathbb{E}[\partial_3\varphi(t,z,X_t^y)\partial_y X_t^y]_{z=X_t^y}\Big\|\Big]\\
&\quad\leq \mathbb{E}\Big[\Big\|\mathbb{E}[\partial_2\varphi(t,z_1,X_t^x)-\partial_2\varphi(t,z_2,X_t^y)]_{z_1=X_t^x;z_2=X_t^y}\Big\|\,\|\partial_x X_t^x\|\Big]\\
&\qquad+\mathbb{E}\Big[\|\partial_x X_t^x-\partial_y X_t^y\|^p\,\Big\|\mathbb{E}[\partial_2\varphi(t,z,X_t^y)]_{z=X_t^y}\Big\|\Big]\\
&\qquad+\mathbb{E}\Big[\Big\|\mathbb{E}[\|\partial_3\varphi(t,z_1,X_t^x)-\partial_3\varphi(t,z_2,X_t^y)\|\,\|\partial_x X_t^x\|]_{z_1=X_t^x;z_2=X_t^y}\Big\|\Big]\\
&\qquad+\mathbb{E}\Big[\Big\|\mathbb{E}[\|\partial_x X_t^x-\partial_y X_t^y\|\,\|\partial_3\varphi(t,z,X_t^y)\|]_{z=X_t^y}\Big\|\Big]\\
&\lesssim \mathbb{E}\Big[\Big\|\mathbb{E}[\partial_2\varphi(t,z_1,X_t^x)-\partial_2\varphi(t,z_2,X_t^y)]_{z_1=X_t^x;z_2=X_t^y}\Big\|^2\Big]^{\frac{1}{2}}\mathbb{E}\Big[\|\partial_x X_t^x\|^2\Big]^{\frac{1}{2}}\\
&\qquad+\mathbb{E}[\|\partial_x X_t^x-\partial_y X_t^y\|]\\
&\qquad+\mathbb{E}\Big[\Big\|\mathbb{E}\left[\|\partial_3\varphi(t,z_1,X_t^x)-\partial_3\varphi(t,z_2,X_t^y)\|^2\right]^{\frac{1}{2}}_{z_1=X_t^x;z_2=X_t^y}\Big\|\Big]\mathbb{E}\Big[\|\partial_x X_t^x\|^2\Big]^{\frac{1}{2}}\\
&\qquad+\mathbb{E}[\|\partial_x X_t^x-\partial_y X_t^y\|]\\
&\quad\lesssim \mathbb{E}\Big[\|\partial_x X_t^x\|^2\Big]^{\frac{1}{2}}\mathbb{E}\Big[\|X_t^x-X_t^y\|^2\Big]^{\frac{1}{2}}+\mathbb{E}[\|\partial_x X_t^x-\partial_y X_t^y\|].
\end{aligned}$$

Thus, in combination with (4.24) we get

$$\mathbb{E}[\|\partial_x X_t^x-\partial_y X_t^y\|]\lesssim \int_0^t \mathbb{E}\Big[\|X_s^x-X_s^y\|^2\Big]^{\frac{1}{2}}+\mathbb{E}[\|\partial_x X_s^x-\partial_y X_s^y\|]ds.$$

Using equation (4.14), we get

$$\mathbb{E}[\|\partial_x X_t^x - \partial_y X_t^y\|] \lesssim |x-y| + \int_0^t \mathbb{E}[\|\partial_x X_s^x - \partial_y X_s^y\|] ds.$$

Finally, since $\mathbb{E}[\|\partial_x X_s^x - \partial_y X_s^y\|]$ is integrable over $[0,T]$ and Borel measurable, we can apply Jones' generalization of Grönwall's inequality [27, Lemma 5] to get

$$\mathbb{E}[\|\partial_x X_t^x - \partial_y X_t^y\|] \lesssim |x-y|.$$

Thus, $\partial_x X^x$ has an almost surely continuous version in $x \in K$ by Kolmogorov's continuity theorem and consequently X^x is continuously differentiable for every $t \in [0,T]$. □

4.2. **Bismut-Elworthy-Li formula.** In this subsection we turn our attention to the Bismut-Elworthy-Li formula (4.15). With the help of the approximating sequence defined in (4.28) we show in the one-dimensional case, i.e. $d=1$, that $\partial_x \mathbb{E}[\Phi(X_T^x)]$ exists in the strong sense for functionals Φ merely satisfying some integrability condition, i.e. we show that $\mathbb{E}[\Phi(X_T^x)]$ is continuously differentiable.

Lemma 4.2 *Consider $d=1$. Let $(b \diamond \varphi)$ admit a decomposition* (4.20) *and let $b, \varphi \in \mathcal{L}([0,T]\times\mathbb{R}\times\mathbb{R})$. Further, let $(X_t^x)_{t\in[0,T]}$ be the unique strong solution of mean-field SDE* (4.2) *and $\Phi \in \mathcal{C}_b^{1,1}(\mathbb{R})$. Then $\mathbb{E}[\Phi(X_t^x)] \in \mathcal{C}^1(\mathbb{R})$ and*

$$\partial_x \mathbb{E}\left[\Phi(X_t^x)\right] = \mathbb{E}\left[\Phi'(X_t^x)\partial_x X_t^x\right], \tag{4.26}$$

where Φ' denotes the first derivative of Φ and $\partial_x X_t^x$ is the first variation process of X_t^x as given in (4.12).

In order to proof Lemma 4.2, we need to define a sequence of mean-field equations similar to [7] whose unique strong solutions approximate the unique strong solution of (4.2), where $(b \diamond \varphi)$ fulfills the assumptions of Lemma 4.2. More precisely, by standard approximation arguments there exist sequences

$$b_n := \tilde{b}_n + \hat{b}_n, \quad \text{and} \quad \varphi_n := \tilde{\varphi}_n + \hat{\varphi}_n, \quad n \geq 1, \tag{4.27}$$

where $b_n, \varphi_n \in \mathcal{C}_0^\infty([0,T]\times\mathbb{R}\times\mathbb{R})$ with

$$\sup_{n\geq 1} \left(\|\tilde{b}_n\|_\infty + \|\tilde{\varphi}_n\|_\infty\right) \leq C < \infty$$

and

$$\sup_{n\geq 1} \left(|\hat{b}_n(t,y,z)| + |\hat{\varphi}_n(t,y,z)|\right) \leq C(1+|y|+|z|)$$

for every $t \in [0,T]$ and $y,z \in \mathbb{R}$, such that $b_n \to b$ and $\varphi_n \to \varphi$ in a.e. $(t,y,z) \in [0,T]\times\mathbb{R}\times\mathbb{R}$ with respect to the Lebesgue measure, respectively. The original drift coefficients b and φ are denoted by b_0 and φ_0, respectively. Furthermore, we can assume that there exists a constant $C>0$ independent of $n \in \mathbb{N}$ such that

$$b_n, \varphi_n \in \mathcal{L}([0,T]\times\mathbb{R}\times\mathbb{R}),$$

and that $\hat{b}_n$ and $\hat{\varphi}_n$ are Lipschitz continuous in the second variable (4.7) for all $n \geq 0$. Under these conditions the corresponding mean-field SDEs, defined by

$$\begin{aligned} dX_t^{n,x} &= b_n\left(t, X_t^{n,x}, \int_{\mathbb{R}} \varphi_n(t, X_t^{n,x}, z) \mathbb{P}_{X_t^{n,x}}(dz)\right) dt + dB_t, \ t \in [0,T], \\ X_0^{n,x} &= x \in \mathbb{R}, \end{aligned} \tag{4.28}$$

have unique strong solutions which are Malliavin differentiable by Theorem 3.2. Likewise the strong solutions $\{X^{n,x}\}_{n\geq 0}$ are continuously differentiable with respect to the initial condition by Proposition 4.1. Due to Corollary A.1 we have that $(X_t^{n,x})_{t\in[0,T]}$ converges to $(X_t^x)_{t\in[0,T]}$ in $L^2(\Omega)$ as $n \to \infty$ and similar to [7, Lemma 3.10] one can show for any compact subset $K \subset \mathbb{R}$ and $p \geq 1$ that

$$\sup_{n\geq 0} \sup_{t\in[0,T]} \sup_{x\in K} \mathbb{E}[\|\partial_x X_t^{n,x}\|^p]^{\frac{1}{p}} < \infty. \tag{4.29}$$

Proof of Lemma 4.2. Note first that $\mathbb{E}\left[\Phi(X_t^x)\right]$ is weakly differentiable by Corollary 2.7 and equation (4.26) holds by [7, Lemma 4.1]. Hence it suffices to show that $\partial_x \mathbb{E}[\Phi(X_t^x)]$ is continuous. In order to prove this we show that

$$\begin{aligned} \mathbb{E}[\Phi(X_t^{n,x})] &\xrightarrow[n\to\infty]{} \mathbb{E}[\Phi(X_t^x)] \quad \forall x \in \mathbb{R}, \text{ and} \\ \mathbb{E}\left[\Phi'(X_t^{n,x})\partial_x X_t^{n,x}\right] &\xrightarrow[n\to\infty]{} \mathbb{E}\left[\Phi'(X_t^x)\partial_x X_t^x\right] \quad \text{uniformly for } x \in K, \end{aligned}$$

where $\{(X_t^{n,x})_{t\in[0,T]}\}_{n\geq 1}$ is the approximating sequence defined in (4.28) and $K \subset \mathbb{R}$ is a compact subset. Note that

$$\partial_x \mathbb{E}[\Phi(X_t^{n,x})] = \mathbb{E}\left[\Phi'(X_t^{n,x})\partial_x X_t^{n,x}\right]$$

is continuous in x due to Proposition 4.1. The first convergence follows directly by Remark A.5. For the uniform convergence let $K \subset \mathbb{R}$ be a compact set and define for $n \geq 0$

$$\begin{aligned} D_n(s,t,x) &:= \exp\left\{-\int_s^t \int_{\mathbb{R}} b_n(u, y, \varrho_u^{n,x}(y)) L^{B^x}(du, dy)\right\}, \text{ and} \\ E_n(x) &:= \mathcal{E}\left(\int_0^T b_n(s, B_s^x, \varrho_s^{n,x}(B_s^x)) dB_s\right), \end{aligned}$$

where $\varrho_u^{n,x}(y) := \int_{\mathbb{R}} \varphi(u, y, z) \mathbb{P}_{X_u^{n,x}}(dz)$. In a first approximation we get using $\|\Phi'\|_\infty < \infty$ and representation (4.13) that

$$\begin{aligned} &|\mathbb{E}\left[\Phi'(X_t^{n,x})\partial_x X_t^{n,x} - \Phi'(X_t^x)\partial_x X_t^x\right]| \\ &\quad \lesssim \mathbb{E}\Bigg[\Bigg|E_n(x)\left(D_n(0,t,x) + \int_0^t D_n(s,t,x)\partial_x b_n(s,y,\varrho_s^{n,x}(y))|_{y=B_s^x} ds\right) \\ &\qquad - E_0(x)\left(D_0(0,t,x) + \int_0^t D_0(s,t,x)\partial_x b_n(s,y,\varrho_s^{n,x}(y))|_{y=B_s^x} ds\right)\Bigg|\Bigg] \\ &\quad =: A_n(t,x). \end{aligned}$$

Equivalently, we get using $\|\partial_3\varphi\|_\infty < \infty$ that for every $t \in [0,T]$ and $y \in \mathbb{R}$

$$|\partial_x \varrho_t^{n,x}(y) - \partial_x \varrho_t^x(y)| = |\mathbb{E}\left[\partial_3\varphi(t,y,X_t^{n,x})\partial_x X_t^{n,x} - \partial_3\varphi(t,y,X_t^x)\partial_x X_t^x\right]| \lesssim A_n(t,x).$$

Note furthermore that by (4.29) we have for every $y \in \mathbb{R}$ that

$$\begin{aligned}|\partial_x b_n(s,y,\varrho_s^{n,x}(y))| &= |\partial_3 b_n(s,y,\varrho_s^{n,x}(y))\partial_x \varrho_s^{n,x}(y)| \lesssim |\mathbb{E}\left[\partial_3\varphi(t,y,X_t^{n,x})\partial_x X_t^{n,x}\right]| \\ &\leq \mathbb{E}[|\partial_x X_t^{n,x}|] < \infty, \end{aligned} \tag{4.30}$$

and for every $p \geq 1$

$$\begin{aligned}&\mathbb{E}\Big[\Big|\partial_x b_n(t,y,\varrho_t^{n,x}(y))|_{y=B_t^x} - \partial_x b(t,y,\varrho_t^x(y))|_{y=B_t^x}\Big|^p\Big]^{\frac{1}{p}} \\ &\lesssim \mathbb{E}[|\partial_3 b_n(t,B_t^x,\varrho_t^{n,x}(B_t^x)) - \partial_3 b(t,B_t^x,\varrho_t^x(B_t^x))|^p]^{\frac{1}{p}} \\ &\quad + \mathbb{E}[|\partial_x \varrho_t^{n,x}(B_t^x) - \partial_x \varrho_t^x(B_t^x)|^p]^{\frac{1}{p}} \\ &\lesssim \mathbb{E}[|\partial_3 b_n(t,B_t^x,\varrho_t^{n,x}(B_t^x)) - \partial_3 b(t,B_t^x,\varrho_t^x(B_t^x))|^p]^{\frac{1}{p}} + A_n(t,x). \end{aligned} \tag{4.31}$$

Using Hölder's inequality, (4.30), Lemma A.2, and Corollary A.3 we can decompose $A_n(t,x)$ into

$$\begin{aligned}&A_n(t,x) \\ &\lesssim \mathbb{E}\left[|E_n(x) - E_0(x)|\left|D_n(0,t,x) + \int_0^t D_n(s,t,x)\partial_x b_n(s,y,\varrho_s^{n,x}(y))|_{y=B_s^x} ds\right|\right] \\ &\quad + \mathbb{E}\left[E_0(x)\left|D_n(0,t,x) - D_0(0,t,x)\right|\right] \\ &\quad + \mathbb{E}\Big[E_0(x)\Big|\int_0^t D_n(s,t,x)\partial_x b_n(s,y,\varrho_s^{n,x}(y))|_{y=B_s^x} \\ &\qquad - D_0(s,t,x)\partial_x b(s,y,\varrho_s^x(y))|_{y=B_s^x} ds\Big|\Big] \\ &\lesssim \mathbb{E}[|E_n(x) - E_0(x)|^q]^{\frac{1}{q}} + \mathbb{E}[|D_n(0,t,x) - D_0(0,t,x)|^p]^{\frac{1}{p}} \\ &\quad + \mathbb{E}\left[\left|\int_0^t D_n(s,t,x)\partial_x b_n(s,y,\varrho_s^{n,x}(y))|_{y=B_s^x} - D_0(s,t,x)\partial_x b(s,y,\varrho_s^x(y))|_{y=B_s^x} ds\right|^p\right]^{\frac{1}{p}} \\ &=: F_n(x) + G_n(0,t,x) + H_n(t,x), \end{aligned}$$

where $q := \frac{2(1+\varepsilon)}{2+\varepsilon}$ and $p := \frac{1+\varepsilon}{\varepsilon}$. Furthermore, we can bound $H_n(t,x)$ due to Corollary A.3, (4.30), and (4.31) by

$$\begin{aligned}H_n(t,x) &\leq \int_0^t \mathbb{E}[|D_n(s,t,x) - D_0(s,t,x)|^p\,|\partial_x b_n(s,y,\varrho_s^{n,x}(y))|_{y=B_s^x}|^p]^{\frac{1}{p}} ds \\ &\quad + \int_0^t \mathbb{E}\big[|\partial_x b_n(s,y,\varrho_s^{n,x}(y))|_{y=B_s^x} - \partial_x b(s,y,\varrho_s^x(y))|_{y=B_s^x}|^p\,|D_0(s,t,x)|^p\big]^{\frac{1}{p}} ds \\ &\lesssim \int_0^t G_n(s,t,x)ds + \int_0^t \mathbb{E}\Big[|\partial_3 b_n(s,B_s^x,\varrho_s^{n,x}(B_s^x)) - \partial_3 b(s,B_s^x,\varrho_s^x(B_s^x))|^{2p}\Big]^{\frac{1}{2p}} ds \\ &\quad + \int_0^t A_n(s,x)ds \end{aligned}$$

$$=: \int_0^t G_n(s,t,x)ds + \int_0^t K_n(s,x)ds + \int_0^t A_n(s,x)ds,$$

and thus

$$A_n(t,x) \leq C\left(F_n(x) + \sup_{s\in[0,t]} G_n(s,t,x) + \sup_{s\in[0,T]} K_n(s,x)\right) + C\int_0^t A_n(s,x)ds,$$

for some constant $C > 0$ independent of $t \in [0,T]$, $n \geq 0$, and $x \in K$. Consequently we get by Grönwall's inequality

$$A_n(t,x) \lesssim F_n(x) + G_n(0,t,x) + \sup_{s\in[0,T]} K_n(s,x) + \int_0^t\int_0^t G_n(s,r,x)dsdr.$$

F_n converges to 0 uniformly in $x \in K$ by Corollary A.4. Furthermore, we have that $G_n(s,t,x)$ is integrable over t and s by Corollary A.3 and converges to 0 uniformly in $x \in K$ by Corollary A.6. Finally, we get due to $b \in \mathcal{L}([0,T]\times\mathbb{R}\times\mathbb{R})$ that

$$\begin{aligned} K_n(s,x) &\leq \int_0^t \mathbb{E}\Big[\left|\partial_3 b_n(s,B_s^x,\varrho_s^{n,x}(B_s^x)) - \partial_3 b_n(s,B_s^x,\varrho_s^x(B_s^x)\right|^{2p}\Big]^{\frac{1}{2p}} ds \\ &\quad + \int_0^t \mathbb{E}\Big[\left|\partial_3 b_n(s,B_s^x,\varrho_s^x(B_s^x)) - \partial_3 b(s,B_s^x,\varrho_s^x(B_s^x))\right|^{2p}\Big]^{\frac{1}{2p}} ds \\ &\lesssim \int_0^t \left|\varrho_s^{n,x}(B_s^x) - \varrho_s^x(B_s^x)\right| ds \\ &\quad + \int_0^t \mathbb{E}\Big[\left|\partial_3 b_n(s,B_s^x,\varrho_s^x(B_s^x)) - \partial_3 b(s,B_s^x,\varrho_s^x(B_s^x))\right|^{2p}\Big]^{\frac{1}{2p}} ds \end{aligned}$$

Note first that due to Remark A.5 we have that $|\varrho_s^{n,x}(B_s^x) - \varrho_s^x(B_s^x)|$ converges uniformly in $s \in [0,T]$ and $x \in K$ to 0 as n goes to infinity. Moreover,

$$\begin{aligned} &\mathbb{E}\Big[\left|\partial_3 b_n(s,B_s^x,\varrho_s^x(B_s^x)) - \partial_3 b(s,B_s^x,\varrho_s^x(B_s^x))\right|^{2p}\Big]^{\frac{1}{2p}} \\ &= \left(\int_{\mathbb{R}} \left|\partial_3 b_n\left(t,y,\varrho_s^x(y)\right) - \partial_3 b\left(t,y,\varrho_s^x(y)\right)\right|^{2p} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} dy\right)^{\frac{2}{2p}} \\ &\leq e^{\frac{x^2}{2pt}}\left(\int_{\mathbb{R}} \left|\partial_3 b_n\left(t,y,\varrho_s^x(y)\right) - \partial_3 b\left(t,y,\varrho_s^x(y)\right)\right|^{2p} \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{4t}} dy\right)^{\frac{2}{2p}}, \end{aligned}$$

where we have used $e^{-\frac{(y-x)^2}{2t}} = e^{-\frac{y^2}{4t}} e^{-\frac{(y-2x)^2}{4t}} e^{\frac{x^2}{2t}} \leq e^{-\frac{y^2}{4t}} e^{\frac{x^2}{2t}}$. Furthermore, equivalent to (4.25) we can find a constant $C > 0$ by Corollary 2.6 such that for all $t \in [0,T]$ and $x,y \in K$

$$|\varrho_s^x(z) - \varrho_s^y(z)| \leq C|x-y|.$$

Consequently the function $x \mapsto \varrho_s^x(y)$ is continuous uniformly in $t \in [0,T]$. Thus $\Lambda^K := \{\varrho_s^x(y) : x \in K\} \subset \mathbb{R}$ is compact as an image of a compact set under a continuous function. Therefore due to the definition of the approximating sequence

$$\sup_{x\in K} |\partial_3 b_n(s,y,\varrho_s^x(y)) - \partial_3 b(s,y,\varrho_s^x(y))| = \sup_{z\in\Lambda^K} |\partial_3 b_n(s,y,z) - \partial_3 b(s,y,z)| \xrightarrow[n\to\infty]{} 0,$$

and hence $K_n(s,x)$ converges to 0 uniformly in $s \in [0,T]$ and $x \in K$. $\square$

We define the weight function $\omega_T : \mathbb{R} \to \mathbb{R}$ by

$$\omega_T(y) := \exp\left\{-\frac{|y|^2}{4T}\right\}, \quad y \in \mathbb{R}. \tag{4.32}$$

Theorem 4.3 *Consider $d=1$. Let $(b \diamond \varphi)$ admit a decomposition (4.20) and let $b, \varphi \in \mathcal{L}([0,T] \times \mathbb{R} \times \mathbb{R})$. Further, let $(X_t^x)_{t\in[0,T]}$ be the unique strong solution of mean-field SDE (4.2) and $\Phi \in L^{2p}(\mathbb{R};\omega_T)$, where $p := \frac{1+\varepsilon}{\varepsilon}$, $\varepsilon > 0$ sufficiently small with respect to Lemma A.2 and $\omega_T : \mathbb{R} \to \mathbb{R}$ as defined in (4.32). Then*

$$u(x) := \mathbb{E}\left[\Phi(X_T^x)\right]$$

is continuously differentiable in $x \in \mathbb{R}$ and the derivative takes the form

$$u'(x) = \mathbb{E}\left[\Phi(X_T^x)\left(\int_0^T a(s)\partial_x X_s^x + \partial_x b(t,y,\varrho_t^x(y))|_{y=B_t^x}\int_0^s a(u)dudB_s\right)\right], \tag{4.33}$$

where $a : \mathbb{R} \to \mathbb{R}$ is any bounded, measurable function such that

$$\int_0^T a(s)ds = 1.$$

Proof. Due to Lemma 4.2 we already know that in the case $\Phi \in \mathcal{C}_b^{1,1}(\mathbb{R})$ the functional $\mathbb{E}[\Phi(X_T^x)]$ is continuously differentiable and analogously to [7, Theorem 4.2] it can be shown that representation (4.33) holds. Now, using mollification we can approximate $\Phi \in L^{2p}(\mathbb{R};\omega_T)$ by a sequence of smooth functionals $\{\Phi_n\}_{n\geq 1} \subset C_0^\infty(\mathbb{R})$ such that $\Phi_n \to \Phi$ in $L^{2p}(\mathbb{R};\omega_T)$ as $n \to \infty$. We define

$$u_n(x) := \mathbb{E}\left[\Phi_n(X_T^x)\right] \quad \text{and}$$
$$\overline{u}(x) := \mathbb{E}\left[\Phi(X_T^x)\left(\int_0^T a(s)\partial_x X_s^x + \partial_x b(t,y,\varrho_t^x(y))|_{y=B_t^x}\int_0^s a(u)dudB_s\right)\right].$$

Note first that $\overline{u}$ is well-defined. Indeed, due to (4.24), Lemma A.2, and (4.30) we get

$$\begin{aligned}
|\overline{u}(x)| &\leq \mathbb{E}\left[\Phi(X_T^x)^2\right]^{\frac{1}{2}} \mathbb{E}\left[\left(\int_0^T a(s)\partial_x X_s^x + \partial_x b(t,y,\varrho_t^x(y))|_{y=B_t^x}\int_0^s a(u)dudB_s\right)^2\right]^{\frac{1}{2}} \\
&\lesssim \mathbb{E}\left[\Phi(B_T^x)^2 \mathcal{E}\left(\int_0^T b(u,B_u^x,\rho_u^x)dB_u\right)\right]^{\frac{1}{2}} \sup_{s\in[0,T]} \mathbb{E}\left[(\partial_x X_s^x)^2\right]^{\frac{1}{2}} \\
&\lesssim \mathbb{E}\left[|\Phi(B_T^x)|^{2p}\right]^{\frac{1}{2p}} < \infty.
\end{aligned} \tag{4.34}$$

Due to Lemma 4.2, u_n is continuously differentiable for all $n \geq 1$. Thus it remains to show that $u_n'(x)$ converges to $\overline{u}(x)$ compactly in x as $n \to \infty$, where denotes u_n' the first derivative of u_n with respect to x. Exactly in the same way as in equation

(4.34) we can find for any compact subset $K \subset \mathbb{R}$ a constant C such that for every $x \in K$

$$\begin{aligned}
|u'(x) - \overline{u}(x)| &\leq C\mathbb{E}\left[|\Phi_n(B_T^x) - \Phi(B_T^x)|^{2p}\right]^{\frac{1}{2p}} \\
&= C\left(\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi T}} |\Phi_n(y) - \Phi(y)|^{2p} e^{-\frac{(y-x)^2}{2T}} dy\right)^{\frac{1}{2p}} \\
&\leq C\left(\frac{e^{\frac{x^2}{2T}}}{\sqrt{2\pi T}} \int_{\mathbb{R}} |\Phi_n(y) - \Phi(y)|^{2p} e^{-\frac{y^2}{4T}} dy\right)^{\frac{1}{2p}} \\
&= C\left(\frac{e^{\frac{x^2}{2T}}}{\sqrt{2\pi T}}\right)^{\frac{1}{2p}} \|\Phi_n - \Phi\|_{L^{2p}(\mathbb{R};\omega_T)},
\end{aligned}$$

where we have used $e^{-\frac{(y-x)^2}{2t}} = e^{-\frac{y^2}{4t}} e^{-\frac{(y-2x)^2}{4t}} e^{\frac{x^2}{2t}} \leq e^{-\frac{y^2}{4t}} e^{\frac{x^2}{2t}}$. Consequently

$$\lim_{n\to\infty} \sup_{x\in K} |u_n'(x) - \overline{u}(x)| = 0.$$

Thus $u' = \overline{u}$ and u is continuously differentiable. □

Appendix A. Technical Results

The first corollary is due to [7, Proposition 3.8].

Corollary A.1 *Consider $d = 1$. Let $(b \diamond \varphi)$ admit a decomposition* (4.20) *and b be Lipschitz continuous in the third variable* (4.8). *Further let $(X_t^x)_{t\in[0,T]}$ be the unique strong solution of mean-field SDE* (4.2) *and $\{(X_t^{n,x})_{t\in[0,T]}\}_{n\geq 1}$ be the unique strong solutions of* (4.28). *Then, for all $t \in [0,T]$ and every $x \in \mathbb{R}$*

$$\mathbb{E}\Big[|X_t^{n,x} - X_t^x|^2\Big]^{\frac{1}{2}} \xrightarrow[n\to\infty]{} 0.$$

The upcoming lemma is an extension of [7, Lemma A.4] to multi dimensions.

Lemma A.2 *Let $b, \varphi : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be two measurable functions satisfying the linear growth condition* (4.4). *Furthermore, let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, B, X^x)$ be a weak solution of mean-field SDE* (4.2). *Then,*

$$\left\|b\left(t, X_t^x, \int_{\mathbb{R}^d} \varphi(t, X_t^x, z)\mathbb{P}_{X_t^x}(dz)\right)\right\| \leq C\left(1 + \|x\| + \sup_{s\in[0,T]} \|B_s\|\right)$$

for some constant $C > 0$. Consequently, for any compact set $K \subset \mathbb{R}^d$, and $1 \leq p < \infty$, there exist $\varepsilon > 0$ and a constant $C > 0$ such that the following boundaries hold:

$$\sup_{x\in K} \mathbb{E}\left[\sup_{t\in[0,T]} \left\|b\left(t, X_t^x, \int_{\mathbb{R}^d} \varphi(t, X_t^x, z)\mathbb{P}_{X_t^x}(dz)\right)\right\|^p\right] < \infty$$

$$\sup_{t\in[0,T]}\sup_{x\in K}\mathbb{E}\left[\|X_t^x\|^p\right]\leq C(1+\|x\|^p)<\infty$$

$$\sup_{x\in K}\mathbb{E}\left[\mathcal{E}\left(\int_0^T b\left(u,B_u^x,\int_{\mathbb{R}^d}\varphi(u,B_u^x,z)\mathbb{P}_{X_u^x}(dz)\right)dB_u\right)^{1+\varepsilon}\right]<\infty$$

In the following results which are due to [7, Lemma A.5, Lemma A.6 & Lemma A.7] we use the notation $\mathfrak{b}_n(t,y)=b_n\left(t,y,\int_{\mathbb{R}}\varphi(t,y,z)\mathbb{P}_{X_t^{n,x}}(dz)\right)$.

Corollary A.3 *Consider $d=1$. Suppose $(b\diamond\varphi)$ admits a decomposition* (4.20) *and that b and φ are Lipschitz continuous in the third variable* (4.8)*. Let $(X_t^x)_{t\in[0,T]}$ be the unique strong solution of mean-field SDE* (4.2)*. Moreover, $\{b_n\}_{n\geq1}$ is the approximating sequence of b as defined in* (4.27) *and $(X_t^{n,x})_{t\in[0,T]}$, $n\geq1$, the corresponding unique strong solutions of* (4.28)*. Then, for all $\lambda\in\mathbb{R}$ and any compact subset $K\subset\mathbb{R}$,*

$$\sup_{n\geq0}\sup_{s,t\in[0,T]}\sup_{x\in K}\mathbb{E}\left[\exp\left\{-\lambda\int_s^t\int_{\mathbb{R}}\mathfrak{b}_n(s,y)\,L^{B^x}(ds,dy)\right\}\right]<\infty.$$

Corollary A.4 *Consider $d=1$. Suppose $(b\diamond\varphi)$ admits a decomposition* (4.20) *and that b and φ are Lipschitz continuous in the third variable* (4.8)*. Let $(X_t^x)_{t\in[0,T]}$ be the unique strong solution of mean-field SDE* (4.2)*. Furthermore, $\{b_n\}_{n\geq1}$ is the approximating sequence of b as defined in* (4.27) *and $(X_t^{n,x})_{t\in[0,T]}$, $n\geq1$, the corresponding unique strong solutions of* (4.28)*. Then for any compact subset $K\subset\mathbb{R}$ and $q:=\frac{2(1+\varepsilon)}{2+\varepsilon}$, $\varepsilon>0$ sufficiently small with regard to Lemma A.2,*

$$\sup_{x\in K}\mathbb{E}\left[\left|\mathcal{E}\left(\int_0^T\mathfrak{b}_n(t,B_t^x)dB_t\right)-\mathcal{E}\left(\int_0^T\mathfrak{b}(t,B_t^x)dB_t\right)\right|^q\right]^{\frac{1}{q}}\xrightarrow[n\to\infty]{}0.$$

Remark A.5. Note that due to Corollary A.4 it is readily seen that for any $\psi\in\mathrm{Lip}(\mathbb{R})$

$$\mathbb{E}[\psi(X_t^{n,x})]\xrightarrow[n\to\infty]{}\mathbb{E}[\psi(X_t^x)]$$

uniformly in $t\in[0,T]$ and $x\in K$, where $K\subset\mathbb{R}$ is a compact subset.

Corollary A.6 *Consider $d=1$. Suppose $(b\diamond\varphi)$ admits a decomposition* (4.20) *and that b and φ are Lipschitz continuous in the third variable* (4.8)*. Let $(X_t^x)_{t\in[0,T]}$ be the unique strong solution of mean-field SDE* (4.2)*. Furthermore, $\{b_n\}_{n\geq1}$ is the approximating sequence of b as defined in* (4.27) *and $(X_t^{n,x})_{t\in[0,T]}$, $n\geq1$, the corresponding unique strong solutions of* (4.28)*. Then for any compact subset $K\subset\mathbb{R}$, $0\leq s\leq t\leq T$ and $p\geq1$,*

$$\sup_{x\in K}\mathbb{E}\left[\left|\exp\left\{-\int_s^t\int_{\mathbb{R}}\mathfrak{b}_n(u,y)L^{B^x}(du,dy)\right\}-\exp\left\{-\int_s^t\int_{\mathbb{R}}\mathfrak{b}(u,y)L^{B^x}(du,dy)\right\}\right|^p\right]^{\frac{1}{p}}\xrightarrow[n\to\infty]{}0.$$

Appendix B. Skorokhod's representation theorem

The following result is a version of Skorokhod's representation theorem and is due to [38, Ch. 1 Sec.6].

Theorem B.1 *Let $\{(\xi_t^n)_{t\in[0,T]}\}_{n\geq 1}$ be a sequence of $\mathbb{R}^d$-valued stochastic processes defined on probability spaces $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$, respectively, which are stochastically continuous from the right and fulfill for every $\varepsilon > 0$*

$$\lim_{C\to\infty} \lim_{n\to\infty} \sup_{t\in[0,T]} \mathbb{P}^n(\|\xi_t^n\| > C) = 0, \text{ and}$$
$$\lim_{h\to 0} \lim_{n\to\infty} \sup_{|t-s|\leq h} \mathbb{P}^n(\|\xi_t^n - \xi_s^n\| > \varepsilon) = 0.$$

Then, there exists a subsequence $\{n_k\}_{k\geq 1} \subset \mathbb{N}$ and a sequence of $\mathbb{R}^d$-valued stochastic processes $\{(X_t^k)_{t\in[0,T]}\}_{k\geq 0}$ on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

(i) for all $k \geq 1$, finite dimensional distributions of the processes X^k and ξ^{n_k} coincide, and

(ii) X_t^k converges in probability to X_t^0 for every $t \in [0, T]$.

Remark B.2. Note that we say that finite dimensional distributions of two processes X and ξ, defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, respectively, coincide, if for every finite sequence of time points $\{t_k\}_{1\leq k\leq N} \subset [0, T]$, $1 \leq N < \infty$, we have that

$$\mathbb{P}_{(X_{t_1},\dots X_{t_N})} = \widetilde{\mathbb{P}}_{(\xi_{t_1},\dots \xi_{t_N})}.$$

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Chapter 5
Restoration of Well-Posedness of Infinite-dimensional Singular ODE's via Noise

Contribution of the thesis' author

The paper *Restoration of Well-Posedness of Infinite-dimensional Singular ODE's via Noise* is a joint work with Prof. Dr. David Baños, Prof. Dr. Thilo Meyer-Brandis, and Prof. Dr. Frank Proske.

M. Bauer was significantly involved in the development of all parts of the paper. In particular, M. Bauer made major contributions to the editorial work and the proofs of Proposition 4.4, Proposition 4.7, and Lemma 4.9 as well as Theorem 4.10, Theorem 4.11, and Appendix B. Moreover, the example in section 5 is due to M. Bauer.

RESTORATION OF WELL-POSEDNESS OF INFINITE-DIMENSIONAL SINGULAR ODE'S VIA NOISE

DAVID BAÑOS, MARTIN BAUER, THILO MEYER-BRANDIS, AND FRANK PROSKE

Abstract. In this paper we aim at generalizing the results of A. K. Zvonkin [41] and A. Y. Veretennikov [39] on the construction of unique strong solutions of stochastic differential equations with singular drift vector field and additive noise in the Euclidean space to the case of infinite-dimensional state spaces. The regularizing driving noise in our equation is chosen to be a locally non-Hölder continuous Hilbert space valued process of fractal nature, which does not allow for the use of classical construction techniques for strong solutions from PDE or semimartingale theory. Our approach, which does not resort to the Yamada-Watanabe principle for the verification of pathwise uniqueness of solutions, is based on Malliavin calculus.

Keywords. Malliavin calculus · fractional Brownian motion · L^2-compactness criterion · strong solutions of SDEs · irregular drift coefficient.

1. Introduction

The main objective of this paper is the construction of (unique) strong solutions of infinite-dimensional stochastic differential equations (SDEs) with a singular drift and additive noise. In fact, we want to derive our results from the perspective of a rather recently established theory of stochastic regularization (see [19] and the references therein) with respect to a new general method based on Malliavin calculus and another variational technique which can be applied to different types of SDEs and stochastic partial differential equations (SPDEs).

In order to explain the concept of stochastic regularization, let us consider the first-order ordinary differential equation (ODE)

$$\frac{d}{dt}X_t^x = b(t, X_t^x), \quad X_0 = x \in \mathcal{H}, \quad t \in [0,T] \tag{5.1}$$

for a vector field $b : [0,T] \times \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is a separable Hilbert space with norm $\|\cdot\|_{\mathcal{H}}$.

Using Picard iteration, it is fairly straight forward to see that the ODE (5.1) has a unique (global) solution $(X_t^x)_{t\in[0,T]}$, if the driving vector field b satisfies a linear growth and Lipschitz condition, that is

$$\|b(t,x)\|_{\mathcal{H}} \le C_1(1 + \|x\|_{\mathcal{H}})$$

The research is partially supported by the FINEWSTOCH (NFR-ISP) project.

and

$$\|b(t,x) - b(t,y)\|_{\mathcal{H}} \leq C_2 \|x-y\|_{\mathcal{H}}$$

for all t, x and y with constants $C_1, C_2 < \infty$.

However, well-posedness in the sense of existence and uniqueness of solutions may fail, if the vector field b lacks regularity, that is if e.g. b is not Lipschitz continuous. In this case, the ODE (5.1) may not even admit the existence of a solution in the case $\mathcal{H} = \mathbb{R}^d$.

On the other hand, the situation changes, if one integrates on both sides of the ODE (5.1) and adds a "regularizing" noise to the right hand side of the resulting integral equation.

More precisely, if $\mathcal{H} = \mathbb{R}^d$, well-posedness of the ODE (5.1) can be restored via regularization by a Brownian (additive) noise, that is by a perturbation of the ODE (5.1) given by the SDE

$$dX_t^x = b(t, X_t^x)dt + \varepsilon dB_t, \quad t \in [0,T], \quad X_0^x = x \in \mathbb{R}^d, \tag{5.2}$$

where $(B_t)_{t\in[0,T]}$ is a Brownian motion in $\mathbb{R}^d$ and $\varepsilon > 0$.

If the vector field b is merely bounded and measurable, it turns out that the SDE (5.2) – regardless how small ε is – possesses a unique (global) strong solution, that is a solution $(X_t^x)_{t\in[0,T]}$, which as a process is a measurable functional of the driving noise $(B_t)_{t\in[0,T]}$. This surprising and remarkable result was first obtained by A. K. Zvonkin [41] in the one-dimensional case, whose proof, using PDE techniques, is based on a transformation ("Zvonkin-transformation"), that converts the SDE (5.2) into a SDE without drift part. Subsequently, this result was generalized by A. Y. Veretennikov [39] to the multi-dimensional case. Much later, that is 35 years later, Zvonkin's and Veretennikov's results were extended by G. Da Prato, F. Flandoli, E. Priola and M. Röckner [13] to the infinite-dimensional setting by using estimates of solutions of Kolmogorov's equation on Hilbert spaces. In fact, the latter authors study mild solutions $(X_t)_{t\in[0,T]}$ to the SDE

$$dX_t = AX_t dt + b(X_t)dt + \sqrt{Q}dW_t, \quad t \in [0,T], \quad X_0 = x \in \mathcal{H},$$

where $(W_t)_{t\in[0,T]}$ is a cylindrical Brownian motion on $\mathcal{H}$, $A : D(A) \to \mathcal{H}$ a negative self-adjoint operator with compact resolvent, $Q : \mathcal{H} \to \mathcal{H}$ a non-negative definite self-adjoint bounded operator and $b : \mathcal{H} \to \mathcal{H}$. Here, the authors prove for $b \in L^\infty(\mathcal{H};\mathcal{H})$ under certain conditions on A and Q the existence of a unique mild solution, which is adapted to a completed filtration generated by $(W_t)_{t\in[0,T]}$. So restoration of well-posedness of the ODE (5.1) with a singular vector field is established via regularization by *both* the cylindrical Brownian noise $(W_t)_{t\in[0,T]}$ and A, which cannot be chosen to be the zero operator.

Other works in this direction in the infinite-dimensional setting based on different methods are e.g. A. S. Sznitman [38], A. Y. Pilipenko, M. V. Tantsyura [36] in connection with systems of McKean-Vlasov equations and G. Ritter, G. Leha

[25] in the case of discontinuous drift vector fields of a rather specific form. We also refer to the references therein.

In this article, we aim at restoring well-posedness of singular ODE's by using a certain non-Hölder continuous additive noise of fractal nature. More specifically, we want to analyze solutions to the following type of SDE:

$$X_t^x = x + \int_0^t b(t, X_s^x)ds + \mathbb{B}_t, \quad t \in [0,T], \tag{5.3}$$

where the $\mathcal{H}-$valued regularizing noise $(\mathbb{B}_t)_{t\in[0,T]}$ is a stationary Gaussian process with locally non-Hölder continuous paths given by

$$\mathbb{B}_t = \sum_{k\geq 1} \lambda_k B_t^{H_k} e_k.$$

Here $\{\lambda_k\}_{k\geq 1} \subset \mathbb{R}$, $\{e_k\}_{k\geq 1}$ is an orthonormal basis of $\mathcal{H}$ and $\{B_\cdot^{H_k}\}_{k\geq 1}$ are independent one-dimensional fractional Brownian motions with Hurst parameters $H_k \in (0, \frac{1}{2})$, $k \geq 1$, such that

$$H_k \searrow 0$$

for $k \to \infty$.

Under certain (rather mild) growth conditions on the Fourier components b_k, $k \geq 1$, of the singular vector field $b : [0,T] \times \mathcal{H} \to \mathcal{H}$ (see (5.22) and (5.23)), which do not necessarily require that all b_k are equal (compare e.g. to [38]), we show in this paper the existence of a unique (global) strong solution to the SDE (5.3) driven by the non-Markovian process $(\mathbb{B}_t)_{t\in[0,T]}$.

Our approach for the construction of strong solutions to (5.3) relies on Malliavin calculus (see e.g. D. Nualart [32]) and another variational technique, which involves the use of spatial regularity of local time of finite-dimensional approximations of $\mathbb{B}_t$. In contrast to the above mentioned works (and most of other related works in the literature), the method in this paper is not based on PDE, Markov or semimartingale techniques. Furthermore, our technique corresponds to a construction principle, which is diametrically opposed to the commonly used Yamada-Watanabe principle (see e.g. [40]): Using the Yamada-Watanabe principle, one combines the existence of a weak solution to a SDE with pathwise uniqueness to obtain strong uniqueness of solutions. So

$$\boxed{\text{Weak existence}} + \boxed{\text{Pathwise uniqueness}} \Rightarrow \boxed{\text{Strong uniqueness}}.$$

This tool is in fact used by many authors in the literature. See e.g. the above mentioned authors or I. Gyöngy, T. Martinez [22], I. Gyöngy, N. V. Krylov [21], N. V. Krylov, M. Röckner [24] or S. Fang, T. S. Zhang [18], just to mention a few.

However, using our approach, verification of the existence of a strong solution, which is unique in law, provides strong uniqueness:

$$\boxed{\text{Strong existence}} + \boxed{\text{Uniqueness in law}} \Rightarrow \boxed{\text{Strong uniqueness}}.$$

See also H. J. Engelbert [17] in the finite-dimensional Brownian case regarding the latter construction principle.

In order to briefly explain our method in the case of time-homogeneous vector fields, we mention that we apply an infinite-dimensional generalization of a compactness criterion for square integrable Brownian functionals in $L^2(\Omega)$, which is originally due to G. Da Prato, P. Malliavin, and D. Nualart [14], to a double-sequence of strong solutions $\{(X_t^{d,\varepsilon})_{t\in[0,T]}\}_{d\geq 1,\varepsilon>0}$ associated with the following SDE's

$$X_t^{d,\varepsilon} = x + \int_0^t b^{d,\varepsilon}(s, X_s^{d,\varepsilon})ds + \mathbb{B}_t, \quad t \in [0,T]. \tag{5.4}$$

Here $\{b^{d,\varepsilon}\}_{d\in\mathbb{N},\varepsilon>0}$ is an approximating double-sequence of vector fields of the singular drift b, which are smooth and live on $d-$dimensional subspaces of $\mathcal{H}$.

The application of the above mentioned compactness criterion (for each fixed t), however, requires certain (uniform) estimates with respect to the Malliavin derivative D_t of $X_t^{d,\varepsilon}$ in the direction of a cylindrical Brownian motion. For this purpose, the Malliavin derivative $D_\cdot : \mathbb{D}^{1,2}(\mathcal{H}) \longrightarrow L^2([0,T]\times\Omega)\otimes\mathcal{L}_{HS}(\mathcal{H},\mathcal{H})$ ($\mathbb{D}^{1,2}(\mathcal{H})$ is the space of $\mathcal{H}-$valued Malliavin differentiable random variables and $\mathcal{L}_{HS}(\mathcal{H},\mathcal{H})$ is the space of Hilbert-Schmidt operators from $\mathcal{H}$ to $\mathcal{H}$) in connection with a chain rule is applied to both sides of (5.4) and one obtains the following linear equation:

$$D_s X_t^{d,\varepsilon} = \int_s^t \left(b^{d,\varepsilon}\right)'(X_u^{d,\varepsilon}) D_s X_u^{d,\varepsilon} du + \sum_{n\geq 1} \lambda_n K_{H_n}(t,s) \langle e_n, \cdot\rangle_{\mathcal{H}} e_n, \; s<t, \tag{5.5}$$

where $\left(b^{d,\varepsilon}\right)'$ is the derivative of $b^{d,\varepsilon}$, $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ the inner product and K_H a certain kernel function defined for Hurst parameters $H_n \in (0,\frac{1}{2})$.

We remark here that this type of linearization based on a stochastic derivative D_t actually corresponds to the Nash-Moser principle, which is used for the construction of solutions of (non-linear) PDE's by means of linearization of equations via classical derivatives. See e.g. J. Moser [31].

In a next step we then can derive a representation of $D_s X_t^{d,\varepsilon}$ (under a Girsanov change of measure) in (5.5) which is not based on derivatives of $b^{d,\varepsilon}$ by using Picard iteration and the following variational argument:

$$\begin{aligned}\int_{t<s_1<\ldots<s_n<u} \kappa(s) D^\alpha f(\mathbb{B}_s^d) ds &= \int_{\mathbb{R}^{dn}} D^\alpha f(z) L_\kappa^n(t,z) dz \\ &= (-1)^{|\alpha|} \int_{\mathbb{R}^{dn}} f(z) D^\alpha L_\kappa^n(t,z) dz,\end{aligned}$$

where $\mathbb{B}_s^d := (B_{s_1}^{H_1}, ..., B_{s_1}^{H_d}, ..., B_{s_n}^{H_1}, ..., B_{s_n}^{H_d})$ and $f:\mathbb{R}^{dn}\longrightarrow\mathbb{R}$ is a smooth function with compact support. Here D^α stands for a partial derivative of order $|\alpha|$ with respect a multi-index α. Further, $L_{\kappa}^{n}(t,z)$ is a spatially differentiable local time of $\mathbb{B}_{\cdot}^{d}$ on a simplex scaled by non-negative integrable function $\kappa(s) = \kappa_1(s)...\kappa_n(s)$.

Then, using the latter we can verify the required estimates for the Malliavin derivative of the approximating solutions in connection with the above mentioned compactness criterion and we finally obtain (under some additional arguments) that for each fixed t

$$X_t^{d,\varepsilon} \longrightarrow X_t \text{ in } L^2(\Omega)$$

for $\varepsilon \searrow 0, d \longrightarrow \infty$, where $(X_t)_{t\in[0,T]}$ is the unique strong solution to (5.3).

Finally, let us also mention a series of papers, from which our construction method gradually evolved: We refer to the works [27], [28], [29], [30] in the case of finite-dimensional Brownian noise. See [20] in the Hilbert space setting in connection with Hölder continuous drift vector fields. In the case of SDEs driven by Lévy processes we mention [23]. Other results can be found in [6], [1] with respect to SDEs driven by fractional Brownian motion and related noise. See also [7] in the case of "skew fractional Brownian motion", [5] with respect to singular delay equations and [8] in the case of Brownian motion driven mean-field equations.

We shall also point to the work of R. Catellier and M. Gubinelli [11], who prove existence and *path by path* uniqueness (in the sense of A. M. Davie [15]) of strong solutions of fractional Brownian motion driven SDEs with respect to (distributional) drift vector fields belonging to the Besov-Hölder space $B^{\alpha}_{\infty,\infty}$, $\alpha \in \mathbb{R}$. The approach of the authors is based inter alia on the theorem of Arzela-Ascoli and a comparison principle based on an average translation operator. In the distributional case, that is $\alpha < 0$, the drift part of the SDE is given by a generalized non-linear Young integral defined via the topology of $B^{\alpha}_{\infty,\infty}$. See also D. Nualart, Y. Ouknine [33] in the one-dimensional case.

The structure of our article is as follows: In Section 2 we introduce the mathematical framework of this paper. Further, in Section 3 we discuss some properties of the process $\mathbb{B}_{\cdot}$ and weak solutions of the SDE (5.3). Section 4 is devoted to the construction of unique strong solutions to the SDE (5.3). Finally, in Section 5 examples of singular vector fields for which strong solutions exist are given.

Notation. For the sake of readability we assume throughout the paper that $1 \leq T < \infty$ is a finite time horizon. We define $\mathcal{H}$ to be an infinite-dimensional separable real-valued Hilbert space with scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and orthonormal basis $\{e_k\}_{k\geq 1}$. Denote by $\|\cdot\|_{\mathcal{H}}$ the induced norm on $\mathcal{H}$ defined by $\|x\|_{\mathcal{H}} := \langle x,x\rangle_{\mathcal{H}}^{\frac{1}{2}}$, $x \in \mathcal{H}$. For every $x \in \mathcal{H}$ and $k \geq 1$ we denote by $x^{(k)} := \langle x, e_k\rangle_{\mathcal{H}}$ the projection onto the subspace spanned by e_k, $k \geq 1$. Loosely speaking we are referring to the subspace spanned by e_k, $k \geq 1$, as the k-th dimension. In line with this notation we denote the projection of the SDE (5.3) on the subspace spanned by e_k, $k \geq 1$, by

$X^{(k)} := \langle X, e_k\rangle_{\mathcal{H}}$. Moreover we can write the SDE (5.3) as an infinite dimensional system of real-valued stochastic differential equations, namely

$$X_t^{(k)} = x^{(k)} + \int_0^t b_k(s, X_s)ds + \mathbb{B}_t^{(k)}, \quad t \in [0,T], \quad k \geq 1,$$

where b_k and $\mathbb{B}^{(k)}$ are the projections on the subspace spanned by e_k, $k \geq 1$, of b and $\mathbb{B}$, respectively. Note here that the function $b_k : [0,T] \times \mathcal{H} \to \mathbb{R}$ has still domain $[0,T] \times \mathcal{H}$. Furthermore, we define the truncation operator π_d, $d \geq 1$, which maps an element $x \in \mathcal{H}$ onto the first d dimensions, by

$$\pi_d x := \sum_{k=1}^d x^{(k)} e_k. \tag{5.6}$$

The truncated space $\pi_d\mathcal{H}$ is denoted by $\mathcal{H}_d$. We define the change of basis operator $\tau : \mathcal{H} \to \ell^2$ by

$$\tau x = \tau \sum_{k\geq 1} x^{(k)} e_k = \sum_{k\geq 1} x^{(k)} \widetilde{e}_k, \tag{5.7}$$

where $\{\widetilde{e}_k\}_{k\geq 1}$ is an orthonormal basis of ℓ^2. It is easily seen that the operator τ is a bijection and we denote its inverse by $\tau^{-1} : \ell^2 \to \mathcal{H}$.

Further frequently used notation:

- Let $(\mathcal{X}, \mathcal{A}, \mu)$ denote a measurable space and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ a normed space. Then $L^2(\mathcal{X};\mathcal{Y})$ denotes the space of square integrable functions X over $\mathcal{X}$ taking values in $\mathcal{Y}$ and is endowed with the norm
$$\|X\|^2_{L^2(\mathcal{X};\mathcal{Y})} = \int_{\mathcal{X}} \|X(\omega)\|^2_{\mathcal{Y}} \mu(d\omega).$$
- The space $L^2(\Omega, \mathcal{F})$ denotes the space of square integrable random variables on the sample space Ω measurable with respect to the σ-algebra $\mathcal{F}$.
- We define $\mathbb{B}^x := x + \mathbb{B}$.
- For any vector u we denote its transposed by $u^\top$.
- We denote by Id the identity operator.
- The Jacobian of a differentiable function is denoted by ∇.
- For any multi-index α of length d and any d-dimensional vector u we define $u^\alpha := \prod_{i=1}^d u_i^{\alpha_i}$.
- For two mathematical expressions $E_1(\theta), E_2(\theta)$ depending on some parameter θ we write $E_1(\theta) \lesssim E_2(\theta)$, if there exists a constant $C > 0$ not depending on θ such that $E_1(\theta) \leq CE_2(\theta)$.
- Let A be some countable set. Then we denote by $\#A$ its cardinality.

2. Preliminaries

2.1. **Shuffles.** Let m and n be two integers. We denote by $\mathcal{S}(m,n)$ the set of *shuffle permutations*, i.e. the set of permutations $\sigma : \{1,\dots,m+n\} \to \{1,\dots,m+n\}$ such that $\sigma(1) < \cdots < \sigma(m)$ and $\sigma(m+1) < \cdots < \sigma(m+n)$. Equivalently we denote for integers k and n by $\mathcal{S}(k;n)$ the set of shuffle permutations of k sets of size n, i.e. the set of permutations $\sigma : \{1,\dots,k\cdot n\} \to \{1,\dots,k\cdot n\}$ such that $\sigma(m\cdot n+1) < \cdots < \sigma((m+1)\cdot n)$ for all $m = 0,\dots,k-1$. Furthermore the n-dimensional simplex Δ^n of the interval (s,t) is defined by

$$\Delta^n_{s,t} := \{(u_1,\dots,u_n) \in [0,T]^n : \ s < u_1 < \cdots < u_n < t\}.$$

Note that the product of two simplices can be written as

$$\Delta^m_{s,t} \times \Delta^n_{s,t} = \bigcup_{\sigma\in\mathcal{S}(m,n)} \{(w_1,\dots,w_{m+n}) \in [0,T]^{m+n} : \ w_\sigma \in \Delta^{m+n}_{s,t}\} \cup \mathfrak{N}, \qquad (5.8)$$

where the set $\mathfrak{N}$ has Lebesgue measure zero and w_σ denotes the shuffled vector $(w_{\sigma(1)},\dots,w_{\sigma(m+n)})$. For the sake of readability we denote throughout the paper the integral over the simplex $\Delta^n_{s,t}$ of the product of integrable functions $f_i : [0,T] \to \mathbb{R}$, $i = 1,\dots,n$, by

$$\int_{\Delta^n_{s,t}} \prod_{j=1}^n f_j(u_j)du := \int_s^t \int_{u_1}^t \cdots \int_{u_{n-1}}^t \prod_{j=1}^n f_j(u_j)du_n \cdots du_2 du_1.$$

Due to (5.8), we get for integrable functions $f_i : [0,T] \to \mathbb{R}$, $i = 1,\dots,m+n$, that

$$\int_{\Delta^m_{s,t}} \prod_{j=1}^m f_j(u_j)du \int_{\Delta^n_{s,t}} \prod_{j=m+1}^{m+n} f_j(u_j)du = \sum_{\sigma\in\mathcal{S}(m,n)} \int_{\Delta^{m+n}_{s,t}} \prod_{j=1}^{m+n} f_{\sigma(j)}(w_j)dw. \qquad (5.9)$$

For a proof of a more general result we refer the reader to [6, Lemma 2.1].

2.2. **Fractional Calculus.** In the following we give some basic definitions and properties on fractional calculus. For more insights on the general theory we refer the reader to [34] and [37].

Let $a,b \in \mathbb{R}$ with $a < b$, $f,g \in L^p([a,b])$ with $p \geq 1$ and $\alpha > 0$. We define the *left-* and *right-sided Riemann-Liouville fractional integrals* by

$$I^\alpha_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y)dy,$$

and

$$I^\alpha_{b^-} g(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} g(y)dy,$$

for almost all $x \in [a,b]$. Here Γ denotes the gamma function.

Furthermore, for any given integer $p \geq 1$, let $I^\alpha_{a^+}(L^p)$ and $I^\alpha_{b^-}(L^p)$ denote the images of $L^p([a,b])$ by the operator $I^\alpha_{a^+}$ and $I^\alpha_{b^-}$, respectively. If $0 < \alpha < 1$ as well as $f \in I^\alpha_{a^+}(L^p)$ and $g \in I^\alpha_{b^-}(L^p)$, we define the *left-* and *right-sided Riemann-Liouville fractional derivatives* by

$$D^\alpha_{a^+} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(y)}{(x-y)^\alpha} dy, \qquad (5.10)$$

and

$$D_{b-}^{\alpha} g(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b \frac{g(y)}{(y-x)^{\alpha}} dy, \tag{5.11}$$

respectively. The left- and right-sided derivatives of f and g defined in (5.10) and (5.11) admit moreover the representations

$$D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}} dy \right),$$

and

$$D_{b-}^{\alpha} g(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{g(x)}{(b-x)^{\alpha}} + \alpha \int_x^b \frac{g(x)-g(y)}{(y-x)^{\alpha+1}} dy \right).$$

Last, we get by construction that similar to the fundamental theorem of calculus

$$I_{a+}^{\alpha}(D_{a+}^{\alpha} f) = f, \tag{5.12}$$

for all $f \in I_{a+}^{\alpha}(L^p)$, and

$$D_{a+}^{\alpha}(I_{a+}^{\alpha} g) = g, \tag{5.13}$$

for all $g \in L^p([a,b])$. Equivalent results hold for I_{b-}^{α} and D_{b-}^{α}.

2.3. **Fractional Brownian motion.** The one-dimensional *fractional Brownian motion*, in short fBm, $B^H = \left(B_t^H\right)_{t\in[0,T]}$ with Hurst parameter $H \in (0, \frac{1}{2})$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is defined as a centered Gaussian process with covariance function

$$R_H(t,s) := \mathbb{E}\left[B_t^H B_s^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

Note that $\mathbb{E}\left[\left|B_t^H - B_s^H\right|^2\right] = |t-s|^{2H}$ and hence B^H has stationary increments and almost surely Hölder continuous paths of order $H - \varepsilon$ for all $\varepsilon \in (0, H)$. However, the increments of B^H, $H \in (0, \frac{1}{2})$, are not independent and B^H is not a semimartingale, see e.g. [32, Proposition 5.1.1].

Subsequently we give a brief outline of how a fractional Brownian motion can be constructed from a standard Brownian motion. For more details we refer the reader to [32].

Recall the following result (see [32, Proposition 5.1.3]) which gives the kernel of a fractional Brownian motion and an integral representation of $R_H(t,s)$ in the case of $H < \frac{1}{2}$.

Proposition 2.1 *Let $H < \frac{1}{2}$. The kernel*

$$K_H(t,s) := c_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} + \left(\frac{1}{2} - H\right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right], \tag{5.14}$$

where $c_H = \sqrt{\frac{2H}{(1-2H)\beta\left(1-2H,H+\frac{1}{2}\right)}}$ *and* β *is the beta function, satisfies*

$$R_H(t,s) = \int_0^{t\wedge s} K_H(t,u)K_H(s,u)du. \tag{5.15}$$

Subsequently, we denote by W a standard Brownian motion on the complete filtered probability space $(\Omega,\mathcal{F},\mathbb{F}^W,\mathbb{P})$, where $\mathbb{F}^W := (\mathcal{F}_t^W)_{t\in[0,T]}$ is the natural filtration of W augmented by all $\mathbb{P}$-null sets. Using the kernel given in (5.14) it is well known that the fractional Brownian motion B^H has a representation

$$B_t^H = \int_0^t K_H(t,s)dW_s,\ H \in \left(0,\frac{1}{2}\right). \tag{5.16}$$

Note that due to representation (5.16) the natural filtration generated by B^H is identical to $\mathbb{F}^W$. Furthermore, equivalent to the case of a standard Brownian motion, it exists a version of Girsanov's theorem for fractional Brownian motion which is due to [16, Theorem 4.9]. In the following we state the version given in [33, Theorem 3.1].

But first let us define the isomorphism K_H from $L^2([0,T])$ onto $I_{0+}^{H+\frac{1}{2}}(L^2)$ (see [16, Theorem 2.1]) given by

$$(K_H\varphi)(s) = I_{0+}^{2H} s^{\frac{1}{2}-H} I_{0+}^{\frac{1}{2}-H} s^{H-\frac{1}{2}}\varphi, \quad \varphi \in L^2([0,T]). \tag{5.17}$$

From (5.17) and the properties of the Riemann-Liouville fractional integrals and derivatives (5.12) and (5.13), the inverse of K_H is given by

$$(K_H^{-1}\varphi)(s) = s^{\frac{1}{2}-H} D_{0+}^{\frac{1}{2}-H} s^{H-\frac{1}{2}} D_{0+}^{2H}\varphi(s), \quad \varphi \in I_{0+}^{H+\frac{1}{2}}(L^2).$$

It can be shown (see [33]) that if φ is absolutely continuous

$$(K_H^{-1}\varphi)(s) = s^{H-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H} s^{\frac{1}{2}-H}\varphi'(s), \tag{5.18}$$

where φ' denotes the weak derivative of φ.

Theorem 2.2 (Girsanov's theorem for fBm) *Let* $u = (u_t)_{t\in[0,T]}$ *be a process with integrable trajectories and set* $\widetilde{B}_t^H = B_t^H + \int_0^t u_s ds,\ t \in [0,T]$. *Assume that*

(i) $\int_0^\cdot u_s ds \in I_{0+}^{H+\frac{1}{2}}(L^2([0,T]))$, $\mathbb{P}$*-a.s., and*
(ii) $\mathbb{E}[\mathcal{E}_T] = 1$, *where*

$$\mathcal{E}_T := \exp\left\{-\int_0^T K_H^{-1}\left(\int_0^\cdot u_r dr\right)(s)dW_s - \frac{1}{2}\int_0^T K_H^{-1}\left(\int_0^\cdot u_r dr\right)^2(s)ds\right\}.$$

Then the shifted process $\widetilde{B}^H$ *is an* $\mathbb{F}^W$*– fractional Brownian motion with Hurst parameter* H *under the new probability measure* $\widetilde{\mathbb{P}}$ *defined by* $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}_T$.

Remark 2.3. Theorem 2.2 can be extended to the multi- and infinite-dimensional cases, which will be considered in this paper primarily. Indeed, note first that the measure change in Girsanov's theorem acts dimension-wise. In particular, consider the two dimensional shifted process

$$\begin{aligned} X_t^{(1)} &= B_t^{H_1} + \int_0^t u_s^{(1)} ds, \\ X_t^{(2)} &= B_t^{H_2} + \int_0^t u_s^{(2)} ds, \ t \in [0,T], \end{aligned}$$

where B^{H_1} and B^{H_2} are two fractional Brownian motions with Hurst parameters H_1 and H_2 generated by the independent standard Brownian motions $W^{(1)}$ and $W^{(2)}$, respectively, and $u^{(1)}$ and $u^{(2)}$ are two shifts fulfilling the conditions of Theorem 2.2. Then the measure change with respect to the stochastic exponential

$$\mathcal{E}_T^{(1)} := \exp\left\{ -\int_0^T K_{H_1}^{-1}\left(\int_0^\cdot u_r^{(1)} dr\right)(s) dW_s^{(1)} - \frac{1}{2}\int_0^T K_{H_1}^{-1}\left(\int_0^\cdot u_r^{(1)} dr\right)^2 (s) ds \right\}$$

yields the two dimensional process

$$\begin{aligned} X_t^{(1)} &= \widetilde{B}_t^{H_1}, \\ X_t^{(2)} &= B_t^{H_2} + \int_0^t u_s^{(2)} ds, \ t \in [0,T]. \end{aligned}$$

Here, $\widetilde{B}^{H_1}$ is a fractional Brownian motions with respect to the measure $\widetilde{\mathbb{P}}$ defined by $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}_T^{(1)}$. Note that B^{H_2} is still a fractional Brownian motion under $\widetilde{\mathbb{P}}$, since $W^{(1)}$ and $W^{(2)}$ are independent. Applying Girsanov's theorem again with respect to the stochastic exponential

$$\mathcal{E}_T^{(2)} := \exp\left\{ -\int_0^T K_{H_2}^{-1}\left(\int_0^\cdot u_r^{(2)} dr\right)(s) dW_s^{(2)} - \frac{1}{2}\int_0^T K_{H_2}^{-1}\left(\int_0^\cdot u_r^{(2)} dr\right)^2 (s) ds \right\},$$

yields the two dimensional process

$$\begin{aligned} X_t^{(1)} &= \widetilde{B}_t^{H_1}, \\ X_t^{(2)} &= \widetilde{B}_t^{H_2}, \ t \in [0,T], \end{aligned}$$

where $\widetilde{B}^{H_1}$ and $\widetilde{B}^{H_2}$ are independent fractional Brownian motions with respect to the measure $\hat{\mathbb{P}}$ defined by

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = \frac{d\hat{\mathbb{P}}}{d\widetilde{\mathbb{P}}}\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}_T^{(2)}\mathcal{E}_T^{(1)}.$$

Repeating iteratively yields the stochastic exponential – if well-defined –

$$\mathcal{E}_T := \prod_{k\geq 1} \mathcal{E}_T^{(k)}$$

acting on infinite dimensions.

Finally, we give the property of strong local non-determinism of the fractional Brownian motion B^H with Hurst parameter $H \in (0, \frac{1}{2})$ which was proven in [35, Lemma 7.1]. This property will essentially help us to overcome the limitations of not having independent increments of the underlying noise.

Lemma 2.4 *Let B^H be a fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2})$. Then there exists a constant $\mathfrak{K}_H$ dependent merely on H such that for every $t \in [0, T]$ and $0 < r \le t$*

$$\mathrm{Var}\Big(B_t^H \,\Big|\, B_s^H : |t-s| \ge r\Big) \ge \mathfrak{K}_H r^{2H}.$$

3. Cylindrical fractional Brownian motion and weak solutions

We start this section by defining the driving noise $(\mathbb{B}_t)_{t\in[0,T]}$ in SDE (5.3). Let $\{W^{(k)}\}_{k\ge1}$ be a sequence of independent one-dimensional standard Brownian motions on a joint complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We define the cylindrical Brownian motion W taking values in $\mathcal{H}$ by

$$W_t := \sum_{k\ge1} W_t^{(k)} e_k, \quad t \in [0, T],$$

and denote by $\mathbb{F}^W := \left(\mathcal{F}_t^W\right)_{t\in[0,T]}$ its natural filtration augmented by the $\mathbb{P}$-null sets. Moreover, we define a sequence of Hurst parameters $H := \{H_k\}_{k\ge1} \subset \left(0, \frac{1}{2}\right)$ with the following properties:

(i) $\sum_{k\ge1} H_k < \frac{1}{6}$
(ii) $\sup_{k\ge1} H_k < \frac{1}{12}$

Using H we construct the sequence of fractional Brownian motions $\{B^{H_k}\}_{k\ge1}$ associated to $\{W^{(k)}\}_{k\ge1}$ by

$$B_t^{H_k} := \int_0^t K_{H_k}(t, s) dW_s^{(k)}, \quad t \in [0, T], \quad k \ge 1,$$

where the kernel $K_{H_k}(\cdot, \cdot)$ is defined as in (5.14). Note that the fractional Brownian motions $\{B^{H_k}\}_{k\ge1}$ are independent by construction. Consequently, we define the cylindrical fractional Brownian motion B^H with associated sequence of Hurst parameters H by

$$B_t^H := \sum_{k\ge1} B_t^{H_k} e_k, \quad t \in [0, T]. \tag{5.19}$$

Nevertheless, the cylindrical fractional Brownian motion B^H is not in the space $L^2(\Omega; \mathcal{H})$. That is why we consider the operator $Q : \mathcal{H} \to \mathcal{H}$ defined by

$$Qx = \sum_{k\ge1} \lambda_k^2 x^{(k)} e_k,$$

for a given sequence of non-negative real numbers $\lambda := \{\lambda_k\}_{k\geq 1} \in \ell^2$ such that $\frac{\lambda}{\sqrt{H}} := \left\{\frac{\lambda_k}{\sqrt{H_k}}\right\}_{k\geq 1} \in \ell^1$. In particular, Q is a self-adjoint operator and we have that the *weighted cylindrical fractional Brownian motion*

$$\mathbb{B}_t := \sqrt{Q} B_t^H = \sum_{k\geq 1} \lambda_k B_t^{H_k} e_k, \tag{5.20}$$

lies in $L^2(\Omega; \mathcal{H})$ for every $t \in [0,T]$. Due to the following lemma the stochastic process $(\mathbb{B}_t)_{t\in[0,T]}$ is continuous in time.

Lemma 3.1 *The stochastic process* $(\mathbb{B}_t)_{t\in[0,T]}$ *defined in* (5.20) *has almost surely continuous sample paths on* $[0,T]$.

Proof. Note first that due to [10][Theorem 1] for any fractional Brownian motion B^H with Hurst parameter $H \in (0, \frac{1}{2})$ there exists a constant $C > 0$ independent of H such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|B_t^H\right|\right] \leq \frac{C}{\sqrt{H}}. \tag{5.21}$$

Using monotone convergence and (5.21) we have that

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]} \|\mathbb{B}_t\|_{\mathcal{H}}\right] &\leq \mathbb{E}\left[\sup_{t\in[0,T]} \sum_{k\geq 1} |\lambda_k| \left|B_t^{H_k}\right|\right] \leq \sum_{k\geq 1} \lambda_k \mathbb{E}\left[\sup_{t\in[0,T]} \left|B_t^{H_k}\right|\right] \\ &\leq \sum_{k\geq 1} \lambda_k \frac{C}{\sqrt{H_k}} < \infty.\end{aligned}$$

Thus, $(\sqrt{Q}B_t^H)_{t\in[0,T]}$ is almost surely finite and $\{(\pi_d \sqrt{Q} B_t^H)_{t\in[0,T]}\}_{d\geq 1}$ is a Cauchy sequence in $L^1(\Omega; \mathcal{C}([0,T]; \mathcal{H}))$ which converges almost surely to $(\sqrt{Q}B_t^H)_{t\in[0,T]}$. $\square$

Before we come to the next result, let us recall the notion of a weak solution and uniqueness in law.

Definition 3.2 The sextuple $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \mathbb{B}, X)$ is called a weak solution of stochastic differential equation (5.3), if

(i) $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a complete filtered probability space, where $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ satisfies the usual conditions of right-continuity and completeness,

(ii) $\mathbb{B} = (\mathbb{B}_t)_{t\in[0,T]}$ is a weighted cylindrical fractional $(\mathbb{F}, \mathbb{P})$-Brownian motion as defined in (5.20), and

(iii) $X = (X_t)_{t\in[0,T]}$ is a continuous, $\mathbb{F}$-adapted, $\mathcal{H}$-valued process satisfying $\mathbb{P}$-a.s.

$$X_t = x + \int_0^t b(s, X_s) ds + \mathbb{B}_t, \quad t \in [0,T].$$

Remark 3.3. For notational simplicity we refer solely to the process X as a weak solution (or later on as a strong solution) in the case of an unambiguous stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, B)$.

Definition 3.4 We say a weak solution X^1 with respect to the stochastic basis $(\Omega^1, \mathcal{F}^1, \mathbb{F}^1, \mathbb{P}^1, \mathbb{B}^1)$ of the SDE (5.3) is *weakly unique* or *unique in law*, if for any other weak solution X^2 of (5.3) on a potential other stochastic basis $(\Omega^2, \mathcal{F}^2, \mathbb{F}^2, \mathbb{P}^2, \mathbb{B}^2)$ it holds that

$$\mathbb{P}^1_{X^1} = \mathbb{P}^2_{X^2},$$

whenever $\mathbb{P}^1_{X_0^1} = \mathbb{P}^2_{X_0^2}$.

Proposition 3.5 *Let $b : [0,T] \times \mathcal{H} \to \mathcal{H}$ be a measurable and bounded function with $\|b_k\|_\infty \leq C_k \lambda_k < \infty$ for every $k \geq 1$ where $C := \{C_k\}_{k\geq 1} \in \ell^1$. Then SDE (5.3) has a weak solution $(X_t)_{t\in[0,T]}$ such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t\|^2_{\mathcal{H}}\right] < \infty.$$

Moreover, the solution is unique in law.

Proof. Let $\{W^{(k)}\}_{k\geq 1}$ be a sequence of independent standard Brownian motions on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$. Consider the cylindrical fractional Brownian motion $\widehat{B}^H$ generated by $\{W^{(k)}\}_{k\geq 1}$ as defined in (5.19) with associated sequence of Hurst parameters H. We define the stochastic exponential $\mathcal{E}$ by

$$\begin{aligned}\mathcal{E}_t := \exp\Bigg\{\sum_{k\geq 1} \bigg(\int_0^t K_{H_k}^{-1}\left(\int_0^\cdot b_k\left(u, x + \sqrt{Q}\widehat{B}^H_u\right)\lambda_k^{-1} du\right)(s) dW_s^{(k)} \\ -\frac{1}{2}\int_0^t K_{H_k}^{-1}\left(\int_0^\cdot b_k\left(u, x + \sqrt{Q}\widehat{B}^H_u\right)\lambda_k^{-1} du\right)^2(s) ds\bigg)\Bigg\}.\end{aligned}$$

In order to show that the stochastic exponential $\mathcal{E}$ is well-defined we first have to verify that for every $k \geq 1$

$$\int_0^\cdot b_k\left(u, x + \sqrt{Q}\widehat{B}^H_u\right)\lambda_k^{-1} du \in I_{0+}^{H_k + \frac{1}{2}}\left(L^2([0,T])\right), \ \mathbb{P} - \text{a.s.}.$$

Due to (5.18) this property is fulfilled, if for all $k \geq 1$

$$\int_0^T \left(b_k\left(u, x + \sqrt{Q}\widehat{B}^H_u\right)\lambda_k^{-1}\right)^2 du < \infty,$$

which holds since $\|b_k\|_\infty \leq C_k \lambda_k$. Furthermore, we can find a constant $C > 0$ such that

$$\begin{aligned}&\exp\left\{\frac{1}{2}\sum_{k\geq 1}\int_0^T K_{H_k}^{-1}\left(\int_0^\cdot b_k\left(u, x + \sqrt{Q}\widehat{B}^H_u\right)\lambda_k^{-1} du\right)^2(s) ds\right\} \\ &\quad \leq \exp\left\{CT^2 \sum_{k\geq 1} C_k^2\right\} < \infty.\end{aligned}$$

Hence, by Novikov's criterion $\mathcal{E}_t$ is a martingale, in particular $\mathbb{E}[\mathcal{E}_t] = 1$ for all $t \in [0,T]$. Consequently, under the probability measure $\mathbb{P}$, defined by $\frac{d\mathbb{P}}{d\mathbb{Q}} := \mathcal{E}_T$, the process $B_t^H := \widehat{B}_t^H - \int_0^t \sqrt{Q}^{-1} b\left(u, x + \sqrt{Q}\widehat{B}_u^H\right) du,\ t \in [0,T]$, is a cylindrical fractional Brownian motion due to Theorem 2.2 and Remark 2.3. Therefore, $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \sqrt{Q}B^H, X)$, where $X_t := x + \sqrt{Q}\widehat{B}_t^H$, is a weak solution of SDE (5.3). Since the probability measures $\mathbb{Q} \approx \mathbb{P}$ are equivalent, the solution is unique in law. □

4. Strong Solutions and Malliavin Derivative

After establishing the existence of a weak solution, we investigate under which conditions SDE (5.3) has a strong solution. Therefore, let us first recall the notion of a strong solution and moreover the notion of pathwise uniqueness.

Definition 4.1 A weak solution $(\Omega, \mathcal{F}, \mathbb{F}^{\mathbb{B}}, \mathbb{P}, \mathbb{B}, X^x)$ of the stochastic differential equation (5.3) is called *strong solution*, if $\mathbb{F}^{\mathbb{B}}$ is the filtration generated by the driving noise $\mathbb{B}$ and augmented with the $\mathbb{P}$-null sets.

Definition 4.2 We say a weak solution $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \mathbb{B}, X^1)$ of (5.3) is *pathwise unique*, if for any other weak solution $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \mathbb{B}, X^2)$ on the same stochastic basis,

$$\mathbb{P}\left(\omega \in \Omega : X_t^1(\omega) = X_t^2(\omega)\ \forall t \geq 0\right) = 1.$$

The cause of this paper is to establish the existence of strong solutions of stochastic differential equation (5.3) for *singular* drift coefficients b. More precisely, we define the class $\mathfrak{B}([0,T] \times \mathcal{H}; \mathcal{H})$ of measurable functions $b : [0,T] \times \mathcal{H} \to \mathcal{H}$ for which there exist sequences $C \in \ell^1$ and $D \in \ell^1$ such that for every $k \geq 1$

$$\begin{aligned} \sup_{y \in \mathcal{H}} \sup_{t \in [0,T]} |b_k(t,y)| &\leq C_k \lambda_k, \text{ and} \\ \sup_{d \geq 1} \int_{\mathbb{R}^d} \sup_{t \in [0,T]} \left|b_k\left(t, \sqrt{Q}\sqrt{\mathcal{K}}\tau^{-1} y\right)\right| dy &\leq D_k \lambda_k, \end{aligned} \tag{5.22}$$

where $y = (y_1, \ldots, y_d)$ and $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ is the defined by

$$\mathcal{K}x = \sum_{k \geq 1} \mathfrak{K}_{H_k} x^{(k)} e_k,\ x \in \mathcal{H}, \tag{5.23}$$

for $\{\mathfrak{K}_{H_k}\}_{k \geq 1}$ being the local non-determinism constant of $\{B^{H_k}\}_{k \geq 1}$ as given in Lemma 2.4.

In order to prove the existence of a strong solution for drift coefficients of class $\mathfrak{B}([0,T] \times \mathcal{H}; \mathcal{H})$ we proceed in the following way:

1) We define an approximating double-sequence $\{b^{d,\varepsilon}\}_{d \geq 1, \varepsilon > 0}$ for drift coefficients of type (5.22) which merely act on d dimensions and are sufficiently smooth
2) For every $d \geq 1$ and $\varepsilon > 0$, we prove that the SDE

$$X_t^{d,\varepsilon} = x + \int_0^t b^{d,\varepsilon}(s, X_s^{d,\varepsilon}) ds + \mathbb{B}_t,\ t \in [0,T], \tag{5.24}$$

has a unique strong solution which is Malliavin differentiable

3) We show that the double-sequence of strong solutions $X_t^{d,\varepsilon}$ converges weakly to $\mathbb{E}\left[X_t|\mathcal{F}_t^W\right]$, where X_t is the unique weak solution of SDE (5.3)
4) Applying a compactness criterion based on Malliavin calculus, we prove that the double-sequence is relatively compact in $L^2(\Omega, \mathcal{F}_t^W)$
5) Last, we show that X_t is adapted to the filtration $\mathbb{F}^{\mathbb{B}}$ and thus is a strong solution of SDE (5.3)

4.1. **Approximating double-sequence.** Recall the truncation operator π_d, $d \geq 1$, defined in (5.6) and the change of basis operator τ defined in (5.7). We define the operator $\widetilde{\pi}_d : \mathcal{H} \to \mathbb{R}^d$ as $\widetilde{\pi}_d := \tau \circ \pi_d$. For every $k \geq 1$ let the function $\widetilde{b}^d : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be defined by

$$\widetilde{b}^d(t,z) = \widetilde{\pi}_d b\left(t, \tau^{-1} z\right). \tag{5.25}$$

Let φ_ε, $\varepsilon > 0$, be a mollifier on $\mathbb{R}^d$ such that for any locally integrable function $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and for every $t \in [0,T]$ the convolution $f(t,\cdot) * \varphi_\varepsilon$ is smooth and

$$f(t,\cdot) * \varphi_\varepsilon \to f(t,\cdot), \ \varepsilon \to 0,$$

almost everywhere with respect to the Lebesgue measure. Finally, we define for every $d \geq 1$ and $\varepsilon > 0$ the double-sequence $b^{d,\varepsilon} : [0,T] \times \mathcal{H} \to \mathcal{H}$ by

$$b^{d,\varepsilon}(t,y) := \tau^{-1}\left(\widetilde{b}^d(t, \widetilde{\pi}_d y) * \varphi_\varepsilon(\widetilde{\pi}_d y)\right). \tag{5.26}$$

Analogously to (5.25), we define for $t \in [0,T]$ and $z \in \mathbb{R}^d$

$$\widetilde{b}^{d,\varepsilon}(t,z) := \tau b^{d,\varepsilon}(t, \tau^{-1} z) = \widetilde{b}^d(t,z) * \varphi_\varepsilon(z). \tag{5.27}$$

Due to the definition of the mollifier φ_ε we have that for every $d \geq 1$

$$b^{d,\varepsilon}(t,\tau^{-1}z) = \tau^{-1}\left(\widetilde{b}^d(t,z) * \varphi_\varepsilon(z)\right) \underset{\varepsilon \to 0}{\longrightarrow} \tau^{-1}\widetilde{b}^d(t,z) = b^d(t,\tau^{-1}z) \tag{5.28}$$

for almost every $(t,z) \in [0,T] \times \mathbb{R}^d$ with respect to the Lebesgue measure. Thus, due to (5.28) and the canonical properties of the truncation operator we have that

$$b^{d,\varepsilon}(t,y) \underset{\varepsilon \to 0}{\longrightarrow} b^d(t,y) \underset{d \to \infty}{\longrightarrow} b(t,y)$$

pointwise in $[0,T] \times \mathcal{H}$, where $b^d := \pi_d b$. Due to the assumptions on b we further get for every $p \geq 2$ using dominated convergence that

$$\lim_{d \to \infty} \lim_{\varepsilon \to 0} \mathbb{E}\left[\int_0^T \left\| b^{d,\varepsilon}(t, \mathbb{B}_t^x) - b(t, \mathbb{B}_t^x) \right\|_{\mathcal{H}}^p dt\right]^{\frac{1}{p}} = 0.$$

Hence, we can speak of an approximating double-sequence $\{b^{d,\varepsilon}\}_{d \geq 1, \varepsilon > 0}$ of the drift coefficient b. In line with the previously used notation we define

$$b_k^{d,\varepsilon}(t,y) := \langle b^{d,\varepsilon}(t,y), e_k \rangle_{\mathcal{H}} = \langle \widetilde{b}^{d,\varepsilon}(t,\tau y), \widetilde{e}_k \rangle =: \widetilde{b}_k^{d,\varepsilon}(t,\tau y),$$
$$b_k^d(t,y) := \langle b^d(t,y), e_k \rangle_{\mathcal{H}} = \langle \widetilde{b}^d(t,\tau y), \widetilde{e}_k \rangle =: \widetilde{b}_k^d(t,\tau y).$$

Moreover, note that $b^{d,\varepsilon}, b^d \in \mathfrak{B}([0,T]\times\mathcal{H};\mathcal{H})$.

Remark 4.3. Note that we needed to truncate and shift the domain of the function b to $\mathbb{R}^d$ merely in order to apply mollification.

4.2. **Malliavin differentiable strong solutions for regular drifts.** In the following proposition we establish the existence of a unique strong solution for a class of drift coefficients which contains the approximating sequence $\{b^{d,\varepsilon}\}_{d\geq 1,\varepsilon>0}$. More specifically, we consider drift coefficients $b \in \mathfrak{B}([0,T]\times\mathcal{H};\mathcal{H})$ such that for all $k\geq 1$ and all $t\in[0,T]$

$$b_k(t,\cdot)\in \mathrm{Lip}_{L_k}(\mathcal{H};\mathbb{R}),$$

where $L\in\ell^2$. We denote the space of such functions by $\mathfrak{L}([0,T]\times\mathcal{H};\mathcal{H})$.

Proposition 4.4 *Let $b\in\mathfrak{L}([0,T]\times\mathcal{H};\mathcal{H})$. Then SDE* (5.3) *has a pathwise unique strong solution.*

Proof. In order to prove the existence of a strong solution we use Picard iteration and proceed similar to the well-known case of finite dimensional SDEs. More precisely, we define inductively the sequence $Y^0 := x+\mathbb{B}$ and for all $n\geq 1$

$$Y_t^n = x+\int_0^t b\left(s, Y_s^{n-1}\right)ds + \mathbb{B}_t,\ t\in[0,T]. \tag{5.29}$$

We show next that $\{Y^n\}_{n\geq 0}$ is a Cauchy sequence in $L^2([0,T]\times\Omega)$. Indeed, due to monotone convergence we get for every $n\geq 1$ and $t\in[0,T]$

$$\begin{aligned}\mathbb{E}\left[\left\|Y_t^{n+1}-Y_t^n\right\|_{\mathcal{H}}^2\right]^{\frac12} &= \mathbb{E}\left[\left\|\int_0^t b(s,Y_s^n)-b(s,Y_s^{n-1})ds\right\|_{\mathcal{H}}^2\right]^{\frac12} \qquad (5.30)\\ &\leq \int_0^t \left(\sum_{k\geq 1}\mathbb{E}\left[\left|b_k(s,Y_s^n)-b_k(s,Y_s^{n-1})\right|^2\right]\right)^{\frac12} ds\\ &\leq \|L\|_{\ell^2}\int_0^t \mathbb{E}\left[\left\|Y_s^n-Y_s^{n-1}\right\|_{\mathcal{H}}^2\right]^{\frac12} ds,\end{aligned}$$

and

$$\mathbb{E}\left[\left\|Y_t^1-Y_t^0\right\|_{\mathcal{H}}^2\right]^{\frac12} = \mathbb{E}\left[\left\|\int_0^t b(s,x+\mathbb{B}_s)ds\right\|_{\mathcal{H}}^2\right]^{\frac12}\leq t\|C\lambda\|_{\ell^2}.$$

By induction we obtain for every $n\geq 0$ a constant A depending on C, λ and L such that

$$\mathbb{E}\left[\left\|Y_t^{n+1}-Y_t^n\right\|_{\mathcal{H}}^2\right]^{\frac12}\leq \frac{A^{n+1}}{(n+1)!}t^{n+1}.$$

Hence, for every $m,n\geq 0$

$$\|Y^m-Y^n\|_{L^2([0,T]\times\Omega;\mathcal{H})}\leq \sum_{k=n}^{m-1}\|Y^{k+1}-Y^k\|_{L^2([0,T]\times\Omega;\mathcal{H})}$$

$$
\begin{aligned}
&= \sum_{k=n}^{m-1} \mathbb{E}\left[\int_0^T \left\|Y_t^{k+1} - Y_t^k\right\|_{\mathcal{H}}^2 dt\right]^{\frac{1}{2}} \\
&\leq \sum_{k=n}^{m-1} \frac{A^{k+1}}{(k+1)!} T^{k+\frac{3}{2}} =: B(n,m).
\end{aligned}
$$

Since $B(n,m)$ is bounded by $T^{\frac{1}{2}} e^{AT}$, the series converges and

$$
B(n,m) \xrightarrow[n,m\to\infty]{} 0.
$$

Therefore $\{Y^n\}_{n\geq 0}$ is a Cauchy sequence in $L^2([0,T]\times\Omega;\mathcal{H})$. Define

$$
X_t := \lim_{n\to\infty} Y_t^n
$$

as the $L^2([0,T]\times\Omega;\mathcal{H})$ limit of $\{Y^n\}_{n\geq 0}$. Then X_t is $\mathcal{F}_t^{\mathbb{B}}$ adapted for all $t\in[0,T]$ since this holds for all Y_t^n, $n \geq 0$. We prove that X_t solves SDE (5.3):

We have for all $n \geq 0$ and $t\in[0,T]$ that

$$
Y_t^{n+1} = x + \int_0^t b(s, Y_s^n) ds + \mathbb{B}_t.
$$

Using the Lipschitz continuity of b, we get

$$
\begin{aligned}
\mathbb{E}\left[\left\|\int_0^t b(s,Y_s^n) - b(s,X_s) ds\right\|_{\mathcal{H}}^2\right]^{\frac{1}{2}} &\leq \int_0^t \left(\sum_{k\geq 1} \mathbb{E}\Big[|b_k(s,Y_s^n) - b_k(s,X_s)|^2\Big]\right)^{\frac{1}{2}} ds \\
&\leq \|L\|_{\ell^2} \int_0^t \mathbb{E}\Big[\|Y_s^n - X_s\|_{\mathcal{H}}^2\Big]^{\frac{1}{2}} ds \xrightarrow[n\to\infty]{} 0.
\end{aligned}
$$

Hence, $(X_t)_{t\in[0,T]}$ is a strong solution of SDE (5.3).

In order to show pathwise uniqueness, let X and Y be two strong solutions on the same stochastic basis $(\Omega,\mathcal{F},\mathbb{P},\mathbb{B})$ with the same initial condition. Then for all $t\in[0,T]$ we get similar to (5.30) that

$$
\mathbb{E}\Big[\|X_t - Y_t\|_{\mathcal{H}}^2\Big]^{\frac{1}{2}} \leq \|L\|_{\ell^2} \int_0^t \mathbb{E}\Big[\|X_s - Y_s\|_{\mathcal{H}}^2\Big]^{\frac{1}{2}} ds.
$$

Using Grönwall's inequality yields that $\mathbb{E}\Big[\|X_t - Y_t\|_{\mathcal{H}}^2\Big] = 0$ for all $t\in[0,T]$, and therefore $X_t = Y_t$ $\mathbb{P}$-a.s. for all $t\in[0,T]$. But since X and Y are almost surely continuous we get

$$
\mathbb{P}\left(\omega\in\Omega : X_t^1(\omega) = X_t^2(\omega)\ \forall t\geq 0\right) = 1.
$$

□

Next we investigate under which conditions the unique strong solution is Malliavin differentiable. But let us start with a definition of Malliavin differentiability of a random variable in the space $\mathcal{H}$.

Definition 4.5 Let X be an $\mathcal{H}$-valued square integrable functional of the cylindrical Brownian motion $(W_t)_{t\in[0,T]}$. We define the operator D^m, $m \geq 1$, such that

$$D^m X = \sum_{k\geq 1} D^m X^{(k)} e_k,$$

as the Malliavin derivative in the direction of the m-th Brownian motion $W^{(m)}$. Here, $D^m X^{(k)}$, $m, k \geq 1$, is the (standard) Malliavin derivative with respect to the Brownian motion $W^{(m)}$ of the square integrable random variable $X^{(k)}$ taking values in $\mathbb{R}$. We say a random variable X with values in $\mathcal{H}$ is in the space $\mathbb{D}^{1,2}(\mathcal{H})$ of Malliavin differentiable functions in $L^2(\Omega)$ if and only if

$$\|X\|^2_{\mathbb{D}^{1,2}(\mathcal{H})} := \sum_{m\geq 1} \int_0^T \mathbb{E}\Big[\|D_s^m X\|^2_{\mathcal{H}}\Big] ds < \infty.$$

Moreover, a stochastic process $(X_t)_{t\in[0,T]}$ with values in $\mathcal{H}$ is said to be in the space $\mathbb{D}^{1,2}([0,T]\times\mathcal{H})$ if and only if for every $t \in [0,T]$

$$\|X_t\|^2_{\mathbb{D}^{1,2}(\mathcal{H})} := \sum_{m\geq 1} \int_0^T \mathbb{E}\Big[\|D_s^m X_t\|^2_{\mathcal{H}}\Big] ds < \infty.$$

By means of Definition 4.5 we extend the well-known chain rule in Malliavin Calculus, cf. [32, Proposition 1.2.4], to Malliavin differentiable random variables taking values in $\mathcal{H}$. But first we define the class $\mathcal{L}_0(\mathcal{H})$ of Lipschitz continuous functions on $\mathcal{H}$ with vanishing Lipschitz constants.

We say a function $f : \mathcal{H} \to \mathcal{H}$ is in the space $\mathcal{L}_0(\mathcal{H})$ if there exist sequences of constants $L, M \in \ell^2$ such that for all $k \geq 1$ and $x, y \in \mathcal{H}$

$$|\langle f(x) - f(y), e_k\rangle_{\mathcal{H}}| \leq L_k \sum_{i\geq 1} M_i |\langle x - y, e_i\rangle_{\mathcal{H}}|. \tag{5.31}$$

Lemma 4.6 *Let $f \in \mathcal{L}_0(\mathcal{H})$ with associated Lipschitz sequences $L, M \in \ell^2$ and $Y \in \mathbb{D}^{1,2}(\mathcal{H})$. Then, $f(Y) \in \mathbb{D}^{1,2}(\mathcal{H})$ and there exists a double-sequence $\{G_i^{(k)}\}_{k,i\geq 1}$ of random variables with $G_i^{(k)} \leq L_k \cdot M_i$ $\mathbb{P}$-a.s. for all $k, i \geq 1$ such that for every $m \geq 1$*

$$D^m f(Y) = \sum_{k\geq 1}\sum_{i\geq 1} G_i^{(k)} D^m \langle Y, e_i\rangle_{\mathcal{H}} e_k. \tag{5.32}$$

Moreover,

$$\|f(Y)\|_{\mathbb{D}^{1,2}(\mathcal{H})} \leq \|L\|_{\ell^2} \cdot \|M\|_{\ell^2} \cdot \|Y\|_{\mathbb{D}^{1,2}(\mathcal{H})}.$$

Proof. First, consider the case $f : \mathbb{R}^d \to \mathbb{R}^d$ for some $d \geq 1$, where Y is taking values in $\mathbb{R}^d$. Using the chain rule, see [32, Proposition 1.2.4], and the notion of Malliavin Differentiability in Definition 4.5, there exists a double-sequence $\{G_i^{(k)}\}_{1\leq k,i\leq d}$

of random variables with $G_i^{(k)} \leq L_k \cdot M_i$ $\mathbb{P}$-a.s. for all $1 \leq k, i \leq d$ such that for every $m \geq 1$

$$D^m f(Y) = \sum_{k=1}^{d} D^m f_k(Y)\widetilde{e}_k = \sum_{k=1}^{d}\sum_{i=1}^{d} G_i^{(k)} D^m \langle Y, \widetilde{e}_i \rangle \widetilde{e}_k. \tag{5.33}$$

Recall the change of basis operator $\tau : \mathcal{H} \to \ell^2$ defined in (5.7). Let now $f : \mathcal{H}_d \to \mathcal{H}_d$, where Y is taking values in $\mathcal{H}_d$. Define $g : \mathbb{R}^d \to \mathbb{R}^d$ by $g := \tau \circ f \circ \tau^{-1}$. Then g is Lipschitz continuous in the sense of (5.31) with associated Lipschitz sequences $L, M \in \ell^2$ and due to equality (5.33) we get the identity

$$\begin{aligned} \tau D^m f(Y) &= \tau \sum_{k=1}^{d} D^m f_k(Y) e_k = \sum_{k=1}^{d} D^m g_k(\tau Y)\widetilde{e}_k \\ &= \sum_{k=1}^{d}\sum_{i=1}^{d} G_i^{(k)} D^m \langle \tau Y, \widetilde{e}_i \rangle \widetilde{e}_k = \sum_{k=1}^{d}\sum_{i=1}^{d} G_i^{(k)} D^m \langle Y, e_i \rangle_{\mathcal{H}} \widetilde{e}_k \\ &= \tau \sum_{k=1}^{d}\sum_{i=1}^{d} G_i^{(k)} D^m \langle Y, e_i \rangle_{\mathcal{H}} e_k. \end{aligned}$$

Thus, equation (5.32) holds for $f : \mathcal{H}_d \to \mathcal{H}_d$. Let finally $f : \mathcal{H} \to \mathcal{H}$, where Y is taking values in $\mathcal{H}$. Recall the truncation operator $\pi_d : \mathcal{H} \to \mathcal{H}_d$ defined in (5.6). Since f is Lipschitz continuous, $f(\pi_d Y)$ converges to $f(Y)$ in $L^2(\Omega)$. Furthermore, we have for every $d \geq 1$ that

$$\begin{aligned} &\|\pi_d f(\pi_d Y)\|^2_{\mathbb{D}^{1,2}(\mathcal{H})} = \sum_{m\geq 1} \int_0^T \mathbb{E}\Big[\|D_s^m(\pi_d f(\pi_d Y))\|^2_{\mathcal{H}}\Big] ds \qquad (5.34) \\ &= \sum_{m\geq 1}\sum_{k=1}^{d} \int_0^T \mathbb{E}\left[\left|\sum_{i=1}^{d} G_i^{d,(k)} D_s^m \langle Y, e_i\rangle_{\mathcal{H}}\right|^2\right] ds \\ &\leq \|L\|^2_{\ell^2} \sum_{m\geq 1} \int_0^T \mathbb{E}\left[\left|\sum_{i=1}^{d} M_i D_s^m \langle Y, e_i\rangle_{\mathcal{H}}\right|^2\right] ds \\ &\leq \|L\|^2_{\ell^2} \cdot \|M\|^2_{\ell^2} \sum_{m\geq 1} \int_0^T \mathbb{E}\Big[\|D_s^m Y\|^2_{\mathcal{H}}\Big] ds = \|L\|^2_{\ell^2} \cdot \|M\|^2_{\ell^2} \cdot \|Y\|^2_{\mathbb{D}^{1,2}(\mathcal{H})} < \infty. \end{aligned}$$

Note that the double-sequence $\{G_i^{d,(k)}\}_{i\geq 1, k\geq 1}$ depends on $d \geq 1$. Nevertheless, $\|\pi_d f(\pi_d Y)\|_{\mathbb{D}^{1,2}(\mathcal{H})}$ is uniformly bounded in $d \geq 1$. Thus, due to [32, Lemma 1.2.3] and dominated convergence we have $f(Y) \in \mathbb{D}^{1,2}(\mathcal{H})$ and $D^m(\pi_d f(\pi_d Y))$ converges weakly to $D^m f(Y)$ for every $m \geq 1$. Moreover, the sequence $\{G_i^{d,(k)}\}_{d\geq 1}$ is bounded by $L_k \cdot M_i$ for every $k, i \geq 1$. Hence, for every $k, i \geq 1$ there exists a subsequence $\{G_i^{d_n,(k)}\}_{n\geq 1}$ which converges weakly to some random variable $\widetilde{G}_i^{(k)}$

which is bounded by $L_k \cdot M_i$. Summarizing we get that in $L^2([0,T]\times\Omega;\mathcal{H})$

$$
\begin{aligned}
D^m f(Y) &= \lim_{n\to\infty} \pi_{d_n} D^m f(\pi_{d_n} Y) = \lim_{n\to\infty} \sum_{k=1}^{d_n}\sum_{i=1}^{d_n} G_i^{d_n,(k)} D^m \langle Y, e_i\rangle_{\mathcal{H}} e_k \\
&= \sum_{k\geq 1}\sum_{i\geq 1} \widetilde{G}_i^{(k)} D^m \langle Y, e_i\rangle_{\mathcal{H}} e_k,
\end{aligned}
$$

where the last equality holds due to (5.34) and dominated convergence. $\square$

Define the class $\mathfrak{L}_0([0,T]\times\mathcal{H};\mathcal{H})$ by

$$
\begin{aligned}
&\mathfrak{L}_0([0,T]\times\mathcal{H};\mathcal{H}) = \\
&\qquad \left\{f \in \mathfrak{B}([0,T]\times\mathcal{H};\mathcal{H}) : f(t,\cdot) \in \mathcal{L}_0(\mathcal{H}) \text{ uniformly in } t\in[0,T]\right\},
\end{aligned}
$$

and note that $f(t,\cdot)\in\mathcal{L}_0(\mathcal{H})$ uniformly in $t\in[0,T]$ implies $f_k(t,\cdot)\in \mathrm{Lip}_{L_k}(\mathcal{H};\mathbb{R})$, $k\geq 1$, uniformly in $t\in[0,T]$ for some sequence $L\in\ell^2$. Thus, $\mathfrak{L}_0([0,T]\times\mathcal{H};\mathcal{H})\subset \mathfrak{L}([0,T]\times\mathcal{H};\mathcal{H})$.

Proposition 4.7 *Let $b\in\mathfrak{L}_0([0,T]\times\mathcal{H};\mathcal{H})$. Then the unique strong solution $(X_t)_{t\in[0,T]}$ of (5.3) is Malliavin differentiable.*

Proof. Recall the Picard iteration defined in (5.29)

$$
Y_t^n = x + \int_0^t b\left(s, Y_s^{n-1}\right) ds + \mathbb{B}_t,\ t\in[0,T],\ n\geq 1, \tag{5.35}
$$

and $Y^0 = x + \mathbb{B}$. We denote the k-th dimension of the infinite dimensional system (5.35) by $Y^{n,(k)} := \langle Y^n, e_k\rangle_{\mathcal{H}}$.

Using the Picard iteration (5.35), we show that for every step $n\geq 0$ the process Y^n is Malliavin differentiable. We prove this using induction. For $n=0$ we have that for all $t\in[0,T]$ using (5.15)

$$
\begin{aligned}
\left\|Y_t^0\right\|^2_{\mathbb{D}^{1,2}(\mathcal{H})} &= \sum_{m\geq 1}\int_0^T \mathbb{E}\left[\left\|D_s^m Y_t^0\right\|_{\mathcal{H}}^2\right] ds \\
&= \sum_{m\geq 1}\int_0^T \mathbb{E}\left[\left\|\sum_{k\geq 1}\lambda_k D_s^m B_t^{H_k} e_k\right\|_{\mathcal{H}}^2\right] ds \\
&= \sum_{m\geq 1}\int_0^T \mathbb{E}\left[\left\|\lambda_m D_s^m B_t^{H_m} e_m\right\|_{\mathcal{H}}^2\right] ds \\
&= \sum_{m\geq 1}\int_0^T \lambda_m^2 K_{H_m}^2(t,s) ds \\
&= \sum_{m\geq 1}\lambda_m^2 R_{H_m}(t,t) = \sum_{m\geq 1}\lambda_m^2 t^{2H_m} < \infty.
\end{aligned}
$$

Now suppose that $\|Y_t^n\|_{\mathbb{D}^{1,2}(\mathcal{H})} < \infty$ for $n \geq 0$. Due to Lemma 4.6 $b(t, Y_t^n)$ is in $\mathbb{D}^{1,2}(\mathcal{H})$ and we have for every $t \in [0,T]$ that

$$\|b(t, Y_t^n)\|_{\mathbb{D}^{1,2}(\mathcal{H})} \leq \|L\|_{\ell^2} \cdot \|M\|_{\ell^2} \cdot \|Y_t^n\|_{\mathbb{D}^{1,2}(\mathcal{H})} < \infty,$$

for some $L, M \in \ell^2$ independent of $n \geq 0$. Moreover, $\int_0^T b(r, Y_r^n) dr$ is Malliavin differentiable admitting for all $0 \leq s \leq T$ the representation

$$D_s^m \left(\int_0^T b(r, Y_r^n) dr \right) = \int_s^T D_s^m b(r, Y_r^n) dr.$$

Thus, we get for Y^{n+1} that

$$\begin{aligned}
\left\| Y_t^{n+1} \right\|_{\mathbb{D}^{1,2}(\mathcal{H})} &= \left\| \left(\int_0^T b(s, Y_s^n) ds + Y_t^0 \right) \right\|_{\mathbb{D}^{1,2}(\mathcal{H})} \\
&\leq \int_0^T \|b(s, Y_s^n)\|_{\mathbb{D}^{1,2}(\mathcal{H})} \, ds + \left\| Y_t^0 \right\|_{\mathbb{D}^{1,2}(\mathcal{H})} \\
&\leq \|L\|_{\ell^2} \cdot \|M\|_{\ell^2} \cdot \int_0^T \|Y_s^n\|_{\mathbb{D}^{1,2}(\mathcal{H})} \, ds + \left\| Y_t^0 \right\|_{\mathbb{D}^{1,2}(\mathcal{H})} < \infty.
\end{aligned}$$

Hence, Y^{n+1} is Malliavin differentiable in the sense of Definition 4.5. Moreover, we can find a positive constant A depending on L, M, λ and T such that

$$\|Y_t^n\|_{\mathbb{D}^{1,2}(\mathcal{H})} \leq \sum_{k=0}^n \frac{A^{k+1}}{k!} t^k \leq A \cdot e^{At}.$$

Consequently, $\|Y_t^n\|_{\mathbb{D}^{1,2}(\mathcal{H})}^2$ is uniformly bounded in $n \geq 0$ and therefore, since $Y^n \to X$ in $L^2([0,T] \times \Omega)$ and the Malliavin derivative is a closable operator, also X is Malliavin differentiable in the sense of Definition 4.5. □

Let us finally put the previous results together and show that SDE (5.24) has a unique Malliavin differentiable strong solution.

Corollary 4.8 *Let* $b^{d,\varepsilon} : [0,T] \times \mathcal{H} \to \mathcal{H}$ *be defined as in* (5.26). *Then, SDE* (5.24) *has a unique strong solution* $\left(X_t^{d,\varepsilon} \right)_{t \in [0,T]}$ *which is Malliavin differentiable. Furthermore, the Malliavin derivative* $D_s^m X_t^{d,\varepsilon}$ *has for* $0 \leq s < t \leq T$ *a.s. the representation*

$$\begin{aligned}
D_s^m X_t^{d,\varepsilon} &= \lambda_m K_{H_m}(t,s) e_m \\
&\quad + \lambda_m \sum_{n \geq 1} \int_{\Delta_{s,t}^n} K_{H_m}(u_1, s) \sum_{\eta_0, \dots \eta_{n-1}=1}^d \left(\prod_{j=1}^n \partial_{\eta_j} \widetilde{b}_{\eta_{j-1}}^{d,\varepsilon} \left(u_j, \tau X_{u_j}^{d,\varepsilon} \right) \right) e_{\eta_0} du,
\end{aligned} \tag{5.36}$$

where $\eta_n = m$ *and* $\widetilde{b}^{d,\varepsilon} : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ *is defined as in* (5.27).

Proof. If the drift function $b^{d,\varepsilon}$ is in the class $\mathfrak{L}_0([0,T] \times \mathcal{H}, \mathcal{H})$, then SDE (5.24) has a unique Malliavin differentiable strong solution by Proposition 4.4 and Proposition 4.7. Thus we merely need to show that $b^{d,\varepsilon}(t, \cdot) \in \mathcal{L}_0(\mathcal{H})$ uniformly in

$t \in [0,T]$. Let $t \in [0,T]$ and $y, z \in \mathcal{H}$. Then, using the triangular inequality and the mean-value theorem we get for all $1 \leq k \leq d$ that

$$\begin{aligned}\left|\left\langle b^{d,\varepsilon}(t,y) - b^{d,\varepsilon}(t,z), e_k\right\rangle_{\mathcal{H}}\right| &= \left|b_k^{d,\varepsilon}(t,y) - b_k^{d,\varepsilon}(t,z)\right| = \left|\widetilde{b}_k^{d,\varepsilon}(t,\tau^{-1}y) - \widetilde{b}_k^{d,\varepsilon}(t,\tau^{-1}z)\right| \\ &\leq \sum_{i=1}^d \left|\widetilde{b}_k^{d,\varepsilon}\left(t, \sum_{j=1}^{i-1} z_j \widetilde{e}_j + \sum_{j=i}^d y_j \widetilde{e}_j\right) - \widetilde{b}_k^{d,\varepsilon}\left(t, \sum_{j=1}^{i} z_j \widetilde{e}_j + \sum_{j=i+1}^d y_j \widetilde{e}_j\right)\right| \\ &\leq \sum_{i=1}^d \sup_{\xi \in \mathbb{R}^d} |\partial_i \widetilde{b}_k^{d,\varepsilon}(t,\xi)||y_i - z_i| = \sum_{i=1}^d \sup_{\xi \in \mathbb{R}^d} |\partial_i \widetilde{b}_k^{d,\varepsilon}(t,\xi)||\langle y - z, e_i\rangle|.\end{aligned}$$

Note that we can find sequences $\{L_k\}_{1 \leq k \leq d}$ and $\{M_i\}_{1 \leq i \leq d}$ such that for all $1 \leq k, i \leq d$ we have $\sup_{\xi \in \mathbb{R}^d} |\partial_i \widetilde{b}_k^{d,\varepsilon}(t,\xi)| \leq L_k \cdot M_i$. Hence, $b^{d,\varepsilon} \in \mathfrak{L}_0([0,T] \times \mathcal{H}; \mathcal{H})$.

It is left to show that representation (5.36) holds. First note that due to the definition of the Malliavin derivative of a random variable Y with values in $\mathcal{H}$, see Definition 4.5, we have that $D^m(\tau Y) = \tau D^m Y$, for all $m \geq 1$. Consequently, we get for $0 \leq s < t \leq T$ using Lemma 4.6 that the Malliavin derivative $D_s^m X_t^{d,\varepsilon}$ can be written as

$$D_s^m X_t^{d,\varepsilon} = \tau^{-1} D_s^m \widetilde{X}_t^{d,\varepsilon} = \int_s^t \nabla \widetilde{b}^{d,\varepsilon}\left(u, \widetilde{X}_u^{d,\varepsilon}\right) D_s^m X_u^{d,\varepsilon} du + D_s^m \mathbb{B}_t.$$

Iterating this step yields

$$D_s^m X_t^{d,\varepsilon} = \sum_{n \geq 1} \int_{\Delta_{s,t}^n} \left(\prod_{j=1}^n \nabla \widetilde{b}^{d,\varepsilon}\left(u_j, \widetilde{X}_{u_j}^{d,\varepsilon}\right)\right) \lambda_m K_{H_m}(u_1, s) e_m du + \lambda_m K_{H_m}(t,s) e_m.$$

Further note that

$$\nabla \widetilde{b}^{d,\varepsilon}\left(u_j, \widetilde{X}_{u_j}^{d,\varepsilon}\right) = \nabla\left(\sum_{k=1}^d \widetilde{b}_k^{d,\varepsilon}\left(u_j, \widetilde{X}_{u_j}^{d,\varepsilon}\right) e_k\right) = \sum_{l=1}^d \sum_{k=1}^d \partial_l \widetilde{b}_k^{d,\varepsilon}\left(u_j, \widetilde{X}_{u_j}^{d,\varepsilon}\right) e_k e_l^\top.$$

Thus, we get for every $n \geq 1$

$$\prod_{j=1}^n \nabla \widetilde{b}^{d,\varepsilon}\left(u_j, \widetilde{X}_{u_j}^{d,\varepsilon}\right) = \sum_{l=1}^d \sum_{k=1}^d \left(\sum_{\eta_1,\dots\eta_{n-1}=1}^d \prod_{j=1}^n \partial_{\eta_j} \widetilde{b}_{\eta_{j-1}}^{d,\varepsilon}\left(u_j, \widetilde{X}_{u_j}^{d,\varepsilon}\right)\right) e_k e_l^\top, \tag{5.37}$$

where $\eta_0 = k$ and $\eta_n = l$ and consequently, representation (5.36) holds. □

4.3. **Weak convergence.** In this step we show that the sequence of unique strong solutions $\{X^{d,\varepsilon}\}_{d \geq 1, \varepsilon > 0}$ of the approximating SDEs (5.24) converge weakly to the weak solution of (5.3) where $b \in \mathfrak{B}([0,T] \times \mathcal{H}; \mathcal{H})$.

Lemma 4.9 *Let* $b \in \mathfrak{B}([0,T] \times \mathcal{H}; \mathcal{H})$. *Furthermore, let* $(X_t)_{t \in [0,T]}$ *be the weak solution of* (5.3). *Consider the approximating sequence of strong solutions* $\{(X_t^{d,\varepsilon})_{t \in [0,T]}\}_{d \geq 1, \varepsilon > 0}$ *of SDEs* (5.24), *where* $b^{d,\varepsilon} : [0,T] \times \mathcal{H} \to \mathcal{H}$ *is defined as* *in* (5.26). *Then, for every* $t \in [0,T]$ *and for any bounded continuous function* $\phi : \mathcal{H} \to \mathbb{R}$

$$\phi(X_t^{d,\varepsilon}) \xrightarrow[d\to\infty,\varepsilon\to 0]{} \mathbb{E}\Big[\phi(X_t)\Big|\mathcal{F}_t^W\Big],$$

weakly in $L^2(\Omega, \mathcal{F}_t^W)$.

Proof. Using the Wiener transform

$$\mathcal{W}(Z)(f) := \mathbb{E}\left[Z\mathcal{E}\left(\int_0^T \langle f(s), dW_s\rangle_{\mathcal{H}}\right)\right],$$

of some random variable $Z \in L^2(\Omega, \mathcal{F}_T^W)$ in $f \in L^2([0,T];\mathcal{H})$, it suffices to show for any arbitrary $f \in L^2([0,T];\mathcal{H})$ that

$$\mathcal{W}(\phi(X_t^{d,\varepsilon}))(f) \xrightarrow[d\to\infty,\varepsilon\to 0]{} \mathcal{W}\left(\mathbb{E}\Big[\phi(X_t)\Big|\mathcal{F}_t^W\Big]\right)(f).$$

So, let $f \in L^2([0,T];\mathcal{H})$ be arbitrary, then by using Girsanov's theorem we get

$$\begin{aligned}
&\left|\mathcal{W}(\phi(X_t^{d,\varepsilon}))(f) - \mathcal{W}\left(\mathbb{E}\Big[\phi(X_t)\Big|\mathcal{F}_t^W\Big]\right)(f)\right| \\
&\quad= \Bigg|\mathbb{E}\left[\phi(\mathbb{B}_t^x)\mathcal{E}\left(\int_0^T \left\langle f(s) + \left(\sum_{k=1}^d K_{H_k}^{-1}\left(\int_0^\cdot b_k^{d,\varepsilon}(u,\mathbb{B}_u^x)\lambda_k^{-1}du\right)(s)e_k\right), dW_s\right\rangle_{\mathcal{H}}\right)\right] \\
&\qquad - \mathbb{E}\left[\phi(\mathbb{B}_t^x)\mathcal{E}\left(\int_0^T \left\langle f(s) + \left(\sum_{k\geq 1} K_{H_k}^{-1}\left(\int_0^\cdot b_k(u,\mathbb{B}_u^x)\lambda_k^{-1}du\right)(s)e_k\right), dW_s\right\rangle_{\mathcal{H}}\right)\right]\Bigg| \\
&\quad\lesssim \mathbb{E}\Bigg[\Bigg|\mathcal{E}\left(\int_0^T \left\langle f(s) + \left(\sum_{k=1}^d K_{H_k}^{-1}\left(\int_0^\cdot b_k^{d,\varepsilon}(u,\mathbb{B}_u^x)\lambda_k^{-1}du\right)(s)e_k\right), dW_s\right\rangle_{\mathcal{H}}\right) \\
&\qquad - \mathcal{E}\left(\int_0^T \left\langle f(s) + \left(\sum_{k\geq 1} K_{H_k}^{-1}\left(\int_0^\cdot b_k(u,\mathbb{B}_u^x)\lambda_k^{-1}du\right)(s)e_k\right), dW_s\right\rangle_{\mathcal{H}}\right)\Bigg|\Bigg].
\end{aligned}$$

Using the inequality

$$|e^x - e^y| \leq |x-y|\,(e^x + e^y) \quad \forall x, y \in \mathbb{R},$$

we get

$$\begin{aligned}
&\left|\mathcal{W}(\phi(X_t^{d,\varepsilon}))(f) - \mathcal{W}\left(\mathbb{E}\Big[\phi(X_t)\Big|\mathcal{F}_t^W\Big]\right)(f)\right| \\
&\quad\lesssim \mathbb{E}\Bigg[\Bigg|\int_0^T \left\langle f(s) + \left(\sum_{k=1}^d K_{H_k}^{-1}\left(\int_0^\cdot b_k^{d,\varepsilon}(u,\mathbb{B}_u^x)\lambda_k^{-1}du\right)(s)e_k\right), dW_s\right\rangle_{\mathcal{H}} \\
&\qquad - \int_0^T \left\langle f(s) + \left(\sum_{k\geq 1} K_{H_k}^{-1}\left(\int_0^\cdot b_k(u,\mathbb{B}_u^x)\lambda_k^{-1}du\right)(s)e_k\right), dW_s\right\rangle_{\mathcal{H}}\Bigg|\Bigg] \\
&\qquad + \mathbb{E}\Bigg[\Bigg|\int_0^T \left\langle \left(f(s) + \left(\sum_{k=1}^d K_{H_k}^{-1}\left(\int_0^\cdot b_k^{d,\varepsilon}(u,\mathbb{B}_u^x)\lambda_k^{-1}du\right)(s)e_k\right)\right)^2, ds\right\rangle_{\mathcal{H}}
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^T \left\langle \left(f(s) + \left(\sum_{k\geq 1} K_{H_k}^{-1} \left(\int_0^{\cdot} b_k(u, \mathbb{B}_u^x) \lambda_k^{-1} du \right)(s) e_k \right) \right)^2, ds \right\rangle_{\mathcal{H}} \Bigg| \Bigg] \\
\leq\ & \mathbb{E}\left[\left| \sum_{k=1}^d \int_0^T K_{H_k}^{-1} \left(\int_0^{\cdot} b_k^{d,\varepsilon}(u, \mathbb{B}_u^x)\lambda_k^{-1} - b_k(u,\mathbb{B}_u^x)\lambda_k^{-1} du \right)(s) dW_s^{(k)} \right.\right. \\
&\left.\left. - \sum_{k\geq d+1} \int_0^T K_{H_k}^{-1} \left(\int_0^{\cdot} b_k(u, \mathbb{B}_u^x) \lambda_k^{-1} du \right)(s) dW_s^{(k)} \right| \right] \\
&+ A_{d,\varepsilon}(f),
\end{aligned}
$$

where

$$
\begin{aligned}
A_{d,\varepsilon}(f) := \mathbb{E}\Bigg[\Bigg| & \int_0^T \left\langle \left(f(s) + \left(\sum_{k=1}^d K_{H_k}^{-1} \left(\int_0^{\cdot} b_k^{d,\varepsilon}(u, \mathbb{B}_u^x) \lambda_k^{-1} du \right)(s) e_k \right) \right)^2, ds \right\rangle_{\mathcal{H}} \\
&- \int_0^T \left\langle \left(f(s) + \left(\sum_{k\geq 1} K_{H_k}^{-1} \left(\int_0^{\cdot} b_k(u, \mathbb{B}_u^x) \lambda_k^{-1} du \right)(s) e_k \right) \right)^2, ds \right\rangle_{\mathcal{H}} \Bigg| \Bigg].
\end{aligned}
$$

For every $k \geq 1$, we get with representation (5.18) that

$$
\begin{aligned}
\mathcal{K}_{H_k}^{-1}(d,\varepsilon,s) &:= K_{H_k}^{-1} \left(\int_0^{\cdot} b_k^{d,\varepsilon}(u, \mathbb{B}_u^x)\lambda_k^{-1} - b_k(u,\mathbb{B}_u^x)\lambda_k^{-1} du \right)(s) \\
&= s^{H_k - \frac{1}{2}} I_{0+}^{\frac{1}{2}-H_k} s^{\frac{1}{2}-H_k} \left(b_k^{d,\varepsilon}(s, \mathbb{B}_s^x) - b_k(s, \mathbb{B}_s^x) \right) \lambda_k^{-1} \\
&= \frac{\lambda_k^{-1}}{\Gamma\left(\frac{1}{2} - H_k\right)} \int_0^s \left(\frac{u}{s} \right)^{\frac{1}{2}-H_k} (s-u)^{-\frac{1}{2}-H_k} \left(b_k^{d,\varepsilon}(u, \mathbb{B}_u^x) - b_k(u, \mathbb{B}_u^x) \right) du,
\end{aligned}
$$

which is bounded by

$$
\begin{aligned}
\left| \mathcal{K}_{H_k}^{-1}(d,\varepsilon,s) \right| &\leq 2 \frac{C_k}{\Gamma\left(\frac{1}{2} - H_k\right)} \int_0^s \left(\frac{u}{s} \right)^{\frac{1}{2}-H_k} (s-u)^{-\frac{1}{2}-H_k} du \\
&= 2 \frac{C_k}{\Gamma\left(\frac{1}{2} - H_k\right)} s^{\frac{1}{2}-H_k} \beta\left(\frac{3}{2} - H_k, \frac{1}{2} - H_k \right) \lesssim C_k.
\end{aligned}
$$

Consequently, we get for every $d \geq 1$ using the Burkholder-Davis-Gundy inequality that

$$
\mathbb{E}\left[\left| \sum_{k=1}^d \int_0^T \mathcal{K}_{H_k}^{-1}(d,\varepsilon,s) dW_s^{(k)} \right| \right] \leq \sum_{k=1}^d \mathbb{E}\left[\int_0^T \left| \mathcal{K}_{H_k}^{-1}(d,\varepsilon,s) \right|^2 ds \right]^{\frac{1}{2}} \lesssim \sum_{k\geq 1} C_k < \infty.
$$

Hence, by dominated convergence

$$
\lim_{d\to\infty} \lim_{\varepsilon\to 0} \mathbb{E}\left[\left| \sum_{k=1}^d \int_0^T \mathcal{K}_{H_k}^{-1}(d,\varepsilon,s) dW_s^{(k)} \right| \right] = 0.
$$

Equivalently, we have

$$\mathbb{E}\left[\left|\int_0^T \sum_{k\geq d+1} K_{H_k}^{-1}\left(\int_0^\cdot b_k(u,\mathbb{B}_u^x)\lambda_k^{-1}du\right)(s)dW_s^{(k)}\right|\right] \lesssim \sum_{k\geq 1} C_k < \infty.$$

Thus, again by dominated convergence

$$\lim_{d\to\infty}\lim_{\varepsilon\to 0}\mathbb{E}\left[\left|\int_0^T \sum_{k\geq d+1} K_{H_k}^{-1}\left(\int_0^\cdot b_k(u,\mathbb{B}_u^x)\lambda_k^{-1}du\right)(s)dW_s^{(k)}\right|\right] = 0.$$

Similarly, one can show that $A_{d,\varepsilon}(f)$ vanishes for every $f\in L^2([0,T];\mathcal{H})$ as $\varepsilon\to 0$ and $d\to\infty$. Consequently, $\phi(X_t^{d,\varepsilon}) \xrightarrow[d\to\infty,\varepsilon\to 0]{} \mathbb{E}\Big[\phi(X_t)\Big|\mathcal{F}_t^W\Big]$ weakly in $L^2(\Omega,\mathcal{F}_t^W)$. $\square$

4.4. **Application of the compactness criterion.**

Theorem 4.10 *The double-sequence* $\left\{X_t^{d,\varepsilon}\right\}_{d\geq 1,\varepsilon>0}$ *of strong solutions of SDE* (5.24) *is relatively compact in* $L^2(\Omega,\mathcal{F}_t^W)$.

Proof. We are aiming at applying the compactness criterion given in Theorem A.3. Therefore, let $0<\alpha_m<\beta_m<\frac{1}{2}$ and $\gamma_m>0$ for all $m\geq 1$ and define the sequence $\mu_{s,m}=2^{-i\alpha_m}\gamma_m$, if $s=2^i+j$, $i\geq 0$, $0\leq j\leq 2^i$, $m\geq 1$ where $\mu_{s,m}\longrightarrow 0$ for $s,m\longrightarrow\infty$. We have to check that there exists a uniform constant C such that for all $\{X_t^{d,\varepsilon}\}_{d\geq 1,\varepsilon>0}$

$$\left\|X_t^{d,\varepsilon}\right\|_{L^2(\Omega;\mathcal{H})}\leq C, \tag{5.38}$$

$$\sum_{m\geq 1}\gamma_m^{-2}\left\|D^m X_t^{d,\varepsilon}\right\|^2_{L^2(\Omega;L^2([0,T];\mathcal{H}))}\leq C,$$

and

$$\sum_{m\geq 1}\frac{1}{(1-2^{-2(\beta_m-\alpha_m)})\gamma_m^2}\int_0^T\int_0^T\frac{\left\|D_s^m X_t^{d,\varepsilon}-D_u^m X_t^{d,\varepsilon}\right\|^2_{L^2(\Omega;\mathcal{H})}}{|s-u|^{1+2\beta_m}}dsdu\leq C. \tag{5.39}$$

Note first that (5.38) is fulfilled due to the uniform boundedness of $\{b^{d,\varepsilon}\}_{d\geq 1,\varepsilon>0}$ and the definition of the process $(\mathbb{B}_t)_{t\in[0,T]}$, see (5.20).

Next we show uniform boundedness of (5.39). Note first that under the assumption $u\leq s$ we have

$$\begin{aligned}
D_s^m X_t^{d,\varepsilon}-D_u^m X_t^{d,\varepsilon} &= \lambda_m\left(K_{H_m}(t,s)-K_{H_m}(t,u)\right)e_m\\
&\quad+\int_s^t\nabla\widetilde{b}^{d,\varepsilon}(v,\widetilde{X}_v^{d,\varepsilon})D_s^m X_v^{d,\varepsilon}dv-\int_u^t\nabla\widetilde{b}^{d,\varepsilon}(v,\widetilde{X}_v^{d,\varepsilon})D_u^m X_v^{d,\varepsilon}dv\\
&=\lambda_m\left(K_{H_m}(t,s)-K_{H_m}(t,u)\right)e_m-\int_u^s\nabla\widetilde{b}^{d,\varepsilon}(v,\widetilde{X}_v^{d,\varepsilon})D_u^m X_v^{d,\varepsilon}dv\\
&\quad+\int_s^t\nabla\widetilde{b}^{d,\varepsilon}(v,\widetilde{X}_v^{d,\varepsilon})\left(D_s^m X_v^{d,\varepsilon}-D_u^m X_v^{d,\varepsilon}\right)dv
\end{aligned}$$

$$= \lambda_m \left(K_{H_m}(t,s) - K_{H_m}(t,u)\right) e_m - D_u^m X_s^{d,\varepsilon} + \lambda_m K_{H_m}(s,u) e_m$$
$$+ \int_s^t \nabla \widetilde{b}^{d,\varepsilon}(v, \widetilde{X}_v^{d,\varepsilon}) \left(D_s^m X_v^{d,\varepsilon} - D_u^m X_v^{d,\varepsilon}\right) dv.$$

Using iteration we obtain the representation

$$\begin{aligned} D_s^m X_t^{d,\varepsilon} - D_u^m X_t^{d,\varepsilon} &= \lambda_m \left(K_{H_m}(t,s) - K_{H_m}(t,u)\right) e_m \\ &+ \lambda_m \sum_{n\geq 1} \int_{\Delta_{s,t}^n} \prod_{j=1}^n \nabla \widetilde{b}^{d,\varepsilon}(v_j, \widetilde{X}_{v_j}^{d,\varepsilon}) \left(K_{H_m}(v_1,s) - K_{H_m}(v_1,u)\right) e_m dv \\ &+ \left(\mathrm{Id} + \sum_{n\geq 1} \int_{\Delta_{s,t}^n} \prod_{j=1}^n \nabla \widetilde{b}^{d,\varepsilon}(v_j, \widetilde{X}_{v_j}^{d,\varepsilon}) dv\right) \left(\lambda_m K_{H_m}(s,u) e_m - D_u^m X_s^{d,\varepsilon}\right), \end{aligned}$$

where by Corollary 4.8

$$\begin{aligned} &\left(\lambda_m K_{H_m}(s,u) e_m - D_u^m X_s^{d,\varepsilon}\right) = \\ &\quad - \lambda_m \sum_{n\geq 1} \int_{\Delta_{u,s}^n} K_{H_m}(v_1,u) \sum_{\eta_0,\dots,\eta_{n-1}=1}^d \prod_{j=1}^n \partial_{\eta_j} \widetilde{b}_{\eta_{j-1}}^{d,\varepsilon}(v_j, \widetilde{X}_{v_j}^{d,\varepsilon}) e_{\eta_0} dv. \end{aligned}$$

Consequently, we get due to (5.37) that

$$D_s^m X_t^{d,\varepsilon} - D_u^m X_t^{d,\varepsilon} = \lambda_m \left(\mathfrak{I}_1 + \mathfrak{I}_2 + \mathfrak{I}_3\right),$$

where

$$\begin{aligned} \mathfrak{I}_1 &:= \left(K_{H_m}(t,s) - K_{H_m}(t,u)\right) e_m, \\ \mathfrak{I}_2 &:= \sum_{n\geq 1} \int_{\Delta_{s,t}^n} \left(K_{H_m}(v_1,s) - K_{H_m}(v_1,u)\right) \sum_{\eta_0,\dots,\eta_{n-1}=1}^d \prod_{j=1}^n \partial_{\eta_j} \widetilde{b}_{\eta_{j-1}}^{d,\varepsilon}(v_j, \widetilde{X}_{v_j}^{d,\varepsilon}) e_{\eta_0} dv, \\ \mathfrak{I}_3 &:= - \left(\mathrm{Id} + \sum_{n\geq 1} \int_{\Delta_{s,t}^n} \sum_{\eta_0,\dots,\eta_{n-1}=1}^d \prod_{j=1}^n \partial_{\eta_j} \widetilde{b}_{\eta_{j-1}}^{d,\varepsilon}(v_j, \widetilde{X}_{v_j}^{d,\varepsilon}) dv\right) \\ &\quad \times \sum_{n\geq 1} \int_{\Delta_{u,s}^n} K_{H_m}(v_1,u) \sum_{\eta_0,\dots,\eta_{n-1}=1}^d \prod_{j=1}^n \partial_{\eta_j} \widetilde{b}_{\eta_{j-1}}^{d,\varepsilon}(v_j, \widetilde{X}_{v_j}^{d,\varepsilon}) e_{\eta_0} dv. \end{aligned}$$

In the following we consider each $\mathfrak{I}_i$, $i = 1,2,3$, separately starting with the first. Due to Lemma B.3 there exists $\beta_1 \in \left(0, \frac{1}{2}\right)$ and a constant $K_1 > 0$ such that

$$\int_0^t \int_0^t \frac{\|\mathfrak{I}_1\|_{L^2(\Omega;\mathcal{H})}^2}{|s-u|^{1+2\beta_1}} ds du = \int_0^t \int_0^t \frac{|K_{H_m}(t,s) - K_{H_m}(t,u)|}{|s-u|^{1+2\beta_1}} ds du \leq K_1 < \infty.$$

Consider now $\mathfrak{I}_2$. Define the density $\mathcal{E}_t^d$ by

$$\mathcal{E}_t^d := \exp\left\{\sum_{k=1}^d \left(\int_0^t K_{H_k}^{-1} \left(\int_0^\cdot b_k^{d,\varepsilon}\left(u, X_u^{d,\varepsilon}\right) \lambda_k^{-1} du\right)(s) dW_s^{(k)}\right.\right.$$

$$-\frac{1}{2}\int_0^t K_{H_k}^{-1}\left(\int_0^{\cdot} b_k^{d,\varepsilon}\left(u, X_u^{d,\varepsilon}\right)\lambda_k^{-1}du\right)^2(s)ds\right)\Bigg\}.$$

Then applying Girsanov's theorem 2.2, monotone convergence and noting that $\sup_{d\geq 1}\sup_{t\in[0,T]]}\|\mathcal{E}_t^d\|_{L^4(\Omega)} < \infty$ yields

$$\begin{aligned}
&\|\mathfrak{I}_2\|^2_{L^2(\Omega;\mathcal{H})}\\
&\leq \sum_{n\geq 1}\sum_{\eta_0,\ldots\eta_{n-1}=1}^{d}\left\|\mathcal{E}_t^d\int_{\Delta^n_{s,t}}\left(K_{H_m}(v_1,s)-K_{H_m}(v_1,u)\right)\prod_{j=1}^n\partial_{\eta_j}\widetilde{b}^{d,\varepsilon}_{\eta_{j-1}}\left(v_j,\tau\mathbb{B}^x_{v_j}\right)dv\right\|^2_{L^2(\Omega)}\\
&\lesssim \sum_{n\geq 1}\sum_{\eta_0,\ldots\eta_{n-1}=1}^{d}\left\|\int_{\Delta^n_{s,t}}\left(K_{H_m}(v_1,s)-K_{H_m}(v_1,u)\right)\prod_{j=1}^n\partial_{\eta_j}\widetilde{b}^{d,\varepsilon}_{\eta_{j-1}}\left(v_j,\tau\mathbb{B}^x_{v_j}\right)dv\right\|^2_{L^4(\Omega)}.
\end{aligned}$$

Using equation (5.9) yields that

$$|\mathfrak{A}_2|^2 := \left|\int_{\Delta^n_{s,t}}\left(K_{H_m}(v_1,s)-K_{H_m}(v_1,u)\right)\prod_{j=1}^n\partial_{\eta_j}\widetilde{b}^{d,\varepsilon}_{\eta_{j-1}}\left(v_j,\tau\mathbb{B}^x_{v_j}\right)dv\right|^2$$

can be written as

$$|\mathfrak{A}_2|^2 = \sum_{\sigma\in\mathcal{S}(n,n)}\int_{\Delta^{2n}_{s,t}}\left(\prod_{j=1}^{2n}g_{[\sigma(j)]}\left(v_j,\tau\mathbb{B}^x_{v_j}\right)\right)\left(\prod_{i=0}^{1}\left(K_{H_m}(v_{(in+1)},s)-K_{H_m}(v_{(in+1)},u)\right)\right)dv$$

where for $j=1,\ldots,n$

$$g_j\left(\cdot,\tau\mathbb{B}^x_{\cdot}\right) = \partial_{\eta_j}\widetilde{b}^{d,\varepsilon}_{\eta_{j-1}}\left(\cdot,\tau\mathbb{B}^x_{\cdot}\right)$$

Repeating the application of (5.9) yields

$$|\mathfrak{A}_2|^4 = \sum_{\sigma\in\mathcal{S}(4;n)}\int_{\Delta^{4n}_{s,t}}\left(\prod_{j=1}^{4n}g_{[\sigma(j)]}\left(v_j,\tau\mathbb{B}^x_{v_j}\right)\right)\left(\prod_{i=0}^{3}\left(K_{H_m}(v_{(in+1)},s)-K_{H_m}(v_{(in+1)},u)\right)\right)dv.$$

Defining $f_j^{d,\varepsilon}(t,\widetilde{y}) := \widetilde{b}^{d,\varepsilon}_{\eta_{j-1}}\left(t,\sqrt{Q}\sqrt{\mathcal{K}}\widetilde{y}\right)$ permits the use of Proposition B.2 with $\sum_{j=1}^{4n}\varepsilon_j = 4$, $|\alpha_j| = 1$ for all $1\leq j\leq 4n$ and thus $|\alpha| = 4n$. Consequently, we get using the assumptions on H and b that

$$\begin{aligned}
\mathbb{E}\left[|\mathfrak{A}_2|^4\right] &= \left\|\int_{\Delta^n_{s,t}}\left(K_{H_m}(v_1,s)-K_{H_m}(v_1,u)\right)\prod_{j=1}^n\partial_{\eta_j}b^{d,\varepsilon}_{\eta_{j-1}}\left(v_j,\tau\mathbb{B}^x_{v_j}\right)dv\right\|^4_{L^4(\Omega)}\\
&\leq \#\mathcal{S}(4;n)\frac{K^{4n}_{d,H}\cdot T^{\frac{|\alpha|}{12}}}{\sqrt{2\pi}^{4dn}}\left(C_{H_m,T}\left(\frac{s-u}{su}\right)^{\gamma_m}s^{(H_m-\frac{1}{2}-\gamma_m)}\right)^{\sum_{j=1}^{4n}\varepsilon_j}\\
&\quad\times\prod_{j=1}^n\left\|\widetilde{b}^{d,\varepsilon}_{\eta_{j-1}}(\cdot,\sqrt{Q}\sqrt{\mathcal{K}}z_j)\right\|^4_{L^1(\mathbb{R}^d;L^\infty([0,T]))}
\end{aligned}$$

$$
\begin{aligned}
&\times \frac{\left(\prod_{k=1}^{d}\left(2\left|\alpha^{(k)}\right|\right)!\right)^{\frac{1}{4}}(t-s)^{-\sum_{k=1}^{d}H_k\left(4n+2\left|\alpha^{(k)}\right|\right)+\left(H_m-\frac{1}{2}-\gamma_m\right)\sum_{j=1}^{4n}\varepsilon_j+4n}}{\Gamma(8n-\sum_{k=1}^{d}H_k(8n+4\left|\alpha^{(k)}\right|)+2(H_m-\frac{1}{2}-\gamma_m)\sum_{j=1}^{4n}\varepsilon_j)^{\frac{1}{2}}} \\
&\leq 2^{8n}\frac{K_{d,H}^{4n}\cdot T^{\frac{n}{3}}}{\sqrt{2\pi}^{4dn}}C_{H_m,T}^{4}\prod_{j=1}^{n}D_{\eta_{j-1}}^{4}\lambda_{\eta_{j-1}}^{4} \\
&\times\left(\frac{s-u}{su}\right)^{4\gamma_m}s^{4\left(H_m-\frac{1}{2}-\gamma_m\right)}(t-s)^{4\left(H_m-\frac{1}{2}-\gamma_m\right)}T^{4n}S_n,
\end{aligned}
$$

where

$$
S_n=\sup_{\eta}\frac{\left(\prod_{k=1}^{d}\left(2\left|\alpha^{(k)}\right|\right)!\right)^{\frac{1}{4}}}{\Gamma\left(8n-\sum_{k=1}^{d}H_k\left(8n+4\left|\alpha^{(k)}\right|\right)+8\left(H_m-\frac{1}{2}-\gamma_m\right)\right)^{\frac{1}{2}}}.
$$

For $n\geq 1$ we have due to the assumptions on H that

$$
\begin{aligned}
A_n&:=8n-\sum_{k=1}^{d}H_k\left(8n+4\left|\alpha^{(k)}\right|\right)+8\left(H_m-\frac{1}{2}-\gamma_m\right) \\
&\geq 8n-8n\|H\|_{\ell^1}-16n\sup_{k\geq 1}|H_k|-4>\frac{16}{3}n-4>0.
\end{aligned}
$$

Thus, we have for n sufficiently large that

$$
\Gamma(A_n)\geq\Gamma\left(\frac{16}{3}n-4\right)\sim\Gamma\left(\frac{16}{3}n+1\right)\left(\frac{16}{3}n\right)^{-4},
$$

and therefore by the approximations in Remark B.7

$$
\begin{aligned}
S_n&\leq\frac{\left(\prod_{k=1}^{d}\left(2\left|\alpha^{(k)}\right|\right)!\right)^{\frac{1}{4}}}{\Gamma\left(8n-\sum_{k=1}^{d}H_k\left(8n+4\left|\alpha^{(k)}\right|\right)+8\left(H_m-\frac{1}{2}-\gamma_m\right)\right)^{\frac{1}{2}}} \\
&\sim\frac{(2\pi)^{\frac{d}{8}}e^{\frac{n}{2}}((10n)!)^{\frac{1}{4}}\left(\frac{16}{3}n\right)^2}{(20\pi n)^{\frac{1}{8}}\,\Gamma\left(\frac{16}{3}n+1\right)^{\frac{1}{2}}} \\
&\leq C^n\frac{(2\pi)^{\frac{d}{8}}((10n)!)^{\frac{1}{4}}n^{\frac{15}{8}}}{\Gamma\left(\frac{16}{3}n+1\right)^{\frac{1}{2}}},
\end{aligned}
$$

where $C>0$ is a constant which may in the following vary from line to line. Using Stirling's formula we have moreover that

$$
\frac{(10n)!}{\Gamma\left(\frac{16}{3}n+1\right)^2}\leq\frac{e^{\frac{1}{120n}}\sqrt{20\pi n}\left(\frac{10n}{e}\right)^{10n}}{\frac{32}{3}\pi n\left(\frac{\frac{16}{3}n}{e}\right)^{\frac{32}{3}n}}\leq\frac{C^n}{\sqrt{\frac{4}{3}n}}\left(\frac{2}{3}n\right)^{-\frac{2}{3}n}\leq\frac{C^n}{\Gamma\left(\frac{2}{3}n+1\right)}.
$$

Consequently, we have for S_n that

$$S_n \sim C^n \left(2\pi\right)^{\frac{d}{8}} n^{\frac{15}{8}} \left(\frac{1}{\Gamma\left(\frac{2}{3}n+1\right)}\right)^{\frac{1}{4}}.$$

Furthermore, using Lemma C.4 we have for every $n \geq 1$ that

$$\sum_{\eta_0,\dots\eta_{n-1}=1}^{d} \prod_{j=1}^{n} D^4_{\eta_{j-1}} \lambda^4_{\eta_{j-1}} = \left(\sum_{k=1}^{d} D^4_k \lambda^4_k\right)^n.$$

Moreover, due to the assumptions on H there exists a finite constant $K > 0$ which is independent of d and H such that $K_{d,H} \leq K$, cf. (5.61). Consequently, there exists a constant $C > 0$ independent of d, ε and n such that for n sufficiently large

$$\begin{aligned}\mathcal{D}_n^2 :=& \sum_{\eta_0,\dots\eta_{n-1}=1}^{d} 2^{8n} \frac{K_{d,H}^{4n} \cdot T^{\frac{n}{3}}}{\sqrt{2\pi}^{4dn}} \left(\prod_{j=1}^{n} D^4_{\eta_{j-1}} \lambda^4_{\eta_{j-1}}\right) T^{4n} S_n \\ &\sim \left(\frac{n^{\frac{15}{2}} C^n}{\Gamma\left(\frac{2}{3}n+1\right)}\right)^{\frac{1}{4}}\end{aligned}$$

and thus due to the comparison test

$$\sum_{n\geq 1} \mathcal{D}_n < \infty.$$

Hence, there exists a constant $C_2 > 0$ independent of d and ε such that

$$\|\mathfrak{I}_2\|^2_{L^2(\Omega;\mathcal{H})} \leq C_2 C^4_{H_m,T} \left(\frac{s-u}{su}\right)^{2\gamma_m} s^{2(H_m-\frac{1}{2}-\gamma_m)} (t-s)^{2\left(H_m-\frac{1}{2}-\gamma_m\right)},$$

and thus we can find a $\beta_2 \in \left(0, \frac{1}{2}\right)$ sufficiently small such that

$$\int_0^t \int_0^t \frac{\|\mathfrak{I}_2\|^2_{L^2(\Omega;\mathcal{H})}}{|s-u|^{1+2\beta_2}} ds du \lesssim C^4_{H_m,T} < \infty.$$

Equivalently, we can show for $\mathfrak{I}_3$ that there exists a $\beta_3 \in \left(0, \frac{1}{2}\right)$ such that

$$\int_0^t \int_0^t \frac{\|\mathfrak{I}_3\|^2_{L^2(\Omega;\mathcal{H})}}{|s-u|^{1+2\beta_2}} ds du \lesssim C^4_{H_m,T} < \infty,$$

where $C_{H_m,T} = C \cdot c_{H_m}$ due to Lemma B.4. Here, c_{H_m} is the constant in (5.14). Thus, we can find a constant $\tilde{C} > 0$ independent of H_m such that $\sup_{H\in(0,\frac{1}{6})} C_{H,T} \leq C < \infty$. Finally, we get with $\beta_m := \min\{\beta_1, \beta_2, \beta_3\}$ that we can find γ_m, $m \geq 1$, such that

$$\sum_{m\geq 1} \frac{1}{(1-2^{-2(\beta_m-\alpha_m)})\gamma_m^2} \int_0^t \int_0^t \frac{\left\|D^m_s X^{d,\varepsilon}_t - D^m_u X^{d,\varepsilon}_t\right\|^2_{L^2(\Omega;\mathcal{H})}}{|s-u|^{1+2\beta_m}} ds du$$

$$\leq \sum_{m\geq 1} \frac{1}{(1-2^{-2(\beta_m-\alpha_m)})\gamma_m^2} \int_0^t \int_0^t \frac{\lambda_m^2 \sum_{l=1}^3 \|\mathfrak{I}_l\|^2_{L^2(\Omega;\mathcal{H})}}{|s-u|^{1+2\beta_m}} ds du$$
$$\lesssim \sum_{m\geq 1} \frac{\lambda_m^2 \tilde{C}^4}{(1-2^{-2(\beta_m-\alpha_m)})\gamma_m^2} < \infty,$$

uniformly in $d \geq 1$ and $\varepsilon > 0$. Similarly, we can show that

$$\sum_{m\geq 1} \gamma_m^{-2} \left\| D^m X_t^{d,\varepsilon} \right\|^2_{L^2(\Omega;L^2([0,1];\mathcal{H}))} < \infty \tag{5.40}$$

uniformly in $d \geq 1$ and $\varepsilon > 0$ and consequently the compactness criterion Theorem A.3 yields the result. □

4.5. $\mathbb{F}^{\mathbb{B}}$ **adaptedness and strong solution.** Finally, we can state and prove the main statement of this paper

Theorem 4.11 *Let* $b \in \mathfrak{B}([0,T]\times\mathcal{H};\mathcal{H})$. *Then SDE* (5.3) *has a unique Malliavin differentiable strong solution.*

Proof. Let $(X_t)_{t\in[0,T]}$ be a weak solution of SDE (5.3) which is unique in law due to Proposition 3.5. Due to Lemma 4.9 we know that for every bounded globally Lipschitz continuous function $\phi : \mathcal{H} \to \mathbb{R}$

$$\phi(X_t^{d,\varepsilon}) \xrightarrow[\varepsilon\to 0,\ d\to\infty]{} \mathbb{E}\Big[\phi(X_t)|\mathcal{F}_t^W\Big]$$

weakly in $L^2(\Omega, \mathcal{F}_t^W)$. Furthermore, by Theorem 4.10 there exist subsequences $\{d_k\}_{k\geq 1}$ and $\{\varepsilon_n\}_{n\geq 1}$ such that

$$\phi(X_t^{d_k,\varepsilon_n}) \xrightarrow[n\to\infty,\ d\to\infty]{} \phi\left(\mathbb{E}\Big[X_t|\mathcal{F}_t^W\Big]\right)$$

strongly in $L^2(\Omega, \mathcal{F}_t^W)$. Uniqueness of the limit yields that X_t is $\mathcal{F}_t^W$–measurable for all $t \in [0,T]$. Since $\mathbb{F}^W = \mathbb{F}^{\mathbb{B}}$, we get that $(X_t)_{t\in[0,T]}$ is a unique strong solution of SDE (5.3). Malliavin differentiability follows by (5.40) and noting that the estimate holds also for $\gamma_m \equiv 1$. □

5. Example

In this section we give an example of a drift function $b \in \mathfrak{B}([0,T]\times\mathcal{H};\mathcal{H})$ to show that the class does not merely contain the null function.

Let $f_k \in L^1(\ell^2; L^\infty([0,T];\ell^2))$, $k \geq 1$, i.e. for all $k \geq 1$ we have for all $z \in \ell^2$

$$\sup_{t\in[0,T]} |f_k(t,z)| \leq C_k^f < \infty \qquad \sup_{d\geq 1} \int_{\mathbb{R}^d} \sup_{t\in[0,T]} |f_k(t,z)| dz \leq D_k^f < \infty, \tag{5.41}$$

such that $C^f, D^f \in \ell^1$ and define for every $k \geq 1$ an operator $A_k : \mathcal{H} \to \mathcal{H}$ which is invertible on $A_k\mathcal{H}$ such that for all $k \geq 1$

$$\det\left(A_k^{-1}\sqrt{Q}^{-1}\sqrt{\mathcal{K}}^{-1}\right) \leq \mathcal{D}_k^A < \infty,$$

where $\mathcal{D}^A \in \ell^1$. Then, we define

$$b_k(t,y) := f_k(t, \tau^{-1} A_k y).$$

This yields

$$\begin{aligned}
&\sup_{t\in[0,T]} |b_k(t,y)| = \sup_{t\in[0,T]} |f_k(t,\tau^{-1}A_k y)| \leq C_k^f, \\
&\int_{\mathcal{H}} \sup_{t\in[0,T]} \left| b_k\left(t, \sqrt{Q}\sqrt{\mathcal{K}} y\right)\right| dy = \int_{\mathcal{H}} \sup_{t\in[0,T]} \left| f_k\left(t, \tau^{-1} A_k \sqrt{Q}\sqrt{\mathcal{K}} y\right)\right| dy \\
&\quad = \int_{\tau^{-1}A_k\mathcal{H}} \sup_{t\in[0,T]} |f_k(t,z)| \det\left(A_k^{-1}\sqrt{Q}^{-1}\sqrt{\mathcal{K}}^{-1}\right) dz \leq D_k^f \mathcal{D}_k^A.
\end{aligned}$$

Due to the definition $C^f \in \ell^1$ and $D^f \cdot \mathcal{D}^A \in \ell^1$ and thus $b \in \mathfrak{B}([0,T]\times\mathcal{H};\mathcal{H})$.

A possible choice for f is

$$f_k(t,z) = C_k^f \cdot e^{-t} \cdot e^{-D_k^f \frac{|z|}{2}} \left(a \mathbb{1}_{\{z\in A\}} + b \mathbb{1}_{\{z \in A^c\}}\right),$$

where $a, b \in \mathbb{R}$ and $A \subset \mathcal{H}$, which obviously fulfills the assumptions (5.41). The operator A_k, $k \geq 1$, can for example be chosen such that there exists a finite subset $N_k \subset \mathbb{N}$ such that for all $k \geq 1$

$$\prod_{n\in N_k} \lambda_k^{-1} \sqrt{\mathfrak{K}_{H_k}}^{-1} \leq C.$$

and we have for every $x \in \mathcal{H}$

$$A_k x = \mathcal{D}_k^A \sum_{n\in N_k} x^{(n)} e_n.$$

Then A_k is invertible on $A_k\mathcal{H}$ for every $k \geq 1$ and

$$\det\left(A_k^{-1}\sqrt{Q}^{-1}\sqrt{\mathcal{K}}^{-1}\right) = \mathcal{D}_k^A \prod_{n\in N_k} \lambda_k^{-1}\sqrt{\mathfrak{K}_{H_k}}^{-1} \leq C\mathcal{D}_k^A.$$

Appendix A. Compactness Criterion

The following result which is originally due to [14] in the finite dimensional case and which can be e.g. found in [9], provides a compactness criterion of square integrable cylindrical Wiener processes on a Hilbert space.

Theorem A.1 *Let $(B_t)_{t\in[0,T]}$ be a cylindrical Wiener process on a separable Hilbert space $\mathcal{H}$ with respect to a complete probability space $(\Omega, \mathcal{F}, \mu)$, where $\mathcal{F}$ is generated by $(B_t)_{t\in[0,T]}$. Further, let $\mathcal{L}_{HS}(\mathcal{H},\mathbb{R})$ be the space of Hilbert-Schmidt operators from $\mathcal{H}$ to $\mathbb{R}$ and let $D : \mathbb{D}^{1,2} \longrightarrow L^2(\Omega; L^2([0,T]) \otimes \mathcal{L}_{HS}(\mathcal{H},\mathbb{R}))$ be the Malliavin derivative in the direction of $(B_t)_{t\in[0,T]}$, where $\mathbb{D}^{1,2}$ is the space of Malliavin differentiable random variables in $L^2(\Omega)$.*

Suppose that C is a self-adjoint compact operator on $L^2([0,T]) \otimes \mathcal{L}_{HS}(\mathcal{H},\mathbb{R})$ with dense image. Then for any $c > 0$ the set

$$\mathcal{G} = \left\{ G \in \mathbb{D}^{1,2} : \|G\|_{L^2(\Omega)} + \left\| C^{-1} DG \right\|_{L^2(\Omega; L^2([0,T]) \otimes \mathcal{L}_{HS}(\mathcal{H},\mathbb{R}))} \le c \right\}$$

is relatively compact in $L^2(\Omega)$.

In this paper we aim at using a special case of the previous theorem, which is more suitable for explicit estimations. To this end we need the following auxiliary result from [14].

Lemma A.2 *Denote by $v_s, s \ge 0$, with $v_0 = 1$ the Haar basis of $L^2([0,1])$. Define for any $0 < \alpha < \frac{1}{2}$ the operator A_α on $L^2([0,1])$ by*

$$A_\alpha v_s = 2^{i\alpha} v_s, \quad \text{if } s = 2^i + j, \quad i \ge 0, \quad 0 \le j \le 2^i,$$

and

$$A_\alpha 1 = 1.$$

Then for $\alpha < \beta < \frac{1}{2}$ we have that

$$\|A_\alpha f\|^2_{L^2([0,1])} \le 2(\|f\|^2_{L^2([0,1])} + \frac{1}{1 - 2^{-2(\beta-\alpha)}} \int_0^1 \int_0^1 \frac{|f(t) - f(u)|^2}{|t-u|^{1+2\beta}} dt du).$$

Theorem A.3 *Let D^k be the Malliavin derivative in the direction of the k-th component of $(B_t)_{t \in [0,T]}$. In addition, let $0 < \alpha_k < \beta_k < \frac{1}{2}$ and $\gamma_k > 0$ for all $k \ge 1$. Define the sequence $\mu_{s,k} = 2^{-i\alpha_k}\gamma_k$, if $s = 2^i + j$, $i \ge 0$, $0 \le j \le 2^i$, $k \ge 1$. Assume that $\mu_{s,k} \longrightarrow 0$ for $s, k \longrightarrow \infty$. Let $c > 0$ and $\mathcal{G}$ the collection of all $G \in \mathbb{D}^{1,2}$ such that*

$$\|G\|_{L^2(\Omega)} \le c,$$

$$\sum_{k \ge 1} \gamma_k^{-2} \left\| D^k G \right\|^2_{L^2(\Omega; L^2([0,1]))} \le c,$$

and

$$\sum_{k \ge 1} \frac{1}{(1 - 2^{-2(\beta_k - \alpha_k)})\gamma_k^2} \int_0^1 \int_0^1 \frac{\left\| D_t^k G - D_u^k G \right\|^2_{L^2(\Omega)}}{|t-u|^{1+2\beta_k}} dt du \le c.$$

Then $\mathcal{G}$ is relatively compact in $L^2(\Omega)$.

Proof. As before denote by $v_s, s \ge 0$, with $v_0 = 1$ the Haar basis of $L^2([0,1])$ and by $e_k^* = \langle e_k, \cdot \rangle_H, k \ge 1$, an orthonormal basis of $\mathcal{L}_{HS}(\mathcal{H},\mathbb{R})$, where $e_k, k \ge 0$, is an orthonormal basis of $\mathcal{H}$. Define a self-adjoint compact operator C on $L^2([0,1]) \otimes \mathcal{L}_{HS}(\mathcal{H},\mathbb{R})$ with dense image by

$$C(v_s \otimes e_k^*) = \mu_{s,k} v_s \otimes e_k^*, \quad s \ge 0, \quad k \ge 1.$$

Then it follows for $G \in \mathbb{D}^{1,2}$ from Lemma A.2 that

$$\left\| C^{-1} DG \right\|^2_{L^2(\Omega; L^2([0,1]) \otimes \mathcal{L}_{HS}(\mathcal{H},\mathbb{R}))}$$

$$
\begin{aligned}
&= \sum_{k\geq 1}\sum_{s\geq 0} \mu_{s,k}^{-2} E[\langle DG, v_s \otimes e_k^*\rangle^2_{L^2([0,1])\otimes \mathcal{L}_{HS}(\mathcal{H},\mathbb{R}))}] \\
&= \sum_{k\geq 1} \gamma_k^{-2} \left\| A_{\alpha_k} D^k G \right\|^2_{L^2(\Omega;L^2([0,1]))} \\
&\leq 2 \sum_{k\geq 1} \gamma_k^{-2} \left\| D^k G \right\|^2_{L^2(\Omega;L^2([0,1]))} \\
&\quad + 2 \sum_{k\geq 1} \frac{1}{(1-2^{-2(\beta_k-\alpha_k)})\gamma_k^2} \int_0^1 \int_0^1 \frac{\left\| D_t^k G - D_u^k G \right\|^2_{L^2(\Omega)}}{|t-u|^{1+2\beta_k}} dt du \\
&\leq M
\end{aligned}
$$

for a constant $M < \infty$. So using Theorem A.1 we obtain the result. □

Appendix B. Integration by parts formula

In this section we derive an integration by parts formula similar to [6] which is used in the proof of Theorem 4.10 to verify the conditions of the compactness criterion Theorem A.3. Before stating the integration by parts formula, we start by giving some definitions and notations frequently used during the course of this section.

Let n be a given integer. We consider the function $f : [0,T]^n \times (\mathbb{R}^d)^n \to \mathbb{R}$ of the form

$$
f(s,z) = \prod_{j=1}^n f_j(s_j, z_j), \quad s = (s_1, \ldots, s_n) \in [0,T]^n, \quad z = (z_1, \ldots, z_n) \in (\mathbb{R}^d)^n, \tag{5.42}
$$

where $f_j : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, $j = 1, \ldots, n$, are compactly supported smooth functions. Further, we deal with the function $\varkappa : [0,T]^n \to \mathbb{R}$ which is of the form

$$
\varkappa(s) = \prod_{j=1}^n \varkappa_j(s_j), \quad s \in [0,T]^n, \tag{5.43}
$$

with integrable factors $\varkappa_j : [0,T] \to \mathbb{R}$, $j = 1, \ldots, n$.

Let α_j be a multi-index and D^{α_j} its corresponding differential operator. For $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^{d\times n}$ we define the norm $|\alpha| = \sum_{j=1}^n \sum_{k=1}^d \alpha_j^{(k)}$ and write

$$
D^\alpha f(s,z) = \prod_{j=1}^n D^{\alpha_j} f_j(s_j, z_j).
$$

Let k be an arbitrary integer. Given $(s,z) = (s_1, \ldots, s_{kn}, z_1, \ldots, z_n) \in [0,T]^{kn} \times (\mathbb{R}^d)^n$ and a shuffle permutation $\sigma \in \mathcal{S}(n,n)$ we define the shuffled functions

$$
f_\sigma(s,z) := \prod_{j=1}^{kn} f_{[\sigma(j)]}(s_j, z_{[\sigma(j)]})
$$

and

$$\varkappa_\sigma(s) := \prod_{j=1}^{kn} \varkappa_{[\sigma(j)]}(s_j),$$

where $[j]$ is equal to $(j - in)$ if $(in+1) \le j \le (i+1)n$, $i = 0, \dots, (k-1)$. For a multi-index α, we define

$$\Psi_\alpha^f(\theta, t, z, H, d) := \left(\prod_{k=1}^{d} \sqrt{(2\,|\alpha^{(k)}|)!}\right) \sum_{\sigma \in \mathcal{S}(n,n)} \int_{\Delta_{\theta,t}^{2n}} |f_\sigma(s,z)|\, |\Delta s|^{-H\left(1+\alpha_{[\sigma(\Delta)]}\right)} ds, \tag{5.44}$$

and

$$\Psi_\alpha^\varkappa(\theta, t, H, d) := \left(\prod_{k=1}^{d} \sqrt{(2\,|\alpha^{(k)}|)!}\right) \sum_{\sigma \in \mathcal{S}(n,n)} \int_{\Delta_{\theta,t}^{2n}} |\varkappa_\sigma(s)|\, |\Delta s|^{-H\left(1+\alpha_{[\sigma(\Delta)]}\right)} ds, \tag{5.45}$$

where for any $a, b \in \mathbb{R}$

$$|\Delta s|^{H_k\left(a+b\cdot\alpha^{(k)}_{[\sigma(\Delta)]}\right)} := |s_1|^{H_k\left(a+b\left(\alpha^{(k)}_{[\sigma(1)]}+\alpha^{(k)}_{[\sigma(2n)]}\right)\right)} \prod_{j=2}^{2n} |s_j - s_{j-1}|^{H_k\left(a+b\left(\alpha^{(k)}_{[\sigma(j)]}+\alpha^{(k)}_{[\sigma(j-1)]}\right)\right)},$$

$$|\Delta s|^{H\left(a+b\cdot\alpha_{[\sigma(\Delta)]}\right)} := \prod_{k=1}^{d} |\Delta s|^{H_k\left(a+b\cdot\alpha^{(k)}_{[\sigma(\Delta)]}\right)}.$$

Theorem B.1 *Suppose the functions $\Psi_\alpha^f(\theta,t,z,H,d)$ and $\Psi_\alpha^\varkappa(\theta,t,H,d)$ defined in (5.44) and (5.45), respectively, are finite. Then,*

$$\Lambda_\alpha^f(\theta,t,z) := (2\pi)^{-dn} \int_{(\mathbb{R}^d)^n} \int_{\Delta_{\theta,t}^n} \prod_{j=1}^{n} f_j(s_j,z_j)(-iu_j)^{\alpha_j} e^{-i\left\langle u_j, \widehat{B}^{d,H}_{s_j} - z_j\right\rangle} ds du, \tag{5.46}$$

where $\widehat{B}_t^{d,H} := \left(\frac{B_t^{H_1}}{\sqrt{\mathfrak{K}_{H_1}}}, \dots, \frac{B_t^{H_d}}{\sqrt{\mathfrak{K}_{H_d}}}\right)^\top$ and $\mathfrak{K}_{H_k}$ is the constant in Lemma 2.4, is a square integrable random variable in $L^2(\Omega)$ and

$$\mathbb{E}\left[\left|\Lambda_\alpha^f(\theta,t,z)\right|^2\right] \le \frac{T^{\frac{|\alpha|}{6}}}{(2\pi)^{dn}} \Psi_\alpha^f(\theta,t,z,H,d). \tag{5.47}$$

Furthermore,

$$\mathbb{E}\left[\left|\int_{(\mathbb{R}^d)^n} \Lambda_\alpha^{\varkappa f}(\theta,t,z)dz\right|\right] \le \frac{T^{\frac{|\alpha|}{12}}}{\sqrt{2\pi}^{dn}} (\Psi_\alpha^\varkappa(\theta,t,H,d))^{\frac{1}{2}} \prod_{j=1}^{n} \|f_j\|_{L^1(\mathbb{R}^d;L^\infty([0,T]))}, \tag{5.48}$$

and the integration by parts formula

$$\int_{\Delta_{\theta,t}^n} D^\alpha f\left(s, \widehat{B}_s^{d,H}\right) ds = \int_{(\mathbb{R}^d)^n} \Lambda_\alpha^f(\theta,t,z)dz, \tag{5.49}$$

holds.

Proof. For notational simplicity we consider merely the case $\theta = 0$ and write $\Lambda_\alpha^f(t,z) := \Lambda_\alpha^f(0,t,z)$. For any integrable function $g : (\mathbb{R}^d)^n \longrightarrow \mathbb{C}$ we have that

$$\begin{aligned}
&\left| \int_{(\mathbb{R}^d)^n} g(u_1,...,u_n) du_1...du_n \right|^2 \\
&\quad = \int_{(\mathbb{R}^d)^n} g(u_1,...,u_n) du_1...du_n \int_{(\mathbb{R}^d)^n} \overline{g(u_{n+1},...,u_{2n})} du_{n+1}...du_{2n} \\
&\quad = \int_{(\mathbb{R}^d)^n} g(u_1,...,u_n) du_1...du_n (-1)^{dn} \int_{(\mathbb{R}^d)^n} \overline{g(-u_{n+1},...,-u_{2n})} du_{n+1}...du_{2n},
\end{aligned}$$

where the change of variables $(u_{n+1},...,u_{2n}) \longmapsto (-u_{n+1},...,-u_{2n})$ was applied in the last equality. Thus,

$$\begin{aligned}
\left|\Lambda_\alpha^f(t,z)\right|^2 &= (2\pi)^{-2dn}(-1)^{dn} \int_{(\mathbb{R}^d)^{2n}} \int_{\Delta_{0,t}^n} \prod_{j=1}^n f_j(s_j,z_j)(-iu_j)^{\alpha_j} e^{-i\left\langle u_j, \widehat{B}_{s_j}^{d,H} - z_j\right\rangle} ds \\
&\quad \times \int_{\Delta_{0,t}^n} \prod_{j=n+1}^{2n} f_{[j]}(s_j, z_{[j]})(-iu_j)^{\alpha_{[j]}} e^{-i\left\langle u_j, \widehat{B}_{s_j}^{d,H} - z_{[j]}\right\rangle} dsdu \\
&= (2\pi)^{-2dn}(-1)^{dn}\, i^{|\alpha|} \sum_{\sigma\in\mathcal{S}(n,n)} \int_{(\mathbb{R}^d)^{2n}} \left(\prod_{j=1}^n e^{-i\langle z_j, u_j + u_{j+n}\rangle}\right) \\
&\quad \times \int_{\Delta_{0,t}^{2n}} f_\sigma(s,z) \left(\prod_{j=1}^{2n} u_{\sigma(j)}^{\alpha_{[\sigma(j)]}}\right) \exp\left\{-i\sum_{j=1}^{2n} \left\langle u_{\sigma(j)}, \widehat{B}_{s_j}^{d,H}\right\rangle\right\} dsdu,
\end{aligned}$$

where we applied shuffling in the sense of (5.9). Taking the expectation on both sides together with the independence of the fractional Brownian motions B^{H_k}, $k = 1,...,d$, yields that

$$\begin{aligned}
&\mathbb{E}\left[\left|\Lambda_\alpha^f(t,z)\right|^2\right] \\
&\quad = (2\pi)^{-2dn}(-1)^{dn}\, i^{|\alpha|} \sum_{\sigma\in\mathcal{S}(n,n)} \int_{(\mathbb{R}^d)^{2n}} \left(\prod_{j=1}^n e^{-i\langle z_j, u_j + u_{j+n}\rangle}\right) \\
&\quad \times \int_{\Delta_{0,t}^{2n}} f_\sigma(s,z) \left(\prod_{j=1}^{2n} u_{\sigma(j)}^{\alpha_{[\sigma(j)]}}\right) \exp\left\{-\frac{1}{2}\operatorname{Var}\left(\sum_{j=1}^{2n} \left\langle u_{\sigma(j)}, \widehat{B}_{s_j}^{d,H}\right\rangle\right)\right\} dsdu \\
&\quad = (2\pi)^{-2dn}(-1)^{dn}\, i^{|\alpha|} \sum_{\sigma\in\mathcal{S}(n,n)} \int_{(\mathbb{R}^d)^{2n}} \left(\prod_{j=1}^n e^{-i\langle z_j, u_j + u_{j+n}\rangle}\right) \\
&\quad \times \int_{\Delta_{0,t}^{2n}} f_\sigma(s,z) \left(\prod_{j=1}^{2n} u_{\sigma(j)}^{\alpha_{[\sigma(j)]}}\right) \exp\left\{-\frac{1}{2}\sum_{k=1}^d \operatorname{Var}\left(\sum_{j=1}^{2n} u_{\sigma(j)}^{(k)} \frac{B_{s_j}^{H_k}}{\sqrt{\mathfrak{K}_{H_k}}}\right)\right\} dsdu \\
&\quad = (2\pi)^{-2dn}(-1)^{dn}\, i^{|\alpha|} \sum_{\sigma\in\mathcal{S}(n,n)} \int_{(\mathbb{R}^d)^{2n}} \left(\prod_{j=1}^n e^{-i\langle z_j, u_j + u_{j+n}\rangle}\right)
\end{aligned}$$

$$\times \int_{\Delta_{0,t}^{2n}} f_\sigma(s,z) \left(\prod_{j=1}^{2n} u_{\sigma(j)}^{\alpha_{[\sigma(j)]}} \right) \prod_{k=1}^{d} \exp\left\{ -\frac{1}{2\mathfrak{K}_{H_k}} (u_\sigma^{(k)})^\top \Sigma_k u_\sigma^{(k)} \right\} ds du, \quad (5.50)$$

where $u_\sigma^{(k)} = \left(u_{\sigma(1)}^{(k)}, \dots, u_{\sigma(2n)}^{(k)}\right)^\top$ and

$$\Sigma_k = \Sigma_k(s) := \left(\mathbb{E}\left[B_{s_i}^{H_k} B_{s_j}^{H_k} \right] \right)_{1 \le i,j \le 2n}.$$

Moreover, we obtain for every $\sigma \in \mathcal{S}(n,n)$ that

$$\begin{aligned} &\int_{\Delta_{0,t}^{2n}} |f_\sigma(s,z)| \int_{(\mathbb{R}^d)^{2n}} \prod_{k=1}^{d} \left(\left(\prod_{j=1}^{2n} \left| u_{\sigma(j)}^{(k)} \right|^{\alpha_{[\sigma(j)]}^{(k)}} \right) e^{-\frac{1}{2\mathfrak{K}_{H_k}} (u_\sigma^{(k)})^\top \Sigma_k u_\sigma^{(k)}} \right) du ds \\ &= \int_{\Delta_{0,t}^{2n}} |f_\sigma(s,z)| \prod_{k=1}^{d} \left(\int_{\mathbb{R}^{2n}} \left(\prod_{j=1}^{2n} \left| u_j^{(k)} \right|^{\alpha_{[\sigma(j)]}^{(k)}} \right) e^{-\frac{1}{2}\left\langle \frac{\Sigma_k}{\mathfrak{K}_{H_k}} u^{(k)}, u^{(k)} \right\rangle} du^{(k)} \right) ds, \quad (5.51) \end{aligned}$$

where $u^{(k)} := \left(u_1^{(k)}, \dots, u_{2n}^{(k)}\right)^\top$. For every $1 \le k \le d$ we have by using substitution that

$$\begin{aligned} &\int_{\mathbb{R}^{2n}} \left(\prod_{j=1}^{2n} \left| u_j^{(k)} \right|^{\alpha_{[\sigma(j)]}^{(k)}} \right) e^{-\frac{1}{2}\left\langle \frac{\Sigma_k}{\mathfrak{K}_{H_k}} u^{(k)}, u^{(k)} \right\rangle} du^{(k)} \quad (5.52) \\ &= \frac{\mathfrak{K}_{H_k}^n}{(\det \Sigma_k)^{1/2}} \int_{\mathbb{R}^{2n}} \left(\prod_{j=1}^{2n} \left| \left\langle \sqrt{\mathfrak{K}_{H_k}} \Sigma_k^{-1/2} u^{(k)}, \widetilde{e}_j \right\rangle \right|^{\alpha_{[\sigma(j)]}^{(k)}} \right) e^{-\frac{1}{2}\left\langle u^{(k)}, u^{(k)} \right\rangle} du^{(k)}. \end{aligned}$$

Considering a standard Gaussian random vector $Z \sim \mathcal{N}(0, \mathrm{Id}_{2n})$, we get that

$$\begin{aligned} &\int_{\mathbb{R}^{2n}} \left(\prod_{j=1}^{2n} \left| \left\langle \Sigma_k^{-1/2} u^{(k)}, \widetilde{e}_j \right\rangle \right|^{\alpha_{[\sigma(j)]}^{(k)}} \right) e^{-\frac{1}{2}\left\langle u^{(k)}, u^{(k)} \right\rangle} du^{(k)} \quad (5.53) \\ &= (2\pi)^n \mathbb{E}\left[\prod_{j=1}^{2n} \left| \left\langle \Sigma_k^{-1/2} Z, \widetilde{e}_j \right\rangle \right|^{\alpha_{[\sigma(j)]}^{(k)}} \right]. \end{aligned}$$

Using a Brascamp-Lieb type inequality which is due to Lemma C.1, we further get that

$$\mathbb{E}\left[\prod_{j=1}^{2n} \left| \left\langle \Sigma_k^{-1/2} Z, \widetilde{e}_j \right\rangle \right|^{\alpha_{[\sigma(j)]}^{(k)}} \right] \le \sqrt{\mathrm{perm}(A_k)} = \sqrt{\sum_{\pi \in S_{2\left|\alpha^{(k)}\right|}} \prod_{i=1}^{2\left|\alpha^{(k)}\right|} a_{i,\pi(i)}^{(k)}},$$

where $\left|\alpha^{(k)}\right| := \sum_{j=1}^n \alpha_j^{(k)}$ and $\mathrm{perm}(A_k)$ is the permanent of the covariance matrix $A_k = (a_{i,j}^{(k)})_{1 \le i,j \le 2\left|\alpha^{(k)}\right|}$ of the Gaussian random vector

$$\Bigg(\underbrace{\left\langle \Sigma_k^{-1/2} Z, \widetilde{e}_1 \right\rangle, \dots, \left\langle \Sigma_k^{-1/2} Z, \widetilde{e}_1 \right\rangle}_{\alpha_{[\sigma(1)]}^{(k)} \text{ times}}, \dots, \underbrace{\left\langle \Sigma_k^{-1/2} Z, \widetilde{e}_{2n} \right\rangle, \dots, \left\langle \Sigma_k^{-1/2} Z, \widetilde{e}_{2n} \right\rangle}_{\alpha_{[\sigma(2n)]}^{(k)} \text{ times}} \Bigg),$$

and S_m denotes the permutation group of size m. Using an upper bound for the permanent of positive semidefinite matrices which is due to [3], we find that

$$\operatorname{perm}(A_k) = \sum_{\pi \in S_{2\left|\alpha^{(k)}\right|}} \prod_{i=1}^{2\left|\alpha^{(k)}\right|} a_{i,\pi(i)}^{(k)} \leq (2\left|\alpha^{(k)}\right|)! \prod_{i=1}^{2\left|\alpha^{(k)}\right|} a_{i,i}^{(k)}. \tag{5.54}$$

Now let $\sum_{l=1}^{j-1} \alpha_{[\sigma(l)]}^{(k)} + 1 \leq i \leq \sum_{l=1}^{j} \alpha_{[\sigma(l)]}^{(k)}$ for some fixed $j \in \{1, ..., 2n\}$. Then

$$a_{i,i}^{(k)} = \mathbb{E}\Big[\Big\langle \Sigma_k^{-1/2} Z, \widetilde{e}_j \Big\rangle \Big\langle \Sigma_k^{-1/2} Z, \widetilde{e}_j \Big\rangle\Big].$$

Substitution gives moreover that

$$\mathbb{E}\Big[\Big\langle \Sigma_k^{-1/2} Z, \widetilde{e}_j \Big\rangle \Big\langle \Sigma_k^{-1/2} Z, \widetilde{e}_j \Big\rangle\Big] = (\det \Sigma_k)^{1/2} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} u_j^2 \exp\left\{ -\frac{1}{2} \langle \Sigma_k u, u \rangle \right\} du. \tag{5.55}$$

Applying Lemma C.2 we get

$$\begin{aligned} \int_{\mathbb{R}^{2n}} u_j^2 \exp\left\{ -\frac{1}{2} \langle \Sigma_k u, u \rangle \right\} du &= \frac{(2\pi)^{(2n-1)/2}}{(\det \Sigma_k)^{1/2}} \int_{\mathbb{R}} v^2 \exp\left\{ -\frac{1}{2} v^2 \right\} dv \frac{1}{\sigma_j^2} \\ &= \frac{(2\pi)^n}{(\det \Sigma_k)^{1/2}} \frac{1}{\sigma_j^2}, \end{aligned} \tag{5.56}$$

where $\sigma_j^2 := \operatorname{Var}\Big(B_{s_j}^{H_k} \Big| B_{s_1}^{H_k}, ..., B_{s_{2n}}^{H_k} \text{ without } B_{s_j}^{H_k} \Big)$.

Subsequently, we aim at the application of the strong local non-determinism property of the fractional Brownian motions, cf. Lemma 2.4, i.e. for all $0 < r < t \leq T$ exists a constant $\mathfrak{K}_{H_k}$ depending on H_k and T such that

$$\operatorname{Var}\Big(B_t^{H_k} \Big| B_s^{H_k}, |t-s| \geq r \Big) \geq \mathfrak{K}_{H_k} r^{2H_k}.$$

Hence, we get due to Lemma C.5 and Lemma C.6 that

$$(\det \Sigma_k(s))^{1/2} \geq \mathfrak{K}_{H_k}^{\frac{(2n-1)}{2}} \left| s_1 \right|^{H_k} \left| s_2 - s_1 \right|^{H_k} ... \left| s_{2n} - s_{2n-1} \right|^{H_k}, \tag{5.57}$$

and

$$\begin{aligned} \sigma_1^2 &\geq \mathfrak{K}_{H_k} \left| s_2 - s_1 \right|^{2H_k}, \\ \sigma_j^2 &\geq \mathfrak{K}_{H_k} \min\left\{ \left| s_j - s_{j-1} \right|^{2H_k}, \left| s_{j+1} - s_j \right|^{2H_k} \right\},\ 2 \leq j \leq 2n-1, \\ \sigma_{2n}^2 &\geq \mathfrak{K}_{H_k} \left| s_{2n} - s_{2n-1} \right|^{2H_k}. \end{aligned}$$

Thus,

$$\prod_{j=1}^{2n} \sigma_j^{-2\alpha_{[\sigma(j)]}^{(k)}} \leq \mathfrak{K}_{H_k}^{-2|\alpha^{(k)}|} T^{4H_k|\alpha^{(k)}|} |\Delta s|^{-2H_k \alpha_{[\sigma(\Delta)]}^{(k)}}. \tag{5.58}$$

Concluding from (5.54), (5.55), (5.56), and (5.58) we have that

$$\begin{aligned}\operatorname{perm}(A_k) &\leq \left(2\left|\alpha^{(k)}\right|\right)! \prod_{i=1}^{2\left|\alpha^{(k)}\right|} a_{i,i}^{(k)} \\ &\leq \left(2\left|\alpha^{(k)}\right|\right)! \prod_{j=1}^{2n} \left((\det \Sigma_k)^{1/2} \frac{1}{(2\pi)^n} \frac{(2\pi)^n}{(\det \Sigma_k)^{1/2}} \frac{1}{\sigma_j^2} \right)^{\alpha^{(k)}_{[\sigma(j)]}} \\ &\leq \left(2\left|\alpha^{(k)}\right|\right)! \mathfrak{K}_{H_k}^{-2|\alpha^{(k)}|} T^{4H_k|\alpha^{(k)}|} |\Delta s|^{-2H_k \alpha^{(k)}_{[\sigma(\Delta)]}}.\end{aligned}$$

Consequently,

$$\mathbb{E}\left[\prod_{j=1}^{2n} \left|\left\langle \Sigma_k^{-1/2} Z, \widetilde{e}_j \right\rangle\right|^{\alpha^{(k)}_{[\sigma(j)]}}\right] \leq \sqrt{(2\,|\alpha^{(k)}|)!}\,\mathfrak{K}_{H_k}^{-|\alpha^{(k)}|} T^{2H_k|\alpha^{(k)}|} |\Delta s|^{-H_k \alpha^{(k)}_{[\sigma(\Delta)]}}.$$

Therefore we get from (5.50), (5.51), (5.52), (5.53), and (5.57) that

$$\begin{aligned}&\mathbb{E}\left[\left|\Lambda^f_\alpha(t,z)\right|^2\right] \\ &\leq (2\pi)^{-2dn} \sum_{\sigma\in\mathcal{S}(n,n)} \int_{\Delta^{2n}_{0,t}} |f_\sigma(s,z)| \prod_{k=1}^d \left(\int_{\mathbb{R}^{2n}} \left|u^{(k)}\right|^{\alpha^{(k)}} e^{-\frac{1}{2\mathfrak{K}_{H_k}} \left\langle \Sigma_k u^{(k)}, u^{(k)} \right\rangle} du^{(k)} \right) ds \\ &\leq (2\pi)^{-dn} \sum_{\sigma\in\mathcal{S}(n,n)} \int_{\Delta^{2n}_{0,t}} |f_\sigma(s,z)| \prod_{k=1}^d \left(\frac{\mathfrak{K}_{H_k}^{n+|\alpha^{(k)}|}}{(\det \Sigma_k(s))^{\frac{1}{2}}} \mathbb{E}\left[\prod_{j=1}^{2n} \left|\left\langle \Sigma_k^{-\frac{1}{2}} Z, \widetilde{e}_j \right\rangle\right|^{\alpha^{(k)}_{\sigma(j)}}\right]\right) ds \\ &\leq (2\pi)^{-dn} \sum_{\sigma\in\mathcal{S}(n,n)} \int_{\Delta^{2n}_{0,t}} |f_\sigma(s,z)| \left(\prod_{k=1}^d |\Delta s|^{-H_k} \mathfrak{K}_{H_k}^{|\alpha^{(k)}|+\frac{1}{2}} \right) \\ &\quad\times \prod_{k=1}^d \left(\sqrt{(2\,|\alpha^{(k)}|)!}\,\mathfrak{K}_{H_k}^{-|\alpha^{(k)}|} T^{2H_k|\alpha^{(k)}|} |\Delta s|^{-H_k \alpha^{(k)}_{[\sigma(\Delta)]}} \right) ds \\ &\leq (2\pi)^{-dn} T^{\frac{|\alpha|}{6}} \left(\prod_{k=1}^d \sqrt{\mathfrak{K}_{H_k}} \sqrt{(2\,|\alpha^{(k)}|)!} \right) \sum_{\sigma\in\mathcal{S}(n,n)} \int_{\Delta^{2n}_{0,t}} |f_\sigma(s,z)|\, |\Delta s|^{-H\left(1+\alpha_{[\sigma(\Delta)]}\right)} ds.\end{aligned}$$

Since $\sup_{k\geq 1} \mathfrak{K}_{H_k} \in (0,1)$, inequality (5.47) holds.

Next we prove the estimate (5.48). With inequality (5.47), we get that

$$\begin{aligned}\mathbb{E}\left[\left|\int_{(\mathbb{R}^d)^n} \Lambda^{\varkappa f}_\alpha(\theta,t,z) dz\right|\right] &\leq \int_{(\mathbb{R}^d)^n} \mathbb{E}\left[\left|\Lambda^{\varkappa f}_\alpha(\theta,t,z)\right|^2\right]^{\frac{1}{2}} dz \\ &\leq \frac{T^{\frac{|\alpha|}{12}}}{\sqrt{2\pi}^{dn}} \int_{(\mathbb{R}^d)^n} (\Psi^{\varkappa f}_\alpha(\theta,t,z,H,d))^{\frac{1}{2}} dz.\end{aligned}$$

Taking the supremum over $[0,T]$ with respect to each function f_j, i.e.

$$\left|f_{[\sigma(j)]}(s_j, z_{[\sigma(j)]})\right| \leq \sup_{s_j\in[0,T]} \left|f_{[\sigma(j)]}(s_j, z_{[\sigma(j)]})\right|,\ j=1,...,2n,$$

yields that

$$
\begin{aligned}
&\mathbb{E}\left[\left|\int_{(\mathbb{R}^d)^n} \Lambda_\alpha^{\varkappa f}(\theta,t,z)dz\right|\right] \\
&\quad\leq \frac{T^{\frac{|\alpha|}{12}}}{\sqrt{2\pi}^{dn}} \max_{\sigma\in\mathcal{S}(n,n)} \int_{(\mathbb{R}^d)^n} \left(\prod_{j=1}^{2n} \left\|f_{[\sigma(j)]}(\cdot, z_{[\sigma(j)]})\right\|_{L^\infty([0,T])}\right)^{\frac{1}{2}} dz \\
&\qquad \times \left(\prod_{k=1}^{d} \sqrt{(2\,|\alpha^{(k)}|)!} \sum_{\sigma\in\mathcal{S}(n,n)} \int_{\Delta_{\theta,t}^{2n}} |\varkappa_\sigma(s)|\,|\Delta s|^{-H\left(1+\alpha_{[\sigma(\Delta)]}\right)} ds\right)^{\frac{1}{2}} \\
&\quad= \frac{T^{\frac{|\alpha|}{12}}}{\sqrt{2\pi}^{dn}} \max_{\sigma\in\mathcal{S}(n,n)} \int_{(\mathbb{R}^d)^n} \left(\prod_{j=1}^{2n} \left\|f_{[\sigma(j)]}(\cdot, z_{[\sigma(j)]})\right\|_{L^\infty([0,T])}\right)^{\frac{1}{2}} dz\ (\Psi_\alpha^{\varkappa}(\theta,t,H,d))^{\frac{1}{2}} \\
&\quad= \frac{T^{\frac{|\alpha|}{12}}}{\sqrt{2\pi}^{dn}} \int_{(\mathbb{R}^d)^n} \prod_{j=1}^{n} \|f_j(\cdot, z_j)\|_{L^\infty([0,T])}\, dz\ (\Psi_\alpha^{\varkappa}(\theta,t,H,d))^{\frac{1}{2}} \\
&\quad= \frac{T^{\frac{|\alpha|}{12}}}{\sqrt{2\pi}^{dn}} \left(\prod_{j=1}^{n} \|f_j\|_{L^1(\mathbb{R}^d;L^\infty([0,T]))}\right) (\Psi_\alpha^{\varkappa}(\theta,t,H,d))^{\frac{1}{2}}.
\end{aligned}
$$

Finally, we show the integration by parts formula (5.49). Note that *a priori* one cannot interchange the order of integration in (5.46), since e.g. for $m=1$, $f\equiv 1$ one gets an integral of the Donsker-Delta function which is not a random variable in the usual sense. Therefore, we define for $R>0$,

$$
\Lambda_{\alpha,R}^{f}(\theta,t,z) := (2\pi)^{-dn} \int_{B(0,R)} \int_{\Delta_{\theta,t}^{n}} \prod_{j=1}^{n} f_j(s_j,z_j)(-iu_j)^{\alpha_j} e^{-i\langle u_j, \widehat{B}_{s_j}^{d,H} - z_j\rangle} ds dv,
$$

where $B(0,R) := \{v\in(\mathbb{R}^d)^n : |v|<R\}$. This yields

$$
|\Lambda_{\alpha,R}^{f}(\theta,t,z)| \leq C_R \int_{\Delta_{\theta,t}^{n}} \prod_{j=1}^{n} |f_j(s_j,z_j)| ds
$$

for a sufficient constant C_R. Under the assumption that the above right-hand side is integrable over $(\mathbb{R}^d)^n$, similar computations as above show that $\Lambda_{\alpha,R}^{f}(\theta,t,z) \to \Lambda_{\alpha}^{f}(\theta,t,z)$ in $L^2(\Omega)$ as $R\to\infty$ for all θ,t and z. By Lebesgue's dominated convergence theorem and the fact that the Fourier transform is an automorphism on the Schwarz space, we obtain

$$
\begin{aligned}
&\int_{(\mathbb{R}^d)^n} \Lambda_\alpha^{f}(\theta,t,z)dz = \lim_{R\to\infty} \int_{(\mathbb{R}^d)^n} \Lambda_{\alpha,R}^{f}(\theta,t,z)dx \\
&\quad= \lim_{R\to\infty} (2\pi)^{-dn} \int_{(\mathbb{R}^d)^n} \int_{B(0,R)} \int_{\Delta_{\theta,t}^{n}} \prod_{j=1}^{n} f_j(s_j,z_j)(-iu_j)^{\alpha_j} e^{-i\left\langle u_j, \widehat{B}_{s_j}^{d,H} - z_j\right\rangle} dzduds \\
&\quad= \lim_{R\to\infty} \int_{\Delta_{\theta,t}^{n}} \int_{B(0,R)} (2\pi)^{-dn} \int_{(\mathbb{R}^d)^n} \prod_{j=1}^{n} f_j(s_j,z_j) e^{i\langle u_j, z_j\rangle} dz (-iu_j)^{\alpha_j} e^{-i\left\langle u_j, \widehat{B}_{s_j}^{d,H}\right\rangle} duds
\end{aligned}
$$

$$= \lim_{R\to\infty} \int_{\Delta^n_{\theta,t}} \int_{B(0,R)} \prod_{j=1}^n \widehat{f}_j(s,-u_j)(-iu_j)^{\alpha_j} e^{-i\langle u_j, \widehat{B}^{d,H}_{s_j}\rangle} du ds$$

$$= \int_{\Delta^n_{\theta,t}} D^\alpha f\left(s, \widehat{B}^{d,H}_s\right) ds$$

which is exactly the integration by parts formula (5.49). □

Applying Theorem B.1 we obtain the following crucial estimate (compare [1], [2], [6], and [7]):

Proposition B.2 *Let the functions f and $\varkappa$ be defined as in* (5.42) *and* (5.43)*, respectively. Further, let $0 \le \theta' < \theta < t \le T$ and for some $m \ge 1$*

$$\varkappa_j(s) = (K_{H_m}(s,\theta) - K_{H_m}(s,\theta'))^{\varepsilon_j}, \ \theta < s < t,$$

for every $j = 1, ..., n$ with $(\varepsilon_1, ..., \varepsilon_n) \in \{0,1\}^n$. Let $\alpha \in (\mathbb{N}_0^d)^n$ be a multi-index. Assume there exists δ such that

$$-\sum_{k=1}^d H_k\left(1 + 2\alpha_j^{(k)}\right) + \left(H_m - \frac{1}{2} - \gamma_m\right) \ge \delta > -1$$

for all $j = 1, \dots n$ and $d \ge 1$, where $\gamma_m \in (0, H_m)$ is sufficiently small. Then there exist constants C_T (depending on T) and $K_{d,H}$ (depending on d and H), such that for any $0 \le \theta < t \le T$ we have

$$\mathbb{E}\left[\left|\int_{\Delta^n_{\theta,t}} \left(\prod_{j=1}^n D^{\alpha_j} f_j(s_j, \widehat{B}_{s_j})\varkappa_j(s_j)\right) ds\right|\right]$$
$$\le \frac{K^n_{d,H} \cdot T^{\frac{|\alpha|}{12}}}{\sqrt{2\pi}^{dn}} \left(C_T \left(\frac{\theta-\theta'}{\theta\theta'}\right)^{\gamma_m} \theta^{(H_m - \frac{1}{2} - \gamma_m)}\right)^{\sum_{j=1}^n \varepsilon_j} \prod_{j=1}^n \|f_j(\cdot, z_j)\|_{L^1(\mathbb{R}^d; L^\infty([0,T]))}$$
$$\times \frac{\left(\prod_{k=1}^d \left(2\left|\alpha^{(k)}\right|\right)!\right)^{\frac{1}{4}} (t-\theta)^{-\sum_{k=1}^d H_k\left(n + 2\left|\alpha^{(k)}\right|\right) + \left(H_m - \frac{1}{2} - \gamma_m\right)\sum_{j=1}^n \varepsilon_j + n}}{\Gamma(2n - \sum_{k=1}^d H_k(2n + 4\left|\alpha^{(k)}\right|) + 2(H_m - \frac{1}{2} - \gamma_m)\sum_{j=1}^n \varepsilon_j)^{\frac{1}{2}}}.$$

In order to prove this result we need the following two auxiliary results.

Lemma B.3 *Let $H \in \left(0, \frac{1}{2}\right)$ and $t \in [0,T]$ be fixed. Then, there exists $\beta \in \left(0, \frac{1}{2}\right)$ and a constant $C > 0$ independent of H such that*

$$\int_0^t \int_0^t \frac{|K_H(t,\theta') - K_H(t,\theta)|^2}{|\theta' - \theta|^{1+2\beta}} d\theta d\theta' \le C < \infty.$$

Proof. Let $0 \le \theta' < \theta \le t$ be fixed. Write

$$K_H(t,\theta) - K_H(t,\theta') = c_H\left[f_t(\theta) - f_t(\theta') + \left(\frac{1}{2} - H\right)(g_t(\theta) - g_t(\theta'))\right],$$

where $f_t(\theta) := \left(\frac{t}{\theta}\right)^{H-\frac{1}{2}} (t-\theta)^{H-\frac{1}{2}}$ and $g_t(\theta) := \int_\theta^t \frac{f_u(\theta)}{u} du$.

We continue with the estimation of $K_H(t,\theta) - K_H(t,\theta')$. First, observe that there exists a constant $0 < C < 1$ such that

$$\frac{y^{-\alpha} - x^{-\alpha}}{(x-y)^\gamma} \le C y^{-\alpha-\gamma}, \tag{5.59}$$

for every $0 < y < x < \infty$ and $\alpha := (\frac{1}{2} - H) \in \left(0, \frac{1}{2}\right)$ as well as $0 < \gamma < \frac{1}{2} - \alpha$. Indeed, rewriting (5.59) yields using the substitution $z := \frac{x}{y}$, $z \in (1, \infty)$,

$$\frac{y^{-\alpha} - x^{-\alpha}}{(x-y)^\gamma} y^{\alpha+\gamma} = \frac{1 - z^{-\alpha}}{(z-1)^\gamma} =: g(z).$$

Furthermore, since $\alpha + \gamma < 1$ we get that

$$\lim_{z\to 1} g(z) = \lim_{z \to 1} \frac{1 - z^{-\alpha}}{(z-1)^\gamma} = \lim_{z\to 1} \frac{1 + \alpha z^{-\alpha-1}}{\gamma (z-1)^{\gamma-1}} = 0,$$

and

$$\lim_{z\to\infty} g(z) = 0.$$

Moreover, for $2 \le z \le \infty$ we get the upper bound

$$0 \le g(z) \le \frac{1 - z^{-\alpha}}{(z-1)^\gamma} < \frac{1}{1} = 1,$$

and for $1 < z < 2$ we have that

$$g(z) = \frac{z^\alpha - 1}{(z-1)^\gamma z^\alpha} < \frac{z - 1}{(z-1)^\gamma (z-1)^\alpha} = (z-1)^{1-\gamma-\alpha} \le 1.$$

This shows inequality (5.59) which then implies for $0 < \gamma < H$ that

$$\begin{aligned} f_t(\theta) - f_t(\theta') &= \left(\frac{t}{\theta}(t-\theta)\right)^{H-\frac{1}{2}} - \left(\frac{t}{\theta'}(t-\theta')\right)^{H-\frac{1}{2}} \\ &\lesssim \left(\frac{t}{\theta}(t-\theta)\right)^{H-\frac{1}{2}-\gamma} t^{2\gamma} \frac{(\theta-\theta')^\gamma}{(\theta\theta')^\gamma} \lesssim (t-\theta)^{H-\frac{1}{2}-\gamma} \frac{(\theta-\theta')^\gamma}{(\theta\theta')^\gamma}. \end{aligned}$$

Further,

$$\begin{aligned} g_t(\theta) - g_t(\theta') &= \int_\theta^t \frac{f_u(\theta) - f_u(\theta')}{u} du - \int_{\theta'}^\theta \frac{f_u(\theta')}{u} du \\ &\le \int_\theta^t \frac{f_u(\theta) - f_u(\theta')}{u} du \\ &\lesssim \frac{(\theta-\theta')^\gamma}{(\theta\theta')^\gamma} \int_\theta^t \frac{(u-\theta)^{H-\frac{1}{2}-\gamma}}{u} du \\ &\le \frac{(\theta-\theta')^\gamma}{(\theta\theta')^\gamma} \theta^{H-\frac{1}{2}-\gamma} \int_1^\infty \frac{(v-1)^{H-\frac{1}{2}-\gamma}}{v} dv \\ &\lesssim \frac{(\theta-\theta')^\gamma}{(\theta\theta')^\gamma} \theta^{H-\frac{1}{2}-\gamma} \end{aligned}$$

$$\lesssim \frac{(\theta-\theta')^{\gamma}}{(\theta\theta')^{\gamma}}\theta^{H-\frac{1}{2}-\gamma}(t-\theta)^{H-\frac{1}{2}-\gamma}.$$

Consequently, we get for $\gamma \in (0,H)$, $0<\theta'<\theta<t\leq T$, that

$$K_H(t,\theta)-K_H(t,\theta') \leq C\cdot c_H \frac{(\theta-\theta')^{\gamma}}{(\theta\theta')^{\gamma}}\theta^{H-\frac{1}{2}-\gamma}(t-\theta)^{H-\frac{1}{2}-\gamma},$$

where $C>0$ is a constant merely depending on T. Thus

$$\begin{aligned}
&\int_0^t\int_0^\theta \frac{(K_H(t,\theta)-K_H(t,\theta'))^2}{|\theta-\theta'|^{1+2\beta}}d\theta' d\theta \\
&\quad \lesssim \int_0^t\int_0^\theta \frac{|\theta-\theta'|^{-1-2\beta+2\gamma}}{(\theta\theta')^{2\gamma}}\theta^{2H-1-2\gamma}(t-\theta)^{2H-1-2\gamma}d\theta' d\theta \\
&\quad = \int_0^t \theta^{2H-1-4\gamma}(t-\theta)^{2H-1-2\gamma}\int_0^\theta |\theta-\theta'|^{-1-2\beta+2\gamma}(\theta')^{-2\gamma}d\theta' d\theta \\
&\quad = \int_0^t \theta^{2H-1-4\gamma-2\beta}(t-\theta)^{2H-1-2\gamma}\frac{\Gamma(-2\beta+2\gamma)\Gamma(-2\gamma+1)}{\Gamma(-2\beta+1)}d\theta \\
&\quad \lesssim \int_0^t \theta^{2H-1-4\gamma-2\beta}(t-\theta)^{2H-1-2\gamma}d\theta \\
&\quad = \frac{\Gamma(2H-2\gamma)\Gamma(2H-4\gamma-2\beta)}{\Gamma(4H-6\gamma-2\beta)}t^{4H-6\gamma-2\beta-1} < \infty,
\end{aligned}$$

for sufficiently small γ and β. On the other hand, we have that

$$\begin{aligned}
&\int_0^t\int_\theta^t \frac{(K_H(t,\theta)-K_H(t,\theta'))^2}{|\theta-\theta'|^{1+2\beta}}d\theta' d\theta \\
&\quad \lesssim \int_0^t \theta^{2H-1-4\gamma}(t-\theta)^{2H-1-2\gamma}\int_\theta^t \frac{|\theta-\theta'|^{-1-2\beta+2\gamma}}{(\theta')^{2\gamma}}d\theta' d\theta \\
&\quad \leq \int_0^t \theta^{2H-1-6\gamma}(t-\theta)^{2H-1-2\gamma}\int_\theta^t |\theta-\theta'|^{-1-2\beta+2\gamma}d\theta' d\theta \\
&\quad \lesssim \int_0^t \theta^{2H-1-6\gamma}(t-\theta)^{2H-1-2\beta}d\theta \lesssim t^{4H-6\gamma-2\beta-1}.
\end{aligned}$$

Therefore,

$$\int_0^t\int_0^t \frac{(K_H(t,\theta)-K_H(t,\theta'))^2}{|\theta-\theta'|^{1+2\beta}}d\theta' d\theta < \infty.$$

□

Lemma B.4 *Let $H\in\left(0,\frac{1}{2}\right)$, $0\leq\theta<t\leq T$ and $(\varepsilon_1,\dots,\varepsilon_n)\in\{0,1\}^n$ be fixed. Assume $w_j+\left(H-\frac{1}{2}-\gamma\right)\varepsilon_j>-1$ for all $j=1,\dots,n$. Then there exists a finite constant $C_{H,T}>0$ depending only on H and T such that for $\gamma\in(0,H)$*

$$\int_{\Delta^n_{\theta,t}}\prod_{j=1}^n (K_H(s_j,\theta)-K_H(s_j,\theta'))^{\varepsilon_j}|s_j-s_{j-1}|^{w_j}ds$$

$$\leq \left(C_{H,T} \left(\frac{\theta - \theta'}{\theta\theta'} \right)^{\gamma} \theta^{\left(H-\frac{1}{2}-\gamma\right)} \right)^{\sum_{j=1}^{n} \varepsilon_j} \Pi_{\gamma}(n)\,(t-\theta)^{\sum_{j=1}^{n}\left(w_j+\left(H-\frac{1}{2}-\gamma\right)\varepsilon_j\right)+n},$$

where

$$\Pi_{\gamma}(m) := \frac{\prod_{j=1}^{n} \Gamma(w_j+1)}{\Gamma\left(\sum_{j=1}^{n} w_j + \left(H-\frac{1}{2}-\gamma\right)\sum_{j=1}^{n}\varepsilon_j + n\right)}. \tag{5.60}$$

Proof. Recall, that for given exponents $a, b > -1$ and some fixed $s_{j+1} > s_j$ we have

$$\int_{\theta}^{s_{j+1}} (s_{j+1}-s_j)^a (s_j-\theta)^b ds_j = \frac{\Gamma(a+1)\,\Gamma(b+1)}{\Gamma(a+b+2)} (s_{j+1}-\theta)^{a+b+1}.$$

Due to Lemma B.3 we have that for every $\gamma \in (0, H)$, $0 < \theta' < \theta < s_j \leq T$,

$$K_H(s_j,\theta) - K_H(s_j,\theta') \leq C_{H,T} \frac{(\theta-\theta')^{\gamma}}{(\theta\theta')^{\gamma}} \theta^{H-\frac{1}{2}-\gamma}(s_j-\theta)^{H-\frac{1}{2}-\gamma},$$

for $C_{H,T} := C \cdot c_H$, where c_H is the constant in (5.14) and $C > 0$ is some constant merely depending on T. Consequently, we get that

$$\begin{aligned}
&\int_{\theta}^{s_2} |K_H(s_1,\theta) - K_H(s_1,\theta')|^{\varepsilon_1} |s_2-s_1|^{w_2} |s_1-\theta|^{w_1} ds_1 \\
&\leq C_{H,T}^{\varepsilon_1} \frac{(\theta-\theta')^{\gamma\varepsilon_1}}{(\theta\theta')^{\gamma\varepsilon_1}} \theta^{\left(H-\frac{1}{2}-\gamma\right)\varepsilon_1} \int_{\theta}^{s_2} |s_2-s_1|^{w_2} |s_1-\theta|^{w_1+\left(H-\frac{1}{2}-\gamma\right)\varepsilon_1} ds_1 \\
&= C_{H,T}^{\varepsilon_1} \frac{(\theta-\theta')^{\gamma\varepsilon_1}}{(\theta\theta')^{\gamma\varepsilon_1}} \theta^{\left(H-\frac{1}{2}-\gamma\right)\varepsilon_1} \frac{\Gamma(\hat{w}_1)\,\Gamma(\hat{w}_2)}{\Gamma(\hat{w}_1+\hat{w}_2)} (s_2-\theta)^{w_1+w_2+\left(H-\frac{1}{2}-\gamma\right)\varepsilon_1+1},
\end{aligned}$$

where

$$\hat{w}_1 := w_1 + \left(H - \frac{1}{2} - \gamma\right)\varepsilon_1 + 1, \quad \hat{w}_2 := w_2 + 1.$$

Noting that

$$\prod_{j=1}^{n-1} \frac{\Gamma\left(\sum_{l=1}^{j} w_l + \left(H-\frac{1}{2}-\gamma\right)\sum_{l=1}^{j}\varepsilon_l + j\right)\Gamma(w_{j+1}+1)}{\Gamma\left(\sum_{l=1}^{j+1} w_l + \left(H-\frac{1}{2}-\gamma\right)\sum_{l=1}^{j}\varepsilon_l + j + 1\right)} \leq \Pi_{\gamma}(n).$$

and iterative integration yields the desired formula. □

Finally, we are able to give the proof of Proposition B.2.

Proof of Proposition B.2. The integration by parts formula (5.49) yields that

$$\int_{\Delta_{\theta,t}^n} \left(\prod_{j=1}^{n} D^{\alpha_j} f_j(s_j, \widehat{B}_{s_j}) \varkappa_j(s_j) \right) ds = \int_{\mathbb{R}^{dn}} \Lambda_{\alpha}^{\varkappa f}(\theta,t,z) dz.$$

Taking the expectation and applying Theorem B.1 we get that

$$\mathbb{E}\left[\left| \int_{\Delta_{\theta,t}^n} \left(\prod_{j=1}^{n} D^{\alpha_j} f_j(s_j, \widehat{B}_{s_j}) \varkappa_j(s_j) \right) ds \right| \right]$$

$$\leq \frac{T^{\frac{|\alpha|}{12}}}{\sqrt{2\pi}^{dn}} (\Psi_\alpha^\varkappa(\theta,t,H,d))^{\frac{1}{2}} \prod_{j=1}^n \|f_j\|_{L^1(\mathbb{R}^d;L^\infty([0,T]))},$$

where

$$\Psi_\alpha^\varkappa(\theta,t,H,d) := \left(\prod_{k=1}^d \sqrt{(2\,|\alpha^{(k)}|)!}\right)$$
$$\times \sum_{\sigma\in\mathcal{S}(n,n)} \int_{\Delta_{0,t}^{2n}} |\Delta s|^{-H\left(1+\alpha_{[\sigma(\Delta)]}\right)} \prod_{j=1}^{2n} (K_{H_m}(s_j,\theta) - K_{H_m}(s_j,\theta'))^{\varepsilon_{[\sigma(j)]}} ds.$$

Under the assumption $-\sum_{k=1}^d H_k(1+\alpha_{[\sigma(j)]}^{(k)}+\alpha_{[\sigma(j-1)]}^{(k)})+(H_m-\frac{1}{2}-\gamma_m)\varepsilon_{[\sigma(j)]} > -1$ for all $j=1,...,2n$, we can apply Lemma B.4 and thus get

$$\Psi_\alpha^\varkappa(\theta,t,H,d)$$
$$\leq \sum_{\sigma\in\mathcal{S}(n,n)} \left(C_T\left(\frac{\theta-\theta'}{\theta\theta'}\right)^{\gamma_m} \theta^{(H_m-\frac{1}{2}-\gamma_m)}\right)^{\sum_{j=1}^{2n}\varepsilon_{[\sigma(j)]}} \Pi_\gamma(2n)$$
$$\times \left(\prod_{k=1}^d \sqrt{(2\,|\alpha^{(k)}|)!}\right)(t-\theta)^{-\sum_{k=1}^d H_k\left(2n+4\left|\alpha^{(k)}\right|\right)+(H_m-\frac{1}{2}-\gamma_m)\sum_{j=1}^{2n}\varepsilon_{[\sigma(j)]}+2n},$$

where $\Pi_\gamma(2n)$ is defined as in (5.60). We define the constant $K_{d,H}$ by

$$K_{d,H} := 2 \sup_{j=1,\dots,2n} \Gamma\left(1-\sum_{k=1}^d H_k\left(1+\alpha_{[\sigma(j)]}^{(k)}+\alpha_{[\sigma(j-1)]}^{(k)}\right)\right) \tag{5.61}$$

and thus an upper bound of $\Pi_\gamma(2n)$ is given by

$$\Pi_\gamma(2n) \leq \frac{K_{d,H}^{2n}}{2^{2n}\Gamma\left(-\sum_{k=1}^d H_k\left(2n+4\,|\alpha^{(k)}|\right)+\left(H_m-\frac{1}{2}-\gamma_m\right)\sum_{j=1}^{2n}\varepsilon_{[\sigma(j)]}+2n\right)}.$$

Note that $\sum_{j=1}^{2n}\varepsilon_{[\sigma(j)]} = 2\sum_{j=1}^n \varepsilon_j$ and

$$\#\mathcal{S}(n,n) = \binom{2n}{n} = \frac{2^{2n}}{\sqrt{\pi}}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} \leq 2^{2n}.$$

Hence, it follows that

$$(\Psi_k^\varkappa(\theta,t,H,d))^{\frac{1}{2}}$$
$$\leq K_{d,H}^n\left(C_T\left(\frac{\theta-\theta'}{\theta\theta'}\right)^{\gamma_m}\theta^{(H_m-\frac{1}{2}-\gamma_m)}\right)^{\sum_{j=1}^n\varepsilon_j}$$
$$\times \frac{\left(\prod_{k=1}^d\left(2\left|\alpha^{(k)}\right|\right)!\right)^{\frac{1}{4}}(t-\theta)^{-\sum_{k=1}^d H_k\left(n+2\left|\alpha^{(k)}\right|\right)+\left(H_m-\frac{1}{2}-\gamma_m\right)\sum_{j=1}^n\varepsilon_j+n}}{\Gamma\left(2n-\sum_{k=1}^d H_k\left(2n+4\,|\alpha^{(k)}|\right)+2\left(H_m-\frac{1}{2}-\gamma_m\right)\sum_{j=1}^n\varepsilon_j\right)^{\frac{1}{2}}},$$

□

Proposition B.5 *Let the functions f and $\varkappa$ be defined as in* (5.42) *and* (5.43), *respectively. Let $0 \leq \theta < t \leq T$ and*

$$\varkappa_j(s) = (K_{H_m}(s,\theta))^{\varepsilon_j}, \theta < s < t,$$

for every $j = 1, \dots, n$ with $(\varepsilon_1, \dots, \varepsilon_n) \in \{0,1\}^n$. Let $\alpha \in (\mathbb{N}_0^d)^n$ be a multi-index and suppose that there exists δ such that

$$-\sum_{k=1}^d H_k \left(1 + 2\alpha_j^{(k)}\right) + \left(H_m - \frac{1}{2}\right) \geq \delta > -1$$

for all $j = 1, \dots, n$ and $d \geq 1$. Then there exist constants C_T (depending on T) and $K_{d,H}$ (depending on d and H) such that for any $0 \leq \theta < t \leq T$ we have

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_{\Delta^n_{\theta,t}} \left(\prod_{j=1}^n D^{\alpha_j} f_j(s_j, \widehat{B}_{s_j}) \varkappa_j(s_j)\right) ds\right|\right] \\
&\leq \frac{K_{d,H}^n \cdot T^{\frac{|\alpha|}{12}}}{\sqrt{2\pi}^{dn}} \left(C_T \theta^{(H_m - \frac{1}{2})}\right)^{\sum_{j=1}^n \varepsilon_j} \prod_{j=1}^n \|f_j(\cdot, z_j)\|_{L^1(\mathbb{R}^d; L^\infty([0,T]))} \\
&\times \frac{\left(\prod_{k=1}^d \left(2\left|\alpha^{(k)}\right|\right)!\right)^{\frac{1}{4}} (t-\theta)^{-\sum_{k=1}^d H_k\left(n + 2\left|\alpha^{(k)}\right|\right) + \left(H_m - \frac{1}{2}\right)\sum_{j=1}^n \varepsilon_j + n}}{\Gamma(2n - \sum_{k=1}^d H_k(2n + 4\left|\alpha^{(k)}\right|) + 2(H_m - \frac{1}{2})\sum_{j=1}^n \varepsilon_j)^{\frac{1}{2}}}.
\end{aligned}$$

The proof of Proposition B.5 is similar to the one of Proposition B.2 by using the subsequent lemma instead of Lemma B.4 and thus it is omitted in this manuscript.

Lemma B.6 *Let $H \in \left(0, \frac{1}{2}\right)$, $0 \leq \theta < t \leq T$ and $(\varepsilon_1, \dots, \varepsilon_n) \in \{0,1\}^n$ be fixed. Assume $w_j + \left(H - \frac{1}{2}\right)\varepsilon_j > -1$ for all $j = 1, \dots, n$. Then there exists a finite constant $C_{H,T} > 0$ depending only on H and T such that*

$$\begin{aligned}
&\int_{\Delta^n_{\theta,t}} \prod_{j=1}^n (K_H(s_j,\theta))^{\varepsilon_j} |s_j - s_{j-1}|^{w_j} ds \\
&\qquad \leq \left(C_{H,T}\theta^{\left(H-\frac{1}{2}\right)}\right)^{\sum_{j=1}^n \varepsilon_j} \Pi_0(n)\,(t-\theta)^{\sum_{j=1}^n \left(w_j + \left(H - \frac{1}{2}\right)\varepsilon_j\right) + n},
\end{aligned}$$

where Π_0 is defined in (5.60).

Proof. Using similar arguments as in the proof of Lemma B.3 we get the following estimate

$$|K_H(s_j,\theta)| \leq C_{H,T}|s_j - \theta|^{H-\frac{1}{2}}\theta^{H-\frac{1}{2}}$$

for every $0 < \theta < s_j < T$ and $C_{H,T} := C \cdot c_H$, where c_H is the constant in (5.14) and $C > 0$ is some constant merely depending on T. Thus,

$$\begin{aligned}
&\int_\theta^{s_2} (K_H(s_1,\theta))^{\varepsilon_1} |s_2 - s_1|^{w_2} |s_1 - \theta|^{w_1} ds_1 \\
&\quad \leq C_{H,T}^{\varepsilon_1} \theta^{\left(H-\frac{1}{2}\right)\varepsilon_1} \int_\theta^{s_2} |s_2 - s_1|^{w_2} |s_1 - \theta|^{w_1 + \left(H - \frac{1}{2}\right)\varepsilon_1} ds_1
\end{aligned}$$

$$= C_{H,T}^{\varepsilon_1}\,\theta^{\left(H-\frac{1}{2}\right)\varepsilon_1}\frac{\Gamma\left(w_1+\left(H-\frac{1}{2}\right)\varepsilon_1+1\right)\Gamma\left(w_2+1\right)}{\Gamma\left(w_1+w_2+\left(H-\frac{1}{2}\right)\varepsilon_1+2\right)}(s_2-\theta)^{w_1+w_2+\left(H-\frac{1}{2}\right)\varepsilon_1+1}.$$

Proceeding similar to the proof of Lemma B.4 yields the desired estimate. □

Remark B.7. Note that

$$\prod_{k=1}^{d}\left(2\left|\alpha^{(k)}\right|\right)! \le \sqrt{2\pi}^{d} e^{\frac{|\alpha|}{2}}\frac{\Gamma\left(\frac{5}{2}|\alpha|+1\right)}{\sqrt{5\pi|\alpha|}}.$$

Indeed, since for $n \ge 1$ sufficiently large we have by Stirling's formula that

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n} \le n! \le e^{\frac{1}{12n}}\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n},$$

we get by assuming without loss of generality that $|\alpha^{(k)}| \ge 1$ for all $1 \le k \le d$, that

$$\begin{aligned}
\prod_{k=1}^{d}\left(2|\alpha^{(k)}|\right)! &\le \prod_{k=1}^{d} e^{\frac{1}{24|\alpha^{(k)}|}}\sqrt{4\pi|\alpha^{(k)}|}\left(\frac{2|\alpha^{(k)}|}{e}\right)^{2|\alpha^{(k)}|}\\
&\le e^{\frac{d}{24}}\sqrt{\frac{8}{5}\pi}^{d}\prod_{k=1}^{d}\left(\frac{5}{2}|\alpha^{(k)}|\right)^{\frac{|\alpha^{(k)}|}{2}}\left(\frac{\frac{5}{2}|\alpha^{(k)}|}{e}\right)^{2|\alpha^{(k)}|}\\
&\le \sqrt{2\pi}^{d}\prod_{k=1}^{d}\left(\frac{\frac{5}{2}|\alpha|}{e}\right)^{\frac{5}{2}|\alpha^{(k)}|}e^{\frac{|\alpha^{(k)}|}{2}}\\
&\le \sqrt{2\pi}^{d}e^{\frac{|\alpha|}{2}}\left(\frac{\frac{5}{2}|\alpha|}{e}\right)^{\frac{5}{2}|\alpha|} \le \sqrt{2\pi}^{d}e^{\frac{|\alpha|}{2}}\frac{\Gamma\left(\frac{5}{2}|\alpha|+1\right)}{\sqrt{5\pi|\alpha|}}.
\end{aligned}$$

Appendix C. Technical Results

The following technical result can be found in [26].

Lemma C.1 *Assume that $X_1, ..., X_n$ are real centered jointly Gaussian random variables, and $\Sigma = (\mathbb{E}[X_jX_k])_{1\le j,k\le n}$ is the covariance matrix, then*

$$\mathbb{E}[|X_1|\ldots|X_n|] \le \sqrt{\mathrm{perm}(\Sigma)},$$

where perm(A) *is the permanent of a matrix $A = (a_{ij})_{1\le i,j\le n}$ defined by*

$$\mathrm{perm}(A) = \sum_{\pi\in\mathcal{S}_n}\prod_{j=1}^{n} a_{j,\pi(j)}$$

for the symmetric group $\mathcal{S}_n$.

The next lemma corresponds to [12, Lemma 2]:

Lemma C.2 *Let $Z_1, ..., Z_n$ be mean zero Gaussian random variables which are linearly independent. Then for any measurable function $g : \mathbb{R} \longrightarrow \mathbb{R}_+$ we have that*

$$\int_{\mathbb{R}^n} g(v_1) e^{-\frac{1}{2}\operatorname{Var}\left(\sum_{j=1}^n v_j Z_j\right)} dv_1 ... dv_n = \frac{(2\pi)^{\frac{n-1}{2}}}{(\det \operatorname{Cov}(Z_1, ..., Z_n))^{\frac{1}{2}}} \int_{\mathbb{R}} g\left(\frac{v}{\sigma_1}\right) e^{-\frac{v^2}{2}} dv,$$

where $\sigma_1^2 := \operatorname{Var}(Z_1 \,|Z_2, ..., Z_n)$.

Remark C.3. Note that here linearly independence is meant in the sense that $\det \operatorname{Cov}(Z_1, ..., Z_n) \neq 0$.

Lemma C.4 *Let $a \in \ell^p$, $1 \leq p < \infty$. Then, for every $n \geq 1$ and $d \geq 1$*

$$\sum_{k_1,...,k_n=1}^{d} \prod_{j=1}^{n} a_{k_j} = \left(\sum_{k=1}^{d} a_k\right)^n, \tag{5.62}$$

and

$$\lim_{d\to\infty} \sum_{k_1,...,k_n}^{d} \prod_{j=1}^{n} |a_{k_j}|^p = (\|a\|_{\ell^p})^n . \tag{5.63}$$

Proof. We proof equation (5.62) by induction. For $n = 1$ the result holds. Therefore we assume that (5.62) holds for n and we show that it also holds for $n + 1$. Thus, we get by the induction hypothesis that

$$\begin{aligned} \sum_{k_1,...,k_{n+1}=1}^{d} \prod_{j=1}^{n+1} a_{k_j} &= \sum_{k_{n+1}=1}^{d} a_{k_{n+1}} \left(\sum_{k_1,...,k_n=1}^{d} \prod_{j=1}^{n} a_{k_j} \right) \\ &= \sum_{k_{n+1}=1}^{d} a_{k_{n+1}} \left(\sum_{k=1}^{d} a_k\right)^n = \left(\sum_{k=1}^{d} a_k\right)^{n+1}. \end{aligned}$$

Equation (5.63) is an immediate consequence of (5.62) and the continuity of the function $f(x) = x^n$ for fixed $n \geq 1$. ☐

The subsequent lemmas are due to [4].

Lemma C.5 *Let $(X_1, \dots, X_n)$ be a mean-zero Gaussian random vector. Then,*

$$\det \operatorname{Cov}(X_1, \dots, X_n) = \operatorname{Var}(X_1)\operatorname{Var}(X_2|X_1) \cdots \operatorname{Var}(X_n|X_{n-1}, \dots, X_1).$$

Lemma C.6 *For any square integrable random variable X and σ-algebras $\mathcal{G}_1 \subset \mathcal{G}_2$*

$$\operatorname{Var}(X|\mathcal{G}_1) \geq \operatorname{Var}(X|\mathcal{G}_2).$$

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Chapter 6

McKean-Vlasov equations on infinite-dimensional Hilbert spaces with irregular drift and additive fractional noise

Contribution of the thesis' author

The paper *McKean-Vlasov equations on infinite-dimensional Hilbert spaces with irregular drift and additive fractional noise* is a joint work with Prof. Dr. Thilo Meyer-Brandis.

M. Bauer was significantly involved in the development of all parts of the paper. In particular, M. Bauer made major contributions to the editorial work and the proofs of Theorem 3.4, Theorem 3.8, and Theorem 4.4.

MCKEAN-VLASOV EQUATIONS ON INFINITE-DIMENSIONAL HILBERT SPACES WITH IRREGULAR DRIFT AND ADDITIVE FRACTIONAL NOISE

MARTIN BAUER AND THILO MEYER-BRANDIS

Abstract. This paper establishes results on the existence and uniqueness of solutions to McKean-Vlasov equations, also called mean-field stochastic differential equations, in an infinite-dimensional Hilbert space setting with irregular drift. Here, McKean-Vlasov equations with additive noise are considered where the driving noise is cylindrical (fractional) Brownian motion. The existence and uniqueness of weak solutions are established for drift coefficients that are merely measurable, bounded, and continuous in the law variable. In particular, the drift coefficient is allowed to be singular in the spatial variable. Further, we discuss existence of a pathwisely unique strong solution as well as Malliavin differentiability.

Keywords. McKean-Vlasov equation · mean-field stochastic differential equation · weak solution · strong solution · uniqueness in law · pathwise uniqueness · singular coefficients · fractional Brownian motion · fractional calculus · Malliavin derivative.

1. Introduction

Throughout the paper let $T > 0$ be a finite time horizon and let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space. McKean-Vlasov (for short MKV) equations, also called mean-field stochastic differential equations, are an extension of stochastic differential equations, where the coefficients in addition to time and space are depending on the law of the solution. More precisely, a finite-dimensional McKean-Vlasov equation is commonly defined as

$$dX_t = b\left(t, X_t, \mathbb{P}_{X_t}\right) dt + \sigma\left(t, X_t, \mathbb{P}_{X_t}\right) dB_t,\ t \in [0,T],\ X_0 = x \in \mathbb{R}^d, \tag{6.1}$$

where $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R}^{d\times n}$ are measurable functions, $\mathcal{P}_1(\mathbb{R}^d)$ is the set of probability measures over $\mathbb{R}^d$ with finite first moment, $(\mathbb{P}_{X_t})_{t\in[0,T]}$ denotes the law of $(X_t)_{t\in[0,T]}$ under the probability measure $\mathbb{P}$, and $B = (B_t)_{t\in[0,t]}$ is n-dimensional Brownian motion.

The field of MKV equations is a research area that currently gains broad attention. Developing historically from the works of Vlasov [36], Kac [21], and McKean [26] on the modeling of particle systems in mathematical physics, an increased interest in MKV equations emerged following the work of Lasry and Lions [23] who applied the mean-field approach to topics in Economics and Finance. Later Carmona and Delarue transfered this approach on mean-field games to a probabilistic environment, cf. the manuscript [14] and the cited sources therein.

In this paper we extend the finite-dimensional MKV equation (6.1) to infinite dimensions and further consider cylindrical fractional Brownian motion as additive driving noise, i.e. we look at MKV equations of the form

$$X_t = x + \int_0^t b(s, X_s, \mathbb{P}_{X_s})ds + \mathbb{B}_t, \quad t \in [0,T], \quad x \in \mathcal{H}, \tag{6.2}$$

on a separable Hilbert space $\mathcal{H}$. Here, $\mathbb{B} = (\mathbb{B}_t)_{t\in[0,T]}$ is (weighted) cylindrical fractional Brownian motion defined as

$$\mathbb{B}_t = \sum_{k\geq 1} \lambda_k B_t^{H_k} e_k, \quad t \in [0,T],$$

where $\lambda = \{\lambda_k\}_{k\geq 1} \in \ell^1$, $\{e_k\}_{k\geq 1}$ is an orthonormal basis of $\mathcal{H}$, and $\{B^{H_k}\}_{k\geq 1}$ a sequence of independent one-dimensional fractional Brownian motions with Hurst parameters $\mathbb{H} := \{H_k\}_{k\geq 1} \subset (0,1)$. Note that Hurst parameters in the entire range $(0,1)$ are admitted, and we introduce the following partition: $I_- := \{k : H_k \in (0,1/2)\}$, $I_0 := \{k : H_k = 1/2\}$, and $I_+ := \{k : H_k \in (1/2,1)\}$. The main objective of this paper is to study existence and uniqueness of a solution to the infinite-dimensional MKV equation (6.2) for irregular drift coefficients b.

In the literature existence and uniqueness of solutions of the finite-dimensional MKV equation (6.1) is examined in several papers with respect to various assumptions on the coefficients b and σ, c.f. [7], [6], [8], [11], [12], [13], [15], [16], [19], [20], [24], [25], [28], and [32]. In particular, in [24] Li and Min show the existence of a weak solution of a path dependent finite-dimensional MKV equation by the means of Girsanov's theorem and Schauder's fixed point theorem, where they assume that b is merely measurable and bounded as well as continuous in the law variable. Further, uniqueness in law is proven under the additional assumption that b admits a modulus of continuity. Mishura and Veretennikov show in [28] inter alia the existence of a pathwise unique strong solution to a finite-dimensional MKV equation (6.1), where they assume the drift coefficient b to be merely measurable, of at most linear growth, and continuous in the law variable in the topology of weak convergence. For their proof they use an approximational approach based on techniques applied by Krylov in the theory of stochastic differential equations, cf. [22]. In [6], we consider MKV equation (6.1) with additive noise, i.e. $\sigma \equiv 1$, and singular drift coefficients b. More precisely, for b being bounded and continuous in the law variable with respect to the Kantorovich-Rubinstein metric, it is shown that there exists a Malliavin differentiable strong solution of MKV equation (6.1). For one-dimensional solutions of (6.1) we even allow for certain linear growth behavior of the drift in [8]. In [7] mean-field SDEs are considered where the dependence on the law is in form of a Lebesgue integral. In this case existence of a unique strong solution is shown for singular drift coefficients that might allow for discontinuities in the law variable. We also remark here that the existence of a weak solution for another class of mean-field SDEs that are related to Fokker-Plank equations where the drift coefficient might allow for discontinuities in the law variable is shown in [3], [4], and [5].

Using similar approaches as in [6] and [8], in this paper existence of a weak solution to the infinite-dimensional MKV equation (6.2) is established under the assumption that the drift coefficient b is in the space $L^\infty(\mathcal{H})$, i.e. there exists a sequence $C \in \ell^1$ such that $\|b_k\|_\infty \leq C_k$ for every $b_k := \langle b, e_k \rangle_{\mathcal{H}}$, $k \geq 1$, and for $k \in I_+$ the projection of the drift b_k is Hölder continuous, i.e.

$$|b_k(t,x,\mu) - b_k(s,y,\nu)| \leq C_k \left(|t-s|^{\gamma_k} + \|x-y\|_{\mathcal{H}}^{\alpha_k} + \mathcal{K}(\mu,\nu)^{\beta_k} \right),$$

for suitable constants $C_k, \gamma_k, \alpha_k, \beta_k > 0$, and $\mathcal{K}$ denotes the Kantorovich-Rubinstein metric, cf. (6.4). For $k \in I_- \cup I_0$ it is assumed that the projection b_k is merely continuous with respect to the law variable. More precisely, in order to show existence of a weak solution we first apply Girsanov's theorem to show the existence of a weak solution to the stochastic differential equation, for short SDE,

$$dX_t^\mu = b\left(t, X_t^\mu, \mu_t\right) dt + d\mathbb{B}_t, \; t \in [0,T], \; X_0 = x \in \mathcal{H},$$

where $\mu \in \mathcal{C}([0,T]; \mathcal{P}_1(\mathcal{H}))$ is an arbitrary measure process continuous with respect to time. Afterwards Schauder's fixed point theorem [33] is applied to the function

$$\varphi(\mu) = \mathbb{P}_{X_t^\mu}$$

to show the existence of a fixed point and in particular, to conclude existence of a weak solution to MKV equation (6.2).

Assuming additionally that the drift coefficient b is Lipschitz continuous in the law variable, it is shown that the solution of the infinite-dimensional MKV equation (6.2) is unique in law. In order to show uniqueness in law, we apply similar to [6] and [8] Girsanov's theorem and a Grönwall type argument.

Existence of a strong solution to MKV equation (6.2) is then a consequence of results on ordinary SDEs. Indeed, we can associate the following SDE to MKV equation (6.2):

$$dY_t = b^{\mathbb{P}_X}\left(t, Y_t\right) dt + d\mathbb{B}_t, \; t \in [0,T], \; Y_0 = x \in \mathcal{H}, \tag{6.3}$$

where $b^{\mathbb{P}_X}\left(t,y\right) := b\left(t,y,\mathbb{P}_{X_t}\right)$ and X is a weak solution of MKV equation (6.2). In order to show that (6.2) has a strong solution, it suffices to show that there exists a weak solution that is measurable with respect to the filtration generated by the driving noise $\mathbb{B}$. Since X is as a weak solution to MKV equation (6.2) also a weak solution of SDE (6.3), it is sufficient to show that every weak solution Y of SDE (6.3) is a strong solution. Furthermore, if MKV equation (6.2) has a weakly unique solution, the associated SDE (6.3) is uniquely determined and consequently, pathwise uniqueness of the solution Y of SDE (6.3) implies pathwise uniqueness of the solution X of MKV equation (6.2). Thus, applying existence results on SDEs as for example stated in [2], [27], [30], and [34], yields existence of a (pathwisely unique) strong solution of MKV equation (6.2). Analogously, Malliavin differentiability of the solution to MKV equation (6.2) is deduced from results on SDEs, cf. [6] and [8].

The paper is structured as follows. In Section 2 we give a brief introduction to measure spaces, fractional calculus, and fractional Brownian motion. After introducing the driving noise $\mathbb{B}$ and a version of Girsanov's theorem, we present in Section 3 the main results of this paper on existence and uniqueness of a weak solution to the infinite-dimensional MKV equation (6.2). Concluding, existence of a unique strong solution to MKV equation (6.2) and Malliavin differentiability are discussed in Section 4.

Notation: Subsequently, we give some of the most frequently used notations. Throughout the paper, let $\mathcal{H}$ be a separable Hilbert space with scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and orthonormal basis $\{e_k\}_{k\geq 1} \subset \mathcal{H}$. Denote by $\|\cdot\|_{\mathcal{H}}$ the induced norm on $\mathcal{H}$ defined by $\|x\|_{\mathcal{H}} := \langle x, x\rangle_{\mathcal{H}}^{\frac{1}{2}}$, $x \in \mathcal{H}$. For every $x \in \mathcal{H}$ and $k \geq 1$ we denote by $x^{(k)} := \langle x, e_k\rangle_{\mathcal{H}}$ the projection onto the subspace spanned by e_k. We denote by $b_k : [0,T] \times \mathcal{H} \times \mathcal{P}_1(\mathcal{H}) \to \mathbb{R}$, the projection of b onto the subspace spanned by e_k, $k \geq 1$. Furthermore, we assume for technical reasons that without loss of generality $T \geq 1$.

Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}}), (\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be two normed spaces.

- $L^p(\mathcal{X}; \mathcal{Y})$ denotes the space of functions $f : \mathcal{X} \to \mathcal{Y}$ with existing p-th moment, i.e.
$$\int_{\mathcal{X}} \|f(x)\|_{\mathcal{Y}}^p dx < \infty.$$
If $\mathcal{X} = [a,b]$ is an interval on the real line and $\mathcal{Y} = \mathbb{R}$, we write $L^p[a,b]$.
- $\mathcal{C}^\kappa([0,T]; \mathcal{X})$, $\kappa > 0$, is defined as the space of κ-Hölder continuous functions $f : [0,T] \to \mathcal{X}$, i.e. for all $t, s \in [0,T]$
$$\|f(t) - f(s)\|_{\mathcal{X}} \leq |t-s|^\kappa.$$
- We denote by $\mathrm{Lip}_C(\mathcal{X}; \mathcal{Y})$, $C > 0$ the space of C-Lipschitz continuous functions $f : \mathcal{X} \to \mathcal{Y}$, i.e. for all $x_1, x_2 \in \mathcal{X}$
$$\|f(x_1) - f(x_2)\|_{\mathcal{Y}} \leq C\|x_1, x_2\|_{\mathcal{X}}.$$
- For a function $f : \mathcal{X} \to \mathcal{Y}$ define $\|f\|_{\mathrm{Lip}} := \inf\{C > 0 : f \in \mathrm{Lip}_C(\mathcal{X}; \mathcal{Y})\}$ and $\|f\|_\infty := \sup_{x\in\mathcal{X}} \|f(x)\|_{\mathcal{Y}}$. We define the bounded Lipschitz norm of f as $\|f\|_{\mathrm{BL}} := \|f\|_\infty + \|f\|_{\mathrm{Lip}}$. We say $f \in \mathrm{BL}(\mathcal{X}; \mathcal{Y})$, if $\|f\|_{\mathrm{BL}} \leq 1$.
- The Beta function β is defined by
$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt.$$
- The Gamma function Γ is defined by
$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

- We write $E_1(\theta) \lesssim E_2(\theta)$ for two mathematical expressions $E_1(\theta), E_2(\theta)$ depending on some parameter θ, if there exists a constant $C > 0$ not depending on θ such that $E_1(\theta) \leq CE_2(\theta)$.
- Let $C = \{C_k\}_{k\geq 1}$ and $D = \{D_k\}_{k\geq 1}$ be two sequences. Then, we denote $\frac{C}{D} := \{\frac{C_k}{D_k}\}_{k\geq 1}$.

2. Framework

2.1. **Measure Spaces.** For a general introduction to (probability) measures on metric spaces we refer the reader e.g. to [1]. Let $(\mathcal{S}, d)$ be a complete separable metric space, in particular, $(\mathcal{S}, d)$ is a Radon space. We define the space $\mathcal{M}(\mathcal{S})$ as the space of finite signed Radon measures on $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$, where $\mathcal{B}(\mathcal{S})$ is the Borel-σ-algebra on $\mathcal{S}$. Moreover, let

$$\mathcal{M}_p(\mathcal{S}) := \left\{\mu \in \mathcal{M}(\mathcal{S}) : \int_{\mathcal{S}} d(x, x_0)^p |\mu|(dx) < \infty \text{ for some } x_0 \in \mathcal{S}\right\},$$

be the set of finite signed Radon measures over $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ with finite p-th moment. $\mathcal{M}_1(\mathcal{S})$ equipped with the *Kantorovich norm* $\|\cdot\|_{\mathcal{K}}$, also called *dual bounded Lipschitz norm*, defined by

$$\|\mu\|_{\mathcal{K}} := \sup\left\{\int_{\mathcal{S}} f(x)\mu(dx) : \|f\|_{\mathrm{BL}} \leq 1\right\}, \quad \mu \in \mathcal{M}_1(\mathcal{S}),$$

defines a separable Banach space. Analogously, define the according Kantorovich-Rubinstein metric $\mathcal{K}$ by

$$\mathcal{K}(\mu, \nu) := \|\mu - \nu\|_{\mathcal{K}}, \quad \mu, \nu \in \mathcal{M}_1(\mathcal{S}). \tag{6.4}$$

Let $\mathcal{P}_p(\mathcal{S}) \subset \mathcal{M}_p(\mathcal{S})$ be the set of probability measures over $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ such that the p-th moment exists, i.e.

$$\mathcal{P}_p(\mathcal{S}) := \{\mu \in \mathcal{M}_p(\mathcal{S}) : \mu(\mathcal{S}) = 1 \text{ and } \mu(A) \geq 0 \text{ for all } A \in \mathcal{B}(\mathcal{S})\}.$$

Lastly, define the set of continuous functions $\mathcal{C}([0,T]; \mathcal{M}_1(\mathcal{S}))$ from the time interval $[0,T]$ to the space $\mathcal{M}_1(\mathcal{S})$ and equip it with the norm $\|\mu\|_{\mathcal{K}^*} := \sup_{t\in[0,T]} \|\mu_t\|_{\mathcal{K}}$, $\mu \in \mathcal{C}([0,T]; \mathcal{M}_1(\mathcal{S}))$. It can be shown that $(\mathcal{C}([0,T]; \mathcal{M}_1(\mathcal{S})), \|\cdot\|_{\mathcal{K}^*})$ is a linear separable Banach space.

2.2. **Fractional Calculus.** We give some basic definitions and properties on fractional calculus. For a general theory on this subject we refer the reader to [31].

Let $f \in L^p[a,b]$ for some real numbers $a < b$, where $p \geq 1$, and let $\alpha > 0$. The *left–sided Riemann–Liouville fractional integral* is defined for almost all $x \in [a,b]$ by

$$I^\alpha_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) dy.$$

Moreover, we denote by $I^\alpha_{a+}(L^p[a,b])$ the image of $L^p[a,b]$ by the operator I^α_{a+}.

For $g \in I^{\alpha}_{a+}(L^p[a,b])$ and $0 < \alpha < 1$, the *left–sided Riemann–Liouville fractional derivative* is defined by

$$D^{\alpha}_{a+} g(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial x} \int_a^x \frac{g(y)}{(x-y)^{\alpha}} dy. \tag{6.5}$$

The left–sided derivative of g defined in (6.5) can further be written as

$$D^{\alpha}_{a+} g(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{g(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{g(x)-g(y)}{(x-y)^{\alpha+1}} dy \right).$$

Similar to the fundamental theorem of calculus the following formulas hold

$$I^{\alpha}_{a+}(D^{\alpha}_{a+} f) = f$$

for all $f \in I^{\alpha}_{a+}(L^p[a,b])$ and

$$D^{\alpha}_{a+}(I^{\alpha}_{a+} f) = f$$

for all $f \in L^p[a,b]$.

2.3. **Fractional Brownian motion.** In this section we recall the definition of a fractional Brownian motion and how it can be constructed from a standard Brownian motion using fractional calculus. For a more detailed introduction to this subject we refer the reader to [9] and [29, Chapter 5]

Definition 2.1 We say $B^H = \left(B^H_t\right)_{t\in[0,T]}$ is a one-dimensional fractional Brownian motion with Hurst parameter $H \in (0,1)$, if it is a continuous and centered Gaussian process with covariance function

$$R_H(t,s) := \mathbb{E}\left[B^H_t B^H_s\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

It is well-known that B^H has stationary increments and $(H-\varepsilon)$–Hölder continuous trajectories for all $\varepsilon > 0$. Furthermore, B^H is not a semimartingale and its increments are not independent for all $H \in (0,1)$ but $H = \frac{1}{2}$. For $H = \frac{1}{2}$ the process B^H is a standard Brownian motion.

In the following we divide fractional Brownian motions into three classes by their Hurst parameters. The first class, $H \in (0, \frac{1}{2})$, is referred to as the *singular case*, the second class, $H \in (\frac{1}{2}, 1)$, is referred to as the *regular case*, and the third class, $H = \frac{1}{2}$, is the class of Brownian motions. Subsequently, we define for each class the kernels K_H as well as the related operators $\mathbf{K}_H$ and $\mathbf{K}_H^{-1}$ which allow us to construct a fractional Brownian motion with Hurst parameter $H \in (0,1)$ from a standard Brownian motion. For more details see [17] and [30]. Let $W = (W_t)_{t\in[0,T]}$ be a standard Brownian motion on the complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

Singular Case: Let $H \in (0, \frac{1}{2})$ and define the kernel

$$K_H(t,s) = b_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} + \left(\frac{1}{2} - H\right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right], \tag{6.6}$$

where $b_H = \sqrt{\frac{2H}{(1-2H)\beta(1-2H, H+\frac{1}{2})}}$. Then

$$B_t^H := \int_0^t K_H(t,s) dW_s$$

is a fractional Brownian motion with Hurst parameter H. Furthermore, the kernel K_H yields an operator $\mathbf{K}_H : L^2[0,T] \to I_{0+}^{H+\frac{1}{2}}(L^2[0,T])$ defined by

$$(\mathbf{K}_H f)(s) = \int_0^t K_H(t,s) f(s) ds = I_{0+}^{2H} s^{\frac{1}{2}-H} I_{0+}^{\frac{1}{2}-H} s^{H-\frac{1}{2}} f,$$

where $f \in L^2[0,T]$. Finally, the inverse operator $\mathbf{K}_H^{-1}$ of $\mathbf{K}_H$ is defined by

$$\mathbf{K}_H^{-1} f = s^{\frac{1}{2}-H} D_{0+}^{\frac{1}{2}-H} s^{H-\frac{1}{2}} D_{0+}^{2H} f, \tag{6.7}$$

where $f \in I_{0+}^{H+\frac{1}{2}}(L^2[0,T])$. If f is absolutely continuous, we can write

$$\mathbf{K}_H^{-1} f = s^{H-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H} s^{\frac{1}{2}-H} f'.$$

Regular Case: Let $H \in (\frac{1}{2}, 1)$ and define the kernel

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{1}{2}} (u-s)^{H-\frac{3}{2}} du, \tag{6.8}$$

where $c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$. Then

$$B_t^H := \int_0^t K_H(t,s) dW_s$$

is a fractional Brownian motion with Hurst parameter H. Furthermore, the kernel K_H yields an operator $\mathbf{K}_H : L^2[0,T] \to I_{0+}^{H+\frac{1}{2}}(L^2[0,T])$ defined by

$$(\mathbf{K}_H f)(s) = \int_0^t K_H(t,s) f(s) ds = I_{0+}^{1} s^{H-\frac{1}{2}} I_{0+}^{H-\frac{1}{2}} s^{\frac{1}{2}-H} f,$$

where $f \in L^2[0,T]$. Finally, the inverse operator $\mathbf{K}_H^{-1}$ of $\mathbf{K}_H$ is defined by

$$\mathbf{K}_H^{-1} f = s^{H-\frac{1}{2}} D_{0+}^{H-\frac{1}{2}} s^{\frac{1}{2}-H} f', \tag{6.9}$$

where $f \in I_{0+}^{H+\frac{1}{2}}(L^2[0,T])$.

Brownian case: Let $H = \frac{1}{2}$. Obviously, in the case $H = \frac{1}{2}$ the kernel is given by $K_H(t,s) \equiv 1$. Thus the operator $\mathbf{K}_H$ is defined as

$$(\mathbf{K}_H f)(s) = \int_0^t K_H(t,s) f(s) ds = I_{0+}^1 f,$$

where $f \in L^2[0,T]$, and thus its inverse operator $\mathbf{K}_H^{-1}$ is given by

$$\mathbf{K}_H^{-1} f = f', \tag{6.10}$$

where $f \in I_{0+}^1(L^2[0,T])$.

Remark 2.2. Consider a sequence $\mathbb{H} = \{H_k\}_{k\geq 1}$ of Hurst parameters. For the Hilbert space $\mathcal{H}$ with basis $\{e_k\}_{k\geq 1}$ and $f \in L^2([0,T];\mathcal{H})$, we define the operator $\mathbf{K}_\mathbb{H} : L^2([0,T];\mathcal{H}) \to I_{0+}^{\mathbb{H}+1/2}(L^2([0,T];\mathcal{H}))$ componentwise by

$$(\mathbf{K}_\mathbb{H} f)(s) := \sum_{k\geq 1} (\mathbf{K}_{H_k} f_k)(s) e_k,$$

where $f_k(s) := \langle f(s), e_k \rangle$, $k \geq 1$. Here, we say $f \in I_{0+}^{\mathbb{H}+1/2}(L^2([0,T];\mathcal{H}))$, if for every $k \geq 1$ the projection f_k is in $I_{0+}^{H_k+1/2}(L^2[0,T])$. Similarly we define the inverse $\mathbf{K}_\mathbb{H}^{-1}$ of $\mathbf{K}_\mathbb{H}$ by

$$\mathbf{K}_\mathbb{H}^{-1} f := \sum_{k\geq 1} \mathbf{K}_{H_k}^{-1} f_k e_k,$$

where $f \in I_{0+}^{\mathbb{H}+1/2}(L^2([0,T];\mathcal{H}))$.

2.4. **The weighted cylindrical fractional Brownian motion $\mathbb{B}$.** Let us now define the driving noise $\mathbb{B}$ and afterwards derive a version of Girsanov's theorem for cylindrical fractional Brownian motion. Let $\{W^{(k)}\}_{k\geq 1}$ be a sequence of independent Brownian motions defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Similar to [2] we define the *cylindrical Brownian motion* $W := (W_t)_{t\in[0,T]}$ taking values in $\mathcal{H}$ by

$$W_t := \sum_{k\geq 1} W_t^{(k)} e_k, \quad t \in [0,T].$$

The natural filtration of W augmented by the $\mathbb{P}$-null sets is denoted by $\mathbb{F}^W := (\mathcal{F}_t^W)_{t\in[0,T]}$. Moreover, we consider a sequence of Hurst parameters $\mathbb{H} = \{H_k\}_{k\geq 1}$ and the associated partition $\{I_-, I_0, I_+\}$ of $\mathbb{N}$ defined by

(i) $k \in I_- : \ H_k \in \left(0, \frac{1}{2}\right)$,
(ii) $k \in I_0 : \ H_k = \frac{1}{2}$,
(iii) $k \in I_+ : \ H_k \in \left(\frac{1}{2}, 1\right)$.

For $\{H_k\}_{k\geq 1}$ we construct the sequence of fractional Brownian motions $\{B^{H_k}\}_{k\geq 1}$ associated to $\{W^{(k)}\}_{k\geq 1}$ by

$$B_t^{H_k} := \int_0^t K_{H_k}(t,s)dW_s^{(k)}, \quad t\in[0,T], \quad k\geq 1,$$

where the kernel $K_{H_k}(\cdot,\cdot)$ is defined in (6.6) and (6.8), respectively. Note that by construction the fractional Brownian motions $\{B^{H_k}\}_{k\geq 1}$ are independent. We then define the *cylindrical fractional Brownian motion* $B^{\mathbb{H}}$ with associated sequence of Hurst parameters $\mathbb{H} = \{H_k\}_{k\geq 1}$ by

$$B_t^{\mathbb{H}} := \sum_{k\geq 1} B_t^{H_k} e_k, \quad t\in[0,T].$$

Observe that the natural filtration of $B^{\mathbb{H}}$ augmented by the $\mathbb{P}$-null sets and $\mathbb{F}^W$ coincide. Furthermore, for a given sequence $\lambda := \{\lambda_k\}_{k\geq 1} \in \ell^1$ such that $\sum_{k\in I_-} \frac{\lambda_k}{\sqrt{H_k}} < \infty$, we define the self-adjoint operator $Q : \mathcal{H} \to \mathcal{H}$ by

$$Qx = \sum_{k\geq 1} \lambda_k^2 x^{(k)} e_k,$$

and thereby construct the *weighted cylindrical fractional Brownian motion* $\mathbb{B}$ by

$$\mathbb{B}_t := \sqrt{Q} B_t^{\mathbb{H}} = \sum_{k\geq 1} \lambda_k B_t^{H_k} e_k, \quad t\in[0,T]. \tag{6.11}$$

Due to the following lemma, the process $\mathbb{B}$ is continuous in time and is in $L^2(\Omega;\mathcal{H})$.

Lemma 2.3 *The weighted cylindrical fractional Brownian motion $\mathbb{B}$ defined in* (6.11) *has almost surely continuous sample paths on* $[0,T]$ *and*

$$\sup_{t\in[0,T]} \mathbb{E}\left[\|\mathbb{B}_t\|_{\mathcal{H}}^2\right] < \infty.$$

Proof. Note first that for every $k\in I_-$ and time points $s,t\in[0,T]$, the fractional Brownian motion B^{H_k} fulfills

$$\mathbb{E}\left[\left|\left|B_t^{H_k}\right| - \left|B_s^{H_k}\right|\right|^2\right]^{\frac{1}{2}} \leq \mathbb{E}\left[\left|B_t^{H_k} - B_s^{H_k}\right|^2\right]^{\frac{1}{2}} = |t-s|^{H_k}.$$

Hence due to [10, Theorem 1] the expected maximum of $|B^{H_k}|$ is bounded by

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|B_t^{H_k}\right|\right] = T^{H_k}\mathbb{E}\left[\sup_{t\in[0,1]} \left|B_t^{H_k}\right|\right] \lesssim \frac{T^{H_k}}{\sqrt{H_k}}.$$

In the case of a standard Brownian motion, i.e. $H = \frac{1}{2}$, the exact value of the expected maxima is known and is equal to $\sqrt{\frac{2T}{\pi}}$. Using Sudakov-Fernique's inequality (see [35, Theorem 1]) we thus get for $k\in I_0\cup I_+$ the of H_k independent upper bound

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|B_t^{H_k}\right|\right] \leq T^{H_k - \frac{1}{2}}\mathbb{E}\left[\sup_{t\in[0,T]} \left|W_t^{(k)}\right|\right] = T^{H_k}\sqrt{\frac{2}{\pi}} \leq T\sqrt{\frac{2}{\pi}}.$$

Let us now consider the weighted cylindrical fractional Brownian motion $\mathbb{B}$ defined in (6.11). Using the previous bounds we have that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|\mathbb{B}_t\|_{\mathcal{H}}\right] = \mathbb{E}\left[\sup_{t\in[0,T]} \left\|\sum_{k\geq 1} \lambda_k B_t^{H_k} e_k\right\|_{\mathcal{H}}\right] \leq \sum_{k\geq 1} \lambda_k \mathbb{E}\left[\sup_{t\in[0,T]} \left|B_t^{H_k}\right|\right]$$
$$\lesssim \sum_{k\in I_-} \frac{\lambda_k T^{H_k}}{\sqrt{H_k}} + \sum_{k\in I_0\cup I_+} \lambda_k \lesssim \sum_{k\in I_-} \frac{\lambda_k}{\sqrt{H_k}} + \|\lambda\|_{\ell^1} < \infty.$$

Consequently, the stochastic process $\mathbb{B}$ is almost surely finite and the sequence of projections $\{\sum_{k=1}^n \langle \mathbb{B}, e_k\rangle_{\mathcal{H}} e_k\}_{n\geq 1}$ is a Cauchy sequence in $L^1(\Omega; \mathcal{C}([0,T];\mathcal{H}))$ converging almost surely to the process $\mathbb{B}$. Thus, $t \mapsto \mathbb{B}_t$ is continuous on $[0,T]$. Furthermore, using Parseval's identity we get

$$\mathbb{E}\left[\|\mathbb{B}_t\|_{\mathcal{H}}^2\right] = \mathbb{E}\left[\left\|\sum_{k\geq 1} \lambda_k B_t^{H_k} e_k\right\|_{\mathcal{H}}^2\right] = \sum_{k\geq 1} \lambda_k^2 \mathbb{E}\left[\left|B_t^{H_k}\right|^2\right] = \sum_{k\geq 1} \lambda_k^2 t^{2H_k} \leq \|\lambda\|_{\ell^2}^2 T^2 < \infty.$$

□

2.5. **Girsanov's theorem for cylindrical fractional Brownian motions.** Due to [2, Theorem 2.2 and Remark 2.3] we get the following version of Girsanov's theorem for cylindrical fractional Brownian motions.

Theorem 2.4 (Girsanov's theorem for fBm) *Let $u = \{u_t, t \in [0,T]\}$ be an $\mathbb{F}^W$-adapted process with values in $\mathcal{H}$ and integrable trajectories. If*

(i) $\int_0^\cdot u_s^{(k)} ds \in I_{0+}^{H_k+\frac{1}{2}}(L^2[0,T])$, *$\mathbb{P}$-a.s. for every $k \geq 1$, and*
(ii) $\mathbb{E}\left[\exp\left\{\sum_{k\geq 1}\int_0^T \mathbf{K}_{H_k}^{-1}\left(\int_0^\cdot u_r^{(k)} dr\right)^2 (s) ds\right\}\right] < \infty$,

where $\mathbf{K}_{H_k}^{-1}$ is defined as in (6.7), (6.9), *and* (6.10), *respectively, then the shifted process*

$$\widetilde{B}_t^{\mathbb{H}} := B_t^{\mathbb{H}} + \int_0^t u_s ds = \sum_{k\geq 1}\left(B_t^{H_k} + \int_0^t u_s^{(k)} ds\right) e_k,$$

is a cylindrical fractional Brownian motion with associated sequence of Hurst parameters $\mathbb{H} = \{H_k\}_{k\geq 1}$ under the new probability measure $\widetilde{\mathbb{P}}$ defined by $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} := \mathcal{E}_T$, where

$$\mathcal{E}_T := \exp\left\{\sum_{k\geq 1}\left(\int_0^T \mathbf{K}_{H_k}^{-1}\left(\int_0^\cdot u_r^{(k)} dr\right)(s) dW_s^{(k)} - \frac{1}{2}\int_0^T \mathbf{K}_{H_k}^{-1}\left(\int_0^\cdot u_r^{(k)} dr\right)^2 (s) ds\right)\right\}. \tag{6.12}$$

It is shown in [30] that in the case $k \in I_- \cup I_0$ it is sufficient to assume $\int_0^T |u_s^{(k)}|^2 ds < \infty$ such that for $u^{(k)}$ condition (i) in Theorem 2.4 is fulfilled. In the case $k \in I_+$ condition (i) in Theorem 2.4 is fulfilled if the process $u^{(k)}$ is assumed to have Hölder continuous trajectories of order $H_k - \frac{1}{2} + \varepsilon$ for some $\varepsilon > 0$. If we assume further that

(ii*) $\int_0^T \mathbf{K}_{H_k}^{-1} \left(\int_0^{\cdot} u_r^{(k)} dr \right)^2 (s) ds \leq D_k$ $\mathbb{P}$-a.s. for all $k \geq 1$,

where $D = \{D_k\}_{k \geq 1} \in \ell^1$ is a sequence of constants, then assumption (ii) is also fulfilled and thus Girsanov's theorem is applicable. We summarize these observations in the following corollary.

Corollary 2.5 *Let $(u_t)_{t\in[0,T]}$ be an $\mathbb{F}^W$-adapted process such that $\int_0^T |u_s^{(k)}|^2 ds < \infty$ for all $k \in I_- \cup I_0$, and for $k \in I_+$ the process $u^{(k)}$ has Hölder continuous trajectories of order $H_k - \frac{1}{2} + \varepsilon$ for some $\varepsilon > 0$. Furthermore, assume that condition (ii*) is fulfilled. Then, conditions (i) and (ii) in Theorem 2.4 are satisfied, and thus the stochastic exponential* (6.12) *defines the Radon-Nikodym density of a probability measure. Moreover, for every $p \in [0, \infty)$*

$$\mathbb{E}[|\mathcal{E}_T|^p] < \infty.$$

3. Existence and Uniqueness of Weak Solutions

In this section we proof under sufficient conditions on the drift function b the existence and uniqueness of weak solutions to the MKV equation (6.2), where the weighted cylindrical fractional Brownian motion is characterized by a given sequence of Hurst parameters $\mathbb{H}$ and the weighting operator Q. We show first existence of a weak solution using Theorem 2.4 and Schauder's fixed point theorem. Afterwards weak uniqueness of the solution is proven. Let us first recall the definition of a weak solution and uniqueness in law, and then state the main result of this section.

Definition 3.1 We say the six-tuple $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \mathbb{B}, X)$ is a *weak solution* of MKV equation (6.2), if

(i) $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and $\mathbb{F} := \{\mathcal{F}_t\}_{t\in[0,T]}$ is a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the usual conditions of right-continuity and completeness,
(ii) $X = (X_t)_{t\in[0,T]}$ is a continuous, $\mathbb{F}$-adapted, $\mathcal{H}$-valued process; $\mathbb{B} := (\mathbb{B}_t)_{t\in[0,T]}$ is a weighted cylindrical fractional Brownian motion with respect to $(\mathbb{F}, \mathbb{P})$,
(iii) X satisfies $\mathbb{P}$-a.s. MKV equation (6.2), where $\mathbb{P}_{X_t} \in \mathcal{P}_1(\mathcal{H})$ denotes for all $t \in [0, T]$ the law of X_t with respect to $\mathbb{P}$.

Remark 3.2. We merely say that X is a weak solution of MKV equation (6.2), if there is no ambiguity about the filtered stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \mathbb{B})$.

Definition 3.3 A weak solution $(\Omega^1, \mathcal{F}^1, \mathbb{F}^1, \mathbb{P}^1, \mathbb{B}^1, X^1)$ of MKV equation (6.2) is called *unique in law*, if for any other weak solution $(\Omega^2, \mathcal{F}^2, \mathbb{F}^2, \mathbb{P}^2, \mathbb{B}^2, X^2)$ of (6.2) it holds that $\mathbb{P}^1_{X^1} = \mathbb{P}^2_{X^2}$, whenever $\mathbb{P}^1_{X^1_0} = \mathbb{P}^2_{X^2_0}$.

Theorem 3.4 *Let $b : [0,T] \times \mathcal{H} \times \mathcal{P}_1(\mathcal{H}) \to \mathcal{H}$ be a measurable function such that $\|b_k\|_\infty \leq C_k\lambda_k$ for all $k \geq 1$, where $\frac{C}{\sqrt{1-\mathbb{H}}} \in \ell^1$ for $C := \{C_k\}_{k\geq 1}$ and assume that*

$$\left(\sum_{k\geq 1} \lambda_k^2 (t-s)^{2H_k}\right)^{\frac{1}{2}} \leq \rho|t-s|^\kappa,$$

where $\rho > 0$ and $0 < \kappa < 1$ are constants. Furthermore, assume that in the case $k \in I_+$,

$$|b_k(t,x,\mu) - b_k(s,y,\nu)| \leq C_k\lambda_k \left(|t-s|^{\gamma_k} + \|x-y\|_{\mathcal{H}}^{\alpha_k} + \mathcal{K}(\mu,\nu)^{\beta_k}\right), \tag{6.13}$$

where $\gamma_k > H_k - \frac{1}{2}$, $2 \geq \kappa\alpha_k > 2H_k - 1$, and $\kappa\beta_k > H_k - \frac{1}{2}$, and in the case $k \in I_- \cup I_0$ that for every $\mu \in \mathcal{C}([0,T];\mathcal{P}_1(\mathcal{H}))$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $k \geq 1$ and $\nu \in \mathcal{C}([0,T];\mathcal{P}_1(\mathcal{H}))$

$$\sup_{t\in[0,T]} \mathcal{K}(\mu_t,\nu_t) < \delta \;\Rightarrow \sup_{t\in[0,T],\; y\in\mathcal{H}} |b_k(t,y,\mu_t) - b_k(t,y,\nu_t)| < \varepsilon C_k\lambda_k. \tag{6.14}$$

Then, MKV equation (6.2) has a weak solution.

The proof of Theorem 3.4 is divided into two main steps. First we show using Theorem 2.4 that for every $\mu \in \mathcal{C}^\kappa([0,T];\mathcal{P}_1(\mathcal{H}))$, for some suitable $\kappa > 0$, the (distribution dependent) SDE

$$dX_t^\mu = b\left(t, X_t^\mu, \mu_t\right) dt + d\mathbb{B}_t, \quad t \in [0,T], \quad X_0^\mu = x, \tag{6.15}$$

has a weak solution. Second, we apply Schauder's fixed point theorem, see [33], to find a solution of MKV equation (6.2). Let us start with the application of Girsanov's theorem in the following lemma.

Lemma 3.5 *Let $b : [0,T] \times \mathcal{H} \times \mathcal{P}_1(\mathcal{H}) \to \mathcal{H}$ be a measurable function such that $\|b_k\|_\infty \leq C_k\lambda_k$ for all $k \geq 1$, where $\frac{C}{\sqrt{1-\mathbb{H}}} \in \ell^1$. Furthermore, assume that for every $k \in I_+$ the function b_k fulfills assumption (6.13). Then for every $\mu \in \mathcal{C}^\kappa([0,T];\mathcal{P}_1(\mathcal{H}))$, SDE (6.15) has a weak solution which is unique in law.*

Proof. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space with a sequence of independent Brownian motions $\{W^{(k)}\}_{k\geq 1}$ defined thereon. Following the constructions in Section 2.4, we define the cylindrical fractional Brownian motion $B^{\mathbb{H}}$ with associated sequence of Hurst parameters $\mathbb{H} = \{H_k\}_{k\geq 1}$ generated by W. Further, we define the process $X_t^\mu := x + \sqrt{Q}B_t^{\mathbb{H}}$, $t \in [0,T]$. If $u_t := \sqrt{Q}^{-1} b(t, X_t^\mu, \mu_t)$, $t \in [0,T]$, fulfills the assumptions of Corollary 2.5, we get due to Theorem 2.4 that the process

$$B_t^{\mathbb{H},\mu} := B_t^{\mathbb{H}} - \int_0^t \sqrt{Q}^{-1} b\left(u, x + B_u^{\mathbb{H}}, \mu_u\right) du, \quad t \in [0,T],$$

is a cylindrical fractional Brownian motion with respect to the probability measure $\mathbb{P}^\mu$ defined by $\frac{d\mathbb{P}^\mu}{d\mathbb{P}} := \mathcal{E}_T^\mu$, where

$$\mathcal{E}_T^\mu := \exp\left\{\sum_{k\geq 1}\left(\int_0^T \mathbf{K}_{H_k}^{-1}\left(\int_0^\cdot u_r^{(k)}dr\right)(s)dW_s^{(k)} - \frac{1}{2}\int_0^T \mathbf{K}_{H_k}^{-1}\left(\int_0^\cdot u_r^{(k)}dr\right)^2(s)ds\right)\right\}. \tag{6.16}$$

Consequently, the sextuple $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^\mu, \sqrt{Q}B^{\mathbb{H},\mu}, X^\mu)$ is a weak solution of SDE (6.15). Thus it is left to show that u fulfills the assumptions of Corollary 2.5.

Let $k \in I_- \cup I_0$. Then,

$$\int_0^T |u_s^{(k)}|^2 ds = \int_0^T |\lambda_k^{-1} b_k(s, X_s^\mu, \mu_s)|^2 ds \leq TC_k^2 < \infty,$$

where we have used that b_k is bounded by $\lambda_k C_k$. Consider now the case $k \in I_+$, then we get for $t, s \in [0,T]$ that

$$\begin{aligned}
\mathbb{E}\left[\left|u_t^{(k)} - u_s^{(k)}\right|\right] &= \lambda_k^{-1}\mathbb{E}[|b_k(t, X_t^\mu, \mu_t) - b_k(s, X_s^\mu, \mu_s)|] \\
&\leq C_k\left(|t-s|^{\gamma_k} + \mathbb{E}\left[\left\|\sqrt{Q}B_t^{\mathbb{H}} - \sqrt{Q}B_s^{\mathbb{H}}\right\|_{\mathcal{H}}^{\alpha_k}\right] + \mathcal{K}(\mu_t, \mu_s)^{\beta_k}\right) \\
&\leq C_k\left(|t-s|^{\gamma_k} + \left(\sum_{j\geq 1}\mathbb{E}\left[\lambda_j^2\left|B_t^{H_j} - B_s^{H_j}\right|^2\right]\right)^{\frac{\alpha_k}{2}} + |t-s|^{\kappa\beta_k}\right) \\
&\leq C_k\left(|t-s|^{\gamma_k} + \left(\sum_{j\geq 1}\lambda_j^2|t-s|^{2H_j}\right)^{\frac{\alpha_k}{2}}\right) \\
&\lesssim C_k\left(|t-s|^{\gamma_k} + |t-s|^{\frac{\kappa\alpha_k}{2}}\right) \lesssim |t-s|^{\gamma_k} + |t-s|^{\frac{\kappa\alpha_k}{2}},
\end{aligned} \tag{6.17}$$

where we have assumed without loss of generality that $\gamma_k = \kappa\beta_k$. Due to Kolmogorov's continuity theorem and the assumptions $\gamma_k > H_k - \frac{1}{2}$ and $2 \geq \kappa\alpha_k > 2H_k - 1$, we get that $u^{(k)}$ is $(H_k - \frac{1}{2} + \varepsilon)$–Hölder continuous in $t \in [0,T]$ for some $\varepsilon > 0$ and hence, assumption (i) of Theorem 2.4 is fulfilled for all $k \geq 1$ due to Corollary 2.5. Next, we show that assumption (ii*) holds, i.e. for all $k \geq 1$

$$\int_0^T \mathbf{K}_{H_k}^{-1}\left(\int_0^\cdot u_r^{(k)}dr\right)^2(s)ds \leq D_k,$$

where $D = \{D_k\}_{k\geq 1} \in \ell^1$. Consider first the case $k \in I_0$, then

$$\int_0^T \mathbf{K}_{H_k}^{-1}\left(\int_0^\cdot u_r^{(k)}dr\right)^2(s)ds = \int_0^T |\lambda_k^{-1}b_k(s, X_s^\mu, \mu_s)|^2 ds \leq TC_k^2,$$

and thus we define $D_k := TC_k^2$ for $k \in I_0$. In the case $k \in I_-$ it is shown in [2] that

$$\int_0^T \mathbf{K}_{H_k}^{-1}\left(\int_0^\cdot u_r^{(k)}dr\right)^2(s)ds \lesssim T^2C_k^2,$$

and thus we define $D_k := T^2 C_k^2$ for $k \in I_-$. Last, we consider the case $k \in I_+$ and get that

$$
\begin{aligned}
&\left|\mathbf{K}_{H_k}^{-1}\left(\int_0^\cdot u_r^{(k)} dr\right)(s)\right| = \left|\mathbf{K}_{H_k}^{-1}\left(\int_0^\cdot \lambda_k^{-1} b_k(r, X_r^\mu, \mu_r) dr\right)(s)\right| \\
&\quad \le \frac{C_k s^{\frac{1}{2}-H_k}}{\Gamma\left(\frac{3}{2}-H_k\right)} + \frac{\left(H_k - \frac{1}{2}\right) s^{H_k - \frac{1}{2}}}{\lambda_k \Gamma\left(\frac{3}{2}-H_k\right)} \left|\int_0^s \frac{b_k(s, X_s^\mu, \mu_s) s^{\frac{1}{2}-H_k} - b_k(r, X_r^\mu, \mu_r) r^{\frac{1}{2}-H_k}}{(s-r)^{H_k+\frac{1}{2}}} dr\right| \\
&\quad \le \frac{C_k s^{\frac{1}{2}-H_k}}{\Gamma\left(\frac{3}{2}-H_k\right)} + \frac{\left(H_k - \frac{1}{2}\right) s^{H_k - \frac{1}{2}}}{\lambda_k \Gamma\left(\frac{3}{2}-H_k\right)} \left(\int_0^s |b_k(s, X_s^\mu, \mu_s)| \frac{r^{\frac{1}{2}-H_k} - s^{\frac{1}{2}-H_k}}{(s-r)^{H_k+\frac{1}{2}}} dr\right. \\
&\quad \left. + \int_0^s r^{\frac{1}{2}-H_k} \frac{|b_k(s, X_s^\mu, \mu_s) - b_k(r, X_r^\mu, \mu_r)|}{(s-r)^{H_k+\frac{1}{2}}} dr\right). \qquad (6.18)
\end{aligned}
$$

Due to (6.17) there exists $\varepsilon > 0$ such that for all $k \in I_+$

$$|b_k(s, X_s^\mu, \mu_s) - b_k(r, X_r^\mu, \mu_r)| \lesssim C_k \lambda_k |s-r|^{H_k - \frac{1}{2} + \varepsilon}.$$

Thus, (6.18) can be further bounded by

$$
\begin{aligned}
\left|\mathbf{K}_{H_k}^{-1}\left(\int_0^\cdot u_r^{(k)} dr\right)(s)\right| &\lesssim \frac{C_k s^{\frac{1}{2}-H_k}}{\Gamma\left(\frac{3}{2}-H_k\right)} + \frac{C_k \left(H_k - \frac{1}{2}\right) s^{H_k - \frac{1}{2}}}{\Gamma\left(\frac{3}{2}-H_k\right)} \\
&\quad \times \left(\int_0^s \frac{r^{\frac{1}{2}-H_k} - s^{\frac{1}{2}-H_k}}{(s-r)^{H_k+\frac{1}{2}}} dr + \int_0^s r^{\frac{1}{2}-H_k} (s-r)^{\varepsilon - 1} dr\right) \\
&\le \frac{C_k s^{\frac{1}{2}-H_k}}{\Gamma\left(\frac{3}{2}-H_k\right)} + \frac{C_k \left(H_k - \frac{1}{2}\right) s^{H_k - \frac{1}{2}}}{\Gamma\left(\frac{3}{2}-H_k\right)} \\
&\quad \times \left(s^{1-2H_k} \int_0^1 \frac{u^{\frac{1}{2}-H_k} - 1}{(1-u)^{\frac{1}{2}+H_k}} du + s^{\frac{1}{2}-H_k+\varepsilon} \beta\left(\frac{3}{2} - H_k, \varepsilon\right)\right) \\
&\le \frac{C_k s^{\frac{1}{2}-H_k}}{\Gamma\left(\frac{3}{2}-H_k\right)} + \frac{C_k \left(H_k - \frac{1}{2}\right) s^{\frac{1}{2}-H_k}}{\Gamma\left(\frac{3}{2}-H_k\right)} + \frac{C_k \left(H_k - \frac{1}{2}\right) s^{\varepsilon}}{\Gamma\left(\frac{3}{2}-H_k\right)} \beta\left(\frac{3}{2} - H_k, \varepsilon\right) \\
&\lesssim C_k s^{\frac{1}{2}-H_k} + C_k.
\end{aligned}
$$

Here, we have used that

$$\sup_{\alpha \in \left(0, \frac{1}{2}\right)} \int_0^1 \frac{u^{-\alpha} - 1}{(1-u)^{\alpha+1}} du < \infty.$$

Integrating the squared of the inverse kernel over the time interval $[0, T]$ yields

$$\int_0^T \left|\mathbf{K}_{H_k}^{-1}\left(\int_0^\cdot u_r^{(k)} dr\right)(s)\right|^2 ds \le 2C_k^2 \left(\int_0^T s^{1-2H_k} ds + 1\right) \lesssim \frac{1}{1-H_k} C_k^2,$$

and thus we define $D_k := \frac{C_k^2}{1-H_k}$ for $k \in I_+$. Finally, we see that $D \in \ell^1$. Indeed,

$$\sum_{k\geq 1} D_k = T\sum_{k\in I_0} C_k^2 + T^2 \sum_{k\in I_-} C_k^2 + \sum_{k\in I_+} \frac{C_k^2}{1-H_k} \lesssim \sum_{k\geq 1} \frac{C_k^2}{1-H_k},$$

which is finite by assumption. Thus the stochastic exponential $\mathcal{E}_T^\mu$ is well-defined and gives the probability measure $\mathbb{P}^\mu$. If $\mathcal{E}_T^\mu$ is invertible, the solution of SDE (6.15) is unique in law. Indeed, let X and Y be two solutions of SDE(6.15) with respect to the measures $\mathbb{P}$ and $\mathbb{Q}$, respectively. Then, we have for every bounded functional $f : \mathcal{H} \to \mathbb{R}$ that

$$\mathbb{E}_{\mathbb{P}}[f(X)] = \mathbb{E}_{\mathbb{P}^\mu}\left[f\left(x + \sqrt{Q}B^{\mathbb{H},\mu}\right)\eta_T\right] = \mathbb{E}_{\mathbb{Q}}[f(Y)],$$

and thus X and Y have the same law. Here,

$$\eta_T := \exp\left\{\sum_{k\geq 1}\left(-\int_0^T \mathbf{K}_{H_k}^{-1}\left(\int_0^{\cdot} u_r^{(k)}dr\right)(s)d\widetilde{W}_s^{(k)} - \frac{1}{2}\int_0^T \mathbf{K}_{H_k}^{-1}\left(\int_0^{\cdot} u_r^{(k)}dr\right)^2(s)ds\right)\right\},$$

is the inverse of $\mathcal{E}_T^\mu$, where $\widetilde{W} = \{\widetilde{W}^{(k)}\}_{k\geq 1}$ is a sequence of independent Brownian motions with respect to the measure $\mathbb{P}^\mu$ which generate the fractional Brownian motions $\{B^{H_k,\mu}\}_{k\geq 1}$.

In order to show that η_T is well-defined it suffices by Corollary 2.5 to prove that the assumptions (i) and (ii*) are fulfilled. Due to the proof of the existence of a weak solution of SDE (6.15), in particular the derivation in (6.17), it suffices to show that for every $k \in I_+$

$$\mathbb{E}\Big[|X_t^{(k),\mu} - X_s^{(k),\mu}|^2\Big] \lesssim |t-s|^{2H_k}.$$

Using Hölder's inequality and the fact that X^μ solves the SDE (6.15) we get for every $k \in I_+$ that

$$\begin{aligned}\mathbb{E}_{\mathbb{P}^\mu}\Big[|X_t^{(k),\mu} - X_s^{(k),\mu}|^2\Big] &= \mathbb{E}_{\mathbb{P}^\mu}\left[\left|\int_s^t b_k(r, X_r^\mu, \mu_r)dr + \lambda_k B_t^{H_k,\mu} - \lambda_k B_s^{H_k,\mu}\right|^2\right] \\ &\lesssim \left(C_k^2\lambda_k^2|t-s|^2 + \lambda_k^2|t-s|^{2H_k}\right) \lesssim |t-s|^{2H_k}.\end{aligned}$$

Consequently, $\mathcal{E}_T^\mu$ is invertible and thus the solution is unique in law. □

As a direct consequence of the proof of Lemma 3.5 we get under the assumption that there are no Hurst parameters of the regular case, i.e. $I_+ = \emptyset$, existence and uniqueness (in law) of a solution for an even broader class of drift coefficients b and measures μ.

Corollary 3.6 *Assume $I_+ = \emptyset$. Let $b : [0,T]\times\mathcal{H}\times\mathcal{P}_1(\mathcal{H}) \to \mathcal{H}$ be a measurable function such that $\|b_k\|_\infty \leq C_k\lambda_k$ for all $k \geq 1$, where $C \in \ell^1$. Then SDE (6.15) has a weak solution which is unique in law for every $\mu \in \mathcal{C}([0,T];\mathcal{P}_1(\mathcal{H}))$.*

Next, we come to the second step of the proof of Theorem 3.4, namely the application of Schauder's fixed point theorem, see [33].

Proof of Theorem 3.4. Define $E := \mathcal{C}^\kappa([0,T];\mathcal{P}_1(\mathcal{H})) \subset \mathcal{C}([0,T];\mathcal{M}_1(\mathcal{H}))$. Then Lemma 3.5 yields that SDE (6.15) has a weak solution X^μ which is unique in law for every $\mu \in E$.

Consider the function $\psi : E \to \mathcal{C}([0,T];\mathcal{M}_1(\mathcal{H}))$ defined by

$$\psi_s(\mu) := \mathbb{P}^{\mu}_{X_s^\mu}, \quad s \in [0,T].$$

If ψ has a fixed point, i.e. $\mu_s^* = \psi_s(\mu^*) = \mathbb{P}^{\mu^*}_{X_s^{\mu^*}}$, $s \in [0,T]$, we can insert μ^* in SDE (6.15) and consequently get a weak solution of MKV equation (6.2). In order to apply Schauder's fixed point theorem we have to verify that $(E, \|\cdot\|_{\mathcal{K}^*})$ is convex, ψ is continuous, and it exists a compact subset G of E such that $\psi(E) \subset G \subset E$.

$(E, \|\cdot\|_{\mathcal{K}^*})$ is convex. This is an immediate consequence of the definition of E and the fact that the Kantorovich-Rubinstein metric $\mathcal{K}$ is induced by the Kantorovich norm $\|\cdot\|_{\mathcal{K}}$.

ψ is continuous. Consider an arbitrary $\mu \in E$ and let $\varepsilon > 0$. Due to the continuity assumption (6.14) on b, we can find $\delta > 0$ such that for every $\nu \in E$ with $\sup_{t\in[0,T]} \mathcal{K}(\mu_t, \nu_t) < \delta$

$$\sup_{t\in[0,T], y\in\mathcal{H}} |b_k(t,y,\mu_t) - b_k(t,y,\nu_t)| < C_k \lambda_k \varepsilon, \quad k \geq 1.$$

Consequently, we get by the measure change defined in (6.16) and Cauchy-Schwarz' inequality that

$$\begin{aligned}
\mathcal{K}(\psi_t(\mu), \psi_t(\nu)) &= \sup_{h\in \mathrm{BL}(\mathcal{H};\mathbb{R})} \left| \int_{\mathcal{H}} h(y) \mathbb{P}^{\mu}_{X_t^\mu}(dy) - \int_{\mathcal{H}} h(y) \mathbb{P}^{\nu}_{X_t^\nu}(dy) \right| \\
&= \sup_{h\in \mathrm{BL}(\mathcal{H};\mathbb{R})} \left| \mathbb{E}\left[\left(h\left(\mathbb{B}_t^x\right) - h(x) \right) \mathcal{E}_T^\mu \right] - \mathbb{E}\left[\left(h\left(\mathbb{B}_t^x\right) - h(x) \right) \mathcal{E}_T^\nu \right] \right| \\
&\leq \mathbb{E}\left[\|\mathbb{B}_t\|_{\mathcal{H}} \left| \mathcal{E}_T^\mu - \mathcal{E}_T^\nu \right| \right] \leq \mathbb{E}\left[\|\mathbb{B}_t\|_{\mathcal{H}}^2 \right]^{\frac{1}{2}} \mathbb{E}\left[\left| \mathcal{E}_T^\mu - \mathcal{E}_T^\nu \right|^2 \right]^{\frac{1}{2}}.
\end{aligned}$$

Note that $\sup_{t\in[0,T]} \mathbb{E}\left[\|\mathbb{B}_t\|_{\mathcal{H}}^2 \right]$ is finite due to Lemma 2.3. We now employ the inequality

$$|e^x - e^y| \leq |x-y| \left(e^x + e^y \right), \quad x, y \in \mathbb{R}. \tag{6.19}$$

Since $\mathcal{E}_T^\mu \in L^p(\Omega)$ for every $\mu \in E$ and $1 \leq p < \infty$ by Lemma 2.3, we get again by Cauchy-Schwarz' and Minkowski's inequality that

$$\begin{aligned}
\mathbb{E}\left[\left| \mathcal{E}_T^\mu - \mathcal{E}_T^\nu \right|^2 \right]^{\frac{1}{2}} \lesssim \mathbb{E}\Bigg[\Bigg| \sum_{k\geq 1} \int_0^T \lambda_k^{-1} \bigg(&\mathbf{K}_{H_k}^{-1} \left(\int_0^\cdot b_k\left(u, \mathbb{B}_u^x, \mu_u\right) du \right)(s) \\
&- \mathbf{K}_{H_k}^{-1} \left(\int_0^\cdot b_k\left(u, \mathbb{B}_u^x, \nu_u\right) du \right)(s) \bigg) dW_s^{(k)} \Bigg|^4 \Bigg]^{\frac{1}{4}}
\end{aligned}$$

$$+\frac{1}{2}\mathbb{E}\left[\left|\sum_{k\geq 1}\int_0^T \lambda_k^{-2}\left(\mathbf{K}_{H_k}^{-1}\left(\int_0^\cdot b_k\left(u,\mathbb{B}_u^x,\mu_u\right)du\right)^2(s)\right.\right.\right.$$
$$\left.\left.\left.-\mathbf{K}_{H_k}^{-1}\left(\int_0^\cdot b_k\left(u,\mathbb{B}_u^x,\nu_u\right)du\right)^2(s)\right)ds\right|^4\right]^{\frac{1}{4}} =: A+B.$$

For A we get equivalently to Lemma 3.5 using the linearity of $\mathbf{K}_H^{-1}$ for every $H\in(0,1)$ and Burkholder-Davis-Gundy's inequality that

$$A \lesssim \mathbb{E}\left[\sum_{k\geq 1}\left(\int_0^T \frac{1}{\lambda_k^2}\mathbf{K}_{H_k}^{-1}\left(\int_0^\cdot b_k\left(u,\mathbb{B}_u^x,\mu_u\right)-b_k\left(u,\mathbb{B}_u^x,\nu_u\right)du\right)^2(s)ds\right)^2\right]^{\frac{1}{4}}$$
$$\lesssim \left(\sum_{k\geq 1} D_k\varepsilon^2\right)^{\frac{1}{2}} \lesssim \varepsilon.$$

For B note that

$$B \lesssim \mathbb{E}\left[\left|\sum_{k\geq 1}\frac{1}{\lambda_k^2}\int_0^T\left(\mathbf{K}_{H_k}^{-1}\left(\int_0^\cdot b_k\left(u,\mathbb{B}_u^x,\mu_u\right)+b_k\left(u,\mathbb{B}_u^x,\nu_u\right)du\right)(s)\right)\right.\right.$$
$$\left.\left.\times\left(\mathbf{K}_{H_k}^{-1}\left(\int_0^\cdot b_k\left(u,\mathbb{B}_u^x,\mu_u\right)-b_k\left(u,\mathbb{B}_u^x,\nu_u\right)du\right)(s)\right)ds\right|^4\right]^{\frac{1}{4}},$$

which can be bounded equivalently to A. Hence, ψ is continuous.

ψ maps E onto itself. It suffices to show that for every $\mu\in E$

$$\mathcal{K}(\psi_t(\mu),\psi_s(\mu)) \lesssim |t-s|^\kappa.$$

Let $\mu\in E$ be arbitrary and without loss of generality $s<t$. Then we get

$$\begin{aligned}\mathcal{K}(\psi_t(\mu),\psi_s(\mu)) &= \sup_{h\in \mathrm{BL}(\mathcal{H};\mathbb{R})}\left|\mathbb{E}\left[h(X_t^\mu)-h(X_s^\mu)\right]\right| \leq \mathbb{E}\left[\|X_t^\mu-X_s^\mu\|_{\mathcal{H}}^2\right]^{\frac{1}{2}}\\
&= \mathbb{E}\left[\left\|\int_s^t b\left(u,X_u^\mu,\mu_u\right)du+\mathbb{B}_t-\mathbb{B}_s\right\|_{\mathcal{H}}^2\right]^{\frac{1}{2}}\\
&\leq \left(\sum_{k\geq 1}C_k^2\lambda_k^2\right)^{\frac{1}{2}}(t-s)+\left(\sum_{k\geq 1}\lambda_k^2\mathbb{E}\left[\left|B_t^{H_k,\mu}-B_s^{H_k,\mu}\right|^2\right]\right)^{\frac{1}{2}}\\
&\leq \left(\sum_{k\geq 1}C_k^2\lambda_k^2\right)^{\frac{1}{2}}(t-s)+\left(\sum_{k\geq 1}\lambda_k^2(t-s)^{2H_k}\right)^{\frac{1}{2}} \lesssim |t-s|^\kappa.\end{aligned}$$

$\exists G\subset E$ compact such that $\psi(E)\subset G\subset E$. Define

$$\Delta := \left\{\mathbb{P}_{X_s^\mu}^\mu,\ s\in[0,T],\ \mu\in\mathcal{C}^\kappa([0,T];\mathcal{P}_1(\mathcal{H}))\right\}\subset\mathcal{P}_1(\mathcal{H}).$$

By the last step, we already know that for $s,t \in [0,T]$ and $\mu \in \mathcal{C}^\kappa([0,T];\mathcal{P}_1(\mathcal{H}))$,

$$\mathcal{K}(\mathbb{P}^\mu_{X^\mu_t}, \mathbb{P}^\mu_{X^\mu_s}) \lesssim |t-s|^\kappa.$$

Hence, $\psi(E) \subset G := \mathcal{C}^\kappa([0,T];\overline{\Delta}) \subset E$, where $\overline{\Delta}$ is the closure of Δ with respect to the Kantorovich-Rubinstein metric. If we can show that Δ is relatively compact, then G will be compact.

Indeed, note first that G is a closed set of equicontinuous functions. Moreover, for every $s \in [0,T]$ the set

$$G_s := \left\{\mathbb{P}^\mu_{X^\mu_s},\ \mu \in \mathcal{C}^\kappa([0,T];\mathcal{P}_1(\mathcal{H}))\right\} \subset \overline{\Delta}$$

is relatively compact due to the compactness of $\overline{\Delta}$. Hence, we can apply Arzelá-Ascoli's theorem which shows the compactness of G with respect to the metric induced by $\|\cdot\|_{\mathcal{K}^*}$.

In order to show relatively compactness of Δ, note first that relatively compactness of Δ is equivalent to tightness of Δ. Tightness of Δ then again is implied by uniformly integrability of the set

$$\mathcal{X} := \{X^\mu_s,\ s \in [0,T],\ \mu \in \mathcal{C}^\kappa([0,T];\mathcal{P}_1(\mathcal{H}))\}.$$

Hence, it suffices to show that

$$\sup_{s\in[0,T]} \sup_{\mu\in\mathcal{C}^\kappa([0,T];\mathcal{P}_1(\mathcal{H}))} \mathbb{E}\left[\|X^\mu_s\|^2_{\mathcal{H}}\right] < \infty,$$

but this follows directly due to Lemma 2.3 and the observation

$$\mathbb{E}\left[\|X^\mu_s\|^2_{\mathcal{H}}\right] = \mathbb{E}\left[\|x + \int_0^s b(r, X^\mu_r, \mu_r)dr + \mathbb{B}_s\|^2_{\mathcal{H}}\right] \lesssim \|x\|^2_{\mathcal{H}} + T^2\|C\lambda\|_{\ell^2} + \|\mathbb{B}_s\|^2_{\mathcal{H}}.$$

Finally, we can apply Schauder's fixed point theorem, which yields a fixed point $\mu^* = \psi(\mu^*) = \mathbb{P}^{\mu^*}_{X^{\mu^*}}$. Define $\mathbb{P} := \mathbb{P}^{\mu^*}$, $X := X^{\mu^*}$ and $B^{\mathbb{H}} := B^{\mathbb{H},\mu^*}$. Then, $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, B^{\mathbb{H}}, X)$ is a weak solution of MKV equation (6.2). □

For the case $I_+ = \emptyset$ we get an immediate extension of Theorem 3.4.

Corollary 3.7 *Assume $I_+ = \emptyset$. Let $b : [0,T]\times\mathcal{H}\times\mathcal{P}_1(\mathcal{H}) \to \mathcal{H}$ be a measurable function such that $\|b_k\|_\infty \le C_k\lambda_k$ for all $k \ge 1$, where $C \in \ell^1$, and assume that b is continuous in the sense of* (6.14). *Then, MKV equation* (6.2) *has a weak solution.*

Proof. The proof is analog to the proof of Theorem 3.4, where we define the sets

$$E := \left\{\mu \in \mathcal{C}([0,T];\mathcal{P}_1(\mathcal{H})) : \mathcal{K}(\mu_t,\mu_s) \le \left(\sum_{k\ge1} C_k^2\lambda_k^2\right)^{\frac{1}{2}}(t-s) + \left(\sum_{k\ge1}\lambda_k^2(t-s)^{2H_k}\right)^{\frac{1}{2}}\right\},$$

and

$$G := \left\{\mu \in \mathcal{C}([0,T];\overline{\Delta}) : \mathcal{K}(\mu_t,\mu_s) \le \left(\sum_{k\ge1} C_k^2\lambda_k^2\right)^{\frac{1}{2}}(t-s) + \left(\sum_{k\ge1}\lambda_k^2(t-s)^{2H_k}\right)^{\frac{1}{2}}\right\}.$$

□

Concluding this section we show that under slightly more regularity in the law variable of the drift b we get a solution which is unique in law.

Theorem 3.8 *Suppose the assumptions of Theorem 3.4 are fulfilled and in addition that* $\sup_{k\in I_+} H_k < 1$. *Furthermore, for every* $k \geq 1$ *assume that for all* $\mu, \nu \in \mathcal{P}_1(\mathcal{H})$

$$\sup_{t\in[0,T],y\in\mathcal{H}} |b_k(t,y,\mu) - b_k(t,y,\nu)| \leq C_k \lambda_k \mathcal{K}(\mu,\nu). \tag{6.20}$$

Then, MKV equation (6.2) *has a weak solution which is unique in law.*

Proof. In this proof we proceed similar to [8, Theorem 2.7]. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \mathbb{B}, X)$ and $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{F}}, \widetilde{\mathbb{P}}, \widetilde{\mathbb{B}}, Y)$ be two weak solutions of MKV equation (6.2) such that $X_0 = Y_0 = x \in \mathcal{H}$. For the sake of readability we assume x to be the Null element in $\mathcal{H}$ whereas the general case can be shown analogously. Furthermore we denote by $B^{\mathbb{H}}$ and $\widetilde{B}^{\mathbb{H}}$ the cylindrical fractional Brownian motions related to $\mathbb{B}$ and $\widetilde{\mathbb{B}}$, respectively. Lastly, we denote by $\{W^{(k)}\}_{k\geq 1}$ and $\{\widetilde{W}^{(k)}\}_{k\geq 1}$ the generating sequences of Brownian motions of $B^{\mathbb{H}}$ and $\widetilde{B}^{\mathbb{H}}$, respectively.

Due to the proof of Theorem 2.4 and Theorem 3.4 there exist probability measures $\mathbb{Q}$ and $\widetilde{\mathbb{Q}}$ such that X and Y are weighted cylindrical fractional Brownian motions of the form (6.11) under $\mathbb{Q}$ and $\widetilde{\mathbb{Q}}$, respectively. Furthermore, we define the probability measure $\widehat{\mathbb{Q}} \approx \widetilde{\mathbb{P}}$ by

$$\frac{d\widehat{\mathbb{Q}}}{d\widetilde{\mathbb{P}}} := \exp\left\{ -\sum_{k\geq 1} \int_0^t \lambda_k^{-1} \mathbf{K}_{H_k}^{-1} \left(\int_0^\cdot b_k\left(u, Y_u, \widetilde{\mathbb{P}}_{Y_u}\right) - b_k\left(u, Y_u, \mathbb{P}_{X_u}\right) du \right)(s) d\widetilde{W}_s^{(k)} \right.$$
$$\left. -\frac{1}{2} \sum_{k\geq 1} \int_0^t \lambda_k^{-2} \mathbf{K}_{H_k}^{-1} \left(\int_0^\cdot b_k\left(u, Y_u, \widetilde{\mathbb{P}}_{Y_u}\right) - b_k\left(u, Y_u, \mathbb{P}_{X_u}\right) du \right)^2 (s) ds \right\},$$

and the $\widehat{\mathbb{Q}}$ cylindrical fractional Brownian motion

$$\widehat{B}_t^{\mathbb{H}} := \widetilde{B}_t^{\mathbb{H}} + \int_0^t \sqrt{Q}^{-1} \left(b\left(s, Y_s, \widetilde{\mathbb{P}}_{Y_s}\right) - b\left(s, Y_s, \mathbb{P}_{X_s}\right) \right) ds, \quad t \in [0,T].$$

Note that we can find a measurable function $\Phi : [0,T] \times \mathcal{C}([0,T];\mathcal{H}) \to \mathcal{H}$ such that

$$B_t^{\mathbb{H}} = \Phi_t(X) \quad \text{and} \quad \widehat{B}_t^{\mathbb{H}} = \Phi_t(Y),$$

since

$$B_t^{\mathbb{H}} = \sqrt{Q}^{-1} \left(X_t - \int_0^t b\left(s, X_s, \mathbb{P}_{X_s}\right) ds \right), \text{ and}$$
$$\widehat{B}_t^{\mathbb{H}} = \sqrt{Q}^{-1} \left(Y_t - \int_0^t b\left(s, Y_s, \mathbb{P}_{X_s}\right) ds \right).$$

Consequently,

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left[F(B^{\mathbb{H}},X)\right] &= \mathbb{E}_{\mathbb{Q}}\left[\mathcal{E}\left(\int_0^T \sqrt{Q}^{-1} b\left(t,X_t,\mathbb{P}_{X_t}\right)dX_t\right)F(\Phi(X),X)\right]\\
&= \mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\mathcal{E}\left(\int_0^T \sqrt{Q}^{-1} b\left(t,Y_t,\mathbb{P}_{X_t}\right)dY_t\right)F(\Phi(Y),Y)\right]\\
&= \mathbb{E}_{\widehat{\mathbb{Q}}}\left[F(\widehat{B}^{\mathbb{H}},Y)\right],
\end{aligned}$$

for every bounded measurable functional $F:\mathcal{C}([0,T];\mathcal{H})\times\mathcal{C}([0,T];\mathcal{H})\to\mathbb{R}$ and thus $\mathbb{P}_{(B^{\mathbb{H}},X)}=\widehat{\mathbb{Q}}_{(\widehat{B}^{\mathbb{H}},Y)}$. Therefore it is left to show that $\sup_{t\in[0,T]}\mathcal{K}\left(\widehat{\mathbb{Q}}_{Y_t},\widetilde{\mathbb{P}}_{Y_t}\right)=0$ from which we can conclude that $\frac{d\widehat{\mathbb{Q}}}{d\widetilde{\mathbb{P}}}=1$ and in particular that $\mathbb{P}_X=\widetilde{\mathbb{P}}_Y$.

Applying a measure change, inequality (6.19), Burkholder-Davis-Gundy's inequality, and assumption (6.20), yield

$$\begin{aligned}
\mathcal{K}\left(\mathbb{P}_{X_t},\widetilde{\mathbb{P}}_{Y_t}\right) &= \sup_{h\in \mathrm{BL}(\mathcal{H};\mathbb{R})}\left|\mathbb{E}_{\widehat{\mathbb{Q}}}\left[h(Y_t)-h(0)\right]-\mathbb{E}_{\widetilde{\mathbb{P}}}\left[h(Y_t)-h(0)\right]\right|\\
&\leq \sup_{h\in \mathrm{BL}(\mathcal{H};\mathbb{R})}\mathbb{E}_{\widetilde{\mathbb{P}}}\left[\left|\frac{d\widehat{\mathbb{Q}}}{d\widetilde{\mathbb{P}}}-1\right|\left|h\left(Y_t\right)-h(0)\right|\right]\\
&\leq \mathbb{E}_{\widetilde{\mathbb{P}}}\left[\left|\frac{d\widehat{\mathbb{Q}}}{d\widetilde{\mathbb{P}}}-1\right|^2\right]^{\frac12}\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\left(\frac{d\widetilde{\mathbb{P}}}{d\widetilde{\mathbb{Q}}}\right)^2\right]^{\frac14}\mathbb{E}_{\widetilde{\mathbb{Q}}}\left[\left\|\widetilde{B}^{\mathbb{H}}_t\right\|_{\mathcal{H}}^4\right]^{\frac14}\\
&\lesssim \mathbb{E}\left[\left|\sum_{k\geq1}\int_0^t \lambda_k^{-2}\mathbf{K}_{H_k}^{-1}\left(\int_0^{\cdot} b_k\left(u,\mathbb{B}_u,\widetilde{\mathbb{P}}_{Y_u}\right)-b_k\left(u,\mathbb{B}_u,\mathbb{P}_{X_u}\right)du\right)^2(s)ds\right|^2\right]^{\frac14}\\
&\quad+\mathbb{E}\left[\left|\sum_{k\geq1}\int_0^t \lambda_k^{-2}\mathbf{K}_{H_k}^{-1}\left(\int_0^{\cdot} b_k\left(u,\mathbb{B}_u,\widetilde{\mathbb{P}}_{Y_u}\right)-b_k\left(u,\mathbb{B}_u,\mathbb{P}_{X_u}\right)du\right)^2(s)ds\right|^4\right]^{\frac14}=:A.
\end{aligned}$$

Consider first the Brownian case $k\in I_0$. Then, we get

$$\int_0^t \mathbf{K}_{H_k}^{-1}\left(\int_0^{\cdot} b_k\left(u,\mathbb{B}_u,\widetilde{\mathbb{P}}_{Y_u}\right)-b_k\left(u,\mathbb{B}_u,\mathbb{P}_{X_u}\right)du\right)^2(s)ds\leq C_k^2\lambda_k^2\int_0^t\mathcal{K}(\mathbb{P}_{X_s},\widetilde{\mathbb{P}}_{Y_s})^2ds.$$

In the singular case $k\in I_-$, we have

$$\begin{aligned}
&\int_0^t \mathbf{K}_{H_k}^{-1}\left(\int_0^{\cdot} b_k\left(u,\mathbb{B}_u,\widetilde{\mathbb{P}}_{Y_u}\right)-b_k\left(u,\mathbb{B}_u,\mathbb{P}_{X_u}\right)du\right)^2(s)ds\\
&\leq \frac{C_k^2\lambda_k^2}{\Gamma\left(\frac12-H_k\right)^2}\int_0^t s^{2H_k-1}\mathcal{K}(\mathbb{P}_{X_s},\widetilde{\mathbb{P}}_{Y_s})^2\left(\int_0^s (s-u)^{-H_k-\frac12}u^{\frac12-H_k}du\right)^2ds\\
&\leq \frac{C_k^2\lambda_k^2}{\Gamma\left(\frac12-H_k\right)^2}\int_0^t s^{1-2H_k}\mathcal{K}(\mathbb{P}_{X_s},\widetilde{\mathbb{P}}_{Y_s})^2\beta\left(\frac32-H_k,\frac12-H_k\right)^2ds
\end{aligned}$$

$$\leq \frac{C_k^2 \lambda_k^2 T^{1-2H_k} \Gamma\left(\frac{3}{2} - H_k\right)^2}{\Gamma(2-2H_k)^2} \int_0^t \mathcal{K}(\mathbb{P}_{X_s}, \widetilde{\mathbb{P}}_{Y_s})^2 ds$$
$$\lesssim C_k^2 \lambda_k^2 \int_0^t \mathcal{K}(\mathbb{P}_{X_s}, \widetilde{\mathbb{P}}_{Y_s})^2 ds.$$

Lastly we get in the regular case $k \in I_+$ equivalent to (6.18) that

$$\int_0^t \mathbf{K}_{H_k}^{-1} \left(\int_0^{\cdot} b_k\left(u, \mathbb{B}_u, \widetilde{\mathbb{P}}_{Y_u}\right) - b_k\left(u, \mathbb{B}_u, \mathbb{P}_{X_u}\right) du \right)^2 (s) ds$$
$$\lesssim C_k^2 \lambda_k^2 \int_0^t \mathcal{K}\left(\mathbb{P}_{X_s}, \widetilde{\mathbb{P}}_{Y_s}\right)^2 s^{1-2H_k} ds.$$

Using Hölder's inequality with $1 < p < \frac{1}{2\sup_{k \in I_+} H_k - 1}$ and its conjugate $q > 1$ yields

$$\int_0^t \mathcal{K}\left(\mathbb{P}_{X_s}, \widetilde{\mathbb{P}}_{Y_s}\right)^2 s^{1-2H_k} ds$$
$$\leq \left(\int_0^t \mathcal{K}\left(\mathbb{P}_{X_s}, \widetilde{\mathbb{P}}_{Y_s}\right)^{2q} ds \right)^{\frac{1}{q}} \left(\int_0^t s^{p(1-2H_k)} ds \right)^{\frac{1}{p}}$$
$$\leq \left(\int_0^t \mathcal{K}\left(\mathbb{P}_{X_s}, \widetilde{\mathbb{P}}_{Y_s}\right)^{2q} ds \right)^{\frac{1}{q}} \left(\frac{1}{p(1-2H_k)+1} t^{p(1-2H_k)+1} \right)^{\frac{1}{p}}$$
$$\lesssim \left(\int_0^t \mathcal{K}\left(\mathbb{P}_{X_s}, \widetilde{\mathbb{P}}_{Y_s}\right)^{2q} ds \right)^{\frac{1}{q}}.$$

Consequently,

$$\mathcal{K}(\mathbb{P}_{X_t}, \widetilde{\mathbb{P}}_{Y_t}) \lesssim \left(\sum_{k \geq 1} C_k^2 \left(\int_0^t \mathcal{K}\left(\mathbb{P}_{X_s}, \widetilde{\mathbb{P}}_{Y_s}\right)^{2q} ds \right)^{\frac{1}{q}} \right)^{\frac{1}{2}} + \sum_{k \geq 1} C_k^2 \left(\int_0^t \mathcal{K}\left(\mathbb{P}_{X_s}, \widetilde{\mathbb{P}}_{Y_s}\right)^{2q} ds \right)^{\frac{1}{q}}$$
$$\lesssim \left(\int_0^t \mathcal{K}\left(\mathbb{P}_{X_s}, \widetilde{\mathbb{P}}_{Y_s}\right)^{2q} ds \right)^{\frac{1}{2q}} + \left(\int_0^t \mathcal{K}\left(\mathbb{P}_{X_s}, \widetilde{\mathbb{P}}_{Y_s}\right)^{2q} ds \right)^{\frac{1}{q}}.$$

Assume $\int_0^t \mathcal{K}(\mathbb{P}_{X_s}, \widetilde{\mathbb{P}}_{Y_s})^{2q} ds \geq 1$. Then,

$$\mathcal{K}\left(\mathbb{P}_{X_t}, \widetilde{\mathbb{P}}_{Y_t}\right)^q \lesssim \int_0^t \mathcal{K}\left(\mathbb{P}_{X_s}, \widetilde{\mathbb{P}}_{Y_s}\right)^{2q} ds.$$

In the case $0 \leq \int_0^t \mathcal{K}(\mathbb{P}_{X_s}, \widetilde{\mathbb{P}}_{Y_s})^{2q} ds < 1$, we get

$$\mathcal{K}\left(\mathbb{P}_{X_t}, \widetilde{\mathbb{P}}_{Y_t}\right)^{2q} \lesssim \int_0^t \mathcal{K}\left(\mathbb{P}_{X_s}, \widetilde{\mathbb{P}}_{Y_s}\right)^{2q} ds.$$

Next we show that $t \mapsto \mathcal{K}\left(\mathbb{P}_{X_t}, \widetilde{\mathbb{P}}_{Y_t}\right)$ is continuous. Since $t \mapsto X_t$ and $t \mapsto Y_t$ are almost surely continuous, we immediately get that $t \mapsto \mathbb{P}_{X_t}$ and $t \mapsto \widetilde{\mathbb{P}}_{Y_t}$ are weakly continuous. Furthermore, it can be shown as in the proof of Theorem 3.4

that $\{\mathbb{P}_{X_t} : t \in [0,T]\}$ and $\left\{\widetilde{\mathbb{P}}_{Y_t} : t \in [0,T]\right\}$ are relatively compact with respect to the Kantorovich-Rubinstein metric and consequently, that $t \mapsto \mathcal{K}\left(\mathbb{P}_{X_t}, \widetilde{\mathbb{P}}_{Y_t}\right)$ is continuous. Hence, using Grönwall's inequality in the first case and a non-linear Grönwall type inequality by Stachurska [18, Theorem 25] in the second, yields $\mathcal{K}\left(\mathbb{P}_{X_u}, \widetilde{\mathbb{P}}_{Y_t}\right) = 0$ for all $t \in [0,T]$ and thus the proof is complete. □

4. Strong Solutions and Pathwise Uniqueness

In this section we examine under which assumptions MKV equation (6.2) has a pathwisely unique strong solution. Therefore, we first recall the definitions of a strong solution and pathwise uniqueness.

Definition 4.1 A *strong solution* of MKV equation (6.2) is a weak solution $(\Omega, \mathcal{F}, \mathbb{F}^{\mathbb{B}}, \mathbb{P}, \mathbb{B}, X)$ where $\mathbb{F}^{\mathbb{B}}$ is the filtration generated by the weighted cylindrical fractional Brownian motion $\mathbb{B}$ and augmented with the $\mathbb{P}$-null sets.

Definition 4.2 We say a weak solution $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \mathbb{B}, X)$ of MKV equation (6.2) is *pathwisely unique*, if for any other weak solution $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, \mathbb{B}, Y)$ on the same stochastic basis with the same initial condition $X_0 = Y_0$,

$$\mathbb{P}\left(\forall t \geq 0 : X_t = Y_t\right) = 1.$$

Remark 4.3. In the following we speak of a unique solution, if the solution is unique in law and pathwisely unique.

Provided that a weak solution of MKV equation (6.2) exists, the task of proving the existence of a strong solution becomes a problem in the field of SDEs. More precisely, the difference between a weak and a strong solution lies in the measurability with respect to the filtration of the driving noise. Since the dependence on the law is mere deterministic, it does not effect adaptedness of the solution. Therefore, the SDE

$$Y_t = Y_0 + \int_0^t b^{\mathbb{P}_X}(s, Y_s)dt + \mathbb{B}_t, \quad t \in [0,T], \tag{6.21}$$

can be considered, where $b^{\mathbb{P}_X}(s,y) = b\left(s, y, \mathbb{P}_{X_s}\right)$ and $(X_s)_{s\in[0,T]}$ is a weak solution of MKV equation (6.2). For more details on this transition we refer the reader to [8]. Subsequently we give a general result regarding strong solutions of MKV equation (6.2).

Theorem 4.4 *Suppose the assumptions of Theorem 3.4 are fulfilled and SDE* (6.21) *has a unique strong solution* $(Y_t)_{t\in[0,T]}$. *Then, MKV equation* (6.2) *has a strong solution. More precisely, any weak solution* $(X_t)_{t\in[0,T]}$ *of MKV equation* (6.2) *is a strong solution. If in addition* $\sup_{k\in I_+} H_k < 1$ *and condition* (6.20) *is fulfilled, the solution of MKV equation* (6.2) *is unique.*

Proof. Due to Theorem 3.4 there exists a weak solution X of MKV equation (6.2). Moreover, X can be seen as a weak solution of the associated SDE (6.21). Since SDE (6.21) has a unique strong solution Y, i.e. in particular Y is a weak solution which is unique in law, we have that X and Y have the same law. Thus, equations (6.2) and (6.21) coincide and Y is a strong solution of MKV equation (6.2).

Under the additional assumptions $\sup_{k\in I_+} H_k < 1$ and condition (6.20), we know by Theorem 3.8 that the weak solution X of MKV equation (6.2) is unique in law. Consequently, there exists a unique associated SDE (6.21), which has by assumption the unique strong solution Y. In particular, Y is also a strong solution of MKV equation (6.2) due to the first part. Since the associated SDE is uniquely determined, the pathwise uniqueness of a solution to SDE (6.21) transfers to the solution of MKV equation (6.2). Thus, Y is the unique strong solution of MKV equation (6.2). ☐

In the following we link Theorem 4.4 to results in the literature on the existence of strong solutions of SDEs. We start with a corollary in the infinite-dimensional case applying the result of [2]. Subsequently, we consider the finite-dimensional case applying the result of [30].

Corollary 4.5 *Assume* $I_0 \cup I_+ = \emptyset$, $\sum_{k\in I_-} H_k < \frac{1}{6}$, *and* $\sup_{k\in I_-} H_k < \frac{1}{12}$. *Let* $b : [0,T]\times\mathcal{H}\times\mathcal{P}_1(\mathcal{H}) \to \mathcal{H}$ *be a measurable function fulfilling the Lipschitz condition* (6.20) *and for which there exist sequences* $C \in \ell^1$ *and* $D \in \ell^1$ *such that for every* $k \geq 1$

$$\sup_{y\in\mathcal{H}} \sup_{t\in[0,T]} |b_k(t,y,\mu)| \leq C_k\lambda_k, \ \ \textit{and}$$

$$\sup_{d\geq 1} \int_{\mathbb{R}^d} \sup_{t\in[0,T]} |b_k\left(t, \sqrt{Q}\sqrt{\mathcal{K}}\tau^{-1}y, \mu\right)| dy \leq D_k\lambda_k,$$

where $y = (y_1,\ldots,y_d)$ *and* $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ *is defined by*

$$\mathcal{K}x = \sum_{k\geq 1} \mathfrak{K}_{H_k} x^{(k)} e_k, \ \ x \in \mathcal{H},$$

for $\{\mathfrak{K}_{H_k}\}_{k\geq 1}$ *being the local non-determinism constant of* $\{B^{H_k}\}_{k\geq 1}$, *i.e. a constant merely dependent on* H *such that for every* $t \in [0,T]$ *and* $0 < r \leq t$

$$\operatorname{Var}\left(B_t^H \,\middle|\, B_s^H : |t-s| \geq r\right) \geq \mathfrak{K}_H r^{2H}.$$

Then, MKV equation (6.2) *has a Malliavin differentiable unique strong solution.*

Proof. The result is an immediate consequence of Theorem 4.4 and [2, Theorem 4.11]. ☐

Consider now the one-dimensional real-valued MKV equation

$$X_t = x + \int_0^t b\left(s, X_s, \mathbb{P}_{X_s}\right) ds + B_t^H, \quad t \in [0,T], \tag{6.22}$$

where $b : [0,T] \times \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ and B_t^H one-dimensional fractional Brownian motion with Hurst parameter H.

Corollary 4.6 *Let $b : [0,T]\times\mathbb{R}\times\mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ be a bounded measurable function. If $H > 1/2$ suppose that*

$$|b(t,x,\mu) - b(s,y,\nu)| \le C\left(|t-s|^\gamma + |x-y|^\alpha + \mathcal{K}(\mu,\nu)^\beta\right),$$

where $C > 0$, $\gamma > H - \frac{1}{2}$, $2 \ge \alpha > 2H-1$, and $\beta > H - \frac{1}{2}$, and if $H \le 1/2$ suppose that for every $\mu \in \mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}))$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\nu \in \mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}))$

$$\sup_{t\in[0,T]} \mathcal{K}(\mu_t,\nu_t) < \delta \;\Rightarrow\; \sup_{t\in[0,T],\; y\in\mathcal{H}} |b(t,y,\mu_t) - b(t,y,\nu_t)| < \varepsilon.$$

Then, MKV equation (6.22) *has a strong solution. If in addition condition* (6.20) *is fulfilled, the solution is unique.*

Proof. This result is a direct consequence of [30] together with Theorem 3.4 and Theorem 3.8, respectively. □

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